

Self-adjoint extensions of third order differential operators

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Abstract

This thesis is devoted to the study of self-adjoint extensions of certain symmetric differential operators. Such operators play a fundamental role in functional analysis and mathematical physics, as they describe observables and generate well-defined dynamics.

We investigate both classical and new examples of symmetric, or formally symmetric, differential operators on the real line, half-lines, and punctured domains. In particular, we analyse first-order and second-order operators, and introduce new classes of third-order homogeneous and inhomogeneous differential operators, as well as systems of differential operators.

For each class of operators, we determine the deficiency indices and use them to characterize the existence of self-adjoint extensions. Whenever such extensions exist, they are described in terms of boundary conditions or parametrized via unitary maps between the corresponding deficiency subspaces.

A key aspect of this work is the establishment of an equivalence between different operator formulations. This allows us to transfer results from one problem to another, providing a unified framework in which properties that are difficult to analyse for a given operator can be deduced from a related one.

While classical methods based on boundary conditions are effective for lower-order operators, they turn out to be insufficient in the case of third-order differential operators. To overcome this difficulty, we adopt an alternative approach inspired by the theory of point interactions and singular perturbations. This requires the development of auxiliary results, some of which are established here for the first time.

Using this approach, we obtain a complete characterization of all self-adjoint extensions for several classes of third-order differential operators on punctured domains. These results constitute the main contribution of the thesis.

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Author's Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for a degree or other qualification at this University or elsewhere. All sources are acknowledged as references.

*To my beloved husband, Salem Alqarni, and my children
Lamar, Abdulaziz, and Faisal.
Because of you, I found the strength to continue and
complete this journey.*

Chapter 1

Introduction

1.1 Overview

The subject of this thesis is the theory of self-adjoint extensions of certain symmetric operators. Self-adjoint operators play a fundamental role in functional analysis and quantum mechanics, as they describe observables and generate unitary dynamics. From the theoretical point of view, this work is mainly based on the classical framework developed in the monographs by Reed and Simon, see [35, 34], and also on the theory of linear operators in Hilbert spaces, see [4, 3].

The mathematical tools used throughout this thesis rely heavily on functional analysis, Sobolev spaces, and partial differential equations; see, for instance, [36, 29, 1, 20, 25, 2, 18, 28]. Additional background on real and complex analysis can be found in [38, 37, 17].

In the physics literature, self-adjoint extensions appear under the name of *point interactions*, and they have been extensively studied by many authors. A systematic treatment of such models is given in [15, 16], while further developments and applications can be found in [14, 6, 7, 8, 9, 11, 12, 10, 13, 5]. These works provide important connections between operator theory, stochastic analysis, and mathematical physics.

The objective of this thesis is to analyse both classical and new examples of symmetric, or formally symmetric, differential operators. We begin by studying well-known cases, including first-order and second-order differential operators, considered on the full real line, on half-lines, and on punctured domains.

We then investigate new classes of operators, namely third-order homogeneous and inhomogeneous differential operators, again on the full line, half-lines, and punctured lines. Furthermore, we extend the analysis to systems of differential operators.

In all these cases, we determine the deficiency indices of the operators and use this information to decide whether self-adjoint extensions exist. Whenever such extensions exist, they are characterized by means of a fundamental theorem from extension theory, see [34], which establishes a correspondence between self-adjoint extensions and unitary maps between the positive and negative deficiency subspaces.

In several cases, we follow the classical method presented in [34], describing self-adjoint extensions in terms of suitable boundary conditions. However, this method turns out to be insufficient for the class of third-order differential operators considered in this thesis.

To overcome this difficulty, we adopt an alternative approach developed in the literature on point interactions and singular perturbations, see [22, 23]. In order to apply this method, we first establish the auxiliary results required for its implementation, some of which were previously stated without proof.

We then apply this approach to study self-adjoint extensions of third-order homogeneous differential operators, as well as operators involving both first-order and third-order terms, under the assumption of formal symmetry on punctured domains. This leads to a complete characterization of all self-adjoint extensions in terms of boundary conditions.

These results constitute the main and most advanced contribution of this thesis. A more detailed account of these results can be found in [23] and [22].

After submission of this thesis, we became aware of the paper by Mugnolo, Noja and Seifert [32], which studies Airy-type operators closely related to those considered in Chapters 5 and 6. In particular, the half-line deficiency-index results that overlap with [32] are regarded here as known results, whereas the contribution of the present treatment lies in the operator-theoretic derivation and the adjoint-domain characterization developed in this thesis. A detailed comparison between their results and those obtained in this thesis is presented in Remark 8.36 in Chapter 8.

It is worth pointing out that our approach is motivated by the paper [22] in which the whole class of self-adjoint extensions of one-dimensional Dirac operators on what is nowadays called the punctured real line is determined.

1.2 Thesis Structure

This thesis comprises eight chapters, including the introduction, which are described as the follows:

Chapter 2 provides the necessary mathematical background for this thesis. It reviews key concepts and results from Sobolev spaces, spectral theory of bounded and unbounded operators, and the theory of adjoints and self-adjoint extensions.

The chapter also includes a discussion of unitary operators, ellipticity in one-dimensional domains, and properties of closed and closable operators. Relevant remarks and results that are used later in the thesis are highlighted throughout.

Chapter 3 is devoted to the study of first-order differential operators. We begin by analysing the operator on a bounded interval, where all self-adjoint extensions are explicitly characterized using classical results.

We then consider the first-order differential operator on the punctured real line. In this setting, we study the symmetry and determine the adjoint operator. We also compute the deficiency indices and describe the corresponding deficiency subspaces, which allows us to characterize all self-adjoint extensions of the operator.

Chapter 4 is devoted to the study of second-order differential operators (the Laplacian). We begin with the analysis of the operator on the real line, where we determine the domain of the adjoint operator and show that the deficiency indices are equal to zero, implying that the operator is essentially self-adjoint.

We then consider the Laplacian on the punctured real line. In this setting, we study symmetry and determine the adjoint operator, as well as compute the deficiency indices. We further analyse the closure of the operator, describe its domain, and investigate its properties.

Finally, we obtain a complete characterization of all self-adjoint extensions using von Neumann's theory. In particular, the extensions are parametrized by unitary operators in $U(2)$ and described explicitly in terms of boundary conditions.

Chapter 5 is devoted to the study of third-order linear homogeneous differential operators in several settings. We begin with the operator on the real line and determine its basic properties, including symmetry, the adjoint, and the deficiency indices.

We then consider the corresponding operators on the positive and negative half-lines, where the adjoint and the deficiency indices are analysed separately. After that, we study the punctured real line, which forms the main part of the chapter. In this setting, we investigate symmetry, the adjoint, the deficiency indices, and the closure of the operator.

The chapter also includes a characterization of the domain of the closure, the adjoint of the closure, and the deficiency indices of the closed operator. Using von Neumann's theory, we obtain a complete classification of all self-adjoint extensions. These extensions

are parametrized by unitary operators in $U(3)$ and are further described explicitly in terms of boundary conditions.

Finally, several concrete examples are presented to illustrate how the general theory applies to particular choices of unitary matrices.

Chapter 6 extends the analysis to third-order linear inhomogeneous differential operators. We study these operators on the real line, as well as on the positive and negative half-lines, where symmetry, the adjoint, and the deficiency indices are determined.

We then focus on the punctured real line, where we investigate the adjoint operator, establish trace-type results, and compute the deficiency indices. The closure of the operator and its properties are also analysed.

Although the adjoint and deficiency indices can be characterized, the classical approach based on von Neumann's theory does not lead to a complete description of all self-adjoint extensions in this setting. This motivates the introduction of an alternative method based on abstract boundary formalism, which is developed in Chapter 8.

Chapter 7 is concerned with several equivalence results between different operator problems. We first establish unitary equivalences between scalar differential operators on punctured domains and corresponding vector-valued operators on the half-line. This is carried out for the Laplacian, first-order differential operators, third-order homogeneous differential operators, and third-order inhomogeneous differential operators.

These equivalence results provide a unified framework that allows properties of one problem to be transferred directly to another. In particular, symmetry, density of the domain, adjoint operators, closure properties, deficiency indices, and self-adjoint extensions can be deduced in one setting from the corresponding results in the other.

The chapter then extends the analysis to systems of differential operators on the half-line. We study systems of first-order differential operators, as well as systems of third-order homogeneous and inhomogeneous differential operators. For these systems, we investigate symmetry, closure properties, adjoint operators, and deficiency indices, together with the corresponding results for their closures.

Chapter 8 introduces an alternative approach to the study of self-adjoint extensions, based on abstract boundary formalism. This method is motivated by the limitations of the classical von Neumann approach encountered in Chapter 6 for third-order inhomogeneous differential operators.

The chapter begins with the abstract boundary framework, including boundary maps, Green identities, neutral subspaces, and the general classification of self-adjoint extensions

via the unitary group $U(d)$. These abstract results are then applied to a punctured third-order differential operator with an additional first-order term.

We first consider the special case $\alpha = 0$, where the boundary maps and Green identity are described explicitly. We then treat the general case $\alpha \in \mathbb{R}$, determining the adjoint operator, the closure of the minimal operator, and the kernel and surjectivity properties of the associated boundary maps.

Finally, we obtain a complete characterization of all self-adjoint extensions of the operator in terms of boundary conditions, parametrized by unitary matrices in $U(3)$. This chapter contains one of the main results of the thesis.

Chapter 2

Preliminaries

In this chapter, we present the principal preliminary results and definitions from the various areas of mathematics that are needed in the rest of the thesis.

2.1 Sobolev spaces

Definition 2.1. Assume that $\mathcal{O}$ is an open subset of $\mathbb{R}$, e.g. the whole line $\mathbb{R}$, the positive half-line $(0, \infty)$, negative half-line $(-\infty, 0)$ or any open interval, or a union, finite or infinite, of open intervals.

(o) Assume that $p \in [1, \infty)$. We say that $u \in L_{loc}^p(\mathcal{O})$ if and only if $u : \mathcal{O} \rightarrow \mathbb{C}$ is Lebesgue measurable and for every closed and bounded interval $[c, d] \subset \mathcal{O}$

$$\int_c^d |u(x)|^p dx < \infty.$$

If $\mathcal{O} = (a, b)$ with $a < b \in \mathbb{R}$, then the condition $[c, d] \subset \mathcal{O}$, which reads $[c, d] \subset (a, b)$, is equivalent to

$$a < c \leq d < b.$$

Remark 2.2. It follows from the Hölder inequality [38, Theorem 3.5] that if $1 \leq p \leq q$, then

$$L_{loc}^q(\mathcal{O}) \subset L_{loc}^p(\mathcal{O}).$$

Definition 2.1. We use the notation from Definition 2.1.

(i) Assume that $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Assume that $u \in L_{loc}^1(\mathcal{O})$. We say that function u is n -times weakly differentiable if and only if there exists a function $v \in L_{loc}^1(\mathcal{O})$ such that

for every $\phi \in C_0^\infty(\mathcal{O})$

$$\int_{\mathcal{O}} u(x) \phi^{(n)}(x) dx = (-1)^n \int_{\mathcal{O}} v(x) \phi(x) dx. \quad (2.1.1)$$

Proposition 2.3. *If a function $u : \mathcal{O} \rightarrow \mathbb{C}$ is of C^n -class and $u^{(n)}(x)$ denotes the classical n -th order derivative of u at $x \in \mathcal{O}$, then obviously $u \in L_{loc}^1(\mathcal{O})$ and $u^{(n)} \in L_{loc}^1(\mathcal{O})$ and u is n -times weakly differentiable with the equality (2.1.1) satisfied with function $v = u^{(n)}$. Hence, using the language and notation of the Proposition 2.12, if a function u is of C^n -class (classically), then u is n -times weakly differentiable and the weak derivative is equal to the classical derivative:*

$$D^n u = u^{(n)}.$$

Proof. The proof follows by applying n -times the Integration by parts formula from [37, Theorem 6.22]. $\square$

Let us recall the following definition

Definition 2.4. [19, Definition 6.8]. Two functions $v_1, v_2 \in L_{loc}^1(\mathcal{O})$ are equal almost everywhere if and only if there exists a set $A \subset \mathcal{O}$ with Lebesgue measure zero such that $v_1(x) = v_2(x)$ for all $x \in \mathcal{O} \setminus A$.

We also say/write that

$$v_1(x) = v_2(x) \text{ for almost all } x \in \mathcal{O}.$$

Equivalently, $v_1 = v_2$ almost everywhere if and only if the Lebesgue measure of the set $\{x \in \mathcal{O} : v_1(x) \neq v_2(x)\}$ is equal to 0.

2.1.1 Absolutely continuous functions

We recall the definition and some basic properties of absolutely continuous functions. These properties will be used later in the characterization of the domains of adjoint and closed differential operators. We recall the definition from [17, Section 6.11].

Definition 2.5. Let $a, b \in \mathbb{R}$, with $a < b$. A function

$$f : [a, b] \longrightarrow \mathbb{C}$$

is said to be *absolutely continuous* on $[a, b]$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every finite collection of pairwise disjoint intervals

$$(a_k, b_k) \subset [a, b], \quad k = 1, \dots, n,$$

satisfying

$$\sum_{k=1}^n (b_k - a_k) < \delta,$$

we have

$$\sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon.$$

The space of all absolutely continuous functions on $[a, b]$ is denoted by

$$AC([a, b]).$$

Remark 2.6. Every absolutely continuous function is uniformly continuous, and hence continuous. However, the converse is not true in general.

The following characterization is particularly useful in the study of differential operators.

Theorem 2.7. *Let $f : [a, b] \rightarrow \mathbb{C}$. Then $f \in AC([a, b])$ if and only if there exists a function $g \in L^1(a, b)$ such that*

$$f(x) = f(a) + \int_a^x g(t) dt, \quad x \in [a, b].$$

In this case, f is differentiable almost everywhere,

$$f' \in L^1(a, b),$$

and

$$f(x) = f(a) + \int_a^x f'(t) dt, \quad x \in [a, b].$$

Definition 2.8. Let $I \subseteq \mathbb{R}$ be an open interval. A function

$$f : I \longrightarrow \mathbb{C}$$

is said to be *locally absolutely continuous* on I , written

$$f \in AC_{\text{loc}}(I),$$

if

$$f \in AC([c, d])$$

for every compact interval $[c, d] \subset I$.

Remark 2.9. In particular, if $f \in AC_{\text{loc}}(I)$, then f' exists almost everywhere on I , belongs to $L^1_{\text{loc}}(I)$, and

$$f(x) - f(y) = \int_y^x f'(t) dt$$

whenever x, y belong to the same compact subinterval of I .

Remark 2.10. For an interval $I \subseteq \mathbb{R}$, a function f belongs to $W_{\text{loc}}^{1,1}(I)$ if and only if it has a representative which belongs to $AC_{\text{loc}}(I)$. For this representative, the classical derivative agrees almost everywhere with the weak derivative.

2.1.2 Weak derivatives

We recall the standard definition of the weak derivative from [1, Definition 1.62].

Definition 2.11 (Weak derivative). Let $\mathcal{O} \subset \mathbb{R}$ be an open set and let $u \in L_{\text{loc}}^1(\mathcal{O})$. We say that u is weakly differentiable if there exists a function $Du \in L_{\text{loc}}^1(\mathcal{O})$ such that for every $\phi \in C_0^\infty(\mathcal{O})$

$$\int_{\mathcal{O}} u(x)\phi'(x) dx = - \int_{\mathcal{O}} Du(x)\phi(x) dx.$$

The function Du is called the *weak derivative* of u .

We also have the following result

Proposition 2.12. Assume that $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Assume that $u \in L_{\text{loc}}^1(\mathcal{O})$. If two functions $v_1, v_2 \in L_{\text{loc}}^1(\mathcal{O})$ satisfy the above condition (2.1.1), then $v_1 = v_2$ almost everywhere. Hence we can write

$$D^n u := v$$

and call $D^n u$ the weak derivative of order n of function u .

Proposition 2.13. If a function $u \in L_{\text{loc}}^1(\mathcal{O})$ is n -times weakly differentiable and $1 \leq k < n$, then u is k -times weakly differentiable and the weak derivative $D^k u$ is $(n-k)$ -times weakly differentiable. Moreover,

$$D^{n-k}(D^k u) = D^n u \quad \text{almost everywhere.}$$

The next result is a one-dimensional version of Sobolev embedding Theorem, see e.g. the paper [28] by Jacques Simon.

Theorem 2.14. If a function $u \in L_{\text{loc}}^1(\mathcal{O})$ is n -times weakly differentiable, so in particular $D^n u \in L_{\text{loc}}^1(\mathcal{O})$, then there exists a function $\tilde{u} : \mathcal{O} \rightarrow \mathbb{C}$ of C^{n-1} class such that

$$u(x) = \tilde{u}(x) \text{ for almost all } x \in \mathcal{O}. \quad (2.1.2)$$

Moreover, if a function $u \in L_{\text{loc}}^1(\mathcal{O})$ is n -times weakly differentiable and $D^n u \in L_{\text{loc}}^p(\mathcal{O})$ with $p \in (1, \infty)$, then the function $\tilde{u}$ from the first part of Theorem, is of C^{n-1} class and the $\tilde{u}^{(n-1)}$ is locally Hölder continuous with exponent $1 - \frac{1}{p}$.

Since Hölder continuity appears in Theorem 2.14, we recall the following definition.

Definition 2.15. Let $n \in \mathbb{N}_0$ and let $\alpha \in (0, 1)$. A function $f : [a, b] \rightarrow \mathbb{C}$ belongs to the Hölder space $C^{n+\alpha}([a, b])$ if $f \in C^n([a, b])$ and its n -th derivative satisfies the Hölder condition of exponent α , i.e.,

$$|f^{(n)}(x) - f^{(n)}(y)| \leq C|x - y|^\alpha, \quad x, y \in [a, b],$$

for some constant $C > 0$.

Remark 2.16. The notation $C^{n+\alpha}([a, b])$ is equivalent to the notation $C^{n,\alpha}([a, b])$ used, for example, in Adams and Fournier [1, Section 1.29].

Definition 2.17 (Locally Hölder continuous function). Let $\mathcal{O} \subset \mathbb{R}$ be an open set and let $\alpha \in (0, 1]$. A function $f : \mathcal{O} \rightarrow \mathbb{R}$ is said to be *locally Hölder continuous with exponent α* if for every compact set $K \subset \mathcal{O}$ there exists a constant $C_K > 0$ such that

$$|f(x) - f(y)| \leq C_K|x - y|^\alpha \quad \text{for all } x, y \in K.$$

An interesting question is what happens when $\mathcal{O}$ is an open interval and the spaces $L_{loc}^p(\mathcal{O})$ are defined in a slightly different way.

then there exists a function $\tilde{u} : \overline{\mathcal{O}} \rightarrow \mathbb{C}$ of C^{n-1} such that condition (2.1.2) is satisfied.

Definition 2.18. Assume that $\mathcal{O}$ is an open interval of $\mathbb{R}$, i.e. one of the three following forms

- (i) $\mathcal{O} = (a, b)$ for some $a < b \in \mathbb{R}$.
- (ii) $\mathcal{O} = (a, \infty)$ for some $a \in \mathbb{R}$.
- (iii) $\mathcal{O} = (-\infty, b)$ for some $b \in \mathbb{R}$.

Assume that $f : \mathcal{O} \rightarrow \mathbb{C}$ is Lebesgue measurable and $p \in [1, \infty)$. We say that $u \in L_{loc}^p(\mathcal{O})$ if and only if for every closed and bounded interval $[c, d] \subset \mathcal{O}$,

$$\int_a^d |u(x)|^p dx < \infty.$$

Moreover,

- (i) We say that $u \in L_{loc}^p([a, b])$ if and only if for every closed and bounded interval $[a, d] \subset [a, b]$, i.e. $a \leq d < b$

$$\int_a^d |u(x)|^p dx < \infty.$$

We say that $u \in L_{loc}^p((a, b])$ if and only if for every closed and bounded interval $[c, b] \subset (a, b]$, i.e. $a < c \leq b$

$$\int_c^b |u(x)|^p dx < \infty.$$

(ii) We say that $u \in L_{loc}^p([a, \infty))$ if and only if for every closed and bounded interval $[a, d] \subset [a, \infty)$, i.e. $a \leq d < \infty$

$$\int_a^d |u(x)|^p dx < \infty.$$

(iii) We say that $u \in L_{loc}^p((-\infty, b])$ if and only if for every closed and bounded interval $[c, b] \subset (-\infty, b]$, i.e. $-\infty < c \leq b$,

$$\int_c^b |u(x)|^p dx < \infty.$$

Remark 2.19. Let us point out that the definition of what it means that $D^3u \in L^2((0, \infty); \mathbb{C})$ is given in part (iii) of the Definition 2.18 above. The same comment applies to condition $D^3u \in L^2((-\infty, 0); \mathbb{C})$.

The following result is obvious.

Proposition 2.20. *In the framework of Definition 2.18, every space defined there is a proper subset of the corresponding $L_{loc}^p(\mathcal{O})$, i.e.*

$$\begin{aligned} L_{loc}^p([a, b)) \cup L_{loc}^p((a, b]) &\subset L_{loc}^p((a, b)), \\ L_{loc}^p([a, \infty)) &\subset L_{loc}^p((a, \infty)) \\ L_{loc}^p((-\infty, b]) &\subset L_{loc}^p((-\infty, b)) \end{aligned}$$

Proof. To show that the sets on left are proper subsets of the right we only consider the case (ii) with $a = 0$ and $p = 1$. We will prove that a function f defined by

$$f(x) = \frac{1}{x}, \quad x \in (0, \infty),$$

belongs to $L_{loc}^p((0, \infty))$ but it does not belong to $L_{loc}^p([0, \infty))$. Indeed, if $0 < c < d < \infty$, then

$$\int_c^d |f(x)| dx = \int_c^d \frac{1}{x} dx = \ln d - \ln c = \ln\left(\frac{d}{c}\right) < \infty$$

so that $f \in L_{loc}^p((0, \infty))$ but, if $0 < d < \infty$, then

$$\int_0^d |f(x)| dx = \lim_{\varepsilon \rightarrow 0+} \int_\varepsilon^d |f(x)| dx = \lim_{\varepsilon \rightarrow 0+} (\ln d - \ln \varepsilon) = \infty.$$

so that $f \notin L_{loc}^p([0, \infty))$. □

Example 2.21. If $\mathcal{O}$ is an open interval in $\mathbb{R}$ then it is of one of the four forms:

- (i) $\mathcal{O} = (a, b)$ for some $a < b \in \mathbb{R}$. Then $\overline{\mathcal{O}} = [a, b]$.
- (ii) $\mathcal{O} = (a, \infty)$ for some $a \in \mathbb{R}$. Then $\overline{\mathcal{O}} = [a, \infty)$.
- (iii) $\mathcal{O} = (-\infty, b)$ for some $b \in \mathbb{R}$. Then $\overline{\mathcal{O}} = (-\infty, b]$.
- (iv) $\mathcal{O} = \mathbb{R}$. Then $\overline{\mathcal{O}} = \mathbb{R}$.

Hence, the second part of Proposition 2.14 in case (ii) can be written as follows.

If a function $u \in L_{loc}^1((a, \infty))$, for some $a \in \mathbb{R}$, is n -times weakly differentiable, then there exists a function $\tilde{u} : [a, \infty) \rightarrow \mathbb{C}$ of C^{n-1} class such that (2.1.2) is satisfied.

Note that function $\tilde{u}$ satisfies the following.

- (o) $\lim_{x \rightarrow a+} \tilde{u}(x)$ exists (and is denoted $\tilde{u}(a+)$).
- (i) If $k \in \{1, \dots, n-1\}$, then $\lim_{x \rightarrow a+} \tilde{u}^{(k)}(x)$ exists (and is denoted $\tilde{u}^{(k)}(a+)$).

This definitions allow us to work with functions that may not be smooth but still have well-defined derivatives in the weak sense.

In what follows we use the following assumption.

Assumption 2.22. We assume that $\mathcal{O}$ is an open interval of $\mathbb{R}$.

Remark 2.23. This is a special case of the Sobolev embedding Theorem. Moreover, the classical $n-1$ order derivative of function $\tilde{u}$, i.e. $\tilde{u}$ is locally absolutely continuous in the sense of [38, Definition 7.17], see also Theorem 7.18 therein.

The above Sobolev embedding Theorem can be strengthened in the following way. If a function $u \in L_{loc}^1(\mathcal{O})$ is n -times weakly differentiable, $p \in (1, \infty)$, then C^{n-1} class function $\tilde{u} : \mathcal{O} \rightarrow \mathbb{C}$ from above is locally Hölder continuous of order $\alpha < 1 - \frac{1}{p}$, see [39].

Proposition 2.3 can be generalized using absolutely continuous functions as follows.

Proposition 2.24. If $\mathcal{O} \subset \mathbb{R}$ is an open interval and a function $u : \mathcal{O} \rightarrow \mathbb{C}$ is of C^{n-1} -class and $u^{(n-1)}$ is absolutely continuous, then obviously $u \in L_{loc}^1(\mathcal{O})$ and, by [38, Theorem 7.18], $u^{(n)}(x)$ exists for almost all $x \in \mathcal{O}$ and $u^{(n)} \in L_{loc}^1(\mathcal{O})$, and then u is n -times weakly differentiable with the equality (2.1.1) satisfied with function $v = u^{(n)}$, i.e.

$$D^n u = u^{(n)}.$$

Proof. The proof follows by applying $n-1$ -times the Integration by parts formula from [37, Theorem 6.22] and then [38, Theorem 7.18]. $\square$

Definition 2.25. Assume that $n \in \mathbb{N}^*$. If $\mathcal{O} \subset \mathbb{R}$ is an open interval, then function $u : \mathcal{O} \rightarrow \mathbb{C}$ belongs to $H^n(\mathcal{O})$ if and only if $u \in L^2(\mathcal{O})$ and all weak derivatives $D^k u$ for $k \leq n$ exist and belong to the space $L^2(\mathcal{O})$.

The following result can be proved (and is well known).

Proposition 2.26. *If $\mathcal{O} \subset \mathbb{R}$ is an open interval, then the space $H^n(\mathcal{O})$ is a Hilbert space endowed with the following inner product,*

$$\langle u, v \rangle_{H^n} := \sum_{k=0}^n \int_{\mathcal{O}} D^k u(x) \overline{D^k v(x)} dx.$$

In particular, space $H^0(\mathcal{O})$ is equal to $L^2(\mathcal{O})$.

The above mentioned Sobolev Embedding Theorem, i.e. Theorem 2.14, takes the following form

Theorem 2.27. *If $u \in H^n(\mathcal{O})$ then $\tilde{u} \in C^{n-1+\alpha}(\mathcal{O})$ with $\alpha \in [0, \frac{1}{2})$ and $D^{n-1}u$ is absolutely continuous, and the map*

$$H^n(\mathcal{O}) \ni u \mapsto \tilde{u} \in C^{n-1+\alpha}(\mathcal{O})$$

is linear and bounded.

Usually, the function $\tilde{u}$ is identified with u and denoted simply by u .

Definition 2.28. If $\mathcal{O} \subset \mathbb{R}$ is an open interval, then a function $u : \mathcal{O} \rightarrow \mathbb{C}$ belongs to $H_{\text{loc}}^n(\mathcal{O})$ if and only if $u \in L_{\text{loc}}^2(\mathcal{O})$ and all weak derivatives $D^k u$ for $k \leq n$ exist and belong to the space $L_{\text{loc}}^2(\mathcal{O})$, see Definition 2.18.

Remark 2.29. Let us point out that if $\mathcal{O}$ is an open and bounded domain, then the following two conditions are equivalent.

- (i) $u \in H^n(\mathcal{O})$,
- (ii) $u \in L^2(\mathcal{O})$ and the weak derivative $D^n u$ belongs to the space $L^2(\mathcal{O})$.

But this is not true when $\mathcal{O}$ is an open but unbounded domain.

On the other hand, if $\mathcal{O}$ is an open domain, then the following two conditions are equivalent.

- (iii) $u \in H_{\text{loc}}^n(\mathcal{O})$;
- (iv) $u \in L_{\text{loc}}^2(\mathcal{O})$ and the weak derivative $D^n u$ belongs to the space $L_{\text{loc}}^2(\mathcal{O})$.

The above definition needs to be generalized to the setup from Definition 2.18 in the following way.

Definition 2.30. Assume that $\mathcal{O}$ is an open interval of $\mathbb{R}$, i.e. one of the three forms listed in Definition 2.18. A function $u : \mathcal{O} \rightarrow \mathbb{C}$ belongs to

- (i) $H_{\text{loc}}^n([a, b))$ if and only if $u \in L_{\text{loc}}^2([a, b))$ and all weak derivatives $D^k u$ for $k \leq n$ exist and belong to the space $L_{\text{loc}}^2([a, b))$.
- (i') $H_{\text{loc}}^n((a, b])$ if and only if $u \in L_{\text{loc}}^2((a, b])$ and all weak derivatives $D^k u$ for $k \leq n$ exist and belong to the space $L_{\text{loc}}^2((a, b])$.
- (ii) $H_{\text{loc}}^n([a, \infty))$ if and only if $u \in L_{\text{loc}}^2([a, \infty))$ and all weak derivatives $D^k u$ for $k \leq n$ exist and belong to the space $L_{\text{loc}}^2([a, \infty))$.
- (iii) $H_{\text{loc}}^n((-\infty, b])$ if and only if $u \in L_{\text{loc}}^2((-\infty, b])$ and all weak derivatives $D^k u$ for $k \leq n$ exist and belong to the space $L_{\text{loc}}^2((-\infty, b])$.

It follows from the above mentioned Sobolev embedding theorem that

Theorem 2.31. *if $u \in H_{\text{loc}}^n(\mathcal{O})$ is then $\tilde{u} \in C_{\text{loc}}^{n+\frac{1}{2}}(\mathcal{O})$. The function $\tilde{u}$ is usually denoted simply by u .*

Similarly,

- (i) *if $u \in H_{\text{loc}}^n([a, b))$ then $\tilde{u} \in C_{\text{loc}}^{n+\frac{1}{2}}([a, b))$*
- (i') *if $u \in H_{\text{loc}}^n((a, b])$ then $\tilde{u} \in C_{\text{loc}}^{n+\frac{1}{2}}((a, b])$*
- (ii) *if $u \in H_{\text{loc}}^n([a, \infty))$ then $\tilde{u} \in C_{\text{loc}}^{n+\frac{1}{2}}([a, \infty))$*
- (iii) *if $u \in H_{\text{loc}}^n((-\infty, b])$ then $\tilde{u} \in C_{\text{loc}}^{n+\frac{1}{2}}((-\infty, b])$*

In what follows we will identify $\tilde{u}$ and write only u .

We have the following Corollary.

Corollary 2.32. *Assume that $\mathcal{O}$ is an open interval in $\mathbb{R}$, i.e. one of the three forms listed in Definition 2.18. If $u \in H_{\text{loc}}^n([a, \infty))$ or $u \in H_{\text{loc}}^n([a, b))$ then*

- (o) $\lim_{x \rightarrow a+} u(x)$ exists (and is denoted $u(a+)$).
- (i) *If $k \in \{1, \dots, n-1\}$, then $\lim_{x \rightarrow a+} u^{(k)}(x)$ exists (and is denoted $\tilde{u}^{(k)}(a+)$).*

If $u \in H_{\text{loc}}^n((-\infty, b])$ or $u \in H_{\text{loc}}^n((a, b])$, then

(o) $\lim_{x \rightarrow b-} u(x)$ exists (and is denoted $u(b-)$).

(i) If $k \in \{1, \dots, n-1\}$, then $\lim_{x \rightarrow b-} u^{(k)}(x)$ exists (and is denoted $\tilde{u}^{(k)}(b-)$).

A special case is the following result.

Proposition 2.33. *A function u belongs to $H_{loc}^3(\mathcal{O})$ if and only if the following conditions satisfies*

(α) *A function u is of C^2 -class,*

(β) *The second order (classical) derivative u'' is absolutely continuous, and hence classically differentiable at almost every $x \in \mathbb{R}$ belongs to the space L_{loc}^2 .*

Example 2.34. *The following definitions are equivalent.*

(i) *We say that $u \in H^1(\mathbb{R}, \mathbb{C})$ if and only if $u \in L^2(\mathbb{R}, \mathbb{C})$ and the weak derivative Du exists and belongs to $L^2(\mathbb{R}, \mathbb{C})$.*

(ii) *We say that $u \in H^1(\mathbb{R}, \mathbb{C})$ if and only if $u \in L^2(\mathbb{R}, \mathbb{C})$ and u is absolutely continuous on every compact subset of $\mathbb{R}$, and its derivative u' exists almost everywhere, see [38, Theorem 7.18], and it belongs to $L^2(\mathbb{R}, \mathbb{C})$.*

(iii) *We say that $u \in H^1(\mathbb{R}, \mathbb{C})$ if and only if the Fourier transform $\mathcal{F}u = \hat{u}$ satisfies the following condition*

$$\int_{\mathbb{R}} (1 + |\xi|^2) |\hat{u}(\xi)|^2 d\xi < \infty.$$

See [36, Definition 8.8].

Definition 2.35 (Definition 3.2(c), [1]). Let $\Omega \subset \mathbb{R}$ be an open set. We define the Sobolev space $H_0^1(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$.

Remark 2.36. The following facts are useful in the applications considered later.

(i) One has

$$H_0^1(\mathbb{R}) = H^1(\mathbb{R}),$$

see [1, Corollary 3.23].

(ii) In contrast,

$$H_0^1(\mathbb{R} \setminus \{0\}) \neq H^1(\mathbb{R} \setminus \{0\}).$$

(iii) The space $H^1(\mathbb{R} \setminus \{0\})$ may be identified with

$$H^1((-\infty, 0)) \oplus H^1((0, \infty)).$$

In particular, a function $u \in H^1(\mathbb{R} \setminus \{0\})$ has, in general, two unrelated traces at the origin, namely $u(0-)$ and $u(0+)$.

(iv) The space $H_0^1(\mathbb{R} \setminus \{0\})$ consists of those functions in $H^1(\mathbb{R} \setminus \{0\})$ which can be approximated in $H^1(\mathbb{R} \setminus \{0\})$ by functions from $C_0^\infty(\mathbb{R} \setminus \{0\})$.

One can replace $H^1(\mathbb{R}, \mathbb{C})$ space by space $H^n(\mathbb{R}, \mathbb{C})$.

Example 2.37. *The following definitions are equivalent.*

(i) We say that $u \in H^n(\mathbb{R}, \mathbb{C})$ if and only if $u \in L^2(\mathbb{R}, \mathbb{C})$ and the weak derivative Du exists and belongs to $L^2(\mathbb{R}, \mathbb{C})$.

(ii) We say that $u \in H^n(\mathbb{R}, \mathbb{C})$ if and only if $u \in L^2(\mathbb{R}, \mathbb{C})$ and u is absolutely continuous on every compact subset of $\mathbb{R}$, and its derivative u' exists almost everywhere, see [38, Theorem 7.18], and it belongs to $L^2(\mathbb{R}, \mathbb{C})$.

(iii) We say that $u \in H^n(\mathbb{R}, \mathbb{C})$ if and only if the Fourier transform $\mathcal{F}u = \hat{u}$ satisfies the following condition

$$\int_{\mathbb{R}} (1 + |\xi|^2) |\hat{u}(\xi)|^2 d\xi < \infty.$$

See [36, Definition 8.8].

One can replace $H^1(\mathbb{R}, \mathbb{C})$ space by space $H^1(\mathcal{O}, \mathbb{C})$.

Example 2.38. *The following definitions are equivalent.*

(i) We say that $u \in H^1(\mathbb{R}, \mathbb{C})$ if and only if $u \in L^2(\mathbb{R}, \mathbb{C})$ and the weak derivative Du exists and belongs to $L^2(\mathbb{R}, \mathbb{C})$.

(ii) We say that $u \in H^1(\mathbb{R}, \mathbb{C})$ if and only if $u \in L^2(\mathbb{R}, \mathbb{C})$ and u is absolutely continuous on every compact subset of $\mathbb{R}$, and its derivative u' exists almost everywhere, see [38, Theorem 7.18], and it belongs to $L^2(\mathbb{R}, \mathbb{C})$.

One can replace $H^1(\mathcal{O}, \mathbb{C})$ space by space $H^n(\mathcal{O}, \mathbb{C})$.

Example 2.39. *The following definitions are equivalent.*

(i) We say that $u \in H^1(\mathbb{R}, \mathbb{C})$ if and only if $u \in L^2(\mathbb{R}, \mathbb{C})$ and the weak derivative Du exists and belongs to $L^2(\mathbb{R}, \mathbb{C})$.

(ii) We say that $u \in H^1(\mathbb{R}, \mathbb{C})$ if and only if $u \in L^2(\mathbb{R}, \mathbb{C})$ and u is absolutely continuous on every compact subset of $\mathbb{R}$, and its derivative u' exists almost everywhere, see [38, Theorem 7.18], and it belongs to $L^2(\mathbb{R}, \mathbb{C})$.

2.2 The spectrum of bounded and unbounded linear operators

The following material is based on [35, section VIII.1].

Definition 2.40. Assume that H is a Hilbert space with inner product denoted by $\langle \cdot, \cdot \rangle_H$ and $T : D(T) \rightarrow H$, where $D(T) \subset H$, is a linear transformation (or map or function). The graph of T is the following set

$$\begin{aligned}\Gamma(T) &:= \{(x, Tx) \in H \times H : x \in D(T)\} \\ &= \{(x, y) \in H \times H : x \in D(T), y = Tx\}.\end{aligned}$$

A linear map T is called closed if and only if the graph $\Gamma(T)$ is a closed subset of $H \times H$.

We have the following basic result which we do not prove, see [35, ch. VIII, p.250].

Proposition 2.41. *The set $H \times H$ is a vector space under component-wise addition and scalar multiplication, i.e.*

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2). \\ \alpha(x, y) &= (\alpha x, \alpha y)\end{aligned}$$

Moreover, $H \times H$ is a Hilbert space with the inner product defined by

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{H \times H} = \langle x_1, x_2 \rangle_H + \langle y_1, y_2 \rangle_H.$$

The following based on [35, section VI.3. p. 188].

Definition 2.42. Let X be a normed vector space. We denote by

$$\mathcal{L}(X)$$

the space of all bounded linear operators from X into X .

Definition 2.43. Assume that X is normed vector space. Let $T \in \mathcal{L}(X)$.

(a) An element $x \in X \setminus \{0\}$ which satisfies

$$Tx = \lambda x$$

for some $\lambda \in \mathbb{C}$ is called an eigenvector of T . The corresponding number λ is called an eigenvalue. The set of all eigenvalues is called the point spectrum of T and denoted by $\sigma_p(T)$.

- (b) The resolvent set of T is the set, denoted by $\rho(T)$, of all $\lambda \in \mathbb{C}$ such that the linear map $\lambda I - T$ is an isomorphism from X onto X .
- (c) The complement of the resolvent set of T , denoted by $\sigma(T)$, is called the spectrum of T , i.e. denoted by $\rho(T)$, i.e.

$$\sigma(T) := \mathbb{C} \setminus \rho(T).$$

Note, that if λ is an eigenvalue, then $\lambda I - T$ is not injective and therefore λ belongs in the spectrum of T . In other words,

$$\sigma_p(T) \subset \sigma(T).$$

Definition 2.44. Assume that X is a normed vector space and let $L : D(L) \subset X \rightarrow X$ be a linear map.

- (a) An element $x \in D(L) \setminus \{0\}$ satisfying

$$Lx = \lambda x$$

for some $\lambda \in \mathbb{C}$ is called an **eigenvector** of L . The corresponding number λ is called an **eigenvalue**. The set of all eigenvalues is called the **point spectrum** of L and is denoted by $\sigma_p(L)$.

- (b) The **resolvent set** of L is the set

$$\rho(L) := \{\lambda \in \mathbb{C} : (\lambda I - L) : D(L) \rightarrow X \text{ is bijective and } (\lambda I - L)^{-1} \in \mathcal{L}(X)\}.$$

- (c) The complement of the resolvent set is called the **spectrum** of L , namely

$$\sigma(L) := \mathbb{C} \setminus \rho(L).$$

This definition follows [35, Chapter VIII, p. 253].

2.3 Adjoints and self-adjoint extensions

Let us recall the following definition.

Definition 2.45. Assume that H is a Hilbert space with norm $\|\cdot\|$ and E is a subset of H .

- (i) The set E is said to be dense in H if and only if for every $u \in H$ and every $\delta > 0$ there exists $\phi \in E$ such that

$$\|u - \phi\| < \delta.$$

- (ii) The closure $\bar{E}$ in H is the smallest closed set that contains E .

Example 2.46. One can prove the following result. The closure $\bar{E}$ in H is the set of all elements $u \in H$ such that there exists a sequence $(\phi_n) \subset E$ with $\phi_n \rightarrow u$ in H , i.e.,

$$\bar{E} := \left\{ u \in H : \exists (\phi_n) \subset E \text{ such that } \lim_{n \rightarrow \infty} \|\phi_n - u\| = 0 \right\}.$$

Example 2.47. $\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q} \subset \mathbb{R}$ are dense, but $\mathbb{N} \subset \mathbb{R}$ is not dense.

The following result is stated as [34, Ex.10- p.194].

Proposition 2.48. Assume that $-\infty \leq a < b \leq \infty$. Then the set $C_0^\infty(a, b)$ is a dense subset of $L^2(a, b)$. In particular $C_0^\infty(\mathbb{R})$ is a dense subset of $L^2(\mathbb{R})$. Moreover, $C_0^\infty(\mathbb{R} \setminus \{0\})$ is also a dense subset of $L^2(\mathbb{R})$.

Proof. We only prove the last assertion. Let $f \in L^2(\mathbb{R})$ and let $\varepsilon > 0$. Since $C_0^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, there exists $\phi \in C_0^\infty(\mathbb{R})$ such that

$$\|f - \phi\|_{L^2(\mathbb{R})} < \frac{\varepsilon}{2}.$$

Choose a smooth cut-off function $\chi_\delta \in C^\infty(\mathbb{R})$ such that

$$0 \leq \chi_\delta \leq 1, \quad \chi_\delta(x) = 0 \quad \text{for } |x| \leq \delta,$$

and

$$\chi_\delta(x) = 1 \quad \text{for } |x| \geq 2\delta.$$

Set

$$\psi_\delta(x) := \chi_\delta(x)\phi(x).$$

Then

$$\psi_\delta \in C_0^\infty(\mathbb{R} \setminus \{0\}).$$

Moreover,

$$\|\phi - \psi_\delta\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} |1 - \chi_\delta(x)|^2 |\phi(x)|^2 dx \leq \int_{-2\delta}^{2\delta} |\phi(x)|^2 dx.$$

Since $\phi \in L^2(\mathbb{R})$, the last integral tends to zero as $\delta \rightarrow 0$. Hence, for sufficiently small $\delta > 0$,

$$\|\phi - \psi_\delta\|_{L^2(\mathbb{R})} < \frac{\varepsilon}{2}.$$

Therefore,

$$\|f - \psi_\delta\|_{L^2(\mathbb{R})} \leq \|f - \phi\|_{L^2(\mathbb{R})} + \|\phi - \psi_\delta\|_{L^2(\mathbb{R})} < \varepsilon.$$

Thus $C_0^\infty(\mathbb{R} \setminus \{0\})$ is dense in $L^2(\mathbb{R})$. □

The following characterization of dense subsets is standard; see, for example, [36, Chapter 1].

Remark 2.49. Assume that

$$H = L^2(\mathbb{R}), \quad E = C_0^\infty(\mathbb{R}).$$

Then following conditions are equivalent.

- (i) The set E is a dense subset of H .
- (ii) For every $u \in H$ and every $\delta > 0$ there exists $\phi \in E$ such that

$$\|u - \phi\| < \delta.$$

- (iii) for every $u \in L^2(\mathbb{R})$ and every $\delta > 0$ there exists

$$\phi \in C_0^\infty(\mathbb{R}) : \text{such that } \phi \in B(u, \delta),$$

where

$$B(u, \delta) := \{v \in L^2(\mathbb{R}) : \|u - v\|_{L^2(\mathbb{R})} < \delta\}$$

is the open ball in $L^2(\mathbb{R})$ centred at u with radius δ .

- (iv) for every $u \in L^2(\mathbb{R})$ there exists a sequence $\{\phi_n\}_{n=1}^\infty$ such that

$$\phi_n \in C_0^\infty(\mathbb{R}) \text{ and } \phi_n \rightarrow u \text{ in } L^2(\mathbb{R}).$$

The last condition means that $\|u - \phi_n\| \rightarrow 0$ as $n \rightarrow \infty$.

- (v) The closure $\bar{E}$ in H is equal to H .

Densely defined operators play an important role in operator theory, since the adjoint of an operator can be defined only when the operator is densely defined.

Definition 2.50 (Densely defined [35], p.249). Let H be a Hilbert space. A linear operator A , with domain $D(A)$, on a Hilbert space H , i.e. $D(A)$ is a subspace of H and $A : D(A) \rightarrow H$, is called densely defined if and only if, $D(A)$ is a dense subset of H , i.e.

$$\overline{D(A)} = H.$$

Definition 2.51 (Adjoint operator [35], VIII, p.252). Let $A : D(A) \subset H \rightarrow H$ be a densely defined linear operator on a Hilbert space H . An element $u \in H$ is said to belong to the domain $D(A^*)$ of the adjoint operator if and only if there exists $z \in H$ such that

$$\langle A\phi, u \rangle_H = \langle \phi, z \rangle_H, \quad \text{for every } \phi \in D(A). \quad (2.3.1)$$

Exercise 2.52. Prove, in the framework of Definition 2.51, that if two elements $z_1, z_2 \in H$ satisfy condition (2.3.1), then $z_1 = z_2$.

Solution of Exercise 2.52. Assume z_1, z_2 satisfy (2.3.1). Then for all $\phi \in D(A)$

$$\langle \phi, z_1 \rangle = \langle A\phi, u \rangle = \langle \phi, z_2 \rangle.$$

Hence

$$\langle \phi, z_1 - z_2 \rangle = 0 \quad \text{for all } \phi \in D(A).$$

Since $D(A)$ is dense in H , it follows that $z_1 - z_2 = 0$, so $z_1 = z_2$. □

This Exercise implies that the following definition makes sense.

Definition 2.53. Let A be a densely defined linear operator on a Hilbert space H . For $u \in D(A^*)$, the element $z \in H$ satisfying (2.3.1) is uniquely determined. We define

$$A^*u = z.$$

The operator A^* is called the **adjoint** of A .

One can prove the following result, see [35, VIII, p.252].

Proposition 2.54. Let A be a densely defined linear operator on a Hilbert space H . An element $v \in H$ belongs to $D(A^*)$ if and only if there exists a constant $C > 0$ such that

$$|\langle Au, v \rangle| \leq C\|u\| \quad \text{for all } u \in D(A). \quad (2.3.2)$$

The next result is a standard consequence of the Riesz representation theorem.

Proposition 2.55. *Let A be a densely defined linear operator on a Hilbert space H . Then the adjoint operator A^* exists.*

Proof of Proposition 2.55. Let $u \in D(A^*)$. By Definition 2.51, the map

$$\phi \mapsto \langle A\phi, u \rangle$$

defines a linear functional on $D(A)$.

If (2.3.2) holds, this functional is bounded. Since H is a Hilbert space, the Riesz representation theorem implies the existence of a unique element $z \in H$ such that

$$\langle A\phi, u \rangle = \langle \phi, z \rangle \quad \text{for all } \phi \in D(A).$$

We therefore define $A^*u = z$, which proves that the adjoint operator exists. $\square$

Definition 2.56 (Symmetric [35], VIII.2, p.255). Assume that A is a densely defined linear operator on a Hilbert space H . We say that A is **symmetric** if

$$\langle A\phi, \psi \rangle = \langle \phi, A\psi \rangle \quad \text{for all } \phi, \psi \in D(A).$$

Proposition 2.57. *Assume that A is a densely defined linear operator in a Hilbert space H . If A is symmetric, then*

$$A \subset A^*,$$

that is,

$$D(A) \subset D(A^*) \quad \text{and} \quad A\phi = A^*\phi \quad \text{for all } \phi \in D(A).$$

Proof of Proposition 2.57. If A is symmetric, then

$$\langle A\phi, \psi \rangle = \langle \phi, A\psi \rangle \quad \text{for all } \phi, \psi \in D(A).$$

Fix $\phi \in D(A)$. The map

$$\psi \mapsto \langle A\phi, \psi \rangle$$

defines a bounded linear functional on $D(A)$. Hence, by the definition of the adjoint, $\phi \in D(A^*)$ and

$$A^*\phi = A\phi.$$

Therefore $D(A) \subset D(A^*)$ and $A\phi = A^*\phi$, which means $A \subset A^*$. $\square$

Next, we recall several fundamental concepts concerning self-adjoint extensions of symmetric operators in Hilbert spaces. In particular, we introduce deficiency subspaces and deficiency indices and state the classical von Neumann theorem describing all self-adjoint extensions.

Definition 2.58 ([35], p.39). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. They are said to be **isomorphic** if there exists a linear operator

$$U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$$

which is onto and satisfies

$$\langle Ux, Uy \rangle_{\mathcal{H}_2} = \langle x, y \rangle_{\mathcal{H}_1}, \quad \text{for all } x, y \in \mathcal{H}_1.$$

Such an operator U is called a **unitary operator**.

The following definition based on [34, Section X, Theorem X.2, v.2, p. 138].

Definition 2.59. Suppose that A_0 is a symmetric operator on a Hilbert space H . Define the *deficiency subspaces* of A_0 by

$$H_+ = \ker(i - A_0^*) = \text{Ran}(i + A_0)^\perp, \quad H_- = \ker(i + A_0^*) = \text{Ran}(-i + A_0)^\perp.$$

The pair of numbers $n_+(A_0)$ and $n_-(A_0)$ defined by

$$n_+(A_0) = \dim(H_+), \quad n_-(A_0) = \dim(H_-),$$

are called the *deficiency indices* of linear map A_0 .

Let us formulate the following fundamental result, see [34, Section X, Theorem X.2, v.2, p. 141].

Theorem 2.60. Assume that $A_0 : D(A_0) \subset H \rightarrow H$ be a densely defined closed symmetric operator, and let n_+ and n_- be its deficiency indices. Then

1. A_0 is self-adjoint if and only if $n_+ = n_- = 0$.
- 2.(i) There exists an operator A such that

$$A : D(A) \rightarrow H : D(A_0) \subset D(A), \quad \text{and } A_0x = Ax, \quad x \in D(A_0)$$

and $A = (A)^*$ if and only if $n_+ = n_-$.

(ii) *There exists a bijective correspondence between the set of all self-adjoint extensions of A_0 and the set of all unitary maps from H_+ onto H_- . To be precise, for each unitary operator*

$$C : H_+ \longrightarrow H_-, \quad H_{\pm} := \ker(A_0^* \mp iI),$$

we define the operator A^C by

$$\begin{aligned} D(A^C) &:= \left\{ u = u_0 + u_+ + \Gamma u_+ : u_0 \in D(A_0), u_+ \in H_+ \right\}, \\ A^C(u_0 + u_+ + \Gamma u_+) &:= Au_0 + i u_+ - i \Gamma u_+. \end{aligned} \tag{2.3.3}$$

Then A^C is self-adjoint and satisfies

$$A_0 \subset A^C \subset A_0^*.$$

Conversely, if A is a self-adjoint extension of a symmetric densely defined linear map A_0 , then there exists a unique unitary operator $C : H_+ \rightarrow H_-$ such that $A = A^C$.

(iii) *In other words, the family of all self-adjoint extensions of A_0 is equal to the family $\{A^C : C \in U(H_+, H_-)\}$*

We used the following notation.

Let us formulate the following two important consequences of the above result.

Corollary 2.61. *If A_0 is symmetric operator and $n_+ = n_- = 0$, then A_0 has a unique self-adjoint extension.*

Corollary 2.62. *If A_0 is symmetric operator and $n_+ \neq n_-$, then A_0 has no self-adjoint extension.*

2.4 The Cayley Transform

In this section, we recall the Cayley transform and explain its connection with von Neumann's extension theory. The Cayley transform is a classical tool which relates symmetric and self-adjoint operators to isometric and unitary operators. We follow the presentation in Akhiezer and Glazman [3, Section 67].

Definition 2.63. [Cayley Transform] Let A be a closed symmetric operator on a Hilbert space H . The *Cayley transform* of A is the operator

$$V_A = (A - iI)(A + iI)^{-1},$$

defined on

$$D(V_A) = \text{Ran}(A + iI),$$

with

$$\text{Ran}(V_A) = \text{Ran}(A - iI).$$

Equivalently,

$$V_A(A + iI)u = (A - iI)u, \quad u \in D(A).$$

Remark 2.64. The Cayley transform V_A is an isometry. Indeed, for every $u \in D(A)$,

$$\|(A + iI)u\|^2 = \|Au\|^2 + \|u\|^2 = \|(A - iI)u\|^2,$$

which implies

$$\|V_A(A + iI)u\| = \|(A + iI)u\|.$$

Hence the Cayley transform converts the study of the possibly unbounded operator A into the study of an isometric operator V_A .

Definition 2.65. [Inverse Cayley Transform] Let V be an isometric operator such that $I - V$ is injective. The *inverse Cayley transform* is defined by

$$A = i(I + V)(I - V)^{-1}.$$

Equivalently, if

$$h = \frac{1}{2i}(I - V)f,$$

then

$$Ah = \frac{1}{2}(I + V)f.$$

Remark 2.66. The Cayley transform provides an equivalent formulation of von Neumann's extension theory. A closed symmetric operator is self-adjoint if and only if its Cayley transform is unitary. Consequently, the problem of describing self-adjoint extensions is equivalent to extending the Cayley transform from an isometry to a unitary operator. This is equivalent to von Neumann's classification by unitary maps between the deficiency subspaces.

Proposition 2.67 (Connection with von Neumann's Theory). *Let A be a densely defined closed symmetric operator on a Hilbert space H , and let*

$$H_+ = \ker(A^* - iI), \quad H_- = \ker(A^* + iI)$$

be its deficiency subspaces.

By von Neumann's theorem, every self-adjoint extension of A is uniquely determined by a unitary operator

$$U : H_+ \rightarrow H_-.$$

On the other hand, the Cayley transform

$$V_A = (A - iI)(A + iI)^{-1}$$

is an isometric operator from $\text{Ran}(A + iI)$ onto $\text{Ran}(A - iI)$.

The deficiency subspaces satisfy the orthogonal decompositions

$$H = \text{Ran}(A + iI) \oplus H_-,$$

and

$$H = \text{Ran}(A - iI) \oplus H_+.$$

Therefore, extending the isometry V_A to a unitary operator on H is equivalent to choosing a unitary operator

$$U : H_+ \rightarrow H_-.$$

Hence the Cayley transform provides an equivalent formulation of von Neumann's extension theory: self-adjoint extensions of A correspond precisely to unitary extensions of the Cayley transform V_A .

2.5 Unitary operators

Remark 2.68. In what follows, the term *operator* will always mean a bounded linear map. The abbreviation *ONB* denotes an orthonormal basis.

Definition 2.69. [35, p.39]. If X and Y are two complex Hilbert spaces, a linear surjective, i.e. onto, map $T : X \rightarrow Y$ is called unitary if and only if for all $x_1, x_2 \in X$,

$$\langle T(x_1), T(x_2) \rangle_Y = \langle x_1, x_2 \rangle_X.$$

Two complex Hilbert spaces X and Y are said to be isomorphic if and only if there exists a unitary operator $T : X \rightarrow Y$.

Note that if a linear map $T : X \rightarrow Y$ is unitary then not only it is surjective but also injective, hence bijective.

Exercise 2.70. Prove that if a linear map $T : X \rightarrow Y$ is unitary then it is injective.

Solution to Exercise 2.70. To show that T is injective, we need to prove that if $T(x_1) = T(x_2)$, then $x_1 = x_2$.

Assume $T(x_1) = T(x_2)$ for some $x_1, x_2 \in X$, and we put $x = x_1 - x_2$. Then we have:

$$T(x_1) = T(x_2) \implies T(x_1 - x_2) = T(0) = 0.$$

Thus,

$$T(x) = 0. \tag{2.5.1}$$

Since T is unitary, we can apply the inner product such that

$$\langle T(x), T(x) \rangle_Y = \langle x, x \rangle_X. \tag{2.5.2}$$

Substituting by (2.5.1) into the equation (2.5.2), we get

$$\langle 0, 0 \rangle_Y = \langle x, x \rangle_X.$$

Since the zero vector with itself is zero, this leads to

$$0 = \langle x, x \rangle_X.$$

Since the inner product is zero if and only if the vector itself is the zero vector in a Hilbert space, this implies that

$$x = 0$$

. Therefore $x_1 - x_2 = 0$ and hence we infer that $x_1 = x_2$. $\square$

Proposition 2.71. Assume that X and Y are two complex Hilbert spaces and a map $T : X \rightarrow Y$ is unitary. Assume that X_1 and Y_1 are two complex Hilbert spaces such that X_1 is isomorphic to X and Y_1 is isomorphic to Y and let

$$S : X_1 \rightarrow X \text{ and } S : Y_1 \rightarrow Y$$

are the corresponding unitary operators. Then the map

$$R^{-1} \circ T \circ S : X_1 \rightarrow Y_1$$

is also unitary.

Proposition 2.72 (Proof of Proposition 2.71 is trivial and hence omitted).

Definition 2.73. Assume that X is a vector space. A finite set $\{e_1, \dots, e_n\} \subset X$ is called a *basis* of X if and only if

(i) the vectors $e_1, \dots, e_n$ are linearly independent, i.e.

$$\alpha_1 e_1 + \dots + \alpha_n e_n = 0 \implies \alpha_1 = \dots = \alpha_n = 0,$$

for all scalars $\alpha_1, \dots, \alpha_n \in \mathbb{K}$, where $\mathbb{K}$ denotes the scalar field $\mathbb{R}$ or $\mathbb{C}$. and

(ii) they span X , i.e.

$$X = \text{span}\{e_1, \dots, e_n\} = \{\alpha_1 e_1 + \dots + \alpha_n e_n : \alpha_1, \dots, \alpha_n \in \mathbb{K}\}.$$

One can show the following result.

Proposition 2.74. *If $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_k\}$ are two bases of X , then $n = k$.*

Hence the following definition is well posed.

Definition 2.75. If X is a vector space. Then $\dim X = n$ if and only if X has a basis $\{e_1, \dots, e_n\}$ consisting of n elements. If no finite basis exists, then X is said to be infinite-dimensional.

Proposition 2.76. *Assume that X and Y are two complex Hilbert spaces and a map $T : X \rightarrow Y$ is unitary.*

If (e_n) is (a finite or infinite) ONB of X , then (Te_n) is (a finite or infinite) ONB of Y .

If (e_n) is (a finite or infinite) ONB of X and (f_n) is (a finite or infinite) ONB of Y and $T : X \rightarrow Y$ is a linear map such that

$$f_n = T(e_n), \quad n \in \mathbb{N},$$

then T is unitary.

Definition 2.77. [21, p.4]. A unitary matrix of dimension n is the matrix of a unitary operator $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$ with respect to the canonical basis $\{e_1, \dots, e_n\}$.

Proposition 2.78. *Assume that X and Y are two complex Hilbert spaces of dimension n .*

(i) *Let $T : X \rightarrow Y$ be a unitary operator. Let $(e_j)_{j=1}^n$ be an ONB of X and $(f_j)_{j=1}^n$ be an ONB of Y . If A is the matrix of T with respect to these bases, then A is a unitary matrix.*

(ii) *Conversely, assume that A is a unitary $n \times n$ matrix. Let $(e_j)_{j=1}^n$ be an ONB of X and $(f_j)_{j=1}^n$ be an ONB of Y . If $T : X \rightarrow Y$ is a linear operator whose matrix with respect to these bases is A , then T is a unitary operator.*

Example 2.79. For $\alpha \in \mathbb{C}$, define

$$U_\alpha : \mathbb{C} \rightarrow \mathbb{C}, \quad U_\alpha z = \alpha z.$$

Then U_α is unitary if and only if

$$|\alpha| = 1.$$

Thus the unitary operators on the one-dimensional Hilbert space $\mathbb{C}$ are parametrized by

$$\mathbb{T} = \{\alpha \in \mathbb{C} : |\alpha| = 1\}.$$

2.6 Closed and closable operators, adjoints, and operator extensions

Let us recall some basic definitions.

Definition 2.80. Assume that H is a Hilbert space and A and B are linear operators in H with domains $D(A)$ and $D(B)$. We write $A \subset B$ and say that B is an extension of A , if and only if

$$D(A) \subset D(B), \quad \text{and} \quad Ax = Bx, \quad x \in D(A).$$

Definition 2.81. Assume that H is a Hilbert space and A , with domain $D(A)$, is a symmetric operator. We say that A has a self-adjoint extension if and only if there exists a linear self-adjoint operator B in H with domain $D(B)$ such that $A \subset B$.

Definition 2.82. Assume that H be a Hilbert space and let A be a linear operator

$$A : D(A) \subset H \rightarrow H.$$

The *graph* of A is the subset of $H \times H$ defined by

$$\Gamma(A) := \{(x, Ax) \in H \times H : x \in D(A), y = Ax\}. \quad (2.6.1)$$

The operator A is called *closed* if and only if $\Gamma(A)$ is a closed subset of $H \times H$.

The following result gives an equivalent form of the definition of a closed operator.

Proposition 2.83. Assume that A is a linear operator in a Hilbert space H . Then A is closed if and only if the following condition is satisfied.

If a sequence $(u_n) \subset D(A)$ satisfies

$$u_n \rightarrow u \text{ in } H \quad \text{and} \quad Au_n \rightarrow v \text{ in } H.$$

then

$$u \in D(A) \quad \text{and} \quad Au = v.$$

Proof of Proposition 2.83. Recall that A is closed if and only if its graph $\Gamma(A)$ defined in (2.6.1) is a closed subset of $H \times H$. Let $(u_n) \subset D(A)$ be such that

$$u_n \rightarrow u \text{ in } H \quad \text{and} \quad Au_n \rightarrow v \text{ in } H.$$

Then

$$(u_n, Au_n) \rightarrow (u, v) \text{ in } H \times H.$$

Since $(u_n, Au_n) \in \Gamma(A)$ and $\Gamma(A)$ is closed, we conclude that

$$(u, v) \in \Gamma(A).$$

By (2.6.1) this implies $u \in D(A)$ and $Au = v$. □

Definition 2.84 (Closable definition p.250 [35]). Assume that A is a linear operator in a Hilbert space H . We say that A is closable if and only if it has a closed extension.

Definition 2.85. [37, Definition 2.26] Let $E \subset X$, where X is a metric space. The *closure* of E , denoted by $\overline{E}$, is defined

$$\overline{E} := \{x \in X : \forall \varepsilon > 0, \quad B(x, \varepsilon) \cap E \neq \emptyset\}.$$

Equivalently, $\overline{E}$ is the union of E and the set of all limit points of E , i.e. $\overline{E} = E \cup E'$, where E' is the set of all limit points of E .

Equivalently, $x \in \overline{E}$ if and only if there exists a sequence (x_n) of elements E such that $x_n \rightarrow x$ in X .

Remark 2.86. According to the definition of closure in a topological space, see [35, Theorem IV.1, p. 96]), we have the following characterization. Here $X = H \times H$ and $E = \Gamma(A)$.

Assume that A , with domain $D(A)$, is a linear operator in a Hilbert space H . An element (x, y) belongs to $\overline{\Gamma(A)}$ if and only if there exists a sequence (x_n, y_n) of elements $\Gamma(A)$ such that

$$(x_n, y_n) \rightarrow (x, y) \text{ in } H \times H.$$

Proposition 2.87. Assume that H is a Hilbert space and $A_0 : D(A_0) \rightarrow H$, where $D(A_0) \subset H$, is a closable linear operator. Then there exists a unique closed operator S in H such that the following two conditions are satisfied:

(i) $A_0 \subset S$,

(ii) if A_1 is a closed extension of A_0 , then $S \subset A_1$.

Definition 2.88. The unique operator S from Proposition 2.87 is called *the smallest closed extension* or the *closure* of A_0 and is denoted by $\overline{A_0}$.

Proof of Proposition 2.87. **To prove uniqueness,** suppose that S and M are two closed operators in H satisfying both properties (i) and (ii) with respect to A_0 .

Since S satisfies property (ii), we can apply it with $A_1 = M$, because M satisfies property (i). Thus, we conclude that $S \subset M$. Similarly, since M satisfies property (ii), we can apply it with $A_1 = S$, because S satisfies property (i). Hence, $M \subset S$. Therefore, we conclude that $S = M$, proving uniqueness. $\square$

To prove existence, consider the closure of the graph of A_0 , i.e.

$$\overline{\Gamma(A_0)} \subset H \times H.$$

Since A_0 is closable, this closure is the graph of some operator. We need to show that, since S is closed and extends A_0 , the limits (x, z_1) and (x, z_2) belong to $\Gamma(S)$. Hence $z_1 = Sx = z_2$.

let us choose and fix $(x, z_1), (x, z_2) \in \overline{\Gamma(A_0)}$. By the previous result in Remark 2.86, there exists two $\Gamma(A_0)$ -valued sequences (x_n, y_n) and (x'_n, y'_n) such that $(x_n, y_n) \rightarrow (x, z_1)$ and $(x'_n, y'_n) \rightarrow (x, z_2)$. Let S be a closed extension of A_0 , i.e. $\Gamma(A_0) \subset \Gamma(S)$. The above implies that sequences (x_n, y_n) and (x'_n, y'_n) are $\Gamma(S)$ -valued and $(x, z_1), (x, z_2) \in \Gamma(S)$. This implies $z_1 = S(x)$ and $z_2 = S(x)$.

Let us denote by B a function $B : D(B) \rightarrow H$ such that $\Gamma(B) = \overline{\Gamma(A_0)}$. Then one can show that B is a linear operator. Since $\Gamma(B) = \overline{\Gamma(A_0)}$ and $\Gamma(A_0)$ is the graph of a linear operator, it is a linear subspace of $H \times H$. The closure of a linear subspace is again a linear subspace by [36, Theorem 1.13 p. 11], so $\Gamma(B)$ is a linear subspace. Therefore, B is a linear operator.

Hence B is closed because its graph is closed. By construction, this operator B extends A_0 . Moreover, B is contained in every closed extension of A_0 . Hence, B is the smallest closed extension of A_0 . The proof is complete. $\square$

The proof of the next result can be found in [33, Proposition 10.34].

Proposition 2.89. *If A is a densely defined symmetric operator in a Hilbert space H , then A is closable.*

Exercise 2.90.(i) Assume that A_0 is a linear operator in a Hilbert space H . Show that the set $\Gamma(A_0)$ is a subspace of $H \times H$.

(ii) Assume that A_0 is a bounded linear operator in a Hilbert space H with $D(A_0) = H$. Show that A_0 is closed.

Solution of Exercise 2.90. Proof of Part (i), let A_0 be a linear operator with domain and $D(A_0) \subset H$. Define its graph as

$$\Gamma(A_0) := \{(x, A_0x) \in H \times H \mid x \in D(A_0)\}.$$

By [36, Definition subspace p.6], Since $D(A_0)$ is a linear subspace, $0 \in D(A_0)$. Because A_0 is linear, we have $A_0(0) = 0$. Therefore, $(0, 0) \in \Gamma(A_0)$.

Let $(x_1, A_0x_1), (x_2, A_0x_2) \in \Gamma(A_0)$, and let $\alpha, \beta \in \mathbb{C}$ be scalars. Then $x_1, x_2 \in D(A_0)$, and since $D(A_0)$ is a subspace, $\alpha x_1 + \beta x_2 \in D(A_0)$. Moreover, by linearity of A_0 , we have:

$$A_0(\alpha x_1 + \beta x_2) = \alpha A_0x_1 + \beta A_0x_2.$$

Thus,

$$(\alpha x_1 + \beta x_2, \alpha A_0x_1 + \beta A_0x_2) \in \Gamma(A_0).$$

Since $\Gamma(A_0)$ contains $(0, 0)$ and is closed under linear combinations, so $\Gamma(A_0)$ is a subspace of $H \times H$.

Proof of Part (ii), by Definition 2.82, A_0 is closed if and only if its graph $\Gamma(A_0)$ defined in (2.6.1) is closed subset in $H \times H$ with the inner product.

Let $(x_n, A_0x_n) \rightarrow (x, y)$ in $H \times H$, such that

$$x_n \rightarrow x \quad \text{and} \quad A_0x_n \rightarrow y \quad \text{in } H.$$

Since A_0 is bounded and $D(A_0) = H$, the operator is continuous. Therefore, if $x_n \rightarrow x$, then $A_0x_n \rightarrow A_0x$. By uniqueness of limits we obtain $y = A_0x$. Therefore $(x, y) \in \Gamma(A_0)$ and $\Gamma(A_0)$ is closed.

□

The following Theorem based on [35, Section VIII.1, p.250-253].

Theorem 2.91. *Let A be a densely defined operator on a Hilbert space H . Then:*

(a) A^* exists and is closed.

(b) A is closable if and only if $D(A^*)$ is dense. In this case, $A^{**} := (A^*)^*$ exists, is closed (by (a)) and

$$\overline{A} = A^{**}.$$

(c) If A is closable, then

$$(\overline{A})^* = A^*.$$

Corollary 2.92. *Let A be a densely defined symmetric operator on a Hilbert space H . Then A is closable, its closure $\overline{A}$ is symmetric, and*

$$(\overline{A})^* = A^*.$$

Consequently,

$$\ker(\overline{A}^* - iI) = \ker(A^* - iI),$$

and

$$\ker(\overline{A}^* + iI) = \ker(A^* + iI).$$

Therefore,

$$n_+(\overline{A}) = n_+(A), \quad n_-(\overline{A}) = n_-(A).$$

Proof. By Proposition 2.89, the operator A is closable. Since A is symmetric, its closure $\overline{A}$ is also symmetric. Moreover, by Theorem 2.91,

$$(\overline{A})^* = A^*.$$

The equalities of the deficiency subspaces and deficiency indices follow immediately. $\square$

2.7 Ellipticity on one-dimensional domains

In this section we recall a basic regularity result for elliptic differential operators on the real line. The result we use is a one-dimensional special case of elliptic regularity, see [36, Theorem 8.12, p. 220].

Definition 2.93. Suppose that $\mathcal{O}$ is an open interval in $\mathbb{R}$, i.e. $\mathcal{O} = (a, b)$. Let $N \in \mathbb{N}$ denote the order of the differential operator, and let

$$L = \sum_{k=0}^N a_k D^k \tag{2.7.1}$$

where $a_k \in C^\infty(\mathcal{O})$ for $k = 0, \dots, N$, and the leading coefficient a_N is a non-zero constant. The operator L is called *elliptic* if and only if $a_N \neq 0$.

The formal adjoint of L is a differential operator defined by

$$L^* \phi := \sum_{k=0}^N (-1)^k D^k (a_k \phi).$$

Remark 2.94. We use the standard notion of ellipticity for one-dimensional differential operators adopted in [36, Theorem 8.12, p. 220]. In particular, the operator

$$L = \sum_{k=0}^N a_k D^k$$

is called elliptic whenever its principal coefficient satisfies

$$a_N(x) \neq 0, \quad x \in \mathcal{O}.$$

Theorem 2.95. Suppose that $\mathcal{O}$ is an open interval in $\mathbb{R}$. Assume that the operator L defined by (2.7.1) is elliptic. Let $u, v \in L_{\text{loc}}^2(\mathcal{O})$ and suppose that

$$Lu = v$$

in the weak sense, i.e. for every $\phi \in C_0^\infty(\mathcal{O})$

$$\int_{\mathcal{O}} v(x) \overline{\phi(x)} dx = \int_{\mathcal{O}} u(x) \overline{(L^* \phi)(x)} dx,$$

Then $u \in H_{\text{loc}}^N(\mathcal{O})$, see Definition 2.28.

We will use this theorem in finding the domain of adjoints of certain in-homogenous partial differential operators in L^2 spaces.

Theorem 2.96. Suppose that $\mathcal{O}$ is an open interval in $\mathbb{R}$ of one of the three forms listed in Definition 2.18. Assume that operator L defined by (2.7.1) is elliptic and assume that $u, v \in L_{\text{loc}}^2(\mathcal{O})$ satisfy

$$Lu = v$$

in the weak sense. Then, see Definition 2.30,

- (i) if $\mathcal{O} = (a, b)$ and $u, v \in L_{\text{loc}}^2([a, b))$, then $u \in H_{\text{loc}}^N([a, b))$.
- (i') if $\mathcal{O} = (a, b)$ and $u, v \in L_{\text{loc}}^2((a, b])$, then $u \in H_{\text{loc}}^N((a, b])$.
- (ii) if $\mathcal{O} = (a, \infty)$ and $u, v \in L_{\text{loc}}^2([a, \infty))$, then $u \in H_{\text{loc}}^N([a, \infty))$,
- (iii) if $\mathcal{O} = (-\infty, b)$ and $u, v \in L_{\text{loc}}^2((-\infty, b])$, then $u \in H_{\text{loc}}^N((-\infty, b])$.

Chapter 3

First order differential operator

In this chapter, we study first-order differential operators in various settings. We begin with the analysis of such operators on a bounded interval, where we characterize all self-adjoint extensions of a symmetric operator.

We then extend the study to unbounded domains, including the real line and the punctured real line, where additional difficulties arise due to the presence of singular points. In these cases, we investigate symmetry, determine the adjoint operator, and compute the deficiency indices.

The results obtained in this chapter provide a foundation for the analysis of higher-order differential operators and more general settings considered in the subsequent chapters.

3.1 First-order differential operator on a bounded interval

3.1.1 symmetry and Characterization of all self-adjoint extensions of first-order operator

We will use the above result to determine all self-adjoint operators which are extensions of symmetric operators in the cases we studied before.

Before we do this, let us analyze in detail [34, Example 1, p. 141].

Example 3.1. *We consider the Hilbert space*

$$H = L^2(0, 1; \mathbb{C})$$

with the standard inner product and norm and a linear operator A_0 defined by

$$D(A_0) = C_0^\infty((0, 1); \mathbb{C})$$

$$A_0 u = iu' = iDu, \text{ if } u \in D(A_0).$$

It is known that $C_0^\infty(0, 1)$ is dense in $L^2(0, 1)$, see Proposition 2.48 and [34, Ex. 10, p. 194]; for another approach, one may apply Lusin's theorem [38, Theorem 2.24]). Hence A_0 is densely defined.

To check symmetry, let $u, v \in C_0^\infty(0, 1)$. Then

$$(A_0 u, v) - (u, A_0 v) = \int_0^1 iu'(x) \overline{v(x)} dx - \int_0^1 u(x) \overline{iv'(x)} dx.$$

By Integration by parts formula from [37, Theorem 6.22],

$$\int_0^1 iu'(x) \overline{v(x)} dx = i[u(x)\overline{v(x)}]_0^1 - \int_0^1 u(x) \overline{iv'(x)} dx,$$

so

$$(A_0 u, v) - (u, A_0 v) = i[u(x)\overline{v(x)}]_0^1.$$

Since $u, v \in C_0^\infty(0, 1)$ vanish at 0 and 1, the boundary term is zero, hence

$$(A_0 u, v) - (u, A_0 v) = 0.$$

Thus A_0 is symmetric and densely defined.

Remark 3.2. Since A_0 is densely defined and symmetric, Proposition 2.89 and Theorem 2.91 implies that A_0 is closable, $\overline{A_0}$ is symmetric, and

$$(\overline{A_0})^* = A_0^*.$$

Consequently,

$$n_\pm(\overline{A_0}) = n_\pm(A_0).$$

We claim that

$$D(A_0^*) = \{u : (0, 1) \rightarrow \mathbb{C} : u' \text{ exist,}$$

$$\text{and } \int_0^1 |u'(x)|^2 dx < \infty \text{ and } \int_0^1 |u(x)|^2 dx < \infty\}$$

$$= \{u : [0, 1] \rightarrow \mathbb{C} : u \text{ is absolutely continues and } \int_0^1 |u'(x)|^2 dx < \infty\}$$

Moreover

$$A_0^*u = iu', \text{ if } u \in D(A_0^*).$$

Proposition 3.3. *If $S^* = S : S \supset A_0$, then $\exists A^\alpha : |\alpha| = 1$ and $S = A^\alpha$.*

Proof of Proposition 3.3. Suppose that S is a self-adjoint operator, i.e. $S^* = S$, such that $S \supset A_0$. Then, one can show that

$$D(S^*) = D(A_0^*)$$

and

$$S^*u = iu', \text{ if } u \in D(S^*).$$

Indeed, since $A_0 \subset S$, we know from general operator theory that $S^* \subset A_0^*$, see [35, Definition p.252, VIII.1]). But S is self-adjoint, i.e. $S = S^*$, then $S \subset A_0^*$, so we obtain

$$D(S) \subset D(A_0^*), \quad Su = iu', \quad u \in D(S).$$

On the other hand, because S is a maximal symmetric operator, its adjoint cannot be smaller than A_0^* , hence $A_0^* \subset S^*$. Therefore

$$D(S^*) = D(A_0^*), \quad S^*u = iu', \quad u \in D(S^*).$$

Take two arbitrary elements $\phi \in D(S)$ and $\psi \in D(S)$. Then by the Integration by parts formula from [37, Theorem 6.22] for absolutely continuous functions, see [38], we have the following equalities

$$\begin{aligned} (S\phi, \psi)_H - (\phi, S\psi)_H &= \int_0^1 i\phi'(x)\overline{\psi'(x)} dx - \int_0^1 \phi(x)\overline{i\psi'(x)} dx \\ &= i\left[\int_0^1 \phi'(x)\overline{\psi(x)} dx + \int_0^1 \phi(x)\overline{\psi'(x)} dx\right] \\ &= i\left[-\int_0^1 \phi(x)\overline{\psi'(x)} dx + \phi(1)\overline{\psi(1)} \right. \\ &\quad \left. - \phi(0)\overline{\psi(0)} + \int_0^1 \phi(x)\overline{\psi'(x)} dx\right] \\ &= i[\phi(1)\overline{\psi(1)} - \phi(0)\overline{\psi(0)}]. \end{aligned} \tag{3.1.1}$$

Since S is self-adjoint and hence symmetric, we get

$$(S\phi, \phi)_H - (\phi, S\phi)_H = 0. \tag{3.1.2}$$

Let now assume further $\phi \in D(S) \setminus D(A_0)$. Hence by (3.1.1) and (3.1.2) we infer that

$$i[|\phi(1)|^2 - |\phi(0)|^2] = 0$$

This implies that $|\phi(1)|^2 = |\phi(0)|^2$ and therefore we deduce that

$$|\phi(1)| = |\phi(0)|. \quad (3.1.3)$$

Next we claim that

$$\phi(0) \neq 0. \quad (3.1.4)$$

Indeed, if $\phi(0) = 0$, then by (3.1.3) we would get $\phi(1) = 0$ what would imply that $\phi \in D(A_0)$. This is impossible.

It follows from (3.1.4) that there exists $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and

$$\phi(1) = \alpha\phi(0). \quad (3.1.5)$$

It is enough to put $\alpha = \frac{\phi(1)}{\phi(0)} \in \mathbb{C}$. Then indeed, $|\alpha| = \frac{|\phi(1)|}{|\phi(0)|} = 1$ and equality (3.1.5) holds.

By the same argument, if $\psi \in D(S) \setminus D(A_0)$, then there exists β such that $\psi(1) = \beta\psi(0)$. We claim that

$$\beta = \alpha. \quad (3.1.6)$$

Indeed, by apply identity (3.1.1) to ϕ, ψ we deduce

$$\begin{aligned} 0 &= \phi(1)\overline{\psi(1)} - \phi(0)\overline{\psi(0)} \\ &= \alpha\phi(0)\overline{\beta\psi(0)} - \phi(0)\overline{\psi(0)} \\ &= (\alpha\overline{\beta} - 1)\phi(0)\overline{\psi(0)} \end{aligned}$$

Therefore, we infer that $\alpha\overline{\beta} = 1$ and hence

$$\overline{\beta} = \frac{1}{\alpha} = \frac{\overline{\alpha}}{\alpha\overline{\alpha}} = \frac{\overline{\alpha}}{|\alpha|^2} = \overline{\alpha}$$

what obviously implies (3.1.6). Summing up, we proved

$$\exists \alpha \in \mathbb{C} : \alpha = 1 : \forall \phi \in D(S), \phi(1) = \alpha\phi(0).$$

Hence, we proved the following proposition which is preceded by a relevant definition.

Definition 3.4. In the framework of Proposition 3.8, if $\alpha \in \mathbb{T}$, then we put

$$\begin{aligned} D(A^\alpha) &= \{\phi \in D(A_0^*) : \phi(1) = \alpha\phi(0)\}, \\ A^\alpha\phi &= A_0^*\phi, \quad \text{if } \phi \in D(A^\alpha). \end{aligned} \tag{3.1.7}$$

By Proposition 3.3 together with Definition 3.4, every self-adjoint extension S of A_0 is of the form A^α for some $\alpha \in \mathbb{T}$. $\square$

Therefore, we immediately obtain the following theorem.

Theorem 3.5. *The set of all self-adjoint extensions of A_0 is*

$$\{A^\alpha : \alpha \in \mathbb{T}\}.$$

3.1.2 An alternative proof of Theorem 3.5

In the above we gave a "bare-hand" proof of Theorem 3.5. We now give a second proof of Theorem 3.5 using Theorem 2.60, i.e. von Neumann theory.

The next step based on [35, Example p. 257].

To calculate the deficiency indices for the Example 3.1, as follows

$$\begin{aligned} D(A_0) &= C_0^\infty((0, 1); \mathbb{C}) \\ A_0u &= iu' = iDu, \quad \text{if } u \in D(A_0). \end{aligned}$$

It is known that A_0 is symmetric densely defined by

$$\begin{aligned} D(A_0^*) &= \{u : [0, 1] \rightarrow \mathbb{C} : u \text{ is absolutely continuous and} \\ &\int_0^1 |Du(x)|^2 dx < \infty \text{ and } A_0^*u = iDu\}. \end{aligned}$$

Note that if $u : [0, 1] \rightarrow \mathbb{C}$ is absolutely continuous then u is continuous and since $[0, 1]$ is compact, u is a bounded function, i.e.

$$\exists M > 0 : |u(x)| \leq M, \quad \forall x \in [0, 1].$$

Hence

$$\int_0^1 |u(x)|^2 dx \leq M^2, \infty.$$

We take $A_0^*u = iDu$, that is, means

$$A_0^*e^x = ie^x$$

then

$$(A_0^* - iI)e^x = 0.$$

And

$$A_0^*e^{-x} = ie^{-x}$$

then

$$(A_0^* + iI)e^{-x} = 0.$$

Hence

$$\begin{aligned} e^x &\in \ker(A_0^* - iI) \\ e^{-x} &\in \ker(A_0^* + iI). \end{aligned}$$

One can show that

$$\begin{aligned} H_+ &= \ker(A_0^* - iI) = \mathbb{C} \cdot e^x \\ H_- &= \ker(A_0^* + iI) = \mathbb{C} \cdot e^{-x}. \end{aligned} \tag{3.1.8}$$

Hence the deficiency indices are

$$n_+(A_0) = 1, \quad n_-(A_0) = 1.$$

We need to find all unitary operators from H_+ to H_- . By the von Neumann theory, see Theorem 2.60, the self-adjoint extensions are parametrized by unitary operators $\Gamma : H_+ \rightarrow H_-$. Choosing orthonormal bases

$$\phi_+(x) = \frac{\sqrt{2}}{\sqrt{e^2 - 1}}e^x, \quad \phi_-(x) = \frac{\sqrt{2}e}{\sqrt{e^2 - 1}}e^{-x}.$$

Since H_+ and H_- are one-dimensional, Example 2.79 shows that every unitary map $\Gamma : H_+ \rightarrow H_-$ is determined by a parameter $\gamma \in \mathbb{T}$.

Remark 3.6. Note that the function $\{(0, 1) \ni x \mapsto e^x \in L^2(0, 1)\}$, because

$$\int_0^1 e^{2x} dx = \frac{1}{2}[e^{2x}|_0^1] = \frac{1}{2}(e^2 - 1) < \infty.$$

However, its norm square is $\neq 1$. Similarly,

$$\|e^{-x}\|_{L^2}^2 = \int_0^1 e^{-2x} dx = \frac{-1}{2}[e^{-2x}|_0^1] = \frac{1}{2}(1 - e^{-2}) \neq 1.$$

So we normalize these two functions, i.e. define functions ϕ_+ and ϕ_- by

$$\begin{aligned} \phi_+(x) &= \frac{\sqrt{2}}{\sqrt{e^2 - 1}}e^x, \quad x \in (0, 1), \\ \phi_-(x) &= \frac{\sqrt{2}}{\sqrt{1 - e^{-2}}}e^{-x} = \frac{\sqrt{2}e}{\sqrt{e^2 - 1}}e^{-x}, \quad x \in (0, 1), \end{aligned} \tag{3.1.9}$$

Indeed, we have

$$\|\phi_+\|_{L^2} = 1, \quad \|\phi_-\|_{L^2} = 1.$$

Hence, by (3.1.8), ϕ_+ is an orthogonal basis of H_+ and ϕ_- is a orthogonal basis of H_- . Therefore, by Example 2.79, if $\Gamma : H_+ \rightarrow H_-$ is a unitary operator, then $\gamma \in \mathbb{T}$ such that

$$\Gamma : H_+ \ni s\phi_+ \mapsto \gamma s\phi_- \in H_-, \quad s \in \mathbb{C}. \quad (3.1.10)$$

Next, $\Gamma : H_+ \rightarrow H_-$ is unitary operator and $\gamma \in \mathbb{T}$ is the the corresponding number, then we define a linear operator A_Γ in the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$ by the following formula

$$\begin{aligned} D(A_\Gamma) &:= \{u = u_0 + u_+ + \Gamma(u_+) : u_0 \in D(A_0), u_+ \in H_+\} \\ A_\Gamma(u) &= A_0 u_0 + s\phi_+ - \gamma s\phi_-, \quad u \in D(A_\Gamma). \end{aligned} \quad (3.1.11)$$

Note that in view of (3.1.10), we have

$$D(A_\Gamma) = \{u = u_0 + s\phi_+ + \gamma s\phi_- : u_0 \in D(A_0), s \in \mathbb{C}\}.$$

Let us point out that according to Theorem 2.60, we have the following result.

Theorem 3.7. *In the above framework, the family of all self-adjoint extensions of A_0 is equal to the family $\{A_\Gamma : \Gamma \in U(H_+, H_-)\}$.*

We continue with the previous Remark.

Proposition 3.8. *If $\Gamma : H_+ \rightarrow H_-$ is unitary operator, $\gamma \in \mathbb{T}$ is the corresponding number and A_Γ is the linear operator in the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$ defined by formula (3.1.11), then*

$$\boxed{D(A_\Gamma) = \{\psi \in D(A_0^*) : \psi(1) = \alpha\psi(0)\}} \quad (3.1.12)$$

where

$$\alpha = \frac{\gamma + e}{1 + \gamma e} \quad (3.1.13)$$

Note that $\alpha \in \mathbb{T}$, and

$$\phi \in D(A_0), \quad \psi \in D(A_\Gamma),$$

so we change the symbol to indicate that ψ is any function in the extended domain.

Therefore, Proposition 3.8 can be equivalently written in the following way.

Proposition 3.9. *If $\Gamma : H_+ \rightarrow H_-$ is unitary operator, $\gamma \in \mathbb{T}$ is the corresponding number, and $\alpha \in \mathbb{T}$ is given by (3.1.13), and A_Γ is the linear operator in the Hilbert space*

$L^2(\mathbb{R}, \mathbb{C})$ defined by formula (3.1.11) and A^α is defined by formula (3.1.7), then

$$\begin{aligned} D(A_\Gamma) &= D(A^\alpha), \\ A_\Gamma \phi &= A^\alpha \phi, \text{ for } \phi \in D(A^\alpha). \end{aligned} \tag{3.1.14}$$

The following result is a consequence of the above Proposition. But before we do so let us write down the following exercise.

Exercise 3.10. The map

$$\mathbb{T} \ni \gamma \mapsto \alpha = \frac{\gamma + e}{1 + \gamma e} \in \mathbb{T} \tag{3.1.15}$$

is a well defined bijection.

Solution. To prove that the map (3.1.15) is well defined we need to show the following result.

Lemma 3.11. *If $\gamma \in \mathbb{T}$ and α is defined by formula (3.1.15), then $\alpha \in \mathbb{T}$.*

Proof of Lemma 3.11. Let us choose and fix $\gamma \in \mathbb{T}$. We will prove that $|\alpha|^2 = 1$. We have

$$\begin{aligned} |\alpha|^2 &= \alpha \bar{\alpha} = \frac{\gamma + e}{1 + \gamma e} \overline{\left(\frac{\gamma + e}{1 + \gamma e} \right)} = \frac{\gamma + e}{1 + \gamma e} \frac{\bar{\gamma} + \bar{e}}{\overline{1 + \gamma e}} = \frac{\gamma + e}{1 + \gamma e} \frac{\bar{\gamma} + \bar{e}}{1 + \bar{\gamma} \bar{e}} \\ &= \frac{\gamma + e}{1 + \gamma e} \frac{\bar{\gamma} + e}{1 + \bar{\gamma} e} = \frac{(\gamma + e)(\bar{\gamma} + e)}{(1 + \gamma e)(1 + \bar{\gamma} e)} \\ &= \frac{\gamma \bar{\gamma} + \gamma e + \bar{\gamma} e + e^2}{1 + \gamma e + \bar{\gamma} e + e^2} = \frac{1 + (\gamma + \bar{\gamma})e + e^2}{1 + (\gamma + \bar{\gamma})e + e^2} = 1, \end{aligned}$$

because $\gamma \in \mathbb{T}$, so that $\gamma \bar{\gamma} = 1$. □

To prove that the map (3.1.15) is an injection we need to show that if $\gamma_1, \gamma_2 \in \mathbb{T}$ are such $\alpha_1 = \alpha_2$ then $\gamma_1 = \gamma_2$.

Suppose that $\gamma_1, \gamma_2 \in \mathbb{T}$ are such that

$$\alpha(\gamma_1) = \alpha(\gamma_2), \quad \text{where} \quad \alpha(\gamma) = \frac{\gamma + e}{1 + \gamma e}.$$

Thus we infer that

$$\frac{\gamma_1 + e}{1 + e\gamma_1} = \frac{\gamma_2 + e}{1 + e\gamma_2}.$$

Cross-multiplying and then simplifying gives

$$\begin{aligned}(\gamma_1 + e)(1 + e\gamma_2) &= (\gamma_2 + e)(1 + e\gamma_1). \\ \gamma_1 + e\gamma_1\gamma_2 + e + e^2\gamma_2 &= \gamma_2 + e\gamma_2\gamma_1 + e + e^2\gamma_1 \\ \gamma_1 + e^2\gamma_2 &= \gamma_2 + e^2\gamma_1 \\ (1 - e^2)(\gamma_1 - \gamma_2) &= 0.\end{aligned}$$

Since $e \neq \pm 1$, we have $1 - e^2 \neq 0$, we infer that

$$\gamma_1 = \gamma_2.$$

Hence the map $\gamma \mapsto \alpha$ is injective.

To prove that the map (3.1.15) is a surjection we need to show that if $\alpha \in \mathbb{T}$ then we can find $\gamma \in \mathbb{T}$ such that

$$\alpha = \boxed{\frac{\gamma + e}{1 + \gamma e}}. \quad (3.1.16)$$

Let $\alpha \in \mathbb{T}$ be arbitrary. Suppose γ satisfies the above equation (3.1.16). By multiply both sides by $1 + \gamma e$ and performing some basic transformations we deduce that

$$\gamma(\alpha e - 1) = e - \alpha.$$

Since $|\alpha| = 1$, we infer that $|\alpha e| = 1 \times e = e > 1$. Hence $\alpha e \neq 1$ and therefore

$$\gamma = \frac{e - \alpha}{\alpha e - 1} = \boxed{\frac{-\alpha + e}{1 + (-\alpha)e}}.$$

Since obviously $-\alpha \in \mathbb{T}$, by Lemma 3.11 we infer that $\gamma \in \mathbb{T}$, i.e. $|\gamma| = 1$.

Therefore, we proved that the map (3.1.15) is surjective. $\square$

The following result, i.e. Proposition 3.12 now follows immediately from Proposition 3.8 and Exercise 3.10, since for each $\alpha \in \mathbb{T}$ there exists a unique $\gamma \in \mathbb{T}$ such that $\alpha = \frac{\gamma + e}{1 + \gamma e}$. It can be seen as the converse to Proposition 3.9.

Proposition 3.12. *Assume $\alpha \in \mathbb{T}$ and let A^α be defined by formula (3.1.7). If $\gamma \in \mathbb{T}$ is related to α by equation (3.1.13) (and such a number γ exists because of Exercise 3.10) and $\Gamma : H_+ \rightarrow H_-$ is the corresponding unitary operator, and A_Γ is the linear operator in the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$ defined by (3.1.11), then the identity (3.1.14) holds, i.e. $A^\alpha = A_\Gamma$.*

Proof of Proposition 3.12. Fix $\alpha \in \mathbb{T}$ and choose $\gamma \in \mathbb{T}$ with $\alpha = \frac{\gamma + e}{1 + \gamma e}$, see Exercise 3.10, and let $\Gamma : H_+ \rightarrow H_-$ be the corresponding unitary. By Proposition 3.8,

$$D(A_\Gamma) = \{\psi \in D(A_0^*) : \psi(1) = \alpha\psi(0)\} = D(A^\alpha).$$

On this common domain both operators act as A_0^* , hence $A^\alpha = A_\Gamma$. $\square$

The following Corollary follows from Propositions 3.9 and 3.12, Exercise 3.10 and Theorem 2.60.

Corollary 3.13. *Let $\text{SA}(A_0)$ denote the set of all self-adjoint extensions of the operator A_0 . The family or the set of all self-adjoint extensions of operator A_0 is equal the family or the set*

$$\{A^\alpha : \alpha \in \mathbb{T}\}, \quad (3.1.17)$$

where operator A^α is defined in formula (3.1.7).

Let us recall that sets X and Y are equal if and only if the following assertions hold.

- (i) If $x \in X$ then $x \in Y$.
- (ii) If $y \in Y$ then $y \in X$.

Proof of Corollary 3.13. we need to prove the following assertions hold.

- (i) If A is a self-adjoint extension of operator A_0 , then there exists $\alpha \in \mathbb{T}$ such that $A = A^\alpha$.
- (ii) If $\alpha \in \mathbb{T}$ then A^α is self-adjoint extension of operator A_0 .

Proof of (i). If A is a self-adjoint extension of operator A_0 , by Theorem 2.60, there exists $\Gamma \in U(H_+, H_-)$ such that $A = A_\Gamma$. Let next $\gamma \in \mathbb{T}$ is the number as in Proposition 3.8. Then, by Proposition 3.9 we infer $A_\Gamma = A^\alpha$, where $\alpha = \frac{\gamma + e}{1 + \gamma e} \in \mathbb{T}$. Hence $A = A^\alpha$.

Proof of (ii). By Exercise 3.10, there exists $\gamma \in \mathbb{T}$ with $\alpha = \frac{\gamma + e}{1 + \gamma e}$. Let $\Gamma : H_+ \rightarrow H_-$ be the unitary corresponding to γ . Proposition 3.12 yields that $A^\alpha = A_\Gamma$. Since A_Γ is a self-adjoint extension of A_0 , see Theorem 2.60, so is A^α . Both (i) and (ii) hold. Therefore the set of all self-adjoint extensions of A_0 is precisely the set (3.1.17). $\square$

Proof of Proposition 3.8. To prove formula (3.1.12), we proceed in two steps.

Step 1: We will show that

$$D(A_\Gamma) \subset \{\psi \in D(A_0^*) : \psi(1) = \alpha\psi(0)\}. \quad (3.1.18)$$

Let us choose and fix a unitary operator $\Gamma : H_+ \rightarrow H_-$, the corresponding number $\gamma \in \mathbb{T}$. and the linear operator A_Γ in the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$ defined by formula (3.1.11). Note that if $u \in D(A_\Gamma)$, then

$$u = \phi + s\phi_+ + \gamma s\phi_-,$$

where $\phi \in D(A_0)$, $s \in \mathbb{C}$ and ϕ_+, ϕ_- as in (3.1.9), Since $\phi(0) = \phi(1) = 0$, we compute

$$\begin{aligned} u(0) &= \phi(0) + s\phi_+(0) + \gamma s\phi_-(0) \\ &= 0 + s \frac{\sqrt{2}}{\sqrt{e^2 - 1}} + s\gamma \frac{\sqrt{2}e}{\sqrt{e^2 - 1}} = \frac{\sqrt{2}(1 + \gamma e)}{\sqrt{e^2 - 1}} s. \end{aligned}$$

Similarly,

$$\begin{aligned} u(1) &= \phi(1) + s\phi_+(1) + \gamma s\phi_-(1) \\ &= 0 + s \frac{\sqrt{2}}{\sqrt{e^2 - 1}} e + s\gamma \frac{\sqrt{2}}{\sqrt{e^2 - 1}} = \frac{\sqrt{2}(\gamma + e)}{\sqrt{e^2 - 1}} s. \end{aligned}$$

Since $1 + \gamma e \neq 0$ (because $|\gamma| = 1$ and $e > 0$), division gives

$$\frac{u(1)}{u(0)} = \frac{\gamma + e}{1 + \gamma e} =: \alpha.$$

Thus

$$u(1) = \alpha u(0), \quad \alpha = \frac{\gamma + e}{1 + \gamma e}.$$

Moreover,

$$|\alpha|^2 = \frac{\gamma + e}{1 + \gamma e} \cdot \frac{\bar{\gamma} + e}{1 + \bar{\gamma}e} = \frac{1 + e(\gamma + \bar{\gamma}) + e^2}{1 + e(\gamma + \bar{\gamma}) + e^2} = 1,$$

so $\alpha \in \mathbb{T}$.

In other words we proved

$$D(A_\Gamma) \subset \{\psi \in D(A_0^*) : \psi(1) = \alpha \psi(0)\}.$$

where $\alpha = \frac{\gamma + e}{1 + \gamma e}$ such that $|\alpha| = 1$. Therefore (3.1.18) holds.

Step 2: Reverse inclusion. We will show that

$$\psi = \phi + s\phi_+ + \gamma s\phi_- \in D(A_\Gamma)$$

Assume that $\psi \in D(A_0^*)$ satisfies $\psi(1) = \alpha \psi(0)$, where α and γ are related by (3.1.13).

Define

$$s := \frac{\sqrt{e^2 - 1}}{\sqrt{2}} \frac{\psi(0)}{1 + \gamma e}, \quad \phi := \psi - s\phi_+ - \gamma s\phi_-.$$

Then, using the same boundary values as above, we find

$$\phi(0) = \psi(0) - \frac{\sqrt{2}}{\sqrt{e^2-1}}(1 + \gamma e) s = 0, \quad \phi(1) = \psi(1) - \frac{\sqrt{2}}{\sqrt{e^2-1}}(\gamma + e) s = 0,$$

where the second equality uses $\psi(1) = \alpha\psi(0)$ and $\alpha = (\gamma + e)/(1 + \gamma e)$. Thus $\phi \in D(A_0)$ and therefore

$$\psi = \phi + s\phi_+ + \gamma s\phi_- \in D(A_\Gamma).$$

This proves the reverse inclusion.

From step 1 and step 2, with $\alpha = \frac{\gamma + e}{1 + \gamma e} \in \mathbb{T}$, we conclude that

$$\boxed{D(A_\Gamma) = \{\psi \in D(A_0^*) : \psi(1) = \alpha\psi(0)\}}.$$

□

Remark 3.14. In the proof of Proposition 3.8, the coefficient s in the decomposition

$$\psi = \phi + s\phi_+ + \gamma s\phi_-$$

is chosen so that the function $\phi \in D(A_0)$, i.e. so that ϕ satisfies the boundary conditions

$$\phi(0) = \phi(1) = 0.$$

Since

$$\phi(0) = \psi(0) - s\phi_+(0) - \gamma s\phi_-(0),$$

and

$$\phi_+(0) = \frac{\sqrt{2}}{\sqrt{e^2-1}}, \quad \phi_-(0) = \frac{\sqrt{2}e}{\sqrt{e^2-1}},$$

we obtain

$$\phi(0) = \psi(0) - s \frac{\sqrt{2}}{\sqrt{e^2-1}} (1 + \gamma e).$$

Therefore, imposing $\phi(0) = 0$ determines the value of s uniquely as

$$s = \frac{\sqrt{e^2-1}}{\sqrt{2}} \frac{\psi(0)}{1 + \gamma e}.$$

With this choice, one verifies using $\psi(1) = \alpha\psi(0)$ and $\alpha = \frac{\gamma + e}{1 + \gamma e}$ that also $\phi(1) = 0$, and thus $\phi \in D(A_0)$. In other words, the choice of s ensures precisely that

$$\psi = \phi + s\phi_+ + \gamma s\phi_- \in D(A_\Gamma).$$

3.2 First order differential operator on the punctured line

3.2.1 Symmetry and the adjoint

In this section we consider the Hilbert space $H = L^2(\mathbb{R}, \mathbb{C})$ and we define an operator A_1 as follows

$$\begin{aligned} D(A_1) &= C_0^\infty((-\infty, 0) \cup (0, \infty); \mathbb{C}) = C_0^\infty(\mathbb{R} \setminus \{0\}; \mathbb{C}) \\ A_1 u &= iu' = iDu \text{ if } u \in D(A_1). \end{aligned} \tag{3.2.1}$$

Note that

$$\begin{aligned} u \in D(A_1) &\iff u : \mathbb{R} \rightarrow \mathbb{C} \text{ of } C^\infty \text{ class and} \\ &\exists 0 < a < b : \text{supp } u \subset [-b, -a] \cup [a, b]. \end{aligned}$$

Hence, if $u \in D(A_1)$ then $\text{supp } u$ is compact and there exist numbers $0 < a < b$:

$$u(t) = 0 \text{ if } t \geq b \text{ or } t \leq -b \text{ or } t \in [-a, a].$$

Remark 3.15. Although

$$D(A_1) = C_0^\infty(\mathbb{R} \setminus \{0\}) \subsetneq C_0^\infty(\mathbb{R}),$$

it follows from Proposition 2.48 that $D(A_1)$ is dense in $L^2(\mathbb{R})$.

Moreover, A_1 is a linear operator. Indeed, $C_0^\infty(\mathbb{R} \setminus \{0\})$ is a vector space, and for $u, v \in D(A_1)$ and $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned} A_1(\alpha u + \beta v) &= i(\alpha u + \beta v)' = i(\alpha u' + \beta v') \\ &= \alpha(iu') + \beta(iv') = \alpha A_1 u + \beta A_1 v. \end{aligned}$$

We begin with the following result.

Proposition 3.16. *The operator A_1 defined by (3.2.1) is symmetric, i.e.*

$$\langle A_1 u, v \rangle_{L^2} = \langle u, A_1 v \rangle_{L^2}; \quad \forall u, v \in D(A_1)$$

and densely defined, i.e. $D(A_1)$ is a dense subspace of the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$.

The next result follows by applying Proposition 2.89.

Proposition 3.17. *The operator A_1 is closable.*

Proof of Proposition 3.16. We need to prove that A_1 is symmetric. In what follows we consider the case when $a = -\infty$ and $b = \infty$. The proof in a general case is similar.

Let us choose and fix $u, v \in C_0^\infty(\mathbb{R})$. Thus

$$\begin{aligned}\langle A_1 u, v \rangle &= \langle i u', v \rangle = \int_{-\infty}^{\infty} i \frac{d}{dx} u(x) \overline{v(x)} dx = -i \int_{-\infty}^{\infty} u(x) \overline{v'(x)} dx \\ &= \int_{-\infty}^{\infty} u(x) \overline{i v'(x)} dx = \langle u, A_1 v \rangle.\end{aligned}$$

Hence A_1 is symmetric. □

3.2.2 Deficiency indices of the first order differential operator on the punctured line

The main result of this section is the following one.

Theorem 3.18. *Assume that A_1 is the operator defined by formula (3.2.1). Then*

$$n_+(A_1) = 1$$

$$n_-(A_1) = 1$$

Moreover, the space $\ker(iI - A_1^*)$ is generated by function e_1 and the space $\ker(iI + A_1^*)$ is generated by function e_2 , where

$$e_1(x) = \begin{cases} 0 & \text{if } x > 0 \\ ce^x & \text{if } x < 0 \end{cases}$$

and

$$e_2(x) = \begin{cases} ce^{-x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \quad (3.2.2)$$

The following result is a consequences of Theorem 3.18. In fact, we investigated the previous result in order to prove or disprove the next corollary.

Corollary 3.19. *The operator A_1 on $L^2(\mathbb{R} \setminus \{0\}, \mathbb{C})$ defined by (3.2.1) has at least one self-adjoint extension.*

Proof of Corollary 3.19. According to Theorem 2.60, for a closed symmetric operator A to have a self-adjoint extension, it is necessary that the deficiency indices satisfy $n_+ = n_-$.

Since we have $n_+(A_1) = 1$ and $n_-(A_1) = 1$ from Theorem 3.18, it follows that

$$n_+ = n_-.$$

Based on the result of Theorem 2.60, the operator A_1 has at least one self-adjoint extension.

Thus, we conclude that the operator A_1 defined by equation (3.2.1) has at least one self-adjoint extension. This completes the proof of the corollary.

□

In the remaining of this section we will occupy ourselves proving the theorem above. For this aim we will calculate the deficiency indices $n_+(A_1)$ and $n_-(A_1)$.

In order to calculate the deficiency indices of A_1 , we first need to find the domain of the adjoint operator A_1^* .

Proposition 3.20. *The following two conditions are equivalent.*

$$(i) \ u \in D(A_1^*)$$

(ii) $u \in L^2(\mathbb{R}, \mathbb{C})$ and the restriction of u to $(0, \infty)$, i.e. $u : (0, \infty) \rightarrow \mathbb{C}$ is locally absolutely continuous and

$$\int_0^\infty |Du(t)|^2 dt < \infty,$$

and the restriction of u to $(-\infty, 0)$, i.e. $u : (-\infty, 0) \rightarrow \mathbb{C}$ is locally absolutely continuous and

$$\int_{-\infty}^0 |Du(t)|^2 dt < \infty.$$

Moreover,

$$A_1^*u = iDu, \text{ if } u \in D(A_1^*).$$

It follows that the domain of the adjoint operator can be characterized in a more detailed way. For this purpose we will need the following result that is a consequence of the Sobolev embedding Theorem.

Proposition 3.21. *If $u \in D(A_1^*)$, then the following limits exist:*

$$\begin{aligned} \lim_{x \rightarrow 0+} u(x) &=: u(0+) \\ \lim_{x \rightarrow 0-} u(x) &=: u(0-) \end{aligned}$$

Remark 3.22. Note that the limits above are usually different. Hence a function $u \in D(A_1^*)$ doesn't need to be continuous.

Let us observe that by Proposition 3.20 we have the following result.

Corollary 3.23. *The following two conditions are equivalent.*

$$(i) \ u \in \ker(iI - A_1^*)$$

(ii) the restrictions of function u to the intervals $(0, \infty)$ and $(-\infty, 0)$ are square integrable and locally absolutely continuous and

$$\int_0^\infty |Du(x)|^2 dx < \infty, \quad \int_{-\infty}^0 |Du(x)|^2 dx < \infty \quad (3.2.3)$$

and

$$Du = u \text{ on } (0, \infty) \text{ and } Du = u \text{ on } (-\infty, 0).$$

Proof of Theorem 3.18. Part I. Let choose and fix $u \in \ker(iI - A_1^*)$. Then by Corollary 3.23, u satisfies the following conditions

$$Du = u \quad \text{on } (0, \infty), \quad (3.2.4)$$

$$Du = u \quad \text{on } (-\infty, 0). \quad (3.2.5)$$

The general solution of (3.2.4) is

$$u(x) = c_+ e^x, \quad x > 0,$$

and the general solution of (3.2.5) is

$$u(x) = c_- e^x, \quad x < 0,$$

where $c_+, c_- \in \mathbb{C}$.

Claim 3.24. We have $e^x \notin L^2(0, \infty)$ and $e^x \in L^2(-\infty, 0)$.

Proof. We compute

$$\int_0^\infty |e^x|^2 dx = \int_0^\infty e^{2x} dx = \infty,$$

hence $e^x \notin L^2(0, \infty)$.

On the other hand,

$$\int_{-\infty}^0 |e^x|^2 dx = \int_{-\infty}^0 e^{2x} dx = \frac{1}{2} < \infty,$$

so $e^x \in L^2(-\infty, 0)$. □

By Claim 3.24, square integrability implies $c_+ = 0$. Hence

$$u(x) = \begin{cases} 0, & x > 0, \\ c_- e^x, & x < 0. \end{cases}$$

Define

$$e_1(x) = \begin{cases} 0, & x > 0, \\ e^x, & x < 0. \end{cases}$$

Thus

$$u = c_- e_1.$$

Conversely, it is straightforward to check that $e_1 \in \ker(iI - A_1^*)$.

Therefore, $\ker(iI - A_1^*) = \text{span}\{e_1\}$ and hence

$$n_+(A_1) = 1.$$

□

Proof of Theorem 3.18. Part II. Let us choose and fix $u \in \ker(iI + A_1^*)$. Then by Corollary 3.23, u satisfies the following conditions

$$\begin{aligned} Du &= -u & \text{on } (0, \infty), \\ Du &= -u & \text{on } (-\infty, 0). \end{aligned}$$

The general solutions are

$$u(x) = c_+ e^{-x}, \quad x > 0, \quad u(x) = c_- e^{-x}, \quad x < 0.$$

Claim 3.25. We have $e^{-x} \in L^2(0, \infty)$ and $e^{-x} \notin L^2(-\infty, 0)$.

Proof. We compute

$$\int_0^\infty |e^{-x}|^2 dx = \int_0^\infty e^{-2x} dx = \frac{1}{2} < \infty,$$

hence $e^{-x} \in L^2(0, \infty)$.

On the other hand,

$$\int_{-\infty}^0 |e^{-x}|^2 dx = \int_{-\infty}^0 e^{-2x} dx = \infty,$$

so $e^{-x} \notin L^2(-\infty, 0)$.

□

By Claim 3.25, square integrability implies $c_- = 0$. Hence

$$u(x) = \begin{cases} c_+ e^{-x}, & x > 0, \\ 0, & x < 0. \end{cases}$$

Define

$$e_2(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & x < 0. \end{cases}$$

Thus

$$u = c_+ e_2.$$

Conversely, function e_2 defined in (3.2.2) is square integrable and locally absolutely continuous on $(-\infty, 0)$ and $(0, \infty)$ and satisfies conditions (3.2.3). Hence, $e_2 \in \ker(iI + A_1^*)$. Therefore, $\ker(iI + A_1^*) = \text{span}\{e_2\}$ and hence

$$n_-(A_1) = 1.$$

□

3.2.3 Finding the adjoint of the closure of a linear operator

Let A_1 be the operator defined by formula (3.2.1). Since A_1 is densely defined and symmetric, by Proposition 2.89, it is closable. Hence the closure $\overline{A_1}$ exists.

We want to modify Proposition 3.20 about the adjoint of A_1 by finding the adjoint of $\overline{A_1}$.

Proposition 3.26. *Let A_1 be the operator defined by (3.2.1), and let $\overline{A_1}$ be its closure in $L^2(\mathbb{R}, \mathbb{C})$. Then the adjoint of $\overline{A_1}$ coincides with the adjoint of A_1 , i.e.,*

$$(\overline{A_1})^* = A_1^*.$$

A measurable function $u : \mathbb{R} \rightarrow \mathbb{C}$ belongs to $D(\overline{A_1})^*$ if and only if the following conditions hold:

1. $u \in L^2(\mathbb{R}, \mathbb{C})$,
2. The restriction $u : (0, \infty) \rightarrow \mathbb{C}$ is locally absolutely continuous and

$$\int_0^\infty |Du(t)|^2 dt < \infty,$$

3. The restriction $u : (-\infty, 0) \rightarrow \mathbb{C}$ is locally absolutely continuous and

$$\int_{-\infty}^0 |Du(t)|^2 dt < \infty.$$

Moreover, for such u , the adjoint acts as

$$(\overline{A_1})^* u = iDu.$$

Remark 3.27. By Corollary 2.92, the closure of a densely defined symmetric operator is again symmetric and satisfies

$$(\overline{A})^* = A^*.$$

Hence the deficiency subspaces, and consequently the deficiency indices, of A and $\overline{A}$ coincide.

The following result is an immediate consequence of Corollary 2.92 in the Preliminaries. Since the closure of a densely defined symmetric operator is again symmetric and

$$(\overline{A})^* = A^*,$$

the deficiency subspaces of A and $\overline{A}$ coincide. Hence their deficiency indices are equal.

Corollary 3.28. *Let A_1 be the operator defined by (3.2.1), and let $\overline{A_1}$ be its closure in $L^2(\mathbb{R}, \mathbb{C})$. Then both A_1 and $\overline{A_1}$ are symmetric and densely defined, and the deficiency indices of $\overline{A_1}$ coincide with those of A_1 . In particular, by Theorem 3.18 we have*

$$n_+(\overline{A_1}) = n_+(A_1) = 1, \quad n_-(\overline{A_1}) = n_-(A_1) = 1.$$

3.2.4 Results for the first order operator on the punctured line

A preliminary result of this section is the following one.

Proposition 3.29. *Assume that A_1 is the operator defined by formula (3.2.1). Then A_1 is not closed.*

Proof of Proposition 3.29. We will prove that the linear map A_1 is not closed. For this aim, by Proposition 2.83 it is sufficient to show that

- (i) there exists a function $u \in H \setminus D(A_1)$
- (ii) and a $D(A_1)$ -valued sequence (u_n) such that $u_n \rightarrow u$ in H
- (iii) and $A_1 u_n$ is convergent in H .

For the Aim (i) we consider a function $u : \mathbb{R} \rightarrow \mathbb{C}$ defined by

$$u(x) = \begin{cases} e^{-x^{-2}} e^{-x} = e^{-(x^{-2}+x)} = & \text{if } x \in (0, \infty) \\ 0, & \text{if } x \in (-\infty, 0]. \end{cases} \quad (3.2.6)$$

Let us state some useful properties of function u .

Lemma 3.30. *The function u defined by formula (3.2.6) satisfies the following properties.*

(i) *Function u is of C^1 -class and*

$$u'(x) = \begin{cases} e^{-(x^{-2}+x)}(2x^{-3} - 1) & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

(ii) *Function u belongs to $H^1(\mathbb{R})$.*

(iii) *Function u does not belong to $D(A_1)$.*

Proof. The proof of part (i) is standard and hence omitted. Part (ii) follows from the observation that since u is of C^1 -class it is measurable and showing that

$$\int_{\mathbb{R}} |u(x)|^2 dx < \infty$$

and

$$\int_{\mathbb{R}} |u'(x)|^2 dx < \infty.$$

Part (iii) follows since u has not compact support. □

From the above Lemma we deduce that function u satisfies Aim (i) above. **Now we will deal with Aim (ii).**

From Definition 2.35 and Remark 2.36-(i), we deduce that the following results.

Lemma 3.31.

✓ $C_0^\infty(\mathbb{R})$ is dense in $H^1(\mathbb{R})$.

✗ $C_0^\infty(\mathbb{R} \setminus \{0\})$ is not dense in $H^1(\mathbb{R})$.

Lemma 3.32. *Let u be the function defined in (3.2.6). Then there exists a sequence $(u_n)_{n=1}^\infty$ such that*

$$u_n \in D(A_1) \quad \text{and} \quad u_n \rightarrow u \quad \text{in } H^1(\mathbb{R}).$$

Before embarking on the proof of Lemma 3.32, let us conclude the proof that A_1 is not closed.

Lemma 3.33. *Assume that $u \in H^1(\mathbb{R})$. For $n \in \mathbb{N}$ define the translation*

$$v_n(x) := u\left(x - \frac{1}{n}\right).$$

Then $v_n \in H^1(\mathbb{R})$ with weak derivative

$$Dv_n(x) = Du\left(x - \frac{1}{n}\right) \in L^2(\mathbb{R}).$$

Proof. The proof can be found in [20, p. 301]. □

Thus each v_n lies in $H^1(\mathbb{R})$. Moreover, since the support of v_n is contained in $[1/n, \infty)$, the restriction $\tilde{v}_n$ to $(0, \infty)$ belongs to $H_0^1(0, \infty)$. By Proposition 2.14 and Example 2.21, for every $n \in \mathbb{N}$ there exists $\tilde{u}_n \in C_0^\infty(0, \infty)$ such that

$$\|\tilde{u}_n - \tilde{v}_n\|_{H^1(0, \infty)} \leq \|v_n - u\|_{H^1(\mathbb{R})}.$$

Extending $\tilde{u}_n$ by 0 on $(-\infty, 0]$ we obtain $u_n \in C_0^\infty(\mathbb{R} \setminus \{0\}) = D(A_1)$ and

$$\lim_{n \rightarrow \infty} \|u_n - u\|_{H^1(\mathbb{R})} = 0.$$

Moreover,

$$A_1 u_n = i D u_n,$$

so $(u_n, i D u_n) \in \Gamma(A_1)$ and $(u_n, i D u_n) \rightarrow (u, i D u)$ in $H \times H$. But $(u, i D u) \notin \Gamma(A_1)$ because $u \notin D(A_1)$.

This completes the proof of **Part 1** of Proposition 3.29.

Proof of Lemma 3.32. Define, for every $n \in \mathbb{N}$, a function

$$v_n : \mathbb{R} \ni x \mapsto u\left(x - \frac{1}{n}\right) \in \mathbb{C},$$

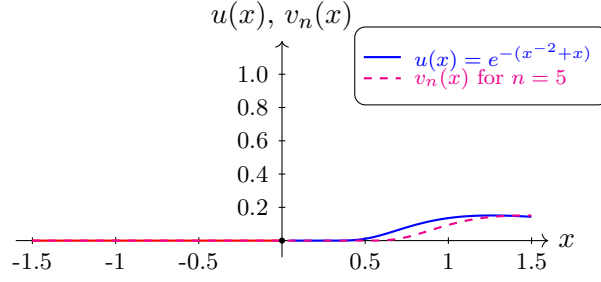
i.e. see Figure 3.1,

$$v_n(x) = \begin{cases} e^{-[(x-\frac{1}{n})^{-2} + (x-\frac{1}{n})]} & \text{if } x > \frac{1}{n}, \\ 0, & \text{if } x \leq \frac{1}{n}. \end{cases}$$

For $x \leq 1/n$ we clearly have $v_n(x) = 0$. For $x > 1/n$, we note that

$$v_n(x) = e^{-(x-\frac{1}{n})^{-2}} e^{1/n} e^{-x} \leq e e^{-x} = e^{1-x} = g(x).$$

FIGURE 3.1: Graphs of $u(x) = e^{-(x^{-2}+x)}$ and the approximating function $v_n(x)$ with $n = 5$.



Thus

$$0 \leq v_n(x) \leq g(x), \quad \text{for every } x \in \mathbb{R}, \quad (3.2.7)$$

Moreover, since u is continuous and $x - \frac{1}{n} \rightarrow x$, therefore by [37, Theorem 4.6, p.86], we have

$$v_n(x) \rightarrow u(x) \quad \text{for all } x \in \mathbb{R}. \quad (3.2.8)$$

by applying the Lebesgue dominated convergence Theorem, see [1, Thm 1.50, page 18], we infer that

$$\|v_n - u\|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} |v_n(x) - u(x)|^2 dx \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.2.9)$$

Let us show why the LDCT can be used. Firstly we observe that (3.2.7) and (3.2.8) we have

$$0 \leq u(x) \leq g(x), \quad \text{for every } x \in \mathbb{R},$$

Secondly, we observe that v_n is Lebesgue measurable, see [19] and that by the Change of Variables Theorem, see Theorems 6.19 and 10.9 in [37],

$$\int_{-\infty}^{\infty} |v_n(x)|^2 dx = \int_{-\infty}^{\infty} |u(x - \frac{1}{n})|^2 dx = \int_{-\infty}^{\infty} |u(y)|^2 dy < \infty.$$

Thirdly, by the triangle inequality we observe that

$$|v_n(x) - u(x)|^2 \leq 4g(x)^2 =: h(x), \quad x \in \mathbb{R}.$$

Fourthly, using properties of function g we can show that

$$\int_{\mathbb{R}} h(x) dx < \infty.$$

Thus we have shown that all assumption of the LDCT are satisfied and hence we deduce that (3.2.9) holds.

After having proved (3.2.9) we embark on proving the convergence in $H^1(\mathbb{R})$, i.e.

$$\lim_{n \rightarrow \infty} \|v_n - u\|_{H^1(\mathbb{R})} = 0. \quad (3.2.10)$$

It is sufficient to show that $Dv_n \rightarrow Du$ in $L^2(\mathbb{R})$. This can be proved in two different ways. We will use a shorter one. Since $Du \in L^2(\mathbb{R})$, by applying the previous Step to function Du we have

$$\|Dv_n - Du\|_{L^2(\mathbb{R})} = \|Du(\cdot - \frac{1}{n}) - Du(\cdot)\|_{L^2(\mathbb{R})} \xrightarrow{n \rightarrow \infty} 0.$$

Hence the proof of (3.2.10) is complete.

Thus we have shown that $v_n \rightarrow u$ in $H^1(\mathbb{R})$. But $v_n \notin D(A_1) = C_0^\infty(\mathbb{R} \setminus \{0\})$. To find a $D(A_1) = C_0^\infty(\mathbb{R} \setminus \{0\})$ -valued sequence, we need to modify v_n . For this aim we choose a function $\eta_n \in C_0^\infty(\mathbb{R})$ such that

$$\begin{aligned} 0 \leq \eta_n \leq 1, \quad \eta_n &\equiv 0 \text{ on } \{|x| \leq \frac{1}{2n}\}, \\ \eta_n &\equiv 1 \text{ on } \{\frac{1}{n} \leq |x| \leq n\}, \quad \text{supp} \eta_n \subset \{\frac{1}{2n} < |x| < 2n\}. \end{aligned}$$

Set $u_n := \eta_n v_n$. Then

$$u_n \in C_0^\infty(\mathbb{R} \setminus \{0\}) = D(A_1).$$

Using that multiplication by a bounded Lipchitz function is bounded on H^1 ([20, Prop. 9.4]) and that the supports where $\eta_n \neq 1$ shrink to the origin (and to the far tails), we have

$$\|u_n - v_n\|_{H^1(\mathbb{R})} \rightarrow 0.$$

Therefore,

$$\|u_n - u\|_{H^1(\mathbb{R})} \leq \|u_n - v_n\|_{H^1} + \|v_n - u\|_{H^1} \xrightarrow{n \rightarrow \infty} 0.$$

This finishes the proof of Lemma 3.32 because there exists a sequence a $D(A_1)$ -valued sequence u_n such that $u_n \rightarrow u$ in $H^1(\mathbb{R})$. $\square$

We now deal with Aim (iii). By Lemma 3.32, there exists a sequence $(u_n) \subset D(A_1)$ such that

$$u_n \rightarrow u \quad \text{in } H^1(\mathbb{R}).$$

In particular,

$$u_n \rightarrow u \quad \text{and} \quad Du_n \rightarrow Du \quad \text{in } L^2(\mathbb{R}).$$

Since

$$A_1 u_n = i D u_n,$$

it follows that

$$A_1 u_n \rightarrow iDu \quad \text{in } L^2(\mathbb{R}).$$

Thus

$$(u_n, A_1 u_n) \rightarrow (u, iDu) \quad \text{in } H \times H.$$

However, $u \notin D(A_1)$ by Lemma 3.30. Hence $(u, iDu) \notin \Gamma(A_1)$ and therefore the graph $\Gamma(A_1)$ is not closed.

This completes the proof of Proposition 3.29. $\square$

Proposition 3.34. *Assume that A_1 is the operator defined by formula (3.2.1). A function $u \in D(\overline{A_1})$ if and only if the following conditions are satisfied*

(i) $u \in L^2(\mathbb{R}, \mathbb{C})$,

(ii) the function u belongs to the Sobolev space $H_{\text{loc}}^1(\mathbb{R} \setminus \{0\}, \mathbb{C})$ and the function $u : \mathbb{R} \rightarrow \mathbb{C}$ is locally absolutely continuous and

$$\int_{\mathbb{R}} |Du(t)|^2 dt < \infty,$$

(iii) $u(0) = 0$.

Moreover,

$$\overline{A_1} u = iDu, \text{ if } u \in D(\overline{A_1}). \quad (3.2.11)$$

Proof of Proposition 3.34, Part 1. Let us choose and fix $u \in D(\overline{A_1})$.

We will prove that u satisfies the three conditions (i)-(iii) listed above.

By definition of the closure of an operator, see Remark 2.86, there exist a sequence $(u_n) \subset D(A_1)$ and a function $z \in L^2(\mathbb{R}, \mathbb{C})$ such that

$$u_n \rightarrow u \quad \text{in } L^2(\mathbb{R}), \text{ and } A_1 u_n \rightarrow z \quad \text{in } L^2(\mathbb{R}). \quad (3.2.12)$$

Since $A_1 u_n = i u_n'$ we infer that

$$u_n' \rightarrow -iz \quad \text{in } L^2(\mathbb{R}). \quad (3.2.13)$$

We first show that u is locally absolutely continuous and that $Du \in L^2(\mathbb{R})$.

Let $T > 0$. Since $u_n, u_m \in C_0^\infty(\mathbb{R} \setminus \{0\})$, we have, for $t \in [0, T]$ and $n, m \in \mathbb{N}$,

$$u_n(t) - u_m(t) = \int_0^t (u'_n(s) - u'_m(s)) ds.$$

By the Cauchy–Schwarz inequality, see [38, Theorem 3.5 p.63]

$$|u_n(t) - u_m(t)| \leq \sqrt{t} \|u'_n - u'_m\|_{L^2(\mathbb{R})} \leq \sqrt{T} \|u'_n - u'_m\|_{L^2(\mathbb{R})}.$$

Taking the supremum over $t \in [0, T]$, we obtain

$$\|u_n - u_m\|_{C([0, T])} \leq \sqrt{T} \|u'_n - u'_m\|_{L^2(\mathbb{R})}.$$

Since (3.2.13) implies that (u'_n) is Cauchy in $L^2(\mathbb{R})$, it follows that (u_n) is Cauchy in $C([0, T])$, see [37, Theorem 7.8]. Hence there exists $v_T \in C([0, T])$ such that

$$u_n \rightarrow v_T \quad \text{uniformly on } [0, T].$$

On the other hand, by (3.2.12), we have $u_n \rightarrow u$ in $L^2([0, T])$. By the uniqueness of the limit in $L^2([0, T])$, we conclude that

$$v_T(x) = u(x) \quad \text{Leb a.e. on } [0, T].$$

Thus, after redefining u on a null set if necessary, we may assume that u is continuous on $[0, T]$. Since $T > 0$ is arbitrary, u is continuous on $[0, \infty)$. Repeating the same argument on $(-\infty, 0]$, there exists a continuous function $v : \mathbb{R} \rightarrow \mathbb{C}$ such that

$$v(x) = u(x) \quad \text{Leb a.e. on } \mathbb{R}.$$

and for every $a < b$

$$\sup_{x \in [a, b]} |u_n(x) - v(x)| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.2.14)$$

We conclude that u is continuous on each half-line.

Next we prove that u is weakly differentiable and that $Du = -iz$, see Definition 2.1. Let $\phi \in C_0^\infty(\mathbb{R} \setminus \{0\})$. Since $u_n \in C_0^\infty(\mathbb{R} \setminus \{0\})$, integration by parts gives

$$\int_{\mathbb{R}} u_n(x) \phi'(x) dx = - \int_{\mathbb{R}} u'_n(x) \phi(x) dx. \quad (3.2.15)$$

Passing to the limit in (3.2.15) using (3.2.12) and (3.2.13), we obtain

$$\int_{\mathbb{R}} u(x) \phi'(x) dx = - \int_{\mathbb{R}} (-iz)(x) \phi(x) dx.$$

Hence, since ϕ is arbitrary, we infer that u is weakly differentiable and

$$Du = -iz \in L^2(\mathbb{R}).$$

Therefore $u \in H_{\text{loc}}^1(\mathbb{R} \setminus \{0\}, \mathbb{C})$, and u is locally absolutely continuous on each half-line.

It remains to prove that $u(0) = 0$. Since each $u_n \in D(A_1) = C_0^\infty(\mathbb{R} \setminus \{0\})$, we have $u_n(0) = 0$. Moreover, by (3.2.14), we know that $u_n \rightarrow u$ uniformly on every compact interval. In particular, the convergence is uniform on $[-1, 1]$. Hence

$$u(0) = \lim_{n \rightarrow \infty} u_n(0) = 0.$$

We have proved conditions (i), (ii), and (iii). Finally, since $Du = -iz$, we have $z = iDu$. By definition of the closure, this implies (3.2.11). This completes the proof of Part 1. $\square$

Proof of Proposition 3.34, Part 2. We prove the converse implication. Let $u \in L^2(\mathbb{R}, \mathbb{C})$ satisfy conditions (i), (ii), and (iii) of the proposition. We will show that $u \in D(\overline{A_1})$.

By (ii), the restrictions $u|_{(-\infty, 0)}$ and $u|_{(0, \infty)}$ belong to the Sobolev spaces

$$H^1((-\infty, 0), \mathbb{C}) \quad \text{and} \quad H^1((0, \infty), \mathbb{C}),$$

respectively. since $u \in H^1((0, \infty))$, by Corollary 2.32, the limit $\lim_{x \rightarrow 0+} u(x)$ exists and coincides with $u(0)$. Since $u(0) = 0$, it follows that

$$u|_{(0, \infty)} \in H_0^1((0, \infty), \mathbb{C}), \quad \text{and} \quad u|_{(-\infty, 0)} \in H_0^1((-\infty, 0), \mathbb{C}).$$

By Definition 2.35, the space $H_0^1(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. In particular, for $\Omega = (-\infty, 0)$ and $\Omega = (0, \infty)$, we deduce that $C_0^\infty((-\infty, 0))$ is dense in $H_0^1((-\infty, 0))$ and $C_0^\infty((0, \infty))$ is dense in $H_0^1((0, \infty))$. Thus there exist sequences (f_n) and (g_n) such that

$$f_n \in C_0^\infty((-\infty, 0)), \quad g_n \in C_0^\infty((0, \infty))$$

and

$$f_n \rightarrow u|_{(-\infty, 0)} \quad \text{in } H^1((-\infty, 0)), \quad g_n \rightarrow u|_{(0, \infty)} \quad \text{in } H^1((0, \infty)).$$

Define $u_n : \mathbb{R} \rightarrow \mathbb{C}$ by

$$u_n(x) = \begin{cases} f_n(x), & x < 0, \\ g_n(x), & x > 0, \\ 0, & x = 0. \end{cases}$$

Then $u_n \in C_0^\infty(\mathbb{R} \setminus \{0\}) = D(A_1)$ for every n , since f_n and g_n are smooth with compact support in $(-\infty, 0)$ and $(0, \infty)$, respectively, and the point $x = 0$ is excluded from the domain. Moreover, by construction,

$$u_n \rightarrow u \quad \text{in } L^2(\mathbb{R}),$$

and

$$Du_n \rightarrow Du \quad \text{in } L^2(\mathbb{R}).$$

Therefore,

$$A_1 u_n = iDu_n \rightarrow iDu \quad \text{in } L^2(\mathbb{R}).$$

Hence, by definition of the closure of an operator,

$$u \in D(\overline{A_1}) \quad \text{and} \quad \overline{A_1} u = iDu.$$

This completes the proof of Part 2. □

Remark 3.35. If one compares Proposition 3.29 with the first part of Proposition 3.20, one can see that

$$D(\overline{A_1}) \subset D(A_1^*).$$

Moreover, by comparing Propositions 3.34 and 3.20, we obtain

$$\overline{A_1} u = A_1^* u \quad \text{for all } u \in D(\overline{A_1}).$$

In other words, in the sense of Definition 2.80,

$$\overline{A_1} \subset A_1^*.$$

Secondly, item (iii) in Proposition 3.34 makes sense in view of Proposition 3.21.

We also need the following result in the next Section 3.2.5.

3.2.5 Finding all self-adjoint extensions operators for the first order closure on the punctured line

The main result of this section is a consequence of the general von Neumann theory, see [34, Section X, Theorem X.2, v.2, p. 141]) and Theorem 2.60.

By Proposition 3.26 and Corollary 3.28, the closure $\overline{A_1}$ is a closed symmetric operator in $L^2(\mathbb{R}, \mathbb{C})$ satisfying the following proposition.

Proposition 3.36. *Assume that A_1 is the operator defined by formula (3.2.1). Then*

- (i) *the closure $\overline{A_1}$ is a symmetric operator in $H = L^2(\mathbb{R}, \mathbb{C})$,*
- (ii) *the adjoint of $\overline{A_1}$ is equal to the adjoint of A_1 , i.e.*

$$\overline{A_1}^* = A_1^*,$$

- (ii) *the deficiency indices of $\overline{A_1}$ are equal to the deficiency indices of A_1 , i.e.*

$$n_+(\overline{A_1}) = n_-(\overline{A_1}) = 1.$$

We must work with the closure $\overline{A_1}$, because Theorem 2.60 requires closed operators.

Theorem 3.37. *Assume that A_1 is the operator defined by formula (3.2.1), and let $\overline{A_1}$ be its closure. Then $\overline{A_1}$ is a closed symmetric operator with deficiency indices*

$$n_+(\overline{A_1}) = n_-(\overline{A_1}) = 1 < \infty.$$

Set

$$H_+ = \ker(iI - \overline{A_1}^*) = \ker(iI - A_1^*), \quad H_- = \ker(iI + \overline{A_1}^*) = \ker(iI + A_1^*). \quad (3.2.16)$$

Then every self-adjoint extension of $\overline{A_1}$ can be characterized in the following way

- (i) *If $\Gamma : H_+ \rightarrow H_-$ is a unitary operator and the operator A_Γ is defined by*

$$\begin{aligned} D(A_\Gamma) &= \left\{ u \in H : u = u_0 + u_+ + \Gamma(u_+), \ u_0 \in D(\overline{A_1}), \ u_+ \in H_+ \right\}, \\ A_\Gamma(u_0 + u_+ + \Gamma(u_+)) &= \overline{A_1}u_0 + iu_+ - i\Gamma(u_+), \end{aligned}$$

then A_Γ is self-adjoint and $\overline{A_1} \subset A_\Gamma$, i.e.,

$$D(\overline{A_1}) \subset D(A_\Gamma), \quad \overline{A_1}(u_0) = A_\Gamma(u_0) \text{ for all } u_0 \in D(\overline{A_1}).$$

- (ii) *Conversely, every self-adjoint operator A with $\overline{A_1} \subset A$ is of the form $A = A_\Gamma$ for a unique unitary map*

$$\Gamma : H_+ \rightarrow H_-.$$

Proof of Theorem 3.37. Follows from Theorem 2.60. □

Applying of Theorem 3.37. Since $n_{\pm}(\overline{A_1}) = 1$, the spaces H_+ and H_- are one-dimensional. By (3.2.16) and Theorem 3.18, these two spaces are generated respectively by the following functions

$$\begin{aligned} e_1(x) &= \sqrt{2}e^x \cdot \mathbf{1}_{(-\infty, 0]}(x) \in H_+, \\ e_2(x) &= \sqrt{2}e^{-x} \cdot \mathbf{1}_{[0, \infty)}(x) \in H_-, \end{aligned}$$

where $\mathbf{1}_{(-\infty, 0]}(x)$ and $\mathbf{1}_{[0, \infty)}(x)$ are the indicator functions of the negative and positive real axes, respectively.

These functions satisfy

$$\|e_1\| = \|e_2\| = 1, \quad \langle e_1, e_2 \rangle_{L^2} = 0, \quad (3.2.17)$$

and their boundary values are

$$e_1(0-) = \sqrt{2}, \quad e_1(0+) = 0, \quad e_2(0+) = \sqrt{2}, \quad e_2(0-) = 0.$$

The vectors e_1 and e_2 form orthogonal bases of the corresponding spaces. Moreover they are perpendicular to each other. For the completeness of the presentation let check the second condition in (3.2.17).

$$\langle e_1, e_2 \rangle = \int_{\mathbb{R}} e_1(x) \overline{e_2(x)} dx = \int_{\mathbb{R}} 0 dx = 0,$$

because the function $e_1 = 0$ on $[0, \infty)$ and $e_2 = 0$ on $(-\infty, 0]$.

We also compute their L^2 -norms. By the first condition in (3.2.17)

$$\|e_1\|^2 = 2 \int_{-\infty}^0 |e^x|^2 dx = 2 \int_{-\infty}^0 e^{2x} dx = 2 \left[\frac{1}{2} e^{2x} \right]_{-\infty}^0 = 2 \cdot \frac{1}{2} = 1.$$

and similarly for e_2 .

By Theorem 3.37, all self-adjoint extensions of $\overline{A_1}$ are parametrized by unitary maps $\Gamma : H_+ \rightarrow H_-$. Since both spaces are one-dimensional, every such Γ is a "multiplication" operator by a constant $\alpha \in \mathbb{C}$ of length one, i.e. $\Gamma = \Gamma_{\alpha}$, where

$$\Gamma_{\alpha}(e_1) = \alpha e_2, \quad \alpha \in \mathbb{T}. \quad (3.2.18)$$

Let us now choose and fix such $\alpha \in \mathbb{T}$. By Theorem 3.37 we have

$$\begin{aligned} D(A_{\Gamma_\alpha}) &= \left\{ u_0 + u_+ + \Gamma_\alpha u_+ : u_0 \in D(\overline{A_1}), u_+ \in H_+ \right\}, \\ A_{\Gamma_\alpha}(u_0 + u_+ + \Gamma_\alpha u_+) &= \overline{A_1}u_0 + iu_+ - i\Gamma_\alpha u_+. \end{aligned}$$

Since the space H_+ is one dimensional and generated by e_1 , using the formula

$$\begin{aligned} D(A_{\Gamma_\alpha}) &= \left\{ u_0 + ce_1 + c\alpha e_2 : u_0 \in D(\overline{A_1}), c \in \mathbb{C} \right\}. \\ A_{\Gamma_\alpha}(u_0 + ce_1 + c\alpha e_2) &= \overline{A_1}u_0 + ice_1 - ic\alpha e_2. \end{aligned}$$

Let us choose and fix $u_0 \in D(\overline{A_1})$ and $c \in \mathbb{C}$ and put $u = u_0 + ce_1 + c\alpha e_2$.

Since, by Proposition 3.34-(iii), $u_0(0\pm) = 0$, using the properties of functions e_1 and e_2 , we deduce that

$$u(0-) = u_0(0-) + ce_1(0-) + c\alpha e_2(0-) = 0 + c\sqrt{2} + 0 = \sqrt{2}c,$$

and

$$u(0+) = u_0(0+) + ce_1(0+) + c\alpha e_2(0+) = 0 + 0 + c\alpha\sqrt{2} = \sqrt{2}c\alpha.$$

Therefore, we deduce

$$\boxed{u(0+) = \alpha u(0-).} \tag{3.2.19}$$

We proved that if $u \in D(A_{\Gamma_\alpha})$ then u satisfies condition (3.2.19).

Conversely, if $\alpha \in \mathbb{T}$ and operator A^α is defined by

$$\begin{aligned} D(A^\alpha) &:= \{u \in D(A_1^*) : u(0+) = \alpha u(0-)\} \\ A^\alpha u &:= A_1^* u, \text{ if } u \in D(A^\alpha). \end{aligned}$$

then this operator is a self-adjoint extension of the operator A_1 and

$$A^\alpha = A_{\Gamma_\alpha}, \tag{3.2.20}$$

where $\Gamma_\alpha : H_+ \rightarrow H_-$ is defined in (3.2.18).

These considerations can be summarized in the following theorem. □

Theorem 3.38. *The set of self-adjoint extensions of operator A_1 coincides with the family*

$$\text{SA}(A_1) = \{A^\alpha : \alpha \in \mathbb{T}\}$$

where every operator A^α is determined by the boundary condition

$$u(0+) = \alpha u(0-).$$

Remark 3.39. Equality (3.2.19) is the boundary condition defining the operator A^α .

Finish the proof of Theorem 3.38. By Theorem 3.37, every self-adjoint extension of $\overline{A_1}$ is of the form A_Γ for a unique unitary map $\Gamma : H_+ \rightarrow H_-$. Since H_+ and H_- are one-dimensional, every such Γ is multiplication by a scalar $\alpha \in \mathbb{T}$.

In the construction above, see (3.2.18)–(3.2.19). We showed that, for every $\alpha \in \mathbb{T}$,

$$D(A_{\Gamma_\alpha}) = \{ u \in D(A_1^*) : u(0+) = \alpha u(0-) \} = D(A^\alpha),$$

and the formula (3.2.20) holds.

Conversely, if $\alpha \in \mathbb{T}$ and A^α is defined by the boundary condition (3.2.19), then the same computation shows that $A^\alpha = A_{\Gamma_\alpha}$ for the corresponding unitary map $\Gamma_\alpha : H_+ \rightarrow H_-$, and therefore A^α is self-adjoint.

Hence the family

$$\{A^\alpha : \alpha \in \mathbb{T}\}$$

is exactly the set of all self-adjoint extensions of A_1 , that is

$$\text{SA}(A_1) = \{A^\alpha : \alpha \in \mathbb{T}\}.$$

□

Chapter 4

Second-order differential operator (Laplacian)

In this chapter, we study second-order differential operators, in particular the Laplacian. We begin with operators defined on the real line, where we analyse symmetry and describe the adjoint operator.

We then consider the corresponding operators on the punctured real line, where we investigate the closure, compute the deficiency indices, and characterize all self-adjoint extensions. These results extend the analysis of first-order operators to higher-order cases.

4.1 Second-order differential operator (Laplacian) on the real line

We consider the Hilbert space

$$H = L^2(\mathbb{R}, \mathbb{C})$$

with the standard inner product and norm, and define the operator A_0 by

$$\begin{aligned} D(A_0) &= C_0^\infty(\mathbb{R}) \\ A_0 u &= D^2 u = \Delta u \text{ if } u \in D(A_0). \end{aligned} \tag{4.1.1}$$

We begin with the following result.

Proposition 4.1. *The operator A_0 is symmetric in the space H , i.e.*

$$\langle A_0 u, v \rangle_{L^2} = \langle u, A_0 v \rangle_{L^2}; \quad \forall u, v \in D(A_0). \tag{4.1.2}$$

Proof of Proposition 4.1. We need to prove that A_0 is symmetric. In what follows we consider the case when $a = -\infty$ and $b = \infty$. The proof in a general case is similar.

Let us choose and fix $u, v \in C_0^\infty(\mathbb{R})$. Then by the definition (4.1.1) of the linear map A_0 we have

$$\langle u, A_0 v \rangle_{L^2} = \langle u, v'' \rangle_{L^2} = \int_{-\infty}^{\infty} u(x) \overline{v''(x)} dt \quad (4.1.3)$$

To prove (4.1.2), since both u, v have compact support, we can find interval $[a, b]$ and $[c, d]$ such that $u(x) = 0$ if $t \notin [a, b]$ and $v(x) = 0$ if $t \notin [c, d]$. Then we choose $\alpha = \min\{a, c\}$ and $\beta = \max\{b, d\}$. Then $[a, b] \subset [\alpha, \beta]$ and $[c, d] \subset [\alpha, \beta]$, so that

$$[a, b] \cup [c, d] \subset [\alpha, \beta].$$

Therefore,

$$\text{if } t \notin (\alpha, \beta), \text{ the } u(x) = 0, \text{ and } v(x) = 0.$$

Hence the integral in (4.1.3) is equal to the corresponding integral over a bounded interval $[\alpha, \beta]$ (over which I can apply the Integration by Parts [37, Theorem 6.22]).

Thus, by equality (4.1.3), we get

$$\begin{aligned} L.H.S &= \langle u, A_0 v \rangle = \int_{-\infty}^{\infty} u(x) \overline{v''(x)} dx = \int_{-\infty}^{\infty} u(x) \overline{(v'(x))'} dt \\ &= \int_{\alpha}^{\beta} u(x) \overline{(v'(x))'} dx = [u(x) \overline{v'(x)}]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} u'(x) \overline{v'(x)} dx \\ &= [0 - 0] - \int_{\alpha}^{\beta} u'(x) \overline{v'(x)} dx = -[u'(x) \overline{v(x)}]_{\alpha}^{\beta} + \int_{\alpha}^{\beta} u''(x) \overline{v(x)} dx \\ &= -[0 - 0] + \int_{\alpha}^{\beta} u''(x) \overline{v(x)} dx = \int_{-\infty}^{\infty} u''(x) \overline{v(x)} dx = \langle A_0 u, v \rangle = R.H.S. \end{aligned}$$

Hence we infer that A_0 is symmetric in H . □

We need to determine $D(A_0^*)$ and next find a formula for $A_0^* v$ if $v \in D(A_0^*)$.

We begin with the following characterization of the domain of the adjoint operator A_0^* , which will be proven later on.

Proposition 4.2. *Suppose that A_0 is the linear operator defined by (4.1.1). Then*

$$\begin{aligned} D(A_0^*) &= \{v : \mathbb{R} \rightarrow \mathbb{C} : v \text{ is of } C^1\text{-class such that } \int_{-\infty}^{\infty} |v(x)|^2 dt < \infty \\ &\text{and the derivative } v' = Dv \text{ is locally absolutely continuous function} \\ &\int_{-\infty}^{\infty} |v''(x)|^2 dt < \infty\}, \end{aligned}$$

and

$$A_0^* u = D^2 u, \quad u \in D(A_0^*).$$

In the above expression, $v'' = D^2v$ denotes the derivative of $v' = Dv$. Since v' is locally absolutely continuous, in the sense of Definition 2.8, the derivative v'' exists almost everywhere.

4.1.1 Deficiency indices of the second-order differential operator (Laplacian) on the real line

In this subsection we compute the deficiency indices $n_+(A_0)$ and $n_-(A_0)$ of the operator A_0 . Using Definition 2.59, we have

$$n_+(A_0) = \dim \ker(iI - A_0^*), \quad n_-(A_0) = \dim \ker(iI + A_0^*).$$

We begin with the computation of $n_+(A_0)$.

Lemma 4.3. *Assume that A_0 is the operator defined by (4.1.1). Then*

$$\dim \ker(iI - A_0^*) = 0.$$

Proof of Lemma 4.3. By [34, Definition of kernel] and Proposition 4.2, we have

$$\begin{aligned} u \in \ker(iI - A_0^*) &\iff u \in D(A_0^*) \text{ and } (iI - A_0^*)u = 0 \\ &\iff u \in D(A_0^*) \text{ and } D^2u = iu \text{ on } \mathbb{R}. \end{aligned}$$

Let us choose and fix $u \in \ker(iI - A_0^*)$. Then u satisfies

$$u''(x) = iu(x), \quad x \in \mathbb{R}. \tag{4.1.4}$$

To solve this equation, we look for solutions of the form

$$u(x) = e^{\lambda x}.$$

Substituting into (4.1.4) gives

$$\lambda^2 = i.$$

Since

$$i = e^{i\pi/2},$$

the two square roots are

$$\lambda = \pm \frac{\sqrt{2}}{2}(1 + i).$$

Hence the general solution is

$$u(x) = c_1 e^{\frac{\sqrt{2}}{2}(1+i)x} + c_2 e^{-\frac{\sqrt{2}}{2}(1+i)x}, \quad x \in \mathbb{R},$$

for some constants $c_1, c_2 \in \mathbb{C}$.

We claim that such a function belongs to $L^2(\mathbb{R})$ only if $c_1 = c_2 = 0$. Indeed, since

$$\operatorname{Re}\left(\frac{\sqrt{2}}{2}(1+i)\right) = \frac{\sqrt{2}}{2} > 0,$$

we have

$$e^{\frac{\sqrt{2}}{2}(1+i)x} \notin L^2(0, \infty), \quad e^{-\frac{\sqrt{2}}{2}(1+i)x} \notin L^2(-\infty, 0).$$

Therefore, if $u \in L^2(\mathbb{R})$, then necessarily $c_1 = c_2 = 0$, so $u = 0$.

Thus

$$\ker(iI - A_0^*) = \{0\},$$

and hence

$$n_+(A_0) = \dim \ker(iI - A_0^*) = 0.$$

This proves the lemma. □

We now compute $n_-(A_0)$.

Lemma 4.4. *Assume that A_0 is the operator defined by (4.1.1). Then*

$$\dim \ker(iI + A_0^*) = 0.$$

Proof of Lemma 4.4. By [34, Definition of kernel] and Proposition 4.2, we have

$$\begin{aligned} u \in \ker(iI + A_0^*) &\iff u \in D(A_0^*) \text{ and } (iI + A_0^*)u = 0 \\ &\iff u \in D(A_0^*) \text{ and } D^2u = -iu \text{ on } \mathbb{R}. \end{aligned}$$

Let us choose and fix $u \in \ker(iI + A_0^*)$. Then u satisfies

$$u''(x) = -iu(x), \quad x \in \mathbb{R}. \tag{4.1.5}$$

Again, looking for solutions of the form $u(x) = e^{\lambda x}$, we obtain

$$\lambda^2 = -i.$$

Since

$$-i = e^{-i\pi/2},$$

the two square roots are

$$\lambda = \pm \frac{\sqrt{2}}{2}(1 - i).$$

Hence the general solution is

$$u(x) = c_1 e^{\frac{\sqrt{2}}{2}(1-i)x} + c_2 e^{-\frac{\sqrt{2}}{2}(1-i)x}, \quad x \in \mathbb{R},$$

for some constants $c_1, c_2 \in \mathbb{C}$.

Since

$$\operatorname{Re}\left(\frac{\sqrt{2}}{2}(1 - i)\right) = \frac{\sqrt{2}}{2} > 0,$$

we have

$$e^{\frac{\sqrt{2}}{2}(1-i)x} \notin L^2(0, \infty), \quad e^{-\frac{\sqrt{2}}{2}(1-i)x} \notin L^2(-\infty, 0).$$

Therefore, if $u \in L^2(\mathbb{R})$, then necessarily $c_1 = c_2 = 0$, so $u = 0$.

Thus

$$\ker(iI + A_0^*) = \{0\},$$

and hence

$$n_-(A_0) = \dim \ker(iI + A_0^*) = 0.$$

This proves the lemma. □

We conclude with the following result.

Proposition 4.5. *Assume that A_0 is the operator defined by (4.1.1). Then A_0 is a densely defined symmetric operator on $H = L^2(\mathbb{R}, \mathbb{C})$ and its deficiency indices satisfy*

$$n_+(A_0) = n_-(A_0) = 0.$$

In particular, A_0 is essentially self-adjoint, i.e. it has a unique self-adjoint extension. Moreover, the unique self-adjoint extension A of A_0 is as follows

$$D(A) = H^2(\mathbb{R}), \quad Au = D^2u = \Delta u.$$

Proof of Proposition 4.5. The density of $D(A_0) = C_0^\infty(\mathbb{R})$ in $L^2(\mathbb{R})$ follows from Proposition 2.48, and the symmetry of A_0 was proved in Proposition 4.1. Moreover, by Lemmas 4.3 and 4.4, its deficiency indices satisfy

$$n_+(A_0) = n_-(A_0) = 0.$$

Hence, by Corollary 2.61, A_0 is essentially self-adjoint. □

4.2 Second-order differential operator (Laplacian) on the punctured line

In this section, we define an operator A_1 on $H = L^2(\mathbb{R}, \mathbb{C})$ and

$$\begin{aligned} D(A_1) &= C_0^\infty((-\infty, 0) \cup (0, \infty); \mathbb{C}) = C_0^\infty(\mathbb{R} \setminus \{0\}; \mathbb{C}), \\ A_1 u &= D^2 u = \Delta u \text{ if } u \in D(A_1). \end{aligned} \tag{4.2.1}$$

We begin with the following result.

Proposition 4.6. *The operator A_1 is symmetric in the space H , i.e.*

$$\langle A_1 u, v \rangle_{L^2} = \langle u, A_1 v \rangle_{L^2}; \quad \forall u, v \in D(A_1).$$

Proof of Proposition 4.6. The proof is analogous to Proposition 4.1. Since functions in $C_0^\infty(\mathbb{R} \setminus \{0\})$ have compact support away from the origin, integration by parts on $(-\infty, 0)$ and $(0, \infty)$ gives

$$\langle A_1 u, v \rangle = \langle u, A_1 v \rangle.$$

Hence A_1 is symmetric. □

Now, we want to calculate the deficiency indices of A_1 . For this purpose we need first to find the domains of the adjoint operator A_1^* . We have the following analog of Proposition 4.2.

Proposition 4.7. *Let A_1 be defined by (4.2.1). Then $u \in D(A_1^*)$ if and only if*

$$u \in L^2(\mathbb{R})$$

and the restrictions

$$u_- := u|_{(-\infty, 0)}, \quad u_+ := u|_{(0, \infty)}$$

satisfy

$$u_\pm \in H_{\text{loc}}^2, \quad D^2 u_\pm \in L^2.$$

Equivalently, $u_\pm$ and $Du_\pm$ are locally absolutely continuous and

$$\int_{-\infty}^0 |D^2 u(x)|^2 dx < \infty, \quad \int_0^\infty |D^2 u(x)|^2 dx < \infty.$$

Moreover,

$$A_1^* u = D^2 u$$

on each half-line.

Proposition 4.8. *A measurable function $v : \mathbb{R} \rightarrow \mathbb{C}$ belongs to $D(A_1^*)$ if and only if*

$$\int_{-\infty}^{\infty} |v(x)|^2 dt < \infty$$

and the restriction $v : (0, \infty) \rightarrow \mathbb{C}$ is of C^1 – class such that

the derivative $v' = Dv$ is locally absolutely continuous function and

$$\int_0^{\infty} |v''(x)|^2 dt < \infty$$

and the restriction $v : (-\infty, 0) \rightarrow \mathbb{C}$ is of C^1 – class such that

the derivative $v' = Dv$ is locally absolutely continuous function and

$$\int_{-\infty}^0 |v''(x)|^2 dt < \infty.$$

Moreover,

$$A_1^* u = D^2 u = u'', \quad u \in D(A_1^*).$$

In the above expression, $v'' = D^2 v$ denotes the derivative of $v' = Dv$. Since v' is locally absolutely continuous, in the sense of Definition 2.8, the derivative v'' exists almost everywhere.

Proposition 4.9. *If $u \in D(A_1^*)$ then the following limits exist:*

$$\lim_{x \rightarrow 0+} u(x) =: u(0+)$$

$$\lim_{x \rightarrow 0-} u(x) =: u(0-)$$

$$\lim_{x \rightarrow 0+} u'(x) =: u'(0+)$$

$$\lim_{x \rightarrow 0-} u'(x) =: u'(0-).$$

Remark 4.10. The traces from the two half-lines are independent. In general,

$$u(0+) \neq u(0-)$$

and

$$u'(0+) \neq u'(0-).$$

Thus neither u nor u' is required to be continuous at the origin.

4.2.1 Deficiency indices of the second-order differential operator (Laplacian) on the punctured line

The main result of this section is the following one.

Theorem 4.11. *Assume that A_1 is the operator defined by formula (4.2.1). Then*

$$n_+(A_1) = 2$$

$$n_-(A_1) = 2$$

The following result is a consequences of Theorem 4.11. In fact, we investigated the previous result in order to prove or disprove the next corollary.

Corollary 4.12. *The operator A_1 is closable, and its closure $\overline{A_1}$ admits self-adjoint extensions.*

Proof of Corollary 4.12. Since A_1 is densely defined and symmetric, it is closable. By Corollary 2.92,

$$n_{\pm}(\overline{A_1}) = n_{\pm}(A_1).$$

Therefore, Theorem 4.11 gives

$$n_+(\overline{A_1}) = n_-(\overline{A_1}) = 2.$$

The existence assertion now follows from Theorem 2.60. $\square$

In the remaining of this section we will we occupy ourselves proving the theorem above. For this aim we will calculate the deficiency indices $n_+(A_1)$ and $n_-(A_1)$.

Proposition 4.13. *Assume that A_1 is the operator defined by (4.2.1). Then*

$$n_+(A_1) = 2.$$

Proof of Proposition 4.13. Let us observe that by Proposition 4.6

$$\begin{aligned} u \in \ker(iI - A_1^*) &\iff u \in D(A_1^*) \text{ and } (iI - A_1^*)u = 0 \\ &\iff u \in D(A_1^*) \text{ and } iu = D^2u \iff u \in D(A_1^*) \text{ and } D^2u = iu. \\ &\iff \text{the restrictions of function } u \text{ to the intervals } (0, \infty) \text{ and } (-\infty, 0) \text{ are} \\ &\quad \text{of } C^1\text{-class and their derivatives are locally absolutely continuous and} \\ &\quad \int_0^\infty |D^2u(x)|^2 dx < \infty, \int_{-\infty}^0 |D^2u(x)|^2 dx < \infty \\ &\quad \text{and } D^2u = iu \text{ on } (0, \infty) \text{ and } D^2u = iu \text{ on } (-\infty, 0). \end{aligned} \tag{4.2.2}$$

The above equivalence is analogous to the earlier one in (4.2.2).

Let us choose and fix $u \in \ker(iI - A_1^*)$, then by (4.2.2), u satisfies the following two conditions.

(1) $D^2u = iu$ on $(0, \infty)$ and

(2) $D^2u = iu$ on $(-\infty, 0)$.

Let us recall that function u in $D(A_1^*)$ need not be continuous at 0.

To solve $D^2u = iu$ on $(0, \infty)$, we note that the general solution is of the form, where $c_1, c_2 \in \mathbb{C}$ is a constant

$$u(x) = c_1 e^{-\frac{\sqrt{2}}{2}(1+i)x} + c_2 e^{\frac{\sqrt{2}}{2}(1+i)x}, \quad x > 0,$$

Since $\operatorname{Re}(\frac{\sqrt{2}}{2}(1+i)) > 0$ and $\operatorname{Re}(-\frac{\sqrt{2}}{2}(1+i)) < 0$, in order that $u \in L^2$, we need to have $c_2 = 0$. Hence

$$u(x) = c_1 e^{-\frac{\sqrt{2}}{2}(1+i)x}, \quad x > 0.$$

To solve $D^2u = iu$ on $(-\infty, 0)$, we note that a general solution is of the form, where $c_1, c_2 \in \mathbb{C}$ is a constant

$$u(x) = c_1 e^{-\frac{\sqrt{2}}{2}(1+i)x} + c_2 e^{\frac{\sqrt{2}}{2}(1+i)x}, \quad x < 0,$$

Since $\operatorname{Re}(\frac{\sqrt{2}}{2}(1+i)) > 0$ and $\operatorname{Re}(-\frac{\sqrt{2}}{2}(1+i)) < 0$, in order that $u \in L^2$, we need to have $c_1 = 0$. Hence

$$u(x) = c_2 e^{\frac{\sqrt{2}}{2}(1+i)x}, \quad x < 0.$$

Hence we proved that if $u \in \ker(iI - A_1^*)$, then there exists numbers $c_1, c_2 \in \mathbb{C}$ such that the function u satisfies

$$u(x) = \begin{cases} c_1 e^{-\frac{\sqrt{2}}{2}(1+i)x} & \text{if } x > 0 \\ c_2 e^{\frac{\sqrt{2}}{2}(1+i)x} & \text{if } x < 0 \end{cases} \quad (4.2.3)$$

One can show easily that every function u defined by (4.2.3) belongs to $\ker(iI - A_1^*)$.

Thus, the space $\ker(iI - A_1^*)$ has dimension 2 because it contains precisely two linearly independent functions

$$e_1(x) = \begin{cases} e^{-\frac{\sqrt{2}}{2}(1+i)x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \quad (4.2.4)$$

and

$$e_2(x) = \begin{cases} 0 & \text{if } x > 0 \\ e^{\frac{\sqrt{2}}{2}(1+i)x} & \text{if } x < 0 \end{cases} \quad (4.2.5)$$

Thus

$$\dim \ker(iI - A_1^*) = 2$$

Thus, the proof is complete. $\square$

In the same way, we prove that

Proposition 4.14. *Assume that A_1 is the operator defined by (4.2.1). Then*

$$n_-(A_1) = 2.$$

Proof of Proposition 4.14. Let us observe that by Proposition 4.6

$$\begin{aligned} u \in \ker(iI + A_1^*) &\iff u \in D(A_1^*) \text{ and } (iI + A_1^*)u = 0 \\ &\iff u \in D(A_1^*) \text{ and } iu = -D^2u \iff u \in D(A_1^*) \text{ and } D^2u = -iu. \\ &\iff \text{the restrictions of function } u \text{ to the intervals } (0, \infty) \text{ and } (-\infty, 0) \text{ are} \\ &\quad \text{of } C^1\text{-class and their derivatives are locally absolutely continuous and} \\ &\quad \int_0^\infty |D^2u(x)|^2 dx < \infty, \int_{-\infty}^0 |D^2u(x)|^2 dx < \infty \\ &\quad \text{and } D^2u = -iu \text{ on } (0, \infty) \text{ and } D^2u = -iu \text{ on } (-\infty, 0). \end{aligned} \tag{4.2.6}$$

The above equivalence is analogous to the earlier one in (4.2.6).

Let us choose and fix $u \in \ker(iI + A_1^*)$, then by (4.2.6), u satisfies the following conditions.

- (1) $D^2u = -iu$ on $(0, \infty)$ and
- (2) $D^2u = -iu$ on $(-\infty, 0)$.

Let us recall that function u is not even continuous at 0.

To solve $D^2u = -iu$ on $(0, \infty)$, we note that the general solution is of the form, where $c_1, c_2 \in \mathbb{C}$ is a constant

$$u(x) = c_1 e^{-\frac{\sqrt{2}}{2}(1-i)x} + c_2 e^{\frac{\sqrt{2}}{2}(1-i)x}, \quad x > 0,$$

We found the values of λ in the previous Section 4.1, by solved the equation (4.1.5).

Since $\operatorname{Re}(\frac{\sqrt{2}}{2}(1-i)) > 0$ and $\operatorname{Re}(-\frac{\sqrt{2}}{2}(1-i)) < 0$, in order that $u \in L^2$, we need to have $c_2 = 0$. Hence

$$u(x) = c_1 e^{-\frac{\sqrt{2}}{2}(1-i)x}, \quad x > 0.$$

To solve $D^2u = -iu$ on $(-\infty, 0)$.

we note that the general solution is of the form, where $c_1, c_2 \in \mathbb{C}$ is a constant

$$u(x) = c_1 e^{-\frac{\sqrt{2}}{2}(1-i)x} + c_2 e^{\frac{\sqrt{2}}{2}(1-i)x}, \quad x < 0,$$

Since $\operatorname{Re}(\frac{\sqrt{2}}{2}(1-i)) > 0$ and $\operatorname{Re}(-\frac{\sqrt{2}}{2}(1-i)) < 0$, in order that $u \in L^2$, we need to have $c_1 = 0$. Hence

$$u(x) = c_2 e^{\frac{\sqrt{2}}{2}(1-i)x}, \quad x < 0.$$

Hence we proved that if $u \in \ker(iI + A_1^*)$, then there exists numbers $c_1, c_2 \in \mathbb{C}$ such that the function u satisfies

$$u(x) = \begin{cases} c_1 e^{-\frac{\sqrt{2}}{2}(1-i)x} & \text{if } x > 0 \\ c_2 e^{\frac{\sqrt{2}}{2}(1-i)x} & \text{if } x < 0 \end{cases} \quad (4.2.7)$$

One can show easily that every function u defined by (4.2.7) belongs to $\ker(iI + A_1^*)$.

Thus, the space $\ker(iI + A_1^*)$ has dimension 2 because it contains precisely two linearly independent functions

$$e_1(x) = \begin{cases} e^{-\frac{\sqrt{2}}{2}(1-i)x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \quad (4.2.8)$$

and

$$e_2(x) = \begin{cases} 0 & \text{if } x > 0 \\ e^{\frac{\sqrt{2}}{2}(1-i)x} & \text{if } x < 0 \end{cases} \quad (4.2.9)$$

Thus

$$\dim \ker(iI + A_1^*) = 2$$

Thus, the proof is complete. $\square$

4.2.2 The adjoint of the closure of second-order differential operator (Laplacian) on the punctured line

Let A_1 be the operator defined by formula (4.2.1). Since A_1 is densely defined and symmetric, Corollary 2.92 implies that A_1 is closable, that $\overline{A_1}$ is symmetric. The closure $\overline{A_1}$ exists.

We now extend Proposition 4.7 by determining the adjoint of the closure $\overline{A_1}$.

Proposition 4.15. *Let A_1 be the operator defined by formula (4.2.1), and let $\overline{A_1}$ denote its closure in $L^2(\mathbb{R}, \mathbb{C})$. Then,*

$$(\overline{A_1})^* = A_1^*.$$

A measurable function $u : \mathbb{R} \rightarrow \mathbb{C}$ belongs to $D((\overline{A_1})^*)$ if and only if:

(i) $u \in L^2(\mathbb{R}, \mathbb{C})$;

(ii) The restriction $u : (0, \infty) \rightarrow \mathbb{C}$ is twice weakly differentiable with

$$\boxed{D^2 u \in L^2((0, \infty))},$$

i.e.

$$\int_0^\infty |D^2 u(t)|^2 dt < \infty.$$

(iii) The restriction $u : (-\infty, 0) \rightarrow \mathbb{C}$ is twice weakly differentiable with $D^2 u \in L^2((-\infty, 0))$,

i.e.

$$\int_{-\infty}^0 |D^2 u(t)|^2 dt < \infty.$$

Moreover, if $u \in D((\overline{A_1})^*)$, then

$$(\overline{A_1})^* u = D^2 u = \Delta u.$$

4.2.3 Deficiency indices of the closure of second-order differential operator (Laplacian) on the punctured line

From Proposition 4.15, we already know that the closure $\overline{A_1}$ of the operator defined by (4.2.1) exists and that its adjoint coincides with the adjoint of A_1 , i.e.

$$(\overline{A_1})^* = A_1^*.$$

Corollary 4.16. *The operators A_1 and $\overline{A_1}$ have the same deficiency subspaces and deficiency indices. More precisely,*

$$\ker(iI - (\overline{A_1})^*) = \ker(iI - A_1^*)$$

and

$$\ker(iI + (\overline{A_1})^*) = \ker(iI + A_1^*).$$

Consequently,

$$n_+(\overline{A_1}) = n_+(A_1) = 2, \quad n_-(\overline{A_1}) = n_-(A_1) = 2.$$

Proof. This follows from Corollary 2.92 and Theorem 4.11. □

4.2.4 Results for the second-order differential operator (Laplacian) on the punctured line

We begin with the following preliminary result. Our first step is to establish the result below.

Proposition 4.17. *The operator A_1 defined by (4.2.1) is not closed.*

Proof of Proposition 4.17. The proof follows the same method as in Proposition 3.29.

We will show that the graph $\Gamma(A_1)$ is not closed in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

Let $u : \mathbb{R} \rightarrow \mathbb{C}$ be the function defined in (3.2.6). By Lemma 3.30, we have

$$u \in H^1(\mathbb{R}) \subset L^2(\mathbb{R}), \quad u \notin D(A_1).$$

Moreover, by a direct computation one checks that $D^2u \in L^2(\mathbb{R})$, hence $u \in H^2(\mathbb{R})$.

Using the same construction as in Lemma 3.32, but working in $H^2(\mathbb{R})$ instead of $H^1(\mathbb{R})$, there exists a sequence $(u_n) \subset C_0^\infty(\mathbb{R} \setminus \{0\}) = D(A_1)$ such that

$$u_n \rightarrow u \quad \text{in } H^2(\mathbb{R}).$$

In particular,

$$u_n \rightarrow u, \quad D^2u_n \rightarrow D^2u \quad \text{in } L^2(\mathbb{R}).$$

Since $A_1u_n = D^2u_n$, we obtain

$$A_1u_n \rightarrow D^2u \quad \text{in } L^2(\mathbb{R}).$$

Thus

$$(u_n, A_1u_n) \rightarrow (u, D^2u) \quad \text{in } L^2 \times L^2.$$

However, $u \notin D(A_1)$, hence $(u, D^2u) \notin \Gamma(A_1)$. Therefore, the graph $\Gamma(A_1)$ is not closed, and A_1 is not closed. $\square$

The following proposition gives an explicit characterization of $D(\overline{A_1})$.

Proposition 4.18. *Assume that A_1 is the operator defined by (4.2.1). A function u belongs to $D(\overline{A_1})$ if and only if the following conditions hold:*

$$(i) \quad u \in L^2(\mathbb{R}, \mathbb{C}),$$

(ii) $u \in H_{\text{loc}}^2(\mathbb{R} \setminus \{0\}, \mathbb{C})$ and $u : \mathbb{R} \rightarrow \mathbb{C}$ is locally absolutely continuous together with its first derivative, and

$$\int_{\mathbb{R}} |D^2 u(t)|^2 dt < \infty,$$

(iii) The boundary conditions

$$u(0+) = u(0-) = 0, \quad u'(0+) = u'(0-) = 0.$$

Moreover,

$$\overline{A_1} u = D^2 u.$$

Proof of Proposition 4.18, Part 1. The proof follows the same method as in Proposition 3.34, Part 1. Let $u \in D(\overline{A_1})$. Then there exist $(u_n) \subset D(A_1)$ and $z \in L^2(\mathbb{R})$ such that

$$u_n \rightarrow u, \quad A_1 u_n = D^2 u_n \rightarrow z \quad \text{in } L^2(\mathbb{R}).$$

Thus

$$D^2 u_n \rightarrow z \quad \text{in } L^2(\mathbb{R}).$$

Using the same argument as in the first-order case (based on the Cauchy–Schwarz inequality and uniform convergence on compact sets), we deduce that $u \in H_{\text{loc}}^2(\mathbb{R} \setminus \{0\})$ and

$$D^2 u = z.$$

Since each $u_n \in C_0^\infty(\mathbb{R} \setminus \{0\})$, we have

$$u_n(0) = 0, \quad u_n'(0) = 0.$$

Passing to the limit uniformly on compact intervals yields

$$u(0\pm) = 0, \quad u'(0\pm) = 0.$$

This proves (i)–(iii), and the identity

$$\overline{A_1} u = D^2 u$$

follows directly from the definition of the closure. □

Proof of Proposition 4.18, Part 2. The proof follows the same method as in Proposition 3.34, Part 2. Let u satisfy conditions (i)–(iii). Then

$$u|_{(-\infty, 0)} \in H^2((-\infty, 0)), \quad u|_{(0, \infty)} \in H^2((0, \infty)).$$

Since $u(0\pm) = 0$ and $u'(0\pm) = 0$, it follows that

$$u|_{(-\infty, 0)} \in H_0^2((-\infty, 0)), \quad u|_{(0, \infty)} \in H_0^2((0, \infty)).$$

By definition of H_0^2 , there exist sequences

$$f_n \in C_0^\infty((-\infty, 0)), \quad g_n \in C_0^\infty((0, \infty))$$

such that

$$f_n \rightarrow u|_{(-\infty, 0)} \quad \text{in } H^2((-\infty, 0)), \quad g_n \rightarrow u|_{(0, \infty)} \quad \text{in } H^2((0, \infty)).$$

Define

$$u_n(x) = \begin{cases} f_n(x), & x < 0, \\ g_n(x), & x > 0, \\ 0, & x = 0. \end{cases}$$

Then $u_n \in C_0^\infty(\mathbb{R} \setminus \{0\}) = D(A_1)$.

Moreover,

$$u_n \rightarrow u, \quad D^2 u_n \rightarrow D^2 u \quad \text{in } L^2(\mathbb{R}).$$

Hence

$$A_1 u_n = D^2 u_n \rightarrow D^2 u,$$

and therefore

$$u \in D(\overline{A_1}), \quad \overline{A_1} u = D^2 u.$$

□

4.2.5 Finding all self-adjoint operators extensions for the second-order differential operator (Laplacian) on the punctured line

We now apply von Neumann's theorem to the closed symmetric operator $\overline{A_1}$.

By Corollary 4.16,

$$n_+(\overline{A_1}) = n_-(\overline{A_1}) = 2.$$

Hence the self-adjoint extensions of $\overline{A_1}$ are in one-to-one correspondence with unitary operators

$$\Gamma : H_+ \longrightarrow H_-,$$

where

$$H_+ = \ker(iI - A_1^*), \quad H_- = \ker(iI + A_1^*).$$

Since

$$\dim H_+ = \dim H_- = 2,$$

the unitary maps $H_+ \rightarrow H_-$ may be represented, after choosing orthonormal bases, by matrices in $U(2)$.

Moreover, the orthonormal bases of the deficiency subspaces are the same for A_1 and $\overline{A_1}$, i.e.

$$\begin{aligned} H_+ &:= \ker(iI - \overline{A_1}^*) = \text{span}\{\tilde{e}_1, \tilde{e}_2\}, \\ H_- &:= \ker(iI + \overline{A_1}^*) = \text{span}\{\tilde{f}_1, \tilde{f}_2\}, \end{aligned} \tag{4.2.10}$$

with $\tilde{e}_j, \tilde{f}_j$, $j = 1, 2$ as in (4.2.15) and (4.2.17), see also Tables 4.1–4.2 for their boundary values.

Recall the following definition

Definition 4.19. (i) The symbol $\mathcal{U}(H_+, H_-)$ denotes the set of all unitary operators $\Gamma : H_+ \rightarrow H_-$, i.e. by Definition 2.69, $\Gamma : H_+ \rightarrow H_-$ is a bijection satisfying

$$\langle \Gamma u, \Gamma v \rangle = \langle u, v \rangle \text{ for all } u, v \in H_+.$$

(ii) The symbol $U(2)$ denotes the group of all 2×2 unitary complex matrices. A matrix $M = [m_{j\ell}] \in \mathbb{C}^{2 \times 2}$ belongs to $U(2)$ if and only if

$$M^* M = M M^* = I_2,$$

where M^* denotes the conjugate transpose of M , i.e.

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad M^* = \overline{M}^T = \begin{pmatrix} \overline{m_{11}} & \overline{m_{21}} \\ \overline{m_{12}} & \overline{m_{22}} \end{pmatrix}.$$

Equivalently, the columns (or rows) of M form an orthonormal basis of $\mathbb{C}^2$.

We now apply Theorem 2.60 to $\overline{A_1}$. Let us emphasize that we need to use operator $\overline{A_1}$ because Theorem 2.60 is only applicable to closed operators.

Note that Theorem 4.20 is closely related to Theorem 3.37.

In fact, by the extension theorem for closed symmetric operators with equal deficiency indices, see Theorem 2.60, every self-adjoint extension of $\overline{A_1}$ is of the form A_Γ , where $\Gamma : H_+ \rightarrow H_-$ is a unitary operator. Since $\dim H_+ = \dim H_- = 2$, we have $\mathcal{U}(H_+, H_-) \cong U(2)$, see Definition 4.19.

Theorem 4.20 (Self-adjoint extensions of $\overline{A_1}$). *Let $M = (m_{jk}) \in U(2)$ be the matrix and let $\Gamma : H_+ \rightarrow H_-$ be the unitary determined by M with respect to the bases in (4.2.10); i.e.,*

$$\Gamma(\tilde{e}_k) = m_{1k}\tilde{f}_1 + m_{2k}\tilde{f}_2.$$

Equivalently, see (4.2.18) or (4.2.19). Then the von Neumann formula (2.3.3) is as follows

$$\begin{aligned} D(A_\Gamma) &:= \left\{ u = u_0 + u_+ + \Gamma u_+ : u_0 \in D(\overline{A_1}), u_+ \in H_+ \right\}, \\ A_\Gamma(u_0 + u_+ + \Gamma u_+) &:= \overline{A_1} u_0 + i u_+ - i \Gamma u_+. \end{aligned} \tag{4.2.11}$$

Then A_Γ is self-adjoint in $L^2(\mathbb{R})$ and $\overline{A_1} \subset A_\Gamma$.

Moreover, every self-adjoint extension A_1 of $\overline{A_1}$ equals A_Γ for a unique

$$\boxed{\Gamma \in \mathcal{U}(H_+, H_-) \cong U(2)}.$$

Idea of proof. By Corollary 2.92 we have $(\overline{A_1})^* = A_1^*$ and $n_\pm(\overline{A_1}) = 2$. Hence the hypotheses of Theorem 2.60 are satisfied. Applying it is von Neumann parametrization with the two-dimensional deficiency spaces in (4.2.10), yielding (4.2.11). $\square$

Our Objective is to formulate and proof a result similar to Corollary 3.13.

Remark 4.21 (Explicit form of elements of $D(A_\Gamma)$ via boundary values). Using the explicit form of the orthonormal bases $\{\tilde{e}_1, \tilde{e}_2\}$ and $\{\tilde{f}_1, \tilde{f}_2\}$ and their boundary values (Tables 4.1–4.2), every $u \in D(A_\Gamma)$ can be written in the following concrete form

$$u = u_0 + s_1(\tilde{e}_1 + m_{11}\tilde{f}_1 + m_{21}\tilde{f}_2) + s_2(\tilde{e}_2 + m_{12}\tilde{f}_1 + m_{22}\tilde{f}_2),$$

with $u_0 \in D(\overline{A_1})$ and $(s_1, s_2) \in \mathbb{C}^2$.

We now state a result completely analogous to Theorem 3.7 for the first-order operator.

Corollary 4.22 (Classification of all self-adjoint extensions of $\overline{A_1}$). *Let $\text{SA}(\overline{A_1})$ denote the set of all self-adjoint extensions of the closure of the Laplacian A_1 on $\mathbb{R} \setminus \{0\}$. Then*

$$\boxed{\text{SA}(\overline{A_1}) = \{ A_\Gamma : \Gamma \in \mathcal{U}(H_+, H_-) \} \cong U(2).} \quad (4.2.12)$$

In other words, every self-adjoint extension of $\overline{A_1}$ is uniquely determined by a unitary matrix $M \in U(2)$, and conversely every $M \in U(2)$ produces a (unique) self-adjoint extension A_Γ .

Proof of Corollary 4.22. (i) Let A be any self-adjoint extension of $\overline{A_1}$. By Theorem 2.60, there exists a unique unitary operator $\Gamma : H_+ \rightarrow H_-$ such that $A = A_\Gamma$.

(ii) Conversely, Theorem 4.20 asserts that for every unitary map $\Gamma : H_+ \rightarrow H_-$, the operator A_Γ defined by (4.2.11) is self-adjoint.

Combining (i) and (ii) yields the formula (4.2.12). Since $\dim H_+ = \dim H_- = 2$, we have $\mathcal{U}(H_+, H_-) \cong U(2)$, which completes the proof. □

In what follows we plan to present a generalization of Corollary 3.13 to the current setting.

With the abstract description of all self-adjoint extensions of $\overline{A_1}$ in place, we now turn to an explicit realization of these extensions for the Laplacian on $\mathbb{R} \setminus \{0\}$.

Our next goal is to compute the deficiency bases, their boundary values at $0\pm$, and the corresponding boundary conditions satisfied by functions in $D(A_\Gamma)$.

More precisely:

- (i) We recall the deficiency subspaces H_+ and H_- from Subsection 4.2 and construct orthonormal bases adapted to the problem.
- (ii) We verify orthogonality and normalization and compute the boundary values of the basis functions and their derivatives at 0^+ and 0^- .
- (iii) These explicit computations will allow us, in the next subsection, to express the domains $D(A_\Gamma)$ in terms of concrete boundary conditions coupling $u(0\pm)$ and $u'(0\pm)$.

Deficiency subspace H_+ . From (4.2.4)–(4.2.5) in Subsection 4.2, we found

$$H_+ = \text{span}\{e_1, e_2\}$$

where

$$e_1(x) = 1_{(0,\infty)}(x)e^{-\frac{\sqrt{2}}{2}(1+i)x}, \quad e_2(x) = 1_{(-\infty,0)}(x)e^{\frac{\sqrt{2}}{2}(1+i)x},$$

and

$$\mathbf{1}_{(0,\infty)}(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \quad \mathbf{1}_{(-\infty,0)}(x) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases}$$

Now, we will check if this an orthogonal basis, which means

$$\langle e_1, e_2 \rangle_{L^2} = 0 \tag{4.2.13}$$

$$\|e_1\| = \|e_2\| = 1. \tag{4.2.14}$$

We will start with the first condition (4.2.13) is true, i.e. if we take

$$\begin{aligned} \langle e_1, e_2 \rangle_{L^2} &= \int_{-\infty}^{\infty} e_1 \bar{e}_2 \, dx \\ &= \int_{-\infty}^{\infty} 1_{(0,\infty)}(x)e^{-\frac{\sqrt{2}}{2}(1+i)x} \cdot 1_{(-\infty,0)}(x)e^{\frac{\sqrt{2}}{2}(1+i)x} \, dx \\ &= \int_{-\infty}^{\infty} 0 \, dx = 0. \end{aligned}$$

So, the condition (4.2.13) holds.

Now, to check the second condition (4.2.14), we take

$$\begin{aligned} \|e_1\|_{L^2}^2 &= \int_{-\infty}^{\infty} |e_1(x)|^2 \, dx = \int_0^{\infty} |e^{-\frac{\sqrt{2}}{2}x} e^{\frac{i\sqrt{2}}{2}x}| \, dx \\ &= \int_0^{\infty} |e^{-\frac{\sqrt{2}}{2}x}|^2 |e^{\frac{i\sqrt{2}}{2}x}|^2 \, dx = \int_0^{\infty} e^{-\sqrt{2}x} \, dx \\ &= \frac{-1}{\sqrt{2}} e^{-\sqrt{2}x} \Big|_0^{\infty} = \frac{1}{\sqrt{2}}. \end{aligned}$$

Similarly for $\|e_2\|_{L^2}^2$. Hence,

$$\|e_1\|_{L^2} = \|e_2\|_{L^2} = 2^{-1/4}.$$

We define the normalized functions

$$\begin{cases} \tilde{e}_1(x) = 2^{\frac{1}{4}} e_1(x) = 2^{\frac{1}{4}} 1_{(0,\infty)}(x) e^{-\frac{\sqrt{2}}{2}(1+i)x}, \\ \tilde{e}_2(x) = 2^{\frac{1}{4}} e_2(x) = 2^{\frac{1}{4}} 1_{(-\infty,0)}(x) e^{\frac{\sqrt{2}}{2}(1+i)x}, \end{cases} \quad x \in \mathbb{R}. \tag{4.2.15}$$

So that

$$\|\tilde{e}_1\|_{L^2} = \|\tilde{e}_2\|_{L^2} = 1, \quad \langle \tilde{e}_1, \tilde{e}_2 \rangle_{L^2} = 0.$$

Thus $\{\tilde{e}_1, \tilde{e}_2\}$ is an orthonormal basis of H_+ .

Note that the derivatives of $\tilde{e}_1(x)$ and $\tilde{e}_2(x)$ are

$$\begin{aligned}\tilde{e}'_1(x) &= -2^{-\frac{1}{4}}1_{(0,\infty)}(x)(1+i)e^{-\frac{\sqrt{2}}{2}(1+i)x}, \\ \tilde{e}'_2(x) &= 2^{-\frac{1}{4}}1_{(-\infty,0)}(x)(1+i)e^{\frac{\sqrt{2}}{2}(1+i)x}.\end{aligned}$$

Evaluating at $0\pm$ we get the boundary values summarized in Table 4.1.

	$\tilde{e}_1$	$\tilde{e}_2$	$\tilde{e}'_1$	$\tilde{e}'_2$
0^+	$2^{1/4}$	0	$-2^{-1/4}(1+i)$	0
0^-	0	$2^{1/4}$	0	$2^{-1/4}(1+i)$

TABLE 4.1: Boundary values of the functions $\tilde{e}_1$, $\tilde{e}_2$ and their derivatives.

Every $u_+ \in H_+$ has a unique representation

$$u_+ = s_1\tilde{e}_1 + s_2\tilde{e}_2, \quad s_1, s_2 \in \mathbb{C}. \quad (4.2.16)$$

Deficiency subspace H_- . From (4.2.8)–(4.2.9) in Subsection 4.2 we similarly have

$$H_- = \text{span}\{f_1, f_2\}$$

where

$$\begin{cases} f_1(x) = 1_{(0,\infty)}(x)e^{-\frac{\sqrt{2}}{2}(1+i)x} \\ f_2(x) = 1_{(-\infty,0)}(x)e^{\frac{\sqrt{2}}{2}(1+i)x}. \end{cases} \quad x \in \mathbb{R}.$$

As above,

$$\|f_1\|_{L^2}^2 = \|f_2\|_{L^2}^2 = \frac{1}{\sqrt{2}}, \quad \langle f_1, f_2 \rangle_{L^2} = 0.$$

We define

$$\begin{cases} \tilde{f}_1(x) = 2^{\frac{1}{4}}f_1(x) = 2^{\frac{1}{4}}1_{(0,\infty)}(x)e^{-\frac{\sqrt{2}}{2}(1+i)x} \\ \tilde{f}_2(x) = 2^{\frac{1}{4}}f_2(x) = 2^{\frac{1}{4}}1_{(-\infty,0)}(x)e^{\frac{\sqrt{2}}{2}(1+i)x}. \end{cases} \quad x \in \mathbb{R}. \quad (4.2.17)$$

so that $\{\tilde{f}_1, \tilde{f}_2\}$ is an orthonormal basis of H_- .

Differentiating, we get

$$\begin{cases} \tilde{f}'_1(x) = -2^{-\frac{1}{4}}1_{(0,\infty)}(x)(1+i)e^{-\frac{\sqrt{2}}{2}(1+i)x} \\ \tilde{f}'_2(x) = 2^{-\frac{1}{4}}(1+i)1_{(-\infty,0)}(x)e^{\frac{\sqrt{2}}{2}(1+i)x}. \end{cases} \quad x \in \mathbb{R}.$$

The boundary values at $0\pm$ coincide with those in Table 4.2.

	$\tilde{f}_1$	$\tilde{f}_2$	$\tilde{f}'_1$	$\tilde{f}'_2$
0^+	$2^{1/4}$	0	$-2^{-1/4}(1+i)$	0
0^-	0	$2^{1/4}$	0	$2^{-1/4}(1+i)$

TABLE 4.2: Boundary values of the functions $\tilde{f}_1$, $\tilde{f}_2$ and their derivatives.

Thus

$$H_- = \text{span}\{\tilde{f}_1, \tilde{f}_2\}, \quad \dim H_- = 2.$$

From Case 1 and 2: we proved that the space H_+ has orthonormal basis $\tilde{e}_1, \tilde{e}_2$, and the space H_- with orthonormal basis $\tilde{f}_1, \tilde{f}_2$.

We proved that every element of H_+ is of the form

$$\sum_{k=1}^2 s_k \tilde{e}_k = s_1 \tilde{e}_1 + s_2 \tilde{e}_2.$$

Unitary maps $\Gamma : H_+ \rightarrow H_-$. Let us recall that by Theorem 4.20, every self-adjoint extension of $\overline{A_1}$ is determined by a unitary operator

$$\Gamma : H_+ \rightarrow H_-,$$

which, relative to the orthonormal bases $\{\tilde{e}_1, \tilde{e}_2\}$ and $\{\tilde{f}_1, \tilde{f}_2\}$, is represented by a unitary matrix $M \in U(2)$:

$$M = \begin{pmatrix} a & b \\ -e^{i\phi}\bar{b} & e^{i\phi}\bar{a} \end{pmatrix}, \quad |a|^2 + |b|^2 = 1, \quad \phi \in [0, 2\pi). \quad (4.2.18)$$

the adjoint matrix is

$$M^* = \begin{pmatrix} \bar{a} & -e^{-i\phi}b \\ \bar{b} & e^{-i\phi}a \end{pmatrix}.$$

A direct computation shows that

$$M^*M = MM^* = I_2,$$

Thus $M \in U(2)$.

This is one of the standard parametrization of $U(2)$; see https://en.wikipedia.org/wiki/Unitary_matrix.

Another representation is the following one

$$M = e^{i\phi/2} \begin{pmatrix} e^{i\alpha} \cos \theta & e^{i\beta} \sin \theta \\ -e^{-i\beta} \sin \theta & e^{-i\alpha} \cos \theta \end{pmatrix} \quad (4.2.19)$$

where $\alpha, \beta, \phi, \theta \in [0, 2\pi)$.

If I have matrix M in (4.2.18), then

$$\begin{aligned} M \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} &= \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ -e^{i\phi}\bar{b} & e^{i\phi}\bar{a} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} as_1 + bs_2 \\ -e^{i\phi}\bar{b}s_1 + e^{i\phi}\bar{a}s_2 \end{pmatrix}. \end{aligned}$$

Therefore

$$\begin{aligned} \Gamma \left(\sum_{k=1}^2 s_k \tilde{e}_k \right) &= (m_{11}s_1 + m_{12}s_2) \tilde{f}_1 + (m_{21}s_1 + m_{22}s_2) \tilde{f}_2 \\ &= (as_1 + bs_2) \tilde{f}_1 + (-e^{i\phi}\bar{b}s_1 + e^{i\phi}\bar{a}s_2) \tilde{f}_2. \end{aligned} \quad (4.2.20)$$

In terms of the bases, this means

$$\begin{aligned} \Gamma(\tilde{e}_1) &= m_{11}\tilde{f}_1 + m_{21}\tilde{f}_2, & \Gamma(\tilde{e}_2) &= m_{12}\tilde{f}_1 + m_{22}\tilde{f}_2. \\ \Gamma(\tilde{e}_1) &= a\tilde{f}_1 + e^{-i\phi}\bar{b}\tilde{f}_2, & \Gamma(\tilde{e}_2) &= b\tilde{f}_1 + e^{i\phi}\bar{a}\tilde{f}_2. \end{aligned} \quad (4.2.21)$$

If $u_+ \in H_+$ is written as in (4.2.16),

$$u_+ = s_1 \tilde{e}_1 + s_2 \tilde{e}_2,$$

then

$$\begin{aligned} \Gamma u_+ &= s_1 \Gamma(\tilde{e}_1) + s_2 \Gamma(\tilde{e}_2) \\ &= (as_1 + bs_2) \tilde{f}_1 + (-e^{i\phi}\bar{b}s_1 + e^{i\phi}\bar{a}s_2) \tilde{f}_2. \end{aligned} \quad (4.2.22)$$

Decomposition of elements of $D(A_\Gamma)$. By Theorem 4.20, the domain of the self-adjoint extension A_Γ is given by

$$D(A_\Gamma) = \{u = u_0 + u_+ + \Gamma u_+ : u_0 \in D(\overline{A_1}), u_+ \in H_+\}.$$

Lemma 4.23. Assume that $s_1, s_2 \in \mathbb{C}$ and $x_+, x_-, y_+, y_- \in \mathbb{C}$, and assume that $\Delta \neq 0$. Then the following system of equations

$$\begin{aligned} y_+ &= -2^{-1/4}(1+i)((1+a)s_1 + bs_2), \\ y_- &= 2^{-1/4}(1+i)(-e^{i\phi}\bar{b}s_1 + (1+e^{i\phi}\bar{a})s_2), \\ x_+ &= 2^{1/4}((1+a)s_1 + bs_2), \\ x_- &= 2^{1/4}(-e^{i\phi}\bar{b}s_1 + (1+e^{i\phi}\bar{a})s_2). \end{aligned}$$

is satisfied if and only if the next system

$$\begin{aligned} s_1 &= \frac{(1+e^{i\phi}\bar{a})x_+ - bx_-}{2^{1/4}\Delta}, \\ s_2 &= \frac{e^{i\phi}\bar{b}x_+ + (1+a)x_-}{2^{1/4}\Delta}, \\ (1+i)((1+e^{i\phi}\bar{a})x_+ - bx_-) + \sqrt{2}((1+e^{i\phi}\bar{a})y_+ + by_-) &= 0, \\ (1+i)(e^{i\phi}\bar{b}x_+ + (1+a)x_-) + \sqrt{2}(e^{i\phi}\bar{b}y_+ - (1+a)y_-) &= 0. \end{aligned}$$

is also satisfied.

Proof of Lemma 4.23. Denote $y = (y_+, y_-) \in \mathbb{C}^2$, $x = (x_+, x_-) \in \mathbb{C}^2$ and $s = (s_1, s_2) \in \mathbb{C}^2$. Then the 1st system of 4 equations is equivalent to

$$\begin{aligned} y &= Bs, \\ x &= Cs \end{aligned}$$

for some concrete 2×2 matrices B and C determined by the above coefficient. Since $\Delta \neq 0$, the matrix C is invertible.

The 2nd system of 4 equations is equivalent to

$$\begin{aligned} s &= C^{-1}x, \\ C^{-1}x - B^{-1}y &= 0 \end{aligned}$$

for the same matrices B and C .

One can easily prove that these two systems are equivalent.

- (i) If the first system holds, then $x = Cs$ implies $s = C^{-1}x$, and applying B^{-1} to $y = Bs$ yields $C^{-1}x - B^{-1}y = 0$.
- (ii) Conversely, if the second system holds, then $s = C^{-1}x$ implies $x = Cs$, and applying B to $C^{-1}x - B^{-1}y = 0$ gives $y = Bs$.

Thus the two systems are equivalent.

The proof is complete. $\square$

Assume that $u \in D(A_\Gamma)$. Using (4.2.16) and (4.2.22), u can be written as

$$\begin{aligned} u &= u_0 + s_1 \tilde{e}_1 + s_2 \tilde{e}_2 + \Gamma(s_1 \tilde{e}_1 + s_2 \tilde{e}_2) \\ &= u_0 + s_1 (\tilde{e}_1 + \Gamma(\tilde{e}_1)) + s_2 (\tilde{e}_2 + \Gamma(\tilde{e}_2)) \\ &= u_0 + s_1 (\tilde{e}_1 + m_{11} \tilde{f}_1 + m_{21} \tilde{f}_2) + s_2 (\tilde{e}_2 + m_{12} \tilde{f}_1 + m_{22} \tilde{f}_2), \end{aligned} \quad (4.2.23)$$

for some $u_0 \in D(\overline{A_1})$ and $s_1, s_2 \in \mathbb{C}$.

From the Tables 4.1–4.2, we have the following boundary values of the ONB's $\tilde{e}_1, \tilde{e}_2$ of H_+ and $\tilde{f}_1, \tilde{f}_2$ of H_-

$$\begin{aligned} \tilde{e}_1(0+) &= 2^{1/4}, \quad \tilde{e}_2(0+) = 0, \quad \tilde{f}_1(0+) = 2^{1/4}, \quad \tilde{f}_2(0+) = 0, \\ \tilde{e}_1(0-) &= 0, \quad \tilde{e}_2(0-) = 2^{1/4}, \quad \tilde{f}_1(0-) = 0, \quad \tilde{f}_2(0-) = 2^{1/4}, \end{aligned}$$

Hence, by (4.2.23) and we infer that

$$\begin{aligned} u(0+) &= u_0(0+) + u_+(0+) + (\Gamma u_+)(0+) \\ &= 0 + s_1 (\tilde{e}_1(0+) + m_{11} \tilde{f}_1(0+) + m_{21} \tilde{f}_2(0+)) \\ &\quad + s_2 (\tilde{e}_2(0+) + m_{12} \tilde{f}_1(0+) + m_{22} \tilde{f}_2(0+)) \\ &= s_1 (2^{\frac{1}{4}} + 2^{\frac{1}{4}} a - e^{i\phi} \bar{b} \cdot (0)) + s_2 (0 + 2^{\frac{1}{4}} b + e^{i\phi} \bar{a} \cdot (0)) \\ &= 2^{\frac{1}{4}} (1 + a) s_1 + 2^{\frac{1}{4}} b s_2 = 2^{\frac{1}{4}} \boxed{((1 + a) s_1 + b s_2)}. \end{aligned}$$

and

$$\begin{aligned} u(0-) &= u_0(0-) + u_+(0-) + (\Gamma u_+)(0-) \\ &= 0 + s_1 (\tilde{e}_1(0-) + m_{11} \tilde{f}_1(0-) + m_{21} \tilde{f}_2(0-)) \\ &\quad + s_2 (\tilde{e}_2(0-) + m_{12} \tilde{f}_1(0-) + m_{22} \tilde{f}_2(0-)) \\ &= s_1 (0 + a \cdot (0) - 2^{\frac{1}{4}} e^{i\phi} \bar{b}) + s_2 (2^{\frac{1}{4}} + b \cdot (0) + 2^{\frac{1}{4}} e^{i\phi} \bar{a}) \\ &= -2^{\frac{1}{4}} e^{i\phi} \bar{b} s_1 + 2^{\frac{1}{4}} s_2 + 2^{\frac{1}{4}} e^{i\phi} \bar{a} s_2 = 2^{\frac{1}{4}} \boxed{(-e^{i\phi} \bar{b} s_1 + (1 + e^{i\phi} \bar{a}) s_2)} \end{aligned}$$

From the above computations we obtained the boundary values

$$\boxed{\begin{aligned} u(0+) &= 2^{1/4} ((1 + a) s_1 + b s_2), \\ u(0-) &= 2^{1/4} (-e^{i\phi} \bar{b} s_1 + (1 + e^{i\phi} \bar{a}) s_2). \end{aligned}}$$

These two relations form a linear system for the unknowns s_1 and s_2 . We now rewrite this system in a convenient algebraic form and solve it.

Step 1: Divide out the common factor. Put

$$x := \frac{u(0+)}{2^{1/4}}, \quad y := \frac{u(0-)}{2^{1/4}}.$$

Then the boundary system becomes

$$\begin{cases} (1+a)s_1 + bs_2 = x, \\ -e^{i\phi\bar{b}}s_1 + (1+e^{i\phi\bar{a}})s_2 = y. \end{cases}$$

In matrix form,

$$A \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} 1+a & b \\ -e^{i\phi\bar{b}} & 1+e^{i\phi\bar{a}} \end{pmatrix}.$$

Since $M \in U(2)$, the matrix A is invertible.

Step 2: Compute the determinant.

$$\Delta = \det(A) = (1+a)(1+e^{i\phi\bar{a}}) - b(-e^{i\phi\bar{b}}) = (1+a)(1+e^{i\phi\bar{a}}) + e^{i\phi}|b|^2.$$

Since A is invertible (because the matrix comes from a unitary matrix $M \in U(2)$, and satisfy $M^*M = MM^* = I_2$), we infer that $\Delta \neq 0$.

Step 3: Solve using Cramer's rule. (Cramer's rule: see Wikipedia or ChiliMath.)

$$s_1 = \frac{1}{\Delta} \begin{vmatrix} x & b \\ y & 1+e^{i\phi\bar{a}} \end{vmatrix} = \frac{(1+e^{i\phi\bar{a}})x - by}{\Delta},$$

$$s_2 = \frac{1}{\Delta} \begin{vmatrix} 1+a & x \\ -e^{i\phi\bar{b}} & y \end{vmatrix} = \frac{e^{i\phi\bar{b}}x + (1+a)y}{\Delta}.$$

Step 4: Substitute back x, y in terms of $u(0\pm)$.

Since

$$x = \frac{u(0+)}{2^{1/4}}, \quad y = \frac{u(0-)}{2^{1/4}},$$

we obtain

$$\boxed{\begin{aligned} s_1 &= \frac{(1+e^{i\phi\bar{a}})u(0+) - bu(0-)}{2^{1/4}\Delta}, \\ s_2 &= \frac{e^{i\phi\bar{b}}u(0+) + (1+a)u(0-)}{2^{1/4}\Delta}. \end{aligned}} \quad (4.2.24)$$

where

$$\Delta = (1 + a)(1 + e^{i\phi}\bar{a}) + e^{i\phi}|b|^2.$$

Derivative boundary system. Using the boundary values in Tables 4.1–4.2, we get

$$\tilde{e}'_1(0+) = -2^{-1/4}(1 + i), \quad \tilde{e}'_2(0+) = 0,$$

$$\tilde{f}'_1(0+) = -2^{-1/4}(1 + i), \quad \tilde{f}'_2(0+) = 0.$$

Hence, by (4.2.23) and we infer that

$$\begin{aligned} u'(0+) &= s_1(\tilde{e}'_1(0+) + m_{11}\tilde{f}'_1(0+) + m_{21}\tilde{f}'_2(0+)) \\ &\quad + s_2(\tilde{e}'_2(0+) + m_{12}\tilde{f}'_1(0+) + m_{22}\tilde{f}'_2(0+)) \\ &= s_1[-2^{-1/4}(1 + i) - m_{11}2^{-1/4}(1 + i)] + s_2[-m_{12}2^{-1/4}(1 + i)]. \end{aligned}$$

Since $m_{11} = a$ and $m_{12} = b$, this simplifies to

$$u'(0+) = -2^{-1/4}(1 + i)((1 + a)s_1 + bs_2). \quad (4.2.25)$$

Analogously, since

$$\tilde{e}'_1(0-) = 0, \quad \tilde{e}'_2(0-) = 2^{-1/4}(1 + i),$$

$$\tilde{f}'_1(0-) = 0, \quad \tilde{f}'_2(0-) = 2^{-1/4}(1 + i),$$

we infer

$$\begin{aligned} u'(0-) &= s_1(\tilde{e}'_1(0-) + m_{11}\tilde{f}'_1(0-) + m_{21}\tilde{f}'_2(0-)) \\ &\quad + s_2(\tilde{e}'_2(0-) + m_{12}\tilde{f}'_1(0-) + m_{22}\tilde{f}'_2(0-)) \\ &= s_1[m_{21}2^{-1/4}(1 + i)] + s_2[2^{-1/4}(1 + i) + m_{22}2^{-1/4}(1 + i)]. \end{aligned}$$

Using $m_{21} = -e^{i\phi}\bar{b}$ and $m_{22} = e^{i\phi}\bar{a}$, we find

$$u'(0-) = 2^{-1/4}(1 + i)(-e^{i\phi}\bar{b}s_1 + (1 + e^{i\phi}\bar{a})s_2). \quad (4.2.26)$$

Combining equations (4.2.25) and (4.2.26), we have

$$\boxed{\begin{aligned} u'(0+) &= -2^{-1/4}(1 + i)((1 + a)s_1 + bs_2), \\ u'(0-) &= 2^{-1/4}(1 + i)(-e^{i\phi}\bar{b}s_1 + (1 + e^{i\phi}\bar{a})s_2). \end{aligned}} \quad (4.2.27)$$

These formulas will later be combined with the corresponding expressions for $u(0\pm)$ to obtain the boundary conditions defining $D(A_\Gamma)$.

From the formulas (4.2.27), the equations are completely analogous to the boundary relations for $u(0\pm)$; the same linear-algebraic method gives explicit expressions for s_1 and s_2 .

Step 1: Divide out the common factor. Put

$$x' := \frac{u'(0+)}{-2^{-1/4}(1+i)}, \quad y' := \frac{u'(0-)}{2^{-1/4}(1+i)}. \quad (4.2.28)$$

Then the system becomes

$$\begin{cases} (1+a)s_1 + bs_2 = x', \\ -e^{i\phi}\bar{b}s_1 + (1+e^{i\phi}\bar{a})s_2 = y'. \end{cases}$$

In matrix form,

$$A \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}, \quad A = \begin{pmatrix} 1+a & b \\ -e^{i\phi}\bar{b} & 1+e^{i\phi}\bar{a} \end{pmatrix}.$$

As before, A is invertible because it comes from a unitary matrix $M \in U(2)$.

Step 2: Determinant.

$$\Delta = \det(A) = (1+a)(1+e^{i\phi}\bar{a}) + e^{i\phi}|b|^2.$$

Step 3: Solve using Cramer's rule.

$$s_1 = \frac{(1+e^{i\phi}\bar{a})x' - by'}{\Delta},$$

$$s_2 = \frac{e^{i\phi}\bar{b}x' + (1+a)y'}{\Delta}.$$

Step 4: Substitute back the definitions of x' and y' from (4.2.28).

We deduce that

$$s_1 = \frac{(1+e^{i\phi}\bar{a}) \frac{u'(0+)}{-2^{-1/4}(1+i)} - b \frac{u'(0-)}{2^{-1/4}(1+i)}}{\Delta},$$

$$s_2 = \frac{e^{i\phi}\bar{b} \frac{u'(0+)}{-2^{-1/4}(1+i)} + (1+a) \frac{u'(0-)}{2^{-1/4}(1+i)}}{\Delta}.$$

Thus

$$\boxed{\begin{aligned} s_1 &= \frac{-(1 + e^{i\phi} \bar{a}) u'(0+) - b u'(0-)}{2^{-1/4}(1+i) \Delta}, \\ s_2 &= \frac{-e^{i\phi} \bar{b} u'(0+) + (1+a) u'(0-)}{2^{-1/4}(1+i) \Delta}, \end{aligned}} \quad (4.2.29)$$

where

$$\Delta = (1+a)(1 + e^{i\phi} \bar{a}) - e^{-i\phi} |b|^2.$$

From (4.2.24) and (4.2.29) we know that the same constants s_1, s_2 can be expressed both in terms of $(u(0+), u(0-))$ and in terms of $(u'(0+), u'(0-))$.

Indeed, for s_1 we have

$$\frac{(1 + e^{i\phi} \bar{a}) u(0+) - b u(0-)}{2^{1/4} \Delta} = \frac{-(1 + e^{i\phi} \bar{a}) u'(0+) - b u'(0-)}{2^{-1/4}(1+i) \Delta}.$$

Cancelling Δ and multiplying both sides by $2^{1/4} \cdot 2^{-1/4} = \sqrt{2}$, we obtain

$$(1+i)((1 + e^{i\phi} \bar{a})u(0+) - b u(0-)) + \sqrt{2}((1 + e^{i\phi} \bar{a})u'(0+) + b u'(0-)) = 0.$$

Similarly, equating the two formulas for s_2 gives

$$\frac{e^{i\phi} \bar{b} u(0+) + (1+a) u(0-)}{2^{1/4} \Delta} = \frac{-e^{i\phi} \bar{b} u'(0+) + (1+a) u'(0-)}{2^{-1/4}(1+i) \Delta},$$

hence

$$(1+i)(e^{i\phi} \bar{b} u(0+) + (1+a) u(0-)) + \sqrt{2}(e^{i\phi} \bar{b} u'(0+) - (1+a) u'(0-)) = 0.$$

Therefore, the following pair of linear boundary conditions

$$\boxed{\begin{aligned} (1+i)((1 + e^{i\phi} \bar{a})u(0+) - b u(0-)) + \sqrt{2}((1 + e^{i\phi} \bar{a})u'(0+) + b u'(0-)) &= 0, \\ (1+i)(e^{i\phi} \bar{b} u(0+) + (1+a) u(0-)) + \sqrt{2}(e^{i\phi} \bar{b} u'(0+) - (1+a) u'(0-)) &= 0. \end{aligned}} \quad (4.2.30)$$

To simplify the boundary conditions, we put $u(0+) = u(0-) = x$ and $u'(0+) = u'(0-) = y$, we get

$$\begin{aligned} (1+i)((1 + e^{i\phi} \bar{a}) - b)x + \sqrt{2}((1 + e^{i\phi} \bar{a}) + b)y &= 0, \\ (1+i)(e^{i\phi} \bar{b} + (1+a))x + \sqrt{2}(e^{i\phi} \bar{b} - (1+a))y &= 0. \end{aligned}$$

4.2.6 Nontrivial Solutions of a Complex Linear System

We determine all coefficients

$$\phi \in [0, 2\pi), \quad a, b \in \mathbb{C}, \quad |a|^2 + |b|^2 = 1,$$

for which the linear system with unknowns x, y ,

$$\begin{aligned} (1+i)((1+e^{i\phi}\bar{a})-b)x + \sqrt{2}((1+e^{i\phi}\bar{a})+b)y &= 0, \\ (1+i)(e^{i\phi}\bar{b}+(1+a))x + \sqrt{2}(e^{i\phi}\bar{b}-(1+a))y &= 0, \end{aligned} \tag{4.2.31}$$

admits a nontrivial solution $(x, y) \neq (0, 0)$.

To determine when the system (4.2.31) admits a nontrivial solution, we examine the determinant of its coefficient matrix. A nontrivial solution exists if and only if this determinant vanishes. Factoring out the nonzero scalar $(1+i)\sqrt{2}$, the condition reduces to

$$\det \begin{pmatrix} (1+e^{i\phi}\bar{a})-b & (1+e^{i\phi}\bar{a})+b \\ e^{i\phi}\bar{b}+(1+a) & e^{i\phi}\bar{b}-(1+a) \end{pmatrix} = 0.$$

A direct computation yields

$$-2\left((1+e^{i\phi}\bar{a})(1+a)+be^{i\phi}\bar{b}\right) = 0,$$

and hence

$$(1+e^{i\phi}\bar{a})(1+a)+e^{i\phi}|b|^2 = 0.$$

Using the constraint $|a|^2 + |b|^2 = 1$, this reduces to

$$(1+a)+e^{i\phi}(1+\bar{a}) = 0. \tag{4.2.32}$$

We have two cases as follow:

- (i) **Case 1:** $a = -1$: Then $|a|^2 = 1$ forces $b = 0$. Equation (4.2.32) is identically satisfied for every $\phi \in [0, 2\pi)$. Thus

$$(a, b, \phi) = (-1, 0, \phi), \quad \phi \in [0, 2\pi),$$

is a family of solutions.

(ii) **Case 2:** $a \neq -1$: Then $1 + \bar{a} \neq 0$, and (4.2.32) uniquely determines ϕ :

$$e^{i\phi} = -\frac{1+a}{1+\bar{a}} = -\frac{1+a}{\overline{(1+a)}}.$$

Since the right-hand side has modulus 1, such ϕ always exists and is unique modulo 2π . Writing $\theta = \arg(1+a)$, this can be expressed as

$$\phi \equiv \pi + 2\theta \pmod{2\pi}.$$

Here a is arbitrary with $|a| \leq 1$, $a \neq -1$, and b is arbitrary subject only to

$$|b| = \sqrt{1 - |a|^2}.$$

The system (4.2.31) admits a nontrivial solution if and only if

$$(1+a) + e^{i\phi}(1+\bar{a}) = 0 \quad \text{and} \quad |a|^2 + |b|^2 = 1.$$

This condition yields exactly the two families described above.

characterize exactly the domain $D(A_\Gamma)$ corresponding to the unitary matrix M in (4.2.18). We proved the following result.

Proposition 4.24. *Let*

$$M = [m_{kj}]_{k,j=1}^2 = \begin{pmatrix} a & b \\ -e^{i\phi}\bar{b} & e^{i\phi}\bar{a} \end{pmatrix} \in U(2), \quad |a|^2 + |b|^2 = 1,$$

and let $\Gamma : H_+ \rightarrow H_-$ be the unitary operator defined by

$$\Gamma(\tilde{e}_j) = \sum_{k=1}^2 m_{kj} \tilde{f}_k.$$

If $u \in D(A_\Gamma)$, then u satisfies the boundary conditions (4.2.30).

4.2.7 Converse construction for the Laplacian

In this subsection we prove the converse direction of the characterization of self-adjoint extensions of the Laplacian. Starting from a unitary map $\Gamma : H_+ \rightarrow H_-$, we derive the corresponding boundary conditions satisfied by functions in $D(A_\Gamma)$, and conversely show that any function satisfying these boundary conditions admits a decomposition of the form prescribed by von Neumann's theory, see Theorem 2.60.

The argument relies on expressing the boundary values of u and u' at the origin linearly in terms of the coefficients (s_1, s_2) associated with the deficiency subspaces, and on the invertibility of the resulting boundary matrices.

$$\begin{aligned}
D(A_\Gamma) &= \left\{ u \in D(\bar{A}_1^*) : u = u_0 + u_+ + \Gamma(u_+), \text{ for some } u_+ \in H_+ \right\}, \\
&= \left\{ u \in D(\bar{A}_1^*) : u = u_0 + \sum_{j=1}^2 s_j \tilde{e}_j + \sum_{j=1}^2 \Gamma(\tilde{e}_j) s_j, \text{ for some } s = (s_1, s_2) \in \mathbb{C}^2 \right\}, \\
&= \left\{ u \in D(\bar{A}_1^*) : u = u_0 + \sum_{j=1}^2 s_j (\tilde{e}_j + \Gamma \tilde{e}_j), \text{ for some } s = (s_1, s_2) \in \mathbb{C}^2 \right\}.
\end{aligned} \tag{4.2.33}$$

where $\{\tilde{e}_1, \tilde{e}_2\}$ is a basis of H_+ .

Definition 4.25. Assume that $\Gamma : H_+ \rightarrow H_-$ is a linear isomorphism. Let us also denote by $\{\tilde{f}_1, \tilde{f}_2\}$ is a basis of H_- . Let us denote by M the matrix of operator Γ in the basis on $\{\tilde{e}_1, \tilde{e}_2\}$ and $\{\tilde{f}_1, \tilde{f}_2\}$, i.e.

$$\Gamma \left(\sum_{k=1}^2 s_k \tilde{e}_k \right) = \Gamma (s_1 \tilde{e}_1 + s_2 \tilde{e}_2) = \left(\sum_{j=1}^2 m_{1j} s_j \right) \tilde{f}_1 + \left(\sum_{j=1}^2 m_{2j} s_j \right) \tilde{f}_2. \tag{4.2.34}$$

Let $u \in D(A_\Gamma)$. Then the boundary values of u at $0\pm$ satisfy

$$u(0\pm) = \sum_{j=1}^2 s_j \left[\tilde{e}_j(0\pm) + (\Gamma \tilde{e}_j)(0\pm) \right]. \tag{4.2.35}$$

Similarly, for the first derivative we have

$$u'(0\pm) = \sum_{j=1}^2 s_j \left[\tilde{e}'_j(0\pm) + (\Gamma \tilde{e}_j)'(0\pm) \right]. \tag{4.2.36}$$

Equations (4.2.35) can be written in a matrix form as

$$\begin{pmatrix} u(0+) \\ u(0-) \end{pmatrix} = C \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \tag{4.2.37}$$

where $C \in \mathbb{C}^{2 \times 2}$ is the matrix whose entries are determined by the boundary values of $\tilde{e}_j$ and $\Gamma \tilde{e}_j$ at $0\pm$.

Likewise, equations (4.2.36) can be written as

$$\begin{pmatrix} u'(0+) \\ u'(0-) \end{pmatrix} = B \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \quad (4.2.38)$$

where $B \in \mathbb{C}^{2 \times 2}$ is the matrix determined by the boundary values of $\tilde{e}'_j$ and $(\Gamma \tilde{e}_j)'$ at $0\pm$.

Using Tables 4.1–4.2 and (4.2.20), the matrices C and B take the explicit forms (4.2.39) and (4.2.40). Therefore, the matrix C is given by

$$C = 2^{1/4} \begin{pmatrix} 1+a & b \\ -e^{i\phi\bar{b}} & 1+e^{i\phi\bar{a}} \end{pmatrix}, \quad (4.2.39)$$

and the derivative matrix is

$$B = 2^{-1/4}(1+i) \begin{pmatrix} -(1+a) & -b \\ -e^{i\phi\bar{b}} & 1+e^{i\phi\bar{a}} \end{pmatrix}. \quad (4.2.40)$$

Assume that the matrices B and C are invertible. Then from (4.2.37) we obtain

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = C^{-1} \begin{pmatrix} u(0+) \\ u(0-) \end{pmatrix}, \quad (4.2.41)$$

and from (4.2.38) we obtain

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = B^{-1} \begin{pmatrix} u'(0+) \\ u'(0-) \end{pmatrix}. \quad (4.2.42)$$

We can write the boundary values at $0\pm$ in matrix form (4.2.41) as

$$\begin{pmatrix} u(0+) \\ u(0-) \end{pmatrix} = C \begin{pmatrix} s_1 \\ s_2 \end{pmatrix},$$

where

$$C = \begin{pmatrix} \tilde{e}_1(0+) + (\Gamma \tilde{e}_1)(0+) & \tilde{e}_2(0+) + (\Gamma \tilde{e}_2)(0+) \\ \tilde{e}_1(0-) + (\Gamma \tilde{e}_1)(0-) & \tilde{e}_2(0-) + (\Gamma \tilde{e}_2)(0-) \end{pmatrix}.$$

Similarly, for the derivatives (4.2.42),

$$\begin{pmatrix} u'(0+) \\ u'(0-) \end{pmatrix} = B \begin{pmatrix} s_1 \\ s_2 \end{pmatrix},$$

where

$$B = \begin{pmatrix} \tilde{e}'_1(0+) + (\Gamma\tilde{e}_1)'(0+) & \tilde{e}'_2(0+) + (\Gamma\tilde{e}_2)'(0+) \\ \tilde{e}'_1(0-) + (\Gamma\tilde{e}_1)'(0-) & \tilde{e}'_2(0-) + (\Gamma\tilde{e}_2)'(0-) \end{pmatrix}.$$

Hence from (4.2.37) and (4.2.42), we infer that

$$C^{-1} \begin{pmatrix} u(0+) \\ u(0-) \end{pmatrix} = B^{-1} \begin{pmatrix} u'(0+) \\ u'(0-) \end{pmatrix}. \quad (4.2.43)$$

These two matrices B and C depend on the unitary map Γ .

Since the unitary map Γ depends on a unitary matrix $M \in U(2)$, the boundary conditions (4.2.43) can be expressed in terms of M .

Define the operator A^M by

$$D(A^M) = \{u \in D(\bar{A}_1^*) : u \text{ satisfies the boundary conditions (4.2.43)}\}. \quad (4.2.44)$$

Claim 4.26. If $u \in D(A_\Gamma)$, then $u \in D(A^M)$.

Proof. Let $u \in D(A_\Gamma)$. By the definition of $D(A_\Gamma)$ in (4.2.33), there exist $u_0 \in D(\bar{A}_1)$ and $u_+ \in H_+$ such that

$$u = u_0 + u_+ + \Gamma u_+.$$

Put $u_+ = s_1 \tilde{e}_1 + s_2 \tilde{e}_2$ for some $(s_1, s_2) \in \mathbb{C}^2$.

By Proposition 4.18, the boundary traces of u_0 vanish:

$$u_0(0+) = u_0(0-) = 0, \quad u'_0(0+) = u'_0(0-) = 0.$$

Hence, by linearity of the trace maps,

$$\begin{pmatrix} u(0+) \\ u(0-) \end{pmatrix} = \begin{pmatrix} u_+(0+) + (\Gamma u_+)(0+) \\ u_+(0-) + (\Gamma u_+)(0-) \end{pmatrix}, \quad \begin{pmatrix} u'(0+) \\ u'(0-) \end{pmatrix} = \begin{pmatrix} u'_+(0+) + (\Gamma u_+)'(0+) \\ u'_+(0-) + (\Gamma u_+)'(0-) \end{pmatrix}.$$

By the definitions of the matrices C and B in (4.2.37) and (4.2.38), these identities can be written as

$$\begin{pmatrix} u(0+) \\ u(0-) \end{pmatrix} = C \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \quad \begin{pmatrix} u'(0+) \\ u'(0-) \end{pmatrix} = B \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}.$$

Since B and C are invertible, we obtain

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = C^{-1} \begin{pmatrix} u(0+) \\ u(0-) \end{pmatrix} = B^{-1} \begin{pmatrix} u'(0+) \\ u'(0-) \end{pmatrix}.$$

Therefore the boundary condition (4.2.43) holds, and hence $u \in D(A^M)$ by definition (4.2.44). $\square$

Now we prove the converse implication. Assume that $M = [m_{ij}]_{i,j=1}^2$ is an invertible matrix and Γ is defined by (4.2.34). Then

Claim 4.27. If $u \in D(A^M)$, then $u \in D(A_\Gamma)$.

Proof of Claim 4.27. Take $u \in D(A^M)$. Then $u \in D(A_1^*)$, and so the boundary values

$$(u(0+), u(0-)), \quad (u'(0+), u'(0-))$$

are well defined and satisfy the condition (4.2.43).

Hence, we can define the coefficients $(s_1, s_2) \in \mathbb{C}^2$ in two equivalent ways:

$$s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = C^{-1} \begin{pmatrix} u(0+) \\ u(0-) \end{pmatrix}, \quad (4.2.45)$$

and

$$s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = B^{-1} \begin{pmatrix} u'(0+) \\ u'(0-) \end{pmatrix}. \quad (4.2.46)$$

We want to show that these coefficients (s_1, s_2) give rise to the decomposition required for membership of u in $D(A_\Gamma)$.

For convenience, we introduce the notation

$$x = (x_+, x_-) := (u(0+), u(0-)),$$

and similarly

$$y = (y_+, y_-) := (u'(0+), u'(0-)).$$

With this notation, the formulas (4.2.45) and (4.2.46) can be rewritten as

$$s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = C^{-1} \begin{pmatrix} x_+ \\ x_- \end{pmatrix} = C^{-1}x, \quad (4.2.47)$$

and

$$s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = B^{-1} \begin{pmatrix} y_+ \\ y_- \end{pmatrix} = B^{-1}y. \quad (4.2.48)$$

We aim to show that the two definitions of the coefficients (s_1, s_2) given in (4.2.47) and (4.2.48) coincide.

Lemma 4.28. *Assume the matrices B and C are invertible. We Consider the linear systems*

$$\begin{pmatrix} x_+ \\ x_- \end{pmatrix} = C \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \implies x = Cs, \quad (4.2.49)$$

and

$$\begin{pmatrix} y_+ \\ y_- \end{pmatrix} = B \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \implies y = Bs. \quad (4.2.50)$$

Then the system (4.2.49)–(4.2.50) is equivalent to

$$\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = C^{-1} \begin{pmatrix} x_+ \\ x_- \end{pmatrix} \implies s = C^{-1}x, \quad (4.2.51)$$

together with

$$C^{-1} \begin{pmatrix} x_+ \\ x_- \end{pmatrix} = B^{-1} \begin{pmatrix} y_+ \\ y_- \end{pmatrix} \implies C^{-1}x = B^{-1}y. \quad (4.2.52)$$

Proof of Lemma 4.28. This result is an immediate consequence of Lemma 4.23. Indeed, equations (4.2.49) and (4.2.50) correspond to the relations $x = Cs$ and $y = Bs$, while (4.2.51) and (4.2.52) express the equivalent system $s = C^{-1}x$ together with $C^{-1}x = B^{-1}y$. $\square$

With $\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ defined by (4.2.45), or equivalent (4.2.46), we put

$$u_+ := \sum_{j=1}^2 s_j \tilde{e}_j \in H_+ \subset D(\bar{A}_1^*),$$

and

$$u_- := \sum_{j=1}^2 s_j \Gamma \tilde{e}_j \in H_- \subset D(\bar{A}_1^*).$$

Put

$$u_0 := u - (u_+ + u_-) \in D(\bar{A}_1^*). \quad (4.2.53)$$

Claim 4.29.

$$u_0 \in D(\bar{A}_1).$$

Proof. By Proposition 4.18, it is sufficient to prove that

$$u_0(0+) = u_0(0-) = 0, \quad u'_0(0+) = u'_0(0-) = 0.$$

Or equivalently, that

$$\begin{pmatrix} u_0(0+) \\ u_0(0-) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u'_0(0+) \\ u'_0(0-) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

But the definition of u_0 (4.2.53),

$$\begin{aligned} u_0(0+) &= u(0+) - (u_+(0+) + u_-(0+)), \\ u_0(0-) &= u(0-) - (u_+(0-) + u_-(0-)). \end{aligned}$$

On the other hand,

$$\begin{aligned} u_+(0+) + u_-(0+) &= \sum_{j=1}^2 s_j \tilde{e}_j(0+) + \sum_{j=1}^2 s_j (\Gamma \tilde{e}_j)(0+), \\ &= \sum_{j=1}^2 [\tilde{e}_j(0+) + \Gamma(\tilde{e}_j)(0+)] s_j. \end{aligned}$$

Similarly,

$$u_+(0-) + u_-(0-) = \sum_{j=1}^2 [\tilde{e}_j(0-) + \Gamma(\tilde{e}_j)(0-)] s_j.$$

Using Tables 4.1–4.2 and the representation of Γ in (4.2.20), the matrix C in (4.2.54) is exactly the explicit matrix (4.2.39).

Therefore, we infer that

$$\begin{pmatrix} u_+(0+) + u_-(0+) \\ u_+(0-) + u_-(0-) \end{pmatrix} = C \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}. \quad (4.2.54)$$

Therefore, by condition the above with (4.2.47), we get

$$\begin{pmatrix} u(0+) \\ u(0-) \end{pmatrix} = \begin{pmatrix} u_+(0+) + u_-(0+) \\ u_+(0-) + u_-(0-) \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} u_0(0+) \\ u_0(0-) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In a similar way, we prove

$$\begin{pmatrix} u'_0(0+) \\ u'_0(0-) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By definition of u_0 , see (4.2.53), we also have

$$\begin{aligned} u'_0(0+) &= u'(0+) - (u'_+(0+) + u'_-(0+)), \\ u'_0(0-) &= u'(0-) - (u'_+(0-) + u'_-(0-)). \end{aligned} \quad (4.2.55)$$

On the other hand, using linearity and the representation $u_+ = \sum_{j=1}^2 s_j \tilde{e}_j$ and $u_- = \Gamma u_+$, we obtain

$$\begin{aligned} u'_+(0+) + u'_-(0+) &= \sum_{j=1}^2 s_j \tilde{e}'_j(0+) + \sum_{j=1}^2 s_j (\Gamma \tilde{e}_j)'(0+) \\ &= \sum_{j=1}^2 [\tilde{e}'_j(0+) + (\Gamma \tilde{e}_j)'(0+)] s_j, \end{aligned}$$

and similarly

$$u'_+(0-) + u'_-(0-) = \sum_{j=1}^2 [\tilde{e}'_j(0-) + (\Gamma \tilde{e}_j)'(0-)] s_j.$$

Therefore, the derivative traces can be written in matrix form as

$$\begin{pmatrix} u'_+(0+) + u'_-(0+) \\ u'_+(0-) + u'_-(0-) \end{pmatrix} = B \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \quad (4.2.56)$$

where B is exactly the explicit matrix (4.2.40) (obtained from Tables 4.1–4.2 and (4.2.20)).

Now, by the definition of the coefficients $s = (s_1, s_2)$ via the derivative data, see (4.2.46), we have

$$\begin{pmatrix} u'(0+) \\ u'(0-) \end{pmatrix} = B \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}. \quad (4.2.57)$$

Comparing (4.2.56) and (4.2.57), we deduce

$$\begin{pmatrix} u'(0+) \\ u'(0-) \end{pmatrix} = \begin{pmatrix} u'_+(0+) + u'_-(0+) \\ u'_+(0-) + u'_-(0-) \end{pmatrix}. \quad (4.2.58)$$

Substituting (4.2.58) into (4.2.55) gives

$$\begin{pmatrix} u'_0(0+) \\ u'_0(0-) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, $u_0 \in D(\bar{A}_1)$. □

Therefore, $D(A^M) \subset D(A_\Gamma)$. This completes the proof. □

4.2.8 Abstract Characterization of self-adjoint extensions via Unitary maps

We now reformulate the converse construction in an abstract form, using boundary conditions parameterized by a unitary matrix.

Definition 4.30. Fix a unitary matrix $\alpha \in U(2)$, and define the operator A^α by

$$\begin{aligned} D(A^\alpha) &:= \left\{ u \in D(A_1^*) : u \text{ satisfies the boundary conditions (4.2.30)} \right\}, \\ A^\alpha u &:= A_1^* u, \quad u \in D(A^\alpha). \end{aligned} \tag{4.2.59}$$

Theorem 4.31. *Let $\alpha \in U(2)$. Then the operator A^α defined by (4.2.59) coincides with the abstract self-adjoint extension A_{Γ_α} given in Theorem 2.60. In particular, we have*

$$A^\alpha = A_{\Gamma_\alpha},$$

where $\Gamma_\alpha : H_+ \rightarrow H_-$ is the unitary map corresponding to α . In particular,

$$D(A^\alpha) = D(A_{\Gamma_\alpha}),$$

and the boundary conditions (4.2.30) uniquely determine the self-adjoint extension.

The proof follows directly from the converse construction established in the previous Subsection 4.2.7 and is therefore omitted.

Remark 4.32. This result is the exact analogue of the first-order case, see Theorem 3.38. In the first-order setting, the deficiency spaces are one-dimensional, and the self-adjoint extensions are parametrized by $\alpha \in \mathbb{T}$, requiring a single boundary condition. In contrast, for the Laplacian, the deficiency spaces are two-dimensional, and the self-adjoint extensions are parametrized by $\alpha \in U(2)$, requiring two linear boundary conditions are required.

Chapter 5

Third-order linear homogeneous differential operators

In this chapter, we study third-order linear homogeneous differential operators. Using the classical von Neumann approach, we analyse symmetry, determine the adjoint operator, and compute the deficiency indices.

This allows us to obtain a complete characterization of all self-adjoint extensions of the operators under consideration. We also provide several examples and applications illustrating the abstract theory.

5.1 Third-order linear homogeneous differential operators on the real line

5.1.1 Symmetry and the adjoint

The third-order linear homogeneous differential operators studied here have applications in research on nonlinear Korteweg–De Vries (KdV) equations. We consider the Hilbert space

$$H = L^2(\mathbb{R}, \mathbb{C})$$

with the standard inner product and norm. We define the operator A_0 by

$$\begin{aligned} D(A_0) &= C_0^\infty(\mathbb{R}) \\ A_0 u &= (iD)^3 u = -iu''', \end{aligned} \tag{5.1.1}$$

where $D^3 u$ denotes the classical third derivative of u .

Proposition 5.1. *The operator A_0 defined by formula (5.1.1) is symmetric in the space H , i.e.*

$$\langle A_0 u, v \rangle_{L^2} = \langle u, A_0 v \rangle_{L^2}; \quad \forall u, v \in D(A_0). \quad (5.1.2)$$

Proof of Proposition 5.1. We need to prove that A_0 is symmetric. In what follows we consider the case when $a = -\infty$ and $b = \infty$. The proof in a general case is similar. Let us choose and fix $u, v \in C_0^\infty(\mathbb{R})$. Then by the definition (5.1.1) of the operator A_0 we have

$$\langle u, A_0 v \rangle_{L^2} = \langle u, -iv''' \rangle_{L^2} = \int_{-\infty}^{\infty} u(x) \overline{(-iv'''(x))} dt \quad (5.1.3)$$

To prove (5.1.2), since both u, v have compact support, we can find interval $[a, b]$ and $[c, d]$ such that $u(x) = 0$ if $t \notin [a, b]$ and $v(x) = 0$ if $t \notin [c, d]$. Then we choose $\alpha = \min\{a, c\}$ and $\beta = \max\{b, d\}$. Then $[a, b] \subset [\alpha, \beta]$ and $[c, d] \subset [\alpha, \beta]$, so that

$$[a, b] \cup [c, d] \subset [\alpha, \beta].$$

Therefore,

$$\text{if } x \notin (\alpha, \beta), \text{ then } u(x) = 0, \text{ and } v(x) = 0.$$

Hence the integral in (5.1.3) is equal to the corresponding integral over a bounded interval $[\alpha, \beta]$ (over which I can apply the Integration by Parts [37, Theorem 6.22]). Thus, by equality (5.1.3), we get

$$\begin{aligned} RHS &= \langle u, A_0 v \rangle = \int_{-\infty}^{\infty} u(x) \overline{(-iv'''(x))} dx = i \int_{-\infty}^{\infty} u(x) \overline{(v'''(x))'} dt \\ &= i \int_{\alpha}^{\beta} u(x) \overline{(v'''(x))'} dx = i [u(t) \overline{v'''(x)}]_{\alpha}^{\beta} - \int_{\alpha}^{\beta} u'(x) \overline{v'''(x)} dx \\ &= 0 - i \int_{\alpha}^{\beta} u'(x) \overline{v'''(x)} dx = -i [u'(t) \overline{v'''(x)}]_{\alpha}^{\beta} - i \int_{\alpha}^{\beta} u''(x) \overline{v'''(x)} dx \\ &= 0 + i \int_{\alpha}^{\beta} u''(x) \overline{v'''(x)} dx = i [u''(t) \overline{v'''(x)}]_{\alpha}^{\beta} - i \int_{\alpha}^{\beta} u'''(x) \overline{v'''(x)} dx = \\ &= \int_{\alpha}^{\beta} -iu'''(x) \overline{v'''(x)} dx = \int_{-\infty}^{\infty} -iu'''(x) \overline{v'''(x)} dx = \langle A_0 u, v \rangle = LHS. \end{aligned}$$

Hence we infer that A_0 is symmetric in H . □

We need to determine $D(A_0^*)$ and next find a formula for $A_0^* v$ if $v \in D(A_0^*)$.

We begin with the following characterization of the domain of the adjoint operator A_0^* . We will prove the following result.

Lemma 5.2. *Let A_0 be the operator defined by (5.1.1). Then*

$$D(A_0^*) = \left\{ u \in L^2(\mathbb{R}, \mathbb{C}) : D^3 u \in L^2(\mathbb{R}, \mathbb{C}) \right\}, \quad (5.1.4)$$

where D^3u is understood in the weak sense. Moreover,

$$A_0^*u = -iD^3u, \quad u \in D(A_0^*). \quad (5.1.5)$$

Proof of Lemma 5.2. Part I. We will prove that the set on the RHS of equality (5.1.4) is a subset of the LHS, i.e.

$$\{u \in L^2(\mathbb{R}, \mathbb{C}) : D^3u \in L^2(\mathbb{R}; \mathbb{C})\} \subseteq D(A_0^*).$$

Let $u \in L^2(\mathbb{R}, \mathbb{C})$ such that the weak derivative D^3u exists and $D^3u \in L^2(\mathbb{R}; \mathbb{C})$. We want to prove that $u \in D(A_0^*)$. Thus, by the definition 2.51, we want to prove that $\exists u^* \in L^2(\mathbb{R}, \mathbb{C})$ such that

$$\int u \overline{A_0 v} = \int u^* \overline{v}, \quad \forall v \in D(A_0). \quad (5.1.6)$$

Let us put

$$u^* := -iD^3u$$

By the assumptions about u , we infer that $u^* \in L^2(\mathbb{R}, \mathbb{C})$. Now we will prove (5.1.6). Let us choose and fix $v \in D(A_0) = C_0^\infty(\mathbb{R}, \mathbb{C})$. We have,

$$\begin{aligned} LHS &= \int u(x) \overline{A_0 v(x)} dx = \int u(x) \overline{[-iv''(x)]} dx = \\ &= i \int u(x) \overline{v'''(x)} dx = i \int D^3u(x) \overline{v(x)} dx = i \int u^*(x) \overline{v(x)} dx = RHS. \end{aligned}$$

Hence (5.1.6) follows. $\square$

Proof of Lemma (5.2). Part II. We will prove that the set on the LHS of equality (5.1.4) is a subset of the RHS, i.e.

$$D(A_0^*) \subseteq \{u \in L^2(\mathbb{R}, \mathbb{C}) : D^3u \in L^2(\mathbb{R}; \mathbb{C})\}.$$

Let us choose and fix an element $u \in D(A_0^*)$. Since obviously, $u \in L^2(\mathbb{R}, \mathbb{C})$, we need to prove that the weak derivative D^3u exists and $D^3u \in L^2(\mathbb{R}; \mathbb{C})$, see Definition 2.11.

Since, $u \in D(A_0^*)$ and $C_0^\infty(\mathbb{R}, \mathbb{C}) = D(A_0)$, by the Definition 2.51, there exists an element $A_0^*u \in L^2(\mathbb{R}, \mathbb{C})$ such that

$$\langle A_0 \phi, u \rangle_{L^2} = \langle \phi, A_0^* u \rangle_{L^2}, \quad \forall \phi \in C_0^\infty(\mathbb{R}, \mathbb{C}).$$

Since $A_0 \phi = -i\phi'''$ we infer that

$$\int_{\mathbb{R}} (-i)\phi'''(x) \overline{u(x)} dx = \int_{\mathbb{R}} \phi(x) \overline{A_0^* u(x)} dx \quad \forall \phi \in C_0^\infty(\mathbb{R}, \mathbb{R}).$$

Hence

$$-i \int_{\mathbb{R}} u(x) \phi'''(x) dx = \int_{\mathbb{R}} A_0^* u(x) \phi(x) dx \quad \forall \phi \in C_0^\infty(\mathbb{R}, \mathbb{R}).$$

Hence, multiplying by i

$$\int_{\mathbb{R}} u(x) \phi'''(x) dx = - \int_{\mathbb{R}} (-i) A_0^* u(x) \phi(x) dx \quad \forall \phi \in C_0^\infty(\mathbb{R}, \mathbb{R}).$$

Therefore, the third order weak derivative $D^3 u$ exists and

$$D^3 u = -i A_0^* u$$

Because $A_0^* u \in L^2(\mathbb{R})$, we infer that $D^3 u \in L^2(\mathbb{R})$. The result follows.

We also proved equality (5.1.5). □

5.1.2 Deficiency indices of the third-order linear homogeneous differential operators on the real line

In this section we will calculate the deficiency indices $n_+(A_0)$ and $n_-(A_0)$ of the operator A_0 defined by (5.1.1). To be precise we will calculate the following numbers

$$n_+(A_0) = \dim \ker(iI - A_0^*), \quad n_-(A_0) = \dim \ker(iI + A_0^*),$$

see Definition 2.59. The main results in this subsection as follow

Theorem 5.3. *Assume that A_0 is the operator defined by formula (5.1.1). Then*

$$n_-(A_0) = 0$$

$$n_+(A_0) = 0$$

The following main result of this section is a consequence of Theorem 5.3.

From Proposition (5.5) and Proposition (5.7) we deduce

Corollary 5.4. *The operator A_0 on $L^2(\mathbb{R}, \mathbb{C})$ defined by equation (5.1.1) has precisely one self-adjoint extension. This extension is characterized by the Corollary 2.61. Moreover, the unique self-adjoint extension A of A_0 is as follows*

$$D(A) = H^3(\mathbb{R}), \quad Au = -iD^3 u.$$

We begin with the following Proposition.

Proposition 5.5. Assume that A_0 is the operator defined by formula (5.1.1). Then we have

$$n_+(A_0) = 0.$$

Proof of Proposition 5.5. In order to calculate $n_+(A_0)$, By [34, Definition $\ker(B)$] and Lemma 5.2 we have

$$\begin{aligned} u \in \ker(iI - A_0^*) &\iff u \in D(A_0^*) \text{ and } (iI - A_0^*)u = 0 \\ &\iff u \in D(A_0^*) \text{ and } iu = -iD^3u \\ &\iff u \in D(A_0^*) \text{ and } D^3u = -u \\ &\iff u : \mathbb{R} \rightarrow \mathbb{C} : u \text{ is of } C^2 - \text{ class} \end{aligned}$$

and the derivative $u'' = D^2u$ is locally absolutely continuous function

$$\begin{aligned} &\text{and } \int_{-\infty}^{\infty} |u'''(x)|^2 dx < \infty \\ &\text{and } D^3u = u''' = -u \text{ on } \mathbb{R}. \end{aligned} \tag{5.1.7}$$

Let us choose and fix $u \in \ker(iI - A_0^*)$. Then by (5.1.7),

$$\frac{d^3}{dx^3}u(x) = -u(x) \text{ for almost every } x \in \mathbb{R}. \tag{5.1.8}$$

The general solution of the above equation (5.1.8), see [24, Section 2.9], is

$$u(x) = c_1 e^{-x} + c_2 e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right), \quad x \in \mathbb{R}, \tag{5.1.9}$$

for some constant $c_1, c_2, c_3 \in \mathbb{C}$. Therefore the general solution can equivalently be written as

$$u(x) = a_1 e^{-x} + a_2 e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} + a_3 e^{\frac{1}{2}x - i\frac{\sqrt{3}}{2}x}, \quad x \in \mathbb{R}, \tag{5.1.10}$$

for some constant $a_1, a_2, a_3 \in \mathbb{C}$.

Claim 5.6. Let u be a function of the form (5.1.10) belongs to $H = L^2(\mathbb{R}, \mathbb{C})$, then $a_1 = a_2 = a_3 = 0$. Hence $u = 0$.

Proof of Claim 5.6. We begin with the following trivial observation

$$|e^{\frac{1}{2}x \pm i\frac{\sqrt{3}}{2}x}|^2 = e^x.$$

Therefore

$$\int_0^{\infty} |e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x}|^2 dx = \int_0^{\infty} |e^{\frac{1}{2}x - i\frac{\sqrt{3}}{2}x}|^2 dx = \int_0^{\infty} e^x dx = \infty. \tag{5.1.11}$$

Since $u \in L^2(0, \infty)$, necessarily

$$a_2 = a_3 = 0.$$

Next, on $(-\infty, 0)$ we have

$$\int_{-\infty}^0 |e^{-x}|^2 dx = \int_{-\infty}^0 e^{-2x} dx = \int_0^{\infty} e^{2x} dx = \infty,$$

while

$$\int_0^{\infty} |e^{-\frac{1}{2}x}|^2 dx < \infty.$$

Thus $u \in L^2(\mathbb{R})$ implies that $u - a_1 e^{-x} \in L^2([0, \infty))$. This allows us to rewrite on $(0, \infty)$

$$u(x) = a_2 e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} + a_3 e^{\frac{1}{2}x - i\frac{\sqrt{3}}{2}x}, \quad x \in (0, \infty), \quad (5.1.12)$$

for some constant $a_2, a_3 \in \mathbb{C}$.

Then, by employing basic properties of complex numbers we have

$$\begin{aligned} |u(x)|^2 &= |a_2|^2 e^x + |a_3|^2 e^x + 2e^x \operatorname{Re}(a_2 \bar{a}_3) = (|a_2|^2 + |a_3|^2 + 2\operatorname{Re}(a_2 \bar{a}_3))e^x \\ &= |a_2 + a_3|^2 e^x, \quad x \in \mathbb{R}. \end{aligned}$$

Therefore, in view (5.1.11), if $|a_2 + a_3|^2 > 0$

$$\int_0^{\infty} |u(x)|^2 dx = |a_2 + a_3|^2 \int_0^{\infty} e^x dx = \infty.$$

But, since by assumptions LHS is finite, we deduce that $|a_2 + a_3|^2 = 0$ and so $a_3 = -a_2$.

Hence, by (5.1.12), we get

$$\begin{aligned} u(x) &= a_2 (e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} - e^{\frac{1}{2}x - i\frac{\sqrt{3}}{2}x}) \\ &= a_2 e^{\frac{1}{2}x} (e^{i\frac{\sqrt{3}}{2}x} - e^{-i\frac{\sqrt{3}}{2}x}) \\ &= a_2 e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right), \quad x \in \mathbb{R}, \end{aligned}$$

Now, suppose for the time being that

$$\int_0^{\infty} e^x \sin^2\left(\frac{\sqrt{3}}{2}x\right) dx = \infty. \quad (5.1.13)$$

Then, we deduce that $a_2 = 0$, and therefore, $a_3 = 0$

To prove (5.1.13) it is equivalent to prove that

$$\boxed{\int_0^{\infty} e^x \sin^2 x dx = \infty.} \quad (5.1.14)$$

It remains to prove that $a_1 = 0$. Since we already have that $a_2 = a_3 = 0$, we get

$$u(x) = a_1 e^{-x}, \quad x \in \mathbb{R}.$$

Since $u \in H$,

$$\|u\|_H^2 = \int_{-\infty}^{\infty} |u(x)|^2 dx < \infty.$$

Hence,

$$\int_{-\infty}^0 |u(x)|^2 dx \leq \int_{-\infty}^{\infty} |u(x)|^2 dx < \infty.$$

On the other hand,

$$\int_{-\infty}^0 |e^{-x}|^2 dx = \int_{-\infty}^0 e^{-2x} dx = \int_0^{\infty} e^{2x} dx = \infty.$$

Hence, if $a_1 \neq 0$, then also

$$\int_{-\infty}^0 |u(x)|^2 dx = \infty$$

what leads to a contradiction. Hence we infer that also $a_1 = 0$.

Therefore, $a_1 = a_2 = a_3 = 0$ and so $u = 0$. □

The proof of Proposition 5.5 is complete. □

The following result is a companion result to Proposition 5.5.

Proposition 5.7. *Assume that A_0 is the operator defined by formula (5.1.1). Then we have*

$$n_-(A_0) = 0.$$

Proof of Proposition 5.7. In order to calculate $n_-(A_0)$, we need to find the space

$$n_-(A_0) = \dim \ker(iI + A_0^*).$$

By [34, Definition $\ker(B)$] and Lemma 5.2 we have

$$\begin{aligned} u \in \ker(iI + A_0^*) &\iff u \in D(A_0^*) \text{ and } (iI + A_0^*)u = 0 \\ &\iff u \in D(A_0^*) \text{ and } iu = iD^3u \\ &\iff u \in D(A_0^*) \text{ and } D^3u = u \\ &\iff u : \mathbb{R} \rightarrow \mathbb{C} : u \text{ is of } C^2\text{-class} \end{aligned}$$

and the second derivative D^2u is locally absolutely continuous function

$$\text{and } \int_{-\infty}^{\infty} |D^3(x)|^2 dx < \infty$$

$$\text{and } D^3u = u \text{ on } \mathbb{R}.$$

(5.1.15)

Let us choose and fix $u \in \ker(iI + A_0^*)$. Then by (5.1.15),

$$\frac{d^3}{dx^3}u(x) = u(x) \text{ for almost every } x \in \mathbb{R} \quad (5.1.16)$$

How do you solve this equation? You look for a solution of the form

$$u(x) = e^{\lambda x}$$

You substitute it into equation (5.1.16) and you get

$$\lambda^3 e^{\lambda x} = e^{\lambda x} \Rightarrow \lambda^3 = 1.$$

This equation has three roots

$$\lambda_1 = 1, \quad \lambda_2 = -\frac{1}{2} + i\frac{2\pi}{3}, \quad \lambda_3 = -\frac{1}{2} - i\frac{2\pi}{3}$$

The general solution of the above equation (5.1.16), see [24, Section 2.9], is

$$u(x) = a_1 e^{\lambda_1 x} + a_2 e^{\lambda_2 x} + a_3 e^{\lambda_3 x}$$

i.e.

$$u(x) = a_1 e^x + a_2 e^{-\frac{1}{2}x + i\frac{2\pi}{3}x} + a_3 e^{-\frac{1}{2}x - i\frac{2\pi}{3}x}, \quad x \in \mathbb{R}, \quad (5.1.17)$$

for some constant $a_1, a_2, a_3 \in \mathbb{C}$.

Claim 5.8. If function u is of the form (5.1.17) belongs to $H = L^2(\mathbb{R}, \mathbb{C})$, then $a_1 = a_2 = a_3 = 0$, and so $u = 0$.

Proof of Claim 5.8. We begin with the following two trivial observations

$$\begin{aligned} \int_0^\infty |e^{-\frac{1}{2}x + i\frac{2\pi}{3}x}|^2 dx &= \infty, & \int_{-\infty}^0 |e^{-\frac{1}{2}x - i\frac{2\pi}{3}x}|^2 dx &= \infty, \\ \int_{-\infty}^0 |e^x|^2 dx &< \infty, \end{aligned} \quad (5.1.18)$$

Let us choose and fix a function u of the form (5.1.17) which belongs to $H = L^2(\mathbb{R}, \mathbb{C})$. Hence, by equality (5.1.18), the function $u - a_1 e^x$ belongs to the space $H = L^2((-\infty, 0], \mathbb{C})$. Hence we can assume that $a_1 = 0$, i.e. we can assume that u is of the form, for some constant $a_2, a_3 \in \mathbb{C}$,

$$u(x) = a_2 e^{-\frac{1}{2}x + i\frac{2\pi}{3}x} + a_3 e^{-\frac{1}{2}x - i\frac{2\pi}{3}x}, \quad x \in (-\infty, 0), \quad (5.1.19)$$

Then, by employing basic properties of complex numbers we have

$$\begin{aligned} |u(x)|^2 &= |a_2|^2 e^{-x} + |a_3|^2 e^{-x} + 2e^{-x} \operatorname{Re}(a_2 \bar{a}_3) \\ &= (|a_2|^2 + |a_3|^2 + 2\operatorname{Re}(a_2 \bar{a}_3)) e^{-x} = |a_2 + a_3|^2 e^{-x}, \quad x \in \mathbb{R}. \end{aligned}$$

Therefore, if $|a_2 + a_3|^2 > 0$

$$\int_{-\infty}^0 |u(x)|^2 dx = |a_2 + a_3|^2 \int_{-\infty}^0 |e^{-x}|^2 dx = \infty.$$

But, since by assumptions LHS is finite, we deduce that $|a_2 + a_3|^2 = 0$ and so $a_3 = -a_2$. Hence, by (5.1.19), we get

$$\begin{aligned} u(x) &= a_2 (e^{-\frac{1}{2}x + i\frac{2\pi}{3}x} - e^{-\frac{1}{2}x - i\frac{2\pi}{3}x}) \\ &= a_2 e^{-\frac{1}{2}x} (e^{i\frac{2\pi}{3}x} - e^{-i\frac{2\pi}{3}x}) = a_2 e^{-\frac{1}{2}x} \sin(\frac{2\pi}{3}x), \quad x \in \mathbb{R}, \end{aligned}$$

Now, suppose for the time being that

$$\int_{-\infty}^0 e^{-x} \sin^2(\frac{2\pi}{3}x) dx = \infty. \quad (5.1.20)$$

Then, we deduce that $a_2 = 0$, and therefore, $a_3 = 0$

To prove (5.1.20) it is equivalent to prove that

$$\boxed{\int_{-\infty}^0 e^{-x} \sin^2 x dx = \infty}$$

what in turn is equivalent to (5.1.14). □

It remains to prove that $a_1 = 0$. Since we already that $a_2 = a_3 = 0$, we get

$$u(x) = a_1 e^x, \quad x \in \mathbb{R}.$$

Since $u \in H$

$$u(x) = a_1^2 \int_{-\infty}^0 e^{2x} dx < \infty,$$

so $a_1 = 0$. The proof of Proposition 5.7 is complete. □

5.2 Third-order linear homogeneous differential operators on the positive real line

In this section we consider the Hilbert space

$$H = L^2(\mathbb{R}_+, \mathbb{C}) = L^2((0, \infty), \mathbb{C})$$

equipped with the standard inner product and norm.

We define the following operator

$$\begin{aligned} D(A_1) &= C_0^\infty((0, \infty); \mathbb{C}) \\ A_1 u &= (iD)^3 u = -iu''' \end{aligned} \tag{5.2.1}$$

Proposition 5.9. *The operator A_1 is symmetric in the space H , i.e.*

$$\langle A_1 u, v \rangle_{L^2} = \langle u, A_1 v \rangle_{L^2}, \quad \forall u, v \in D(A_1).$$

Remark 5.10. The proof of symmetry of the operator A_1 defined by (5.2.1) is analogous to the proof of symmetry of the operator A_0 , see Proposition 5.1.

Moreover, the symmetry of A_1 follows directly from the symmetry of A_0 .

Proof of Remark 5.10. Recall that

$$\begin{aligned} D(A_0) &= C_0^\infty(\mathbb{R}; \mathbb{C}), \\ D(A_1) &= C_0^\infty((0, \infty); \mathbb{C}). \end{aligned}$$

Hence, $D(A_1) \subset D(A_0)$. Moreover,

$$\begin{aligned} A_0 u &= (iD)^3 u = -iu''', \quad u \in D(A_0), \\ A_1 u &= (iD)^3 u = -iu''', \quad u \in D(A_1). \end{aligned}$$

Thus, $A_1 u = A_0 u$ for all $u \in D(A_1)$, and therefore

$$A_1 \subset A_0.$$

Since A_0 is symmetric, it follows that A_1 is also symmetric. Indeed, for $u, v \in D(A_1)$,

$$\langle A_1 u, v \rangle = \langle A_0 u, v \rangle = \langle u, A_0 v \rangle = \langle u, A_1 v \rangle.$$

□

We now compute the deficiency indices of the operator A_1 . For this purpose, we first characterize the domain of the adjoint operator A_1^* .

Proposition 5.11. *A measurable function $v : (0, \infty) \rightarrow \mathbb{C}$ belongs to $D(A_1^*)$ if and only if the following conditions hold:*

$$\begin{aligned} v &\in L^2(0, \infty), \text{ i.e. } \int_0^\infty |v(x)|^2 dx < \infty, \\ v'' &= D^2v \text{ is locally absolutely continuous on } (0, \infty), \text{ and} \\ \int_0^\infty |v'''(x)|^2 dx &< \infty. \end{aligned}$$

Moreover,

$$A_1^*v = (iD)^3v = -iv''', \quad v \in D(A_1^*).$$

Next, we compute the deficiency indices $n_+(A_1)$ and $n_-(A_1)$ of A_1 , see Definition of deficiency subspaces [34, p. 138], i.e.

$$n_+(A_1) = \dim \ker(iI - A_1^*), \quad n_-(A_1) = \dim \ker(iI + A_1^*).$$

Thus, in order to calculate $n_+(A_1)$, we need to find the space $\ker(iI - A_1^*)$.

Proposition 5.12. *Assume that A_1 is the operator defined by formula (5.2.1). Then*

$$n_+(A_1) = 1.$$

Proof of Proposition 5.12. To find $n_+(A_1)$, let us observe that by Proposition 5.11

$$\begin{aligned} u \in \ker(iI - A_1^*) &\iff u \in D(A_1^*) \text{ and } (iI - A_1^*)u = 0 \\ &\iff u \in D(A_1^*) \text{ and } iu = -iD^3u \iff u \in D(A_1^*) \text{ and } D^3u = -u. \\ &\iff \text{the restriction of } u \text{ to the interval } (0, \infty) \text{ is of class } C^2 \\ &\quad \text{and the derivatives are locally absolutely continuous} \\ &\quad \text{and } D^3u = -u \text{ on } (0, \infty). \end{aligned} \tag{5.2.2}$$

We solved this equation before, see (5.1.8).

Let us choose and fix $u \in \ker(iI - A_1^*)$. Then by (5.2.2), u satisfies

$$D^3u = -u \text{ on } (0, \infty).$$

The general solution of $D^3u = -u$ is of the form (5.1.9), i.e.

$$u(x) = c_1 e^{-x} + c_2 e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_3 e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right), \quad x > 0,$$

for some constant $c_1, c_2, c_3 \in \mathbb{C}$. Equivalent form is

$$u(x) = a_1 e^{-x} + a_2 e^{(\frac{1}{2} + i\frac{\sqrt{3}}{2})x} + a_3 e^{(\frac{1}{2} - i\frac{\sqrt{3}}{2})x}, \quad x > 0. \quad (5.2.3)$$

for some constant $a_1, a_2, a_3 \in \mathbb{C}$.

Remark 5.13. From numbers -1 , $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\frac{1}{2} - i\frac{\sqrt{3}}{2}$, only the first one has negative real part.

Claim 5.14. If function u is of the form (5.2.3) belongs to $H = L^2((0, \infty), \mathbb{C})$, then $a_2 = 0$ and $a_3 = 0$, and so $u = a_1 e^{-x}$.

Proof of Claim 5.14. We begin with the following two trivial observations

$$\int_0^\infty |e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x}|^2 dx = \infty, \quad \int_0^\infty |e^{\frac{1}{2}x - i\frac{\sqrt{3}}{2}x}|^2 dx = \infty, \quad (5.2.4)$$

$$\int_0^\infty |e^{-x}|^2 dx < \infty, \quad (5.2.5)$$

$$\int_0^\infty |e^x|^2 dx = \infty.$$

Let us choose and fix a function u of the form (5.2.3) which belongs to $H = L^2((0, \infty), \mathbb{C})$. Hence, by equality (5.2.4), the function $u - a_1 e^{-x}$ belongs to the space $H = L^2((0, \infty), \mathbb{C})$. Hence we can assume that $a_1 = 0$, i.e. we can assume that u is of the form

$$u(x) = a_2 e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} + a_3 e^{\frac{1}{2}x - i\frac{\sqrt{3}}{2}x}, \quad x \in (0, \infty), \quad (5.2.6)$$

for some constant $a_2, a_3 \in \mathbb{C}$. We want to show that $a_2 = a_3 = 0$. $\square$

Then, by employing basic properties of complex numbers we have

$$\begin{aligned} |u(x)|^2 &= |a_2|^2 e^x + |a_3|^2 e^x + 2e^x \operatorname{Re}(a_2 \bar{a}_3) = (|a_2|^2 + |a_3|^2 + 2\operatorname{Re}(a_2 \bar{a}_3))e^x \\ &= |a_2 + a_3|^2 e^x, \quad x \in (0, \infty). \end{aligned}$$

Therefore, in view (5.2.5), if $|a_2 + a_3|^2 > 0$

$$\int_0^\infty |u(x)|^2 dx = |a_2 + a_3|^2 \int_0^\infty |e^x|^2 dx = \infty.$$

But, since by assumptions, the *LHS* is finite, we deduce that $|a_2 + a_3|^2 = 0$. Hence $a_3 = -a_2$, and by (5.2.6), since

$$\int_0^\infty |u(x)|^2 dx = a_2^2 \int_0^\infty e^x \sin^2\left(\frac{\sqrt{3}}{2}x\right) dx$$

we get

$$\begin{aligned} u(x) &= a_2(e^{\frac{1}{2}x+i\frac{\sqrt{3}}{2}x} - e^{\frac{1}{2}x-i\frac{\sqrt{3}}{2}x}) = a_2e^{\frac{1}{2}x}(e^{i\frac{\sqrt{3}}{2}x} - e^{-i\frac{\sqrt{3}}{2}x}) \\ &= a_2e^{\frac{1}{2}x}\sin(\frac{\sqrt{3}}{2}x), \quad x \in \mathbb{R}, \end{aligned}$$

by (5.1.13) we deduce that $a_2 = 0$. □

Proposition 5.15. *Assume that A_1 is the operator defined by formula (5.2.1). Then*

$$n_-(A_1) = 2.$$

Proof of Proposition 5.15. To find $n_-(A_1)$, let us observe that by Proposition 5.11

$$\begin{aligned} u \in \ker(iI + A_1^*) &\iff u \in D(A_1^*) \text{ and } (iI + A_1^*)u = 0 \\ \iff u \in D(A_1^*) \text{ and } iu = iD^3u &\iff u \in D(A_1^*) \text{ and } D^3u = u. \\ \iff \text{the function } u \text{ to the intervals } (0, \infty) \text{ is of } C^2\text{-class} & \tag{5.2.7} \\ \text{and the derivatives are locally absolutely continuous} & \\ \text{and } D^3u = u \text{ on } (0, \infty). & \end{aligned}$$

We solved this equation before, see (5.1.15).

Let us choose and fix $u \in \ker(iI + A_1^*)$, then by (5.2.7), u satisfies the following conditions.

$$D^3u = u \text{ on } (0, \infty).$$

The general solution of $D^3u = u$ is of the form (5.1.16), i.e.

$$u(x) = c_1e^x + c_2e^{\frac{1}{2}x}\cos(\frac{\sqrt{3}}{2}x) + c_3e^{\frac{1}{2}x}\sin(\frac{\sqrt{3}}{2}x), \quad x > 0,$$

for some constant $c_1, c_2, c_3 \in \mathbb{C}$.

$$u(x) = a_1e^x + a_2e^{-\frac{1}{2}x+i\frac{\sqrt{3}}{2}x} + a_3e^{-\frac{1}{2}x-i\frac{\sqrt{3}}{2}x}, \quad x > 0, \tag{5.2.8}$$

for some constant $a_1, a_2, a_3 \in \mathbb{C}$.

Claim 5.16. If function u is of the form (5.2.8) belongs to $H = L^2((0, \infty), \mathbb{C})$, then $a_1 = 0$ and so

$$u = a_2e^{-\frac{1}{2}x+i\frac{\sqrt{3}}{2}x} + a_3e^{-\frac{1}{2}x-i\frac{\sqrt{3}}{2}x}.$$

Proof of Claim 5.16. We begin with the following two trivial observations

$$\int_0^\infty |e^{-\frac{1}{2}x+i\frac{\sqrt{3}}{2}x}|^2 dx < \infty, \quad (5.2.9)$$

$$\int_0^\infty |e^{-\frac{1}{2}x-i\frac{\sqrt{3}}{2}x}|^2 dx < \infty, \quad (5.2.10)$$

$$\int_0^\infty |e^x|^2 dx = \infty. \quad (5.2.11)$$

Let us choose and fix a function u of the form (5.2.8) which belongs to $H = L^2((0, \infty), \mathbb{C})$. Hence by equality (5.2.9) and (5.2.10), the function

$$u - a_2 e^{-\frac{1}{2}x+i\frac{\sqrt{3}}{2}x} - a_3 e^{-\frac{1}{2}x-i\frac{\sqrt{3}}{2}x}$$

belongs to the space $H = L^2((0, \infty), \mathbb{C})$. Hence we can assume that $a_2 = a_3 = 0$, i.e. we can assume that u is of the form

$$u(x) = a_1 e^x, \quad x \in (0, \infty), \quad (5.2.12)$$

for some constant $a_1 \in \mathbb{C}$. Therefore, in view (5.2.11), if $|a_1| \neq 0$

$$\int_0^\infty |u(x)|^2 dx = |a_1|^2 \int_0^\infty |e^x|^2 dx = \infty.$$

But, since by assumptions, the *LHS* is finite, we deduce that $|a_1|^2 = 0$. Hence $a_1 = 0$. Hence $a_1 = 0$. Hence, by (5.2.12), we get

$$u(x) = a_2 e^{-\frac{1}{2}x+i\frac{\sqrt{3}}{2}x} + a_3 e^{-\frac{1}{2}x-i\frac{\sqrt{3}}{2}x}.$$

□

Hence the proof of Proposition 5.15 is complete. □

From Proposition (5.12) and Proposition (5.15) we deduce the following immediate consequence.

Theorem 5.17. *Assume that A_1 is the operator defined by formula (5.2.1). Then,*

$$n_-(A_1) = 2$$

$$n_+(A_1) = 1$$

The following is a consequences of Theorem 5.17.

Corollary 5.18. *The operator A_1 on $L^2((0, \infty), \mathbb{C})$ defined by equation (5.2.1) has no self-adjoint extensions.*

5.3 Third-order linear homogeneous differential operators on the negative real line

We now consider the corresponding operator on the negative half-line. Let

$$H_- = L^2((-\infty, 0), \mathbb{C}),$$

and define

$$D(A_-) = C_0^\infty((-\infty, 0); \mathbb{C}), \quad A_- u = -iD^3 u.$$

Remark 5.19. The analysis on the negative half-line follows directly from that on the positive half-line by reflection. Define the unitary operator

$$U : L^2((-\infty, 0), \mathbb{C}) \longrightarrow L^2((0, \infty), \mathbb{C})$$

by

$$(Uf)(x) = f(-x), \quad x > 0.$$

If A_+ denotes the operator considered in Section 5.2, then

$$UA_-U^{-1} = -A_+.$$

Indeed, the third derivative changes sign under the reflection $x \mapsto -x$.

Consequently, the positive and negative deficiency indices are interchanged:

$$n_+(A_-) = n_-(A_+), \quad n_-(A_-) = n_+(A_+).$$

By Theorem 5.17,

$$n_+(A_+) = 1, \quad n_-(A_+) = 2.$$

Therefore,

$$n_+(A_-) = 2, \quad n_-(A_-) = 1.$$

Hence, since the deficiency indices are unequal, A_- has no self-adjoint extension.

5.4 Third-order linear homogeneous differential operators on the punctured line

5.4.1 Symmetry and the adjoint

In this section we consider the Hilbert space

$$H = L^2(\mathbb{R}, \mathbb{C})$$

with the standard inner product and norm. We define the operator A_1 as follows

$$\begin{aligned} D(A_1) &= C_0^\infty((-\infty, 0) \cup (0, \infty); \mathbb{C}) = C_0^\infty(\mathbb{R} \setminus \{0\}), \\ A_1 u &= (iD)^3 u = -iu''' = -iD^3 u \end{aligned} \tag{5.4.1}$$

Proposition 5.20. *The operator A_1 is symmetric in the space H , i.e.*

$$\langle A_1 u, v \rangle_{L^2} = \langle u, A_1 v \rangle_{L^2}; \quad \forall u, v \in D(A_1).$$

Remark 5.21. The proof of Symmetry of operator A_1 defined by (5.4.1) does not differ proof of Symmetry of operator A_0 , see Proposition 5.1.

Moreover, the Symmetry of A_1 is a consequence of Symmetry of A_0 .

Proof of Proposition 5.20. Let us recall that by definitions of these operators

$$\begin{aligned} D(A_0) &= C_0^\infty(\mathbb{R}; \mathbb{C}), \\ D(A_1) &= C_0^\infty(\mathbb{R} \setminus \{0\}; \mathbb{C}), \end{aligned}$$

Hence $D(A_1) \subset D(A_0)$. Moreover,

$$\begin{aligned} A_0 u &= (iD)^3 u = -iu''', \quad u \in D(A_0), \\ A_1 u &= (iD)^3 u = -iu''', \quad u \in D(A_1). \end{aligned}$$

Hence, $A_1 u = A_0 u$, if $u \in D(A_1)$. In other words,

$$A_1 \subset A_0.$$

It is trivial to observe that if A_0 is symmetric then A_1 is symmetric as well. Indeed, if $u, v \in D(A_1)$ then $u, v \in D(A_0)$ and

$$\begin{aligned} \langle A_1 u, v \rangle &= \langle A_0 u, v \rangle = \langle u, A_0 v \rangle \\ &= \langle u, A_1 v \rangle. \end{aligned}$$

□

We begin with the following characterization of the domain of the adjoint operator A_1^* . We will prove the following result.

Lemma 5.22. *Let A_1 be the operator defined by (5.4.1). Then*

$$D(A_1^*) = \left\{ u \in L^2(\mathbb{R}, \mathbb{C}) : D^3u \in L^2((-\infty, 0), \mathbb{C}) \text{ and } D^3u \in L^2((0, \infty), \mathbb{C}) \right\},$$

where D^3u is understood in the weak sense on each interval $(-\infty, 0)$ and $(0, \infty)$. Moreover,

$$A_1^*u = -iD^3u, \quad u \in D(A_1^*),$$

where the operator acts separately on $(0, \infty)$ and $(-\infty, 0)$.

Proof of Lemma 5.22, Part I. We prove that

$$\left\{ u \in L^2(\mathbb{R}) : D^3u \in L^2((0, \infty)) \text{ and } D^3u \in L^2((-\infty, 0)) \right\} \subseteq D(A_1^*).$$

Let $u \in L^2(\mathbb{R})$ be such that $D^3u \in L^2((0, \infty))$ and $D^3u \in L^2((-\infty, 0))$, where the derivatives are understood in the weak sense.

Define

$$u^* := -iD^3u.$$

Then $u^* \in L^2(\mathbb{R})$.

Let us choose and fix $v \in D(A_1) = C_0^\infty(\mathbb{R} \setminus \{0\})$. Then

$$\langle u, A_1v \rangle = \int_{\mathbb{R}} u(x) \overline{(-iv^{(3)}(x))} dx = i \int_{\mathbb{R}} u(x) \overline{v^{(3)}(x)} dx.$$

Since v has compact support contained in $\mathbb{R} \setminus \{0\}$, we can split the integral and integrate by parts separately on $(0, \infty)$ and $(-\infty, 0)$, obtaining

$$\int_{\mathbb{R}} u(x) \overline{v^{(3)}(x)} dx = \int_{\mathbb{R}} D^3u(x) \overline{v(x)} dx.$$

Thus,

$$\langle u, A_1v \rangle = \int_{\mathbb{R}} (-iD^3u(x)) \overline{v(x)} dx = \langle u^*, v \rangle.$$

Hence $u \in D(A_1^*)$ and $A_1^*u = -iD^3u$. □

Proof of Lemma 5.22, Part II. We prove that

$$D(A_1^*) \subseteq \left\{ u \in L^2(\mathbb{R}) : D^3u \in L^2((0, \infty)) \text{ and } D^3u \in L^2((-\infty, 0)) \right\}.$$

Let $u \in D(A_1^*)$. Then there exists $w \in L^2(\mathbb{R})$ such that

$$\langle A_1\phi, u \rangle = \langle \phi, w \rangle, \quad \forall \phi \in C_0^\infty(\mathbb{R} \setminus \{0\}).$$

Since $A_1\phi = -i\phi^{(3)}$, we obtain

$$-i \int_{\mathbb{R}} \phi^{(3)}(x) \overline{u(x)} dx = \int_{\mathbb{R}} \phi(x) \overline{w(x)} dx.$$

Multiplying by i gives

$$\int_{\mathbb{R}} u(x) \phi^{(3)}(x) dx = - \int_{\mathbb{R}} (-iw(x)) \phi(x) dx.$$

Now take $\phi \in C_0^\infty((0, \infty))$. Then

$$\int_{(0, \infty)} u(x) \phi^{(3)}(x) dx = - \int_{(0, \infty)} (-iw(x)) \phi(x) dx.$$

Thus, u admits a third weak derivative on $(0, \infty)$ given by

$$D^3u = -iw \quad \text{on } (0, \infty),$$

and since $w \in L^2(\mathbb{R})$, we have $D^3u \in L^2((0, \infty))$.

The same argument applied to test functions in $C_0^\infty((-\infty, 0))$ shows that

$$D^3u = -iw \quad \text{on } (-\infty, 0),$$

and hence $D^3u \in L^2((-\infty, 0))$. This completes the proof of Lemma 5.22. $\square$

5.4.2 Deficiency indices of the third-order linear homogeneous differential operators on the punctured line

The main result of this section is the following one.

Theorem 5.23. *Assume that A_1 is the operator defined by formula (5.4.1). Then*

$$n_+(A_1) = 3$$

$$n_-(A_1) = 3$$

The following result is a consequences of Theorem 5.23. In fact, we investigated the previous result in order to prove or disprove the next corollary.

Corollary 5.24. *The operator A_1 on $L^2(\mathbb{R} \setminus \{0\}, \mathbb{C})$ defined by formula (5.4.1) has at least two self-adjoint extension.*

Proof of Corollary 5.24. According to Theorem 2.60, for a closed symmetric operator A to have a self-adjoint extension, it is necessary that the deficiency indices satisfy $n_+ = n_-$.

Since we have $n_+(A_1) = 3$ and $n_-(A_1) = 3$ from Theorem 3.18, it follows that

$$n_+ = n_-.$$

Based on the result of Theorem 2.60, the operator A_1 has at least one self-adjoint extension.

Thus, we conclude that the operator A_1 defined by formula (5.4.1) has at least one self-adjoint extension. This completes the proof of the corollary. $\square$

In the remaining of this section we will calculate the deficiency indices $n_+(A_1)$ and $n_-(A_1)$. In order to calculate the deficiency indices of A_1 , we first need to find the domain of the adjoint operator A_1^* .

Proposition 5.25. *A measurable function $v : \mathbb{R} \rightarrow \mathbb{C}$ belongs to $D(A_1^*)$ if and only if*

$$\int_{-\infty}^{\infty} |v(x)|^2 dt < \infty$$

and the restriction $v : (0, \infty) \rightarrow \mathbb{C}$ is of C^2 - class such that

the derivative $v'' = D^2v$ is locally absolutely continuous function and

$$\int_0^{\infty} |v'''(x)|^2 dt < \infty$$

and the restriction $v : (-\infty, 0) \rightarrow \mathbb{C}$ is of C^2 - class such that

the derivative $v'' = D^2v$ is locally absolutely continuous function and

$$\int_{-\infty}^0 |v'''(x)|^2 dt < \infty.$$

and

$$A_1^*v = (iD)^3v, \quad u \in D(A_1^*).$$

In the above expression, $v''' = D^3v$ denotes the derivative of $v'' = D^2v$. Since v'' is locally absolutely continuous, in the sense of Definition 2.8, the derivative v''' exists almost everywhere.

Proposition 5.26. *If $u \in D(A_1^*)$ then the following limits exist:*

$$\begin{aligned} \lim_{x \rightarrow 0^+} u(x) &=: u(0+) \\ \lim_{x \rightarrow 0^-} u(x) &=: u(0-) \\ \lim_{x \rightarrow 0^+} u'(x) &=: u'(0+) \\ \lim_{x \rightarrow 0^-} u'(x) &=: u'(0-) \\ \lim_{x \rightarrow 0^+} u''(x) &=: u''(0+) \\ \lim_{x \rightarrow 0^-} u''(x) &=: u''(0-). \end{aligned}$$

Remark 5.27. Note that the limits above are usually different. Hence a functions $u \in D(A_1^*)$, u' and u'' do not need to be continuous.

The following definition makes sense by Proposition 5.26.

Definition 5.28 (Boundary traces). Define the right and left trace of an arbitrary element $u \in D(\bar{A}_1^*)$ by

$$\mathbf{tr}_+(u) := \begin{pmatrix} u(0+) \\ u'(0+) \\ u''(0+) \end{pmatrix}, \quad \mathbf{tr}_-(u) := \begin{pmatrix} u(0-) \\ u'(0-) \\ u''(0-) \end{pmatrix}.$$

Proof of Proposition 5.25. Part 1. We assume that $u \in D(A_1^*)$.

By Definition 2.51, this means that $u \in L^2(\mathbb{R})$ and there exists a function $z \in L^2(\mathbb{R})$ such that

$$\langle A_1 \phi, u \rangle = \langle \phi, z \rangle, \quad \text{for all } \phi \in D(A_1) = C_0^\infty(\mathbb{R} \setminus \{0\}).$$

By the definition (5.4.1) of the operator A_1 , we infer that the inner product becomes

$$\langle A_1 \phi, u \rangle = \langle -iD^3 \phi, u \rangle = \int_{\mathbb{R}} -i D^3 \phi(x) \overline{u(x)} dx.$$

On the other hand,

$$\langle \phi, z \rangle = \int_{\mathbb{R}} \phi(x) \overline{z(x)} dx.$$

Therefore, the equation

$$\int_{\mathbb{R}} -i D^3 \phi(x) \overline{u(x)} dx = \int_{\mathbb{R}} \phi(x) \overline{z(x)} dx \tag{5.4.2}$$

holds for all test functions $\phi \in C_0^\infty(\mathbb{R} \setminus \{0\})$.

We now aim to show that the function u satisfies the following conditions:

- (i) The restriction $u|_{(0,\infty)}$ belongs to $C^2((0,\infty))$, it's second derivative is locally absolutely continuous, and $D^3u \in L^2(0,\infty)$,
- (ii) The restriction $u|_{(-\infty,0)}$ belongs to $C^2((-\infty,0))$, it's second derivative is locally absolutely continuous, and $D^3u \in L^2(-\infty,0)$.

To prove that, let us fix and choose any compact interval $[a,b] \subset (0,\infty)$. Assume that $\phi \in C_0^\infty((a,b)) \subset D(A_1)$. Then equation (5.4.2) becomes

$$\int_a^b -i\phi^{(3)}(x) \cdot \overline{u(x)} dx = \int_a^b \phi(x) \cdot \overline{z(x)} dx.$$

Since $z \in L^2(\mathbb{R})$, it follows that the weak derivative D^3u exists and belongs to $L^2((a,b))$. Hence, the 1st part of Remark 2.29, we deduce that $u \in H^3((a,b))$.

Therefore, by the Sobolev Embedding Theorem 2.27, this implies that $u \in C^2((a,b))$ and D^2u is absolutely continuous on (a,b) .

Moreover, applying Proposition 2.24, we conclude that the third classical derivative $u^{(3)}$ exists almost everywhere, belongs to $L^2((a,b))$, and coincides with the weak derivative D^3u .

Since (a,b) was arbitrary, we deduce that $u \in H_{\text{loc}}^3((0,\infty))$, $u \in C^2((0,\infty))$, D^2u is locally absolutely continuous, and $D^3u \in L^2((0,\infty))$.

Similarly we argue on $(-\infty,0)$ and deduce that $u \in H_{\text{loc}}^3((-\infty,0))$, $u \in C^2((-\infty,0))$, D^2u is locally absolutely continuous, and $D^3u \in L^2((-\infty,0))$.

In other words, $D^3u = -iz$ on both intervals $(0,\infty)$ and $(-\infty,0)$. This ensures that $u \in H_{\text{loc}}^3(\mathbb{R} \setminus \{0\})$, u is of class C^2 on $\mathbb{R} \setminus \{0\}$ and the second derivative D^2u is locally absolutely continuous. Therefore, conditions (i) and (ii) of the proposition are satisfied. $\square$

Proof of Proposition 5.25. Part 2. Assume that the measurable function $u : \mathbb{R} \rightarrow \mathbb{C}$ satisfies the following conditions:

- (i) $u \in L^2(\mathbb{R})$,
- (ii) The restriction $u_1 := u|_{(0,\infty)}$ is of class C^2 , its second derivative D^2u_1 is locally absolutely continuous on $(0,\infty)$, and

$$\int_0^\infty |D^3u_1(x)|^2 dx < \infty,$$

- (iii) The restriction $u_2 := u|_{(-\infty, 0)}$ is of class C^2 , its second derivative D^2u_2 is locally absolutely continuous on $(-\infty, 0)$, and

$$\int_{-\infty}^0 |D^3u_2(x)|^2 dx < \infty.$$

Let us define a function $z : \mathbb{R} \rightarrow \mathbb{C}$

$$z(x) := \begin{cases} -iD^3u_1(x), & \text{if } x > 0, \\ -iD^3u_2(x), & \text{if } x < 0, \end{cases}$$

and arbitrary $z(0)$. Since the Lebesgue measure of the singleton $\{0\}$ is equal to 0, we deduce from Conditions (ii) and (iii), that $z \in L^2(\mathbb{R})$.

We now verify that $u \in D(A_1^*)$. By definition 2.51, it suffices to show that for all $\phi \in D(A_1) = C_0^\infty(\mathbb{R} \setminus \{0\})$,

$$\langle A_1\phi, u \rangle = \langle \phi, z \rangle.$$

Let $\phi \in C_0^\infty(\mathbb{R} \setminus \{0\})$. Then

$$\langle A_1\phi, u \rangle = \langle -iD^3\phi, u \rangle = \int_{\mathbb{R}} -iD^3\phi(x) \cdot \overline{u(x)} dx.$$

We split the integral into two parts:

$$\int_{\mathbb{R}} -iD^3\phi(x) \cdot \overline{u(x)} dx = \int_0^\infty -iD^3\phi(x) \cdot \overline{u_1(x)} dx + \int_{-\infty}^0 -iD^3\phi(x) \cdot \overline{u_2(x)} dx.$$

Since ϕ has compact support, and both u_1 and u_2 are C^2 -class with D^2u locally absolutely continuous, we may integrate by parts three times on each interval.

On $(0, \infty)$, integration by parts from [37, Theorem 6.22] gives

$$\int_0^\infty -iD^3\phi(x) \overline{u_1(x)} dx = \int_0^\infty \phi(x) \overline{iD^3u_1(x)} dx = \int_0^\infty \phi(x) \overline{z(x)} dx.$$

Similarly, on $(-\infty, 0)$:

$$\int_{-\infty}^0 -iD^3\phi(x) \overline{u_2(x)} dx = \int_{-\infty}^0 \phi(x) \cdot \overline{iD^3u_2(x)} dx = \int_{-\infty}^0 \phi(x) \overline{z(x)} dx.$$

Thus, by adding both parts, we infer that

$$\langle A_1\phi, u \rangle = \int_{\mathbb{R}} \phi(x) \cdot \overline{z(x)} dx = \langle \phi, z \rangle.$$

Therefore, the adjoint identity holds for all $\phi \in D(A_1)$, and by the definition of A_1^* , we conclude that $u \in D(A_1^*)$ and

$$A_1^*u = z = \begin{cases} -iD^3u_1(x), & \text{if } x > 0, \\ -iD^3u_2(x), & \text{if } x < 0. \end{cases}$$

This completes the proof of Part 2. $\square$

Proposition 5.29. *Assume that A_1 is the operator defined in formula (5.4.1). Then*

$$n_+(A_1) = 3.$$

Proof of Proposition 5.29. To find $n_+(A_1)$, let us observe that by Proposition 5.25

$$\begin{aligned} u \in \ker(iI - A_1^*) &\iff u \in D(A_1^*) \text{ and } (iI - A_1^*)u = 0 \\ \iff u \in D(A_1^*) \text{ and } iu = -iD^3u &\iff u \in D(A_1^*) \text{ and } D^3u = -u. \\ \iff \text{the restrictions of function } u \text{ to the intervals } (0, \infty) \text{ and } (-\infty, 0) & \\ \text{are of } C^2\text{-class and their derivatives are locally absolutely continuous} & \quad (5.4.3) \\ \text{and } \int_0^\infty |D^3u(x)|^2 dx < \infty, \int_{-\infty}^0 |D^3u(x)|^2 dx < \infty & \\ \text{and } D^3u = -u \text{ on } (0, \infty) \text{ and } D^3u = -u \text{ on } (-\infty, 0). & \end{aligned}$$

The above equivalence is analogous to the earlier one in (5.4.3).

Let us choose and fix $u \in \ker(iI - A_1^*)$, then by (5.4.3), u satisfies the following conditions.

$$D^3u = -u \text{ on } (0, \infty), \quad (5.4.4)$$

$$D^3u = -u \text{ on } (-\infty, 0). \quad (5.4.5)$$

In section 5.2 we proved that if a function $u \in D(A_1^*)$ is solution to equation (5.4.4) then it has to be of the following form

$$u = a_1 e^{-x}, \quad x > 0$$

for some $a_1 \in \mathbb{C}$.

In section 5.3 we proved that if a function $u \in D(A_1^*)$ is solution to equation (5.4.5) then it has to be of the following form

$$u(x) = a_2 e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} + a_3 e^{\frac{1}{2}x - i\frac{\sqrt{3}}{2}x}, \quad x < 0.$$

If $u \in \ker(iI - A_1^*)$, then a function u defined by

$$u(x) = \begin{cases} a_1 e^{-x} & \text{if } x > 0 \\ a_2 e^{\frac{1}{2}x} \cos(\frac{\sqrt{3}}{2}x) + a_3 e^{\frac{1}{2}x} \sin(\frac{\sqrt{3}}{2}x) & \text{if } x < 0 \end{cases} \quad (5.4.6)$$

$\ker(iI - A_1^*)$ consists of function given by (5.4.6).

Thus the space $\ker(iI - A_1^*)$ have dimension 3 because it contains precisely three linearly independent functions

$$e_1(x) = \begin{cases} e^{-x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \quad (5.4.7)$$

and

$$e_2(x) = \begin{cases} 0 & \text{if } x > 0 \\ a_2 e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} & \text{if } x < 0 \end{cases}$$

and

$$e_3(x) = \begin{cases} 0 & \text{if } x > 0 \\ a_3 e^{\frac{1}{2}x - i\frac{\sqrt{3}}{2}x} & \text{if } x < 0 \end{cases} \quad (5.4.8)$$

Therefore, we complete the proof. $\square$

Now, in order to calculate $n_-(A_1)$, we need to find the space $\ker(iI + A_1^*)$.

Proposition 5.30. *Assume that A_1 is the operator defined by formula (5.4.1). Then*

$$n_-(A_1) = 3. \quad (5.4.9)$$

Proof of Proposition 5.30. To find $n_-(A_1)$, let us observe that by Proposition 5.20

$$\begin{aligned} u \in \ker(iI + A_1^*) &\iff u \in D(A_1^*) \text{ and } (iI + A_1^*)u = 0 \\ &\iff u \in D(A_1^*) \text{ and } iu = iD^3u \iff u \in D(A_1^*) \text{ and } D^3u = u. \\ &\iff \text{the restrictions of function } u \text{ to the intervals } (0, \infty) \text{ and } (-\infty, 0) \\ &\quad \text{are of } C^2\text{-class and their derivatives are locally absolutely continuous} \\ &\quad \text{and } \int_0^\infty |D^3u(x)|^2 dx < \infty, \int_{-\infty}^0 |D^3u(x)|^2 dx < \infty \\ &\quad \text{and } D^3u = u \text{ on } (0, \infty) \text{ and } D^3u = u \text{ on } (-\infty, 0). \end{aligned} \quad (5.4.10)$$

The above equivalence is analogous to the earlier one in (5.4.10).

Let us choose and fix $u \in \ker(iI + A_1^*)$, then by (5.4.10), u satisfies the following conditions.

- (1) $D^3u = u$ on $(0, \infty)$ and

(2) $D^3u = u$ on $(-\infty, 0)$.

In section 5.2 we solved that a function $u \in D(A_1^*)$ and solution (1) has to be of the form

$$u(x) = a_2 e^{-\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} + a_3 e^{-\frac{1}{2}x - i\frac{\sqrt{3}}{2}x}, \quad x > 0$$

In section 5.3 we solved that a function $u \in D(A_1^*)$ and solution (2) has to be of the form

$$u(x) = a_1 e^x, \quad x < 0$$

for some constant $a_1, a_2, a_3 \in \mathbb{C}$.

If $u \in \ker(iI + A_1^*)$, then a function u defined by

$$u(x) = \begin{cases} a_2 e^{-\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} + a_3 e^{-\frac{1}{2}x - i\frac{\sqrt{3}}{2}x} & \text{if } x > 0 \\ a_1 e^x & \text{if } x < 0 \end{cases} \quad (5.4.11)$$

$\ker(iI + A_1^*)$ consists of function given by (5.4.11).

Thus the space $\ker(iI + A_1^*)$ have dimension 3 because it contains precisely three linearly independent functions

$$e_1(x) = \begin{cases} e^{-\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \quad (5.4.12)$$

and

$$e_2(x) = \begin{cases} e^{-\frac{1}{2}x - i\frac{\sqrt{3}}{2}x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

and

$$e_3(x) = \begin{cases} 0 & \text{if } x > 0 \\ e^x & \text{if } x < 0 \end{cases} \quad (5.4.13)$$

Therefore, we infer identity (5.4.9) as claimed. $\square$

As a direct consequence of Propositions 5.29 and 5.30, we obtain the following description of the deficiency subspaces.

Corollary 5.31. *Assume that A_1 is the operator defined by (5.4.1). Then the deficiency subspaces*

$$H_+ := \ker(iI - A_1^*), \quad H_- := \ker(iI + A_1^*)$$

satisfy

$$\dim H_+ = 3, \quad \dim H_- = 3.$$

Moreover, the space H_+ is spanned by the functions

$$\begin{aligned} e_1(x) &= \begin{cases} e^{-x}, & x > 0, \\ 0, & x < 0, \end{cases} \\ e_2(x) &= \begin{cases} 0, & x > 0, \\ e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x}, & x < 0, \end{cases} \\ e_3(x) &= \begin{cases} 0, & x > 0, \\ e^{\frac{1}{2}x - i\frac{\sqrt{3}}{2}x}, & x < 0. \end{cases} \end{aligned}$$

Similarly, the space H_- is spanned by the functions

$$\begin{aligned} f_1(x) &= \begin{cases} e^{-\frac{1}{2}x + i\frac{\sqrt{3}}{2}x}, & x > 0, \\ 0, & x < 0, \end{cases} \\ f_2(x) &= \begin{cases} e^{-\frac{1}{2}x - i\frac{\sqrt{3}}{2}x}, & x > 0, \\ 0, & x < 0, \end{cases} \\ f_3(x) &= \begin{cases} 0, & x > 0, \\ e^x, & x < 0. \end{cases} \end{aligned}$$

5.4.3 Results for the third-order linear homogeneous differential operators on the punctured line

As a starting point, we prove the following.

Proposition 5.32. *The operator A_1 defined by (5.4.1) is not closed.*

Proof of Proposition 5.32. The proof follows the same method as in Proposition 3.29 and Proposition 4.17.

We show that the graph $\Gamma(A_1)$ is not closed in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Let $u : \mathbb{R} \rightarrow \mathbb{C}$ be defined by

$$u(x) = \begin{cases} e^{-x^{-2}}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Then $u \in C^\infty(\mathbb{R})$ and $u^{(k)}(0) = 0$ for all $k \in \mathbb{N}$, see [37, Chapter 6]. In particular, u is weakly differentiable of all orders.

Moreover, a standard argument, see, e.g., the proof of Proposition 3.29, shows that

$$u \in H^3(\mathbb{R}),$$

while clearly

$$u \notin D(A_1) = C_0^\infty(\mathbb{R} \setminus \{0\}),$$

since u is not compactly supported. Using the same approximation argument as in Lemma 3.32, but in $H^3(\mathbb{R})$, there exists a sequence $(u_n) \subset D(A_1)$ such that

$$u_n \rightarrow u \quad \text{in } H^3(\mathbb{R}).$$

In particular,

$$u_n \rightarrow u, \quad D^3 u_n \rightarrow D^3 u \quad \text{in } L^2(\mathbb{R}).$$

Since $A_1 u_n = -iD^3 u_n$, we obtain

$$A_1 u_n \rightarrow -iD^3 u \quad \text{in } L^2(\mathbb{R}).$$

Thus

$$(u_n, A_1 u_n) \rightarrow (u, -iD^3 u) \quad \text{in } L^2(\mathbb{R}) \times L^2(\mathbb{R}).$$

However, $u \notin D(A_1)$, hence $(u, -iD^3 u) \notin \Gamma(A_1)$. Therefore, the graph $\Gamma(A_1)$ is not closed, and A_1 is not closed. $\square$

Proposition 5.33. *Assume that A_1 is the operator defined by formula (5.4.1). A function $u \in D(\overline{A_1})$ if and only if the following conditions are satisfied*

(i) $u \in L^2(\mathbb{R}, \mathbb{C})$,

(ii) the function u belongs to the Sobolev space $H_{\text{loc}}^3(\mathbb{R} \setminus \{0\}, \mathbb{C})$ and

$u^{(2)} : \mathbb{R} \rightarrow \mathbb{C}$ is locally absolutely continuous and

$$\int_{\mathbb{R}} |D^3 u(t)|^2 dt < \infty,$$

(iii) $u(0) = 0$, $u'(0) = 0$ and $u''(0) = 0$.

Moreover,

$$\overline{A_1} u = -iD^3 u, \quad \text{if } u \in D(\overline{A_1}). \quad (5.4.14)$$

Proof of Proposition 5.33. Part 1. Let us choose and fix $u \in D(\overline{A_1})$. Then obviously $u \in L^2(\mathbb{R}, \mathbb{C})$ and so condition (i) is satisfied. In order to show that condition (ii) is satisfied we need to show that

$$u \in H_{\text{loc}}^3(\mathbb{R} \setminus \{0\}, \mathbb{C}).$$

By the Definition 2.85 of the operator closure, there exists a sequence (u_n) in $D(A_1)$ and $z \in L^2(\mathbb{R})$ such that:

$$\begin{aligned} u_n &\rightarrow u \quad \text{in } L^2(\mathbb{R}), \\ A_1 u_n &= -i u_n^{(3)} \rightarrow z \quad \text{in } L^2(\mathbb{R}). \end{aligned} \quad (5.4.15)$$

Thus, in particular

$$u_n^{(3)} \rightarrow iz \text{ in } L^2(\mathbb{R}). \quad (5.4.16)$$

(i) **Uniform convergence on compacts.** Since $u_n \in C_0^\infty(\mathbb{R} \setminus \{0\})$, we may integrate three times and deduce that for every $t \in \mathbb{R}$,

$$\begin{aligned} u_n''(t) &= u_n''(0) + \int_0^t u_n^{(3)}(s) ds = \int_0^t u_n^{(3)}(s) ds, \\ u_n'(t) &= u_n'(0) + \int_0^t u_n''(s) ds = \int_0^t u_n''(s) ds, \\ u_n(t) &= u_n(0) + \int_0^t u_n'(s) ds = \int_0^t u_n'(s) ds. \end{aligned} \quad (5.4.17)$$

By using [1, Fubini's Theorem, p.19] and change the variables. We consider $0 < \sigma < r < s < t$, then

$$\begin{aligned} u_n(t) &= u_n(0) + \int_0^t u_n'(s) ds = \int_0^t \left[\int_0^s u_n''(r) dr \right] ds \\ &= \int_0^t \left[\int_0^s \left[\int_0^r u_n'''(\sigma) d\sigma \right] dr \right] ds = \int_0^t u_n'''(\sigma) \left[\int_\sigma^t dr \left(\int_r^t ds \right) \right] d\sigma \\ &= \int_0^t u_n'''(\sigma) \left[\int_\sigma^t (t-r) dr \right] d\sigma \end{aligned}$$

By using change the variables and with Lebesgue measure, we have

$$r' = t - r, \quad dr' = dr$$

where $0 < r' < t - \sigma$. So,

$$\int_\sigma^t (t-r) dr = \int_0^{t-\sigma} r' dr' = \frac{1}{2} (t-\sigma)^2.$$

Hence,

$$u_n(t) = \int_0^t u_n'''(\sigma) \left[\frac{1}{2} (t-\sigma)^2 \right] d\sigma. \quad (5.4.18)$$

Now, we will prove $\{u_n(t)\}$ is a Cauchy sequences.

Using the above identity we prove that

$$\begin{aligned} u_n(t) - u_m(t) &= \int_0^t [u_n'''(s) - u_m'''(s)] \frac{(t-s)^2}{2} ds = \int_{-\infty}^{\infty} [u_n'''(s) - u_m'''(s)] k_t(s) ds \\ &= \langle D^3 u_n - D^3 u_m, k_t \rangle_{L^2(\mathbb{R})}. \end{aligned}$$

where

$$k_t(s) = 1_{(0,t)}(s) \frac{(t-s)^2}{2}, \quad s \in \mathbb{R}.$$

Note that

$$\begin{aligned} \|k_t\|_{L^2(\mathbb{R})}^2 &= \int_{-\infty}^{\infty} |k_t(s)|^2 ds = \int_0^t |k_t(s)|^2 ds + \int_{\mathbb{R} \setminus (0,t)} |k_t(s)|^2 ds \\ &= \int_0^t \frac{(t-s)^4}{4} ds + \int_{\mathbb{R} \setminus (0,t)} 0 ds = \frac{1}{4} \int_0^t s^4 ds = \frac{1}{20} t^5. \end{aligned}$$

By applying the Cauchy-Schwarz inequality yields

$$\begin{aligned} |u_n(t) - u_m(t)| &\leq \|D^3 u_n - D^3 u_m\|_{L^2(\mathbb{R})} \|k_t\|_{L^2(\mathbb{R})} \\ &\leq \|D^3 u_n - D^3 u_m\|_{L^2(\mathbb{R})} \cdot \frac{t^5}{20}. \end{aligned}$$

We choose and fix $T > 0$. Then by taking the supremum, we have

$$\sup_{0 < t \leq T} |u_n - u_m| \leq \|D^3 u_n - D^3 u_m\|_{L^2(\mathbb{R})} \cdot \frac{T^5}{20}.$$

Since by (5.4.16), the sequence $(D^3 u_n)_{n=1}^{\infty}$ is Cauchy in $L^2(\mathbb{R}, \mathbb{C})$, by [37, Theorem 7.8], we infer there exists $\tilde{u} \in C([0, T], \mathbb{C})$ such that

$$u_n \rightarrow \tilde{u} \quad \text{uniformly on} \quad [0, T]. \quad (5.4.19)$$

In particular, $u_n \rightarrow \tilde{u}$ in $L^2[0, T]$. On the other hand, $u_n \rightarrow u$ in $L^2(\mathbb{R})$ and hence

$$u_n \rightarrow u \quad \text{in} \quad L^2[0, T].$$

Therefore, by the uniqueness of the limit we infer that $u = \tilde{u}$ as elements of $L^2[0, T]$. Hence, by the definition of the space $L^2[0, T]$ we deduce that

$$u = \tilde{u} \text{ almost everywhere on } [0, T].$$

In particular, we can assume that u is continuous on $[0, T]$. Since T was arbitrary, we can assume that u is continuous on $[0, \infty)$.

Repeating the above argument on the interval $[-T, 0]$, we deduce that the function u is continuous, and

$$\forall T > 0 : u_n \rightarrow u \quad \text{uniformly on} \quad [-T, T].$$

(ii) **Existence of the third weak derivative of u .** We will show now that u is weakly differentiable up to third order and that $D^3u = iz$. For this purpose let us choose and fix a test function $\phi \in C_0^\infty(\mathbb{R})$. Since every $u_n \in C_0^\infty(\mathbb{R})$, integration by parts gives

$$\int u_n(x) \phi^{(3)}(x) dx = (-1)^3 \int u_n^{(3)}(x) \phi(x) dx.$$

Taking the limit $n \rightarrow \infty$, by (5.4.15) and (5.4.16), we infer that

$$\begin{aligned} \int u(x) \phi^{(3)}(x) dx &= \lim_{n \rightarrow \infty} \int u_n(x) \phi^{(3)}(x) dx = - \lim_{n \rightarrow \infty} \int u_n^{(3)}(x) \phi(x) dx \\ &= - \int (iz(x)) \phi(x) dx. \end{aligned}$$

By the definition of weak derivatives, we infer that the third weak derivative of u exists and equals $D^3u = iz \in L^2(\mathbb{R})$. Hence, if $u \in D(\overline{A_1})$, then $u \in L_{\text{loc}}^2(\mathbb{R})$, u is weakly 3-times differentiable, and $D^3u = iz \in L^2(\mathbb{R})$. In particular, by the local characterization of Sobolev spaces, see, Remark 2.29) we infer that $u \in H_{\text{loc}}^3(\mathbb{R})$. By the Sobolev Embedding Theorem we deduce that $u \in C^2(\mathbb{R})$ and u satisfies the following integral identity, similar to (5.4.18)

$$u(t) = \int_0^t u'''(\sigma) \frac{(t-\sigma)^2}{2} d\sigma, \quad t \in \mathbb{R}.$$

We will show the following result.

Lemma 5.34. *Assume that $T > 0$. Then the following holds:*

$$u'_n \rightarrow u' \quad \text{uniformly on} \quad [0, T]. \quad (5.4.20)$$

$$u''_n \rightarrow u'' \quad \text{uniformly on} \quad [0, T]. \quad (5.4.21)$$

Proof of Lemma 5.34. **To proof equality (5.4.20).** By (5.4.17), we write:

$$u'_n(t) = \int_0^t u''_n(r) dr = \int_0^t \left[\int_0^r u'''_n(\sigma) d\sigma \right] dr.$$

Using the Fubini Theorem to switch the order of integration, we obtain:

$$u'_n(t) = \int_0^t u'''_n(\sigma) \left(\int_\sigma^t dr \right) d\sigma = \int_0^t u'''_n(\sigma) (t - \sigma) d\sigma.$$

The same identity holds for function u . Hence we can estimate the difference

$$|u'_n(t) - u'(t)| = \left| \int_0^t (u_n''' - u''')(\sigma)(t - \sigma) d\sigma \right|.$$

By the Cauchy–Schwarz inequality:

$$|u'_n(t) - u'(t)| \leq \left(\int_0^t |u_n''' - u'''|^2 d\sigma \right)^{1/2} \left(\int_0^t (t - \sigma)^2 d\sigma \right)^{1/2}.$$

We compute $\int_0^t (t - \sigma)^2 d\sigma = \frac{t^3}{3}$.

Therefore,

$$|u'_n(t) - u'(t)| \leq \sqrt{\frac{t^3}{3}} \cdot \|u_n''' - u'''\|_{L^2(0,t)}.$$

Taking the supremum over $t \in [0, T]$, we obtain:

$$\|u'_n - u'\|_{C([0,T])} \leq \sqrt{T^3/3} \cdot \|D^3u_n - D^3u\|_{L^2(0,T)}.$$

Since $D^3u_n \rightarrow D^3u$ in $L^2(0, T)$ we deduce that

$$u'_n \rightarrow u' \quad \text{uniformly on } [0, T].$$

Hence (5.4.20) follows. But in fact we can deduce a stronger assertion.

Taking $T < 0$, we deduce that $u'_n \rightarrow u'$ locally uniformly on $(-\infty, \infty)$, i.e. for every $T > 0$,

$$u'_n \rightarrow u' \quad \text{uniformly on } [-T, T].$$

To proof equality (5.4.21). By (5.4.17), we begin with

$$u_n''(t) = \int_0^t u_n'''(\sigma) d\sigma.$$

Therefore, for any $n, m \in \mathbb{N}$,

$$u_n''(t) - u_m''(t) = \int_0^t (u_n'''(\sigma) - u_m'''(\sigma)) d\sigma.$$

Taking absolute value and applying the Cauchy–Schwarz inequality:

$$\begin{aligned} |u_n''(t) - u_m''(t)| &\leq \left(\int_0^t 1^2 d\sigma \right)^{1/2} \left(\int_0^t |u_n'''(\sigma) - u_m'''(\sigma)|^2 d\sigma \right)^{1/2} \\ &= \sqrt{t} \cdot \|u_n''' - u_m'''\|_{L^2(0,t)}. \end{aligned}$$

In particular, taking supremum over $t \in [0, T]$, we get:

$$\|u_n'' - u_m''\|_{C([0, T])} \leq \sqrt{T} \cdot \|u_n''' - u_m'''\|_{L^2(0, T)}.$$

Since $D^3 u_n \rightarrow D^3 u$ in $L^2(0, T)$ we deduce that

$$u_n'' \rightarrow u'' \quad \text{uniformly on } [0, T].$$

Hence (5.4.21) follows. But in fact we can deduce a stronger assertion.

Taking $T < 0$, we deduce that $u_n'' \rightarrow u''$ locally uniformly on $(-\infty, \infty)$, i.e. for every $T > 0$,

$$u_n'' \rightarrow u'' \quad \text{uniformly on } [-T, T].$$

□

(iii) Now we are ready to finish the proof of part (iii) of Proposition 5.33. Since all $u_n \in C_0^\infty(\mathbb{R} \setminus \{0\})$, we have for all n :

$$u_n(0) = u_n'(0) = u_n''(0) = 0.$$

By uniform convergence on $[0, T]$ in equations (5.4.19), (5.4.20) and (5.4.21) of u_n , u_n' and u_n'' , we deduce that

$$u(0) = \lim_{n \rightarrow \infty} u_n(0) = 0,$$

$$u'(0) = \lim_{n \rightarrow \infty} u_n'(0) = 0,$$

$$u''(0) = \lim_{n \rightarrow \infty} u_n''(0) = 0.$$

Hence u satisfies the three conditions in Proposition 5.33, and $\overline{A_1}u = -iD^3u$. This completes the proof of **Part 1**.

□

Before proving the converse implication, we clarify the Sobolev spaces used on the half-lines.

Remark 5.35. We briefly clarify the Sobolev spaces used in the proof below.

- The spaces $H^3((0, \infty))$ and $H^3([0, \infty))$ are identified in the usual way, since changing the value of a function at one point does not affect its L^2 -class or its weak derivatives.
- The space $H_0^3((0, \infty))$ is defined as the closure of $C_0^\infty((0, \infty))$ in the H^3 -norm.

- A function $u \in H^3((0, \infty))$ belongs to $H_0^3((0, \infty))$ if, in the trace sense,

$$u(0) = u'(0) = u''(0) = 0.$$

- Accordingly, when we write $H_0^3([0, \infty))$, we mean the subspace of $H^3([0, \infty))$ consisting of functions whose traces at 0 vanish up to order two, namely

$$H_0^3([0, \infty)) = \{ u \in H^3([0, \infty)) : u(0) = u'(0) = u''(0) = 0 \}.$$

Proof of Proposition 5.33. Part 2. We now prove the converse implication. Let us choose and fix $u \in L^2(\mathbb{R}, \mathbb{C})$ satisfying conditions (i)–(iii). We will show that $u \in D(\overline{A_1})$.

By assumptions (ii) and (iii), the restrictions

$$u_- := u|_{(-\infty, 0]}, \quad u_+ := u|_{[0, \infty)}$$

belong to the Sobolev spaces

$$u_- \in H_0^3((-\infty, 0]), \quad u_+ \in H_0^3([0, \infty)),$$

see Remark 5.35.

Therefore, by [1, Theorem 3.22, p. 68], there exist sequences

$$f_n \in C_0^\infty((-\infty, 0)), \quad g_n \in C_0^\infty((0, \infty))$$

such that

$$f_n \rightarrow u_- \quad \text{in } H^3((-\infty, 0]), \quad g_n \rightarrow u_+ \quad \text{in } H^3([0, \infty)).$$

For each $n \in \mathbb{N}$, define $u_n : \mathbb{R} \rightarrow \mathbb{C}$ by

$$u_n(x) := \begin{cases} f_n(x), & x < 0, \\ g_n(x), & x > 0, \\ 0, & x = 0. \end{cases}$$

Then

$$u_n \in C_0^\infty(\mathbb{R} \setminus \{0\}) = D(A_1).$$

Moreover, the classical third-order derivative of u_n is given by

$$u_n^{(3)}(x) = \begin{cases} D^3 f_n(x), & x < 0, \\ D^3 g_n(x), & x > 0, \\ 0, & x = 0. \end{cases} \quad (5.4.22)$$

Hence, by Proposition 2.3, the weak derivative $D^3 u_n$ exists and is equal to the right-hand side of (5.4.22).

We claim that

$$u_n \rightarrow u \quad \text{in } L^2(\mathbb{R}), \quad A_1 u_n = -iD^3 u_n \rightarrow -iD^3 u \quad \text{in } L^2(\mathbb{R}).$$

Indeed, since $f_n \rightarrow u_-$ in $H^3((-\infty, 0])$ and $g_n \rightarrow u_+$ in $H^3([0, \infty))$, in particular we have

$$f_n \rightarrow u_- \quad \text{in } L^2((-\infty, 0]), \quad g_n \rightarrow u_+ \quad \text{in } L^2([0, \infty)).$$

Therefore

$$\begin{aligned} \|u_n - u\|_{L^2(\mathbb{R})}^2 &= \int_{-\infty}^{\infty} |u_n(x) - u(x)|^2 dx \\ &= \int_{-\infty}^0 |f_n(x) - u_-(x)|^2 dx + \int_0^{\infty} |g_n(x) - u_+(x)|^2 dx \\ &= \|f_n - u_-\|_{L^2((-\infty, 0))}^2 + \|g_n - u_+\|_{L^2((0, \infty))}^2 \rightarrow 0. \end{aligned}$$

Similarly, since convergence in H^3 implies convergence of the third weak derivatives in L^2 , we obtain

$$\begin{aligned} \|D^3 u_n - D^3 u\|_{L^2(\mathbb{R})}^2 &= \int_{-\infty}^{\infty} |D^3 u_n(x) - D^3 u(x)|^2 dx \\ &= \int_{-\infty}^0 |D^3 f_n(x) - D^3 u_-(x)|^2 dx + \int_0^{\infty} |D^3 g_n(x) - D^3 u_+(x)|^2 dx \\ &= \|D^3 f_n - D^3 u_-\|_{L^2((-\infty, 0))}^2 + \|D^3 g_n - D^3 u_+\|_{L^2((0, \infty))}^2 \rightarrow 0. \end{aligned}$$

Hence

$$(u_n, A_1 u_n) = (u_n, -iD^3 u_n) \rightarrow (u, -iD^3 u) \quad \text{in } L^2(\mathbb{R}) \times L^2(\mathbb{R}).$$

By Remark 2.86, we infer that

$$(u, -iD^3 u) \in \overline{\Gamma(A_1)} = \Gamma(\overline{A_1}).$$

Therefore

$$u \in D(\overline{A_1}), \quad \overline{A_1}u = -iD^3u.$$

In particular, equality (5.4.14) holds. This completes the proof of Part 2.

Therefore, the proof of Proposition 5.33 is complete. $\square$

5.4.4 The adjoint of the closure

Let A_1 be the operator defined by formula (5.4.1). Since A_1 is densely defined and symmetric, it follows from Proposition 2.89 that it is closable. Hence, the closure $\overline{A_1}$ exists.

We now extend Proposition 5.25 by determining the adjoint of the closure $\overline{A_1}$.

Proposition 5.36. *Let A_1 be the operator defined by formula (5.4.1), and let $\overline{A_1}$ denote its closure in $L^2(\mathbb{R}, \mathbb{C})$. Then,*

$$(\overline{A_1})^* = A_1^*.$$

A measurable function $v : \mathbb{R} \rightarrow \mathbb{C}$ belongs to $D((\overline{A_1})^*)$ if and only if:

(i) $v \in L^2(\mathbb{R}, \mathbb{C})$;

(ii) *The restriction $v : (0, \infty) \rightarrow \mathbb{C}$ is three times weakly differentiable, with third derivative*

$$D^3v \in L^2((0, \infty)), \quad \text{i.e.} \quad \int_0^\infty |D^3v(t)|^2 dt < \infty;$$

equivalently, v , Dv , and D^2v are locally absolutely continuous on $(0, \infty)$;

(iii) *The restriction $v : (-\infty, 0) \rightarrow \mathbb{C}$ is three times weakly differentiable, with third derivative*

$$D^3v \in L^2((-\infty, 0)), \quad \text{i.e.} \quad \int_{-\infty}^0 |D^3v(t)|^2 dt < \infty;$$

equivalently, v , Dv , and D^2v are locally absolutely continuous on $(-\infty, 0)$.

Moreover, for every $v \in D((\overline{A_1})^*)$, the adjoint acts as

$$(\overline{A_1})^*v = A_1^*v = -iD^3v.$$

Remark 5.37. The characterization above coincides with that of A_1^* in Proposition 5.25. This agreement follows from Corollary 2.92, which guarantees that taking the closure of a symmetric operator does not affect its adjoint.

5.4.5 Deficiency indices of the closure of the third-order linear homogeneous differential operators on the punctured line

From Proposition 5.36, we already know that the closure $\overline{A_1}$ of the third-order linear homogeneous differential operators defined by (5.4.1) exists and that

$$(\overline{A_1})^* = A_1^*.$$

By Corollary 2.92, the closure of a densely defined symmetric operator remains symmetric, and its deficiency subspaces coincide with those of the original operator:

$$\ker(i - A_1^*) = \ker(i - \overline{A_1}^*), \quad \ker(i + A_1^*) = \ker(i + \overline{A_1}^*).$$

Consequently, the deficiency indices of A_1 and $\overline{A_1}$ coincide:

$$n_+(\overline{A_1}) = n_+(A_1), \quad n_-(\overline{A_1}) = n_-(A_1).$$

Corollary 5.38. *Let A_1 be the third-order differential operator defined by (5.4.1), and let $\overline{A_1}$ denote its closure in $L^2(\mathbb{R}, \mathbb{C})$. Then both A_1 and $\overline{A_1}$ are symmetric and densely defined, and their deficiency indices coincide. In particular, by Theorem 5.23, we have*

$$n_+(\overline{A_1}) = n_+(A_1) = 3, \quad n_-(\overline{A_1}) = n_-(A_1) = 3.$$

Remark 5.39. Corollary 2.92 guarantees that for any densely defined symmetric operator including higher-order differential operators such as the third-order operator taking the closure preserves both symmetry and the deficiency indices. Therefore, the spectral properties and self-adjoint extensions of $\overline{A_1}$ are identical to those of A_1 .

5.4.6 Characterization of all self-adjoint extensions of the third-order linear homogeneous differential operators on the punctured line

We will characterize all self-adjoint extensions of operator A_1 in $L^2(\mathbb{R}, \mathbb{C})$ defined by equation (5.4.1). In this section we apply Theorem 2.60 in order to classify all self-adjoint extensions of the Airy operator A_1 on $\mathbb{R} \setminus \{0\}$, using the deficiency space in Theorem 5.23. Recall that A_1 is defined by (5.4.1). From Theorem 5.23 we know that the deficiency indices of A_1 are

$$n_+(A_1) = n_-(A_1) = 3.$$

Hence,

$$n_+(\overline{A_1}) = n_-(\overline{A_1}) = 3.$$

Let

$$H_+ := \ker(iI - A_1^*), \quad H_- := \ker(iI + A_1^*).$$

Then

$$\dim H_+ = \dim H_- = 3.$$

Let $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ be an orthonormal basis of H_+ , and $\{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3\}$ an orthonormal basis of H_- , as constructed in Section 5.4.2.

Definition 5.40.(i) The symbol $\mathcal{U}(H_+, H_-)$ denotes the set of all unitary operators

$$\mathcal{C} : H_+ \longrightarrow H_-,$$

that is, by Definition 2.69, $\mathcal{C}$ is a linear bijection satisfying

$$\boxed{\langle \mathcal{C}u, \mathcal{C}v \rangle = \langle u, v \rangle \quad \text{for all } u, v \in H_+.$$

(ii) The symbol $U(3)$ denotes the group of all 3×3 complex unitary matrices.

A matrix $M = [m_{j\ell}] \in \mathbb{C}^{3 \times 3}$ belongs to $U(3)$ if and only if

$$M^*M = MM^* = I_3,$$

where

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the 3×3 identity matrix, and M^* is the conjugate transpose of M , i.e.

$$\text{if } M = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}, \text{ then } M^* = \overline{M}^T = \begin{pmatrix} \overline{m_{11}} & \overline{m_{21}} & \overline{m_{31}} \\ \overline{m_{12}} & \overline{m_{22}} & \overline{m_{32}} \\ \overline{m_{13}} & \overline{m_{23}} & \overline{m_{33}} \end{pmatrix}.$$

Equivalently, $A \in \mathbb{C}^{3 \times 3}$ the columns (or, equivalently, the rows) of M form an orthonormal basis of $\mathbb{C}^3$.

Since by Corollary 5.31, in our case

$$\dim H_+ = \dim H_- = 3,$$

we have the following result.

Proposition 5.41. *Assume that H_+ and H_- are Hilbert spaces such that*

$$\dim H_+ = \dim H_- = 3.$$

Then the set of all unitary operator from H_+ to H_- , denoted by $\mathcal{U}(H_+, H_-)$ isomorphic to the unitary group $U(3)$, i.e.

$$\mathcal{U}(H_+, H_-) \cong U(3).$$

Proof of Proposition 5.41. Since $\dim H_+ = \dim H_- = 3$, both H_+ and H_- are finite-dimensional, i.e. 3-dimensional Hilbert spaces.

Let $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ be an orthonormal basis of H_+ , and $\{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3\}$ an orthonormal basis of H_- .

Let $\Gamma \in \mathcal{U}(H_+, H_-)$. Then Γ is uniquely determined by its action on the basis of H_+ :

$$\Gamma \tilde{e}_j = \sum_{k=1}^3 m_{kj} \tilde{f}_k, \quad j = 1, 2, 3,$$

for some coefficients $m_{kj} \in \mathbb{C}$.

Thus, Γ is represented with respect to these bases by a matrix $M = (m_{kj}) \in \mathbb{C}^{3 \times 3}$.

Since Γ is unitary, it preserves the inner product, i.e.

$$\langle \Gamma \tilde{e}_j, \Gamma \tilde{e}_\ell \rangle = \langle \tilde{e}_j, \tilde{e}_\ell \rangle = \delta_{j\ell}.$$

Hence, the columns of M form an orthonormal system in $\mathbb{C}^3$, which implies

$$M^*M = I_3.$$

Similarly, since Γ is unitary, it is bijective, and hence $MM^* = I_3$. Therefore, $M \in U(3)$.

Conversely, given any matrix $M \in U(3)$, we define a linear operator $\Gamma : H_+ \rightarrow H_-$ by

$$\Gamma \tilde{e}_j = \sum_{k=1}^3 m_{kj} \tilde{f}_k.$$

Extending linearly, we obtain a unitary operator Γ .

Thus, we obtain a bijective correspondence between $\mathcal{U}(H_+, H_-)$ and $U(3)$, which complete the proof. $\square$

We now apply Theorem 2.60 to $\overline{A_1}$. Let us recall that $\overline{A_1}$ is a closed linear and densely defined linear operator. Hence, self-adjoint extensions of $\overline{A_1}$ exist because $n_+ = n_- = 3$. Moreover the family of all self-adjoint extension of $\overline{A_1}$ are uniquely parametrized by the family of all unitary operator $\mathbf{C} : H_+ \rightarrow H_-$.

Theorem 5.42 (Self-adjoint extensions of $\overline{A_1}$). *Let $\mathbf{C} \in \mathcal{U}(H_+, H_-)$. Define the corresponding von Neumann extension $A_{\mathbf{C}}$ as follows.*

$$\begin{aligned} D(A_{\mathbf{C}}) &= \left\{ u = u_0 + u_+ + \mathbf{C}u_+ : u_0 \in D(\overline{A_1}), u_+ \in H_+ \right\}, \\ A_{\mathbf{C}}(u_0 + u_+ + \mathbf{C}u_+) &= \overline{A_1}u_0 + i u_+ - i \mathbf{C}u_+. \end{aligned} \quad (5.4.23)$$

Then $A_{\mathbf{C}}$ is self-adjoint and satisfies

$$\overline{A_1} \subset A_{\mathbf{C}} \subset A_1^*.$$

Moreover, if A is a self-adjoint extension of $\overline{A_1}$, then $A = A_{\mathbf{C}}$ for a unique $\mathbf{C} \in \mathcal{U}(H_+, H_-)$.

Idea of proof. Since $n_+(\overline{A_1}) = n_-(\overline{A_1}) = 3 < \infty$, the hypotheses of Theorem 2.60 apply. Thus all self-adjoint extensions are of the form (5.4.23), with $\mathbf{C}$ ranging over all unitary maps $H_+ \rightarrow H_-$. $\square$

Corollary 5.43 (Classification of all self-adjoint extensions of $\overline{A_1}$). *Let $\text{SA}(\overline{A_1})$ denote the family of all self-adjoint extensions of $\overline{A_1}$ on $\mathbb{R} \setminus \{0\}$. Then*

$$\boxed{\text{SA}(\overline{A_1}) = \{ A_{\mathbf{C}} : \mathbf{C} \in \mathcal{U}(H_+, H_-) \} \cong U(3).}$$

In particular, every extension corresponds to a unique 3×3 unitary matrix.

Remark 5.44. The situation is fully analogous to the Laplacian case (Subsection 4.2), except that there the deficiency spaces were 2-dimensional and parametrization used $U(2)$, whereas for the Airy operator we encounter 3-dimensional deficiency spaces and hence $U(3)$.

5.4.6.1 Finding an ONB of the deficiency Subspace H_+

The objective of this subsection is to construct an orthonormal basis of the deficiency subspace H_+ , starting from the natural basis given in (5.4.24) and applying the Gram–Schmidt orthonormalization procedure.

Let us recall that from (5.4.7)–(5.4.8), we have

$$H_+ := \ker(iI - A_1^*) = \text{span}\{e_1, e_2, e_3\},$$

where

$$\begin{aligned} e_1(x) &= \mathbf{1}_{(0,\infty)}(x)e^{-x}, \\ e_2(x) &= \mathbf{1}_{(-\infty,0)}(x)e^{\frac{1}{2}x+i\frac{\sqrt{3}}{2}x}, \\ e_3(x) &= \mathbf{1}_{(-\infty,0)}(x)e^{\frac{1}{2}x-i\frac{\sqrt{3}}{2}x}. \end{aligned} \tag{5.4.24}$$

The indicator functions are

$$\mathbf{1}_{(0,\infty)}(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \quad \mathbf{1}_{(-\infty,0)}(x) = \begin{cases} 1, & x < 0, \\ 0, & x > 0. \end{cases}$$

We will check if $\{e_1, e_2, e_3\}$ form an orthonormal basis of H_+ by checking whether the following conditions are satisfied.

$$\|e_j\| = 1 \text{ for } j = 1, 2, 3 \tag{5.4.25}$$

$$\langle e_j, e_k \rangle_{L^2} = 0 \text{ if } j, k = 1, 2, 3 \text{ and } j \neq k \tag{5.4.26}$$

Let us check the first condition (5.4.25)

$$\|e_1\|^2 = \int_0^\infty |e_1(x)|^2 dx = \int_0^\infty e^{-2x} dx = \frac{1}{2}.$$

Thus

$$\|e_1\| = \frac{1}{\sqrt{2}}.$$

Hence, $\|e_1\| \neq 1$ and we need to normalize e_1 , i.e. $\tilde{e}_1 = e_1\|e_1\|^{-1}$. Thus we put

$$\tilde{e}_1(x) = \sqrt{2}e^{-x} = \sqrt{2}e_1(x).$$

Next, for $\|e_2\|$

$$\|e_2\|^2 = \int_{-\infty}^0 |e_2(x)|^2 dx = \int_{-\infty}^0 e^x dx = 1.$$

For e_3

$$\|e_3\|^2 = \int_{-\infty}^0 |e_3(x)|^2 dx = \int_{-\infty}^0 e^x dx = 1.$$

Claim 5.45. The vectors $\{e_1, e_2, e_3\}$ do not form an orthonormal basis of H_+ .

Proof of 5.45. Recall that (5.4.24), we will check the inner products between e_1 and e_2 and between e_1 and e_3 in the second condition (5.4.26), as follow

$$\begin{aligned} \langle e_1, e_2 \rangle &= \int_{-\infty}^{\infty} e_1(x) \overline{e_2(x)} dx \\ &= \int_{-\infty}^{\infty} 1_{(0,\infty)} e^{-x} \cdot 1_{(-\infty,0)}(x) e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} dx = \int_{-\infty}^{\infty} 0 dx = 0. \end{aligned}$$

and similarly

$$\begin{aligned} \langle e_1, e_3 \rangle &= \int_{-\infty}^{\infty} e_1(x) \overline{e_3(x)} dx \\ &= \int_{-\infty}^{\infty} 1_{(0,\infty)} e^{-x} \cdot 1_{(-\infty,0)}(x) e^{\frac{1}{2}x - i\frac{\sqrt{3}}{2}x} dx = \int_{-\infty}^0 0 dx = 0 \end{aligned}$$

where e_1 is perpendicular to e_2 and e_3 . Thus e_1 is orthogonal to both e_2 and e_3 . Finally, for the inner products between e_2 and e_3 , we have

$$\begin{aligned} \langle e_2, e_3 \rangle &= \int_{-\infty}^{\infty} e_2(x) \overline{e_3(x)} dx \\ &= \int_{-\infty}^{\infty} 1_{(-\infty,0)}(x) e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} \cdot 1_{(-\infty,0)}(x) e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} dx \\ &= \int_{-\infty}^0 e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} \cdot e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} dx = \int_{-\infty}^0 e^{(1+i\sqrt{3})x} dx = \frac{1}{1+i\sqrt{3}} \\ &= \frac{1-i\sqrt{3}}{(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{1-i\sqrt{3}}{4} = \frac{1}{4} - i\frac{\sqrt{3}}{4} =: \lambda. \end{aligned} \tag{5.4.27}$$

Since $\lambda \neq 0$, the vectors e_2 and e_3 are not perpendicular, and therefore the set $\{e_1, e_2, e_3\}$ is not an orthonormal basis of H_+ as claimed. $\square$

In order to find an orthonormal basis of the space H_+ , we apply the Gram-Schmidt orthonormalization procedure to vectors e_2, e_3 , see [17, Exercise 11.3, p.335]. We keep e_3 and orthogonalize e_2 against e_3 , and we also normalize e_1 , i.e. with $\lambda = \langle e_2, e_3 \rangle$, see (5.4.27).

$$\begin{aligned}
\tilde{e}_1 &= \sqrt{2}e_1, \\
\tilde{e}_2 &= \frac{e_2 - \lambda e_3}{\|e_2 - \lambda e_3\|}, \\
\tilde{e}_3 &= e_3.
\end{aligned} \tag{5.4.28}$$

Claim 5.46. The vectors $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ is an orthonormal basis of H_+ .

Proof of 5.46. We verify the two orthonormality conditions (5.4.25) and (5.4.26). We already know

$$\begin{aligned}
\|e_1\|_{L^2}^2 &= \int_0^\infty e^{-2x} dx = \frac{1}{2}, & \|e_2\|_{L^2}^2 &= \int_{-\infty}^0 e^x dx = 1, \\
\|e_3\|_{L^2}^2 &= \int_{-\infty}^0 e^x dx = 1.
\end{aligned}$$

Thus

$$\|\tilde{e}_1\|_{L^2}^2 = 1 \quad \text{and} \quad \|\tilde{e}_3\|_{L^2}^2 = 1.$$

For $\tilde{e}_2$, we use the previously computed inner product $\langle e_2, e_3 \rangle = \lambda$ and note that $\|e_2\|_{L^2} = \|e_3\|_{L^2} = 1$. Then

$$\begin{aligned}
\|e_2 - \lambda e_3\|_{L^2}^2 &= \langle e_2 - \lambda e_3, e_2 - \lambda e_3 \rangle \\
&= \|e_2\|^2 - \bar{\lambda} \langle e_2, e_3 \rangle - \lambda \langle e_3, e_2 \rangle + |\lambda|^2 \|e_3\|^2 \\
&= 1 - \bar{\lambda} \lambda - \lambda \bar{\lambda} + |\lambda|^2 = 1 + |\lambda|^2 - 2|\lambda|^2 = 1 - |\lambda|^2.
\end{aligned}$$

Since

$$|\lambda|^2 = \left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2 = \frac{1}{4},$$

we get

$$\|e_2 - \lambda e_3\|_{L^2}^2 = 1 - \frac{1}{4} = \frac{3}{4}, \quad \|e_2 - \lambda e_3\|_{L^2} = \frac{\sqrt{3}}{2}.$$

By (5.4.28),

$$\tilde{e}_2 = \frac{e_2 - \lambda e_3}{\|e_2 - \lambda e_3\|} = \frac{2}{\sqrt{3}}(e_2 - \lambda e_3),$$

so

$$\|\tilde{e}_2\|_{L^2}^2 = \left\| \frac{2}{\sqrt{3}}(e_2 - \lambda e_3) \right\|_{L^2}^2 = \frac{4}{3} \|e_2 - \lambda e_3\|_{L^2}^2 = \frac{4}{3} \cdot \frac{3}{4} = 1.$$

This proves (5.4.25).

Firstly, since we already showed $\langle e_1, e_3 \rangle = 0$

$$\langle \tilde{e}_1, \tilde{e}_3 \rangle = \langle \sqrt{2}e_1, e_3 \rangle = \sqrt{2} \langle e_1, e_3 \rangle = 0.$$

Next, since $\langle e_1, e_2 \rangle = \langle e_1, e_3 \rangle = 0$

$$\begin{aligned}\langle \tilde{e}_1, \tilde{e}_2 \rangle &= \left\langle \sqrt{2} e_1, \frac{2}{\sqrt{3}}(e_2 - \lambda e_3) \right\rangle \\ &= \frac{2\sqrt{2}}{\sqrt{3}}(\langle e_1, e_2 \rangle - \lambda \langle e_1, e_3 \rangle) = 0.\end{aligned}$$

Finally,

$$\begin{aligned}\langle \tilde{e}_2, \tilde{e}_3 \rangle &= \left\langle \frac{2}{\sqrt{3}}(e_2 - \lambda e_3), e_3 \right\rangle \\ &= \frac{2}{\sqrt{3}}(\langle e_2, e_3 \rangle - \lambda \langle e_3, e_3 \rangle) = 0.\end{aligned}$$

Thus the condition 2 (5.4.26) is holds.

From (5.4.28), we deduce that the system $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ defined by

$$\begin{aligned}\tilde{e}_1 &= \sqrt{2}e_1, \\ \tilde{e}_2 &= \frac{2}{\sqrt{3}}(e_2 - \lambda e_3), \\ \tilde{e}_3 &= e_3.\end{aligned}\tag{5.4.29}$$

Hence $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ is an orthonormal basis of H_+ .

Therefore, the proof of Claim 5.46 is complete. $\square$

This completes the construction of an orthonormal basis of H_+ .

5.4.6.2 Derivatives and Boundary Values of the ONB of H_+

Let us recall, see (5.4.29), that from this formula we have

$$\tilde{e}'_1(x) = -\sqrt{2} \mathbf{1}_{(0,\infty)}(x) e^{-x}, \quad x > 0,$$

$$\tilde{e}''_1(x) = \sqrt{2} \mathbf{1}_{(0,\infty)}(x) e^{-x}, \quad x > 0,$$

and

$$\tilde{e}'_2(x) = \frac{2}{\sqrt{3}} \left[\left(\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) e^{\frac{1}{2}x + i \frac{\sqrt{3}}{2}x} - \lambda \left(\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) e^{\frac{1}{2}x - i \frac{\sqrt{3}}{2}x} \right], \quad x < 0,$$

$$\tilde{e}'_3(x) = \left(\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) e^{\frac{1}{2}x - i \frac{\sqrt{3}}{2}x}, \quad x < 0.$$

$$\tilde{e}_2''(x) = \frac{2}{\sqrt{3}} \left[\left(\frac{1}{2} + i\frac{\sqrt{3}}{2} \right)^2 e^{\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} - \lambda \left(\frac{1}{2} - i\frac{\sqrt{3}}{2} \right)^2 e^{\frac{1}{2}x - i\frac{\sqrt{3}}{2}x} \right], \quad x < 0,$$

$$\tilde{e}_3''(x) = \left(\frac{1}{2} - i\frac{\sqrt{3}}{2} \right)^2 e^{\frac{1}{2}x - i\frac{\sqrt{3}}{2}x}, \quad x < 0.$$

5.4.6.3 Finding an ONB of the deficiency Subspace H_-

We now will find an ONB of the deficiency space H_- from (5.4.12)–(5.4.13). Define

$$H_- := \text{span}\{f_1, f_2, f_3\},$$

where

$$H_- := \ker(iI + A_1^*).$$

where

$$f_1(x) = \mathbf{1}_{(0,\infty)}(x) e^{-\frac{1}{2}x + i\frac{\sqrt{3}}{2}x},$$

$$f_2(x) = \mathbf{1}_{(0,\infty)}(x) e^{-\frac{1}{2}x - i\frac{\sqrt{3}}{2}x},$$

$$f_3(x) = \mathbf{1}_{(-\infty,0)}(x) e^x.$$

We want to construct an orthonormal basis of H_- . We begin by checking whether $\{f_1, f_2, f_3\}$ satisfy the following conditions

$$\begin{aligned} \|f_j\| &= 1 \text{ for } j = 1, 2, 3 \\ \langle f_j, f_k \rangle_{L^2} &= 0 \text{ if } j, k = 1, 2, 3 \text{ and } j \neq k \end{aligned} \tag{5.4.30}$$

Claim 5.47. The vectors $\{f_1, f_2, f_3\}$ do not form an orthonormal basis of H_- .

Proof of Claim 5.47. Norms. Firstly, we have

$$\|f_1\|_{L^2}^2 = \int_{-\infty}^{\infty} |f_1(x)|^2 dx = \int_0^{\infty} |e^{-\frac{1}{2}x + i\frac{\sqrt{3}}{2}x}|^2 dx = \int_0^{\infty} e^{-x} dx = 1,$$

Similarly,

$$\|f_2\|_{L^2}^2 = \int_0^{\infty} |e^{-\frac{1}{2}x - i\frac{\sqrt{3}}{2}x}|^2 dx = \int_0^{\infty} e^{-x} dx = 1,$$

For f_3 we have

$$\|f_3\|_{L^2}^2 = \int_{-\infty}^{\infty} |f_3(x)|^2 dx = \int_{-\infty}^0 e^{2x} dx = \frac{1}{2},$$

hence

$$\|f_3\|_{L^2} = \frac{1}{\sqrt{2}}, \quad \text{and we set} \quad \tilde{f}_3 := \sqrt{2} f_3$$

so that $\|\tilde{f}_3\|_{L^2} = 1$.

Orthogonality. Because f_1 and f_2 are supported on $(0, \infty)$ while f_3 is supported on $(-\infty, 0)$, we immediately have

$$\langle f_1, f_3 \rangle_{L^2} = \langle f_2, f_3 \rangle_{L^2} = 0.$$

However, f_1 and f_2 both have support in $(0, \infty)$, and in fact are not perpendicular

$$\begin{aligned} \langle f_1, f_2 \rangle_{L^2} &= \int_{-\infty}^{\infty} f_1(x) \overline{f_2(x)} dx = \int_0^{\infty} e^{-\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} \overline{e^{-\frac{1}{2}x - i\frac{\sqrt{3}}{2}x}} dx \\ &= \int_0^{\infty} e^{(-1+i\sqrt{3})x} dx = \frac{1}{1-i\sqrt{3}} = \frac{1+i\sqrt{3}}{4}. \end{aligned}$$

It is convenient to denote

$$\mu := \langle f_2, f_1 \rangle_{L^2} = \overline{\langle f_1, f_2 \rangle_{L^2}} = \frac{1-i\sqrt{3}}{4}. \quad (5.4.31)$$

Then

$$|\mu|^2 = \frac{1}{16} + \frac{3}{16} = \frac{4}{16} = \frac{1}{4} \neq 0.$$

Since $\mu \neq 0$, the vectors f_1 and f_2 are not orthogonal, hence $\{f_1, f_2, f_3\}$ is not an orthonormal set.

This completes the proof of Claim 5.47. $\square$

To construct an orthonormal basis of H_- , we apply the Gram–Schmidt procedure to f_1, f_2, f_3 and get

$$\begin{aligned} \tilde{f}_1 &= \boxed{f_1} \\ \tilde{f}_2 &= \boxed{\frac{f_2 - \mu f_1}{\|f_2 - \mu f_1\|}}, \\ \tilde{f}_3 &= \boxed{\sqrt{2}f_3}. \end{aligned}$$

Claim 5.48. The vectors $\{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3\}$ form an orthonormal basis of H_- .

Proof of Claim 5.48. We set

$$\tilde{f}_1 := f_1.$$

Since $\|f_3\| = 1/\sqrt{2}$, we define

$$\tilde{f}_3 := \sqrt{2} f_3, \quad \|\tilde{f}_3\|_{L^2} = 1.$$

Next, let $\mu = \langle f_2, f_1 \rangle_{L^2}$ be defined in (5.4.31). Define

$$g_2 := f_2 - \mu f_1.$$

Then

$$\begin{aligned} \|g_2\|_{L^2}^2 &= \langle f_2 - \mu f_1, f_2 - \mu f_1 \rangle_{L^2} \\ &= \|f_2\|^2 - \bar{\mu} \langle f_2, f_1 \rangle_{L^2} - \mu \langle f_1, f_2 \rangle_{L^2} + |\mu|^2 \|f_1\|^2 \\ &= 1 + |\mu|^2 - 2|\mu|^2 = 1 - |\mu|^2 = 1 - \frac{1}{4} = \frac{3}{4}. \end{aligned}$$

Therefore $\|g_2\|_{L^2} = \frac{\sqrt{3}}{2}$.

We normalize and define

$$\begin{aligned} \tilde{f}_1 &:= f_1, \\ \tilde{f}_2 &:= \frac{g_2}{\|g_2\|} = \frac{f_2 - \mu f_1}{\|f_2 - \mu f_1\|} = \frac{2}{\sqrt{3}}(f_2 - \mu f_1), \\ \tilde{f}_3 &:= \sqrt{2} f_3. \end{aligned} \tag{5.4.32}$$

where

$$\lambda = \langle f_2, f_1 \rangle_{L^2}, \quad |\lambda|^2 = \frac{1}{4}.$$

such that $\lambda = \bar{\mu}$.

By construction, $\|\tilde{f}_1\| = \|\tilde{f}_2\| = \|\tilde{f}_3\| = 1$, and because f_3 is supported on $(-\infty, 0)$ and f_1, f_2 on $(0, \infty)$

$$\langle \tilde{f}_1, \tilde{f}_2 \rangle = \langle f_1, g_2 \rangle = 0, \quad \langle \tilde{f}_1, \tilde{f}_3 \rangle = 0, \quad \langle \tilde{f}_2, \tilde{f}_3 \rangle = 0.$$

Thus the conditions in (5.4.30) hold.

Hence, $\{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3\}$ is an orthonormal basis for H_- . This completes the proof of Claim 5.48. $\square$

Remark 5.49. From Claim 5.46 it follows that every element of the deficiency space H_+ has the following unique form

$$u_+ = s_1 \tilde{e}_1 + s_2 \tilde{e}_2 + s_3 \tilde{e}_3, \quad s_1, s_2, s_3 \in \mathbb{C}.$$

From Claim 5.48 that every $u_- \in H_-$ has a unique representation

$$u_- = t_1 \tilde{f}_1 + t_2 \tilde{f}_2 + t_3 \tilde{f}_3, \quad t_1, t_2, t_3 \in \mathbb{C}.$$

5.4.6.4 Derivatives and Boundary Values of the ONB of the space H_-

Let us recall, see (5.4.32), that from these formula, for $\tilde{f}_1(x)$ we have

$$\tilde{f}_1'(x) = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \tilde{f}_1(x), \quad x > 0.$$

$$\tilde{f}_1''(x) = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 \tilde{f}_1(x), \quad x > 0.$$

For $\tilde{f}_2(x)$, we get

$$\tilde{f}_2'(x) = \frac{2}{\sqrt{3}} \left[\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) e^{-\frac{1}{2}x - i\frac{\sqrt{3}}{2}x} - \lambda \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) e^{-\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} \right], \quad x > 0$$

$$\tilde{f}_2''(x) = \frac{2}{\sqrt{3}} \left[\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2 e^{-\frac{1}{2}x - i\frac{\sqrt{3}}{2}x} - \lambda \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 e^{-\frac{1}{2}x + i\frac{\sqrt{3}}{2}x} \right], \quad x > 0.$$

For $\tilde{f}_3(x) = \sqrt{2}e^x$, we have

$$\tilde{f}_3'(x) = \sqrt{2}e^x, \quad \tilde{f}_3''(x) = \sqrt{2}e^x, \quad x < 0,$$

Unitary maps $\mathcal{C} : H_+ \rightarrow H_-$. Let us recall that by Theorem 5.42, every self-adjoint extension of the third-order operator $\overline{A_1}$ corresponds to a unitary operator

$$\mathcal{C} : H_+ \longrightarrow H_-,$$

in a way described there. Relative to the orthonormal bases $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ of H_+ , and $\{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3\}$ of H_- .

The map $\mathcal{C}$ is represented by a 3×3 unitary matrix

$$U = (u_{jk})_{j,k=1}^3 \in U(3),$$

in such a way that

$$\boxed{\mathcal{C}(\tilde{e}_k) = \sum_{j=1}^3 u_{jk} \tilde{f}_j}, \quad k = 1, 2, 3. \quad (5.4.33)$$

And via-versa, see equality (4.2.21) for a similar argument.

With the notation introduced above, we record the action of the unitary map $\mathcal{C}$ on a general element of the deficiency subspace H_+ .

Proposition 5.50. *Assume that $\mathcal{C} : H_+ \rightarrow H_-$ is a unitary map represented, with respect to the above orthonormal bases, by a unitary matrix $U = (u_{jk}) \in U(3)$, see (5.4.33). Assume that $s_1, s_2, s_3 \in \mathbb{C}$, and*

$$u_+ = \sum_{k=1}^3 s_k \tilde{e}_k \in H_+.$$

Then

$$\mathcal{C}u_+ = \sum_{j=1}^3 t_j \tilde{f}_j, \quad (5.4.34)$$

where

$$t_j = \sum_{k=1}^3 u_{jk} s_k, \quad j = 1, 2, 3.$$

The relationship between $\mathbf{t} = (t_1, t_2, t_3)$ and $\mathbf{s} = (s_1, s_2, s_3)$ can be written as

$$\mathbf{t} = U \mathbf{s}.$$

Proof. By linearity of $\mathcal{C}$ and using (5.4.33), we compute

$$\begin{aligned} \mathcal{C}u_+ &= \mathcal{C} \left(\sum_{k=1}^3 s_k \tilde{e}_k \right) = \sum_{k=1}^3 s_k \mathcal{C}(\tilde{e}_k) = \sum_{k=1}^3 s_k \sum_{j=1}^3 u_{jk} \tilde{f}_j \\ &= \sum_{j=1}^3 \left(\sum_{k=1}^3 u_{jk} s_k \right) \tilde{f}_j. \end{aligned}$$

This prove the result. □

Let us choose and fix an element $u \in D(A_{\mathcal{C}})$. By Theorem 5.42, there exists $\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{C}^3$, such that u has the form

$$u = u_0 + u_+ + \mathcal{C}u_+$$

where

$$u_0 \in D(\overline{A_1}), \quad u_+ = \sum_{k=1}^3 s_k \tilde{e}_k \in H_+. \quad (5.4.35)$$

Therefore, by formula (5.4.34),

$$u = u_0 + \sum_{k=1}^3 s_k \tilde{e}_k + \sum_{j=1}^3 t_j \tilde{f}_j,$$

where

$$\mathbf{t} = U\mathbf{s}. \quad (5.4.36)$$

The following claim establishes the connection between the abstract von Neumann parametrization and the concrete realization of the self-adjoint extension in terms of the unitary matrix U .

We aim to prove the following result.

Claim 5.51. Let $\mathcal{C} : H_+ \rightarrow H_-$ be a unitary map represented by $U \in U(3)$ with respect to the bases $\{\tilde{e}_k\}$ and $\{\tilde{f}_j\}$. If $u \in D(A_{\mathcal{C}})$, then $u \in D(A^U)$.

However, a complete proof of this result remains open in this setting. We therefore formulate it as a conjecture. On the other hand, in the last section, i.e. Section 8 we proposed a different approach to this question.

5.5 Boundary conditions and trace representation

5.5.1 General argument

By Proposition 5.33 and Definition 5.28, For $u_0 \in D(A_1^*)$, the following are equivalent:

- (i) $u_0 \in D(\bar{A}_1)$,
- (ii) $\mathbf{tr}_+(u_0) = \mathbf{0}$ and $\mathbf{tr}_-(u_0) = \mathbf{0}$.

Hence, in the decomposition (5.4.35), only the deficiency components u_+ and $\mathcal{C}u_+$ contribute to the boundary traces.

For convenience, and using $\lambda = \frac{1}{4} - i\frac{\sqrt{3}}{4} = \bar{\mu}$ from (5.4.27), we introduce

$$\alpha := \frac{2}{\sqrt{3}}(1 - \lambda).$$

Using (A.3.2), (A.3.3), and (A.3.4), we obtain

$$u(0+) = \sqrt{2}s_1 + t_1 + \alpha t_2, \quad (5.5.1)$$

$$u'(0+) = -\sqrt{2}s_1 + Dt_1 + Ct_2,$$

$$u''(0+) = \sqrt{2}s_1 + Ft_1 + Gt_2, \quad (5.5.2)$$

and

$$u(0-) = \alpha s_2 + s_3 + \sqrt{2} t_3, \quad (5.5.3)$$

$$\begin{aligned} u'(0-) &= A s_2 + B s_3 + \sqrt{2} t_3, \\ u''(0-) &= E s_2 + F s_3 + \sqrt{2} t_3. \end{aligned} \quad (5.5.4)$$

Let us introduce the following four 3×3 matrices $S_{\pm}, T_{\pm} \in \mathbb{C}^{3 \times 3}$

$$S_+ := \begin{pmatrix} \sqrt{2} & 0 & 0 \\ -\sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix}, \quad T_+ := \begin{pmatrix} 1 & \alpha & 0 \\ D & C & 0 \\ F & G & 0 \end{pmatrix},$$

and

$$S_- := \begin{pmatrix} 0 & \alpha & 1 \\ 0 & A & B \\ 0 & E & F \end{pmatrix}, \quad T_- := \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} \end{pmatrix},$$

Using the boundary trace operators $\mathbf{tr}_{\pm}$ defined earlier 5.28 as follows.

Definition 5.52 (Boundary traces). Define the full trace map by

$$\mathbf{tr} : D(\bar{A}_1^*) \ni u \mapsto \mathbf{tr}(u) := \begin{pmatrix} u(0+) \\ u'(0+) \\ u''(0+) \\ u(0-) \\ u'(0-) \\ u''(0-) \end{pmatrix} \in \mathbb{C}^6.$$

We have the following fundamental result.

Proposition 5.53. *The restriction of the map $\mathbf{tr}$ to $H_+ \oplus H_-$ is injective.*

Proof of Proposition 5.53. Since $H_+ \cap H_- = \{0\}$, it is enough to show that the restrictions $\mathbf{tr}|_{H_+}$ and $\mathbf{tr}|_{H_-}$ are injective.

Injectivity on H_+ . Let $u \in H_+$ and assume that

$$\mathbf{tr}(u) = \mathbf{0} \in \mathbb{C}^6. \quad (5.5.5)$$

Since $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ is a basis of H_+ , there exist $s_1, s_2, s_3 \in \mathbb{C}$ such that

$$u = s_1 \tilde{e}_1 + s_2 \tilde{e}_2 + s_3 \tilde{e}_3.$$

Using the explicit boundary identities (5.5.1)–(5.5.2) and (5.5.3)–(5.5.4) for elements of H_+ , we get

$$u(0+) = \sqrt{2} s_1, \quad u'(0+) = -\sqrt{2} s_1, \quad u''(0+) = \sqrt{2} s_1,$$

and

$$u(0-) = \alpha s_2 + s_3, \quad u'(0-) = A s_2 + B s_3, \quad u''(0-) = E s_2 + F s_3.$$

Since $\sqrt{2} \neq 0$, from (5.5.5) and the first three equalities, we get

$$s_1 = 0.$$

The second three equalities and (5.5.5) imply that

$$\begin{pmatrix} \alpha & 1 \\ A & B \\ E & F \end{pmatrix} \begin{pmatrix} s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We claim that the matrix

$$M_- = \begin{pmatrix} \alpha & 1 \\ A & B \\ E & F \end{pmatrix}$$

has rank 2. For this purpose, it is sufficient to find at least one 2×2 minor that is $\neq 0$.

For this aim the minor formed by the first two rows, i.e.

$$\det \begin{pmatrix} \alpha & 1 \\ A & B \end{pmatrix} = \alpha B - A.$$

Substituting the explicit constants (5.5.15)

$$\alpha = \frac{\sqrt{3}}{2} + \frac{i}{2}, \quad A = \frac{\sqrt{3}}{2} + \frac{3i}{2}, \quad B = \frac{1}{2} - \frac{i\sqrt{3}}{2},$$

we compute

$$\alpha B = \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right) \left(\frac{1}{2} - \frac{i\sqrt{3}}{2} \right) = \frac{\sqrt{3}}{2} - \frac{i}{2},$$

hence

$$\alpha B - A = \left(\frac{\sqrt{3}}{2} - \frac{i}{2} \right) - \left(\frac{\sqrt{3}}{2} + \frac{3i}{2} \right) = -2i \neq 0.$$

Therefore $\text{rank } M_- = 2$.

Hence the kernel of M_- is trivial and therefore we infer $s_2 = s_3 = 0$.

Consequently $u = 0$, and thus $\mathbf{tr}|_{H_+}$ is injective.

Injectivity on H_- . Let $u \in H_-$ and assume that $\mathbf{tr}(u) = \mathbf{0}$. Write

$$u = t_1 \tilde{f}_1 + t_2 \tilde{f}_2 + t_3 \tilde{f}_3, \quad t_1, t_2, t_3 \in \mathbb{C}.$$

Using the boundary identities (5.5.1)–(5.5.2) and (5.5.3)–(5.5.4) for elements of H_- , we obtain

$$u(0-) = \sqrt{2} t_3, \quad u'(0-) = \sqrt{2} t_3, \quad u''(0-) = \sqrt{2} t_3.$$

Hence $\sqrt{2} t_3 = 0$, so $t_3 = 0$.

The remaining three equations from $\mathbf{tr}(u) = 0$ yield

$$\begin{pmatrix} 1 & \alpha \\ D & C \\ F & G \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Again, by construction of the deficiency basis, the two columns of this 3×2 matrix are linearly independent, so its kernel is trivial. Hence $t_1 = t_2 = 0$.

Thus $u = 0$, and $\mathbf{tr}|_{H_-}$ is injective.

Since $\mathbf{tr}|_{H_+}$ and $\mathbf{tr}|_{H_-}$ are injective, it follows that $\mathbf{tr}$ restricted to $H_+ \oplus H_-$ is injective. $\square$

Equations (5.5.1)–(5.5.2) and (5.5.3)–(5.5.4) form a system of 6 linear equations, i.e.

$$\begin{aligned} u(0+) &= \sqrt{2} s_1 + t_1 + \alpha t_2, \\ u'(0+) &= -\sqrt{2} s_1 + D t_1 + C t_2, \\ u''(0+) &= \sqrt{2} s_1 + F t_1 + G t_2, \\ u(0-) &= \alpha s_2 + s_3 + \sqrt{2} t_3, \\ u'(0-) &= A s_2 + B s_3 + \sqrt{2} t_3, \\ u''(0-) &= E s_2 + F s_3 + \sqrt{2} t_3. \end{aligned} \tag{5.5.6}$$

From (5.5.6), for appropriately defined 6×3 matrices S and T

$$S = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ -\sqrt{2} & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & A & B \\ 0 & E & F \end{pmatrix} \in \mathbb{C}^{6 \times 3}, \quad T = \begin{pmatrix} 1 & \alpha & 0 \\ D & C & 0 \\ F & G & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} \end{pmatrix} \in \mathbb{C}^{6 \times 3}. \tag{5.5.7}$$

The above system can be written in a form

$$\mathbf{tr}(u) = S\mathbf{s} + T\mathbf{t} = S\mathbf{s} + TU\mathbf{s} = P(U)\mathbf{s}.$$

where

$$P(U) := S + TU \in \mathbb{C}^{6 \times 3}.$$

Note that $P(U)$ is a 6×3 matrix. We have

Proposition 5.54. *The rank of matrix $P = P(U)$ is equal to 3.*

Proof of Proposition 5.54. Let us choose and fix the matrix U and so the corresponding unitary matrix $\mathbf{C} : H_+ \rightarrow H_-$.

Using the standard canonical basis' of $\mathbb{C}^3$ and $\mathbb{C}^6$ and the basis $\tilde{e}_j$ of H_+ and $\tilde{e}_j + \tilde{f}_k$ of $H_+ \oplus H_-$ we can view U as a map

$$U : \mathbb{C}^3 \rightarrow H_+$$

and $P(U)$ as a map

$$H_+ \oplus H_- \rightarrow \mathbb{C}^6$$

It is sufficient to prove that $P(U)$ is injective.

Let us define an auxiliary linear map

$$\Phi : H_+ \ni \phi \mapsto \phi + \mathbf{C}\phi \in H_+ \oplus H_-.$$

Obviously, this auxiliary map is injective (indeed, if $\phi + \mathbf{C}\phi = 0$, then $\mathbf{C}\phi = -\phi \in H_+$, so $\mathbf{C}\phi \in H_+ \cap H_- = \{0\}$ and hence $\phi = 0$). So the dimension of its range is equal to 3.

We claim that

$$\mathbf{tr} \circ \Phi \circ U = P(U). \quad (5.5.8)$$

Proof of (5.5.8) Let $\mathbf{s} \in \mathbb{C}^3$ be arbitrary and set

$$u_+ := U\mathbf{s} = \sum_{k=1}^3 s_k \tilde{e}_k \in H_+, \quad u_- := \mathbf{C}u_+ = \sum_{j=1}^3 t_j \tilde{f}_j \in H_-.$$

where $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{C}^3$ satisfies $\mathbf{t} = U\mathbf{s}$, see (5.4.36). By Definition 5.52 and the boundary identities (5.5.6), we obtain

$$\mathbf{tr}(u_+ + u_-) = S\mathbf{s} + T\mathbf{t}.$$

Since $\mathbf{t} = U\mathbf{s}$, this becomes

$$\mathbf{tr}(u_+ + u_-) = S\mathbf{s} + TU\mathbf{s} = (S + TU)\mathbf{s} = P(U)\mathbf{s}.$$

On the other hand, by definition of Φ we have

$$(\mathbf{tr} \circ \Phi \circ U)(\mathbf{s}) = \mathbf{tr}(\Phi(U\mathbf{s})) = \mathbf{tr}(u_+ + \textcolor{red}{C}u_+) = \mathbf{tr}(u_+ + u_-).$$

Combining the last two displays gives (5.5.8).

Since $U : \mathbb{C}^3 \rightarrow H_+$ is injective since U is unitary, Φ is injective by the argument above, and $\mathbf{tr}$ is injective on $H_+ \oplus H_-$ by Proposition 5.53. Therefore their composition $\mathbf{tr} \circ \Phi \circ U = P(U)$ is injective. Hence $\ker P(U) = \{0\}$.

Since $P(U) : \mathbb{C}^3 \rightarrow \mathbb{C}^6$ is a linear map, the rank–nullity theorem, see [27, Section 0.2.3. p.6], we infer that

$$\text{rank } P(U) + \dim \ker P(U) = 3.$$

Hence, if $P(U)$ is injective, then $\ker P(U) = \{0\}$ and therefore

$$\text{rank } P(U) = 3.$$

This completes the proof. □

Proposition 5.54 implies that the dimension of the space

$$\text{Range } P(U) := \{P(U)\mathbf{s} : \mathbf{s} \in \mathbb{C}^3\}$$

is equal to 3. Hence, Since we are in a finite-dimensional Hilbert space, by [36, Theorem 12.10], we have

$$\ker(T^*) = (\text{Range}(T))^\perp.$$

Hence,

$$\text{Range}(T) \oplus \ker(T^*) = H.$$

Applying this to $T = P(U)$ yields

$$\text{Range}(P(U)) \oplus \ker[(P(U))^*] = \mathbb{C}^6.$$

Remark 5.55. In $\mathbb{C}^6$ we have the fundamental identity

$$(\text{Ran } P(U))^\perp = \ker(P(U)^*),$$

In particular, if $\text{rank } P(U) = 3$, then $\dim \ker(P(U)^*) = 6 - 3 = 3$.

We infer the following result.

Corollary 5.56. Assume that $\text{rank } P(U) = 3$, and let $l_1 = l_1(U), l_2 = l_2(U), l_3 = l_3(U)$ be an orthonormal basis of $\ker(P(U)^*)$. Assume that a vector $\vec{u} \in \mathbb{C}^6$. Then $\vec{u}$ belongs to $\text{Range}(P(U))$ if and only if

$$\langle l_j, \vec{u} \rangle_{\mathbb{C}^6} = 0 \text{ for } j = 1, 2, 3.$$

The above allows us to define the following boundary conditions

$$\boxed{\langle l_j, \mathbf{tr}(u) \rangle_{\mathbb{C}^6} = 0, \text{ for } j = 1, 2, 3.} \quad (5.5.9)$$

Remark 5.57. In view of the above result from [30], Proposition 5.54 is equivalent to the following assertion.

$$\dim \ker [(P(U))^*] = 3$$

Definition 5.58. Assume that $U = (u_{jk})_{j,k=1}^3 \in U(3)$. Then operator A^U is defined by

$$\begin{aligned} D(A^U) &:= \left\{ u \in D(A_1^*) : u \text{ satisfies the boundary conditions (5.5.9)} \right\}, \\ A^U u &:= A_1^* u, \quad u \in D(A^U). \end{aligned}$$

Lemma 5.59. Let $U \in U(3)$ represent $\mathbf{C} : H_+ \rightarrow H_-$ with respect to the fixed bases. For $\mathbf{s} \in \mathbb{C}^3$, set $u_+ := \sum_{k=1}^3 s_k \tilde{e}_k \in H_+$. Then

$$\mathbf{tr}(u_+ + \mathbf{C}u_+) = P(U)\mathbf{s}.$$

Proof of Lemma 5.59. Write $\mathbf{t} = U\mathbf{s}$ and use the six boundary identities (5.5.6). Collecting the coefficients of $\mathbf{s}$ and $\mathbf{t}$ gives

$$\mathbf{tr}(u_+ + \mathbf{C}u_+) = S\mathbf{s} + T\mathbf{t} = S\mathbf{s} + TU\mathbf{s} = (S + TU)\mathbf{s} = P(U)\mathbf{s}.$$

□

Claim 5.60. Assume that $\mathbf{C} : H_+ \rightarrow H_-$ is a unitary map represented by $U \in U(3)$ with respect to the bases $\{\tilde{e}_k\}$ and $\{\tilde{f}_j\}$. Assume also that $\text{rank } P(U) = 3$. If $u \in D(A_{\mathbf{C}})$, then $u \in D(A^U)$.

Proof of Claim 5.60. Let $u \in D(A_{\mathbf{C}})$. By definition of $D(A_{\mathbf{C}})$, there exist $u_0 \in D(\overline{A_1})$ and $u_+ \in H_+$ such that

$$u = u_0 + u_+ + \mathbf{C}u_+.$$

Since $u_0 \in D(\overline{A_1})$, we have $\mathbf{tr}(u_0) = \mathbf{0}$, hence

$$\mathbf{tr}(u) = \mathbf{tr}(u_+ + \mathbf{C}u_+).$$

Write $u_+ = \sum_{k=1}^3 s_k \tilde{e}_k$ for some $\mathbf{s} \in \mathbb{C}^3$. By Lemma 5.59,

$$\mathbf{tr}(u) = \mathbf{tr}(u_+ + \mathbf{C}u_+) = P(U)\mathbf{s} \in \text{Ran } P(U).$$

By Corollary 5.56, the condition $\mathbf{tr}(u) \in \text{Ran } P(U)$ is equivalent to the three conditions (5.5.9), where $\{\ell_1(U), \ell_2(U), \ell_3(U)\}$ is an orthonormal basis of $\ker((P(U))^*)$. Therefore $u \in D(A^U)$. $\square$

Claim 5.61. Assume that $\mathbf{C} : H_+ \rightarrow H_-$ is a unitary map represented by $U \in U(3)$ with respect to the bases $\{\tilde{e}_k\}_{k=1}^3$ and $\{\tilde{f}_j\}_{j=1}^3$. Assume that $\text{rank } P(U) = 3$. If $u \in D(A^U)$, then $u \in D(A_{\mathbf{C}})$.

Remark 5.62. Since $P(U) \in \mathbb{C}^{6 \times 3}$, the condition $\text{rank } P(U) = 3$ means that $P(U)$ has full column rank. In particular, the determinant of $P(U)$ is not defined in this setting.

Proof of Claim 5.61. Let $u \in D(A^U)$. Then $u \in D(A_1^*)$ and u satisfies the boundary conditions (5.5.9). By Corollary 5.56, this is equivalent to

$$\mathbf{tr}(u) \in \text{Ran } P(U).$$

Hence there exists $\mathbf{s} \in \mathbb{C}^3$ such that

$$\mathbf{tr}(u) = P(U)\mathbf{s}. \quad (5.5.10)$$

Define

$$u_+ := \sum_{k=1}^3 s_k \tilde{e}_k \in H_+, \quad u_0 := u - (u_+ + \mathbf{C}u_+) \in D(A_1^*).$$

By Lemma 5.59, we have

$$\mathbf{tr}(u_+ + \mathbf{C}u_+) = P(U)\mathbf{s}.$$

Together with (5.5.10) and linearity of $\mathbf{tr}$, this yields

$$\mathbf{tr}(u_0) = \mathbf{tr}(u) - \mathbf{tr}(u_+ + \mathbf{C}u_+) = P(U)\mathbf{s} - P(U)\mathbf{s} = \mathbf{0}.$$

By Proposition 5.33 and Definition 5.52, $\mathbf{tr}(u_0) = \mathbf{0}$ implies $u_0 \in D(\overline{A_1})$. Therefore

$$u = u_0 + u_+ + \mathbf{C}u_+, \quad u_0 \in D(\overline{A_1}), \quad u_+ \in H_+,$$

so $u \in D(A_{\mathbf{C}})$ by the definition of $D(A_{\mathbf{C}})$. $\square$

5.5.2 Applications

In this subsection we will show how previous results work in concrete examples.

5.5.2.1 Case $U = I_3$ (identity coupling)

Let

$$U = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\mathbf{t} = U\mathbf{s} = \mathbf{s}$, and using (5.5.7), the full-trace matrix becomes

$$P(I_3) = S + T = \begin{pmatrix} \sqrt{2} + 1 & \alpha & 0 \\ -\sqrt{2} + D & C & 0 \\ \sqrt{2} + F & G & 0 \\ 0 & \alpha & 1 + \sqrt{2} \\ 0 & A & B + \sqrt{2} \\ 0 & E & F + \sqrt{2} \end{pmatrix} \in \mathbb{C}^{6 \times 3}. \quad (5.5.11)$$

The adjoint matrix $P(I_3)^*$. Since $P(I_3)$ has (in general) complex entries, its adjoint is the conjugate transpose

$$P(I_3)^* = \overline{P(I_3)}^\top \in \mathbb{C}^{3 \times 6}.$$

Therefore,

$$P(I_3)^* = \begin{pmatrix} \overline{\sqrt{2} + 1} & \overline{-\sqrt{2} + D} & \overline{\sqrt{2} + F} & 0 & 0 & 0 \\ \bar{\alpha} & \bar{C} & \bar{G} & \bar{\alpha} & \bar{A} & \bar{E} \\ 0 & 0 & 0 & \overline{1 + \sqrt{2}} & \overline{B + \sqrt{2}} & \overline{F + \sqrt{2}} \end{pmatrix}.$$

(Here $\overline{\sqrt{2} + 1} = \sqrt{2} + 1$ and similarly for the other real constants, but we keep the conjugation to emphasize the definition of $*$.)

Computing $\ker(P(I_3)^*)$. Let $\ell = (\ell_1, \dots, \ell_6) \in \mathbb{C}^6$. Then $\ell \in \ker(P(I_3)^*)$ if and only if

$$P(I_3)^*\ell = 0.$$

Writing this out row-by-row gives the following linear system:

$$(\sqrt{2} + 1)\ell_1 + (-\sqrt{2} + D)\ell_2 + (\sqrt{2} + F)\ell_3 = 0, \quad (5.5.12)$$

$$\bar{\alpha}\ell_1 + \bar{C}\ell_2 + \bar{G}\ell_3 + \bar{\alpha}\ell_4 + \bar{A}\ell_5 + \bar{E}\ell_6 = 0, \quad (5.5.13)$$

$$(1 + \sqrt{2})\ell_4 + (B + \sqrt{2})\ell_5 + (F + \sqrt{2})\ell_6 = 0. \quad (5.5.14)$$

Since $P(I_3) \in \mathbb{C}^{6 \times 3}$ has full column rank, $\text{rank } P(I_3) = 3$, and hence

$$\dim \ker(P(I_3)^*) = 6 - \text{rank } P(I_3) = 3.$$

Recall Proposition 5.54, which asserts that $\text{rank } P(U) = 3$. We now verify this statement explicitly in the present case.

Proof of Proposition 5.54 for $U = I_3$. Recall that for $U = I_3$ the full trace matrix of $P(I_3)$ defined by (5.5.11). where the constants $\alpha, A, B, C, D, E, F, G$ are determined by from Tables A.3–A.4, and the Airy deficiency basis and depend on

$$\lambda = \frac{1}{4} - i\frac{\sqrt{3}}{4}.$$

A direct computation yields the explicit values

$$\begin{aligned} \alpha &= \frac{\sqrt{3}}{2} + \frac{i}{2}, & A &= \frac{\sqrt{3}}{2} + \frac{3i}{2}, & B &= \frac{1}{2} - \frac{i\sqrt{3}}{2}, \\ C &= -\frac{\sqrt{3}}{2} - \frac{3i}{2}, & D &= -\frac{1}{2} + \frac{i\sqrt{3}}{2}, & E &= i, \\ F &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}, & G &= -\frac{\sqrt{3}}{2} - \frac{3i}{2}. \end{aligned} \tag{5.5.15}$$

We consider the matrix $P(I_3)$ defined by (5.5.11). Write it in column form

$$P(I_3) = (c_1 \ c_2 \ c_3),$$

where

$$c_1 = \begin{pmatrix} 1 + \sqrt{2} \\ D - \sqrt{2} \\ F + \sqrt{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} \alpha \\ C \\ G \\ \alpha \\ A \\ E \end{pmatrix}, \quad c_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 + \sqrt{2} \\ B + \sqrt{2} \\ F + \sqrt{2} \end{pmatrix}.$$

The vectors c_1 and c_3 have disjoint supports:

$$c_1 \text{ is supported in rows 1--3,} \quad c_3 \text{ is supported in rows 4--6.}$$

Since $1 + \sqrt{2} \neq 0$, both are nonzero and therefore linearly independent. Hence

$$\text{rank}(P(I_3)) \geq 2.$$

The rank drops to 2 if and only if

$$c_2 \in \text{span}\{c_1, c_3\}.$$

To characterize of Rank 2, suppose

$$c_2 = xc_1 + yc_3$$

for some scalars x, y .

From rows 1–3 we obtain

$$(\alpha, C, G) = x(1 + \sqrt{2}, D - \sqrt{2}, F + \sqrt{2}).$$

From the first component:

$$x = \frac{\alpha}{1 + \sqrt{2}}.$$

Therefore

$$C = \frac{\alpha}{1 + \sqrt{2}}(D - \sqrt{2}), \quad G = \frac{\alpha}{1 + \sqrt{2}}(F + \sqrt{2}).$$

From rows 4–6 we obtain

$$(\alpha, A, E) = y(1 + \sqrt{2}, B + \sqrt{2}, F + \sqrt{2}).$$

From the first component:

$$y = \frac{\alpha}{1 + \sqrt{2}}.$$

Thus necessarily $x = y$, and we must also have

$$A = \frac{\alpha}{1 + \sqrt{2}}(B + \sqrt{2}), \quad E = \frac{\alpha}{1 + \sqrt{2}}(F + \sqrt{2}).$$

Hence

$$\text{rank}(P(I_3)) = 2 \iff \begin{cases} C = \frac{\alpha}{1 + \sqrt{2}}(D - \sqrt{2}), \\ G = \frac{\alpha}{1 + \sqrt{2}}(F + \sqrt{2}), \\ A = \frac{\alpha}{1 + \sqrt{2}}(B + \sqrt{2}), \\ E = \frac{\alpha}{1 + \sqrt{2}}(F + \sqrt{2}). \end{cases}$$

If any of these relations fails, then

$$\text{rank}(P(I_3)) = 3.$$

Substituting the explicit values of $\alpha, A, B, C, D, E, F, G$, we obtain for example

$$C - \frac{\alpha}{1 + \sqrt{2}}(D - \sqrt{2}) = -i\sqrt{2} \neq 0.$$

Hence the necessary proportionality conditions fail, and $c_2 \notin \text{span}\{c_1, c_3\}$. Consequently, the three columns of $P(I_3)$ are linearly independent, and therefore

$$\boxed{\text{rank}(P(I_3)) = 3.}$$

□

Having established that $\text{rank } P(I_3) = 3$, it follows that

$$\dim \ker(P(I_3)^*) = 6 - \text{rank } P(I_3) = 3.$$

Consequently, the kernel of $P(I_3)^*$ is a three-dimensional subspace of $\mathbb{C}^6$.

In order to describe the boundary conditions explicitly, we now construct a basis of $\ker(P(I_3)^*)$ by solving the linear system (5.5.12)–(5.5.14).

Construction of a spanning set. Choose (ℓ_3, ℓ_5, ℓ_6) as free parameters and solve for (ℓ_1, ℓ_2, ℓ_4) . For instance, take

$$\ell_3 = 1, \quad \ell_5 = 0, \quad \ell_6 = 0.$$

Then (5.5.14) becomes

$$(1 + \sqrt{2})\ell_4 = 0 \implies \ell_4 = 0,$$

since $1 + \sqrt{2} \neq 0$.

With this choice, (5.5.12)–(5.5.13) reduce to the 2×2 system

$$\begin{pmatrix} \sqrt{2} + 1 & -\sqrt{2} + D \\ \alpha & C \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = -\begin{pmatrix} \sqrt{2} + F \\ G \end{pmatrix}. \quad (5.5.16)$$

Set

$$\Delta := \det \begin{pmatrix} \sqrt{2} + 1 & -\sqrt{2} + D \\ \alpha & C \end{pmatrix} = (\sqrt{2} + 1)C - \alpha(-\sqrt{2} + D).$$

Using the explicit values

$$\alpha = \frac{\sqrt{3}}{2} + \frac{i}{2}, \quad C = -\frac{\sqrt{3}}{2} - \frac{3i}{2}, \quad D = -\frac{1}{2} + \frac{i\sqrt{3}}{2},$$

one checks that

$$\Delta = -i\sqrt{2} \neq 0.$$

Hence (5.5.16) has a unique solution, and by Cramer's rule

$$\ell_1 = \frac{-(\sqrt{2} + F)C + (-\sqrt{2} + D)G}{\Delta}, \quad \ell_2 = \frac{(\sqrt{2} + F)\alpha - (\sqrt{2} + 1)G}{\Delta}. \quad (5.5.17)$$

Therefore

$$\ell^{(1)} := (\ell_1, \ell_2, 1, 0, 0, 0) \in \ker(P(I_3)^*). \quad (5.5.18)$$

Since $\mathbf{tr}(u) \in \text{Rang } P(I_3)$ if and only if $\mathbf{tr}(u) \perp \ker(P(I_3)^*)$, each $\ell \in \ker(P(I_3)^*)$ gives rise to one scalar condition

$$\ell^* \mathbf{tr}(u) = 0,$$

(where ℓ^* denotes the conjugate transpose of ℓ). For $\ell = \ell^{(1)}$ in (5.5.18), this becomes

$$\boxed{\overline{\ell_1} u(0+) + \overline{\ell_2} u'(0+) + u''(0+) = 0},$$

with ℓ_1, ℓ_2 given by (5.5.17).

Remark. To obtain *three* boundary conditions (as required), one repeats the same construction to find two further linearly independent vectors $\ell^{(2)}, \ell^{(3)} \in \ker(P(I_3)^*)$, for example by choosing $(\ell_3, \ell_5, \ell_6) = (0, 1, 0)$ and $(0, 0, 1)$, and solving for the remaining components. Optionally, applying Gram–Schmidt to $\{\ell^{(1)}, \ell^{(2)}, \ell^{(3)}\}$ produces an orthonormal basis $\{\ell_1(I_3), \ell_2(I_3), \ell_3(I_3)\}$, and the boundary conditions can be written as

$$\langle \ell_j(I_3), \mathbf{tr}(u) \rangle_{\mathbb{C}^6} = 0, \quad j = 1, 2, 3,$$

equivalently $\ell_j(I_3)^* \mathbf{tr}(u) = 0$ for $j = 1, 2, 3$.

5.5.2.2 Case $U = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3})$

Assume that

$$U = \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{i\theta_3} \end{pmatrix} \in U(3). \quad (5.5.19)$$

To compute $\mathbf{t} = U\mathbf{s}$, let

$$\mathbf{s} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = U\mathbf{s}.$$

Then

$$t_1 = e^{i\theta_1} s_1, \quad t_2 = e^{i\theta_2} s_2, \quad t_3 = e^{i\theta_3} s_3.$$

Recall the full boundary trace vector

$$\mathbf{tr}(u) = \begin{pmatrix} u(0+) \\ u'(0+) \\ u''(0+) \\ u(0-) \\ u'(0-) \\ u''(0-) \end{pmatrix} \in \mathbb{C}^6.$$

From (5.5.6) we have

$$\mathbf{tr}(u) = S\mathbf{s} + T\mathbf{t}, \tag{5.5.20}$$

where the matrices $S, T \in \mathbb{C}^{6 \times 3}$ are defined in (5.5.7).

Since $\mathbf{t} = U\mathbf{s}$, (5.5.20) becomes

$$\mathbf{tr}(u) = (S + TU)\mathbf{s} =: P(U)\mathbf{s},$$

where $P(U) \in \mathbb{C}^{6 \times 3}$.

Using (5.5.19), we obtain

$$P(U) = S + T \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) = [p_1(\theta_1) \ p_2(\theta_2) \ p_3(\theta_3)], \tag{5.5.21}$$

where the columns are

$$p_1(\theta_1) = \begin{pmatrix} \sqrt{2} + e^{i\theta_1} \\ -\sqrt{2} + D e^{i\theta_1} \\ \sqrt{2} + F e^{i\theta_1} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_2(\theta_2) = \begin{pmatrix} \alpha e^{i\theta_2} \\ C e^{i\theta_2} \\ G e^{i\theta_2} \\ \alpha \\ A \\ E \end{pmatrix},$$

$$p_3(\theta_3) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 + \sqrt{2}e^{i\theta_3} \\ B + \sqrt{2}e^{i\theta_3} \\ F + \sqrt{2}e^{i\theta_3} \end{pmatrix}.$$

Hence the six boundary equalities (5.5.6) can be written compactly as

$$\mathbf{tr}(u) = P(U)\mathbf{s}.$$

The adjoint matrix $P(U)^*$. Since $P(U)$ has (in general) complex entries, its adjoint is the conjugate transpose

$$P(U)^* = \overline{P(U)}^\top \in \mathbb{C}^{3 \times 6}.$$

Therefore, from (5.5.21),

$$P(U)^* = \begin{pmatrix} \overline{\sqrt{2} + e^{i\theta_1}} & \overline{-\sqrt{2} + De^{i\theta_1}} & \overline{\sqrt{2} + Fe^{i\theta_1}} & 0 & 0 & 0 \\ \overline{\alpha e^{i\theta_2}} & \overline{Ce^{i\theta_2}} & \overline{Ge^{i\theta_2}} & \overline{\alpha} & \overline{A} & \overline{E} \\ 0 & 0 & 0 & \overline{1 + \sqrt{2}e^{i\theta_3}} & \overline{B + \sqrt{2}e^{i\theta_3}} & \overline{F + \sqrt{2}e^{i\theta_3}} \end{pmatrix}.$$

Here we keep the conjugation signs to emphasize the definition of $*$.

Let $\ell = (\ell_1, \dots, \ell_6) \in \mathbb{C}^6$. Then $\ell \in \ker(P(U)^*)$ if and only if

$$P(U)^*\ell = 0.$$

Writing this out row-by-row gives the following linear system:

$$\overline{(\sqrt{2} + e^{i\theta_1})} \ell_1 + \overline{(-\sqrt{2} + De^{i\theta_1})} \ell_2 + \overline{(\sqrt{2} + Fe^{i\theta_1})} \ell_3 = 0, \quad (5.5.22)$$

$$\overline{\alpha e^{i\theta_2}} \ell_1 + \overline{Ce^{i\theta_2}} \ell_2 + \overline{Ge^{i\theta_2}} \ell_3 + \overline{\alpha} \ell_4 + \overline{A} \ell_5 + \overline{E} \ell_6 = 0, \quad (5.5.23)$$

$$\overline{(1 + \sqrt{2}e^{i\theta_3})} \ell_4 + \overline{(B + \sqrt{2}e^{i\theta_3})} \ell_5 + \overline{(F + \sqrt{2}e^{i\theta_3})} \ell_6 = 0. \quad (5.5.24)$$

Write $P(U)$ in column form (5.5.21). The vectors $p_1(\theta_1)$ and $p_3(\theta_3)$ have disjoint supports: $p_1(\theta_1)$ is supported in rows 1–3 and $p_3(\theta_3)$ in rows 4–6. If

$$p_1(\theta_1) \neq 0 \quad \text{and} \quad p_3(\theta_3) \neq 0,$$

then $p_1(\theta_1)$ and $p_3(\theta_3)$ are linearly independent and hence

$$\text{rank}(P(U)) \geq 2.$$

The rank drops to 2 if and only if

$$p_2(\theta_2) \in \text{span}\{p_1(\theta_1), p_3(\theta_3)\}.$$

Suppose that

$$p_2(\theta_2) = x p_1(\theta_1) + y p_3(\theta_3) \quad \text{for some } x, y \in \mathbb{C}.$$

Comparing rows 1–3 gives

$$(\alpha e^{i\theta_2}, C e^{i\theta_2}, G e^{i\theta_2}) = x(\sqrt{2} + e^{i\theta_1}, -\sqrt{2} + D e^{i\theta_1}, \sqrt{2} + F e^{i\theta_1}),$$

hence (from the first component)

$$x = \frac{\alpha e^{i\theta_2}}{\sqrt{2} + e^{i\theta_1}} \quad (\text{provided } \sqrt{2} + e^{i\theta_1} \neq 0).$$

Similarly, comparing rows 4–6 gives

$$(\alpha, A, E) = y(1 + \sqrt{2}e^{i\theta_3}, B + \sqrt{2}e^{i\theta_3}, F + \sqrt{2}e^{i\theta_3}),$$

hence (from the first component)

$$y = \frac{\alpha}{1 + \sqrt{2}e^{i\theta_3}} \quad (\text{provided } 1 + \sqrt{2}e^{i\theta_3} \neq 0).$$

Therefore, $\text{rank}(P(U)) = 2$ would force the proportionality conditions

$$\begin{aligned} C e^{i\theta_2} &= \frac{\alpha e^{i\theta_2}}{\sqrt{2} + e^{i\theta_1}} (-\sqrt{2} + D e^{i\theta_1}), \\ G e^{i\theta_2} &= \frac{\alpha e^{i\theta_2}}{\sqrt{2} + e^{i\theta_1}} (\sqrt{2} + F e^{i\theta_1}), \\ A &= \frac{\alpha}{1 + \sqrt{2}e^{i\theta_3}} (B + \sqrt{2}e^{i\theta_3}), \\ E &= \frac{\alpha}{1 + \sqrt{2}e^{i\theta_3}} (F + \sqrt{2}e^{i\theta_3}). \end{aligned} \tag{5.5.25}$$

If any of the relations in (5.5.25) fails, then $p_2(\theta_2) \notin \text{span}\{p_1(\theta_1), p_3(\theta_3)\}$ and hence

$$\boxed{\text{rank}(P(U)) = 3.}$$

Checking whether the rank can drop to 2. Cancel the nonzero factor $e^{i\theta_2}$ in the first identity of (5.5.25) and put

$$z := e^{i\theta_1} \quad (|z| = 1).$$

Then the condition becomes

$$C = \frac{\alpha}{\sqrt{2} + z}(-\sqrt{2} + Dz).$$

Multiply both sides by $(\sqrt{2} + z)$ and rearranging to get

$$\begin{aligned} C(\sqrt{2} + z) &= \alpha(-\sqrt{2} + Dz), \\ (C - \alpha D)z &= -(\alpha + C)\sqrt{2}, \\ z &= \frac{-(\alpha + C)\sqrt{2}}{C - \alpha D}. \end{aligned} \tag{5.5.26}$$

Using the explicit constants from (5.5.15):

$$\alpha = \frac{\sqrt{3}}{2} + \frac{i}{2}, \quad C = -\frac{\sqrt{3}}{2} - \frac{3i}{2}, \quad D = -\frac{1}{2} + \frac{i\sqrt{3}}{2}.$$

A direct computation gives

$$\alpha + C = -i, \quad C - \alpha D = -i\sqrt{2},$$

and hence (5.5.26) becomes

$$z = \frac{-(-i)\sqrt{2}}{-i\sqrt{2}} = -\frac{\sqrt{2}}{2}.$$

But $-\frac{\sqrt{2}}{2}$ does *not* lie on the unit circle, since

$$\left| -\frac{\sqrt{2}}{2} \right| = \frac{\sqrt{2}}{2} \neq 1.$$

This contradicts the fact that $z = e^{i\theta_1}$ must satisfy $|z| = 1$.

Therefore the first proportionality condition in (5.5.25) fails for every $\theta_1 \in \mathbb{R}$. Consequently,

$$p_2(\theta_2) \notin \text{span} \{p_1(\theta_1), p_3(\theta_3)\} \quad \text{for all } \theta_1, \theta_2, \theta_3,$$

and we conclude that the columns of $P(U)$ are always linearly independent.

$$\text{rank } P(U) = 3 \quad \text{for all } U = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \in U(3).$$

To obtain three boundary conditions, one can proceed exactly as in the case $U = I_3$: write $P(U)^*\ell = 0$ for $\ell \in \mathbb{C}^6$, solve for a basis $\ell^{(1)}, \ell^{(2)}, \ell^{(3)}$ of $\ker(P(U)^*)$, and then impose

$$\ell^{(j)*}\mathbf{tr}(u) = 0, \quad j = 1, 2, 3,$$

(or orthonormalize the kernel basis via Gram–Schmidt and use inner-product form).

Dimension of $\ker(P(U)^*)$. Assume from now on that $\text{rank}(P(U)) = 3$. Then

$$\dim \ker(P(U)^*) = 6 - \text{rank}(P(U)) = 3.$$

Construction of a spanning set. Choose (ℓ_3, ℓ_5, ℓ_6) as free parameters and solve for (ℓ_1, ℓ_2, ℓ_4) .

First kernel vector. Take

$$\ell_3 = 1, \quad \ell_5 = 0, \quad \ell_6 = 0.$$

Then (5.5.24) becomes

$$\overline{(1 + \sqrt{2}e^{i\theta_3})} \ell_4 = 0.$$

Assuming $1 + \sqrt{2}e^{i\theta_3} \neq 0$, we obtain $\ell_4 = 0$. With this choice, (5.5.22)–(5.5.23) reduce to

$$\begin{pmatrix} \frac{\sqrt{2} + e^{i\theta_1}}{\alpha e^{i\theta_2}} & \frac{-\sqrt{2} + D e^{i\theta_1}}{C e^{i\theta_2}} \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = - \begin{pmatrix} \frac{\sqrt{2} + F e^{i\theta_1}}{G e^{i\theta_2}} \end{pmatrix}.$$

Set

$$\Delta_1(U) := \det \begin{pmatrix} \frac{\sqrt{2} + e^{i\theta_1}}{\alpha e^{i\theta_2}} & \frac{-\sqrt{2} + D e^{i\theta_1}}{C e^{i\theta_2}} \end{pmatrix}.$$

Assume $\Delta_1(U) \neq 0$ (this holds for generic angles). Then by Cramer's rule

$$\begin{aligned} \ell_1 &= \frac{-\sqrt{2} + F e^{i\theta_1} \overline{C e^{i\theta_2}} + \sqrt{2} + D e^{i\theta_1} \overline{G e^{i\theta_2}}}{\Delta_1(U)}, \\ \ell_2 &= \frac{\sqrt{2} + F e^{i\theta_1} \overline{\alpha e^{i\theta_2}} - \sqrt{2} + e^{i\theta_1} \overline{G e^{i\theta_2}}}{\Delta_1(U)}. \end{aligned}$$

Therefore

$$\ell^{(1)} := (\ell_1, \ell_2, 1, 0, 0, 0) \in \ker(P(U)^*).$$

Second kernel vector. Take

$$\ell_3 = 0, \quad \ell_5 = 1, \quad \ell_6 = 0.$$

Then (5.5.24) gives

$$\overline{(1 + \sqrt{2}e^{i\theta_3})} \ell_4 + \overline{(B + \sqrt{2}e^{i\theta_3})} = 0,$$

so (assuming $1 + \sqrt{2}e^{i\theta_3} \neq 0$)

$$\ell_4 = -\frac{\overline{(B + \sqrt{2}e^{i\theta_3})}}{\overline{(1 + \sqrt{2}e^{i\theta_3})}}.$$

Now (5.5.22) becomes

$$\overline{(\sqrt{2} + e^{i\theta_1})} \ell_1 + \overline{(-\sqrt{2} + De^{i\theta_1})} \ell_2 = 0,$$

and (5.5.23) becomes

$$\overline{\alpha e^{i\theta_2}} \ell_1 + \overline{Ce^{i\theta_2}} \ell_2 + \overline{\alpha} \ell_4 + \overline{A} = 0.$$

This is again a 2×2 system for (ℓ_1, ℓ_2) , with a unique solution if $\Delta_1(U) \neq 0$. Define

$$\ell^{(2)} := (\ell_1, \ell_2, 0, \ell_4, 1, 0) \in \ker(P(U)^*).$$

Third kernel vector. Take

$$\ell_3 = 0, \quad \ell_5 = 0, \quad \ell_6 = 1.$$

Then (5.5.24) gives

$$\overline{(1 + \sqrt{2}e^{i\theta_3})} \ell_4 + \overline{(F + \sqrt{2}e^{i\theta_3})} = 0,$$

so (assuming $1 + \sqrt{2}e^{i\theta_3} \neq 0$)

$$\ell_4 = -\frac{\overline{(F + \sqrt{2}e^{i\theta_3})}}{\overline{(1 + \sqrt{2}e^{i\theta_3})}}.$$

Now (5.5.22) becomes

$$\overline{(\sqrt{2} + e^{i\theta_1})} \ell_1 + \overline{(-\sqrt{2} + De^{i\theta_1})} \ell_2 = 0,$$

and (5.5.23) becomes

$$\overline{\alpha e^{i\theta_2}} \ell_1 + \overline{Ce^{i\theta_2}} \ell_2 + \overline{\alpha} \ell_4 + \overline{E} = 0.$$

Again, this is a 2×2 system for (ℓ_1, ℓ_2) with unique solution if $\Delta_1(U) \neq 0$. Define

$$\ell^{(3)} := (\ell_1, \ell_2, 0, \ell_4, 0, 1) \in \ker(P(U)^*).$$

Since $\mathbf{tr}(u) \in \text{Rang}(P(U))$ if and only if $\mathbf{tr}(u) \perp \ker(P(U)^*)$, each $\ell \in \ker(P(U)^*)$ gives one scalar boundary condition

$$\ell^* \mathbf{tr}(u) = 0.$$

In particular, using the spanning set $\{\ell^{(1)}, \ell^{(2)}, \ell^{(3)}\}$ we obtain

$$(\ell^{(k)})^* \mathbf{tr}(u) = 0, \quad k = 1, 2, 3.$$

Optionally, applying Gram–Schmidt to $\{\ell^{(1)}, \ell^{(2)}, \ell^{(3)}\}$ produces an orthonormal basis $\{\ell_1(U), \ell_2(U), \ell_3(U)\}$ of $\ker(P(U)^*)$, and the boundary conditions can be written as

$$\langle \ell_j(U), \mathbf{tr}(u) \rangle_{\mathbb{C}^6} = 0, \quad j = 1, 2, 3,$$

equivalently $\ell_j(U)^* \mathbf{tr}(u) = 0$ for $j = 1, 2, 3$.

5.5.2.3 The Fourier-type unitary matrix

Let $\omega = e^{2\pi i/3}$ be a primitive cubic root of unity. Then

$$\omega^3 = 1, \quad \omega \neq 1, \quad 1 + \omega + \omega^2 = 0, \quad \bar{\omega} = \omega^2.$$

We consider the unitary matrix

$$U_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \in U(3).$$

Let $\mathbf{s} = (s_1, s_2, s_3)$ and $\mathbf{t} = (t_1, t_2, t_3) = U_3 \mathbf{s}$. Then

$$\begin{aligned} t_1 &= \frac{1}{\sqrt{3}}(s_1 + s_2 + s_3), \\ t_2 &= \frac{1}{\sqrt{3}}(s_1 + \omega s_2 + \omega^2 s_3), \\ t_3 &= \frac{1}{\sqrt{3}}(s_1 + \omega^2 s_2 + \omega s_3). \end{aligned}$$

Recall the full boundary trace

$$\mathbf{tr}(u) := \begin{pmatrix} u(0+) \\ u'(0+) \\ u''(0+) \\ u(0-) \\ u'(0-) \\ u''(0-) \end{pmatrix} \in \mathbb{C}^6,$$

and the matrices $S, T \in \mathbb{C}^{6 \times 3}$ defined by (5.5.7). For any $U \in U(3)$ we have

$$\mathbf{tr}(u) = S\mathbf{s} + T\mathbf{t} = S\mathbf{s} + TU\mathbf{s} = \underbrace{(S + TU)}_{=:P(U)} \mathbf{s}.$$

In particular, for $U = U_3$,

$$P(U_3) = S + TU_3 \in \mathbb{C}^{6 \times 3}.$$

Write $T = (T_1 \ T_2 \ T_3)$ in columns to compute $P(U_3)$, where

$$T_1 = \begin{pmatrix} 1 \\ D \\ F \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \alpha \\ C \\ G \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{pmatrix}.$$

Since $U_3 = \frac{1}{\sqrt{3}}[u^{(1)} \ u^{(2)} \ u^{(3)}]$ with

$$u^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad u^{(2)} = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}, \quad u^{(3)} = \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix},$$

we obtain (column-by-column)

$$TU_3 = \frac{1}{\sqrt{3}}[T_1 + T_2 + T_3, \ T_1 + \omega T_2 + \omega^2 T_3, \ T_1 + \omega^2 T_2 + \omega T_3].$$

Hence

$$TU_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 + \alpha & 1 + \omega\alpha & 1 + \omega^2\alpha \\ D + C & D + \omega C & D + \omega^2 C \\ F + G & F + \omega G & F + \omega^2 G \\ \sqrt{2} & \omega^2 \sqrt{2} & \omega \sqrt{2} \\ \sqrt{2} & \omega^2 \sqrt{2} & \omega \sqrt{2} \\ \sqrt{2} & \omega^2 \sqrt{2} & \omega \sqrt{2} \end{pmatrix}.$$

Therefore, from $P(U_3) = S + TU_3$ we get the explicit 6×3 matrix

$$P(U_3) = \begin{pmatrix} \sqrt{2} + \frac{1+\alpha}{\sqrt{3}} & \frac{1+\omega\alpha}{\sqrt{3}} & \frac{1+\omega^2\alpha}{\sqrt{3}} \\ -\sqrt{2} + \frac{D+C}{\sqrt{3}} & \frac{D+\omega C}{\sqrt{3}} & \frac{D+\omega^2 C}{\sqrt{3}} \\ \sqrt{2} + \frac{F+G}{\sqrt{3}} & \frac{F+\omega G}{\sqrt{3}} & \frac{F+\omega^2 G}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} & \alpha + \frac{\omega^2\sqrt{2}}{\sqrt{3}} & 1 + \frac{\omega\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} & A + \frac{\omega^2\sqrt{2}}{\sqrt{3}} & B + \frac{\omega\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} & E + \frac{\omega^2\sqrt{2}}{\sqrt{3}} & F + \frac{\omega\sqrt{2}}{\sqrt{3}} \end{pmatrix}. \quad (5.5.27)$$

Equivalently, in column form

$$P(U_3) = (p_1 \ p_2 \ p_3),$$

where

$$p_1 = \begin{pmatrix} \sqrt{2} + \frac{1+\alpha}{\sqrt{3}} \\ -\sqrt{2} + \frac{D+C}{\sqrt{3}} \\ \sqrt{2} + \frac{F+G}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}, \quad p_2 = \begin{pmatrix} \frac{1+\omega\alpha}{\sqrt{3}} \\ \frac{D+\omega C}{\sqrt{3}} \\ \frac{F+\omega G}{\sqrt{3}} \\ \alpha + \frac{\omega^2\sqrt{2}}{\sqrt{3}} \\ A + \frac{\omega^2\sqrt{2}}{\sqrt{3}} \\ E + \frac{\omega^2\sqrt{2}}{\sqrt{3}} \end{pmatrix}, \quad p_3 = \begin{pmatrix} \frac{1+\omega^2\alpha}{\sqrt{3}} \\ \frac{D+\omega^2 C}{\sqrt{3}} \\ \frac{F+\omega^2 G}{\sqrt{3}} \\ 1 + \frac{\omega\sqrt{2}}{\sqrt{3}} \\ B + \frac{\omega\sqrt{2}}{\sqrt{3}} \\ F + \frac{\omega\sqrt{2}}{\sqrt{3}} \end{pmatrix}.$$

The adjoint matrix $P(U_3)^*$. Since $P(U_3)$ has complex entries, its adjoint is the conjugate transpose:

$$P(U_3)^* = \overline{P(U_3)}^\top \in \mathbb{C}^{3 \times 6}.$$

Using $\bar{\omega} = \omega^2$, we obtain directly from (5.5.27)

$$P(U_3)^* = \begin{pmatrix} \overline{p_{11}} & \overline{p_{21}} & \overline{p_{31}} & \overline{p_{41}} & \overline{p_{51}} & \overline{p_{61}} \\ \overline{p_{12}} & \overline{p_{22}} & \overline{p_{32}} & \overline{p_{42}} & \overline{p_{52}} & \overline{p_{62}} \\ \overline{p_{13}} & \overline{p_{23}} & \overline{p_{33}} & \overline{p_{43}} & \overline{p_{53}} & \overline{p_{63}} \end{pmatrix},$$

where p_{jk} denote the entries of $P(U_3)$ in (5.5.27). (We keep the conjugation symbols for clarity.)

Rank of $P(U_3)$. Recall Proposition 5.54, which asserts that $\text{rank } P(U) = 3$. We verify it here for $U = U_3$ by exhibiting a 3×3 minor with nonzero determinant.

Let $\beta := \frac{\sqrt{2}}{\sqrt{3}}$ and we consider the 3×3 sub-matrix obtained from rows 4, 5, 6 of $P(U_3)$, i.e.

$$M := \begin{pmatrix} \beta & \alpha + \omega^2\beta & 1 + \omega\beta \\ \beta & A + \omega^2\beta & B + \omega\beta \\ \beta & E + \omega^2\beta & F + \omega\beta \end{pmatrix}.$$

Denote by R_1, R_2, R_3 the first, second, and third rows of M , respectively. Explicitly,

$$\begin{aligned} R_1 &= (\beta, \alpha + \omega^2\beta, 1 + \omega\beta), \\ R_2 &= (\beta, A + \omega^2\beta, B + \omega\beta), \\ R_3 &= (\beta, E + \omega^2\beta, F + \omega\beta). \end{aligned}$$

Since subtracting one row from another does not change the value of the determinant, we perform the elementary row operations

$$R_2 \longleftarrow R_2 - R_1, \quad R_3 \longleftarrow R_3 - R_1.$$

After these operations, the matrix M is transformed into

$$\widetilde{M} = \begin{pmatrix} \beta & \alpha + \omega^2\beta & 1 + \omega\beta \\ 0 & A - \alpha & B - 1 \\ 0 & E - \alpha & F - 1 \end{pmatrix},$$

and we have

$$\det(M) = \det(\widetilde{M}).$$

Expanding the determinant along the first column, we get

$$\det(M) = \beta \det \begin{pmatrix} A - \alpha & B - 1 \\ E - \alpha & F - 1 \end{pmatrix}.$$

A direct computation of the 2×2 determinant gives

$$\det(M) = \beta[(A - \alpha)(F - 1) - (B - 1)(E - \alpha)].$$

We now substitute by using the explicit values (5.5.15).

First compute each factor:

$$\begin{aligned} A - \alpha &= \left(\frac{\sqrt{3}}{2} + \frac{3i}{2} \right) - \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right) = i, \\ F - 1 &= \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2} \right) - 1 = -\frac{3}{2} - \frac{i\sqrt{3}}{2}, \\ B - 1 &= \left(\frac{1}{2} - \frac{i\sqrt{3}}{2} \right) - 1 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}, \end{aligned}$$

$$E - \alpha = i - \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right) = -\frac{\sqrt{3}}{2} + \frac{i}{2}.$$

Hence

$$\begin{aligned} (A - \alpha)(F - 1) &= i \left(-\frac{3}{2} - \frac{i\sqrt{3}}{2} \right) = -\frac{3i}{2} + \frac{\sqrt{3}}{2}, \\ (B - 1)(E - \alpha) &= \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2} \right) \left(-\frac{\sqrt{3}}{2} + \frac{i}{2} \right) \\ &= \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} \right) + \left(-\frac{i}{4} + \frac{3i}{4} \right) = \frac{\sqrt{3}}{2} + \frac{i}{2}. \end{aligned}$$

Therefore the bracket simplifies to

$$\begin{aligned} (A - \alpha)(F - 1) - (B - 1)(E - \alpha) &= \left(-\frac{3i}{2} + \frac{\sqrt{3}}{2} \right) - \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right) \\ &= -2i. \end{aligned}$$

Substituting back, we obtain

$$\det(M) = \beta(-2i).$$

In the present Fourier-type case one has $\beta = \frac{\sqrt{2}}{\sqrt{3}}$, hence

$$\det(M) = -2\frac{\sqrt{2}}{\sqrt{3}}i = -\frac{2\sqrt{6}}{3}i \neq 0.$$

Therefore $\text{rank}(M) = 3$, hence $\text{rank}(P(U_3)) \geq 3$. Since $P(U_3) \in \mathbb{C}^{6 \times 3}$, we conclude

$$\boxed{\text{rank}(P(U_3)) = 3.}$$

In particular,

$$\dim \ker(P(U_3)^*) = 6 - \text{rank}(P(U_3)) = 3.$$

Computing $\ker(P(U_3)^*)$. Let $\ell = (\ell_1, \dots, \ell_6) \in \mathbb{C}^6$. Then $\ell \in \ker(P(U_3)^*)$ if and only if

$$P(U_3)^*\ell = 0,$$

i.e. if and only if the following three equations hold:

$$\overline{p_{11}} \ell_1 + \overline{p_{21}} \ell_2 + \overline{p_{31}} \ell_3 + \overline{p_{41}} \ell_4 + \overline{p_{51}} \ell_5 + \overline{p_{61}} \ell_6 = 0, \quad (5.5.28)$$

$$\overline{p_{12}} \ell_1 + \overline{p_{22}} \ell_2 + \overline{p_{32}} \ell_3 + \overline{p_{42}} \ell_4 + \overline{p_{52}} \ell_5 + \overline{p_{62}} \ell_6 = 0,$$

$$\overline{p_{13}} \ell_1 + \overline{p_{23}} \ell_2 + \overline{p_{33}} \ell_3 + \overline{p_{43}} \ell_4 + \overline{p_{53}} \ell_5 + \overline{p_{63}} \ell_6 = 0. \quad (5.5.29)$$

Here the coefficients p_{jk} are those in (5.5.27). Since $\dim \ker(P(U_3)^*) = 3$, we may choose three free parameters and solve the resulting 3×3 system for the remaining three variables.

Construction of a spanning set. Choose (ℓ_3, ℓ_5, ℓ_6) as free parameters and solve for (ℓ_1, ℓ_2, ℓ_4) . As in the identity case, we build three linearly independent solutions by taking

$$(\ell_3, \ell_5, \ell_6) = (1, 0, 0), \quad (0, 1, 0), \quad (0, 0, 1).$$

(a) *First kernel vector.* Set $(\ell_3, \ell_5, \ell_6) = (1, 0, 0)$. Then (5.5.28)–(5.5.29) become a 3×3 linear system for (ℓ_1, ℓ_2, ℓ_4) :

$$\begin{pmatrix} \overline{p_{11}} & \overline{p_{21}} & \overline{p_{41}} \\ \overline{p_{12}} & \overline{p_{22}} & \overline{p_{42}} \\ \overline{p_{13}} & \overline{p_{23}} & \overline{p_{43}} \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_4 \end{pmatrix} = - \begin{pmatrix} \overline{p_{31}} \\ \overline{p_{32}} \\ \overline{p_{33}} \end{pmatrix}. \quad (5.5.30)$$

Denote the determinant of the coefficient matrix by

$$\Delta_{(1)} := \det \begin{pmatrix} \overline{p_{11}} & \overline{p_{21}} & \overline{p_{41}} \\ \overline{p_{12}} & \overline{p_{22}} & \overline{p_{42}} \\ \overline{p_{13}} & \overline{p_{23}} & \overline{p_{43}} \end{pmatrix}.$$

Since $\text{rank}(P(U_3)) = 3$, we can (and do) choose this 3×3 system so that $\Delta_{(1)} \neq 0$ (for example, one may verify numerically using the explicit Airy constants). Hence (5.5.30) has a unique solution, and by Cramer's rule we obtain explicit formulae for (ℓ_1, ℓ_2, ℓ_4) . We define

$$\ell^{(1)} := (\ell_1, \ell_2, 1, \ell_4, 0, 0) \in \ker(P(U_3)^*).$$

(b) *Second kernel vector.* Set $(\ell_3, \ell_5, \ell_6) = (0, 1, 0)$. Then we solve

$$\begin{pmatrix} \overline{p_{11}} & \overline{p_{21}} & \overline{p_{41}} \\ \overline{p_{12}} & \overline{p_{22}} & \overline{p_{42}} \\ \overline{p_{13}} & \overline{p_{23}} & \overline{p_{43}} \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_4 \end{pmatrix} = - \begin{pmatrix} \overline{p_{51}} \\ \overline{p_{52}} \\ \overline{p_{53}} \end{pmatrix},$$

and define

$$\ell^{(2)} := (\ell_1, \ell_2, 0, \ell_4, 1, 0) \in \ker(P(U_3)^*).$$

(c) *Third kernel vector.* Set $(\ell_3, \ell_5, \ell_6) = (0, 0, 1)$. Then we solve

$$\begin{pmatrix} \overline{p_{11}} & \overline{p_{21}} & \overline{p_{41}} \\ \overline{p_{12}} & \overline{p_{22}} & \overline{p_{42}} \\ \overline{p_{13}} & \overline{p_{23}} & \overline{p_{43}} \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_4 \end{pmatrix} = - \begin{pmatrix} \overline{p_{61}} \\ \overline{p_{62}} \\ \overline{p_{63}} \end{pmatrix},$$

and define

$$\ell^{(3)} := (\ell_1, \ell_2, 0, \ell_4, 0, 1) \in \ker(P(U_3)^*).$$

By construction, $\ell^{(1)}, \ell^{(2)}, \ell^{(3)}$ are linearly independent, hence they form a spanning set of $\ker(P(U_3)^*)$. Optionally, applying Gram–Schmidt to $\{\ell^{(1)}, \ell^{(2)}, \ell^{(3)}\}$ produces an orthonormal basis $\{\ell_1(U_3), \ell_2(U_3), \ell_3(U_3)\}$ of $\ker(P(U_3)^*)$.

Since $\mathbf{tr}(u) \in \text{Ran } P(U_3)$ if and only if $\mathbf{tr}(u) \perp \ker(P(U_3)^*)$, each $\ell \in \ker(P(U_3)^*)$ yields one boundary condition $\ell^* \mathbf{tr}(u) = 0$. Therefore, the self-adjoint domain corresponding to U_3 can be written as

$$D(A^{U_3}) = \left\{ u \in D(A_1^*) : \langle \ell_j(U_3), \mathbf{tr}(u) \rangle_{\mathbb{C}^6} = 0, \ j = 1, 2, 3 \right\}.$$

Equivalently,

$$\boxed{\ell_j(U_3)^* \mathbf{tr}(u) = 0, \quad j = 1, 2, 3.}$$

5.5.2.4 Case U : The cyclic permutation matrix

We consider the unitary (permutation) matrix

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in U(3), \quad (5.5.31)$$

which corresponds to the cyclic permutation $(1\ 2\ 3)$, i.e. $Ue_1 = e_2$, $Ue_2 = e_3$, $Ue_3 = e_1$, and satisfies $U^3 = I_3$.

Recall that $\mathbf{t} = U\mathbf{s}$, where

$$\mathbf{s} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}.$$

Hence

$$t_1 = s_2, \quad t_2 = s_3, \quad t_3 = s_1.$$

Recall the full boundary trace vector

$$\mathbf{tr}(u) = \begin{pmatrix} u(0+) \\ u'(0+) \\ u''(0+) \\ u(0-) \\ u'(0-) \\ u''(0-) \end{pmatrix} \in \mathbb{C}^6.$$

and the matrices $S, T \in \mathbb{C}^{6 \times 3}$ defined by (5.5.7).

For any $U \in U(3)$ we have

$$\mathbf{tr}(u) = S\mathbf{s} + T\mathbf{t} = S\mathbf{s} + TU\mathbf{s} = (S + TU)\mathbf{s} =: P(U)\mathbf{s},$$

where $P(U) \in \mathbb{C}^{6 \times 3}$. For the cyclic permutation matrix (5.5.31), the multiplication TU permutes the columns of T :

$$TU = [T_2 \ T_3 \ T_1],$$

where T_k denotes the k th column of T . Thus

$$P(U) = S + TU = [S_1 + T_2 \ S_2 + T_3 \ S_3 + T_1],$$

where S_k denotes the k th column of S .

Writing this explicitly gives

$$P(U) = \begin{pmatrix} \sqrt{2} + \alpha & 0 & 1 \\ -\sqrt{2} + C & 0 & D \\ \sqrt{2} + G & 0 & F \\ 0 & \alpha + \sqrt{2} & 1 \\ 0 & A + \sqrt{2} & B \\ 0 & E + \sqrt{2} & F \end{pmatrix} \in \mathbb{C}^{6 \times 3}. \quad (5.5.32)$$

Equivalently, if we denote the columns by

$$P(U) = [p_1 \ p_2 \ p_3],$$

then

$$p_1 = \begin{pmatrix} \sqrt{2} + \alpha \\ -\sqrt{2} + C \\ \sqrt{2} + G \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \alpha + \sqrt{2} \\ A + \sqrt{2} \\ E + \sqrt{2} \end{pmatrix}, \quad p_3 = \begin{pmatrix} 1 \\ D \\ F \\ 1 \\ B \\ F \end{pmatrix}.$$

Step 3: Rank of $P(U)$. The vectors p_1 and p_2 have disjoint supports: p_1 is supported in rows 1–3 and p_2 is supported in rows 4–6. Moreover, $\sqrt{2} + \alpha \neq 0$ and $\alpha + \sqrt{2} \neq 0$, hence $p_1 \neq 0$ and $p_2 \neq 0$. Therefore p_1 and p_2 are linearly independent and

$$\text{rank } P(U) \geq 2.$$

The rank drops to 2 if and only if $p_3 \in \text{span}\{p_1, p_2\}$, i.e. if and only if there exist $x, y \in \mathbb{C}$ such that

$$p_3 = x p_1 + y p_2. \quad (5.5.33)$$

Comparing rows 1–3 in (5.5.33) yields

$$(1, D, F) = x(\sqrt{2} + \alpha, -\sqrt{2} + C, \sqrt{2} + G),$$

so

$$x = \frac{1}{\sqrt{2} + \alpha}, \quad D = \frac{-\sqrt{2} + C}{\sqrt{2} + \alpha}, \quad F = \frac{\sqrt{2} + G}{\sqrt{2} + \alpha}. \quad (5.5.34)$$

Similarly, comparing rows 4–6 yields

$$(1, B, F) = y(\alpha + \sqrt{2}, A + \sqrt{2}, E + \sqrt{2}),$$

so

$$y = \frac{1}{\alpha + \sqrt{2}}, \quad B = \frac{A + \sqrt{2}}{\alpha + \sqrt{2}}, \quad F = \frac{E + \sqrt{2}}{\alpha + \sqrt{2}}. \quad (5.5.35)$$

Consequently, we have the implication

$$\text{rank } P(U) = 2 \implies \text{all identities in (5.5.34) and (5.5.35) hold.} \quad (5.5.36)$$

Using the explicit constants from (5.5.15), and substituting these values into. For example, the second identity in (5.5.34), using the explicit values

$$\alpha = \frac{\sqrt{3}}{2} + \frac{i}{2}, \quad C = -\frac{\sqrt{3}}{2} - \frac{3i}{2}, \quad D = -\frac{1}{2} + \frac{i\sqrt{3}}{2},$$

we compute

$$D - \frac{-\sqrt{2} + C}{\sqrt{2} + \alpha} = -\frac{1}{2} + \frac{i\sqrt{3}}{2} - \frac{-\sqrt{2} - \frac{\sqrt{3}}{2} - \frac{3i}{2}}{\sqrt{2} + \frac{\sqrt{3}}{2} + \frac{i}{2}} = -\frac{i}{\sqrt{2}} \neq 0.$$

hence the conditions in (5.5.36) fail. Therefore $p_3 \notin \text{span}\{p_1, p_2\}$ and the three columns of $P(U)$ are linearly independent. We conclude that

$$\boxed{\text{rank } P(U) = 3.} \quad (5.5.37)$$

The adjoint matrix $P(U)^*$. Since $P(U)$ may have complex entries, its adjoint is the conjugate transpose:

$$P(U)^* = \overline{P(U)}^\top \in \mathbb{C}^{3 \times 6}.$$

Using (5.5.32), we infer that

$$P(U)^* = \begin{pmatrix} \overline{\sqrt{2} + \alpha} & \overline{-\sqrt{2} + C} & \overline{\sqrt{2} + G} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{\alpha + \sqrt{2}} & \overline{A + \sqrt{2}} & \overline{E + \sqrt{2}} \\ \overline{1} & \overline{D} & \overline{F} & \overline{1} & \overline{B} & \overline{F} \end{pmatrix}.$$

Let $\ell = (\ell_1, \dots, \ell_6) \in \mathbb{C}^6$. Then $\ell \in \ker(P(U)^*)$ if and only if $P(U)^*\ell = 0$, i.e. the following system holds:

$$\overline{(\sqrt{2} + \alpha)} \ell_1 + \overline{(-\sqrt{2} + C)} \ell_2 + \overline{(\sqrt{2} + G)} \ell_3 = 0, \quad (5.5.38)$$

$$\overline{(\alpha + \sqrt{2})} \ell_4 + \overline{(A + \sqrt{2})} \ell_5 + \overline{(E + \sqrt{2})} \ell_6 = 0, \quad (5.5.39)$$

$$\ell_1 + \overline{D} \ell_2 + \overline{F} \ell_3 + \ell_4 + \overline{B} \ell_5 + \overline{F} \ell_6 = 0. \quad (5.5.40)$$

Since $\text{rank } P(U) = 3$ by (5.5.37), we have

$$\dim \ker(P(U)^*) = 6 - \text{rank } P(U) = 3.$$

Constructing a spanning set. A convenient way to parameterize the solutions is to choose (ℓ_3, ℓ_5, ℓ_6) as free parameters. First, (5.5.39) determines ℓ_4 in terms of (ℓ_5, ℓ_6) :

$$\ell_4 = -\frac{\overline{(A + \sqrt{2})} \ell_5 + \overline{(E + \sqrt{2})} \ell_6}{\overline{(\alpha + \sqrt{2})}}. \quad (5.5.41)$$

Next, (5.5.38) gives a linear relation between (ℓ_1, ℓ_2) and ℓ_3 :

$$\overline{(\sqrt{2} + \alpha)} \ell_1 + \overline{(-\sqrt{2} + C)} \ell_2 = -\overline{(\sqrt{2} + G)} \ell_3. \quad (5.5.42)$$

Finally, substituting (5.5.41) into (5.5.40) yields a second equation linking (ℓ_1, ℓ_2) to (ℓ_3, ℓ_5, ℓ_6) :

$$\ell_1 + \overline{D} \ell_2 = -\overline{F} \ell_3 + \frac{\overline{(A + \sqrt{2})} \ell_5 + \overline{(E + \sqrt{2})} \ell_6}{(\alpha + \sqrt{2})} - \overline{B} \ell_5 - \overline{F} \ell_6. \quad (5.5.43)$$

The pair (5.5.42) and (5.5.43) is a 2×2 linear system for (ℓ_1, ℓ_2) . Solving it for three independent choices of (ℓ_3, ℓ_5, ℓ_6) (e.g. $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$) produces three linearly independent vectors $\ell^{(1)}, \ell^{(2)}, \ell^{(3)} \in \ker(P(U)^*)$. Optionally, applying Gram–Schmidt implies an orthonormal basis $\{\ell_1(U), \ell_2(U), \ell_3(U)\}$ of $\ker(P(U)^*)$.

Since $\mathbf{tr}(u) \in \text{Ran } P(U)$ if and only if $\mathbf{tr}(u) \perp \ker(P(U)^*)$, the boundary conditions for the self-adjoint extension corresponding to the cyclic permutation matrix are:

$$\boxed{\langle \ell_j(U), \mathbf{tr}(u) \rangle_{\mathbb{C}^6} = 0, \quad j = 1, 2, 3,}$$

equivalently $\ell_j(U)^* \mathbf{tr}(u) = 0$ for $j = 1, 2, 3$.

5.5.2.5 The unitary matrix U_5

We consider the self-adjoint extension corresponding to the unitary matrix

$$U_5 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & e^{i\phi} \end{pmatrix} \in U(3).$$

Set $a := \frac{1}{\sqrt{2}}$. For $\mathbf{s} = (s_1, s_2, s_3)$ and $\mathbf{t} = U_5 \mathbf{s}$, we have

$$t_1 = a(s_1 + s_2), \quad t_2 = a(s_1 - s_2), \quad t_3 = e^{i\phi} s_3.$$

Recall that for any $U \in U(3)$,

$$\mathbf{tr}(u) = S\mathbf{s} + T\mathbf{t} = S\mathbf{s} + TU\mathbf{s} = (S + TU)\mathbf{s} =: P(U)\mathbf{s},$$

where $P(U) \in \mathbb{C}^{6 \times 3}$. For U_5 we compute TU_5 column-by-column. Denote by T_k the k th column of T :

$$T_1 = \begin{pmatrix} 1 \\ D \\ F \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} \alpha \\ C \\ G \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{pmatrix}.$$

Since the columns of U_5 are $(a, a, 0)$, $(a, -a, 0)$, $(0, 0, e^{i\phi})$, we obtain

$$TU_5 = [a(T_1 + T_2) \quad a(T_1 - T_2) \quad e^{i\phi}T_3],$$

hence

$$TU_5 = \begin{pmatrix} a(1 + \alpha) & a(1 - \alpha) & 0 \\ a(D + C) & a(D - C) & 0 \\ a(F + G) & a(F - G) & 0 \\ 0 & 0 & \sqrt{2}e^{i\phi} \\ 0 & 0 & \sqrt{2}e^{i\phi} \\ 0 & 0 & \sqrt{2}e^{i\phi} \end{pmatrix}.$$

Therefore,

$$P(U_5) = S + TU_5 = \begin{pmatrix} \sqrt{2} + a(1 + \alpha) & a(1 - \alpha) & 0 \\ -\sqrt{2} + a(D + C) & a(D - C) & 0 \\ \sqrt{2} + a(F + G) & a(F - G) & 0 \\ 0 & \alpha & 1 + \sqrt{2}e^{i\phi} \\ 0 & A & B + \sqrt{2}e^{i\phi} \\ 0 & E & F + \sqrt{2}e^{i\phi} \end{pmatrix} \in \mathbb{C}^{6 \times 3}. \quad (5.5.44)$$

The adjoint matrix $P(U_5)^*$. Since $P(U_5)$ has complex entries, its adjoint is the conjugate transpose

$$P(U_5)^* = \overline{P(U_5)}^\top \in \mathbb{C}^{3 \times 6}.$$

Writing $P(U_5) = (p_{jk})_{1 \leq j \leq 6, 1 \leq k \leq 3}$, we have

$$P(U_5)^* = \begin{pmatrix} \overline{p_{11}} & \overline{p_{21}} & \overline{p_{31}} & \overline{p_{41}} & \overline{p_{51}} & \overline{p_{61}} \\ \overline{p_{12}} & \overline{p_{22}} & \overline{p_{32}} & \overline{p_{42}} & \overline{p_{52}} & \overline{p_{62}} \\ \overline{p_{13}} & \overline{p_{23}} & \overline{p_{33}} & \overline{p_{43}} & \overline{p_{53}} & \overline{p_{63}} \end{pmatrix}.$$

Using (5.5.44), this becomes explicitly

$$P(U_5)^* = \begin{pmatrix} \overline{\sqrt{2} + a(1 + \alpha)} & \overline{-\sqrt{2} + a(D + C)} & \overline{\sqrt{2} + a(F + G)} & 0 & 0 & 0 \\ \overline{a(1 - \alpha)} & \overline{a(D - C)} & \overline{a(F - G)} & \overline{\alpha} & \overline{A} & \overline{E} \\ 0 & 0 & 0 & \overline{1 + \sqrt{2}e^{i\phi}} & \overline{B + \sqrt{2}e^{i\phi}} & \overline{F + \sqrt{2}e^{i\phi}} \end{pmatrix}.$$

Let $\ell = (\ell_1, \dots, \ell_6) \in \mathbb{C}^6$. Then $\ell \in \ker(P(U_5)^*)$ if and only if

$$P(U_5)^* \ell = 0.$$

Writing the three equations row-by-row gives

$$\overline{\sqrt{2} + a(1 + \alpha)} \ell_1 + \overline{-\sqrt{2} + a(D + C)} \ell_2 + \overline{\sqrt{2} + a(F + G)} \ell_3 = 0, \quad (5.5.45)$$

$$\overline{a(1 - \alpha)} \ell_1 + \overline{a(D - C)} \ell_2 + \overline{a(F - G)} \ell_3 + \overline{\alpha} \ell_4 + \overline{A} \ell_5 + \overline{E} \ell_6 = 0, \quad (5.5.46)$$

$$\overline{1 + \sqrt{2}e^{i\phi}} \ell_4 + \overline{B + \sqrt{2}e^{i\phi}} \ell_5 + \overline{F + \sqrt{2}e^{i\phi}} \ell_6 = 0. \quad (5.5.47)$$

As in the cases $U = I_3$ and $U = U_3$, the first and third columns of $P(U_5)$ have disjoint supports: the first column is supported in rows 1–3 and the third column is supported in rows 4–6. Moreover,

$$1 + \sqrt{2}e^{i\phi} \neq 0 \quad (\text{since } |e^{i\phi}| = 1),$$

so the third column is nonzero, and similarly the first column is nonzero because its first entry $\sqrt{2} + a(1 + \alpha) \neq 0$ (in fact, with the explicit α below it cannot vanish).

Hence the first and third columns are linearly independent, so $\text{rank } P(U_5) \geq 2$. The rank can drop to 2 only if the second column lies in the span of the first and third columns. We show that this never happens, hence $\text{rank } P(U_5) = 3$ for all ϕ .

Proposition 5.63. *For every $\phi \in \mathbb{R}$, the matrix $P(U_5)$ has full column rank:*

$$\boxed{\text{rank } P(U_5) = 3.}$$

Consequently $\dim \ker(P(U_5)^*) = 6 - \text{rank } P(U_5) = 3$.

Proof. Write $P(U_5) = (c_1 \ c_2 \ c_3)$ for its columns. From (5.5.44):

$$c_1 = \begin{pmatrix} \sqrt{2} + a(1 + \alpha) \\ -\sqrt{2} + a(D + C) \\ \sqrt{2} + a(F + G) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad c_2 = \begin{pmatrix} a(1 - \alpha) \\ a(D - C) \\ a(F - G) \\ \alpha \\ A \\ E \end{pmatrix}, \quad c_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 + \sqrt{2}e^{i\phi} \\ B + \sqrt{2}e^{i\phi} \\ F + \sqrt{2}e^{i\phi} \end{pmatrix}.$$

Then c_1 is supported in rows 1–3 and c_3 is supported in rows 4–6, hence c_1 and c_3 are linearly independent. Thus $\text{rank } P(U_5) \geq 2$.

Assume $\text{rank } P(U_5) = 2$. Then $c_2 \in \text{span } \{c_1, c_3\}$, so there exist $x, y \in \mathbb{C}$ such that $c_2 = xc_1 + yc_3$. Comparing rows 4–6 gives

$$(\alpha, A, E) = y(1 + \sqrt{2}e^{i\phi}, B + \sqrt{2}e^{i\phi}, F + \sqrt{2}e^{i\phi}),$$

so necessarily

$$y = \frac{\alpha}{1 + \sqrt{2}e^{i\phi}}$$

and hence

$$A = \frac{\alpha}{1 + \sqrt{2}e^{i\phi}}(B + \sqrt{2}e^{i\phi}), \quad E = \frac{\alpha}{1 + \sqrt{2}e^{i\phi}}(F + \sqrt{2}e^{i\phi}).$$

Now compare rows 1–3. Since c_3 has zeros in rows 1–3, we get

$$(a(1 - \alpha), a(D - C), a(F - G)) = x(\sqrt{2} + a(1 + \alpha), -\sqrt{2} + a(D + C), \sqrt{2} + a(F + G)),$$

so in particular

$$x = \frac{a(1 - \alpha)}{\sqrt{2} + a(1 + \alpha)}. \quad (5.5.48)$$

Thus a necessary condition for rank 2 is

$$a(D - C) = x(-\sqrt{2} + a(D + C)), \quad a(F - G) = x(\sqrt{2} + a(F + G)), \quad (5.5.49)$$

with x as in (5.5.48).

We now show that (5.5.49) fails. Recall (as before) the explicit values (5.5.15).

A direct computation gives

$$D - C = \frac{\sqrt{3} - 1}{2} + i\frac{\sqrt{3} + 3}{2}, \quad D + C = -\frac{1 + \sqrt{3}}{2} + i\frac{\sqrt{3} - 3}{2}.$$

Also,

$$\sqrt{2} + a(1 + \alpha) = \sqrt{2} + \frac{1}{\sqrt{2}} \left(1 + \frac{\sqrt{3}}{2} + \frac{i}{2} \right) = \frac{1}{2\sqrt{2}} (4 + 2 + \sqrt{3} + i) \neq 0,$$

so x in (5.5.48) is well-defined. Substituting these values into the first identity in (5.5.49), one obtains a nonzero residual (hence the equality cannot hold). Therefore $c_2 \notin \text{span} \{c_1, c_3\}$, and the three columns are linearly independent.

Thus $\text{rank } P(U_5) = 3$. The dimension statement follows from

$$\dim \ker(P(U_5)^*) = 6 - \text{rank } P(U_5).$$

□

Since $\text{rank } P(U_5) = 3$, we have $\dim \ker(P(U_5)^*) = 3$. We construct three linearly independent vectors in the kernel as in the case $U = I_3$.

First kernel vector. Choose $(\ell_3, \ell_5, \ell_6) = (1, 0, 0)$. Then (5.5.47) gives

$$\overline{1 + \sqrt{2}e^{i\phi}} \ell_4 = 0 \implies \ell_4 = 0 \quad (\text{since } 1 + \sqrt{2}e^{i\phi} \neq 0).$$

With $\ell_3 = 1$, $\ell_4 = \ell_5 = \ell_6 = 0$, equations (5.5.45)–(5.5.46) reduce to a 2×2 system for (ℓ_1, ℓ_2) :

$$\begin{pmatrix} \overline{\sqrt{2} + a(1 + \alpha)} & \overline{-\sqrt{2} + a(D + C)} \\ \overline{a(1 - \alpha)} & \overline{a(D - C)} \end{pmatrix} \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} = - \begin{pmatrix} \overline{\sqrt{2} + a(F + G)} \\ \overline{a(F - G)} \end{pmatrix}. \quad (5.5.50)$$

Set the determinant

$$\Delta_{U_5}^{(1)} := \det \begin{pmatrix} \overline{\sqrt{2} + a(1 + \alpha)} & \overline{-\sqrt{2} + a(D + C)} \\ \overline{a(1 - \alpha)} & \overline{a(D - C)} \end{pmatrix}.$$

Since $\text{rank } P(U_5) = 3$, the two rows in (5.5.50) are independent and $\Delta_{U_5}^{(1)} \neq 0$, hence (5.5.50) has a unique solution. By Cramer's rule,

$$\begin{aligned} \ell_1 &= \frac{\overline{-\sqrt{2} + a(F + G)} \overline{a(D - C)} + \overline{-\sqrt{2} + a(D + C)} \overline{a(F - G)}}{\Delta_{U_5}^{(1)}}, \\ \ell_2 &= \frac{\overline{\sqrt{2} + a(F + G)} \overline{a(1 - \alpha)} - \overline{\sqrt{2} + a(1 + \alpha)} \overline{a(F - G)}}{\Delta_{U_5}^{(1)}}. \end{aligned}$$

Therefore,

$$\ell^{(1)} := (\ell_1, \ell_2, 1, 0, 0, 0) \in \ker(P(U_5)^*).$$

Second and third kernel vectors. Repeat the same construction with $(\ell_3, \ell_5, \ell_6) = (0, 1, 0)$ and $(0, 0, 1)$. In each case, (5.5.47) determines ℓ_4 uniquely in terms of (ℓ_5, ℓ_6) , and then (5.5.45)–(5.5.46) give a 2×2 system for (ℓ_1, ℓ_2) . This yields two more vectors

$$\ell^{(2)}, \ell^{(3)} \in \ker(P(U_5)^*)$$

which are linearly independent from $\ell^{(1)}$.

Optionally, apply Gram–Schmidt to $\{\ell^{(1)}, \ell^{(2)}, \ell^{(3)}\}$ to obtain an orthonormal basis $\{\ell_1(U_5), \ell_2(U_5), \ell_3(U_5)\}$ of $\ker(P(U_5)^*)$.

As before, $\mathbf{tr}(u) \in \text{Ran } P(U_5)$ if and only if $\mathbf{tr}(u) \perp \ker(P(U_5)^*)$. Therefore, if $\{\ell_j(U_5)\}_{j=1}^3$ is any orthonormal basis of $\ker(P(U_5)^*)$, then the domain $D(A^{U_5})$ can be described by the three scalar boundary conditions

$$\boxed{\langle \ell_j(U_5), \mathbf{tr}(u) \rangle_{\mathbb{C}^6} = 0, \quad j = 1, 2, 3,}$$

equivalently $\ell_j(U_5)^* \mathbf{tr}(u) = 0$ for $j = 1, 2, 3$.

Remark 5.64. If one prefers a non-orthonormal spanning set $\{\ell^{(1)}, \ell^{(2)}, \ell^{(3)}\}$, then the boundary conditions can be written simply as

$$(\ell^{(j)})^* \mathbf{tr}(u) = 0, \quad j = 1, 2, 3,$$

and orthonormalization is optional.

Chapter 6

Third order linear in-homogeneous differential operators

In this chapter, we study third-order linear inhomogeneous differential operators. We analyse their symmetry properties, determine the adjoint operator, and investigate the corresponding deficiency indices on various domains.

Although the classical von Neumann approach can be applied in several cases, it does not lead to a complete description of all self-adjoint extensions in full generality. This limitation motivates the need for an alternative framework, which will be developed in Chapter 8.

6.1 Third order linear in-homogeneous differential operators on real line

Recall, that by a real line we understand the set $\mathbb{R}$ of all real numbers.

We assume that $\mathcal{O}$ is an open subset of $\mathbb{R}$. We consider the Hilbert space

$$H = L^2(\mathcal{O}, \mathbb{C})$$

with the standard inner product and norm. Assume that $a_3, a_1 \in \mathbb{R} \setminus \{0\}$, and define operator A_0

$$\begin{aligned} D(A_0) &= C_0^\infty(\mathcal{O}, \mathbb{C}) \\ A_0 u &= a_3(iD)^3 u + a_1 i D u = -i a_3 u''' + i a_1 u', \end{aligned} \tag{6.1.1}$$

where u''' and u' are the classical derivative.

We begin with the following standard result. One can prove the above result by employing the following general result.

Lemma 6.1. *Suppose that B_0 and C_0 are symmetric operators on a Hilbert space H with the same domain*

$$D(B_0) = D(C_0) = D.$$

Let $\alpha, \beta \in \mathbb{R}$ and define

$$A_0 u = \alpha B_0 u + \beta C_0 u, \quad u \in D.$$

Then A_0 is symmetric.

Proof of Lemma 6.1. Let $u, v \in D$. Then

$$\begin{aligned} \langle A_0 u, v \rangle &= \langle \alpha B_0 u + \beta C_0 u, v \rangle \\ &= \alpha \langle B_0 u, v \rangle + \beta \langle C_0 u, v \rangle. \end{aligned}$$

Since B_0 and C_0 are symmetric,

$$\langle B_0 u, v \rangle = \langle u, B_0 v \rangle, \quad \langle C_0 u, v \rangle = \langle u, C_0 v \rangle.$$

Since $(\alpha B_0 + \beta C_0)$ is symmetric iff $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$ such that $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned} \langle A_0 u, v \rangle &= \alpha \langle u, B_0 v \rangle + \beta \langle u, C_0 v \rangle \\ &= \langle u, \alpha B_0 v + \beta C_0 v \rangle \\ &= \langle u, A_0 v \rangle. \end{aligned}$$

Therefore, A_0 is symmetric. □

Lemma 6.2. *The operator A_0 defined by (6.1.1) is symmetric.*

Proof of Lemma 6.2. The operators iD and $(iD)^3$ with domain $C_0^\infty(\mathcal{O})$ are symmetric, see previous sections 3.1 and 5.1, in Proposition 5.1.

Hence, by Lemma 6.1, the operator $A_0 = a_3(iD)^3 + a_1 iD$ is symmetric. □

We need to determine $D(A_0^*)$, we begin with the following characterization of the domain of the adjoint operator A_0^* .

Lemma 6.3. *Suppose that $\mathcal{O} \subset \mathbb{R}$ is open and $a_3 \neq 0$. Let A_0 be defined by (6.1.1). Then*

$$D(A_0^*) = \{u \in L^2(\mathcal{O}, \mathbb{C}) : Du, D^3u \text{ exist and belong to } L_{\text{loc}}^2(\mathcal{O}), \\ (-ia_3D^3u + ia_1Du) \in L^2(\mathcal{O})\}. \quad (6.1.2)$$

Moreover, for every $u \in D(A_0^*)$ we have

$$A_0^*u = -ia_3D^3u + ia_1Du.$$

Proof of Lemma 6.3. "⊂". Assume that $u \in D(A_0^*)$. Then, by definition of the adjoint operator, there exists $z \in L^2(\mathcal{O}, \mathbb{C})$ such that

$$\langle u, A_0\phi \rangle = \langle z, \phi \rangle \quad \text{for every } \phi \in C_0^\infty(\mathcal{O}).$$

Expanding A_0 , we obtain

$$\int_{\mathcal{O}} u(x) \overline{(-ia_3\phi^{(3)}(x) + ia_1\phi'(x))} dx = \int_{\mathcal{O}} z(x) \overline{\phi(x)} dx.$$

This means that u satisfies

$$Lu = z \quad \text{in the weak sense,}$$

where the differential operator L is given by

$$L = -ia_3D^3 + ia_1D.$$

Since $u, z \in L^2(\mathcal{O}, \mathbb{C})$, we also have

$$u, z \in L_{\text{loc}}^2(\mathcal{O}, \mathbb{C}).$$

Moreover, the identity above shows that

$$Lu = z$$

in the weak sense, where

$$L = -ia_3D^3 + ia_1D.$$

Since $a_3 \neq 0$, the principal coefficient of L does not vanish, and hence L is an elliptic differential operator of order 3. Therefore, by Theorem 2.95; see also [36, Theorem 8.12], we infer that

$$u \in H_{\text{loc}}^3(\mathcal{O}).$$

By the definition of $H_{\text{loc}}^3(\mathcal{O})$, the weak derivatives $D^k u$, $k = 1, 2, 3$, exist and belong to $L_{\text{loc}}^2(\mathcal{O})$. In particular,

$$Du, D^3 u \in L_{\text{loc}}^2(\mathcal{O}).$$

Finally, returning to the weak identity, we obtain

$$z = -ia_3 D^3 u + ia_1 Du \in L^2(\mathcal{O}, \mathbb{C}).$$

This completes the proof of the inclusion " $\subset$ ". □

Proof of Lemma 6.3. " $\supset$ ". Assume that

$$u \in L^2(\mathcal{O}, \mathbb{C}), \quad Du, D^3 u \in L_{\text{loc}}^2(\mathcal{O}, \mathbb{C}),$$

and

$$-ia_3 D^3 u + ia_1 Du \in L^2(\mathcal{O}, \mathbb{C}).$$

Define

$$z := -ia_3 D^3 u + ia_1 Du \in L^2(\mathcal{O}, \mathbb{C}).$$

Let $\phi \in C_0^\infty(\mathcal{O}) = D(A_0)$. Since Du and $D^3 u$ exist in the weak sense, by Definition 2.1 we have

$$\int_{\mathcal{O}} u(x) \overline{\phi'(x)} dx = - \int_{\mathcal{O}} Du(x) \overline{\phi(x)} dx$$

and

$$\int_{\mathcal{O}} u(x) \overline{\phi^{(3)}(x)} dx = - \int_{\mathcal{O}} D^3 u(x) \overline{\phi(x)} dx.$$

Therefore,

$$\begin{aligned} \langle u, A_0 \phi \rangle &= \int_{\mathcal{O}} u(x) \overline{(-ia_3 \phi^{(3)}(x) + ia_1 \phi'(x))} dx \\ &= \int_{\mathcal{O}} (-ia_3 D^3 u(x) + ia_1 Du(x)) \overline{\phi(x)} dx \\ &= \int_{\mathcal{O}} z(x) \overline{\phi(x)} dx = \langle z, \phi \rangle. \end{aligned}$$

Since this holds for every $\phi \in D(A_0)$, we conclude from the definition of the adjoint that

$$u \in D(A_0^*) \quad \text{and} \quad A_0^* u = z = -ia_3 D^3 u + ia_1 Du.$$

This completes the proof of the inclusion " $\supset$ ". □

Remark 6.4. The main point in the proof of Lemma 6.3 is to establish that, if $u \in D(A_0^*)$, then the weak derivatives Du and $D^3 u$ exist. This follows from the elliptic regularity

argument used in the proof, which yields

$$u \in H^3(\mathcal{O}).$$

Consequently,

$$Du, D^3u \in L^2_{\text{loc}}(\mathcal{O}).$$

In general, these derivatives need not belong to $L^2(\mathcal{O})$.

Remark 6.5. Lemma 6.3 above is formulated in terms of weak derivatives. In one dimension, weak differentiability has an equivalent classical interpretation. More precisely, by [38, Theorem 7.18], the following statements are equivalent:

- (i) $u \in L^2_{\text{loc}}(\mathcal{O})$ and the weak derivative Du exists and belongs to $L^2_{\text{loc}}(\mathcal{O})$.
- (ii) u is locally absolutely continuous on $\mathcal{O}$, hence differentiable almost everywhere, and its classical derivative u' belongs to $L^2_{\text{loc}}(\mathcal{O})$.

Using this characterization, the domain in (6.1.2) can be reformulated as follows:

$$\begin{aligned} D(A_0^*) = \{u \in L^2(\mathcal{O}, \mathbb{C}) : u \in C^2(\mathcal{O}), \quad u'' \text{ is locally absolutely continuous,} \\ u^{(3)} \in L^2_{\text{loc}}(\mathcal{O}), \quad -ia_3u^{(3)} + ia_1u' \in L^2(\mathcal{O})\}. \end{aligned} \quad (6.1.3)$$

Remark 6.6. Lemma 6.3 applies to the cases $\mathcal{O} = (0, \infty)$, $\mathcal{O} = (-\infty, 0)$ and $\mathcal{O} = \mathbb{R} \setminus \{0\}$. In particular

- (i) If $\mathcal{O} = (0, \infty)$, then $u \in D(A_0^*)$ if and only if the following conditions are satisfied:

- Weak derivatives D^3u and Du exist.
- $D^3u \in L^2_{\text{loc}}((0, \infty))$ and $Du \in L^2_{\text{loc}}((0, \infty))$.
- The expression $(-ia_3D^3u + ia_1Du) \in L^2((0, \infty))$.

- (ii) $\mathcal{O} = (-\infty, 0)$, then $u \in D(A_0^*)$ if and only if the following conditions are satisfied:

- Weak derivatives D^3u and Du exist.
- $D^3u \in L^2_{\text{loc}}((-\infty, 0))$ and $Du \in L^2_{\text{loc}}((-\infty, 0))$.
- The expression $(-ia_3D^3u + ia_1Du) \in L^2((-\infty, 0))$.

- (iii) if $\mathcal{O} = (-\infty, 0) \cup (0, \infty)$, then $u \in D(A_0^*)$ if and only if the following conditions are satisfied:

- D^3u and Du exist.
- Both belong to $L^2_{loc}((-\infty, 0))$ and $L^2_{loc}((0, \infty))$.
- The combination $(-ia_3D^3u + ia_1Du) \in L^2((-\infty, 0) \cup (0, \infty))$.

These conditions hold. Therefore, Lemma 6.3 is proved in all cases.

Proposition 6.7. *Assume that A_0 is defined by (6.1.1) on $L^2(\mathbb{R}, \mathbb{C})$. Then*

$$D(A_0^*) = \{u \in L^2(\mathbb{R}, \mathbb{C}) : Du, D^3u \text{ exist and belong to } L^2_{loc}(\mathbb{R}), \\ (-ia_3D^3u + ia_1Du) \in L^2(\mathbb{R}, \mathbb{C})\}.$$

Proof of Proposition 6.7. This is a direct consequence of Lemma 6.3 applied to $\mathcal{O} = \mathbb{R}$. $\square$

Next we will formulate and prove a results which generalizes Propositions 5.32 and 5.33 to the present setting.

Proposition 6.8. *Assume that A_1 is the operator defined by*

$$D(A_1) := C_0^\infty(\mathbb{R} \setminus \{0\}, \mathbb{C}), \\ A_1u := -iD^3u + iDu, \quad u \in D(A_1).$$

Then A_1 is not closed.

Moreover, a function $u \in D(\overline{A_1})$ if and only if the following conditions hold:

(i) $u \in L^2(\mathbb{R}, \mathbb{C});$

(ii) $u \in H^3_{loc}(\mathbb{R} \setminus \{0\}, \mathbb{C}), D^2u$ is locally absolutely continuous, and

$$\int_{\mathbb{R}} |D^3u(t)|^2 dt < \infty, \quad \int_{\mathbb{R}} |Du(t)|^2 dt < \infty;$$

(iii) $u(0+) = u(0-) = 0, \quad u'(0+) = u'(0-) = 0, \quad u''(0+) = u''(0-) = 0.$

In addition,

$$\overline{A_1}u = -iD^3u + iDu, \quad \text{for } u \in D(\overline{A_1}).$$

Proof of Proposition 6.8. Part 1. We prove that A_1 is not closed. As in the proof of Proposition 5.32, it is sufficient to show that the graph $\Gamma(A_1)$ is not closed in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Consider the function

$$u(x) = \begin{cases} e^{-x^{-2}}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

It is well known, see [37, Chapter 6], that $u \in C^\infty(\mathbb{R})$ and

$$u(0) = u'(0) = u''(0) = 0.$$

Moreover, by the same argument as in the proof of Proposition 5.32, one has

$$u \in H^3(\mathbb{R}).$$

In particular,

$$u, Du, D^3u \in L^2(\mathbb{R}).$$

On the other hand, since $u(x) \neq 0$ for all $x > 0$, the function u is not compactly supported, and therefore

$$u \notin D(A_1) = C_0^\infty(\mathbb{R} \setminus \{0\}, \mathbb{C}).$$

Using the same translation-and-cutoff approximation method as in the proof of Lemma 3.32, but now in the Sobolev space $H^3(\mathbb{R})$, we can find a sequence $(u_n) \subset C_0^\infty(\mathbb{R} \setminus \{0\}, \mathbb{C})$ such that

$$u_n \rightarrow u \quad \text{in } H^3(\mathbb{R}).$$

Hence

$$u_n \rightarrow u, \quad Du_n \rightarrow Du, \quad D^3u_n \rightarrow D^3u \quad \text{in } L^2(\mathbb{R}).$$

By definition of A_1 ,

$$A_1u_n = -iD^3u_n + iDu_n,$$

and therefore

$$A_1u_n \rightarrow -iD^3u + iDu \quad \text{in } L^2(\mathbb{R}).$$

Thus

$$(u_n, A_1u_n) \rightarrow (u, -iD^3u + iDu) \quad \text{in } L^2(\mathbb{R}) \times L^2(\mathbb{R}).$$

However, since $u \notin D(A_1)$, we have

$$(u, -iD^3u + iDu) \notin \Gamma(A_1).$$

Hence $\Gamma(A_1)$ is not closed, and therefore A_1 is not closed. □

Proof of Proposition 6.8. Part 2. Let us choose and fix $u \in D(\overline{A_1})$. We prove that u satisfies conditions (i)–(iii).

By definition of the closure of an operator, see Remark 2.86, there exist a sequence $(u_n) \subset D(A_1)$ and a function $z \in L^2(\mathbb{R}, \mathbb{C})$ such that

$$u_n \rightarrow u \quad \text{in } L^2(\mathbb{R}), \quad (6.1.4)$$

$$A_1 u_n = -iD^3 u_n + iD u_n \rightarrow z \quad \text{in } L^2(\mathbb{R}). \quad (6.1.5)$$

In particular,

$$D^3 u_n - D u_n \rightarrow iz \quad \text{in } L^2(\mathbb{R}).$$

We now use the same method as in the proof of Proposition 5.33, replacing the operator $-iD^3$ by $-iD^3 + iD$. Since each $u_n \in C_0^\infty(\mathbb{R} \setminus \{0\})$, we have

$$u_n(0) = u_n'(0) = u_n''(0) = 0.$$

Hence, for every $t \in \mathbb{R}$,

$$u_n(t) = \int_0^t u_n^{(3)}(\sigma) \frac{(t-\sigma)^2}{2} d\sigma.$$

As in the proof of Proposition 5.33, this identity, together with the Cauchy–Schwarz inequality, implies that (u_n) is uniformly Cauchy on every compact interval whenever $(D^3 u_n)$ is Cauchy in $L^2(\mathbb{R})$. Moreover, arguing exactly as before with u_n' and u_n'' , one also obtains local uniform convergence of (u_n') and (u_n'') on compact intervals.

On the other hand, from (6.1.5) and the same argument used in Proposition 5.33, we infer that $(D u_n)$ and $(D^3 u_n)$ are Cauchy sequences in $L^2(\mathbb{R})$. Hence there exist functions $Du, D^3 u \in L^2(\mathbb{R})$ such that

$$D u_n \rightarrow Du, \quad D^3 u_n \rightarrow D^3 u \quad \text{in } L^2(\mathbb{R}).$$

Therefore,

$$u_n \rightarrow u, \quad u_n' \rightarrow u', \quad u_n'' \rightarrow u''$$

uniformly on compact subsets of each half-line. In particular, the one-sided limits at 0 exist and satisfy

$$u(0+) = u(0-) = 0, \quad u'(0+) = u'(0-) = 0, \quad u''(0+) = u''(0-) = 0.$$

Next we identify the third weak derivative of u . Let $\phi \in C_0^\infty(\mathbb{R} \setminus \{0\})$. Since $u_n \in C_0^\infty(\mathbb{R} \setminus \{0\})$, integration by parts gives

$$\int_{\mathbb{R}} u_n(x) \phi^{(3)}(x) dx = - \int_{\mathbb{R}} u_n^{(3)}(x) \phi(x) dx.$$

Passing to the limit, using (6.1.4) and the convergence $D^3u_n \rightarrow D^3u$ in $L^2(\mathbb{R})$, we obtain

$$\int_{\mathbb{R}} u(x) \phi^{(3)}(x) dx = - \int_{\mathbb{R}} D^3u(x) \phi(x) dx.$$

Hence D^3u is the third weak derivative of u and belongs to $L^2(\mathbb{R})$. Since also $Du \in L^2(\mathbb{R})$, we conclude that

$$u \in H_{\text{loc}}^3(\mathbb{R} \setminus \{0\}, \mathbb{C}).$$

Finally, from (6.1.5) and the convergences of Du_n and D^3u_n , we obtain

$$z = -iD^3u + iDu.$$

Therefore,

$$\overline{A_1}u = -iD^3u + iDu.$$

This proves Part 2. □

Proof of Proposition 6.8. Part 3. We prove the converse implication.

Assume that $u \in L^2(\mathbb{R}, \mathbb{C})$ satisfies:

- (i) $u \in L^2(\mathbb{R}, \mathbb{C})$;
- (ii) $u \in H_{\text{loc}}^3(\mathbb{R} \setminus \{0\}, \mathbb{C})$ and

$$Du \in L^2(\mathbb{R}), \quad D^3u \in L^2(\mathbb{R});$$

- (iii) $u(0+) = u(0-) = 0, \quad u'(0+) = u'(0-) = 0, \quad u''(0+) = u''(0-) = 0.$

We show that $u \in D(\overline{A_1})$.

Let

$$u_- := u|_{(-\infty, 0)}, \quad u_+ := u|_{(0, \infty)}.$$

By assumption,

$$u_- \in H^3((-\infty, 0)), \quad u_+ \in H^3((0, \infty)),$$

and the boundary conditions at 0 imply

$$u_- \in H_0^3((-\infty, 0)), \quad u_+ \in H_0^3((0, \infty)),$$

where H_0^3 denotes the closure of C_0^∞ in H^3 .

Hence, by the definition of H_0^3 , there exist sequences

$$f_n \in C_0^\infty((-\infty, 0)), \quad g_n \in C_0^\infty((0, \infty))$$

such that

$$f_n \rightarrow u_- \quad \text{in } H^3((-\infty, 0)), \quad g_n \rightarrow u_+ \quad \text{in } H^3((0, \infty)).$$

Define

$$u_n(x) = \begin{cases} f_n(x), & x < 0, \\ g_n(x), & x > 0, \\ 0, & x = 0. \end{cases}$$

Then

$$u_n \in C_0^\infty(\mathbb{R} \setminus \{0\}, \mathbb{C}) = D(A_1).$$

Moreover, by construction,

$$u_n \rightarrow u \quad \text{in } L^2(\mathbb{R}),$$

and also

$$Du_n \rightarrow Du, \quad D^3u_n \rightarrow D^3u \quad \text{in } L^2(\mathbb{R}).$$

Therefore,

$$A_1u_n = -iD^3u_n + iDu_n \rightarrow -iD^3u + iDu \quad \text{in } L^2(\mathbb{R}).$$

Thus

$$(u_n, A_1u_n) \rightarrow (u, -iD^3u + iDu) \quad \text{in } L^2(\mathbb{R}) \times L^2(\mathbb{R}).$$

By Remark 2.86, this implies

$$u \in D(\overline{A_1}), \quad \overline{A_1}u = -iD^3u + iDu.$$

This completes the proof of Part 3.

The proof of Proposition 6.8 is complete. $\square$

6.1.1 Deficiency indices of the third-order linear inhomogeneous differential operators on real line

We consider the following differential operator

$$A_0 = a_3(iD)^3 + a_1(iD), \quad \text{with } a_3 \neq 0. \quad (6.1.6)$$

Note that the above does not contain differentiation of the second order.

Without loss of generality, we will assume that

$$a_3 = 1 \tag{6.1.7}$$

and will denote

$$a_1 = -3p \tag{6.1.8}$$

Thus, in view of (6.1.7) and (6.1.8), the operator A_0 takes the form

$$A_0 = (iD)^3 - 3p(iD) = -iD^3 - 3piD.$$

The characteristic polynomial of the operator A_0 is

$$w(z) = a_3(iz)^3 + a_1(iz), \quad z \in \mathbb{C}.$$

The following result is well known.

Lemma 6.9. *If $z \in \mathbb{C}$, then*

$$A_0(e^{zx}) = w(z)e^{zx}, \quad x \in \mathbb{R}.$$

In this section, we will calculate the deficiency indices $n_+(A_0)$ and $n_-(A_0)$ of the operator A_0 defined by (6.1.6). To be precise we will calculate the following numbers

$$\begin{aligned} n_+(A_0) &= \dim \ker(iI - A_0^*) \\ n_-(A_0) &= \dim \ker(iI + A_0^*), \end{aligned}$$

see Definition deficiency sub-spaces [34, p.138].

We begin with the following Lemma.

Lemma 6.10. *Assume that A_1 is the operator defined by (6.1.6). Then*

$$\ker(A_0^* + iI) = \{0\}.$$

Proof of Lemma 6.10. Let $u \in \ker(A_0^* + iI)$. We will show that $u = 0$. Then by Lemma 6.10

$$\begin{aligned} (A_0^* + iI)u &= 0. \\ -ia_3D^3u + ia_1Du + iu &= 0 \\ a_3D^3u - a_1Du - u &= 0 \end{aligned} \tag{6.1.9}$$

By (6.1.7) and (6.1.8), the last equation becomes

$$D^3u + 3pDu - u = 0 \quad (6.1.10)$$

The characteristic polynomial of equation (6.1.10) is

$$w(z) = z^3 + 3pz - 1.$$

We consider the equation

$$w(z) = 0. \quad (6.1.11)$$

Now we will investigate how many roots of equation (6.1.11) have negative or positive real part. Define

$$\Delta = (-1)^2 + 4p^3$$

By [31, Theorem 3 in VII.4.2], there are three cases

(C1) $\Delta < 0$. Then the equation (6.1.11) has three distinct real roots

$$\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}.$$

(C2) $\Delta = 0$. Then the equation (6.1.11) has three real roots, one of multiplicity 2

$$\lambda_1, \lambda_2 = \lambda_1, \lambda_3 \in \mathbb{R}.$$

(C3) $\Delta > 0$. Then the equation (6.1.11) has one real root and two non-real conjugate roots

$$\lambda_1 \in \mathbb{R}, \quad \lambda_2 \in \mathbb{C} \setminus \mathbb{R}, \quad \lambda_3 = \overline{\lambda_2}.$$

In cases (C1) and (C3), the general solution of equation (6.1.9) has the form

$$u(x) = \sum_{j=1}^3 \alpha_j e^{\lambda_j x} = \alpha_1 e^{\lambda_1 x} + \alpha_2 e^{\lambda_2 x} + \alpha_3 e^{\lambda_3 x}, \quad x \in \mathbb{R}. \quad (6.1.12)$$

Because,

In case (C3), write

$$\lambda_2 = a + ib, \quad \lambda_3 = \overline{\lambda_2} = a - ib,$$

where

$$a = \operatorname{Re}(\lambda_2), \quad b = \operatorname{Im}(\lambda_2).$$

Then

$$e^{\lambda_2 x} = e^{ax} (\cos(bx) + i \sin(bx)),$$

and

$$e^{\lambda_3 x} = e^{ax} (\cos(bx) - i \sin(bx)).$$

Equivalently,

$$e^{\lambda_2 x} = e^{\operatorname{Re}(\lambda_2)x} (\cos(\operatorname{Im}(\lambda_2)x) + i \sin(\operatorname{Im}(\lambda_2)x)),$$

and

$$e^{\lambda_3 x} = e^{\operatorname{Re}(\lambda_2)x} (\cos(\operatorname{Im}(\lambda_2)x) - i \sin(\operatorname{Im}(\lambda_2)x)).$$

Thus, in case (C3), we may write

$$\begin{aligned} u(x) &= \alpha_1 e^{\lambda_1 x} + \alpha_2 e^{\lambda_2 x} + \alpha_3 e^{\lambda_3 x} \\ &= \alpha_1 e^{\lambda_1 x} + e^{ax} [(\alpha_2 + \alpha_3) \cos(bx) + i(\alpha_2 - \alpha_3) \sin(bx)], \quad x \in \mathbb{R}. \end{aligned}$$

In case (C2), the characteristic polynomial has a double root $\lambda_1 = \lambda_2$ and a simple root λ_3 . Therefore, the general solution of equation (6.1.9), is

$$u(x) = (\alpha_1 + \alpha_2 x) e^{\lambda_1 x} + \alpha_3 e^{\lambda_3 x}, \quad x \in \mathbb{R},$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$. □

Lemma 6.10 is a consequence of the following result.

Lemma 6.11. *In each case, if u is solution of equation (6.1.9) such that $u \in L^2(\mathbb{R})$ then $u = 0$.*

Before we embark on the proof of Lemma 6.11 we state the following simple Exercise.

Exercise 6.12. If $\lambda \in \mathbb{C}$, then

$$\int_{-\infty}^{\infty} |e^{\lambda x}|^2 dx = \infty.$$

Solution of Exercise 6.12. Let us observe that

$$\int_{-\infty}^{\infty} |e^{\lambda x}|^2 dx = \int_{-\infty}^{\infty} e^{2\operatorname{Re}\lambda x} dx$$

If $\operatorname{Re}\lambda = 0$, $\operatorname{Re}\lambda > 0$ or $\operatorname{Re}\lambda < 0$, then we have respectively

$$\int_{-\infty}^{\infty} e^{0x} dx = \int_{-\infty}^{\infty} 1 dx = \infty,$$

$$\int_{-\infty}^{\infty} e^{2\operatorname{Re}\lambda x} dx \geq \int_0^{\infty} e^{2\operatorname{Re}\lambda x} dx = \infty,$$

$$\int_{-\infty}^{\infty} e^{2\operatorname{Re}\lambda x} dx \geq \int_{-\infty}^0 e^{2\operatorname{Re}\lambda x} dx = \infty.$$

The solution is complete. $\square$

Proof of Lemma 6.11. Case (C1). Let us consider function u of the form (6.1.12). By use Exercise 6.12

$$\int_{-\infty}^{\infty} |e^{\lambda_j x}|^2 dx = \infty, \quad j = 1, 2, 3. \quad (6.1.13)$$

So every functions $e^{\lambda_j x}$ does not belong to the space $L^2(\mathbb{R})$.

Our aim is to prove that $\alpha_1 = \alpha_2 = \alpha_3 = 0$. If only one of the numbers $\alpha_1, \alpha_2, \alpha_3$ are no zero, e.g. $\alpha_1 \neq 0$ and $\alpha_2 = \alpha_3 = 0$, then

$$u(x) = \alpha_1 e^{\lambda_1 x}, \quad x \in \mathbb{R},$$

and so by (6.1.13), $|u|_{L^2} = \infty$. This is contradiction.

Now, assume that $\alpha_1 \neq 0$, $\alpha_2 \neq 0$ and $\alpha_3 \neq 0$. Since $\lambda_1, \lambda_2, \lambda_3$ are distinct, we can order them, i.e.

$$\lambda_1 < \lambda_2 < \lambda_3.$$

Suppose that $\lambda_3 > 0$. Then

$$u(x) = \alpha_3 e^{\lambda_3 x} \left(1 + \frac{\alpha_2}{\alpha_3} e^{(\lambda_2 - \lambda_3)x} + \frac{\alpha_1}{\alpha_3} e^{(\lambda_1 - \lambda_3)x} \right)$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x)|^2 dx &\geq \int_0^{\infty} |u(x)|^2 dx \\ &= |\alpha_3|^2 \int_0^{\infty} |e^{2\lambda_3 x}| \left| 1 + \frac{\alpha_2}{\alpha_3} e^{(\lambda_2 - \lambda_3)x} + \frac{\alpha_1}{\alpha_3} e^{(\lambda_1 - \lambda_3)x} \right|^2 dx. \end{aligned}$$

Since, $\lambda_2 - \lambda_3 < 0$, and $\lambda_1 - \lambda_3 < 0$. So,

$$e^{(\lambda_2 - \lambda_3)x} \rightarrow 0 \quad \text{when } x \rightarrow \infty,$$

and

$$e^{(\lambda_1 - \lambda_3)x} \rightarrow 0 \quad \text{when } x \rightarrow \infty.$$

Hence,

$$\exists R_0 > 0 : \left| 1 + \frac{\alpha_2}{\alpha_3} e^{(\lambda_2 - \lambda_3)x} + \frac{\alpha_1}{\alpha_3} e^{(\lambda_1 - \lambda_3)x} \right|^2 > \frac{1}{4}, \quad \forall x \in [R_0, \infty).$$

Therefore,

$$|u|_{L^2}^2 \geq |\alpha_3|^2 \int_{R_0}^{\infty} e^{2\lambda_3 x} \frac{1}{4} dx = \frac{|\alpha_3|^2}{4} \int_{R_0}^{\infty} e^{2\lambda_3 x} dx = \infty.$$

The other cases i.e. precisely two numbers $\alpha_1, \alpha_2, \alpha_3$ are $\neq 0$. follow similar route.

In the case (C2), we argue as in the case (C1). In the case (C3), u is of the form (6.1.12) and $\lambda_1 \in \mathbb{R}$, and $\lambda_3 = \overline{\lambda_2} \in \mathbb{C} \setminus \mathbb{R}$. Arguing as in case (C1), but replacing λ_2, λ_3 by $\operatorname{Re} \lambda_2 = \operatorname{Re} \lambda_3$, we can prove that. $\square$

The following result is a companion result to Lemma 6.10.

Lemma 6.13. *Assume that A_1 is the operator defined by (6.1.6). Then*

$$\ker(iI - A_0^*) = \{0\}.$$

Proof. The proof is as of Lemma 6.10. $\square$

From Lemmata 6.11 and 6.13, we deduce the following immediate consequence A .

Corollary 6.14. *Assume that A_0 is the third-order operator defined by formula (6.1.6). Then*

$$n_-(A_0) = 0$$

$$n_+(A_0) = 0$$

The following main result of this section is a consequence of Corollary 6.14.

Theorem 6.15. *The operator A_0 on $L^2(\mathbb{R}, \mathbb{C})$ defined by equation (6.1.6) has one self-adjoint extension. Moreover, the unique self-adjoint extension A of A_0 is as follows*

$$D(A) = H^3(\mathbb{R}), \quad Au = a_3(iD)^3u + a_1(iD)u.$$

Remark 6.16. In this case one can show that the closure A_0 is equal to A .

6.2 Third-order linear inhomogeneous differential operators on positive half line

6.2.1 Symmetry and the adjoint

In this section we consider the following Hilbert space

$$H = L^2((0, \infty); \mathbb{C}).$$

We define the operator of A_1 as follows

$$\begin{aligned} D(A_1) &= C_0^\infty((0, \infty); \mathbb{C}) \\ A_1 u &= a_3(iD)^3 u + a_1 iD u = -ia_3 u''' + ia_1 u', \end{aligned} \tag{6.2.1}$$

Recall that we assume that $a_3 \neq 0$.

Lemma 6.17. *The operator A_1 defined by (6.2.1) is symmetric.*

Remark 6.18. The proof of symmetry of the operator A_1 is analogous to the proof of symmetry of the operator A_0 , see Lemma 6.2. Moreover, the symmetry of A_1 follows directly from the symmetry of A_0 .

Proof of Lemma 6.17. Recall that

$$\begin{aligned} D(A_0) &= C_0^\infty(\mathbb{R}; \mathbb{C}), \\ D(A_1) &= C_0^\infty((0, \infty); \mathbb{C}). \end{aligned}$$

Hence

$$D(A_1) \subset D(A_0).$$

Moreover, for $u \in D(A_1)$ we have

$$\begin{aligned} A_0 u &= a_3(iD)^3 u + a_1(iD)u, \\ A_1 u &= a_3(iD)^3 u + a_1(iD)u, \end{aligned}$$

and therefore

$$A_1 u = A_0 u \quad \text{for all } u \in D(A_1),$$

which means that $A_1 \subset A_0$.

Since A_0 is symmetric, it follows that A_1 is symmetric as well. Indeed, for all $u, v \in D(A_1)$ we have $u, v \in D(A_0)$ and

$$\langle A_1 u, v \rangle = \langle A_0 u, v \rangle = \langle u, A_0 v \rangle = \langle u, A_1 v \rangle.$$

Thus, A_1 is symmetric. □

To find $D(A_1^*)$ using the formula (6.1.3) from the Section 6.1 with the set $\mathcal{O} = (0, \infty)$ and so we get the following three equivalent results.

Lemma 6.19. *Suppose that A_1 is the linear operator defined by (6.2.1). Then,*

$$\begin{aligned} D(A_1^*) := & \{u \in L^2((0, \infty), \mathbb{C}) : u \text{ is of } C^2 \text{ class on } [0, \infty), \\ & u'' \text{ is absolutely continuous and hence differentiable} \\ & \text{almost everywhere on } [0, \infty) \\ & \text{with respect to the Lebesgue measure,} \\ & u''' \text{ is Lebesgue measurable and belongs to } L^2_{loc}([0, \infty), \mathbb{C}) : \\ & \int_0^\infty |a_3 u'''(x) - a_1 u'(x)|^2 dx < \infty\} \end{aligned}$$

Moreover, if $u \in D(A_1^*)$, then

$$A_1^* u = i (a_3 u''' - a_1 u').$$

Lemma 6.20. *Suppose that A_1 is the linear operator defined by (6.2.1). Then,*

$$\begin{aligned} D(A_1^*) := & \{u \in L^2((0, \infty), \mathbb{C}) \cap H^3_{loc}([0, \infty); \mathbb{C}) : \\ & \int_0^\infty |a_3 D^3 u(x) - a_1 Du(x)|^2 dx < \infty.\} \end{aligned}$$

Moreover, if $u \in D(A_1^*)$, then

$$A_1^* u = i (a_3 D^3 u - a_1 Du)$$

Lemma 6.21. *Suppose that A_1 is the linear operator defined by (6.2.1). Then,*

$$\begin{aligned} D(A_1^*) := & \{u \in L^2(0, \infty; \mathbb{C}) : \text{weak derivatives } D^3 u \text{ and } Du \text{ exists and} \\ & \in L^2_{loc}[0, \infty; \mathbb{C}) \text{ and } \{-ia_3 D^3 u + ia_1 Du \in L^2(0, \infty; \mathbb{C})\}\}. \end{aligned}$$

6.2.2 Deficiency indices of the third-order linear inhomogeneous differential operators on the positive real line

In this section we calculate the deficiency indices of the operator A_1 defined by (6.2.1). Using Definition 2.59, we have

$$n_+(A_1) = \dim \ker(iI - A_1^*) \quad n_-(A_1) = \dim \ker(iI + A_1^*)$$

The main result of this section is the following one.

Theorem 6.22. *Assume that A_1 is the operator defined by formula (6.2.1). Then, if $a_3 > 0$, then*

$$n_-(A_1) = 2,$$

$$n_+(A_1) = 1,$$

if $a_3 < 0$, then

$$n_-(A_1) = 1,$$

$$n_+(A_1) = 2,$$

In particular, the deficiency indices depend only on the sign of a_3 and do not depend on a_1 .

Remark 6.23. The above result is a generalization of Corollary 5.17 in Subsection 5.2, which we consider as the special case $a_3 = 1$ and $a_1 = 0$.

Let us rewrite A_1 in formula (6.2.1) as follows

$$A_1 = a_3 \left((iD)^3 + \frac{a_1}{a_3} iD \right).$$

Assume A_0 defined as follows

$$A_0 = (iD)^3 - 3piD.$$

Then

$$A_1 = a_3 A_0,$$

where $-3p = \frac{a_1}{a_3}$. If $A_1 = a_3 A_0$, then

$$A_1^* = a_3 A_0^*.$$

If choose and fix the constant $\alpha > 0$ and by [34, Theorem X.1 p.136], then

$$n_+(A_1) = \dim \ker(iI - A_1^*) = \dim \ker(\alpha iI - A_1^*).$$

We have two cases

(i) **Case 1:** If $\alpha = a_3 > 0$, then

$$n_+(A_1) = \dim \ker(a_3(iI - A_1^*)) = n_+(A_0).$$

Similarly,

$$n_-(A_1) = n_-(A_0).$$

(ii) **Case 2:** If $\alpha = a_3 < 0$,

$$\begin{aligned} n_+(A_1) &= \dim \ker(iI - a_3 A_1^*) = \dim \ker\left(a_3 \left(\frac{i}{a_3} I - A_0^*\right)\right) \\ &= \dim \ker\left(\frac{i}{a_3} I - A_0^*\right), \quad \operatorname{Im}\left(\frac{i}{a_3}\right) < 0 \\ &= n_-(A_0). \end{aligned}$$

Similarly,

$$\begin{aligned} n_-(A_1) &= \dim \ker(-iI - a_3 A_1^*) = \dim \ker\left(a_3 \left(\frac{-i}{a_3} I - A_0^*\right)\right) \\ &= \dim \ker\left(\frac{-i}{a_3} I - A_0^*\right), \quad \operatorname{Im}\left(\frac{-i}{a_3}\right) > 0 \\ &= n_+(A_0). \end{aligned}$$

Summarizing we proved.

Proposition 6.24. *If $\alpha = a_3 > 0$ in formula (6.2.1), then*

$$n_+(A_1) = n_+(A_0), \quad n_-(A_1) = n_-(A_0).$$

Moreover, if $\alpha = a_3 < 0$, then

$$n_+(A_1) = n_-(A_0), \quad n_-(A_1) = n_+(A_0).$$

The following result is a consequences of Theorem 6.22. In fact, we investigated the previous result in order to prove or disprove the next corollary.

Corollary 6.25. *Assume that A_1 is an operator on $L^2((0, \infty), \mathbb{C})$ defined by equation (6.2.1) has no self-adjoint extensions.*

Proof of Corollary 6.25. Since we have $n_+(A_1) = 1$ and $n_-(A_1) = 2$ from the Theorem 6.22, it follows that

$$n_+ \neq n_-.$$

Therefore, applying the conclusion from Theorem 2.60, we can assert that the operator A_1 has no self-adjoint extensions.

Thus, we conclude that the operator A_1 defined by equation (6.2.1) has no self-adjoint extensions. This completes the proof. $\square$

In the remaining of this section we will occupy ourselves proving the theorem above. For this aim we will calculate the deficiency indices $n_-(A_1)$ and $n_+(A_1)$.

In order to calculate $n_-(A_1)$, we need to find the space $\ker(iI + A_1^*)$. We begin with the following Lemma.

Lemma 6.26. *Assume that A_1 is an operator on $L^2((0, \infty), \mathbb{C})$ defined by equation (6.2.1), then*

$$\dim \ker(A_1^* + iI) = 2.$$

Proof of Lemma 6.26. Part I. Let $u \in \ker(A_1^* + iI) \setminus \{0\}$, i.e.

$$A_1^*u + iu = 0.$$

We infer that

$$a_3 D^3 u - a_1 Du - u = 0.$$

In what follows we assume that

$$a_3 = 1$$

and we denote

$$-a_1 = 3p \in \mathbb{R}.$$

In this case we have

$$\begin{aligned} D(A_1) &= C_0^\infty((0, \infty); \mathbb{C}) \\ A_1 u &= -iD^3 u - 3piDu. \end{aligned}$$

Hence the last equation becomes

$$D^3 u + 3pDu - u = 0 \text{ on } (0, \infty) \tag{6.2.2}$$

The characteristic polynomial of equation (6.2.2) is

$$w(\lambda) = \lambda^3 + 3p\lambda - 1 \quad (6.2.3)$$

whose characteristic equation is

$$\lambda^3 + 3p\lambda - 1 = 0 \quad (6.2.4)$$

is equal to¹

This equation has three roots

$$\lambda_1, \lambda_2, \lambda_3$$

As in section 6.1.1, we have to consider the following three cases.

(C1) $\Delta < 0$, equation (6.2.4) has three distinct real roots

$$\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}.$$

(C2) $\Delta = 0$, equation (6.2.4) has three real roots, one of multiplicity 2

$$\lambda_1, \lambda_2 = \lambda_1, \lambda_3 \in \mathbb{R}.$$

(C3) $\Delta > 0$, equation (6.2.4) has one real root and two non-real conjugate roots

$$\lambda_1 \in \mathbb{R}, \quad \lambda_2 \in \mathbb{C} \setminus \mathbb{R}, \quad \lambda_3 = \overline{\lambda_2}.$$

In case (C1), the general solution of equation (6.2.2) is given by

$$u(x) = \sum_{j=1}^3 \alpha_j e^{\lambda_j x} = \alpha_1 e^{\lambda_1 x} + \alpha_2 e^{\lambda_2 x} + \alpha_3 e^{\lambda_3 x}, \quad x \in \mathbb{R}_+. \quad (6.2.5)$$

In case (C3), we can also write

$$\begin{aligned} u(x) &= \alpha_1 e^{\lambda_1 x} + \alpha_2 e^{\lambda_2 x} + \alpha_3 e^{\overline{\lambda_2} x} \\ &= \alpha_1 e^{\lambda_1 x} + [(\alpha_2 + \alpha_3) \cos(\operatorname{Re} \lambda_2 x) + i(\alpha_2 - \alpha_3) \sin(\operatorname{Im} \lambda_2 x)], \quad x \in \mathbb{R}_+. \end{aligned}$$

¹Note that we have replaced p by $3p$, hence in our formula we do not have factor 27.

In case (C2), the general solution u of equation (6.2.2) has the following form:

$$u(x) = \sum_{j=1}^3 \alpha_j x^{\delta_{2j}} e^{\lambda_j x} = \alpha_1 e^{\lambda_1 x} + \alpha_2 x e^{\lambda_2 x} + \alpha_3 e^{\lambda_3 x}, \quad x \in \mathbb{R}_+.$$

We will need the following result proven earlier in different contexts.

Lemma 6.27. *Function u of the form (6.2.5) belongs to the space $L^2(0, \infty)$, i.e. $\int_0^\infty |u(x)|^2 dx$ is finite, if and only if the following condition is satisfied. If the coefficient $a_k \neq 0$, then $\lambda_k < 0$. Equivalently, if and only if $\lambda_k \geq 0$ then $a_k = 0$.*

Now we formulate a slightly more general result.

Lemma 6.28. *Assume that $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{C}$. Then a function u of the form*

$$u(x) = (a_1 + b_1 x) e^{\lambda_1 x} + (a_2 + b_2 x) e^{\lambda_2 x} + (a_3 + b_3 x) e^{\lambda_3 x}, \quad x \in [0, \infty)$$

belongs to the space $L^2(0, \infty)$, i.e. $\int_0^\infty |u(x)|^2 dx < \infty$, if and only if one of the following equivalent conditions is satisfied.

- (i) *If for some $k = 1, 2, 3$, $a_k \neq 0$ or $b_k \neq 0$, then $\lambda_k < 0$.*
- (ii) *If $\lambda_k \geq 0$, then $a_k = 0$ and $b_k = 0$.*

The discriminant of equation (6.2.4) is

$$\Delta = 1 + 4p^3$$

In order to continue with the proof of Lemma 6.26 we need to investigate how many roots of equation (6.2.4) have negative real part. $\square$

We make a break in the proof of Lemma 6.26 in order to formulate the following auxiliary result. The proof will be given on the next page.

Proposition 6.29. *If*

$$p \in \mathbb{R}$$

then

- (i) *the number of roots of equation (6.2.4) whose real part is negative, is equal to 2,*
- (ii) *the number of roots of equation (6.2.4) whose real part is positive, is equal to 1.*

Proof of Lemma 6.26. Part II. We need to find the basis of the space $\ker(A_1^* + iI)$.

For this purpose let us recall that we had proved that if $u \in \ker(A_1^* + iI)$ then there exists numbers a_1, a_2, a_3 such that (6.2.5) holds. By Proposition 6.29 precisely 2 eigenvalues λ_k have negative real values. Without loss of generality we can denote those eigenvalues by λ_2 and λ_3 .

In other words, $\operatorname{Re}\lambda_2 < 0$, $\operatorname{Re}\lambda_3 < 0$ and $\operatorname{Re}\lambda_1 > 0$. Since $u \in \ker(A_1^* + iI) \subset D(A_1^*)$, we have $u \in L^2(0, \infty)$. Hence, by Lemma 6.27 we infer that $a_1 = 0$. Therefore, we deduce that

$$u(x) = a_2 e^{\lambda_2 x} + a_3 e^{\lambda_3 x}, \quad x \in [0, \infty). \quad (6.2.6)$$

On the other hand, since $\operatorname{Re}\lambda_2 < 0$, $\operatorname{Re}\lambda_3 < 0$, it is easy to verify that every function of the form (6.2.6) satisfies equation (6.2.2) and belongs to $L^2(0, \infty)$, hence belongs to $\ker(A_1^* + iI)$. We conclude that the space $\ker(A_1^* + iI)$ is equal to the set of all functions u represented by formula (6.2.6). Since $\lambda_2 \neq \lambda_3$, the functions $e^{\lambda_2 x}$ and $e^{\lambda_3 x}$ are linearly independent. Therefore, they form a basis of $\ker(A_1^* + iI)$.

Thus,

$$\dim \ker(A_1^* + iI) = 2.$$

Hence the proof of Lemma 6.26 is complete. $\square$

Proof of Proposition 6.29. We consider three cases. **Case (C1):** $\Delta < 0$.

In this case, $\Delta = 1 + 4p^3 < 0$, also

$$p < -\left(\frac{1}{4}\right)^{1/3}.$$

We consider a auxiliary quadratic equation

$$z^2 - z - p^3 = 0. \quad (6.2.7)$$

Since $\Delta < 0$, equation (6.2.7) has two non-real conjugate solutions. By [31, Theorem 3 in VII.4.2], these solutions can be written in the form

$$z = r e^{i\phi}, \quad z' = r e^{-i\phi}, \quad (6.2.8)$$

for some $r > 0$ and $\phi \in (0, \pi)$.

On the other hand, by the formula for solutions of quadratic equation,

$$z = \frac{1 + \sqrt{1 + 4p^3}}{2} = \frac{1}{2} + \frac{i}{2}\sqrt{-1 - 4p^3}, \quad (6.2.9)$$

$$z' = \frac{1 - \sqrt{1 + 4p^3}}{2} = \frac{1}{2} - \frac{i}{2}\sqrt{-1 - 4p^3}. \quad (6.2.10)$$

Hence

$$\operatorname{Re} z = \frac{1}{2} > 0, \quad \operatorname{Im} z = \frac{1}{2}\sqrt{-1 - 4p^3} > 0.$$

Therefore z lies in the first quadrant, and so in (6.2.8) we may choose

$$\phi \in \left(0, \frac{\pi}{2}\right).$$

Moreover, from (6.2.8) and (6.2.9)–(6.2.10), we obtain

$$z + z' = 1, \quad (6.2.11)$$

$$zz' = -p^3. \quad (6.2.12)$$

Since

$$z + z' = r(e^{i\phi} + e^{-i\phi}) = 2r \cos \phi,$$

it follows from (6.2.11) that

$$2r \cos \phi = 1, \quad \text{i.e.} \quad \cos \phi = \frac{1}{2r}. \quad (6.2.13)$$

Also, (6.2.12) gives

$$r^2 = -p^3. \quad (6.2.14)$$

Since in Case (C1), we have $p^3 < -\frac{1}{4}$, we infer that

$$-p^3 > \frac{1}{4}.$$

Hence, by (6.2.14),

$$r^2 > \frac{1}{4} \quad \text{and therefore, since } r > 0 \quad r > \frac{1}{2}.$$

Consequently,

$$0 < \frac{1}{2r} < 1.$$

Using (6.2.13), we obtain

$$0 < \cos \phi < 1.$$

Since $\phi \in (0, \pi)$, this implies again that

$$\phi \in \left(0, \frac{\pi}{2}\right). \quad (6.2.15)$$

Now we apply the Cardan formula, see [31, Chapter VII, P.291]. Let

$$y_1 = \sqrt[3]{r} e^{i\phi/3}, \quad y'_1 = \sqrt[3]{r} e^{-i\phi/3}, \quad \varepsilon = e^{2\pi i/3}.$$

Then the three roots of equation (6.2.4) are

$$\lambda_1 = y_1 + y'_1, \quad \lambda_2 = \varepsilon y_1 + \varepsilon^2 y'_1, \quad \lambda_3 = \varepsilon^2 y_1 + \varepsilon y'_1.$$

Hence

$$\begin{aligned} \operatorname{Re} \lambda_1 &= 2\sqrt[3]{r} \cos\left(\frac{\phi}{3}\right), \\ \operatorname{Re} \lambda_2 &= 2\sqrt[3]{r} \cos\left(\frac{\phi}{3} + \frac{2\pi}{3}\right), \\ \operatorname{Re} \lambda_3 &= 2\sqrt[3]{r} \cos\left(\frac{\phi}{3} + \frac{4\pi}{3}\right). \end{aligned} \quad (6.2.16)$$

Since (6.2.15), we have

$$0 < \frac{\phi}{3} < \frac{\pi}{6}.$$

Therefore

$$\cos\left(\frac{\phi}{3}\right) > 0.$$

Also,

$$\frac{2\pi}{3} < \frac{\phi}{3} + \frac{2\pi}{3} < \frac{5\pi}{6},$$

so

$$\cos\left(\frac{\phi}{3} + \frac{2\pi}{3}\right) < 0.$$

Similarly,

$$\frac{4\pi}{3} < \frac{\phi}{3} + \frac{4\pi}{3} < \frac{3\pi}{2},$$

and hence

$$\cos\left(\frac{\phi}{3} + \frac{4\pi}{3}\right) < 0.$$

Thus, by (6.2.16), we deduce that

$$\operatorname{Re} \lambda_1 > 0, \quad \operatorname{Re} \lambda_2 < 0, \quad \operatorname{Re} \lambda_3 < 0.$$

Therefore, in Case (C1), exactly two roots of equation (6.2.4) have negative real part and exactly one root has positive real part.

Case (C2): $\Delta = 0$. In this case, $\Delta = 1 + 4p^3 = 0$, so

$$p = -\left(\frac{1}{4}\right)^{1/3}.$$

Hence the auxiliary quadratic equation (6.2.7) has a double root, namely

$$z = z' = \frac{1}{2}.$$

Thus, in the Cardan formula, we may take

$$y_1 = y'_1 = \sqrt[3]{\frac{1}{2}}.$$

Therefore, the roots of equation (6.2.4) are

$$\lambda_1 = y_1 + y'_1 = 2\sqrt[3]{\frac{1}{2}} > 0,$$

and

$$\lambda_2 = \varepsilon y_1 + \varepsilon^2 y'_1 = (\varepsilon + \varepsilon^2)\sqrt[3]{\frac{1}{2}} = -\sqrt[3]{\frac{1}{2}} < 0,$$

$$\lambda_3 = \varepsilon^2 y_1 + \varepsilon y'_1 = (\varepsilon^2 + \varepsilon)\sqrt[3]{\frac{1}{2}} = -\sqrt[3]{\frac{1}{2}} < 0.$$

Hence, in Case (C2), exactly two roots of equation (6.2.4) are negative and exactly one root is positive.

Case (C3): $\Delta > 0$. In this case, $\Delta = 1 + 4p^3 > 0$, so

$$p > -\left(\frac{1}{4}\right)^{1/3}.$$

Hence equation (6.2.4) has exactly one real root and two non-real conjugate roots. Let us denote them by

$$\lambda_1 \in \mathbb{R}, \quad \lambda_2, \lambda_3 = \overline{\lambda_2} \in \mathbb{C} \setminus \mathbb{R}.$$

We first show that $\lambda_1 > 0$. Let w defined by (6.2.3), hence

$$w(0) = -1 < 0$$

and

$$\lim_{\lambda \rightarrow +\infty} w(\lambda) = +\infty.$$

Thus by intermediate value theorem implies that w has a positive real root. Since there is only one real root in Case (C3), this root must be λ_1 . Hence

$$\lambda_1 > 0.$$

Next, since λ_2 and λ_3 are conjugate, they have the same real part. By the Vieta formula, see [31, Chapter IX. p.347]

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

Taking real parts, we obtain

$$\lambda_1 + 2\operatorname{Re}(\lambda_2) = 0.$$

Therefore,

$$\operatorname{Re}(\lambda_2) = \operatorname{Re}(\lambda_3) = -\frac{\lambda_1}{2} < 0.$$

Thus, in Case (C3), exactly two roots of equation (6.2.4) have negative real part and exactly one root has positive real part.

In all three cases (C1), (C2), and (C3), equation (6.2.4) has exactly two roots whose real part is negative and exactly one root whose real part is positive. This completes the proof of Proposition 6.29. $\square$

In order to calculate $n_+(A_1)$, we need to determine the space $\ker(iI - A_1^*)$.

Lemma 6.30. *Assume that A_1 is the operator on $L^2((0, \infty), \mathbb{C})$ defined by (6.2.1). Then*

$$\dim \ker(iI - A_1^*) = 1.$$

Proof of Lemma 6.30. Part I. Let us choose and fix

$$u \in \ker(iI - A_1^*) \setminus \{0\}.$$

Then

$$(iI - A_1^*)u = 0,$$

that is,

$$iu - A_1^*u = 0.$$

Using the formula for A_1^* , we infer that

$$a_3 D^3 u - a_1 D u + u = 0. \tag{6.2.17}$$

As before, we normalize the coefficients by assuming

$$a_3 = 1 \quad \text{and} \quad -a_1 = 3p.$$

Then (6.2.17) becomes

$$D^3u + 3pDu + u = 0. \quad (6.2.18)$$

For solutions of the form, see [24, Section 2.9],

$$u(x) = e^{\lambda x},$$

we are led to the characteristic equation

$$\lambda^3 + 3p\lambda + 1 = 0. \quad (6.2.19)$$

We denote the corresponding polynomial by

$$w(\lambda) = \lambda^3 + 3p\lambda + 1. \quad (6.2.20)$$

Let

$$\lambda_1, \lambda_2, \lambda_3$$

be the roots of equation (6.2.19). Its discriminant is

$$\Delta = 1 + 4p^3.$$

According to the multiplicity of the roots, the general solution of (6.2.18) is of exponential type, and is covered by Lemmas 6.27 and 6.28. In particular, every solution $u \in L^2(0, \infty)$ is obtained by keeping only those terms corresponding to roots whose real part is negative.

Thus, in order to determine $\ker(iI - A_1^*)$, it remains to know how many roots of equation (6.2.19) have negative real part. This is the content of the next proposition. $\square$

Proposition 6.31. *If*

$$p \in \mathbb{R},$$

then

- (i) *the number of roots of equation (6.2.20) whose real part is negative is equal to 1;*
- (ii) *the number of roots of equation (6.2.20) whose real part is positive is equal to 2.*

Proof of Proposition 6.31. We divide the proof into three cases.

Case (C1): $\Delta < 0$.

In this case,

$$\Delta = 1 + 4p^3 < 0,$$

hence

$$p < -\left(\frac{1}{4}\right)^{1/3}.$$

We consider the auxiliary quadratic equation

$$z^2 + z - p^3 = 0. \quad (6.2.21)$$

Since $\Delta < 0$, equation (6.2.21) has two non-real conjugate solutions. By [31, Theorem 3 in VII.4.2], these solutions can be written in the form

$$z = re^{i\phi}, \quad z' = re^{-i\phi},$$

for some $r > 0$ and $\phi \in (0, \pi)$.

On the other hand, by the quadratic formula,

$$z_{1,2} = \frac{-1 \pm \sqrt{1 + 4p^3}}{2} = -\frac{1}{2} \pm \frac{i}{2} \sqrt{-1 - 4p^3}.$$

Hence

$$\operatorname{Re} z = -\frac{1}{2} < 0, \quad \operatorname{Im} z = \frac{1}{2} \sqrt{-1 - 4p^3} > 0,$$

so we may choose

$$\phi \in \left(\frac{\pi}{2}, \pi\right).$$

By the Cardan's formula, the roots of equation (6.2.19) are

$$\lambda_1 = y_1 + y_1', \quad \lambda_2 = \varepsilon y_1 + \varepsilon^2 y_1', \quad \lambda_3 = \varepsilon^2 y_1 + \varepsilon y_1',$$

where

$$y_1 = \sqrt[3]{r} e^{i\phi/3}, \quad y_1' = \sqrt[3]{r} e^{-i\phi/3}, \quad \varepsilon = e^{2\pi i/3}.$$

Hence

$$\begin{aligned} \operatorname{Re} \lambda_1 &= 2\sqrt[3]{r} \cos\left(\frac{\phi}{3}\right), \\ \operatorname{Re} \lambda_2 &= 2\sqrt[3]{r} \cos\left(\frac{\phi}{3} + \frac{2\pi}{3}\right), \\ \operatorname{Re} \lambda_3 &= 2\sqrt[3]{r} \cos\left(\frac{\phi}{3} + \frac{4\pi}{3}\right). \end{aligned} \quad (6.2.22)$$

Since

$$\frac{\pi}{2} < \phi < \pi,$$

we have

$$\frac{\pi}{6} < \frac{\phi}{3} < \frac{\pi}{3}.$$

Therefore,

$$\cos\left(\frac{\phi}{3}\right) > 0.$$

Also,

$$\frac{5\pi}{6} < \frac{\phi}{3} + \frac{2\pi}{3} < \pi,$$

hence

$$\cos\left(\frac{\phi}{3} + \frac{2\pi}{3}\right) < 0.$$

Finally,

$$\frac{3\pi}{2} < \frac{\phi}{3} + \frac{4\pi}{3} < \frac{5\pi}{3},$$

so

$$\cos\left(\frac{\phi}{3} + \frac{4\pi}{3}\right) > 0.$$

Thus, by (6.2.22),

$$\operatorname{Re}\lambda_1 > 0, \quad \operatorname{Re}\lambda_2 < 0, \quad \operatorname{Re}\lambda_3 > 0.$$

Therefore, in Case (C1), exactly one root has negative real part and exactly two roots have positive real part.

Case (C2): $\Delta = 0$.

In this case,

$$\Delta = 1 + 4p^3 = 0 \iff p = -\left(\frac{1}{4}\right)^{1/3}.$$

Hence the auxiliary quadratic equation

$$z^2 + z - p^3 = 0$$

has a double root

$$z = z' = -\frac{1}{2}.$$

Thus, in the Cardan's formula, we may take

$$y_1 = \sqrt[3]{-\frac{1}{2}},$$

and

$$y_1' = \sqrt[3]{-\frac{1}{2}}.$$

Therefore, the roots of equation (6.2.19) are

$$\begin{aligned}\lambda_1 &= y_1 + y'_1 = -2\sqrt[3]{\frac{1}{2}} < 0, \\ \lambda_2 &= \varepsilon y_1 + \varepsilon^2 y'_1 = (\varepsilon + \varepsilon^2)\sqrt[3]{-\frac{1}{2}} = \sqrt[3]{\frac{1}{2}} > 0, \\ \lambda_3 &= \varepsilon^2 y_1 + \varepsilon y'_1 = (\varepsilon^2 + \varepsilon)\sqrt[3]{-\frac{1}{2}} = \sqrt[3]{\frac{1}{2}} > 0.\end{aligned}$$

Hence, in Case (C2), exactly one root is negative and two roots are positive.

Case (C3): $\Delta > 0$.

In this case,

$$\Delta = 1 + 4p^3 > 0 \iff p > -\left(\frac{1}{4}\right)^{1/3}.$$

Hence equation (6.2.19) has exactly one real root and two non-real conjugate roots. Let us denote them by

$$\lambda_1 \in \mathbb{R}, \quad \lambda_2, \lambda_3 = \overline{\lambda_2} \in \mathbb{C} \setminus \mathbb{R}.$$

We first show that $\lambda_1 < 0$. Since

$$w(0) = 1 > 0$$

and

$$\lim_{\lambda \rightarrow -\infty} w(\lambda) = -\infty,$$

the intermediate value theorem implies that w has a negative real root. Since there is only one real root in Case (C3), this root must be λ_1 . Hence

$$\lambda_1 < 0.$$

Next, since λ_2 and λ_3 are conjugate, they have the same real part. By Vieta's formulas,

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

Taking real parts, we obtain

$$\lambda_1 + 2\operatorname{Re}(\lambda_2) = 0.$$

Therefore,

$$\operatorname{Re}(\lambda_2) = \operatorname{Re}(\lambda_3) = -\frac{\lambda_1}{2} > 0.$$

Thus, in Case (C3), exactly one root has negative real part and exactly two roots have positive real part.

In all three cases (C1), (C2), and (C3), equation (6.2.19) has exactly one root whose real part is negative and exactly two roots whose real part is positive. This completes the proof of Proposition 6.31. $\square$

Proof of Lemma 6.30. Part II. By Proposition 6.31, equation (6.2.19) has exactly one root whose real part is negative. Without loss of generality, we denote this root by λ_1 . Thus

$$\operatorname{Re}\lambda_1 < 0, \quad \operatorname{Re}\lambda_2 > 0, \quad \operatorname{Re}\lambda_3 > 0.$$

Let $u \in \ker(iI - A_1^*)$. Since

$$\ker(iI - A_1^*) \subset D(A_1^*),$$

we have

$$u \in L^2(0, \infty).$$

Moreover, by Part I, u is a solution of equation (6.2.18). Hence, using Lemmata 6.27 and 6.28, all terms corresponding to roots with non-negative real part must vanish. Therefore,

$$u(x) = a_1 e^{\lambda_1 x}, \quad x \in [0, \infty), \quad (6.2.23)$$

for some $a_1 \in \mathbb{C}$.

Conversely, since $\operatorname{Re}\lambda_1 < 0$, every function of the form (6.2.23) belongs to $L^2(0, \infty)$ and satisfies equation (6.2.18). Hence it belongs to $\ker(iI - A_1^*)$.

Thus $\ker(iI - A_1^*)$ is the one-dimensional space generated by the function $e^{\lambda_1 x}$. Therefore,

$$\dim \ker(iI - A_1^*) = 1.$$

This completes the proof of Lemma 6.30. $\square$

We now compare the general third-order inhomogeneous setting with the homogeneous third-order operator studied earlier. In the special case $p = 0$, the characteristic equations reduce to those of the homogeneous problem, and the corresponding root distributions agree with the previously computed deficiency indices.

Remark 6.32.(i) When $p = 0$, equation (6.2.2) reduces to

$$D^3 u = u.$$

Indeed, when $p = 0$, the characteristic equation (6.2.4) becomes

$$\lambda^3 = 1.$$

Its roots are

$$\lambda_1 = 1, \quad \lambda_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \lambda_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Hence

$$\operatorname{Re}\lambda_2 = \operatorname{Re}\lambda_3 < 0, \quad \operatorname{Re}\lambda_1 > 0.$$

Thus, in this special case, two roots have negative real part and one root has positive real part, in agreement with Proposition 6.29.

(ii) When $p = 0$, equation (6.2.18) becomes

$$D^3u = -u.$$

Indeed, equation (6.2.20) becomes

$$\lambda^3 = -1,$$

whose roots are

$$\lambda_1 = -1, \quad \lambda_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \lambda_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2},$$

Hence

$$\operatorname{Re}\lambda_1 < 0, \quad \operatorname{Re}\lambda_2 = \operatorname{Re}\lambda_3 > 0.$$

Thus, in this special case, one root has negative real part and two roots have positive real part, in agreement with Proposition 6.31.

This is exactly the homogeneous third-order case considered earlier for the operator defined in (5.2.1). In that setting, the corresponding deficiency indices were computed in Propositions 5.12 and 5.15, see also Theorem 5.17.

6.3 Third-order linear inhomogeneous differential operators on negative half line

6.3.1 Symmetry and the adjoint

In this subsection we consider the Hilbert space

$$H = L^2((-\infty, 0); \mathbb{C}).$$

We define the operator A_1 by

$$\begin{aligned} D(A_1) &= C_0^\infty((-\infty, 0); \mathbb{C}), \\ A_1u &= a_3(iD)^3u + a_1(iD)u = -ia_3D^3u + ia_1Du = -i(a_3D^3u - a_1Du). \end{aligned} \tag{6.3.1}$$

Recall that we assume that $a_3 \neq 0$.

Lemma 6.33. *The operator A_1 defined by (6.3.1) is symmetric.*

Proof. The proof is identical to the proof of Lemma 6.17. Indeed,

$$D(A_1) = C_0^\infty((-\infty, 0); \mathbb{C}) \subset C_0^\infty(\mathbb{R}; \mathbb{C}) = D(A_0),$$

and $A_1 u = A_0 u$ for every $u \in D(A_1)$. Hence $A_1 \subset A_0$, and since A_0 is symmetric, A_1 is symmetric as well. $\square$

To find $D(A_1^*)$ using the formula (6.1.3) from the Section 6.1, i.e. with the set $\mathcal{O} = (0, \infty)$ and so we get the following three equivalent results.

Lemma 6.34. *Suppose that A_1 is the linear operator defined by (6.3.1). Then,*

$$\begin{aligned} D(A_1^*) := & \{u \in L^2((-\infty, 0), \mathbb{C}) : u \text{ is of } C^2 \text{ class on } (-\infty, 0], \\ & u'' \text{ is absolutely continuous and hence differentiable almost everywhere} \\ & \text{on } (-\infty, 0] \text{ (with respect to the Lebesgue measure),} \\ & u''' \text{ is Lebesgue measurable and belongs to } L_{loc}^2((-\infty, 0], \mathbb{C}).\} \end{aligned}$$

Lemma 6.35. *Suppose that A_1 is the linear operator defined by (6.3.1). Then,*

$$\begin{aligned} D(A_1^*) := & \{u \in L^2((-\infty, 0), \mathbb{C}) \cap H_{loc}^3((-\infty, 0]; \mathbb{C}) : \\ & \int_{-\infty}^0 |a_3 D^3 u(x) - a_1 Du(x)|^2 dx < \infty.\} \end{aligned}$$

Moreover, if $u \in D(A_1^*)$, then

$$A_1^* u = i(a_3 D^3 u - a_1 Du).$$

Lemma 6.36. *Suppose that A_1 is the linear operator defined by (6.3.1). Then,*

$$\begin{aligned} D(A_1^*) := & \{u \in L^2((-\infty, 0); \mathbb{C}) : \text{weak derivatives } D^3 u \text{ and } Du \text{ exists and} \\ & \in L_{loc}^2((-\infty, 0]; \mathbb{C}) \text{ and } \{-ia_3 D^3 u + ia_1 Du \in L^2((-\infty, 0); \mathbb{C})\}\}. \end{aligned}$$

6.3.2 Deficiency indices of the third-order linear inhomogeneous differential operators on negative half line

Let us recall that the deficiency indices $n_+(A_1)$ and $n_-(A_1)$ of the operator A_1 defined by (6.3.1). Using Definition 2.59, we have

$$\begin{aligned} n_+(A_1) &= \dim \ker(iI - A_1^*) \\ n_-(A_1) &= \dim \ker(iI + A_1^*) \end{aligned}$$

The main result of this section is the following one.

Theorem 6.37. *Assume that A_1 is the operator defined by formula (6.3.1). Then*

$$\begin{aligned} n_-(A_1) &= 1 \\ n_+(A_1) &= 2 \end{aligned}$$

The following result is a consequences of Theorem 6.37. In fact, we investigated the previous result in order to prove or disprove the next corollary.

Corollary 6.38. *The operator A_1 on $L^2((-\infty, 0), \mathbb{C})$ defined by equation (6.3.1) has no self-adjoint extensions.*

Proof of Corollary 6.38. Since we have $n_+(A_1) = 1$ and $n_-(A_1) = 2$, from Theorem 6.37, it follows that

$$n_+ \neq n_-.$$

Therefore, applying the conclusion from Theorem 2.60, we can assert that the operator A_1 has no self-adjoint extensions.

Thus, we conclude that the operator A_1 defined by equation (6.3.1) has no self-adjoint extensions. This completes the proof. $\square$

In the remaining of this section we will occupy ourselves proving the theorem above. For this aim we will calculate the deficiency indices $n_+(A_1)$ and $n_-(A_1)$.

In order to calculate $n_-(A_1)$, we need to find the space $\ker(iI + A_1^*)$. We begin with the following Lemma.

Lemma 6.39. *Assume that A_1 is the operator defined by formula (6.3.1). Then*

$$\dim \ker(A_1^* + iI) = 1.$$

Proof of Lemma 6.39. Part I. Let $u \in \ker(A_1^* + iI)$, i.e.

$$u \in D(A_1^*) \text{ and } A_1^*u + iu = 0.$$

This implies that

$$u \in D(A_1^*) \text{ and } -i(a_3 D^3 v - a_1 Dv) + iu = 0$$

what implies that $u \in D(A_1^*)$ and

$$a_3 D^3 u - a_1 Du - u = 0$$

In what follows we assume that

$$a_3 = 1$$

and we denote

$$-a_1 = 3p.$$

Hence the last equation becomes

$$D^3 u + 3pDu - u = 0 \tag{6.3.2}$$

The solution of the form

$$u(x) = e^{\lambda x}$$

we substitute it into equation (6.3.2) and you get

$$\lambda^3 e^{\lambda x} + 3p\lambda e^{\lambda x} - e^{\lambda x} = 0$$

Hence

$$\lambda^3 + 3p\lambda - 1 = 0$$

This equation has three roots

$$\lambda_1, \lambda_2, \lambda_3$$

The general solution of the above equation (6.3.2), see [24, Section 2.9], is

$$u(x) = a_1 e^{\lambda_1 x} + a_2 e^{\lambda_2 x} + a_3 e^{\lambda_3 x}$$

The normalised characteristic polynomial of equation (6.3.2) is

$$w(\lambda) = \lambda^3 + 3p\lambda - 1$$

Define, see [31, chapter VII, page 293], the discriminant of equation

$$\lambda^3 + 3p\lambda - 1 = 0 \tag{6.3.3}$$

is equal to²

$$\Delta = 1 + 4p^3$$

In order to continue with the proof of Lemma 6.39 we need to know how many roots of equation (6.3.3) have positive real part.

The following result follows from Proposition 6.29 proven in the earlier parts of the paper.

Proposition 6.40. *If*

$$p \in \mathbb{R}$$

the number of roots of equation (6.3.3) whose real part is positive, is equal to 1.

Hence the proof of Lemma 6.39 is complete. $\square$

Remark 6.41. We now compare the general third-order inhomogeneous operator on the negative half-line with the homogeneous operator considered in Remark 5.19.

When $p = 0$, the first-order term vanishes and the operator reduces to the homogeneous third-order operator. For the positive deficiency subspace, the corresponding equation becomes

$$D^3u = -u \quad \text{on } (-\infty, 0).$$

Its characteristic equation is

$$\lambda^3 = -1,$$

with roots

$$\lambda_1 = -1, \quad \lambda_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \lambda_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Since

$$e^{\lambda x} \in L^2((-\infty, 0)) \quad \text{if and only if} \quad \operatorname{Re} \lambda > 0,$$

the two roots λ_2 and λ_3 give square-integrable solutions. Hence

$$n_+(A_1) = 2.$$

For the negative deficiency subspace, the equation becomes

$$D^3u = u \quad \text{on } (-\infty, 0),$$

with characteristic equation

$$\lambda^3 = 1.$$

²Note that we have replaced p by $3p$, hence in our formula we do not have factor 27.

Its roots are

$$\lambda_1 = 1, \quad \lambda_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \lambda_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Only λ_1 has positive real part, and therefore only one corresponding solution belongs to $L^2((-\infty, 0))$. Thus

$$n_-(A_1) = 1.$$

Consequently, when $p = 0$, the general inhomogeneous calculation reduces to the homogeneous case and gives

$$n_+(A_1) = 2, \quad n_-(A_1) = 1,$$

in agreement with Remark 5.19.

In order to calculate $n_+(A_1)$, we need to find the space $\ker(iI - A_1^*)$.

Lemma 6.42.

$$\dim \ker(iI - A_1^*) = 2.$$

Proof of Lemma 6.42. Part I. Let $u \in \ker(iI - A_1^*)$. Then

$$\begin{aligned} iu - A_1^*u &= 0 \\ iu - i(a_3D^3u - a_1Du) &= 0 \\ a_3D^3u - a_1Du + u &= 0 \end{aligned}$$

In what follows we assume that

$$a_3 = 1 \text{ and we denote } -a_1 = 3p.$$

Hence the last equation becomes

$$D^3u + 3pDu + u = 0 \tag{6.3.4}$$

This equation has three roots

$$\lambda_1, \lambda_2, \lambda_3$$

The general solution of the above equation (6.3.4), see [24, Section 2.9], is

$$u(x) = a_1e^{\lambda_1x} + a_2e^{\lambda_2x} + a_3e^{\lambda_3x}$$

The normalized characteristic polynomial of equation (6.3.4) is

$$w(\lambda) = \lambda^3 + 3p\lambda + 1$$

Define, see [31, chapter VII, page 293], the discriminant of equation

$$\lambda^3 + 3p\lambda + 1 = 0 \quad (6.3.5)$$

is equal to³

$$\Delta = 1 + 4p^3$$

In order to continue with the proof of Lemma 6.42 we need to know how many roots of equation (6.3.5) have positive real part.

The following result follows from Proposition 6.31 proven in earlier parts of our paper.

Proposition 6.43. *If*

$$p \in \mathbb{R}$$

the number of roots of equation (6.3.5) whose real part is positive, is equal to 2.

Hence the proof of Lemma 6.42 is complete. $\square$

6.4 Third-order linear inhomogeneous differential operators on the punctured line

6.4.1 Symmetry and the adjoint

In this section we consider the Hilbert space

$$H = L^2(\mathbb{R}, \mathbb{C})$$

with the standard inner product and norm. We study the following operator

$$\begin{aligned} D(A_1) &= C_0^\infty((-\infty, 0) \cup (0, \infty); \mathbb{C}) = C_0^\infty(\mathbb{R} \setminus \{0\}; \mathbb{C}), \\ A_1 u &= a_3(iD)^3 + a_1(iD) = -ia_3 D^3 + ia_1 D = -i(a_3 D^3 - a_1 D). \end{aligned} \quad (6.4.1)$$

Proposition 6.44. *The operator A_1 defined by (6.4.1) is symmetric in the space $H = L^2(\mathbb{R}, \mathbb{C})$, i.e.*

$$\langle A_1 u, v \rangle_{L^2} = \langle u, A_1 v \rangle_{L^2}, \quad \forall u, v \in D(A_1).$$

Remark 6.45. The proof of symmetry of A_1 is analogous to the proof of symmetry of A_0 . see Lemma 6.2. Moreover, it follows directly from the symmetry of A_0 .

³Note that we have replaced p by $3p$, hence in our formula we do not have factor 27.

Proof of Proposition 6.44. Since

$$D(A_1) = C_0^\infty(\mathbb{R} \setminus \{0\}; \mathbb{C}) \subset C_0^\infty(\mathbb{R}; \mathbb{C}) = D(A_0),$$

and

$$A_1 u = a_3(iD)^3 u + a_1(iD)u = A_0 u \quad \text{for every } u \in D(A_1),$$

we have $A_1 \subset A_0$. Since A_0 is symmetric, it follows that A_1 is symmetric. $\square$

Proposition 6.46. *Assume that A_1 is the operator defined by formula (6.4.1). Then A_1 is not closed.*

Moreover, a function u belongs to $D(\overline{A_1})$ if and only if the following conditions are satisfied:

(i) $u \in L^2(\mathbb{R}, \mathbb{C}),$

(ii) $u \in H_{\text{loc}}^3(\mathbb{R} \setminus \{0\}, \mathbb{C})$ and $u'' : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{C}$ is locally absolutely continuous. Moreover,

$$\int_{\mathbb{R}} |a_3 D^3 u(t) - a_1 D u(t)|^2 dt < \infty,$$

(iii) $u(0) = 0, u'(0) = 0,$ and $u''(0) = 0.$

Moreover,

$$\overline{A_1} u = -i(a_3 D^3 u - a_1 D u), \quad u \in D(\overline{A_1}).$$

Proof of Proposition 6.46. The proof is analogous to the proof of Proposition 5.32 and Proposition 5.33. More precisely, one repeats the same argument with the operator $-iD^3$ replaced everywhere by $-i(a_3 D^3 - a_1 D)$.

In the proof that A_1 is not closed, one uses the same test function as in Proposition 5.32 and the fact that this function belongs to $H^3(\mathbb{R})$, hence also satisfies $a_3 D^3 u - a_1 D u \in L^2(\mathbb{R})$.

For the characterization of $D(\overline{A_1})$, one follows the same steps as in Proposition 5.33: if $u_n \rightarrow u$ in $L^2(\mathbb{R})$ and $A_1 u_n \rightarrow z$ in $L^2(\mathbb{R})$, then

$$-i(a_3 D^3 u_n - a_1 D u_n) \rightarrow z \quad \text{in } L^2(\mathbb{R}),$$

which implies that $a_3 D^3 u - a_1 D u \in L^2(\mathbb{R})$, while the same compactness and uniform-convergence arguments yield

$$u(0) = u'(0) = u''(0) = 0.$$

Conversely, if u satisfies conditions (i)–(iii), then one approximates the restrictions of u on $(-\infty, 0)$ and $(0, \infty)$ by functions in $C_0^\infty((-\infty, 0))$ and $C_0^\infty((0, \infty))$, respectively, exactly as in the proof of Proposition 5.33, and concludes that

$$u \in D(\overline{A_1}) \quad \text{and} \quad \overline{A_1}u = -i(a_3D^3u - a_1Du).$$

Therefore, the proof follows by the same method as in the homogeneous third-order case, and we omit the repeated details. $\square$

To proceed, it is convenient to denote the operators A_1 from the previous sections by a different notation. The operator defined by formula (6.2.1) will be denoted by B_+ . The operator defined by formula (6.3.1) will be denoted by B_- .

We proved in previous Sections 6.2.2 and 6.3.2 that

$$\begin{aligned} n_-(B_+) &= 2, \quad n_+(B_+) = 1 \\ n_-(B_-) &= 1, \quad n_+(B_-) = 2 \end{aligned}$$

To characterize the domain $D(A_1^*)$, we apply Lemma 6.3 and formula (6.1.3) from the Section 6.1, i.e. we consider $\mathcal{O} = (-\infty, 0) \cup (0, \infty)$, so we get

Proposition 6.47. *A Lebesgue measurable function*

$$v : (-\infty, 0) \cup (0, \infty) \rightarrow \mathbb{C}$$

belongs to $D(A_1^*)$ if and only if the following conditions are satisfied.

(i) *Function v belongs to $L^2((-\infty, 0) \cup (0, \infty); \mathbb{C})$ and it is of C^2 -class,*

(ii) *the classical second-order derivative v'' belongs to $AC_{\text{loc}}((-\infty, 0) \cup (0, \infty))$, see Definition 2.8, in the sense that it is absolutely continuous on every interval of the form $(0, a)$ or $(-a, 0)$, $a > 0$. Hence, by Theorem 2.7, the third-order derivative v''' exists almost everywhere, and the function v''' is Lebesgue measurable and it belongs to $L_{\text{loc}}^2((-\infty, 0] \cup [0, \infty), \mathbb{C})$,*

(iii)

$$\int_{\mathbb{R} \setminus \{0\}} |a_3D^3v(x) - a_1Dv(x)|^2 dx < \infty.$$

Moreover, if $v \in D(A_1^*)$, then

$$A_1^*v = a_1iv' + a_3(-i)v''' = -i(a_3D^3v - a_1Dv), \quad v \in D(A_1^*).$$

The above characterization can be equivalently written as follows.

$$\begin{aligned} D(A_1^*) &:= \{v \in L^2((-\infty, 0) \cup (0, \infty); \mathbb{C}) : \\ &\quad v \in H_{loc}^3((-\infty, 0]) \text{ and } v \in H_{loc}^3([0, \infty); \mathbb{C}) \\ &\quad \int_{\mathbb{R} \setminus \{0\}} |a_3 D^3 v(x) - a_1 Dv(x)|^2 dx < \infty.\} \end{aligned}$$

Proof. The equivalence follows directly from Lemma 6.3 applied to the open set $\mathcal{O} = (-\infty, 0) \cup (0, \infty)$, together with the characterization of H_{loc}^3 functions given in (6.1.3).

The equivalence between the two formulations follows from the standard relation between classical differentiability, local absolute continuity, and weak derivatives; see Proposition 2.24. $\square$

We use here a non-standard definition of the space $H_{loc}^3((-\infty, 0) \cup (0, \infty); \mathbb{C})$. According to the standard definition, see [36, Definition 8.10],

$u \in H_{loc}^3((-\infty, 0) \cup (0, \infty); \mathbb{C})$ if and only if for every closed interval $[a, b]$ such that $[a, b] \subset (-\infty, 0) \cup (0, \infty)$, the restriction of u to that interval belongs to the Sobolev space $H^3(a, b)$. A consequence of this definition is that we cannot take intervals of the form $(0, b)$, $b > 0$ nor $(a, 0)$, $a < 0$ because their's closure are not contained in $(-\infty, 0) \cup (0, \infty)$.

Definition 6.48. Assume that $n \in \mathbb{N}$, including 0. A Lebesgue measurable function

$$u : (-\infty, 0) \cup (0, \infty) \rightarrow \mathbb{C}$$

belongs to the space $H_{loc}^n((-\infty, 0) \cup (0, \infty); \mathbb{C})$ if and only if for every $a > 0$, the restriction of u to the interval $(0, a)$ or $(-a, 0)$ belongs to the Sobolev space $H^n(0, a)$, respectively $H^n(-a, 0)$.

Similarly, we can define the spaces $H_{loc}^n(-\infty, \beta)$ and $H_{loc}^n(\alpha, \infty)$, where $\alpha, \beta \in \mathbb{R}$.

We have the following obvious but important result.

Proposition 6.49. The space $H_{loc}^n((-\infty, 0) \cup (0, \infty); \mathbb{C})$ defined by Definition 2.28 is contained in the corresponding space defined in [36, Definition 8.10]. Moreover, the former space is strictly smaller than the latter.

Remark 6.50. If we consider a special case $n = 0$, then a function

$$u(x) = \frac{1}{x}, \quad x \in (-\infty, 0) \cup (0, \infty)$$

This function belongs to the classical $L_{loc}^2((-\infty, 0) \cup (0, \infty))$ but it does not belong to the space $H_{loc}^0((-\infty, 0) \cup (0, \infty); \mathbb{C})$ according to Definition 2.30. Indeed, if $[a, b] \subset (0, \infty)$ or

$[a, b] \subset (-\infty, 0)$, then

$$\int_{(a,b)} |u(x)|^2 dx < \infty$$

but, for every $b > 0$,

$$\int_{(0,b)} |u(x)|^2 dx = \infty.$$

An important consequence of the fact that in the previous Lemma we used the newly defined space $H_{loc}^3(-\infty, 0) \cup (0, \infty)$ lies in the following "trace" type result.

Proposition 6.51. *If $u \in D(A_1^*)$ then the following limits exist:*

$$\begin{aligned} \lim_{x \rightarrow 0^+} u(x) &=: u(0+) \\ \lim_{x \rightarrow 0^-} u(x) &=: u(0-) \\ \lim_{x \rightarrow 0^+} u'(x) &=: u'(0+) \\ \lim_{x \rightarrow 0^-} u'(x) &=: u'(0-) \\ \lim_{x \rightarrow 0^+} u''(x) &=: u''(0+) \\ \lim_{x \rightarrow 0^-} u''(x) &=: u''(0-). \end{aligned}$$

Remark 6.52. Note that the limits from the right and left above are, in general, different. Thus, if $u \in D(A_1^*)$, then neither u nor u' need to be continuous at 0. For instance, the function

$$u(x) = \operatorname{sgn}(x)e^{-|x|}, \quad x \in \mathbb{R} \setminus \{0\},$$

belongs to $D(A_1^*)$, but it is discontinuous at $x = 0$, since

$$u(0+) = 1, \quad u(0-) = -1.$$

Remark 6.53. We had to modify the definition of the spaces $H_{loc}^n((-\infty, 0) \cup (0, \infty); \mathbb{C})$, see Definition 6.48, so that for every

$$u \in H_{loc}^n((-\infty, 0) \cup (0, \infty); \mathbb{C}),$$

the limits in Proposition 6.51 exist.

Proposition 6.54. *A measurable function $v : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{C}$ belongs to $D(A_1^*)$ if and only if the following two conditions are satisfied.*

- (i) *The restriction v^+ of the function v to the interval $(0, \infty)$ belongs to $D(B_+^*)$;*
- (ii) *The restriction v^- of the function v to the interval $(-\infty, 0)$ belongs to $D(B_-^*)$;*

Moreover, if $v \in D(A_1^*)$, then the restriction of the function A_1^*v to the interval $(0, \infty)$ is equal to $B_+^*v^+$ and the restriction of the function A_1^*v to the interval $(-\infty, 0)$ is equal to $B_-^*v^-$.

Lemma 6.47 can be given an equivalent form using the operators $B_\pm$.

Proposition 6.55 (Decomposition of the adjoint domain). *The domain of the adjoint operator A_1^* admits the decomposition*

$$D(A_1^*) = \{v : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{C} \mid v^+ \in D(B_+^*), v^- \in D(B_-^*)\}.$$

Moreover,

$$(A_1^*v)(x) = \begin{cases} (B_+^*v^+)(x), & x > 0, \\ (B_-^*v^-)(x), & x < 0. \end{cases}$$

Proof. The result follows directly from Proposition 6.54 and the fact that the operator A_1 acts independently on the positive and negative half-lines. $\square$

6.4.2 Deficiency indices of the third-order linear inhomogeneous differential operators on the punctured line

In this section, we will calculate the deficiency indices $n_+(A_1)$ and $n_-(A_1)$ of the operator A_1 defined by (6.4.1). Using Definition 2.59, we have

$$\begin{aligned} n_-(A_1) &= \dim \ker (iI + A_1^*) \\ n_+(A_1) &= \dim \ker (iI - A_1^*). \end{aligned}$$

The main result of this section is the following one.

Theorem 6.56. *Assume that A_1 is the operator defined by formula (6.4.1). Then*

$$\begin{aligned} n_-(A_1) &= 3 \\ n_+(A_1) &= 3 \end{aligned}$$

The following result is a consequences of Theorem 6.56. In fact, we investigated the previous result in order to prove or disprove the next corollary.

Corollary 6.57. *The operator A_1 on $L^2(\mathbb{R} \setminus \{0\}, \mathbb{C})$ defined by (6.4.1) has at least two self-adjoint extension. In fact, every self-adjoint extension of A_1 can be characterize via unitary maps between $\ker (iI + A_1^*)$ and $\ker (iI - A_1^*)$.*

Proof of Corollary 6.57. According to Theorem 2.60, for a closed symmetric operator A to have a self-adjoint extension, it is necessary that the deficiency indices satisfy $n_+ = n_-$.

Since we have $n_+(A_1) = 3$ and $n_-(A_1) = 3$ from Theorem 6.56, it follows that

$$n_+ = n_-.$$

By the result from Theorem 2.60, the operator A_1 has at least two self-adjoint extension.

Thus, we conclude that the operator A_1 defined by equation (6.4.1) has at least one self-adjoint extension. This completes the proof of the corollary. $\square$

In the remaining of this section we will occupy ourselves proving the theorem above. For this aim we will calculate the deficiency indices $n_+(A_1)$ and $n_-(A_1)$.

In order to calculate $n_-(A_1)$, we need to find the space $\ker(iI + A_1^*)$. We begin with the following Lemma.

Lemma 6.58. *Assume that A_1 is the operator defined by formula (6.4.1). Then*

$$n_-(A_1) = 3.$$

Proof of Lemma 6.58. To find $n_-(A_1)$, let us denote

$$u^+ = u|_{(0,\infty)}, \quad \text{and} \quad u^- = u|_{(-\infty,0)},$$

and let us observe that by Proposition 6.44

$$\begin{aligned} u \in \ker(iI + A_1^*) &\iff u \in D(A_1^*) \text{ and } (iI + A_1^*)u = 0, \\ &\iff u \in D(A_1^*) \text{ and } a_3 D^3 u - a_1 D u - u = 0 \iff \dots \end{aligned}$$

We have, by Proposition 6.54,

$$\begin{aligned} \dots &\iff u^+ \in D(B_+) \text{ then } iu^+ + (B_+^*)u^+ = 0, \\ &\text{and } u^- \in D(B_-) \text{ then } iu^- + (B_-^*)u^- = 0, \\ &\iff u^+ \in \ker(iI + B_+^*) \text{ and } u^- \in \ker(iI + B_-^*). \end{aligned}$$

Hence the map

$$\ker(iI + A_1^*) \ni u \mapsto (u^+, u^-) \in \ker(iI + B_+^*) \times \ker(iI + B_-^*), \quad (6.4.2)$$

is a well defined map. In fact we have

Lemma 6.59. *The map defined by (6.4.2) is a linear isomorphism.*

Hence, by Definition linear isomorphism [26, p.35], some basic linear algebra [26], we get

$$\dim(\ker(iI + A_1^*)) = \dim \ker(iI + B_+^*) + \dim \ker(iI + B_-^*).$$

Hence we proved

$$n_-(A_1) = n_-(B_+) + n_-(B_-).$$

Remark 6.60. In Section 6.2 we proved that if a function $u \in D(A_1^*)$ is solution to equation (6.2.4), from Proposition 6.29, the number of roots of the equation (6.2.4) whose real part is negative, is equal to 2.

In Section 6.3 we proved that if a function $u \in D(A_1^*)$ is solution to equation (6.3.3), from Proposition 6.40, the number of roots of the equation (6.3.3) whose real part is negative, is equal to 1.

Similarly,

$$n_+(A_1) = n_+(B_+) + n_+(B_-).$$

We deduce that the following lemma.

Lemma 6.61. *Assume that A_1 is the operator defined by formula (6.4.1). Then*

$$n_+(A_1) = 3.$$

Therefore, we complete the proof. □

Remark 6.62. Since A_1 is densely defined and symmetric, its adjoint coincides with the adjoint of its closure, that is,

$$A_1^* = (\overline{A_1})^*.$$

Consequently, the deficiency subspaces and deficiency indices of A_1 and $\overline{A_1}$ coincide:

$$n_{\pm}(A_1) = n_{\pm}(\overline{A_1}).$$

Therefore, the results of this section also describe the deficiency indices of the closed symmetric operator $\overline{A_1}$.

From Lemmata 6.58 and 6.61, we deduce the following immediate consequence.

Corollary 6.63. *Let A_1 symmetric and densely defined by (6.4.1), and let $\overline{A_1}$ denote its closure in $L^2(\mathbb{R}, \mathbb{C})$. Then*

$$n_+(\overline{A_1}) = n_+(A_1) = 3, \quad n_-(\overline{A_1}) = n_-(A_1) = 3.$$

6.4.3 Finding all self-adjoint operators extensions for the third-order linear inhomogeneous on punctured line

We will use the characterization of the adjoint operator and the computation of the deficiency indices obtained in Section 6.4, together with von Neumann's extension theory, to determine all self-adjoint extensions of the symmetric operator A_1 .

We define an operator A_1 on the Hilbert space $H = L^2(\mathbb{R}, \mathbb{C})$ by

$$\begin{aligned} D(A_1) &= C_0^\infty((-\infty, 0) \cup (0, \infty); \mathbb{C}) = C_0^\infty(\mathbb{R} \setminus \{0\}; \mathbb{C}), \\ A_1 u &= a_3(iD)^3 + a_1(iD) = -ia_3 D^3 + ia_1 D \text{ if } u \in D(A_1). \end{aligned}$$

We assume that $a_1, a_3 \in \mathbb{R}$.

Without loss of generality, we will assume that

$$a_3 = 1$$

and will denote

$$a_1 = -3p$$

In this case we have

$$A_1 u = -iD^3 u - 3piDu.$$

We showed earlier, in a different context in Subsection 6.4, that the operator has deficiency indices $(3, 3)$. So it has self-adjoint extensions. We will find all these self-adjoint extensions.

By von Neumann's theorem, see Theorem 2.60, the deficiency spaces are

$$H_- = \ker(iI + A_1^*), \quad H_+ = \ker(iI - A_1^*),$$

and $n_-(A_1) = \dim H_-$, $n_+(A_1) = \dim H_+$.

Remark 6.64. It follows from Theorem 6.56, that

$$\begin{aligned} n_-(A_1) &= 3 \\ n_+(A_1) &= 3. \end{aligned}$$

The objective is to find orthonormal bases of spaces H_+ and H_- .

We start with H_- . The discriminant of equation

$$\lambda^3 + 3p\lambda - 1 = 0$$

is equal to

$$\Delta = 1^2 + 4p^3$$

For this purpose we consider the following three cases.

(i) Assume that $\Delta < 0$. Let $r > 0$ is such that

$$r^2 = -p^3.$$

Let us also recall the definition of the parameter ϕ :

$$\cos \phi = \frac{1}{2r}$$

We showed that

$$r > \frac{1}{2}, \quad p < -\frac{1}{4} \quad \text{and} \quad \cos\left(\frac{\phi}{3}\right)$$

Hence, we have

$$H_- = \text{span}\{e_1, e_2, e_3\}$$

where

$$\begin{aligned} e_1(x) &= 1_{(-\infty, 0)} e^{\lambda_1 x} = 1_{(-\infty, 0)} e^{2\sqrt[3]{r} \cos\left(\frac{\phi}{3}\right)x} \\ e_2(x) &= 1_{(0, \infty)} e^{\lambda_2 x} = 1_{(0, \infty)} e^{\sqrt[3]{r} \left(-\cos\left(\frac{\phi}{3}\right) + i\frac{\sqrt{3}}{3} \sin\left(\frac{\phi}{3}\right) \right)x} \\ e_3(x) &= 1_{(0, \infty)} e^{\lambda_3 x} = 1_{(0, \infty)} e^{\sqrt[3]{r} \left(-\cos\left(\frac{\phi}{3}\right) - i\frac{\sqrt{3}}{3} \sin\left(\frac{\phi}{3}\right) \right)x}. \end{aligned}$$

where λ_1, λ_2 and λ_3 are roots of the characteristic equation, i.e.

$$\begin{aligned} \lambda_1 &= 2\sqrt[3]{r} \cos\left(\frac{\phi}{3}\right). \\ \lambda_2 &= \sqrt[3]{r} \left(-\cos\left(\frac{\phi}{3}\right) + i\frac{\sqrt{3}}{3} \sin\left(\frac{\phi}{3}\right) \right). \\ \lambda_3 &= \sqrt[3]{r} \left(-\cos\left(\frac{\phi}{3}\right) - i\frac{\sqrt{3}}{3} \sin\left(\frac{\phi}{3}\right) \right). \end{aligned}$$

We will check if $\{e_1, e_2, e_3\}$ forms an orthonormal basis by verifying the conditions (5.4.30).

Let us begin with the first condition. Since $4\sqrt[3]{r} \cos\left(\frac{\phi}{3}\right) > 0$ we infer that

$$\begin{aligned}\|e_1\|^2 &= \int_{-\infty}^{\infty} |e_1(x)|^2 dx = \int_{-\infty}^0 |e_1(x)|^2 dx \\ &= \int_{-\infty}^0 e^{4\sqrt[3]{r} \cos\left(\frac{\phi}{3}\right)x} dx \\ &= \left[\frac{1}{4\sqrt[3]{r} \cos\left(\frac{\phi}{3}\right)} e^{4\sqrt[3]{r} \cos\left(\frac{\phi}{3}\right)x} \right]_{-\infty}^0 = \frac{1}{4\sqrt[3]{r} \cos\left(\frac{\phi}{3}\right)} = \frac{1}{4 \cos\left(\frac{\phi}{3}\right)} r^{-1/3}\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|e_2\|^2 &= \int_{-\infty}^{\infty} |e_2(x)|^2 dx = \int_0^{\infty} |e^{\sqrt[3]{r}\left(-\cos\left(\frac{\phi}{3}\right)+i\frac{\sqrt{3}}{3}\sin\left(\frac{\phi}{3}\right)\right)x}|^2 dx \\ &= \int_0^{\infty} e^{2\sqrt[3]{r}\left(-\cos\left(\frac{\phi}{3}\right)\right)x} dx = \frac{1}{2\sqrt[3]{r} \cos\left(\frac{\phi}{3}\right)} = \frac{1}{2 \cos\left(\frac{\phi}{3}\right)} r^{-1/3}\end{aligned}$$

Since $|e_3(x)|^2 = |e_2(x)|^2$ for all $x \in \mathbb{R}$ we infer that

$$\|e_3\|^2 = \int_{-\infty}^{\infty} |e_3(x)|^2 dx = \int_{-\infty}^{\infty} |e_2(x)|^2 dx = \frac{1}{2\sqrt[3]{r} \cos\left(\frac{\phi}{3}\right)} = \frac{1}{2 \cos\left(\frac{\phi}{3}\right)} r^{-1/3}.$$

Now we can normalize each function

$$\begin{aligned}\tilde{e}_1(x) &= \frac{e_1(x)}{\|e_1\|} = 2 \sqrt{\cos\left(\frac{\phi}{3}\right)} r^{-1/6} \cdot e_1(x), \\ \tilde{e}_2(x) &= \frac{e_2(x)}{\|e_2\|} = \sqrt{2 \cos\left(\frac{\phi}{3}\right)} r^{-1/6} \cdot e_2(x), \\ \tilde{e}_3(x) &= \frac{e_3(x)}{\|e_3\|} = \sqrt{2 \cos\left(\frac{\phi}{3}\right)} r^{-1/6} \cdot e_3(x).\end{aligned}$$

Now, we need to verify the condition $\langle e_j, e_k \rangle_{L^2} = 0$ for $j \neq k$.

$$\begin{aligned}\langle e_1, e_2 \rangle_{L^2} &= \int_{-\infty}^{\infty} e_1(x) e_2(x) dx \\ &= \int_{-\infty}^0 1_{(-\infty, 0)} e^{2\sqrt[3]{r} \cos\left(\frac{\phi}{3}\right)x} \cdot 1_{(0, \infty)} e^{\sqrt[3]{r}\left(-\cos\left(\frac{\phi}{3}\right)+i\frac{\sqrt{3}}{3}\sin\left(\frac{\phi}{3}\right)\right)x} dx = 0.\end{aligned}$$

$$\begin{aligned}\langle e_1, e_3 \rangle_{L^2} &= \int_{-\infty}^{\infty} e_1(x) e_3(x) dx \\ &= \int_{-\infty}^0 1_{(-\infty, 0)} e^{2\sqrt[3]{r} \cos\left(\frac{\phi}{3}\right)x} \cdot 1_{(0, \infty)} e^{\sqrt[3]{r}\left(-\cos\left(\frac{\phi}{3}\right)-i\frac{\sqrt{3}}{3}\sin\left(\frac{\phi}{3}\right)\right)x} dx = 0.\end{aligned}$$

$$\begin{aligned}
\langle e_2, e_3 \rangle_{L^2} &= \int_{-\infty}^{\infty} e_2(x) e_3(x) dx \\
&= \int_0^{\infty} e^{\sqrt[3]{r} \left(-\cos\left(\frac{\phi}{3}\right) + i \frac{\sqrt{3}}{3} \sin\left(\frac{\phi}{3}\right) \right) x} \cdot e^{\sqrt[3]{r} \left(-\cos\left(\frac{\phi}{3}\right) - i \frac{\sqrt{3}}{3} \sin\left(\frac{\phi}{3}\right) \right) x} dx \\
&= \int_0^{\infty} e^{\sqrt[3]{r} (-2 \cos(\frac{\phi}{3})) x} dx = \frac{1}{2 \cos\left(\frac{\phi}{3}\right)} r^{-1/3}.
\end{aligned}$$

We conclude that the basis e_1, e_2, e_3 of the space H_- is not an orthonormal basis. We will construct an orthonormal basis starting with the basis e_1, e_2, e_3 as before. Let us summarize previous calculations.

(i) Vector e_1 is perpendicular to vectors e_2 and e_3 .

So in order to construct an ONB, using a the Gramm-Schmidt method, we put

$$\lambda := \langle e_2, e_3 \rangle \|e_3\|^{-2} \quad (6.4.3)$$

and

$$\begin{aligned}
\tilde{e}_1 &= \|e_1\|^{-1} e_1 \\
\tilde{e}_3 &= \|e_3\|^{-1} e_3 \\
\tilde{e}_2 &= \frac{e_2 - \lambda e_3}{\|e_2 - \lambda e_3\|}.
\end{aligned}$$

Let us formulate the following basic result.

Proposition 6.65. *The vectors $\tilde{e}_1$, $\tilde{e}_2$ and $\tilde{e}_3$ form an orthormal basis of H_- .*

Proof of Proposition 6.65. Obviously $\|\tilde{e}_j\| = 1$ for each j and Vector $\tilde{e}_1$ is perpendicular to vectors $\tilde{e}_3$ and $\tilde{e}_2$ (because $\tilde{e}_2$ is a linear combination of vectors e_2 and e_3 . We only need to show that $\langle \tilde{e}_2, \tilde{e}_3 \rangle$. We have, by the definitions above, in particular by (6.4.3),

$$\begin{aligned}
\langle \tilde{e}_2, \tilde{e}_3 \rangle &= \|e_2 - \lambda e_3\|^{-1} \|e_3\|^{-1} \langle e_2 - \lambda e_3, e_3 \rangle \\
&= \|e_2 - \lambda e_3\|^{-1} \|e_3\|^{-1} \left(\langle e_2, e_3 \rangle - \lambda \|e_3\|^2 \right) = 0.
\end{aligned}$$

The proof is complete. □

Remark 6.66. It follows from above that

$$\begin{aligned}
\tilde{e}_1(x) &= \frac{e_1(x)}{\|e_1\|} = 2 \sqrt{\cos\left(\frac{\phi}{3}\right)} r^{-1/6} \cdot e_1(x), \\
\tilde{e}_3(x) &= \frac{e_3(x)}{\|e_3\|} = \sqrt{2 \cos\left(\frac{\phi}{3}\right)} r^{-1/6} \cdot e_3(x).
\end{aligned}$$

We will also find a similar expression for $\tilde{e}_2(x)$.

Now we compute $\|e_2 - \lambda e_3\|^2$:

$$\begin{aligned}\|e_2 - \lambda e_3\|^2 &= \langle e_2 - \lambda e_3, e_2 - \lambda e_3 \rangle \\ &= \langle e_2, e_2 \rangle - \bar{\lambda} \langle e_2, e_3 \rangle - \lambda \langle e_3, e_2 \rangle + |\lambda|^2 \langle e_3, e_3 \rangle \\ &= 1 - |\lambda|^2 - |\lambda|^2 + |\lambda|^2 \\ &= 1 - |\lambda|^2.\end{aligned}$$

Substituting the value of $|\lambda|^2$

$$\|e_2 - \lambda e_3\|^2 = 1 - \frac{1}{4(\sqrt[3]{r})^2 \cos^2\left(\frac{\phi}{3}\right)}.$$

Thus, we find

$$\begin{aligned}\|e_2 - \lambda e_3\| &= \sqrt{1 - \frac{1}{4(\sqrt[3]{r})^2 \cos^2\left(\frac{\phi}{3}\right)}} \\ &= \frac{\sqrt{4(\sqrt[3]{r})^2 \cos^2\left(\frac{\phi}{3}\right) - 1}}{2\sqrt[3]{r} \cos^2\left(\frac{\phi}{3}\right)} \\ &= \frac{\sqrt{(\sqrt[3]{r})^2 \cos^2\left(\frac{\phi}{3}\right) - \frac{1}{4}}}{\sqrt[3]{r} \cos^2\left(\frac{\phi}{3}\right)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{e}_2(x) &= \frac{e_2(x) - \lambda e_3(x)}{\|e_2 - \lambda e_3\|} \\ &= \frac{\sqrt[3]{r} \cos^2\left(\frac{\phi}{3}\right)}{\sqrt{(\sqrt[3]{r})^2 \cos^2\left(\frac{\phi}{3}\right) - \frac{1}{4}}}(e_2(x) - \lambda e_3(x))\end{aligned}$$

In the context of the case where $\Delta < 0$, the roots of the characteristic equation

$$\lambda^3 + 3p\lambda + 1 = 0$$

are given by:

$$\begin{aligned}\lambda_1 &= -2\sqrt[3]{r} \cos\left(\frac{\phi}{3}\right), \\ \lambda_2 &= \sqrt[3]{r} \left(\cos\left(\frac{\phi}{3}\right) + i\frac{\sqrt{3}}{3} \sin\left(\frac{\phi}{3}\right) \right), \\ \lambda_3 &= \sqrt[3]{r} \left(\cos\left(\frac{\phi}{3}\right) - i\frac{\sqrt{3}}{3} \sin\left(\frac{\phi}{3}\right) \right).\end{aligned}$$

Here, λ_1 is the real root, while λ_2 and λ_3 are complex conjugate roots.

(ii) Assume that $\Delta = 0$, then

$$\begin{aligned}\lambda_1 &= 2\sqrt[3]{\frac{1}{2}} \cos\left(\frac{0}{3}\right) = 2\sqrt[3]{\frac{1}{2}}, \\ \lambda_2 &= 2\sqrt[3]{\frac{1}{2}} \cos\left(\frac{0}{3} + \frac{2\pi}{3}\right) = 2\sqrt[3]{\frac{1}{2}} \cos\left(\frac{2\pi}{3}\right) \\ &= 2\sqrt[3]{\frac{1}{2}} \left(-\frac{1}{2}\right) = -\sqrt[3]{\frac{1}{2}}, \\ \lambda_3 &= 2\sqrt[3]{\frac{1}{2}} \cos\left(\frac{0}{3} + \frac{4\pi}{3}\right) = 2\sqrt[3]{\frac{1}{2}} \cos\left(\frac{4\pi}{3}\right) \\ &= 2\sqrt[3]{\frac{1}{2}} \left(-\frac{1}{2}\right) = -\sqrt[3]{\frac{1}{2}}.\end{aligned}$$

(iii) Assume that $\Delta > 0$, then

$$\begin{aligned}\lambda_1 &= 2\sqrt[3]{r} \cos\left(\frac{\phi}{3}\right), \\ \lambda_2 &= -\sqrt[3]{r} \left(\cos\left(\frac{\phi}{3}\right) + i\sqrt{3} \sin\left(\frac{\phi}{3}\right) \right), \\ \lambda_3 &= -\sqrt[3]{r} \left(\cos\left(\frac{\phi}{3}\right) - i\sqrt{3} \sin\left(\frac{\phi}{3}\right) \right).\end{aligned}$$

This indicates that both roots have the same real part, which is negative, assuming r is a positive number.

We were not able to complete the above approach in full generality. This led us to consider a different method based on abstract boundary formalism. The details of this alternative approach are presented in chapter 8, Brzeźniak and Jefferies in [22].

Chapter 7

Equivalence between scalar and system differential operators

In this chapter, we study the equivalence between different differential operator problems. We present several concrete examples showing how a scalar problem can be transformed into an equivalent system formulation.

In each case, one of the problems has already been analysed, and using this equivalence we deduce that the second problem shares analogous properties. In particular, properties such as symmetry, dense domain, and deficiency indices are preserved under this correspondence.

This approach provides a unified framework for treating both scalar and system differential operators.

7.1 Equivalence between two problems related to the Laplacian

We consider two Hilbert spaces

$$X := H_2 := L^2(0, \infty; \mathbb{C}^2)$$

and

$$Y := H_1 := L^2((-\infty, 0) \cup (0, \infty); \mathbb{C}).$$

Elements of the space X are vector-valued functions $\vec{u} = (u_1, u_2) : (0, \infty) \rightarrow \mathbb{C}^2$. Elements of the space Y are scalar-valued functions $u : (-\infty, 0) \cup (0, \infty) \rightarrow \mathbb{C}$.

We define two operators: B_0 in X and A_0 in Y by

$$\begin{aligned} D(B_0) &= C_0^\infty(0, \infty; \mathbb{C}^2) \subset X, \\ B_0 \vec{u} &= (\Delta u_1, \Delta u_2), \quad \vec{u} \in D(B_0). \end{aligned} \tag{7.1.1}$$

and

$$\begin{aligned} D(A_0) &= C_0^\infty((-\infty, 0) \cup (0, \infty); \mathbb{C}) \subset Y, \\ A_0 u &= \Delta u, \quad u \in D(A_0). \end{aligned} \tag{7.1.2}$$

These objects are isomorphic in the following sense.

Proposition 7.1. *We consider two maps*

$$T : X \rightarrow Y \text{ and } \Lambda : Y \rightarrow X$$

defined by

$$(T\vec{u})(x) := \begin{cases} u_1(x) & \text{if } x > 0 \\ u_2(-x) & \text{if } x < 0 \end{cases} \tag{7.1.3}$$

for $\vec{u} \in X$ and

$$(\Lambda u)(x) := (u(x), u(-x)), \quad x \in (0, \infty). \tag{7.1.4}$$

Then

- (i) Both maps are well defined and isometric.
- (ii) $T \circ \Lambda = \text{id}$ on Y and $\Lambda \circ T = \text{id}$ on X , i.e. where $\text{id}_X : X \rightarrow X$ and $\text{id}_Y : Y \rightarrow Y$ denote the identity maps.

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \text{id} \uparrow & & \downarrow \text{id} \\ X & \xleftarrow{\Lambda} & Y \end{array}$$

FIGURE 7.1: Id diagram

- (iii) Both maps are unitary.

- (iv) Denote

$$\begin{aligned} X_1 &= C_0^\infty(0, \infty; \mathbb{C}^2), \\ Y_1 &= C_0^\infty((-\infty, 0) \cup (0, \infty); \mathbb{C}), \end{aligned}$$

then $T(X_1) \subset Y_1$ and $\Lambda(Y_1) \subset X_1$.

(v) $T_1 \circ \Lambda_1 = \text{id}$ on X_1 and $\Lambda_1 \circ T_1 = \text{id}$ on Y_1 , where

$$T_1 := T|_{X_1} \text{ and } \Lambda_1 = \Lambda|_{Y_1}.$$

Proof of Proposition 7.1. We prove each item separately. **Proof of (i).** The maps T and Λ are obviously well defined and linear. Now we will show that T is isometric. We need to verify that

$$\langle T(\vec{u}), T(\vec{v}) \rangle_Y = \langle \vec{u}, \vec{v} \rangle_X, \quad \text{for all } \vec{u}, \vec{v} \in X.$$

The inner products are defined for $X = L^2(0, \infty; \mathbb{C}^2)$ as follows

$$\langle \vec{u}, \vec{v} \rangle_X = \int_0^\infty \left(\overline{u_1(x)} v_1(x) + \overline{u_2(x)} v_2(x) \right) dx.$$

and for $Y = L^2(\mathbb{R} \setminus \{0\}, \mathbb{C})$ as follows

$$\langle h, w \rangle_Y = \int_{-\infty}^0 \overline{w(x)} h(x) dx + \int_0^\infty \overline{w(x)} h(x) dx.$$

Now, we compute

$$\langle T(\vec{u}), T(\vec{v}) \rangle_Y = \int_{-\infty}^0 \overline{u_2(-x)} v_2(-x) dx + \int_0^\infty \overline{u_1(x)} v_1(x) dx.$$

Changing the variable in the first integral ($y = -x$, so $dx = -dy$) gives

$$\int_{-\infty}^0 \overline{u_2(-x)} v_2(-x) dx = \int_0^\infty \overline{u_2(y)} v_2(y) dy.$$

Thus, we have

$$\langle T(\vec{u}), T(\vec{v}) \rangle_Y = \int_0^\infty \overline{u_1(x)} v_1(x) dx + \int_0^\infty \overline{u_2(x)} v_2(x) dx = \langle \vec{u}, \vec{v} \rangle_X.$$

Hence we proved that T is isometric.

To show that $\Lambda : Y \rightarrow X$ is isometric, we need to verify that:

$$\langle \Lambda(u), \Lambda(v) \rangle_X = \langle u, v \rangle_Y \quad \text{for all } u, v \in Y.$$

The inner product in X is defined as

$$\langle \vec{u}, \vec{v} \rangle_X = \int_0^\infty \left(u_1(x) \overline{v_1(x)} + u_2(x) \overline{v_2(x)} \right) dx,$$

and similarly for Y

$$\langle u, v \rangle_Y = \int_{-\infty}^0 u(x) \overline{v(x)} dx + \int_0^{\infty} u(x) \overline{v(x)} dx.$$

Using the definition of Λ , we have

$$\Lambda(u) = (u(x), u(-x)) \quad \text{and} \quad \Lambda(v) = (v(x), v(-x)).$$

Thus,

$$\langle \Lambda(u), \Lambda(v) \rangle_X = \int_0^{\infty} (\overline{u(x)}v(x) + \overline{u(-x)}v(-x)) dx.$$

We can split this into two integrals

$$\langle \Lambda(u), \Lambda(v) \rangle_X = \int_0^{\infty} \overline{u(x)}v(x) dx + \int_0^{\infty} \overline{u(-x)}v(-x) dx.$$

For the second integral, we change of the variable. Let $y = -x$ (thus $dx = -dy$), which gives us

$$\int_0^{\infty} \overline{u(-x)}v(-x) dx = \int_{-\infty}^0 \overline{u(y)}v(y)(-dy) = \int_{-\infty}^0 \overline{u(y)}v(y) dy.$$

Combining both parts

$$\langle \Lambda(u), \Lambda(v) \rangle_X = \int_0^{\infty} \overline{u(x)}v(x) dx + \int_{-\infty}^0 \overline{u(y)}v(y) dy.$$

Thus, we have

$$\langle \Lambda(u), \Lambda(v) \rangle_X = \int_0^{\infty} \overline{u(x)}v(x) dx + \int_{-\infty}^0 \overline{u(y)}v(y) dy = \langle u, v \rangle_Y.$$

Therefore, Λ is indeed isometric.

Proof of (ii). Proof that $T \circ \Lambda = \text{id}$ on Y .

Let $u \in Y$ be arbitrary, where $u : (-\infty, 0) \cup (0, \infty) \rightarrow \mathbb{C}$.

Define $\vec{v} = \Lambda u \in X$. By definition of Λ , see (7.1.4), we have

$$\vec{v}(x) = (v_1(x), v_2(x)) = (u(x), u(-x)), \quad x \in (0, \infty).$$

Applying the operator T to $\vec{v}$ and using (7.1.3), we obtain

$$T(\vec{v})(x) = \begin{cases} v_1(x), & x > 0, \\ v_2(-x), & x < 0. \end{cases}$$

Substituting the expressions of v_1 and v_2 gives

$$T(\vec{v})(x) = \begin{cases} u(x), & x > 0, \\ u(-(-x)) = u(x), & x < 0. \end{cases}$$

Hence, for all $x \in (-\infty, 0) \cup (0, \infty)$,

$$(T(\Lambda u))(x) = u(x).$$

Since u was arbitrary in Y , we conclude that

$$T(\Lambda u) = u \quad \text{for all } u \in Y.$$

Therefore,

$$T \circ \Lambda = \text{id}_Y.$$

Proof that $\Lambda \circ T = \text{id}$ on X . Let $\vec{u} = (u_1, u_2) \in X = L^2(0, \infty; \mathbb{C}^2)$ be arbitrary. Define

$$v = T(\vec{u}) \in Y.$$

By definition of T , see (7.1.3), we have

$$v(x) = \begin{cases} u_1(x), & x > 0, \\ u_2(-x), & x < 0. \end{cases}$$

Now apply Λ to v . By definition (7.1.4),

$$(\Lambda v)(x) = (v(x), v(-x)), \quad x \in (0, \infty).$$

Substituting the expressions of $v(x)$ and $v(-x)$ gives

$$(\Lambda v)(x) = (u_1(x), u_2(x)).$$

Therefore,

$$(\Lambda(T\vec{u}))(x) = \vec{u}(x) \quad \text{for all } x \in (0, \infty).$$

Since $\vec{u}$ was arbitrary in X , we conclude that

$$\Lambda \circ T = \text{id}_X.$$

Proof of (iii). To prove that the maps T and Λ defined by the proposition 7.1 are unitary operators, by Definition 2.69, it is enough to show that both T and Λ are surjective isometries.

We already proved in part (i) that both T and Λ are isometric. Hence, we only need to prove that both maps are bijections.

Since $T \circ \Lambda = \text{id}$, we infer that T is surjective. Indeed, if $y \in Y$, then with $x := \Lambda y \in X$ we infer that

$$y = T(\Lambda y) = T(x).$$

Similarly, since $\Lambda \circ T = \text{id}$ we infer that Λ is surjective.

Let us take $x_1, x_2 \in X$ such that $T(x_1) = T(x_2)$. Thus $\Lambda(T(x_1)) = \Lambda(T(x_2))$. Since $\Lambda \circ T = \text{id}$ we infer that $x_1 = x_2$ and hence we deduce that T is injective.

Similarly, using $T \circ \Lambda = \text{id}$, we deduce that Λ is injective. Therefore, both T and Λ are bijections and isometries.

Proof of (iv). Let $\vec{u} = (u_1, u_2) \in X_1 = C_0^\infty(0, \infty; \mathbb{C}^2)$. Then both u_1 and u_2 are smooth and compactly supported in $(0, \infty)$. By the definition of T , the function $T\vec{u}$ is smooth on $(-\infty, 0) \cup (0, \infty)$ and has compact support there. Hence $T(X_1) \subset Y_1$.

Similarly, if $u \in Y_1$, then $u|_{(0, \infty)}$ and $x \mapsto u(-x)$ belong to $C_0^\infty(0, \infty)$, so $\Lambda(Y_1) \subset X_1$.

When we apply the operator T to a function $\vec{u} = (u_1, u_2) \in X_1$, you obtain a function defined by (7.1.3).

Since u_1 and u_2 both vanish at infinity, $T(u)$ will also vanish as $x \rightarrow \pm\infty$. Thus, $T(u)$ has compact support and

$$T(X_1) \subset Y_1.$$

Similarly, when we apply Λ to a function $u \in Y_1$ by (7.1.4).

Therefore,

$$\Lambda(Y_1) \subset X_1.$$

Proof of (v). Recall from item (ii) that

$$T \circ \Lambda = \text{id}_Y, \quad \Lambda \circ T = \text{id}_X.$$

Let $T_1 := T|_{X_1}$ and $\Lambda_1 := \Lambda|_{Y_1}$. Since $T(X_1) \subset Y_1$ and $\Lambda(Y_1) \subset X_1$, the compositions $T_1 \circ \Lambda_1$ and $\Lambda_1 \circ T_1$ are well defined.

Restricting the identities from item (ii) gives

$$T_1 \circ \Lambda_1 = \text{id}_{Y_1}, \quad \Lambda_1 \circ T_1 = \text{id}_{X_1}.$$

Thus we proved the Proposition 7.1. □

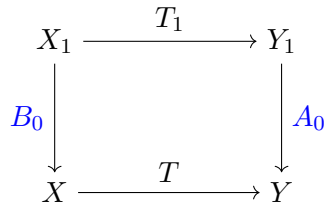


FIGURE 7.2: Commutative-diagram

The main result in this subsection reads as follows.

Proposition 7.2. *Let $T : X \rightarrow Y$ be the map defined by the equations (7.1.3). Let the maps B_0 and A_0 be defined by (7.1.1) and (7.1.2) respectively. Then the diagram 7.2 above is commuting, i.e.*

$$A_0 \circ T_1 = T \circ B_0. \tag{7.1.5}$$

Proof of Proposition 7.2. We want to show that equation (7.1.5) holds, that is

$$A_0(T_1(\vec{u})) = T(B_0(\vec{u})) \tag{7.1.6}$$

for all $\vec{u} \in X_1$. Recall that the map T_1 is the restriction of T to X_1 , namely

$$T_1 = T|_{X_1}.$$

Let $\vec{u} = (u_1, u_2) \in X_1 = C_0^\infty(0, \infty; \mathbb{C}^2)$. Denote $v = T_1(\vec{u}) \in Y$.

By definition of T , see (7.1.3), we have

$$v(x) := \begin{cases} u_1(x) & \text{if } x \in (0, \infty), \\ u_2(-x) & \text{if } x \in (-\infty, 0). \end{cases}$$

Applying the operator A_0 to v , we obtain

$$A_0 v(x) := \begin{cases} \frac{d^2}{dx^2} u_1(x) = u_1''(x) & \text{if } x \in (0, \infty), \\ \frac{d^2}{dx^2} u_2(-x) = u_2''(-x) & \text{if } x \in (-\infty, 0). \end{cases} \quad (7.1.7)$$

On the other hand, applying the operator B_0 to $\vec{u}$ gives

$$B_0(\vec{u}) = (u_1''(x), u_2''(x)).$$

Now we apply the operator T to $B_0(\vec{u})$. Using again the definition (7.1.3), we obtain

$$[T(B_0 \vec{u})](x) := \begin{cases} u_1''(x) & \text{if } x > 0, \\ u_2''(-x) & \text{if } x < 0. \end{cases} \quad (7.1.8)$$

Comparing the two expressions (7.1.7) and (7.1.8), we see that

$$A_0(T_1(\vec{u}))(x) = [T(B_0(\vec{u}))](x)$$

for all $x \in (-\infty, 0) \cup (0, \infty)$.

Hence equation (7.1.6) holds. Therefore, the diagram 7.2 is commuting as stated in Proposition 7.2. $\square$

We have already studied the second-order differential operator $A_1 = A_0$, acting on the Hilbert space

$$Y = L^2((-\infty, 0) \cup (0, \infty); \mathbb{C})$$

and defined by (4.2.1) in Section 4.2. In particular, Proposition 4.8 gives a characterization of the domain of the adjoint A_1^* , namely

$$A_1^* u = D^2 u = u'' \quad \text{for } u \in D(A_1^*),$$

while Theorem 4.11 shows that

$$n_+(A_1) = 2, \quad n_-(A_1) = 2.$$

On the other hand, Proposition 7.1 shows that the map $T : X \rightarrow Y$ is unitary, with inverse $\Lambda : Y \rightarrow X$. Moreover, in the Laplacian case the corresponding commutative relation proved earlier reads $A_0 \circ T_1 = T \circ B_0$.

Therefore the operators A_0 and B_0 are unitarily equivalent. Consequently, the adjoints, deficiency subspaces and deficiency indices of these operators correspond under the unitary map T . This allows us to deduce the properties of B_0 from the corresponding properties of A_0 .

Corollary 7.3. *Let the operators B_0 and A_0 be defined by the corresponding second-order formulas in $X = L^2(0, \infty; \mathbb{C}^2)$ and $Y = L^2((-\infty, 0) \cup (0, \infty); \mathbb{C})$. Then the following two conditions are equivalent.*

(i) $\vec{u} = (u_1, u_2) \in D(B_0^*)$.

(ii) $\vec{u} \in L^2((0, \infty); \mathbb{C}^2)$ and the functions $u_1, u_2 : (0, \infty) \rightarrow \mathbb{C}$ are of class C^1 , their derivatives u_1', u_2' are locally absolutely continuous, and

$$\int_0^\infty |u_1''(x)|^2 dx < \infty, \quad \int_0^\infty |u_2''(x)|^2 dx < \infty.$$

Moreover,

$$B_0^* \vec{u} = (D^2 u_1, D^2 u_2) = (u_1'', u_2'') \quad \text{for } \vec{u} \in D(B_0^*).$$

Proof. By Proposition 7.1, the map $T : X \rightarrow Y$ is unitary and its inverse is Λ . Moreover, by the commutative diagram for the Laplacian case, we have

$$A_0 \circ T_1 = T \circ B_0.$$

Let $\vec{u} \in X$. Then

$$\vec{u} \in D(B_0^*) \iff T\vec{u} \in D(A_0^*).$$

By Proposition 4.8, the domain $D(A_0^*)$ consists of functions $v \in L^2((-\infty, 0) \cup (0, \infty); \mathbb{C})$ such that the restrictions of v to $(0, \infty)$ and $(-\infty, 0)$ are of class C^1 , their derivatives are locally absolutely continuous, and

$$\int_0^\infty |v''(x)|^2 dx < \infty, \quad \int_{-\infty}^0 |v''(x)|^2 dx < \infty.$$

Since

$$(T\vec{u})(x) = \begin{cases} u_1(x), & x > 0, \\ u_2(-x), & x < 0, \end{cases}$$

these conditions are equivalent to requiring that u_1, u_2 are of class C^1 on $(0, \infty)$, their derivatives are locally absolutely continuous, and

$$\int_0^\infty |u_1''(x)|^2 dx < \infty, \quad \int_0^\infty |u_2''(x)|^2 dx < \infty.$$

This proves the characterization of $D(B_0^*)$.

Finally, using again the intertwining relation $A_0 T_1 = T B_0$, we obtain

$$B_0^* \vec{u} = (D^2 u_1, D^2 u_2).$$

□

Theorem 7.4. *Let B_0 be the operator defined by (7.1.1) in the Hilbert space*

$$X = L^2(0, \infty; \mathbb{C}^2).$$

Then B_0 is symmetric and its deficiency indices satisfy

$$n_+(B_0) = 2,$$

$$n_-(B_0) = 2.$$

Proof. By Proposition 7.1, the map $T : X \rightarrow Y$ is unitary. Furthermore the operators A_0 and B_0 satisfy the intertwining relation

$$A_0 \circ T_1 = T \circ B_0.$$

Hence the operators A_0 and B_0 are unitarily equivalent.

Unitary equivalence preserves the adjoint operator and the deficiency subspaces. Therefore

$$n_+(B_0) = n_+(A_0), \quad n_-(B_0) = n_-(A_0).$$

By Theorem 4.11 we know that

$$n_+(A_0) = 2, \quad n_-(A_0) = 2.$$

Hence

$$n_+(B_0) = 2, \quad n_-(B_0) = 2.$$

□

Since the deficiency indices of B_0 are equal, von Neumann's extension theorem, see Theorem 2.60, implies that B_0 admits at least one self-adjoint extension.

7.2 Equivalence between two problems related to the first order differential operators

We consider two Hilbert spaces

$$X = H_2 = L^2(0, \infty; \mathbb{C}^2)$$

and

$$Y = H_1 = L^2((-\infty, 0) \cup (0, \infty), \mathbb{C})$$

Let us define two operators: B_0 in X and A_0 in Y by

$$\begin{aligned} D(B_0) &= C_0^\infty(0, \infty; \mathbb{C}^2), \\ B_0 \vec{u} &= (iDu_1, -iDu_2) \quad \text{on } (0, \infty). \end{aligned} \tag{7.2.1}$$

where $\vec{u} = (u_1, u_2) : (0, \infty) \rightarrow \mathbb{C}^2$,

$$\begin{aligned} D(A_0) &= C_0^\infty((-\infty, 0) \cup (0, \infty); \mathbb{C}), \\ A_0 u &= iDu \text{ if } u \in D(A_0). \end{aligned} \tag{7.2.2}$$

$$\begin{array}{ccc} X_1 & \xrightarrow{T_1} & Y_1 \\ B_0 \downarrow & & \downarrow A_0 \\ X & \xrightarrow{T} & Y \end{array}$$

FIGURE 7.3: Commutative-diagram

Proposition 7.5. *Let $T : X \rightarrow Y$ be the map defined by the equations (7.1.3). Let the maps B_0 and A_0 be defined by (7.2.1) and (7.2.2) respectively. Then the diagram 7.3 above is commuting, i.e.*

$$A_0 \circ T_1 = T \circ B_0. \tag{7.2.3}$$

Proof of Proposition 7.5. We want to show that equation (7.2.3) holds, that is,

$$A_0(T_1(\vec{u})) = T(B_0(\vec{u})) \tag{7.2.4}$$

for all $\vec{u} \in X_1$.

Let $\vec{u} = (u_1, u_2) \in X_1 = C_0^\infty(0, \infty; \mathbb{C}^2)$ and define

$$v := T_1(\vec{u}) \in Y_1.$$

By definition of T ,

$$v(x) := \begin{cases} u_1(x), & x > 0, \\ u_2(-x), & x < 0. \end{cases}$$

Applying the operator A_0 to v , we obtain

$$A_0 v(x) := \begin{cases} iv'(x) = iu_1'(x), & x > 0, \\ iv'(x) = -iu_2'(-x), & x < 0. \end{cases}$$

On the other hand,

$$B_0(\vec{u}) = (iu_1', -iu_2').$$

Applying T to $B_0(\vec{u})$, we get

$$[T(B_0 \vec{u})](x) := \begin{cases} iu_1'(x), & x > 0, \\ -iu_2'(-x), & x < 0. \end{cases}$$

Comparing the two expressions above, we conclude that

$$A_0(T_1(\vec{u}))(x) = [T(B_0 \vec{u})](x)$$

for all $x \in (-\infty, 0) \cup (0, \infty)$.

Therefore, (7.2.4) holds, and hence the diagram 7.3 is commuting. $\square$

We have already studied in the previous section the first-order differential operator, see Section 3.2, and $A_1 = A_0$, acting on the Hilbert space

$$Y = L^2((-\infty, 0) \cup (0, \infty); \mathbb{C}),$$

with domain

$$D(A_0) = C_0^\infty((-\infty, 0) \cup (0, \infty); \mathbb{C}), \quad A_0 u = iDu.$$

In particular, Proposition 3.20 gives a characterization of the domain of the adjoint A_1^* , namely

$$A_1^* u = iDu \quad \text{for } u \in D(A_1^*),$$

where u is locally absolutely continuous on both half-lines $(-\infty, 0)$ and $(0, \infty)$ and satisfies

$$\int_{-\infty}^0 |Du(x)|^2 dx < \infty, \quad \int_0^\infty |Du(x)|^2 dx < \infty.$$

Moreover, Theorem 3.18 determines the deficiency indices of the operator $A_1 = A_0$, i.e.

$$n_+(A_1) = 1, \quad n_-(A_1) = 1.$$

We now show that the operator B_0 introduced in (7.2.1) has the same properties. The key observation is that the operators A_0 and B_0 are related through the unitary transformation constructed in Proposition 7.1.

More precisely, Proposition 7.1 shows that the map

$$T : X \rightarrow Y$$

is unitary and has inverse $\Lambda : Y \rightarrow X$. Furthermore, Proposition 7.5 proves that A_0 and B_0 satisfy the relation

$$A_0 \circ T_1 = T \circ B_0, \tag{7.2.5}$$

where $T_1 = T|_{X_1}$.

Relation (7.2.5) means that the operators A_0 and B_0 are *unitarily equivalent*. Consequently, the adjoints, deficiency subspaces and deficiency indices of these operators correspond under the unitary map T .

Proposition 7.6. *Let the operators B_0 and A_0 be defined by (7.2.1) and (7.2.2). Then the following two conditions are equivalent.*

$$(i) \quad \vec{u} = (u_1, u_2) \in D(B_0^*).$$

$$(ii) \quad \vec{u} \in L^2((0, \infty); \mathbb{C}^2) \text{ and } u_1, u_2 \text{ are locally absolutely continuous on } (0, \infty) \text{ and}$$

$$\int_0^\infty |Du_1(x)|^2 dx < \infty, \quad \int_0^\infty |Du_2(x)|^2 dx < \infty.$$

Moreover

$$B_0^* \vec{u} = (iDu_1, -iDu_2).$$

Proof of Proposition 7.6. By Proposition 7.1, the map $T : X \rightarrow Y$ is unitary and $\Lambda = T^{-1}$. Moreover, by Proposition 7.5 we have

$$A_0 \circ T_1 = T \circ B_0.$$

Let $\vec{u} \in X$. Then $\vec{u} \in D(B_0^*)$ if and only if

$$T\vec{u} \in D(A_0^*).$$

By Proposition 3.20, the domain $D(A_0^*)$ consists of functions $v \in L^2((-\infty, 0) \cup (0, \infty))$ that are locally absolutely continuous on both half-lines and satisfy

$$\int_{-\infty}^0 |Dv(x)|^2 dx < \infty, \quad \int_0^{\infty} |Dv(x)|^2 dx < \infty.$$

Since

$$(T\vec{u})(x) = \begin{cases} u_1(x), & x > 0, \\ u_2(-x), & x < 0, \end{cases}$$

these conditions are equivalent to

$$\int_0^{\infty} |Du_1(x)|^2 dx < \infty, \quad \int_0^{\infty} |Du_2(x)|^2 dx < \infty,$$

with u_1, u_2 locally absolutely continuous on $(0, \infty)$. This proves the characterization of $D(B_0^*)$.

Finally, using again the relation $A_0 T_1 = T B_0$, we obtain

$$B_0^* \vec{u} = (iDu_1, -iDu_2).$$

□

Corollary 7.7. *Assume that the operator B_0 defined by (7.2.1). Then the deficiency indices of the operator B_0 satisfy*

$$\begin{aligned} n_+(B_0) &= 1, \\ n_-(B_0) &= 1. \end{aligned}$$

In particular, B_0 admits at least one self-adjoint extension.

Proof of Corollary 7.7. By Theorem 3.18,

$$n_+(A_0) = 1, \quad n_-(A_0) = 1.$$

Since T is unitary and $A_0 \circ T_1 = T \circ B_0$, the operators A_0 and B_0 are unitarily equivalent. Therefore their deficiency spaces are isomorphic and

$$n_+(B_0) = n_+(A_0), \quad n_-(B_0) = n_-(A_0).$$

Hence

$$n_+(B_0) = 1, \quad n_-(B_0) = 1.$$

Since the deficiency indices are equal, von Neumann's extension theorem, see Theorem 2.60, implies that B_0 admits at least one self-adjoint extension. $\square$

7.3 Equivalence between two problems related to third-order homogeneous differential operators

We consider two Hilbert spaces

$$X = H_2 = L^2(0, \infty; \mathbb{C}^2)$$

and

$$Y = H_1 = L^2((-\infty, 0) \cup (0, \infty), \mathbb{C})$$

Let define operator B_0 in X and A_0 in Y by

$$\begin{aligned} D(B_0) &= C_0^\infty(0, \infty; \mathbb{C}^2), \\ B_0 \vec{u} &= \begin{pmatrix} -iD^3 u_1, iD^3 u_2 \end{pmatrix} \quad \text{on } (0, \infty), \end{aligned} \tag{7.3.1}$$

where $\vec{u} = (u_1, u_2) : (0, \infty) \rightarrow \mathbb{C}^2$,

$$\begin{aligned} D(A_0) &= C_0^\infty((-\infty, 0) \cup (0, \infty); \mathbb{C}), \\ A_0 u &= -iD^3 u \text{ if } u \in D(A_0). \end{aligned} \tag{7.3.2}$$

The following remark explains why the operator A_0 is homogeneous of order 3.

Remark 7.8. Let $v = \mathcal{F}u$ denote the Fourier transform of u . Using the property of the Fourier transform of derivatives we infer that

$$\mathcal{F}(-iD^3 u)(\xi) = -i\xi^3 v(\xi).$$

Thus the symbol of the operator $A_0 = -iD^3$ is

$$\sigma(\xi) = -i\xi^3.$$

Since

$$\sigma(\lambda\xi) = \lambda^3 \sigma(\xi) \quad \forall \lambda \in \mathbb{R},$$

the operator A_0 is homogeneous of order 3.

Similarly, for the first-order operator iD we have

$$\mathcal{F}(iDu)(\xi) = i\xi v(\xi),$$

hence this operator is homogeneous of order 1.

More generally, if

$$A = -iD^3 + iD,$$

then its symbol is

$$\sigma(\xi) = -i\xi^3 + i\xi,$$

which is not homogeneous since

$$\sigma(\lambda\xi) \neq \lambda^m \sigma(\xi)$$

for any fixed m .

Finally, the Laplacian

$$A = \Delta$$

has symbol $|\xi|^2$ and therefore is homogeneous of order 2.

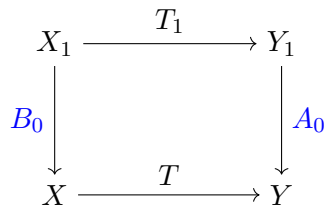


FIGURE 7.4: Commutative diagram

Proposition 7.9. *Let $T : X \rightarrow Y$ be the map defined by the equations (7.1.3). Let the maps B_0 and A_0 be defined by (7.3.1) and (7.3.2) respectively. Then the diagram 7.4 above is commuting, i.e.*

$$A_0 \circ T_1 = T \circ B_0. \quad (7.3.3)$$

Proof of Proposition 7.9. We want to prove that equation (7.3.3) holds, i.e.

$$A_0(T_1(\vec{u})) = T(B_0(\vec{u})) \quad (7.3.4)$$

for every $\vec{u} \in X_1$.

Let $\vec{u} = (u_1, u_2) \in X_1 = C_0^\infty(0, \infty; \mathbb{C}^2)$. Denote $v = T_1 \vec{u}$. By definition of the map T we have

$$v(x) = \begin{cases} u_1(x) & \text{if } x > 0, \\ u_2(-x) & \text{if } x < 0. \end{cases}$$

Applying the operator $A_0 = -iD^3$, we have

$$A_0 v(x) = \begin{cases} -iD^3 u_1(x) & \text{if } x > 0, \\ -iD^3(u_2(-x)) & \text{if } x < 0. \end{cases}$$

Using the chain rule we obtain

$$D^3(u_2(-x)) = -u_2'''(-x),$$

hence

$$A_0 v(x) = \begin{cases} -i u_1'''(x) & \text{if } x > 0, \\ i u_2'''(-x) & \text{if } x < 0. \end{cases}$$

Now compute the right-hand side of (7.3.4). First we apply B_0 to $\vec{u}$:

$$B_0(\vec{u}) = (-iu_1''', iu_2''').$$

Applying T we obtain

$$[T(B_0 \vec{u})](x) = \begin{cases} -iu_1'''(x) & \text{if } x > 0, \\ iu_2'''(-x) & \text{if } x < 0. \end{cases} \quad (7.3.5)$$

Comparing the expressions for $A_0 v$ and $T(B_0 \vec{u})$ we obtain

$$A_0(T_1 \vec{u})(x) = [T(B_0 \vec{u})](x) \quad \text{for all } x \in (-\infty, 0) \cup (0, \infty).$$

Therefore

$$A_0 \circ T_1 = T \circ B_0,$$

which proves that the diagram 7.4 is commuting.

□

We have already studied the third-order differential operator $A_1 = A_0$, acting on the Hilbert space

$$Y = L^2((-\infty, 0) \cup (0, \infty); \mathbb{C})$$

and defined by (5.4.1) in Section 5.4. In particular, Proposition 5.25 gives a characterization of the domain of the adjoint A_1^* , i.e.

$$A_1^* v = (iD)^3 v = -iv''' \quad \text{for } v \in D(A_1^*),$$

while Theorem 5.23 shows that

$$n_+(A_1) = 3, \quad n_-(A_1) = 3.$$

On the other hand, Proposition 7.1 shows that the map $T : X \rightarrow Y$ is unitary, with inverse $\Lambda : Y \rightarrow X$. Moreover, in the third-order case the corresponding commutative relation proved earlier reads $A_0 \circ T_1 = T \circ B_0$.

Therefore the operators A_0 and B_0 correspond under the unitary map T . Consequently, their adjoints, deficiency subspaces and deficiency indices correspond as well.

Corollary 7.10. *Let the operators B_0 and A_0 be defined by the corresponding third-order formulas in $X = L^2(0, \infty; \mathbb{C}^2)$ and $Y = L^2((-\infty, 0) \cup (0, \infty); \mathbb{C})$. Then the following two conditions are equivalent.*

$$(i) \quad \vec{u} = (u_1, u_2) \in D(B_0^*).$$

(ii) $\vec{u} \in L^2((0, \infty); \mathbb{C}^2)$ and the functions $u_1, u_2 : (0, \infty) \rightarrow \mathbb{C}$ are of class C^2 , their second derivatives u_1'', u_2'' are locally absolutely continuous, and

$$\int_0^\infty |u_1'''(x)|^2 dx < \infty, \quad \int_0^\infty |u_2'''(x)|^2 dx < \infty.$$

Moreover,

$$B_0^* \vec{u} = ((iD)^3 u_1, (iD)^3 u_2) = (-iu_1''', iu_2''') \quad \text{for } \vec{u} \in D(B_0^*).$$

Proof. By Proposition 7.1, the map $T : X \rightarrow Y$ is unitary and its inverse is Λ . Moreover, by the commutative diagram for the third-order case, we have

$$A_0 \circ T_1 = T \circ B_0.$$

Let $\vec{u} \in X$. Then $\vec{u} \in D(B_0^*)$ if and only if

$$T\vec{u} \in D(A_0^*).$$

By Proposition 5.25, the domain $D(A_0^*)$ consists of functions $v \in L^2((-\infty, 0) \cup (0, \infty); \mathbb{C})$ such that the restrictions of v to $(0, \infty)$ and $(-\infty, 0)$ are of class C^2 , their second derivatives are locally absolutely continuous, and

$$\int_0^\infty |v'''(x)|^2 dx < \infty, \quad \int_{-\infty}^0 |v'''(x)|^2 dx < \infty.$$

Since

$$(T\vec{u})(x) = \begin{cases} u_1(x), & x > 0, \\ u_2(-x), & x < 0, \end{cases}$$

these conditions are equivalent to requiring that u_1, u_2 are of class C^2 on $(0, \infty)$, their second derivatives are locally absolutely continuous, and

$$\int_0^\infty |u_1'''(x)|^2 dx < \infty, \quad \int_0^\infty |u_2'''(x)|^2 dx < \infty.$$

This proves the characterization of $D(B_0^*)$.

Finally, using again the relation $A_0 T_1 = T B_0$, we obtain

$$B_0^* \vec{u} = ((iD)^3 u_1, (iD)^3 u_2) = (-iu_1''', iu_2''').$$

□

Theorem 7.11. *Let B_0 be the operator defined by (7.3.1) in the Hilbert space*

$$X = L^2(0, \infty; \mathbb{C}^2).$$

Then B_0 is symmetric and its deficiency indices satisfy

$$n_+(B_0) = 3,$$

$$n_-(B_0) = 3.$$

Proof of Theorem 7.11. By Proposition 7.1, the map $T : X \rightarrow Y$ is unitary. Furthermore the operators A_0 and B_0 satisfy the intertwining relation

$$A_0 \circ T_1 = T \circ B_0.$$

Hence the operators A_0 and B_0 are unitarily equivalent.

Unitary equivalence preserves the adjoint operator and the deficiency subspaces. Therefore

$$n_+(B_0) = n_+(A_0), \quad n_-(B_0) = n_-(A_0).$$

By Theorem 5.23 we know that

$$n_+(A_0) = 3, \quad n_-(A_0) = 3.$$

Hence

$$n_+(B_0) = 3, \quad n_-(B_0) = 3.$$

□

Since the deficiency indices of B_0 are equal, von Neumann's extension theorem, see Theorem 2.60, implies that B_0 admits at least one self-adjoint extension.

7.4 Equivalence between two problems related to the third-order linear in-homogeneous differential operators

We consider two Hilbert spaces

$$X = H_2 = L^2(0, \infty; \mathbb{C}^2)$$

and

$$Y = H_1 = L^2((-\infty, 0) \cup (0, \infty), \mathbb{C})$$

Let define operator B_0 in X and A_0 in Y by respectively

$$\begin{aligned} D(B_0) &= C_0^\infty(0, \infty; \mathbb{C}^2), \\ B_0 \vec{u} &= \left(a_1 i D u_1 + a_3 (-i) D^3 u_1, -a_1 i D u_2 + a_3 i D^3 u_2 \right) \quad \text{on } (0, \infty). \end{aligned} \tag{7.4.1}$$

when $\vec{u} = (u_1, u_2) : (0, \infty) \rightarrow \mathbb{C}^2$,

$$\begin{aligned} D(A_0) &= C_0^\infty((-\infty, 0) \cup (0, \infty); \mathbb{C}), \\ A_0 u &= a_1 i D u + a_3 (-i) D^3 u \text{ if } u \in D(A_0). \end{aligned} \tag{7.4.2}$$

Let us define auxiliary operators

$$\begin{aligned} D(B_{01}) &= C_0^\infty(0, \infty; \mathbb{C}^2), \\ B_{01} \vec{u} &= (i D u_1, -i D u_2) \quad \text{on } (0, \infty). \end{aligned}$$

and

$$\begin{aligned} D(B_{02}) &= C_0^\infty(0, \infty; \mathbb{C}^2), \\ B_{02} \vec{u} &= (-i D^3 u_1, i D^3 u_2) \quad \text{on } (0, \infty). \end{aligned}$$

Similarly, define the operators A_{01} and A_{02} in Y by

$$\begin{aligned} D(A_{01}) &= C_0^\infty((-\infty, 0) \cup (0, \infty); \mathbb{C}), \\ A_{01} u &= i D u. \end{aligned}$$

and

$$D(A_{02}) = C_0^\infty((-\infty, 0) \cup (0, \infty); \mathbb{C}),$$

$$A_{02}u = -iD^3u.$$

Lemma 7.12. *The operators B_0 and A_0 admit the decompositions*

$$B_0 = a_1 B_{01} + a_3 B_{02},$$

and

$$A_0 = a_1 A_{01} + a_3 A_{02}.$$

The operators A_{01} and B_{01} are those considered in Subsection 3.1. Hence, by Proposition 7.5 we have

$$A_{01} \circ T_1 = T \circ B_{01}. \quad (7.4.3)$$

Similarly, from the third-order homogeneous case, see Proposition 7.9, we obtain

$$A_{02} \circ T_1 = T \circ B_{02}. \quad (7.4.4)$$

$$\begin{array}{ccc} X_1 & \xrightarrow{T_1} & Y_1 \\ B_0 \downarrow & & \downarrow A_0 \\ X & \xrightarrow{T} & Y \end{array}$$

FIGURE 7.5: Commutative diagram

Proposition 7.13. *Let $T : X \rightarrow Y$ be the map defined by the equations (7.1.3). Let the maps B_0 and A_0 be defined by (7.4.1) and (7.4.2) respectively. Then the diagram 7.5 above is commuting, i.e.*

$$A_0 \circ T_1 = T \circ B_0. \quad (7.4.5)$$

Proof of Proposition 7.13. By Lemma 7.12, we have

$$B_0 = a_1 B_{01} + a_3 B_{02}, \quad A_0 = a_1 A_{01} + a_3 A_{02}.$$

Using the commuting relations (7.4.3) and (7.4.4), we obtain

$$\begin{aligned}
 A_0 \circ T_1 &= (a_1 A_{01} + a_3 A_{02}) \circ T_1 \\
 &= a_1 A_{01} \circ T_1 + a_3 A_{02} \circ T_1 \\
 &= a_1 T \circ B_{01} + a_3 T \circ B_{02} \\
 &= T \circ (a_1 B_{01} + a_3 B_{02}) \\
 &= T \circ B_0.
 \end{aligned}$$

Hence $A_0 \circ T_1 = T \circ B_0$, which proves (7.4.5). $\square$

We have already studied the third-order linear inhomogeneous differential operator $A_1 = A_0$, acting on the Hilbert space

$$Y = L^2((-\infty, 0) \cup (0, \infty); \mathbb{C})$$

and defined by (6.4.1) in Section 6.4. In particular, Lemma 6.47 gives a characterization of the domain of the adjoint A_1^* , i.e.

$$A_1^* v = a_1 i v' + a_3 (-i) v''' = -i(a_3 D^3 v - a_1 Dv), \quad \text{for } v \in D(A_1^*),$$

while Theorem 6.56 shows that

$$n_+(A_1) = 3, \quad n_-(A_1) = 3.$$

On the other hand, Proposition 7.1 shows that the map $T : X \rightarrow Y$ is unitary, with inverse $\Lambda : Y \rightarrow X$. Moreover, in the present case the corresponding commutative relation proved earlier reads

$$A_0 \circ T_1 = T \circ B_0.$$

Therefore the operators A_0 and B_0 correspond under the unitary map T . Consequently, their adjoints, deficiency subspaces and deficiency indices correspond as well.

Corollary 7.14. *Let the operators B_0 and A_0 be defined by the corresponding third-order inhomogeneous formulas in $X = L^2(0, \infty; \mathbb{C}^2)$ and $Y = L^2((-\infty, 0) \cup (0, \infty); \mathbb{C})$. Then the following two conditions are equivalent.*

(i) $\vec{u} = (u_1, u_2) \in D(B_0^*)$.

(ii) $\vec{u} \in L^2((0, \infty); \mathbb{C}^2)$ and the functions $u_1, u_2 : (0, \infty) \rightarrow \mathbb{C}$ are of class C^2 , their second derivatives u_1'', u_2'' are locally absolutely continuous, and

$$a_3 D^3 u_1 - a_1 D u_1 \in L^2(0, \infty; \mathbb{C}), \quad a_3 D^3 u_2 - a_1 D u_2 \in L^2(0, \infty; \mathbb{C}).$$

Moreover,

$$B_0^* \vec{u} = (a_1 i D u_1 + a_3 (-i) D^3 u_1, -a_1 i D u_2 + a_3 i D^3 u_2)$$

for all $\vec{u} \in D(B_0^*)$.

Proof. By Proposition 7.1, the map $T : X \rightarrow Y$ is unitary and its inverse is Λ . Moreover, by Proposition 7.13, we have $A_0 \circ T_1 = T \circ B_0$.

Let $\vec{u} \in X$. Then $\vec{u} \in D(B_0^*)$ if and only if

$$T \vec{u} \in D(A_0^*).$$

By Lemma 6.47, the domain $D(A_0^*)$ consists of functions $v \in L^2((-\infty, 0) \cup (0, \infty); \mathbb{C})$ such that:

- v is of class C^2 on each half-line,
- v'' is locally absolutely continuous on each half-line,
-

$$a_3 D^3 v - a_1 D v \in L^2((-\infty, 0) \cup (0, \infty); \mathbb{C}).$$

Since

$$(T \vec{u})(x) = \begin{cases} u_1(x), & x > 0, \\ u_2(-x), & x < 0, \end{cases}$$

these conditions are equivalent to requiring that u_1, u_2 are of class C^2 on $(0, \infty)$, their second derivatives are locally absolutely continuous, and

$$a_3 D^3 u_1 - a_1 D u_1 \in L^2(0, \infty; \mathbb{C}), \quad a_3 D^3 u_2 - a_1 D u_2 \in L^2(0, \infty; \mathbb{C}).$$

This proves the characterization of $D(B_0^*)$.

Finally, using again the relation $A_0 T_1 = T B_0$, we obtain

$$B_0^* \vec{u} = (a_1 i D u_1 + a_3 (-i) D^3 u_1, -a_1 i D u_2 + a_3 i D^3 u_2).$$

□

Theorem 7.15. Let B_0 be the operator defined by (7.4.1) in the Hilbert space

$$X = L^2(0, \infty; \mathbb{C}^2).$$

Then B_0 is symmetric and its deficiency indices satisfy

$$\begin{aligned} n_+(B_0) &= 3, \\ n_-(B_0) &= 3. \end{aligned}$$

Proof of Theorem 7.15. By Proposition 7.1, the map $T : X \rightarrow Y$ is unitary. Moreover, by Proposition 7.13, the operators A_0 and B_0 correspond under T . Hence their deficiency subspaces correspond, and therefore

$$n_+(B_0) = n_+(A_0), \quad n_-(B_0) = n_-(A_0).$$

By Theorem 6.56,

$$n_+(A_0) = 3, \quad n_-(A_0) = 3.$$

Consequently,

$$n_+(B_0) = 3, \quad n_-(B_0) = 3.$$

□

Since the deficiency indices of B_0 are equal, von Neumann's extension theorem implies that B_0 admits at least two self-adjoint extensions.

We now summarize the above results and their consequences.

We have already studied the third-order linear inhomogeneous differential operator $A_1 = A_0$ acting on the Hilbert space

$$Y = L^2((-\infty, 0) \cup (0, \infty); \mathbb{C})$$

and defined by (6.4.1) in Section 6.4. In particular, Lemma 6.47 characterizes the domain of the adjoint $A_1^* = A_0^*$, namely

$$A_1^*v = a_1iv' + a_3(-i)v''' = -i(a_3D^3v - a_1Dv), \quad v \in D(A_1^*),$$

while Theorem 6.56 shows that

$$n_+(A_1) = 3, \quad n_-(A_1) = 3.$$

On the other hand, Proposition 7.1 shows that the map $T : X \rightarrow Y$ is unitary, with inverse $\Lambda : Y \rightarrow X$. Moreover, by Proposition 7.13, we have

$$A_0 \circ T_1 = T \circ B_0.$$

Therefore the operators A_0 and B_0 correspond under the unitary map T . Consequently, their adjoints, deficiency subspaces and deficiency indices correspond as well.

In what follows, we extend the equivalence framework developed above to systems of differential operators. This allows us to treat vector-valued functions and to obtain analogous results for first- and third-order operators, both in the homogeneous and in-homogeneous settings.

7.5 Systems of first-order differential operators

This section can also be interpreted as a study of first order differential operators on certain graphs. This section studies a generalization of the operator B_0 studied in subsection 7.2.

We assume that $n \in \mathbb{N}$ such that $n \geq 2$. We consider the following Hilbert space

$$X = L^2(0, \infty; \mathbb{C}^n)$$

Let

$$a_1, \dots, a_n \in \mathbb{R},$$

when $\vec{u} = (u_1, u_2, \dots, u_n) : (0, \infty) \rightarrow \mathbb{C}^n$. Let us define the operator B_0 in X by

$$\begin{aligned} D(B_0) &= C_0^\infty(0, \infty; \mathbb{C}^n), \\ B_0 \vec{u} &= (a_1 i D u_1, a_2 i D u_2, \dots, a_n i D u_n) \quad \text{on } (0, \infty). \end{aligned} \tag{7.5.1}$$

The first property is now trivial, as it generalizes immediately from the scalar case $n = 1$, which was treated in Subsection 3.2.1.

Proposition 7.16. *Let B_0 be the operator defined by (7.5.1). Then B_0 is a linear symmetric operator in the Hilbert space X .*

Proof of Proposition 7.16. We must show that the operator B_0 is symmetric in X , that is,

$$\langle B_0 \vec{u}, \vec{v} \rangle_X = \langle \vec{u}, B_0 \vec{v} \rangle_X \quad \text{for all } \vec{u}, \vec{v} \in D(B_0). \tag{7.5.2}$$

Let us choose arbitrary $\vec{u}, \vec{v} \in D(B_0) = C_0^\infty(0, \infty; \mathbb{C}^n)$. Then

$$\begin{aligned} \langle B_0 \vec{u}, \vec{v} \rangle_X &= \sum_{j=1}^n \int_0^\infty (a_j i u'_j(x)) \overline{v_j(x)} dx \\ &= i \sum_{j=1}^n a_j \int_0^\infty u'_j(x) \overline{v_j(x)} dx. \end{aligned} \quad (7.5.3)$$

Using integration by parts we obtain

$$\int_0^\infty u'_j(x) \overline{v_j(x)} dx = \left[u_j(x) \overline{v_j(x)} \right]_0^\infty - \int_0^\infty u_j(x) \overline{v'_j(x)} dx.$$

Since $u_j, v_j \in C_0^\infty(0, \infty)$, the boundary terms vanish, and therefore

$$\int_0^\infty u'_j(x) \overline{v_j(x)} dx = - \int_0^\infty u_j(x) \overline{v'_j(x)} dx. \quad (7.5.4)$$

Substituting (7.5.4) into (7.5.3) gives

$$\langle B_0 \vec{u}, \vec{v} \rangle_X = -i \sum_{j=1}^n a_j \int_0^\infty u_j(x) \overline{v'_j(x)} dx.$$

Observing that

$$\overline{a_j i v'_j(x)} = -i a_j \overline{v'_j(x)},$$

we obtain

$$\langle B_0 \vec{u}, \vec{v} \rangle_X = \sum_{j=1}^n \int_0^\infty u_j(x) \overline{a_j i v'_j(x)} dx = \langle \vec{u}, B_0 \vec{v} \rangle_X.$$

Thus (7.5.2) holds, and therefore the operator B_0 is symmetric. $\square$

The following result generalizes Propositions 3.29 and 3.34.

Proposition 7.17. *Assume that B_0 is the operator defined by formula (7.5.1). Then B_0 is not closed.*

Moreover, a function $\vec{u} \in D(\overline{B_0})$ if and only if the following conditions are satisfied

- (i) $\vec{u} \in L^2(0, \infty; \mathbb{C}^n)$,
- (ii) $\vec{u}$ is locally absolutely continuous on $(0, \infty)$, and

$$\int_0^\infty |D\vec{u}(t)|_{\mathbb{C}^n}^2 dt < \infty,$$

- (iii) $\vec{u}(0) = 0 \in \mathbb{C}^n$.

Moreover, for any $\vec{u} \in D(\overline{B_0})$, we have:

$$\overline{B_0} \vec{u} = (a_1 i D u_1, \dots, a_n i D u_n).$$

Proof of Proposition 7.17. We show that B_0 is not closed. The proof follows the same method as in Proposition 3.29. To do that with more details, it suffices to find a pair $(\vec{u}, \vec{z}) \in L^2 \times L^2$ and sequence $(\vec{u}_k) \subset D(B_0)$ such that

$$\vec{u}_k \rightarrow \vec{u}, \quad B_0 \vec{u}_k \rightarrow \vec{z},$$

but $\vec{u} \notin D(B_0)$.

Let us choose a function

$$\varphi(x) = \begin{cases} e^{-x^{-2}}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

It is well known, see [37, Chapter 6], that

- $\varphi \in C^\infty(0, \infty) \cap H^1(0, \infty)$,
- $\varphi(0) = 0$ and $\varphi^{(k)}(0) = 0$ for all $k \geq 1$,
- but $\varphi \notin C_0^\infty(0, \infty)$ (i.e., not compactly supported).

Define

$$\vec{u}(x) := (\varphi(x), 0, \dots, 0).$$

Clearly, $\vec{u} \in L^2(0, \infty; \mathbb{C}^n)$ and $D\vec{u}$ exists and is equal to $(D\varphi, 0, \dots, 0)$. Therefore, $D\vec{u} \in L^2(0, \infty; \mathbb{C}^n)$.

Since $C_0^\infty(0, \infty)$ is dense in $H_0^1(0, \infty)$, see [1, Section 3.2], there exists $\varphi_k \in C_0^\infty(0, \infty)$ such that

$$\varphi_k \rightarrow \varphi \quad \text{in } H^1(0, \infty). \quad (7.5.5)$$

Define

$$\vec{u}_k := (\varphi_k, 0, \dots, 0) \in D(B_0).$$

Then by (7.5.5)

$$\vec{u}_k \rightarrow \vec{u} \quad \text{in } L^2$$

and

$$B_0 \vec{u}_k = (a_1 i D \varphi_k, 0, \dots, 0) \rightarrow (a_1 i D \varphi, 0, \dots, 0) =: \vec{z}.$$

However $\vec{u} \notin D(B_0)$ because φ is not compactly supported.

Hence the graph $\Gamma(B_0)$ is not closed and therefore B_0 is not closed.

Part II: the case $n = 2$. Recall that in Subsection 7.2, we introduced the operator A_0 acting on

$$Y = L^2((-\infty, 0) \cup (0, \infty); \mathbb{C}),$$

and we proved in Proposition 7.5 that the mapping

$$T : X \rightarrow Y, \quad X = L^2(0, \infty; \mathbb{C}^2),$$

defined in (7.1.3), is an isometric isomorphism satisfying

$$A_0 \circ T = T \circ B_0.$$

By Proposition 3.34, applied to the operator A_0 , the domain of $\overline{A_0}$ is characterized by:

- (i) $u \in L^2((-\infty, 0) \cup (0, \infty))$,
- (ii) u is absolutely continuous on each half-line and $Du \in L^2$,
- (iii) $u(0) = 0$.

Since T is an isomorphism and intertwines the operators, we have

$$\vec{u} \in D(\overline{B_0}) \iff T(\vec{u}) \in D(\overline{A_0}).$$

Therefore, $\vec{u} = (u_1, u_2)$ belongs to $D(\overline{B_0})$ if and only if:

- (i) $\vec{u} \in L^2(0, \infty; \mathbb{C}^2)$,
- (ii) each u_j is absolutely continuous on $(0, \infty)$ and $Du_j \in L^2(0, \infty)$,
- (iii) $\vec{u}(0) = 0 \in \mathbb{C}^2$.

Moreover, using the commutation relation (7.2.3), we obtain

$$\overline{B_0} \vec{u} = T^{-1}(\overline{A_0} T(\vec{u})) = (a_1 i D u_1, a_2 i D u_2).$$

Conversely, if $\vec{u}$ satisfies conditions (i)–(iii), then $u = T(\vec{u})$ belongs to $D(\overline{A_0})$, and therefore $\vec{u} \in D(\overline{B_0})$.

This completes Part II for $n = 2$.

Part III: the case $n \geq 3$. We prove the result directly for general $n \geq 3$, following the same method as in the proof of Proposition 3.34, applied componentwise.

Assume that $\vec{u} = (u_1, \dots, u_n) \in D(\overline{B_0})$. By Definition 2.85, there exists a sequence $(\vec{u}_k) \subset D(B_0) = C_0^\infty(0, \infty; \mathbb{C}^n)$ such that

$$\vec{u}_k \rightarrow \vec{u} \quad \text{in } L^2(0, \infty; \mathbb{C}^n),$$

and

$$B_0 \vec{u}_k = (a_1 i Du_1^{(k)}, \dots, a_n i Du_n^{(k)}) \rightarrow \vec{z} \quad \text{in } L^2(0, \infty; \mathbb{C}^n).$$

For each $j = 1, \dots, n$, we have

$$u_j^{(k)} \rightarrow u_j \quad \text{in } L^2(0, \infty), \quad a_j i Du_j^{(k)} \rightarrow z_j \quad \text{in } L^2(0, \infty).$$

Hence,

$$Du_j^{(k)} \rightarrow -\frac{i}{a_j} z_j \quad \text{in } L^2(0, \infty).$$

Since $u_j^{(k)} \in C_0^\infty(0, \infty)$, we may write

$$u_j^{(k)}(t) = \int_0^t Du_j^{(k)}(s) ds.$$

Thus, for $k, m \in \mathbb{N}$,

$$u_j^{(k)}(t) - u_j^{(m)}(t) = \int_0^t (Du_j^{(k)}(s) - Du_j^{(m)}(s)) ds.$$

By the Cauchy–Schwarz inequality [38, Theorem 3.5, p. 63],

$$|u_j^{(k)}(t) - u_j^{(m)}(t)| \leq \sqrt{t} \|Du_j^{(k)} - Du_j^{(m)}\|_{L^2(0,t)}.$$

Taking the supremum over $t \in [0, T]$, we obtain

$$\|u_j^{(k)} - u_j^{(m)}\|_{C([0,T])} \leq \sqrt{T} \|Du_j^{(k)} - Du_j^{(m)}\|_{L^2(0,T)}.$$

Since $(Du_j^{(k)})$ is Cauchy in $L^2(0, \infty)$, it follows that $(u_j^{(k)})$ is Cauchy in $C([0, T])$ for every $T > 0$. Hence,

$$u_j^{(k)} \rightarrow u_j \quad \text{uniformly on compact intervals.}$$

Moreover,

$$Du_j^{(k)} \rightarrow Du_j \quad \text{in } L^2(0, \infty),$$

which implies that u_j is locally absolutely continuous and

$$Du_j = -\frac{i}{a_j} z_j \in L^2(0, \infty).$$

Since each $u_j^{(k)} \in C_0^\infty(0, \infty)$, we have

$$u_j^{(k)}(0) = 0.$$

Passing to the limit using uniform convergence on $[0, T]$, we obtain

$$u_j(0) = 0.$$

Combining the above results, we conclude that

- (i) $\vec{u} \in L^2(0, \infty; \mathbb{C}^n)$,
- (ii) each u_j is locally absolutely continuous on $(0, \infty)$ and $Du_j \in L^2(0, \infty)$,
- (iii) $\vec{u}(0) = 0 \in \mathbb{C}^n$.

Finally, passing to the limit in $B_0 \vec{u}_k$, we obtain

$$\overline{B_0} \vec{u} = (a_1 i Du_1, \dots, a_n i Du_n).$$

This concludes the proof of part III.

Hence the proof of Proposition 7.17 is complete. $\square$

The following result characterizes the domain of the adjoint operator.

Proposition 7.18. *Assume that $\vec{u} = (u_1, u_2, \dots, u_n) : (0, \infty) \rightarrow \mathbb{C}^n$ belongs to the space X . Then $\vec{u}$ belongs to the domain $D(B_0^*)$ if and only if for every $k = 1, \dots, n$, the component u_k is locally absolutely continuous and*

$$\int_0^\infty |Du_k(t)|^2 dt < \infty.$$

Proof of Proposition 7.18. Let $X := L^2(0, \infty; \mathbb{C}^n)$ and recall that B_0 is defined by (7.5.1).

We characterize the domain of the adjoint operator B_0^* . By the Riesz Representation Theorem, see Proposition 2.54, a function $\vec{v} \in X$ belongs to $D(B_0^*)$ if and only if there

exists a constant $C > 0$ such that

$$|\langle B_0 \vec{u}, \vec{v} \rangle_X| \leq C \|\vec{u}\|_X, \quad \forall \vec{u} \in D(B_0).$$

Let $\vec{v} = (v_1, \dots, v_n) \in X$ and take $\vec{u} \in D(B_0) = C_0^\infty(0, \infty; \mathbb{C}^n)$. Then

$$\langle B_0 \vec{u}, \vec{v} \rangle_X = \sum_{j=1}^n \int_0^\infty a_j i Du_j(t) \overline{v_j(t)} dt.$$

Since each $u_j \in C_0^\infty(0, \infty)$, integration by parts yields

$$\int_0^\infty Du_j(t) \overline{v_j(t)} dt = - \int_0^\infty u_j(t) \overline{Dv_j(t)} dt,$$

with no boundary terms.

Thus,

$$\langle B_0 \vec{u}, \vec{v} \rangle_X = - \sum_{j=1}^n a_j i \int_0^\infty u_j(t) \overline{Dv_j(t)} dt = \langle \vec{u}, (-a_1 i Dv_1, \dots, -a_n i Dv_n) \rangle_X.$$

Therefore, the mapping $\vec{u} \mapsto \langle B_0 \vec{u}, \vec{v} \rangle_X$ extends to a bounded linear functional on X if and only if each $Dv_j \in L^2(0, \infty)$.

Moreover, the above integration by parts is valid if and only if each v_j is locally absolutely continuous on $(0, \infty)$.

Hence,

$$\vec{v} \in D(B_0^*) \iff \begin{cases} v_j \in AC_{\text{loc}}(0, \infty), \\ Dv_j \in L^2(0, \infty), \\ \text{for all } j = 1, \dots, n. \end{cases}$$

This completes the proof. □

Remark 7.19. The condition that each component u_k is locally absolutely continuous is not accidental. It is required in order to justify the use of integration by parts in the characterization of the adjoint operator.

Indeed, for functions $u \in C_0^\infty(0, \infty)$ and $v \in L^2(0, \infty)$, the identity

$$\int_0^\infty Du(t) \overline{v(t)} dt = - \int_0^\infty u(t) \overline{Dv(t)} dt$$

holds if and only if v admits a weak derivative Dv which belongs to $L^2(0, \infty)$. This is equivalent to v being locally absolutely continuous.

Therefore, the requirement

$$v_k \in AC_{\text{loc}}(0, \infty), \quad Dv_k \in L^2(0, \infty),$$

ensures that the formal integration by parts used in the computation of the adjoint is mathematically justified.

In other words, the domain of the adjoint operator consists precisely of those functions for which the differential expression defining B_0 can be transferred to the other side of the inner product.

7.5.1 Deficiency indices of first-order differential systems on the half-line

In this subsection, we compute the deficiency indices of the operator B_0 defined in (7.5.1). The computation is based on solving the deficiency equations

$$(iI \mp B_0^*)\vec{u} = 0$$

Let us introduce two subsets of $\{1, \dots, n\}$ according to the sign of the coefficients a_k :

$$K_+ := \{k \in \{1, \dots, n\} : a_k < 0\}, \quad (7.5.6)$$

$$K_- := \{k \in \{1, \dots, n\} : a_k > 0\}. \quad (7.5.7)$$

These sets are disjoint and satisfy

$$K_+ \cup K_- = \{1, \dots, n\}, \quad \#(K_+) + \#(K_-) = n.$$

Note that this convention differs from the third-order case, since the L^2 -integrability of the exponential solutions depends on the sign of $\frac{1}{a_k}$.

The following result provides an explicit formula for the deficiency indices of B_0 .

Proposition 7.20. *The deficiency indices of the operator B_0 are given by*

$$n_-(B_0) = \#K_-, \quad (7.5.8)$$

$$n_+(B_0) = \#K_+, \quad (7.5.9)$$

where the sets K_+ and K_- are defined in (7.5.6) and (7.5.7), respectively.

The special case of this result when $n = 2$ is given (to be done) in subsection 7.2.

Corollary 7.21. *The operator B_0 on $L^2(0, \infty; \mathbb{C}^n)$ defined by (7.5.1) has at least one self-adjoint operator if and only if*

$$\#K_- = \#K_+ \quad (7.5.10)$$

In particular, if n is an odd number, the operator B_0 on $L^2(0, \infty; \mathbb{C}^n)$ has no self-adjoint extensions.

If n is an even number, i.e. $n = 2p$ and $\#K_- = \#K_+ = p$, then B_0 on $L^2(0, \infty; \mathbb{C}^n)$ has self-adjoint extensions. But, if n is an even number and $\#K_- \neq \#K_+ = p$, then B_0 on $L^2(0, \infty; \mathbb{C}^n)$ has no self-adjoint extensions.

Proof of Corollary 7.21. From Proposition 7.20, If n is odd, then

$$\#K_+ + \#K_- = n$$

is odd, hence

$$\#K_+ \neq \#K_-.$$

Therefore, B_0 has no self-adjoint extensions.

If n is even, say $n = 2p$, then

$$\#K_+ = \#K_- = p.$$

Then B_0 has self-adjoint extensions exist. □

If the condition (7.5.10) is satisfied then every self-adjoint operator of B_0 is of the following form.

Proof of Proposition 7.20. To find $n_+(B_0) = \dim (iI - B_0^*)$, let us assume that that $\vec{u}$ belongs to the domain $D(B_0^*)$ and

$$\vec{u} \in \ker(iI - B_0^*). \quad (7.5.11)$$

Note that

$$\begin{aligned} \vec{u} \in \ker(iI - B_0^*) &\iff \vec{u} \in D(B_0^*) \text{ and } (iI - B_0^*)\vec{u} = 0 \\ &\iff \vec{u} \in D(B_0^*) \text{ and } a_k i D u_k = i u_k \quad \forall k = 1, 2, \dots, n \end{aligned}$$

Thus, by Proposition 7.18, for every $k = 1, \dots, n$ the component u_k is locally absolutely continuous and

$$\int_0^\infty [|u_k(x)|^2 + |Du_k(x)|^2] dt < \infty. \quad (7.5.12)$$

and

$$\iff \vec{u} \in D(B_0^*) \text{ and } Du_k = \frac{1}{a_k} u_k \quad \forall k$$

Each equation

$$Dv_k(x) = \frac{1}{a_k} v_k(x), \quad x > 0.$$

implies that the general solution for each component v_k is of the form:

$$v_k(x) = c_k e^{\frac{x}{a_k}}, \quad x > 0.$$

where c_k is a constant.

Hence we proved that if $\vec{u}$ satisfies (7.5.11) then there exists a vector $\vec{c} = (c_1, \dots, c_n) \in \mathbb{C}^n$ such that

$$\vec{u}(x) = (c_1 e^{\frac{x}{a_1}}, \dots, c_n e^{\frac{x}{a_n}}), \quad x > 0$$

We need to check when function $\vec{u}$ satisfies condition (7.5.12). We need to examine the integrability of each component $u_k(x) = c_k e^{\frac{x}{a_k}}$. Let us choose and fix k . We calculate the L^2 -norm of u_k :

$$\int_0^\infty |u_k(x)|^2 dx = \int_0^\infty |c_k e^{\frac{x}{a_k}}|^2 dx = \int_0^\infty |c_k|^2 e^{\frac{2x}{a_k}} dx = |c_k|^2 \int_0^\infty e^{\frac{2x}{a_k}} dx.$$

The integral is finite if and only if $\frac{2}{a_k} < 0$, i.e., $a_k < 0$.

In other words, we proved

- **If** $a_k < 0$, then $u_k \in L^2(0, \infty)$.
- **If** $a_k > 0$, $u_k \notin L^2(0, \infty)$.

This allows us to find the basis of $\ker(iI - B_0^*)$. Namely, let us define, for $k \in K_+$, a function

$$e_k(x) := e^{\frac{x}{a_k}}, \quad x \in (0, \infty).$$

We proved above that every $e_k \in \ker(iI - B_0^*)$ and moreover, every function $\vec{u} \in \ker(iI - B_0^*)$ is a linear combination of these functions e_k , $k \in K_+$. Moreover, these functions are

linearly independent. This all implies that

$$\dim \ker(iI - B_0^*) = \#K_+.$$

This proves the first part of Proposition, i.e. equality (7.5.9).

To find $n_-(B_0) = \dim(iI + B_0^*)$, we assume $\vec{u} \in \ker(iI + B_0^*)$ and follow a similar argument.

Thus, we have

$$\vec{u} \in D(B_0^*) \text{ and } (iI + B_0^*)\vec{u} = 0.$$

This leads to

$$a_k i D u_k = -i u_k \quad \forall k.$$

Thus, we find

$$D u_k = -\frac{1}{a_k} u_k \quad \forall k.$$

The general solution is given by

$$v_k(x) = c_k e^{-\frac{x}{a_k}}, \quad x > 0.$$

To check integrability, let us choose and fix k . We calculate the L^2 -norm of u_k :

$$\int_0^\infty |u_k(x)|^2 dx = |c_k|^2 \int_0^\infty e^{-\frac{2x}{a_k}} dx.$$

The integral is finite if and only if $-\frac{2}{a_k} > 0$, i.e., $a_k > 0$.

In the other words, we proved

- **If** $a_k > 0$, then $u_k \in L^2(0, \infty)$.
- **If** $a_k < 0$, then $u_k \notin L^2(0, \infty)$.

This allows us to find the basis of $\ker(iI + B_0^*)$. Let us define, for $k \in K_-$, a function:

$$e_k(x) := e^{-\frac{x}{a_k}}, \quad x \in (0, \infty).$$

Hence, every $e_k \in \ker(iI + B_0^*)$ for $k \in K_-$ and every function $\vec{u} \in \ker(iI + B_0^*)$ is a linear combination of these functions. These functions are also linearly independent,

which implies:

$$\dim \ker(iI + B_0^*) = \#K_-.$$

Thus, we conclude:

$$n_-(B_0) = \#K_-,$$

which proves the second part of Proposition 7.20, i.e., equality (7.5.8).

Hence the prove of Proposition 7.20 is complete. □

Conclusion

B_0 defined by (7.5.1) on $L^2(0, \infty; \mathbb{C}^n)$ has self-adjoint extensions
if and only if $\#(K_+) = \#(K_-)$.

We now apply Theorem 2.60 to characterize all self-adjoint extensions of the operator B_0 .

Corollary 7.22. *Assume that condition (7.5.10) is satisfied, i.e.*

$$\#K_- = \#K_+.$$

Let

$$H_+ := \ker(B_0^* - iI), \quad H_- := \ker(B_0^* + iI)$$

be the deficiency subspaces of B_0 .

Then

$$\dim H_+ = \dim H_- = \#K_+ = \#K_-.$$

Moreover, by Theorem 2.60, the self-adjoint extensions of B_0 are in one-to-one correspondence with the unitary operators

$$\Gamma : H_+ \rightarrow H_-.$$

More precisely, for each unitary operator $\Gamma : H_+ \rightarrow H_-$, the corresponding self-adjoint extension B_0^Γ is defined by

$$\begin{aligned} D(B_0^\Gamma) &:= \left\{ \vec{u}_0 + \vec{u}_+ + \Gamma \vec{u}_+ : \vec{u}_0 \in D(B_0), \vec{u}_+ \in H_+ \right\}, \\ B_0^\Gamma(\vec{u}_0 + \vec{u}_+ + \Gamma \vec{u}_+) &:= B_0 \vec{u}_0 + i \vec{u}_+ - i \Gamma \vec{u}_+. \end{aligned}$$

Conversely, every self-adjoint extension of B_0 is of this form.

Proof of Corollary 7.22. By Proposition 7.20, we have

$$n_+(B_0) = \#K_+, \quad n_-(B_0) = \#K_-.$$

Hence condition (7.5.10) implies

$$n_+(B_0) = n_-(B_0).$$

Since

$$n_+(B_0) = \dim H_+, \quad n_-(B_0) = \dim H_-,$$

we obtain

$$\dim H_+ = \dim H_- = \#K_+ = \#K_-.$$

Therefore, Theorem 2.60 applies and yields the claimed classification of all self-adjoint extensions of B_0 . $\square$

7.5.2 Deficiency indices of the closure of a system of first-order differential operators

We already know that the closure $\overline{B_0}$ of the operator defined by (7.5.1) exists and that its adjoint coincides with the adjoint of B_0 , i.e.

$$(\overline{B_0})^* = B_0^*.$$

By Corollary 2.92, this equality implies that the deficiency subspaces of $\overline{B_0}$ and B_0 coincide:

$$\ker(i - B_0^*) = \ker(i - \overline{B_0}^*), \quad \ker(i + B_0^*) = \ker(i + \overline{B_0}^*).$$

Consequently, the deficiency indices of B_0 and $\overline{B_0}$ are the same:

$$n_+(\overline{B_0}) = n_+(B_0), \quad n_-(\overline{B_0}) = n_-(B_0).$$

Corollary 7.23. *Let B_0 be the operator defined by formula (7.5.1), and let $\overline{B_0}$ denote its closure in $L^2(0, \infty; \mathbb{C}^n)$.*

Then B_0 is densely defined and symmetric, its closure $\overline{B_0}$ is again symmetric, and their deficiency indices coincide.

In particular, by Corollary 2.92 and Proposition 7.20, we have

$$\begin{aligned} n_-(\overline{B_0}) &= n_-(B_0) = \#K_-, \\ n_+(\overline{B_0}) &= n_+(B_0) = \#K_+. \end{aligned}$$

Corollary 7.24. *Let $\overline{B_0}$ be the closure of the operator B_0 .*

Then the following statements hold:

(i) *If*

$$n_+(B_0) \neq n_-(B_0),$$

then $\overline{B_0}$ has no self-adjoint extensions.

(ii) *If*

$$n_+(B_0) = n_-(B_0),$$

then $\overline{B_0}$ admits self-adjoint extensions.

This follows from Corollary 2.92 and Corollaries 2.61 and 2.62.

Corollary 7.25. *The operator $\overline{B_0}$ admits self-adjoint extensions if and only if*

$$\#K_+ = \#K_-.$$

Proof. This follows from Corollary 7.24 and Proposition 7.20. □

7.6 Systems of third-order homogeneous differential operators

Assume that

$$n \in \mathbb{N} \setminus \{0, 1\}.$$

and

$$a_1, \dots, a_n \in \mathbb{R} \setminus \{0\}. \tag{7.6.1}$$

We consider a Hilbert space

$$X = L^2(0, \infty; \mathbb{C}^n)$$

with inner product

$$\langle \vec{u}, \vec{v} \rangle_X = \sum_{k=1}^n \int_0^\infty u_k \overline{v_k} dx$$

when $\vec{v} \in X$ and

$$\vec{u} = \{(0, \infty) \ni x \mapsto (u_1, u_2, \dots, u_n) \in \mathbb{C}^n\} \in X.$$

Let define an operator B_0 in X by respectively

$$\begin{aligned} D(B_0) &= C_0^\infty(0, \infty; \mathbb{C}^n), \\ B_0 \vec{u} &= (a_1 i D^3 u_1, a_2 i D^3 u_2, \dots, a_n i D^3 u_n) \quad \text{on } (0, \infty). \end{aligned} \quad (7.6.2)$$

Proposition 7.26. *Operator B_0 is a linear symmetric operator in the Hilbert space X .*

Proof. To show that B_0 is symmetric, we need to verify that

$$\langle B_0 \vec{u}, \vec{v} \rangle_X = \langle \vec{u}, B_0 \vec{v} \rangle_X, \quad \text{for all } \vec{u}, \vec{v} \in D(B_0).$$

Let us choose and fix $\vec{u}, \vec{v} \in D(B_0)$. Using the assumption (7.6.1) and calculating both sides, we find:

$$\langle B_0 \vec{u}, \vec{v} \rangle_X = \sum_{k=1}^n \int_0^\infty (a_k i D^3 u_k) \overline{v_k} dx = i \sum_{k=1}^n a_k \int_0^\infty D^3 u_k(x) \overline{v_k(x)} dx$$

and

$$\langle \vec{u}, B_0 \vec{v} \rangle_X = \sum_{k=1}^n \int_0^\infty u_k \overline{(a_k i D^3 v_k)} dx = -i \sum_{k=1}^n a_k \int_0^\infty u_k \overline{D^3 v_k(x)} dx.$$

Using the assumption that functions $\vec{u}$ and $\vec{v}$ have compact supports, by applying integration by parts, see [37, Theorem 6.22], we can show that these two integrals are equal. Thus, B_0 is symmetric. $\square$

The following result is a generalization of Proposition 5.33.

Proposition 7.27. *Assume that B_0 is the operator defined by formula (7.6.2).*

Then B_0 is not closed.

Moreover, a function $\vec{u} \in D(\overline{B_0})$ if and only if the following conditions are satisfied:

- (i) $\vec{u} \in L^2(0, \infty; \mathbb{C}^n)$,
- (ii) each component u_j is locally absolutely continuous on $(0, \infty)$ up to order 2, that is, u_j , Du_j , and $D^2 u_j$ are locally absolutely continuous, and

$$\int_0^\infty |D^3 \vec{u}(t)|_{\mathbb{C}^n}^2 dt < \infty,$$

(iii) $u_j(0) = Du_j(0) = D^2u_j(0) = 0$ for all $j = 1, \dots, n$.

Moreover, for any $\vec{u} \in D(\overline{B_0})$, we have:

$$\overline{B_0}\vec{u} = (a_1 i D^3 u_1, \dots, a_n i D^3 u_n).$$

Proof of Proposition 7.27. The proof follows the same method as in Proposition 5.32. Hence we omit the proof that the graph $\Gamma(B_0)$ is not closed in $X \times X = L^2(0, \infty; \mathbb{C}^n) \times L^2(0, \infty; \mathbb{C}^n)$.

Proof of Part II: the case $n = 2$.

We follow the method used in Proposition 7.17.

Let $X = L^2(0, \infty; \mathbb{C}^2)$ and $Y = L^2((-\infty, 0) \cup (0, \infty); \mathbb{C})$, and let

$$T : X \rightarrow Y$$

be the isometric isomorphism defined in (7.3.5). By construction, the operators B_0 and A_0 satisfy

$$T \circ B_0 = A_0 \circ T. \quad (7.6.3)$$

Since T is an isometric isomorphism, identity (7.6.3) extends to closures, and therefore

$$T(D(\overline{B_0})) = D(\overline{A_0}), \quad \overline{B_0} = T^{-1} \circ \overline{A_0} \circ T.$$

Let $\vec{u} \in D(\overline{B_0})$ and set $u = T(\vec{u})$. Then $u \in D(\overline{A_0})$.

Let $\vec{u} \in D(\overline{B_0})$ and set $u := T(\vec{u})$. Then $u \in D(\overline{A_1})$. By Proposition 5.33, we obtain:

- $u \in L^2((-\infty, 0) \cup (0, \infty))$,
- $u \in H_{\text{loc}}^3(\mathbb{R} \setminus \{0\})$,
- $D^3 u \in L^2(\mathbb{R})$,
- $u(0) = u'(0) = u''(0) = 0$.

Transferring these properties through T , we deduce that $\vec{u} = (u_1, u_2)$ satisfies:

- (i) $\vec{u} \in L^2(0, \infty; \mathbb{C}^2)$,
- (ii) $u_j \in H_{\text{loc}}^3(0, \infty)$ and $D^3 u_j \in L^2(0, \infty)$,

(iii) $u_j(0) = Du_j(0) = D^2u_j(0) = 0$ for $j = 1, 2$.

Moreover,

$$\overline{B_0} \vec{u} = T^{-1}(\overline{A_1}u) = (a_1 i D^3 u_1, a_2 i D^3 u_2).$$

Conversely. Assume that $\vec{u} = (u_1, u_2)$ satisfies conditions (i)–(iii). Define

$$u := T(\vec{u}).$$

Then $u \in L^2((-\infty, 0) \cup (0, \infty))$.

Moreover, the assumptions on $\vec{u}$ imply that u satisfies the scalar conditions of Proposition 5.33. Hence,

$$u \in D(\overline{A_1}).$$

Therefore,

$$\vec{u} = T^{-1}u \in D(\overline{B_0}).$$

This completes the proof of Part II.

Proof of Part III: the case $n \geq 3$.

We argue componentwise, following the scalar result in Proposition 5.32.

Forward implication. Let $\vec{u} = (u_1, \dots, u_n) \in D(\overline{B_0})$. Then there exist $\vec{u}^{(k)} \in D(B_0)$ and $\vec{z} \in X$ such that

$$\vec{u}^{(k)} \rightarrow \vec{u}, \quad B_0 \vec{u}^{(k)} \rightarrow \vec{z} \quad \text{in } X.$$

Thus for each $j = 1, \dots, n$,

$$u_j^{(k)} \rightarrow u_j, \quad D^3 u_j^{(k)} \rightarrow \frac{1}{a_j i} z_j \quad \text{in } L^2(0, \infty).$$

Applying Proposition 5.33 componentwise, we obtain:

- $u_j \in H_{\text{loc}}^3(0, \infty)$,
- $D^3 u_j \in L^2(0, \infty)$,
- $u_j(0) = Du_j(0) = D^2 u_j(0) = 0$.

Since j was arbitrary, these properties hold for all components.

Moreover,

$$\overline{B_0} \vec{u} = (a_1 i D^3 u_1, \dots, a_n i D^3 u_n).$$

Conversely. Assume that $\vec{u} = (u_1, \dots, u_n)$ satisfies conditions (i)–(iii). Then, for each j , the function u_j satisfies the scalar assumptions of Proposition 5.33. Hence,

$$u_j \in D(\overline{A_{0,j}}), \quad A_{0,j} = a_j i D^3.$$

Therefore, for each j , there exists a sequence $u_j^{(k)} \in C_0^\infty(0, \infty)$ such that

$$u_j^{(k)} \rightarrow u_j, \quad D^3 u_j^{(k)} \rightarrow D^3 u_j \quad \text{in } L^2(0, \infty).$$

Define

$$\vec{u}^{(k)} = (u_1^{(k)}, \dots, u_n^{(k)}).$$

Then $\vec{u}^{(k)} \in D(\overline{B_0})$ and

$$\vec{u}^{(k)} \rightarrow \vec{u}, \quad \overline{B_0} \vec{u}^{(k)} \rightarrow \overline{B_0} \vec{u}.$$

Hence,

$$\vec{u} \in D(\overline{B_0}).$$

This completes the proof of Part III.

This completes the proof of Proposition 7.27.

□

We now characterize the domain of the adjoint operator $\overline{B_0}^*$.

Proposition 7.28. *Assume that $\vec{u} = (u_1, u_2, \dots, u_n) : (0, \infty) \rightarrow \mathbb{C}^n$ belongs to the space X . Then $\vec{u}$ belongs to the domain $D(\overline{B_0}^*)$ if and only if for every $k = 1, \dots, n$, the component u_k is locally absolutely continuous and*

$$\int_0^\infty |D^3 u_k(t)|^2 dt < \infty.$$

Proof of Proposition 7.28. The proof follows the same argument as in the first-order case, see Proposition 7.18, with D replaced by D^3 .

Let $\vec{u} \in D(\overline{B_0}^*)$. By definition of the adjoint operator, each component u_k satisfies the corresponding scalar condition.

Thus each u_k is locally absolutely continuous up to order two and satisfies

$$D^3 u_k \in L^2(0, \infty).$$

Conversely, if these conditions hold, then $\vec{u} \in D(B_0^*)$. $\square$

7.6.1 Deficiency indices of third-order homogeneous differential systems on the half-line

In this section we find the deficiency indices of the operator B_0 as follows.

Let us define two sets.

$$k \in K_+ \text{ if and only if } a_k > 0,$$

$$k \in K_- \text{ if and only if } a_k < 0.$$

It follows from our assumptions that these two disjoint sets satisfy

$$K_+ \cup K_- = \{1, \dots, n\}.$$

Hence,

$$\#(K_+) + \#(K_-) = n.$$

Proposition 7.29. *The deficiency indices of the operator B_0 in (7.6.2) are given by*

$$n_-(B_0) = 2n - \#(K_+) \tag{7.6.4}$$

$$n_+(B_0) = 2n - \#(K_-), \tag{7.6.5}$$

Corollary 7.30. *The operator B_0 on $L^2(0, \infty; \mathbb{C}^n)$ defined by (7.6.2) has at least one self-adjoint extension if and only if the condition (7.6.12) is satisfied, then every self-adjoint extension of B_0 can be characterized similarly to the first-order case in Section 7.5.*

Proof of Proposition 7.29. To find $n_-(B_0) = \dim(iI + B_0^*)$, we assume

$$\vec{u} \in \ker(iI + B_0^*).$$

For $n_-(B_0) = \dim(iI + B_0^*)$, we have

$$a_k i D^3 u_k = -i u_k \quad \forall k = 1, 2, \dots, n.$$

i.e.

$$D^3 u_k = \frac{-1}{a_k} u_k \quad \forall k = 1, 2, \dots, n. \tag{7.6.6}$$

Assume a solution of the form $u_k(x) = e^{\lambda x}$. Substituting into the ODE gives

$$D^3(e^{\lambda x}) = \lambda^3 e^{\lambda x},$$

thus we deduce that

$$\lambda^3 = -\frac{1}{a_k}.$$

By standard methods, the cube roots of $-\frac{1}{a_k}$ are as follows

$$\lambda_m = r^{1/3} e^{i(\theta+2\pi m)/3}, \quad m = 0, 1, 2.$$

where

$$r = \frac{1}{|a_k|}, \quad \theta = \begin{cases} \pi, & \text{if } a_k > 0, \\ 0, & \text{if } a_k < 0. \end{cases}$$

I.e.

- Divide the angle θ by 3,
- Add $\frac{2\pi}{3}$ each time to get the other roots.

Case 1. Assume that $a_k > 0$. Then we have $\theta = \pi$, so

$$\lambda_{k,m} = \left(\frac{1}{a_k}\right)^{1/3} e^{i(\pi+2\pi m)/3}, \quad m = 0, 1, 2.$$

Again, by tedious but standard calculations we infer that

$$\begin{aligned} \lambda_{k,0} &= \left(\frac{1}{a_k}\right)^{1/3} \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\ \lambda_{k,1} &= \lambda_{k,1}(a_k) = -\left(\frac{1}{a_k}\right)^{1/3} \\ \lambda_{k,2} &= \left(\frac{1}{a_k}\right)^{1/3} \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right). \end{aligned} \tag{7.6.7}$$

Thus, the general solution of (7.6.6) is

$$u_k(x) = c_1 e^{\lambda_0 x} + c_2 e^{\lambda_1 x} + c_3 e^{\lambda_2 x}, \quad \text{for } x > 0.$$

where $c_1, c_2, c_3 \in \mathbb{C}$.

We analyze the real parts of $\lambda_{k,m}$

$$\operatorname{Re}(\lambda_0) = \frac{1}{2} \left(\frac{1}{a_k}\right)^{1/3} > 0,$$

$$\begin{aligned}\operatorname{Re}(\lambda_1) &= -\left(\frac{1}{a_k}\right)^{1/3} < 0, \\ \operatorname{Re}(\lambda_2) &= \frac{1}{2}\left(\frac{1}{a_k}\right)^{1/3} > 0.\end{aligned}$$

Thus, only $e^{\lambda_1 x}$ belong to $L^2(0, \infty)$.

Case 2. Assume that $a_k < 0$. Then we have $\theta = 0$, so

$$\lambda_m = \left(\frac{1}{a_k}\right)^{1/3} e^{i(0+2\pi m)/3} = \left(\frac{1}{a_k}\right)^{1/3} e^{2\pi m/3}, \quad m = 0, 1, 2.$$

As before,

$$\begin{aligned}\lambda_{k,0} &= \left(\frac{1}{|a_k|}\right)^{1/3}, \\ \lambda_{k,1} &= \left(\frac{1}{|a_k|}\right)^{1/3} \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right),\end{aligned}\tag{7.6.8}$$

$$\lambda_{k,2} = \left(\frac{1}{|a_k|}\right)^{1/3} \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right).\tag{7.6.9}$$

The general solution of (7.6.6) is

$$u_k(x) = c_1 e^{\lambda_0 x} + c_2 e^{\lambda_1 x} + c_3 e^{\lambda_2 x}, \quad \text{for } x > 0,$$

where $c_1, c_2, c_3 \in \mathbb{C}$.

We analyze the real parts of λ_m

$$\begin{aligned}\operatorname{Re}(\lambda_0) &= \left(\frac{1}{|a_k|}\right)^{1/3} > 0, \\ \operatorname{Re}(\lambda_1) &= -\frac{1}{2}\left(\frac{1}{|a_k|}\right)^{1/3} < 0, \\ \operatorname{Re}(\lambda_2) &= -\frac{1}{2}\left(\frac{1}{|a_k|}\right)^{1/3} < 0.\end{aligned}$$

Thus we infer that

- $e^{\lambda_0 x} \notin L^2(0, \infty)$,
- $e^{\lambda_1 x}, e^{\lambda_2 x} \in L^2(0, \infty)$.

Now we define functions e_k , when $a_k > 0$, i.e. $k \in K_+$ by

$$e_k(x) = e^{\lambda_1(a_k)x}, \quad x \in (0, \infty),$$

where $\lambda_1(a_k)$ is defined by formula (7.6.7).

When, $a_k < 0$, i.e. $k \in K_-$ we define two functions e_k^j , $j = 1, 2$ by

$$e_k^j(x) = e^{\lambda_j(a_k)x}, \quad x \in (0, \infty),$$

where $\lambda_j(a_k)$ are defined by formula (7.6.8) and (7.6.9).

The functions e_k , $k \in K_+$ and e_k^j , $k \in K_-$, $j = 1, 2$, are linearly independent and we proved that if $\vec{u} \in D(B_0^*)$, then there exists complex numbers c_k , $k \in K_+$ and c_k^j , $k \in K_-$, $j = 1, 2$ such that

$$\vec{u} = \sum_{k \in K_+} c_k e_k + \sum_{k \in K_-} \sum_{j=1}^2 c_k^j e_k^j$$

The converse is obviously true. Hence

$$\begin{aligned} n_-(B_0) &= \dim \ker D(B_0^*) = \#(K_+) + 2\#(K_-) \\ &= \#(K_+) + 2(n - \#(K_+)) = 2n - \#(K_+) \\ n_-(B_0) &= \dim \ker D(B_0^*) = \#(K_+) + 2\#(K_-) \\ &= n - \#(K_-) + 2(\#(K_+)) = n + \#(K_-) \end{aligned} \tag{7.6.10}$$

This proves the first part of Proposition, i.e. equality (7.6.4).

Similarly, you can solve for $n_+(B_0)$, following an analogous process.

To find $n_+(B_0) = \dim(iI - B_0^*)$, we assume

$$\vec{u} \in \ker(iI - B_0^*).$$

This leads to

$$a_k i D^3 u_k(x) = i u_k(x) \text{ for } x > 0 \quad \forall k = 1, 2, \dots, n.$$

i.e.

$$D^3 u_k = \frac{1}{a_k} u_k \quad \forall k = 1, 2, \dots, n.$$

Assuming $u_k(x) = e^{\lambda x}$, we get

$$\lambda^3 = \frac{1}{a_k}.$$

In polar form, the roots are

$$\lambda_m = \left(\frac{1}{|a_k|} \right)^{1/3} e^{i\phi_m}, \quad \phi_m = \frac{\theta + 2\pi m}{3}.$$

Case 1: $a_k > 0$. Then $\theta = 0$, so the roots are:

$$\begin{aligned} \lambda_0 &= \left(\frac{1}{a_k} \right)^{1/3}, \\ \lambda_1 &= \left(\frac{1}{a_k} \right)^{1/3} \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2} \right), \\ \lambda_2 &= \left(\frac{1}{a_k} \right)^{1/3} \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2} \right). \end{aligned}$$

Only λ_1 and λ_2 have negative real parts, so belong to $L^2(0, \infty)$.

Case 2: $a_k < 0$. Then $\theta = \pi$, so the roots are

$$\begin{aligned} \lambda_0 &= \left(\frac{1}{|a_k|} \right)^{1/3} \left(\frac{1}{2} + i\frac{\sqrt{3}}{2} \right), \\ \lambda_1 &= -\left(\frac{1}{|a_k|} \right)^{1/3}, \\ \lambda_2 &= \left(\frac{1}{|a_k|} \right)^{1/3} \left(\frac{1}{2} - i\frac{\sqrt{3}}{2} \right). \end{aligned}$$

Only λ_1 has negative real part, so $e^{\lambda_1 x} \in L^2(0, \infty)$.

Now, we define

$$\begin{aligned} f_k^j(x) &= e^{\lambda_j(a_k)x}, \quad j = 1, 2, \quad \text{for } a_k > 0, \\ f_k(x) &= e^{\lambda_1(a_k)x}, \quad \text{for } a_k < 0. \end{aligned}$$

Let

$$K_+ = \{k : a_k > 0\}, \quad K_- = \{k : a_k < 0\},$$

with

$$\#(K_+) + \#(K_-) = n.$$

Thus, we obtain

$$\begin{aligned} n_+(B_0) &= 2\#(K_+) + \#(K_-) = 2n - \#(K_-) \\ n_+(B_0) &= 2\#(K_+) + (n - \#(K_+)) = n + \#(K_+). \end{aligned} \tag{7.6.11}$$

This proves the second part of Proposition, i.e. equality (7.6.5).

Therefore, from (7.6.10) and (7.6.11), we take these results

$$n_+(B_0) = 2n - \#(K_-), \quad n_-(B_0) = 2n - \#(K_+)$$

So

$$\begin{aligned} n_+(B_0) = n_-(B_0) &\iff 2n - \#(K_-) = 2n - \#(K_+) \\ &\iff \#(K_-) = \#(K_+) \end{aligned}$$

□

We the following result

Corollary 7.31. *The operator B_0 on $L^2(0, \infty; \mathbb{C}^n)$ defined by (7.6.2) has at least one self-adjoint operator if and only if*

$$\#K_- = \#K_+ \tag{7.6.12}$$

In particular, if n is an odd number, the operator B_0 on $L^2(0, \infty; \mathbb{C}^n)$ has no self-adjoint extensions.

Proof of Corollary 7.31. From previous results, B_0 has self-adjoint extensions if and only if $\#K_+ = \#K_- = \frac{n}{2}$.

If n is odd, then $\frac{n}{2} \notin \mathbb{Z}$, so such an equality between two integers is impossible. Therefore, B_0 cannot have self-adjoint extensions in this case.

□

Conclusion

B_0 defined by (7.6.2) on $L^2(0, \infty; \mathbb{C}^n)$ has self-adjoint extensions
if and only if $\#(K_+) = \#(K_-) = \frac{n}{2}$.

We now apply Theorem 2.60 to characterize all self-adjoint extensions of the operator B_0 .

Corollary 7.32. *Assume that n is even and that condition (7.6.12) is satisfied, i.e. $\#(K_+) = \#(K_-)$.*

Let

$$H_+ := \ker(B_0^* - iI), \quad H_- := \ker(B_0^* + iI)$$

be the deficiency subspaces of B_0 . Since $n_{\pm}(B_0) = \frac{3n}{2}$,

$$\dim H_+ = \dim H_- = \frac{3n}{2}.$$

Moreover, by Theorem 2.60, the self-adjoint extensions of B_0 are in one-to-one correspondence with the unitary operators

$$\Gamma : H_+ \rightarrow H_-.$$

More precisely, for each unitary operator Γ , the corresponding self-adjoint extension B_0^Γ is given by

$$\begin{aligned} D(B_0^\Gamma) &= \left\{ \vec{u} = \vec{u}_0 + \vec{u}_+ + \Gamma \vec{u}_+ : \vec{u}_0 \in D(B_0), \vec{u}_+ \in H_+ \right\}, \\ B_0^\Gamma(\vec{u}_0 + \vec{u}_+ + \Gamma \vec{u}_+) &= B_0 \vec{u}_0 + i \vec{u}_+ - i \Gamma \vec{u}_+. \end{aligned}$$

Conversely, every self-adjoint extension of B_0 is of this form.

7.6.2 Deficiency indices of the closure of a system of third-order homogeneous differential operators

We already know that the closure $\overline{B_0}$ of the operator defined by (7.6.2) exists and that its adjoint coincides with the adjoint of B_0 , i.e.

$$(\overline{B_0})^* = B_0^*.$$

By Corollary 2.92, this equality implies that the deficiency subspaces of $\overline{B_0}$ and B_0 coincide:

$$\ker(i - B_0^*) = \ker(i - \overline{B_0}^*), \quad \ker(i + B_0^*) = \ker(i + \overline{B_0}^*).$$

Consequently, the deficiency indices of B_0 and $\overline{B_0}$ are the same:

$$n_+(\overline{B_0}) = n_+(B_0), \quad n_-(\overline{B_0}) = n_-(B_0).$$

Corollary 7.33. *Let B_0 be the operator defined by formula (7.6.2), and let $\overline{B_0}$ denote its closure in $L^2(0, \infty; \mathbb{C}^n)$.*

Then B_0 is densely defined and symmetric, its closure $\overline{B_0}$ is again symmetric, and their deficiency indices coincide.

In particular, by Corollary 2.92 and Proposition 7.29, we have

$$n_-(\overline{B_0}) = n_-(B_0) = 2n - \#(K_+),$$

$$n_+(\overline{B_0}) = n_+(B_0) = 2n - \#(K_-).$$

Corollary 7.34. *Let $\overline{B_0}$ be the closure of the operator B_0 .*

Then the following statements hold:

(i) *If*

$$n_+(B_0) \neq n_-(B_0),$$

then $\overline{B_0}$ has no self-adjoint extensions.

(ii) *If*

$$n_+(B_0) = n_-(B_0),$$

then $\overline{B_0}$ admits self-adjoint extensions.

This follows from Corollary 2.92 and Corollaries 2.61 and 2.62.

Corollary 7.35. *The operator $\overline{B_0}$ admits self-adjoint extensions if and only if*

$$\#(K_+) = \#(K_-).$$

Proof. This follows from Corollary 7.34 and Proposition 7.29. □

7.7 Systems of third-order in-homogeneous differential operators

Assume that $n \in \mathbb{N} \setminus \{0, 1\}$ and

$$a_1, b_1, \dots, a_n, b_n \in \mathbb{R} \setminus \{0\}.$$

We consider a Hilbert space

$$X = L^2(0, \infty; \mathbb{C}^n) \tag{7.7.1}$$

Let define an operator B_0 in X by

$$\begin{aligned} D(B_0) &= C_0^\infty(0, \infty; \mathbb{C}^n), \\ B_0 \vec{u} &= \left((a_1(iD)^3 + b_1(iD))u_1, \dots, (a_n(iD)^3 + b_n(iD))u_n \right), \end{aligned} \tag{7.7.2}$$

when $\vec{u} = (u_1, u_2, \dots, u_n) : (0, \infty) \rightarrow \mathbb{C}^n$.

Proposition 7.36. *Assume that all coefficients a_k, b_k , for $k = 1, \dots, n$, satisfy Assumption 7.7. Then the operator B_0 defined by (7.7.2) is a linear symmetric operator in the Hilbert space X in (7.7.1).*

Proof of Proposition 7.36. Let us fix $\vec{u}, \vec{v} \in D(B_0) = C_0^\infty(0, \infty; \mathbb{C}^n)$.

We compute the two inner products:

$$\langle B_0 \vec{u}, \vec{v} \rangle_X = \sum_{k=1}^n \int_0^\infty \left(a_k(-i)D^3 u_k + b_k i D u_k \right) \overline{v_k(x)} dx,$$

$$\langle \vec{u}, B_0 \vec{v} \rangle_X = \sum_{k=1}^n \int_0^\infty u_k(x) \cdot \overline{a_k(-i)D^3 v_k(x) + b_k i D v_k(x)} dx.$$

By linearity and the fact that $a_k, b_k \in \mathbb{R}$, we obtain

$$\begin{aligned} \langle B_0 \vec{u}, \vec{v} \rangle_X &= i \sum_{k=1}^n \left[-a_k \int_0^\infty D^3 u_k(x) \overline{v_k(x)} dx + b_k \int_0^\infty D u_k(x) \overline{v_k(x)} dx \right], \\ \langle \vec{u}, B_0 \vec{v} \rangle_X &= i \sum_{k=1}^n \left[a_k \int_0^\infty u_k(x) \overline{D^3 v_k(x)} dx - b_k \int_0^\infty u_k(x) \overline{D v_k(x)} dx \right]. \end{aligned}$$

Using integration by parts exactly as in Proposition 7.16 and Proposition 7.26, and since $u_k, v_k \in C_0^\infty(0, \infty)$, all boundary terms vanish. In particular,

for each $k = 1, \dots, n$,

$$\int_0^\infty D^3 u_k \overline{v_k} dx = - \int_0^\infty u_k \overline{D^3 v_k} dx, \quad \int_0^\infty D u_k \overline{v_k} dx = - \int_0^\infty u_k \overline{D v_k} dx.$$

Substituting these identities, we obtain

$$\langle B_0 \vec{u}, \vec{v} \rangle_X = \langle \vec{u}, B_0 \vec{v} \rangle_X.$$

Thus, B_0 is symmetric. □

The following proposition provides a characterization of the domain of $\overline{B_0}$, extending the corresponding results obtained earlier for scalar third-order operators on the punctured line, see Section 6.4.

Proposition 7.37. *Assume that B_0 is the operator defined by formula (7.7.2) in the Hilbert space (7.7.1). Then the following assertions hold true.*

Part I *The operator B_0 is not closed.*

Part II *A function $\vec{u} \in D(\overline{B_0})$ if and only if the following conditions are satisfied:*

- (i) $\vec{u} \in L^2(0, \infty; \mathbb{C}^n)$,
- (ii) each component u_j is of class C^2 on $(0, \infty)$, the derivatives u_j and Du_j are locally absolutely continuous, and D^3u_j exists almost everywhere, with

$$\int_0^\infty |a_j D^3u_j(t) - b_j Du_j(t)|^2 dt < \infty, \quad \text{for all } j = 1, \dots, n.$$

- (iii) $u_j(0) = Du_j(0) = D^2u_j(0) = 0$ for all $j = 1, \dots, n$.

Moreover, for any $\vec{u} \in D(\overline{B_0})$, we have:

$$\overline{B_0} \vec{u} = (a_1(iD)^3u_1 + b_1iDu_1, \dots, a_n(iD)^3u_n + b_niDu_n). \quad (7.7.3)$$

Proof of Proposition 7.37. Part I. We prove that B_0 is not closed. By Proposition 2.83, it is sufficient to construct

- (i) a function $\vec{u} \in X \setminus D(B_0)$,
- (ii) and a sequence $(\vec{u}_k) \subset D(B_0)$ such that

$$\vec{u}_k \rightarrow \vec{u} \quad \text{in } X,$$

- (iii) and $(B_0 \vec{u}_k)$ converges in X .

We follow the same construction as in the proof of Propositions 3.29 and 5.32. Let

$$\varphi(x) = \begin{cases} e^{-x^{-2}}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

As is well known, see [37, Chapter 6], the function φ belongs to $C^\infty(\mathbb{R})$ and all its derivatives vanish at 0. Moreover, as in the construction used in Proposition 3.29, one verifies that

$$\varphi \in H^3(0, \infty), \quad D\varphi, D^3\varphi \in L^2(0, \infty).$$

Define

$$\vec{u}(x) := (\varphi(x), 0, \dots, 0).$$

Then $\vec{u} \in X$, but $\vec{u} \notin D(B_0)$ since $\varphi \notin C_0^\infty(0, \infty)$.

Since $\varphi \in H^3(0, \infty)$ and vanishes at 0 together with its derivatives, by the density of $C_0^\infty(0, \infty)$ in $H_0^3(0, \infty)$, see Definition 2.35 and standard Sobolev theory, e.g. [1], there

exists a sequence

$$\varphi_k \in C_0^\infty(0, \infty)$$

such that

$$\varphi_k \rightarrow \varphi \quad \text{in } H^3(0, \infty).$$

Set

$$\vec{u}_k(x) := (\varphi_k(x), 0, \dots, 0).$$

Then $\vec{u}_k \in D(B_0)$ and

$$\vec{u}_k \rightarrow \vec{u} \quad \text{in } X.$$

From the convergence in $H^3(0, \infty)$ we obtain

$$D\varphi_k \rightarrow D\varphi, \quad D^3\varphi_k \rightarrow D^3\varphi \quad \text{in } L^2(0, \infty).$$

Hence

$$B_0\vec{u}_k = \left((a_1(iD)^3 + b_1(iD))\varphi_k, 0, \dots, 0 \right) \rightarrow \left((a_1(iD)^3 + b_1(iD))\varphi, 0, \dots, 0 \right)$$

in X .

We have constructed a sequence $(\vec{u}_k, B_0\vec{u}_k)$ converging in $X \times X$ to a limit $(\vec{u}, \vec{z})$ with $\vec{u} \notin D(B_0)$. Therefore, by Proposition 2.83, the graph $\Gamma(B_0)$ is not closed, and hence B_0 is not closed. $\square$

Proof of Proposition 7.37. Part II. The case $n = 2$. We follow the same argument as in Proposition 7.27 using a unitary isomorphism. Let $T : X \rightarrow Y$ be defined by formula (7.1.3). We will use here the operator A_0 , which was defined in (7.4.2) as follows

$$\begin{aligned} D(A_0) &= C_0^\infty((-\infty, 0) \cup (0, \infty); \mathbb{C}), \\ A_0u &= a_1iDu + a_3(-i)D^3u \text{ if } u \in D(A_0). \end{aligned}$$

Then, as in Proposition 7.13, one can prove that

$$T \circ B_0 = A_0 \circ T.$$

This identity extends to closures of the operators A_0 and B_0 , i.e.

$$T(D(\overline{B_0})) = D(\overline{A_0}), \quad \overline{B_0} = T^{-1} \circ \overline{A_0} \circ T.$$

Let $\vec{u} = (u_1, u_2) \in D(\overline{B_0})$ and $u := T(\vec{u})$. Then by the above $u \in D(\overline{A_0})$ and

Then, as in the scalar third-order case, see Proposition 6.46, we obtain

- $u \in H_{\text{loc}}^3(\mathbb{R} \setminus \{0\})$,
- $aD^3u - bDu \in L^2(\mathbb{R})$,
- $u(0) = u'(0) = u''(0) = 0$.

Transferring these properties via T , exactly as in Proposition 7.27, we obtain

- $u_j \in H_{\text{loc}}^3([0, \infty))$, $j = 1, 2$,
- $a_j D^3 u_j - b_j D u_j \in L^2(0, \infty)$, $j = 1, 2$,
- $u_j(0) = D u_j(0) = D^2 u_j(0) = 0$, $j = 1, 2$.

This implies that $\vec{u}$ satisfies conditions (i)–(iii). Moreover, using the commutation relation established in Proposition 7.13, we obtain (7.7.3).

Conversely, assume that $\vec{u} = (u_1, u_2)$ satisfies conditions (i)–(iii). Define

$$u := T(\vec{u}).$$

Then, by construction of T , we have $u \in L^2((-\infty, 0) \cup (0, \infty))$.

Moreover, the assumptions on $\vec{u}$ imply that u satisfies the scalar conditions (i)–(iii) in Proposition 6.46. Hence, by that proposition, we obtain

$$u \in D(\overline{A_0}).$$

Therefore, using the relation

$$D(\overline{B_0}) = T^{-1}(D(\overline{A_0})),$$

we conclude that

$$\vec{u} = T^{-1}u \in D(\overline{B_0}).$$

This proves the converse implication in the case $n = 2$.

This completes the proof of part II. □

Sketch of the proof of Proposition 7.37. Part III. The case $n \geq 2$. We argue component-wise, following Proposition 6.46 and the scalar argument in Proposition 5.33.

Let $\vec{u} \in D(\overline{B_0})$. Then there exists $\vec{u}^{(k)} \in D(B_0)$ such that

$$\vec{u}^{(k)} \rightarrow \vec{u}, \quad B_0 \vec{u}^{(k)} \rightarrow \vec{z}.$$

Thus for each j ,

$$u_j^{(k)} \rightarrow u_j, \quad a_j(iD)^3 u_j^{(k)} + b_j(iD) u_j^{(k)} \rightarrow z_j.$$

Arguing as in Proposition 7.27, we deduce that

- $u_j \in H_{\text{loc}}^3([0, \infty))$,
- $a_j D^3 u_j - b_j D u_j \in L^2(0, \infty)$,
- $u_j(0) = D u_j(0) = D^2 u_j(0) = 0$.

Conversely, assume that $\vec{u} = (u_1, \dots, u_n)$ satisfies conditions (i)–(iii). Then, for each $j = 1, \dots, n$, the component u_j satisfies the scalar assumptions of Proposition 6.46. Hence,

$$u_j \in D(\overline{A_{0,j}}),$$

where $A_{0,j} = a_j(iD)^3 + b_j(iD)$.

Therefore, for each j , there exists a sequence $u_j^{(k)} \in C_0^\infty(0, \infty)$ such that

$$u_j^{(k)} \rightarrow u_j, \quad (a_j(iD)^3 + b_j(iD)) u_j^{(k)} \rightarrow (a_j(iD)^3 + b_j(iD)) u_j$$

in $L^2(0, \infty)$. Define

$$\vec{u}^{(k)} = (u_1^{(k)}, \dots, u_n^{(k)}),$$

we obtain $\vec{u}^{(k)} \in D(B_0)$ and

$$\vec{u}^{(k)} \rightarrow \vec{u}, \quad B_0 \vec{u}^{(k)} \rightarrow \overline{B_0} \vec{u}.$$

Hence, $\vec{u} \in D(\overline{B_0})$.

This completes the proof. □

We argue componentwise and use the characterization of the adjoint operator obtained in Proposition 6.47.

Proposition 7.38. Assume that $\vec{u} = (u_1, u_2, \dots, u_n) : (0, \infty) \rightarrow \mathbb{C}^n$ belongs to the space X . Then $\vec{u}$ belongs to the domain $D(B_0^*)$ if and only if for every $k = 1, \dots, n$, the component u_k satisfies the following conditions:

(i) $u_k \in L^2(0, \infty)$ and $u_k \in C^2(0, \infty)$,

(ii) u_k'' is locally absolutely continuous on $(0, \infty)$, hence $u_k^{(3)}$ exists almost everywhere and belongs to $L_{\text{loc}}^2(0, \infty)$,

(iii)

$$\int_0^\infty \left| a_k D^3 u_k(t) - b_k D u_k(t) \right|^2 dt < \infty.$$

Moreover, for every $\vec{u} \in D(B_0^*)$, we have

$$B_0^* \vec{u} = \left(-i(a_1 D^3 u_1 - b_1 D u_1), \dots, -i(a_n D^3 u_n - b_n D u_n) \right).$$

7.7.1 Deficiency indices of third-order inhomogeneous differential systems on the half-line

In this section we find the deficiency indices of the operator B_0 as follows.

Let us temporarily define auxiliary operators $A_{0,k}$, $k = 1, \dots, n$,

$$\begin{aligned} A_{0,k} &= a_k (iD)^3 + b_k iD, \\ D(A_{0,k}) &= C_0^\infty(0, \infty; \mathbb{C}). \end{aligned}$$

To find $n_+(B_0) = \dim \ker(iI - B_0^*)$, let us assume that

$$\vec{u} \in \ker(iI - B_0^*),$$

i.e.

$$\vec{u} \in D(B_0^*),$$

and $B_0^* \vec{u} = i \vec{u}$ is equivalent to

$$a_k (iD)^3 u_k + b_k i D u_k = i u_k, \quad k = 1, \dots, n$$

which is equivalent to

$$u_k \in D(A_{0,k}^*), \quad A_{0,k}^* = a_k (iD)^3 u_k + b_k i D u_k, \quad k = 1, \dots, n,$$

which is equivalent to

$$u_k \in \ker(iI - A_{0,k}^*), \quad k = 1, \dots, n. \quad (7.7.4)$$

The long argument proves the following result.

Proposition 7.39. *A function $\vec{u}$ satisfies*

$$\vec{u} \in \ker(iI - B_0^*).$$

if and only if condition (7.7.4) holds.

On the other hand, by Theorem 6.22 in Subsection 6.2.2, we have

$$\dim \ker(iI - A_{0,k}^*) = n_+(A_{0,k}) = \begin{cases} 1 & \text{if } a_k > 0 \\ 2 & \text{if } a_k < 0. \end{cases}$$

Therefore we deduce the following result.

Corollary 7.40.

$$\dim \ker(iI - B_0^*) = \#\{k \in \{1, \dots, n\} : a_k > 0\} + 2 \cdot \#\{k \in \{1, \dots, n\} : a_k < 0\}.$$

By a similar argument, we can prove the following assertions.

Proposition 7.41. *A function $\vec{u}$ satisfies*

$$\vec{u} \in \ker(iI + B_0^*).$$

if and only if the following condition is satisfied

$$u_k \in \ker(iI + A_{0,k}^*), \quad k = 1, \dots, n.$$

Moreover,

$$\dim \ker(iI + A_{0,k}^*) = n_-(A_{0,k}) = \begin{cases} 2 & \text{if } a_k > 0 \\ 1 & \text{if } a_k < 0. \end{cases}$$

Corollary 7.42.

$$\dim \ker(iI + B_0^*) = 2 \cdot \#\{k \in \{1, \dots, n\} : a_k > 0\} + \#\{k \in \{1, \dots, n\} : a_k < 0\}.$$

Summarizing we proved.

Proposition 7.43. *The deficiency indices of the operator B_0 defined by (7.7.2) are given by*

$$\begin{aligned} n_+(B_0) &= \#\{k \in \{1, \dots, n\} : a_k > 0\} + 2 \cdot \#\{k \in \{1, \dots, n\} : a_k < 0\}, \\ n_-(B_0) &= 2 \cdot \#\{k \in \{1, \dots, n\} : a_k > 0\} + \#\{k \in \{1, \dots, n\} : a_k < 0\}. \end{aligned}$$

In fact $n_+(B_0) > 0$ and $n_-(B_0) > 0$.

Proposition 7.44. *Assume that $a_k \in \mathbb{R} \setminus \{0\}$ for every $k = 1, \dots, n$.*

Then

$$\{1, \dots, k\} = \{j : a_j > 0\} \cup \{j : a_j < 0\}.$$

Moreover, the above union is disjoint.

Proof of Proposition 7.44. Put

$$N_+ := \#\{j : a_j > 0\}, \quad N_- := \#\{j : a_j < 0\}.$$

Then

$$N_+ + N_- = k.$$

By Proposition 7.43, we have

$$\begin{aligned} n_+(B_0) &= N_+ + 2N_-, \\ n_-(B_0) &= N_- + 2N_+. \end{aligned}$$

Using $N_+ = k - N_-$, we obtain

$$\begin{aligned} n_+(B_0) &= N_+ + 2N_- \\ &= (k - N_-) + 2N_- \\ &= k + N_- \geq k. \end{aligned}$$

Similarly, using $N_- = k - N_+$, we obtain

$$\begin{aligned} n_-(B_0) &= N_- + 2N_+ \\ &= (k - N_+) + 2N_+ \\ &= k + N_+ \geq k. \end{aligned}$$

Moreover,

$$\begin{aligned} n_+(B_0) = n_-(B_0) &\iff k + N_- = N_+ + k \\ &\iff N_- = N_+. \end{aligned}$$

Since $N_+ = N_-$, this is equivalent to

$$N_+ = N_- = \frac{k}{2}.$$

Therefore, if k is odd, then $\frac{k}{2} \notin \mathbb{Z}$, and hence

$$n_+(B_0) \neq n_-(B_0),$$

so B_0 has no self-adjoint extensions.

If k is even, i.e. $k = 2p$, then self-adjoint extensions exist if and only if

$$n_+(B_0) = n_-(B_0) = \frac{k}{2},$$

which is equivalent to

$$\#\{j : a_j > 0\} = \#\{j : a_j < 0\}.$$

In this case, all self-adjoint extensions of B_0 can be characterized by unitary maps between the deficiency subspaces

$$\ker(B_0^* - iI) \quad \text{and} \quad \ker(B_0^* + iI),$$

see Theorem 2.60. □

We conclude with the following important and unexpected observation.

Remark 7.45. The deficiency indices of operators B_0 depend only on the coefficients a_k but not on the coefficients b_k . They are the same as the corresponding deficiency indices when all $b_k = 0$, see Theorem 6.22 in Subsection 6.2.2.

7.7.2 Deficiency indices of the closure of a system of third-order inhomogeneous differential operators

We already know that the closure $\overline{B_0}$ of the operator defined by (7.7.2) exists and that its adjoint coincides with the adjoint of B_0 , i.e.

$$(\overline{B_0})^* = B_0^*.$$

By Corollary 2.92, this equality implies that the deficiency subspaces of $\overline{B_0}$ and B_0 are identical:

$$\ker(i - B_0^*) = \ker(i - \overline{B_0}^*), \quad \ker(i + B_0^*) = \ker(i + \overline{B_0}^*).$$

Consequently, the deficiency indices of B_0 and $\overline{B_0}$ coincide:

$$n_+(\overline{B_0}) = n_+(B_0), \quad n_-(\overline{B_0}) = n_-(B_0).$$

Corollary 7.46. *Let B_0 be the operator defined by formula (7.7.2), and let $\overline{B_0}$ denote its closure in $L^2(0, \infty; \mathbb{C}^n)$. Then both B_0 and $\overline{B_0}$ are densely defined and symmetric, and their deficiency indices coincide. In particular, by Corollary 2.92 and Proposition 7.43, we have*

$$\begin{aligned} n_+(\overline{B_0}) &= n_+(B_0) \\ &= \#\{k \in \{1, \dots, n\} : a_k > 0\} + 2 \#\{k \in \{1, \dots, n\} : a_k < 0\}, \\ n_-(\overline{B_0}) &= n_-(B_0) \\ &= 2 \#\{k \in \{1, \dots, n\} : a_k > 0\} + \#\{k \in \{1, \dots, n\} : a_k < 0\}. \end{aligned}$$

Corollary 7.47. *Let B_0 be the operator defined by (7.7.2), and let $\overline{B_0}$ denote its closure in X .*

Then the following statements hold:

(i) *If*

$$n_+(B_0) \neq n_-(B_0),$$

then $\overline{B_0}$ has no self-adjoint extensions.

(ii) *If*

$$n_+(B_0) = n_-(B_0),$$

then $\overline{B_0}$ admits self-adjoint extensions.

This follows from Corollary 2.92 and Corollaries 2.61 and 2.62.

$$(\overline{B_0})^* = B_0^* \quad \text{and} \quad n_{\pm}(\overline{B_0}) = n_{\pm}(B_0).$$

Corollary 7.48. *The operator $\overline{B_0}$ admits self-adjoint extensions if and only if*

$$\#\{k \in \{1, \dots, n\} : a_k > 0\} = \#\{k \in \{1, \dots, n\} : a_k < 0\}.$$

Proof. This follows immediately from Corollary 7.47 and Proposition 7.43. $\square$

Chapter 8

Self-adjoint extensions of a punctured third-order operator with a first-order term

In the previous chapter 6, we investigated self-adjoint extensions of third-order differential operators using the classical von Neumann approach. Although this method is effective in several situations, we were not able to complete the analysis in full generality for the class of operators under consideration.

This limitation motivates the use of an alternative framework based on abstract boundary formalism. This approach allows one to describe self-adjoint extensions in terms of boundary conditions encoded via suitable trace mappings.

8.1 Abstract boundary formalism

The abstract boundary formalism allows one to describe self-adjoint extensions in terms of boundary conditions encoded via suitable trace mappings. Our goal is to characterize all self-adjoint extensions of the operator in a systematic way and to express them explicitly through these boundary conditions.

The approach adopted here is inspired by the work of Brzeźniak and Jefferies [22], where similar techniques were developed in the context of point interactions.

Assumption 8.1. Let A_0 be a densely defined symmetric and closed operator in a Hilbert space H . Denote by A_0^* its adjoint operator.

Definition 8.2 (Green form / boundary form). The Green form associated with A_0^* is defined by

$$[f, g]_{A_0^*} := \langle A_0^* f, g \rangle_H - \langle f, A_0^* g \rangle_H, \quad f, g \in D(A_0^*). \quad (8.1.1)$$

Definition 8.3 (Boundary maps). Let

$$\Gamma_+, \Gamma_- : D(A_0^*) \rightarrow \mathbb{C}^d$$

be linear maps. We call $\Gamma_\pm$ the boundary maps and $\Gamma_\pm(f) \in \mathbb{C}^d$ the boundary values of f .

Assumption 8.4 (Diagonal Green identity). Assume that there exist boundary maps Γ_+, Γ_- such that

$$[f, g]_{A_0^*} = -i \left(\langle \Gamma_+(f), \Gamma_+(g) \rangle_{\mathbb{C}^d} - \langle \Gamma_-(f), \Gamma_-(g) \rangle_{\mathbb{C}^d} \right), \quad f, g \in D(A_0^*), \quad (8.1.2)$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^d}$ denotes the standard inner product in $\mathbb{C}^d$, linear in the first argument.

Definition 8.5 (Sesquilinear inner product on $\mathbb{C}^d$). For $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d) \in \mathbb{C}^d$, we define

$$\langle x, y \rangle_{\mathbb{C}^d} := \sum_{j=1}^d x_j \overline{y_j}.$$

Remark 8.6. Let us notice that A_0^* is in general not symmetric.

Definition 8.7 (The unitary group $U(d)$). The unitary group of degree d is the set

$$U(d) := \{U \in M_d(\mathbb{C}) : U^* U = I_d\},$$

where U^* denotes the conjugate transpose of U and I_d is the $d \times d$ identity matrix.

Definition 8.8 (Conjugate transpose). Let $U = (u_{jk})_{j,k=1}^d \in M_d(\mathbb{C})$. The conjugate transpose (adjoint) of U is the matrix $U^* = (u_{jk}^*)_{j,k=1}^d := \overline{U}^T$ defined by

$$u_{jk}^* = \overline{u_{kj}}, \quad j, k \in \{1, \dots, d\},$$

Let us recall the following fundamental result.

Proposition 8.9. Assume that $U \in M_d(\mathbb{C})$. Then the following conditions are equivalent.

- (i) $\langle Ux, Uy \rangle_{\mathbb{C}^d} = \langle x, y \rangle_{\mathbb{C}^d}$ for all $x, y \in \mathbb{C}^d$
- (ii) the columns of U form an orthonormal basis of $\mathbb{C}^d$;
- (iii) the rows of U form an orthonormal basis of $\mathbb{C}^d$;

$$(iv) \quad UU^* = I_d;$$

$$(v) \quad U^*U = I_d.$$

Remark 8.10. Throughout this section we assume the following:

- H is a complex Hilbert space.
- A_0 is a densely defined, closed, symmetric linear operator in H .
- There exist the boundary maps

$$\{\Gamma_+, \Gamma_- : D(A_0^*) \longrightarrow \mathbb{C}^d : \text{the condition (8.1.2) holds} \}.$$

In particular, all boundary constructions are performed on $D(A_0^*)$.

Definition 8.11. Assume Assumption 8.4. Let $U \in U(d)$. Define an operator A_U by

$$\begin{aligned} D(A_U) &:= \{f \in D(A_0^*) : \Gamma_+(f) = U\Gamma_-(f)\}, \\ A_U f &:= A_0^* f. \end{aligned} \tag{8.1.5}$$

Lemma 8.12 (Unitary boundary condition gives symmetry). *Assume Assumption 8.4 and that $U \in U(d)$. Then A_U is symmetric.*

Proof of Lemma 8.12. Let us choose and fix $f, g \in D(A_U)$. Using equation (8.1.2), $\Gamma_+(f) = U\Gamma_-(f)$ and $\Gamma_+(g) = U\Gamma_-(g)$, we obtain

$$\begin{aligned} \langle A_U f, g \rangle - \langle f, A_U g \rangle &= [f, g]_{A_0^*} = -i \left(\langle \Gamma_+(f), \Gamma_+(g) \rangle_{\mathbb{C}^d} - \langle \Gamma_-(f), \Gamma_-(g) \rangle_{\mathbb{C}^d} \right) \\ &= -i \left(\langle U\Gamma_-(f), U\Gamma_-(g) \rangle - \langle \Gamma_-(f), \Gamma_-(g) \rangle \right) \\ &= -i \left(\langle \Gamma_-(f), \Gamma_-(g) \rangle - \langle \Gamma_-(f), \Gamma_-(g) \rangle \right) = 0, \end{aligned}$$

by item (i) of Proposition 8.9, since U is unitary. Hence

$$\langle A_U f, g \rangle = \langle f, A_U g \rangle.$$

Since f and g are arbitrary elements of $D(A)$, the result follows. $\square$

Definition 8.13. Assume that X and Y be vector spaces. By direct sum $X \oplus Y$ we understand the cartesian product $X \times Y$, where every element $(x, y) \in X \times Y$ is denoted by $x \oplus y$. This set is endowed with the standard operations

$$\begin{aligned} (x_1 \oplus y_1) + (x_2 \oplus y_2) &:= (x_1 + x_2 \oplus y_1 + y_2), \\ \alpha(x_1 \oplus y_1) &:= (\alpha x_1 \oplus \alpha y_1) \end{aligned}$$

for all $\alpha \in \mathbb{C}$ and $x_1 \oplus y_1, x_2 \oplus y_2 \in X \oplus Y$.

Proposition 8.14. *Assume that X and Y be vector spaces. The direct sum $X \oplus Y$ is a vector space.*

If X and Y are finite dimensional vector spaces then the direct sum $X \oplus Y$ is a finite dimensional vector space and

$$\dim(X \oplus Y) = \dim X + \dim Y.$$

In particular, $\mathbb{C}^d \oplus \mathbb{C}^d$ is isomorphic to $\mathbb{C}^{2d}$.

Definition 8.15 (neutral subspaces are graphs of unitaries). Let

$$\mathcal{G} := \mathbb{C}^d \oplus \mathbb{C}^d$$

and equip $\mathcal{G}$ with the **indefinite sesqui-linear**, i.e. linear w.r.t. to the first variable and anti-linear w.r.t. to the second variable, bilinear map

$$[\cdot, \cdot]_{\mathcal{G}} : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{C} \quad (8.1.6)$$

defined by

$$[x, y]_{\mathcal{G}} := \langle x_+, y_+ \rangle_{\mathbb{C}^d} - \langle x_-, y_- \rangle_{\mathbb{C}^d}, \quad (8.1.7)$$

where $x = (x_+, x_-) \in \mathcal{G}$ and $y = (y_+, y_-) \in \mathcal{G}$.

A linear subspace $M \subset \mathcal{G}$ is called neutral if and only if

$$[\xi, \eta]_{\mathcal{G}} = 0 \quad \text{for all } \xi, \eta \in M.$$

A linear subspace $M \subset \mathcal{G}$ is called maximal neutral if and only if it is neutral and not properly contained in any other neutral subspace.

Exercise 8.16. Show that

$$[x + \tilde{x}, y]_{\mathcal{G}} = [x, y]_{\mathcal{G}} + [\tilde{x}, y]_{\mathcal{G}},$$

$$[x, y + \tilde{y}]_{\mathcal{G}} = [x, y]_{\mathcal{G}} + [x, \tilde{y}]_{\mathcal{G}},$$

$$[\alpha x, y]_{\mathcal{G}} = \alpha [x, y]_{\mathcal{G}},$$

$$[x, \alpha y]_{\mathcal{G}} = \bar{\alpha} [x, y]_{\mathcal{G}},$$

for all $x, y, \tilde{x}, \tilde{y} \in \mathcal{G}$ and $\alpha \in \mathbb{C}$. Show that it is not true that for every $x \in \mathcal{G}$

$$[x, x]_{\mathcal{G}} = 0 \iff x = 0. \quad (8.1.8)$$

Solution. Write $x = (x_+, x_-)$, $y = (y_+, y_-)$, $\tilde{x} = (\tilde{x}_+, \tilde{x}_-)$, $\tilde{y} = (\tilde{y}_+, \tilde{y}_-) \in \mathcal{G} = \mathbb{C}^d \oplus \mathbb{C}^d$. Recall **indefinite** sesqui-linear form

$$[x, y]_{\mathcal{G}} = \langle x_+, y_+ \rangle_{\mathbb{C}^d} - \langle x_-, y_- \rangle_{\mathbb{C}^d},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{C}^d}$ is sesqui-linear, linear in the first argument and conjugate-linear in the second.

Additivity in the first argument.

$$\begin{aligned} [x + \tilde{x}, y]_{\mathcal{G}} &= \langle x_+ + \tilde{x}_+, y_+ \rangle - \langle x_- + \tilde{x}_-, y_- \rangle \\ &= (\langle x_+, y_+ \rangle + \langle \tilde{x}_+, y_+ \rangle) - (\langle x_-, y_- \rangle + \langle \tilde{x}_-, y_- \rangle) \\ &= (\langle x_+, y_+ \rangle - \langle x_-, y_- \rangle) + (\langle \tilde{x}_+, y_+ \rangle - \langle \tilde{x}_-, y_- \rangle) \\ &= [x, y]_{\mathcal{G}} + [\tilde{x}, y]_{\mathcal{G}}. \end{aligned}$$

Additivity in the second argument.

$$\begin{aligned} [x, y + \tilde{y}]_{\mathcal{G}} &= \langle x_+, y_+ + \tilde{y}_+ \rangle - \langle x_-, y_- + \tilde{y}_- \rangle \\ &= (\langle x_+, y_+ \rangle + \langle x_+, \tilde{y}_+ \rangle) - (\langle x_-, y_- \rangle + \langle x_-, \tilde{y}_- \rangle) \\ &= [x, y]_{\mathcal{G}} + [x, \tilde{y}]_{\mathcal{G}}. \end{aligned}$$

Homogeneity in the first argument.

$$\begin{aligned} [\alpha x, y]_{\mathcal{G}} &= \langle \alpha x_+, y_+ \rangle - \langle \alpha x_-, y_- \rangle \\ &= \alpha \langle x_+, y_+ \rangle - \alpha \langle x_-, y_- \rangle = \alpha [x, y]_{\mathcal{G}}. \end{aligned}$$

Conjugate-homogeneity in the second argument.

$$\begin{aligned} [x, \alpha y]_{\mathcal{G}} &= \langle x_+, \alpha y_+ \rangle - \langle x_-, \alpha y_- \rangle \\ &= \bar{\alpha} \langle x_+, y_+ \rangle - \bar{\alpha} \langle x_-, y_- \rangle = \bar{\alpha} [x, y]_{\mathcal{G}}. \end{aligned}$$

to disprove (8.1.8), we just need one non-zero vector x such that $[x, x]_{\mathcal{G}} = 0$.

Let $z \in \mathbb{C}^d$ be any non-zero vector and define

$$x := (z, z) \in \mathcal{G} = \mathbb{C}^d \oplus \mathbb{C}^d.$$

Since $z \neq 0$, we clearly have $x \neq 0$. Now compute

$$\begin{aligned} [x, x]_{\mathcal{G}} &= \langle z, z \rangle_{\mathbb{C}^d} - \langle z, z \rangle_{\mathbb{C}^d} \\ &= \|z\|^2 - \|z\|^2 \\ &= 0. \end{aligned}$$

Thus we have found a non-zero vector x satisfying $[x, x]_{\mathcal{G}} = 0$.

Therefore, the implication

$$[x, x]_{\mathcal{G}} = 0 \Rightarrow x = 0$$

is false in general. □

Lemma 8.17 (Maximal neutral subspaces are graphs of unitaries). *Assume that a linear subspace $M \subset \mathcal{G}$ is maximal neutral. Then there exists a unique $U \in U(d)$ such that*

$$M = \{(Uz, z) : z \in \mathbb{C}^d\}.$$

Exercise 8.18 (Maximal neutral subspaces are graphs of unitaries). Assume that $U \in U(d)$. Define a set M by

$$M = \{(Uz, z) : z \in \mathbb{C}^d\}.$$

Show that $M \subset \mathcal{G}$ is a linear subspace and it is maximal neutral.

Proof of Lemma 8.17. (Sketch following the text) Step 1: The coordinate projections $\pi_{\pm}$ restricted to M are injective. Hence $\dim M \leq d$.

Step 2: If $\dim M \leq 2$, one can enlarge M while preserving neutrality, contradicting maximality. Hence $\dim M = d$.

Step 3: Since $\pi_-|_M$ is bijective, define $U : \mathbb{C}^d \rightarrow \mathbb{C}^d$ by

$$\xi_z = (Uz, z) \in M.$$

Linearity follows from linearity of M . Neutrality implies

$$\langle Uz, Uw \rangle = \langle z, w \rangle,$$

hence U is unitary. Uniqueness follows immediately from representation (1.8). □

Assumption 8.19 (Boundary map is onto and has correct kernel). Assume that Γ_+, Γ_- are boundary maps such that the map

$$\Gamma := (\Gamma_+, \Gamma_-) : D(A_0^*) \rightarrow \mathbb{C}^d \oplus \mathbb{C}^d$$

is surjective and

$$\ker \Gamma = D(A_0).$$

Theorem 8.20 (Classification of self-adjoint extensions via $U(d)$). *Assume Assumptions 8.4 and 8.19. Then*

Part I. *For each $U \in U(d)$, the operator A_U defined by (8.1.5) is self-adjoint and satisfies*

$$A_0 \subset A_U \subset A_0^*.$$

Part II. *Conversely, if A is self-adjoint and*

$$A_0 \subset A \subset A_0^*,$$

then there exists a unique $U \in U(d)$ such that $A = A_U$.

Remark 8.21. For every $U \in U(d)$ the operator A_U satisfies

$$A_0 \subset A_U \subset A_0^*.$$

In particular,

$$D(A_0) \subset D(A_U) \subset D(A_0^*)$$

and

$$A_U f = A_0 f \quad \text{for } f \in D(A_0),$$

and

$$A_U f = A_0^* f \quad \text{for } f \in D(A_U).$$

Proof of Theorem 8.20. Part I: We want to prove that A_U is self-adjoint, i.e. $A_U = A_U^*$. From Lemma 8.12 we already know that A_U is symmetric, hence

$$A_U \subset A_U^*. \tag{8.1.9}$$

It remains to prove the reverse inclusion

$$A_U^* \subset A_U.$$

Let us choose and fix $u \in D(A_U^*)$. Then, by definition of the adjoint operator, just proven formula (8.1.9) and definition (8.1.1), we infer that

$$[u, v]_{A_0^*} = 0 \quad \text{for every } v \in D(A_U).$$

Let us recall the Green identity (8.1.2)

$$[u, v]_{A_0^*} = -i \left(\langle \Gamma_+(u), \Gamma_+(v) \rangle - \langle \Gamma_-(u), \Gamma_-(v) \rangle \right). \quad (8.1.10)$$

Therefore we infer

$$\langle \Gamma_+(u), \Gamma_+(v) \rangle - \langle \Gamma_-(u), \Gamma_-(v) \rangle = 0 \quad \text{for every } v \in D(A_U).$$

Since by definition of A_U we have

$$\Gamma_+(v) = U\Gamma_-(v), \text{ for every } v \in D(A_U), \quad (8.1.11)$$

we infer from (8.1.11) and (8.1.10) that

$$0 = \langle \Gamma_+(u), U\Gamma_-(v) \rangle - \langle \Gamma_-(u), \Gamma_-(v) \rangle. \quad (8.1.12)$$

Using the unitarity of U , we rewrite

$$\langle \Gamma_+(u), U\Gamma_-(v) \rangle = \langle U^*\Gamma_+(u), \Gamma_-(v) \rangle.$$

Hence (8.1.12) becomes

$$0 = \langle U^*\Gamma_+(u) - \Gamma_-(u), \Gamma_-(v) \rangle \quad \text{for all } v \in D(A_U). \quad (8.1.13)$$

Now let us choose and fix an arbitrary vector $z \in \mathbb{C}^d$. By surjectivity of the boundary map Γ_- , there exists $v \in D(A_0^*)$ such that

$$\Gamma_-(v) = z, \quad \Gamma_+(v) = Uz.$$

Then (8.1.11) holds, so

$$v \in D(A_U).$$

We note that in the above argument, it is sufficient to assume that the map Γ_- is surjective.

Substituting $\Gamma_-(v) = z$ into (8.1.13) gives

$$\langle U^*\Gamma_+(u) - \Gamma_-(u), z \rangle = 0 \quad \text{for all } z \in \mathbb{C}^d.$$

Therefore,

$$U^*\Gamma_+(u) - \Gamma_-(u) = 0.$$

Applying U to both sides yields

$$\Gamma_+(u) = U\Gamma_-(u). \quad (8.1.14)$$

By definition of $D(A_U)$ defined by (8.1.5), equation (8.1.14) implies

$$u \in D(A_U).$$

Hence we have proved

$$D(A_U^*) \subset D(A_U).$$

Moreover, for $u \in D(A_U^*)$, we have

$$A_U^*u = A_0^*u = A_Uu,$$

since both operators are restrictions of A_0^* . Combining this with (8.1.9), we conclude

$$A_U = A_U^*.$$

Therefore A_U is self-adjoint.

Part II: Conversely, let us assume that $A = A^*$ is self-adjoint and

$$A_0 \subset A \subset A_0^*.$$

Define

$$M := \Gamma(D(A)) \subset \mathbb{C}^d \oplus \mathbb{C}^d, \quad (8.1.15)$$

where

$$\Gamma(u) = (\Gamma_+(u), \Gamma_-(u)).$$

Since $D(A) \subset D(A_0^*)$, by Lemma 8.21, the set M is well-defined.

Next we will show that M is neutral. For this aim let us choose and fix $x, y \in M$. Then, by the definition of the set M , there exist $u, v \in D(A)$ such that

$$x = \Gamma(u) = (\Gamma_+(u), \Gamma_-(u))$$

and

$$y = \Gamma(v) = (\Gamma_+(v), \Gamma_-(v)).$$

We consider the indefinite form on the space $\mathcal{G} := \mathbb{C}^d \oplus \mathbb{C}^d$ defined by (8.1.6) and (8.1.7).

Using the Green identity, we compute:

$$[x, y]_{\mathcal{G}} = \langle \Gamma_+(u), \Gamma_+(v) \rangle - \langle \Gamma_-(u), \Gamma_-(v) \rangle = i[u, v]_{A_0^*}.$$

Since $u, v \in D(A)$, $A \subset A_0^*$ and A is symmetric we infer that

$$[u, v]_{A_0^*} = \langle A_0^* u, v \rangle - \langle u, A_0^* v \rangle = \langle Au, v \rangle - \langle u, Av \rangle = 0.$$

Therefore,

$$[x, y]_{\mathcal{G}} = 0 \quad \text{for all } x, y \in M.$$

what implies that M is a neutral subspace as claimed.

Since A is self-adjoint, it follows from Lemma 8.22 below, that M is maximal neutral.

By Lemma 8.17, there exists a unique unitary operator $U \in U(d)$ such that

$$M = \{(Uz, z) : z \in \mathbb{C}^d\}. \quad (8.1.16)$$

Next we need to prove that

$$A = A_U. \quad (8.1.17)$$

Firstly, we will prove the inclusion

$$A \subset A_U. \quad (8.1.18)$$

For this aim, let us take $u \in D(A)$. Then by definition of M , $\Gamma(u) \in M$, so by (8.1.16) there exists $z \in \mathbb{C}^d$ such that

$$(\Gamma_+(u), \Gamma_-(u)) = \Gamma(u) = (Uz, z).$$

Hence, we deduce that

$$\Gamma_+(u) = Uz, \quad \Gamma_-(u) = z.$$

Therefore,

$$\Gamma_+(u) = U\Gamma_-(u).$$

and so, by the definition (8.1.5) of A_U , $u \in D(A_U)$. Therefore, we infer that

$$D(A) \subset D(A_U). \quad (8.1.19)$$

If $u \in D(A)$, then $Au = A_0^*u$ because $A \subset A_0^*$. Moreover, by just proved inclusion (8.1.19) and the definition of A_U , $A_U u = A_0^*u$. Hence, we infer that $Au = A_U u$. In this way, we finished the proof of (8.1.18).

Secondly, we will prove the inclusion

$$A_U \subset A. \quad (8.1.20)$$

We begin by showing that

$$D(A_U) \subset D(A), \quad (8.1.21)$$

For this aim let us choose $u \in D(A_U)$. Then by (8.1.5), $u \in D(A_0^*)$ and

$$\Gamma_+(u) = U\Gamma_-(u).$$

Thus, by (8.1.16) with $z = \Gamma_-(u)$, we deduce that

$$(\Gamma_+(u), \Gamma_-(u)) = (U\Gamma_-(u), \Gamma_-(u)) \in M.$$

By definition (8.1.15) of M , we infer that there exists $v \in D(A)$ such that

$$\Gamma(u) = \Gamma(v).$$

Therefore, by linearity of the map Γ we infer that

$$\Gamma(u - v) = 0.$$

i.e. $u - v \in \ker \Gamma$. By Assumption 8.19,

$$u - v \in D(A_0).$$

Since $D(A_0) \subset D(A)$ and $v \in D(A)$ we deduce that

$$u = (u - v) + v \in D(A).$$

Hence, we proved (8.1.21).

As before we can prove that if $u \in D(A_U)$, then $Au = A_0^*u$. Consequently (8.1.20) follows.

Combining (8.1.18) and (8.1.20), we conclude the proof of equality (8.1.17).

The unitary operator U is unique by Lemma (8.17).

This completes the proof of Theorem 8.20. $\square$

Lemma 8.22 (Self-adjointness implies maximal neutrality). *Under the assumptions of Theorem 8.20 we define a sesqui-linear map on $\mathcal{G} := \mathbb{C}^d \oplus \mathbb{C}^d$ by the following formula*

$$[(x_+, x_-), (y_+, y_-)]_{\mathcal{G}} := \langle x_+, y_+ \rangle_{\mathbb{C}^3} - \langle x_-, y_- \rangle_{\mathbb{C}^3}.$$

Let

$$M := \Gamma(D(A)) \subset \mathcal{G}.$$

Then

(i) M is neutral, i.e.

$$[\xi, \eta]_{\mathcal{G}} = 0 \quad \text{for all } \xi, \eta \in M.$$

(ii) If A is self-adjoint, then M is maximal neutral; equivalently,

$$M^\perp = M,$$

where $M^\perp$ is defined by

$$M^\perp := \{\omega \in \mathcal{G} : [\omega, \xi]_{\mathcal{G}} = 0 \quad \forall \xi \in M\}.$$

8.2 Application: A punctured third-order differential operator with a first-order term

8.2.1 Special case $\alpha = 0$

We consider the operator

$$A = -iD^3 - 3\alpha iD.$$

In what follows, we study the special case $\alpha = 0$. Let the Hilbert space be

$$H = L^2(\mathbb{R}). \tag{8.2.1}$$

We study the following operator

$$A_0 u := (-iD)^3 u = -iD^3 u = -iu^{(3)}. \tag{8.2.2}$$

Its domain is defined by

$$\begin{aligned} D(A_0) = \Big\{ u \in L^2(\mathbb{R}) : u|_{(0,\infty)} \in H^3(0,\infty), \quad u|_{(-\infty,0)} \in H^3(-\infty,0), \\ \text{and } u(0+) = u(0-) = 0, \quad u'(0+) = u'(0-) = 0, \quad u''(0+) = u''(0-) = 0 \Big\}. \end{aligned}$$

This operator A_0 is not the same as operator A_1 defined at the beginning of subsection 5.4.1. Indeed, they have different domains. Moreover, this operator A_0 is equal to the closure of operator A_1 , see Proposition 5.33. Hence the adjoint of this new operator A_0 is equal to the adjoint operator A_1 defined at the beginning of subsection 5.4.1. So we have the following results, see Proposition 5.36 for the latter.

Proposition 8.23. *The operator A_0 defined by (8.2.2) is symmetric, closed, and densely defined.*

Lemma 8.24. *The adjoint domain is*

$$D(A_0^*) = \left\{ u \in L^2(\mathbb{R}) : u|_{(0,\infty)} \in H^3(0,\infty), u|_{(-\infty,0)} \in H^3(-\infty,0) \right\},$$

and

$$A_0^* u = -iu^{(3)} = -iD^3 u.$$

8.2.2 Diagonalized boundary maps

Define boundary maps for $u \in D(A_0^*)$ as follows

$$\begin{aligned} p_{\pm}^0(u) &= \frac{1}{\sqrt{2}} \left(u(0\pm) + u''(0\pm) \right), \\ q_{\pm}^0(u) &= \frac{1}{\sqrt{2}} \left(-u(0\pm) + u''(0\pm) \right), \\ r_{\pm}^0(u) &= u'(0\pm). \end{aligned}$$

Define boundary operators $\Gamma_{\pm}^0$ by the following formulae

$$\Gamma_+^0(u) = \begin{pmatrix} p_+^0(u) \\ q_-^0(u) \\ r_-^0(u) \end{pmatrix} \in \mathbb{C}^3, \quad (8.2.3)$$

$$\Gamma_-^0(u) = \begin{pmatrix} q_+^0(u) \\ r_+^0(u) \\ p_-^0(u) \end{pmatrix} \in \mathbb{C}^3. \quad (8.2.4)$$

It will become clear from later where do these operators come from.

Proposition 8.25 (Third order Green identity). *For all $u, v \in D(A_0^*)$, then*

$$\langle A_0^* u, v \rangle_{L^2} - \langle u, A_0^* v \rangle_{L^2} = -i \left(W_+^0(u, v) - W_-^0(u, v) \right),$$

where the bilinear boundary maps $W_{\pm}^0$ are defined by

$$W_{\pm}^0(u, v) := u''(0\pm) \overline{v(0\pm)} - u'(0\pm) \overline{v'(0\pm)} + u(0\pm) \overline{v''(0\pm)}. \quad (8.2.5)$$

Remark 8.26. Note that if $u, v \in D(A_0^*)$, then

$$\langle \Gamma_+^0(u), \Gamma_+^0(v) \rangle_{\mathbb{C}^3} - \langle \Gamma_-^0(u), \Gamma_-^0(v) \rangle_{\mathbb{C}^3} = W_+^0(u, v) - W_-^0(u, v),$$

Proof of Remark 8.26. Let us choose and fix $u, v \in D(A_0^*)$. Recall the definitions (8.2.3) and (8.2.4). Since the inner product in $\mathbb{C}^3$ is defined by

$$\langle x, y \rangle_{\mathbb{C}^3} = x_1 \overline{y_1} + x_2 \overline{y_2} + x_3 \overline{y_3}. \quad (8.2.6)$$

$$\begin{aligned} \langle \Gamma_+^0(u), \Gamma_+^0(v) \rangle_{\mathbb{C}^3} &= p_+^0(u) \overline{p_+^0(v)} + q_+^0(u) \overline{q_+^0(v)} + r_+^0(u) \overline{r_+^0(v)}, \\ \langle \Gamma_-^0(u), \Gamma_-^0(v) \rangle_{\mathbb{C}^3} &= q_-^0(u) \overline{q_-^0(v)} + r_-^0(u) \overline{r_-^0(v)} + p_-^0(u) \overline{p_-^0(v)}. \end{aligned}$$

Therefore, by the definitions (8.2.3) and (8.2.4) of the boundary operators $\Gamma_{\pm}^0$ we infer that

$$\begin{aligned} LHS &= \langle \Gamma_+^0(u), \Gamma_+^0(v) \rangle_{\mathbb{C}^3} - \langle \Gamma_-^0(u), \Gamma_-^0(v) \rangle_{\mathbb{C}^3} \\ &= \left(p_+^0(u) \overline{p_+^0(v)} - p_-^0(u) \overline{p_-^0(v)} \right) + \left(q_+^0(u) \overline{q_+^0(v)} - q_-^0(u) \overline{q_-^0(v)} \right) \\ &\quad + \left(r_+^0(u) \overline{r_+^0(v)} - r_-^0(u) \overline{r_-^0(v)} \right). \end{aligned} \quad (8.2.7)$$

On the other hand, it follows from (8.2.5) that

$$\begin{aligned} RHS &= W_+^0(u, v) - W_-^0(u, v) = \left(p_+(u) \overline{p_+(v)} - p_-(u) \overline{p_-(v)} \right) \\ &\quad - \left(q_+(u) \overline{q_+(v)} - q_-(u) \overline{q_-(v)} \right) - \left(r_+(u) \overline{r_+(v)} - r_-(u) \overline{r_-(v)} \right). \end{aligned} \quad (8.2.8)$$

Comparing (8.2.7) and (8.2.8), we obtain

$$\langle \Gamma_+^0(u), \Gamma_+^0(v) \rangle_{\mathbb{C}^3} - \langle \Gamma_-^0(u), \Gamma_-^0(v) \rangle_{\mathbb{C}^3} = W_+^0(u, v) - W_-^0(u, v).$$

The proof of Remark 8.26 is complete. $\square$

Before we embark on a proof of Proposition 8.25, we will formulate and prove a corresponding result for first order operator.

Exercise 8.27 (First order Green identity). If $f, g \in D(A_0^*)$, then

$$\langle (-iD)f, g \rangle - \langle f, (-iD)g \rangle = -i \left(-f(0+) \overline{g(0+)} + f(0-) \overline{g(0-)} \right). \quad (8.2.9)$$

Solution of Exercise 8.27. Let us choose and fix $f, g \in D(A_0^*)$. Then

$$\begin{aligned} \langle (-iD)f, g \rangle &= -i \int_{-\infty}^{\infty} f'(x) \overline{g(x)} dx \\ &= -i \left[\int_0^{\infty} f'(x) \overline{g(x)} dx + \int_{-\infty}^0 f'(x) \overline{g(x)} dx \right]. \end{aligned} \quad (8.2.10)$$

Integrating by parts on $(0, \infty)$ and on $(-\infty, 0)$ yields

$$\int_0^{\infty} f'(x) \overline{g(x)} dx = \left[f(x) \overline{g(x)} \right]_0^{\infty} - \int_0^{\infty} f(x) \overline{g'(x)} dx. \quad (8.2.11)$$

and

$$\int_{-\infty}^0 f'(x) \overline{g(x)} dx = \left[f(x) \overline{g(x)} \right]_{-\infty}^0 - \int_{-\infty}^0 f(x) \overline{g'(x)} dx. \quad (8.2.12)$$

Substitute (8.2.11)–(8.2.12) into (8.2.10):

$$\begin{aligned} \langle (-iD)f, g \rangle &= -i \left(\left[f(x) \overline{g(x)} \right]_0^{\infty} - \int_0^{\infty} f(x) \overline{g'(x)} dx \right. \\ &\quad \left. + \left[f(x) \overline{g(x)} \right]_{-\infty}^0 - \int_{-\infty}^0 f(x) \overline{g'(x)} dx \right). \end{aligned}$$

Using the (vanishing) limits at infinity, we keep only the boundary terms at $0\pm$:

$$\left[f(x) \overline{g(x)} \right]_0^{\infty} + \left[f(x) \overline{g(x)} \right]_{-\infty}^0 = -f(0+) \overline{g(0+)} + f(0-) \overline{g(0-)}.$$

Hence,

$$\begin{aligned} \langle (-iD)f, g \rangle &= -i \left(-f(0+) \overline{g(0+)} + f(0-) \overline{g(0-)} \right) \\ &\quad + i \left(\int_0^{\infty} f(x) \overline{g'(x)} dx + \int_{-\infty}^0 f(x) \overline{g'(x)} dx \right) \\ &= -i \left(-f(0+) \overline{g(0+)} + f(0-) \overline{g(0-)} \right) + \int_{-\infty}^{\infty} f(x) \overline{ig'(x)} dx. \end{aligned}$$

Since

$$\langle f, (-iD)g \rangle = \int_{-\infty}^{\infty} f(x) \overline{(-ig'(x))} dx,$$

we deduce (8.2.9) as claimed. $\square$

We are now ready to prove Proposition 8.25 as follows.

Proof of Proposition 8.25. Let us choose and fix $u, v \in D(A_0^*)$. We compute

$$\begin{aligned} LHS &= \langle A_0^* u, v \rangle - \langle u, A_0^* v \rangle = \int_{-\infty}^{\infty} (-iu^{(3)}(x)) \overline{v(x)} dx - \int_{-\infty}^{\infty} u(x) \overline{(-iv^{(3)}(x))} dx. \\ &= \int_0^{\infty} (-i) u^{(3)}(x) \overline{v(x)} dx + \int_{-\infty}^0 (-i) u^{(3)}(x) \overline{v(x)} dx \\ &\quad - \int_0^{\infty} u(x) (i) \overline{v^{(3)}(x)} dx - \int_{-\infty}^0 u(x) (i) \overline{v^{(3)}(x)} dx. \end{aligned}$$

Using Integration by part, we infer that

$$\begin{aligned} LHS &= (-i) \left[\int_0^{\infty} D(D^2 u(x)) \overline{v(x)} dx + \int_{-\infty}^0 D(D^2 u(x)) \overline{v(x)} dx \right. \\ &\quad \left. + \int_0^{\infty} u(x) \overline{D(D^2 v(x))} dx + \int_{-\infty}^0 u(x) \overline{D(D^2 v(x))} dx \right] \\ &= (-i) \left[\boxed{- \int_0^{\infty} D^2 u(x) \overline{Dv(x)} dx} + D^2 u(x) \overline{v(x)} \Big|_0^{\infty} \right. \\ &\quad \boxed{- \int_{-\infty}^0 D^2 u(x) \overline{Dv(x)} dx} + D^2 u(x) \overline{v(x)} \Big|_{-\infty}^0 \\ &\quad \boxed{- \int_0^{\infty} Du(x) \overline{D^2 v(x)} dx} + u(x) \overline{D^2 v(x)} \Big|_0^{\infty} \\ &\quad \left. \boxed{- \int_{-\infty}^0 Du(x) \overline{D^2 v(x)} dx} + u(x) \overline{D^2 v(x)} \Big|_{-\infty}^0 \right] \end{aligned}$$

Applying integration by parts again we get

$$\begin{aligned} &\boxed{- \int_0^{\infty} D^2 u(x) \overline{Dv(x)} dx - \int_{-\infty}^0 D^2 u(x) \overline{Dv(x)} dx} \\ &= \left[\boxed{\int_0^{\infty} Du(x) \overline{D^2 v(x)} dx} - Du(x) \overline{Dv(x)} \Big|_0^{\infty} \right. \\ &\quad \left. \boxed{+ \int_{-\infty}^0 Du(x) \overline{D^2 v(x)} dx} - Du(x) \overline{Dv(x)} \Big|_{-\infty}^0 \right]. \end{aligned}$$

Hence the blue terms cancel with the green ones and therefore we get

$$\begin{aligned} LHS = (-i) & \left[D^2 u(x) \overline{v(x)} \Big|_0^\infty + D^2 u(x) \overline{v(x)} \Big|_{-\infty}^0 \right. \\ & + u(x) \overline{D^2 v(x)} \Big|_0^\infty + u(x) \overline{D^2 v(x)} \Big|_{-\infty}^0 \\ & \left. - Du(x) \overline{Dv(x)} \Big|_0^\infty - Du(x) \overline{Dv(x)} \Big|_{-\infty}^0 \right] \end{aligned}$$

$$D^2 u(x) \overline{v(x)} \Big|_0^\infty = \lim_{x \rightarrow \infty} D^2 u(x) \overline{v(x)} - \lim_{x \rightarrow 0+} D^2 u(x) \overline{v(x)} = -u''(0+) \overline{v(0+)}.$$

and

$$D^2 u(x) \overline{v(x)} \Big|_{-\infty}^0 = \lim_{x \rightarrow 0-} D^2 u(x) \overline{v(x)} - \lim_{x \rightarrow -\infty} D^2 u(x) \overline{v(x)} = u''(0-) \overline{v(0-)}.$$

Similarly

$$\begin{aligned} u(x) \overline{D^2 v(x)} \Big|_0^\infty &= \lim_{x \rightarrow \infty} u(x) \overline{v''(x)} - \lim_{x \rightarrow 0+} u(x) \overline{v''(x)} = -u(0+) \overline{v''(0+)}, \\ u(x) \overline{D^2 v(x)} \Big|_{-\infty}^0 &= \lim_{x \rightarrow 0-} u(x) \overline{v''(x)} - \lim_{x \rightarrow -\infty} u(x) \overline{v''(x)} = u(0-) \overline{v''(0-)}, \\ -Du(x) \overline{Dv(x)} \Big|_0^\infty &= -\left(\lim_{x \rightarrow \infty} u'(x) \overline{v'(x)} - \lim_{x \rightarrow 0+} u'(x) \overline{v'(x)} \right) = u'(0+) \overline{v'(0+)}, \\ -Du(x) \overline{Dv(x)} \Big|_{-\infty}^0 &= -\left(\lim_{x \rightarrow 0-} u'(x) \overline{v'(x)} - \lim_{x \rightarrow -\infty} u'(x) \overline{v'(x)} \right) = -u'(0-) \overline{v'(0-)}. \end{aligned}$$

Therefore

$$\begin{aligned} LHS = (-i) & \left[-u''(0+) \overline{v(0+)} + u''(0-) \overline{v(0-)} - u(0+) \overline{v''(0+)} + u(0-) \overline{v''(0-)} \right. \\ & \left. + u'(0+) \overline{v'(0+)} - u'(0-) \overline{v'(0-)} \right]. \end{aligned}$$

Collecting the terms corresponding to 0+ and the terms corresponding to 0-, we obtain

$$\begin{aligned} LHS = -i & \left[(u''(0+) \overline{v(0+)} - u'(0+) \overline{v'(0+)} + u(0+) \overline{v''(0+)}) \right. \\ & \left. - (u''(0-) \overline{v(0-)} - u'(0-) \overline{v'(0-)} + u(0-) \overline{v''(0-)}) \right]. \end{aligned}$$

By the definition of $W_{\pm}^0(u, v)$ in (8.2.5), we obtain

$$LHS = -i \left(W_+^0(u, v) - W_-^0(u, v) \right).$$

The proof of Proposition 8.25 is complete. $\square$

Using Remark 8.26, we deduce the following corollary.

Corollary 8.28 (Green identity-2). *If $u, v \in D(A_0^*)$, then*

$$\langle A_0^* u, v \rangle_{L^2} - \langle u, A_0^* v \rangle_{L^2} = -i \left(\langle \Gamma_+^0(u), \Gamma_+^0(v) \rangle_{\mathbb{C}^3} - \langle \Gamma_-^0(u), \Gamma_-^0(v) \rangle_{\mathbb{C}^3} \right).$$

8.2.3 General Case $\alpha \in \mathbb{R}$

We assume that $\alpha \in \mathbb{R}$. We put $H := L^2(\mathbb{R})$ as in (8.2.1). Define the densely defined operator $A_{\alpha,0}$ by

$$\begin{aligned} D(A_{\alpha,0}) &:= C_0^\infty((-\infty, 0) \cup (0, \infty)), \\ A_{\alpha,0}u &:= -iu^{(3)} - 3\alpha iu'. \end{aligned} \quad (8.2.16)$$

We can prove as in earlier parts of the thesis the following result.

Proposition 8.29. *The operator $A_{\alpha,0}$ defined by (8.2.16) is symmetric and densely defined.*

We now describe the adjoint operator as follows.

Lemma 8.30. *The adjoint of $A_{\alpha,0}^*$ is*

$$D(A_{\alpha,0}^*) = H_{\text{loc}}^3((-\infty, 0] \oplus H_{\text{loc}}^3[0, \infty)). \quad (8.2.17)$$

and

$$A_{\alpha,0}^*u = -iu^{(3)} - 3\alpha iu' \quad \text{on } \mathbb{R} \setminus \{0\}, \quad (8.2.18)$$

Proof of Lemma 8.30. This results was proved earlier. $\square$

Lemma 8.31 (Closure of $A_{\alpha,0}$). *The closure $\overline{A_{\alpha,0}}$ of $A_{\alpha,0}$ is characterized as follows.*

$$D(\overline{A_{\alpha,0}}) = \{u \in D(A_{\alpha,0}^*) : u(0\pm) = u'(0\pm) = u''(0\pm) = 0\}.$$

Proof of Lemma 8.31. This results also was proved earlier. $\square$

Proposition 8.32 (Green identity). *There exists boundary maps*

$$\Gamma_{\alpha,+} : D(A_{\alpha,0}^*) \ni f \mapsto \begin{pmatrix} p_{\alpha,+}(f) \\ q_{\alpha,-}(f) \\ r_{\alpha,-}(f) \end{pmatrix} \in \mathbb{C}^3 \quad (8.2.19)$$

and

$$\Gamma_{\alpha,-} : D(A_{\alpha,0}^*) \ni f \mapsto \begin{pmatrix} q_{\alpha,+}(f) \\ r_{\alpha,+}(f) \\ p_{\alpha,-}(f) \end{pmatrix} \in \mathbb{C}^3. \quad (8.2.20)$$

such that for all $f, g \in D(A_{\alpha,0}^*)$,

$$\langle A_{\alpha,0}^*f, g \rangle_{L^2} - \langle f, A_{\alpha,0}^*g \rangle_{L^2} = -i \left(\langle \Gamma_{\alpha,+}(f), \Gamma_{\alpha,+}(g) \rangle_{\mathbb{C}^3} - \langle \Gamma_{\alpha,-}(f), \Gamma_{\alpha,-}(g) \rangle_{\mathbb{C}^3} \right). \quad (8.2.21)$$

Remark 8.33. We will see during the proof that these maps are of the following form. For $u \in D(A_{\alpha,0}^*)$,

$$p_{\alpha,\pm}(u) := \sqrt{\lambda_{\alpha,+}} \frac{\lambda_{\alpha,+}u(0\pm) + u''(0\pm)}{\sqrt{1 + \lambda_{\alpha,+}^2}}, \quad (8.2.22)$$

$$q_{\alpha,\pm}(u) := \sqrt{-\lambda_{\alpha,-}} \frac{\lambda_{\alpha,-}u(0\pm) + u''(0\pm)}{\sqrt{1 + \lambda_{\alpha,-}^2}}, \quad (8.2.23)$$

$$r_{\alpha,\pm}(u) := u'(0\pm), \quad (8.2.24)$$

where $\lambda_{\alpha,\pm}$ are defined as follow

$$\begin{aligned} \Delta_\alpha &:= 9\alpha^2 + 4, \\ \lambda_{\alpha,\pm} &:= \frac{3\alpha \pm \sqrt{\Delta_\alpha}}{2}. \end{aligned} \quad (8.2.25)$$

Note that for $\alpha = 0$, we have

$$\Delta_0 = \sqrt{9\alpha^2 + 4} = 2, \quad \lambda_{0,\pm} = \frac{\pm 2}{2} = \pm 1.$$

Proof of Proposition 8.32. We choose and fix $f, g \in D(A_{\alpha,0}^*)$. Using (8.2.18), by Proposition 8.25 and Exercise 8.27 we have

$$\begin{aligned} LHS &= \langle A_{\alpha,0}^* f, g \rangle - \langle f, A_{\alpha,0}^* g \rangle = \langle -iD^3 f, g \rangle - \langle f, -iD^3 g \rangle \\ &\quad + 3\alpha [\langle -iDf, g \rangle - \langle f, -iDg \rangle] \\ &= -i [W_+^0(f, g) - W_-^0(f, g) + 3\alpha (-f(0+) \overline{g(0+)} + f(0-) \overline{g(0-)})]. \end{aligned} \quad (8.2.26)$$

where the maps $W_\pm^0$ have been defined in (8.2.5). Next we introduce

$$\begin{aligned} W_\pm^\alpha(f, g) &= f''(0\pm) \overline{g(0\pm)} - f'(0\pm) \overline{g'(0\pm)} \\ &\quad + f(0\pm) \overline{g''(0\pm)} + 3\alpha f(0\pm) \overline{g(0\pm)}. \end{aligned} \quad (8.2.27)$$

Combining (8.2.26) with (8.2.5) we get

$$\langle A_{\alpha,0}^* f, g \rangle - \langle f, A_{\alpha,0}^* g \rangle = -i (W_+^\alpha(f, g) - W_-^\alpha(f, g)),$$

Let us put

$$v_\pm(f) := \begin{pmatrix} f(0\pm) \\ f''(0\pm) \end{pmatrix} \in \mathbb{C}^2, \quad f \in D(A_{\alpha,0}).$$

In view of Lemma 8.30, the maps $v_{\pm}$ are well define, where B_{α} is 2×2 matrix defined by

$$B_{\alpha} := \begin{pmatrix} 3\alpha & 1 \\ 1 & 0 \end{pmatrix}.$$

Observe that the first, third and fourth terms in (8.2.27) combine as

$$f''(0\pm) \overline{g(0\pm)} + f(0\pm) \overline{g''(0\pm)} + 3\alpha f(0\pm) \overline{g(0\pm)} = \langle B_{\alpha} v_{\pm}(f), v_{\pm}(g) \rangle_{\mathbb{C}^2}. \quad (8.2.28)$$

Indeed, we have

$$\begin{aligned} \langle B_{\alpha} v_{\pm}(f), v_{\pm}(g) \rangle &= \left\langle \begin{pmatrix} 3\alpha & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f(0\pm) \\ f''(0\pm) \end{pmatrix}, \begin{pmatrix} g(0\pm) \\ g''(0\pm) \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} 3\alpha f(0\pm) + f''(0\pm) \\ f(0\pm) \end{pmatrix}, \begin{pmatrix} g(0\pm) \\ g''(0\pm) \end{pmatrix} \right\rangle \\ &= 3\alpha f(0\pm) \overline{g(0\pm)} + f''(0\pm) \overline{g(0\pm)} + f(0\pm) \overline{g''(0\pm)}. \end{aligned}$$

Let us observe that B_{α} is a real symmetric matrix. Hence by linear algebra it admits an orthonormal basis of eigenvectors. To find this Its characteristic polynomial of B_{α}

$$\begin{aligned} \det(B_{\alpha} - \lambda I) &= \det \begin{pmatrix} 3\alpha - \lambda & 1 \\ 1 & -\lambda \end{pmatrix} \\ &= (3\alpha - \lambda)(-\lambda) - 1 \\ &= \lambda^2 - 3\alpha\lambda - 1. \end{aligned}$$

Hence the eigenvalues are given by formula (8.2.25). Notice that

$$\lambda_{\alpha,-} < 0 < \lambda_{\alpha,+}.$$

Eigenvectors corresponding to these eigenvalues are proportional to

$$e_{\alpha,+} = \begin{pmatrix} \lambda_{\alpha,+} \\ 1 \end{pmatrix}, \quad e_{\alpha,-} = \begin{pmatrix} \lambda_{\alpha,-} \\ 1 \end{pmatrix}.$$

Moreover,

$$\|e_{\alpha,+}\|^2 = \lambda_{\alpha,+}^2 + 1, \quad \|e_{\alpha,-}\|^2 = \lambda_{\alpha,-}^2 + 1. \quad (8.2.29)$$

Since B_{α} is symmetric, by linear algebra, these eigenvectors are orthogonal. We can check this directly.

$$\begin{aligned}
\langle e_{\alpha,+}, e_{\alpha,-} \rangle &= \lambda_{\alpha,+} \lambda_{\alpha,-} + 1 = \frac{9\alpha^2 - \Delta_\alpha^2}{4} + 1 \\
&= \frac{9\alpha^2 - (9\alpha^2 + 4)}{4} + 1 = \frac{-4}{4} + 1 = -1 + 1 = 0.
\end{aligned}$$

After normalization, they form an orthonormal basis of $\mathbb{C}^2$, i.e. the normalized eigenvectors are as follows

$$\widehat{e}_{\alpha,+} := \frac{e_{\alpha,+}}{\|e_{\alpha,+}\|}, \quad \widehat{e}_{\alpha,-} := \frac{e_{\alpha,-}}{\|e_{\alpha,-}\|}. \quad (8.2.30)$$

Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{C}^2$. Using the ONB (8.2.30) we have

$$x = x_{\alpha,+} \widehat{e}_{\alpha,+} + x_{\alpha,-} \widehat{e}_{\alpha,-}, \quad x_{\alpha,+} = \langle x, \widehat{e}_{\alpha,+} \rangle_{\mathbb{C}^2}, \quad x_{\alpha,-} = \langle x, \widehat{e}_{\alpha,-} \rangle_{\mathbb{C}^2},$$

and similarly

$$y = y_{\alpha,+} \widehat{e}_{\alpha,+} + y_{\alpha,-} \widehat{e}_{\alpha,-}, \quad y_{\alpha,+} = \langle y, \widehat{e}_{\alpha,+} \rangle_{\mathbb{C}^2}, \quad y_{\alpha,-} = \langle y, \widehat{e}_{\alpha,-} \rangle_{\mathbb{C}^2}.$$

Using (8.2.29), (8.2.30) and the sesqui-linear inner product (8.2.6) in $\mathbb{C}^2$, we can compute

$$\begin{aligned}
x_{\alpha,+} &= \langle x, \widehat{e}_{\alpha,+} \rangle_{\mathbb{C}^2} = \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \frac{1}{\sqrt{1 + \lambda_{\alpha,+}^2}} \begin{pmatrix} \lambda_{\alpha,+} \\ 1 \end{pmatrix} \right\rangle_{\mathbb{C}^2} \\
&= \frac{1}{\sqrt{1 + \lambda_{\alpha,+}^2}} (\lambda_{\alpha,+} x_1 + x_2),
\end{aligned} \quad (8.2.31)$$

Similarly,

$$\begin{aligned}
x_{\alpha,-} &= \langle x, \widehat{e}_{\alpha,-} \rangle_{\mathbb{C}^2} = \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \frac{1}{\sqrt{1 + \lambda_{\alpha,-}^2}} \begin{pmatrix} \lambda_{\alpha,-} \\ 1 \end{pmatrix} \right\rangle_{\mathbb{C}^2} \\
&= \frac{1}{\sqrt{1 + \lambda_{\alpha,-}^2}} (\lambda_{\alpha,-} x_1 + x_2).
\end{aligned} \quad (8.2.32)$$

Using orthonormality by linear algebra, we deduce that

$$\langle B_\alpha x, y \rangle_{\mathbb{C}^2} = \lambda_{\alpha,+} x_{\alpha,+} \overline{y_{\alpha,+}} + \lambda_{\alpha,-} x_{\alpha,-} \overline{y_{\alpha,-}}. \quad (8.2.33)$$

Since $\lambda_{\alpha,+} > 0$ and $\lambda_{\alpha,-} < 0$, we can define

$$p_\alpha(x) := \sqrt{\lambda_{\alpha,+}} x_{\alpha,+}, \quad q_\alpha(x) := \sqrt{-\lambda_{\alpha,-}} x_{\alpha,-},$$

Hence we infer that with this notation formula (8.2.33) becomes

$$\langle B_\alpha x, y \rangle_{\mathbb{C}^2} = p_\alpha(x) \overline{p_\alpha(y)} - q_\alpha(x) \overline{q_\alpha(y)}. \quad (8.2.34)$$

Next recall that

$$v_\pm(f) := \begin{pmatrix} f(0\pm) \\ f''(0\pm) \end{pmatrix} \in \mathbb{C}^2, \quad v_\pm(g) := \begin{pmatrix} g(0\pm) \\ g''(0\pm) \end{pmatrix} \in \mathbb{C}^2.$$

Case 1. We consider first the (+) case. Put

$$x := v_+(f) := \begin{pmatrix} f(0+) \\ f''(0+) \end{pmatrix} \in \mathbb{C}^2, \quad y := v_+(g) := \begin{pmatrix} g(0+) \\ g''(0+) \end{pmatrix} \in \mathbb{C}^2.$$

Define for $f \in D(A_{\alpha,0}^*)$

$$p_{\alpha,+}(f) := p_\alpha(v_+(f)), \quad q_{\alpha,+}(f) := q_\alpha(v_+(f))$$

Hence we have for $g \in D(A_{\alpha,0}^*)$

$$p_{\alpha,+}(g) := p_\alpha(v_+(g)), \quad q_{\alpha,+}(g) := q_\alpha(v_+(g))$$

Then formula (8.2.34) becomes

$$\langle B_\alpha v_+(f), v_+(g) \rangle_{\mathbb{C}^2} = p_{\alpha,+}(f) \overline{p_{\alpha,+}(g)} - q_{\alpha,+}(f) \overline{q_{\alpha,+}(g)}.$$

Case 2. Next we consider the (−) case. Put

$$x := v_-(f) := \begin{pmatrix} f(0-) \\ f''(0-) \end{pmatrix} \in \mathbb{C}^2, \quad y := v_-(g) := \begin{pmatrix} g(0-) \\ g''(0-) \end{pmatrix} \in \mathbb{C}^2.$$

We now define

$$p_{\alpha,-}(f) := p_\alpha(v_-(f)), \quad q_{\alpha,-}(f) := q_\alpha(v_-(f))$$

and deduce that

$$p_{\alpha,-}(g) := p_\alpha(v_-(g)), \quad q_{\alpha,-}(g) := q_\alpha(v_-(g))$$

As in the Case 1, Then formula (8.2.34) becomes

$$\langle B_\alpha v_-(f), v_-(g) \rangle_{\mathbb{C}^2} = p_{\alpha,-}(f) \overline{p_{\alpha,-}(g)} - q_{\alpha,-}(f) \overline{q_{\alpha,-}(g)}.$$

The two formulae obtained in Cases 1 and 2 can be written in a compact form as follows.

$$\langle B_\alpha v_\pm(f), v_\pm(g) \rangle_{\mathbb{C}^2} = p_{\alpha,\pm}(f) \overline{p_{\alpha,\pm}(g)} - q_{\alpha,\pm}(f) \overline{q_{\alpha,\pm}(g)}.$$

Using formulae (8.2.31) and (8.2.32) we get the following explicit formulas

$$\begin{aligned} p_{\alpha,\pm}(f) &= \sqrt{\lambda_{\alpha,+}} x_{1,\pm} = \sqrt{\lambda_{\alpha,+}} \frac{\lambda_{\alpha,+} f(0\pm) + f''(0\pm)}{\sqrt{1 + \lambda_{\alpha,+}^2}}, \\ q_{\alpha,\pm}(f) &= \sqrt{-\lambda_{\alpha,-}} x_{2,\pm} = \sqrt{-\lambda_{\alpha,-}} \frac{\lambda_{\alpha,-} f(0\pm) + f''(0\pm)}{\sqrt{1 + \lambda_{\alpha,-}^2}}. \end{aligned}$$

and the same with f replaced by g .

Note that these formulae agree with (8.2.22) and (8.2.23).

It follows from (8.2.27), (8.2.28) and (8.2.24) that

$$W_\pm^\alpha(f, g) = p_\pm(f) \overline{p_\pm(g)} - q_\pm(f) \overline{q_\pm(g)} - r_\pm(f) \overline{r_\pm(g)}.$$

Using formulas (8.2.19)–(8.2.20), we obtain

$$W_+^\alpha(f, g) - W_-^\alpha(f, g) = \langle \Gamma_{\alpha,+}(f), \Gamma_{\alpha,+}(g) \rangle - \langle \Gamma_{\alpha,-}(f), \Gamma_{\alpha,-}(g) \rangle, \quad (8.2.35)$$

and substituting (8.2.35) into (8.2.27) gives (8.2.21). The proof is complete. $\square$

Lemma 8.34 (Kernel and surjectivity of $(\Gamma_{\alpha,+}, \Gamma_{\alpha,-})$). *Let*

$$\Gamma_\alpha := (\Gamma_{\alpha,+}, \Gamma_{\alpha,-}) : D(A_{\alpha,0}^*) \rightarrow \mathbb{C}^3 \oplus \mathbb{C}^3.$$

Then

$$\ker \Gamma_\alpha = D(A_{\alpha,0}),$$

and Γ_α is surjective.

Proof of Lemma 8.34. Kernel. If $\Gamma_\alpha(u) = 0$, then by formulas (8.2.19) and (8.2.20),

$$p_\pm(u) = q_\pm(u) = r_\pm(u) = 0.$$

By these formulas (8.2.22) and (8.2.23)

$$(u(0\pm), u''(0\pm)) \mapsto (p_\pm(u), q_\pm(u))$$

is invertible. Indeed, if we replace $(u(0\pm), u''(0\pm))$ with (x, y) , is equal to the follows one

$$C^2 \ni (x, y) \mapsto \left(\sqrt{\lambda_{\alpha,+}} \frac{\lambda_{\alpha,+} x + y}{\sqrt{1 + \lambda_{\alpha,+}^2}}, \sqrt{\lambda_{\alpha,+}} \frac{-\lambda_{\alpha,-} x + y}{\sqrt{1 + \lambda_{\alpha,-}^2}} \right).$$

Since the determinant of this map is non-zero, it is invertible. We infer that

$$u(0\pm) = u''(0\pm) = 0$$

Moreover, by the definition of $\Gamma_{\alpha,\pm}$, and (8.2.24) we deduce that

$$r_{\pm}(u) = 0 \Rightarrow u'(0\pm) = 0.$$

Hence $u \in D(A_{\alpha,0})$ by Lemma 8.31. The reverse inclusion is obvious.

Surjectivity. The trace map

$$\mathbf{tr} : D(A_{\alpha,0}^*) \ni u \mapsto (u(0+), u'(0+), u''(0+), u(0-), u'(0-), u''(0-)) \in \mathbb{C}^6,$$

is surjective.

We choose two functions $\eta_{\pm} \in C_0^\infty(\mathbb{R}_{\pm})$, where

$$\mathbb{R}_+ = [0, \infty), \quad \mathbb{R}_- = (-\infty, 0],$$

such that

$$\eta_+(x) = 1 \quad \text{for } x \in [0, 1], \quad \eta_-(x) = 1 \quad \text{for } x \in [-1, 0].$$

We define $\phi_k^\pm$, $k = \{0, 1, 2\}$ by

$$\phi_0^+(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

$$\phi_0^-(x) = \begin{cases} 0, & x \geq 0, \\ 1, & x < 0, \end{cases}$$

$$\phi_1^+(x) = \begin{cases} x \eta_+(x), & x \geq 0, \\ 0, & x < 0, \end{cases}$$

$$\phi_1^-(x) = \begin{cases} 0 & x \geq 0, \\ -x \eta_+(x), & x < 0, \end{cases}$$

$$\phi_2^+(x) = \begin{cases} \frac{x^2}{2} \eta_+(x), & x \geq 0, \\ 0, & x < 0. \end{cases}$$

$$\phi_2^-(x) = \begin{cases} 0 & x \geq 0, \\ \frac{x^2}{2} \eta_+(x), & x < 0. \end{cases}$$

It follows from the formula (8.2.17) that each of those six functions belong to $D(A_{\alpha,0}^*)$. Moreover, note that

$$(\phi_1^-)'(0-) := \lim_{x \rightarrow 0^-} \frac{\phi_1^-(x) - \phi_1^-(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{-x - 0}{x - 0} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = \lim_{x \rightarrow 0^-} (-1) = -1.$$

The behaviour of these functions near the origin is illustrated in Figure 8.1.

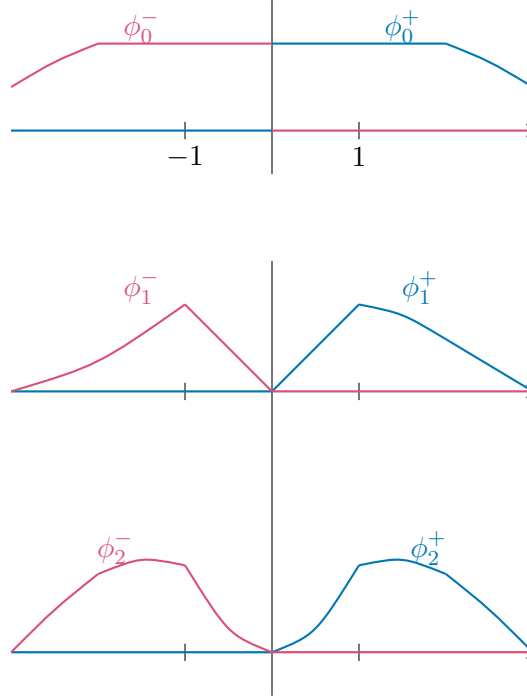


FIGURE 8.1: The functions $\phi_j^\pm$. The functions ϕ_j^+ (blue) are supported on $\mathbb{R}_+$ and vanish on $\mathbb{R}_-$, while ϕ_j^- (pink) are supported on $\mathbb{R}_-$ and vanish on $\mathbb{R}_+$.

We note that

$$(\phi_j^\pm)^{(k)}(0\pm) = \delta_{jk}, \quad j, k \in \{0, 1, 2\}.$$

From Table 8.1, we obtain

$$\begin{aligned} \mathbf{tr}(\phi_0^+) &= (1, 0, 0, 0, 0, 0), & \mathbf{tr}(\phi_1^+) &= (0, 1, 0, 0, 0, 0), & \mathbf{tr}(\phi_2^+) &= (0, 0, 1, 0, 0, 0), \\ \mathbf{tr}(\phi_0^-) &= (0, 0, 0, -1, 0, 0), & \mathbf{tr}(\phi_1^-) &= (0, 0, 0, 0, -1, 0), & \mathbf{tr}(\phi_2^-) &= (0, 0, 0, 0, 0, -1). \end{aligned}$$

	0^+ -values			0^- -values		
	ϕ	ϕ'	ϕ''	ϕ	ϕ'	ϕ''
ϕ_0^+	1	0	0	0	0	0
ϕ_1^+	0	1	0	0	0	0
ϕ_2^+	0	0	1	0	0	0
ϕ_0^-	0	0	0	-1	0	0
ϕ_1^-	0	0	0	0	-1	0
ϕ_2^-	0	0	0	0	0	-1

TABLE 8.1: Boundary values of $\phi_0^\pm, \phi_1^\pm, \phi_2^\pm$ under the boundary map $\mathbf{tr}$.

So the range of $\mathbf{tr}$ contain the vectors form a basis of $\mathbb{C}^6$. Therefore, by linear algebra the boundary map Γ_α is surjective. $\square$

We conclude this section by formulating the following result on the parametrization of self-adjoint extensions of $A_{\alpha,0}$ by $U(3)$.

8.2.4 Self-adjoint extensions

Theorem 8.35 (Self-adjoint extensions parametrized by $U(3)$). *For every $U \in U(3)$ define*

$$D(A_U) := \{ u \in D(A_{\alpha,0}^*) : \Gamma_{\alpha,+}(u) = U\Gamma_{\alpha,-}(u) \}, \quad A_U u := A_{\alpha,0}^* u.$$

Then A_U is self-adjoint and satisfies

$$A_{\alpha,0} \subset A_U \subset A_{\alpha,0}^*.$$

Conversely, if A is self-adjoint and $A_{\alpha,0} \subset A \subset A_{\alpha,0}^*$, then there exists a unique $U \in U(3)$ such that $A = A_U$.

Proof of Theorem 8.35. Proposition 8.32 provides the Green identity for the operator $A_{\alpha,0}^*$, and therefore coincides with Assumption 8.4.

Moreover, by Lemma 8.34, the boundary map Γ_α is surjective. This follows from the explicit construction of the functions $\phi_0^\pm, \phi_1^\pm, \phi_2^\pm \in D(A_{\alpha,0}^*)$, whose boundary values form a basis of $\mathbb{C}^6$. Consequently, the range of Γ_α coincides with $\mathbb{C}^6$, and hence Assumption 8.19 is satisfied.

Furthermore, again by Lemma 8.34, we have

$$\ker \Gamma_\alpha = D(A_{\alpha,0}).$$

Therefore, all the assumptions of Theorem 8.20 are fulfilled, and the claim follows. In particular, every unitary matrix $U \in U(3)$ defines a self-adjoint extension A_U via

$$D(A_U) = \{ u \in D(A_{\alpha,0}^*) : \Gamma_{\alpha,+}(u) = U\Gamma_{\alpha,-}(u) \}, \quad A_U u = A_{\alpha,0}^* u,$$

Conversely, every self-adjoint extension A such that

$$A_{\alpha,0} \subset A \subset A_{\alpha,0}^*$$

is of this form for a unique unitary matrix $U \in U(3)$. $\square$

This shows that all self-adjoint extensions of $A_{\alpha,0}$ are completely determined by boundary conditions at the origin encoded by a unitary matrix $U \in U(3)$.

8.3 Some remarks

In this section, we conclude the chapter with some remarks on the relation between the results obtained in this thesis and related results in the literature. In particular, we discuss the connection with the work of Mugnolo, Noja and Seifert [32].

Remark 8.36. In particular, Lemma 2.1 of [32] treats the operator

$$\alpha \frac{d^3}{dx^3} + \beta \frac{d}{dx}$$

on the positive and negative half-lines. The authors obtain deficiency indices $(1, 2)$ on $(0, \infty)$ and $(2, 1)$ on $(-\infty, 0)$, respectively, and consequently show that the corresponding symmetric operators on a single half-line have no self-adjoint extensions.

The homogeneous third-order operator studied in Chapter 5 is the special case in which the first-order coefficient vanishes. Thus, the deficiency index computations in Theorem 5.17 and Corollary 5.19 agree with the corresponding half-line results of [32]. Likewise, the third- and first-order operators studied in Chapter 6 belong to the same Airy-type class considered in [32]; in particular, the half-line deficiency indices obtained in Theorems 6.22 and 6.37 are consistent with their results. Moreover, for the operator on the punctured line, the equality of the deficiency indices,

$$n_+ = n_- = 3,$$

obtained in Theorem 6.56 is also consistent with the observation in [32] that the direct sum of the two half-line operators has deficiency indices $(3, 3)$.

The approach taken in this thesis is, however, different in several respects. In [32], the adjoint on each half-line is obtained directly and the deficiency equations are analysed by what the authors describe as a “tedious but elementary computation”. In the present thesis, the domain of the adjoint is characterized in a more general one-dimensional setting using weak derivatives and elliptic regularity, following the framework of [36]; see, in particular, Lemma 6.3 and its proof. Our approach will enable us to study operators of any order and this will be presented in a following publication.

This provides a systematic adjoint-domain characterization that is then used throughout Chapter 6 for the real line, the half-lines, and the punctured line. In particular, we avoid a mistake made in [32], see paragraph “Extensions of A_0 generating unitary groups” on page 1634, where the authors claim that roughly speaking the domain of the adjoint operator consists of functions belonging to the Sobolev class $H^3(0, \infty)$, while in the correct statement is, see Proposition 6.7, it should be $H_{\text{loc}}^3([0, \infty))$.

Accordingly, the half-line deficiency-index results and the fact that the direct-sum operator has equal deficiency indices should be regarded as known results consistent with [32]. The contribution of the present treatment lies in the detailed operator-theoretic derivation used here, including the weak-derivative and elliptic-regularity characterization of the adjoint domain and its subsequent application within the framework developed in this thesis.

Appendix A

Tables

A.1 Boundary values at $0\pm$ for H_+ .

Define for convenience:

$$A := \frac{2}{\sqrt{3}} \left[\left(\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) - \lambda \left(\frac{1}{2} - i\frac{\sqrt{3}}{2} \right) \right], \quad B := \frac{1}{2} - i\frac{\sqrt{3}}{2},$$

$$E := \frac{2}{\sqrt{3}} \left[\left(\frac{1}{2} + i\frac{\sqrt{3}}{2} \right)^2 - \lambda \left(\frac{1}{2} - i\frac{\sqrt{3}}{2} \right)^2 \right], \quad F := \left(\frac{1}{2} - i\frac{\sqrt{3}}{2} \right)^2.$$

	u -values			u' -values			u'' -values		
	$\tilde{e}_1$	$\tilde{e}_2$	$\tilde{e}_3$	$\tilde{e}'_1$	$\tilde{e}'_2$	$\tilde{e}'_3$	$\tilde{e}''_1$	$\tilde{e}''_2$	$\tilde{e}''_3$
0^+	$\sqrt{2}$	0	0	$-\sqrt{2}$	0	0	$\sqrt{2}$	0	0
0^-	0	$\frac{2}{\sqrt{3}}(1 - \lambda)$	1	0	A	B	0	E	F

TABLE A.1: Boundary values of $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ and their first and second derivatives at $0\pm$.

A.2 Boundary values at $0\pm$.

Evaluating the above expressions at $x = 0+$ (so that $e^0 = 1$), we obtain the boundary constants

$$D := \tilde{f}'_1(0+), \quad C := \tilde{f}'_2(0+), \quad F := \tilde{f}''_1(0+), \quad G := \tilde{f}''_2(0+).$$

Define for convenience:

$$D := -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad C := \frac{2}{\sqrt{3}} \left[\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2} \right) - \lambda \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2} \right) \right],$$

$$F := \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2, \quad G := \frac{2}{\sqrt{3}} \left[\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2 - \lambda \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 \right].$$

	u -values			u' -values			u'' -values		
	$\tilde{f}_1$	$\tilde{f}_2$	$\tilde{f}_3$	$\tilde{f}'_1$	$\tilde{f}'_2$	$\tilde{f}'_3$	$\tilde{f}''_1$	$\tilde{f}''_2$	$\tilde{f}''_3$
0+	1	$\frac{2}{\sqrt{3}}(1-\lambda)$	0	D	C	0	F	G	0
0-	0	0	$\sqrt{2}$	0	0	$\sqrt{2}$	0	0	$\sqrt{2}$

TABLE A.2: Boundary values of $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$ and their first and second derivatives at $0\pm$.

A.3 Boundary values $v(0\pm)$, $v'(0\pm)$, and $v''(0\pm)$.

From Tables A.1 and A.2, the boundary values of the basis of H_+ are

	$\tilde{e}_1$	$\tilde{e}_2$	$\tilde{e}_3$
$v(0+)$	$\sqrt{2}$	0	0
$v'(0+)$	$-\sqrt{2}$	0	0
$v''(0+)$	$\sqrt{2}$	0	0
$v(0-)$	0	$\frac{2}{\sqrt{3}}(1-\lambda)$	1
$v'(0-)$	0	A	B
$v''(0-)$	0	E	F

TABLE A.3: Boundary values of $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ and their first and second derivatives at $0\pm$.

A.3.1

where, for convenience, we put

$$\begin{aligned} A &:= \frac{2}{\sqrt{3}} \left[\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) - \lambda \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \right], \\ B &:= \frac{1}{2} - i\frac{\sqrt{3}}{2}, \\ E &:= \frac{2}{\sqrt{3}} \left(\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 - \lambda \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2 \right), \\ F &:= \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}. \end{aligned}$$

For the basis of H_- we have

	$\tilde{f}_1$	$\tilde{f}_2$	$\tilde{f}_3$
$v(0+)$	1	$\frac{2}{\sqrt{3}}(1-\lambda)$	0
$v'(0+)$	D	C	0
$v''(0+)$	F	G	0
$v(0-)$	0	0	$\sqrt{2}$
$v'(0-)$	0	0	$\sqrt{2}$
$v''(0-)$	0	0	$\sqrt{2}$

TABLE A.4: Boundary values of $\tilde{f}_1, \tilde{f}_2, \tilde{f}_3$ and their first and second derivatives at $0\pm$.

where

$$\begin{aligned}
D &:= -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \\
C &:= \frac{2}{\sqrt{3}} \left[\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) - \lambda \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \right], \\
G &:= \frac{2}{\sqrt{3}} \left(\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^2 - \lambda \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2 \right).
\end{aligned}$$

Remark A.1. Let us emphasize that all these seven constants $A - G$ depend only on λ , and λ depends on the choice of the ONB $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3$ of the space H_+ . Hence these constants depend on this choice of the ONB of H_+ .

$$u = u_0 + s_1\tilde{e}_1 + s_2\tilde{e}_2 + s_3\tilde{e}_3 + t_1\tilde{f}_1 + t_2\tilde{f}_2 + t_3\tilde{f}_3. \quad (\text{A.3.1})$$

We use Tables A.3–A.4 to compute

$$u(0+), \quad u'(0+), \quad u''(0+), \quad u(0-), \quad u'(0-), \quad u''(0-).$$

By Proposition 5.33, since $u_0 \in D(\overline{A_1})$, we infer that

$$u_0(0+) = u'_0(0+) = u''_0(0+) = 0 \quad \text{and} \quad u_0(0-) = u'_0(0-) = u''_0(0-) = 0.$$

Hence only u_+ and Γu_+ contribute to the boundary values.

Boundary values $u(0\pm)$. Using the equation (A.3.1) and Tables A.3–A.4, we have

$$\begin{aligned}
u(0+) &= s_1\tilde{e}_1(0+) + s_2\tilde{e}_2(0+) + s_3\tilde{e}_3(0+) + t_1\tilde{f}_1(0+) + t_2\tilde{f}_2(0+) + t_3\tilde{f}_3(0+) \\
&= \sqrt{2}s_1 + t_1 + \frac{2}{\sqrt{3}}(1-\lambda)t_2,
\end{aligned}$$

and similarly

$$u(0-) = \frac{2}{\sqrt{3}}(1 - \lambda) s_2 + s_3 + \sqrt{2} t_3.$$

Thus,

$$\begin{aligned} u(0+) &= \sqrt{2} s_1 + t_1 + \frac{2}{\sqrt{3}}(1 - \lambda) t_2, \\ u(0-) &= \frac{2}{\sqrt{3}}(1 - \lambda) s_2 + s_3 + \sqrt{2} t_3. \end{aligned} \tag{A.3.2}$$

Boundary values $u'(0\pm)$. Using the same previous tables,

$$\begin{aligned} u'(0+) &= -\sqrt{2} s_1 + D t_1 + C t_2, \\ u'(0-) &= A s_2 + B s_3 + \sqrt{2} t_3. \end{aligned} \tag{A.3.3}$$

Boundary values $u''(0\pm)$. Again from Tables A.3–A.4,

$$\begin{aligned} u''(0+) &= \sqrt{2} s_1 + F t_1 + G t_2, \\ u''(0-) &= E s_2 + F s_3 + \sqrt{2} t_3. \end{aligned} \tag{A.3.4}$$

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