

Symplectic Structures On Normed Spaces And SL(2,R) Action On The Operator Projective Line

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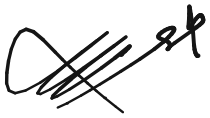
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Introduction

The study of finite-dimensional symplectic spaces plays a significant role in linear algebra [1]. Since functional analysis can be seen as a generalisation of linear algebra to infinite-dimensional vector spaces, it would be interesting to work on an infinite-dimensional version of symplectic theory, see references [8], [10] and [5]. This thesis continues this theme.

In the 19th century, Sophus Lie introduced an important class of groups, which are now called Lie groups, establishing a foundation for the study of continuous transformation groups. A fundamental example is the $\mathrm{SL}(2,\mathbb{R})$ group, consisting of all 2×2 real matrices with determinant equal to one, which plays a central role in many areas of mathematics [20]. In addition, Felix Klein, in his Erlangen programme, proposed that geometry is the study of invariants of a space under a certain transformation group [22].

In the 20th century, functional analysis and some related fields witnessed foundational developments. Around 1909, David Hilbert developed the theory of infinite-dimensional inner product spaces. Later, the concept of an abstract Hilbert space was axiomatically formalised by von Neumann, who also continued to develop operator theory. Within the same century, significant progress in Lie theory and representation theory was achieved through the contributions of Von Neumann and Hermann Weyl. This allows wider classes of mathematical objects, such as functions and operators, to be considered as sets on which groups act. This reflects the philosophy (attributed to Israel Gelfand) that many mathematical objects can be understood through their representation in another space.

More recently, Vladimir Kisil systematically applied this philosophy to analytic functions, functional calculi and relevant spectra and called it the Erlangen programme at Large [20].

Consider the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}$. Any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ defines the Mobius transformation given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : x \mapsto \frac{ax+b}{cx+d} \quad \text{for all} \quad x \in \mathbb{R}. \tag{0.0.1}$$

One immediately addresses an issue when $cx + d = 0$. A naive solution [21] is to extend the real numbers by adding infinity with the following rules:

$$\frac{x}{0} = \infty, \qquad \frac{x}{\infty} = 0, \qquad \frac{\infty}{\infty} = 1, \qquad \infty + x = \infty \qquad \text{for any} \qquad x \in \mathbb{R}\backslash\{0\}.$$

The extended space $\mathbb{R} \cup \{\infty\}$ is represented by the real projective line $P\mathbb{R}$ whose elements are viewed as geometric objects, cf. section 4.1 below.

The issue becomes further complicated when we take such an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ and attempt to define the Mobius transformation on a set of linear operators acting on a Hilbert

space via the formula:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : T \mapsto (aT + b)(cT + d)^{-1}. \tag{0.0.2}$$

In this case, we also encounter an issue when $c \neq 0$ and $-\frac{d}{c}$ is in the spectrum of T. Clearly, imposing the assumption that $-\frac{d}{c}$ is excluded from the spectra of all operators makes the situation overly restrictive. For this reason, an alternative approach is needed to handle this issue.

In this thesis, we will use the symplectic structure to provide a solution to this issue by constructing an extension of the real projective line $P\mathbb{R}$ to an operator setting, call it the operator projective line.

The purpose of this thesis is to introduce:

- An overview of symplectic spaces of finite or infinite dimensionality.
- The concept of the operator projective line.
- The $\mathrm{SL}(2,\mathbb{R})$ action on the operator projective line.
- A connection between the $\mathrm{SL}(2,\mathbb{R})$ action on the operator projective line and spectral properties.

Through these objectives, we establish a consistent and rigorous framework for Mobius transformations of operators to handle the formula (0.0.2) even when $-\frac{d}{c}$ is in the spectrum of the operator.

Broadly speaking, the approach is to extend the concept of an operator in such a way that its graph can be represented by an element of the operator projective line. Then, $\mathrm{SL}(2,\mathbb{R})$ acts on such operators through their graph.

This thesis is organised as follows. Chapter 1 develops the theory of symplectic Banach spaces of either even finite or infinite dimension. In particular, we will emphasise the infinite-dimensional case as it has received less attention in the literature than the finite-dimensional one. This establishes the necessary background and structural results that will be used throughout the thesis. Chapter 2 introduces general results on group actions as a foundational tool for the later chapters. Chapter 3 is devoted to $\mathrm{SL}(2,\mathbb{R})$ group, its properties and its subgroups since the $\mathrm{SL}(2,\mathbb{R})$ action will play a central role in the main results of this thesis. In Chapter 4, we will describe how $\mathrm{SL}(2,\mathbb{R})$ acts on the real projective line $P\mathbb{R}$ and Hilbert spaces. In Chapter 5, we will define the concept of a bi-equidefect operator and demonstrate its connection with the Lagrangian subspaces. In particular, we will show that the graph of a bi-equidefect operator can be represented by a Lagrangian subspace. This connection will establish a consistent framework to define $\mathrm{SL}(2,\mathbb{R})$ action on bi-equidefect operators. In the last chapter, we will introduce the main contribution of this thesis: the concept of the operator projective line. Then, we will proceed to define $\mathrm{SL}(2,\mathbb{R})$ action on bi-equidefect operators and its connection with their spectrum.

Chapter 1

Symplectic Spaces

In this chapter, we will discuss general results about symplectic spaces of even finite and infinite dimensions. Throughout this chapter, V denotes a real normed space, and V^* is its dual space unless otherwise stated.

1.1 Symplectic Form

Definition 1.1.1. *[8, p.2] A symplectic form ω is a bilinear map $V \times V \to \mathbb{R}$ satisfying the following conditions:*

(i) It is bounded; that is there exists $C > 0$ such that $|\omega(x, y)| \leqslant C\|x\|\|y\|$ for every $x, y \in V$,

(ii) It is skew-symmetric; that is, $\omega(x, y) = -\omega(y, x)$ for every $x, y \in V$,

(iii) It is weakly nondegenerate; that is, $\omega(x, y) = 0, \forall y \in V \implies x = 0$.

The pair (V, ω) is a symplectic space.

Remark 1.1.2. *Since ω is skew-symmetric, then for every $x \in V$ we have $\omega(x, x) = 0$. This is because for any $x \in V$, we have*

$$\omega(x, x) = -\omega(x, x) \implies \omega(x, x) = 0.$$

The symplectic form ω induces a bounded linear map [10, p.3]

$$L_\omega : V \to V^*$$

$$x \mapsto L_\omega(x)$$

such that

$$L_\omega(x)(y) = \omega(x, y) \quad \text{for all} \quad x, y \in V.$$

This map is called the induced map of ω.

Note that weak nondegeneracy of symplectic form ω is equivalent to that its induced map L_ω being injective because the following statements are equivalent:

- $\ker(L_\omega) = \{0\}$,
- $L_\omega(x)(y) = 0, \forall y \in V \implies x = 0$,

- $\omega(x, y) = 0, \forall y \in V \implies x = 0.$

Moreover, if L_ω is bijective, the symplectic form ω is said to be strongly nondegenerate.

It is easy to see that if a symplectic form ω is strongly nondegenerate, then it will also be weakly nondegenerate. However, in this thesis, whenever we say weakly nondegenerate, we mean that the induced map $L_\omega : V \to V^*$ is only injective and not surjective. Also, in cases where the distinction between weak nondegeneracy and strong nondegeneracy is unnecessary, we might shortly say ω is nondegenerate.

Since the map $L_\omega^{-1} : V^* \to V$ exists in case of strongly nondegenerate symplectic form ω, this pushes us to investigate whether L_ω^{-1} is bounded or not. The next theorem and its corollary give us the answer.

Theorem 1.1.3. *If (V, ω) is a real symplectic Banach space and ω is strongly nondegenerate, then the induced map $L_\omega : V \to V^*$ is open.*

Proof. Let U be an open subset of V, and we need to show that $L_\omega(U)$ is an open subset of V^*. To this end, assume that $y \in L_\omega(U)$. Thus, we can find $x \in U$ so that $y = L_\omega(x)$. Since U is open, we can find a ball centred at x with radius $r > 0$ such that $B(x, r) \subset U$. Since ω is strongly nondegenerate, then L_ω is surjective. Consequently, by applying the open mapping Theorem [5, p.35], there exists $R > 0$ such that

$$B_{V^*}(0, R) \subset L_\omega(B_V(0, 1)). \tag{1.1.1}$$

Hence

$$B_{V^*}(y, rR) \subset L_\omega(B_V(x, r)). \tag{1.1.2}$$

Therefore, $B_{V^*}(y, rR)$ is an open ball centered at y and contained in $L_\omega(U)$, and this implies $L_\omega(U)$ is open. □

Corollary 1.1.4. *Let (V, ω) be a real symplectic Banach space, ω be strongly nondegenerate and*

$$L_\omega : V \to V^*; \qquad L_\omega(x)(y) = \omega(x, y) \quad \textit{for all} \quad x, y \in V$$

be the induced map of the symplectic form ω. Then, the inverse of this induced map $L_\omega^{-1} : V^* \to V$ is bounded.

Proof. By ((1.1.1)), we can find $R > 0$ such that for every $x \in V$,

$$\|L_\omega(x)\| < R \implies \|x\| < 1.$$

By putting $y = L_\omega(x)$, we obtain

$$\|y\| < R \implies \|L_\omega^{-1}(y)\| < 1.$$

If we put $y' = \frac{2}{R}y$, then

$$\|y'\| < 2 \implies \|L_\omega^{-1}(y')\| < \frac{2}{R}.$$

This implies

$$\|L_\omega^{-1}\| = \sup_{\|y'\|=1} \|L_\omega^{-1}(y')\| < \frac{2}{R}$$

which means L_ω^{-1} is bounded. □

If (V,ω) is even finite-dimensional, then the induced map $L_\omega : V \to V^*$ is always bijective since $V \cong V^*$ always holds. This means the symplectic form ω is always strongly nondegenerate. For this reason, the distinction between weak and strong nondegeneracy will only be considered if the space (V,ω) is infinite-dimensional. As a result, we will have three cases. Firstly, an even finite-dimensional space. Secondly, an infinite-dimensional space with a weakly nondegenerate symplectic form. Thirdly, an infinite-dimensional space with a strongly nondegenerate symplectic form. It might be useful to introduce three examples, one for each case.

Example 1.1.5. *Consider $(\mathbb{R}^{2n},\omega)$ with a basis $S = \bigcup_{i=1}^{n}\{e_i, e_{n+i}\}$ where e_i and e_{n+i} are the vectors whose components are 1 in the i-th position (respectively, the $(n+i)$-th position) and zeroes elsewhere. The symplectic form is defined as*

$$\omega(e_i,e_j) = \omega(e_{n+i},e_{n+j}) = 0 \quad \text{and} \quad \omega(e_i,e_{j+n}) = \delta_{ij} \quad \text{where} \quad i,j = 1,2,\ldots,n. \tag{1.1.3}$$

Remark 1.1.6. *The above example is a typical example of a real even finite-dimensional symplectic space because, as we will see later, any $2n$-dimensional symplectic space equipped with a certain symplectic form ω looks a lot like $(\mathbb{R}^{2n},\omega)$, nothing more, cf Corollary 1.3.9 below. Therefore, an infinite-dimensional symplectic structure will be more interesting. For this reason, our investigation in this chapter will focus more on the infinite-dimensional case than the even finite-dimensional one.*

Example 1.1.7. *Consider the sequence space (ℓ^1,ω) where*

$$\ell^1 = \left\{ x = (x_k)_{k\in\mathbb{N}} \mid x_k \in \mathbb{R},\ \forall k \in \mathbb{N} \ \text{ and } \ \|x\|_1 = \sum_{k=1}^{\infty} |x_k| < \infty \right\},$$

$$\omega(x,y) = \sum_{k=1}^{\infty} (x_{2k-1}y_{2k} - y_{2k-1}x_{2k}),$$

and $x = (x_k)_k$, $y = (y_k)_k \in \ell^1$. We can verify the conditions of the symplectic form ω as follows:

- *ω is bounded since*

$$\begin{aligned}
|\omega(x,y)| &\leqslant \sum_{k=1}^{\infty} (|x_{2k-1}y_{2k}| + |y_{2k-1}x_{2k}|) \\
&= \sum_{k=1}^{\infty} |x_{2k-1}y_{2k}| + \sum_{k=1}^{\infty} |y_{2k-1}x_{2k}| \\
&\leqslant \left(\sum_{k=1}^{\infty} |x_{2k-1}|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |y_{2k}|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} |y_{2k-1}|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |x_{2k}|^2\right)^{\frac{1}{2}} \\
&\leqslant 2\left(\sum_{k=1}^{\infty} |x_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |y_k|^2\right)^{\frac{1}{2}} \\
&= 2\|x\|_2\|y\|_2 \\
&\leqslant 2\|x\|_1\|y\|_1,
\end{aligned}$$

- *it is skew-symmetric since*

$$\omega(x,y) = \sum_{k=1}^{\infty} (x_{2k-1}y_{2k} - y_{2k-1}x_{2k}) = -\sum_{k=1}^{\infty} (y_{2k-1}x_{2k} - x_{2k-1}y_{2k}) = -\omega(y,x).$$

- *it is weakly nondegenerate since*

$$\begin{aligned}\omega(x,y)=0,\ \forall y\in\ell^1 &\implies \sum_{k=1}^{\infty}(x_{2k-1}y_{2k}-y_{2k-1}x_{2k})=0,\ \forall y\in\ell^1\\ &\implies \sum_{k=1}^{\infty}\frac{|x_k|}{k^2}=0,\\ &\text{by choosing}\quad y_n=\begin{cases}\frac{-1}{n^2}\frac{|x_{n+1}|}{x_{n+1}} & \text{if}\quad n=2k-1\ \text{and}\ x_{n+1}\neq 0\\ \frac{1}{n^2}\frac{|x_{n-1}|}{x_{n-1}} & \text{if}\quad n=2k\ \text{and}\ x_{n-1}\neq 0\\ 0 & \text{if}\quad \text{otherwise}\end{cases}\\ &\implies x=0.\end{aligned}$$

To show that ω is not strongly nondegenerate, we need to prove that there is no bounded bijection from ℓ^1 to $(\ell^1)^*$. To this end, let us assume, by contradiction, that $L:\ell^1\to(\ell^1)^*$ is a bounded bijection. Now, let $\epsilon>0$. Since ℓ^1 is separable, then there exists a dense countable subset $A\subset\ell^1$; that is,

$$\forall x\in\ell^1,\exists a\in A\quad \text{such that}\quad \|x-a\|<\frac{\epsilon}{\|L\|}.$$

Therefore, $L(A)$ will be a countable subset of $(\ell^1)^*$ as well. Moreover, from the surjectivity of L, we have

$$\forall y\in(\ell^1)^*,\exists x\in\ell^1\quad \text{such that}\quad L(x)=y.$$

Hence

$$\forall y\in(\ell^1)^*,\exists a\in A\quad \text{such that}\quad \|y-L(a)\|=\|L(x-a)\|\leqslant\|L\|\|x-a\|<\|L\|\frac{\epsilon}{\|L\|}=\epsilon.$$

This means that the countable subset $L(A)$ is dense in $(\ell^1)^*$. This implies $(\ell^1)^*$ is separable, which is a contradiction because

$$(\ell^1)^*\cong\ell^\infty=\left\{x=(x_k)_{k\in\mathbb{N}}\mid x_k\in\mathbb{R},\ \forall k\in\mathbb{N}\ \text{ and }\ \|x\|_\infty=\sup_{k\in\mathbb{N}}|x_k|<\infty\right\}.$$

Note that this contradiction follows from the surjectivity of L. Therefore, we conclude that there is no bounded bijection from ℓ^1 to $(\ell^1)^*$, as required.

Moreover, $L_\omega:\ell^1\to\ell^\infty$ cannot be an open map. To prove that, assume, by contradiction, that it is open. Thus, $L_\omega(\ell^1)$ is open in $(\ell^\infty,\|\cdot\|_\infty)$ because ℓ^1 is open in $(\ell^1,\|\cdot\|_1)$. This implies the existence of an open ball centred at 0 with radius $R>0$

$$B_{\ell^\infty}(0,R)\subseteq L_\omega(\ell^1). \tag{1.1.4}$$

We claim that this leads to

$$\operatorname{Clin}\left(B_{\ell^\infty}(0,R)\right)=\ell^\infty. \tag{1.1.5}$$

To prove this claim, we already have

$$\operatorname{Clin}\left(B_{\ell^\infty}(0,R)\right)\subseteq\ell^\infty.$$

For the other direction, suppose that a nonzero vector $x\in\ell^\infty$. Then, by scalling x, we obtain

$$\left\|\frac{Rx}{2\|x\|_\infty}\right\|_\infty<R.$$

Hence

$$\frac{Rx}{2\|x\|_\infty} \in B_{\ell^\infty}(0,R) \implies x \in \mathrm{Clin}\left(B_{\ell^\infty}(0,R)\right),$$

which implies

$$\ell^\infty \subseteq \mathrm{Clin}\left(B_{\ell^\infty}(0,R)\right).$$

This proves the claim (1.1.5). But, the line (1.1.4) with (1.1.5) leads to that

$$L_\omega(\ell^1) = \ell^\infty,$$

which means that L_ω is surjective. This contradicts what we showed above. Therefore, we conclude that L_ω cannot be open.

Next, let us see an example of an infinite-dimensional symplectic space endowed with a strongly nondegenerate symplectic form.

Example 1.1.8. *Consider (ℓ^2, ω) where*

$$\ell^2 = \left\{ x = (x_k)_{k\in\mathbb{N}} \mid x_k \in \mathbb{R},\ \forall k \in \mathbb{N} \ \text{ and } \ \|x\|_2^2 = \sum_{k=1}^{\infty} |x_k|^2 < \infty \right\},$$

$$\omega(x,y) = \sum_{k=1}^{\infty} (x_{2k-1}y_{2k} - y_{2k-1}x_{2k}),$$

and $x = (x_k)_k$, $y = (y_k)_k \in \ell^2$. We can verify the conditions of the symplectic form ω as follows:

- *ω is bounded since*

$$\begin{aligned}
|\omega(x,y)| &\leqslant \sum_{k=1}^{\infty} \left(|x_{2k-1}y_{2k}| + |y_{2k-1}x_{2k}|\right) \\
&= \sum_{k=1}^{\infty} |x_{2k-1}y_{2k}| + \sum_{k=1}^{\infty} |x_{2k}y_{2k-1}| \\
&\leqslant \left(\sum_{k=1}^{\infty} |x_{2k-1}|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |y_{2k}|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty} |x_{2k}|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |y_{2k-1}|^2\right)^{\frac{1}{2}} \\
&\leqslant 2\left(\sum_{k=1}^{\infty} |x_k|^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |y_k|^2\right)^{\frac{1}{2}} \\
&= 2\|x\|_2\|y\|_2,
\end{aligned}$$

- *it is skew-symmetric since*

$$\omega(x,y) = \sum_{k=1}^{\infty} (x_{2k-1}y_{2k} - y_{2k-1}x_{2k}) = -\sum_{k=1}^{\infty} (y_{2k-1}x_{2k} - x_{2k-1}y_{2k}) = -\omega(y,x),$$

- *it is weakly nondegenerate since*

$$\begin{aligned}\omega(x,y)=0,\ \forall y\in\ell^2 &\implies \sum_{k=1}^{\infty}(x_{2k-1}y_{2k}-y_{2k-1}x_{2k})=0,\ \forall y\in\ell^2\\ &\implies \sum_{k=1}^{\infty}\frac{|x_k|}{k}=0,\\ &\textit{by choosing}\quad y_n=\begin{cases}\frac{-1}{n}\frac{|x_{n+1}|}{x_{n+1}} & \textit{if}\ \ n=2k-1\ \textit{and}\ x_{n+1}\neq 0\\ \frac{1}{n}\frac{|x_{n-1}|}{x_{n-1}} & \textit{if}\ \ n=2k\ \textit{and}\ x_{n-1}\neq 0\\ 0 & \textit{if}\quad \textit{otherwise}\end{cases}\\ &\implies x=0\end{aligned}$$

To induce the map $L_\omega:\ell^2\to(\ell^2)^*$*, we can write* $x\in\ell^2$ *in the form*

$$x=(x_1,x_2,x_3,\dots)\quad \textit{where}\quad x_i\in\mathbb{R},\forall i\in\mathbb{N}.$$

Also, if we consider a decomposition

$$\ell^2=M\oplus M^\perp$$

where

$$M=\{(x_n)_n\in\ell^2\mid x_{2k}=0,\ \forall k\in\mathbb{N}\}\quad \textit{and}\quad M^\perp=\{(x_n)_n\in\ell^2\mid x_{2k-1}=0,\ \forall k\in\mathbb{N}\},$$

then each vector x *in* ℓ^2 *can be written as*

$$x=(x_M,x'_M)$$

where

$$x_M=P_Mx=(x_1,0,x_3,0,\dots)\in M\quad \textit{and}\quad x'_M=P_{M^\perp}x=(0,x_2,0,x_4,\dots)\in M^\perp.$$

Let $R:M\to M^\perp$ *and* $L:M^\perp\to M$ *be the right shift operator and the left shift operator, respectively. Thus,*

$$\begin{aligned}L_\omega(x)(y)&=\langle(-x_2,x_1,-x_4,x_3,\dots),(y_1,y_2,y_3,y_4,\dots)\rangle\\ &=\langle(0,x_1,0,x_3,\dots),(0,y_2,0,y_4,\dots)\rangle_{M^\perp}-\langle(x_2,0,x_4,0,\dots),(y_1,0,y_3,0,\dots)\rangle_M\\ &=\langle P_{M^\perp}Rx,P_{M^\perp}y\rangle_{M^\perp}-\langle P_MLx,P_My\rangle_M\\ &=\left\langle Rx_M,y'_M\right\rangle_{M^\perp}-\left\langle Lx'_M,y_M\right\rangle_M\\ &=\left\langle\begin{pmatrix}-Lx'_M\\ Rx_M\end{pmatrix},\begin{pmatrix}y_M\\ y'_M\end{pmatrix}\right\rangle\\ &=\left\langle\begin{pmatrix}0 & -L\\ R & 0\end{pmatrix}\begin{pmatrix}x_M\\ x'_M\end{pmatrix},\begin{pmatrix}y_M\\ y'_M\end{pmatrix}\right\rangle\\ &=\langle Tx,y\rangle\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ *is the standard inner product,* $\langle\cdot,\cdot\rangle_M$ *and* $\langle\cdot,\cdot\rangle_{M^\perp}$ *are its restrictions to* M *and* $M^\perp$*, respectively, and*

$$T:\ell^2\to\ell^2$$
$$T\begin{pmatrix}x_M\\ x'_M\end{pmatrix}=\begin{pmatrix}0 & -L\\ R & 0\end{pmatrix}\begin{pmatrix}x_M\\ x'_M\end{pmatrix}.\tag{1.1.6}$$

The injectivity of L_ω was proved above from the weak nondegeneracy of ω, while its surjectivity follows from the Riesz representation Theorem and invertibility of T. To elaborate further, assume that $\phi \in (\ell^2)^*$ and then, by Riesz representation Theorem, there exists $x \in \ell^2$ such that

$$\phi(y) = \langle Tx, y\rangle \quad \text{for every} \quad y \in \ell^2. \tag{1.1.7}$$

Thus, from the above argument,

$$\phi(y) = L_\omega(x)(y) \quad \text{for every} \quad y \in \ell^2, \tag{1.1.8}$$

which means L_ω is surjective. Therefore, ω is strongly nondegenerate.

Remark 1.1.9. *In the example 1.1.8, the relation (1.1.7) and (1.1.8) give us*

$$\langle Tx, y\rangle = \omega(x, y) \quad \text{for every} \quad y \in \ell^2 \tag{1.1.9}$$

where the operator T is defined in (1.1.6). Later, we will see that the identity (1.1.9) is true for every real Hilbert space V, not just for ℓ^2. In fact, the operator $T : V \to V$ is significant in establishing the connection between Hilbert and symplectic structures, cf. Section 1.5 below.

Example 1.1.10. *Let V be a real reflexive Banach space and V^* be its dual space. Consider the space $(V \oplus V^*, \|\cdot\|_V + \|\cdot\|_{V^*})$ where $\|\cdot\|_V$ and $\|\cdot\|_{V^*}$ are the norms of V and V^* respectively. Also, let the space $V \oplus V^*$ be equipped with the symplectic form*

$$\omega((x, f), (x', f')) = f'(x) - f(x') \quad \text{for every} \quad x, x' \in V,\ f, f' \in V^*.$$

We can verify the conditions of the symplectic form as follows:

- *ω is bounded since*

$$\begin{aligned} |\omega((x, f), (x', f'))| &= |f'(x) - f(x')| \\ &\leqslant |f'(x)| + |f(x')| \\ &\leqslant \|f'\|_{V^*}\|x\|_V + \|f\|_{V^*}\|x'\|_V \\ &\leqslant \|f\|_{V^*}\|f'\|_{V^*} + \|f\|_{V^*}\|x'\|_V + \|x\|_V\|f'\|_{V^*} + \|x\|_V\|x'\|_V \\ &\leqslant (\|f\|_{V^*} + \|x\|_V)\left(\|f'\|_{V^*} + \|x'\|_V\right). \end{aligned}$$

- *It is skew-symmetric since*

$$\omega((x, f), (x', f')) = f'(x) - f(x') = -\left(f(x') - f'(x)\right) = -\omega((x', f'), (x, f)),$$

- *To prove that it is weakly nondegenerate, we need to prove that the following holds:*

$$\omega((x, f), (x', f')) = 0,\ \forall (x', f') \in V \oplus V^* \implies (x, f) = (0, 0).$$

  or, equivalently and more practically,

  (i) $\omega((x, f), (0, f')) = f'(x) = 0,\ \forall f' \in V^* \implies x = 0.$

  (ii) $\omega((x, f), (x', 0)) = -f(x') = 0,\ \forall x' \in V \implies f = 0.$

  Observe that the implication (i) holds because V^* separates the vectors of V. Also, for the implication (ii), it is straightforward. This shows that the form ω is weakly nondegenerate.

Furthermore, since V is reflexive, then the form ω is strongly nondegenerate. To clarify this, note that

$$(V \oplus V^*)^* \cong V^* \oplus V^{**}.$$

This means every $\phi \in (V \oplus V^*)^*$ can be written as

$$\phi : V \oplus V^* \to \mathbb{R},\ \phi(x,f)_{(g,\hat{y})} = g(x) + \hat{y}(f) \ \text{ for every } \ (x,f) \in V \oplus V^*$$

where $g \in V^*$ and $\hat{y} \in V^{**}$. Now, from reflexivity of V, we have

$$V^{**} \cong V.$$

That is, every $\hat{y} \in V^{**}$ corresponds to a unique $y \in V$ such that

$$\hat{y}(f) = f(y) \ \text{ for every } \ f \in V^*.$$

Thus, the above $\phi \in (V \oplus V^*)^*$ can be written as

$$\phi(x,f)_{(y,g)} = g(x) + f(y) \ \text{ for every } \ (x,f),(y,g) \in V \oplus V^*.$$

By putting

$$g = f' \quad \text{and} \quad y = -x',$$

we obtain

$$\begin{aligned} \phi(x,f)_{(-x',f')} &= f'(x) + f(-x') \\ &= f'(x) - f(x') \\ &= L_\omega(x,f)(x',f'). \end{aligned}$$

Therefore, the induced map L_ω is bijective.

In conclusion, the form ω is a strongly nondegenerate symplectic form.

Theorem 1.1.11. *If (V,ω) is a real symplectic space and $x,y \in V$ are nonzero vectors satisfying $\omega(x,y) \neq 0$, then x and y are linearly independent.*

Proof. Assume that x and y are nonzero vectors in V satisfying $\omega(x,y) \neq 0$. By contradiction, if x and y are dependent, then there exists a nonzero scalar λ such that $y = \lambda x$. Hence,

$$\omega(x,y) = \omega(x,\lambda x) = \lambda\omega(x,x) = 0,$$

which contradicts the assumption that $\omega(x,y) \neq 0$. □

Remark 1.1.12. *The converse of Theorem 1.1.11 is not true; that is, we can find two independent vectors $x,y \in V$ such that $\omega(x,y) = 0$. See Example 1.1.5 above.*

1.2 Symplectic Complement and Subspaces of a Symplectic Space

After we introduce the concept of a symplectic space, there is an important type of subspaces that look a lot like the orthogonal complements in Hilbert spaces. However, as we will see shortly, this type of subspace has some properties that differ from those of orthogonal complements.

Definition 1.2.1 (Symplectic complement). *[8, p.4] Let M be a subspace of a real symplectic space V endowed with symplectic form ω. A subspace*

$$M^\omega = \{x \in V \mid \omega(x, y) = 0,\ \forall y \in M\}$$

is called a symplectic complement of M.

Note that if ω is strongly nondegenerate, then we can apply the induced map L_ω on symplectic space V to obtain a subspace of V^* corresponding to the aforementioned symplectic complement M^ω:

$$\{x^* \in V^* \mid x^*(y) = 0,\ \forall y \in M\}.$$

This subspace is called the annihilator of M, denoted $ann(M)$. Since ω is strongly nondegenerate (L_ω is bijective), the above annihilator ann(M) will be isomorphic to the symplectic complement M^ω. However, the following example demonstrates that if ω is weakly nondegenerate, the annihilator and symplectic complement need not be isomorphic.

Example 1.2.2. *Consider the sequence space*

$$\ell^1 = \left\{ x = (x_n)_{n\in\mathbb{N}} \mid x_n \in \mathbb{R}, \forall \in \mathbb{N} \ \text{ and } \ \sum_{n\in\mathbb{N}} |x_n| < \infty \right\}$$

with the symplectic form

$$\omega(x, y) = \sum_{k=1}^{\infty} (x_{2k-1}y_{2k} - x_{2k}y_{2k-1})$$

where $x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in \ell^1$. Then, the induced linear map will be

$$L_\omega : \ell^1 \to (\ell^1)^*$$

where

$$L_\omega(x)(y) = \omega(x, y) \quad \text{for all} \quad x, y \in \ell^1.$$

Now, if we consider the subspace

$$L = \{x = (x_n)_n \in \ell^1 \mid x_{2k} = 0,\ \forall k \in \mathbb{N}\},$$

then

$$\begin{aligned}
L_\omega(L^\omega) &= \{L_\omega(x) \mid x \in L^\omega\} \\
&= \{L_\omega(x) \mid x \in \ell^1 \ \text{ with } \ L_\omega(x)(y) = 0, \forall (y_n)_n \in L\} \\
&= \left\{ L_\omega(x) \mid x \in \ell^1 \ \text{ with } \ \sum_{k=1}^{\infty} (x_{2k-1}y_{2k} - x_{2k}y_{2k-1}) = 0, \forall (y_n)_n \in L \right\} \\
&= \left\{ L_\omega(x) \mid x \in \ell^1 \ \text{ with } \ \sum_{k=1}^{\infty} x_{2k}y_{2k-1} = 0, \forall (y_n)_n \in L \right\}.
\end{aligned}$$

Also, since

$$(\ell^1)^* = \ell^\infty,$$

then

$$\begin{aligned}
\text{ann}(L) &= \{\phi_x \in (\ell^1)^* \mid x = (x_n)_n \in \ell^\infty,\ \phi_x(y) = 0,\ \forall y \in L\} \\
&= \left\{ \phi_x \in (\ell^1)^* \mid x = (x_n)_n \in \ell^\infty,\ \phi_x(y) = \sum_{n=1}^{\infty} x_n y_n = 0,\ \forall y = (y_n)_n \in L \right\}
\end{aligned}$$

Therefore, the annihilator $\operatorname{ann}(L)$ *is larger than* $L_\omega(L^\omega)$*. For instance, if we put*

$$x_0 = (0, 1, 0, 1, \dots),$$

then

$$\phi_{x_0} \in \operatorname{ann}(L)$$

since

$$\phi_{x_0}(y) = \sum_{n=1}^{\infty} x_n y_n = 0, \quad \forall y = (y_n)_n \in L.$$

But

$$\phi_{x_0} \notin L_\omega(L^\omega).$$

This means that the annihilator $\operatorname{ann}(L)$ *and the symplectic complement* L^ω *are not isomorphic.*

Theorem 1.2.3. *Let* M *be a subspace of a real symplectic space* V *endowed with symplectic form* ω*. The above symplectic complement* M^ω *is a closed subspace of* V*.*

Proof. Let $(x_n)_n$ be a convergent sequence in M^ω and $\lim_{n\to\infty} x_n = x$. Since the map ω is bounded, then there exists a positive number C such that for every $x, y \in V$

$$|\omega(x, y)| \leqslant C\|x\|\|y\|.$$

Thus, for every $y \in M$,

$$\begin{aligned} |\omega(x, y)| &= |\omega(x, y) - \omega(x_n, y)| \\ &\leqslant |\omega(x - x_n, y)| \\ &\leqslant C\|x - x_n\|\|y\| \end{aligned}$$

Take $n \to \infty$ to obtain $\omega(x, y) = 0$ for all $y \in M$ and this implies $x \in M^\omega$.

□

One of the main differences between the orthogonal complement and the symplectic complement is that the symplectic complement does not necessarily satisfy

$$M \cap M^\omega = \{0\}.$$

Therefore, we will obtain four special cases (see the definition below).

Definition 1.2.4. *[28, p.6] Let* (V, ω) *be a real symplectic space, and* M *be a subspace of* V*. The subspace* M *is called*

(i) symplectic if $M \cap M^\omega = \{0\}$*,*

(ii) isotropic if $M \subseteq M^\omega$*,*

(iii) Lagrangian if $M = M^\omega$*,*

(iv) co-isotropic if $M^\omega \subseteq M$*.*

Theorem 1.2.5. *If* M *is a Lagrangian subspace of a real symplectic space* (V, ω)*, then it is closed.*

Proof. Since M is Lagrangian, it satisfies $M = M^\omega$. By Theorem 1.2.3, M is closed. □

Theorem 1.2.7 below provides us with some common properties between symplectic and orthogonal complements. However, we first need the following lemma as a helpful preliminary to prove the properties that require the symplectic form to be strongly nondegenerate.

Lemma 1.2.6. *If (V,ω) is an even finite-dimensional or separable infinite-dimensional real symplectic Banach space with a strongly nondegenerate symplectic form ω, M is a subspace of V, and $x_0 \notin \overline{M}$, then the form ω separates the vector x_0 from the subspace M; that is, there exists $y_0 \in V$ such that*

$$\omega(y_0, x_0) \neq 0 \quad \textit{and} \quad \omega(y_0, m) = 0, \ \forall m \in \overline{M}.$$

Proof. Consider the induced map of ω

$$L_\omega : V \to V^*$$

$$x \mapsto L_\omega(x).$$

Since ω is strongly nondegenerate, then $L_\omega : V \to V^*$ is bijective. In addition, since $x_0 \notin \overline{M}$, then, from [31, Theorem 3.5, p.60], there exists $\phi \in V^*$ such that

$$\phi(x_0) \neq 0 \quad \text{and} \quad \phi(m) = 0, \ \forall m \in \overline{M}.$$

Now, from bijectivity of $L_\omega : V \to V^*$, there exists $y_0 \in V$ such that

$$\omega(y_0, \cdot) = L_\omega(y_0) = \phi.$$

Consequently, we obtain

$$\omega(y_0, x_0) = \phi(x_0) \neq 0 \quad \text{and} \quad \omega(y_0, m) = \phi(m) = 0, \ \forall m \in \overline{M}.$$

This completes the proof.

□

Theorem 1.2.7. *Let (V,ω) be an even finite-dimensional or separable infinite-dimensional real symplectic Banach space, and let M and N be subspaces of V. Thus, we have the following statements hold:*

(i) if $M \subseteq N$, then $N^\omega \subseteq M^\omega$,

(ii) $M^\omega \cap N^\omega = (M+N)^\omega$,

(iii) if ω is strongly nondegenerate, then $(M^\omega)^\omega = \overline{M}$

(iv) if ω is strongly nondegenerate and M, N are closed, then $\overline{M^\omega + N^\omega} = (M \cap N)^\omega$.

(v) symplectic complement of M and symplectic complement of its closure $\overline{M}$ are the same; that is, $M^\omega = (\overline{M})^\omega$.

Proof. (i) Assume that $M \subseteq N$. Then

$$\begin{aligned} x \in N^\omega &\implies \omega(x,y) = 0, \ \forall y \in N \\ &\implies \omega(x,y) = 0, \ \forall y \in M \quad \text{since} \quad M \subseteq N \\ &\implies x \in M^\omega. \end{aligned}$$

Therefore,

$$N^\omega \subseteq M^\omega.$$

(ii) On one hand,

$$\begin{aligned} x \in M^\omega \cap N^\omega &\implies \omega(x, y_1) = \omega(x, y_2) = 0, \ \forall y_1 \in M, \ \forall y_2 \in N \\ &\implies \omega(x, y_1 + y_2) = 0, \ \forall y_1 + y_2 \in M + N \\ &\implies x \in (M + N)^\omega. \end{aligned}$$

Therefore,

$$M^\omega \cap N^\omega \subseteq (M + N)^\omega.$$

On the other hand,

$$\begin{aligned} x \in (M + N)^\omega &\implies \omega(x, y_1 + y_2) = 0, \ \forall y_1 + y_2 \in M + N \\ &\implies \omega(x, y_1) = 0, \ \forall y_1 \in M \ \text{ and } \ \omega(x, y_2) = 0, \ \forall y_2 \in N \\ &\implies x \in M^\omega \quad \text{and} \quad x \in N^\omega \\ &\implies x \in M^\omega \cap N^\omega. \end{aligned}$$

Therefore,

$$(M + N)^\omega \subseteq M^\omega \cap N^\omega.$$

(iii) The double-symplectic complement of M is:

$$(M^\omega)^\omega = \{x \in V \mid \omega(x, y) = 0 \ \text{ for all } \ y \in M^\omega\}.$$

To prove that

$$(M^\omega)^\omega = \overline{M},$$

assume that $x \in \overline{M}$, then there exists $(x_n)_n \subset M$ such that $x_n \to x$. Now, by definition of M^ω, for every $n \in \mathbb{N}$ we obtain

$$\omega(x_n, y) = 0 \ \text{ for all } \ y \in M^\omega.$$

Hence, $(x_n)_n \subset (M^\omega)^\omega$. Since $(M^\omega)^\omega$ is closed, it contains the limit point of $(x_n)_n$; that is, $x \in (M^\omega)^\omega$. Therefore,

$$\overline{M} \subseteq (M^\omega)^\omega.$$

To show the equality, assume, for contradiction, that there exists a nonzero $x_0 \in (M^\omega)^\omega$ such that $x_0 \notin \overline{M}$. Then, we will have the subspace

$$U = \overline{M} \oplus span\{x_0\}.$$

Since ω is strongly nondegenerate, then by Lemma 1.2.6, we can find a nonzero $y_0 \in V$ such that

$$\omega(x_0, y_0) \neq 0 \quad \text{and} \quad \omega(x, y_0) = 0 \ \text{ for all } \ x \in \overline{M}.$$

This implies $y_0 \in M^\omega$. But $x_0 \in (M^\omega)^\omega$ which leads to $\omega(x_0, y_0) = 0$. This is a contradiction with $\omega(x_0, y_0) \neq 0$. Therefore, the equality

$$(M^\omega)^\omega = \overline{M}$$

must hold.

(iv) Assume that M and N are closed and the form ω is strongly nondegenerate. Then,

$$\overline{M} = M \quad \text{and} \quad \overline{N} = N.$$

In (ii), we have

$$(M+N)^\omega = M^\omega \cap N^\omega.$$

Thus, if we replace M and N in (ii) by M^ω and N^ω respectively and use (iii), we obtain

$$(M^\omega + N^\omega)^\omega = (M^\omega)^\omega \cap (N^\omega)^\omega = \overline{M} \cap \overline{N} = M \cap N.$$

Hence

$$[(M^\omega + N^\omega)^\omega]^\omega = (M \cap N)^\omega.$$

By using (iii) again, we obtain

$$\overline{M^\omega + N^\omega} = (M \cap N)^\omega.$$

(v) We have

$$M \subseteq \overline{M}.$$

By (i), this implies

$$(\overline{M})^\omega \subseteq M^\omega.$$

On the other hand,

$$x \in M^\omega \implies \omega(x,y) = 0, \ \forall y \in M.$$

By boundedness of ω,

$$\omega(x,y) = 0, \ \forall y \in \overline{M} \implies x \in \left(\overline{M}\right)^\omega,$$

implying

$$M^\omega \subseteq \left(\overline{M}\right)^\omega.$$

We conclude that

$$M^\omega = \left(\overline{M}\right)^\omega.$$

□

In the proof of (iii), we can note that the inequality

$$(M^\omega)^\omega \supseteq \overline{M}$$

always holds in general, even if the symplectic form ω is not strongly nondegenerate. However, the equality

$$(M^\omega)^\omega = \overline{M}$$

may fail to hold if the nondegeneracy of ω is not strong. The following example illustrates that.

Example 1.2.8. *Consider the symplectic* $(\ell^1 \oplus (\ell^1)^*, \omega)$ *where*

$$\omega((x,f),(x',f')) = f'(x) - f(x') \quad \textit{for every} \quad x, x' \in \ell^1, \ f, f' \in (\ell^1)^*.$$

We equip this space with the norm:

$$\|(x,f)\| = \|x\|_{\ell^1} + \|f\|_\infty \quad \textit{for every} \quad (x,f) \in \ell^1 \oplus (\ell^1)^*.$$

Note that since ℓ^1 is not reflexive, the induced map

$$L_\omega : \ell^1 \oplus (\ell^1)^* \to (\ell^1 \oplus (\ell^1)^*)^*$$

is not surjective. For this reason, the symplectic form is weakly nondegenerate. Since $(\ell^1)^*$ can be identified with ℓ^∞,

$$\ell^1 \oplus (\ell^1)^* = \ell^1 \oplus \ell^\infty.$$

Now, consider

$$M = c_{00} \oplus \ell^1 = \{((x_n)_n, (y_n)_n) \in \ell^1 \oplus \ell^1 \mid \exists N \in \mathbb{N} \text{ such that } x_n = 0 \text{ for all } n > N\}$$

as a subspace of $\ell^1 \oplus \ell^\infty$ where ℓ^1 is viewed as a subspace of ℓ^∞. We know that the closure of c_{00} in $(\ell^1, \|\cdot\|_1)$ is ℓ^1 and the closure of ℓ^1 in $(\ell^\infty, \|\cdot\|_\infty)$ is c_0 where

$$c_0 = \left\{x = (x_n)_n \in \ell^\infty \mid \lim_{n\to\infty} x_n = 0\right\}.$$

Thus, the closure of M is:

$$\overline{M} = \ell^1 \oplus c_0$$

Now, let us compute the symplectic complement of M:

$$\begin{aligned} M^\omega &= \{(x, f) \in \ell^1 \oplus (\ell^1)^* \mid \omega((x, f), (x', f')) = 0,\ \forall (x', f') \in c_{00} \oplus \ell^1\} \\ &= \{(x, f) \in \ell^1 \oplus (\ell^1)^* \mid f'(x) - f(x') = 0,\ \forall (x', f') \in c_{00} \oplus \ell^1\} \\ &= \{(0, 0)\}. \end{aligned}$$

We may need to clarify that the last equality is indeed $\{(0,0)\}$. To do that, assume that $(x, f) \in \ell^1 \oplus (\ell^1)^*$ such that

$$f'(x) = f(x'),\ \forall (x', f') \in c_{00} \oplus \ell^1.$$

Thereafter, since the equality holds for every $(x', f') \in c_{00} \oplus \ell^1$, we can choose $f' = 0$ to obtain

$$f'(x) = 0 = f(x'),\ \forall x' \in c_{00}.$$

Since c_{00} is dense in $(\ell^1, \|\cdot\|_1)$, then

$$f(x') = 0,\ \forall x' \in \ell^1.$$

This implies

$$f = 0.$$

Similarly, we can choose $x' = 0$ to obtain

$$f'(x) = f(x') = 0,\ \forall f' \in \ell^1 \implies x = 0.$$

Therefore, the equality

$$M^\omega = \{(0, 0)\}$$

indeed holds. Consequently,

$$(M^\omega)^\omega = \ell^1 \oplus (\ell^1)^* = \ell^1 \oplus \ell^\infty.$$

But,

$$\overline{M} = \ell^1 \oplus c_0.$$

This means

$$(M^\omega)^\omega \neq \overline{M},$$

as needed.

Moreover, since the equality

$$\overline{M^\omega + N^\omega} = (M \cap N)^\omega$$

follows from the equality

$$(M^\omega)^\omega = \overline{M}$$

as shown in the proof of Theorem 1.2.7, then the inequality

$$M^\omega + N^\omega \subseteq (M \cap N)^\omega$$

holds for any symplectic form ω, while the strong nondegeneracy of the symplectic form ω requires to guarantee the equality

$$\overline{M^\omega + N^\omega} = (M \cap N)^\omega.$$

To demonstrate that, observe,

$$(M^\omega + N^\omega)^\omega = (M^\omega)^\omega \cap (N^\omega)^\omega \supseteq \overline{M} \cap \overline{N} = M \cap N.$$

Hence

$$[(M^\omega + N^\omega)^\omega]^\omega \subseteq (M \cap N)^\omega.$$

Thus,

$$\overline{M^\omega + N^\omega} \subseteq (M \cap N)^\omega.$$

Then,

$$M^\omega + N^\omega \subseteq (M \cap N)^\omega.$$

For the equality, let us consider the following example to show

$$\overline{M^\omega + N^\omega} = (M \cap N)^\omega$$

may fail to hold.

Example 1.2.9. *Consider the symplectic $(\ell^1 \oplus c_0, \omega)$ where*

$$\omega((x,y),(x',y')) = \sum_{n=1}^{\infty} (x_n y'_n - x'_n y_n)$$

for every

$$x = (x_n)_n, x' = (x'_n)_n \in \ell^1,\ y = (y_n)_n, y' = (y'_n)_n \in c_0.$$

The norm equipped with this space is:

$$\|(x,y)\| = \|x\|_{\ell^1} + \|y\|_{c_0} \quad \textit{for every} \quad (x,f) \in \ell^1 \oplus c_0.$$

Note that since ℓ^1 is not reflexive, the induced map

$$L_\omega : \ell^1 \oplus c_0 \to (\ell^1 \oplus c_0)^*$$

is not surjective. For this reason, the symplectic form is weakly nondegenerate. Now, consider the closed subspaces

$$M = \{(x,x) \in \ell^1 \oplus c_0 \mid x \in \ell^1\} \quad \textit{and} \quad N = \ell^1 \oplus \{0\}.$$

Therefore,

$$\begin{aligned}
M^\omega &= \big\{(x,y) \in \ell^1 \oplus c_0 \mid \omega((x,y),(x',y')) = 0,\ \forall (x',y') \in M\big\} \\
&= \big\{(x,y) \in \ell^1 \oplus c_0 \mid \omega((x,y),(x',x')) = 0,\ \forall x' \in \ell^1\big\} \\
&= \left\{(x,y) \in \ell^1 \oplus c_0 \mid \sum_{n=1}^{\infty} (x_n x'_n - x'_n y_n) = 0,\ \forall x' \in \ell^1\right\} \\
&= \left\{(x,y) \in \ell^1 \oplus c_0 \mid \sum_{n=1}^{\infty} x'_n (x_n - y_n) = 0,\ \forall x' \in \ell^1\right\} \\
&= \big\{(x,y) \in \ell^1 \oplus c_0 \mid x_n - y_n = 0 \ \textit{for every}\ n \in \mathbb{N}\big\} \\
&= \big\{(x,y) \in \ell^1 \oplus c_0 \mid x_n = y_n \ \textit{for every}\ n \in \mathbb{N}\big\} \\
&= \big\{(x,x) \in \ell^1 \oplus c_0 \mid x \in \ell^1\big\} \\
&= M
\end{aligned}$$

Also, the symplectic complement of N is:

$$\begin{aligned}
N^\omega &= \big\{(x,y) \in \ell^1 \oplus c_0 \mid \omega((x,y),(x',y')) = 0,\ \forall (x',y') \in \ell^1 \oplus \{0\}\big\} \\
&= \big\{(x,y) \in \ell^1 \oplus c_0 \mid \omega((x,y),(x',0)) = 0,\ \forall x' \in \ell^1\big\} \\
&= \left\{(x,y) \in \ell^1 \oplus c_0 \mid \sum_{n=1}^{\infty} x'_n y_n = 0,\ \forall x' \in \ell^1\right\} \\
&= \big\{(x,y) \in \ell^1 \oplus c_0 \mid y_n = 0\ \forall n \in \mathbb{N}\big\} \\
&= \big\{(x,y) \in \ell^1 \oplus c_0 \mid y = 0\big\} \\
&= \ell^1 \oplus \{0\} \\
&= N
\end{aligned}$$

In conclusion, we have

$$\overline{M^\omega + N^\omega} = \ell^1 \oplus c_0. \tag{1.2.1}$$

On the other hand, we have

$$(M \cap N)^\omega = N^\omega = \ell^1 \oplus \{0\}. \tag{1.2.2}$$

This implies

$$\overline{M^\omega + N^\omega} \neq (M \cap N)^\omega.$$

In the following corollary, we will see that the symplectic structure is preserved when taking closures.

Corollary 1.2.10. *Let (V,ω) be an even finite-dimensional or separable infinite-dimensional real symplectic Banach space with a strongly nondegenerate symplectic form ω, and let M be a subspace of V. Then*

(i) M is isotropic $\iff$ $\overline{M}$ is isotropic,

(ii) M is co-isotropic $\iff$ $\overline{M}$ is co-isotropic,

(iii) M is symplectic $\iff$ $\overline{M}$ is symplectic,

where $\overline{M}$ is the closure of M.

Before we start proving this Corollary, we first need the following Lemma, which is helpful to prove the statement (*iii*):

Lemma 1.2.11. *If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are real Banach spaces and $F : X \to Y$ is a closed surjective linear operator, then F is bounded below on D, where*

$$D = \{x \in X \mid F(x) \neq 0\}.$$

Formally, there exists $C > 0$ such that for every $x \in X$ with $\|F(x)\|_Y \neq 0$,

$$C\|x\|_X \leqslant \|F(x)\|_Y. \tag{1.2.3}$$

Proof. Since F is a closed operator, then $(X, \|\cdot\|_F)$ is a Banach space where $\|\cdot\|_F$ is the graph-norm and for every $x \in X$ it is defined as

$$\|x\|_F = \|x\|_X + \|F(x)\|_Y.$$

The key idea of the proof is to equip X with $\|\cdot\|_F$ instead of $\|\cdot\|_X$, and then we show that for every $x \in D$, the operator

$$F : (X, \|\cdot\|_F) \to (Y, \|\cdot\|_Y)$$

satisfies

$$C\|x\|_F \leqslant \|F(x)\|_Y \quad \text{where} \quad C > 0.$$

After that, we immediately derive that (1.2.3) holds. Let us carefully achieve that in more detail. Note that the operator

$$F : (X, \|\cdot\|_F) \to (Y, \|\cdot\|_Y)$$

still maps a Banach space to a Banach space. Also, it is still bounded since for every $x \in X$,

$$\|F(x)\|_Y \leqslant \|x\|_F.$$

Therefore,

$$F : (X, \|\cdot\|_F) \to (Y, \|\cdot\|_Y)$$

is a linear surjective bounded operator between two Banach spaces. From the open mapping Theorem [5, p.35], there exists $R > 0$ such that

$$B_Y(0, R) \subset F(B_F(0, 1))$$

where $B_Y(0, R)$ is an open ball in $(Y, \|\cdot\|_Y)$ with radius $R > 0$ and $B_F(0, 1)$ is the unit open ball in $(X, \|\cdot\|_F)$. This implies that for every $x \in X$ with $\|F(x)\|_Y < R$, we will obtain $\|x\|_F < 1$. Now, take an arbitrary $x_0 \in X$ with $0 < \|F(x_0)\|_Y < R$. Thus, $\|x_0\|_F < 1$. By scalling

$$x_0' = \frac{R}{2\|F(x_0)\|_Y} x_0, \tag{1.2.4}$$

we will have

$$F(x_0') = \frac{R}{2\|F(x_0)\|_Y} F(x_0) \implies \|F(x_0')\|_Y = \frac{R}{2} < R \implies \|x_0'\|_F < 1. \tag{1.2.5}$$

Hence, from (1.2.4) and (1.2.5), we have

$$\|x_0\|_F = \frac{2\|F(x_0)\|_Y}{R} \|x_0'\|_F < \frac{2}{R} \|F(x_0)\|_Y.$$

By putting $C = \frac{R}{2} > 0$, we obtain

$$\|x_0\|_F < \frac{1}{C}\|F(x_0)\|_Y,$$

which implies

$$C\|x_0\|_F < \|F(x_0)\|_Y. \tag{1.2.6}$$

Now, we can extend the inequality (1.2.6) to all $x \in D$ as follows: for every $x \in X$ with $\|F(x)\|_Y \neq 0$, we can write

$$x = \frac{2\|F(x)\|_Y}{R} \cdot \frac{Rx}{2\|F(x)\|_Y}$$

and then

$$\left\| F\left(\frac{Rx}{2\|F(x)\|_Y}\right)\right\|_Y = \frac{R}{2} < R.$$

Thus, we can apply the above argument on $\frac{Rx}{2\|F(x)\|_Y}$ instead of x_0 to obtain

$$C\left\|\frac{Rx}{2\|F(x)\|_Y}\right\|_F < \left\|F\left(\frac{Rx}{2\|F(x)\|_Y}\right)\right\|_Y.$$

Now, we can multiply both sides by $\frac{2\|F(x)\|_Y}{R}$, and thus, by using linearity, we obtain

$$C\|x\|_F < \|F(x)\|_Y.$$

Finally, since every $x \in D$ we have

$$\|x\|_X < \|x\|_X + \|F(x)\|_Y = \|x\|_F \quad \text{and} \quad C\|x\|_F < \|F(x)\|_Y,$$

then

$$C\|x\|_X < \|F(x)\|_Y \quad \text{for every} \quad x \in D.$$

This completes the proof. □

Proof of Corollary 1.2.10. (i) Since M^ω is closed, the following statements are equivalent:

- M is isotropic.
- $M \subseteq M^\omega$.
- $\overline{M} \subseteq M^\omega$.
- $\overline{M}$ is isotropic.

(ii) From Theorem 1.2.7, we have the following:

$$\begin{aligned} M \text{ is co-isotropic} &\implies M^\omega \subseteq M \\ &\implies M^\omega \subseteq (M^\omega)^\omega \\ &\implies \left(\overline{M}\right)^\omega \subseteq \overline{M} \\ &\implies \overline{M} \text{ is co-isotropic.} \end{aligned}$$

On the other hand, assume that $\overline{M}$ is co-isotropic, and we want to prove that M is co-isotropic. For contradiction, suppose that M is not co-isotropic. Then $M^\omega \nsubseteq M$, meaning, there exists a nonzero $x \in M^\omega$ such that $x \notin M$. Set

$$W = M \oplus \operatorname{span}\{x\}.$$

From Lemma 1.2.6, the strong nondegeneracy of ω results in existence of a nonzero $y \in V$ such that

$$\omega(x, y) \neq 0 \quad \text{and} \quad \omega(y, m) = 0 \ \text{ for all } \ m \in M. \tag{1.2.7}$$

By boundedness of ω, we can extend the vanishing property of $\omega|_M$ to $\overline{M}$; i.e,

$$\omega(y, m) = 0 \ \text{ for all } \ m \in \overline{M}. \tag{1.2.8}$$

Recall Theorem 1.2.7, we have

$$M^\omega = \left(\overline{M}\right)^\omega. \tag{1.2.9}$$

Now, since $\overline{M}$ is co-isotropic (i.e $\left(\overline{M}\right)^\omega \subseteq \overline{M}$) and $x \in M^\omega$, then the equality (1.2.9) gives

$$x \in M^\omega \subseteq \overline{M}.$$

Thus, from (1.2.8), we have $\omega(y, x) = 0$ which contradicts (1.2.7). Therefore, we can conclude that the subspace M must satisfy $M^\omega \subseteq M$, meaning M is co-isotropic.

(iii) First of all, recall the induced map of ω:

$$L_\omega : V \to V^*$$

$$x \mapsto L_\omega(x).$$

Also, remember that the subspace $M \subset V$ is symplectic if and only if the restriction of the induced map to M is injective. For simplicity, put

$$f = L_\omega|_M \quad \text{and} \quad F = L_\omega|_{\overline{M}}$$

where $L_\omega|_M$ is the restriction of the induced map to M and $L_\omega|_{\overline{M}}$ is the restriction of the induced map to $\overline{M}$. We aim to prove that

$$M \text{ is symplectic} \iff \overline{M} \text{ is symplectic}$$

by showing that

$$f \text{ is injective} \iff F \text{ is injective.}$$

Assume that $\overline{M}$ is symplectic. Then F will be injective on $\overline{M}$. Since

$$\forall x \in \overline{M}, \ \exists (x_n)_n \subset M \quad \text{such that} \quad \lim_{n\to\infty} x_n = x,$$

by boundedness of L_ω,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} F(x_n) = F(x).$$

This means F is the extension of f by adding the limit points. Since the limit of a zero sequence must be zero, we have the implications

$$f(x) = 0, \ \forall x \in M \implies F(x) = 0, \ \forall x \in \overline{M} \implies x = 0.$$

This means f is injective, and then M is symplectic.

The converse is more subtle. Assume that M is symplectic. Thus, f is injective. We aim to show that F is injective via showing it is bounded below. This can be achieved by showing that $F : \overline{M} \to \mathrm{Ran}(F)$ is a closed operator with a closed range $\mathrm{Ran}(F)$. To this end, assume that $(x_n)_n \subset \overline{M}$ such that $x_n \to x$ and $F(x_n) \to y$. Then $x \in \overline{M}$ and, by boundedness of F, $F(x_n) \to F(x)$, which implies $y = F(x)$. This proves that F is a closed

operator. It remains to prove that $\mathrm{Ran}(F)$ is closed. Suppose that $(g_n)_n \subset \mathrm{Ran}(F)$ and $g_n \to g$. Then, there exists $(x_n)_n \subset \overline{M}$ such that for every n, $F(x_n) = g_n$. Thereafter,

$$g = \lim_{n\to\infty} g_n = \lim_{n\to\infty} F(x_n) = F(\lim_{n\to\infty} x_n) = F(x).$$

Since $(x_n)_n \subset \overline{M}$, then $x \in \overline{M}$ which means $g \in \mathrm{Ran}(F)$. Therefore, $\mathrm{Ran}(F)$ is closed, meaning $\mathrm{Ran}(F)$ is a Banach space. Now, since $F : \overline{M} \to \mathrm{Ran}(F)$ is a closed operator between two Banach spaces, then, from Lemma 1.2.11, there exists $C > 0$ such that for every $x \in \overline{M}$ with $\|F(x)\| \neq 0$, we have

$$C\|x\| \leqslant \|F(x)\|. \tag{1.2.10}$$

Now, we can prove the injectivity of F as follows: suppose that $x \in \overline{M}$ with $F(x) = 0$, and we aim to show that $x = 0$. To this end, note that $x \in \overline{M}$ implies the existence of $(x_n)_n \subset M$ such that $x_n \to x$. By bounededness of F, we have $F(x_n) \to F(x)$, meaning $F(x_n) \to 0$. Now, we have two cases:

- Case 1: $(F(x_n))_n$ is eventially zero; that is, there exists $N \in \mathbb{N}$ such that $n \geqslant N \implies F(x_n) = 0$ and thus $f(x_n) = 0$ for all $n \geqslant N$. From injectivity of f, we have $x_n = 0$ for all $n \geqslant N$, implying $x_n \to 0$. Therefore, $x = 0$.
- Case 2: $(F(x_n))_n$ is not eventially zero; that is, there exists a subsequence $(x_{n_k})_k$ such that $\|F(x_{n_k})\| \neq 0$ for all k. Since F statisfies (1.2.10), then $C\|x_{n_k}\| \leqslant \|F(x_{n_k})\|$. Consequently, $\|x_{n_k}\| \to 0$ since $\|F(x_{n_k})\| \to 0$. Therefore, $x = 0$.

In conclusion, both cases state that if $F(x) = 0$, then $x = 0$. Hence, F is injective, meaning $\overline{M}$ is symplectic.

□

Corollary 1.2.12. *Let (V, ω) be an even finite-dimensional or separable infinite-dimensional real symplectic Banach space with strongly nondegenerate symplectic form ω, and let M be a closed subspace of V. Then*

$$M \text{ is symplectic} \iff M^\omega \text{ is symplectic.}$$

Proof. Since M^ω is closed, from Theorem 1.2.7 (*iii*), we can deduce that

$$[(M^\omega)^\omega]^\omega = \overline{M^\omega} = M^\omega.$$

Hence, by Corollary 1.2.10, the following statements are equivalent:

- M is symplectic,
- $\overline{M}$ is symplectic,
- $(M^\omega)^\omega$ is symplectic,
- $(M^\omega)^\omega \cap [(M^\omega)^\omega]^\omega = \{0\}$,
- $(M^\omega)^\omega \cap M^\omega = \{0\}$,
- M^ω is symplectic.

□

Corollary 1.2.13. *If (V,ω) is a real symplectic space, N is a Lagrangian subspace of V and M is a subspace of V, then*

(i) $N \subset M \implies M$ *is a co-isotropic subspace.*

(ii) $M \subset N \implies M$ *is an isotropic subspace.*

Proof. (i)

$$N \subset M \implies M^\omega \subset N^\omega = N \subset M \implies M \text{ is a co-isotropic subspace of } V.$$

(ii)

$$M \subset N \implies M \subset N = N^\omega \subset M^\omega \implies M \text{ is a isotropic subspace of } V.$$

□

Informally, Corollary 1.2.13 tells us that every Lagrangian subspace is half of the whole space. Therefore, if a subspace is contained in a Lagrangian subspace, it is isotropic, while if a subspace contains a Lagrangian subspace, it is co-isotropic.

Example 1.2.14. *Consider $(\mathbb{R}^6,\omega)$ such that*

$$\omega(a_i,a_j)=\omega(a_{3+i},a_{3+j})=0 \quad \textit{and} \quad \omega(a_i,a_{j+3})=\delta_{ij}, \quad i,j=1,2,3$$

where a_n is the vector whose components are all zero, except the n-th component equals 1. Note that the subspace

$$\mathbb{R}^3 = \operatorname{Clin}\{a_1,a_2,a_3\}$$

is Lagrangian, and one can easily check that all the subspaces of $\mathbb{R}^3$ are isotropic and the subspaces contained $\mathbb{R}^3$ are co-isotropic.

In addition to the aforementioned properties of the symplectic complement, we can also add the following: the relationship between the adjoint operator and the orthogonal complement still holds for the symplectic complement as well. See the next theorem.

Theorem 1.2.15. *If (V,ω) is a real symplectic space with a strongly nondegenerate symplectic form ω and L_ω is its induced map and $T: V \to V$ is a bounded linear operator, then we have*

$$\left[L_\omega^{-1}(\operatorname{Ran} T^*)\right]^\omega = \ker T \quad \textit{and} \quad \ker T^* = \operatorname{ann}(\operatorname{Ran} T).$$

Proof. For the first equality,

$$\begin{aligned}
\left[L_\omega^{-1}(\operatorname{Ran} T^*)\right]^\omega &= \{x \in V \mid \omega(x,y)=0,\ \forall y \in L_\omega^{-1}(\operatorname{Ran} T^*)\} \\
&= \{x \in V \mid \omega(y,x)=0,\ \forall y \in L_\omega^{-1}(\operatorname{Ran} T^*)\} \\
&= \{x \in V \mid L_\omega(y)(x)=0,\ \forall y \in L_\omega^{-1}(\operatorname{Ran} T^*)\} \\
&= \{x \in V \mid L_\omega(y)(x)=0,\ \forall L_\omega(y) \in \operatorname{Ran} T^*\} \\
&= \{x \in V \mid T^*L_\omega(z)(x)=0,\ \forall L_\omega(z) \in \operatorname{Dom} T^*\} \\
&= \{x \in V \mid L_\omega(z)(Tx)=0,\ \forall L_\omega(z) \in V^*\} \\
&= \{x \in V \mid \omega(z,Tx)=0,\ \forall z \in V\} \\
&= \{x \in V \mid Tx=0\} \qquad \text{(from nondegeneracy of } \omega) \\
&= \ker T.
\end{aligned}$$

For the second equality,

$$\begin{aligned}\ker T^* &= \{x^* \in V^* \mid T^*(x^*)(z) = 0,\ \forall z \in V\} \\ &= \{x^* \in V^* \mid x^*(Tz) = 0,\ \forall z \in V\} \\ &= \{x^* \in V^* \mid x^*(y) = 0,\ \forall y \in \operatorname{Ran} T\} \\ &= \operatorname{ann}(\operatorname{Ran} T).\end{aligned}$$

□

1.3 Basic Symplectic Spaces and Symplectic Basis

An isomorphism between two symplectic spaces that preserves the structure of the symplectic space is called a symplectomorphism. See the following definition.

Definition 1.3.1. *[28, p.5] An isomorphism $\phi : (V_1, \omega_1) \to (V_2, \omega_2)$ is called a symplectomorphism if for every $x, y \in V_1$ we have*

$$\phi^*\omega_2(x, y) = \omega_2(\phi x, \phi y) = \omega_1(x, y).$$

In this case, the two symplectic spaces (V_1, ω_1) and (V_2, ω_2) are called symplectomorphic spaces.

The following definition provides us with an important type of symplectic space.

Definition 1.3.2. *[8] A Banach symplectic space (V, ω) is called basic if there exists a reflexive Banach space Y and a symplectomorphism Φ such that*

$$\Phi : V \to Y \oplus Y^* \quad \text{and} \quad \omega(u, v) = \omega_0(\Phi u, \Phi v) \quad \text{for every} \quad u, v \in V$$

where Y^* is the dual space of Y and ω_0 is the canonical symplectic form, which is defined as

$$\omega_0((x, x^*), (y, y^*)) = y^*(x) - x^*(y) \quad \text{for every} \quad x, y \in Y, x^*, y^* \in Y^*. \tag{1.3.1}$$

Now, correspondingly to the concept of orthogonal basis in Hilbert spaces, we have a favourite type of basis for symplectic spaces. This basis is called a symplectic basis. See the next definition.

Definition 1.3.3 (Symplectic basis)**.** *[28, p.7] Given a symplectic space (V, ω) with a countable basis*

$$\bigcup_{i \in I \subseteq \mathbb{N}} \{e_i, f_i\} \tag{1.3.2}$$

where I is the indexing set of the basis and its cardinality is:

- $\operatorname{Card} I = \frac{1}{2} \dim V$ *if* $\dim V < \infty$
- $\operatorname{Card} I = \infty$ *if* $\dim V = \infty$.

This basis is called symplectic if it satisfies

$$\omega(e_i, e_j) = \omega(f_i, f_j) = 0 \quad \text{and} \quad \omega(e_i, f_j) = \delta_{ij} \ \text{ for every } \ i, j \in I. \tag{1.3.3}$$

Remark 1.3.4. *Since we are dealing with a symplectic space that can be either even finite or infinite-dimensional spaces, we will henceforth adopt the expression (1.3.2) to write the basis of the space. Consequently, the cardinality of the indexing set $I \subseteq \mathbb{N}$ will depend on the dimension of V as described in the above definition. Therefore, from now on, whenever we write*

$$S = \bigcup_{i \in I \subseteq \mathbb{N}} \{e_i, f_i\}$$

as a basis of a symplectic space V in a statement, we mean that the statement holds for V whether V is even finite or infinite dimensional.

The Theorem 1.3.8 below states the conditions that guarantee the existence of such a basis. But, before we are prepared to prove this theorem, we need first to recall a Schauder basis of a Banach space.

Definition 1.3.5. *[26] We say that $\{u_k\}_{k\in\mathbb{N}}$ is a Schauder basis of a Banach space V if for every $v \in V$ there exists a unique sequence of scalars $\{a_k\}_{k\in\mathbb{N}}$ such that*

$$v = \sum_{k=1}^{\infty} a_k u_k. \tag{1.3.4}$$

Note that the order of the summation (1.3.4) might affect the convergence of the series. Therefore, if we want to have a freedom to reorder the summation (1.3.4), we must assume that the basis $\{u_k\}_{k\in\mathbb{N}}$ is symmetric, which means [26, p.15] for any permutation π of $\mathbb{N}$, the convergence of the series

$$v = \sum_{k=1}^{\infty} a_{\pi(k)} u_{\pi(k)} \tag{1.3.5}$$

still holds.

Also, it is worth mentioning that the existence of a Schauder basis of a Banach space V is more restrictive than V being separable. For example, Enflo, in his paper [11], has constructed a separable Banach space without a Schauder basis.

Also, there is a type of Schauder bases called a shrinking basis. See the following definition.

Definition 1.3.6. *[26] Let V be a Banach space with a Schauder basis $\{u_k\}_{k\in\mathbb{N}}$ and V^* be its dual space. For every k, we define $u_k^* \in V^*$ as*

$$u_k^*(u_n) = \delta_{kn}.$$

The Schauder basis $\{u_k\}_{k\in\mathbb{N}}$ is called shrinking if its dual set $\{u_k^*\}_{k\in\mathbb{N}} \subset V^*$ is also a Schauder basis of V^*.

The following theorem is an immediate consequence of James' Theorem [26] that provides the relationship between a shrinking basis and the reflexivity of a Banach space.

Theorem 1.3.7. *Let V be a Banach space with a Schauder basis $\{u_k\}_{k\in\mathbb{N}}$. Then,*

$$V \text{ is reflexive} \implies \{u_k\}_{k\in\mathbb{N}} \text{ is a shrinking basis.}$$

Now, we are prepared to introduce the main theorem of this section.

Theorem 1.3.8 (Existence of symplectic basis)**.** *Assume that (V,ω) is an even finite-dimensional or infinite-dimensional real basic symplectic Banach space; that is,*

$$V \cong Y \oplus Y^*$$

where Y is a reflexive Banach space. Also, assume that Y has a symmetric Schauder basis $\{u_k\}_{k\in\mathbb{N}}$ and Y^* is the dual space of Y and $Y\oplus Y^*$ is equipped with the canonical symplectic form (1.3.1). Then V possesses a symplectic basis.

Proof. We will start with the case of an infinite-dimensional symplectic space, and then the case of finite dimension will be derived immediately.

First of all, consider

$$\Phi : V \to Y \oplus Y^*.$$

to be a symplectomorphism. Now, let $\{u_k^*\}_{k\in\mathbb{N}} \subset Y^*$ be the dual set of the basis $\{u_k\}_{k\in\mathbb{N}}$, which defined as

$$u_i^*(u_j) = \delta_{ij}$$

Since V is reflexive, from Theorem 1.3.7, the basis $\{u_k\}_{k\in\mathbb{N}}$ is shrinking, which means the dual set $\{u_k^*\}_{k\in\mathbb{N}}$ forms a Schauder basis of Y^*. Furthermore, since $\{u_k\}_{k\in\mathbb{N}}$ is symmetric, then $\{u_k^*\}_{k\in\mathbb{N}}$ is also symmetric. Hence

$$A = \bigcup_{k\in\mathbb{N}} \{u_k, u_k^*\}$$

is a symmetric Schauder basis of $Y \oplus Y^*$. Consequently,

$$B = \bigcup_{k\in\mathbb{N}} \{e_k, f_k\}$$

is also a symmetric Schauder basis of V where

$$\Phi^{-1}(u_k, 0) = e_k \qquad \text{and} \qquad \Phi^{-1}(0, u_k^*) = f_k.$$

This means for every $v \in V$ there exist unique scalar sequences $(a_k)_k$ and $(b_k)_k$ such that

$$v = \sum_{k=1}^{\infty} a_k e_k + \sum_{k=1}^{\infty} b_k f_k.$$

Furthermore, for every $i, j \in \mathbb{N}$, we have the following:

- $\omega(e_i, e_j) = \omega\left(\Phi^{-1}(u_i, 0), \Phi^{-1}(u_j, 0)\right) = \omega\left((u_i, 0), (u_j, 0)\right) = 0^*(u_i) - 0^*(u_j) = 0.$
- $\omega(f_i, f_j) = \omega\left(\Phi^{-1}(0, u_i^*), \Phi^{-1}(0, u_j^*)\right) = \omega\left((0, u_i^*), (0, u_j^*)\right) = u_j^*(0) - u_i^*(0) = 0.$
- $\omega(e_i, f_j) = \omega\left(\Phi^{-1}(u_i, 0), \Phi^{-1}(0, u_j^*)\right) = \omega\left((0, u_i), (0, u_j^*)\right) = u_j^*(u_i) = \delta_{ij}.$

Therefore,

$$B = \bigcup_{k\in\mathbb{N}} \{e_k, f_k\}$$

is a symplectic basis.

In the finite-dimensional case, the above approach is still valid. Indeed, by the same approach, one can easily construct a basis

$$B_n = \bigcup_{k=1}^{n} \{e_k, f_k\}$$

for the $2n$-dimensional space.

This completes the proof. □

Corollary 1.3.9. *[28, p.7] Every 2n-dimensional real symplectic space (V,ω) is symplectomorphic to $(\mathbb{R}^{2n},\omega_0)$ where ω_0 is standard symplectic form, which is defined as*

$$\omega_0(a_i,a_j)=\omega_0(a_{n+i},a_{n+j})=0 \quad \textit{and} \quad \omega_0(a_i,a_{j+n})=\delta_{ij},\ \forall i,j=1,2,\dots,n.$$

Here, $\{a_i\}_{k=1}^{2n}$ is the standard basis of $\mathbb{R}^{2n}$.

Proof. Assume that

$$S=\{e_1,f_1,\dots,e_n,f_n\}.$$

is a symplectic basis of V. Define the isomorphism

$$\phi:\mathbb{R}^{2n}\to V,\quad \phi(a_k)=\begin{cases} e_k & \text{if} \quad k=1,2,\cdots,n \\ f_{k-n} & \text{if} \quad k=n+1,n+2,\cdots,2n. \end{cases}$$

Then, for every $i,j=1,2,\cdots,n$, we have

- $\omega_0(a_i,a_j)=0=\omega(e_i,e_j)=\omega(\phi(a_i),\phi(a_j))$,
- $\omega_0(a_{n+i},a_{n+j})=0=\omega(f_i,f_j)=\omega(\phi(a_{n+i}),\phi(a_{n+j}))$,
- $\omega_0(a_i,a_{j+n})=\delta_{ij}=\omega(e_i,f_j)=\omega(\phi(a_i),\phi(a_{n+j}))$.

By using the linearity of ω and ϕ, this implies that for every $x,y\in\mathbb{R}^{2n}$

$$\omega_0(x,y)=\omega(\phi(x),\phi(y)),$$

which means the isomorphism ϕ is a symplectomorphism. Therefore, the symplectic spaces $(\mathbb{R}^{2n},\omega_0)$ and (V,ω) are symplectomorphic. □

Example 1.3.10. *Consider $(\ell^2\oplus(\ell^2)^*,\omega)$ such that*

$$\omega((x,x^*),(y,y^*))=y^*(x)-x^*(y)=\langle x,y^*\rangle-\langle y,x^*\rangle \quad \textit{for every} \quad (x,x^*),(y,y^*)\in\ell^2\oplus(\ell^2)^*.$$

Since $\ell^2\cong(\ell^2)^*$, we can have an isomorphism such that

$$x'=\Phi(x^*)\in\ell^2 \qquad \textit{and} \qquad y'=\Phi(y^*)\in\ell^2.$$

and

$$\langle x,y^*\rangle=\langle x,y'\rangle=\sum_{k=1}^{\infty}x_ky'_k \quad \textit{and} \quad \langle y,x^*\rangle=\langle y,x'\rangle=\sum_{k=1}^{\infty}x'_ky_k$$

where $x=(x_k)_k$, $x'=(x'_k)_k$, $y=(y_k)_k$, $y'=(y'_k)_k\in\ell^2$. Hence

$$\omega((x,x^*),(y,y^*))=\sum_{k=1}^{\infty}(x_ky'_k-x'_ky_k).$$

Recall the standard basis of ℓ^2

$$B=\{u_k=(0,\dots,0,1^{k^{th}},0,\dots)\mid k=1,2,\dots\}.$$

The symplectic basis of $\ell^2\oplus\ell^2$ will be

$$S=\bigcup_{i=1}^{\infty}\{e_i,f_i\}$$

where

$$e_1 = ((1,0,0,\dots);(0,0,0,\dots)), \qquad f_1 = ((0,0,0,\dots);(1,0,0,\dots),$$
$$e_2 = ((0,1,0,\dots);(0,0,0,\dots)), \qquad f_2 = ((0,0,0,\dots);(0,1,0,\dots)),$$
$$e_3 = ((0,0,1,\dots);(0,0,0,\dots)), \qquad f_3 = ((0,0,0,\dots);(0,0,1,\dots)),$$

For any $k = 2,3,4,\dots$, the general formulas are

$$e_k = ((0,\dots,0,1^{k^{th}},0,\dots);(0,0,0,\dots) \quad \textit{and} \quad f_k = ((0,0,0,\dots);(0,\dots,0,1^{k^{th}},0,\dots))$$

which is the same as the standard basis of $\ell^2 \oplus \ell^2$.

1.4 Quotient Symplectic Spaces

Let $(V, \|\cdot\|_V)$ be a real normed space and let

$$\omega : V \times V \to \mathbb{R}$$

be a bounded skew-symmetric bilinear form, which may be degenerate. In this section, we will study how the form ω inherits a nondegenerate bounded skew-symmetric bilinear form (symplectic form)

$$\tilde{\omega}([x],[y]) = \omega(x,y) \tag{1.4.1}$$

where $[x],[y]$ are equivalence classes in the quotient space

$$V/M = \{[v] = v + M \mid v \in V\}$$

for a closed subspace $M \subset V$. The norm of V/M is the quotient norm, which is defined as:

$$\|[v]\| = \inf_{m \in M} \|v - m\|_V \quad \text{for every} \quad [v] \in V/M. \tag{1.4.2}$$

The above closed subspace M is not arbitrary; it is chosen in such a way that ensures the form ω inherits a symplectic form $\tilde{\omega}$ on V/M. Therefore, we must determine the restrictions on the subspace M under which:

- $\tilde{\omega}$ is well-defined,
- $\tilde{\omega}$ is bounded,
- $\tilde{\omega}$ is skew-symmetric,
- $\tilde{\omega}$ is nondegenerate.

To avoid writing a long proof for verifying these four conditions, we will examine each of them individually in a separate theorem. In other words, we will separately determine the restrictions on M under which each condition is satisfied. Afterwards, we will combine all these restrictions to characterise when $(V/M, \tilde{\omega})$ becomes a quotient symplectic space.

The first theorem provides the restrictions on the subspace M under which the form $\tilde{\omega}$ in (1.4.1) is well-defined on the quotient space V/M.

Theorem 1.4.1. *Let $(V, \|\cdot\|)$ be either an even finite-dimensional or a separable infinite-dimensional real normed space, and ω is a bounded skew-symmetric bilinear form on the space V. Consider the quotient space V/M equipped with the quotient norm*

$$\|[v]\| = \inf_{m \in M} \|v - m\|_V \quad \textit{for every} \quad [v] \in V/M.$$

Also, consider the form

$$\tilde{\omega}([x],[y]) = \omega(x,y) \quad \textit{for every} \quad [x],[y] \in V/M.$$

This form $\tilde{\omega}$ is well-defined on the quotient space V/M if and only if $M \subseteq V^\omega$.

Proof. To prove that the form $\tilde{\omega}$ is well-defined, we must show that $\tilde{\omega}$ does not depend on the choice of representatives x, y in the cosets $[x] = x + M$ and $[y] = y + M$ in V/M. To achieve this, assume that $[x], [x'], [y], [y'] \in V/M$ such that

$$[x] = [x'] \quad \text{and} \quad [y] = [y'],$$

which means

$$x' = x + m_1 \quad \text{and} \quad y' = y + m_2 \quad \text{for some} \quad m_1, m_2 \in M.$$

Then,

$$\begin{aligned} \tilde{\omega}([x'],[y']) &= \omega(x', y') \\ &= \omega(x + m_1, y + m_2) \\ &= \omega(x,y) + \omega(x,m_2) + \omega(m_1,y) + \omega(m_1,m_2) \\ &= \tilde{\omega}([x],[y]) + \omega(x,m_2) + \omega(m_1,y) + \omega(m_1,m_2) \end{aligned}$$

The form $\tilde{\omega}$ is well-defined if and only if

$$\tilde{\omega}([x'],[y']) = \tilde{\omega}([x],[y]).$$

In other words, it is well-defined if and only if the sum of the three extra terms on the right-hand side vanish:

$$\omega(x,m_2) + \omega(m_1,y) + \omega(m_1,m_2) = 0.$$

Since x, y have been chosen arbitrarily, we require:

$$\omega(x,m_2) = \omega(m_1,y) = \omega(m_1,m_2) = 0 \ \ \forall x, y \in V, \ m_1, m_2 \in M.$$

Now, observe

- $\omega(m_1,m_2) = 0 \ \ \forall m_1, m_2 \in M$ requires $M \subseteq M^\omega$ (i.e. M is isotropic).
- $\omega(x,m_2) = \omega(m_1,y) = 0 \ \ \forall x, y \in V, \ m_1, m_2 \in M$ requires $M \subseteq V^\omega$.

In conclusion, the form $\tilde{\omega}$ is well-defined if and only if M satisfies $M \subseteq V^\omega$.

□

Remark 1.4.2. *Note that, in the above Theorem, V^ω is not necessarily equal to $\{0\}$ because we did not assume that ω is nondegenerate on V.*

In the second theorem, we will examine the boundedness of the form $\tilde{\omega}$ in (1.4.1) on the quotient space V/M.

Theorem 1.4.3. *Let $(V, \|\cdot\|)$ be either an even finite-dimensional or a separable infinite-dimensional real normed space, and ω be a bounded skew-symmetric bilinear form on the space V, and let M be a closed subspace contained in V^ω. Consider the quotient space V/M equipped with the quotient norm*

$$\|[v]\| = \inf_{m\in M} \|v - m\|_V \quad \textit{for every} \quad [v] \in V/M.$$

Also, consider the bilinear form

$$\tilde{\omega}([x],[y]) = \omega(x,y) \quad \textit{for every} \quad [x],[y] \in V/M.$$

Then, the form $\tilde{\omega}$ is bounded on the quotient space V/M.

Proof. To prove that $\tilde{\omega}$ is bounded, we must show existence of $C > 0$ such that

$$|\tilde{\omega}([x],[y])| \leqslant C\|[x]\|\|[y]\| \quad \text{for every} \quad [x],[y] \in V/M.$$

Since for every $[x],[y] \in V/M$

$$\|[x]\| = \inf_{m\in M} \|x - m\|_V \quad \text{and} \quad \|[y]\| = \inf_{m\in M} \|y - m\|_V,$$

then, by this definition, for every $\epsilon > 0$ there exists $m_1, m_2 \in M$ such that

$$\|x + m_1\|_V \leqslant \epsilon + \inf_{m\in M} \|x - m\|_V \ \text{ and } \ \|y + m_2\|_V \leqslant \epsilon + \inf_{m\in M} \|y - m\|_V. \tag{1.4.3}$$

Now, since M is contained in V^ω, then, from Theorem 1.4.1, $\tilde{\omega}$ is well-defined; that is,

$$\omega(x,y) = \omega(x + m_1, y + m_2)$$

Thus, from the boundedness of ω and line (1.4.3), we can obtain the required $C > 0$ as follows: for every $[x],[y] \in V/M$ we can write

$$\begin{aligned}
|\tilde{\omega}([x],[y])| &= |\omega(x,y)| \\
&= |\omega(x + m_1, y + m_2)| \\
&\leqslant C\|x + m_1\|_V \|y + m_2\|_V \\
&\leqslant C(\epsilon + \inf_{m\in M} \|x - m\|_V)(\epsilon + \inf_{m\in M} \|y - m\|_V) \\
&\leqslant C(\epsilon + \|[x]\|)(\epsilon + \|[y]\|).
\end{aligned}$$

Finally, if we take $\epsilon \to 0$, we obtain

$$|\tilde{\omega}([x],[y])| \leqslant C\|[x]\|\|[y]\| \quad \text{for every} \quad [x],[y] \in V/M,$$

which means that $\tilde{\omega}$ is bounded on V/M. □

In the third theorem, we will examine whether the form $\tilde{\omega}$ in (1.4.1) is skew-symmetric on the quotient space V/M.

Theorem 1.4.4. *Let $(V, \|\cdot\|)$ be either an even finite-dimensional or a separable infinite-dimensional real normed space, and ω be a bounded skew-symmetric bilinear form on the space V, and let M be a closed subspace contained in V^ω. Consider the quotient space V/M equipped with the quotient norm*

$$\|[v]\| = \inf_{m\in M} \|v - m\|_V \quad \textit{for every} \quad [v] \in V/M.$$

Also, consider the bilinear form

$$\tilde{\omega}([x],[y]) = \omega(x,y) \quad \textit{for every} \quad [x],[y] \in V/M.$$

Then, the form $\tilde{\omega}$ is skew-symmetric on the quotient space V/M.

Proof. The skew-symmetry of $\tilde{\omega}$ follows from the skew-symmetry of ω. To clarify this, note that for every $[x], [y] \in V/M$, by the definition of $\tilde{\omega}$, we have

$$\tilde{\omega}([x], [y]) = \omega(x, y) = -\omega(y, x) = -\tilde{\omega}([y], [x])$$

□

In the last theorem of this argument, we will examine whether the form $\tilde{\omega}$ in (1.4.1) is non-degenerate on the quotient space V/M.

Theorem 1.4.5. *Let $(V, \|\cdot\|_V)$ be either an even finite-dimensional or a separable infinite-dimensional real normed space, and ω be a bounded skew-symmetric bilinear form on the space V, and let M be a closed subspace contained in V^ω. Consider the quotient space V/M equipped with the quotient norm*

$$\|[v]\| = \inf_{m \in M} \|v - m\|_V \quad \textit{for every} \quad [v] \in V/M.$$

Also, consider the bilinear form

$$\tilde{\omega}([x], [y]) = \omega(x, y) \quad \textit{for every} \quad [x], [y] \in V/M.$$

This form $\tilde{\omega}$ is nondegenerate on the quotient space V/M if and only if $M = V^\omega$.

Proof. Let L_ω be the induced map of ω. Since V^ω is the largest subspace in V on which the form ω vanishes, then

$$V^\omega = \ker(L_\omega).$$

Now, we can define the induced map of $\tilde{\omega}$ on $V/\ker(L_\omega)$ as follows: for every $[x], [y] \in V/M$

$$L_{\tilde{\omega}}([x])([y]) = \tilde{\omega}([x], [y]) = \omega(x, y) = L_\omega(x)(y).$$

Recall the First Isomorphism Theorem for Normed Spaces to obtain

$$V/\ker(L_\omega) \cong \mathrm{Rang}(L_\omega)$$

where $\mathrm{Rang}(L_\omega)$ is the range of L_ω. This theorem ensures that the induced map of $\tilde{\omega}$

$$L_{\tilde{\omega}} : V/\ker(L_\omega) \to \mathrm{Rang}(L_\omega)$$

$$L_{\tilde{\omega}}([x]) = L_\omega(x)$$

is bijective, which implies that $\tilde{\omega}$ is nondegenerate on $V/\ker(L_\omega)$. Now, we have shown that if

$$M = V^\omega = \ker(L_\omega), \tag{1.4.4}$$

then $\tilde{\omega}$ is nondegenerate on V/M. But, if

$$M \subsetneq V^\omega = \ker(L_\omega),$$

then there exists a nonzero $x \in V^\omega \backslash M$ representing the class $[x] = x + M \in V/M$ such that

$$L_{\tilde{\omega}}([x])([y]) = 0 \ \ \forall [y] \in V/M \implies [x] \in \ker(L_{\tilde{\omega}}).$$

This means that $\ker(L_{\tilde{\omega}}) \neq \{[0]\}$, and thus the form $\tilde{\omega}$ is degenerate on V/M. Therefore, we can conclude that $\tilde{\omega}$ is nondegenerate if and only if $M = V^\omega$. □

Remark 1.4.6. *From the above four theorems, we can conclude that a bounded skew-symmetric bilinear form ω on a real normed space V induces a quotient symplectic normed space $(V/M, \tilde{\omega})$ if and only if*

- $M = V^{\omega}$,
- *or equivalently* $M = \ker(L_{\omega})$ *where* L_{ω} *is the induced map of* ω.

It may also be worth to consider the case where U is a subspace of a real symplectic space (V, ω) and M is another subspace contained in U. In this case, we need to examine when the quotient space U/M carries a symplectic structure. See the following Theorem.

Theorem 1.4.7. *Let (V, ω) be either an even finite-dimensional or a separable infinite-dimensional real symplectic space, let U be any closed subspace, and M be another closed subspace contained in U. Consider the quotient space $(U/M, \|\cdot\|)$ where*

$$\|[u]\| = \inf_{m \in M} \|u - m\|_V \quad \textit{for every} \quad [u] \in U/M.$$

Also, consider the form

$$\tilde{\omega}([x], [y]) = \omega(x, y) \quad \textit{for every} \quad [x], [y] \in U/M,$$

This quotient space $(U/M, \tilde{\omega})$ is symplectic if and only if M is isotropic and $M = U \cap U^{\omega}$.

Proof. In Remark 1.4.6, we have stated that for M as a closed subspace of V, the quotient space $(V/M, \tilde{\omega})$ is symplectic if and only if $M = V^{\omega}$. However, the subspace V^{ω} is originally the largest space on which the symplectic form ω vanishes; that is,

$$M = \{x \in V \mid \omega(x, y) = 0 \text{ for all } y \in V\}.$$

By adopting this definition of subspace M in the case of examining $(U/M, \tilde{\omega})$, the subspace M will also be the largest space on which the restriction of the symplectic form $\omega|_{U \times U}$ vanishes. In other words,

$$M = \{x \in U \mid \omega(x, y) = 0 \text{ for all } y \in U\}.$$

This is exactly,

$$M = U \cap U^{\omega}.$$

Thus, from Remark 1.4.6, the pair $(U/M, \tilde{\omega})$ is a symplectic space if and only if $M = U \cap U^{\omega}$. □

Remark 1.4.8. *The above theorem provides an approach to constructing the symplectic quotient space $(U/(U \cap U^{\omega}), \tilde{\omega})$ from the subspace U, under only a single condition, which is the closedness of U. In fact, the closedness of the subspace U is an essential condition to guarantee the closedness of the intersection $U \cap U^{\omega}$. See the next example, which illustrates that $U \cap U^{\omega}$ is not always closed if U is not closed.*

Example 1.4.9. *Consider (ℓ^2, ω) such that*

$$\omega(x, y) = \sum_{k=1}^{\infty} (x_{2k-1} y_{2k} - y_{2k-1} x_{2k})$$

where $x = (x_k)_k$ *and* $y = (y_k)_k$ *are in* ℓ^2*, and consider the symplectic basis*

$$S = \bigcup_{k=1}^{\infty} \{e_k, f_k\}$$

where

$$e_k = (0, \dots, 0, 1^{2k-1}, 0, \dots), \quad \textit{and} \quad f_k = (0, \dots, 0, 1^{2k}, 0, \dots).$$

Thereafter, if we consider the non-closed subspace

$$U = \operatorname{Lin} \bigcup_{k=1}^{\infty} \{e_k, f_{2k}\}$$

and its symplectic complement

$$U^\omega = \operatorname{Clin} \bigcup_{k=1}^{\infty} \{e_{2k-1}\},$$

then the intersection

$$U \cap U^\omega = \operatorname{Lin} \bigcup_{k=1}^{\infty} \{e_{2k-1}\},$$

is a non-closed subspace.

However, in the case of M being a coisotropic subspace of a real symplectic space (V, ω), the intersection $M \cap M^\omega = M^\omega$ is always closed. The following corollary will be a particular case of Theorem 1.4.7, as it provides us with a method to construct a symplectic quotient space from a coisotropic subspace of a symplectic space. This corollary is a generalisation of Lemma 2.7 (i) in [27, p. 41], as it includes separable infinite-dimensional symplectic spaces, not just finite-dimensional spaces.

Corollary 1.4.10. *Let* $(V, \|\cdot\|_V)$ *be either an even finite-dimensional or a separable infinite-dimensional real normed space equipped with a symplectic form* ω*, and let* M *be a coisotropic subspace. Then, the symplectic form* ω *induces another symplectic form* $\tilde{\omega}$ *on the quotient nomed space* $(M/M^\omega, \|\cdot\|)$ *where* $\|\cdot\|$ *is the quotient norm:*

$$\|[v]\| = \inf_{m \in M^\omega} \|v - m\|_V \quad \textit{for every} \quad [v] \in M/M^\omega,$$

and the induced symplectic form is defined as

$$\tilde{\omega}([x], [y]) = \omega(x, y) \quad \textit{for every} \quad [x], [y] \in M/M^\omega.$$

Proof. Since M is a coisotropic, then, by Theorem 1.2.7, then $M^\omega = M \cap M^\omega$ because $M^\omega \subseteq M$. Thus, from Theorem 1.4.7, the quotient space $(M/M^\omega, \tilde{\omega})$ is a symplectic space. □

Furthermore, in [27], D. McDuff and D. Salamon state that if we have a Lagrangian subspace in an even finite-dimensional real symplectic space, then we can construct a corresponding Lagrangian subspace in the induced quotient space. In the following theorem, we will state a similar result, but we will generalise it to cover separable infinite-dimensional real symplectic spaces, not just finite-dimensional spaces.

Theorem 1.4.11. *Let* (V, ω) *be either an even finite-dimensional or a separable infinite-dimensional real symplectic space, let* M *be a coisotropic subspace, and* L *be a Lagrangian subspace such that* $L + M^\omega$ *is closed. Then,*

$$\tilde{L} = ((L \cap M) + M^\omega)/M^\omega$$

is a Lagrangian subspace in the symplectic space $(M/M^\omega, \tilde{\omega})$ *where*

$$\tilde{\omega}([x],[y]) = \omega(x,y) \quad \textit{for every} \quad [x],[y] \in M/M^\omega.$$

Proof. To show that $\tilde{L}$ is Lagrangian, we need to verify the equality

$$\tilde{L}^{\tilde{\omega}} = \tilde{L}. \tag{1.4.5}$$

To achieve that, note that we have

- $L \cap M \subseteq L$,
- L is Lagrangian,
- M is coisotropic.

This implies that

- $L \cap M$ is isotropic
- M^ω is isotropic.

This means

- $\omega(l_1, l_2) = 0$ for all $l_1, l_2 \in L \cap M$,
- $\omega(m_1, m_2) = 0$ for all $m_1, m_2 \in M^\omega$.

Now, since every $[x],[y] \in \tilde{L}$ we can write

- $[x] = l_1 + m_1 + M^\omega = l_1 + M^\omega$ where $l_1 \in L \cap M,\ m_1 \in M^\omega$
- $[y] = l_2 + m_2 + M^\omega = l_2 + M^\omega$ where $l_2 \in L \cap M,\ m_2 \in M^\omega$,

then, by definition of $\tilde{\omega}$, the following statements are equivalent:

- $\omega(l_1, l_2) = 0$ for all $l_1, l_2 \in L \cap M$.
- $\tilde{\omega}([x],[y]) = 0$ for all $[x],[y] \in \tilde{L}$.

This means

$$\tilde{L} = ((L \cap M) + M^\omega)/M^\omega$$

is isotropic. That is,

$$\tilde{L} \subseteq \tilde{L}^{\tilde{\omega}}. \tag{1.4.6}$$

On the other hand. Assume that $[x] \in \tilde{L}^{\tilde{\omega}}$. Thus

$$\begin{aligned}
[x] \in \tilde{L}^{\tilde{\omega}} &\iff \tilde{\omega}([x],[y]) = 0 \ \ \forall [y] \in \tilde{L} \\
&\iff \omega(x,y) = 0 \ \ \forall y \in (L \cap M) + M^\omega \\
&\iff \omega(x, l' + m') = 0 \ \ \forall y = l' + m' \in (L \cap M) + M^\omega \\
&\iff \omega(x, l' + m') = 0 \ \ \forall l' \in L \cap M,\ \forall m' \in M^\omega \\
&\iff \omega(x, l') = 0 \ \ \forall l' \in L \cap M \ \text{ and } \ \omega(x, m') = 0 \ \ \forall m' \in M^\omega \\
&\iff \omega(x, l') = 0 \ \ \forall l' \in L \cap M \\
&\iff x \in (L \cap M)^\omega \cap M.
\end{aligned}$$

Hence

$$[x] \in ((L \cap M)^\omega \cap M)/M^\omega.$$

By Theorem 1.2.7, we have

$$[x] \in ((L \cap M)^\omega \cap M)/M^\omega = ((\overline{L^\omega + M^\omega}) \cap M)/M^\omega = ((\overline{L + M^\omega}) \cap M)/M^\omega.$$

Hence, the following holds.

$$\begin{aligned}
x \in (\overline{L + M^\omega}) \cap M &\implies x \in L + M^\omega \text{ and } x \in M \\
&\implies x = l + m \text{ where } l \in L,\ m \in M^\omega \text{ and } x \in M \\
&\implies m \in M^\omega,\ l = x - m \in L \text{ and } l = x - m \in M \text{ since } M^\omega \subseteq M \\
&\implies l \in L \cap M \text{ and } m \in M^\omega \\
&\implies x = l + m \in (L \cap M) + M^\omega \\
&\implies [x] \in ((L \cap M) + M^\omega)/M^\omega \\
&\implies [x] \in \tilde{L}.
\end{aligned}$$

Therefore, we assumed that $[x] \in \tilde{L}^{\tilde{\omega}}$ and then we ended with $[x] \in \tilde{L}$. This results in

$$\tilde{L}^{\tilde{\omega}} \subseteq \tilde{L}. \tag{1.4.7}$$

We conclude, from (1.4.6) and (1.4.7), that the equality (1.4.5) holds. □

Remark 1.4.12. *In the above theorem, we have proved that, from any coisotropic subspace M and Lagrangian subspace L such that $L + M^\omega$ is closed, we can construct a Lagrangian subspace in the symplectic quotient space M/M^ω. Similarly, we can construct coisotropic subspaces and isotropic subspaces in the quotient space M/M^ω. Further details are provided in the next two theorems.*

The following theorem is devoted to the case of coisotropic subspaces

Theorem 1.4.13. *Let (V, ω) be either an even finite-dimensional or a separable infinite-dimensional real symplectic space, let M be a coisotropic subspace, and U be a closed coisotropic subspace such that $U + M^\omega$ is closed. Then,*

$$\tilde{U} = ((U \cap M) + M^\omega)/M^\omega$$

is a coisotropic subspace in the symplectic space $(M/M^\omega, \tilde{\omega})$ where

$$\tilde{\omega}([x], [y]) = \omega(x, y) \quad \textit{for every} \quad [x], [y] \in M/M^\omega.$$

Proof. We can show that $\tilde{U}$ is coisotropic by verifying the inequality

$$\tilde{U}^{\tilde{\omega}} \subseteq \tilde{U}. \tag{1.4.8}$$

To achieve this, note that

$$\tilde{U}^{\tilde{\omega}} = \{[x] \in M/M^\omega \mid \tilde{\omega}([x], [y]) = 0 \text{ for all } [y] \in \tilde{U}\}.$$

Assume that $[x] \in \tilde{U}^{\tilde{\omega}}$. We aim to show that $[x] \in \tilde{U}$. To this end, observe that

$$\begin{aligned}
[x] \in \tilde{U}^{\tilde{\omega}} &\iff \tilde{\omega}([x],[y]) = 0 \ \ \forall [y] \in \tilde{U} \\
&\iff \omega(x,y) = 0 \ \ \forall y \in (U \cap M) + M^{\omega} \\
&\iff \omega(x, u' + m') = 0 \ \ \forall y = u' + m' \in (U \cap M) + M^{\omega} \\
&\iff \omega(x, u' + m') = 0 \ \ \forall u' \in U \cap M, \ \forall m' \in M^{\omega} \\
&\iff \omega(x,u') = 0 \ \ \forall u' \in U \cap M \ \text{ and } \ \omega(x,m') = 0 \ \ \forall m' \in M^{\omega} \\
&\iff \omega(x,u') = 0 \ \ \forall u' \in U \cap M \\
&\iff x \in (U \cap M)^{\omega} \cap M.
\end{aligned}$$

Thereafter, by Theorem 1.2.7, we have

$$[x] \in ((U \cap M)^{\omega} \cap M)/M^{\omega} = ((\overline{U^{\omega} + M^{\omega}}) \cap M)/M^{\omega} \subseteq ((\overline{U + M^{\omega}}) \cap M)/M^{\omega}.$$

Hence, the following holds.

$$\begin{aligned}
x \in (\overline{U + M^{\omega}}) \cap M &\implies x \in U + M^{\omega} \ \text{ and } \ x \in M \\
&\implies x = u + m \ \text{ where } \ u \in U, \ m \in M^{\omega} \ \text{ and } \ x \in M \\
&\implies m \in M^{\omega}, \ u = x - m \in U \ \text{ and } \ u = x - m \in M \ \text{ since } \ M^{\omega} \subseteq M \\
&\implies u \in U \cap M \ \text{ and } \ m \in M^{\omega} \\
&\implies x = u + m \in (U \cap M) + M^{\omega} \\
&\implies [x] \in ((U \cap M) + M^{\omega})/M^{\omega} \\
&\implies [x] \in \tilde{U}.
\end{aligned}$$

In conclusion, we have shown that

$$[x] \in \tilde{U}^{\tilde{\omega}} \implies [x] \in \tilde{U}.$$

This means $\tilde{U}^{\tilde{\omega}} \subseteq \tilde{U}$; that is, $\tilde{U}$ is coisotropic in $(M/M^{\omega}, \tilde{\omega})$. □

The following theorem concerns the case of isotropic subspaces.

Theorem 1.4.14. *Let (V,ω) be either an even finite-dimensional or a separable infinite-dimensional real symplectic space, let M be a coisotropic subspace, and U be an isotropic subspace. Then,*

$$\tilde{U} = ((U \cap M) + M^{\omega})/M^{\omega}$$

is an isotropic subspace in the symplectic space $(M/M^{\omega}, \tilde{\omega})$ where

$$\tilde{\omega}([x],[y]) = \omega(x,y) \quad \textit{for every} \quad [x],[y] \in M/M^{\omega}.$$

Proof. We can show that $\tilde{U}$ is isotropic by verifying the inequality

$$\tilde{U} \subseteq \tilde{U}^{\tilde{\omega}}. \tag{1.4.9}$$

To achieve this, note that since U is isotropic and $M \cap U \subseteq U$, then $M \cap U$ is isotropic, which means

$$M \cap U \subseteq (M \cap U)^{\omega}.$$

By definition, for every $[x],[y] \in \tilde{U}$, we can write

- $x = u + m$ for some $u \in U \cap M,\ m \in M^\omega$,
- $y = u' + m'$ for some $u' \in U \cap M,\ m' \in M^\omega$.

Now, by definition of $\tilde{\omega}$, the following statements are equivalent:

- $\omega(u, u') = 0 \ \ \forall u, u' \in U \cap M$
- $\omega(u, u') + \omega(u, m') + \omega(m, u') + \omega(m, m') = 0 \ \ \forall u, u' \in U \cap M,\ m, m' \in M^\omega$
- $\omega(u + m, u' + m') = 0 \ \ \forall u, u' \in U \cap M,\ m, m' \in M^\omega$
- $\omega(x, y) = 0$ for all $x, y \in (U \cap M) + M^\omega$
- $\tilde{\omega}([x], [y]) = 0$ for all $[x], [y] \in \tilde{U}$.

This implies $\tilde{U} \subseteq \tilde{U}^{\tilde{\omega}}$; that is, the subspace $\tilde{U}$ is isotropic in $(M/M^\omega, \tilde{\omega})$. □

Furthermore, in a similar way, we can construct a symplectic subspace in an induced quotient space.

Theorem 1.4.15. *Let (V, ω) be either an even finite-dimensional or a separable infinite-dimensional real symplectic space, let M be a coisotropic subspace, and U be a symplectic subspace and contained in M. Then,*

$$\tilde{U} = (U + M^\omega)/M^\omega$$

is a symplectic subspace in the symplectic space $(M/M^\omega, \tilde{\omega})$ where

$$\tilde{\omega}([x], [y]) = \omega(x, y) \quad \textit{for every} \quad [x], [y] \in M/M^\omega.$$

Proof. To prove that $\tilde{U}$ is symplectic, we need to show that the restriction $\omega|_{\tilde{U} \times \tilde{U}}$ is nondegenerate. To achieve this, note that since M is coisotropic and U is symplectic contained in M, then

- $M^\omega \subseteq M$,
- $U^\omega \cap U = \{0\}$,
- $U \subseteq M$.

Assume that $[x] \in \tilde{U}$. Then, we can write

$$x = u + m \ \text{ for some } \ u \in U,\ m \in M^\omega.$$

By definition of $\tilde{\omega}$, observe that

$$\begin{aligned}
\tilde{\omega}([x], [y]) = 0 \ \ \forall [y] \in \tilde{U} \ &\implies \omega(x, y) = 0 \ \ \forall y \in U \\
&\implies \omega(u + m, u' + m') = 0 \ \ \forall u' + m' \in U + M^\omega \\
&\implies \omega(u + m, u' + m') = 0 \ \ \forall u' \in U,\ \forall m' \in M^\omega \\
&\implies \omega(u + m, u') = 0 \ \ \forall u' \in U \ \text{ and } \ \omega(u + m, m') = 0 \ \ \forall m' \in M^\omega \\
&\implies \omega(u, u') = 0 \ \ \forall u' \in U \ \text{ and } \ \omega(m, m') = 0 \ \ \forall m' \in M^\omega \\
&\implies \omega(u, u') = 0 \ \ \forall u' \in U \\
&\implies u \in U^\omega.
\end{aligned}$$

This leads to

$$u \in U^\omega \cap U = \{0\} \implies u = 0.$$

That is,

$$[x] = m + M^\omega \text{ where } m \in M^\omega \implies [x] = [0].$$

Since we have proved that

$$\omega|_{\tilde{U}\times\tilde{U}}([x],[y]) = 0 \text{ for all } [y] \in \tilde{U} \implies [x] = [0],$$

then the restriction $\omega|_{\tilde{U}\times\tilde{U}}$ is nondegenerate. □

Remark 1.4.16. *The symplectic subspace, in Theorem 1.4.15, can be written as*

$$\tilde{U} = ((U \cap M) + M^\omega)/M^\omega,$$

which is the same as the formulas for Lagrangian, coisotropic, and isotropic subspaces in Theorem 1.4.11, Theorem 1.4.13, and Theorem 1.4.14, respectively. However, since, in Theorem 1.4.15, we added an extra assumption ($U \subseteq M$), we wrote $U \cap M$ as U. In fact, without assuming that $U \subseteq M$, the intersection $M \cap U$ may fail to be symplectic in (V, ω). As a result,

$$\tilde{U} = ((M \cap U) + M^\omega)/M^\omega$$

may fail to be symplectic in $(M/M^\omega, \tilde{\omega})$. The following example illustrates that.

Example 1.4.17. *Consider the symplectic space $(\mathbb{R}^4, \omega)$ where the symplectic form is given by:*

$$\omega\big((x_1, y_1, x_2, y_2), (x_1', y_1', x_2', y_2')\big) = x_1y_1' - y_1x_1' + x_2y_2' - y_2x_2'.$$

Consider the coisotropic subspace

$$M = \{(0, y_1, x_2, y_2) \mid x_1, y_1, x_2 \in \mathbb{R}\}.$$

Its symplectic complement is:

$$M^\omega = \{(0, y_1, 0, 0) \mid y_1 \in \mathbb{R}\}.$$

Also, consider the symplectic subspace

$$U = \{(x_1, 0, x_2, 0) \mid x_1, x_2 \in \mathbb{R}\}.$$

Its symplectic complement is:

$$U^\omega = \{(0, y_1, 0, y_2) \mid y_1, y_2 \in \mathbb{R}\}.$$

Now, the intersection is:

$$M \cap U = \{(0, 0, x_2, 0) \mid x_2 \in \mathbb{R}\}.$$

Its symplectic complement is:

$$(M \cap U)^\omega = \{(x_1, y_1, x_2, 0) \mid x_1, y_1, x_2 \in \mathbb{R}\}.$$

Now,

$$(M \cap U) \cap (M \cap U)^\omega = \{(0, 0, x_2, 0) \mid x_2 \in \mathbb{R}\} \neq \{(0, 0, 0, 0)\},$$

which means that the intersection $M \cap U$ is not symplectic. Furthermore, if we consider the quotient space M/M^ω with the symplectic form

$$\tilde{\omega}([x],[y]) = \omega(x, y) \quad \textit{for every} \quad [x],[y] \in M/M^\omega,$$

then

$$\tilde{U} = ((M \cap U) + M^{\omega})/M^{\omega}$$

is not symplectic. This is because $\tilde{\omega}$ is degenerate on $\tilde{U}$. To illustrate this, note, by definition of $\tilde{\omega}$, that

$$(0,0,1,0) \notin M^{\omega} \text{ and } \omega((0,0,1,0),(x_1,y_1,x_2,y_2)) = 0 \ \forall (x_1,y_1,x_2,y_2) \in M \cap U$$

is equivalent to

$$[(0,0,1,0)] \neq [(0,0,0,0)] \text{ and } \tilde{\omega}([(0,0,1,0)],[(x_1,y_1,x_2,y_2)]) = 0 \ \forall [(x_1,y_1,x_2,y_2)] \in \tilde{U}.$$

1.5 Complex Structure and Decomposition of a Symplectic Space

Let (V,ω) be a real symplectic space and ω be strongly nondegenerate. Also, let V be equipped with a norm $\|\cdot\|$ equivalent to a Hilbert norm $\|\cdot\|_H$, which means there exist constants $C_1, C_2 > 0$ such that

$$C_1\|\cdot\|_H \leqslant \|\cdot\| \leqslant C_2\|\cdot\|_H, \tag{1.5.1}$$

and the norm $\|\cdot\|_H$ is inherited from an inner product $h(\cdot,\cdot)$; that is, $(V,h(\cdot,\cdot))$ is a Hilbert space with $h(\cdot,\cdot) = \|\cdot\|_H^2$. In this section, we will introduce the definition of a complex structure. Then, we will construct a complex structure providing a connection between the above Banach structure and the symplectic structure. After that, we will show its important role for the decomposition of symplectic spaces.

We begin by introducing the definition of a complex structure.

Definition 1.5.1. *[6, p.108] An isomorphism $J : V \to V$ is called a complex structure on a real Banach space V if $J^2 = -I$ (or equivalently, $J^{-1} = -J$).*

Since we aim to introduce a complex structure as a link between the above Banach structure and symplectic structure of V, then, as will be seen later, the complex structure J must be chosen to be compatible with the symplectic form ω. See the following definition.

Definition 1.5.2. *[6, p.84] Let (V,ω) be a real symplectic space. We say that a complex structure $J : V \to V$ is compatible with ω or ω-compatible if for every nonzero $u, v \in V$ we have*

$$\omega(u,Ju) > 0 \quad \text{and} \quad \omega(Ju,Jv) = \omega(u,v). \tag{1.5.2}$$

Lemma 1.5.3. *If (V,ω) is a real symplectic space and $J : V \to V$ is a ω-compatible complex structure, then for every $u, v \in V$ we have*

$$\omega(u,Jv) = -\omega(Ju,v) \tag{1.5.3}$$

Proof. For every $u, v \in V$, we have

$$\omega(u,Jv) = \omega(Ju,J^2v) = \omega(Ju,-v) = -\omega(Ju,v).$$

□

Now, let us start constructing the required complex structure $J : V \to V$. As we stated above, the real symplectic Banach space V is equipped with a strongly nondegenerate ω and its norm $\|\cdot\|$ is equivalent to a norm $\|\cdot\|_H$, which is inherited from an inner product $h(\cdot,\cdot)$ on V (i.e. $\|\cdot\|_H^2 = h(\cdot,\cdot)$). Thereafter, from the Riesz Theorem and the strong nondegeneracy of ω, the forms $h(\cdot,\cdot)$ and $\omega(\cdot,\cdot)$ induce two isomorphisms:

$$L_h : V \to V^*;\ L_h(v) = h(v,\cdot) \quad \text{and} \quad L_\omega : V \to V^*;\ L_\omega(v) = \omega(v,\cdot) \quad \text{for every} \quad v \in V. \quad (1.5.4)$$

After that, we can define

$$A := L_h^{-1} \circ L_\omega : V \to V$$

Note that, the isomorphism A is skew-symmetric (i.e. $A^* = -A$) w.r.t $h(\cdot,\cdot)$ because for every $x, y \in V$ we have

$$\begin{aligned} h(Ax, y) &= L_h(Ax)(y) \\ &= L_h \circ L_h^{-1} \circ L_\omega(x)(y) \\ &= L_\omega(x)(y) \\ &= \omega(x,y) \\ &= -\omega(y,x) \\ &= L_\omega(-y)(x) \\ &= L_h \circ L_h^{-1} \circ L_\omega(-y)(x) \\ &= L_h(-Ay)(x) \\ &= h(-Ay, x) \\ &= h(x, -Ay). \end{aligned}$$

Now, we can put

$$J := |A|^{-1} A : V \to V$$

where $|A| = (A^*A)^{\frac{1}{2}}$. It remains to show that J is, indeed, a ω-compatible complex structure. To this end, observe

- $J^2 = (|A|^{-1}A)(|A|^{-1}A) = |A|^{-2}A^2 = (A^*A)^{-1}A^2 = -A^{-2}A^2 = -I.$

- J is a symplectomorphism because for every $x, y \in V$ we have

$$\begin{aligned}
\omega(Jx, Jy) &= L_\omega(Jx)(Jy) \\
&= L_h(AJx)(Jy) \\
&= L_h(A|A|^{-1}Ax)(Jy) \\
&= L_h(A^2|A|^{-1}x)(Jy) \\
&= L_h(-A^*A|A|^{-1}x)(Jy) \\
&= L_h(-|A|^2|A|^{-1}x)(Jy) \\
&= L_h(-|A|x)(Jy) \\
&= h(-|A|x, Jy) \\
&= h(-|A|x, |A|^{-1}Ay) \\
&= h(-A^*(|A|^{-1})^*|A|x, y) \\
&= h(Ax, y) \\
&= L_h(Ax)(y) \\
&= L_h \circ L_h^{-1} \circ L_\omega(x)(y) \\
&= L_\omega(x, y) \\
&= \omega(x, y).
\end{aligned}$$

- The last condition, which states:

$$\omega(x, Jx) > 0 \quad \text{for every nonzero} \quad x \in V \tag{1.5.5}$$

  will be proven later in Lemma 1.5.4.

Note that, in the above argument, the skew-symmetric isomorphism $A : V \to V$ connects the inner product h with the symplectic form ω via the following identity.

$$\omega(x, y) = h(Ax, y) \quad \text{for every} \quad x, y \in V. \tag{1.5.6}$$

Since $J = |A|^{-1}A$, then, for every $x, y \in V$, we can obtain the following identity

$$\omega(x, Jy) = h(|A|x, y). \tag{1.5.7}$$

This is because

$$\begin{aligned}
\omega(x, Jy) &= h(Ax, Jy) \\
&= h(Ax, |A|^{-1}Ay) \\
&= h(A^*|A|^{-1}Ax, y) \\
&= h(A^*A|A|^{-1}x, y) \\
&= h(|A|^2|A|^{-1}x, y) \\
&= h(|A|x, y).
\end{aligned}$$

If we put

$$g(x, y) = h(|A|x, y) \quad \text{for every} \quad x, y \in V, \tag{1.5.8}$$

then we obtain a bilinear form on V satisfying

$$\omega(x, Jy) = g(x, y) \quad \text{for every} \quad x, y \in V. \tag{1.5.9}$$

The following lemma proves that g is indeed an inner product on V.

Lemma 1.5.4. *If $(V, h(\cdot,\cdot))$ is a real Hilbert space and $P : V \to V$ is a positive isomorphism, then*

$$g(\cdot,\cdot) = h(P\cdot,\cdot).$$

is an inner product.

Proof. Since $P \geqslant 0$, then for each $x \in V$, we have $g(x,x) = h(Px,x) \geqslant 0$. The bilinearity of g follows immediately from the bilinearity of h and the linearity of P. Also, the symmetry of g follows from the symmetry of h and the positivity of P. To illustrate that, note that for every $x, y \in V$ we have

$$g(x,y) = h(Px,y) = h(x,Py) = h(Py,x) = g(y,x).$$

It remains to show that

$$g(x,x) = 0 \iff x = 0,$$

which is equivalent to

$$h(Px,x) = 0 \iff x = 0.$$

One direction is straightforward. Thus, we need only to show the other direction, which is

$$h(Px,x) = 0 \implies x = 0.$$

To this end, assume that $h(Px,x) = 0$ and consider

$$y = \|x\|P^{-1}x - \lambda x$$

where $\|x\| = h(x,x)^{\frac{1}{2}}$ and λ is a real number, will be chosen later. Thus,

$$\begin{aligned}
0 \leqslant h(Py,y) &= h\left(\|x\|x - \lambda Px, \|x\|P^{-1}x - \lambda x\right) \\
&= \|x\|^2 h\left(x, P^{-1}x\right) - \lambda\|x\|h(x,x) - \lambda\|x\|h(Px,P^{-1}x) + \lambda^2 h(Px,x) \\
&= \|x\|^2 h\left(x, P^{-1}x\right) - \lambda\|x\|h(x,x) - \lambda\|x\|h(x,PP^{-1}x) \\
&= \|x\|^2 h(x,P^{-1}x) - 2\lambda\|x\|h(x,x) \\
&= \|x\|^2 h(x,P^{-1}x) - 2\lambda\|x\|^3 \quad \text{(By using } h(x,x) = \|x\|^2\text{)} \\
&\leqslant \|x\|^2\|x\|\|P^{-1}x\| - 2\lambda\|x\|^3 \quad \text{(By applying Cauchy Schwarz Inequality on } h(x,P^{-1}x)\text{)} \\
&\leqslant \|P^{-1}\|\|x\|^4 - 2\lambda\|x\|^3 \quad \text{(From boundedness of } P^{-1}\text{)} \\
&= \left(\|P^{-1}\|\|x\| - 2\lambda\right)\|x\|^3.
\end{aligned}$$

Now, if we choose

$$\lambda = \|P^{-1}\|\|x\|,$$

we obtain

$$0 \leqslant -\|P^{-1}\|\|x\|^4 \implies \|x\| = 0 \implies x = 0.$$

We conclude that g is, indeed, an inner product. □

Remark 1.5.5. *One can note that if a complex structure $J : V \to V$ and an inner product $g(\cdot,\cdot)$ satisfying*

$$\omega(\cdot, J\cdot) = g(\cdot,\cdot),$$

then the last condition (1.5.5) above:

$$\omega(x,Jx) = g(x,x) > 0 \quad \textit{for every nonzero} \quad x \in V,$$

always holds. Consequently, J is compatible with ω.

Remark 1.5.6. *The above symplectic form ω will always be strongly nondegenerate because, as a consequence of (1.5.9), the induced map*

$$L_\omega : V \to V^*$$

can be given as

$$L_\omega(v)(\cdot) = L_g(Jv)(\cdot),$$

which is bijective from the Riesz Theorem.

It is common and helpful to write symplectic Banach spaces as a direct sum of a closed subspace and its symplectic complement. More precisely, if (V, ω) is either an even finite-dimensional or a separable infinite-dimensional real symplectic Banach space and its norm satisfies (1.5.1), can V be written as

$$V = M \oplus M^\omega \tag{1.5.10}$$

for every closed subspace $M \subset V$? Obviously, the answer is negative because the necessary condition for (1.5.10) to hold is:

$$M \cap M^\omega = \{0\}.$$

However, this only holds if M is a symplectic subspace. However, this issue can be settled by using a ω-compatible complex structure $J : V \to V$ to obtain

$$JM \cap M^\omega = \{0\} \tag{1.5.11}$$

for every closed subspace $M \subset V$. This enables us to study the possibility of writing

$$V = JM \oplus M^\omega. \tag{1.5.12}$$

for every closed subspace $M \subset V$. Observe, from the first condition in (1.5.2), the property (1.5.11) always holds for any closed subspace $M \subset V$ because

$$\begin{aligned} x \neq 0 \text{ and } x \in JM \cap M^\omega &\implies \exists m_0 \in M \backslash \{0\} \text{ such that } x = Jm_0 \text{ and } x \in M^\omega \\ &\implies \omega(Jm_0, m) = 0, \ \forall m \in M \text{ since } Jm_0 = x \in M^\omega \\ &\implies \omega(Jm_0, m_0) = 0, \end{aligned}$$

which contradicts the first condition of compatibility, which is $\omega(Jm_0, m_0) > 0$.

The following theorem addresses the above question regarding the decomposition (1.5.12).

Theorem 1.5.7. *Let (V, ω) be either an even finite-dimensional or a separable infinite-dimension real symplectic Banach space equipped with a norm $\|\cdot\|$ equivalent to a Hilbert norm $\|\cdot\|_H$, the symplectic form ω be strongly nondegenerate, and M be a closed subspace of V. Then there exists a ω-compatible complex structure $J : V \to V$ such that*

$$V = JM \oplus M^\omega. \tag{1.5.13}$$

Proof. Let h be the inner product that induces $\|\cdot\|_H$; that is,

$$h(x, x) = \|x\|_H^2 \quad \text{for every} \quad x \in H.$$

By the Riesz Theorem, we obtain an isomorphism

$$L_h : V \to V^* \quad \text{such that} \quad L_h(v) = h(v, \cdot) \quad \text{for every} \quad v \in V.$$

Since the symplectic form ω is strongly nondegenerate, we also obtain an isomorphism

$$L_\omega : V \to V^* \quad \text{such that} \quad L_\omega(v) = \omega(v, \cdot) \quad \text{for every} \quad v \in V.$$

As we constructed above, we obtain the ω-compatible complex structure

$$J := |A|^{-1} A : V \to V$$

where

$$A := L_h^{-1} \circ L_\omega : V \to V.$$

By Lemma 1.5.4, we also obtain an inner product on V

$$g(\cdot, \cdot) = h(|A|\cdot, \cdot).$$

Since M is a closed subspace, then there is a g-complement subspace M^g such that

$$V = M \oplus M^g$$

where

$$M^g = \{v \in V \mid g(v, m) = 0 \ \forall m \in M\}.$$

Now, we need to show that the ω-compatible complex structure J satisfies

$$J\left(M^\omega\right) = M^g. \tag{1.5.14}$$

To achieve that, note that, for every $u \in V$, the following statements are equivalent:

- $v \in J\left(M^\omega\right)$.
- $J^{-1}v \in M^\omega$.
- $\omega(J^{-1}v, m) = 0 \ \forall m \in M$.
- $L_\omega(J^{-1}v)(m) = 0 \ \forall m \in M$.
- $L_\omega(A^{-1}|A|v)(m) = 0 \ \forall m \in M$.
- $L_h(|A|v)(m) = 0 \ \forall m \in M$.
- $h(|A|v, m) = 0 \ \forall m \in M$.
- $g(v, m) = 0 \ \forall m \in M$.
- $v \in M^g$.

This means

$$M^g = J\left(M^\omega\right).$$

Therefore,

$$V = M \oplus J\left(M^\omega\right).$$

By applying J on both sides,

$$JV = JM \oplus J^2\left(M^\omega\right),$$

By using the surjectivity of J and $J^2 = -I$,

$$V = JM \oplus M^\omega.$$

This completes the proof. □

Alternatively, we can have a decomposition

$$V = M \oplus (JM)^{\omega}. \tag{1.5.15}$$

But proving this needs the following Lemma

Lemma 1.5.8. *Let (V, ω) be a real symplectic space, M be a subspace of V and $J : V \to V$ be a ω-comptible complex structure. Then*

$$J(M^{\omega}) = (JM)^{\omega}.$$

Proof.

$$\begin{aligned} x \in M^{\omega} &\implies \omega(x, y) = 0, \ \forall y \in M \\ &\implies \omega(Jx, Jy) = 0, \ \forall y \in M \\ &\implies \omega(Jx, z) = 0, \ \forall z \in JM \\ &\implies Jx \in (JM)^{\omega}. \end{aligned}$$

Therefore,

$$J(M^{\omega}) \subseteq (JM)^{\omega}. \tag{1.5.16}$$

On the other hand,

$$\begin{aligned} x \in (JM)^{\omega} &\implies \omega(x, Jy) = 0, \ \forall y \in M \\ &\implies \omega(Jx, y) = 0, \ \forall y \in M \qquad \text{(from the equality (1.5.3))} \\ &\implies Jx \in M^{\omega} \\ &\implies J^2 x \in J(M^{\omega}) \\ &\implies x \in J(M^{\omega}) \qquad \text{(by using } J^2 = -I) \end{aligned}$$

Therefore,

$$(JM)^{\omega} \subseteq J(M^{\omega}). \tag{1.5.17}$$

From (1.5.16) and (1.5.17), we conclude that

$$(JM)^{\omega} = J(M^{\omega}).$$

□

As a result, we apply the complex structure J on both sides of

$$V = JM \oplus M^{\omega}$$

to obtain

$$JV = J^2 M \oplus J\left(M^{\omega}\right).$$

By using the surjectivity of J and $J^2 = -I$ and Lemma 1.5.8, we obtain the required decomposition

$$V = M \oplus (JM)^{\omega}.$$

Moreover, another result can be deduced from Lemma 1.5.8. If M is Lagrangian, then

$$M^{\omega} = M.$$

By Lemma 1.5.8,

$$(JM)^{\omega} = JM. \tag{1.5.18}$$

which means the ω-compatible complex structure preserves the Lagrangian property. Consequently, if M is a Lagrangian subspace, then the decomposition (1.5.15) becomes

$$V = M \oplus JM. \tag{1.5.19}$$

This shows that if M is Lagrangian, then it also has a Lagrangian complement w.r.t. the inner product

$$g(\cdot,\cdot) = \omega(\cdot, J\cdot).$$

Therefore, the space V can be written as a direct sum of two complementary Lagrangian subspaces L and L'

$$V = L \oplus L'. \tag{1.5.20}$$

Here, it may be helpful to present an example.

Example 1.5.9. *Consider the real symplectic Banach space $(\ell^2, \|\cdot\|_{\sim}, \omega)$ such that*

$$\omega(x,y) = \sum_{k=1}^{\infty} (x_{2k-1}y_{2k} - y_{2k-1}x_{2k})$$

and

$$\|x\|_{\sim} = \|x\|_2 + |x_1|$$

where $x = (x_k)_k$ and $y = (y_k)_k$ are in ℓ^2 and $\|\cdot\|_2$ is the standard norm of ℓ^2. Note that

$$\|x\|_2 \leqslant \|x\|_2 + |x_1| \leqslant 2\|x\|_2 \quad \textit{for every} \quad x \in \ell^2.$$

This means the norm $\|\cdot\|_{\sim}$ is equivalent to $\|\cdot\|_2$. However, it is not a Hilbert norm because it does not satisfy the parallelogram law:

$$\|x+y\|_{\sim}^2 + \|x-y\|_{\sim}^2 \neq 2\left(\|x\|_{\sim}^2 + \|y\|_{\sim}^2\right) \quad \textit{for some} \quad x, y \in \ell^2.$$

To check that, take $x = (1,0,0,0,\dots)$ and $y = (0,1,0,0,\dots)$. Then,

- *The lift hand side:*

$$\|x+y\|_{\sim}^2 + \|x-y\|_{\sim}^2 = 2(\sqrt{2}+1)^2 = 6 + 2\sqrt{2}.$$

- *The right-hand side*

$$2\left(\|x\|_{\sim}^2 + \|y\|_{\sim}^2\right) = 2(2^2+1) = 10.$$

Now, consider the symplectic basis

$$S = \bigcup_{k=1}^{\infty} \{e_k, f_k\}$$

where

$$e_k = (0,\dots,0,1^{2k-1},0,\dots), \quad \textit{and} \quad f_k = (0,\dots,0,1^{2k},0,\dots).$$

Thus, the following subspace

$$L = \mathrm{Clin}\{e_k \mid k = 1,2,\dots\}.$$

is Lagrangian in ℓ^2. If we want to double-check this by calculating L^{ω}, we do the following:

for every $u, v \in \ell^2$, we write

$$u = (u_1, u_2, u_3, \dots) \quad \textit{and} \quad v = (v_1, v_2, v_3, \dots).$$

Hence

$$\begin{aligned} L^\omega &= \{u = (u_1, u_2, \dots) \in \ell^2 \mid \omega(u, v) = 0,\ \forall v \in L\} \\ &= \{u = (u_1, u_2, \dots) \in \ell^2 \mid \sum_{k=1}^{\infty} v_{2k-1} u_{2k} = 0,\ \forall v \in L\} \\ &= \{u = (u_1, u_2, \dots) \in \ell^2 \mid u_{2k} = 0,\ \forall k = 1, 2, \dots\} \\ &= \{u = (u_1, 0, u_3, 0, \dots, u_{2k-1}, 0, \dots) \in \ell^2\} \\ &= \mathrm{Clin}\{e_k \mid k = 1, 2, \dots\} \\ &= L. \end{aligned}$$

This ensures that the subspace L is Lagrangian. Also, from (1.5.18), we have

$$(JL)^\omega = JL.$$

Now, from (1.5.19), we can write

$$\ell^2 = L \oplus JL$$

where

$$J : \ell^2 \to \ell^2$$

is a ω-complex structure defined as

$$J = \begin{pmatrix} 0 & -R^* \\ R & 0 \end{pmatrix}.$$

where

$$R : L \to JL;\ R(u_1, 0, u_3, 0, \dots) = (0, u_1, 0, u_3, 0, \dots)$$

and

$$R^* = L : JL \to L;\ L(0, u_1, 0, u_3, 0 \dots) = (u_1, 0, u_3, 0, \dots).$$

Observe that if we put $J = A$, then $|A| = (AA^*)^{\frac{1}{2}} = I$. Thus, we have $J = |A|^{-1}A$, which is the same form of the above constructed ω-compatible complex structure.

In the following theorem, we will introduce a general decomposition of any closed subspace M of a real symplectic Banach space V that is equivalent to a Hilbert space.

Theorem 1.5.10. *Let (V, ω) be either an even finite-dimensional or a separable infinite-dimensional real symplectic Banach space with a norm equivalent to a Hilbert norm, and M be a closed subspace of V. Then, the subspace M can be written as*

$$M = K \oplus W \tag{1.5.21}$$

where K is isotropic and W is symplectic (possibly $K = \{0\}$ or $W = \{0\}$).

Proof. We have three cases:

- Case 1: If the restriction $\omega|_{M \times M}$ is nondegenerate, then the proof will be completed by putting

$$W = M \quad \text{and} \quad K = \{0\}.$$

- Case 2: If the restriction $\omega|_{M\times M} = 0$, then also the proof will be completed by putting

$$W = \{0\} \quad \text{and} \quad K = M.$$

- Case 3: If the restriction $\omega|_{M\times M}$ is neighter identically zero nor nondegenerate, then we set

$$K = M \cap M^{\omega}$$

as the largest isotropic subspace contained in M. Now, since M is closed and the norm of V is equivalent to a Hilbert norm, say $\|\cdot\|_g = \sqrt{g(\cdot,\cdot)}$, then (M, g) is a Hilbert space and K is closed in M. Thereafter, it has a complement W in M (i.e $M = K \oplus W$). Consequently, W will be isomorphic to the quotient space M/K by the isomorphism

$$P : W \to M/K$$

$$x \mapsto [x]$$

where $[x] = x+K$ is an equivalence class in M/K. As we discussed in the previous section, this quotient space is endowed with the symplectic form

$$\tilde{\omega}([x],[y]) = \omega(x,y) \quad \text{for every} \quad [x] = x + K, [y] = y + K \in M/K.$$

Observe,

$$\tilde{\omega}([x],[y]) = 0,\ \forall [y] \in M/K \implies [x] = K = [0].$$

Now, since we have

$$K = M \cap M^{\omega},$$

then, by Theorem 1.4.7, the form $\tilde{\omega}$ is nondegenerate on M/K. Consequently, by definition of $\tilde{\omega}$, the form ω is nondegenerate on the subspace W. This means that W is a symplectic subspace.

In conclusion, the subspace M can be written as

$$M = K \oplus W$$

where K is isotropic and W is symplectic.

□

An immediate consideration is whether the decompistion (1.5.21) is unique. To answer this, note that the isotropic part of (1.5.21) is unique because it is determined by only M and ω. However, the symplectic part, as the complement of K in M, will not be unique except in the trivial cases (i.e., when M is symplectic or isotropic). A simple example to show that there are many possible symplectic parts is:

Example 1.5.11. *Consider* $(\mathbb{R}^4, \omega)$ *with a basis*

$$S = \{e_1, e_2, e_3, e_4\}$$

where

$$e_1 = (1,0,0,0),\ e_2 = (0,1,0,0),\ e_3 = (0,0,1,0),\ e_4 = (0,0,0,1).$$

The symplectic form is defined as

$$\omega(e_i, e_j) = \omega(e_{2+i}, e_{2+j}) = 0 \quad \textit{and} \quad \omega(e_i, e_{j+2}) = \delta_{ij} \quad \textit{where} \quad i,j = 1,2.$$

Now consider:

$$M = \text{Clin}\{e_1, e_2, e_3\}.$$

In this subspace,

$$M^\omega = \text{Clin}\{e_2\}.$$

Then, the isotropic part is uniquely specified as

$$K = M \cap M^\omega = \text{Clin}\{e_2\}.$$

However, for the symplectic part, we have many possible choices, for instance,

$$W = \text{Clin}\{e_1, e_3\} \quad \textit{and} \quad W' = \text{Clin}\{e_1 + e_2, e_3\}.$$

Also, there are many other symplectic parts.

In the end of this section, we will illustrate the value of the existence of an inner product on a symplectic Banach space in the above construction. In particular, we will introduce an interesting example of a symplectic Banach space V equipped with a strongly nondegenerate symplectic form ω, but it is not isomorphic to any Hilbert space. After that, we will see that this space contains a Lagrangian subspace L satisfying $L = L^\omega$, but there is no other Lagrangian subspace L' such that V can be written as a topological direct sum

$$V = L \oplus L',$$

in contrast to (1.5.20).

To avoid confusion, particularly in such spaces, we will introduce the following definition to distinguish the Lagrangian subspace satisfying (1.5.20) from the one that does not. First, we recall the notion of topological direct sum in Banach spaces.

Definition 1.5.12. *Let $(V, \|\cdot\|)$ be a Banach space and let M, N be closed subspaces of V. Also, consider the product space $M \times N$ equipped with the norm*

$$\|(m,n)\|_{M\times N} = \|m\|_M + \|n\|_N \quad \textit{for all} \quad (m,n) \in M \times N.$$

We say that V is a topological sum of M and N, written

$$V = M \oplus N$$

if the canonical map

$$\Phi : M \times N \to V, \quad \Phi(m,n) = m + n$$

is a linear homomorphism, meaning it is a bounded linear bijection with a bounded inverse.

Definition 1.5.13. *Let $(V, \omega, \|\cdot\|)$ be a real symplectic Banach space such that $\|\cdot\|$ satisfying (1.5.1) and L be a subspace of V. We say that L is a strong Lagrangian subspace if*

- $L = L^\omega$.
- *There exists another Lagrangian subspace L' of V such that the space V can be written as a topological direct sum*

$$V = L \oplus L'.$$

Now, we will present the example mentioned above, which has been studied in several papers, for instance, [18], [19] and [7].

Example 1.5.14 (Kalton–Peck Space)**.** *Consider the Kalton–Peck map*

$$KP : \ell^2 \to \ell^2,$$

which is defined as

$$KP\left(\sum_{n=1}^{\infty} x_n e_n\right) = \sum_{n=1}^{\infty} \left(x_n \log\left(\frac{|x_n|}{\|x\|}\right)\right) e_n$$

with the convention $0\log(0) = 0$ *where* $\{e_n\}_{n\in\mathbb{N}}$ *is the standard basis of* ℓ^2*. The Kalton–Peck space is the Banach space*

$$Z_2 = \left\{(y, x) \in \ell^\infty \times \ell^2 \mid y - KP(x) \in \ell^2\right\}$$

with the norm

$$\|(y, x)\|_{Z_2} = \|x\|_2 + \|y - KP(x)\|_2$$

where $\|\cdot\|_2$ *is the standard norm of* ℓ^2*. Now, consider the symplectic form*

$$\omega : Z_2 \times Z_2 \to \mathbb{R};\ \omega((y, x), (y', x')) = \langle x, y' - KP(x')\rangle_{\ell^2} - \langle x', y - KP(x)\rangle_{\ell^2}$$

where $\langle\cdot,\cdot\rangle_{\ell^2}$ *is the standard inner product of* ℓ^2 *and* $(y, x), (y', x') \in Z_2$*. The induced map*

$$L_\omega : Z_2 \to Z_2^*;\ L_\omega(u)(\cdot) = \omega(u, \cdot) \quad \text{for every} \quad u \in Z_2$$

is an isomorphism [19]. Thus, ω *is strongly nondegenerate. Moreover, since* Z_2 *has no unconditional Schauder basis [19], it is not isomorphic to any Hilbert space.*

Also, note that if $(y, x) \in Z_2$ *with* $x = 0$*, then* $y = y - KP(0) \in \ell^2$*. Thus, the vectors of* ℓ^2*, as a subspace of* Z_2*, are expressed as* $(y, 0) \in Z_2$*; that is,*

$$\ell^2 = \{(y', 0) \in Z_2\}.$$

Now, we will show that ℓ^2 *components is a maximal isotropic subspace of* Z_2*. To this end, observe that*

$$\omega((y, 0), (y', 0)) = \langle 0, y'\rangle_{\ell^2} - \langle 0, y\rangle_{\ell^2} = 0,\ \forall y, y' \in \ell^2.$$

Furthermore, let us compute

$$\begin{aligned}
(\ell^2)^\omega &= \{(y', x') \in Z_2 \mid \omega((y, 0), (y', x')) = 0, \forall y \in \ell^2\} \\
&= \{(y', x') \in Z_2 \mid \langle 0, y' - KP(x')\rangle_{\ell^2} - \langle x', y\rangle_{\ell^2} = 0, \forall y \in \ell^2\} \\
&= \{(y', x') \in Z_2 \mid \langle x', y\rangle_{\ell^2} = 0, \forall y \in \ell^2\} \\
&= \{(y', x') \in Z_2 \mid x' = 0\} \\
&= \{(y', 0) \in Z_2\} \\
&= \ell^2.
\end{aligned}$$

According to [19, Theorem 10], every infinite-dimensional isotropic subspace of Z_2 *cannot be complemented. Therefore,* ℓ^2 *components of* Z_2 *will not be complemented, which means* ℓ^2 *is Lagrangian, but not strong Lagrangian. Also,* Z_2 *cannot be decomposed into complementary Lagrangian subspaces.*

1.6 Symplectic Hilbert Spaces

Compared with separable Banach spaces, separable Hilbert spaces have a more manageable structure. The presence of an inner product is a powerful tool that helps to link a symplectic structure to a Hilbert structure. In section 1.5, we have seen that the identity (1.5.9) provides us with helpful advantages such as complementarity of every closed subspace, compatibility of complex structure and the duality identification by the Riesz Theorem. In this section, we will discuss separable real symplectic Hilbert spaces. Since we will introduce some statements that rely directly on the inner product, we will assume that (V,ω) is a separable symplectic real Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$ such that

$$\omega(x,y)=\langle Jx,y\rangle \quad \text{for every} \quad x,y\in V \tag{1.6.1}$$

where $J : V \to V$ is a ω-compatible complex structure (see Definition 1.5.2). Although some of these statements may still hold in a separable real Banach space whose norm is equivalent to a Hilbert norm (possibly with additional work), we will present them for a separable symplectic real Hilbert space to simplify the exposition, as this generalisation is not needed for the purpose of this project.

In Hilbert structure, we guarantee the existence of a symplectic basis without extra assumptions. Unlike Banach spaces, in Theorem 1.3.8, we have constructed a symplectic basis for a real symplectic Banach space (V,ω) under the following assumptions:

- There exists a reflexive Banach space Y and a symplecticomorphism Φ such that

$$\Phi : V \to Y \oplus Y^*.$$

- Y possesses a symmetric Schauder basis.

Based on these assumptions, we obtain two Lagrangian subspaces of V

$$\Phi^{-1}(Y\oplus\{0\})=\{e_k \mid k\in I\subset\mathbb{N}\} \quad \text{and} \quad \Phi^{-1}(\{0\}\oplus Y^*)=\{f_k \mid k\in I\subset\mathbb{N}\}$$

where I is the indexing set of the Schauder basis of Y. After that,

$$S=\bigcup_{k\in I\subset\mathbb{N}}\{e_k,f_k\}$$

forms a symplectic basis of V.

However, a separable real Hilbert space always possesses an orthogonal basis that can be modified to a symplectic basis. This can be done via an approach similar to the Gram–Schmidt process. See the following Theorem.

Theorem 1.6.1. *If (V,ω) is an even finite-dimensional or a separable infinite-dimensional real symplectic Hilbert space, then it possesses a symplectic basis.*

Proof. We will start with the case of a separable infinite-dimensional symplectic space, and then the case of finite dimension will be derived immediately.

Since V is a Hilbert space, then it possesses an orthogonal basis

$$A=\{x_k \mid k\in\mathbb{N}\}.$$

To construct a symplectic basis, we pick out the vector x_1 from A. Then, we can find another vector in A (WLOG, say x_2) such that $\omega(x_1, x_2) \neq 0$; otherwise,

$$\omega(x_1, x_k) = 0, \ \forall x_k \in A.$$

Consequently, for every $v \in V$ we have $\{x_k\}_k \subseteq A$ and $\{\lambda_k\}_k \subseteq \mathbb{R}\backslash\{0\}$ such that

$$v = \sum_{k=1}^{\infty} \lambda_k x_k.$$

and thus, by boundedness and linearity of ω, we have

$$\omega(x_1, v) = \omega\left(x_1, \sum_{k=1}^{\infty} \lambda_k x_k\right) = \sum_{k=1}^{\infty} \lambda_k \omega(x_1, x_k) = 0,$$

which contradicts that the symplectic form ω is nondegenerate. Therefore, we can ensure that the vector x_2, which satisfies $\omega(x_1, x_2) \neq 0$, exists within the basis A. Now, define

$$e_1 = x_1 \qquad \text{and} \qquad f_1 = \frac{x_2}{\omega(e_1, x_2)},$$

which results in $\omega(e_1, f_1) = 1$. Also, pick out other vector x_3 from A, and then

$$e_2 = x_3 - \omega(x_3, f_1)e_1 + \omega(x_3, e_1)f_1.$$

which leads to

$$\omega(e_2, e_1) = 0 \qquad \text{and} \qquad \omega(e_2, f_1) = 0.$$

Then, again, as shown above, by the nondegeneracy of ω, we can find another vector in A (WLOG, say x_4) such that $\omega(x_4, e_2) \neq 0$, and define

$$f_2 = \frac{x_4 - \omega(x_4, f_1)e_1 + \omega(x_4, e_1)f_1}{\omega(e_2, x_4)},$$

which implies to

$$\omega(f_2, e_1) = \omega(f_2, f_1) = 0 \qquad \text{and} \qquad \omega(e_2, f_2) = 1.$$

By continuing to adopt a similar procedure, we obtain the general formula

$$e_k = x_{2k-1} + \sum_{i=1}^{k-1} [\omega(x_{2k-1}, e_i)f_i - \omega(x_{2k-1}, f_i)e_i], \ \forall k = 2, 3, \ldots. \tag{1.6.2}$$

and

$$f_k = \frac{x_{2k} + \sum_{i=1}^{k-1} [\omega(x_{2k}, e_i)f_i - \omega(x_{2k}, f_i)e_i]}{\omega(e_k, x_{2k})}, \ \forall k = 2, 3, \ldots. \tag{1.6.3}$$

Therefore, the new sequence of the vectors $\{e_i, f_i\}_{i=1}^{\infty}$ is still linearly independent, and

$$V = \text{Clin}\{x_k \mid k \in \mathbb{N}\} = \text{Clin} \bigcup_{k \in \mathbb{N}} \{e_k, f_k\}.$$

This means

$$S = \bigcup_{k \in \mathbb{N}} \{e_k, f_k\}.$$

is also a basis of V.

It remains to show that this basis is symplectic. To this end, we will use mathematical induction: at the basic step ($i = 2$), we have

$$\omega(e_2, f_2) = 1 \quad \text{and} \quad \omega(e_2, e_1) = \omega(e_2, f_1) = \omega(e_2, e_2) = \omega(f_2, f_1) = \omega(f_2, e_1) = \omega(f_2, f_2) = 0.$$

Now, let $k \in \{2, 3, 4, \dots\}$, and assume that

$$\omega(e_k, e_i) = \omega(f_k, f_i) = 0 \quad \text{and} \quad \omega(e_k, f_i) = \delta_{ki}, \ \text{ for any } \ i \in \{1, 2, \dots, k\}, \tag{1.6.4}$$

and our aim to show that the line (1.6.4) also holds for $k + 1$. To this end, write

$$e_{k+1} = x_{2k+1} + \sum_{j=1}^{k} [\omega(x_{2k+1}, e_j) f_i - \omega(x_{2k+1}, f_j) e_j],$$

and then for any $i \in \{1, 2, \dots, k\}$ we have

$$\begin{aligned}
\omega(e_{k+1}, e_i) &= \omega(x_{2k+1}, e_i) + \sum_{j=1}^{k} [\omega(x_{2k+1}, e_j)\omega(f_j, e_i) - \omega(x_{2k+1}, f_j)\omega(e_j, e_i)] \\
&= \omega(x_{2k+1}, e_i) + \omega(x_{2k+1}, e_i)\omega(f_i, e_i) && \text{(by induction hypothesis)} \\
&= 0 && \text{(since } \omega(f_i, e_i) = -1),
\end{aligned}$$

and from Remark 1.1.2,

$$\omega(e_{k+1}, e_{k+1}) = 0.$$

Also, since

$$f_{k+1} = \frac{x_{2k+2} + \sum_{i=1}^{k} [\omega(x_{2k+2}, e_i) f_i - \omega(x_{2k+2}, f_i) e_i]}{\omega(e_{k+1}, x_{2k+2})},$$

then for any $i = 1, 2, \dots, k$ we have

$$\begin{aligned}
\omega(f_{k+1}, f_i) &= \frac{\omega(x_{2k+2}, f_i) + \sum_{j=1}^{k} [\omega(x_{2k+2}, e_j)\omega(f_j, f_i) - \omega(x_{2k+2}, f_j)\omega(e_j, f_i)]}{\omega(e_{k+1}, x_{2k+2})} \\
&= \frac{\omega(x_{2k+2}, f_i) - \omega(x_{2k+2}, f_i)\omega(e_i, f_i)}{\omega(e_{k+1}, x_{2k+2})} && \text{(by induction hypothesis)} \\
&= 0 && \text{(since } \omega(e_i, f_i) = 1).
\end{aligned}$$

and from Remark 1.1.2,

$$\omega(f_{k+1}, f_{k+1}) = 0.$$

Finally,

$$\begin{aligned}
\omega(e_{k+1}, f_{k+1}) &= \frac{\omega(e_{k+1}, x_{2k+2}) + \sum_{i=1}^{k} [\omega(x_{2k+2}, e_i)\omega(e_{k+1}, f_i) - \omega(x_{2k+2}, f_i)\omega(e_{k+1}, e_i)]}{\omega(e_{k+1}, x_{2k+2})} \\
&= \frac{\omega(e_{k+1}, x_{2k+2})}{\omega(e_{k+1}, x_{2k+2})} && \text{(by induction hypothesis)} \\
&= 1.
\end{aligned}$$

Furthermore, we can choose e_k and f_k in such a way that they have equal norms. To see that, for every $k \in \mathbb{N}$, put $\lambda_k = \sqrt{\frac{\|f_k\|}{\|e_k\|}}$. Now, if put

$$e'_k = \lambda_k e_k \quad \text{and} \quad f'_k = \lambda_k^{-1} f_k,$$

then the property of the symplectic basis will be preserved; that is, for every $k \in \mathbb{N}$, we still have

- $\omega(e_k', e_k') = \omega(\lambda_k e_k, \lambda_k e_k) = \lambda_k^2 \omega(e_k, e_k) = 0$,
- $\omega(f_k', f_k') = \omega(\lambda_k^{-1} e_k, \lambda_k^{-1} f_k) = \left(\lambda_k^{-1}\right)^2 \omega(f_k, f_k) = 0$,
- $\omega(e_k', f_m') = \omega(\lambda_k e_k, \lambda_m^{-1} f_m) = \lambda_k \lambda_m^{-1} \omega(e_k, f_m) = \delta_{km}$.

Moreover, for every $k \in \mathbb{N}$, we have

$$\|e_k'\| = \sqrt{\frac{\|f_k\|}{\|e_k\|}}\|e_k\| = \sqrt{\|f_k\|\|e_k\|} \quad \text{and} \quad \|f_k'\| = \sqrt{\frac{\|e_k\|}{\|f_k\|}}\|f_k\| = \sqrt{\|e_k\|\|f_k\|},$$

which implies

$$\|e_k'\| = \|f_k'\| \quad \text{for every} \quad k \in \mathbb{N}. \tag{1.6.5}$$

In the finite dimension case, the above approach is still valid. Indeed, one can easily check the formulas (1.6.2) and (1.6.3) still hold for the $2n$-dimensional space; that is,

$$e_k = x_{2k-1} + \sum_{i=1}^{k-1} [\omega(x_{2k-1}, e_i) f_i - \omega(x_{2k-1}, f_i) e_i], \ \forall k \in \{1, 2, \cdots, n\} \tag{1.6.6}$$

and

$$f_k = \frac{x_{2k} + \sum_{i=1}^{k-1} [\omega(x_{2k}, e_i) f_i - \omega(x_{2k}, f_i) e_i]}{\omega(e_k, x_{2k})}, \ \forall k \in \{1, 2, \cdots, n\}. \tag{1.6.7}$$

Also, they can be chosen in such a way that they satisfy

$$\|e_k\| = \|f_k\| \quad \text{for every} \quad k \in \{1, 2, \cdots, n\}. \tag{1.6.8}$$

In conclusion, the symplectic basis exists in either finite-dimensional or separable infinite-dimensional symplectic Hilbert spaces. □

In real Hilbert spaces, there is a two-way relationship between symplectic basis and Lagrangian subspace. On the one hand, we can immediately deduce that if we have a symplectic basis

$$S = \bigcup_{k \in I \subseteq \mathbb{N}} \{e_k, f_k\}$$

of (H, ω), then we can obtain two distinct Lagrangian subspaces

$$E = \text{Clin}\{e_k \mid k \in I \subseteq \mathbb{N}\} \quad \text{and} \quad F = \text{Clin}\{f_k \mid k \in I \subseteq \mathbb{N}\}$$

such that

$$H = E \oplus F.$$

On the other hand, the converse is also valid. Specifically, if we have two distinct Lagrangian subspaces E and F satisfying

$$H = E \oplus F,$$

then it is possible to construct a symplectic basis

$$S = \bigcup_{k \in I \subseteq \mathbb{N}} \{e_k, f_k\}$$

such that

$$E = \text{Clin}\{e_k \mid k \in I \subseteq \mathbb{N}\} \quad \text{and} \quad F = \text{Clin}\{f_k \mid k \in I \subseteq \mathbb{N}\}.$$

The proof of the converse is given in the following corollary.

Corollary 1.6.2. *If E and F are two distinct Lagrangian subspaces of a separable real symplectic Hilbert space (H,ω) of either even finite dimension or a separable infinite-dimension such that*

$$H = E \oplus F,$$

then there exists a symplectic basis

$$S = \bigcup_{k\in I\subseteq\mathbb{N}} \{e_k, f_k\},$$

where I is an index set, satisfying

$$E = \mathrm{Clin}\{e_k \mid k \in I \subseteq \mathbb{N}\} \quad \textit{and} \quad F = \mathrm{Clin}\{f_k \mid k \in I \subseteq \mathbb{N}\}.$$

Proof. We will prove the case of a separable infinite-dimensional space, and then the case of finite dimension will be derived immediately.

Take an orthogonal basis of H:

$$\{x_n \mid n \in \mathbb{N}\}$$

such that

$$E = \mathrm{Clin}\{x_{2k-1} \mid k \in \mathbb{N}\} \quad \text{and} \quad F = \mathrm{Clin}\{x_{2k} \mid k \in \mathbb{N}\}.$$

Now, by formulas (1.6.2) and (1.6.3), we can construct a symplectic basis

$$S = \bigcup_{k\in\mathbb{N}} \{e_k, f_k\}$$

as follows:

$$e_1 = x_1 \quad \text{and} \quad e_k = x_{2k-1} + \sum_{i=1}^{k-1} [\omega(x_{2k-1}, e_i) f_i - \omega(x_{2k-1}, f_i) e_i], \ \forall k = 2, 3, \ldots.$$

and

$$f_1 = \frac{x_2}{\omega(e_1, x_2)} \quad \text{and} \quad f_k = \frac{x_{2k} + \sum_{i=1}^{k-1} [\omega(x_{2k}, e_i) f_i - \omega(x_{2k}, f_i) e_i]}{\omega(e_k, x_{2k})}, \ \forall k = 2, 3, \ldots.$$

Since E is Lagrangian, then for every $k \in \mathbb{N}$

$$\omega(x_{2k-1}, e_i) = 0, \forall i \in \{1, \ldots, k-1\}.$$

Thus,

$$e_k = x_{2k-1} - \sum_{i=1}^{k-1} \omega(x_{2k-1}, f_i) e_i, \ \forall k = 2, 3, \ldots,$$

which implies

$$E = \mathrm{Clin}\{x_{2i-1}\}_{i\in\mathbb{N}} = \mathrm{Clin}\{e_k\}_{k\in\mathbb{N}}.$$

Similarly, since F is Lagrangian, then for every $k \in \mathbb{N}$

$$\omega(x_{2k}, f_i) = 0, \forall i \in \{1, \ldots, k-1\}.$$

Thus,

$$f_k = x_{2k} - \sum_{i=1}^{k-1} \omega(x_{2k}, e_i) f_i, \ \forall k = 2, 3, \ldots,$$

which implies

$$F = \mathrm{Clin}\{x_{2i}\}_{i\in\mathbb{N}} = \mathrm{Clin}\{f_k\}_{k\in\mathbb{N}}.$$

This completes the proof. □

As a result, from now on, we can assume from the beginning that the real separable symplectic Hilbert space H has simultaneously two Lagrangian subspaces E, F and a symplectic basis $\{e_k, f_k\}_{k\in I\subseteq\mathbb{N}}$, where I is an index set, such that

- $H = E \oplus F$.
- $E = \text{Clin}\{e_k \mid k \in I \subseteq \mathbb{N}\}$ and $F = \text{Clin}\{f_k \mid k \in I \subseteq \mathbb{N}\}$.
- $\|e_k\| = \|f_k\|$ for every $k \in I \subseteq \mathbb{N}$.

Furthermore, from the decomposition (1.5.19), we have a ω-compatible complex structure $J : H \to H$ such that

$$E^\perp = JE \qquad \text{or equivalently} \qquad JE^\perp = E \tag{1.6.9}$$

where $E^\perp$ is the orthogonal complement of E. Consequently, we can write

$$JE = \text{Clin}\{f_k \mid k \in I \subseteq \mathbb{N}\} \tag{1.6.10}$$

Since a complex structure provides a connection between the symplectic structure and the Hilbert structure. One may wonder whether this connection can include the symplectic and orthonormal bases. More precisely, whether we can obtain a basis such that it is orthonormal in the Hilbert structure and symplectic in the symplectic structure. The following theorem gives the answer.

Theorem 1.6.3. *Let (H, ω) be an even finite-dimensional or a separable infinite-dimensional real symplectic Hilbert space equipped with an inner product $\langle\cdot,\cdot\rangle$, and let*

$$S = \bigcup_{k\in I\subseteq\mathbb{N}} \{e_k, f_k\}$$

be an orthonormal basis of $(H, \langle\cdot,\cdot\rangle)$. Then

$$J : H \to H; \quad Je_k = f_k \text{ and } Jf_k = -e_k \text{ for every } k \in I \subseteq \mathbb{N} \tag{1.6.11}$$

is a complex structure compatible with ω and satisfies

$$\omega(x, y) = \langle Jx, y\rangle \quad \text{for every} \quad x, y \in H \tag{1.6.12}$$

if and only if S is a symplectic basis for (H, ω).

Proof. Assume that $J : H \to H$ is a ω-compatible complex structure defined as (1.6.11) and satisfying (1.6.12). Then, by the orthonormality of S, we have

- $\omega(e_i, e_j) = \langle Je_i, e_j\rangle = \langle f_i, e_j\rangle = 0 \ \forall i, j \in I$.
- $\omega(f_i, f_j) = \langle Jf_i, f_j\rangle = \langle -e_i, f_j\rangle = 0 \ \forall i, j \in I$.
- $\omega(e_i, f_j) = \langle Je_i, f_j\rangle = \langle f_i, f_j\rangle = \delta_{ij} \ \forall i, j \in I$.

Therefore, S is a symplectic basis.

To prove the converse, assume that S is a symplectic basis for (H, ω). We aim to prove that

- J is bounded,
- $J^2 = -I$,

- J is ω-compatible,

by the definition of J: Firstly, let us demonstrate that J is bounded. Take an arbitrary $x \in H$ and $x_n \in H$ such that $x_n \to x$. Thus, there exist $(a_k)_{k=1}^n$ and $(b_k)_{k=1}^n$ such that

$$x_n = \sum_k^n (a_k e_k + b_k f_k).$$

Since $\{e_k, f_k\}_{k \in I \subseteq \mathbb{N}}$ is an orthonormal basis, then

$$\|x_n\|^2 = \left\| \sum_k^n (a_k e_k + b_k f_k) \right\|^2 = \sum_k^n \left(|a_k|^2 + |b_k|^2 \right).$$

By applying J, we obtain

$$\|Jx_n\|^2 = \left\| \sum_k^n (a_k J e_k + b_k J f_k) \right\|^2 = \left\| \sum_k^n (a_k f_k - b_k e_k) \right\|^2 = \sum_k^n (|a_k|^2 + |b_k|^2) = \|x_n\|^2.$$

By taking n to infinity, we have

$$\|Jx\| = \|x\|, \tag{1.6.13}$$

which means J is bounded (actually, it is isometric). Therefore, we can extend the J on all H as follows: for every $x \in H$ we have

$$x = \sum_{k \in I \subseteq \mathbb{N}} (a_k e_k + b_k f_k) \implies Jx = \sum_{k \in I \subseteq \mathbb{N}} (a_k J e_k + b_k J f_k). \tag{1.6.14}$$

Secondly, it follows from (1.6.14):

$$J^2 x = J^2 \left(\sum_{k \in I \subseteq \mathbb{N}} (a_k e_k + b_k f_k) \right) = \sum_{k \in I \subseteq \mathbb{N}} \left(a_k J^2 e_k + b_k J^2 f_k \right) = \sum_{k \in I \subseteq \mathbb{N}} (-a_k e_k - b_k f_k) = -x.$$

Thirdly, for every $x, y \in H$ we have

$$x = \sum_{k \in I \subseteq \mathbb{N}} (a_k e_k + b_k f_k) \quad \text{and} \quad y = \sum_{k \in I \subseteq \mathbb{N}} (a'_k e_k + b'_k f_k) \quad \text{for some real sequences} \quad a_k, a'_k, b_k, b'_k.$$

Then, the criteria of compatibility of J can be verified as follows:

- the first criterion

$$\begin{aligned} \omega(x, Jx) &= \omega \left(\sum_{i \in I \subseteq \mathbb{N}} (a_i e_i + b_i f_i), J \left(\sum_{i \in I \subseteq \mathbb{N}} (a_i e_i + b_i f_i) \right) \right) \\ &= \omega \left(\sum_{i \in I \subseteq \mathbb{N}} (a_i e_i + b_i f_i), \sum_{i \in I \subseteq \mathbb{N}} (a_i f_i - b_i e_i) \right) \\ &= \sum_{i \in I \subseteq \mathbb{N}} (a_i^2 + b_i^2) > 0. \end{aligned}$$

- the second criterion

$$\begin{aligned}
\omega(Jx, Jy) &= \omega\left(J\left(\sum_{i\in I\subseteq\mathbb{N}}(a_i e_i + b_i f_i)\right), J\left(\sum_{j\in I\subseteq\mathbb{N}}(a'_j e_j + b'_j f_j)\right)\right) \\
&= \omega\left(\sum_{i\in I\subseteq\mathbb{N}}(a_i Je_i + b_i Jf_i), \sum_{j\in I\subseteq\mathbb{N}}(a'_j Je_j + b'_j Jf_j)\right) \\
&= \omega\left(\sum_{i\in I\subseteq\mathbb{N}}(a_i f_i - b_i e_i), \sum_{j\in I\subseteq\mathbb{N}}(a'_j f_j - b'_j e_j)\right) \\
&= \sum_{i\in I\subseteq\mathbb{N}}\sum_{j\in I\subseteq\mathbb{N}}\omega(a_i e_i - b_i f_i, a'_j e_j - b'_j f_j) \\
&= \sum_{i\in I\subseteq\mathbb{N}}\sum_{j\in I\subseteq\mathbb{N}}\left(a_i a'_j\omega(e_i, e_j) - a_i b'_j\omega(e_i, f_j) - b_i a'_j\omega(f_i, e_j) + b_i b'_j\omega(f_i, f_j)\right) \\
&= \sum_{i\in I\subseteq\mathbb{N}}\sum_{j\in I\subseteq\mathbb{N}}\left(-a_i b'_j\omega(e_i, f_j) - b_i a'_j\omega(f_i, e_j)\right) \\
&= \sum_{i\in I\subseteq\mathbb{N}}\left(b_i a'_i - a_i b'_i\right) \\
&= \omega\left(\sum_{i\in I\subseteq\mathbb{N}}(a_i e_i + b_i f_i), \sum_{j\in I\subseteq\mathbb{N}}(a'_j e_j + b'_j f_j)\right) \\
&= \omega(x, y).
\end{aligned}$$

This completes the proof. □

Since a Lagrangian subspace can be defined as a maximal isotropic subspace, it is worth considering whether every isotropic subspace can be extended to a Lagrangian subspace. We will investigate this question in the following theorem.

Theorem 1.6.4. *Let (V, ω) be either an even finite-dimensional or a separable infinite-dimensional real symplectic Hilbert space. Then, every isotropic subspace can be extended to a Lagrangian subspace.*

Proof. Let M be an isotropic subspace. By Theorem 1.2.10, $\overline{M}$ is isotropic. Now, if it is maximal, then M is Lagrangian. However, if it is not maximal, then there exists $x_1 \notin M$ such that $\mathrm{Clin}\{x_1\} \oplus \overline{M}$ is isotropic. Again, if the extended subspace $\mathrm{Clin}\{x_1\} \oplus \overline{M}$ is maximal, then $\mathrm{Clin}\{x_1\} \oplus \overline{M}$ is Lagrangian. But, if it is not maximal, then it can be extended further by taking $x_2 \notin \mathrm{Clin}\{x_1\} \oplus \overline{M}$ such that the subspace $\mathrm{Clin}\{x_1, x_2\} \oplus \overline{M}$ is still isotropic. By continuing, we obtain the following set

$$\mathcal{A} = \{U \subseteq V \mid U \text{ is a closed isotropic subspace of } V\}.$$

This set is partially ordered by inclusion. Now, let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a chain in $\mathcal{A}$ where Λ is the index set. If we define $U = \bigcup_{\lambda\in\Lambda} U_\lambda$, then U will be the upper bound of the chain $\{U_\lambda\}_{\lambda\in\Lambda}$. Therefore, by Zorn lemma, $\mathcal{A}$ has a maximal element, and this maximal element is a Lagrangian subspace. □

Corollary 1.6.5. *If (V, ω) is either an even finite-dimensional or a separable infinite-dimensional real symplectic Hilbert space and M is an isotropic subspace of V, then there exists a symplectic*

basis

$$S = \bigcup_{k \in I \subseteq \mathbb{N}} \{e_k, f_k\},$$

where I is an index set, such that

$$M \subseteq \text{Clin}\{e_k \mid k \in I \subseteq \mathbb{N}\}.$$

Proof. Since M is isotropic, then, from Theorem 1.6.4, we can find a Lagrangian subspace M' containing M, i.e., $M \subseteq M'$. Take any basis of M' and call it $\{x_{2i-1}\}_{i \in I}$. After that, complete this basis to a basis of V:

$$B = \{x_{2i-1}\}_{i \in I} \cup \{x_{2i}\}_{i \in I}.$$

By formulas (1.6.2) and (1.6.3), we can construct a symplectic basis

$$S = \bigcup_{k \in I \subseteq \mathbb{N}} \{e_k, f_k\}$$

as follows:

$$e_k = x_{2k-1} + \sum_{i=1}^{k-1} [\omega(x_{2k-1}, e_i) f_i - \omega(x_{2k-1}, f_i) e_i], \ \forall k = 2, 3, \ldots.$$

and

$$f_k = \frac{x_{2k} + \sum_{i=1}^{k-1} [\omega(x_{2k}, e_i) f_i - \omega(x_{2k}, f_i) e_i]}{\omega(e_k, x_{2k})}, \ \forall k = 2, 3, \ldots.$$

Since M' is Lagrangian, then for every $k \in I$

$$\omega(x_{2k-1}, e_i) = 0, \forall i \in \{1, \ldots, k-1\}.$$

Thus,

$$e_k = x_{2k-1} - \sum_{i=1}^{k-1} \omega(x_{2k-1}, f_i) e_i, \ \forall k = 2, 3, \ldots,$$

which implies

$$\text{Clin}\{x_{2i-1}\}_{i \in I} = \text{Clin}\{e_k\}_{k \in I}.$$

Therefore,

$$M \subseteq \text{Clin}\{e_k \mid k \in I \subseteq \mathbb{N}\},$$

as needed. □

As a consequence, we can deduce that every closed co-isotropic subspace contains a Lagrangian subspace. See the following corollary.

Corollary 1.6.6. *Every closed co-isotropic subspace of an even finite-dimensional or separable infinite-dimensional real symplectic Hilbert space (V, ω) contains a Lagrangian subspace.*

Proof. Let M be a co-isotropic subspace of V, i.e $M^\omega \subseteq M$. Then, by Theorem 1.2.7, the symplectic complement M^ω is isotropic becuase $M^\omega \subseteq (M^\omega)^\omega$. By applying Theorem 1.6.4, the isotropic subspace M^ω can be extended to a Lagrangian subspace M'; that is,

$$M^\omega \subseteq M'.$$

Since M' is Lagrangian and M is closed, by Theorem 1.2.7, we have

$$M' = M'^\omega \subseteq (M^\omega)^\omega = \overline{M} = M.$$

Therefore, M' is Lagrangian and contained in the closed co-isotropic subspace M, as required. □

We observe that the above corollary requires the co-isotropic subspace to be closed as an essential assumption for the validity of its proof. However, as will be demonstrated soon, this assumption can be dropped, and the corollary still holds without it. Nevertheless, this generalisation requires additional effort, as we need to define a Lagrangian subspace as a minimal co-isotropic subspace. See the following definition.

Definition 1.6.7. *Let (V,ω) be a symplectic space and M be a subspace of V. We say that M is a minimal co-isotropic subspace if it does not contain another co-isotropic subspace.*

Theorem 1.6.8. *Let (V,ω) be an even finite-dimensional or a separable infinite-dimensional real symplectic Hilbert space. Then, we can define a Lagrangian subspace of V as a minimal co-isotropic subspace.*

Proof. Let L be a closed subspace of V. On one hand, assume that L satisfies $L = L^\omega$. Thus, L is co-isotropic since $L^\omega \subseteq L$. To show the minimality of L, assume, by contradiction, that there exists a co-isotropic subspace L' such that $L' \subsetneq L$ (and thus $L^\omega \subsetneq L'^\omega$). Hence $L' \subsetneq L = L^\omega \subsetneq L'^\omega$ contradicts the co-isotropy of L'.

On the other hand, if L is a minimal co-isotropic subspace, then $L^\omega \subseteq L$. It remains to show the equality $L = L^\omega$. To this end, assume, by contradiction, that $L^\omega \subsetneq L$. Thus, we can find a nonzero $v \in L \backslash L^\omega$. Then,

$$L^\omega + \mathrm{Clin}\{v\} \subseteq L.$$

Since L is a closed subspace, we have

$$\overline{L^\omega + \mathrm{Clin}\{v\}} \subseteq L. \tag{1.6.15}$$

By Theorem 1.2.7,

$$\mathrm{Clin}\{v\} \subset L \implies L^\omega \subset (\mathrm{Clin}\{v\})^\omega . \tag{1.6.16}$$

Moreover, since $\omega(v,v) = 0$ holds, $v \in (\mathrm{Clin}\{v\})^\omega$. This implies

$$\mathrm{Clin}\{v\} \subset (\mathrm{Clin}\{v\})^\omega . \tag{1.6.17}$$

From (1.6.16) and (1.6.17), we obtain

$$L^\omega + \mathrm{Clin}\{v\} \subseteq (\mathrm{Clin}\{v\})^\omega .$$

Thus,

$$\overline{L^\omega + \mathrm{Clin}\{v\}} \subseteq (\mathrm{Clin}\{v\})^\omega . \tag{1.6.18}$$

From (1.6.15) and (1.6.18),

$$\overline{L^\omega + \mathrm{Clin}\{v\}} \subseteq L \cap (\mathrm{Clin}\{v\})^\omega . \tag{1.6.19}$$

Consequently, by Theorem 1.2.7, the subspace $L \cap (Clin\{v\})^\omega$ is co-isotropic because

$$[L \cap (\mathrm{Clin}\{v\})^\omega]^\omega = \overline{L^\omega + [(\mathrm{Clin}\{v\})^\omega]^\omega} \subseteq \overline{L^\omega + \mathrm{Clin}\{v\}} \subseteq L \cap (\mathrm{Clin}\{v\})^\omega .$$

Finally, since $v \notin L^\omega$, then there exists $u \in L$ such that $\omega(u,v) \neq 0$. This implies that

$$u \in L \quad \text{and} \quad u \notin L \cap (\mathrm{Clin}\{v\})^\omega .$$

Thus, the subspace $L \cap (\mathrm{Clin}\{v\})^\omega$ is co-isotropic and properly contained in L. This contradicts the minimality of L. Therefore, equality $L^\omega = L$ must hold. □

Remark 1.6.9. *If a co-isotopic subspace M is given with $M^\omega \subsetneq M$, the above proof provides a helpful process to obtain another co-isotropic subspace contained in M. More precisely, if M is co-isotopic with $M^\omega \subsetneq M$ and $m_0 \in M \backslash M^\omega$, then $M \cap (\mathrm{Clin}\{m_0\})^\omega$ is co-istropic subspace properly contained in M.*

In Theorem 1.6.6, we imposed the restriction that the co-isotropic subspace must be closed. However, the next corollary demonstrates that this restriction is unnecessary, as the containment of Lagrangian space within co-isotropic space always holds even when the co-isotropic space is not closed.

Corollary 1.6.10. *Every co-isotropic subspace of a finite-dimensional or a separable infinite-dimensional real symplectic Hilbert space (V, ω) contains a Lagrangian subspace.*

Proof. Let M be a co-isotropic subspace; that is, $M^\omega \subseteq M$. If $M^\omega = M$, then M itself is a Lagrangian subspace. This completes the proof. However, if $M^\omega \subsetneq M$, then there exists a nonzero $m_1 \in M$ such that $m_1 \notin M^\omega$. Set

$$U_1 = M \cap [\mathrm{Clin}\{m_1\}]^\omega .$$

Then, by Remark 1.6.9, U_1 is co-isotropic satistying

$$M \supset U_1.$$

Now, if $U_1^\omega = U_1$, then U_1 is Lagrangian, which ends the proof. However, if $U_1^\omega \subsetneq U_1$, then, again, there exists $m_2 \in U_1$ such that $m_2 \notin U_2^\omega$. Thereafter, by Remark 1.6.9,

$$U_2 = U_1 \cap [\mathrm{Clin}\{m_2\}]^\omega$$

is co-isotropic, satisfying

$$U \supset U_1 \supset U_2.$$

By continuing in the same manner, we can obtain the following set

$$\mathcal{A} = \{U \subseteq V \mid U \subseteq M,\ U \text{ is co-isotropic}\}.$$

This set is partially ordered by inclusion. Now, let $\{U_\lambda\}_{\lambda\in\Lambda}$ be a chain in $\mathcal{A}$ where Λ is the index set. If we define $U = \bigcap_{\lambda\in\Lambda} U_\lambda$, then U will be the lower bound of the chain $\{U_\lambda\}_{\lambda\in\Lambda}$. Therefore, by the dual form of Zorn's Lemma [3, p.20], the minimal element exists, which is a Lagrangian subspace by Theorem 1.6.8. □

We have defined above that if real Hilbert spaces (H_1, ω_1) and (H_2, ω_2) are symplecticmorphic, then there exists a symplecticmorphism $\phi : H_1 \to H_2$ such that for every $x, y \in H_1$

$$\omega_2(\phi x, \phi y) = \omega_1(x, y). \tag{1.6.20}$$

This equality can be written in terms of complex structures compatible with ω_1 and ω_2 as follows: if J_1 and J_2 are compatible with ω_1 and ω_2 respectively, then for every $x, y \in H_1$ we have

$$\begin{aligned}
\omega_2(\phi x, \phi y) = \omega_1(x, y) &\iff \langle J_2\phi x, \phi y\rangle = \langle J_1 x, y\rangle \\
&\iff \langle \phi^* J_2 \phi x, y\rangle = \langle J_1 x, y\rangle \\
&\iff \langle (\phi^* J_2 \phi - J_1)x, y\rangle = 0.
\end{aligned}$$

From the nondegeneracy of the inner product, we have

$$\phi^* J_2 \phi = J_1 \tag{1.6.21}$$

This provides an alternative criterion for the real symplectic Hilbert spaces (H_1, ω_1) and (H_2, ω_2) to be symplectomorphic. More precisely, we say that the real symplectic Hilbert spaces (H_1, ω_1) and (H_2, ω_2) are symplectomorphic if there exists an isomorphism $\phi : H_1 \to H_2$ such that

$$\phi^* J_2 \phi = J_1$$

where J_1 and J_2 are compatible with ω_1 and ω_2, respectively.

Example 1.6.11. *Consider the space* $\left(L^2([0,1]), \langle \cdot, \cdot \rangle_{L^2}\right)$ *where*

$$L^2([0,1]) = \left\{ f : [0,1] \to \mathbb{R} \mid \|f\|_2^2 = \int_0^1 |f(x)|^2 dx < \infty \right\},$$

and

$$\langle f, g \rangle_{L^2} = \int_0^1 f(x) g(x) dx \quad \textit{for every} \quad f, g \in L^2([0,1]).$$

Consider the Sobolev space $\left(H^1([0,1]), \langle \cdot, \cdot \rangle_{H^1}\right)$ *where*

$$H^1([0,1]) = \{ f \in L^2([0,1]) \mid f' \in L^2([0,1]) \},$$

and

$$\langle f, g \rangle_{H^1} = \int_0^1 f(x) g(x) dx + \int_0^1 f'(x) g'(x) dx \quad \textit{for every} \quad f, g \in H^1(0,1),$$

which is inherited from the norm:

$$\|f\|_{H^1} = \left(\int_0^1 |f(x)|^2 \right)^{\frac{1}{2}} + \left(\int_0^1 |f'(x)|^2 \right)^{\frac{1}{2}} \quad \textit{for every} \quad f \in H^1(0,1),$$

After that, consider the subspace $H_0^1([0,1])$ *of* $H^1([0,1])$ *as the orthogonal complement of constant functions in* $H^1([0,1])$*, then we obtain the symplectic Hilbert space* $H_0^1([0,1])$ *with the symplectic form*

$$\omega(f, g) = \int_0^1 f(x) g'(x) dx \quad \textit{where} \quad f, g \in H_0^1([0,1]).$$

By using the Fourier series, every $f, g \in H_0^1([0,1])$*, we can write*

$$f(x) = \sum_{n=1}^{\infty} \left[a_n \cos(2n\pi x) + b_n \sin(2n\pi x) \right] \quad \textit{and} \quad g(x) = \sum_{n=1}^{\infty} \left[a'_n \cos(2n\pi x) + b'_n \sin(2n\pi x) \right]$$

where $(a_n)_n$ *and* $(a'_n)_n$ *are coefficients for cosine terms, and* $(b_n)_n$ *and* $(b'_n)_n$ *are coefficients for sin terms. By Parseval's identity, the symplectic form can be rewritten as*

$$\omega(f, g) = \sum_{n=1}^{\infty} n\pi (a_n b'_n - a'_n b_n).$$

Also, one can check that the following basis is symplectic:

$$S = \bigcup_{n \in \mathbb{N}} \{ e_n, f_n \}$$

where

$$e_n(x) = -\frac{\cos(2n\pi x)}{\sqrt{n\pi}} \quad \textit{and} \quad f_n(x) = \frac{\sin(2n\pi x)}{\sqrt{n\pi}}.$$

Hence the subspaces

$$E = \mathrm{Clin}\{e_n \mid n \in \mathbb{N}\} = \{f \in H_0^1([0,1]) \mid f(x) = f(1-x)\}$$

and

$$F = \mathrm{Clin}\{f_n \mid n \in \mathbb{N}\} = \{f \in H_0^1([0,1]) \mid f(x) = -f(1-x)\}.$$

are Lagrangian, and

$$E \cap F = \{0\}.$$

Consequently,

$$H_0^1([0,1]) = E \oplus F.$$

Consider the complex structure

$$J : H_0^1([0,1]) \to H_0^1([0,1])$$

defined as

$$J(e_n) = f_n \quad \textit{and} \quad J(f_n) = -e_n.$$

Also, consider

$$\phi : H_0^1([0,1]) \to H_0^1([0,1])$$

defined as

$$\phi(e_n) = \lambda e_n \quad \textit{and} \quad \phi(f_n) = \lambda^{-1} f_n \quad \textit{for some} \quad \lambda \neq 0$$

or using the matrix form

$$\phi \begin{pmatrix} e_n \\ f_n \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \cdot \begin{pmatrix} e_n \\ f_n \end{pmatrix} = \begin{pmatrix} \lambda e_n \\ \lambda^{-1} f_n \end{pmatrix}.$$

Note that $\phi \in \mathrm{SL}(2,\mathbb{R})$ and it is known that the elements of $\mathrm{SL}(2,\mathbb{R})$ preserve symplectic structure. Therefore, for every $f, g \in H_0^1([0,1])$ we have

$$\omega(\phi f, \phi g) = \omega(f, g)$$

or equivalently,

$$\phi^* J \phi = J.$$

Although we are only considering real spaces, it may be useful to mention that if H is a complex Hilbert space, then it can be deduced from symplectic spaces and vice versa. Specifically,

- if a real symplectic space $(H_{\mathbb{R}}, \omega)$ is given, then the identity (1.6.1) yields the existence of a ω-compatible complex structure $J : H_{\mathbb{R}} \to H_{\mathbb{R}}$ and an inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ such that

  $$\langle J\cdot, \cdot \rangle_{\mathbb{R}} = \omega(\cdot, \cdot),$$

  or equivalently,

  $$\langle \cdot, \cdot \rangle_{\mathbb{R}} = -\langle J^2 \cdot, \cdot \rangle_{\mathbb{R}} = -\omega(J\cdot, \cdot).$$

  To further demonstrate this, we will examine the axioms of the inner product:

  - $\langle x, x \rangle_{\mathbb{R}} = -\omega(Jx, x) > 0$ for every nonzero $x \in H_{\mathbb{R}}$ (from (1.5.2)),
  - $\langle x, x \rangle_{\mathbb{R}} = -\omega(Jx, x) = 0 \iff x = 0$ (from (1.5.2)),

– $\langle x, y\rangle_{\mathbb{R}} = -\omega(Jx, y) = \omega(y, Jx) = -\omega(Jy, x) = \langle y, x\rangle_{\mathbb{R}}$ for every $x, y \in H_{\mathbb{R}}$ (from (1.5.3)),

– For every $x, y, z \in H_{\mathbb{R}}$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned}\langle \alpha x + \beta y, z\rangle_{\mathbb{R}} &= -\omega(J(\alpha x + \beta y), z)\\ &= -\alpha\omega(Jx, z) - \beta\omega(Jy, z)\\ &= \alpha\langle x, z\rangle_{\mathbb{R}} + \beta\langle y, z\rangle_{\mathbb{R}}\end{aligned}$$

,

– $|\langle x, y\rangle_{\mathbb{R}}| = |\omega(Jx, y)| \leqslant \|Jx\|\|y\| = \|x\|\|y\|$ for every $x, y \in H_{\mathbb{R}}$ (from (1.6.13)).

Consequently, the complex inner product on $H = H_{\mathbb{R}} \oplus iH_{\mathbb{R}}$ can be obtained as follows:

$$\langle \cdot, \cdot\rangle = \langle \cdot, \cdot\rangle_{\mathbb{R}} + i\langle \cdot, \cdot\rangle_{\mathbb{R}} = -\omega(J\cdot, \cdot) - i\omega(J\cdot, \cdot).$$

- Conversely, if a complex Hilbert space $(H, \langle \cdot, \cdot\rangle)$ is given, then a real symplectic space $(H_{\mathbb{R}}, \omega)$ can be obtained by defining the symplectic form as the minus of the imaginary part of the inner product $\langle \cdot, \cdot\rangle$; that is,

$$\omega(\cdot, \cdot) = -Im\langle \cdot, \cdot\rangle.$$

To clarify this, let us verify the conditions of the symplectic form:

– $|\omega(x, y)| = |Im\langle x, y\rangle| \leqslant \|x\|\|y\|$ for every $x, y \in H$ by Cauchy Schwarz Inequality,

– $\omega(x, y) = -Im\langle x, y\rangle = \overline{Im\langle x, y\rangle} = Im\langle y, x\rangle = -\omega(y, x)$ for every $x, y \in H$,

– $\omega(x, y) = 0\ \forall y \in H \implies Im\langle x, y\rangle = 0\ \forall y \in H \implies x = 0$.

1.7 Equivalence of Lagrangian Subspaces

Let (V, ω) be either an even finite-dimensional or a separable infinite-dimensional real symplectic normed space. The next theorem tells us that all potential symplectic forms ω' that preserve the property of (V, ω) and (V, ω') as symplectic spaces have the same Lagrangian subspaces. But before introducing the theorem, we will need the following lemma and its corollary.

Lemma 1.7.1. *Let ω be a symplectic form on a real normed space V and ω' be another symplectic form on V. Thus, the property*

$$N \text{ is Lagrangian in } (V, \omega) \iff N \text{ is Lagrangian in } (V, \omega'). \tag{1.7.1}$$

is equivalent to that for every $x, y \in V$

$$\omega(x, y) = 0 \iff \omega'(x, y) = 0, \tag{1.7.2}$$

or equivalently

$$\omega(x, y) \neq 0 \iff \omega'(x, y) \neq 0. \tag{1.7.3}$$

Proof. By contradiction, assume that there are $x, y \in V\backslash\{0\}$ such that

$$\omega(x, y) = 0 \quad \text{and} \quad \omega'(x, y) \neq 0.$$

Hence, $\mathrm{Clin}\{x, y\}$ is an isotropic subspace. From Theorem 1.6.4, it can be extended to a Lagrangian subspace N in (V, ω). Since N contains x and y and $\omega'(x, y) \neq 0$, then N will not be a Lagrangian subspace in (V, ω'). This contradicts the assumption (1.7.1). Also, if (1.7.2) holds, then for every subspace $M \subset V$ we have

$$\begin{aligned} x \in M^{\omega} &\iff \omega(x, y) = 0, \ \forall y \in M \\ &\iff \omega'(x, y) = 0, \ \forall y \in M \\ &\iff x \in M^{\omega'}. \end{aligned}$$

Hence

$$M^{\omega} = M^{\omega'}.$$

This implies that if N is Lagrangian with respect to one of ω or ω', then it will be Lagrangian with respect to the other one because

$$N^{\omega} = N = N^{\omega'}.$$

Therefore, (1.7.1) holds. □

From the above lemma, we obtain the following Corollary.

Corollary 1.7.2. *Let ω and ω' be symplectic forms on a real normed space V and $J : V \to V$ be a complex structure. Then, the property*

$$N \text{ is Lagrangian in } (V, \omega) \iff N \text{ is Lagrangian in } (V, \omega')$$

is equivalent to that

$$J \text{ is } \omega\text{-compatible} \iff J \text{ is } \omega'\text{-compatible.} \tag{1.7.4}$$

Proof. By Lemma 1.7.1 and Lemma 1.5.3, the following statements are equivalent:

- J is ω-compatible,
- for every $u, v \in V\backslash\{0\}$, $\omega(u, Ju) \neq 0$ and $\omega(u, v) = \omega(Ju, Jv)$,
- for every $u, v \in V\backslash\{0\}$, $\omega(u, Ju) \neq 0$ and $\omega(u, v) = -\omega(u, J^2 v)$,
- for every $u, v \in V\backslash\{0\}$, $\omega(u, Ju) \neq 0$ and $\omega(u, v) + \omega(u, J^2 v) = 0$,
- for every $u, v \in V\backslash\{0\}$, $\omega(u, Ju) \neq 0$ and $\omega(u, v + J^2 v) = 0$,
- for every $u, v \in V\backslash\{0\}$, $\omega'(u, Ju) \neq 0$ and $\omega'(u, v + J^2 v) = 0$,
- for every $u, v \in V\backslash\{0\}$, $\omega'(u, Ju) \neq 0$ and $\omega'(u, v) + \omega'(u, J^2 v) = 0$,
- for every $u, v \in V\backslash\{0\}$, $\omega'(u, Ju) \neq 0$ and $\omega'(u, v) = -\omega'(u, J^2 v)$,
- for every $u, v \in V\backslash\{0\}$, $\omega'(u, Ju) \neq 0$ and $\omega'(u, v) = \omega'(Ju, Jv)$,
- J is ω'-compatible.

□

Theorem 1.7.3. *Let ω and ω' be symplectic forms on a real normed space V. Then, the following property:*

$$N \text{ is Lagrangian in } (V, \omega) \iff N \text{ is Lagrangian in } (V, \omega') \tag{1.7.5}$$

is equivalent to that for every $u, v \in V$

$$\omega'(u,v) = \lambda\omega(u,v) \quad \textit{where} \quad \lambda \in \mathbb{R}\backslash\{0\}. \tag{1.7.6}$$

Proof. One direction is straightforward because if there exists $\lambda \in \mathbb{R}\backslash\{0\}$ such that $\omega' = \lambda\omega$, then obviously property (1.7.5) holds. However, the proof of the other direction is more involved. First of all, assume that the property (1.7.5) holds and write

$$V = L \oplus JL$$

where L is a Lagrangian subspace in (V,ω), (thus, from assumption (1.7.5), it will be a Lagrangian subspace in (V,ω')), and $J : V \to V$ is a ω-compatible complex structure. Hence, from Corollary 1.7.2, $J : V \to V$ is also a ω'-compatible complex structure. Our aim now is to show that there exists $\lambda \in \mathbb{R}\backslash\{0\}$ such that for every $x, y \in L$ we have

$$\omega'(x, Jy) = \lambda\omega(x, Jy).$$

To this end, assume that $x, y \in L$ and put

$$z = Jy - \frac{\omega(x,Jy)}{\omega(x,Jx)}Jx.$$

By simple calculations,

$$\omega(x,z) = \omega(x,Jy) - \frac{\omega(x,Jy)}{\omega(x,Jx)}\omega(x,Jx) = 0.$$

Thus, by Lemma 1.7.1, specifically, the property (1.7.2), we obtain

$$0 = \omega'(x,z) = \omega'(x,Jy) - \frac{\omega(x,Jy)}{\omega(x,Jx)}\omega'(x,Jx).$$

Hence, by simple calculations,

$$\omega'(x,Jy) = \frac{\omega'(x,Jx)}{\omega(x,Jx)}\omega(x,Jy). \tag{1.7.7}$$

Since x and y are chosen arbitrarily, then (1.7.7) holds for any $x, y \in L$. It remains to show that $\frac{\omega'(x,Jx)}{\omega(x,Jx)}$ is fixed and does not depend on the choice of $x \in L$. This can be accomplished by using the complex structure J and (1.7.7) as follows:

$$\omega'(Jx,J^2y) = \frac{\omega'(Jx,J^2x)}{\omega(Jx,J^2x)}\omega(Jx,J^2y).$$

Also, by using $J^2 = -I$ and skew-symmetry of ω',

$$\omega'(y,Jx) = \frac{\omega'(x,Jx)}{\omega(x,Jx)}\omega(y,Jx). \tag{1.7.8}$$

But, according to (1.7.7),

$$\omega'(y,Jx) = \frac{\omega'(y,Jy)}{\omega(y,Jy)}\omega(y,Jx). \tag{1.7.9}$$

Therefore, we can compare (1.7.8) with (1.7.9) to obtain

$$\frac{\omega'(x,Jx)}{\omega(x,Jx)} = \frac{\omega'(y,Jy)}{\omega(y,Jy)}.$$

Since x and y are arbitrary independent vectors, then $\frac{\omega'(x,Jx)}{\omega(x,Jx)}$ is fixed for all $x \in L$. Thus, there exists a fixed scalar $\lambda \in \mathbb{R}\backslash\{0\}$ independent from x such that

$$\frac{\omega'(x,Jx)}{\omega(x,Jx)} = \lambda.$$

Consequently, from (1.7.7), we obtain that for every $x, y \in L$

$$\omega'(x,Jy) = \lambda\omega(x,Jy)$$

Finally, for every $x, y \in L$ we have

$$\begin{aligned}
\omega'((x,Jy);(x',Jy')) &= \omega'(x+Jy, x'+Jy') \\
&= \omega'(x,x') + \omega'(x,Jy') + \omega'(Jy,x') + \omega'(Jy,Jy') \\
&= \omega'(x,Jy') - \omega'(x',Jy) \\
&= \lambda\left[\omega(x,Jy') - \omega(x',Jy)\right] \\
&= \lambda\left[\omega(x,x') + \omega(x,Jy') + \omega(Jy,x') + \omega(Jy,Jy')\right] \\
&= \lambda\omega(x+Jy, x'+Jy') \\
&= \lambda\omega((x,Jy);(x',Jy')),
\end{aligned}$$

Therefore, we conclude that

$$\exists\lambda \in \mathbb{R}\backslash\{0\},\ \forall u, v \in V \quad \text{such that} \quad \omega'(u,v) = \lambda\omega(u,v),$$

as required. □

Chapter 2

Group Action

In this chapter, we will introduce general results about the group action and some related concepts. Throughout this chapter, G denotes a group unless otherwise stated.

Definition 2.0.1. *[2, p.12] Let G be a group, and X be a nonempty set. The action of G on the set X is defined as a map*

$$\Phi : G \times X \to X$$

$$\Phi(g, x) = g \cdot x$$

satisfying:

- *Identity: $e \cdot x = x \quad$ for any $\quad x \in X \quad$ where e is the identity of G,*
- *Compatibility: $g \cdot (g' \cdot x) = (g \cdot g') \cdot x \quad$ for any $\quad g, g' \in G,\ x \in X$.*

Geometrically, it describes how the points of X are transformed under the elements of G.

(For simplicity, some times, we may write gx instead of $g \cdot x$).

Example 2.0.2. *Let G, the group of all 2×2 invertible matrices with real entries, and let $g \in G$. Consider the conjugation action*

$$\Phi : G \times G \to G$$

which is defined as a matrix multiplication:

$$g \cdot x = g^{-1}xg \quad \text{for every} \quad x \in G.$$

The map Φ, indeed, satisfies the above two conditions of the group action. This is because

- *the identity condition holds by taking e to the group identity,*
- *the compatibility condition also holds since the group operation follows the associative law.*

Observe, here, the group G acts on itself (i.e. $X = G$).

When we discuss the action of a group, two fundamental concepts help us to describe its behaviour. One of these concepts is called the orbit, and the other is called the stabiliser subgroup (also called the isotropy subgroup). Let us begin with the concept of the orbit:

Definition 2.0.3. *[2, p.12] Let G be a group acting on a nonempty set X. The orbit of $x \in X$ is*

$$Orb(x) = \{gx \mid g \in G\}. \tag{2.0.1}$$

Geometrically, it is the set of all points that can be reached from the point x under the elements of G.

A transitive action is directly related to orbits. See the next definition.

Definition 2.0.4. *We say that the action of group G on a nonempty set X is transitive if*

$$\forall x_1, x_2 \in X,\ \exists g \in G \quad \textit{such that} \quad x_2 = gx_1.$$

In other words, any point in X can be reached from any other point in X by applying an appropriate element of G. In this case, the set X is called a G-homogeneous set.

Accordingly, the action of a group is transitive if and only if it has exactly one orbit. Therefore, orbit and G-homogeneous set carry identical meanings and can henceforth be used interchangeably.

We now turn to the concept of stabiliser subgroup at a point $x \in X$, which is a subgroup of G defined as [2, p.12]:

$$G_x = \{g \in G \mid gx = x\}. \tag{2.0.2}$$

Geometrically, it is the set of elements of G that fix the point x.

The following theorem provides an important relation between the orbit and the stabiliser subgroup.

Theorem 2.0.5. *[20, p.21] If G is a group and x_1 and x_2 are in the same orbit, then the corresponding stabilizers G_{x_1} and G_{x_2} are conjugated, i.e.*

$$\exists g \in G \quad \textit{such that} \quad g^{-1}G_{x_1}g = G_{x_2}.$$

Proof. Since x_1 and x_2 are in the same orbit, then

$$\exists g \in G \quad \text{satisfying} \quad gx_2 = x_1.$$

Now, for every $g_1 \in G_{x_1}$, we have

$$g^{-1}g_1gx_2 = g^{-1}g_1x_1 = g^{-1}x_1 = x_2 \implies g^{-1}g_1g \in G_{x_2}.$$

This implies

$$g^{-1}G_{x_1}g \subseteq G_{x_2}.$$

On the other direction, for every $g_2 \in G_{x_2}$, we have

$$gg_2g^{-1}x_1 = gg_2x_2 = gx_2 = x_1 \implies gg_2g^{-1} \in G_{x_1}.$$

This implies

$$gG_{x_2}g^{-1} \subseteq G_{x_1}.$$

Equivalently,

$$G_{x_2} \subseteq g^{-1}G_{x_1}g.$$

Therefore,

$$g^{-1}G_{x_1}g = G_{x_2},$$

as required. □

Remark 2.0.6. *According to Theorem 2.0.5, it is unnecessary to find the stabiliser subgroup of every point; instead, it suffices to identify a stabiliser subgroup at one representative point from each orbit, because the stabiliser subgroups of points lying in the same orbit are conjugated.*

Let G be a group acting on a nonempty set X. If we define the following equivalence relation on X

$$x_1 \sim x_2 \quad \Longleftrightarrow \quad \exists g \in G \quad \text{suct that} \quad gx_1 = x_2,$$

then the set X will be split into equivalence classes, and each class is a G-homogeneous set (orbit) [20, p.20]. The next theorem states that these equivalent classes can be represented as an isomorphic set to a collection of left cosets of some stabiliser subgroup.

Theorem 2.0.7. *If G is a group acting on X, then every G-homogeneous subset $M \subseteq X$ is isomorphic to a collection of left cosets of some stabiliser subgroup G_x, where $x \in M$:*

$$M \cong G/G_x. \tag{2.0.3}$$

Proof. Since M is a G-homogeneous set, then M is an orbit; that is,

$$M = Orb(x) \quad \text{for some} \quad x \in M.$$

Now, consider the stabiliser subgroup

$$G_x = \{g \in G \mid gx = x\}.$$

Also, consider

$$\phi : G/G_x \to Orb(x)$$
$$\phi(gG_x) = gx.$$

We claim that ϕ is an isomorphism. To prove that, we need to show that it is:

- well-defined,
- bijection,
- $\phi(g'gG_x) = g'\phi(gG_x) \quad$ for every $\quad g, g' \in G.$

Firstly, for every $g, g' \in G_x$

$$\begin{aligned} gG_x = g'G_x &\Longleftrightarrow g'^{-1}gG_x = G_x \\ &\Longleftrightarrow g'^{-1}g \in G_x \\ &\Longleftrightarrow g'^{-1}gx = x \\ &\Longleftrightarrow gx = g'x \\ &\Longleftrightarrow \phi(gG_x) = \phi(g'G_x). \end{aligned}$$

This proves that ϕ is well-defined and injective. Secondly, it is surjective because for every $y \in Orb(x)$

$$\exists g \in G \quad \text{such that} \quad y = gx \implies y = \phi(gG_x).$$

Finally, for every $g', g \in G$

$$\phi(g'gG_x) = (g'g)x = g'(gx) = g'\phi(gG_x).$$

□

Remark 2.0.8. *It is worth noting that many distinct G-homogeneous sets may be isomorphic to the same quotient group. See the following example.*

Example 2.0.9. *Consider the group action of $(\mathbb{R}, +)$ on $\mathbb{R}^2$, which acts as horizontal translation:*

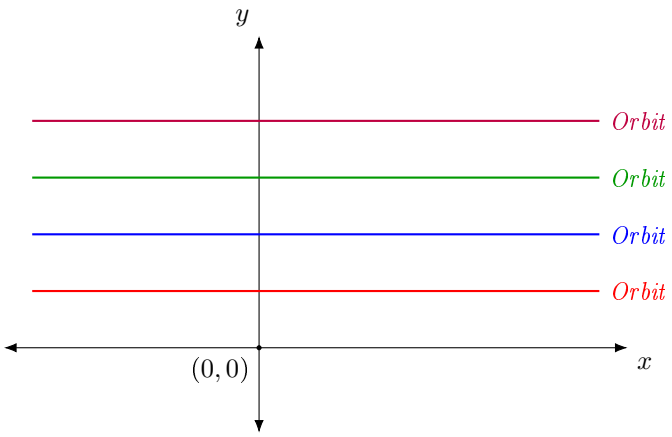
$$r \cdot (x, y) = (x + r, y) \quad \textit{for every} \quad r \in \mathbb{R},\ (x, y) \in \mathbb{R}^2.$$

Note that

$$(\mathbb{R}, +)_0 = \{r \in (\mathbb{R}, +) \mid r \cdot 0 = 0\} = \{0\}.$$

Now, for every fixed $y \in \mathbb{R}$, we have

$$Orb(x, y) = \{(x + r, y) \in \mathbb{R} \mid r \in \mathbb{R}\} \cong \mathbb{R}/\{0\}.$$



This means that for each fixed $y \in \mathbb{R}$, we have a distinct orbit (as a horizontal line at y height), and all such orbits are isomorphic to the quotient group $(\mathbb{R}, +)/(\mathbb{R}, +)_0$.

Chapter 3

Group $\mathrm{SL}(2,\mathbb{R})$

After the introduction to the group action, we also need to write a brief introduction to $\mathrm{SL}(2,\mathbb{R})$ group covering the key definitions and basic facts about $SL(2,\mathbb{R})$.

Definition 3.0.1. *Let $GL(2,\mathbb{R})$ be the group of all 2×2 invertible real matrices under the operation of matrix multiplication, i.e*

$$GL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \quad \& \quad ad-bc \neq 0 \right\},$$

with the group operation being the ordinary matrix multiplication:

$$(A,B) \to AB \quad \textit{for every} \quad A,B \in GL(2,\mathbb{R}).$$

The special linear group $\mathrm{SL}(2,\mathbb{R})$ is a subgroup of $GL(2,\mathbb{R})$ with determinant one, i.e

$$SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R} \quad \& \quad ad-bc = 1 \right\}.$$

One of the main properties of $\mathrm{SL}(2,\mathbb{R})$ is that it is a Lie group, meaning it satisfies the following three conditions:

- it possesses a group structure,
- it possesses a smooth manifold structure,
- group and manifold structures are compatible in the sense that group operations (multiplication and inversion) are smooth maps with respect to the manifold structure.

The group $\mathrm{SL}(2,\mathbb{R})$ is a 3-dimensional smooth manifold [25], and can be factorised into a product of three one-dimensional subgroups:

$$SL(2,\mathbb{R}) = KAN$$

where K geometrically acts as rotations, the subgroup A as dilations or contractions, and the subgroup N as translations. See the next theorem for details.

Theorem 3.0.2. *[16, p.193] The $\mathrm{SL}(2,\mathbb{R})$ has the following factorisation:*

$$SL(2,\mathbb{R}) = K \cdot A \cdot N \tag{3.0.1}$$

where

- $K = \left\{ \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} : t \in (-\pi, \pi] \right\},$
- $A = \left\{ \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} = \exp \begin{pmatrix} -t/2 & 0 \\ 0 & t/2 \end{pmatrix} : t \in \mathbb{R} \right\},$
- $N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} : t \in \mathbb{R} \right\}.$

Proof. Let $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, \mathbb{R})$. We aim to write

$$g = kan$$

where $k \in K$, $a \in A$ and $n \in N$. Note that the upper triangularity is the common property between the subgroups A and N. For this reason, we shall start with picking $k \in K$ so that k^{-1} rotates the first column of g to lie on the x-axis:

$$k^{-1} \cdot \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix} \quad \text{where} \quad p = \sqrt{\alpha^2 + \gamma^2}.$$

This means

$$k^{-1} = \begin{pmatrix} \frac{\alpha}{p} & \frac{\gamma}{p} \\ -\frac{\gamma}{p} & \frac{\alpha}{p} \end{pmatrix} \implies k = \begin{pmatrix} \frac{\alpha}{p} & -\frac{\gamma}{p} \\ \frac{\gamma}{p} & \frac{\alpha}{p} \end{pmatrix}.$$

Thus,

$$g = k(k^{-1}g)$$

where $k^{-1}g$ is upper triangular because, as shown above, its first column lies on the x-axis. Now, since $k^{-1}g$ is upper triangular and belongs to $SL(2, \mathbb{R})$, it has the form:

$$k^{-1}g = \begin{pmatrix} p & q \\ 0 & p^{-1} \end{pmatrix}.$$

This matrix can be factorised as

$$\begin{pmatrix} p & q \\ 0 & p^{-1} \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{q}{p} \\ 0 & 1 \end{pmatrix}.$$

Now, put

$$a = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \in A \quad \text{and} \quad n = \begin{pmatrix} 1 & \frac{q}{p} \\ 0 & 1 \end{pmatrix} \in N.$$

Therefore, we obtain

$$g = kan,$$

as required. □

Remark 3.0.3. *The factorisation (3.0.1) is called the Iwasawa decomposition, which is an important tool in the study of Lie group theory. In general, we can decompose any semisimple Lie group G, not only $SL(2, \mathbb{R})$, into a product of three subgroups*

$$G = KAN$$

where K is a maximal compact subgroup, A is a simply connected abelian subgroup and N is a simply connected nilpotent subgroup. In this thesis, however, this generalisation is unnecessary since we are only considering $\mathrm{SL}(2,\mathbb{R})$*. Nevertheless, if the reader is interested in further details, they are available in many sources about Lie groups, for example, [33] and [23].*

The previous theorem provides the decomposition of $SL(2,\mathbb{R})$ as a product of three Lie subgroups. This makes $SL(2,\mathbb{R})$, as a manifold, diffeomorphic to $S^1 \times \mathbb{R}^+ \times \mathbb{R}$. This can be easily proven by considering the diffeomorphism (smooth bijection with smooth inverse):

$$\phi : S^1 \times \mathbb{R}^+ \times \mathbb{R} \to SL(2,\mathbb{R})$$

$$\phi((cos(t), sin(t)); p; r) = kan$$

where

$$k = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \in K \quad \text{and} \quad a = \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \in A \quad \text{and} \quad n = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \in N.$$

At the end of this chapter, it may be worth mentioning that there are additional important subgroups of $SL(2,\mathbb{R})$ beyond those introduced above [20, p.33]:

- The subgroup
$$A' = \left\{ \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} = exp \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} : t \in \mathbb{R} \right\},$$
which is conjugated to A since
$$\begin{pmatrix} \cosh(\frac{t}{2}) & \sinh(\frac{t}{2}) \\ \sinh(\frac{t}{2}) & \cosh(\frac{t}{2}) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

- The subgroup
$$N' = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$
which is conjugated to N since
$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Moreover, for the two-dimensional subgroups, we have [20, p.29]:

$$F = \left\{ \frac{1}{\sqrt{a}} \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \mid a > 0 \quad \text{and} \quad c \in \mathbb{R} \right\} \quad and \quad F' = \left\{ \frac{1}{\sqrt{a}} \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} \mid a > 0 \quad \text{and} \quad b \in \mathbb{R} \right\}.$$

They are both conjugated since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{a}} \begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{a}} \begin{pmatrix} 1 & -c \\ 0 & a \end{pmatrix} \quad \text{for any } \ a > 0,\ c \in \mathbb{R}.$$

Chapter 4

$\mathrm{SL}(2,\mathbb{R})$ Action On Real Projective Line And Hilbert Spaces

4.1 $\mathrm{SL}(2,\mathbb{R})$ Action On Real Projective Line $P\mathbb{R}$

We will start with $\mathrm{SL}(2,\mathbb{R})$ action on $\mathbb{R}$, and then deduce how $\mathrm{SL}(2,\mathbb{R})$ acts on $P\mathbb{R}$. For every fixed $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ we define the action of group $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}$ as follows:

$$\Phi : \mathrm{SL}(2,\mathbb{R}) \times \mathbb{R} \to \mathbb{R}$$

$$\Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, x\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax+b}{cx+d} \quad \text{where} \quad x \in \mathbb{R} \quad \text{and} \quad cx+d \neq 0. \tag{4.1.1}$$

In this definition, we encounter an evident issue when $cx + d = 0$, which makes the transformation (4.1.1) undefined. This issue can be settled by extending the space $\mathbb{R}$ with the infinity point. The following definition provides us with a space representing the extended space $\mathbb{R} \cup \{\infty\}$.

Definition 4.1.1. *[15, p.33] The real projective space $P\mathbb{R}$ is the set of all lines passing through zero. In other words, it is the collection of all one-dimensional vector subspaces of $\mathbb{R}^2$.*

Since, in $\mathbb{R}^2$, the two vectors (x, y) and $(\lambda x, \lambda y)$, where $\lambda \neq 0$, lie on the same line, then they are equivalent in $P\mathbb{R}$. Thus, every point in $P\mathbb{R}$ can be expressed as a homogeneous coordinate $[x : y]$; that is, the coordinate that satisfies

$$[x : y] = [\lambda x : \lambda y] \quad \text{for every} \quad \lambda \in \mathbb{R}\backslash\{0\}. \tag{4.1.2}$$

Therefore,

- if $y \neq 0$, then we can normalise by setting

$$[x : y] = \left[\frac{x}{y} : 1\right], \tag{4.1.3}$$

  which corresponds to the real number $\frac{x}{y} \in \mathbb{R}$.

- if $y = 0$, then we can also normalise by setting

$$[x : 0] = [1 : 0], \tag{4.1.4}$$

  which corresponds to the infinity point.

The following diagram describes the real projective line $P\mathbb{R}$.

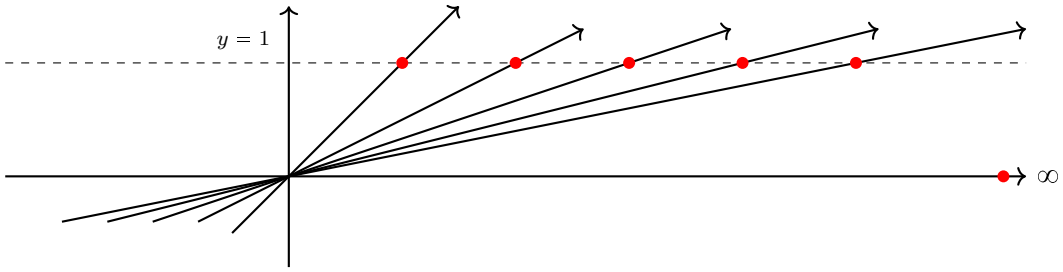


Figure 4.1: The real projective line $P\mathbb{R}$

From the above homogeneity, we can informally state the following rules for infinity:

$$\frac{r}{0} = \infty, \qquad \frac{r}{\infty} = 0, \qquad \frac{\infty}{\infty} = 1, \qquad \infty + r = \infty \qquad \text{for any} \qquad r \in \mathbb{R}\backslash\{0\}. \tag{4.1.5}$$

Therefore, the space $P\mathbb{R}$ allows us to handle the case where the denominator is zero in (4.1.1). More precisely, it allows us to extend (4.1.1) to $\mathbb{R} \cup \{\infty\}$.

Now, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$. To define the $\mathrm{SL}(2, \mathbb{R})$ action on $P\mathbb{R}$, we do the following: since for every $(x, y) \in \mathbb{R}^2$ we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Correspondingly, the $\mathrm{SL}(2, \mathbb{R})$ action on $P\mathbb{R}$ can be defined as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x : y] = [ax + by : cx + dy]. \tag{4.1.6}$$

From the homogeneity, we have

$$[x : y] = \left[\frac{x}{y} : 1\right] \quad \text{where} \quad y \neq 0,$$

which implies the following:

- if $y \neq 0$, then by putting $u = \frac{x}{y}$ we obtain [12, p.14]

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [u : 1] = [au + b : cu + d], \tag{4.1.7}$$

- if $y = 0$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x : 0] = [a : c]. \tag{4.1.8}$$

We will show later that the real projective space $P\mathbb{R}$ is isomorphic to $\mathrm{SL}(2, \mathbb{R})$-homomogeneous space. But we first need to introduce the following lemma.

Lemma 4.1.2. *The stabilizer subgroup of the infinity* $[1:0] \in P\mathbb{R}$ *in* $\mathrm{SL}(2,\mathbb{R})$ *is the upper-triangular matrices of the form:*

$$\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \neq 0 \ \text{ and } \ b \in \mathbb{R} \right\}.$$

Proof. Note that

$$\mathrm{SL}(2,\mathbb{R})_{[1:0]} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [1:0] = [1:0] \right\}$$

With simple computations, we obtain

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \neq 0 \ \text{ and } \ b \in \mathbb{R} \right\}.$$

□

Remark 4.1.3. *Observe*

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \quad \textit{where} \quad a \neq 0 \ \textit{ and } \ b \in \mathbb{R}.$$

This means

$$\mathrm{SL}(2,\mathbb{R})_{[1:0]} = AN$$

where A *and* N *are the subgroups in (3.0.1). Geometrically, the subgroup* $\mathrm{SL}(2,\mathbb{R})_{[1:0]}$ *is interpreted as a composition of scaling and shifting. This is why* $\mathrm{SL}(2,\mathbb{R})_{[1:0]}$ *fixes the infinity point* $[1:0]$*, which represents* x*-axis in* $P\mathbb{R}$*. Intuitively, the subgroup* $\mathrm{SL}(2,\mathbb{R})_{[1:0]}$ *fixes all lines passing through the origin (elements of* $P\mathbb{R}$*). This justifies why* $P\mathbb{R}$ *is isomorphic to the homogeneous space* $\mathrm{SL}(2,\mathbb{R}) / \mathrm{SL}(2,\mathbb{R})_{[1:0]}$*, as will be shown in the following Theorem.*

Theorem 4.1.4. *The projective space* $P\mathbb{R}$ *is isomorphic to* $\mathrm{SL}(2,\mathbb{R}) / \mathrm{SL}(2,\mathbb{R})_{[1:0]}$

$$P\mathbb{R} \cong \mathrm{SL}(2,\mathbb{R}) / \mathrm{SL}(2,\mathbb{R})_{[1:0]}, \tag{4.1.9}$$

which means the real projective space $P\mathbb{R}$ *is a* $\mathrm{SL}(2,\mathbb{R})$*-homogeneous space.*

Proof. For simplicity, put

$$S_0 := \mathrm{SL}(2,\mathbb{R})_{[1:0]}.$$

Now, consider the map

$$\Phi : \mathrm{SL}(2,\mathbb{R}) / S_0 \to P\mathbb{R}; \ \Phi(gS_0) = g \cdot [1:0] \quad \text{where} \quad g \in \mathrm{SL}(2,\mathbb{R}).$$

We aim to show that Φ is a well-defined bijection. To this end, observe, for every $gS_0, g'S_0 \in \mathrm{SL}(2,\mathbb{R}) / S_0$, we have

$$\begin{aligned} gS_0 = g'S_0 &\iff g^{-1}g' \in S_0 \\ &\iff g^{-1}g' \cdot [1:0] = [1:0] \\ &\iff g' \cdot [1:0] = g \cdot [1:0] \\ &\iff \Phi(g'S_0) = \Phi(gS_0). \end{aligned}$$

Hence, Φ is a well-defined and injective function. Moreover, it is surjective because if $[x:y]$ is arbitrary element of $P\mathbb{R}$, then we have two cases:

- Case 1: If $x \neq 0$, then

$$\begin{pmatrix} x & 0 \\ y & \frac{1}{x} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \quad \text{satisfying} \quad \begin{pmatrix} x & 0 \\ y & \frac{1}{x} \end{pmatrix} \cdot [1:0] = [x,y].$$

- Case 2: If $x = 0$ (thus, $y \neq 0$), then

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \quad \text{satisfying} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot [1:0] = [0:1] = [0,y].$$

Therefore, $P\mathbb{R}$ is isomorphic to $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{R})_{[1:0]}$. From Theorem 2.0.7, this implies that $P\mathbb{R}$ is a $\mathrm{SL}(2,\mathbb{R})$-homogeneous space. □

Remark 4.1.5. *The above theorem states that the real projective line can be generated from* $[1:0]$ *under the elements of* $\mathrm{SL}(2,\mathbb{R})$*; that is,*

$$\mathrm{SL}(2,\mathbb{R}) \cdot [1:0] = P\mathbb{R}. \tag{4.1.10}$$

4.2 $\mathrm{SL}(2,\mathbb{R})$ Action On Hilbert Spaces

In this section, we will systematically define $\mathrm{SL}(2,\mathbb{R})$ action on the real Hilbert space H. To this end, let H be either an even finite-dimensional or a separable infinite-dimensional real Hilbert space. Moreover, let M be closed subspaces of H such that its orthogonal complement satisfies

$$\dim M = \dim M^{\perp}. \tag{4.2.1}$$

where $M^{\perp}$ is the orthogonal complements of M. Then, this equality yields the existence of a unitary operator $\mathcal{U} : M \to M^{\perp}$ that extends to the complex structure:

$$J : H \to H; \;\; J(x,y) = \begin{pmatrix} 0 & -\mathcal{U}^{-1} \\ \mathcal{U} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\mathcal{U}^{-1}y \\ \mathcal{U}x \end{pmatrix} \quad \text{for every} \quad x \in M, y \in M^{\perp}. \tag{4.2.2}$$

Now, consider the following symplectic form on H:

$$\omega((x,y);(x',y')) = \langle J(x,y),(x',y')\rangle = \langle \mathcal{U}x, y'\rangle_{M^{\perp}} - \langle \mathcal{U}^{*}y, x'\rangle_{M} \tag{4.2.3}$$

where $x, x' \in M$ and $y, y' \in M^{\perp}$ and $\langle\cdot,\cdot\rangle_{M^{\perp}}$ and $\langle\cdot,\cdot\rangle_{M}$ denote the restrictions of inner product to M and $M^{\perp}$ respectively. From the identity (1.5.14), we have

$$J\left(M^{\omega}\right) = M^{\perp} \quad \text{or equivalently} \quad M^{\omega} = JM^{\perp}, \tag{4.2.4}$$

where M^{ω} is the symplectic complement of M. This implies that M is Lagrangian since

$$M^{\omega} = JM^{\perp} = M.$$

By using the unitary operator $\mathcal{U} : M \to M^{\perp}$, we define the $\mathrm{SL}(2,\mathbb{R})$ action on H as follows: for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}(2,\mathbb{R})$ we have

$$\mathcal{M}_{\mathcal{U}} : \mathrm{SL}(2,\mathbb{R}) \times H \to H$$

$$\mathcal{M}_{\mathcal{U}}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} a & b\mathcal{U}^{-1} \\ c\mathcal{U} & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + b\mathcal{U}^{-1}y \\ c\mathcal{U}x + dy \end{pmatrix}. \tag{4.2.5}$$

Avoiding matrices, the same operator can be written as

$$\Phi_{\mathcal{U}}(g) : H \to H$$

$$\Phi_{\mathcal{U}}(g) = (aI_M + c\mathcal{U})P_M + (dI_{M^\perp} + b\mathcal{U}^{-1})P_{M^\perp} \tag{4.2.6}$$

where I_M and $I_{M^\perp}$ are the identity map on M and $M^\perp$ respectively, and P_M and $P_{M^\perp}$ are the orthogonal projections into M and $M^\perp$ respectively. This will be a linear representation of $\mathrm{SL}(2,\mathbb{R})$ on H by bounded operators:

$$\Phi_{\mathcal{U}}(g_1 g_2) = \Phi_{\mathcal{U}}(g_1)\Phi_{\mathcal{U}}(g_2).$$

Example 4.2.1. *Let*

$$\ell^2 = \mathrm{Clin}\{e_n = (0, 0, ..., 1^{n^{th}}, 0, 0, ...) \mid n \in \mathbb{N}\}.$$

Consider the decomposition

$$\ell^2 = M \oplus M^\perp$$

where

$$M = \mathrm{Clin}\{e_n \mid n \text{ is odd}\} \quad \text{and} \quad M^\perp = \mathrm{Clin}\{e_n \mid n \text{ is even}\}.$$

Let

$$N = \mathrm{Clin}\{(e_n, 2e_{n+1}) \mid n = 1, 3, 5, \dots\}.$$

The unitary operator $\mathcal{U} : M \to M^\perp$ is defined as

$$\mathcal{U}e_n = e_{n+1} \quad \text{for every} \quad n = 1, 3, 5, \dots.$$

If we take $g = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, then, for every $n \in \mathbb{N}$, $\mathrm{SL}(2,\mathbb{R})$ acts as:

$$\begin{aligned} \mathcal{M}_{\mathcal{U}}\left(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} e_n \\ 2e_{n+1} \end{pmatrix}\right) &= \begin{pmatrix} 2 & \mathcal{U}^{-1} \\ \mathcal{U} & 1 \end{pmatrix} \cdot \begin{pmatrix} e_n \\ 2e_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} 2e_n + 2\,\mathcal{U}^{-1}e_{n+1} \\ \mathcal{U}e_n + 2e_{n+1} \end{pmatrix} \\ &= \begin{pmatrix} 4e_n \\ 3e_{n+1} \end{pmatrix} \end{aligned}$$

Thus, the image of the subspace N under the $\mathrm{SL}(2,\mathbb{R})$ action is

$$N^g = \mathrm{Clin}\{(4e_n, 3e_{n+1}) \mid n = 1, 3, 5, \dots\}.$$

As will be shown below, with this action, the real Hilbert space H has two types of orbits:

(i) A non-linear subset of the four-dimensional vector space.

(ii) A two-dimensional subspace.

The following theorem provides us with the first type.

Theorem 4.2.2. *Let H be either an even finite-dimensional or a separable infinite-dimensional real Hilbert space, let a non-zero x be in H and $\mathcal{U} : M \to M^\perp$ be a unitary operator such that the vectors $\mathcal{U}(P_M x)$ and $P_{M^\perp} x$ are not collinear. Then $\mathrm{SL}(2,\mathbb{R})$-orbit of x is a non-linear subset of the four-dimensional vector space spanned by $\{P_M x, \mathcal{U}(P_M x), P_{M^\perp} x, \mathcal{U}^{-1}(P_{M^\perp} x)\}$.*

Proof. From (4.2.6), the orbit of x is:

$$\begin{aligned}\mathrm{Orb}(x) &= \{\Phi_{\mathcal{U}}(g)(P_M x, P_{M^\perp} x) \mid g \in \mathrm{SL}(2,\mathbb{R}))\} \\ &= \{(aP_M x + b\mathcal{U}^{-1}(P_{M^\perp} x), c\mathcal{U}(P_M x) + dP_{M^\perp} x) \mid a,b,c,d \in \mathbb{R} \text{ and } ad - bc = 1\} \\ &\subseteq \mathrm{Lin}\{P_M x, \mathcal{U}(P_M x), P_{M^\perp} x, \mathcal{U}^{-1}(P_{M^\perp} x)\}.\end{aligned}$$

We now show that this span is four-dimensional. By assumption, $\mathcal{U}(P_M x)$ and $P_{M^\perp} x$ are not collinear, and hence they are linearly independent. Since $\mathcal{U}^{-1} : M^\perp \to M$ is unitary (and thus it is linear and injective), it follows that $P_M x$ and $\mathcal{U}^{-1}(P_{M^\perp} x)$ are also linearly independent. Thus,

$$\dim\left(\mathrm{Lin}\{P_M x, \mathcal{U}^{-1}(P_{M^\perp} x)\}\right) = 2 \qquad \text{and} \qquad \dim\left(\mathrm{Lin}\{\mathcal{U}(P_M x), P_{M^\perp} x\}\right) = 2.$$

Moreover,

$$\mathrm{Lin}\{P_M x, \mathcal{U}^{-1}(P_{M^\perp} x)\} \subseteq M \qquad \text{and} \qquad \mathrm{Lin}\{\mathcal{U}(P_M x), P_{M^\perp} x\} \subseteq M^\perp.$$

Since $M \perp M^\perp$, we obtain

$$\mathrm{Lin}\{P_M x, \mathcal{U}(P_M x), P_{M^\perp} x, \mathcal{U}^{-1}(P_{M^\perp} x)\} = \mathrm{Lin}\{P_M x, \mathcal{U}^{-1}(P_{M^\perp} x)\} \oplus \mathrm{Lin}\{\mathcal{U}(P_M x), P_{M^\perp} x\}.$$

Consequently,

$$\dim\left(\mathrm{Lin}\{P_M x, \mathcal{U}(P_M x), P_{M^\perp} x, \mathcal{U}^{-1}(P_{M^\perp} x)\}\right) = 4.$$

It remains to show that $\mathrm{Orb}(x)$ is not a linear subspace. Since

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),$$

we have

$$x = (P_M x, P_{M^\perp} x) \in \mathrm{Orb}(x).$$

Also, suppose that

$$2x = (2P_M x, 2P_{M^\perp} x) \in \mathrm{Orb}(x).$$

Then there exist $a,b,c,d \in \mathbb{R}$, with $ad - bc = 1$, such that

$$aP_M x + b\mathcal{U}^{-1}(P_{M^\perp} x) = 2P_M x \qquad \text{and} \qquad c\mathcal{U}(P_M x) + dP_{M^\perp} x = 2P_{M^\perp} x.$$

Since $P_M x$ and $\mathcal{U}^{-1}(P_{M^\perp} x)$ are linearly independent, the first equality implies

$$a = 2, \qquad b = 0.$$

Similarly, since $\mathcal{U}(P_M x)$ and $P_{M^\perp} x$ are linearly independent, the second equality implies

$$c = 0, \qquad d = 2.$$

Therefore,

$$ad - bc = 4,$$

which contradicts $ad - bc = 1$. Hence,

$$2x \notin \mathrm{Orb}(x).$$

Thus, $\mathrm{Orb}(x)$ is not closed under scalar multiplication and therefore is not a linear subspace. Consequently, $\mathrm{Orb}(x)$ is a non-linear subset of the four-dimensional vector space

$$\mathrm{Lin}\left\{P_M x, \mathcal{U}(P_M x), P_{M^\perp} x, \mathcal{U}^{-1}(P_{M^\perp} x)\right\}.$$

□

The second type of $\mathrm{SL}(2,\mathbb{R})$-orbit is provided by the next theorem.

Theorem 4.2.3. *Let H be either an even finite-dimensional or a separable infinite-dimensional real Hilbert space and let $\mathcal{U} : M \to M^\perp$ be a unitary operator. Then, for every nonzero $x \in M$, the two-dimensional subspace spanned by vectors x and Ux:*

$$\mathbb{R}^2_x = \mathrm{Lin}\{x, \mathcal{U}x\} \tag{4.2.7}$$

is a $\mathrm{SL}(2,\mathbb{R})$-homogeneous space.

Proof. We only need to show that $\mathbb{R}^2_x$ is an orbit of the vector $(x, \mathcal{U}x)$; that is,

$$\mathrm{Orb}\left((x, \mathcal{U}x)\right) = \mathbb{R}^2_x.$$

On the one hand, clearly

$$\begin{aligned}
\mathrm{Orb}((x, \mathcal{U}x)) &= \{\Phi_{\mathcal{U}}(g)(x, \mathcal{U}x) \mid g \in \mathrm{SL}(2,\mathbb{R})\} \\
&= \{((a+b)x, (c+d)\mathcal{U}x) \mid a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1\} \\
&\subseteq \mathbb{R}^2_x.
\end{aligned}$$

On the other hand, if $(\alpha x, \beta \mathcal{U}x) \in \mathbb{R}^2_x$ where $\alpha, \beta \in \mathbb{R}$ (are not both zero), then assume without loss of generality that $\alpha \neq 0$ and consider

$$\begin{pmatrix} \alpha & 0 \\ \beta - \alpha^{-1} & \alpha^{-1} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}),$$

we obtain

$$\begin{pmatrix} \alpha & 0 \\ (\beta - \alpha^{-1})\mathcal{U} & \alpha^{-1} \end{pmatrix} \cdot \begin{pmatrix} x \\ \mathcal{U}x \end{pmatrix} = \begin{pmatrix} \alpha x \\ \beta \mathcal{U}x \end{pmatrix}.$$

This implies $(\alpha x, \beta \mathcal{U}x) \in \mathrm{Orb}\left((x, \mathcal{U}x)\right)$ and thus

$$\mathbb{R}^2_x \subseteq \mathrm{Orb}\left((x, \mathcal{U}x)\right).$$

Therefore,

$$\mathbb{R}^2_x = \mathrm{Orb}\left((x, \mathcal{U}x)\right)$$

as needed. □

Corollary 4.2.4. *Let H be either an even finite-dimensional or a separable infinite-dimensional real Hilbert space, let $\mathcal{U} : M \to M^\perp$ be a unitary operator, and let x be a nonzero $x \in M$. Also, consider the space $\mathbb{R}^2_x$ as defined in (4.2.7). Then, the stabilizer subgroup of $(u, v) \in \mathbb{R}^2_x$ in $\mathrm{SL}(2,\mathbb{R})$ is conjugated to upper-triangular matrices of the form:*

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}$$

Proof. Note that

$$\begin{aligned}
\mathrm{SL}(2,\mathbb{R})_{(x,0)} &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \mid \begin{pmatrix} a & b\mathcal{U}^{-1} \\ c\mathcal{U} & d \end{pmatrix} \cdot \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \mid \begin{pmatrix} ax \\ c\mathcal{U}x \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \mid ax = x \quad \text{and} \quad c\mathcal{U}x = 0 \right\} \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \mid a = 1 \quad \text{and} \quad c = 0 \right\} \\
&= \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \mid b \in \mathbb{R} \right\} \\
&= N.
\end{aligned}$$

Since $\mathbb{R}^2_x$ is homogenous, then, from Theorem 2.0.5, any other stabilizer subgroup of any $(u, v) \in \mathbb{R}^2_x$ is conjugated to $\mathrm{SL}(2,\mathbb{R})_{(x,0)}$. □

As an immediate consequence of Theorem 2.0.7, we will obtain the following corollary.

Corollary 4.2.5. *Let H be either an even finite-dimensional or a separable infinite-dimensional real Hilbert space, let $\mathcal{U} : M \to M^\perp$ be a unitary operator and let x be a non-zero in H such that $\mathcal{U}(P_M x)$ and $P_{M^\perp} x$ are collinear. Then $SL(2,\mathbb{R})$-orbit of x is a two-dimensional subspace isomorphic to $SL(2,\mathbb{R})/N$ where N is the above upper-triangular subgroup of $SL(2,\mathbb{R})$.*

Example 4.2.6. *Consider $H = \mathbb{R}^4$, and its subspaces*

$$M = \mathrm{Clin}\{e_1, e_2\} \quad \textit{and} \quad N = \mathrm{Clin}\{e_1 + 3e_3, e_2 + 4e_4\}$$

where e_n is the vector whose components are all zero, except the n^{th}-component equals 1. Define the isomorphism $\mathcal{U} : M \to M^\perp$ by $\mathcal{U}e_1 = e_3$ and $\mathcal{U}e_2 = e_4$. Now, a generic vector $u \in N$ is

$$u = x(e_1 + 3e_3) + y(e_2 + 4e_4) \quad \textit{for some} \quad x, y \in \mathbb{R}.$$

Thus,

$$P_M u = xe_1 + ye_2, \quad P_{M^\perp} u = 3xe_3 + 4ye_4, \quad \mathcal{U}(P_M u) = xe_3 + ye_4.$$

Observe that the following statements are equivalent:

- *$\mathcal{U}(P_M u)$ and $P_{M^\perp} u$ are collinear.*
- *$\mathcal{U}(P_M u) = \lambda P_{M^\perp} u$ for some nonzero $\lambda \in \mathbb{R}$.*
- *$xe_3 + ye_4 = \lambda(3xe_3 + 4ye_4)$ for some nonzero $\lambda \in \mathbb{R}$.*

- *Either* $x = 0$ *or* $y = 0$.
- *The vector* u *is either scalar multiples of* $e_1 + 3e_3$ *or* $e_2 + 4e_4$.

By applying Corollary 4.2.5, this means that, in the subspace N*, only scalar multiples of* $e_1 + 3e_3$ *or* $e_2 + 4e_4$ *produce a two-dimensional homogeneous space isomorphic to* $SL(2, \mathbb{R})/N$.

Chapter 5

Graph Of Operators And Symplectic Structure

Let M and N be two closed subspaces of a separable real Hilbert space of even finite or infinite dimension $(H, \langle \cdot, \cdot \rangle)$. According to Halmos in his paper [13], these two subspaces are in generic position if the following holds

$$M \cap N = M \cap N^{\perp} = M^{\perp} \cap N = M^{\perp} \cap N^{\perp} = \{0\}. \tag{5.0.1}$$

Thus, as will be demonstrated below, we have

$$\dim(M) = \dim(M^{\perp}). \tag{5.0.2}$$

In this case, Halmos [13], [4] lets M play the role of "the horizontal axis" and $M^{\perp}$ the role of "the vertical axis". Then, he considers a self-adjoint (possibly unbounded) operator

$$T : \mathrm{Dom}(T) \subseteq M \to M,$$

with a trivial kernel and dense range, such that the pairs (M, N) and $(M \oplus \{0\}, \mathrm{Graph}(T))$ are unitarily equivalent. This means that [13] there exists a unitary operator $\mathcal{U} : M \to M^{\perp}$ such that

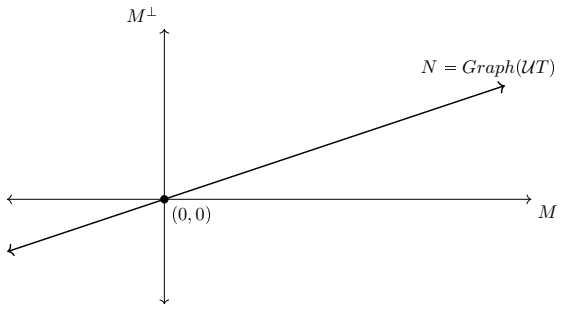
$$\tilde{\mathcal{U}} = I_M \oplus \mathcal{U} : M \oplus M \to M \oplus M^{\perp},$$

$$\tilde{\mathcal{U}}(M \oplus \{0\}) = M \quad \text{and} \quad \tilde{\mathcal{U}}(\mathrm{Graph}(T)) = \{(x, \mathcal{U}Tx) \mid x \in \mathrm{Dom}(T)\} = N$$

where

$$\mathrm{Graph}(T) = \{(x, Tx) \mid x \in \mathrm{Dom}(T)\}.$$

See the following diagram



We aim, in this chapter, to introduce the concept of a bi-equidefect operator, which is an adaptation of the concept of an operator. Also, we will employ Halmos' approach, which helps represent the subspace N as the graph of a bi-equidefect operator. Thereafter, we will define a symplectic form ω from the pair of the subspaces (M, N) such that M and N are Lagrangian subspaces w.r.t. ω.

From now on, if A and B are orthogonal subspaces of a Hilbert space, we will use the symbol $A \oplus B$ to denote their direct sum. Also, the symbol $A \ominus B$ will denote the orthogonal complement of B within A. Moreover, we will denote by P_A and P_B the orthogonal projections into A and B, respectively.

In the following, we will adopt the following notations [4] to denote the special subspaces:

$$M_{00} = M \cap N, \quad M_{01} = M \cap N^{\perp}, \quad M_{10} = M^{\perp} \cap N, \quad M_{11} = M^{\perp} \cap N^{\perp}. \tag{5.0.3}$$

Thus, the subspaces M and $M^{\perp}$ can be written as the following orthogonal decompositions [4]:

$$M = M_{00} \oplus M_{01} \oplus M_0 \quad \text{and} \quad M^{\perp} = M_{10} \oplus M_{11} \oplus M_1. \tag{5.0.4}$$

where M_0 is the orthogonal complement of $M_{00} \oplus M_{01}$ within M, and M_1 is the orthogonal complement of $M_{10} \oplus M_{11}$ within $M^{\perp}$. In addition, we can write the subspace N as a direct sum of orthogonal subspaces:

$$N = N_0 \oplus N_{\infty} \oplus N_{\#} \tag{5.0.5}$$

where $N_0 = M_{00}$, $N_{\infty} = M_{10}$, as defined (5.0.3), and $N_{\#}$ is the orthogonal complement of $N_0 \oplus N_{\infty}$ within N. This decomposition has an important interpretation (will be clarified later).

5.1 Bi-equidefect Operator And Its Graph

Assume that M, N are closed subspaces of a separable real Hilbert space of even finite or infinite dimension H and are in generic position. Then, we can use Halmos' approach [13] to construct a closed linear operator A such that N represents its graph. To this end, we first need the following lemma.

Lemma 5.1.1. *Let H be a separable real Hilbert space of even finite or infinite dimension and $P_M : H \to M$ be the orthogonal projection into the closed subspaces M and N of H such that (M, N) are in generic position. Then, the restriction $P_M|_N$ is injective, and its range is dense in M.*

Proof. To prove that $P_M|_N$ is injective, note

$$\begin{aligned} x \in N \ \text{ and } \ P_M x = 0 &\implies x \in N \ \text{ and } \ x \in \ker(P_M) \\ &\implies x \in N \ \text{ and } \ x \in M^{\perp} \\ &\implies x \in N \cap M^{\perp} \\ &\implies x = 0. \end{aligned}$$

To prove the density, observe

$$\begin{aligned} x \in M \text{ and } x \perp P_M N &\implies x \in M \text{ and } \langle x, P_M y\rangle = 0 \ \forall y \in N \\ &\implies x \in M \text{ and } \langle P_M x, y\rangle = 0 \ \forall y \in N \\ &\implies x \in M \text{ and } \langle x, y\rangle = 0 \ \forall y \in N \\ &\implies x \in M \text{ and } x \in N^\perp \\ &\implies x \in M \cap N^\perp \\ &\implies x = 0. \end{aligned}$$

This completes the proof. □

As Halmos demonstrated in [13], the above two properties of the operator $P_M|_N$ implies that

$$\dim M = \dim N. \tag{5.1.1}$$

Furthermore, it is easy to see that if (M, N) is in generic position, then the pairs

$$(M^\perp, N) \text{ and } (M, N^\perp) \text{ and } (M^\perp, N^\perp)$$

are also in generic position. This implies that

$$\dim M = \dim N = \dim M^\perp = \dim N^\perp. \tag{5.1.2}$$

By these equalities, there exists a unitary operator $\mathcal{R} : M \mapsto M^\perp$. Thereafter, we define

$$A : P_M N \subseteq M \to M$$

$$A(P_M x) = \mathcal{R}^{-1}(P_{M^\perp} x) \quad \text{for every} \quad x \in N \tag{5.1.3}$$

as an operator on the subset $P_M N$ (which is dense in M as shown in Theorem 5.1.1) where P_M and $P_{M^\perp}$ are the orthogonal projections into M and $M^\perp$, respectively. We claim that the operator A is closed and its graph is represented by N. To show that, let $(x_n)_{n\in\mathbb{N}} \subseteq \mathrm{Dom}(A)$ such that

$$x_n \to x \qquad \text{and} \qquad Ax_n \to y.$$

Since $\mathrm{Dom}(A) = P_M N$, then for every $n \in \mathbb{N}$ there exists $z_n \in N$ such that

$$x_n = P_M z_n. \tag{5.1.4}$$

By definition of A,

$$Ax_n = \mathcal{R}^{-1}(P_{M^\perp} z_n) \quad \text{for every} \quad n \in \mathbb{N}. \tag{5.1.5}$$

By applying $\mathcal{R}$ on both sides, we obtain

$$\mathcal{R}Ax_n = P_{M^\perp} z_n.$$

Thus,

$$x_n + \mathcal{R}Ax_n = P_M z_n + P_{M^\perp} z_n = z_n \in N.$$

Since $\mathcal{R}$ is bounded and N is a closed subspace,

$$z := x + \mathcal{R}y = \lim_{n\to\infty} x_n + \mathcal{R}\left(\lim_{n\to\infty} Ax_n\right) \in N.$$

Hence

$$x = P_M z \in P_M N = \mathrm{Dom}(A) \qquad \text{and} \qquad Ax = \mathcal{R}^{-1} P_{M^\perp} z = \mathcal{R}^{-1}\mathcal{R} y = y.$$

Therefore, $(x, y) \in \mathrm{Graph}(A)$ and this means the operator A is closed. Now, define the operator

$$\tilde{\mathcal{R}} : I_M \oplus \mathcal{R} : M \oplus M \to M \oplus M^\perp; \quad \tilde{\mathcal{R}}(u, v) = u + \mathcal{R}v \quad \text{for every} \quad u, v \in M.$$

Since $\mathcal{R} : M \mapsto M^\perp$ is an unitary operator,

$$\|\tilde{\mathcal{R}}(u, v)\|^2 = \|u + \mathcal{R}v\|^2 = \|u\|^2 + \|\mathcal{R}v\|^2 = \|u\|^2 + \|v\|^2 = \|(u, v)\|^2 \quad \text{for every} \quad u, v \in M.$$

This means $\tilde{\mathcal{R}}$ is a unitary operator as well. Furthermore,

$$\tilde{\mathcal{R}}(\mathrm{Graph}(A)) = \{P_M z + \mathcal{R}A(P_M z) \mid z \in N\} = \{P_M z + P_{M^\perp} z \mid z \in N\} = N.$$

Now, we consider its polar decomposition

$$A = WT$$

where

- $W : M \to M$ is a partial isometry with $\ker(W) = \ker(A)$ and $\mathrm{Ran}(W) = \overline{\mathrm{Ran}(A)}$,
- $T = (A^*A)^{\frac{1}{2}}$ is self-adjoint and positive with $\mathrm{Dom}(T) = P_M N$ and $\overline{\mathrm{Ran}(T)} = (\ker(A))^\perp$.

However, since the kernel of A is trivial and its range is dense, then

- $\overline{\mathrm{Ran}(T)} = M$.
- $W : M \to M$ is unitary,

Also, we have

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \left\langle T^2 x, x\right\rangle = \langle A^*Ax, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2,$$

which implies

$$\ker(T) = \ker(A) = \{0\}.$$

Hence, T is a self-adjoint operator defined on $P_M N$ with the trivial kernel and a dense range. Furthermore, if we put $\mathcal{U} := \mathcal{R} \circ W : M \to M^\perp$, then we will obtain the unitary operator mentioned at the beginning of the chapter

$$\tilde{\mathcal{U}} : I_M \oplus \mathcal{U} : M \oplus M \to M \oplus M^\perp; \quad \tilde{\mathcal{U}}(u, v) = u + \mathcal{U}v \quad \text{for every} \quad u, v \in M,$$

which satisfies

$$\begin{aligned}
\tilde{\mathcal{U}}(\mathrm{Graph}(T)) &= \{P_M z + \mathcal{U}T(P_M z) \mid z \in N\} \\
&= \{P_M z + \mathcal{R}WT(P_M z) \mid z \in N\} \\
&= \{P_M z + \mathcal{R}A(P_M z) \mid z \in N\} \\
&= \{P_M z + P_{M^\perp} z \mid z \in N\} \\
&= N.
\end{aligned}$$

Therefore,

$$\tilde{\mathcal{U}}(\mathrm{Graph}(T)) = N. \tag{5.1.6}$$

Up to this stage, everything refers to Halmos' approach [13].

As a generalisation of Halmos' approach, from now on, we relax the condition (5.0.1) by allowing to the intersections

$$M_{00} = M \cap N, \quad M_{01} = M \cap N^{\perp}, \quad M_{10} = M^{\perp} \cap N, \quad M_{11} = M^{\perp} \cap N^{\perp}$$

to be nonzero subspaces with the following assumptions.

Equalities of Dimentionality 5.1.2. *We assume that the following equalities hold:*

- $\dim(M_0) = \dim(M_1)$,
- $\dim(M_{00}) = \dim(M_{11})$,
- $\dim(M_{01}) = \dim(M_{10})$

where M_0 is the orthogonal complement of $M_{00} \oplus M_{01}$ within M, and M_1 is the orthogonal complement of $M_{10} \oplus M_{11}$ within $M^{\perp}$, as decomposed above in (5.0.4).

In this case, Halmos' approach can be applied to the pair of subspaces $(M_0, N_{\#})$ where $N_{\#}$ is described in (5.0.5). This is because $(M_0, N_{\#})$ is in generic position within $M_0 \oplus M_1$, as will be shown in the following theorem.

Theorem 5.1.3. *Let H be a separable real Hilbert space of even finite or infinite dimension and M, N be closed subspaces such that $H = M \oplus M^{\perp}$, where the subspaces M and $M^{\perp}$ have decompositions as in (5.0.4):*

$$M = M_{00} \oplus M_{01} \oplus M_0 \qquad \text{and} \qquad M^{\perp} = M_{10} \oplus M_{11} \oplus M_1.$$

In the subspace $M_0 \oplus M_1$, the pair $(M_0, N_{\#})$ is in generic position; that is, within $M_0 \oplus M_1$

$$M_0 \cap N_{\#} = M_0 \cap N_{\#}^{\perp} = M_0^{\perp} \cap N_{\#} = M_0^{\perp} \cap N_{\#}^{\perp} = \{0\}.$$

Proof. First of all, note that, within $M_0 \oplus M_1$, we have $M_0^{\perp} = M_1$. Then

(i)

$$\begin{aligned} x \in M_0 \cap N_{\#} &\implies x \in M_0 \text{ and } x \in M \cap N = M_{00} \\ &\implies x \in M_0 \cap M_{00} \\ &\implies x = 0. \end{aligned}$$

(ii)

$$\begin{aligned} x \in M_1 \cap N_{\#} &\implies x \in M_1 \text{ and } x \in M^{\perp} \cap N = M_{10} \\ &\implies x \in M_1 \cap M_{10} \\ &\implies x = 0. \end{aligned}$$

(iii)

$$\begin{aligned} x \in M_0 \cap N_{\#}^{\perp} &\implies x \in M_0 \text{ and } x \in M \cap N^{\perp} = M_{01} \\ &\implies x \in M_0 \cap M_{01} \\ &\implies x = 0. \end{aligned}$$

(iv)

$$\begin{aligned} x \in M_1 \cap N_{\#}^{\perp} &\implies x \in M_1 \text{ and } x \in M^{\perp} \cap N^{\perp} = M_{11} \\ &\implies x \in M_1 \cap M_{11} \\ &\implies x = 0. \end{aligned}$$

This completes the proof. □

Consequently, as demonstrated above, Holmas' approach yields the existence of a unitary operator $\mathcal{U}_0 : M_0 \to M_1$ and a densely defined invertible (possibly unbounded) self-adjoint operator

$$T : \mathrm{Dom}(T) \subseteq M_0 \to M_0 \tag{5.1.7}$$

such that

$$N_{\#} = \mathrm{Graph}(\mathcal{U}_0 T) = \{(x, \mathcal{U}_0 T x) \mid x \in \mathrm{Dom}(T)\}. \tag{5.1.8}$$

Moreover, from Equalities of Dimentionality 5.1.2

$$\dim(M_{00}) = \dim(M_{11}) \quad \text{and} \quad \dim(M_{01}) = \dim(M_{10}),$$

we have other unitary operators

$$\mathcal{U}_1 : M_{00} \to M_{11} \quad \text{and} \quad \mathcal{U}_2 : M_{01} \to M_{10}.$$

In this case, we can define

$$\mathcal{U} := \mathcal{U}_0 \oplus \mathcal{U}_1 \oplus \mathcal{U}_2$$

to obtain a unitary operator $\mathcal{U} : M \to M^{\perp}$ such that

$$\mathcal{U} M_0 = M_1 \quad \text{and} \quad \mathcal{U} M_{00} = M_{11} \quad \text{and} \quad \mathcal{U} M_{01} = M_{10}. \tag{5.1.9}$$

Now, let us recall the definition of equidefect operators.

Definition 5.1.4 (Equidefect Operator)**.** *Let $T : \mathrm{Dom}(T) \subseteq X \to Y$ be a closed densely defined linear operator acting from a real Hilbert space X into a real Hilbert space Y. Then T is called an equidefect operator if*

$$\dim(\ker(T)) = \dim(\mathrm{coker}(T)) \tag{5.1.10}$$

where $\mathrm{coker}(T) = Y/\overline{\mathrm{Ran}(T)}$ and the equality allows the dimensions to be infinite.

Define the following operator

$$\hat{T} : \mathrm{Dom}(\hat{T}) \subseteq M_0 \ominus M_{00} \to M_0 \ominus M_{00}$$

$$\hat{T}(x, x_0) = (Tx, 0) \quad \text{for every} \quad (x, x_0) \in M_0 \ominus M_{00} \tag{5.1.11}$$

where $T : \mathrm{Dom}(T) \subseteq M_0 \to M_0$ is the self-adjoint operator in (5.1.7). Also, define the operator

$$\check{T} : \mathrm{Dom}(\check{T}) \subseteq M_1 \ominus M_{10} \to M_1 \ominus M_{10}$$

$$\check{T}(y, y_0) = (\mathcal{U} T^{-1} \mathcal{U}^* y, 0) \quad \text{for every} \quad (y, y_0) \in M_1 \ominus M_{10}. \tag{5.1.12}$$

Here, $T : \mathrm{Dom}(T) \subseteq M_0 \to M_0$ is also the self-adjoint operator in (5.1.7) and $\mathcal{U} : M \to M^{\perp}$ is a unitary operator defined as (5.1.9).

Theorem 5.1.5. *The above operators $\hat{T}$, $\check{T}$ in (5.1.11) and (5.1.12) are densely defined self-adjoint operators.*

Proof. We first prove that $\hat{T}$ is symmetric. Since T is densely defined on M_0,

$$\overline{\mathrm{Dom}(\hat{T})} = \overline{\mathrm{Dom}(T)} \oplus M_{00} = M_0 \oplus M_{00},$$

so $\hat{T}$ is densely defined on $M_0 \oplus M_{00}$ as well. Also, since T is symmetic, for $(x, x_0), (y, y_0) \in \mathrm{Dom}(\hat{T})$, we have

$$\begin{aligned}
\left\langle \hat{T}(x, x_0), (y, y_0) \right\rangle_{M_0 \oplus M_{00}} &= \langle (Tx, 0), (y, y_0) \rangle_{M_0 \oplus M_{00}} \\
&= \langle Tx, y \rangle_{M_0} + \langle 0, y_0 \rangle_{M_{00}} \\
&= \langle x, Ty \rangle_{M_0} + \langle x_0, 0 \rangle_{M_{00}} \\
&= \langle (x, x_0), (Ty, 0) \rangle_{M_0 \oplus M_{00}} \\
&= \left\langle (x, x_0), \hat{T}(y, y_0) \right\rangle_{M_0 \oplus M_{00}}.
\end{aligned}$$

Thus, $\hat{T}$ is symmetric. It remains to prove that

$$\mathrm{Dom}(\hat{T}^*) = \mathrm{Dom}(\hat{T}).$$

From the symmetry of $\hat{T}$, we have

$$\mathrm{Dom}(\hat{T}) \subseteq \mathrm{Dom}(\hat{T}^*). \tag{5.1.13}$$

To show the other side, let $(y, y_0) \in \mathrm{Dom}(\hat{T}^*)$. By the definition of the adjoint, there exists $(z, z_0) \in M_0 \oplus M_{00}$ such that

$$\left\langle \hat{T}(x, x_0), (y, y_0) \right\rangle_{M_0 \oplus M_{00}} = \langle (x, x_0), (z, z_0) \rangle_{M_0 \oplus M_{00}} \quad \text{for every} \quad (x, x_0) \in \mathrm{Dom}(\hat{T}).$$

Hence

$$\langle Tx, y \rangle_{M_0} = \langle x, z \rangle_{M_0} + \langle x_0, z_0 \rangle_{M_{00}} \quad \text{for every} \quad x \in \mathrm{Dom}(T), x_0 \in M_{00}. \tag{5.1.14}$$

This implies that $z_0 = 0$ because if we take $x = 0$ in (5.1.14), we obtain

$$\langle x_0, z_0 \rangle_{M_{00}} = 0 \quad \text{for every} \quad x_0 \in M_{00}.$$

It follows that

$$\langle Tx, y \rangle_{M_0} = \langle x, z \rangle_{M_0} \quad \text{for every} \quad x \in \mathrm{Dom}(T).$$

Thus,

$$y \in \mathrm{Dom}(T^*) \qquad \text{and} \qquad T^* y = z.$$

Since $T = T^*$, we have

$$y \in \mathrm{Dom}(T) \qquad \text{and} \qquad Ty = z.$$

Note that the component y_0 is unrestricted. Thus,

$$(y, y_0) \in \mathrm{Dom}(T) \oplus M_{00} = \mathrm{Dom}(\hat{T}).$$

Consequently,

$$\mathrm{Dom}(\hat{T}^*) \subseteq \mathrm{Dom}(\hat{T}). \tag{5.1.15}$$

From (5.1.13) and (5.1.15), we obtain

$$\mathrm{Dom}(\hat{T}^*) = \mathrm{Dom}(\hat{T}).$$

Therefore, $\hat{T}$ is self-adjoint.

We next prove that $\check{T}$ is self-adjoint. We first show that T^{-1} is self-adjoint. Since T is self-adjoint and injective,

$$\overline{\mathrm{Dom}(T^{-1})} = \overline{\mathrm{Ran}(T)} = \big(\ker T^*\big)^{\perp} = (\ker T)^{\perp} = M_0.$$

Thus, $\mathrm{Dom}(T^{-1})$ is dense in M_0. Let $u, v \in \mathrm{Dom}(T^{-1})$. There exist $x, y \in \mathrm{Dom}(T)$ such that

$$u = Tx, \qquad v = Ty.$$

Using the self-adjointness of T, we obtain

$$\begin{aligned} \langle T^{-1}u, v\rangle_{M_0} &= \langle x, Ty\rangle_{M_0} \\ &= \langle Tx, y\rangle_{M_0} \\ &= \langle u, T^{-1}v\rangle_{M_0}. \end{aligned}$$

Therefore, T^{-1} is symmetric. Thus,

$$\mathrm{Dom}(T^{-1}) \subseteq \mathrm{Dom}((T^{-1})^*) \tag{5.1.16}$$

follows from the symmetry of T^{-1}. To show the reverse inclusion, let $v \in \mathrm{Dom}((T^{-1})^*)$. Then there exists $w \in M_0$ such that

$$\langle T^{-1}u, v\rangle_{M_0} = \langle u, w\rangle_{M_0} \quad \text{for every} \quad u \in \mathrm{Dom}(T^{-1}).$$

Putting $u = Tx$, where $x \in \mathrm{Dom}(T)$, implies

$$\langle x, v\rangle_{M_0} = \langle Tx, w\rangle_{M_0} \quad \text{for every} \quad x \in \mathrm{Dom}(T).$$

Hence

$$w \in \mathrm{Dom}(T^*) \qquad \text{and} \qquad T^*w = v.$$

Since $T = T^*$,

$$w \in \mathrm{Dom}(T) \qquad \text{and} \qquad Tw = v.$$

Therefore,

$$v \in \mathrm{Ran}(T) = \mathrm{Dom}(T^{-1}) \qquad \text{and} \qquad T^{-1}v = w.$$

Thus,

$$\mathrm{Dom}((T^{-1})^*) \subseteq \mathrm{Dom}(T^{-1}). \tag{5.1.17}$$

From (5.1.16) and (5.1.17), we have

$$\mathrm{Dom}((T^{-1})^*) = \mathrm{Dom}(T^{-1}).$$

This means T^{-1} is self-adjoint. Furthermore, if we put

$$A = \mathcal{U}T^{-1}\mathcal{U}^*,$$

where $\mathcal{U} : M \to M^\perp$ is the unitary operator in (5.1.9), then its domain is

$$\begin{aligned}\mathrm{Dom}(A) &= \{y \in M_1 \mid \mathcal{U}^* y \in \mathrm{Dom}(T^{-1})\} \\ &= \mathcal{U}\,\mathrm{Dom}(T^{-1}) \\ &= \mathcal{U}\,\mathrm{Ran}(T).\end{aligned}$$

Since $\mathrm{Dom}(T^{-1})$ is dense and $\mathcal{U}$ is unitary, then $\mathrm{Dom}(A)$ is also dense in M_1. Moreover, since $\mathcal{U}$ is unitary and T^{-1} is self-adjoint, A is also self-adjoint because

$$A^* = \mathcal{U}(T^{-1})^*\mathcal{U}^* = \mathcal{U}T^{-1}\mathcal{U}^* = A,$$

with

$$\mathrm{Dom}(A^*) = \mathcal{U}\,\mathrm{Dom}((T^{-1})^*) = \mathcal{U}\,\mathrm{Dom}(T^{-1}) = \mathrm{Dom}(A).$$

Now, since A is densely defined on M_1,

$$\overline{\mathrm{Dom}(\check{T})} = \overline{\mathrm{Dom}(A)} \oplus M_{10} = M_1 \oplus M_{10}.$$

Thus, $\check{T}$ is densely defined. Furthermore, for $(y, y_0), (v, v_0) \in \mathrm{Dom}(\check{T})$, we have

$$\begin{aligned}\big\langle \check{T}(y, y_0), (v, v_0)\big\rangle_{M_1 \oplus M_{10}} &= \langle (Ay, 0), (v, v_0)\rangle_{M_1 \oplus M_{10}} \\ &= \langle Ay, v\rangle_{M_1} + \langle 0, v_0\rangle_{M_{10}} \\ &= \langle y, Av\rangle_{M_1} + \langle y_0, 0\rangle_{M_{10}} \\ &= \langle (y, y_0), (Av, 0)\rangle_{M_1 \oplus M_{10}} \\ &= \big\langle (y, y_0), \check{T}(v, v_0)\big\rangle_{M_1 \oplus M_{10}}.\end{aligned}$$

Therefore, $\check{T}$ is symmetric. Finally, let $(v, v_0) \in \mathrm{Dom}(\check{T}^*)$. Then there exists $(w, w_0) \in M_1 \oplus M_{10}$ such that

$$\big\langle \check{T}(y, y_0), (v, v_0)\big\rangle_{M_1 \oplus M_{10}} = \langle (y, y_0), (w, w_0)\rangle_{M_1 \oplus M_{10}} \quad \text{for every} \quad (y, y_0) \in \mathrm{Dom}(\check{T}).$$

Equivalently,

$$\langle Ay, v\rangle_{M_1} = \langle y, w\rangle_{M_1} + \langle y_0, w_0\rangle_{M_{10}} \quad \text{for every} \quad y \in \mathrm{Dom}(A), y_0 \in M_{10}.$$

If we take $y = 0$, we obtain $w_0 = 0$. Hence

$$\langle Ay, v\rangle_{M_1} = \langle y, w\rangle_{M_{10}} \quad \text{for every} \quad y \in \mathrm{Dom}(A).$$

It follows that

$$v \in \mathrm{Dom}(A^*) = \mathrm{Dom}(A), \qquad w = A^* v = Av.$$

Since v_0 is unrestricted,

$$(v, v_0) \in \mathrm{Dom}(A) \oplus M_{10} = \mathrm{Dom}(\check{T}).$$

Therefore,

$$\mathrm{Dom}(\check{T}^*) \subseteq \mathrm{Dom}(\check{T}).$$

The reverse inclusion follows from the symmetry of $\check{T}$. Hence

$$\mathrm{Dom}(\check{T}^*) = \mathrm{Dom}(\check{T}) \qquad \text{and} \qquad \check{T}^* = \check{T}.$$

Therefore, $\check{T}$ is self-adjoint. □

In conclusion, we have constructed two self-adjoint operators $\hat{T}$ and $\check{T}$ such that $\mathcal{U}\hat{T}$ and $\mathcal{U}\check{T}$ are equidefect operators satisfying

$$\mathrm{Graph}(\mathcal{U}\hat{T}) = N_0 \ominus N_\# \quad \text{and} \quad \mathrm{Graph}(\mathcal{U}^*\check{T}) = N_\# \ominus N_\infty. \tag{5.1.18}$$

with

$$\ker(\mathcal{U}\hat{T}) = N_0 \quad \text{and} \quad \ker(\mathcal{U}^*\check{T}) = N_\infty. \tag{5.1.19}$$

At the end of this section, we are prepared to introduce the definition of a bi-equidefect operator.

Definition 5.1.6 (Bi-equidefect Operator)**.** *Let H be a separable real Hilbert space of even finite or infinite dimension and M, N be closed subspaces with decomposition (5.0.4):*

$$M = M_{00} \ominus M_{01} \ominus M_0 \qquad \text{and} \qquad M^\perp = M_{10} \ominus M_{11} \ominus M_1.$$

Let $\mathcal{U} : M \to M^\perp$ be a unitary operator such that

$$\mathcal{U}M_0 = M_1 \quad \text{and} \quad \mathcal{U}M_{00} = M_{11} \quad \text{and} \quad \mathcal{U}M_{01} = M_{10}.$$

Also, let $\hat{T} : \mathrm{Dom}(\hat{T}) \subseteq M_0 \ominus M_{00} \to M_0 \ominus M_{00}$ and $\check{T} : \mathrm{Dom}(\check{T}) \subseteq M_1 \ominus M_{10} \to M_1 \ominus M_{10}$ be self-adjoint opeators with dense domains. The pair $(\hat{T}, \check{T})$ is called a bi-equidefect operator if both $\mathcal{U}\hat{T} : \mathrm{Dom}(\hat{T}) \subseteq M_0 \ominus M_{00} \to M_1 \ominus M_{11}$ and $\mathcal{U}^*\check{T} : \mathrm{Dom}(\check{T}) \subseteq M_1 \ominus M_{10} \to M_0 \ominus M_{00}$ are equidefect operators.

Accordingly, the pair $(\hat{T}, \check{T})$ of operators, as defined in (5.1.11) and (5.1.12), is a bi-equidefect operator. Moreover, their graphs overlapping on the subspace $N_\#$, which is the graph of T, see (5.1.8). Therefore, if we glue them together, then the subspace

$$N = N_0 \ominus N_\# \ominus N_\infty$$

can be expressed as

$$N = \ker(\hat{T}) \ominus \mathrm{Graph}(\mathcal{U}T) \ominus \ker(\check{T}). \tag{5.1.20}$$

where $N_0 = \ker(\hat{T})$, $N_\infty = \ker(\check{T})$ and $N_\# = \mathrm{Graph}(\mathcal{U}T)$. Henceforth, whenever we state that the subspace N represents the graph of a bi-equidefect operator $(\hat{T}, \check{T})$, we mean N can be described by gluing two overlapping charts, one as a graph of $\mathcal{U}\hat{T}$ and one as a graph of $\mathcal{U}^*\check{T}$ exactly as in (5.1.20).

The picture can be seen in a way parallel to the construction of the real projective line $P\mathbb{R}$ (see section 4.1) with the two charts:

$$\phi_0 : \mathbb{R} \to P\mathbb{R};\ x \mapsto [x : 1] \quad \text{and} \quad \phi_\infty : \mathbb{R} \to P\mathbb{R};\ x \mapsto [1 : x].$$

Notably, these charts cover $P\mathbb{R} = \mathbb{R} \cup \{\infty\}$ and overlap on $(-\infty, 0) \cup (0, \infty)$ and each contains either $0 = [0 : 1]$ or $\infty = [1 : 0]$.

5.2 Bi-equidefect operators and Symplectic Structure

Recall the real Hilbert space $(H, \langle \cdot, \cdot \rangle)$ mentioned in the preceding section and its subspaces M and N with Equalities of Dimensionality 5.1.2. After introducing a bi-equidefect operator

such that its graph is represented by N, we need to define a symplectic form on H. To this end, recall the unitary operator $\mathcal{U} : M \to M^\perp$ in (5.1.9), which is constructed from the pair of the subspaces (M, N). Thereafter, consider the following complex structure:

$$J_0 = \begin{pmatrix} 0 & -\mathcal{U}^* \\ \mathcal{U} & 0 \end{pmatrix}, \tag{5.2.1}$$

where $\mathcal{U}^*$ is the adjoint of $\mathcal{U}$. This defines the following symplectic form ω on the Hilbert space $H = M \oplus M^\perp$:

$$\omega((x,y),(x',y')) = \langle J_0(x,y),(x',y')\rangle = \langle \mathcal{U}x, y'\rangle_{M^\perp} - \langle x', \mathcal{U}^* y\rangle_M \quad \text{for every} \quad x, x' \in M, y, y' \in M^\perp$$

where $\langle\cdot,\cdot\rangle_{M^\perp}$ and $\langle\cdot,\cdot\rangle_M$ are the restriction of inner product $\langle\cdot,\cdot\rangle$ to $M^\perp$ and M respectively.

In this section, we aim to show that the two subspaces M and N above are Lagrangian subspaces of H w.r.t. the above symplectic form.

From this time forward, we will express the components of H as (y, x) rather than the standard order (x, y). The reason behind that will be clear in the next chapter.

When we switch the order of coordinates, the above symplectic form will be rewritten as

$$\omega((y,x),(y',x')) = \langle J(y,x),(y',x')\rangle = \langle \mathcal{U}x, y'\rangle_{M^\perp} - \langle x', \mathcal{U}^* y\rangle_M \tag{5.2.2}$$

where $x, x' \in M, y, y' \in M^\perp$ and

$$J = -J_0^* = \begin{pmatrix} 0 & \mathcal{U} \\ -\mathcal{U}^* & 0 \end{pmatrix}. \tag{5.2.3}$$

Everything now is prepared to prove that the subspaces M and N are Lagrangian w.r.t the symplectic form (5.2.2). For the subspace M, the proof is straightforward because for every $x \in M$ we have

$$\omega((0,x),(0,x')) = \langle J(0,x),(0,x')\rangle = \langle \mathcal{U}x, 0\rangle_{M^\perp} - \langle x', 0\rangle_M = 0.$$

This means M is an isotropic subspace. To show the maximality, assume, for contradiction, that there exists a nonzero $u_0 \in M^\omega \backslash M$. Thus, $u_0 = (y_0, x_0)$ where $y_0 \neq 0$. Hence,

$$\begin{aligned} \omega((y_0,x_0),(0,x')) = 0 \ \forall x' \in M \iff & \langle \mathcal{U}x_0, 0\rangle_{M^\perp} - \langle x', \mathcal{U}^* y_0\rangle_M = 0 \ \forall x' \in M \\ \iff & \langle x', \mathcal{U}^* y_0\rangle_M = 0 \ \forall x' \in M \\ \iff & \mathcal{U}^* y_0 \perp M \quad \text{and} \quad \mathcal{U}^* y_0 \in M \\ \iff & y_0 = 0. \end{aligned}$$

This means $u_0 = (0, x_0) \in M$, which is contradiction with $u_0 \in M^\omega \backslash M$. Therefore, the equality $M = M^\omega$ must hold, which means M is Lagrangian. Furthermore, by using (1.5.14), $M^\perp = JM$ is also Lagrangian.

For the subspace N, as we decomposed it above in (5.1.20), we have

$$N = \ker(\hat{T}) \oplus \mathrm{Graph}(\mathcal{U}T) \oplus \ker(\check{T})$$

where T, $\hat{T}$ and $\check{T}$ are self-adjoint operators. Thus, $\ker(\hat{T}) \subset M$ and $\ker(\check{T}) \subset M^\perp$ are isotropic as subspaces of Lagrangian subspaces. Also, the isotropy of the subspace $\mathrm{Graph}(\mathcal{U}T)$ can be

shown as follows:

$$\begin{aligned}\omega((\mathcal{U}Tx,x),(\mathcal{U}Tx',x')) &= \langle \mathcal{U}Tx,\mathcal{U}x'\rangle_{M^\perp} - \langle x,Tx'\rangle_M \\ &= \langle Tx,x'\rangle_M - \langle x,Tx'\rangle_M \\ &= \langle Tx,x'\rangle_M - \langle Tx,x'\rangle_M \\ &= 0.\end{aligned}$$

For the maximality, assume, for contradiction, that there exists a nonzero $u_0 \in N^\omega \backslash N$ and put $u_0 = (y_0, x_0)$. Hence,

$$\begin{aligned}\omega((y_0,x_0),(y',x')) = 0\ \forall (y',x') \in N &\iff \langle \mathcal{U}x_0,y'\rangle_{M^\perp} - \langle x',\mathcal{U}^*y_0\rangle_M = 0\ \forall (y',x') \in N \\ &\iff \begin{cases}\langle \mathcal{U}x_0,0\rangle_{M^\perp} - \langle x',\mathcal{U}^*y_0\rangle_M = 0\ \forall (0,x') \in N_0 \\ \langle \mathcal{U}x_0,\mathcal{U}Tx'\rangle_{M^\perp} - \langle x',\mathcal{U}^*y_0\rangle_M = 0\ \forall (\mathcal{U}Tx',x') \in N_\# \\ \langle \mathcal{U}x_0,y'\rangle_{M^\perp} - \langle 0,\mathcal{U}^*y_0\rangle_M = 0\ \forall (y',0) \in N_\infty\end{cases} \\ &\iff \begin{cases}\langle x',\mathcal{U}^*y_0\rangle_M = 0\ \forall x' \in M_{00} \\ \langle x_0,Tx'\rangle_M - \langle x',\mathcal{U}^*y_0\rangle_M = 0\ \forall x' \in M_0 \\ \langle \mathcal{U}x_0,y'\rangle_{M^\perp} = 0\ \forall y' \in M_{10}\end{cases} \\ &\iff \begin{cases}\langle \mathcal{U}x',y_0\rangle_M = 0\ \forall \mathcal{U}x' \in M_{11} \\ \langle x',\hat{T}x_0\rangle_M - \langle x',\mathcal{U}^*y_0\rangle_M = 0\ \forall x' \in M_0 \\ \langle x_0,\mathcal{U}^*y'\rangle_M = 0\ \forall \mathcal{U}^*y' \in M_{01}\end{cases} \\ &\iff \begin{cases}y_0 \perp M_{11} \\ \langle x',\hat{T}x_0 - \mathcal{U}^*y_0\rangle_M = 0\ \forall x' \in M_0 \\ x_0 \perp M_{01}\end{cases} \\ &\iff \begin{cases}y_0 \in M_{10} \oplus M_1 \\ \hat{T}x_0 - \mathcal{U}^*y_0 \perp M_0 \\ x_0 \in M_{00} \oplus M_0\end{cases}\end{aligned}$$

It remains to show that

$$\hat{T}x_0 = \mathcal{U}^*y_0. \tag{5.2.4}$$

First of all, put

$$x_0 = m_0 + m_1 \quad \text{and} \quad y_0 = m_0' + m_1'$$

where $m_0 \in M_{00},\ m_1 \in M_0,\ m_0' \in M_{10}$ and $m_1' \in M_1$. Hence

$$\begin{aligned}\langle x,\hat{T}x_0 - \mathcal{U}^*y_0\rangle_M = 0\ \forall x \in M_0 &\iff \langle x,\hat{T}(m_0+m_1) - \mathcal{U}^*(m_0'+m_1')\rangle_M = 0\ \forall x \in M_0 \\ &\iff \langle x,Tm_1 - \mathcal{U}^*m_0' - \mathcal{U}^*m_1'\rangle_M = 0\ \forall x \in M_0 \\ &\iff \langle x,Tm_1 - \mathcal{U}^*m_1'\rangle_M - \langle x,\mathcal{U}^*m_0'\rangle_M = 0\ \forall x \in M_0.\end{aligned}$$

Since $x \in M_0$ and $\mathcal{U}^*m_0' \in M_{01}$,

$$\langle x,\mathcal{U}^*m_0'\rangle_M = 0\ \forall x \in M_0.$$

Consequently,

$$\langle x,\hat{T}x_0 - \mathcal{U}^*y_0\rangle_M = 0\ \forall x \in M_0 \iff \langle x,Tm_1 - \mathcal{U}^*m_1'\rangle_M = 0\ \forall x \in M_0.$$

Thus,

$$Tm_1 - \mathcal{U}^* m_1' \in M_0 \quad \text{and} \quad (Tm_1 - \mathcal{U}^* m_1') \perp M_0.$$

Hence

$$Tm_1 = \mathcal{U}^* m_1' \implies m_1' = \mathcal{U} T m_1.$$

Therefore,

$$u_0 = x_0 + y_0 = (m_0 + m_1) + (m_0' + m_1') = m_0 + (m_1 + \mathcal{U}Tm_1) + m_0' \in N$$

since $m_0 \in N_0$, $m_1 + \mathcal{U}Tm_1 \in N_\#$ and $m_0' \in N_\infty$. This contradicts that $u_0 \in N^\omega \backslash N$. In conclusion, the subspace N is a maximal isotropic subspace; that is, it is a Lagrangain subspace.

At the end of this chapter, it might be useful to introduce some examples of a bi-equidefect operator whose graph is represented by a Lagrangain subspace to enhance the understanding of the above argument.

Example 5.2.1. *Let* $\mathbb{R}^{12} = M^\perp \oplus M$ *where*

$$M^\perp = Clin\{e_1, e_2, e_3, e_4, e_5, e_6\} \quad \textit{and} \quad M = Clin\{e_7, e_8, e_9, e_{10}, e_{11}, e_{12}\},$$

and $\{e_1, e_2, \ldots, e_{12}\}$ *is the standard basis of* $\mathbb{R}^{12}$. *Define the unitary operator*

$$\mathcal{U} : M \to M^\perp, \ \mathcal{U}(e_{6+k}) = e_k \ \textit{for all} \ k = 1, \ldots, 6.$$

Thus, we can define the symplectic form

$$\omega((y,x),(y',x')) = \langle \mathcal{U}x, y' \rangle_{M^\perp} - \langle \mathcal{U}^* y, x' \rangle_M \quad \textit{for every} \quad (y,x),(y',x') \in M^\perp \oplus M.$$

Consider

$$M_{00} = span\{e_{11}, e_{12}\}, \quad M_{01} = span\{e_7, e_8\}, \quad M_0 = span\{e_9, e_{10}\}.$$

Hence

$$M_{11} = \mathcal{U}M_{00} = span\{e_5, e_6\}, \quad M_{10} = \mathcal{U}M_{01} = span\{e_1, e_2\}, \quad M_1 = \mathcal{U}M_0 = span\{e_3, e_4\}.$$

Also, consider the Lagrangian subspace

$$N = span\{(e_1, 0), (e_2, 0), (3e_3, e_9), (4e_4, e_{10}), (0, e_{11}), (0, e_{12})\}.$$

Define a self-adjoint operator

$$T : M_0 \to M_0$$

$$Te_9 = 3e_9 \quad \textit{and} \quad Te_{10} = 4e_{10}.$$

Then,

$$N_\# = \{(\mathcal{U}Tx, x) \mid x \in M_0\} = span\{(3e_3, e_9), (4e_4, e_{10})\}.$$

By extending T, *we have*

$$\hat{T} : M_0 \oplus M_{00} \to M_0 \oplus M_{00}$$

given by

$$\hat{T}(x, x_0) = (Tx, 0).$$

Thus,

$$\mathcal{U}\hat{T} : M_0 \oplus M_{00} \to M_1 \oplus M_{11}$$

given by

$$\mathcal{U}\hat{T}e_9 = 3e_3, \quad \mathcal{U}\hat{T}e_{10} = 4e_4, \quad \mathcal{U}\hat{T}e_{11} = 0, \quad \mathcal{U}\hat{T}e_{12} = 0$$

and its graph is

$$N_0 \oplus N_\# = \left\{ (\mathcal{U}\hat{T}x, x) \mid x \in M_{00} \oplus M_0 \right\} = span\{(0, e_{11}), (0, e_{12}), (3e_3, e_9), (4e_4, e_{10})\}.$$

Also, $\mathcal{U}\hat{T}$ is a equidefect operator since

$$\dim(\ker(\hat{T})) = \dim M_{00} = 2 \quad and \quad \dim(\operatorname{coker}(\hat{T})) = \dim M_{11} = 2.$$

Dually, define another self-adjoint operator

$$\check{T} : M_1 \oplus M_{10} \to M_1 \oplus M_{10}$$

given by

$$\check{T}(\mathcal{U}x, y_0) = (\mathcal{U}T^{-1}x, 0)$$

where $x \in M_0,\ y_0 \in M_{10}$. Thus,

$$\mathcal{U}^*\check{T} : M_1 \oplus M_{10} \to M_0 \oplus M_{01}$$

given by

$$\mathcal{U}^*\check{T}e_1 = 0, \quad \mathcal{U}^*\check{T}e_2 = 0, \quad \mathcal{U}^*\check{T}e_3 = \frac{1}{3}e_9, \quad \mathcal{U}^*\check{T}e_4 = \frac{1}{4}e_{10}.$$

Then, its graph is

$$\begin{aligned} N_\# \oplus N_\infty = \left\{ (y, \mathcal{U}^*\check{T}y) \mid y \in M_1 \oplus M_{10} \right\} &= span\left\{ (e_3, \frac{1}{3}e_9), (e_4, \frac{1}{4}e_{10}), (e_1, 0), (e_2, 0) \right\} \\ &= span\left\{ (3e_3, e_9), (4e_4, e_{10}), (e_1, 0), (e_2, 0) \right\}. \end{aligned}$$

In addition,

$$\dim(\ker(\mathcal{U}^*\check{T})) = \dim M_{10} = 2 \quad and \quad \dim(\operatorname{coker}(\mathcal{U}^*\check{T})) = \dim M_{01} = 2.$$

This means that $\mathcal{U}^*\check{T}$ is a equidefect operator as well. Now, the pair $(\hat{T}, \check{T})$ is a bi-equidefect operator whose graph is represented by N.

In addition to the previous example, it might be useful to consider another similar example, but on an infinite-dimensional Hilbert space.

Example 5.2.2. *Let*

$$\ell^2 = \operatorname{Clin}\{e_n = (0, 0, ..., 1^{n^{th}}, 0, 0, ...) \mid n \in \mathbb{N}\}$$

and the subspace

$$M^\perp = \operatorname{Clin}\{e_n \mid n \text{ is odd}\} \quad and \quad M = \operatorname{Clin}\{e_n \mid n \text{ is even}\}.$$

Define the unitary operator

$$\mathcal{U} : M \to M^\perp, \ \mathcal{U}(e_n) = e_{n-1} \quad for\ every \quad n = 2, 4, 6, \ldots.$$

Thus, we can define the symplectic form

$$\omega((y, x), (y', x')) = \langle \mathcal{U}x, y' \rangle_{M^\perp} - \langle \mathcal{U}^* y, x' \rangle_M \quad for\ every \quad (x, y), (x', y') \in M \oplus M^\perp.$$

Consider the Lagrangian subspace

$$N = \operatorname{Clin}\{(0,e_2),(e_1,0)\} \oplus \operatorname{Clin}\left\{(\lambda_n e_{n-1}, e_n) \mid 0 < \inf_n |\lambda_n| \leqslant \sup_n |\lambda_n| < \infty,\ n = 4,6,8,\dots\right\}.$$

In this example, we have

$$N_\# = \operatorname{Clin}\left\{(\lambda_n e_{n-1}, e_n) \mid 0 < \inf_n |\lambda_n| \leqslant \sup_n |\lambda_n| < \infty,\ n = 4,6,8,\dots\right\}.$$

If we define the diagonal operator

$$T(x_4, x_6, x_8, \dots) = (\lambda_4 x_4, \lambda_6 x_6, \lambda_8 x_8, \dots),$$

then $N_\#$ will represent the graph of $\mathcal{U}T$. Also, we can set

$$N_0 = \operatorname{Clin}\{e_2\} \quad \textit{and} \quad N_\infty = \operatorname{Clin}\{e_1\}.$$

Therefore, by extending T to

$$\hat{T} : M_0 \oplus M_{00} \to M_0 \oplus M_{00}$$
$$\hat{T}(x, x_0) = (Tx, 0),$$

the subspace $N_0 \oplus N_\#$ will represent the graph of $\mathcal{U}\hat{T}$. Moreover, if we define another self-adjoint operator

$$\check{T} : M_1 \oplus M_{10} \to M_1 \oplus M_{10}$$
$$\check{T}(y, y_0) = (\mathcal{U}T^{-1}\mathcal{U}^* y, 0),$$

then its graph will be represented by $N_\# \oplus N_\infty$. Furthermore, we have

$$\dim(\ker(\mathcal{U}\hat{T})) = 1 = \dim(\operatorname{coker}(\mathcal{U}\hat{T})) \quad \textit{and} \quad \dim(\ker(\mathcal{U}^*\check{T})) = 1 = \dim(\operatorname{coker}(\mathcal{U}^*\check{T})).$$

In conclusion, $\mathcal{U}\hat{T}$ and $\mathcal{U}^*\check{T}$ are equidefect operators, and the pair $(\hat{T}, \check{T})$ is a bi-equidefect operator such that the graph is represented by N.

Chapter 6

$\mathrm{SL}(2,\mathbb{R})$ Action On Bi-equidefect Operators And Their Spectra

We continue working in the same setting with two subspaces M and N of $(H,\langle\cdot,\cdot\rangle)$ as in the preceding chapter, maintaining all previously stated assumptions and statements. As established above, we have constructed a bi-equidefect operator on M whose graph is represented by N. Then, from these subspaces, we have determined the unitary operator $\mathcal{U} : M \to M^{\perp}$, which is defined in (5.1.9). For this unitary operator $\mathcal{U}$, we have the symplectic form:

$$\omega((y,x),(y',x')) = \langle \mathcal{U}x, y'\rangle_{M^{\perp}} - \langle x', \mathcal{U}^{*}y\rangle_{M} \text{ for every } x,x' \in M, y,y' \in M^{\perp}. \tag{6.0.1}$$

With respect to this symplectic form, the subspaces M and N are Lagrangian subspaces. Also, as discussed in Chapter 4, $\mathrm{SL}(2,\mathbb{R})$ action on the Hilbert space H can be given via the same unitary operator $\mathcal{U} : M \to M^{\perp}$ as follows: for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, we define

$$\Phi_{\mathcal{U}}(g) : H \to H;\ \Phi_{\mathcal{U}}(g) = (aI_M + c\mathcal{U})P_M + (dI_{M^{\perp}} + b\mathcal{U}^{-1})P_{M^{\perp}}. \tag{6.0.2}$$

In this chapter, we will introduce the central concept of this thesis. We will define the operator projective line as a generalisation of the real projective line $P\mathbb{R}$. Also, we will define $\mathrm{SL}(2,\mathbb{R})$ action on bi-equidefect operators through the Lagrangian subspaces that represent their graphs. Moreover, we will introduce some connections between $\mathrm{SL}(2,\mathbb{R})$ action and characterisation of the elements of the spectrum of a bi-equidefect operator.

6.1 Operator Projective Line

The core idea of the generalisation of the real projective line $P\mathbb{R}$ to the operator projective line is that we have

$$P\mathbb{R} \cong \mathrm{Lag}\left(\mathbb{R}^2,\omega\right)$$

where

$$\mathrm{Lag}\left(\mathbb{R}^2,\omega\right) = \{L \subset \mathbb{R}^2 \mid L \text{ is a Lagrangian subspace}\},$$

and it is called the Lagrangian Grassmannian [30]. In other words, we can think of the elements of $P\mathbb{R}$ as Lagrangian subspaces with respect to the standard symplectic form:

$$\omega((x,y),(x',y')) = xy' - x'y \quad \text{for every} \quad (x,y),(x',y') \in \mathbb{R}^2.$$

By analogy, we can generalise the above approach from $\mathbb{R}^2$ to the above Hilbert space H by considering the Lagrangian subspaces of H. However, in an infinite-dimensional case, the collection of all Lagrangian subspaces of H is too large, and its structure is not clear. For this reason, we must restrict the class of Lagrangian subspaces to obtain a meaningful structure. To this end, we should introduce the following theorem first.

Theorem 6.1.1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a separable real Hilbert space of either even finite or infinite dimension and M be the subspace which defines the symplectic form:*

$$\omega((y,x),(y',x')) = \langle \mathcal{U}x, y'\rangle_{M^\perp} - \langle x', \mathcal{U}^* y\rangle_M \quad \text{for every} \quad x,x' \in M, y,y' \in M^\perp$$

for $\mathcal{U} : M \to M^\perp$ is the unitary operator in (5.1.9). Then, the action of $\mathrm{SL}(2,\mathbb{R})$ preserves this symplectic form.

Proof. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Thus, for all $(y,x),(y',x') \in H$ such that $y,y' \in M^\perp$ and $x,x' \in M$, we have

$$\begin{aligned}
\omega(g\cdot(y,x); g\cdot(y',x')) &= \omega((ay + b\mathcal{U}x, c\mathcal{U}^*y + dx); (ay' + b\mathcal{U}x', c\mathcal{U}^*y' + dx')) \\
&= \langle ay' + b\mathcal{U}x', cy + d\mathcal{U}x\rangle_{M^\perp} - \langle c\mathcal{U}^*y' + dx', a\mathcal{U}^*y + bx\rangle_M \\
&= ac\langle y,y'\rangle_{M^\perp} + bc\langle y,\mathcal{U}x'\rangle_{M^\perp} + ad\langle \mathcal{U}x, y'\rangle_{M^\perp} + bd\langle \mathcal{U}x,\mathcal{U}x'\rangle_{M^\perp} \\
&- ac\langle \mathcal{U}^*y', \mathcal{U}^*y\rangle_M - ad\langle \mathcal{U}^*y, x'\rangle_M - bc\langle x, \mathcal{U}^*y'\rangle_M - bd\langle x,x'\rangle_M \\
&= \langle \mathcal{U}x, y'\rangle_{M^\perp} - \langle \mathcal{U}^*y, x'\rangle_M \\
&= \omega((y,x);(y',x')).
\end{aligned}$$

This completes the proof. □

Remark 6.1.2. *As an immediate consequence of the above theorem, the $\mathrm{SL}(2,\mathbb{R})$ action (6.0.2) transforms Lagrangian subspaces to Lagrangian subspaces. More precisely, for every $g \in SL(2,\mathbb{R})$ and a Lagrangian subspace N, its transformed subspace $\Phi_{\mathcal{U}}(g)N$ is also Lagrangian.*

Now, we can introduce the definition of the operator projective line, which is the heart of this project.

Definition 6.1.3 (Operator Projective Line)**.** *Let $(H, \langle\cdot,\cdot\rangle)$ be a separable real Hilbert space of even finite or infinite dimension and M, N be closed subspaces with Equalities of Dimensionality 5.1.2. Also, consider the symplectic form (6.0.1) on H:*

$$\omega((y,x),(y',x')) = \langle \mathcal{U}x, y'\rangle_{M^\perp} - \langle x', \mathcal{U}^* y\rangle_M \quad \text{for every} \quad x,x' \in M, y,y' \in M^\perp$$

where the unitary operator $\mathcal{U} : M \to M^\perp$ is determined from M and N, as in (5.1.9). Moreover, assume that $\mathrm{SL}(2,\mathbb{R})$ acts on H via the above transformations (6.0.2), which are defined by the same unitary operator $\mathcal{U}$:

$$\Phi_{\mathcal{U}} : SL(2,\mathbb{R}) \times H \to H;\ \Phi_{\mathcal{U}}(g) = (aI_M + c\mathcal{U})P_M + (dI_{M^\perp} + b\mathcal{U}^{-1})P_{M^\perp}.$$

Then, we define the operator projective line generated by N *as the orbit of N under this action:*

$$PN = \{\Phi_{\mathcal{U}}(g)N \subset H \mid g \in \mathrm{SL}(2,\mathbb{R})\}.$$

Explicitly, for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, the transformed subspace is:

$$\Phi_{\mathcal{U}}(g)N = \{(c\mathcal{U}x + dy, ax + b\mathcal{U}^*y) \mid (y,x) \in N\}.$$

6.2 $\mathrm{SL}(2,\mathbb{R})$ Action On Bi-equidefect Operators

Recall the subspace N mentioned at the beginning of the chapter, and let N^g be its transformed subspace under $g \in \mathrm{SL}(2,\mathbb{R})$. We will introduce a decomposition similar to (5.0.5):

$$N^g = N_0^g \oplus N_\#^g \oplus N_\infty^g \tag{6.2.1}$$

where $N_0^g = M \cap N^g$, $N_\infty^g = M^\perp \cap N^g$, and $N_\#^g$ is the orthogonal complement of $N_0^g \oplus N_\infty^g$ within N^g.

Also, in a similar decomposition to (5.0.4) we can write:

$$M_{00}^g = M \cap N^g, \quad M_{01}^g = M \cap (N^g)^\perp, \quad M_{10}^g = M^\perp \cap N^g, \quad M_{11}^g = M^\perp \cap (N^g)^\perp. \tag{6.2.2}$$

Therefore, we will obtain

$$M = M_{00}^g \oplus M_{01}^g \oplus M_0^g \quad \text{and} \quad M^\perp = M_{10}^g \oplus M_{11}^g \oplus M_1^g. \tag{6.2.3}$$

where M_0^g is the orthogonal complement of $M_{00}^g \oplus M_{01}^g$ within M, and M_1^g is the orthogonal complement of $M_{11}^g \oplus M_{10}^g$ within $M^\perp$.

As a consequence of Theorem 6.1.1, the unitary operator $\mathcal{U} : M \to M^\perp$ still maps each component in (6.2.3) into their corresponding counterpart, as detailed in Theorem 6.2.3 below. To prove this, we first need the next two Lemmas.

Lemma 6.2.1. *The subspace $M^\perp \cap (N^g)^\perp$ is the orthogonal complement of $N_\infty^g \oplus P_{M^\perp} N_\#^g$ within $M^\perp$ where N_∞^g and $N_\#^g$ are the subspaces in (6.2.1).*

Proof. Let $w \in M^\perp \cap (N^g)^\perp$. Then $w \perp N_\infty^g$ since $N_\infty^g \subset N^g$. Also, for any vector $v \in P_{M^\perp} N_\#^g$, there exists $u \in N_\#^g$ such that $v = P_{M^\perp} u$. Thus, $u - v \in M$ and this implies

$$\langle w, v \rangle = \langle w, u + (v - u) \rangle = \langle w, u \rangle + \langle w, v - u \rangle = 0 + 0 = 0$$

since $w \in (N^g)^\perp, u \in N^g$ and $w \in M^\perp, v - u \in M$. Thus, $w \perp P_{M^\perp} N_\#^g$. Therefore, $w \perp N_\infty^g \oplus P_{M^\perp} N_\#^g$.

The converse, if $w \in M^\perp$ and $w \perp N_\infty^g \oplus P_{M^\perp} N_\#^g$, then $w \in (N_\infty^g)^\perp$ and $w \in (N_0^g)^\perp$ because $N_0^g \subset M$. Also, for all $u \in N_\#^g$ we have

$$0 = \langle w, P_{M^\perp} u \rangle = \langle w, u - P_M u \rangle = \langle w, u \rangle - \langle w, P_M u \rangle = \langle w, u \rangle$$

and that means $w \in (N_\#^g)^\perp$. Hence

$$w \in (N_0^g)^\perp \cap (N_\infty^g)^\perp \cap (N_\#^g)^\perp = (N^g)^\perp.$$

Therefore, $w \in M^\perp \cap (N^g)^\perp$. □

In a similar way, we can also prove the following lemma.

Lemma 6.2.2. *$M \cap (N^g)^\perp$ is the orthogonal complement of $N_0^g \oplus P_M N_\#^g$ within M where N_0^g and $N_\#^g$ are the subspace in (6.2.1).*

Proof. Let $w \in M \cap (N^g)^\perp$. Then $w \perp N_0^g$ since $N_0^g \subset N^g$. Also, for any vector $v \in P_M N_\#^g$, there exists $u \in N_\#^g$ such that $v = P_M u$. Thus, $u - v \in M^\perp$ and this implies

$$\langle w, v\rangle = \langle w, u + (v-u)\rangle = \langle w, u\rangle + \langle w, v-u\rangle = 0 + 0 = 0$$

since $w \in (N^g)^\perp, u \in N^g$ and $w \in M, v - u \in M^\perp$. Therefore, $w \perp P_M N_\#^g$ and thus $w \perp N_0^g \oplus P_M N_\#^g$.

The converse, if $w \in M$ and $w \perp N_0^g \oplus P_M N_\#^g$, then $w \in (N_\infty^g)^\perp, w \in (N_0^g)^\perp$ and for all $u \in N_\#^g$ we have

$$0 = \langle w, P_M u\rangle = \langle w, u - P_{M^\perp} u\rangle = \langle w, u\rangle - \langle w, P_{M^\perp} u\rangle = \langle w, u\rangle$$

and that means $w \in (N_\#^g)^\perp$. Hence

$$w \in (N_0^g)^\perp \cap (N_\infty^g)^\perp \cap (N^g)_\#^\perp = (N^g)^\perp.$$

Therefore, $w \in M \cap (N^g)^\perp$. □

Theorem 6.2.3. *Let $g \in \mathrm{SL}(2,\mathbb{R})$ and consider the decomposition*

$$M = M_{00}^g \oplus M_{01}^g \oplus M_0^g \quad \text{and} \quad M^\perp = M_{10}^g \oplus M_{11}^g \oplus M_1^g,$$

as in (6.2.3). Then the above unitary operator $\mathcal{U} : M \to M^\perp$ satisfies

(i) $\mathcal{U}M_{00}^g = M_{11}^g$ *which means* $\dim M_{00}^g = \dim M_{11}^g$,

(ii) $\mathcal{U}M_{01}^g = M_{10}^g$ *which means* $\dim M_{01}^g = \dim M_{10}^g$,

(iii) $\mathcal{U}M_0^g = M_1^g$ *which means* $\dim M_0^g = \dim M_1^g$,

Proof. We have the decomposition

$$H = M^\perp \oplus M.$$

Every element of H is written as (y, x) where $y \in M^\perp$ and $x \in M$. Now, let $(0, x) \in M \cap N^g$. Note that, since N^g is Lagrangian, we have

$$0 = w((0,x);(y',x')) = \langle \mathcal{U}x, y'\rangle_{M^\perp} - \langle 0, x'\rangle_M, \ \forall (y',x') \in N^g.$$

This means

$$\langle \mathcal{U}x, y'\rangle_{M^\perp} = 0 \ \text{ for all } \ (y',0) \in P_{M^\perp} N^g.$$

Therefore, from Lemma 6.2.1,

$$\mathcal{U}(0,x) \in M^\perp \oplus P_{M^\perp}(N_\infty^g \oplus N_\#^g) = M^\perp \cap (N^g)^\perp.$$

This implies

$$\mathcal{U}(M \cap N^g) \subseteq M^\perp \cap (N^g)^\perp. \tag{6.2.4}$$

On the other hand, let $(y,0) \in M^\perp \cap (N^g)^\perp$. Then there exists $(0,u) \in M$ such that

$$\mathcal{U}^*(y,0) = (0,u).$$

Now for all $(y',x') \in N^g$ we have

$$w((0,u);(y',x')) = \langle \mathcal{U}u, y'\rangle_{M^\perp} = \langle y, y'\rangle_{M^\perp} = 0$$

since $y \in (N^g)^\perp$ and $y' \in N^g$. Hence $(0,u) \in (N^g)^\omega$. Since N^g is Lagrangian, then

$$\mathcal{U}^*(y,0) = (0,u) \in M \cap N^g \implies (y,0) \in \mathcal{U}(M \cap N^g).$$

Hence

$$M^\perp \cap (N^g)^\perp \subseteq \mathcal{U}(M \cap N^g). \tag{6.2.5}$$

From (6.2.4) and (6.2.5), the equality (i) holds.

To prove the equality (ii), let $(y,0) \in M^\perp \cap N^g$. Thus, sine N^g is Lagrangian

$$0 = w((y,0);(y',x')) = \langle 0, y' \rangle_{M^\perp} - \langle \mathcal{U}^* y, x' \rangle_M = -\langle \mathcal{U}^* y, x' \rangle_M, \ \forall (y',x') \in N^g.$$

This means

$$\langle \mathcal{U}^* y, x' \rangle_M = 0 \ \text{ for all } \ (x',0) \in P_M N^g.$$

Therefore, from Lemma 6.2.2,

$$\mathcal{U}^*(0,y) \in M \oplus P_M(N_0^g \oplus N_\#^g) = M \cap (N^g)^\perp.$$

Hence

$$\mathcal{U}^*(M^\perp \cap N^g) \subseteq M \cap (N^g)^\perp;$$

that is,

$$M^\perp \cap N^g \subseteq \mathcal{U}(M \cap (N^g)^\perp). \tag{6.2.6}$$

To show the other side, assume that $(0,x) \in M \cap (N^g)^\perp$ and put $\mathcal{U}(0,x) = (v,0)$. Thus, for all $(y',x') \in N^g$ we have

$$w((v,0);(y',x')) = \langle 0, y' \rangle_{M^\perp} - \langle \mathcal{U}^* v, x' \rangle_M = -\langle x, x' \rangle_M = 0$$

since $(0,x) \in (N^g)^\perp$ and $(0,x') \in N^g$. Therefore, $(0,v) \in (N^g)^\omega$. Since N^g is Lagrangian, then

$$\mathcal{U}(0,x) = (v,0) \in M^\perp \cap N^g.$$

This implies

$$\mathcal{U}(M \cap (N^g)^\perp) \subseteq M^\perp \cap N^g. \tag{6.2.7}$$

From (6.2.6) and (6.2.7), the equality (ii) holds.

For (iii), frist of all, we have

$$M^\perp = M_{11}^g \oplus M_{10}^g \oplus M_1^g \quad \text{and} \quad M = M_{00}^g \oplus M_{01}^g \oplus M_0^g.$$

By applying the unitary operator $\mathcal{U}$ on M, we will have

$$M^\perp = M_{11}^g \oplus M_{10}^g \oplus M_1^g \quad \text{and} \quad M^\perp = \mathcal{U}M_{00}^g \oplus \mathcal{U}M_{01}^g \oplus \mathcal{U}M_0^g.$$

By using (i) and (ii), we will have

$$M^\perp = M_{11}^g \oplus M_{10}^g \oplus M_1^g \quad \text{and} \quad M^\perp = M_{11}^g \oplus M_{10}^g \oplus \mathcal{U}M_0^g.$$

This means that $\mathcal{U}M_0^g$ and M_1^g are both orthogonal complements of $M_{11}^g \oplus M_{10}^g$ within $M^\perp$. Since the orthogonal complement is unique, then

$$\mathcal{U}M_0^g = M_1^g,$$

as required. □

Recall the bi-equidefect operator $(\hat{T}, \check{T})$ whose graph is represented by N. As we decomposed in (5.0.5), we have

$$N = N_0 \oplus N_\# \oplus N_\infty.$$

As in Section 5.1, $(\hat{T}, \check{T})$ is the pair of the following operators

$$\hat{T} : \mathrm{Dom}(\hat{T}) \subseteq M_0 \oplus M_{00} \to M_0 \oplus M_{00}$$

$$\hat{T}(x, x_0) = (Tx, 0) \quad \text{for every} \quad (x, x_0) \in M_0 \oplus M_{00}$$

and

$$\check{T} : \mathrm{Dom}(\check{T}) \subseteq M_1 \oplus M_{10} \to M_1 \oplus M_{10}$$

$$\check{T}(y, y_0) = (\mathcal{U}T^{-1}\mathcal{U}^* y, 0) \quad \text{for every} \quad (y, y_0) \in M_1 \oplus M_{10}$$

In the above definitions,

$$T : \mathrm{Dom}(T) \subseteq M_0 \to M_0$$

is an injective self-adjoint operator defined on $P_M N$ with a dense range and a graph is represented by

$$N_\# = \mathrm{Graph}(\mathcal{U}T) = \{(\mathcal{U}Tx, x) \mid x \in \mathrm{Dom}(T)\}.$$

Also,

$$N_0 \oplus N_\# = \mathrm{Graph}(\mathcal{U}\hat{T}) = \{(\mathcal{U}\hat{T}x, x) \mid x \in \mathrm{Dom}(\hat{T})\}$$

and

$$N_\# \oplus N_\infty = \mathrm{Graph}(\mathcal{U}^*\check{T}) = \{(y, \mathcal{U}^*\check{T}y) \mid y \in \mathrm{Dom}(\check{T})\}.$$

This follows from the construction in Section 5.1.

Now, if we take $g = \begin{pmatrix} a & b \\ c & c \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ and apply the $\mathrm{SL}(2, \mathbb{R})$ action $\Phi_\mathcal{U}$ on these graphs, we obtain

$$N^g = N_0^g \oplus N_\#^g \oplus N_\infty^g,$$

as decomposed in (6.2.1). We claim that N^g represents a graph of a new bi-equidefect operator $(\hat{T}_g, \check{T}_g)$ with the same above setting. To prove this claim, we first consider the operator

$$T_g : \mathrm{Dom}(T_g) \subseteq M_0^g \to M_0^g$$

such that

$$N_\#^g = \{(\mathcal{U}T_g x, x) \mid x \in \mathrm{Dom}(T_g)\}.$$

Now, we shall show that T_g is self-adjoint. This can be proven as follows: since $N_\#^g$ is contained in the Lagrangian subspace N^g, then for all $x, x' \in M_0^g$ we have

$$0 = \omega(\mathcal{U}T_g x, x), (\mathcal{U}T_g x', x')) = \langle \mathcal{U}x, \mathcal{U}T_g x' \rangle_{M^\perp} - \langle T_g x, x' \rangle_M = \langle x, T_g x' \rangle_M - \langle T_g x, x' \rangle_M.$$

Therefore,

$$\langle x, T_g x' \rangle_M = \langle T_g x, x' \rangle_M. \tag{6.2.8}$$

Hence T is symmetric. To show that it is self-adjoint, we need to prove the equality of domains:

$$\mathrm{Dom}(T_g) = \mathrm{Dom}(T_g^*) \tag{6.2.9}$$

where $T_g^* : \mathrm{Dom}(T_g^*) \subseteq M_0^g \to M_0^g$ is the adjoint of T_g. To this end, assume, for contradiction, that

$$\mathrm{Dom}(T_g) \subsetneq \mathrm{Dom}(T_g^*).$$

This implies

$$N_\#^g = \mathrm{Graph}(\mathcal{U}T_g) \subsetneq \mathrm{Graph}(\mathcal{U}T_g^*) \subset M_0^g \oplus M_1^g.$$

Now, we aim to show that $\mathrm{Graph}(\mathcal{U}T_g^*)$ is a subspace of N^g. This will lead to a contradiction because we cannot have a subspace of N^g strictly larger than $N_\#^g$ and contained inside $M_0^g \oplus M_1^g$. To accomplish that, take any $(\mathcal{U}T_g x, x) \in \mathrm{Graph}(\mathcal{U}T_g)$ and $(\mathcal{U}T_g^* x', x') \in \mathrm{Graph}(\mathcal{U}T_g^*)$. Then,

$$\begin{aligned}\omega((\mathcal{U}T_g x, x), (\mathcal{U}T_g^* x', x')) &= \langle \mathcal{U}x, \mathcal{U}T_g^* x' \rangle_{M^\perp} - \langle T_g x, x' \rangle_M \\ &= \langle x, T_g^* x' \rangle_M - \langle T_g x, x' \rangle_M \\ &= \langle T_g x, x' \rangle_M - \langle T_g x, x' \rangle_M \\ &= 0.\end{aligned}$$

Also, take any $(0, x) \in N_0^g$ and $(\mathcal{U}T_g^* x', x') \in \mathrm{Graph}(\mathcal{U}T_g^*)$. Then,

$$\omega((0,x), (\mathcal{U}T_g^* x', x')) = \langle \mathcal{U}x, \mathcal{U}T_g^* x' \rangle_{M^\perp} - \langle 0, x' \rangle_M = \langle x, T_g^* x' \rangle_M = 0$$

since $x \in M_{00}^g$ and $T_g^* x' \in M_0^g$. Moreover, take any $(y, 0) \in N_\infty^g$ and $(\mathcal{U}T_g^* x', x') \in \mathrm{Graph}(\mathcal{U}T_g^*)$. Then,

$$\omega((y,0), (\mathcal{U}T_g^* x', x')) = \langle 0, \mathcal{U}T_g^* x' \rangle_{M^\perp} - \langle \mathcal{U}^* y, x' \rangle_M = -\langle \mathcal{U}^* y, x' \rangle_M = 0$$

since $\mathcal{U}^* y \in M_{01}^g$ and $x' \in M_0^g$. Thus, we conclude that

$$N_\#^g \subsetneq \mathrm{Graph}(\mathcal{U}T_g^*) \subset (N^g)^\omega = N^g.$$

Hence, the equality (6.2.9) must hold. Therefore, as required, T_g whose graph is represented by

$$N_\#^g = \{(\mathcal{U}T_g x, x) \mid x \in \mathrm{Dom}(T_g)\}$$

is a self-adjoint operator. Furthermore, the injectivity of T_g and density of its range follow from

$$N_\#^g \cap M = N_\#^g \cap M^\perp = \{0\},$$

as demonstarted in Section 5.1.

Moreover, as constructed in Section 5.1, we can define

$$\mathcal{U}\hat{T}_g : M_0^g \oplus M_{00}^g \to M_1^g \oplus M_{11}^g \tag{6.2.10}$$

where

$$\hat{T}_g(x, x_0) = (T_g x, 0) \quad \text{for every} \quad (x, x_0) \in M_0^g \oplus M_{00}^g. \tag{6.2.11}$$

Moreover, consider another operator

$$\mathcal{U}^*\check{T}_g : M_1^g \oplus M_{10}^g \to M_0^g \oplus M_{01}^g \tag{6.2.12}$$

where

$$\check{T}_g(y, y_0) = (\mathcal{U}T_g^{-1}\mathcal{U}^* y, 0) \quad \text{for every} \quad (y, y_0) \in M_1^g \oplus M_{10}^g. \tag{6.2.13}$$

As a result,

$$\mathrm{Graph}(\mathcal{U}\hat{T}_g) = N_0^g \oplus N_\#^g \quad \text{and} \quad \mathrm{Graph}(\mathcal{U}^*\check{T}_g) = N_\#^g \oplus N_\infty^g \tag{6.2.14}$$

with

$$\ker(\mathcal{U}\hat{T}_g) = N_0^g \quad \text{and} \quad \ker(\mathcal{U}^*\check{T}_g) = N_\infty^g. \tag{6.2.15}$$

Furthermore, from Theorem 6.2.3, we have

$$\mathcal{U}M_{00}^g = M_{11}^g \quad \text{and} \quad \mathcal{U}M_{01}^g = M_{10}^g.$$

This implies $\dim(M_{00}^g)$ and $\dim(M_{11}^g)$ are the same. Also, $\dim(M_{01}^g)$ and $\dim(M_{10}^g)$ are the same. Therefore, the operators $\mathcal{U}\hat{T}_g$ and $\mathcal{U}\check{T}_g$ are indeed equidefect, cf. Definition 5.1.4.

In summary, we can state that the bi-equidefect operator $(\hat{T}, \check{T})$ whose graph is represented by the Lagrangain subspace N is transformed by $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ through transforming N into N^g and determining its intersections with M and $M^\perp$ to obtain a new bi-equidefect operator $(\hat{T}_g, \check{T}_g)$.

Example 6.2.4. *Consider*

$$\mathbb{R}^{12} = M^\perp \oplus M$$

where

$$M = \{e_1, e_2, \ldots, e_6\} \cong \mathbb{R}^6 \quad \textit{and} \quad M^\perp = \{f_1, f_2, \ldots, f_6\} \cong \mathbb{R}^6.$$

Consider the unitary operator

$$\mathcal{U} : M \to M^\perp$$

$$\mathcal{U}e_n = f_n.$$

Now, construct a bi-equidefect operator $(\hat{T}, \check{T})$ on

$$M = \{e_1, e_2, \ldots, e_6\}$$

whose graph is represented by

$$N = \operatorname{span}\big\{(0, e_1), (f_3, e_3), (f_4, e_4), (2f_5, e_5), (3f_6, e_6), (f_2, 0)\big\},$$

which is a Lagrangian subspace with respect to the standard symplectic form. Now, if we take $g = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$, then

$$N^g = \operatorname{span}\big\{(-f_1, e_1),\ (f_2, 0),\ (0, e_3),\ (0, e_4),\ (f_5, e_5),\ (2f_6, e_6)\big\}..$$

In this subspace,

$$N_0^g = \operatorname{span}\{(0, e_3), (0, e_4)\}, \quad N_\#^g = \operatorname{span}\{(-f_1, e_1), (f_5, e_5), (2f_6, e_6)\}, \quad N_\infty^g = \operatorname{span}\{(f_2, 0)\}.$$

If we define the operator T_g on M_0^g

$$T_g e_1 = -e_1, \qquad T_g e_5 = e_5, \qquad T_g e_6 = 2e_6,$$

then

$$N_\#^g = \{(\mathcal{U}T_g x,\ x) : x \in M_0^g\}.$$

The extended operator $\hat{T}_g$ on $M_0^g \ominus M_{00}^g$ as in (6.2.11) is:

$$\hat{T}_g : M_0^g \oplus M_{00}^g \longrightarrow M_0^g \ominus M_{00}^g,$$

$$\hat{T}_g(e_1) = -e_1, \quad \hat{T}_g(e_3) = 0, \quad \hat{T}_g(e_4) = 0, \quad \hat{T}_g(e_5) = e_5, \quad \hat{T}_g(e_6) = 2e_6.$$

The operator $\mathcal{U}\hat{T}_g : M_0^g \oplus M_{00}^g \longrightarrow M_1^g \oplus M_{11}^g$ *is equidefect because*

$$\dim(\ker(\mathcal{U}\hat{T}_g)) = 2 = \dim(\operatorname{coker}(\mathcal{U}\hat{T}_g)).$$

Dually, the operator $\check{T}_g$ *on* $M_1^g \oplus M_{10}^g$ *as in (6.2.13) is:*

$$\check{T}_g : M_1^g \oplus M_{10}^g \longrightarrow M_1^g \oplus M_{10}^g,$$

$$\check{T}_g(f_1) = -f_1, \quad \check{T}_g(f_2) = 0, \quad \check{T}_g(f_5) = f_5, \quad \check{T}_g(f_6) = \tfrac{1}{2} f_6.$$

The operator $\mathcal{U}^*\check{T}_g : M_1^g \ominus M_{10}^g \longrightarrow M_0^g \ominus M_{01}^g$ *is equidefect because*

$$\dim(\ker(\mathcal{U}\check{T}_g)) = 1 = \dim(\operatorname{coker}(\mathcal{U}\check{T}_g)).$$

In conclusion, when we transformed the bi-equidefect operator $(\hat{T}, \check{T})$ *by* $g \in \mathrm{SL}(2,\mathbb{R})$*, we obtained a new bi-equidefect operator* $(\hat{T}_g, \check{T}_g)$*.*

6.3 Spectrum Of Bi-equidefect Operators

Let H be either an even finite-dimensional or a separable infinite-dimensional real Hilbert space and let $T : \mathrm{Dom}(T) \subseteq H \to H$ be a self-adjoint operator, not necessarily bounded. The self-adjointness of T implies that its spectrum is a subset of $\mathbb{R}$. Explicitly, its spectrum is defined as:

$$\sigma(T) = \{\lambda \in \mathbb{R} \mid T - \lambda I \text{ is not boundedly invertible}\}.$$

It is common to write $T - \lambda I$ as $T - \lambda$. Therefore, for simplicity, we may use it in this thesis. Also, it is well-known that

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

where $\sigma_p(T)$ is the point spectrum (set of all eigenvalues of T), $\sigma_c(T)$ is the continuous spectrum and $\sigma_r(T)$ is the residual spectrum [24]. Furthermore, it is known that if T is a self-adjoint operator, then the residual spectrum $\sigma_r(T)$ is empty. Therefore,

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \subseteq \mathbb{R}.$$

Let M, N be closed subspaces of the Hilbert space H with Equalities of Dimensionality 5.1.2. As decomposed in (5.0.4) and (5.0.5),

$$M = M_{00} \ominus M_{01} \ominus M_0, \qquad M^\perp = M_{10} \ominus M_{11} \ominus M_1, \qquad N = N_0 \ominus N_\infty \ominus N_\#. \tag{6.3.1}$$

Moreover, as constructed in (5.1.11) and (5.1.12), a bi-equidefect operator $(\hat{T}, \check{T})$ can be defined as:

$$\hat{T} : M_0 \ominus M_{00} \to M_0 \ominus M_{00}; \;\; \hat{T}(x, x_0) = (Tx, 0) \;\text{ for every }\; (x, x_0) \in M_0 \ominus M_{00}, \tag{6.3.2}$$

and

$$\check{T} : M_1 \ominus M_{10} \to M_1 \ominus M_{10}; \;\; \check{T}(y, y_0) = (\mathcal{U}T^{-1}\mathcal{U}^* y, 0) \;\text{ for every }\; (y, y_0) \in M_1 \ominus M_{10}. \tag{6.3.3}$$

with

$$N_0 \ominus N_\# = \mathrm{Graph}(\mathcal{U}\hat{T}) = \{(\mathcal{U}\hat{T}x, x) \mid x \in \mathrm{Dom}(\hat{T})\}$$

and

$$N_{\#} \oplus N_{\infty} = \mathrm{Graph}(\mathcal{U}^*\check{T}) = \{(y, \mathcal{U}^*\check{T}y) \mid y \in \mathrm{Dom}(\check{T})\}.$$

The following definition introduces the spectrum of this bi-equidefect operator as a subset of the real projective line $P\mathbb{R}$.

Definition 6.3.1. *The spectrum of a bi-equidefect operator $(\hat{T}, \check{T})$ consists of two charts:*

$$\begin{aligned}\sigma(\hat{T}, \check{T}) = &\big\{[\lambda : 1] \in P\mathbb{R} \mid \hat{T} - \lambda I \text{ is not boundedly invertible}\big\} \\ &\cup \big\{[1 : \lambda'] \in P\mathbb{R} \mid \check{T} - \lambda' I \text{ is not boundedly invertible}\big\}.\end{aligned}$$

The former chart will be labelled as $\hat{T}$-chart, while the latter as $\check{T}$-chart.

Remark 6.3.2. *The above two charts overlap on $\sigma(\hat{T}, \check{T}) \setminus \{[0 : 1], [1 : 0]\}$. More precisely, for every $[\lambda : 1] \in \hat{T}$-chart with $\lambda \neq 0$, there exists $[1 : \lambda'] \in \check{T}$-chart such that $\lambda = \frac{1}{\lambda'}$.*

Furthermore, if we assign $[\lambda : 1]$ with λ and $[1 : 0]$ with ∞, then, we conventionally describe the elements of the spectrum of $(\hat{T}, \check{T})$ as:

- $\lambda \in \sigma(\hat{T}, \check{T})$ if $\hat{T} - \lambda I$ is not boundedly invertible,
- $\infty \in \sigma(\hat{T}, \check{T})$ if $\check{T}$ is not boundedly invertible.

Example 6.3.3. *Consider*

$$\mathbb{R}^{12} = M^{\perp} \oplus M$$

where

$$M = \{e_1, e_2, \ldots, e_6\} \cong \mathbb{R}^6 \quad \text{and} \quad M^{\perp} = \{f_1, f_2, \ldots, f_6\} \cong \mathbb{R}^6.$$

In Example 6.2.4, we have constructed a bi-equidefect operator $(\hat{T}, \check{T})$ on

$$M = \{e_1, e_2, \ldots, e_6\}$$

whose graph is represented by

$$N = \mathrm{span}\{(0, e_1), (\mathcal{U}e_3, e_3), (\mathcal{U}e_4, e_4), (2\mathcal{U}e_5, e_5), (3\mathcal{U}e_6, e_6), (\mathcal{U}e_2, 0)\}.$$

One can easily check that the eigenvalues of the bi-equidefect are $\sigma_p(\hat{T}, \check{T}) = \{0,\ 1,\ 2,\ 3,\ \infty\}$. If we transform them by the formula (4.1.1) (with the infinity rules in (4.1.5)),

$$\Phi\left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, x\right) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \cdot x = x - 1,$$

then the transformed set is $\{-1,\ 0,\ 1,\ 2,\ \infty\}$. This set is the eigenvalues of the transformed bi-equidefect operator $(\hat{T}_g, \check{T}_g)$ whose graph is

$$N^g = \mathrm{span}\big\{(-\mathcal{U}e_1, e_1),\ (\mathcal{U}e_2, 0),\ (0, e_3),\ (0, e_4),\ (\mathcal{U}e_5, e_5),\ (2\mathcal{U}e_6, e_6)\big\}.$$

This observation leads us to establish a relationship between the transformation of a bi-equidefect operator and the transformation of its spectrum under $\mathrm{SL}(2, \mathbb{R})$ action as defined in (4.1.7) and (4.1.8). This is what the following theorem states.

Theorem 6.3.4. *Let H be either an even finite-dimensional or a separable infinite-dimensional real Hilbert space and let M, N be closed subspaces with Equalities of Dimensionality 5.1.2. Also, let $(\hat{T}, \check{T})$ be a bi-equidefect operator, as in (6.3.2), (6.3.3), and $(\hat{T}_g, \check{T}_g)$ is its transformed operator under $g \in \mathrm{SL}(2,\mathbb{R})$, see Section 6.2. Moreover, let $\sigma(\hat{T}, \check{T})$ be the spectrum of $(\hat{T}, \check{T})$ and $\sigma(\hat{T}_g, \check{T}_g)$ be the spectrum of $(\hat{T}_g, \check{T}_g)$. Then the spectrum $\sigma(\hat{T}_g, \check{T}_g)$ will be the transformed set of $\sigma(\hat{T}, \check{T})$ under $g \in \mathrm{SL}(2,\mathbb{R})$; that is,*

$$g \cdot \sigma(\hat{T}, \check{T}) = \sigma(\hat{T}_g, \check{T}_g) \quad \textit{for every} \quad g \in \mathrm{SL}(2,\mathbb{R}) \tag{6.3.4}$$

where $\mathrm{SL}(2,\mathbb{R})$ action on the left-hand side is defined in (4.1.6) and,

$$\hat{T}_g = (a\hat{T} + b)(c\hat{T} + d)^{-1} \qquad \textit{and} \qquad \check{T}_g = (c + d\check{T})(a + b\check{T})^{-1}.$$

Proof. As in (6.3.1), assume that

$$N = N_0 \oplus N_\# \oplus N_\infty$$

represents the graph of $(\hat{T}, \check{T})$ such that

$$N_0 \oplus N_\# = \mathrm{Graph}(\mathcal{U}\hat{T}) = \{(\mathcal{U}\hat{T}x, x) \mid x \in \mathrm{Dom}(\hat{T})\}$$

and

$$N_\# \oplus N_\infty = \mathrm{Graph}(\mathcal{U}^*\check{T}) = \{(y, \mathcal{U}^*\check{T}y) \mid y \in \mathrm{Dom}(\check{T})\}.$$

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$. Then, as in Section 6.2, the graph of the bi-equidefect operator $(\hat{T}_g, \check{T}_g)$ is represented by the transformed subspace

$$N^g = \Phi_{\mathcal{U}}(g)N$$

where $\Phi_{\mathcal{U}}(g)$ is in (6.0.2). Also, observe

$$\begin{aligned}
\Phi_{\mathcal{U}}(g)(N_0 \oplus N_\#) &= \Phi_{\mathcal{U}}(g)\,\mathrm{Graph}(\mathcal{U}\hat{T}) \\
&= \{(\mathcal{U}(a\hat{T} + b)x, (c\hat{T} + d)x) \mid x \in \mathrm{Dom}(\hat{T})\} \\
&= \{(\mathcal{U}(a\hat{T} + b)(c\hat{T} + d)^{-1}x, x) \mid x \in \mathrm{Ran}(c\hat{T} + d)\} \\
&\oplus \{(\mathcal{U}(a\hat{T} + b)x, 0) \mid x \in \ker(c\hat{T} + d)\},
\end{aligned}$$

and

$$\begin{aligned}
\Phi_{\mathcal{U}}(g)(N_\# \oplus N_\infty) &= \Phi_{\mathcal{U}}(g)\,\mathrm{Graph}(\mathcal{U}^*\check{T}) \\
&= \{((a + b\check{T})y, \mathcal{U}^*(c + d\check{T})y) \mid y \in \mathrm{Dom}(\check{T})\} \\
&= \{(y, \mathcal{U}^*(c\check{T} + d)(a\check{T} + b)^{-1}y) \mid y \in \mathrm{Ran}(a\check{T} + b)\} \\
&\oplus \{(0, \mathcal{U}^*(c + d\check{T})y) \mid y \in \ker(a\check{T} + b)\}.
\end{aligned}$$

Now, assume that the spectrum of $(\hat{T}, \check{T})$ is

$$\begin{aligned}
\sigma(\hat{T}, \check{T}) = &\left\{[\lambda : 1] \in P\mathbb{R} \mid \hat{T} - \lambda \text{ is not boundedly invertible}\right\} \\
&\cup \left\{[1 : \lambda'] \in P\mathbb{R} \mid \check{T} - \lambda' \text{ is not boundedly invertible}\right\}.
\end{aligned}$$

From (4.1.6), the transformation of $\sigma(\hat{T},\check{T})$ is:

$$\begin{aligned} g\cdot\sigma(\hat{T},\check{T}) &= \left\{[a\lambda+b:c\lambda+d]\in P\mathbb{R}\mid[\lambda:1]\in\sigma(\hat{T},\check{T})\right\} \\ &\cup\left\{[a+b\lambda':c+d\lambda']\in P\mathbb{R}\mid[1:\lambda']\in\sigma(\hat{T},\check{T})\right\} \\ &=\left\{\left[\frac{a\lambda+b}{c\lambda+d}:1\right]\in P\mathbb{R}\mid c\lambda+d\neq 0 \text{ and } [\lambda:1]\in\sigma(\hat{T},\check{T})\right\} \\ &\cup\left\{\left[1:\frac{c+d\lambda'}{a+b\lambda'}\right]\in P\mathbb{R}\mid a+b\lambda'\neq 0 \text{ and } [1:\lambda']\in\sigma(\hat{T},\check{T})\right\}. \end{aligned}$$

We might need to demonstrate the last equality. First of all, we should note that

$$c\lambda+d=0 \qquad \text{and} \qquad a+b\lambda'=0$$

cannot happen because

$$\begin{cases} c\lambda+d=0 \\ a+b\lambda'=0 \end{cases} \implies \begin{cases} c\lambda+d=0 \\ a\lambda+b=0 \end{cases} \implies \begin{cases} \lambda=\frac{-d}{c} \\ \lambda=\frac{-b}{a} \end{cases} \implies \frac{-d}{c}=\frac{-b}{a} \implies ad-bc=0,$$

which contradicts $ad-bc=1$. Next, we should clarify that if

$$c\lambda+d=0 \qquad \text{or} \qquad a+b\lambda'=0,$$

then the transformed element of $[\lambda:1]$ or of $[1:\lambda']$ will be in $g\cdot\sigma(\hat{T},\check{T})$. This follows from the homogeneity of coordinates. To see that, assume that $c\lambda+d=0$ with $\lambda\neq 0$ (thus, $a\lambda+b\neq 0$). Then the transformed element of $[\lambda:1]$ will be

$$\left[1:\frac{c+d\lambda'}{a+b\lambda'}\right]\in g\cdot\sigma(\hat{T},\check{T})$$

because

$$[a\lambda+b:c\lambda+d]=\left[1:\frac{c\lambda+d}{a\lambda+b}\right]=\left[1:\frac{c\frac{1}{\lambda'}+d}{a\frac{1}{\lambda'}+b}\right]=\left[1:\frac{c+d\lambda'}{a+b\lambda'}\right].$$

If $a+b\lambda'=0$ with $\lambda'\neq 0$ (thus, $c+d\lambda'\neq 0$), the transformed element of $[1:\lambda']$ will also be

$$\left[\frac{a\lambda+b}{c\lambda+d}:1\right]\in g\cdot\sigma(\hat{T},\check{T})$$

because

$$[a+b\lambda':c+d\lambda']=\left[\frac{a+b\lambda'}{c+d\lambda'}:1\right]=\left[\frac{a+b\frac{1}{\lambda}}{c+d\frac{1}{\lambda}}:1\right]=\left[\frac{a\lambda+b}{c\lambda+d}:1\right].$$

This shows that the equality

$$\begin{aligned} g\cdot\sigma(\hat{T},\check{T}) &=\left\{\left[\frac{a\lambda+b}{c\lambda+d}:1\right]\in P\mathbb{R}\mid c\lambda+d\neq 0 \text{ and } [\lambda:1]\in\sigma(\hat{T},\check{T})\right\} \\ &\cup\left\{\left[1:\frac{c+d\lambda'}{a+b\lambda'}\right]\in P\mathbb{R}\mid a+b\lambda'\neq 0 \text{ and } [1:\lambda']\in\sigma(\hat{T},\check{T})\right\}. \end{aligned}$$

indeed holds. Now, we need to prove that

$$g\cdot\sigma(\hat{T},\check{T})=\sigma(\hat{T}_g,\check{T}_g) \tag{6.3.5}$$

where

$$\hat{T}_g=(a\hat{T}+b)(c\hat{T}+d)^{-1} \qquad \text{and} \qquad \check{T}_g=(c+d\check{T})(a+b\check{T})^{-1}.$$

But we first need to demonstrate that $\frac{-d}{c}, \frac{-a}{b} \in \sigma(\hat{T}_g, \check{T}_g)$ is not excluded, in order to guarantee the invertibility of

$$c\hat{T} + d \qquad \text{and} \qquad a + b\check{T}.$$

Indeed, note that if $\frac{-d}{c} \in \sigma(\hat{T}_g, \check{T}_g)$, then $\frac{-d}{c} \in \sigma(\check{T}_g)$ and $\frac{-d}{c} \notin \sigma(\hat{T}_g)$ which implies that $c\hat{T} + d$ is invertible. Furthermore, as illustrated above, $ad - bc = 1$ leads to $\frac{-d}{c} \neq \frac{-a}{b}$ which means that $\frac{-d}{c} \in \sigma(\check{T}_g)$ does not affect invertibility of $a + b\check{T}$. Similarly, if $\frac{-a}{b} \in \sigma(\hat{T}_g, \check{T}_g)$, then $\frac{-a}{b} \in \sigma(\hat{T}_g)$ and $\frac{-a}{b} \notin \sigma(\check{T}_g)$ which yields the invertibility of $a + b\check{T}$. Moreover, $\frac{-a}{b} \in \sigma(\hat{T}_g)$ does not prevent $c\hat{T} + d$ from being invertible because $\frac{-d}{c} \neq \frac{-a}{b}$. Therefore, the transformed operators

$$\hat{T}_g = (a\hat{T} + b)(c\hat{T} + d)^{-1} \qquad \text{and} \qquad \check{T}_g = (c + d\check{T})(a + b\check{T})^{-1}$$

are well-defined even when $\frac{-d}{c}, \frac{-a}{b} \in \sigma(\hat{T}_g, \check{T}_g)$. We can now proceed to prove equality (6.3.5). Assume $c\lambda + d \neq 0$ (possibly $a\lambda + b = 0$). We have three cases:

- Case 1: if $c \neq 0$, $d \neq 0$ and $\frac{-d}{c} \notin \sigma(\hat{T})$ (possibly $\frac{-d}{c} \in \sigma(\check{T})$), then $c\hat{T} + d$ is invertible. Thus,

$$\begin{aligned}
\hat{T}_g - \frac{a\lambda + b}{c\lambda + d} &= (a\hat{T} + b)(c\hat{T} + d)^{-1} - \frac{a\lambda + b}{c\lambda + d} \\
&= \Big[(c\lambda + d)(a\hat{T} + b) - (a\lambda + b)(c\hat{T} + d)\Big](c\lambda + d)^{-1}(c\hat{T} + d)^{-1} \\
&= \Big[(bc - ad)\lambda - (bc - ad)\hat{T}\Big](c\lambda + d)^{-1}(c\hat{T} + d)^{-1} \\
&= (c\lambda + d)^{-1}(\hat{T} - \lambda)(c\hat{T} + d)^{-1}.
\end{aligned}$$

- Case 2: if $c \neq 0$, $d = 0$ and $\frac{-d}{c} \notin \sigma(\hat{T})$ (possibly $\frac{-d}{c} \in \sigma(\check{T})$), then $c\hat{T} + d$ is also invertible. Thus, the same argument as in Case 1 remains valid.

- Case 3: if $c = 0$ and $d \neq 0$ (which means $ad = 1$), then

$$\begin{aligned}
\hat{T}_g - \frac{a\lambda + b}{d} &= d^{-1}(a\hat{T} + b) - \frac{a\lambda + b}{d} \\
&= d^{-1}\Big[(a\hat{T} + b) - (a\lambda + b)\Big] \\
&= \big(d^2\big)^{-1}\Big[d(a\hat{T} + b) - d(a\lambda + b)\Big] \\
&= \big(d^2\big)^{-1}\Big[ad\hat{T} - ad\lambda\Big] \\
&= \big(d^2\big)^{-1}\Big(\hat{T} - \lambda\Big).
\end{aligned}$$

Hence, in all the above cases, we have

$$\hat{T}_g - \frac{a\lambda + b}{c\lambda + d} \quad \text{is invertible} \iff \hat{T} - \lambda \quad \text{is invertible.}$$

Equivalently,

$$g \cdot [\lambda : 1] = [a\lambda + b : c\lambda + d] = \left[\frac{a\lambda + b}{c\lambda + d} : 1\right] \in \sigma(\hat{T}_g, \check{T}_g) \iff [\lambda : 1] \in \sigma(\hat{T}, \check{T}). \tag{6.3.6}$$

Now, assume that $a + b\lambda' \neq 0$ (possibly $c + d\lambda' = 0$). Here, we also have three cases:

- Case 1: if $a \neq 0$, $b \neq 0$ and $\frac{-a}{b} \notin \sigma(\check{T})$ (possibly $\frac{-a}{b} \in \sigma(\hat{T})$), then $a + b\check{T}$ is invertible. Thus,

$$\begin{aligned}\check{T}_g - \frac{c + d\lambda'}{a + b\lambda'} &= (c + d\check{T})(a + b\check{T})^{-1} - \frac{c + d\lambda'}{a + b\lambda'} \\ &= \left[(a + b\lambda')(c + d\check{T}) - (c + d\lambda')(a + b\check{T})\right](a + b\lambda')^{-1}(a + b\check{T})^{-1} \\ &= \left[(bc - ad)\lambda' - (bc - ad)\check{T}\right](a + b\lambda')^{-1}(a + b\check{T})^{-1} \\ &= (a + b\lambda')^{-1}(\check{T} - \lambda')(a + b\check{T})^{-1}.\end{aligned}$$

- Case 2: if $a = 0$, $b \neq 0$ and $\frac{-a}{b} \notin \sigma(\check{T})$ (possibly $\frac{-a}{b} \in \sigma(\hat{T})$), then $a + b\check{T}$ is invertible. Thus, the same argument as in Case 1 remains valid.

- Case 3: if $a \neq 0$ and $b = 0$ (which means $ad = 1$), then

$$\begin{aligned}\check{T}_g - \frac{c + d\lambda'}{a} &= (a)^{-1}(c + d\check{T}) - \frac{c + d\lambda'}{a} \\ &= (a)^{-1}\left[(c + d\check{T}) - (c + d\lambda')\right] \\ &= \left(a^2\right)^{-1}\left[a(c + d\check{T}) - a(c + d\lambda')\right] \\ &= \left(a^2\right)^{-1}\left[ad\check{T} - ad\lambda'\right] \\ &= \left(a^2\right)^{-1}(\check{T} - \lambda').\end{aligned}$$

Hence, in all the above cases, we have

$$\check{T}_g - \frac{c + d\lambda'}{a + b\lambda'} \quad \text{is invertible} \iff \check{T} - \lambda' \quad \text{is invertible.}$$

Equivalently,

$$g \cdot [1 : \lambda'] = [a + b\lambda' : c + d\lambda'] = \left[1 : \frac{c + d\lambda'}{a + b\lambda'}\right] \in \sigma(\hat{T}_g, \check{T}_g) \iff [1 : \lambda'] \in \sigma(\hat{T}, \check{T}). \tag{6.3.7}$$

From (6.3.6) and (6.3.7), the equality (6.3.5) holds.

□

This result becomes useful if we recall the spectral decomposition of self-adjoint operators [32], [14]. In this decomposition, we consider the Borel σ-algebra

$$\mathcal{B}(\mathbb{R}) = \sigma\left(\{(a, b] \mid a < b\}\right),$$

with a projection-valued measure

$$E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(H)$$

such that the following properties hold:

- $E(\varnothing) = 0$, $E(\mathbb{R}) = I$.
- For each $A \in \mathcal{B}(\mathbb{R})$, $E(A)$ is an orthogonal projection on H.
- For $A_1, A_2 \in \mathcal{B}(\mathbb{R})$, $E(A_1 \cap A_2) = E(A_1)E(A_2)$.
- E is countably additive; that is, for each sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{B}(\mathbb{R})$, we have

$$E\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} E(A_n).$$

- For each $x \in H$,
$$\mu_x(A) = \langle E(A)x, x\rangle \quad \text{where} \quad A \in \mathcal{B}(\mathbb{R}) \tag{6.3.8}$$
is a measure on the σ-algebra $\mathcal{B}(\mathbb{R})$.

Now, for a self-adjoint operator T on a real Hilbert space $(H, \langle\cdot,\cdot\rangle)$, we have $\sigma(T) \subseteq \mathbb{R}$ and there exists a projection-valued measure

$$E : \mathcal{B}(\sigma(T)) \to \mathcal{L}(H)$$

such that

$$T = \int_{\sigma(T)} \lambda dE(\lambda). \tag{6.3.9}$$

Here,

$$\mathcal{B}(\sigma(T)) = \sigma\left(\{\sigma(T) \cap (a, b] \mid a < b\}\right)$$

and the domain of T is:

$$\mathrm{Dom}(T) = \left\{ x \in H \mid \int_{\sigma(T)} \lambda^2 d\langle E(\lambda)x, x\rangle < \infty \right\}.$$

In this domain, we guarantee that

$$\|Tx\|^2 = \int_{\sigma(T)} \lambda^2 d\langle E(\lambda)x, x\rangle < \infty \ \text{ for every } \ x \in \mathrm{Dom}(T).$$

Now, since a bi-equidefect operator $(\hat{T}, \check{T})$ is a pair of self-adjoint operators, we can write

$$\hat{T} = \int_{\sigma(\hat{T})} \lambda dE(\lambda) \quad \text{and} \quad \check{T} = \int_{\sigma(\check{T})} \lambda' dE(\lambda')$$

where $\sigma(\hat{T})$ and $\sigma(\check{T})$ are the spectra of $\hat{T}$ and $\check{T}$, respective. Thus, as in Definition 6.3.1, the spectrum $\sigma(\hat{T}, \check{T}) \subseteq P\mathbb{R}$ is:

$$\begin{aligned}\sigma(\hat{T}, \check{T}) = &\left\{ [\lambda : 1] \in P\mathbb{R} \mid \hat{T} - \lambda I \text{ is not boundedly invertible} \right\} \\ &\cup \left\{ [1 : \lambda'] \in P\mathbb{R} \mid \check{T} - \lambda' I \text{ is not boundedly invertible} \right\}.\end{aligned}$$

Also, as in Remark 6.3.2, the above two charts overlap on $\sigma(\hat{T}, \check{T}) \setminus \{[0 : 1], [1 : 0]\}$. More precisely, for every $[\lambda : 1] \in \hat{T}$-chart with $\lambda \neq 0$, there exists $[1 : \lambda'] \in \check{T}$-chart such that $\lambda = \frac{1}{\lambda'}$. After that, if we transform this spectrum by $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$, we will obtain

$$\begin{aligned}\sigma(\hat{T}_g, \check{T}_g) = &\left\{ \left[\frac{a\lambda + b}{c\lambda + d} : 1 \right] \in P\mathbb{R} \mid c\lambda + d \neq 0 \text{ and } [\lambda : 1] \in \sigma(\hat{T}, \check{T}) \right\} \\ &\cup \left\{ \left[1 : \frac{c + d\lambda'}{a + b\lambda'} \right] \in P\mathbb{R} \mid a + b\lambda' \neq 0 \text{ and } [1 : \lambda'] \in \sigma(\hat{T}, \check{T}) \right\}.\end{aligned}$$

Remark 6.3.5. *As shown in the proof of Theorem 6.3.4, it is worth emphasising that $a + b\lambda'$ and $c\lambda + d$ cannot both be zero because*

$$\begin{cases} c\lambda + d = 0 \\ a + b\lambda' = 0 \end{cases} \implies \begin{cases} c\lambda + d = 0 \\ a\lambda + b = 0 \end{cases} \implies \begin{cases} \lambda = \frac{-d}{c} \\ \lambda = \frac{-b}{a} \end{cases} \implies \frac{-d}{c} = \frac{-b}{a} \implies ad - bc = 0,$$

which contradicts $ad - bc = 1$.

From (6.3.4), this is the spectrum of the transformed bi-equidefect operator $(\hat{T}_g, \check{T}_g)$ such that

$$\sigma(\hat{T}_g) = \left\{ \frac{a\lambda + b}{c\lambda + d} \in \mathbb{R} \mid c\lambda + d \neq 0 \text{ and } [\lambda : 1] \in \sigma(\hat{T}, \check{T}) \right\}$$

and

$$\sigma(\check{T}_g) = \left\{ \frac{c + d\lambda'}{a + b\lambda'} \in \mathbb{R} \mid a + b\lambda' \neq 0 \text{ and } [1 : \lambda'] \in \sigma(\hat{T}, \check{T}) \right\}$$

Hence, the transformed bi-equidefect operator $(\hat{T}_g, \check{T}_g)$ will have the spectral decompositions:

$$\hat{T}_g = \int_{\sigma(\hat{T})} \frac{a\lambda + b}{c\lambda + d} dE(\lambda) \quad \text{and} \quad \check{T}_g = \int_{\sigma(\check{T})} \frac{c + d\lambda'}{a + b\lambda'} dE(\lambda').$$

As shown above, $c\lambda + d$ and $a + b\lambda'$ cannot both be zero, which guarantees that the operators are well defined.

Example 6.3.6. *Consider*

$$H = L^2([0,1]) \oplus L^2([0,1])$$

where

$$L^2([0,1]) = \left\{ f : [0,1] \to \mathbb{R} \mid \|f\|_2^2 = \int_0^1 |f(x)|^2 dx < \infty \right\}.$$

In this example,

$$M = \text{Clin}\{\phi_n \mid n \text{ is odd}\} \quad \text{and} \quad M^\perp = \text{Clin}\{\phi_n \mid n \text{ is even}\}$$

where

$$\phi_{2k-1} = \left(0, \sqrt{2}\sin\left(\frac{2k-1}{2}\pi x\right)\right) \quad \text{and} \quad \phi_{2k} = \left(\sqrt{2}\sin\left(\frac{2k-1}{2}\pi x\right), 0\right).$$

Define the unitary operator

$$\mathcal{U} : M \to M^\perp, \quad \mathcal{U}(\phi_n) = \phi_{n+1} \quad \text{for every} \quad n = 1, 3, 5, \ldots.$$

Now, consider the integral operator

$$T : M \to M$$
$$Tf(x) = \int_0^1 \min(x, y) f(y) dy$$

where

$$\min(x, y) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant y \leqslant 1 \\ y & \text{if } 0 \leqslant y \leqslant x \leqslant 1 \end{cases}.$$

As T is a compact operator [17], [29], its spectrum is a pure point spectrum. Thus, its spectral decomposition

$$T = \int_{\sigma(T)} \lambda dE(\lambda)$$

can be expressed as a sum:

$$Tf(x) = \sum_{n=1}^{\infty} \lambda_n \langle f, \phi_n \rangle \phi_n(x) \quad \text{for every} \quad n = 1, 3, 5, \ldots.$$

where

$$\lambda_n = \left(\frac{2}{(2n-1)\pi}\right)^2 \quad \text{and} \quad \langle f, \phi_n \rangle = \int_0^1 f(y)\phi_n(y) dy.$$

The operator T, as a bi-equidefect operator, has the spectrum

$$\left\{[\lambda_n : 1] \in P\mathbb{R} \mid \lambda_n = \left(\frac{2}{(2n-1)\pi}\right)^2, \forall n = 1, 3, 5, \dots\right\}.$$

If we take $g = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \in SL(2, \mathbb{R})$ *and transform* $\sigma(T)$*, then, by (4.1.7) and (4.1.8), we obtain*

$$\left\{[\mu_n : 1] \in P\mathbb{R} \mid \mu_n = \frac{\pi^2(2n-1)^2}{\pi^2(2n-1)^2 - 4}, \forall n = 1, 3, 5, \dots\right\}.$$

Therefore, the spectral decomposition of the transformed operator T_g *is*

$$T_g f(x) = \sum_{n=1}^{\infty} \mu_n \langle f, \phi_n \rangle \phi_n(x) \quad \textit{for every} \quad n = 1, 3, 5, \dots$$

where

$$\mu_n = \frac{\pi^2(2n-1)^2}{\pi^2(2n-1)^2 - 4} \quad \textit{and} \quad \langle f, \phi_n \rangle = \int_0^1 f(y)\phi_n(y)dy.$$

We now turn to the connection between $\mathrm{SL}(2, \mathbb{R})$ action and the point spectrum of $(\hat{T}, \check{T})$. We have assumed that H is either an even finite-dimensional or a separable infinite-dimensional real Hilbert space and M, N are closed subspaces with Equalities of Dimensionality 5.1.2. Moreover, we have considered the symplectic form on H:

$$\omega((y, x), (y', x')) = \langle \mathcal{U}x, y' \rangle_{M^\perp} - \langle x', \mathcal{U}^* y \rangle_M \quad \text{for every} \quad x, x' \in M, y, y' \in M^\perp$$

where the unitary operator $\mathcal{U} : M \to M^\perp$ is determined from M and N, as in (5.1.9). According to this symplectic form, we have shown, in section 5.2, that M, N are Lagrangian subspaces. Furthermore, from Remark 6.1.2, the transformed subspace N^g of N under every $g \in \mathrm{SL}(2, \mathbb{R})$ is Lagrangain as well.

In the following theorem, we will denote the Lagrangian subspace by L instead of N to avoid confusion with the subgroup N in (3.0.1).

Theorem 6.3.7. *Let H be either an even finite-dimensional or a separable infinite-dimensional real symplectic Hilbert space equipped with the symplectic form (6.0.1) and let M, L be Lagrangian subspaces satisfying the Equalities of Dimensionality 5.1.2. Also, let $(\hat{T}, \check{T})$ be a bi-equidefect operator, as in (6.3.2), (6.3.3) such that its graph represented by L and $[\lambda : 1] \in P\mathbb{R}$. Then*

$$[\lambda : 1] \textit{ is an eigenvalue of } (\hat{T}, \check{T}) \iff \bigcap_{g \in N_\lambda} L^g \neq \{0\} \tag{6.3.10}$$

where $L^g = \Phi_{\mathcal{U}}(g)L$ with g being an element of

$$N_\lambda = \left\{ \begin{pmatrix} 1 - \lambda t & \lambda^2 t \\ -t & 1 + \lambda t \end{pmatrix} \mid t \in \mathbb{R} \right\}. \tag{6.3.11}$$

For the infinity eigenvalue,

$$[1 : 0] \textit{ is an eigenvalue of } (\hat{T}, \check{T}) \iff \bigcap_{g \in N} L^g \neq \{0\} \tag{6.3.12}$$

where

$$N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

Proof. Since L represents a graph of $(\hat{T}, \check{T})$, we can write

$$L = \{(\mathcal{U}\hat{T}x, x) \mid x \in P_M L\} \oplus \ker(\check{T})$$

where $\mathcal{U} : M \to M^\perp$ is the unitary operator in (5.1.9). We express the elements L as

$$(k + \mathcal{U}\hat{T}x, x) \quad \text{where} \quad x \in P_M L, k \in \ker(\check{T}).$$

Now, observe

$$\begin{aligned}
\bigcap_{g\in N_\lambda} L^g \neq \{0\} &\iff \exists (k + \mathcal{U}\hat{T}x, x) \in H\backslash\{0\} \text{ such that } (k + \mathcal{U}\hat{T}x, x) \in \bigcap_{g\in N_\lambda} L^g \\
&\iff \exists (k + \mathcal{U}\hat{T}x, x) \in H\backslash\{0\} \text{ such that } (k + \mathcal{U}\hat{T}x, x) \in \Phi_{\mathcal{U}}(g)L \text{ for all } g \in N_\lambda \\
&\iff \exists (k + \mathcal{U}\hat{T}x, x) \in H\backslash\{0\} \text{ such that } \Phi_{\mathcal{U}}(g^{-1})(k + \mathcal{U}\hat{T}x, x) \in L \text{ for all } g \in N_\lambda \\
&\iff \exists (k + \mathcal{U}\hat{T}x, x) \in H\backslash\{0\} \text{ such that} \\
&\qquad \begin{pmatrix} 1+\lambda & -\lambda^2 t \\ t & 1-\lambda t \end{pmatrix} \cdot \begin{pmatrix} k + \mathcal{U}\hat{T}x \\ x \end{pmatrix} \in L \text{ for all } t \in \mathbb{R} \\
&\iff \exists (k + \mathcal{U}\hat{T}x, x) \in H\backslash\{0\} \text{ such that} \\
&\qquad \begin{pmatrix} (1+\lambda t)(k + \mathcal{U}\hat{T}x) - \lambda^2 t\mathcal{U}x \\ t(\mathcal{U}^* k + \hat{T}x) + (1-\lambda t)x \end{pmatrix} \in L \text{ for all } t \in \mathbb{R}
\end{aligned}$$

Since L is Lagrangian and $(\mathcal{U}\hat{T}x, x) \in L$ (by taking $t = 0$), we have

$$\omega\big(\big((1+\lambda t)(k + \mathcal{U}\hat{T}x) - \lambda^2 t\mathcal{U}x\,,\, t(\mathcal{U}^* + \hat{T}x) + (1-\lambda t)x\big),\ (k + \mathcal{U}\hat{T}x, x)\big) = 0 \text{ for all } t \in \mathbb{R}$$

By the definition of ω, we have

$$\langle t(k + \mathcal{U}\hat{T}x) + (1-\lambda t)\mathcal{U}x,\ k + \mathcal{U}\hat{T}x\rangle_{M^\perp} - \langle (1+\lambda t)(\mathcal{U}^* k + \hat{T}x) - \lambda^2 t\, x,\ x\rangle_M = 0 \text{ for all } t \in \mathbb{R}.$$

Then

$$t\|k + \mathcal{U}\hat{T}x\|^2 + \langle 1 - \lambda t\mathcal{U}x, k + \mathcal{U}\hat{T}x\rangle_{M^\perp} - \langle (1+\lambda t)(\mathcal{U}^* k + \hat{T}x), x\rangle_M + \lambda^2 t\|x\|^2 = 0 \text{ for all } t \in \mathbb{R}.$$

By simple calculation,

$$\|\mathcal{U}^* k + \hat{T}x - \lambda x\|^2 = 0,$$

which implies

$$\mathcal{U}^* k = \lambda x - \hat{T}x.$$

Since $\mathcal{U}^* k \in M_{01}$, $\lambda x - \hat{T}x \in M_0 \oplus M_{00}$ and $M_{01} \cap M_0 \oplus M_{00} = \{0\}$, then

$$0 = \hat{T}x - \lambda x.$$

Equivalently,

$$\hat{T}x = \lambda x.$$

This means

$$[\lambda : 1] \text{ is an eigenvalue of } (\hat{T}, \check{T}).$$

For the infinity eigenvalue, we write

$$L = \ker(\hat{T}) \oplus \{(y, \mathcal{U}^* \check{T}y) \mid y \in P_{M^\perp} L\}$$

where $\mathcal{U}: M \to M^{\perp}$ is the unitary operator in (5.1.9). Here, we will express the elements L as

$$(y, k + \mathcal{U}^* \check{T} y) \quad \text{where} \quad y \in P_{M^{\perp}} L, k \in \ker(\hat{T}).$$

Now, observe

$$\begin{aligned}
\bigcap_{g \in N} L^g \neq \{0\} &\iff \exists (y, k + \mathcal{U}^* \hat{T} y) \in H \backslash \{0\} \text{ such that } (y, k + \mathcal{U}^* \check{T} y) \in \bigcap_{g \in N} L^g \\
&\iff \exists (y, k + \mathcal{U}^* \hat{T} y) \in H \backslash \{0\} \text{ such that } (y, k + \mathcal{U}^* \check{T} y) \in \Phi_{\mathcal{U}}(g) L \text{ for all } g \in N \\
&\iff \exists (y, k + \mathcal{U}^* \hat{T} y) \in H \backslash \{0\} \text{ such that } \Phi_{\mathcal{U}}(g^{-1})(y, k + \mathcal{U}^* \check{T} y) \in L \text{ for all } g \in N \\
&\iff \exists (y, k + \mathcal{U}^* \hat{T} y) \in H \backslash \{0\} \text{ such that } (y - t\mathcal{U}k - t\check{T} y, k + \mathcal{U}^* \check{T} y) \in L \text{ for all } t \in \mathbb{R}
\end{aligned}$$

Since L is Lagrangian and $(y, k + \mathcal{U}^* \check{T} y) \in L$ (by taking $t = 0$), we have

$$\omega\big((y - t\mathcal{U}k - t\check{T} y, k + \mathcal{U}^* \check{T} y),\ (y, k + \mathcal{U}^* \check{T} y)\big) = 0 \ \text{ for all } \ t \in \mathbb{R}$$

By the definition of ω, we have

$$\langle \mathcal{U}k + \check{T} y, y \rangle_{M^{\perp}} - \langle \mathcal{U}^*(y - t\check{T} y) - tk, k + \mathcal{U}^* \check{T} y \rangle_M = 0 \ \text{ for all } \ t \in \mathbb{R}.$$

Then

$$\langle \mathcal{U}k + \check{T} y, y \rangle_{M^{\perp}} - \langle y, \mathcal{U}k + \check{T} y \rangle_{M^{\perp}} + t\|\mathcal{U}k + \check{T} y\|^2 = 0 \ \text{ for all } \ t \in \mathbb{R}.$$

This implies

$$t\|\mathcal{U}k + \check{T} y\|^2 = 0 \ \text{ for all } \ t \in \mathbb{R}.$$

Hence

$$\mathcal{U}k + \check{T} y = 0.$$

Equivalently,

$$\check{T} y = -\mathcal{U}k.$$

Since $\check{T} y \in M_1 \ominus M_{10}$, $\mathcal{U}k \in M_{11}$ and $M_1 \ominus M_{10} \cap M_{11} = \{0\}$, then $\check{T} y = 0$. This means

$$[1:0] \ \text{ is an eigenvalue of } \ (\hat{T}, \check{T}).$$

This completes the proof. □

Remark 6.3.8. *The above theorem provides us with an alternative criterion for eigenvalues. Indeed, we can geometrically characterise the eigenvalues of a bi-equidefect operator through* $\mathrm{SL}(2, \mathbb{R})$ *action.*

The next theorem concerns the characterisation of elements of the continuous spectrum of a bi-equidefect operator through $\mathrm{SL}(2, \mathbb{R})$ action. But before introducing the theorem, we need the following definition.

Definition 6.3.9. *For a subspace* $N \subset M^{\perp} \ominus M$ *and fixed* $\varepsilon \in (0, 1)$*, we define its* ε*-adjunct neighbourhood* N^{ε} *by*

$$N^{\varepsilon} = \{v + y \mid v \in N,\ y \in M^{\perp} \ \text{ and } \ \|y\| < \varepsilon \|P_M v\|\}.$$

In geometric interpretation, for every vector $v \in N$, we allow perturbation in $P_{M^{\perp}} v$, but the norm of this perturbation is controlled relatively by the norm of $P_M v$. This means that as $\|P_M v\|$ grows, the perturbation of $\|P_{M^{\perp}} v\|$ grows proportionally. The picture can be seen clearly in a two-dimensional space. See the following example.

Example 6.3.10. *Consider the space* $\mathbb{R}^2$*, and let*

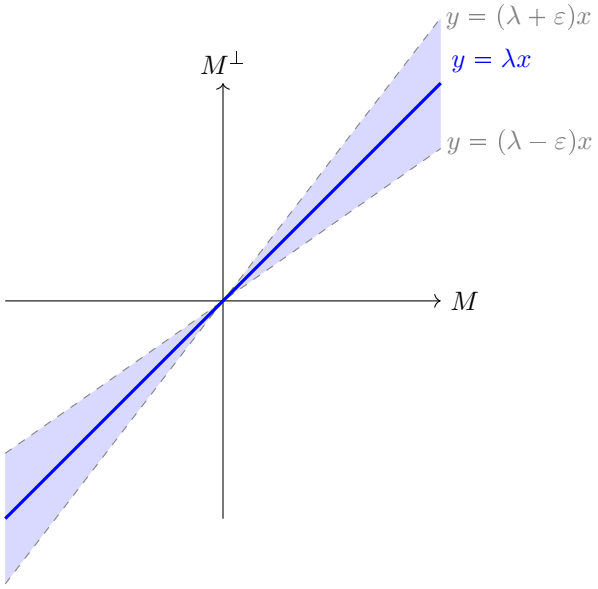
$$M = \{(0, x) \mid x \in \mathbb{R}\} \quad \textit{and} \quad M^\perp = \{(y, 0) \mid y \in \mathbb{R}\}.$$

Also, for fixed $\lambda \in \mathbb{R}\backslash\{0\}$*, consider*

$$N = \{(\lambda x, x) \mid x \in \mathbb{R}\}.$$

Then, for fixed $\varepsilon \in (0, 1)$*, the* ε*-adjunct of* N *is:*

$$N^\varepsilon = \{(\lambda x + \delta, x) \mid |\delta| < \varepsilon |x|\}.$$



The light blue band is ε*-adjunct neighbourhood* N^ε *of* N *:* $y = \lambda x$

Theorem 6.3.11. *Let* H *be either an even finite-dimensional or a separable infinite-dimensional real symplectic Hilbert space equipped with the symplectic form (6.0.1) and let* M, N *be Lagrangian subspaces satisfying the Equalities of Dimensionality 5.1.2 with* $N_\infty = N \cap M^\perp = \{0\}$*. Also, let* $(\hat{T}, \check{T})$ *be a bi-equidefect operator, as in (6.3.2) and (6.3.3), whose graph is represented by* N*. Moreover, let* $\lambda \in \mathbb{R}$ *and assume that the operator* $\hat{T} - \lambda$ *is injective. For fixed* $\varepsilon \in (0, 1)$*, if the* ε*-adjunct neighbourhood of* N *satisfies*

$$N^\varepsilon \cap N^g \neq \{0\} \quad \textit{for every} \quad g \in A_\lambda,$$

then the operator $\hat{T} - \lambda$ *does not have a bounded inverse where* N^g *is the transformed subspaces of* N *under* $g \in A_\lambda$ *and* A_λ *is the semi-group*

$$A_\lambda = \left\{ \begin{pmatrix} e^t & \lambda(e^{-t} - e^t) \\ 0 & e^{-t} \end{pmatrix} \mid t \in (0, \infty) \right\}. \tag{6.3.13}$$

Proof. Set

$$D = \mathrm{Ran}(\hat{T} - \lambda) = \{y \in M \mid y = (\hat{T} - \lambda)x,\ x \in M\}.$$

Then we define the operator S on D as the inverse of $\hat{T} - \lambda$ by letting

$$S : D \to \mathrm{Ran}(S)$$

$$S(y) = (\hat{T} - \lambda)^{-1} y = x.$$

The unboundedness of S follows from $N^\varepsilon \cap N^g$ being not trivial for any $g \in A_\lambda$. To see this, pick any nonzero $(\mathcal{U}\hat{T}x, x) \in N$ and then apply $g_t \in A_\lambda$ to obtain

$$z_t = \Phi_{\mathcal{U}}(g_t)z = (e^t \mathcal{U}\hat{T}x + \lambda(e^{-t} - e^t)\mathcal{U}x, e^{-t}x).$$

Now, consider

$$w_t = (\mathcal{U}\hat{T}e^{-t}x, e^{-t}x) \in N,$$

which implies

$$z_t - w_t = ((e^t - e^{-t})\mathcal{U}(\hat{T} - \lambda)x, 0) \in M^\perp.$$

If $z_t \in N^{g_t} \cap N^\varepsilon$, then, according to the definition of ε-adjunct neighbourhood, we have

$$\|z_t - w_t\| < \varepsilon \|P_M w_t\|.$$

Hence

$$\|(e^t - e^{-t})\mathcal{U}(\hat{T} - \lambda)x\| < \varepsilon e^{-t}\|x\|.$$

Since $\mathcal{U}$ is unitary,

$$\|(\hat{T} - \lambda)x\| < \frac{\varepsilon}{e^{2t} - 1}\|x\|. \tag{6.3.14}$$

Now, since the above argument holds for all $g_t \in A_\lambda$, then

$$\|(\hat{T} - \lambda)x\| < \frac{\varepsilon}{e^{2t} - 1}\|x\| \to 0 \quad \text{as} \quad t \to \infty.$$

Therefore, $\hat{T} - \lambda$ is not bounded below, and thus its inverse $S : D \to \mathrm{Ran}(S)$ will not be bounded. □

Remark 6.3.12. *From the above proof, we can deduce that the existence of nonzero vector $x \in M$ such that (6.3.14) holds for every $t \in (0, \infty)$ is equivalent to $N^\varepsilon \cap N^g \neq \{0\}$ for every $g \in A_\lambda$. More precisely,*

$$\forall t \in (0, \infty),\ \exists x_t \in M \backslash \{0\},\ \|(\hat{T} - \lambda)x_t\| < \frac{\varepsilon}{e^{2t} - 1}\|x_t\| \iff N^\varepsilon \cap N^g \neq \{0\}\ \forall g \in A_\lambda. \tag{6.3.15}$$

Practically, examining the former may be easier than the latter.

Example 6.3.13. *Consider*

$$H = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) = \mathrm{Clin}\{(\phi_i, \phi_j) \mid i, j = 0, 1, 2, \dots\}$$

where

$$\phi_n(x) = c_n e^{\frac{-x^2}{2}} H_n(x)$$

with constants

$$c_n = (2^n n! \sqrt{\pi})^{\frac{-1}{2}}.$$

Here, ϕ_n is called a Hermite function and $H_n(x)$ is a Hermite polynomial [9] and defined as follows

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Let

$$M = \{0\} \oplus L^2(\mathbb{R}) \qquad \text{and} \qquad M^\perp = L^2(\mathbb{R}) \oplus \{0\}.$$

Define the unitary operator

$$\mathcal{U} : M \to M^{\perp}, \;\; \mathcal{U}(0, \phi_n) = (\phi_n, 0) \quad \textit{for every} \quad n = 0, 1, 2, 3, \ldots.$$

Thus, we can define the symplectic form

$$\omega((g, f), (g', f')) = \langle \mathcal{U} f, g' \rangle_{L^2(\mathbb{R})} - \langle \mathcal{U}^* g, f' \rangle_{L^2(\mathbb{R})} \quad \textit{for every} \quad (g, f), (g', f') \in M^{\perp} \oplus M$$

where

$$\langle f, g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x) g(x) dx.$$

Consider the multiplication operator

$$M_{\psi}(f) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

$$M_{\psi}(f)(x) = \psi(x) f(x) \quad \textit{where} \quad \psi(x) = \frac{1}{1 + x^2}.$$

Its graph is represented by the Lagrangian subspace

$$N = \mathrm{Clin}\{(\mathcal{U} M_{\psi} f, f) \mid f \in M\}.$$

If we apply an element

$$g_t = \begin{pmatrix} e^t & \lambda(e^{-t} - e^t) \\ 0 & e^{-t} \end{pmatrix} \in A_{\lambda} \quad \textit{where} \quad \lambda \in \mathbb{R},\, t \in (0, \infty)$$

on N, then we obtain

$$N^{g_t} = \mathrm{Clin}\{((e^t \mathcal{U} M_{\psi} + \lambda(e^{-t} - e^t)\mathcal{U}) f, e^{-t} f) \mid f \in Dom(M_{\psi})\}.$$

Since $N_{\infty} = \{0\}$, Theorem 6.3.11 can be applied to check whether λ lies in the continuous spectrum of M_{ψ}. To this end, we need to compute ε-adjunct neighbourhood N^{ε} for a fixed $\varepsilon \in (0, 1)$:

$$N^{\varepsilon} = \{(\mathcal{U} M_{\psi} f + g, f) \mid f \in Dom(M_{\psi}), g \in M^{\perp} \;\; \textit{and} \;\; \|g\| < \varepsilon \|f\|\},$$

Now, we need to show that

$$N^{\varepsilon} \cap N^{g_t} \neq \{0\} \quad \textit{for every} \quad g_t \in A_{\lambda}.$$

By Remark 6.3.12, this is equivalent to

$$\forall t \in (0, \infty), \, \exists f_t \in M \backslash \{0\}, \, \|(\hat{T} - \lambda) f_t\| < \frac{\varepsilon}{e^{2t} - 1} \|f_t\|. \tag{6.3.16}$$

To achieve that, we put

$$\delta_t := \frac{\varepsilon}{e^{2t} - 1}.$$

For $\lambda \in (0, 1]$, let $t \in (0, \infty)$ and choose $\eta_t > 0$ such that

$$0 < \eta_t < \min\{\delta_t, \lambda\}.$$

Also, consider

$$E_{\eta_t} = \{x \in \mathbb{R} : |\psi(x) - \lambda| < \eta_t\}.$$

Now, we have two cases:

$$\lambda \in (0, 1] \qquad \textit{and} \qquad \lambda = 0.$$

- *Case 1:* $\lambda \in (0,1)$.
  We first show that E_{η_t} *has positive finite Lebesgue measure. Since*

$$\lambda \in \mathrm{Ran}(\psi) = (0,1],$$

  there exists $x_\lambda \in \mathbb{R}$ *such that*

$$\psi(x_\lambda) = \lambda.$$

  Indeed, one may take

$$x_\lambda = \sqrt{\frac{1}{\lambda} - 1}.$$

  Since ψ *is continuous and*

$$|\psi(x_\lambda) - \lambda| = 0 < \eta_t,$$

  there exists $r > 0$ *such that*

$$|\psi(x) - \lambda| < \eta_t \quad \textit{whenever} \quad |x - x_\lambda| < r.$$

  Hence

$$(x_\lambda - r, x_\lambda + r) \subseteq E_{\eta_t}.$$

  Therefore,

$$\mu(E_{\eta_t}) \geqslant 2r > 0.$$

  Now, we turn to prove that $\mu(E_{\eta_t})$ *is finite. If* $x \in E_{\delta_t}$*, then*

$$\begin{aligned}
\left|\frac{1}{1+x^2} - \lambda\right| < \eta_t &\implies \lambda - \eta_t < \frac{1}{1+x^2} < \lambda + \eta_t \\
&\implies \frac{1}{1+x^2} > \lambda - \eta_t \\
&\implies 1 + x^2 < \frac{1}{\lambda - \eta_t} \\
&\implies x^2 < \frac{1}{\lambda - \eta_t} - 1 \\
&\implies |x| < \sqrt{\frac{1}{\lambda - \eta_t} - 1}
\end{aligned}$$

  This means

$$E_{\eta_t} \subseteq \left(-\sqrt{\frac{1}{\lambda - \eta_t} - 1}, \sqrt{\frac{1}{\lambda - \eta_t} - 1}\right);$$

  that is,

$$\mu\left(E_{\eta_t}\right) \leqslant 2\sqrt{\frac{1}{\lambda - \eta_t} - 1}.$$

  Therefore,

$$0 < \mu(E_{\eta_t}) < \infty.$$

  Define

$$f_t : \mathbb{R} \to \mathbb{R}; \quad f_t(x) = \begin{cases} 1 & \textit{if} \quad x \in E_{\eta_t} \\ 0 & \textit{if} \quad x \notin E_{\eta_t} \end{cases}.$$

  Then, $f_t \in L^2(\mathbb{R})$ *and*

$$\|f_t\|_2^2 = \mu(E_{\eta_t}).$$

  Since

$$|\psi(x) - \lambda| < \eta_t \quad \textit{for all} \quad x \in E_{\eta_t},$$

we obtain

$$\begin{aligned}\|(M_\psi - \lambda)f_t\|_2^2 &= \int_{E_{\eta_t}} |\psi(x) - \lambda|^2\, dx \\ &< \eta_t^2 \mu(E_{\eta_t}) \\ &= \eta_t^2 \|f_t\|_2^2.\end{aligned}$$

Thus,

$$\|(M_\psi - \lambda)f_t\|_2 < \eta_t \|f_t\|_2.$$

Since

$$\delta_t = \frac{\varepsilon}{e^{2t} - 1} \quad \textit{and} \quad \eta_t < \delta_t,$$

then

$$\|(M_\psi - \lambda)f_t\|_2 < \frac{\varepsilon}{e^{2t} - 1}\|f_t\|_2.$$

- *Case 2:* $\lambda = 0$.
  In this case,

$$E_{\delta_t} = \left\{ x \in \mathbb{R} : \frac{1}{1 + x^2} < \delta_t \right\}.$$

Assume $0 < \delta_t < 1$. *Then*

$$\begin{aligned}\frac{1}{1+x^2} < \delta_t &\implies 1 + x^2 > \frac{1}{\delta_t} \\ &\implies x^2 > \frac{1}{\delta_t} - 1 \\ &\implies x < -\sqrt{\frac{1}{\delta_t} - 1} \quad \textit{and} \quad x > \sqrt{\frac{1}{\delta_t} - 1} \\ &\implies x \in \left(-\infty, -\sqrt{\frac{1}{\delta_t} - 1}\right) \cup \left(\sqrt{\frac{1}{\delta_t} - 1}, \infty\right).\end{aligned}$$

This means

$$E_{\delta_t} = (-\infty, -R_t) \cup (R_t, \infty) \quad \textit{where} \quad R_t = \sqrt{\frac{1}{\delta_t} - 1}.$$

Thus

$$\mu(E_{\delta_t}) = \infty \implies \chi_{E_{\delta_t}} \notin L^2(\mathbb{R}).$$

Therefore, instead of taking the whole set E_{δ_t}, *consider*

$$F_t = (R_t, R_t + 1).$$

In this case, define

$$g_t(x) = \begin{cases} 1 & \textit{if} \quad x \in F_t \\ 0 & \textit{if} \quad x \notin F_t \end{cases}.$$

Hence

$$g_t \in L^2(\mathbb{R}) \quad \textit{because} \quad \|g_t\|_2^2 = \mu(F_t) = 1.$$

Since $F_t \subset E_{\delta_t}$, *we have*

$$|\psi(x)| < \delta_t \quad \textit{for all} \quad x \in F_t.$$

Therefore,

$$\begin{aligned}\|M_\psi g_t\|_2^2 &= \int_{F_t} |\psi(x)|^2\, dx \\ &< \delta_t^2 \mu(F_t) \\ &= \delta_t^2 \|g_t\|_2^2.\end{aligned}$$

Consequently,

$$\|M_\psi g_t\|_2 < \frac{\varepsilon}{e^{2t}-1}\|g_t\|_2.$$

Therefore, in all cases $\lambda \in [0,1]$*, we have*

$$\|M_\psi g_t\|_2 < \frac{\varepsilon}{e^{2t}-1}\|g_t\|_2.$$

Since t *was chosen arbitrarily, then the statement (6.3.16) holds. This implies that if* $\lambda \in [0,1]$*, then* $M_\psi - \lambda$ *is not bounded below; that is,* $M_\psi - \lambda$ *does not have a bounded inverse.*

For the case where $\lambda \notin [0,1]$*, since* $\mathrm{Ran}(\psi) = (0,1]$*, we will have* $\delta > 0$ *such that*

$$\delta < |\psi(x) - \lambda| \quad \textit{for all} \quad x \in \mathbb{R}.$$

Hence

$$\delta\|f\|_2 < \|(M_\psi - \lambda)f\|_2 \quad \textit{for all} \quad f \in L^2(\mathbb{R}).$$

This means (6.3.16) does not hold for $\lambda \notin [0,1]$*. Therefore, we can conclude that*

- *If* $\lambda \in [0,1]$*, the statement (6.3.16) holds.*
- *If* $\lambda \notin [0,1]$*, the statement (6.3.16) does not hold.*

From Theorem 6.3.11, this is equivalent to

- *If* $\lambda \in [0,1]$*, the operator* $M_\psi - \lambda$ *does not have a bounded inverse.*
- *If* $\lambda \notin [0,1]$*, the operator* $M_\psi - \lambda$ *has a bounded inverse.*

Spectrally, the continuous spectrum of M_ψ *is* $\sigma_c(M_\psi) = [0,1]$*, which coincides with the result in the standard spectral theory.*

As a consequence of Theorem 6.3.7 and Theorem 6.3.11, we have the following corollary.

Corollary 6.3.14. *If* N *is a Lagrangian subspace with* $N \cap M^\perp = N_\infty = \{0\}$*, then the spectrum of a bi-equidefect operator* $(\hat{T}, \check{T})$ *whose graph is represented by* N *can be defined as:*

$$\sigma(\hat{T},\check{T}) = \{[\lambda:1] \in P\mathbb{R} \mid \bigcap_{g\in N_\lambda} N^g \neq \{0\}\} \cup \{[\lambda:1] \in P\mathbb{R} \mid \exists \epsilon \in (0,1),\ \forall g \in A_\lambda,\ N^\epsilon \cap N^g \neq \{0\}\}$$

where N_λ *is in (6.3.11) and* A_λ *is in (6.3.13).*

Conclusion

This thesis presents a framework for the unification of three major themes: symplectic structure, $\mathrm{SL}(2,\mathbb{R})$ action, and operator theory on Hilbert spaces. Their interaction allows us to extend the idea of the real projective line $P\mathbb{R}$ to the operator projective line PN.

Moreover, this thesis introduces the new concept of the bi-equidefect operator as an adaptation of the notion of an operator so that its graph is represented by an element of PN. Furthermore, its spectrum is represented as a subset of the real projective line $P\mathbb{R}$. From this perspective, such operators are not viewed as well-defined maps, but as geometric objects formed by Lagrangian subspaces of a symplectic Hilbert space. This enables $\mathrm{SL}(2,\mathbb{R})$ to cleanly act on these operators by transforming their corresponding Lagrangian subspaces without excluding any elements from their spectrum. In this way, this thesis embeds operator theory in a geometric context, in the sense of the Erlangen Programme at large.

In subsequent work, the spectral theory may be further developed via the operator projective line PN and $\mathrm{SL}(2,\mathbb{R})$ action on it. In the present work, we have already represented an operator by an element of PN and its spectral information appears through the behaviour of this element under $\mathrm{SL}(2,\mathbb{R})$ action. Therefore, it would be interesting to extend this approach by considering several elements of PN that represent a family of operators. After that, we can investigate whether their behaviour under the $\mathrm{SL}(2,\mathbb{R})$ action yields a geometric description of their joint spectrum. This may provide new insight into spectral theory by defining joint spectra via the $\mathrm{SL}(2,\mathbb{R})$ action of operators.

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