

Quantum Magnomechanics at the Nanoscale

Nischal Gajurel

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University of York

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Abstract

The present work offers a comprehensive theoretical analysis of hybrid magnon-phonon-photon systems operating in the nanocavity magnomechanical regime, where magnetostrictive coupling, microwave cavity fields, and mechanical vibrations interact coherently within a single engineered platform. The promising potential of these systems for quantum transduction, phonon-state engineering, and hybrid quantum networks motivates the development of a unified open-quantum-system framework grounded in Heisenberg-Langevin equations and input-output theory, which is applied across a hierarchy of increasingly complex physical regimes.

The analysis begins with the derivation of magnon-photon and magnon-phonon coupling strengths for a thin-film device geometry using the Euler-Bernoulli principle, thereby establishing experimentally accessible parameter targets. Under red-detuned microwave driving, beam-splitter-type magnomechanical interactions enable sideband cooling of the mechanical mode and facilitate coherent quantum-state transfer between magnon and phonon subsystems, with quantitative predictions provided for cooperativity and cooling performance.

Beyond the linear regime, the intrinsic Kerr nonlinearity of the magnon mode is incorporated, revealing a power-dependent renormalization of the effective magnon detuning. This results in phonon frequency shifts and a pronounced transient enhancement of the phonon damping rate at intermediate drive powers, effects that may be exploited to maximize cooling performance without increasing cavity photon occupation.

Finally, a covariance-matrix formalism is developed for evaluating continuous-variable entanglement between the magnon and phonon modes, quantified by the logarithmic negativity. Sideband-selective driving generates robust stationary entanglement that persists against thermal noise across experimentally realized systems. Collectively, these findings establish nanocavity magnomechanics as a versatile platform for coherent quantum control, nonlinear sensing, and the generation of quantum correlations in solid-state hybrid systems.

Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for a degree or other qualification at this University or elsewhere. All sources are acknowledged as references.

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Contents

Abstract	ii
Declaration	iii
Acknowledgements	iv
1 Introduction	1
1.1 Overview of Magnomechanics	1
1.2 Motivation and Objectives	6
1.2.1 Scientific and Technological Motivation	6
1.2.2 Broader Impact and Novel Contributions	7
1.2.3 Challenges and Open Questions	9
1.2.4 Thesis Methodology and Roadmap	9
1.3 Outline of the Thesis	10
2 Magnetism	13
2.1 Ferromagnetism: Order, Origins, and Material Realization	13
2.1.1 The Exchange Interaction as the Quantum Origin of Magnetism	15
2.1.2 Magnetic Free Energy in Ferromagnetic Systems	20
2.2 Magnetization Dynamics and Landau-Lifshitz-Gilbert (LLG) Equation	21
2.3 Ferromagnetic Resonance	24
2.3.1 Kittel Equation	25
2.3.2 FMR Damping	27
3 Optical Cavities and Mechanical Resonators	29
3.1 Theoretical Framework	29
3.2 Electromagnetic Cavities and Resonators	30
3.2.1 Basic Properties	30
3.2.2 Input-Output Formalism for an Optical Cavity	32
3.2.3 Driven Optical Cavity Hamiltonian	32
3.2.4 Heisenberg-Langevin Equations	34
3.2.5 Classical Steady State and Input-Output Relations	34

3.3	Mechanical Resonators	35
3.3.1	Classical Equation of Motion	36
3.3.2	Solution of the Harmonic Oscillator Equation	36
3.3.3	Mechanical Susceptibility	37
3.4	Optomechanical Interaction Hamiltonian	38
3.4.1	Derivation of the Interaction Hamiltonian	38
3.4.2	Transformation to the Rotating Frame	39
3.5	Linearized Equations of Motion	40
3.5.1	Linearization Procedure	40
3.5.2	Full Linearized Hamiltonian	41
3.5.3	Heisenberg-Langevin Equations for Fluctuations	41
3.5.4	Noise Correlators	41
3.5.5	Frequency Domain Equations	42
3.6	Classical Averaged Equations	42
3.6.1	Equations for Mean Fields	42
3.6.2	Steady State and Linearisation around Steady State	43
3.6.3	Optical Spring Effect	43
3.7	State of Art	47
4	Nano-magnomechanics with thin film	48
4.1	Device Architecture and Parameter Constraints	48
4.1.1	Design Constraints and Parameter Selection	50
4.1.2	Proposed Device Parameters	51
4.2	Quantised magnon-photon coupling	52
4.3	Quantised Magnon-Phonon Coupling	58
5	Nano Magnomechanical Interaction Under Red-Detuned Drive	68
5.1	System Hamiltonian	68
5.2	Derivation of the Equations of Motion	70
5.3	Steady-state solutions and mean-field magnon	71
5.4	Linearisation about the steady state and fluctuation dynamics	77
5.5	Reflection Spectrum	81
5.6	Phonon Self-Energy and Magnon Spring Effect	86
5.7	Cooperativity	90
6	Nano Magnomechanical Interaction Under Blue-Detuned Drive	94
6.1	Steady-State Magnon Population	95
6.2	Linearity of the Driven Magnon Mode for Both Detunings	95
6.3	Reflection Spectrum	97
6.4	Magnomechanically Induced Spectral Features	98
6.5	Cooperativity and the Absence of Amplification	100
6.6	Conclusion	101

7	Kerr Non Linear Effect	103
7.1	Nonlinear System Hamiltonian	103
7.2	Derivation of the Nonlinear Equations of Motion	106
7.3	Steady-State Solutions and Mean-Field Magnon	107
7.4	Linearisation About the Steady State and Fluctuation Dynamics	109
7.4.1	Linearized Equations of Motion	110
7.5	Nonlinear Reflection Spectrum with Magnon Self-Kerr	110
7.6	Phonon Self-Energy and Non-Linear magnon spring effect Magnon . .	111
7.7	Results and Discussions	114
8	Magnon-Phonon Entanglement via Sideband-Selective Coupling in Cavity Magnomechanics	120
8.1	Introduction	120
8.2	Mode Entanglement in Hybrid Bosonic Systems	122
8.3	System Hamiltonian	122
8.4	Results	129
8.5	Conclusion	133
	Bibliography	134

List of Figures

1.1	Schematic illustration of the tripartite magnomechanical interaction among the cavity photon, magnon, and phonon modes, described by the annihilation operators $\hat{a}$, $\hat{m}$, and $\hat{b}$ with decay rates κ_a , κ_m , and κ_b , respectively. The coupling strengths g_{ma} and g_{mb} characterise the magnon-photon and magnon-phonon interactions, respectively.	2
1.2	Magnomechanical Interaction between Magnon, Photon and Phonon	4
2.1	Schematic representation of electron spin ordering in (a) ferromagnetic, (b) antiferromagnetic, and (c) ferrimagnetic materials. Arrow length indicates the magnitude of the magnetic moment.	18
2.2	(a) Classical FMR: Larmor precession of the magnetization $\mathbf{M}$ along the effective field vector $\mathbf{H}_{\text{eff}}$ at an angle ϕ to the axis. (b) Quantum FMR: (i) Splitting of electron energy due to Zeeman effect versus the applied magnetic field strength. (ii) Plot of FMR absorption indicating a rise in susceptibility χ	24
3.1	Optomechanical damping rate and optical spring effect vs. detuning and different sideband parameters, with theory (black lines) and experimental data (red dots). Left: Full mechanical damping rate $\Gamma_{\text{eff}}/2\pi$ vs. detuning Δ/κ , with decreasing values of κ/Ω_m from top to bottom ($\kappa/\Omega_m = 3.7, 2.2, 1.4, 0.7$). The regime where Γ_{eff} touches zero is the regime of dynamical instability (mechanical lasing). Right: Optical spring effect, i.e., light-induced mechanical frequency shift $\delta\Omega_m/2\pi$. Data adapted from (Schliesser and Kippenberg, 2010)[2].	45
3.2	Optically induced modification of mechanical frequency and damping due to dynamical backaction. Adapted from Ref. [2].	46

4.1	Schematic of the on-chip magnomechanical device. A YIG thin film (yellow) is grown on a GGG cantilever (grey) that is clamped to a bulk GGG support. A superconducting coplanar-waveguide (CPW) transmission line (blue) is mounted face-down above the film, separated from it by a gap d_c , so that the CPW conductors face the magnetic film. An in-plane static bias field H_e magnetises the film along its long axis. The geometric parameters of the mechanical resonator and the magnetic film are indicated.	49
4.2	Micromagnetic simulation of the fundamental magnon mode obtained using the software Object Orientated MicroMagnetic Framework (OOMMF). (Top) Frequency spectrum obtained from the Fourier transform of the simulated magnetisation dynamics under a spatially uniform transverse microwave drive. The dominant resonance occurs at the magnon frequency ω_m , while the drive frequency ω_d is chosen close to resonance. (Bottom) Spatial distribution of the dynamic magnetisation component M_y at steady state ($t = 5 \mu\text{s}$). Red and blue indicate opposite phases of the oscillation. The mode exhibits a long-wavelength standing-wave profile and is well approximated by the cosine spatial dependence used in the analytical model.	60
5.1	Magnon–photon hybridisation in the linear regime. (a) Two-dimensional map of the magnon population as a function of probe frequency and magnetic field at $P = 1 \text{ nW}$ with $g_{\text{ma}} \neq 0$. (b) Frequency cut at $H = 400 \text{ Oe}$, showing the emergence of two hybridised modes.	75
5.2	Linearity analysis of the driven magnon mode for a fixed drive frequency $\omega_d = \omega_a - 0.5\omega_b$ and input power $P = 0.15 \text{ nW}$. Top panel: Steady-state magnon number n_m as a function of the external magnetic field H . Bottom panel: Corresponding nonlinear ratio $K_m n_m / \kappa_m$ versus H , where the dashed horizontal line denotes the linearity threshold. The obtained values remain well below the threshold over the considered magnetic-field range, confirming the validity of the linearised treatment.	76
5.3	Linearity analysis of the driven magnon mode for a fixed drive frequency $\omega_d = \omega_a - 1.5\omega_b$ and input power $P = 0.3 \text{ nW}$. Top panel: Steady-state magnon number n_m as a function of the external magnetic field H . Bottom panel: Corresponding nonlinear ratio $K_m n_m / \kappa_m$ vs. H , where the dashed horizontal line denotes the linearity threshold. The obtained values remain well below the threshold over the considered magnetic-field range, confirming the validity of the linearised treatment.	77

- 5.4 Reflection spectrum $|S_{11}|$ of the coupled cavity-magnon system as a function of magnetic field H and probe frequency ω . The system is driven at a detuning $\Delta = \omega_d - 1.5\omega_b$ with an input power of $P = 0.2$ nW. The colour map highlights the hybridised polariton modes arising from coherent magnon-photon coupling, where the avoided crossing structure reflects the strong-coupling regime. The narrow magnetic-field window is chosen to ensure operation within the weakly nonlinear regime, where the nonlinear frequency shift remains below half of the magnon linewidth. 83
- 5.5 Reflection spectra $|S_{11}|$ of a hybrid cavity-magnon system as a function of probe frequency for different bias magnetic fields H , under a fixed microwave drive at frequency ω_d . The left panels show the full spectral response, where a broad Lorentzian-like dip corresponds to the lower hybrid magnon-photon polariton mode arising from strong photon-magnon coupling. The field dependence reflects the tuning of the magnon frequency $\omega_m(H)$ and the resulting evolution of the hybridised modes. The right panels show a zoomed-in view in terms of the two-photon detuning $\Delta_{sd} = \omega_s - \omega_d$, revealing a sharp spectral feature near $\Delta_{sd} \approx \omega_b$. This structure originates from coherent magnon-phonon interaction mediated by the cavity field and appears as a Fano-like interference profile superimposed on the hybrid polariton background. The line shape varies with the bias field due to changes in the relative strengths and phases of the phonon-assisted scattering pathways. At specific detunings, the asymmetric profile evolves toward a symmetric transparency window, indicating a crossover to magnomechanically induced transparency (MMIT). . . . 85
- 5.6 The magnon spring effect showing the nonlinear modification of (a) the effective magnon frequency shift $\Delta\omega$ and (b) the linewidth change $\Delta\kappa$ as a function of the normalized drive-cavity detuning $\Delta_d = (\omega_d - \omega_a)/\omega_b$. The dashed black curve shows the corresponding steady-state magnon population n_m . Calculations are performed in the linear regime at an input power of 0.15 nW. 89
- 5.7 Magnetic-field dependence of the cooperativity C and the resulting phonon occupation n_b for a red-detuned drive at $P_d = 0.3$ nW. The bath occupation is determined by the Bose-Einstein distribution at $T_{\text{bath}} = 20$ mK for $\omega_b/2\pi = 2.35$ MHz 92
- 6.1 Linearity analysis for the blue-detuned drive $\omega_d = \omega_a + 0.5\omega_b$ at $P = 0.15$ nW. Top: steady-state magnon number n_m versus H , peaking near $H \simeq 400.05$ Oe. Bottom: Kerr ratio $K_m n_m / \kappa_m$ versus H ; the dashed line is the linearity threshold 0.5 and the shaded region marks the nonlinear regime. The ratio peaks at $\simeq 0.29$, remaining below threshold throughout. 96

6.2	Linearity analysis for the blue-detuned drive $\omega_d = \omega_a + 1.5\omega_b$ at $P = 0.30$ nW. Top: steady-state magnon number n_m versus H , peaking near $H \simeq 400.8$ Oe. Bottom: Kerr ratio $K_m n_m / \kappa_m$ versus H ; the ratio peaks at $\simeq 0.35$, remaining below the linearity threshold of 0.5 over the considered field range.	97
6.3	Reflection spectrum $ S_{11} $ of the coupled cavity-magnon system under blue-detuned driving ($\omega_d = \omega_a + 1.5\omega_b$, $P = 0.3$ nW) as a function of magnetic field H and probe frequency. The avoided crossing near $H \simeq 400$ Oe reflects strong magnon-photon coupling, with the two hybridised polariton branches split by $2g_{ma}$	98
6.4	Reflection spectra $ S_{11} $ under blue-detuned driving ($\omega_d = \omega_a + 1.5\omega_b$, $P = 0.3$ nW) for different bias fields H . Left: full spectral response; the grey dashed line marks the drive frequency ω_d , which lies above both polariton branches, and the shaded band marks the mechanical sideband at $\omega_d - \omega_b$. Right: enlarged view in terms of the two-photon detuning $\Delta_{sd} = \omega_s - \omega_d$, revealing a narrow magnomechanical feature near $\Delta_{sd} \simeq -\omega_b$. The line shape evolves from a Fano-like asymmetric profile toward a symmetric transparency window with bias field, signalling the crossover to magnomechanically induced transparency (MMIT).	99
6.5	Magnetic-field dependence of the cooperativity C for a blue-detuned drive $\omega_d = \omega_a + 1.5\omega_b$ at $P_d = 0.3$ nW and $T_{\text{bath}} = 20$ mK. The cooperativity peaks at $C \simeq 0.47$ near $H \simeq 400.3$ Oe and remains below unity over the entire field window, so the system does not reach the amplification (phonon-lasing) regime; the drive power cannot be raised further without violating the magnon linearity condition.	101
7.1	(a) Phonon frequency shift $\Delta\omega_b/2\pi$ as a function of the normalised drive detuning $(\omega_d - \omega_a)/\omega_b$ for varying drive powers P when $K_m = 0.2$ Hertz. (b) Phonon damping $\Delta\kappa_b/2\pi$ as a function of the normalised drive detuning $(\omega_d - \omega_a)/\omega_b$ for varying drive powers P	115
7.2	(a) Phonon frequency shift $\Delta\omega_b/2\pi$ as a function of the normalised drive detuning $(\omega_d - \omega_a)/\omega_b$ for varying drive powers P when $K_m = 0.2$ Hertz. (b) Phonon damping $\Delta\kappa_b/2\pi$ as a function of the normalised drive detuning $(\omega_d - \omega_a)/\omega_b$ for varying drive powers P	116
7.3	(a) Phonon frequency shift $\Delta\omega_b/2\pi$ and steady-state magnon occupation n_m (grey dashed curve, right axis) as a function of drive power P at fixed detuning $\omega_d = \omega_a - 0.5\omega_b$. (b) Phonon damping $\Delta\kappa_b/2\pi$ and magnon occupation n_m as a function of P for $K_m = 0$ (linear) and $K_m = 2\pi \times 0.2$ Hz (nonlinear).	117

- 8.1 Steady-state magnon–phonon entanglement at a drive power $P = 0.3$ nW and bias field $H \simeq 400$ Oe. **(Top)** Logarithmic negativity $E_{\mathcal{N}}^{m-b}$ as a function of the normalized cavity detuning Δ_a/ω_b and magnon detuning Δ_m/ω_b ; the star marks the global maximum $E_{\mathcal{N}}^{m-b} \simeq 0.017$ at $(\Delta_a/\omega_b, \Delta_m/\omega_b) \simeq (0.49, 0.47)$. **(Bottom)** Line cut at the optimal magnon detuning $\Delta_m/\omega_b \simeq 0.47$, showing $E_{\mathcal{N}}^{m-b}$ versus Δ_a/ω_b . The shaded region indicates the parameter window where $E_{\mathcal{N}}^{m-b} > 0$ 130
- 8.2 Steady-state photon-magnon entanglement at a drive power $P = 0.3$ nW and bias field $H = 400$ Oe. **(Top)** Logarithmic negativity $E_{\mathcal{N}}^{a-m}$ as a function of the normalized cavity detuning Δ_a/ω_b and magnon detuning Δ_m/ω_b ; the star marks the global maximum $E_{\mathcal{N}}^{a-m} \simeq 1.7 \times 10^{-3}$ at $(\Delta_a/\omega_b, \Delta_m/\omega_b) \simeq (0.83, 0.27)$. **(Bottom)** Line cut at the optimal magnon detuning $\Delta_m/\omega_b \simeq 0.27$, showing $E_{\mathcal{N}}^{a-m}$ versus Δ_a/ω_b . The shaded region indicates the parameter window where $E_{\mathcal{N}}^{a-m} > 0$ 132

List of Tables

4.1	Nominal device parameters satisfying the design constraints of Sec. 4.1.1.	52
5.1	System parameters used in the simulation.	74

Chapter 1

Introduction

1.1 Overview of Magnomechanics

Magnomechanics has recently attracted attention as a new field that combines ideas from magnetism, mechanics, and quantum optics. At its centre, this field looks at how different types of excitations, including magnetic, mechanical, and electromagnetic, can interact coherently within solid-state devices [1, 2]. These interactions usually involve three main quasiparticles: magnons, phonons, and photons. Magnons are the quantum counterparts of spin waves and come from the collective precessional motion of spin moments in magnetically ordered materials [3]. Phonons, on the other hand, arise from vibrations of the crystal lattice and represent the quantised modes of mechanical motion in solids [4]. Photons describe quantised electromagnetic fields and often act as carriers of energy or information in cavity-based systems [5]. When these different excitations exist together in the identical structure, their interactions can result in hybrid behaviour, allowing energy and information to be exchanged among magnetic, mechanical, and electromagnetic degrees of freedom [6]. This hybridisation underpins magnomechanical platforms and enables the exploration of coherent dynamics throughout diverse physical domains.

To truly appreciate the significance of magnomechanics, one must first understand the nature of these quasiparticle excitations and why their interactions are so compelling from both fundamental and applied perspectives. Magnons, which emerge from the collective behaviour of billions of electron magnetic moments acting in concert, represent one of the most beautiful manifestations of emergent quantum phenomena in condensed matter [7]. When a magnetic material is excited by a microwave field, for instance, the spins do not simply flip individually; instead, they precess in a coordinated fashion, creating waves of magnetic excitation that propagate through the material much like ripples on a pond [8]. These waves carry both energy and information, and their quantum nature becomes particularly evident when the excitation amplitude is sufficiently small

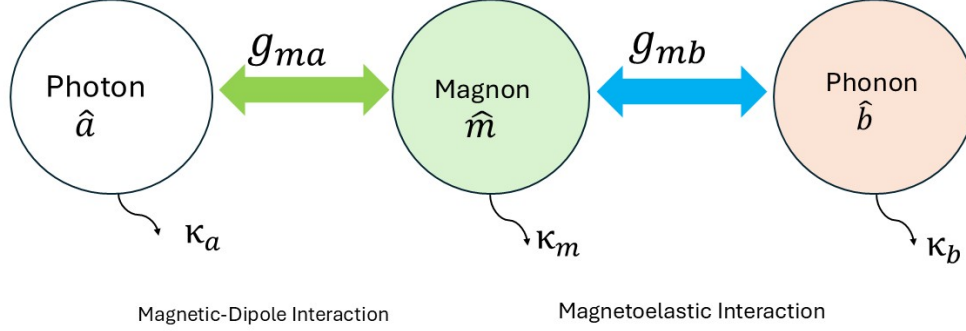


Figure 1.1: Schematic illustration of the tripartite magnomechanical interaction among the cavity photon, magnon, and phonon modes, described by the annihilation operators $\hat{a}$, $\hat{m}$, and $\hat{b}$ with decay rates κ_a , κ_m , and κ_b , respectively. The coupling strengths g_{ma} and g_{mb} characterise the magnon-photon and magnon-phonon interactions, respectively.

[9].

Phonons represent the mechanical vibrations within solids. Atoms in a solid vibrate around their equilibrium positions even at temperature $T = OK$ due to their zero-point energy [7]. These vibrations form collective modes, essentially sound waves, which are described quantum mechanically as phonons [10]. The interaction between magnons and phonons addresses fundamental questions in condensed matter physics regarding the communication between different degrees of freedom in a solid and the mechanisms of energy transfer [11, 12].

In magnomechanical platforms, the coupling between these quasiparticles is typically mediated by magnetostrictive interactions [4, 7]. Magnetostriction describes the tendency of magnetic materials to change shape in response to variations in magnetisation, while mechanical deformation can, in turn, influence the magnetic state of the material [13]. This reciprocal coupling mechanism enables magnons and phonons to interact coherently. When such a magnetic element is placed inside a resonant microwave cavity, the magnetic excitations can also couple strongly to cavity photons through Zeeman interaction [9]. The resulting hybrid configuration, often referred to as cavity magnomechanics, allows all three excitations to interact simultaneously and forms the basis for a wide range of dynamical phenomena [1, 14]. Fig. 1.1 and Fig. 1.2 illustrates a schematic overview of tripartite interaction among cavity photons, magnons, and

phonons.

The conceptual foundations of magnomechanics are closely related to those established in cavity optomechanics, a field that has advanced significantly over the past two decades [2, 15]. In optomechanical systems, radiation pressure within optical or microwave cavities couples electromagnetic fields to mechanical motion [16, 17]. This coupling has enabled achievements such as cooling macroscopic mechanical resonators to their quantum ground states [18], generating squeezed light [19, 20], and observing quantum entanglement between light and motion [21, 22]. Magnomechanical systems extend these concepts to magnetic materials, where magnetoelastic forces serve as the primary interaction mechanism between mechanical motion and internal excitations [14]. Consequently, many theoretical techniques from optomechanics, such as linearisation methods, sideband analysis, and input–output theory, can be adapted to describe magnomechanical interactions [2, 23]. However, the magnetic nature of magnons introduces additional physical phenomena beyond those observed in purely optomechanical systems [24, 25].

Historically, the development of magnomechanics has drawn on several independent research traditions that gradually converged. The theoretical description of spin waves began in the early twentieth century with Bloch’s treatment of ferromagnetism and the subsequent development of quantum theories of magnetic excitations [26]. Bloch’s famous spin-wave theory, proposed in 1930, provided the first systematic framework for understanding the behaviour of collective spin excitations at low temperatures [7, 26]. Later, Holstein and Primakoff introduced a bosonic representation of spin operators that allowed collective spin dynamics to be described in terms of quasiparticles now known as magnons [27]. This transformation, introduced in 1940, remains one of the most important theoretical tools in magnetism, as it allows complex spin systems to be treated using the familiar language of quantum harmonic oscillators [4, 28].

The phenomenon of magnetostriction was first observed by James Joule in 1842, revealing that magnetic ordering could directly influence the mechanical properties of a material [29]. Joule’s original experiment was elegantly simple: he observed that a sample of iron changed its length when magnetised. This seemingly modest observation would eventually lead to a deep understanding of how magnetic and elastic degrees of freedom interact [13]. Over the following decades, theoretical work by researchers such as Kittel and Akhiezer established a comprehensive framework for magnetoelastic coupling [7, 30]. These studies clarified how lattice strain can modify magnetic anisotropy and influence spin dynamics, thereby providing the theoretical basis for magnon–phonon interactions [4]. Charles Kittel, in particular, made seminal contributions to our understanding of ferromagnetic resonance and spin-wave behaviour, laying much of the foundation for modern magnomechanics [7, 31].

More recently, the emergence of cavity optomechanics in the early 2000s provided a powerful conceptual framework for understanding hybrid quantum setups involving mechanical oscillators [2]. Inspired by these developments, researchers began integrating

ferrimagnetic materials, particularly yttrium iron garnet (YIG), with high-quality microwave cavities [9, 32]. YIG has proven to be an especially attractive material for these experiments because it exhibits exceptionally low magnetic damping and supports long-lived magnon modes [33, 34]. The story of YIG is itself worth noting: discovered in the 1950s at Bell Laboratories, this material quickly became the workhorse of microwave magnetism due to its unique combination of low losses and high magnetic ordering temperature [35]. When a small YIG sphere is placed inside a microwave cavity, the collective spin excitations can couple strongly to the cavity photon field [9, 32]. The observation of strong magnon–photon interaction in such systems marked an important milestone and opened the door to the experimental realisation of magnomechanical interactions [1].

The dynamics of magnomechanical systems are governed by several interconnected physical mechanisms [1]. Magnetostrictive coupling links crystal strain to magnetic anisotropy, allowing mechanical vibrations to influence spin-wave excitations and vice versa [4, 7, 31]. At the same time, cavity photons provide an additional interaction channel that connects the magnetic subsystem to external electromagnetic fields [9, 1]. The presence of strong microwave driving fields can enhance the dynamics of those systems by introducing nonlinear effects [36]. Such nonlinearities may lead to phenomena including magnon sideband cooling, self-sustained oscillations, and the generation of quantum entanglements between magnons and phonons [1, 24]. Each of these phenomena provides a window into the rich physics that appears when multiple quantum degrees of freedom are brought into controlled interaction.

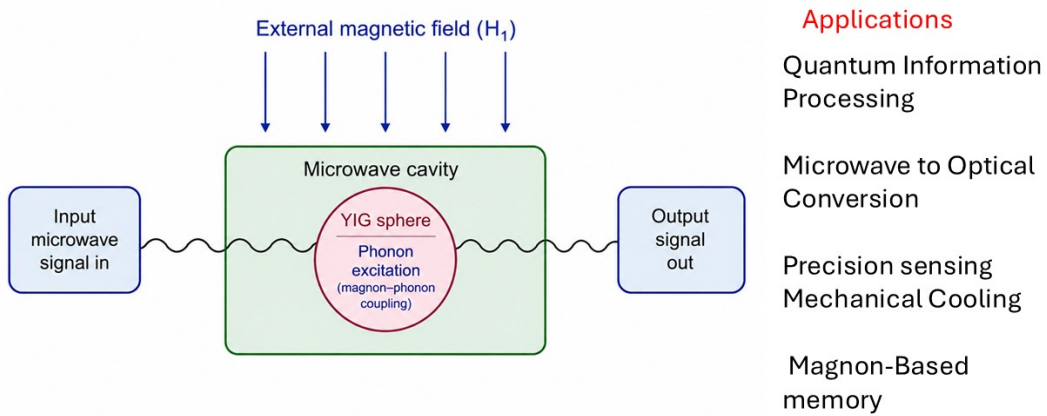


Figure 1.2: Magnomechanical Interaction between Magnon, Photon and Phonon

Due to these capabilities, magnomechanical systems are increasingly regarded as promis-

ing platforms for emerging quantum technologies [6]. A key application is quantum transduction, which consists of transferring information between different physical carriers [37]. For example, superconducting quantum circuits operate in the microwave regime, while long-distance communication utilises optical photons[38]. Magnomechanical devices may serve as intermediate converters uniting these two frequency domains [1, 9]. Magnons can couple efficiently to microwave fields, mechanical vibrations can couple to optical fields through radiation pressure or other mechanisms, and the magnon–phonon interaction delivers the necessary link among these domains [39, 40].

Magnons in high-quality ferrimagnetic materials such as YIG couple strongly and coherently to microwave photons, providing an efficient link between magnetic, optical, and mechanical degrees of freedom [1, 9]. Their coherence times, typically a few hundred nanoseconds [41], are admittedly shorter than the microsecond-to-millisecond coherence times reached by superconducting qubits, so coherence alone is not what makes magnonic systems attractive. Their real appeal lies elsewhere: magnons can be coupled to nanomechanical resonators, and these resonators can then mediate microwave-frequency interactions between modes that are physically separated from one another. This magnon-phonon-microwave route is naturally scalable, as several nanomechanical elements can be coupled together on a single chip, and in principle several such chips could be networked together, something that remains difficult for superconducting qubit architectures given the fabrication and wiring overhead that comes with scaling up. It is this combination of strong, tunable coupling and scalability, both on-chip and potentially between chips, that makes magnonic systems a promising building block for hybrid quantum networks [42, 43]. Meanwhile, the mechanical component of the system can function as a highly sensitive probe of displacement or force, which may enable new sensing technologies capable of detecting extremely small magnetic fields or nanoscale mechanical motion [18, 44].

Beyond technological applications, magnomechanics provides an appealing platform for exploring fundamental questions in quantum physics [2, 25]. In particular, hybrid magnon–phonon systems offer opportunities to investigate macroscopic quantum phenomena, including entanglement between distinct quasiparticles, hybrid squeezing of collective excitations, and the generation of nonclassical mechanical states [1, 24, 25]. Studying such effects may help clarify how quantum behaviour emerges and persists in complex many-body systems [45, 46]. There is something deeply philosophical about these investigations: they probe the boundary between the quantum and classical worlds, exploring how macroscopic mechanical resonators containing billions of atoms can exhibit quantum behaviour when coupled appropriately to other quantum systems [22].

From a materials science perspective, significant research is focused on identifying magnetic compounds with stronger magnetostrictive responses or lower intrinsic damping than conventional YIG [47]. Iron–gallium alloys (Galfenol) and layered magnetic

compounds such as chromium trihalides are being explored as alternative materials [48]. These alternatives present various trade-offs: some exhibit stronger magnetoelastic coupling, which may enhance magnon–phonon interactions, while others are compatible with thin-film fabrication processes that could facilitate the development of integrated devices [49, 50]. Concurrently, advances in microfabrication and nanotechnology are supporting the integration of magnomechanical components into chip-scale architectures [51]. Collectively, these advancements suggest the potential for scalable hybrid devices that integrate magnetic, mechanical, and electromagnetic functionalities on a single platform [6, 52].

Despite these promising developments, several important challenges still exist [6]. Achieving strong magnon–phonon coupling in miniaturised structures requires precise control over material properties and device geometry [1]. The coupling strength scales with the overlap between the magnetic and mechanical modes, and achieving strong coupling often requires confining both excitations to small volumes [14]. Thermal fluctuations in mechanical modes can also introduce significant noise, particularly at higher temperatures [16, 18]. Even at cryogenic temperatures, mechanical modes can remain thermally excited if their frequencies are low, presenting a challenge for quantum experiments that require ground-state cooling [14, 53]. Addressing these challenges will require further progress in materials engineering, cryogenic measurement techniques, and device fabrication [6, 38].

In summary, magnomechanics constitutes a dynamic and rapidly evolving research area at the intersection of condensed matter physics, quantum optics, and nanomechanics [1, 6, 25]. By facilitating coherent interactions among magnons, phonons, and photons, magnomechanical systems provide a robust platform for investigating hybrid quantum dynamics and offer a potential route to new quantum technologies [1, 2]. The present thesis builds upon these advances and seeks to explore new regimes of magnon–phonon–photon interaction in engineered hybrid systems.

1.2 Motivation and Objectives

1.2.1 Scientific and Technological Motivation

Motivated by recent advances, this thesis aims to establish a comprehensive theoretical framework for understanding magnomechanical interactions under experimentally realistic conditions [1]. Special attention is given to regimes where hybrid magnon–phonon–photon dynamics are controlled via external driving fields and engineered coupling mechanisms [1, 25]. The principal objectives of this research are outlined below.

- **Investigation of magnomechanical interactions under detuned microwave drives.**

A primary objective is to examine how externally applied microwave drives influence the dynamics of coupled magnon–phonon systems [1, 25]. The effects of red-detuned and blue-detuned driving fields are specifically analysed. Red-detuned driving is anticipated to enable processes such as sideband cooling and coherent state transfer between magnons and phonons [24]. In contrast, blue-detuned driving can induce parametric amplification, instability thresholds, and the generation of nonclassical states [54, 49]. To understand these processes in magnomechanics is essential for identifying parameter regimes where distinct behaviours dominate [14]. Furthermore, analysing detuned driving offers insight into the fundamental physics of energy exchange between quantum systems with differing characteristic frequencies [2].

- **Characterisation of nonlinear effects in confined magnetic systems.**

Strong microwave excitation can induce nonlinear behaviour in magnonic systems, including Kerr-type frequency shifts resulting from anharmonic interactions between spin excitations [36]. This research quantifies the role of such nonlinearities and explores their consequences for phenomena such as bistability, frequency pulling, and quantum squeezing [36, 49]. The nonlinear behaviour of magnons is notable because it originates from the fact that spins are not true bosons; they obey spin commutation relations, which introduce nonlinearities when the excitation number becomes significant [27, 28]. Understanding these effects is also relevant for designing nonreciprocal microwave components, including magnonic isolators and circulators, which are essential for protecting quantum systems from unwanted back-action [14, 55].

- **Development of a theoretical model for mechanically mediated magnon entanglement.**

An additional objective is to investigate the generation of entanglement between distinct magnon modes via their mutual interaction with a shared mechanical resonator [1, 24]. A quantum master equation framework is used to analyse the dynamics of the hybrid system, accounting for realistic sources of decoherence such as magnon damping and phonon leakage [14, 36, 25]. Strategies for mitigating these effects, including pulsed driving and dynamical decoupling techniques, are also examined [56]. Generating entanglement in solid-state systems remains a central challenge in quantum information science. Magnomechanical systems present a promising approach by combining relatively long coherence times with the capacity to mediate interactions through mechanical motion [6].

1.2.2 Broader Impact and Novel Contributions

The central novel contribution of this thesis is a device design in which the magnon mode is coupled to the flexural mode of a suspended nanomechanical resonator, rather than to the bulk or surface phonon modes considered in most earlier magnomechanical

proposals [1, 25]. Flexural modes are appealing because of their low motional mass and low mechanical frequency, but working with them is not simply a matter of swapping one phonon mode for another: it requires identifying the parameter regimes, mainly resonator geometry, magnon-phonon detuning, and drive strength in which the coupling stays strong and the system remains stable, since flexural modes are inherently softer and more prone to nonlinear behaviour. A large part of this thesis is devoted to mapping out exactly these regimes and studying how the system responds to driving across them, including where bistability and instability set in as the drive power increases. Nonlinearity turns out to matter more here than it typically does in cavity optomechanics: the magnon mode carries its own Kerr-type nonlinearity, rooted in magnetocrystalline anisotropy, and this interacts with the mechanical nonlinearity of the flexural mode in ways that have no real counterpart in purely optomechanical systems. Much of the analytical work in this thesis goes into understanding how these two nonlinear effects combine and what that means for the system's stability. Building on this, the thesis also looks at how well magnon-phonon entanglement survives in this flexural-mode geometry and under what conditions it holds up against the added nonlinearity and thermal noise that come with using a nanomechanical flexural mode. Taken together, this amounts to a concrete device-level design approach, rather than an abstract coupling model, for getting strong and controllable magnomechanical interactions on a chip.

The results presented in this thesis are anticipated to advance several areas within contemporary physics and engineering [2, 6]. From a fundamental perspective, this work investigates strong magnon-phonon coupling at the nanoscale, offering insights into the interplay between spin dynamics and mechanical motion in hybrid quantum systems [1, 25, 36]. Examination of these interactions may also inform broader questions regarding the quantum-to-classical transition in macroscopic physical systems [45, 46]. ...

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From the perspective of quantum engineering, the theoretical models developed in this thesis may inform the design of practical devices that utilise magnomechanical interactions [1, 25]. Potential applications include quantum memories for storing microwave quantum states [1, 9], transducers for converting signals between different frequency domains [1], and hybrid logic components compatible with superconducting or photonic quantum circuits [37, 38, 43]. Storing quantum information in magnon modes is particularly valuable, as it enables interfacing superconducting quantum processors, which operate at microwave frequencies, with other quantum systems functioning at different frequencies [57].

The capacity to control magnetic and mechanical excitations within compact structures

enables the development of ultra-sensitive magnetometers [58], low-noise amplifiers, and advanced microwave signal-processing devices [59]. These technologies have potential applications in quantum metrology as well as in emerging computational paradigms, including neuromorphic architectures and hybrid quantum communication networks [12, 38]. The high sensitivity of mechanical resonators to small forces and displacements, together with their ability to couple to magnetic excitations, creates new possibilities for sensing applications that demand both magnetic-field sensitivity and spatial resolution [18].

1.2.3 Challenges and Open Questions

Despite the considerable promise of magnomechanical platforms, several significant challenges must still be addressed before their full potential can be realised [6]. One major issue concerns the suppression of thermal noise in mechanical modes, particularly in miniaturised devices operating at cryogenic temperatures [18]. Even small amounts of thermal excitation can degrade quantum coherence and limit the performance of hybrid systems [16]. The challenge is compounded by the fact that mechanical modes typically have lower frequencies than magnon modes, which means they are more susceptible to thermal excitation [2]. Achieving ground-state cooling of mechanical modes in magnomechanical systems remains an important experimental goal [60, 61].

Another challenge involves the fabrication of high-quality thin-film magnetic materials [62]. While bulk YIG crystals exhibit exceptionally low damping, thin-film implementations can suffer from increased magnetic losses due to lattice strain, defects, and edge roughness [63]. Improving the material quality and uniformity of such films remains an active area of research. The discrepancy between bulk and thin-film properties is not merely a technical nuisance but points to deeper physics about how disorder and interfaces affect magnetic dynamics [4]. Understanding and mitigating these effects requires close collaboration between theorists, materials scientists, and experimentalists [38, 43].

Finally, questions related to scalability and system design remain open [64, 65]. For example, it may be possible to engineer enhanced coupling strengths using structured magnetic materials such as magnonic crystals or artificial spin-ice lattices [12]. These structures could potentially provide new mechanisms for controlling magnon–phonon interactions, but they also introduce new complexities in terms of fabrication and theoretical modelling [65]. Exploring these possibilities will require both theoretical modelling and experimental investigation and represents an exciting direction for future research [38, 43, 6].

1.2.4 Thesis Methodology and Roadmap

To achieve the stated objectives, this thesis employs a primarily theoretical methodology that integrates analytical modelling with numerical simulations [2, 5]. The analysis

targets parameter regimes accessible to current or imminent experimental capabilities, thereby maintaining the practical relevance of the theoretical predictions [9, 41]. Where applicable, the models are formulated using experimentally realisable device geometries and driving configurations [1, 36, 59]. This focus on experimental applicability underscores the view that theoretical research in quantum physics attains its greatest value when it informs and directs experimental efforts [45].

Alongside analytical developments, numerical techniques are employed to investigate the dynamical behaviour of the hybrid systems examined in this study [14, 55]. These simulations elucidate stability conditions, coupling regimes, and the effects of dissipation on system dynamics [2, 23]. Numerical methods are especially useful for probing parameter regimes where analytical approximations are inadequate, such as in the presence of strong nonlinearities or multiple excitation quanta [36]. The findings reported here may also provide a basis for future experimental collaborations involving microfabrication and cryogenic microwave measurements [43, 6].

1.3 Outline of the Thesis

The remainder of this thesis is organised into two main parts. Part I: Theoretical Background addresses the physical concepts underlying magnetic systems, mechanical vibrations, and hybrid cavity interactions [2, 4, 28]. This section equips readers with the necessary conceptual framework for understanding the original research presented in Part II and serves as an introduction to magnomechanics for those new to the field.

Part II: Results presents the original research contributions of this thesis, focusing on driven magnon–phonon–photon systems and their nonlinear quantum dynamics. Each chapter builds upon the theoretical foundation established in Part I and incrementally explores increasingly complex phenomena.

A brief summary of each chapter is provided below.

- **Chapter 2: Magnetism, Magnetic Interactions and Spin Dynamics**

This chapter introduces the fundamental concepts of magnetism, including ferromagnetic ordering and exchange interactions. It establishes the theoretical background necessary for understanding spin-wave excitations in materials such as yttrium iron garnet. The chapter begins with a historical overview of magnetic theory, followed by discussions of magnetic ordering, exchange interactions, and the various contributions to magnetic free energy. Additionally, it introduces magnetic domains and explains the response of magnetic materials to external fields. This chapter also examines the energetics and dynamical behaviour of magnetic systems [4]. It discusses in detail the contributions to magnetic free energy, including magnetocrystalline anisotropy, magnetostatic energy, magnetoelastic energy, and Zeeman interaction [31, 28]. The Landau–Lifshitz–Gilbert equation is introduced as the fundamental equation governing magnetisation dynamics [66,

67], and its physical implications and limiting cases are explored. The chapter also addresses ferromagnetic resonance, damping mechanisms, magnetostatic modes, and the Holstein–Primakoff transformation, which provides a quantum-mechanical framework for describing spin waves [27, 31]. A brief introduction to phonons as quantised mechanical excitations concludes the chapter, thereby preparing for the subsequent discussion of magnon–phonon coupling [1, 31, 7].

- **Chapter 3: Cavity Optomechanics**

This chapter presents the theoretical framework of cavity optomechanics, describing light–matter interactions in resonant cavities [2, 15]. It introduces the Heisenberg–Langevin formalism and input–output theory as essential tools for analysing open quantum systems [23, 5]. The discussion includes optomechanical coupling in both red- and blue-detuned regimes, highlighting phenomena such as sideband cooling and parametric amplification [16, 18]. This chapter serves as an introduction to optomechanics for readers unfamiliar with the field and provides a foundation for the subsequent chapter on cavity magnomechanics, which extends these concepts to magnetic systems.

Part II: Results

- **Chapter 4: Magnomechanics in a thin film** This chapter depicts the architecture of the device proposed and the calculation of magnon-photon and magnon-phonon coupling in a thin magnetic film using the Euler-Bernoulli principle. Subsequently, this chapter comprises of modelling our hybrid system consisting of three bosonic modes (magnons, photons, and phonons) using open quantum system formalism. The detailed derivation of the expression of reflection spectra and self-energy will be done in this section.
- **Chapter 5: Magnon–Photon–Phonon Interaction Under Red-Detuned Drive**

We investigate the behaviour of the hybrid system under the influence of a red-detuned microwave field [1, 36]. In this regime, beam-splitter-like interactions facilitate coherent energy exchange between subsystems, enabling processes such as sideband cooling and quantum state transfer [1, 59]. We analyse the conditions required for efficient cooling of the mechanical mode and examine the feasibility of transferring quantum states between magnon and phonon modes [14]. The analysis addresses the effects of dissipation and thermal noise and provides quantitative predictions for relevant experimental parameters [2].

- **Chapter 6: Magnon–Photon–Phonon Interaction Under Blue-Detuned Drive**

We examine the dynamics of the system under blue-detuned driving conditions, where parametric amplification and instability phenomena may occur [36]. The

potential for generating non-classical states of the magnetic and mechanical modes is also investigated [49]. We analyse the conditions necessary for squeezing and entanglement and discuss the stability boundaries that delineate different dynamical regimes [1, 25]. This chapter highlights the complex physical phenomena that arise when the driving field amplifies, rather than cools, the system's excitations.

- **Chapter 7: Kerr Nonlinear Effect**

This chapter analyses nonlinear phenomena resulting from strong microwave driving, with a focus on Kerr-type interactions in magnonic systems [36]. The resulting frequency shifts, bistability, and amplitude-dependent dispersion are examined in detail [36]. We develop a theoretical framework to elucidate how nonlinearity alters the system's response to external driving and explore the implications for applications such as nonreciprocal devices and quantum information processing [59]. The analysis demonstrates that Kerr nonlinearities, while often viewed as a complication, can be harnessed for practical functionality when properly controlled.

- **Chapter 8: Magnon–Phonon Entanglement**

This chapter develops a theoretical framework for generating entanglement between magnon modes and a phonon modes mediated by a shared photon mode [1, 24]. The analysis encompasses linearisation procedures, quantum Langevin equations, stability analysis, and covariance matrix evaluation. Entanglement is quantified using logarithmic negativity [24]. We investigate parameter regimes that enable the generation and maintenance of entanglement despite dissipation and discuss strategies for optimising entanglement fidelity [14]. This chapter represents the culmination of the thesis, demonstrating how the preceding chapters collectively achieve a central objective in quantum information science: the creation of robust entanglement between solid-state quantum systems.

Collectively, these chapters establish a unified theoretical framework for nanocavity magnomechanics and demonstrate how hybrid magnon–phonon–photon interactions facilitate coherent control, nonlinear dynamics, and quantum correlations in engineered solid-state systems [1]. The thesis guides readers from foundational concepts to advanced research, serving both newcomers and experienced researchers. Throughout, we emphasise both the mathematical formalism and the underlying physical intuition, providing a comprehensive perspective on this rapidly evolving field.

Chapter 2

Magnetism

2.1 Ferromagnetism: Order, Origins, and Material Realization

Ferromagnetism is a well-known example of collective behaviour in condensed matter. It occurs when unpaired electron spins align in the same direction, producing lasting magnetisation even without an external magnetic field. This order stays stable below the Curie temperature (T_C). When the temperature exceeds T_C , thermal fluctuations disrupt the order, and the material becomes paramagnetic. The alignment of magnetic moments is not due to classical dipole-dipole interactions, which are too weak at room temperature. Instead, it arises from the quantum mechanical exchange interaction. The driving force is electrostatic (Coulomb) repulsion, but the Pauli exclusion principle keeps electrons with parallel spins farther apart, making the repulsion energy depend on spin orientation [68, 69].

The ferromagnetic order was noticed long before it was understood scientifically. Lode-stone, a naturally magnetised form of magnetite (Fe_3O_4), was used as a compass in ancient China. Scientific study of ferromagnetism began in the 19th century. In 1895, Pierre Curie showed that magnetisation depends on temperature and disappears suddenly at a certain point [70]. In 1907, Pierre Weiss proposed the molecular field hypothesis [71], suggesting that an internal field, proportional to the overall magnetisation, aligns neighbouring magnetic moments. Although it started as a hypothesis, this model explained the Curie temperature (T_C) and spontaneous magnetisation, setting the stage for later quantum theories.

A major advance occurred in 1928 when Werner Heisenberg developed a quantum model of ferromagnetism [72]. The Heisenberg model introduced an exchange Hamiltonian to describe how spins interact on neighbouring lattice sites:

$$\mathcal{H} = -2 \sum_{i < j} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j. \quad (2.1)$$

In ferromagnets, the exchange integral J_{ij} is positive, so the lowest-energy state has parallel spin alignment. This Hamiltonian explains ferromagnetic order and serves as the foundation for understanding collective magnetic excitations such as spin waves and magnons, which are key topics in this thesis. Later theories, especially those by Bloch (1932) and Néel (1944), showed that real materials reduce magnetostatic energy by forming regions of uniform magnetisation called magnetic domains [73, 74].

Stable magnetisation in a ferromagnet comes from minimising different free energy contributions. On the microscopic scale, exchange energy favours uniform spin alignment and sets spin-wave stiffness. On the mesoscopic scale, magnetocrystalline anisotropy, caused by spin-orbit coupling and crystal symmetry, defines preferred magnetisation directions called easy axes [75]. The properties of these axes depend on the material and are important for stabilising spin-wave modes in nanostructured devices.

On the macroscopic scale, demagnetising or magnetostatic energy becomes important. This energy comes from long-range dipolar interactions and encourages the formation of domain structures that reduce stray magnetic fields outside the sample. Zeeman energy describes how the bulk magnetic moment interacts with an external magnetic field. This interaction is key for controlling spin-wave resonance, especially in ferromagnetic resonance (FMR) experiments, and for tuning magnon frequencies in real applications.

Among magnetic materials, Yttrium Iron Garnet (YIG, $\text{Y}_3\text{Fe}_5\text{O}_{12}$) is especially important in magnonics and magnomechanics. While YIG is technically a ferrimagnet, with several magnetic sublattices that have unequal and opposite moments, its dynamic behaviour is very similar to that of an ideal ferromagnet [76]. YIG's most notable feature is its extremely low Gilbert damping constant, usually $\alpha \sim 10^{-5}$, which is much lower than in metallic ferromagnets like Permalloy ($\alpha \sim 10^{-2}$ to 10^{-3}) [77]. This very low damping allows for long magnon lifetimes and coherence lengths, making YIG the leading material for studying coherent magnon dynamics and building hybrid quantum systems [78, 79].

When made as thin films or microstructures, YIG has adjustable magnetic anisotropy and clear magnetostatic mode profiles, which are useful for designing specific magnon-phonon interactions [80]. Its high Curie temperature ($T_C \approx 560\text{K}$) allows it to operate at room temperature, a significant advantage over other magnetic insulators or dilute magnetic semiconductors that require very low temperature [81]. In magnomechanics, YIG's magnetostrictive properties let mechanical strain directly change its magnetic anisotropy energy [82, 83], making it efficient for coupling magnons and phonons.

In cavity magnomechanics, YIG's ferrimagnetic order serves as the primary platform for coupling spin and lattice dynamics. Collective magnon modes can mix with phonon

2.1. FERROMAGNETISM: ORDER, ORIGINS, AND MATERIAL REALIZATION

modes through magnetoelastic interactions. This coupling works best when magnetic and mechanical excitations are in the same place and confined, such as in YIG spheres inside high-quality microwave cavities [79, 84] or YIG thin films made as nanomechanical resonators [53, 65].

Recent studies have found ferromagnetic order in new types of materials. Intrinsic ferromagnetism has been seen in two-dimensional van der Waals crystals like CrI_3 and Fe_3GeTe_2 , which keep stable magnetic order even in single layers [85, 86]. These discoveries create new possibilities for nanoscale magnonics and magnomechanics, especially for devices that need precise electrostatic control or can be built into very thin structures. Topological magnetic textures, such as skyrmions, have also been found in materials like MnSi [87]. These spin structures interact strongly with electric currents and mechanical forces, opening new ways to study topological magnon-phonon interactions [88].

Even with these advances, major challenges remain. Magnetic damping, whose microscopic mechanism is still under study [89], limits the coherence time of magnons, which is important for quantum information processing. Although YIG is the standard for low damping, combining it with other materials, like superconducting circuits or quantum dots, often brings extra losses or decoherence from strain [90]. Learning to control damping, especially in thin films, heterostructures, and interfaces, is crucial for making reliable, scalable quantum magnonic devices.

This chapter has covered the basics of ferromagnetism, from its quantum origins to its appearance in materials like YIG. These ideas set the stage for the next chapter, which will examine magnetic free energies and spin dynamics in more detail.

2.1.1 The Exchange Interaction as the Quantum Origin of Magnetism

Magnetism in solids mainly depends on the quantum-mechanical exchange interaction, which comes from the indistinguishability of electrons and the Pauli exclusion principle [91]. Classical magnetic dipole-dipole interactions between electrons are too weak to explain ferromagnetic ordering at room temperature, since they are usually about $\sim 10^{-4}$ eV. In comparison, the exchange interaction is short-ranged but much stronger, often around ~ 0.1 eV, and it sets the preferred alignment of electron spins in solids. As a result, exchange interactions are the primary microscopic mechanism underlying collective magnetic ordering in many materials [72, 92].

The exchange interaction comes from two main effects: Coulomb repulsion and the rule that the total electronic wavefunction must be antisymmetric when particles are swapped, as required by the Pauli exclusion principle for fermions [91, 93]. For two electrons, the total wavefunction must meet the following condition.

2.1. FERROMAGNETISM: ORDER, ORIGINS, AND MATERIAL REALIZATION

$$\Psi(\mathbf{r}_1, \mathbf{s}_1; \mathbf{r}_2, \mathbf{s}_2) = -\Psi(\mathbf{r}_2, \mathbf{s}_2; \mathbf{r}_1, \mathbf{s}_1)$$

In this equation, $\mathbf{r}_i$ and $\mathbf{s}_i$ stand for the position and spin of electron i respectively. The need for antisymmetry links the spatial and spin parts of the wavefunction.

A symmetric spin state (the triplet state with total spin $S = 1$) means the spatial part of the wavefunction must be antisymmetric. This setup reduces the likelihood that the two electrons will be close together, so the Coulomb repulsion energy is lower. On the other hand, an antisymmetric spin state (the singlet state with total spin $S = 0$) needs a symmetric spatial wavefunction, which makes it more likely that the electrons are in the same area and increases the Coulomb energy [72, 93]. The energy difference between these cases leads to the spin-dependent exchange interaction, which is the microscopic reason for magnetic ordering in solids [92, 94].

Consider two electrons occupying orthogonal spatial orbitals $\phi_a(\mathbf{r})$ and $\phi_b(\mathbf{r})$ (e.g., atomic orbitals on neighbouring sites). The total Hamiltonian (ignoring spin-orbit coupling) is

$$\hat{\mathcal{H}} = \hat{h}(\mathbf{r}_1) + \hat{h}(\mathbf{r}_2) + \frac{e^2}{4\pi\epsilon_0|\mathbf{r}_1 - \mathbf{r}_2|}$$

where $\hat{h}$ is the single-electron kinetic plus potential energy. The properly antisymmetrised two-electron states are:

$$\Psi_{\text{singlet}}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} [\phi_a(\mathbf{r}_1)\phi_b(\mathbf{r}_2) + \phi_b(\mathbf{r}_1)\phi_a(\mathbf{r}_2)] \chi_{S=0} \quad (2.2)$$

$$\Psi_{\text{triplet}}(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} [\phi_a(\mathbf{r}_1)\phi_b(\mathbf{r}_2) - \phi_b(\mathbf{r}_1)\phi_a(\mathbf{r}_2)] \chi_{S=1}^{m_S} \quad (2.3)$$

with $\chi_{S=0}$ the singlet spin state and $\chi_{S=1}^{m_S}$ ($m_S = -1, 0, 1$) the triplet states. Their expectation energies are:

$$E_{\text{singlet}} = \langle \Psi_{\text{singlet}} | \hat{\mathcal{H}} | \Psi_{\text{singlet}} \rangle, \quad E_{\text{triplet}} = \langle \Psi_{\text{triplet}} | \hat{\mathcal{H}} | \Psi_{\text{triplet}} \rangle.$$

Evaluating these yields the Coulomb integral K and exchange integral J [92, 94]:

$$K = \iint \frac{e^2}{4\pi\epsilon_0|\mathbf{r}_1 - \mathbf{r}_2|} |\phi_a(\mathbf{r}_1)|^2 |\phi_b(\mathbf{r}_2)|^2 d^3r_1 d^3r_2, \quad (2.4)$$

$$J = \iint \frac{e^2}{4\pi\epsilon_0|\mathbf{r}_1 - \mathbf{r}_2|} \phi_a^*(\mathbf{r}_1)\phi_b^*(\mathbf{r}_2)\phi_b(\mathbf{r}_1)\phi_a(\mathbf{r}_2) d^3r_1 d^3r_2 \quad (2.5)$$

with $J > 0$ for orthogonal orbitals. The energies become

$$E_{\text{singlet}} = 2E_0 + K + J, \quad E_{\text{triplet}} = 2E_0 + K - J$$

where $E_0 = \langle \phi_a | \hat{h} | \phi_a \rangle = \langle \phi_b | \hat{h} | \phi_b \rangle$. The singlet-triplet splitting is

$$E_{\text{singlet}} - E_{\text{triplet}} = 2J.$$

2.1. FERROMAGNETISM: ORDER, ORIGINS, AND MATERIAL REALIZATION

We can map this onto an operator acting only on spins. Using $\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$ eigenvalues,

$$\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 = \begin{cases} -\frac{3}{4}\hbar^2 & (\text{singlet}), \\ +\frac{1}{4}\hbar^2 & (\text{triplet}) \end{cases}$$

one finds the Dirac-Heisenberg form [72, 93]:

$$\hat{\mathcal{H}}_{\text{spin}} = \frac{1}{4}(E_{\text{singlet}} + 3E_{\text{triplet}}) - \frac{E_{\text{singlet}} - E_{\text{triplet}}}{\hbar^2} \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2.$$

substituting $E_{\text{singlet}} - E_{\text{triplet}} = 2J$ and dropping the constant term gives the standard exchange Hamiltonian:

$$\boxed{\hat{\mathcal{H}}_{\text{ex}} = -\frac{2J}{\hbar^2} \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2.}$$

In most condensed matter literature, $\hat{\mathbf{S}}_i$ is taken dimensionless (or absorbing $\hbar$ into J), yielding simply $\hat{\mathcal{H}}_{\text{ex}} = -2J \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$.

The sign of J determines magnetic ordering:

- $J > 0$: Triplet ($S = 1$) lies lower $\rightarrow$ parallel spins $\rightarrow$ **ferromagnetism**.
- $J < 0$: Singlet ($S = 0$) lies lower $\rightarrow$ antiparallel spins $\rightarrow$ **antiferromagnetism**.

Magnetism is not simply an intrinsic property of a single electron spin. Instead, it emerges from the collective behaviour of many electrons in a solid, shaped by quantum statistics, Coulomb interactions, and exchange coupling [68, 92, 95].

In ferromagnetic materials, the exchange integral is positive ($J > 0$), which means the triplet configuration has lower energy than the singlet configuration. This can be seen from the exchange Hamiltonian.

$$\hat{\mathcal{H}}_{\text{ex}} = -2J \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 \tag{2.6}$$

To minimise exchange energy, the scalar product $\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$ must be as large as possible, which means neighbouring spins align in parallel. This alignment causes the spontaneous magnetisation seen in ferromagnets. In contrast, paramagnetic materials do not show long-range spin order because thermal fluctuations are stronger than the exchange coupling [92, 94].

Figure 2.1 shows three common types of magnetic ordering found in solids [94, 95]:

- **Ferromagnetic (FM)**: All spins align parallel, resulting in a large net magnetisation.
- **Antiferromagnetic (AFM)**: Adjacent spins align antiparallel, producing zero net magnetisation.



(a) Ferromagnetic



(b) Antiferromagnetic



(c) Ferrimagnetic

Figure 2.1: Schematic representation of electron spin ordering in (a) ferromagnetic, (b) antiferromagnetic, and (c) ferrimagnetic materials. Arrow length indicates the magnitude of the magnetic moment.

- **Ferrimagnetic (FiM):** Spins align antiparallel but with unequal magnitudes, giving rise to a non-zero net magnetisation.

Even though exchange interactions only act over short distances, large-scale magnetic behaviour comes from the collective ordering of many spins. In real materials with about 10^{23} atoms, describing each spin individually is not practical. Instead, it is better to use a continuum approach, where the magnetic state is described by a continuous magnetisation field [96, 97].

The **magnetisation** is defined as the magnetic moment per unit volume.

$$\mathbf{M}(\mathbf{r}) = \frac{d\boldsymbol{\mu}}{dV} \quad (2.7)$$

In the continuum approximation, magnetisation changes smoothly with position. For a uniformly magnetised ferromagnet, the magnetisation can be written as

$$\mathbf{M}(\mathbf{r}) = M_s \hat{\mathbf{m}}(\mathbf{r}) \quad (2.8)$$

Here, $M_s = |\mathbf{M}|$ is the saturation magnetisation, and $\hat{\mathbf{m}}(\mathbf{r})$ is a unit vector that shows the local direction of magnetisation [96, 94]. Saturation magnetisation means all atomic moments are aligned in the same direction.

In practice, real ferromagnets rarely exist in a single-domain configuration. Instead, the system lowers its total magnetostatic energy by forming magnetic domains separated by domain walls [96]. Achieving full saturation along a given direction therefore typically requires the application of an external magnetic field large enough to overcome both domain-wall pinning and magnetocrystalline anisotropy effects [95, 94]. The continuum description remains valid provided the characteristic length scale exceeds the domain wall width, which is typically in the range 10–100 of nm [96].

2.1. FERROMAGNETISM: ORDER, ORIGINS, AND MATERIAL REALIZATION

A key property of ferromagnetic materials is that there is a critical temperature above which spontaneous magnetisation disappears. This temperature is called the **Curie temperature** T_C , and it marks the point where the material changes from ferromagnetic to paramagnetic as thermal fluctuations become stronger than the exchange interaction [94].

According to Weiss molecular-field theory, each magnetic moment feels an effective internal field that is proportional to the magnetisation.

$$\mathbf{H}_{\text{eff}} = \lambda \mathbf{M} \quad (2.9)$$

Here, λ is the molecular field constant. With this approximation, the Curie temperature can be written as [92, 94]

$$T_C = \frac{g_J \mu_B (J + 1) \lambda M_s}{3k_B} \quad (2.10)$$

In this equation, g_J denotes the Landé g factor, μ_B is the Bohr magneton, k_B is Boltzmann's constant, and J represents the total angular momentum quantum number.

For a lattice where each atom interacts with z nearest neighbours through an exchange interaction J_{ex} , the molecular field constant can be related to the microscopic exchange parameter by

$$\lambda = \frac{2zJ_{\text{ex}}}{g_J^2 \mu_B^2 M_s}. \quad (2.11)$$

If we substitute this into Eq. (2.10), we get

$$T_C = \frac{2zJ_{\text{ex}}J(J + 1)}{3g_J \mu_B k_B}. \quad (2.12)$$

This shows that the Curie temperature depends directly on the strength of the exchange interaction and the number of neighbouring atoms. Materials with strong exchange coupling or many neighbours usually have high Curie temperatures [92, 94]. For technology, it is important that T_C is much higher than room temperature so magnetic order stays stable during use. All the materials studied in this thesis have $T_C \gg 300$ K.

The exchange interaction is the microscopic quantum-mechanical reason for magnetic order. Without the antisymmetry of fermionic wavefunctions and the resulting exchange energy, magnetic order in solids would only come from very weak dipolar interactions. This basic mechanism is important for many topics discussed in later chapters, such as spin-wave excitations, magnon dispersion, and magnetoelastic coupling [68, 95].

2.1.2 Magnetic Free Energy in Ferromagnetic Systems

The stable state of a ferromagnetic system is found by minimizing its total magnetic free energy. In micromagnetics, magnetization is viewed as a continuous vector field, and the system's magnetic state is described by an energy function that includes several physical contributions [97, 96, 98]. The balance of these energy terms determines the stable magnetization direction and how it responds to external influences such as magnetic fields or mechanical strain.

The total magnetic free energy density arises from a number of interactions that depend on the magnetization's orientation and spatial distribution. In ferromagnetic materials the dominant contributions typically include exchange energy, magnetocrystalline anisotropy energy, magnetostatic energy, magnetoelastic energy, and Zeeman energy [82, 83, 94]. Each term corresponds to a distinct physical interaction and contributes to the system's overall magnetic energy landscape.

Exchange energy discourages changes in magnetization from place to place, so it favors neighboring magnetic moments lining up in the same direction. In the continuum model, this energy is described using the gradient of the reduced magnetization vector. It is key to setting magnetic length scales, such as domain wall widths [97, 96].

Magnetocrystalline anisotropy is another important part of the magnetic free energy. It arises from spin-orbit coupling and the crystal lattice's symmetry, which create preferred directions for magnetization within the crystal. These are called easy axes, while directions with higher energy are called hard axes [82, 99]. The anisotropy energy depends on the crystal's symmetry and is usually written as an expansion using the direction cosines of the magnetization vector.

Magnetostatic energy, or demagnetizing energy, comes from the magnetic field created by the magnetization itself. This field produces surface and volume magnetic charges that add to the system's total energy and depend a lot on the shape of the magnetic material [96, 98]. Because of this, the magnetostatic term often favors magnetization patterns that lower stray magnetic fields, like forming magnetic domains.

Magnetoelastic energy is the link between the magnetic and elastic properties of the crystal lattice. When the material is strained, its magnetic anisotropy changes, potentially altering the preferred direction of magnetization. This effect is the basis for magnetostriction in ferromagnetic materials [100].

Zeeman energy describes how magnetization interacts with an external magnetic field. This energy tries to align the magnetization with the applied field, which can compete with the material's own anisotropy [82, 7].

The total magnetic free energy density U in a ferromagnetic system is the sum of all these contributions.

$$U = U_{\text{exchange}} + U_{\text{magnetocrystalline}} + U_{\text{magnetostatic}} + U_{\text{magnetoelastic}} + U_{\text{Zeeman}}. \quad (2.13)$$

Each term in Eq. (2.13) depends on how the magnetization vector is oriented compared to the crystal axes and sometimes on how it changes in space. To study how the total free energy depends on direction, it helps to use spherical polar coordinates and look at how the energy changes with the polar and azimuthal angles.

This way of representing the energy, called the magnetic free-energy surface, helps visualize the system's magnetic anisotropy. The lowest points on the surface show the most stable magnetization directions (easy axes), while the highest points show the least stable ones (hard axes). If the magnetization is uniform and saturated, this method is useful for studying anisotropy and how magnetization reverses in ferromagnetic materials [96].

2.2 Magnetization Dynamics and Landau-Lifshitz-Gilbert (LLG) Equation

Magnetism is not just about static properties; it also involves how magnetization changes over time. Understanding these dynamic processes is crucial in fields such as spintronics and magnonics, where controlling spin dynamics enables the development of new magnetic devices. The way magnetic moments move over time is mainly determined by the link between an electron's magnetic moment and its angular momentum [101, 94].

For a single electron, the magnetic moment $\boldsymbol{\mu}$ is proportional to its total angular momentum $\mathbf{J}$, which combines both orbital angular momentum $\mathbf{L}$ and spin angular momentum $\mathbf{S}$. This relationship is expressed as

$$\boldsymbol{\mu} = \gamma \mathbf{J} \quad (2.14)$$

Here, γ is the gyromagnetic ratio. This ratio shows how magnetic moment and angular momentum are related, and it is important for understanding how magnetization precesses in magnetic materials.

The gyromagnetic ratio for an electron is expressed as

$$\gamma = \frac{g\mu_B}{\hbar} \quad (2.15)$$

In this equation, g is the spectroscopic g -factor, μ_B is the Bohr magneton, and $\hbar$ is the reduced Planck constant. For a free electron, g is about 2. In many transition

metals, especially those with half-filled $3d^5$ shells, the crystal field mostly cancels out the orbital angular momentum, so the effective g -factor stays close to 2 [94].

However, if the orbital contribution is not completely removed, the g -factor can differ from 2. This shows the combined effects of both spin and orbital magnetic moments.

We can estimate how much the orbital (μ_L) and spin (μ_S) parts each contribute to the total magnetic moment using the following formula:

$$g = 2 \left(1 + \frac{\mu_L}{\mu_S} \right) \quad (2.16)$$

This formula lets researchers measure the ratio of orbital to spin magnetic moments in experiments. Methods such as ferromagnetic resonance (FMR) spectroscopy and x-ray magnetic circular dichroism (XMCD) are often used to determine the g -factor and to compare spin and orbital contributions in magnetic materials [28].

When a magnetic field is applied to an atom, the electronic energy levels split depending on the orientation of the magnetic moment relative to the field. This effect, called Zeeman splitting, creates distinct energy levels for different angular-momentum projections.

The amount of splitting depends on the Landé g -factor, which includes both spin and orbital contributions to the total angular momentum. For a state with quantum numbers L , S , and J , the Landé factor is given by

$$g_J = g_L \frac{J(J+1) - S(S+1) + L(L+1)}{2J(J+1)} + g_S \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} \quad (2.17)$$

Here, $g_L = 1$ stands for the orbital part and $g_S = 2$ for the spin part. This formula can be simplified to

$$g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)} \quad (2.18)$$

If the moment is purely orbital ($S = 0$), g_J is close to 1. If it is purely spin ($L = 0$), g_J is close to 2. When a magnetic field B is applied, the Zeeman energy of a state is given by

$$E = g_J \mu_B m_J B \quad (2.19)$$

In this equation, m_J is the magnetic quantum number that shows how much of the total angular momentum points along the direction of the applied field.

2.2. MAGNETIZATION DYNAMICS AND LANDAU-LIFSHITZ-GILBERT (LLG) EQUATION

In ordered magnetic systems, magnetization dynamics come from the way electron spins move together. The overall magnetization vector $\mathbf{M}$ usually lines up with the effective magnetic field $\mathbf{H}_{\text{eff}}$, which is the total of all magnetic fields inside the material. These include applied fields, anisotropy fields, exchange interactions, and demagnetizing fields.

The effective field applies a torque to the magnetization as described by

$$\boldsymbol{\tau} = -\mathbf{M} \times \mathbf{H}_{\text{eff}} \quad (2.20)$$

This torque makes the magnetization precess around the direction of the effective field [101, 28].

The way magnetization changes over time is described by the Landau–Lifshitz–Gilbert (LLG) equation:

$$\frac{d\mathbf{M}}{dt} = -\gamma(\mathbf{M} \times \mathbf{H}_{\text{eff}}) + \frac{\alpha}{M_S} \left(\mathbf{M} \times \frac{d\mathbf{M}}{dt} \right) \quad (2.21)$$

Here, γ is the gyromagnetic ratio and α is the Gilbert damping parameter [102]. The first term shows how magnetization precesses around the effective field, and the second term describes how it slowly lines up with $\mathbf{H}_{\text{eff}}$ due to damping.

If there is no damping ($\alpha = 0$), the magnetization keeps precessing at the Larmor frequency

$$\omega = \gamma H_{\text{eff}} \quad (2.22)$$

In real materials, some damping is always present, so the magnetization spirals toward equilibrium along the effective field direction. This happens over a timescale of about $1/(\alpha\omega)$.

When magnetization moves collectively, it forms spin waves. These are waves of coordinated precession among neighboring magnetic moments. The phase difference between nearby moments sets the wavelength of the spin wave.

For short wavelengths, the exchange interaction is strongest and prefers neighboring atomic moments to be aligned either parallel or antiparallel. If the alignment is disturbed, the system's exchange energy goes up. The typical length scale for this effect is called the exchange length.

$$l_{\text{ex}} = \sqrt{\frac{A}{2\pi M_S^2}} \quad (2.23)$$

In this formula, A is the exchange stiffness constant and M_S is the saturation magnetization, both measured in CGS units [28].

Spin waves dominated by exchange interactions usually have wavelengths shorter than about 100 nm and frequencies in the high GHz to THz range.

For longer wavelengths, around a few micrometers, dipole–dipole interactions become more important. These are called magnetostatic spin waves (MSWs), which come from solving Maxwell’s equations using the magnetostatic approximation [28]. Magnetostatic spin waves usually appear at low GHz frequencies and are especially important in magnonic devices

2.3 Ferromagnetic Resonance

The Ferromagnetic Resonance (FMR) phenomenon refers to the resonant precession of the magnetization of a sample around its equilibrium position through the excitation of electromagnetic radiation. Ferromagnetic resonance can be theoretically described using either a classical or a quantum approach.

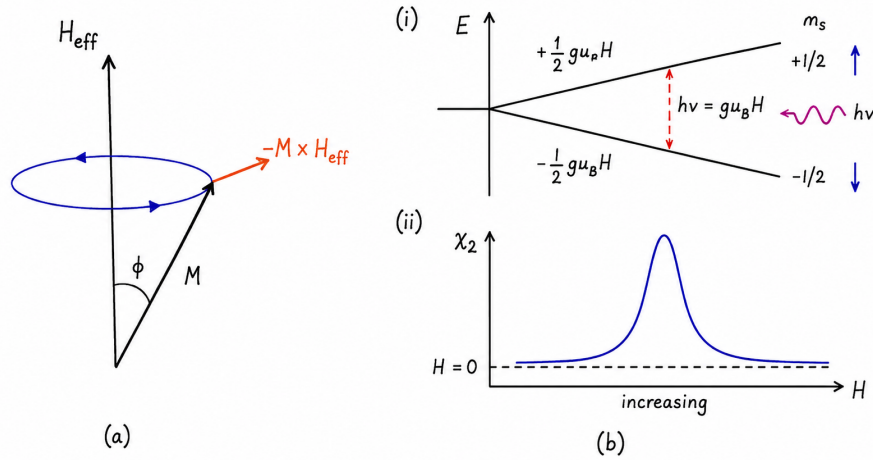


Figure 2.2: (a) Classical FMR: Larmor precession of the magnetization $\mathbf{M}$ along the effective field vector $\mathbf{H}_{\text{eff}}$ at an angle ϕ to the axis. (b) Quantum FMR: (i) Splitting of electron energy due to Zeeman effect versus the applied magnetic field strength. (ii) Plot of FMR absorption indicating a rise in susceptibility χ .

Classical FMR is described originally by Kittel [82] and is based on dynamic considerations presented in Section 2.2, according to which the system’s dynamics are governed by the Landau–Lifshitz–Gilbert (LLG) equation (Equation 2.21). In this description, all magnetic moments of the material are considered in terms of a single magnetization vector. For the magnetization to be in equilibrium, it should be aligned

along the direction of the effective field $\mathbf{H}_{\text{eff}}$, thus minimizing magnetostatic energy. If the magnetization is out of equilibrium because of the component of the external field being perpendicular to $\mathbf{H}_{\text{eff}}$, it is subjected to a torque $-\mathbf{M} \times \mathbf{H}_{\text{eff}}$. In order for precession to occur about the effective field, the frequency of the perturbing field applied to the sample must equal the magnetization's Larmor frequency ($\omega = \gamma H_{\text{eff}}$). The perturbing field is usually in the form of a RF field that works at the resonant frequency. The time evolution of the magnetization vector is described by the LLG equation. Resonant precession goes on indefinitely while the RF field is present. Upon the removal of the RF field, relaxation takes place, whereby the magnetization spirals into equilibrium along the effective field.

The quantum mechanical model of FMR deals with the splitting of energies of electron spins through the Zeeman effect. The energy of the electrons is related to the orientation of the electron spins with respect to the effective field, with the spin pointing opposite to the direction of the field having low energy [56]. When there is a resonance condition, there is an incident RF excitation whose photon energy $\hbar\omega$ corresponds to the Zeeman splitting energy (2.19). With this energy correspondence, there is a magnetically dipolar transition by which the RF photons are absorbed to elevate the electrons to higher energy spins. Later, re-emission of RF photons takes place [103]

2.3.1 Kittel Equation

The Kittel equation is often used to describe how the resonance frequency of ferromagnetic resonance (FMR) changes with an applied magnetic field. This relationship creates a characteristic frequency-field pattern called the Kittel curve. Researchers use these curves to determine key magnetic properties from FMR measurements [82, 28].

In materials like Yttrium Iron Garnet (YIG), the resonance frequency does not increase in a straight line with the applied field at low values. Instead, it roughly follows a square-root relationship ($\sqrt{H}$) at first, then becomes almost linear as the magnetic field increases.

The Kittel equation is based on the Smit-Beljers formalism. This approach defines the resonance condition by looking at how the magnetic free energy changes with the polar (θ_M) and azimuthal (ϕ_M) angles of magnetisation [104].

$$\left(\frac{\omega}{\gamma}\right)^2 = \frac{1}{M_s^2 \sin^2 \theta_M} \left[\frac{\partial^2 U}{\partial \theta_M^2} \frac{\partial^2 U}{\partial \phi_M^2} - \left(\frac{\partial^2 U}{\partial \theta_M \partial \phi_M} \right)^2 \right]. \quad (2.24)$$

Using this approach, Kittel showed that for thin films the resonance condition can be expressed as

$$\omega_0 = \gamma \sqrt{H_{\text{eff}} B_{\text{eff}}}. \quad (2.25)$$

where H_{eff} and B_{eff} represent the effective magnetic field and magnetic induction within the material.

For a (111)-oriented film, the total magnetic free energy density U_{111} includes contributions from Zeeman energy, demagnetising fields, cubic anisotropy, and uniaxial anisotropy:

$$\begin{aligned} U_{111} = & -HM_s \sin \theta_M \cos(\phi_M - \phi_H) \\ & + K_c \left[\frac{1}{3} \cos^4 \theta_M + \frac{1}{4} \sin^4 \theta_M - \frac{\sqrt{2}}{3} \sin^3 \theta_M \cos \theta_M \cos(3\phi_M) \right] \\ & + 2\pi M_s^2 \cos^2 \theta_M - K_u^\perp \cos^2 \theta_M - K_u^\parallel \sin^2 \theta_M \cos^2(\phi_M - \phi_U). \end{aligned} \quad (2.26)$$

Substituting this into the Smit-Beljers Equation, and in the limit that $\theta_M \rightarrow \frac{\pi}{2}$ magnetisation is aligned in-plane, the FMR resonance condition becomes:

$$\begin{aligned} \left(\frac{\omega}{\gamma} \right)_{111}^2 = & \frac{1}{M_s^2} ([HM_s \cos(\phi_M - \phi_H) + 4\pi M_s^2 - K_c - 2K_u^\perp + 2K_u^\parallel \cos^2(\phi_M - \phi_U)] \\ & \cdot [HM_s \cos(\phi_M - \phi_H) + 2K_u^\parallel \cos[2(\phi_M - \phi_U)]] \\ & - 2K_c^2 \sin^2(3\phi_M)) . \end{aligned} \quad (2.27)$$

In the high field limit, such that magnetisation aligns completely with the field ($\phi_M \rightarrow \phi_H = \phi$), further simplifications can be made to the FMR resonance condition:

$$\begin{aligned} \left(\frac{\omega}{\gamma} \right)_{111}^2 = & \frac{1}{M_s^2} ([HM_s + 4\pi M_s^2 - K_c - 2K_u^\perp + 2K_u^\parallel \cos^2(\phi - \phi_U)] \\ & \cdot [HM_s + 2K_u^\parallel \cos[2(\phi - \phi_U)]] \\ & \cdot [-2K_c^2 \sin^2(3\phi)]) . \end{aligned} \quad (2.28)$$

Cubic and uniaxial anisotropy terms can be combined to create a single out-of-plane anisotropy field term, provided the cubic and out-of-plane uniaxial components are parallel; an in-plane easy axis Kittel equation can be obtained thereafter. However, if uniaxial anisotropy is considered negligible compared to cubic anisotropy, the FMR resonance condition simplifies further. In the limit that $K_u^\perp \rightarrow 0$ and $K_u^\parallel \rightarrow 0$:

$$\left(\frac{\omega}{\gamma} \right)_{111}^2 = \frac{1}{M_s^2} ([HM_s] \cdot [HM_s + 4\pi M_s^2 - K_c] - 2K_c^2 \sin^2(3\phi)) \quad (2.29)$$

$$\left(\frac{\omega}{\gamma} \right)_{111}^2 = H(H + 4\pi M_{\text{eff}}) - \frac{2K_c^2}{M_s^2} \sin^2(3\phi). \quad (2.30)$$

A maximum $\left(\frac{\omega}{\gamma}\right)_{111}^2$ is observed in the limit $\phi \rightarrow 0$, due to the $-2K_c^2 \sin^2(3\phi)$ term. From this FMR resonance condition, an in-plane easy-axis Kittel equation can be obtained. The FMR resonant frequency, ω , as a function of applied field, H , for a (111)-oriented film is expressed as:

$$\omega = \gamma \sqrt{H(H + 4\pi M_{\text{eff}})}. \quad (2.31)$$

This relation is widely used to determine magnetic parameters such as the gyromagnetic ratio γ and effective magnetisation M_{eff} from FMR measurements.

2.3.2 FMR Damping

Magnetic damping refers to how energy is lost during the precessional motion of magnetisation. It controls how quickly magnetisation returns to equilibrium, which is important for magnetisation dynamics and how spintronic devices work [66, 67, 102]. In ferromagnetic materials, damping is usually classified as intrinsic or extrinsic, depending on its microscopic origin [105, 106].

Intrinsic damping comes from basic interactions between magnetisation and the electronic or lattice parts of the crystal. Several microscopic processes can cause intrinsic damping, such as eddy-current loss, magnon-phonon coupling, and itinerant-electron relaxation via spin-orbit interactions [107, 108]. In metallic ferromagnets, the main cause is itinerant electron relaxation, often explained by the breathing Fermi surface model [109, 110]. In this model, spin-orbit coupling distorts the Fermi surface as magnetisation precesses, which leads to electron scattering and energy loss [108, 111].

In experiments, magnetic damping is usually measured with ferromagnetic resonance (FMR). The resonance is described by its linewidth ΔH , which is defined as the half width at half maximum (HWHM) of the resonance shape. When intrinsic damping is the main factor, the FMR linewidth increases linearly with the excitation frequency:

$$\Delta H(\omega) = \Delta H(0) + \frac{\alpha\omega}{\gamma} \quad (2.32)$$

$$\Delta H(f) = \Delta H(0) + \frac{2\pi\alpha}{\gamma} f \quad (2.33)$$

Where H is the magnetic field strength, γ is the gyromagnetic ratio, and α is the Gilbert damping parameter. The slope $2\pi\alpha/\gamma$ represents the intrinsic damping behaviour, while the intercept $\Delta H(0)$ describes frequency-independent contributions arising from extrinsic damping.

Extrinsic damping happens when structural imperfections in a material disrupt the uniform movement of magnetisation. One key cause is mosaicity in the crystal structure,

which creates a range of spin directions and thereby widens the resonance linewidth. Magnetic inhomogeneities in the film can also change the local magnetic field, causing nearby spins to lose sync during precession and making the linewidth even broader [112, 113].

Chapter 3

Optical Cavities and Mechanical Resonators

3.1 Theoretical Framework

This chapter presents the essential theoretical framework for the study of cavity optomechanics and, by extension, cavity magnomechanics. It first describes optical resonators, emphasising key physical parameters such as resonance frequency, decay rates, and quality factor. The chapter then introduces the input-output formalism, which governs the coupling between the intracavity field and external radiation channels. This formalism offers a standard approach to describing dissipation and measurement in open quantum optical systems [114, 115, 116].

The discussion next addresses the mechanical degree of freedom, which is initially treated classically as a damped harmonic oscillator and then in its quantised form as a bosonic mode. This quantisation introduces phonon creation and annihilation operators, enabling a unified description of mechanical motion within the quantum harmonic oscillator framework. This approach forms the foundation of quantum optomechanics [2].

Building on these foundations, the optomechanical interaction is introduced through the radiation-pressure coupling Hamiltonian, which links the cavity photon number to mechanical displacement. Linearisation around a strong coherent drive produces effective bilinear interaction terms and results in a set of linearised Heisenberg–Langevin equations that govern the dynamics of the coupled system. These equations constitute the standard model for analysing small fluctuations around the steady state in driven-dissipative cavity systems [2, 116, 5].

Collectively, these results establish the theoretical foundation for cavity magnomechanical systems, in which analogous photon-magnon-phonon interactions arise through

similar linearisation procedures and input-output formalisms.

3.2 Electromagnetic Cavities and Resonators

Optical resonators, also known as cavities, are essential components that confine light within a small volume and enhance its interaction with matter. The most common configuration is the Fabry–Pérot cavity, which consists of two highly reflective parallel mirrors separated by a distance L . The interference of multiple reflections produces a discrete set of resonant frequencies, forming the basis for understanding cavity quantum electrodynamics and optomechanics [117, 118].

While optical cavities operate in the visible or near-infrared regimes, microwave cavity systems are particularly central to circuit quantum electrodynamics and hybrid quantum devices. At microwave frequencies (typically $\sim 1\text{--}10$ GHz), resonators are fabricated using superconducting materials operated at cryogenic temperatures to minimise Ohmic losses and eliminate thermal noise photons. Common architectures include planar microwave cavities—such as coplanar waveguide (CPW) transmission-line resonators and lumped-element LC circuit resonators. These devices achieve exceptionally high quality factors ($Q > 10^5$) and compact, sub-wavelength mode volumes, facilitating strong coupling between confined microwave photons and physical qubits or mechanical modes [119].

3.2.1 Basic Properties

The resonance condition for a Fabry–Pérot cavity requires that the round-trip phase change of the light field be an integer multiple of 2π . Mathematically, this condition is expressed as:

$$2kL = 2m\pi,$$

where $k = 2\pi/\lambda$ is the wave number, λ is the wavelength, and m is an integer known as the mode number. Using the relation $k = \omega/c$, where ω is the angular frequency and c is the speed of light, the resonance condition becomes:

$$2\frac{\omega L}{c} = 2m\pi.$$

Solving for ω gives the resonant frequencies:

$$\omega_m = \frac{m\pi c}{L}.$$

The frequency separation between two consecutive resonant modes is known as the free spectral range (FSR). For modes with indices $m + 1$ and m , the FSR is:

$$\Delta\omega_{\text{FSR}} = \omega_{m+1} - \omega_m = \frac{(m+1)\pi c}{L} - \frac{m\pi c}{L}.$$

This simplifies to the fundamental expression:

$$\Delta\omega_{\text{FSR}} = \frac{\pi c}{L}. \quad (3.1)$$

The corresponding frequency difference in hertz is:

$$\Delta\nu_{\text{FSR}} = \frac{\Delta\omega_{\text{FSR}}}{2\pi} = \frac{c}{2L}.$$

Thus, the free spectral range in hertz is:

$$\Delta\nu_{\text{FSR}} = \frac{c}{2L}.$$

The sharpness of the cavity resonances is characterised by two related quantities: the finesse $\mathcal{F}$ and the quality factor Q . Finesse is defined as the ratio of the free spectral range to the full width at half-maximum (FWHM) of the resonance peak. The FWHM is directly related to the cavity decay rate κ by:

$$\text{FWHM} = \Delta\nu_{\text{FWHM}} = \frac{\kappa}{2\pi},$$

where κ represents the total energy decay rate of the cavity due to all loss mechanisms. Consequently, the finesse can be expressed as:

$$\mathcal{F} = \frac{\Delta\nu_{\text{FSR}}}{\Delta\nu_{\text{FWHM}}} = \frac{c/(2L)}{\kappa/(2\pi)} = \frac{\pi c}{L\kappa}.$$

More elegantly, in angular frequency units:

$$\mathcal{F} = \frac{\Delta\omega_{\text{FSR}}}{\kappa}. \quad (3.2)$$

The quality factor Q of a resonator is associated with the photon lifetime τ within the cavity. By definition, the quality factor relates the stored energy to the energy loss per cycle. For an optical cavity, it is expressed as:

$$Q = \omega_{\text{cav}}\tau = \frac{\omega_{\text{cav}}}{\kappa}. \quad (3.3)$$

A high Q cavity (κ small) stores photons for a long time, enhancing light-matter interactions. The relationship between finesse and quality factor depends on the cavity length, but both fundamentally quantify the resonance's narrowness relative to its central frequency.

3.2.2 Input-Output Formalism for an Optical Cavity

To describe a cavity that is coupled to an external electromagnetic field, such as a laser beam, we employ the input-output formalism. This framework provides a quantum mechanical description of how the cavity field interacts with the external continuum of modes. The formalism is based on the Heisenberg equation of motion, which describes the time evolution of the field amplitude inside the cavity [114, 115, 116].

In the Heisenberg picture, operators evolve in time while quantum states remain constant. We aim to find the equation of motion for the annihilation operator $\hat{a}(t)$ of the cavity mode. The time evolution of $\hat{a}(t)$ is given by:

$$\hat{a}(t) = \hat{U}^\dagger(t) \hat{a} \hat{U}(t)$$

where the time evolution operator is $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$. To derive the Heisenberg equation, we compute the time derivative:

$$\frac{d}{dt} \hat{a}(t) = \frac{d}{dt} \left(\hat{U}^\dagger(t) \hat{a} \hat{U}(t) \right).$$

Applying the product rule:

$$\frac{d}{dt} \hat{a}(t) = \left(\frac{d}{dt} \hat{U}^\dagger(t) \right) \hat{a} \hat{U}(t) + \hat{U}^\dagger(t) \hat{a} \left(\frac{d}{dt} \hat{U}(t) \right).$$

Using the relations $\frac{d}{dt} \hat{U}(t) = -\frac{i}{\hbar} \hat{H} \hat{U}(t)$ and $\frac{d}{dt} \hat{U}^\dagger(t) = \frac{i}{\hbar} \hat{U}^\dagger(t) \hat{H}$, we obtain:

$$\frac{d}{dt} \hat{a}(t) = \frac{i}{\hbar} \hat{U}^\dagger(t) \hat{H} \hat{U}(t) \hat{a} + \hat{U}^\dagger(t) \hat{a} \left(-\frac{i}{\hbar} \hat{H} \hat{U}(t) \right).$$

Noting that $\hat{U}^\dagger(t) \hat{H} \hat{U}(t) = \hat{H}$ (since the Hamiltonian is typically time-independent in the Schrödinger picture) and that $\hat{U}^\dagger(t) \hat{a} \hat{U}(t) = \hat{a}(t)$, the expression simplifies to:

$$\frac{d}{dt} \hat{a}(t) = \frac{i}{\hbar} [\hat{H}, \hat{a}(t)]. \quad (3.4)$$

This is the Heisenberg equation of motion, which forms the starting point for deriving the dynamics of any operator.

3.2.3 Driven Optical Cavity Hamiltonian

The Hamiltonian for an optical cavity driven by a classical laser field is derived as follows. The free Hamiltonian for the cavity mode is:

$$\hat{H}_0 = \hbar \omega_{\text{cav}} \hat{a}^\dagger \hat{a}$$

Here, ω_{cav} denotes the cavity resonance frequency. The driving field, with frequency ω_L , is represented by a classical electric field $E(t) = E_0 \cos(\omega_L t)$. The interaction between this field and the cavity mode is described by:

$$\hat{H}_{\text{drive}} = \hbar\Omega(\hat{a}^\dagger + \hat{a}) \cos(\omega_L t)$$

In this expression, Ω represents the coupling strength between the driving field and the cavity mode. The operator $(\hat{a}^\dagger + \hat{a})$ is proportional to the position operator of the harmonic oscillator that models the cavity field, demonstrating that the driving force couples to the field amplitude. Applying Euler's formula, $\cos(\omega_L t) = (e^{i\omega_L t} + e^{-i\omega_L t})/2$, the interaction Hamiltonian is rewritten as:

$$\hat{H}_{\text{drive}} = \frac{\hbar\Omega}{2}(\hat{a}^\dagger + \hat{a})(e^{i\omega_L t} + e^{-i\omega_L t}).$$

Expanding this expression produces four distinct terms:

$$\hat{H}_{\text{drive}} = \frac{\hbar\Omega}{2} (\hat{a}^\dagger e^{i\omega_L t} + \hat{a} e^{i\omega_L t} + \hat{a}^\dagger e^{-i\omega_L t} + \hat{a} e^{-i\omega_L t}).$$

The total Hamiltonian is given by $\hat{H} = \hat{H}_0 + \hat{H}_{\text{drive}}$. To facilitate analysis, the rotating wave approximation (RWA) is applied. In the interaction picture, terms oscillating at frequencies $\omega_{\text{cav}} + \omega_L$ average to zero over time, whereas terms oscillating at $\omega_{\text{cav}} - \omega_L$ become significant when the cavity is driven near resonance ($\omega_L \approx \omega_{\text{cav}}$) [5]. The terms $\hat{a}^\dagger e^{i\omega_L t}$ and $\hat{a} e^{-i\omega_L t}$ are rapidly oscillating and are neglected, while $\hat{a}^\dagger e^{-i\omega_L t}$ and $\hat{a} e^{i\omega_L t}$ are slowly varying and are retained. With the RWA, the interaction Hamiltonian reduces to:

$$\hat{H}_{\text{drive}} \approx \frac{\hbar\Omega}{2} (\hat{a}^\dagger e^{-i\omega_L t} + \hat{a} e^{i\omega_L t}).$$

To eliminate explicit time dependence, a transformation to a frame rotating at the driving frequency ω_L is performed. The transformed operator is defined as:

$$\hat{a}(t) = \hat{c}(t)e^{-i\omega_L t}.$$

By substituting this expression into the Hamiltonian, together with $\hat{a}^\dagger(t) = \hat{c}^\dagger(t)e^{i\omega_L t}$, the following result is obtained:

$$\hat{H} = \hbar\omega_{\text{cav}}\hat{c}^\dagger\hat{c} + \frac{\hbar\Omega}{2}(\hat{c}^\dagger e^{i\omega_L t} e^{-i\omega_L t} + \hat{c} e^{-i\omega_L t} e^{i\omega_L t}).$$

The exponential factors in the driving term cancel, resulting in:

$$\hat{H} = \hbar\omega_{\text{cav}}\hat{c}^\dagger\hat{c} + \frac{\hbar\Omega}{2}(\hat{c}^\dagger + \hat{c}).$$

The detuning parameter is introduced as $\Delta = \omega_L - \omega_{\text{cav}}$. By rewriting the cavity frequency as $\omega_{\text{cav}} = \omega_L - \Delta$, the Hamiltonian takes the form:

$$\hat{H} = \hbar(\omega_L - \Delta)\hat{c}^\dagger\hat{c} + \frac{\hbar\Omega}{2}(\hat{c}^\dagger + \hat{c}).$$

To transform into the rotating frame of the drive at frequency ω_L , we apply the unitary operator $U(t) = e^{i\omega_L t \hat{c}^\dagger \hat{c}}$. The effective Hamiltonian is given by $\hat{H}_{\text{rot}} = U\hat{H}U^\dagger + i\hbar\frac{\partial U}{\partial t}U^\dagger$. The time derivative generates $-\hbar\omega_L\hat{c}^\dagger\hat{c}$, which explicitly cancels the $\hbar\omega_L\hat{c}^\dagger\hat{c}$ term and yields:

$$\hat{H}_{\text{rot}} = -\hbar\Delta\hat{c}^\dagger\hat{c} + \frac{\hbar\Omega}{2}(\hat{c}^\dagger + \hat{c}). \quad (3.5)$$

3.2.4 Heisenberg-Langevin Equations

The dynamics of the cavity field, including dissipation and input noise, are described by the Heisenberg-Langevin equation. Applying the Heisenberg equation (3.4) with the Hamiltonian (3.5), the commutator is computed as follows:

$$\frac{d\hat{c}}{dt} = \frac{i}{\hbar}[\hat{H}_{\text{rot}}, \hat{c}].$$

Cavity decay and input noise are incorporated by adding dissipative terms according to the standard input-output formalism [114, 115]. The complete quantum Langevin equation is:

$$\frac{d\hat{c}}{dt} = \left(i\Delta - \frac{\kappa}{2}\right)\hat{c} - i\frac{\Omega}{2} + \sqrt{\kappa_{\text{ex}}}\hat{c}_{\text{in}}(t) + \sqrt{\kappa_0}\hat{f}_{\text{in}}(t). \quad (3.6)$$

Here, $\kappa = \kappa_0 + \kappa_{\text{ex}}$ is the total decay rate and $\hat{f}_{\text{in}}(t)$ is the input vacuum noise.

3.2.5 Classical Steady State and Input-Output Relations

The input-output relation links the output field to both the internal cavity field and the input field. For a cavity with a single input port, this relation is given by:

$$\hat{a}_{\text{out}} = \hat{a}_{\text{in}} - \sqrt{\kappa_{\text{ex}}}\hat{a}. \quad (3.7)$$

This equation represents conservation of energy at the cavity mirror, where the outgoing field equals the reflected incoming field minus the field radiated from the cavity.

In the classical limit, expectation values of the operators are considered. For steady-state conditions, $d\langle\hat{a}\rangle/dt = 0$. Using equation (3.6), and neglecting noise terms (which average to zero), transformation to the rotating frame where $\langle\hat{a}\rangle = \langle\hat{c}\rangle e^{-i\omega_L t}$ yields the steady-state condition:

$$0 = \left(i\Delta - \frac{\kappa}{2}\right)\langle\hat{c}\rangle - i\frac{\Omega}{2}.$$

Solving for $\langle \hat{c} \rangle$ gives:

$$\langle \hat{c} \rangle = \frac{i\Omega/2}{i\Delta - \kappa/2} = \frac{\Omega/2}{\Delta + i\kappa/2}.$$

Transforming back to the original frame, $\langle \hat{a} \rangle = \langle \hat{c} \rangle e^{-i\omega_L t}$. The average photon number in the cavity is then:

$$n_{\text{cav}} = |\langle \hat{a} \rangle|^2 = |\langle \hat{c} \rangle|^2 = \frac{\Omega^2/4}{\Delta^2 + (\kappa/2)^2}.$$

The input power is related to the driving amplitude by $P = \hbar\omega_L \langle \hat{a}_{\text{in}}^\dagger \hat{a}_{\text{in}} \rangle$. Typically, the steady-state intra-cavity field is expressed in terms of the input field amplitude. Including the factor $\sqrt{\kappa_{\text{ex}}}$ from the input coupling, the standard result is:

$$\langle \hat{a} \rangle = \frac{\sqrt{\kappa_{\text{ex}}} \langle \hat{a}_{\text{in}} \rangle}{\kappa/2 - i\Delta}, \quad n_{\text{cav}} = \frac{\kappa_{\text{ex}}}{\Delta^2 + (\kappa/2)^2} \frac{P}{\hbar\omega_L}. \quad (3.8)$$

The reflection coefficient $\mathcal{R}$ is defined as the ratio of the output field amplitude to the input field amplitude. Employing the input-output relation (3.7) and the steady-state solution (3.8) yields:

$$\langle \hat{a}_{\text{out}} \rangle = \langle \hat{a}_{\text{in}} \rangle - \sqrt{\kappa_{\text{ex}}} \langle \hat{a} \rangle = \langle \hat{a}_{\text{in}} \rangle - \sqrt{\kappa_{\text{ex}}} \frac{\sqrt{\kappa_{\text{ex}}} \langle \hat{a}_{\text{in}} \rangle}{\kappa/2 - i\Delta} = \langle \hat{a}_{\text{in}} \rangle \left(1 - \frac{\kappa_{\text{ex}}}{\kappa/2 - i\Delta} \right).$$

Upon simplification and using $\kappa = \kappa_0 + \kappa_{\text{ex}}$:

$$\frac{\langle \hat{a}_{\text{out}} \rangle}{\langle \hat{a}_{\text{in}} \rangle} = \frac{\kappa/2 - i\Delta - \kappa_{\text{ex}}}{\kappa/2 - i\Delta} = \frac{(\kappa_0 - \kappa_{\text{ex}})/2 - i\Delta}{(\kappa_0 + \kappa_{\text{ex}})/2 - i\Delta}.$$

Therefore, the reflection coefficient is:

$$\mathcal{R} = \frac{(\kappa_0 - \kappa_{\text{ex}})/2 - i\Delta}{(\kappa_0 + \kappa_{\text{ex}})/2 - i\Delta}. \quad (3.9)$$

The reflection coefficient provides significant insight into the properties of the cavity. At resonance ($\Delta = 0$), $\mathcal{R} = (\kappa_0 - \kappa_{\text{ex}})/(\kappa_0 + \kappa_{\text{ex}})$. For $\kappa_0 = \kappa_{\text{ex}}$ (critical coupling), $\mathcal{R} = 0$, indicating that all incident light is coupled into the cavity. If $\kappa_0 > \kappa_{\text{ex}}$ (over-coupled), the cavity is highly reflective at resonance. If $\kappa_0 < \kappa_{\text{ex}}$ (under-coupled), the reflection coefficient is negative, signifying a phase shift. Far from resonance ($|\Delta| \gg \kappa$), $\mathcal{R} \approx 1$, and the cavity behaves as a simple mirror.

3.3 Mechanical Resonators

Mechanical resonators, including suspended micro-mirrors, vibrating dielectric membranes, and nanomechanical beams, serve as the mechanical degree of freedom in

optomechanical and electromechanical systems [2, 120]. These devices are typically modelled as damped harmonic oscillators characterised by an effective mass m_{eff} , fundamental mechanical frequency Ω_m , and intrinsic damping rate γ_m [121]. Their classical dynamics are governed by Newton's second law subject to elastic restoring forces, viscous or intrinsic damping, and external driving forces, while environmental interactions and thermal fluctuations are incorporated via stochastic Langevin force terms [122, 123].

3.3.1 Classical Equation of Motion

For a mechanical oscillator with effective mass m_{eff} , Newton's second law states that the net force $F(t)$ acting on the oscillator results in acceleration:

$$F(t) = m_{\text{eff}} \frac{d^2 x(t)}{dt^2}.$$

The oscillator is subject to three primary forces. The restoring force is proportional to displacement and opposes motion: $F_{\text{restoring}} = -m_{\text{eff}} \Omega_m^2 x(t)$, where Ω_m denotes the mechanical resonance frequency. The damping force, which accounts for energy dissipation to the environment, is proportional to velocity: $F_{\text{damping}} = -m_{\text{eff}} \Gamma_m \frac{dx(t)}{dt}$, where Γ_m is the damping coefficient per unit mass. An external driving force $F_{\text{ext}}(t)$ may also be present.

Summing these forces gives the equation of motion:

$$m_{\text{eff}} \frac{d^2 x(t)}{dt^2} + m_{\text{eff}} \Gamma_m \frac{dx(t)}{dt} + m_{\text{eff}} \Omega_m^2 x(t) = F_{\text{ext}}(t). \quad (3.10)$$

Dividing by m_{eff} yields:

$$\frac{d^2 x(t)}{dt^2} + \Gamma_m \frac{dx(t)}{dt} + \Omega_m^2 x(t) = \frac{F_{\text{ext}}(t)}{m_{\text{eff}}}.$$

3.3.2 Solution of the Harmonic Oscillator Equation

The general solution to equation (3.10) comprises the homogeneous solution, which describes transient behavior, and a particular solution, representing the steady-state response. For the homogeneous equation, a solution of the form $x_h(t) = e^{rt}$ is assumed. Substitution yields the characteristic equation:

$$r^2 + \Gamma_m r + \Omega_m^2 = 0.$$

The roots are:

$$r = \frac{-\Gamma_m \pm \sqrt{\Gamma_m^2 - 4\Omega_m^2}}{2}.$$

In most mechanical resonators, damping is weak ($\Gamma_m \ll \Omega_m$), resulting in a negative discriminant. By defining the damped resonance frequency $\Omega_d = \sqrt{\Omega_m^2 - (\Gamma_m/2)^2}$, the

roots become complex conjugates:

$$r = -\frac{\Gamma_m}{2} \pm i\Omega_d.$$

The homogeneous solution is therefore a damped oscillation:

$$x_h(t) = Ae^{-\frac{\Gamma_m}{2}t} \cos(\Omega_d t + \phi)$$

Here, A and ϕ are constants determined by the initial conditions. The transient solution decays on a time scale of $2/\Gamma_m$.

For a sinusoidal driving force $F_{\text{ext}}(t) = F_0 \cos(\omega t)$, a particular solution at the driving frequency is sought. Assuming $x_p(t) = X \cos(\omega t - \alpha)$, where X denotes the amplitude and α the phase lag, substitution into the equation of motion and solving for X and α yields:

$$X(\omega) = \frac{F_0/m_{\text{eff}}}{\sqrt{(\Omega_m^2 - \omega^2)^2 + (\Gamma_m \omega)^2}}, \quad \tan \alpha(\omega) = \frac{\Gamma_m \omega}{\Omega_m^2 - \omega^2}.$$

The amplitude reaches its maximum near $\omega \approx \Omega_m$, with the peak width determined by Γ_m . The phase lag increases from 0 at low frequencies to π at high frequencies, passing through $\pi/2$ at resonance.

3.3.3 Mechanical Susceptibility

In the frequency domain, the mechanical response is conveniently described by the susceptibility $\chi_m(\omega)$, defined as the ratio of the displacement response to the applied force. To derive this, we take the Fourier transform of equation (3.10). The Fourier transform and its inverse are defined as:

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt, \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{i\omega t} d\omega.$$

Applying this to each term in equation (3.10):

$$\begin{aligned} \mathcal{F} \left\{ m_{\text{eff}} \frac{d^2 x}{dt^2} \right\} &= m_{\text{eff}} (i\omega)^2 X(\omega) = -m_{\text{eff}} \omega^2 X(\omega), \\ \mathcal{F} \left\{ m_{\text{eff}} \Gamma_m \frac{dx}{dt} \right\} &= m_{\text{eff}} \Gamma_m (i\omega) X(\omega) = i m_{\text{eff}} \Gamma_m \omega X(\omega), \\ \mathcal{F} \{ m_{\text{eff}} \Omega_m^2 x \} &= m_{\text{eff}} \Omega_m^2 X(\omega), \\ \mathcal{F} \{ F_{\text{ext}}(t) \} &= \tilde{F}_{\text{ext}}(\omega). \end{aligned}$$

Assembling these terms:

$$(-m_{\text{eff}} \omega^2 + i m_{\text{eff}} \Gamma_m \omega + m_{\text{eff}} \Omega_m^2) X(\omega) = \tilde{F}_{\text{ext}}(\omega).$$

Solving for $X(\omega)$:

$$X(\omega) = \frac{\tilde{F}_{\text{ext}}(\omega)}{m_{\text{eff}}(\Omega_m^2 - \omega^2 + i\Gamma_m\omega)}.$$

The mechanical susceptibility $\chi_m(\omega)$ is defined as $X(\omega) = \chi_m(\omega)\tilde{F}_{\text{ext}}(\omega)$. Therefore:

$$\chi_m(\omega) = \frac{1}{m_{\text{eff}}(\Omega_m^2 - \omega^2 + i\Gamma_m\omega)}. \quad (3.11)$$

The magnitude $|\chi_m(\omega)|$ quantifies the amplitude of mechanical motion in response to a given force amplitude, while the phase $\arg(\chi_m(\omega))$ specifies the phase lag between displacement and the driving force.

3.4 Optomechanical Interaction Hamiltonian

The coupling between light and mechanical motion arises from the dependence of the cavity resonance frequency on the position of the mechanical resonator. For a Fabry-Pérot cavity of length L with one movable mirror, the resonance frequency is $\omega_{\text{cav}} = m\pi c/L$. A small displacement x of the mirror changes the cavity length to $L \pm x$, thereby shifting the resonance frequency.

3.4.1 Derivation of the Interaction Hamiltonian

The Hamiltonian for the cavity mode with a position-dependent frequency is:

$$\hat{H}_{\text{cav}}(x) = \hbar\omega_{\text{cav}}(x)\hat{a}^\dagger\hat{a}.$$

Assuming the displacement is small compared to the cavity length, we perform a Taylor expansion of $\omega_{\text{cav}}(x)$ around $x = 0$:

$$\omega_{\text{cav}}(x) \approx \omega_{\text{cav}} - G\hat{x},$$

where $G = -\left.\frac{d\omega_{\text{cav}}}{dx}\right|_{x=0}$ is the frequency pull parameter. For a Fabry-Pérot cavity, $\omega_{\text{cav}} = \pi c/L$, so $G = \pi c/L^2 = \omega_{\text{cav}}/L$. Substituting the expansion into the Hamiltonian:

$$\hat{H}_{\text{cav}}(\hat{x}) = \hbar(\omega_{\text{cav}} - G\hat{x})\hat{a}^\dagger\hat{a} = \hbar\omega_{\text{cav}}\hat{a}^\dagger\hat{a} - \hbar G\hat{x}\hat{a}^\dagger\hat{a}.$$

The mechanical oscillator is quantised by expressing the position operator $\hat{x}$ in terms of creation and annihilation operators $\hat{c}^\dagger$ and $\hat{c}$ for the mechanical mode. The zero-point fluctuation amplitude is $x_{\text{ZPF}} = \sqrt{\hbar/(2m_{\text{eff}}\Omega_m)}$, and the position operator is:

$$\hat{x} = x_{\text{ZPF}}(\hat{c} + \hat{c}^\dagger).$$

Substituting this into the interaction term gives:

$$\hat{H}_{\text{int}} = -\hbar G x_{\text{ZPF}} \hat{a}^\dagger \hat{a} (\hat{c} + \hat{c}^\dagger).$$

The optomechanical coupling strength g_0 is defined as:

$$g_0 = Gx_{\text{ZPF}}.$$

This parameter, with units of frequency, characterises the interaction strength between a single photon and a single phonon. It is typically small, ranging from Hz to kHz, but can be increased by raising the intra-cavity photon number. The total Hamiltonian for the system is the sum of the free cavity, free mechanical, and interaction terms:

$$\hat{H} = \hbar\omega_{\text{cav}}\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{c}^\dagger\hat{c} - \hbar g_0\hat{a}^\dagger\hat{a}(\hat{c} + \hat{c}^\dagger). \quad (3.12)$$

3.4.2 Transformation to the Rotating Frame

In experimental situations, the cavity is driven by a laser at frequency ω_L . It is convenient to transform to a frame rotating at the laser frequency to eliminate fast time dependence. The unitary transformation is $\hat{U} = e^{i\omega_L\hat{a}^\dagger\hat{a}t}$. In the rotating frame, the Hamiltonian becomes $\hat{H}_{\text{rot}} = \hat{U}\hat{H}\hat{U}^\dagger - i\hbar\hat{U}\partial_t\hat{U}^\dagger$.

First, we compute the transformation of each term. For the cavity term:

$$\hat{U}(\hbar\omega_{\text{cav}}\hat{a}^\dagger\hat{a})\hat{U}^\dagger = \hbar\omega_{\text{cav}}\hat{a}^\dagger\hat{a},$$

since $\hat{U}$ commutes with $\hat{a}^\dagger\hat{a}$. For the mechanical term, $\hat{U}$ commutes with $\hat{c}$ and $\hat{c}^\dagger$, so:

$$\hat{U}(\hbar\Omega_m\hat{c}^\dagger\hat{c})\hat{U}^\dagger = \hbar\Omega_m\hat{c}^\dagger\hat{c}.$$

For the interaction term, since $\hat{U}$ commutes with $\hat{a}^\dagger\hat{a}$, we have:

$$\hat{U}(-\hbar g_0\hat{a}^\dagger\hat{a}(\hat{c} + \hat{c}^\dagger))\hat{U}^\dagger = -\hbar g_0\hat{a}^\dagger\hat{a}(\hat{c} + \hat{c}^\dagger).$$

Next, we compute the time derivative term. Since $\hat{U}^\dagger = e^{-i\omega_L\hat{a}^\dagger\hat{a}t}$, its time derivative is:

$$\frac{\partial\hat{U}^\dagger}{\partial t} = -i\omega_L\hat{a}^\dagger\hat{a}e^{-i\omega_L\hat{a}^\dagger\hat{a}t} = -i\omega_L\hat{a}^\dagger\hat{a}\hat{U}^\dagger.$$

Therefore:

$$-i\hbar\hat{U}\frac{\partial\hat{U}^\dagger}{\partial t} = -i\hbar\hat{U}(-i\omega_L\hat{a}^\dagger\hat{a}\hat{U}^\dagger) = -\hbar\omega_L\hat{U}\hat{a}^\dagger\hat{a}\hat{U}^\dagger = -\hbar\omega_L\hat{a}^\dagger\hat{a}.$$

Assembling all terms, the Hamiltonian in the rotating frame is:

$$\hat{H}_{\text{rot}} = \hbar\omega_{\text{cav}}\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{c}^\dagger\hat{c} - \hbar g_0\hat{a}^\dagger\hat{a}(\hat{c} + \hat{c}^\dagger) - \hbar\omega_L\hat{a}^\dagger\hat{a}.$$

Combining the cavity terms:

$$\hat{H}_{\text{rot}} = -\hbar(\omega_L - \omega_{\text{cav}})\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{c}^\dagger\hat{c} - \hbar g_0\hat{a}^\dagger\hat{a}(\hat{c} + \hat{c}^\dagger).$$

Defining the detuning $\Delta = \omega_L - \omega_{\text{cav}}$, the final rotating-frame Hamiltonian is:

$$\hat{H}_{\text{rot}} = -\hbar\Delta\hat{a}^\dagger\hat{a} + \hbar\Omega_m\hat{c}^\dagger\hat{c} - \hbar g_0\hat{a}^\dagger\hat{a}(\hat{c} + \hat{c}^\dagger). \quad (3.13)$$

3.5 Linearized Equations of Motion

When the cavity is driven by a strong laser, the intra-cavity field acquires a large coherent amplitude. This condition enables linearization of the equations of motion around the steady state, with quantum fluctuations treated as small perturbations.

3.5.1 Linearization Procedure

We decompose the cavity field operator into a steady-state mean value and a fluctuation operator:

$$\hat{a} = \bar{\alpha} + \delta\hat{a},$$

where $\bar{\alpha} = \langle \hat{a} \rangle$ is the coherent amplitude satisfying $|\bar{\alpha}|^2 = n_{\text{cav}}$, the average number of photons in the cavity. The fluctuation operator $\delta\hat{a}$ satisfies $\langle \delta\hat{a} \rangle = 0$ and represents quantum noise and small classical perturbations. Similarly, for the mechanical mode, we write $\hat{c} = \bar{\beta} + \delta\hat{c}$, where $\bar{\beta}$ is the steady-state mechanical amplitude. In most cases, the steady-state mechanical displacement is zero or can be shifted away by redefining the equilibrium position, so we take $\bar{\beta} = 0$ and $\hat{c} = \delta\hat{c}$.

Substituting these decompositions into the interaction Hamiltonian $-\hbar g_0 \hat{a}^\dagger \hat{a} (\hat{c} + \hat{c}^\dagger)$ and expanding:

$$\hat{a}^\dagger \hat{a} = (\bar{\alpha}^* + \delta\hat{a}^\dagger)(\bar{\alpha} + \delta\hat{a}) = |\bar{\alpha}|^2 + \bar{\alpha} \delta\hat{a}^\dagger + \bar{\alpha}^* \delta\hat{a} + \delta\hat{a}^\dagger \delta\hat{a}.$$

Multiplying by $(\hat{c} + \hat{c}^\dagger)$ gives four categories of terms. The term $-\hbar g_0 |\bar{\alpha}|^2 (\hat{c} + \hat{c}^\dagger)$ represents a constant force on the mechanical oscillator and can be eliminated by shifting the equilibrium position. The term $-\hbar g_0 \delta\hat{a}^\dagger \delta\hat{a} (\hat{c} + \hat{c}^\dagger)$ is quadratic in fluctuations and is neglected in the linearized approximation. The remaining terms are linear in the fluctuations:

$$\hat{H}_{\text{int, lin}} = -\hbar g_0 (\bar{\alpha} \delta\hat{a}^\dagger + \bar{\alpha}^* \delta\hat{a}) (\hat{c} + \hat{c}^\dagger).$$

If we choose the phase of the driving field such that $\bar{\alpha}$ is real and positive (a standard convention), then $\bar{\alpha}^* = \bar{\alpha} = \sqrt{n_{\text{cav}}}$. The linearized interaction Hamiltonian becomes:

$$\hat{H}_{\text{int, lin}} = -\hbar g_0 \sqrt{n_{\text{cav}}} (\delta\hat{a}^\dagger + \delta\hat{a}) (\hat{c} + \hat{c}^\dagger).$$

The enhanced optomechanical coupling rate g is defined as:

$$g = g_0 \sqrt{n_{\text{cav}}}.$$

This is the many-photon coupling strength, which can be significantly larger than the single-photon coupling g_0 when the cavity contains many photons. In terms of g , the linearized interaction Hamiltonian is:

$$\hat{H}_{\text{int, lin}} = -\hbar g (\delta\hat{a}^\dagger + \delta\hat{a}) (\hat{c} + \hat{c}^\dagger). \quad (3.14)$$

3.5.2 Full Linearized Hamiltonian

Including the free evolution terms for the fluctuations, the total linearized Hamiltonian in the rotating frame is:

$$\hat{H}_{\text{lin}} = -\hbar\Delta\delta\hat{a}^\dagger\delta\hat{a} + \hbar\Omega_m\hat{c}^\dagger\hat{c} - \hbar g(\delta\hat{a}^\dagger + \delta\hat{a})(\hat{c} + \hat{c}^\dagger). \quad (3.15)$$

The detuning Δ is measured relative to the laser frequency. This Hamiltonian forms the basis for analysing most cavity optomechanical phenomena in the linear regime.

3.5.3 Heisenberg-Langevin Equations for Fluctuations

Using the Heisenberg equation of motion and including dissipation and noise, we derive the quantum Langevin equations for the fluctuation operators. For the optical fluctuation $\delta\hat{a}$, the equation is:

$$\frac{d}{dt}\delta\hat{a} = \frac{i}{\hbar}[\hat{H}_{\text{lin}}, \delta\hat{a}] - \frac{\kappa}{2}\delta\hat{a} + \sqrt{\kappa_{\text{ex}}}\delta\hat{a}_{\text{in}} + \sqrt{\kappa_0}\delta\hat{f}_{\text{in}}.$$

Evaluating the commutator $[\delta\hat{a}^\dagger\delta\hat{a}, \delta\hat{a}] = -\delta\hat{a}$ and $[(\delta\hat{a}^\dagger + \delta\hat{a})(\hat{c} + \hat{c}^\dagger), \delta\hat{a}] = -(\hat{c} + \hat{c}^\dagger)$ gives:

$$\frac{d}{dt}\delta\hat{a} = \left(i\Delta - \frac{\kappa}{2}\right)\delta\hat{a} + ig(\hat{c} + \hat{c}^\dagger) + \sqrt{\kappa_{\text{ex}}}\delta\hat{a}_{\text{in}} + \sqrt{\kappa_0}\delta\hat{f}_{\text{in}}. \quad (3.16)$$

For the mechanical fluctuation $\delta\hat{c}$, the equation is:

$$\frac{d}{dt}\delta\hat{c} = \frac{i}{\hbar}[\hat{H}_{\text{lin}}, \delta\hat{c}] - \frac{\Gamma_m}{2}\delta\hat{c} + \sqrt{\Gamma_m}\hat{c}_{\text{in}}.$$

Evaluating the commutator: $[\hat{c}^\dagger\hat{c}, \hat{c}] = -\hat{c}$ and $[(\delta\hat{a}^\dagger + \delta\hat{a})(\hat{c} + \hat{c}^\dagger), \hat{c}] = -(\delta\hat{a}^\dagger + \delta\hat{a})$. This yields:

$$\frac{d}{dt}\delta\hat{c} = \left(-i\Omega_m - \frac{\Gamma_m}{2}\right)\delta\hat{c} + ig(\delta\hat{a}^\dagger + \delta\hat{a}) + \sqrt{\Gamma_m}\hat{c}_{\text{in}}. \quad (3.17)$$

3.5.4 Noise Correlators

The noise operators in equations (3.16) and (3.17) are zero-mean Gaussian noise sources characterised by their correlation functions. For the optical input noise, which is typically vacuum noise at optical frequencies (since $\hbar\omega_{\text{cav}} \gg k_B T$ at room temperature), the non-zero correlators are:

$$\langle\delta\hat{a}_{\text{in}}(t)\delta\hat{a}_{\text{in}}^\dagger(t')\rangle = \delta(t-t'), \quad \langle\delta\hat{a}_{\text{in}}^\dagger(t)\delta\hat{a}_{\text{in}}(t')\rangle = 0.$$

For the mechanical thermal noise, which is significant at room temperature, the correlators are:

$$\langle\hat{c}_{\text{in}}(t)\hat{c}_{\text{in}}^\dagger(t')\rangle = (\bar{n}_{\text{th}} + 1)\delta(t-t'), \quad \langle\hat{c}_{\text{in}}^\dagger(t)\hat{c}_{\text{in}}(t')\rangle = \bar{n}_{\text{th}}\delta(t-t'),$$

where $\bar{n}_{\text{th}} = 1/[\exp(\hbar\Omega_m/k_B T) - 1]$ is the mean thermal phonon number of the mechanical bath. At room temperature, $k_B T \gg \hbar\Omega_m$ for typical mechanical frequencies (MHz to GHz range), so $\bar{n}_{\text{th}} \approx k_B T/\hbar\Omega_m \gg 1$.

3.5.5 Frequency Domain Equations

It is often convenient to work in the frequency domain, especially when analyzing the response to external forces or calculating spectra. The Fourier transform of equation (3.16) gives:

$$-i\omega\delta\hat{a}(\omega) = \left(i\Delta - \frac{\kappa}{2}\right)\delta\hat{a}(\omega) + ig(\delta\hat{c}(\omega) + \delta\hat{c}^\dagger(\omega)) + \sqrt{\kappa_{\text{ex}}}\delta\hat{a}_{\text{in}}(\omega) + \sqrt{\kappa_0}\delta\hat{f}_{\text{in}}(\omega).$$

Rearranging:

$$\left(\frac{\kappa}{2} - i(\Delta - \omega)\right)\delta\hat{a}(\omega) = ig(\delta\hat{c}(\omega) + \delta\hat{c}^\dagger(\omega)) + \sqrt{\kappa_{\text{ex}}}\delta\hat{a}_{\text{in}}(\omega) + \sqrt{\kappa_0}\delta\hat{f}_{\text{in}}(\omega).$$

Similarly, the Fourier transform of equation (3.17) gives:

$$-i\omega\delta\hat{c}(\omega) = \left(-i\Omega_m - \frac{\Gamma_m}{2}\right)\delta\hat{c}(\omega) + ig(\delta\hat{a}(\omega) + \delta\hat{a}^\dagger(\omega)) + \sqrt{\Gamma_m}\hat{c}_{\text{in}}(\omega).$$

Rearranging:

$$\left(\frac{\Gamma_m}{2} - i(\Omega_m - \omega)\right)\delta\hat{c}(\omega) = ig(\delta\hat{a}(\omega) + \delta\hat{a}^\dagger(\omega)) + \sqrt{\Gamma_m}\hat{c}_{\text{in}}(\omega).$$

These coupled algebraic equations can be solved to find the mechanical displacement spectrum and the output optical spectrum, revealing the effects of radiation pressure backaction.

3.6 Classical Averaged Equations

In the semiclassical limit, where the photon and phonon numbers are large, we can take expectation values of the quantum Langevin equations to obtain classical equations for the mean fields. While quantum fluctuations are important for understanding noise and entanglement, the classical equations describe the system's deterministic dynamics.

3.6.1 Equations for Mean Fields

Taking the expectation value of equation (3.6) and transforming to the rotating frame, we obtain:

$$\frac{d}{dt}\langle\hat{a}\rangle = \left(i\Delta - \frac{\kappa}{2}\right)\langle\hat{a}\rangle + \sqrt{\kappa_{\text{ex}}}\langle\hat{a}_{\text{in}}\rangle.$$

For the mechanical mode, taking the expectation value of the Heisenberg equation derived from the full Hamiltonian (3.12) yields:

$$\frac{d^2}{dt^2}\langle\hat{x}\rangle + \Gamma_m\frac{d}{dt}\langle\hat{x}\rangle + \Omega_m^2\langle\hat{x}\rangle = \frac{\hbar G}{m_{\text{eff}}}\langle\hat{a}^\dagger\hat{a}\rangle.$$

The right-hand side represents the radiation pressure force, which depends on the intra-cavity photon number.

3.6.2 Steady State and Linearisation around Steady State

In the steady state, $d\langle\hat{a}\rangle/dt = 0$ and $d\langle\hat{x}\rangle/dt = 0$. The steady-state optical amplitude is given by equation (3.8), and the steady-state mechanical displacement is $\langle\hat{x}\rangle_{\text{ss}} = (\hbar G/m_{\text{eff}}\Omega_m^2)\langle\hat{a}^\dagger\hat{a}\rangle_{\text{ss}}$. For small fluctuations around the steady state, we write $\langle\hat{a}\rangle = \bar{\alpha} + \delta\alpha(t)$ and $\langle\hat{x}\rangle = \bar{x} + \delta x(t)$. Substituting into the classical equations and keeping only linear terms in the fluctuations gives:

$$\begin{aligned}\frac{d}{dt}\delta\alpha &= \left(i\Delta - \frac{\kappa}{2}\right)\delta\alpha + iG\bar{\alpha}\delta x, \\ \frac{d^2}{dt^2}\delta x + \Gamma_m\frac{d}{dt}\delta x + \Omega_m^2\delta x &= \frac{\hbar G}{m_{\text{eff}}}(\bar{\alpha}\delta\alpha^* + \bar{\alpha}^*\delta\alpha).\end{aligned}$$

These equations are equivalent to the linearised quantum equations when the noise terms are neglected, confirming the consistency of the linearisation procedure.

3.6.3 Optical Spring Effect

A principal manifestation of dynamical backaction in cavity optomechanics is the optical spring effect, in which the mechanical resonance frequency is altered by radiation pressure forces generated by an intracavity optical field. The underlying mechanism originates from the finite response time of the optical cavity. Because the intracavity photon number does not respond instantaneously to mechanical motion, the radiation-pressure force acquires a phase lag relative to the mechanical displacement. This delayed force modifies both the effective stiffness and damping of the mechanical resonator. The change in the restoring force produces the optical spring effect, while the velocity-dependent component results in optomechanical damping or amplification.

The starting point for the analysis is the linearised optomechanical Hamiltonian.

$$H_{\text{int}} = -\hbar g(\delta a + \delta a^\dagger)(c + c^\dagger), \quad (3.18)$$

where

$$g = g_0\sqrt{n_{\text{cav}}} \quad (3.19)$$

is the enhanced optomechanical coupling rate, where g_0 represents the single-photon optomechanical coupling strength and n_{cav} denotes the average intracavity photon number. Here, the single-photon coupling strength g_0 is a fixed geometric parameter of the system that cannot be altered by the field; rather, it is the field-enhanced coupling rate g that increases with intracavity photon number n_{cav} and laser power. In the frequency domain, the interaction between optical and mechanical fluctuations modifies the intrinsic mechanical susceptibility via the radiation-pressure self-energy.

$$\Sigma(\omega) = -ig^2 [\chi_c(\omega) - \chi_c^*(-\omega)], \quad (3.20)$$

where

$$\chi_c(\omega) = \frac{1}{\kappa/2 - i(\Delta + \omega)} \quad (3.21)$$

is the cavity susceptibility, where κ denotes the cavity decay rate and $\Delta = \omega_L - \omega_c$ specifies the laser detuning from the cavity resonance. The effective mechanical susceptibility is then given by

$$\chi_{\text{eff}}^{-1}(\omega) = \chi_m^{-1}(\omega) + \Sigma(\omega). \quad (3.22)$$

The real component of the self-energy alters the mechanical resonance frequency, whereas the imaginary component modifies the mechanical damping rate. As a result, the effective mechanical resonance frequency is expressed as

$$\Omega_{\text{eff}} = \Omega_m + \delta\Omega_m, \quad (3.23)$$

with the optical spring shift given by

$$\delta\Omega_m = \frac{g^2}{2} \left[\frac{\Delta + \Omega_m}{(\kappa/2)^2 + (\Delta + \Omega_m)^2} + \frac{\Delta - \Omega_m}{(\kappa/2)^2 + (\Delta - \Omega_m)^2} \right]. \quad (3.24)$$

Similarly, the effective mechanical damping rate becomes $\Gamma_{\text{eff}} = \Gamma_m + \Gamma_{\text{opt}}$, where the optomechanical backaction damping rate Γ_{opt} induced by the imaginary part of the self-energy is expressed as

$$\Gamma_{\text{opt}} = g^2 \left[\frac{\kappa/2}{(\kappa/2)^2 + (\Delta + \Omega_m)^2} - \frac{\kappa/2}{(\kappa/2)^2 + (\Delta - \Omega_m)^2} \right]. \quad (3.25)$$

Equation (3.24) demonstrates that the mechanical frequency shift is highly sensitive to both the laser detuning and the optical power, as determined by the enhanced coupling strength g . The sign of the detuning dictates whether the radiation-pressure force increases or decreases the effective spring constant of the mechanical resonator, resulting in mechanical stiffening or softening, respectively.

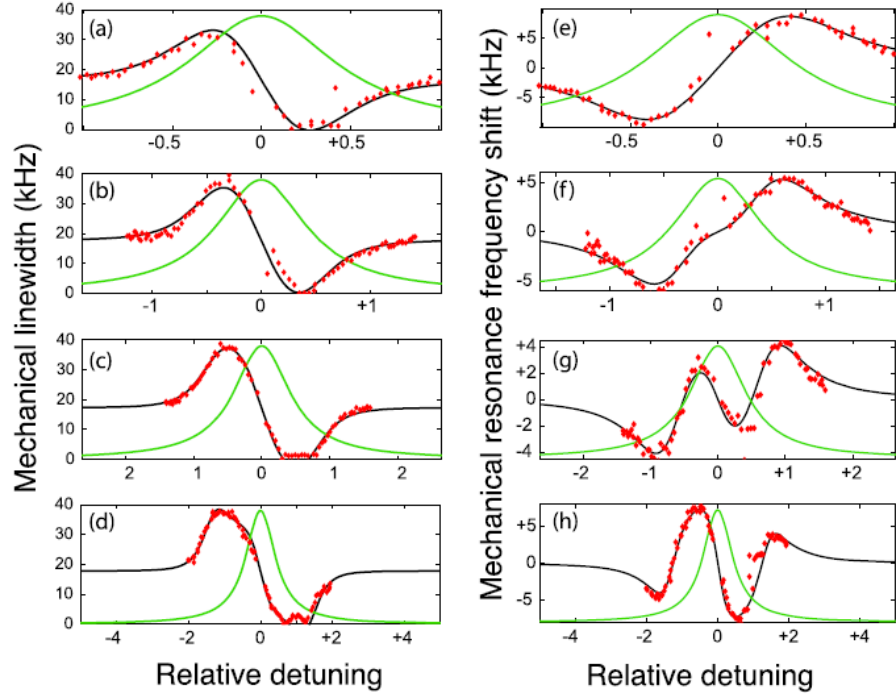


Figure 3.1: Optomechanical damping rate and optical spring effect vs. detuning and different sideband parameters, with theory (black lines) and experimental data (red dots). Left: Full mechanical damping rate $\Gamma_{\text{eff}}/2\pi$ vs. detuning Δ/κ , with decreasing values of κ/Ω_m from top to bottom ($\kappa/\Omega_m = 3.7, 2.2, 1.4, 0.7$). The regime where Γ_{eff} touches zero is the regime of dynamical instability (mechanical lasing). Right: Optical spring effect, i.e., light-induced mechanical frequency shift $\delta\Omega_m/2\pi$. Data adapted from (Schliesser and Kippenberg, 2010)[2].

Figure 3.1 presents the dependence of the optical spring effect on laser detuning. The frequency shift displays a dispersive profile around the cavity resonance, reversing sign as the detuning crosses resonance. This behaviour arises from the changing phase relationship between the mechanical motion and the radiation-pressure force. For red-detuned driving ($\Delta < 0$), the cavity extracts energy from the mechanical oscillator, resulting in positive damping and a characteristic resonance frequency shift. Conversely, blue-detuned driving ($\Delta > 0$) can induce mechanical amplification and self-oscillation if the optically induced negative damping surpasses the intrinsic mechanical losses. The most pronounced frequency shifts occur when the laser detuning is comparable to the mechanical frequency, where the interaction between motional sidebands and the cavity resonance is maximised.

Figure 3.2 offers a comprehensive overview of dynamical backaction by simultaneously illustrating the optically induced modifications in mechanical frequency and damping. The optical spring effect results from the real part of the radiation-pressure self-

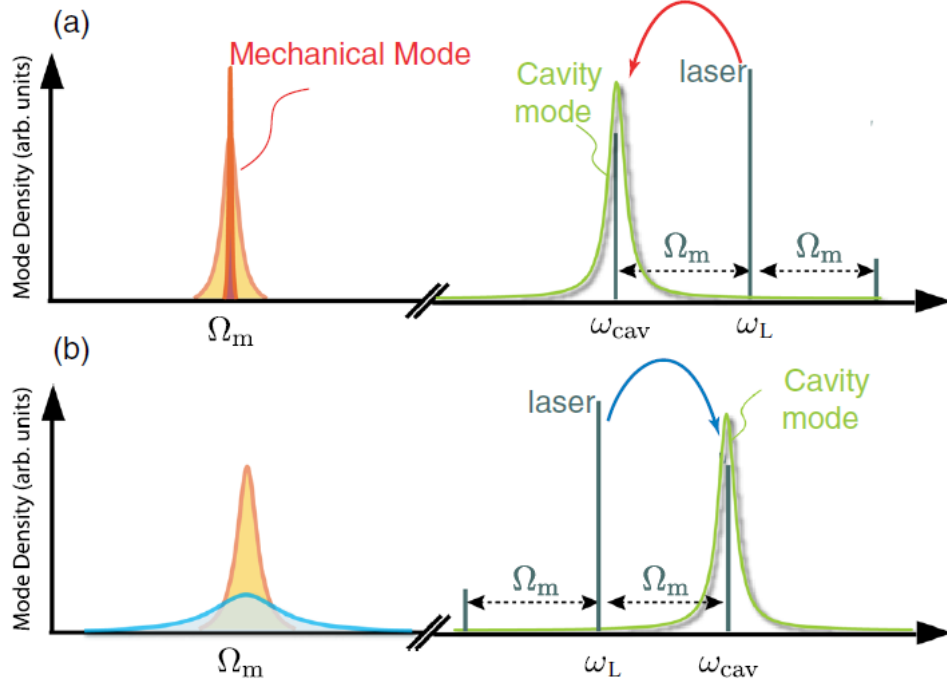


Figure 3.2: Optically induced modification of mechanical frequency and damping due to dynamical backaction. Adapted from Ref. [2].

energy, while the change in damping is governed by its imaginary part. Although both phenomena originate from the same physical interaction, they correspond to distinct quadratures of the delayed radiation-pressure force. Crucially, while the in-phase force quadrature alters the mechanical stiffness (frequency shift), the out-of-phase force component is proportional to mechanical velocity. This out-of-phase force does mechanical work on the oscillator, transferring energy between the optical cavity and mechanical motion to give rise directly to optomechanical backaction damping ($\Gamma_{\text{opt}} > 0$) or amplification ($\Gamma_{\text{opt}} < 0$).

An important observation from Fig. 3.2 is that significant frequency shifts can occur even when optomechanical damping is nearly zero. This finding indicates that changes in mechanical stiffness and mechanical energy exchange represent fundamentally distinct aspects of the radiation-pressure interaction. In the resolved-sideband regime ($\kappa \ll \Omega_m$), the extrema of the optical spring shift are located near $\Delta = \pm \Omega_m$, where the cavity preferentially enhances anti-Stokes or Stokes scattering processes. This interaction can be interpreted as an optically induced renormalisation of the mechanical susceptibility, resulting in a resonance frequency shift analogous to level repulsion in coupled oscillator systems.

Experimentally, the optical spring effect offers a robust method for controlling mechanical resonators exclusively through optical fields. By varying the laser detuning and drive

power, both the resonance frequency and damping rate can be continuously adjusted over a wide range. This technique has become indispensable in cavity optomechanics for manipulating mechanical motion, enhancing force sensitivity, and engineering mechanical dynamics. Additionally, the optical spring effect serves as a direct precursor to similar dynamical backaction phenomena observed in cavity magnomechanics. In magnomechanical systems, the magnetostrictive interaction between magnons and phonons produces a delayed force that alters the effective mechanical stiffness in a manner closely analogous to radiation-pressure coupling. Therefore, understanding the optical spring effect provides essential physical intuition for analysing the magnomechanical spring effect addressed in subsequent chapters.

3.7 State of Art

The current state of the art in cavity magnomechanics relies predominantly on macro-scale platforms utilising millimetre-sized yttrium iron garnet (YIG) spheres, as featured in recent literature [1]. In these bulk implementations, magnetostriction couples collective spin precessions (magnon mode ω_m) to macro-scale mechanical vibrations (phonon mode ω_b). However, because the physical volume (V) of these spheres is large, the zero-point motional amplitude is significantly diluted, restricting the intrinsic single-quanta vacuum coupling rate g_{mb} to the millihertz scale. Achieving operational magnomechanical cooperativity in bulk YIG setups thus requires strong microwave driving, which frequently introduces parasitic thermal noise and Kerr nonlinearities. Moreover, the bulky 3D cavity architecture intrinsically prevents scalable miniaturisation and on-chip integration.

Our work directly addresses these limitations by advancing cavity magnomechanics into a miniaturised, nanoscale on-chip platform. By drastically reducing the physical volume (V), we concentrate both the spin and elastic energy densities, boosting the zero-point strain and substantially enhancing the single-magnon–single-phonon vacuum coupling rate g_{mb} . Specifically, for a mechanical phonon mode operating at $\omega_b/2\pi = 2.35$ MHz, our nanoscale system achieves a vacuum coupling strength of $g_{mb}/2\pi = 60$ Hz which is an improvement of several orders of magnitude over the millimetre-sized spheres established in the state of the art. Beyond this enhanced interaction strength, the nanoscale on-chip architecture enables low-power operation and provides a scalable route toward integrated quantum magnomechanical circuits.

Chapter 4

Nano-magnomechanics with thin film

4.1 Device Architecture and Parameter Constraints

We consider an on-chip magnomechanical system realised on a gadolinium gallium garnet (GGG) substrate that is patterned into a flexural nanobeam operating in the cantilever configuration, with one end clamped to the bulk and the other end free to oscillate as shown in Fig. 4.1. The beam is defined by geometric dimensions L_R , w_R , and t_R , corresponding to its length, width, and thickness, respectively, with the hierarchy $L_R > w_R \gtrsim t_R$. The cross-section is assumed to be either rectangular or isosceles, consistent with standard nanofabrication processes and ensuring a well-defined mechanical mode structure.

A ferrimagnetic yttrium iron garnet (YIG) thin film is deposited on the top surface of the cantilever. The magnetic element is characterised by dimensions L_m , w_m , and t_m , satisfying $L_m \lesssim L_R$, $w_m \gg t_m$, and $t_m \ll t_R$, such that the film remains in the thin-film limit. For definiteness, we restrict to the case $w_m = w_R$, such that the magnetic layer spans the full width of the resonator; this choice simplifies fabrication and maximises mode overlap without loss of generality. The limiting case $L_m = L_R$ corresponds to full surface coverage of the cantilever, whereas for $L_m < L_R$ the magnetic film may be positioned along the beam, provided it remains entirely supported and experiences the strain field associated with the flexural motion. This positional degree of freedom allows optimisation of the magnetoelastic coupling via spatial overlap with the mechanical mode profile.

Unlike conventional monolithic magnomechanical architectures such as YIG spheres or bulk crystals where magnons and phonons are co-bound to the same geometry, this heterostructure decouples magnetic and mechanical engineering. Growing a thin YIG film on a lithographically patterned GGG substrate allows the mechanical frequency,

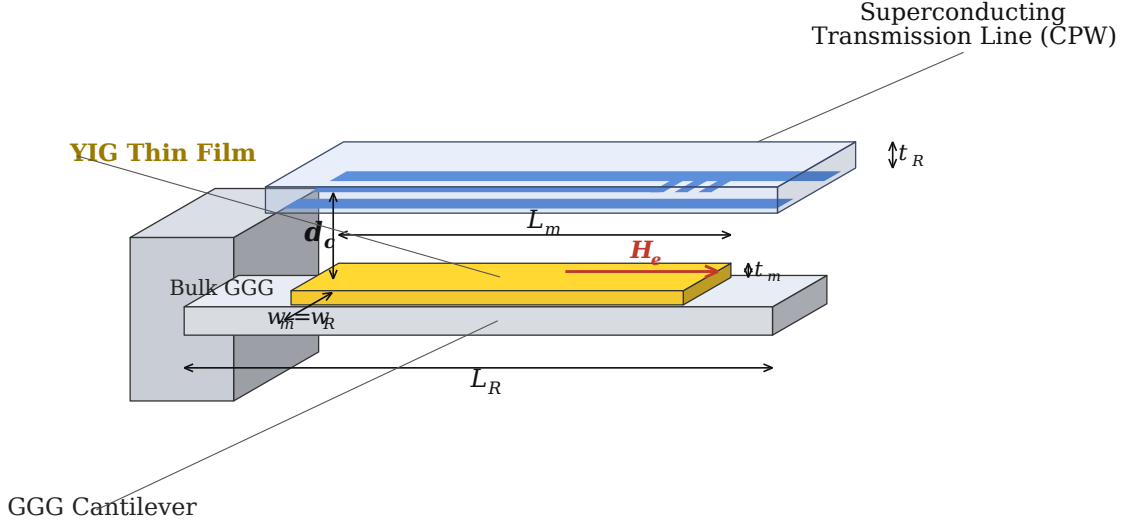


Figure 4.1: Schematic of the on-chip magnomechanical device. A YIG thin film (yellow) is grown on a GGG cantilever (grey) that is clamped to a bulk GGG support. A superconducting coplanar-waveguide (CPW) transmission line (blue) is mounted face-down above the film, separated from it by a gap d_c , so that the CPW conductors face the magnetic film. An in-plane static bias field H_e magnetises the film along its long axis. The geometric parameters of the mechanical resonator and the magnetic film are indicated.

effective mass, and mode profile to be independently engineered via standard nanofabrication, while maintaining the exceptionally low magnetic damping of thin-film YIG. Interface-mediated magnetostriction then couples the spin precession in the YIG film to the engineered flexural phonons of the GGG cantilever.

The cantilever-film composite is suspended at a vertical distance d_c above a superconducting transmission line, defined as the distance from the centre of the magnetic element to the centre conductor of the line. To establish this microscopic standoff distance d_c without mechanically contacting or damping the vibrating beam, the architecture employs a flip-chip fabrication approach. In this assembly, the chip containing the superconducting coplanar waveguide (CPW) transmission line is inverted face-down and aligned over the cantilever chip using lithographic spacers to fix the coupling gap. The transmission line is assumed to be sufficiently long compared to L_m , such that edge-induced inhomogeneities in the microwave magnetic field can be neglected and a quasi-uniform driving field may be assumed across the magnetic structure. This line provides coherent microwave excitation of the uniform precessional (Kittel) mode of the YIG film.

An external static magnetic field H_e is applied along the longitudinal axis of the cantilever, thereby defining the equilibrium magnetisation direction and ensuring saturation

of the magnetic film. In this configuration, the magnetisation dynamics are well described by the uniform mode, while the mechanical subsystem is dominated by the fundamental flexural mode of the cantilever. The resulting architecture establishes a platform in which strain-mediated magnetoelastic interactions couple the mechanical motion to collective spin excitations, forming the basis of the on-chip magnomechanical system considered in this work.

4.1.1 Design Constraints and Parameter Selection

The practical realisation of a viable magnomechanical device requires simultaneous satisfaction of multiple physical constraints that arise from the underlying theoretical frameworks. These constraints collectively define the permissible parameter space for the resonator and magnetic element dimensions.

Mechanical mode validity criterion

The Euler-Bernoulli beam theory, which underpins our description of flexural dynamics, remains valid only when the transverse dimensions are small compared to the wavelength of the mechanical mode. This imposes constraints on the wave vector k_n of the n -th mechanical mode relative to the beam width w_R and thickness t_R as

$$k_n w_R \ll 1, \quad k_n t_R \ll 1, \quad (4.1)$$

which ensures that the Euler-Bernoulli approximation remains accurate for the fundamental mode of interest.

Resolved sideband condition

Efficient magnomechanical interactions in the quantum regime require operation in the resolved sideband limit, where the mechanical frequency exceeds the cavity linewidth:

$$\frac{\kappa}{\omega_R} \ll 1, \quad (4.2)$$

where κ is the total damping rate of the microwave cavity mode.

Kittel mode quantisation condition

The Holstein-Primakoff bosonisation of the uniform magnetisation mode relies on the low-excitation approximation, which requires that the number of magnons excited in the system remain small compared to the total number of spin sites:

$$\frac{n_{\text{mag}}}{N_s} \ll 1, \quad (4.3)$$

where $n_{\text{mag}} = \langle \hat{m}^\dagger \hat{m} \rangle$ is the average magnon occupancy and $N_s = M_s V_m / (\hbar \gamma_g)$ is the total number of spin sites in the magnetic volume $V_m = L_m w_m t_m$. This condition depends nontrivially on both the magnetic volume (which determines N_s) and the resonator dimensions (which influence the magnon population through the detuning $\Delta = \omega_m - \omega_R$). Since N_s scales linearly with V_m while n_{mag} not, larger magnetic volumes are generally preferable for satisfying this criterion.

Single mechanical mode criterion

To realise a well-controlled quantum interface, the magnomechanical interaction should involve only a single mechanical mode. This requires that the frequency spacing Δ_m between adjacent mechanical modes significantly exceeds both the mechanical decay rate γ_m and the coupling strength g ($\Delta_m \gg \gamma_m, g$) to avoid parasitic multi-mode excitation.

For a uniform Euler–Bernoulli cantilever, the frequency of the first excited flexural mode is $\omega_2 \approx 6.27 \omega_R$, yielding a large mode spacing of $\Delta_m = \omega_2 - \omega_R \approx 5.27 \omega_R$. Because this separation is well over five times the fundamental frequency, intrinsic mode selectivity is exceptionally strong. Positioning the magnetic element ($L_m \leq L_R$) near the clamped support maximises coupling to the region of maximum mechanical strain while preventing parasitic overlap with higher-order modes.

Elastic interface condition.

The strain transfer from the mechanical resonator to the magnetic film requires careful consideration of the relative thicknesses. For the magnetic film to experience the strain field of the resonator without mechanical loading or mode distortion, the film thickness must be much smaller than the resonator thickness:

$$\frac{t_m}{t_R} \ll 1. \quad (4.4)$$

4.1.2 Proposed Device Parameters

Synthesising the constraints outlined above, we arrive at a consistent set of device parameters that simultaneously satisfy all physical criteria while remaining within realistic fabrication tolerances. Table 4.1 summarises the nominal dimensions and their corresponding justifications.

For these parameters, the mechanical frequency of the fundamental cantilever mode is:

$$\omega_R = \frac{\lambda_1^2}{L_R^2} \sqrt{\frac{EI}{\rho A_R}} \approx 2\pi \times 2.35 \text{ MHz}, \quad (4.5)$$

where $\lambda_1 \approx 1.875$ is the mode eigenvalue for the cantilever boundary conditions, $E \approx 2.2 \times 10^{11} \text{ N/m}^2$ is the Young’s modulus of GGG, $\rho \approx 7.1 \times 10^3 \text{ kg/m}^3$ is the

Table 4.1: Nominal device parameters satisfying the design constraints of Sec. 4.1.1.

Component	Parameter	Nominal Value	Constraint Justification
Resonator	Length L_R	12.37 μm	$\omega_R/2\pi \approx 2.35$ MHz (cantilever)
	Width w_R	400 nm	$k_n w_R < 0.15$, $w_R \gtrsim t_R$
	Thickness t_R	400 nm	$k_n t_R < 0.15$, $t_R/t_m \gg 1$
Magnetic film	Length L_m	6–12 μm	Mode overlap optimisation
	Width w_m	400 nm	$w_m = w_R$ (full coverage)
	Thickness t_m	25 nm	Thin-film limit, $t_R/t_m \approx 16$
Coupling	Gap d_c	50–100 nm	Inductive microwave coupling strength
	Field H_e	0.01–0.1 T	Kittel mode tuning

mass density, $I = w_R t_R^3/12$ is the area moment of inertia, and $A_R = w_R t_R$ is the cross-sectional area.

The Kittel mode frequency for a thin film magnetised along its plane is:

$$\omega_m = \gamma_g \sqrt{H_e(H_e + 4\pi M_s)}, \quad (4.6)$$

with saturation magnetisation $M_s \approx 140$ emu/cm³ for YIG and the gyromagnetic ratio $\gamma_g/2\pi \approx 2.8$ MHz/Oe. For $H_e = 400$ Oe (0.04 T), this yields $\omega_m/2\pi \approx 2.60$ GHz, placing the system well within the resolved sideband regime given $\omega_R/2\pi \approx 2.35$ MHz and typical cavity linewidths $\kappa/2\pi \sim 0.5$ MHz.

4.2 Quantised magnon-photon coupling

The coherent interaction between a microwave cavity mode and collective spin excitations in a ferrimagnet originates from the Zeeman coupling between the cavity magnetic field and the sample magnetisation. For a magnetic body occupying volume V_m , the interaction Hamiltonian is

$$\hat{H}_{\text{int}} = -\mu_0 \int_{V_m} \hat{M}(r) \cdot \hat{H}_c(r) d^3r, \quad (4.7)$$

where $\hat{M}(r)$ is the magnetisation operator and $\hat{H}_c(r)$ is the quantised cavity magnetic field.

We retain a single cavity mode of angular frequency ω_a described by bosonic operators $\hat{a}$ and $\hat{a}^\dagger$ with $[\hat{a}, \hat{a}^\dagger] = 1$. The magnetic field operator is written as

$$\hat{H}_c(r) = H_{\text{zpf}} h(r) (\hat{a} + \hat{a}^\dagger), \quad (4.8)$$

where $h(r)$ is a dimensionless mode profile and H_{zpf} is the zero-point magnetic-field amplitude that we will determine from energy considerations.

The magnetic energy operator of this mode is

$$\hat{W}_B = \frac{1}{2\mu_0} \int_V \hat{B}_c^2(r) d^3r = \frac{\mu_0}{2} \int_V \hat{H}_c^2(r) d^3r, \quad (4.9)$$

where $\hat{B}_c(r) = \mu_0 \hat{H}_c(r)$.

Substituting Eq. (4.8) into Eq. (4.9):

$$\begin{aligned} \hat{W}_B &= \frac{\mu_0}{2} \int_V [H_{\text{zpf}} h(r) (\hat{a} + \hat{a}^\dagger)]^2 d^3r \\ &= \frac{\mu_0 H_{\text{zpf}}^2}{2} (\hat{a} + \hat{a}^\dagger)^2 \int_V |h(r)|^2 d^3r. \end{aligned} \quad (4.10)$$

The effective magnetic mode volume can be defined as:

$$V_a \equiv \int_V |h(r)|^2 d^3r. \quad (4.11)$$

This quantity represents the spatial volume occupied by the cavity mode, weighted by the field profile.

Substituting Eq. (4.11) into Eq. (4.10):

$$\hat{W}_B = \frac{\mu_0 H_{\text{zpf}}^2 V_a}{2} (\hat{a} + \hat{a}^\dagger)^2. \quad (4.12)$$

For a single electromagnetic normal mode, the time-averaged electric and magnetic energies are equal. Therefore, the magnetic contribution must equal half of the total mode energy:

$$\begin{aligned} \hat{W}_B &= \frac{1}{2} \left[\frac{1}{2} \hbar \omega_a (\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger) \right] \\ &= \frac{1}{2} \left[\frac{1}{2} \hbar \omega_a (2\hat{a}^\dagger \hat{a} + 1) \right] \quad (\text{using } [\hat{a}, \hat{a}^\dagger] = 1) \\ &= \frac{1}{2} \hbar \omega_a \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right). \end{aligned} \quad (4.13)$$

Equating Eq. (4.12) and Eq. (4.13) to determine H_{zpf} :

$$\frac{\mu_0 H_{\text{zpf}}^2 V_a}{2} (\hat{a} + \hat{a}^\dagger)^2 = \frac{1}{2} \hbar \omega_a \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right). \quad (4.14)$$

For a harmonic oscillator, the expectation value of $(\hat{a} + \hat{a}^\dagger)^2$ in the ground state is 1. This normalisation condition yields:

$$\frac{\mu_0 H_{\text{zpf}}^2 V_a}{2} = \frac{1}{2} \hbar \omega_a \cdot \frac{1}{2}. \quad (4.15)$$

Solving for H_{zpf} :

$$\begin{aligned} \mu_0 H_{\text{zpf}}^2 V_a &= \frac{1}{2} \hbar \omega_a \\ H_{\text{zpf}}^2 &= \frac{\hbar \omega_a}{2 \mu_0 V_a} \\ H_{\text{zpf}} &= \sqrt{\frac{\hbar \omega_a}{2 \mu_0 V_a}}. \end{aligned} \quad (4.16)$$

We consider a magnetic element uniformly magnetised by a static bias field applied along $\hat{z}$, such that the relevant collective excitation corresponds to the Kittel mode.

The total spin operator $\hat{\mathbf{S}}$ satisfies the angular momentum algebra,

$$[\hat{S}_i, \hat{S}_j] = i \hbar \varepsilon_{ijk} \hat{S}_k, \quad (4.17)$$

and the magnetisation operator is related to the total spin via

$$\hat{M} = -\frac{\gamma}{V_m} \hat{S}, \quad (4.18)$$

where γ is the gyromagnetic ratio.

Introducing ladder operators to simplify the transverse components:

$$\hat{S}_\pm = \hat{S}_x \pm i \hat{S}_y. \quad (4.19)$$

Expressing $\hat{S}_x$ and $\hat{S}_y$ in terms of the ladder operators:

$$\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-), \quad (4.20)$$

$$\hat{S}_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-). \quad (4.21)$$

Decomposing the magnetisation operator using these relations:

$$\hat{M} = -\frac{\gamma}{V_m} \left[\frac{1}{2}(\hat{S}_+ + \hat{S}_-) \hat{x} + \frac{1}{2i}(\hat{S}_+ - \hat{S}_-) \hat{y} + \hat{S}_z \hat{z} \right]. \quad (4.22)$$

with

$$\begin{aligned}\hat{x} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} && (\text{Unit vector along the } x\text{-axis}) \\ \hat{y} &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} && (\text{Unit vector along the } y\text{-axis}) \\ \hat{z} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} && (\text{Unit vector along the } z\text{-axis})\end{aligned}$$

We denote by N_s the number of spins participating in the uniform mode, such that the total spin quantum number is

$$S = \frac{N_s}{2}. \quad (4.23)$$

To quantise the spin excitations, we employ the Holstein-Primakoff transformation, introducing bosonic operators $\hat{m}$ and $\hat{m}^\dagger$ satisfying

$$[\hat{m}, \hat{m}^\dagger] = 1. \quad (4.24)$$

Expressing the spin operators exactly using the Holstein-Primakoff transformation:

$$\hat{S}_+ = \hbar \sqrt{2S - \hat{m}^\dagger \hat{m}} \hat{m}, \quad (4.25)$$

$$\hat{S}_- = \hbar \hat{m}^\dagger \sqrt{2S - \hat{m}^\dagger \hat{m}}, \quad (4.26)$$

$$\hat{S}_z = \hbar (S - \hat{m}^\dagger \hat{m}). \quad (4.27)$$

Applying the low-excitation approximation. In the regime where

$$\langle \hat{m}^\dagger \hat{m} \rangle \ll 2S, \quad (4.28)$$

we linearise the square root as

$$\sqrt{2S - \hat{m}^\dagger \hat{m}} \simeq \sqrt{2S}. \quad (4.29)$$

Substituting the linearised square root into Eqs. (4.25)–(4.26):

$$\hat{S}_+ \simeq \hbar \sqrt{2S} \hat{m}, \quad (4.30)$$

$$\hat{S}_- \simeq \hbar \sqrt{2S} \hat{m}^\dagger, \quad (4.31)$$

$$\hat{S}_z = \hbar (S - \hat{m}^\dagger \hat{m}) \quad (\text{exact}). \quad (4.32)$$

Substituting Eqs. (4.30)–(4.31) into the expression for $\hat{M}_x$ from Eq. (4.22):

$$\begin{aligned}\hat{M}_x &= -\frac{\gamma}{V_m} \cdot \frac{1}{2}(\hat{S}_+ + \hat{S}_-) \\ &= -\frac{\gamma}{V_m} \cdot \frac{1}{2} \left(\hbar\sqrt{2S} \hat{m} + \hbar\sqrt{2S} \hat{m}^\dagger \right) \\ &= -\frac{\gamma\hbar}{V_m} \sqrt{\frac{S}{2}} (\hat{m} + \hat{m}^\dagger).\end{aligned}\tag{4.33}$$

Similarly, the y -component would be:

$$\hat{M}_y = i\frac{\gamma\hbar}{V_m} \sqrt{\frac{S}{2}} (\hat{m} - \hat{m}^\dagger)\tag{4.34}$$

but for the interaction with a cavity field polarised along x , we focus on $\hat{M}_x$.

Substituting the quantised cavity field [Eq. (4.8)] and the magnetisation [Eq. (4.33)] into the Zeeman interaction [Eq. (4.7)]. Note that for an x -polarised cavity field, only $\hat{M}_x$ contributes:

$$\begin{aligned}\hat{H}_{\text{int}} &= -\mu_0 \int_{V_m} \hat{M}_x(r) \cdot \hat{H}_{c,x}(r) \, d^3r \\ &= -\mu_0 \int_{V_m} \hat{M}_x(r) \cdot H_{\text{zpf}} h_x(r) (\hat{a} + \hat{a}^\dagger) \, d^3r \\ &= -\mu_0 H_{\text{zpf}} (\hat{a} + \hat{a}^\dagger) \int_{V_m} \hat{M}_x(r) h_x(r) \, d^3r.\end{aligned}\tag{4.35}$$

For the uniform (Kittel) mode, $\hat{M}_x$ is spatially constant over V_m . This allows us to take $\hat{M}_x$ out of the integral.

Defining the dimensionless overlap factor that quantifies how well the cavity field profile matches the magnetic sample:

$$\eta = \frac{1}{V_m} \int_{V_m} h_x(r) \, d^3r.\tag{4.36}$$

From this definition, we have:

$$\int_{V_m} h_x(r) \, d^3r = \eta V_m.\tag{4.37}$$

Using this result and taking $\hat{M}_x$ out of the integral, Eq. (4.35) simplifies to:

$$\hat{H}_{\text{int}} = -\mu_0 H_{\text{zpf}} \eta V_m (\hat{a} + \hat{a}^\dagger) \hat{M}_x.\tag{4.38}$$

Substituting Eq. (4.33) for $\hat{M}_x$ into Eq. (4.38):

$$\begin{aligned}\hat{H}_{\text{int}} &= -\mu_0 H_{\text{zpf}} \eta V_m (\hat{a} + \hat{a}^\dagger) \left[-\frac{\gamma \hbar}{V_m} \sqrt{\frac{S}{2}} (\hat{m} + \hat{m}^\dagger) \right] \\ &= \mu_0 H_{\text{zpf}} \eta \gamma \hbar \sqrt{\frac{S}{2}} (\hat{a} + \hat{a}^\dagger) (\hat{m} + \hat{m}^\dagger).\end{aligned}\quad (4.39)$$

Note that the negative signs cancel, yielding a positive prefactor.

Substituting the zero-point field amplitude H_{zpf} from Eq. (4.16):

$$\begin{aligned}\hat{H}_{\text{int}} &= \mu_0 \left(\sqrt{\frac{\hbar \omega_a}{2 \mu_0 V_a}} \right) \eta \gamma \hbar \sqrt{\frac{S}{2}} (\hat{a} + \hat{a}^\dagger) (\hat{m} + \hat{m}^\dagger) \\ &= \sqrt{\frac{\mu_0 \hbar \omega_a}{2 V_a}} \eta \gamma \hbar \sqrt{\frac{S}{2}} (\hat{a} + \hat{a}^\dagger) (\hat{m} + \hat{m}^\dagger).\end{aligned}\quad (4.40)$$

Substituting $S = N_s/2$ from Eq. (4.23):

$$\sqrt{\frac{S}{2}} = \sqrt{\frac{N_s/2}{2}} = \sqrt{\frac{N_s}{4}} = \frac{\sqrt{N_s}}{2}.\quad (4.41)$$

Combining all factors to express the interaction as:

$$\begin{aligned}\hat{H}_{\text{int}} &= \sqrt{\frac{\mu_0 \hbar \omega_a}{2 V_a}} \eta \gamma \hbar \cdot \frac{\sqrt{N_s}}{2} (\hat{a} + \hat{a}^\dagger) (\hat{m} + \hat{m}^\dagger) \\ &= \hbar \left[\eta \gamma \sqrt{\frac{\mu_0 \hbar \omega_a N_s}{8 V_a}} \right] (\hat{a} + \hat{a}^\dagger) (\hat{m} + \hat{m}^\dagger).\end{aligned}\quad (4.42)$$

Thus, we can write:

$$\hat{H}_{\text{int}} = \hbar g_{ma}^{(0)} (\hat{a} + \hat{a}^\dagger) (\hat{m} + \hat{m}^\dagger),\quad (4.43)$$

where the coupling prefactor is

$$g_{ma}^{(0)} = \eta \gamma \sqrt{\frac{\mu_0 \hbar \omega_a N_s}{8 V_a}}.\quad (4.44)$$

Expanding the operator product to see all interaction terms:

$$(\hat{a} + \hat{a}^\dagger)(\hat{m} + \hat{m}^\dagger) = \hat{a}\hat{m} + \hat{a}\hat{m}^\dagger + \hat{a}^\dagger\hat{m} + \hat{a}^\dagger\hat{m}^\dagger.\quad (4.45)$$

Analysing the time dependence of each term (in the interaction picture):

- $\hat{a}\hat{m}^\dagger: \propto e^{-i\omega_a t} e^{i\omega_m t} = e^{-i(\omega_a - \omega_m)t}$ (slowly rotating near resonance)
- $\hat{a}^\dagger\hat{m}: \propto e^{i\omega_a t} e^{-i\omega_m t} = e^{i(\omega_a - \omega_m)t}$ (slowly rotating near resonance)
- $\hat{a}\hat{m}: \propto e^{-i\omega_a t} e^{-i\omega_m t} = e^{-i(\omega_a + \omega_m)t}$ (rapidly rotating)
- $\hat{a}^\dagger\hat{m}^\dagger: \propto e^{i\omega_a t} e^{i\omega_m t} = e^{i(\omega_a + \omega_m)t}$ (rapidly rotating).

In the near-resonant regime where $\omega_a \approx \omega_m$, the counter-rotating terms $\hat{a}\hat{m}$ and $\hat{a}^\dagger\hat{m}^\dagger$ oscillate at approximately $2\omega_a$ and average to zero on the timescales of interest. Neglecting these under the rotating-wave approximation yields:

$$\hat{H}_{\text{int}} \simeq \hbar g_{ma} (\hat{a}\hat{m}^\dagger + \hat{a}^\dagger\hat{m}). \quad (4.46)$$

The final magnon-photon coupling rate is therefore:

$$g_{ma} = \eta \frac{\gamma}{2} \sqrt{\frac{\mu_0 \hbar \omega_a N_s}{2V_a}} \quad (4.47)$$

where we have used $g_{ma}^{(0)} = g_{ma}$ after the RWA (the factor of 2 difference comes from the fact that only half the terms in the expansion survive).

4.3 Quantised Magnon-Phonon Coupling

The uniform precession (Kittel mode) of magnetisation is quantised using the Holstein-Primakoff transformation. For a thin film magnetised in the z -direction, we introduce bosonic operators $\hat{m}$ and $\hat{m}^\dagger$ satisfying $[\hat{m}, \hat{m}^\dagger] = 1$.

The spin raising and lowering operators and transverse spin components are defined in previous section:

Relating magnetisation to spin using $\hat{\mathbf{M}} = -\gamma_g \hat{\mathbf{S}}/V_m$:

$$\hat{M}_x = -\frac{\gamma_g}{V_m} \hat{S}_x = -\frac{\gamma_g \hbar}{V_m} \sqrt{\frac{S}{2}} (\hat{m} + \hat{m}^\dagger) \quad (4.48)$$

$$\hat{M}_y = -\frac{\gamma_g}{V_m} \hat{S}_y = -\frac{\gamma_g \hbar}{V_m} \sqrt{\frac{S}{2}} \left(\frac{1}{i} \right) (\hat{m} - \hat{m}^\dagger) = i \frac{\gamma_g \hbar}{V_m} \sqrt{\frac{S}{2}} (\hat{m} - \hat{m}^\dagger). \quad (4.49)$$

Note the factor of i appears explicitly, and there is no negative sign in $\hat{M}_y$ after simplification.

The spatial profile of the magnon mode used in the quantisation procedure was verified through micromagnetic simulations conducted with OOMMF. The magnetic film was discretised into cells measuring $5 \text{ nm} \times 5 \text{ nm} \times 5 \text{ nm}$ and relaxed to its equilibrium state, accounting for exchange, magnetostatic, and Zeeman interactions. A static bias field of

$H = 400$ Oe was applied along the long axis of the film. Following relaxation, a spatially uniform transverse microwave field was applied at a frequency near the resonance of the lowest magnon mode. The temporal evolution of the magnetisation components was recorded and subjected to Fourier transformation to obtain the spectral response. The resulting resonance spectrum and spatial distribution of the dynamic magnetisation are presented in Fig. 4.2.

The simulated mode profile indicates that the lowest-energy excitation is not perfectly uniform but instead exhibits a long-wavelength standing-wave character. Based on the spatial dependence observed in the micromagnetic simulations, the mode function is approximated as

$$f(y, z) = \cos\left(\frac{\pi y}{w_m}\right) \cos\left(\frac{\pi z}{L_m}\right), \quad (4.50)$$

This function captures the dominant spatial variation of the dynamic magnetisation and serves as the foundation for the quantised description developed in the following sections.

In a thin film magnetised along z , the demagnetising factors are $N_x = 0$, $N_y = 4\pi$, $N_z = 0$. The resulting anisotropy modifies the fluctuation amplitudes. The properly normalised fluctuations become:

$$\delta\hat{m}_x = \sqrt{\frac{2\hbar\gamma_g}{M_s V_m}} \left(1 + \frac{4\pi M_s}{H_e}\right)^{-1/4} (\hat{m} + \hat{m}^\dagger) \cos\left(\frac{\pi y}{w_m}\right) \cos\left(\frac{\pi z}{L_m}\right) \quad (4.51)$$

$$\delta\hat{m}_y = i\sqrt{\frac{2\hbar\gamma_g}{M_s V_m}} \left(1 + \frac{4\pi M_s}{H_e}\right)^{1/4} (\hat{m} - \hat{m}^\dagger) \cos\left(\frac{\pi y}{w_m}\right) \cos\left(\frac{\pi z}{L_m}\right). \quad (4.52)$$

The geometric factors $\left(1 + \frac{4\pi M_s}{H_e}\right)^{\pm 1/4}$ arise from the transformation to the normal modes of precession in the anisotropic field.

Using the constraint $|\hat{\mathbf{m}}| = M_s$ to the second order:

$$\begin{aligned} |\hat{\mathbf{m}}|^2 &= (\delta\hat{m}_x)^2 + (\delta\hat{m}_y)^2 + (M_s + \delta\hat{m}_z)^2 = M_s^2 + \mathcal{O}(\delta m^3) \\ M_s^2 + 2M_s\delta\hat{m}_z + (\delta\hat{m}_x)^2 + (\delta\hat{m}_y)^2 &= M_s^2 \\ 2M_s\delta\hat{m}_z + (\delta\hat{m}_x)^2 + (\delta\hat{m}_y)^2 &= 0 \end{aligned} \quad (4.53)$$

Solving for $\delta\hat{m}_z$:

$$\delta\hat{m}_z = -\frac{1}{2M_s} [(\delta\hat{m}_x)^2 + (\delta\hat{m}_y)^2] + \mathcal{O}(\delta m^3). \quad (4.54)$$

For a beam bending along the x -direction, the displacement field $u_x(z, t)$ follows Euler-Bernoulli beam theory.

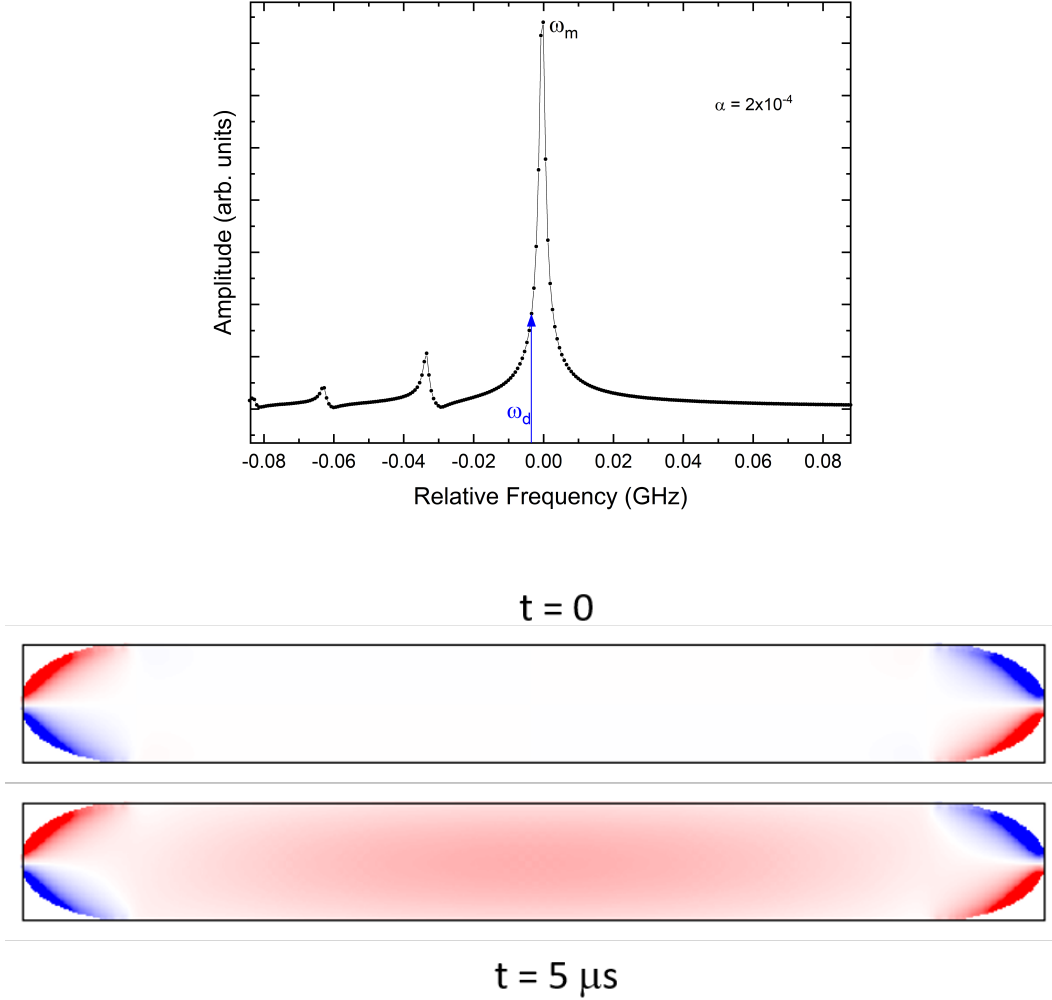


Figure 4.2: Micromagnetic simulation of the fundamental magnon mode obtained using the software Object Orientated MicroMagnetic Framework (OOMMF). (Top) Frequency spectrum obtained from the Fourier transform of the simulated magnetisation dynamics under a spatially uniform transverse microwave drive. The dominant resonance occurs at the magnon frequency ω_m , while the drive frequency ω_d is chosen close to resonance. (Bottom) Spatial distribution of the dynamic magnetisation component M_y at steady state ($t = 5 \mu s$). Red and blue indicate opposite phases of the oscillation. The mode exhibits a long-wavelength standing-wave profile and is well approximated by the cosine spatial dependence used in the analytical model.

For a mode with frequency ω_R and mode shape $\phi_n(z)$:

$$\hat{u}_x(z, t) = \sqrt{\frac{\hbar}{2\rho A_R \omega_R}} \phi_n(z) (\hat{b} e^{-i\omega_R t} + \hat{b}^\dagger e^{i\omega_R t}) \quad (4.55)$$

where ρ is the mass density, A_R is the beam cross-sectional area, and $\hat{b}$, $\hat{b}^\dagger$ are the phonon ladder operators with $[\hat{b}, \hat{b}^\dagger] = 1$.

The strain tensor components are derived from the displacement field using Euler–Bernoulli beam theory. Consider a cantilever aligned along the z -axis with flexural motion in the x -direction. Under pure bending, cross-sections remain planar and normal to the neutral axis ($x = 0$), resulting in a longitudinal displacement $\hat{u}_z = -x \frac{\partial \hat{u}_x}{\partial z}$.

Differentiating $\hat{u}_z$ along the beam axis gives the primary longitudinal strain component $\hat{u}_{zz}$:

$$\hat{u}_{zz} = \frac{\partial \hat{u}_z}{\partial z} = -x \frac{\partial^2 \hat{u}_x}{\partial z^2}. \quad (4.56)$$

This demonstrates that the strain vanishes at the neutral axis ($x = 0$) and varies linearly with the distance x .

Substituting the quantised displacement operator $\hat{u}_x(z) = \sqrt{\frac{\hbar}{2\rho A_R \omega_R}} \phi_n(z) (\hat{b} + \hat{b}^\dagger)$ into the strain expression yields:

$$\begin{aligned} \hat{u}_{zz} &= -x \frac{\partial^2}{\partial z^2} \left[\sqrt{\frac{\hbar}{2\rho A_R \omega_R}} \phi_n(z) (\hat{b} + \hat{b}^\dagger) \right] \\ &= -x \sqrt{\frac{\hbar}{2\rho A_R \omega_R}} \frac{d^2 \phi_n(z)}{dz^2} (\hat{b} + \hat{b}^\dagger). \end{aligned} \quad (4.57)$$

Furthermore, Poisson contraction dictates that longitudinal stretching induces transverse relaxation along the x - and y -axes, with $\hat{u}_{xx} = \hat{u}_{yy} = -\nu \hat{u}_{zz}$, where ν is Poisson's ratio. The non-zero normal components of the quantised strain tensor are therefore:

$$\hat{u}_{xx} = \nu x \sqrt{\frac{\hbar}{2\rho A_R \omega_R}} \frac{d^2 \phi_n(z)}{dz^2} (\hat{b} + \hat{b}^\dagger), \quad (4.58)$$

$$\hat{u}_{yy} = \nu x \sqrt{\frac{\hbar}{2\rho A_R \omega_R}} \frac{d^2 \phi_n(z)}{dz^2} (\hat{b} + \hat{b}^\dagger). \quad (4.59)$$

For a cubic crystal, the magnetoelastic energy density to lowest order in strain and magnetisation fluctuations is:

$$U_{\text{me}} = B_1 [m_x^2 u_{xx} + m_y^2 u_{yy} + m_z^2 u_{zz}] + 2B_2 [m_x m_y u_{xy} + m_y m_z u_{yz} + m_z m_x u_{zx}] \quad (4.60)$$

where B_1 and B_2 are magnetoelastic coupling coefficients.

For the Kittel mode with magnetisation primarily along z , expand around the equilibrium:

$$m_x = \delta \hat{m}_x \quad (4.61)$$

$$m_y = \delta \hat{m}_y \quad (4.62)$$

$$m_z = M_s + \delta \hat{m}_z. \quad (4.63)$$

Substituting these expansions into Eq. (4.60) and keeping terms to second order in fluctuations (first order in strain, second order in magnetisation):

$$U_{\text{me}} = B_1 \left[(\delta\hat{m}_x)^2 u_{xx} + (\delta\hat{m}_y)^2 u_{yy} + (M_s^2 + 2M_s\delta\hat{m}_z + (\delta\hat{m}_z)^2) u_{zz} \right] + 2B_2 [\delta\hat{m}_x\delta\hat{m}_y u_{xy} + (M_s + \delta\hat{m}_z)\delta\hat{m}_y u_{yz} + (M_s + \delta\hat{m}_z)\delta\hat{m}_x u_{zx}]. \quad (4.64)$$

The constant term $B_1 M_s^2 u_{zz}$ averages to zero over a complete oscillation cycle and can be dropped. For flexural modes, the shear strains u_{xy} , u_{yz} , u_{zx} are negligible compared to normal strains. Thus:

$$U_{\text{me}} \approx B_1 \left[(\delta\hat{m}_x)^2 u_{xx} + (\delta\hat{m}_y)^2 u_{yy} + 2M_s\delta\hat{m}_z u_{zz} \right]. \quad (4.65)$$

Substituting $\delta\hat{m}_z$ from Eq. (4.54) into Eq. (4.65):

$$U_{\text{me}} = B_1 \left[(\delta\hat{m}_x)^2 u_{xx} + (\delta\hat{m}_y)^2 u_{yy} + 2M_s \left(-\frac{1}{2M_s} [(\delta\hat{m}_x)^2 + (\delta\hat{m}_y)^2] \right) u_{zz} \right] = B_1 \left[(\delta\hat{m}_x)^2 u_{xx} + (\delta\hat{m}_y)^2 u_{yy} - ((\delta\hat{m}_x)^2 + (\delta\hat{m}_y)^2) u_{zz} \right]. \quad (4.66)$$

Using the Poisson ratio relations $u_{xx} = u_{yy} = -\nu u_{zz}$ from Eq. (4.59):

$$U_{\text{me}} = B_1 \left[(\delta\hat{m}_x)^2 (-\nu u_{zz}) + (\delta\hat{m}_y)^2 (-\nu u_{zz}) - ((\delta\hat{m}_x)^2 + (\delta\hat{m}_y)^2) u_{zz} \right] = -B_1 u_{zz} \left[\nu(\delta\hat{m}_x)^2 + \nu(\delta\hat{m}_y)^2 + (\delta\hat{m}_x)^2 + (\delta\hat{m}_y)^2 \right] = -B_1 (1 + \nu) (\delta\hat{m}_x^2 + \delta\hat{m}_y^2) u_{zz}. \quad (4.67)$$

This is the fundamental magnetoelastic coupling energy density.

Squaring the expressions from Eqs. (4.51) and (4.52):

$$(\delta\hat{m}_x)^2 = \frac{2\hbar\gamma_g}{M_s V_m} \left(1 + \frac{4\pi M_s}{H_e} \right)^{-1/2} (\hat{m} + \hat{m}^\dagger)^2 \cos^2 \left(\frac{\pi y}{w_m} \right) \cos^2 \left(\frac{\pi z}{L_m} \right) \quad (4.68)$$

$$\begin{aligned} (\delta\hat{m}_y)^2 &= i^2 \cdot \frac{2\hbar\gamma_g}{M_s V_m} \left(1 + \frac{4\pi M_s}{H_e} \right)^{1/2} (\hat{m} - \hat{m}^\dagger)^2 \cos^2 \left(\frac{\pi y}{w_m} \right) \cos^2 \left(\frac{\pi z}{L_m} \right) \\ &= -\frac{2\hbar\gamma_g}{M_s V_m} \left(1 + \frac{4\pi M_s}{H_e} \right)^{1/2} (\hat{m} - \hat{m}^\dagger)^2 \cos^2 \left(\frac{\pi y}{w_m} \right) \cos^2 \left(\frac{\pi z}{L_m} \right) \end{aligned} \quad (4.69)$$

Note that $i^2 = -1$ gives the negative sign explicitly.

Expanding the operator products:

$$\begin{aligned} (\hat{m} + \hat{m}^\dagger)^2 &= \hat{m}^2 + \hat{m}^{\dagger 2} + \hat{m}\hat{m}^\dagger + \hat{m}^\dagger\hat{m} \\ &= \hat{m}^2 + \hat{m}^{\dagger 2} + 2\hat{m}^\dagger\hat{m} + 1 \quad (\text{using } [\hat{m}, \hat{m}^\dagger] = 1) \end{aligned} \quad (4.70)$$

$$\begin{aligned} (\hat{m} - \hat{m}^\dagger)^2 &= \hat{m}^2 + \hat{m}^{\dagger 2} - \hat{m}\hat{m}^\dagger - \hat{m}^\dagger\hat{m} \\ &= \hat{m}^2 + \hat{m}^{\dagger 2} - 2\hat{m}^\dagger\hat{m} - 1. \end{aligned} \quad (4.71)$$

Applying the rotating wave approximation. In the interaction picture:

- $\hat{m}^\dagger \hat{m}$ is time-independent (number-conserving)
- $\hat{m}^2$ oscillates as $e^{-2i\omega_m t}$ (rapidly oscillating)
- $\hat{m}^{\dagger 2}$ oscillates as $e^{2i\omega_m t}$ (rapidly oscillating)
- The constant terms (± 1) give $\hat{H}_{\text{int}} \propto (\hat{b} + \hat{b}^\dagger)$, which is a linear drive on the mechanics. This does not couple magnon number to phonons (the dispersive coupling we seek) and averages to zero in thermal equilibrium.

Neglecting the rapidly oscillating terms $\hat{m}^2$ and $\hat{m}^{\dagger 2}$, and dropping the constant terms that don't contribute to dispersive coupling, we keep only the number-conserving terms:

$$(\hat{m} + \hat{m}^\dagger)^2 \xrightarrow{\text{RWA}} 2\hat{m}^\dagger \hat{m} \quad (4.72)$$

$$(\hat{m} - \hat{m}^\dagger)^2 \xrightarrow{\text{RWA}} -2\hat{m}^\dagger \hat{m}. \quad (4.73)$$

Substituting these RWA results back into Eqs. (4.68) and (4.69):

$$(\delta \hat{m}_x)^2 \rightarrow \frac{2\hbar\gamma_g}{M_s V_m} \left(1 + \frac{4\pi M_s}{H_e}\right)^{-1/2} (2\hat{m}^\dagger \hat{m}) \cos^2\left(\frac{\pi y}{w_m}\right) \cos^2\left(\frac{\pi z}{L_m}\right) \quad (4.74)$$

$$\begin{aligned} (\delta \hat{m}_y)^2 &\rightarrow -\frac{2\hbar\gamma_g}{M_s V_m} \left(1 + \frac{4\pi M_s}{H_e}\right)^{1/2} (-2\hat{m}^\dagger \hat{m}) \cos^2\left(\frac{\pi y}{w_m}\right) \cos^2\left(\frac{\pi z}{L_m}\right) \\ &= \frac{4\hbar\gamma_g}{M_s V_m} \left(1 + \frac{4\pi M_s}{H_e}\right)^{1/2} \hat{m}^\dagger \hat{m} \cos^2\left(\frac{\pi y}{w_m}\right) \cos^2\left(\frac{\pi z}{L_m}\right). \end{aligned} \quad (4.75)$$

Adding the x and y contributions:

$$\begin{aligned} (\delta \hat{m}_x)^2 + (\delta \hat{m}_y)^2 &\rightarrow \frac{2\hbar\gamma_g}{M_s V_m} \left[2 \left(1 + \frac{4\pi M_s}{H_e}\right)^{-1/2} + 2 \left(1 + \frac{4\pi M_s}{H_e}\right)^{1/2} \right] \hat{m}^\dagger \hat{m} \\ &\quad \times \cos^2\left(\frac{\pi y}{w_m}\right) \cos^2\left(\frac{\pi z}{L_m}\right) \\ &= \frac{4\hbar\gamma_g}{M_s V_m} \left[\left(1 + \frac{4\pi M_s}{H_e}\right)^{1/2} + \left(1 + \frac{4\pi M_s}{H_e}\right)^{-1/2} \right] \hat{m}^\dagger \hat{m} \\ &\quad \times \cos^2\left(\frac{\pi y}{w_m}\right) \cos^2\left(\frac{\pi z}{L_m}\right). \end{aligned} \quad (4.76)$$

Substituting Eq. (4.76) and the strain expression from Eq. (4.57) into the energy density Eq. (4.67). Note carefully the sign handling:

$$\begin{aligned}
 U_{\text{me}} = & -B_1(1 + \nu) \left\{ \frac{4\hbar\gamma_g}{M_s V_m} \left[\left(1 + \frac{4\pi M_s}{H_e}\right)^{1/2} + \left(1 + \frac{4\pi M_s}{H_e}\right)^{-1/2} \right] \hat{m}^\dagger \hat{m} \cos^2 \left(\frac{\pi y}{w_m} \right) \cos^2 \left(\frac{\pi z}{L_m} \right) \right\} \\
 & \times \left\{ -x \sqrt{\frac{\hbar}{2\rho A_R \omega_R}} \frac{d^2 \phi_n}{dz^2} (\hat{b} + \hat{b}^\dagger) \right\}. \tag{4.77}
 \end{aligned}$$

Simplifying by combining the negative signs (the product of the outer negative and the negative from strain gives a positive):

$$\begin{aligned}
 U_{\text{me}} = & \frac{4B_1(1 + \nu)\hbar\gamma_g}{M_s V_m} \sqrt{\frac{\hbar}{2\rho A_R \omega_R}} \left[\left(1 + \frac{4\pi M_s}{H_e}\right)^{1/2} + \left(1 + \frac{4\pi M_s}{H_e}\right)^{-1/2} \right] \\
 & \times x \cos^2 \left(\frac{\pi y}{w_m} \right) \cos^2 \left(\frac{\pi z}{L_m} \right) \frac{d^2 \phi_n}{dz^2} \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger). \tag{4.78}
 \end{aligned}$$

The total interaction Hamiltonian is obtained by integrating the energy density over the magnetic volume:

$$\hat{H}_{\text{int}} = \int_{V_m} U_{\text{me}} dV. \tag{4.79}$$

Since the magnetic film thickness t_m is much smaller than the resonator thickness t_R ($t_m \ll t_R$), the strain is approximately constant across the magnetic film thickness. We evaluate x at the film position, which is at the top surface of the resonator: $x = t_R/2$.

$$\begin{aligned}
 \hat{H}_{\text{int}} = & \frac{4B_1(1 + \nu)\hbar\gamma_g}{M_s V_m} \sqrt{\frac{\hbar}{2\rho A_R \omega_R}} \left[\left(1 + \frac{4\pi M_s}{H_e}\right)^{1/2} + \left(1 + \frac{4\pi M_s}{H_e}\right)^{-1/2} \right] \frac{t_R}{2} \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger) \\
 & \times \int_{V_m} \cos^2 \left(\frac{\pi y}{w_m} \right) \cos^2 \left(\frac{\pi z}{L_m} \right) \frac{d^2 \phi_n}{dz^2} dV \tag{4.80}
 \end{aligned}$$

Note that V_m appears both in the prefactor and in the denominator of the integral. Write $dV = dy dz dx$ and note that the integrand is independent of x :

$$\int_{V_m} \cos^2 \left(\frac{\pi y}{w_m} \right) \cos^2 \left(\frac{\pi z}{L_m} \right) \frac{d^2 \phi_n}{dz^2} dV = t_m \int_{-w_m/2}^{w_m/2} \int_{-L_m/2}^{L_m/2} \cos^2 \left(\frac{\pi y}{w_m} \right) \cos^2 \left(\frac{\pi z}{L_m} \right) \frac{d^2 \phi_n}{dz^2} dz dy \tag{4.81}$$

Factor out $V_m = w_m L_m t_m$ to create a spatial average:

$$\begin{aligned}
 \hat{H}_{\text{int}} = & \frac{4B_1(1 + \nu)\hbar\gamma_g}{M_s} \sqrt{\frac{\hbar}{2\rho A_R \omega_R}} \left[\left(1 + \frac{4\pi M_s}{H_e}\right)^{1/2} + \left(1 + \frac{4\pi M_s}{H_e}\right)^{-1/2} \right] \frac{t_R}{2} \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger) \\
 & \times \frac{1}{w_m L_m} \int_{-w_m/2}^{w_m/2} \int_{-L_m/2}^{L_m/2} \cos^2 \left(\frac{\pi y}{w_m} \right) \cos^2 \left(\frac{\pi z}{L_m} \right) \frac{d^2 \phi_n}{dz^2} dz dy \tag{4.82}
 \end{aligned}$$

To align with the beam's coordinate system where the film extends from 0 to w_m in y and 0 to L_m in z , shift origins: $y \rightarrow y + w_m/2$, $z \rightarrow z + L_m/2$. This transforms the cosine functions:

$$\cos^2\left(\frac{\pi y}{w_m}\right) \rightarrow \cos^2\left(\frac{\pi(y - w_m/2)}{w_m}\right) = \cos^2\left(\frac{\pi y}{w_m} - \frac{\pi}{2}\right) = \sin^2\left(\frac{\pi y}{w_m}\right) \quad (4.83)$$

$$\cos^2\left(\frac{\pi z}{L_m}\right) \rightarrow \cos^2\left(\frac{\pi(z - L_m/2)}{L_m}\right) = \cos^2\left(\frac{\pi z}{L_m} - \frac{\pi}{2}\right) = \sin^2\left(\frac{\pi z}{L_m}\right) \quad (4.84)$$

Introducing the dimensionless curvature $\bar{\phi}_n''(z)$ where:

$$\frac{d^2\phi_n}{dz^2} = k^2\bar{\phi}_n''(z), \quad k = \frac{2\pi}{\lambda} \quad (4.85)$$

with λ being the wavelength of the flexural mode.

The double integral becomes a spatial average over the magnetic film area:

$$\frac{1}{w_m L_m} \int_0^{w_m} \int_0^{L_m} \sin^2\left(\frac{\pi y}{w_m}\right) \sin^2\left(\frac{\pi z}{L_m}\right) k^2 \bar{\phi}_n''(z) dz dy = k^2 \left\langle \sin^2\left(\frac{\pi y}{w_m}\right) \sin^2\left(\frac{\pi z}{L_m}\right) \bar{\phi}_n''(z) \right\rangle_{S_m} \quad (4.86)$$

where $\langle \dots \rangle_{S_m}$ denotes the average over the film area $S_m = w_m L_m$.

Substituting this result back into Eq. (4.82):

$$\begin{aligned} \hat{H}_{\text{int}} = & \frac{4B_1(1+\nu)\hbar\gamma_g}{M_s} \sqrt{\frac{\hbar}{2\rho A_R \omega_R}} \left[\left(1 + \frac{4\pi M_s}{H_e}\right)^{1/2} + \left(1 + \frac{4\pi M_s}{H_e}\right)^{-1/2} \right] \frac{t_R}{2} k^2 \\ & \times \left\langle \sin^2\left(\frac{\pi y}{w_m}\right) \sin^2\left(\frac{\pi z}{L_m}\right) \bar{\phi}_n''(z) \right\rangle_{S_m} \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger) \end{aligned} \quad (4.87)$$

The interaction Hamiltonian has the standard form for dispersive coupling (coupling proportional to magnon number):

$$\hat{H}_{\text{int}} = \hbar g_{mb} \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger) \quad (4.88)$$

This differs from single magnon-phonon beam-splitter coupling ($\propto \hat{m}^\dagger \hat{b} + \hat{m} \hat{b}^\dagger$). Our derivation yields dispersive coupling because the magnetoelastic energy depends quadratically on the transverse magnetization fluctuations.

Comparing (4.87) with Eq. (4.88) to identify g_{mb} :

$$\hbar g_{mb} = \frac{4B_1(1+\nu)\hbar\gamma_g}{M_s} \sqrt{\frac{\hbar}{2\rho A_R \omega_R}} \left[\left(1 + \frac{4\pi M_s}{H_e}\right)^{1/2} + \left(1 + \frac{4\pi M_s}{H_e}\right)^{-1/2} \right] \frac{t_R}{2} k^2 \langle \dots \rangle_{S_m} \quad (4.89)$$

Cancelling $\hbar$ from both sides:

$$g_{mb} = \frac{4B_1(1+\nu)\gamma_g}{M_s} \sqrt{\frac{\hbar}{2\rho A_R \omega_R}} \left[\left(1 + \frac{4\pi M_s}{H_e}\right)^{1/2} + \left(1 + \frac{4\pi M_s}{H_e}\right)^{-1/2} \right] \frac{t_R}{2} k^2 \langle \dots \rangle_{S_m} \quad (4.90)$$

For flexural beams, the dispersion relation relates k to ω_R . For a beam with Young's modulus E , area moment of inertia I , cross-sectional area A_R , and density ρ :

$$\omega_R = k^2 \sqrt{\frac{EI}{\rho A_R}} \quad (4.91)$$

Solving for k^2 :

$$k^2 = \sqrt{\frac{\rho A_R \omega_R^2}{EI}} = \omega_R \sqrt{\frac{\rho A_R}{EI}} \quad (4.92)$$

Substituting Eq. (4.92) into Eq. (4.90):

$$g_{mb} = \frac{4B_1(1+\nu)\gamma_g}{M_s} \sqrt{\frac{\hbar}{2\rho A_R \omega_R}} \left[\left(1 + \frac{4\pi M_s}{H_e}\right)^{1/2} + \left(1 + \frac{4\pi M_s}{H_e}\right)^{-1/2} \right] \frac{t_R}{2} \left(\omega_R \sqrt{\frac{\rho A_R}{EI}} \right) \langle \dots \rangle_{S_m} \quad (4.93)$$

Expressing $\sqrt{\omega_R}$ in terms of material parameters using Eq. (4.91):

$$\sqrt{\omega_R} = k \left(\frac{EI}{\rho A_R} \right)^{1/4} \quad (4.94)$$

Substitute this back:

$$g_{mb} = \frac{4B_1(1+\nu)\gamma_g}{2\sqrt{2}M_s} \sqrt{\hbar} \left[\left(1 + \frac{4\pi M_s}{H_e}\right)^{1/2} + \left(1 + \frac{4\pi M_s}{H_e}\right)^{-1/2} \right] t_R k \left(\frac{EI}{\rho A_R} \right)^{1/4} \frac{1}{\sqrt{EI}} \langle \dots \rangle_{S_m} \quad (4.95)$$

Simplifying $\left(\frac{EI}{\rho A_R} \right)^{1/4} \frac{1}{\sqrt{EI}} = (EI)^{-1/4} (\rho A_R)^{-1/4}$:

$$g_{mb} = \frac{4B_1(1+\nu)\gamma_g}{2\sqrt{2}M_s} \sqrt{\frac{\hbar}{L_R}} \left(\frac{1}{EI \rho A_R} \right)^{1/4} t_R k \left[\left(1 + \frac{4\pi M_s}{H_e}\right)^{1/2} + \left(1 + \frac{4\pi M_s}{H_e}\right)^{-1/2} \right] \langle \dots \rangle_{S_m} \quad (4.96)$$

where we've introduced L_R (the resonator length) to make the units consistent with the spatial average.

The final dispersive magnon-phonon coupling strength is:

$$g_{mb} = \frac{4B_1(v+1)\gamma_g}{2\sqrt{2}M_s} \sqrt{\frac{\hbar}{L_R}} \left(\frac{1}{EI\rho A_R} \right)^{\frac{1}{4}} t_R k \left(\left(1 + \frac{4\pi M_s}{H_e} \right)^{\frac{1}{2}} + \left(1 + \frac{4\pi M_s}{H_e} \right)^{-\frac{1}{2}} \right) \times \left\langle \sin^2 \left(\frac{\pi y}{w_m} \right) \sin^2 \left(\frac{\pi z}{L_m} \right) \bar{\phi}_n(z)'' \right\rangle_{S_m} \quad (4.97)$$

The term $\propto \hat{m}^\dagger \hat{m}(\hat{b} + \hat{b}^\dagger)$ describes how the magnon number modifies the mechanical resonator's equilibrium position. This is distinct from beam-splitter coupling $\propto \hat{m}^\dagger \hat{b} + \hat{m} \hat{b}^\dagger$ that would describe coherent energy exchange between magnons and phonons.

Chapter 5

Nano Magnomechanical Interaction Under Red-Detuned Drive

5.1 System Hamiltonian

We consider a tripartite hybrid quantum system composed of three interacting bosonic modes: microwave cavity photons, collective spin excitations (magnons), and mechanical vibrations (phonons). The annihilation (creation) operators for the cavity, magnon, and mechanical modes are denoted by $\hat{a}$ ($\hat{a}^\dagger$), $\hat{m}$ ($\hat{m}^\dagger$), and $\hat{b}$ ($\hat{b}^\dagger$), respectively. In the laboratory frame, the total Hamiltonian of the system is given by [1]

$$\begin{aligned}\hat{H} = & \omega_a \hat{a}^\dagger \hat{a} + \omega_b \hat{b}^\dagger \hat{b} + \omega_m \hat{m}^\dagger \hat{m} + g_{ma} (\hat{a} + \hat{a}^\dagger) (\hat{m} + \hat{m}^\dagger) \\ & + g_{mb} \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger) + \epsilon_d \sqrt{2\kappa_{ex}} (\hat{a} e^{i\omega_d t} + \hat{a}^\dagger e^{-i\omega_d t}) \\ & + \epsilon_s \sqrt{2\kappa_{ex}} (\hat{a} e^{i\omega_s t} + \hat{a}^\dagger e^{-i\omega_s t}).\end{aligned}\quad (5.1)$$

The first three terms describe the free Hamiltonians of the uncoupled cavity, mechanical, and magnon modes, modelled as harmonic oscillators with resonance frequencies ω_a , ω_b , and ω_m , respectively. These terms account for the energy stored in the corresponding bosonic degrees of freedom.

The coherent magnon-photon interaction is described by $g_{ma}(\hat{a} + \hat{a}^\dagger)(\hat{m} + \hat{m}^\dagger)$, where g_{ma} denotes the coupling strength arising from the magnetic dipole interaction between the cavity microwave field and the collective spin excitations. Since the Hamiltonian is written in the laboratory frame, both rotating and counter-rotating terms are retained.

The magnon-phonon interaction is captured by the term $g_{mb}\hat{m}^\dagger\hat{m}(\hat{b}+\hat{b}^\dagger)$, which originates from magnetostrictive coupling and has the form of a radiation-pressure interaction [2].

The remaining terms describe coherent microwave drives applied to the cavity mode. A strong drive at frequency ω_d and amplitude ϵ_d populates the cavity, while a weaker

probe at frequency ω_s and amplitude ϵ_s with $\epsilon_s \ll \epsilon_d$ is used to probe the system response. The parameter κ_{ex} denotes the external cavity coupling rate. The dynamics are simplified by eliminating the explicit time dependence due to the coherent driving field through a transformation to a frame rotating at the drive frequency ω_d . This transformation is implemented via the unitary operator.

$$\hat{U}(t) = \exp[i\omega_d t (\hat{a}^\dagger \hat{a} + \hat{m}^\dagger \hat{m})]. \quad (5.2)$$

Under this transformation, the Hamiltonian in the rotating frame is given by

$$\hat{H}' = \hat{U} \hat{H} \hat{U}^\dagger + i \frac{d\hat{U}}{dt} \hat{U}^\dagger$$

After applying the unitary transformation and neglecting the weak signal field whose intensity is much smaller than that of the drive field, the transformed Hamiltonian becomes

$$\begin{aligned} \hat{H}' = & \Delta_a \hat{a}^\dagger \hat{a} + \omega_b \hat{b}^\dagger \hat{b} + \Delta_m \hat{m}^\dagger \hat{m} \\ & + g_{ma} (\hat{a} \hat{m} e^{-2i\omega_d t} + \hat{a} \hat{m}^\dagger + \hat{a}^\dagger \hat{m} + \hat{a}^\dagger \hat{m}^\dagger e^{2i\omega_d t}) \\ & + g_{mb} \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger) + \epsilon_d \sqrt{2\kappa_{ex}} (\hat{a} + \hat{a}^\dagger), \end{aligned} \quad (5.3)$$

where $\Delta_a = \omega_a - \omega_d$ and $\Delta_m = \omega_m - \omega_d$ denote the detunings of the cavity and magnon modes from the drive frequency, respectively.

The transformed Hamiltonian still contains rapidly oscillating terms proportional to $e^{\pm 2i\omega_d t}$. These terms average out over timescales relevant to the system dynamics and can be neglected within the rotating-wave approximation. Applying the RWA

$$\hat{a} \rightarrow \hat{a} e^{-i\omega_d t}, \quad (5.4)$$

$$\hat{a}^\dagger \rightarrow \hat{a}^\dagger e^{i\omega_d t}, \quad (5.5)$$

$$\hat{m} \rightarrow \hat{m} e^{-i\omega_d t}, \quad (5.6)$$

$$\hat{m}^\dagger \rightarrow \hat{m}^\dagger e^{i\omega_d t}. \quad (5.7)$$

yields the simplified Hamiltonian

$$\begin{aligned} \hat{H}_{\text{RWA}} = & \Delta_a \hat{a}^\dagger \hat{a} + \omega_b \hat{b}^\dagger \hat{b} + \Delta_m \hat{m}^\dagger \hat{m} \\ & + g_{ma} (\hat{a} \hat{m}^\dagger + \hat{a}^\dagger \hat{m}) + g_{mb} \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger) \\ & + \epsilon_d \sqrt{2\kappa_{ex}} (\hat{a} + \hat{a}^\dagger). \end{aligned} \quad (5.8)$$

5.2 Derivation of the Equations of Motion

The time evolution of each mode can be determined by the modified Heisenberg equation of motion

$$\dot{\mathcal{O}} = \frac{1}{i\hbar} [\mathcal{O}, \hat{H}] - \kappa \mathcal{O} + \sqrt{\kappa} \mathcal{O}_{in}, \quad (5.9)$$

where $\mathcal{O} = \hat{a}, \hat{m}, \hat{b}$, providing a set of coupled equations that we solve for steady state with the total cavity mode decay rate $\kappa = \kappa_{ex} + \kappa_i$ that includes the decay into the coupling port as well as the intrinsic decay rate κ_i , κ_m the magnon decay rate and κ_b the phonon decay rate.

The equation of motion for $\hat{a}$ is:

$$\begin{aligned} \dot{\hat{a}} &= \frac{1}{i\hbar} [\hat{a}, H] - \kappa_a \hat{a} - i\epsilon_d \sqrt{2\kappa_{ex}} + \sqrt{2\kappa_{ex}} \hat{a}_{in} \\ &= -i\Delta_a [\hat{a}, \hat{a}^\dagger \hat{a}] - ig_{ma} [\hat{a}, (\hat{a} \hat{m}^\dagger + \hat{a}^\dagger \hat{m})] - \kappa_a \hat{a} - i\epsilon_d \sqrt{2\kappa_{ex}} + \sqrt{2\kappa_{ex}} \hat{a}_{in} \\ &= -i\Delta_a \hat{a} - ig_{ma} \hat{m} - \kappa_a \hat{a} - i\epsilon_d \sqrt{2\kappa_{ex}} + \sqrt{2\kappa_{ex}} \hat{a}_{in} \\ &= -[i\Delta_a + \kappa_a] \hat{a} - ig_{ma} \hat{m} - i\epsilon_d \sqrt{2\kappa_{ex}} + \sqrt{2\kappa_{ex}} \hat{a}_{in} \end{aligned}$$

The equation of motion for $\hat{b}$ is:

$$\begin{aligned} \dot{\hat{b}} &= \frac{1}{i\hbar} [\hat{b}, H] - \kappa_b \hat{b} + \sqrt{2\kappa_{ex}} \hat{b}_{in} \\ &= -i\omega_b [\hat{b}, \hat{b}^\dagger \hat{b}] - ig_{mb} [\hat{b}, (\hat{b} + \hat{b}^\dagger)] \hat{m}^\dagger \hat{m} - \kappa_b \hat{b} + \sqrt{2\kappa_{ex}} \hat{b}_{in} \\ &= -i\omega_b \hat{b} - ig_{mb} \hat{m}^\dagger \hat{m} - \kappa_b \hat{b} + \sqrt{2\kappa_{ex}} \hat{b}_{in} \\ &= -(i\omega_b + \kappa_b) \hat{b} - ig_{mb} \hat{m}^\dagger \hat{m} + \sqrt{2\kappa_{ex}} \hat{b}_{in} \end{aligned}$$

The equation of motion for $\hat{m}$ is:

$$\begin{aligned} \dot{\hat{m}} &= \frac{1}{i\hbar} [\hat{m}, H] - \kappa_m \hat{m} + \sqrt{2\kappa_{ex}} \hat{m}_{in} \\ &= -i\Delta_m [\hat{m}, \hat{m}^\dagger \hat{m}] - ig_{ma} [\hat{m}, (\hat{a} \hat{m}^\dagger + \hat{a}^\dagger \hat{m})] - ig_{mb} [\hat{m}, \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger)] - \kappa_m \hat{m} + \sqrt{2\kappa_{ex}} \hat{m}_{in} \\ &= -i\Delta_m \hat{m} - ig_{ma} \hat{a} - ig_{mb} \hat{m} (\hat{b} + \hat{b}^\dagger) - \kappa_m \hat{m} + \sqrt{2\kappa_{ex}} \hat{m}_{in} \\ &= -i(\Delta_m + \kappa_m) \hat{m} - ig_{ma} \hat{a} - ig_{mb} \hat{m} (\hat{b} + \hat{b}^\dagger) + \sqrt{2\kappa_{ex}} \hat{m}_{in} \end{aligned}$$

5.3 Steady-state solutions and mean-field magnon

Under the strong coherent drive, the system is linearised by expanding the mode operators around their classical averages:

$$\hat{a} = \langle a \rangle + \delta a \quad (5.10)$$

$$\hat{m} = \langle m \rangle + \delta m \quad (5.11)$$

$$\hat{b} = \langle b \rangle + \delta b. \quad (5.12)$$

where $\langle a \rangle, \langle m \rangle, \langle b \rangle$ are the classical averages, and $\delta a, \delta m, \delta b$ are the small quantum fluctuations

Then the equations for the classical averages in the steady state are:

$$0 = (-i\Delta_a - \kappa_a)\langle a \rangle - ig_{ma}\langle m \rangle - i\epsilon_d\sqrt{2\kappa_{ex}} \quad (5.13)$$

$$0 = (-i\Delta_m - \kappa_m)\langle m \rangle - ig_{ma}\langle a \rangle - ig_{mb}(\langle b \rangle + \langle b^* \rangle)\langle m \rangle \quad (5.14)$$

$$0 = (-i\omega_b - \kappa_b)\langle b \rangle - ig_{mb}|\langle m \rangle|^2. \quad (5.15)$$

The mean field magnon population can be calculated by solving the coupled equations:

$$0 = (-i\omega_b - \kappa_b)\langle b \rangle - ig_{mb}|\langle m \rangle|^2.$$

Solving for the steady-state values, we obtain

$$\langle b \rangle = \frac{-ig_{mb}|\langle m \rangle|^2}{i\omega_b + \kappa_b}, \quad (5.16)$$

$$\langle b^* \rangle = \frac{ig_{mb}|\langle m \rangle|^2}{-i\omega_b + \kappa_b}. \quad (5.17)$$

$$\langle b \rangle + \langle b^* \rangle = g_{mb}|\langle m \rangle|^2 \left(\frac{-i}{i\omega_b + \kappa_b} + \frac{i}{-i\omega_b + \kappa_b} \right). \quad (5.18)$$

$$= \frac{2g_{mb}\omega_b}{\omega_b^2 + \kappa_b^2}|\langle m \rangle|^2.$$

Substituting (23) into the magnon steady-state equation (28), we finally read,

$$0 = (-i\Delta_m - \kappa_m)\langle m \rangle - ig_{ma}\langle a \rangle - ig_{mb}\langle m \rangle \frac{2g_{mb}\omega_b}{\omega_b^2 + \kappa_b^2}|\langle m \rangle|^2.$$

$$\langle m \rangle = \frac{-ig_{ma}\langle a \rangle}{i\Delta_m + \kappa_m + i\frac{2g_{mb}^2\omega_b}{\omega_b^2 + \kappa_b^2}|\langle m \rangle|^2}. \quad (5.19)$$

Using the approximation that $\frac{2g_{mb}^2|\langle m \rangle|^2}{\omega_b} < \kappa_m \ll \omega_m$ so that the phonon-induced shift of the magnon frequency is small compared to the magnon frequency and linewidth, we can write the average number of steady-state magnons.

$$|\langle m \rangle|^2 = \frac{g_{ma}^2|\langle a \rangle|^2}{\Delta_m^2 + \kappa_m^2} \quad (5.20)$$

In Appendix 1, we will prove that these approximations hold true. Substituting $\langle m \rangle$ into the steady-state solution of $\hat{a}$ given by equation (27):

$$0 = (-i\Delta_a - \kappa_a)\langle a \rangle - ig_{ma}\frac{-ig_{ma}\langle a \rangle}{i\Delta_m + \kappa_m} + \epsilon_d\sqrt{2\kappa_{ex}} \quad (5.21)$$

$$\left[i\Delta_a + \kappa_a + \frac{g_{ma}^2}{i\Delta_m + \kappa_m} \right] \langle a \rangle = \epsilon_d\sqrt{2\kappa_{ex}}$$

$$\langle a \rangle = \frac{\epsilon_d\sqrt{2\kappa_{ex}}}{\left[i\Delta_a + \kappa_a + \frac{g_{ma}^2}{i\Delta_m + \kappa_m} \right]}$$

which simplifies to:

$$\langle a \rangle = \frac{\epsilon_d\sqrt{2\kappa_{ex}}(i\Delta_m + \kappa_m)}{(i\Delta_a + \kappa_a)(i\Delta_m + \kappa_m) + g_{ma}^2} \quad (5.22)$$

The theoretical framework presented in this chapter utilises a linearised description of the magnomechanical system. Accordingly, the validity of the derived interaction Hamiltonian, dynamical equations, and cooperativity analysis depends on several established approximations and specific operating conditions. Notably, the rotating-wave approximation (RWA) is applied, which is justified when the magnomechanical coupling strength is significantly smaller than the bare mode frequencies.

$$g_{ma} \ll \omega_m, \omega_a, \quad (5.23)$$

and when the drive frequency satisfies

$$|\omega_{m,a} - \omega_d| \ll |\omega_{m,a} + \omega_d|. \quad (5.24)$$

Under these conditions, the rapidly oscillating counter-rotating terms average to zero over the relevant timescales and thus do not contribute significantly to the system dynamics.

Additionally, the spin ensemble is assumed to operate in the low-excitation regime, in which the number of excited magnons is much smaller than the total available spin angular momentum. This condition can be expressed as

$$|\langle m \rangle|^2 \ll 2NS, \quad (5.25)$$

where N denotes the total number of spins and $S = 5/2$. This condition ensures the validity of the Holstein–Primakoff transformation and allows the collective spin excitations to be described as bosonic magnon modes.

The analysis further assumes that nonlinear magnon effects are negligible. Specifically, the frequency shift resulting from the magnon self-Kerr nonlinearity must remain small compared to the magnon linewidth. This requirement is quantified by

$$K_m n_m \lesssim \frac{\kappa_m}{2}, \quad (5.26)$$

where K_m denotes the magnon self-Kerr coefficient, n_m is the intracavity magnon population, and κ_m is the magnon energy decay rate. This condition ensures that Kerr-induced frequency shifts do not significantly perturb the linear magnon response. Finally, the weak-coupling condition

$$g_{ma} \ll \omega_m \quad (5.27)$$

must be satisfied, ensuring that the interaction constitutes only a small perturbation to the uncoupled magnon and phonon modes. Collectively, these conditions define the parameter regime in which the linearised magnomechanical theory is valid and form the basis for the analysis presented in this chapter.

The parameter choices in Table 5.1 are directly motivated by and consistent with the physical device architecture designed in Chapter 4. The mechanical frequency $\omega_b/2\pi = 2.35$ MHz corresponds to the fundamental flexural mode of the GGG cantilever ($12.37 \mu\text{m} \times 400 \text{ nm} \times 400 \text{ nm}$). The YIG film volume $V = 1.0 \times 10^{-19} \text{ m}^3$ derives from the thin-film dimensions ($10 \mu\text{m} \times 400 \text{ nm} \times 25 \text{ nm}$), yielding a total spin count of $N = \rho V = 4.22 \times 10^8$ for the spin density $\rho = 4.22 \times 10^{27} \text{ m}^{-3}$. The single-spin-phonon coupling strength $g_{mb} = 65$ Hz reflects the integrated strain overlap across this beam geometry. Crucially, the magnon-photon coupling rate is chosen as $g_{ma} = 1.175$ MHz, which satisfies the condition $2g_{ma} = \omega_b$. This design choice matches the polariton mode splitting directly to the mechanical sideband frequency, establishing optimal conditions for cavity-enhanced backaction.

Parameter	Value	Unit
$\hbar$	1.054×10^{-34}	J·s
M_s	140	emu/cm ³
γ_c	2.80×10^6	Hz Oe ⁻¹
ω_b	2.35×10^6	Hz
g_{ma}	1.175×10^6	Hz
κ_a	5.0×10^5	Hz
κ_{ex}	2.5×10^4	Hz
κ_b	1.0×10^3	Hz
g_{mb}	65	Hz
K_m	0.2	Hz
κ_m	2.604×10^5	Hz
ρ	4.22×10^{27}	m ⁻³
V	1.0×10^{-19}	m ³
$N = \rho V$	4.22×10^8	—

Table 5.1: System parameters used in the simulation.

Figures 5.1–5.3 summarize the steady-state response of the coupled cavity–magnon system, collectively illustrating how coherent driving and magnetic tuning influence excitation strength and the emergence of nonlinear effects.

Figure 5.1 presents the magnon population n_m as a function of detuning Δ_a/ω_b and external magnetic field H . A pronounced resonance ridge is observed near $H \approx 400$ Oe, corresponding to maximal hybridisation between the cavity and magnon modes. In this regime, efficient energy exchange results in a substantial increase in n_m . Outside the resonance region, the response diminishes rapidly due to detuning-induced suppression of cavity-mediated excitation. Notably, the overall behaviour remains smooth, with no evidence of bistability or abrupt spectral distortion. This observation indicates that the system operates within a weakly nonlinear regime, where the Kerr-induced frequency shift is negligible relative to the magnon linewidth.

To investigate dynamical backaction while maintaining operation within the linear limit, two specific drive frequencies are examined: $\omega_d = \omega_a - 0.5\omega_b$ and $\omega_d = \omega_a - 1.5\omega_b$. On resonance ($\omega_m = \omega_a$), the magnon-photon interaction splits the uncoupled resonances into upper and lower polariton branches centred at $\omega_{\pm} = \omega_a \pm g_{ma}$. Because $2g_{ma} = \omega_b$, the polariton eigenmodes are located at $\omega_{\pm} = \omega_a \pm 0.5\omega_b$.

Figures 5.2 and 5.3 provide a quantitative analysis of nonlinear effects under varying driving conditions. In Figure 5.2, for a fixed drive $\omega_d = \omega_a - 0.5\omega_b$, the drive directly matches the lower polariton resonance ω_- . This directly enhances the magnon population n_m to provide significant backaction at low input power. However, because the mode is directly driven on resonance, the nonlinear threshold $K_m n_m / \kappa_m \lesssim 0.5$ is reached at a relatively modest power of 0.15 nW.

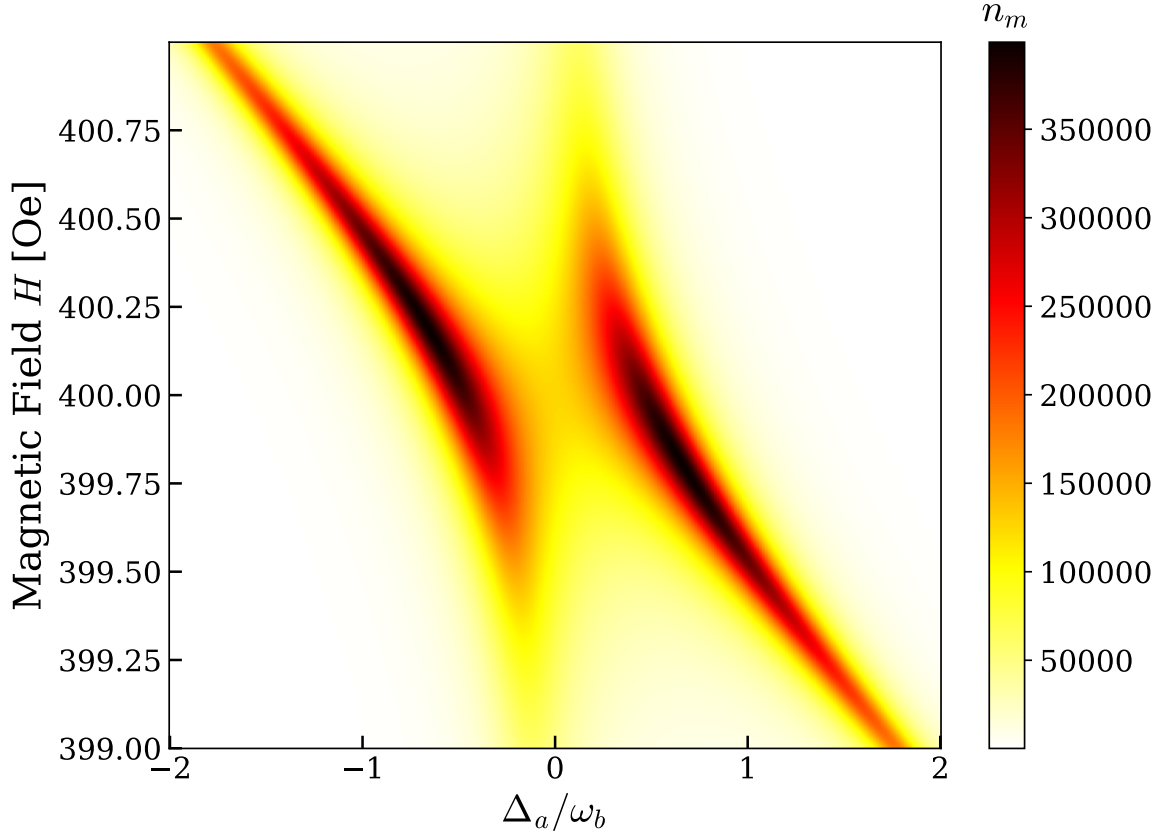


Figure 5.1: Magnon–photon hybridisation in the linear regime. (a) Two-dimensional map of the magnon population as a function of probe frequency and magnetic field at $P = 1 \text{ nW}$ with $g_{\text{ma}} \neq 0$. (b) Frequency cut at $H = 400 \text{ Oe}$, showing the emergence of two hybridised modes.

A direct comparison with Fig. 5.3 reveals a key difference in the accessible operating range. Under the alternative driving condition $\omega_d = \omega_a - 1.5\omega_b$, the drive is red-detuned by exactly one mechanical frequency from the lower polariton mode ($\omega_d = \omega_- - \omega_b$). This places the anti-Stokes scattering sideband ($\omega_d + \omega_b = \omega_-$) directly on the polariton resonance, enhancing mechanical cooling via cavity filtering while keeping direct magnon drive off-resonant. Consequently, the system sustains a larger magnon population at an elevated power of 0.3 nW before nonlinear saturation becomes significant. This improvement results from a reduced effective backaction of the cavity field on the magnon mode, which suppresses Kerr-induced frequency pulling. Consequently, the nonlinear ratio increases more gradually with excitation, enabling a broader stable operating window.

Collectively, the three figures demonstrate that although the system exhibits strong resonant enhancement of magnon excitation, the nonlinear frequency shift remains below

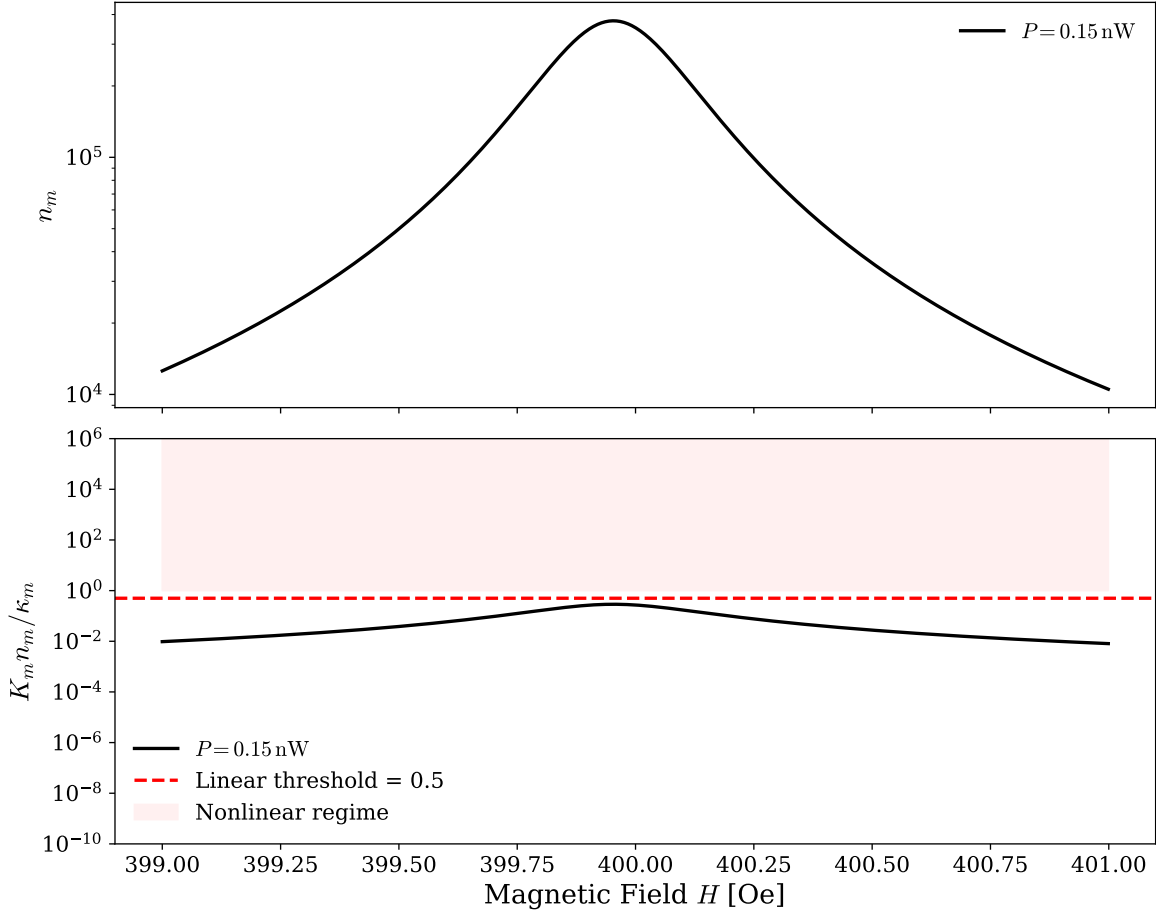


Figure 5.2: Linearity analysis of the driven magnon mode for a fixed drive frequency $\omega_d = \omega_a - 0.5\omega_b$ and input power $P = 0.15$ nW. Top panel: Steady-state magnon number n_m as a function of the external magnetic field H . Bottom panel: Corresponding nonlinear ratio $K_m n_m / \kappa_m$ versus H , where the dashed horizontal line denotes the linearity threshold. The obtained values remain well below the threshold over the considered magnetic-field range, confirming the validity of the linearised treatment.

half of the magnon linewidth in all examined regimes. This observation establishes a clear distinction between the linear response regime and the onset of nonlinear behaviour and underscores the significance of drive detuning in controlling both stability and the maximum achievable excitation.

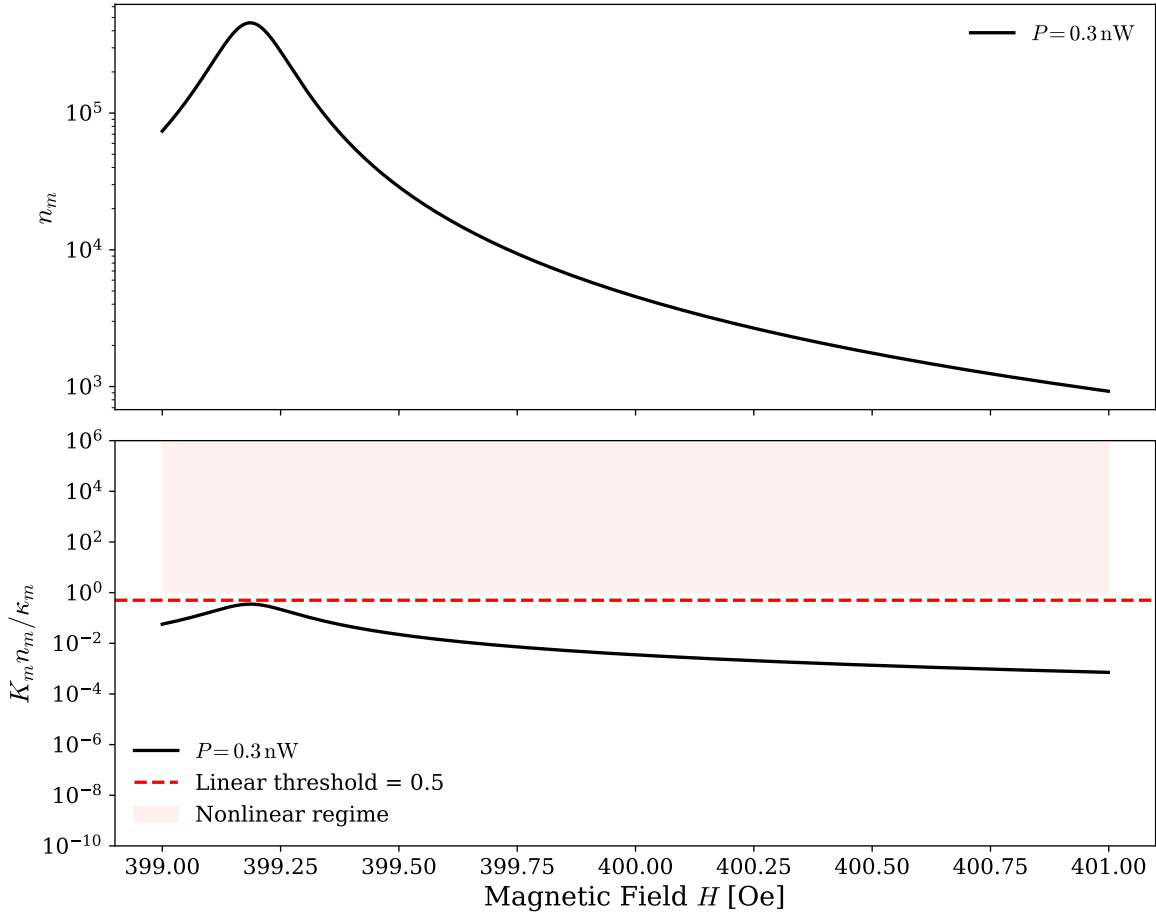


Figure 5.3: Linearity analysis of the driven magnon mode for a fixed drive frequency $\omega_d = \omega_a - 1.5\omega_b$ and input power $P = 0.3 \text{ nW}$. Top panel: Steady-state magnon number n_m as a function of the external magnetic field H . Bottom panel: Corresponding nonlinear ratio $K_m n_m / \kappa_m$ vs. H , where the dashed horizontal line denotes the linearity threshold. The obtained values remain well below the threshold over the considered magnetic-field range, confirming the validity of the linearised treatment.

5.4 Linearisation about the steady state and fluctuation dynamics

We consider the coupled Heisenberg-Langevin equations.

$$\dot{\hat{a}} = -(i\Delta_a + \kappa_a) \hat{a} - ig_{ma} \hat{m} + \sqrt{2\kappa_{ex}} \hat{a}_{in} \quad (5.28)$$

$$\dot{\hat{b}} = -(i\omega_b + \kappa_b) \hat{b} - ig_{mb} \hat{m}^\dagger \hat{m} + \sqrt{2\kappa_{ex}} \hat{b}_{in} \quad (5.29)$$

$$\dot{\hat{m}} = -(i\Delta_m + \kappa_m) \hat{m} - ig_{ma} \hat{a} - ig_{mb} \hat{m} (\hat{b} + \hat{b}^\dagger) + \sqrt{2\kappa_{ex}} \hat{m}_{in}. \quad (5.30)$$

5.4. LINEARISATION ABOUT THE STEADY STATE AND FLUCTUATION DYNAMICS

The nonlinearities arise from the radiation-pressure-type terms $\hat{m}^\dagger \hat{m}$ and $\hat{m}(\hat{b} + \hat{b}^\dagger)$. In typical experiments the system is operated around a stable steady state set by a strong coherent drive; the quantum dynamics of interest are then carried by small fluctuations about this operating point. We therefore linearise the equations of motion about the steady state.

We decompose each intracavity operator into a (possibly complex) steady-state mean field and a zero-mean fluctuation operator:

$$\hat{a}(t) = a + \delta\hat{a}(t), \quad \hat{b}(t) = b + \delta\hat{b}(t), \quad \hat{m}(t) = m + \delta\hat{m}(t), \quad (5.31)$$

and similarly for the input field,

$$\hat{a}_{\text{in}}(t) = a_{\text{in}} + \delta\hat{a}_{\text{in}}(t). \quad (5.32)$$

Here $a = \langle \hat{a} \rangle$, $b = \langle \hat{b} \rangle$, and $m = \langle \hat{m} \rangle$ are c -numbers determined by the steady-state equations (derived in the previous section), while the fluctuation operators satisfy $\langle \delta\hat{a} \rangle = \langle \delta\hat{b} \rangle = \langle \delta\hat{m} \rangle = 0$ by construction.

Substituting (5.31)–(5.32) into (5.28)–(5.30) generates terms of different order in the fluctuation operators. The linearised theory keeps all contributions up to first order in $(\delta\hat{a}, \delta\hat{b}, \delta\hat{m})$ and neglects second-order products such as $\delta\hat{m}^\dagger \delta\hat{m}$ and $\delta\hat{m} \delta\hat{b}$. Formally, this corresponds to an expansion around the operating point in which the fluctuation amplitudes are assumed small compared to the classical steady amplitudes; physically, it is justified when the system is strongly driven so that the mean fields dominate and quantum fluctuations provide only a weak perturbation.

The two nonlinear combinations in (5.29)–(5.30) are expanded as follows.

Using $\hat{m} = m + \delta\hat{m}$,

$$\begin{aligned} \hat{m}^\dagger \hat{m} &= (m^* + \delta\hat{m}^\dagger)(m + \delta\hat{m}) \\ &= |m|^2 + m^* \delta\hat{m} + m \delta\hat{m}^\dagger + \underbrace{\delta\hat{m}^\dagger \delta\hat{m}}_{\mathcal{O}(\delta^2)}. \end{aligned} \quad (5.33)$$

Neglecting the $\mathcal{O}(\delta^2)$ term yields the linearised approximation

$$\hat{m}^\dagger \hat{m} \approx |m|^2 + m^* \delta\hat{m} + m \delta\hat{m}^\dagger. \quad (5.34)$$

Similarly,

$$\begin{aligned} \hat{m}(\hat{b} + \hat{b}^\dagger) &= (m + \delta\hat{m}) \left[(b + b^*) + (\delta\hat{b} + \delta\hat{b}^\dagger) \right] \\ &= m(b + b^*) + m(\delta\hat{b} + \delta\hat{b}^\dagger) + (b + b^*)\delta\hat{m} + \underbrace{\delta\hat{m}(\delta\hat{b} + \delta\hat{b}^\dagger)}_{\mathcal{O}(\delta^2)}. \end{aligned} \quad (5.35)$$

Dropping the $\mathcal{O}(\delta^2)$ term gives

$$\dot{\hat{m}}(\hat{b} + \hat{b}^\dagger) \approx m(b + b^*) + m(\delta\hat{b} + \delta\hat{b}^\dagger) + (b + b^*)\delta\hat{m}. \quad (5.36)$$

We now substitute the decompositions and the linearised products (5.34)–(5.36) into (5.28)–(5.30). Since the steady-state equations have already been solved, we subtract the steady contributions (zeroth order in fluctuations) and retain only first-order fluctuation terms.

Inserting $\hat{a} = a + \delta\hat{a}$ and $\hat{m} = m + \delta\hat{m}$ into (5.28) and removing the steady-state part yields

$$\dot{\delta\hat{a}} = -(i\Delta_a + \kappa_a)\delta\hat{a} - ig_{ma}\delta\hat{m} + \sqrt{2\kappa_{ex}}\delta\hat{a}_{\text{in}}. \quad (5.37)$$

Using (5.34) in (5.29) and subtracting the steady equation gives

$$\dot{\delta\hat{b}} = -(i\omega_b + \kappa_b)\delta\hat{b} - ig_{mb}(m^*\delta\hat{m} + m\delta\hat{m}^\dagger). \quad (5.38)$$

Substituting (5.36) into (5.30) gives the terms linear in $\delta\hat{m}$

$$-(i\Delta_m + \kappa_m)\delta\hat{m} - ig_{mb}(b + b^*)\delta\hat{m} = -\left[i\left(\Delta_m + g_{mb}(b + b^*)\right) + \kappa_m\right]\delta\hat{m}. \quad (5.39)$$

This motivates the definition of the *effective* (steady-state shifted) magnon detuning

$$\Delta'_m \equiv \Delta_m + g_{mb}(b + b^*). \quad (5.40)$$

Physically, $b + b^*$ is proportional to the mean mechanical displacement (a static “position” quadrature), so Δ'_m expresses a dispersive frequency pull of the magnon mode by the static mechanical shift, completely analogous to the static radiation-pressure spring shift in cavity optomechanics[2]. After subtracting the steady-state equation, the remaining first-order dynamics becomes

$$\dot{\delta\hat{m}} = -(i\Delta'_m + \kappa_m)\delta\hat{m} - ig_{ma}\delta\hat{a} - ig_{mb}m(\delta\hat{b} + \delta\hat{b}^\dagger). \quad (5.41)$$

Collecting (5.37), (5.38), and (5.41), the linearised fluctuation equations are

$$\dot{\delta\hat{a}} = -(i\Delta_a + \kappa_a)\delta\hat{a} - ig_{ma}\delta\hat{m} + \sqrt{2\kappa_{ex}}\delta\hat{a}_{\text{in}}, \quad (5.42)$$

$$\dot{\delta\hat{b}} = -(i\omega_b + \kappa_b)\delta\hat{b} - ig_{mb}(m^*\delta\hat{m} + m\delta\hat{m}^\dagger), \quad (5.43)$$

$$\dot{\delta\hat{m}} = -(i\Delta'_m + \kappa_m)\delta\hat{m} - ig_{ma}\delta\hat{a} - ig_{mb}m(\delta\hat{b} + \delta\hat{b}^\dagger), \quad (5.44)$$

with Δ'_m defined in (5.40).

linearisation around the classical steady state changes the nonlinear dynamics into an effective linear Gaussian system that describes the quantum fluctuations[124]. The

5.4. LINEARISATION ABOUT THE STEADY STATE AND FLUCTUATION DYNAMICS

steady-state amplitudes (a, b, m) define the operating point of the system and determine the strength and structure of the resulting linearised couplings between the cavity, magnon, and mechanical modes. The linearised magnon-photon interaction appears as a beam-splitter-type coupling between the corresponding fluctuations, as reflected by the terms proportional to g_{ma} in the linearised cavity and magnon equations of motion. In the rotating frame defined by the detunings, this interaction hybridises the cavity and magnon modes and enables coherent exchange and transduction of quantum and thermal fluctuations between them.

The mechanical fluctuations couple to the magnon subsystem through fluctuations of the magnon number operator, which acts as an effective force on the mechanical mode. After linearisation, this force is proportional to the quadrature-like combination $m^* \delta \hat{m} + m \delta \hat{m}^\dagger$, corresponding to magnon fluctuations projected along the phase defined by the steady coherent magnon amplitude. As a result, the mechanical response is sensitive to a specific magnon quadrature set by the classical operating point.

Conversely, the linearised magnon equation contains a term proportional to $g_{mb}m(\delta \hat{b} + \delta \hat{b}^\dagger)$, which is characteristic of radiation-pressure-type interactions. Here, the magnon fluctuations are driven by the mechanical position quadrature, with an effective coupling strength that is enhanced by the steady magnon amplitude. Consequently, even when the intrinsic single-quantum magnon-phonon coupling g_{mb} is weak, a large coherent magnon population can give rise to a substantial effective linear interaction $G_{mb} \equiv g_{mb}m$.

In addition, the appearance of the shifted detuning Δ'_m reflects a static dispersive effect arising from the mean mechanical displacement, which renormalises the magnon resonance frequency. From a dynamical perspective, the linearised equations may therefore be viewed as being written in a frame where the magnon mode is detuned from its bare resonance by this static frequency pull, while the remaining interaction terms describe fluctuations about the displaced equilibrium configuration.

Taken together, the nonlinear magnon-mechanical interaction gives rise to two distinct effects: a static renormalisation of the magnon frequency encoded in Δ'_m , and a drive-enhanced linear coupling between magnon and mechanical fluctuations. The resulting linearised model provides the natural starting point for frequency-domain analyses, including the calculation of susceptibilities, mode hybridisation, and noise spectra within the Gaussian approximation.

5.5 Reflection Spectrum

Starting from the linearised fluctuation equations,

$$\dot{\delta\hat{a}} = -(i\Delta_a + \kappa_a) \delta\hat{a} - ig_{ma} \delta\hat{m} + \sqrt{2\kappa_{ex}} \delta\hat{a}_{\text{in}}, \quad (5.45)$$

$$\dot{\delta\hat{b}} = -(i\omega_b + \kappa_b) \delta\hat{b} - ig_{mb} (m^* \delta\hat{m} + m \delta\hat{m}^\dagger), \quad (5.46)$$

$$\dot{\delta\hat{m}} = -(i\Delta'_m + \kappa_m) \delta\hat{m} - ig_{ma} \delta\hat{a} - ig_{mb} m (\delta\hat{b} + \delta\hat{b}^\dagger), \quad (5.47)$$

we transform to the frequency domain and compute the cavity reflection spectrum using standard input-output theory.

We adopt the Fourier transform pair

$$\hat{O}(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \hat{O}[\omega] e^{-i\omega t}, \quad \hat{O}[\omega] = \int_{-\infty}^{+\infty} dt \hat{O}(t) e^{+i\omega t}, \quad (5.48)$$

so that $\frac{d}{dt} \hat{O}(t) \longleftrightarrow -i\omega \hat{O}[\omega]$. For Hermitian conjugates, $\hat{O}^\dagger(t) \leftrightarrow \hat{O}^\dagger[\omega]$ with the same convention, and one may use the identity $\hat{O}^\dagger[\omega] = (\hat{O}[-\omega])^\dagger$ when needed.

It is convenient to introduce the (bare) linear response functions

$$\chi_a^{-1}[\omega] \equiv \kappa_a + i(\Delta_a - \omega), \quad (5.49)$$

$$\chi_m^{-1}[\omega] \equiv \kappa_m + i(\Delta'_m - \omega), \quad (5.50)$$

$$\chi_b^{-1}[\omega] \equiv \kappa_b + i(\omega_b - \omega), \quad (5.51)$$

so that a generic single-mode equation $\dot{\hat{c}} = -(i\Omega + \kappa)\hat{c} + \hat{F}$ becomes $\hat{c}[\omega] = \chi_c[\omega] \hat{F}[\omega]$ in the frequency domain. Physically, $\chi_j[\omega]$ quantifies how strongly a mode j responds to a drive at detuning ω ; couplings renormalize these susceptibilities via self-energies, leading to spectral features such as normal-mode splitting and mechanically induced transparency/absorption.

Applying (5.48) to (5.45)–(5.47) yields

$$\chi_a^{-1}[\omega] \delta\hat{a}[\omega] = -ig_{ma} \delta\hat{m}[\omega] + \sqrt{2\kappa_{ex}} \delta\hat{a}_{\text{in}}[\omega], \quad (5.52)$$

$$\chi_b^{-1}[\omega] \delta\hat{b}[\omega] = -ig_{mb} (m^* \delta\hat{m}[\omega] + m \delta\hat{m}^\dagger[\omega]), \quad (5.53)$$

$$\chi_m^{-1}[\omega] \delta\hat{m}[\omega] = -ig_{ma} \delta\hat{a}[\omega] - ig_{mb} m (\delta\hat{b}[\omega] + \delta\hat{b}^\dagger[\omega]). \quad (5.54)$$

Let the cavity be probed at frequency ω_s (probe amplitude ϵ_s) in the presence of a strong drive at frequency ω_d (which sets the steady magnon amplitude $m \equiv \langle \hat{m} \rangle$ and hence the linearised coupling $G_{mb} = g_{mb}m$). We define the probe-drive frequency offset as

$$\Omega \equiv \omega_s - \omega_d. \quad (5.55)$$

In the drive-rotating frame, the linear susceptibilities (inverse response functions) are

$$\chi_a^{-1}(\Omega) = \kappa_a + i(\Delta_a - \Omega), \quad \Delta_a \equiv \omega_a - \omega_d, \quad (5.56)$$

$$\chi_m^{-1}(\Omega) = \kappa_m + i(\Delta'_m - \Omega), \quad \Delta'_m \equiv \omega'_m - \omega_d, \quad (5.57)$$

$$\chi_b^{-1}(\Omega) = \kappa_b + i(\omega_b - \Omega). \quad (5.58)$$

The mechanical “position” that enters the magnon equation is $(\delta\hat{b} + \delta\hat{b}^\dagger)$, hence in frequency space the magnon experiences the *sum* of the mechanical response at $+\Omega$ and at $-\Omega$. Equivalently, the mechanics-induced magnon self-energy is

$$\Sigma_{mb}(\Omega) = |G_{mb}|^2 \left[\chi_b(\Omega) + \chi_b^*(-\Omega) \right] = g_{mb}^2 |m|^2 \left[\frac{1}{\kappa_b + i(\omega_b - \Omega)} + \frac{1}{\kappa_b - i(\omega_b + \Omega)} \right]. \quad (5.59)$$

The magnon-dressed susceptibility is then

$$\chi_m^{\text{eff}}(\Omega) = \frac{1}{\chi_m^{-1}(\Omega) + \Sigma_{mb}(\Omega)}. \quad (5.60)$$

Eliminating the magnon yields the cavity effective susceptibility

$$\chi_a^{\text{eff}}(\Omega) = \frac{1}{\chi_a^{-1}(\Omega) + g_{ma}^2 \chi_m^{\text{eff}}(\Omega)} = \frac{1}{\kappa_a + i(\Delta_a - \Omega) + \frac{g_{ma}^2}{\kappa_m + i(\Delta'_m - \Omega) + \Sigma_{mb}(\Omega)}}. \quad (5.61)$$

Using the standard single-port input-output relation [114],

$$\delta\hat{a}_{\text{out}} = \delta\hat{a}_{\text{in}} - \sqrt{2\kappa_{ex}} \delta\hat{a}, \quad (5.62)$$

the coherent reflection amplitude is

$$r(\Omega) \equiv \frac{\delta a_{\text{out}}(\Omega)}{\delta a_{\text{in}}(\Omega)} = 1 - 2\kappa_{ex} \chi_a^{\text{eff}}(\Omega). \quad (5.63)$$

Substituting (5.61)–(5.59) gives the fully explicit “simplest” form

$$r(\Omega) = 1 - \frac{2\kappa_{ex}}{\kappa_a + i(\Delta_a - \Omega) + \frac{g_{ma}^2}{\kappa_m + i(\Delta'_m - \Omega) + |G_{mb}|^2 \left[\frac{1}{\kappa_b + i(\omega_b - \Omega)} + \frac{1}{\kappa_b - i(\omega_b + \Omega)} \right]}}.$$

(5.64)

The drive frequency is set below the cavity resonance in a red-detuned configuration, $\omega_d = \omega_a - 1.5\omega_b$. This configuration positions the system near the triple-resonance condition, which ensures efficient excitation of the magnon mode and establishes a steady-state magnon population that mediates the effective magnomechanical interaction.

The complete magnon-mediated reflection spectrum is presented as a function of magnetic field and probe frequency in Fig. 5.4. The colour map displays the magnitude of the reflection coefficient $|S_{11}|$, which reveals the hybridised cavity–magnon–phonon response of the system. The contour plot illustrates reflectivity as a function of the applied magnetic field H and probe frequency ω . The cavity resonance appears as a bright horizontal band and remains largely independent of the magnetic field outside the hybridisation region. In contrast, the Kittel-like magnon mode demonstrates a clear field dependence and intersects the cavity mode near $H \approx 400$ Oe, where both frequencies become resonant.

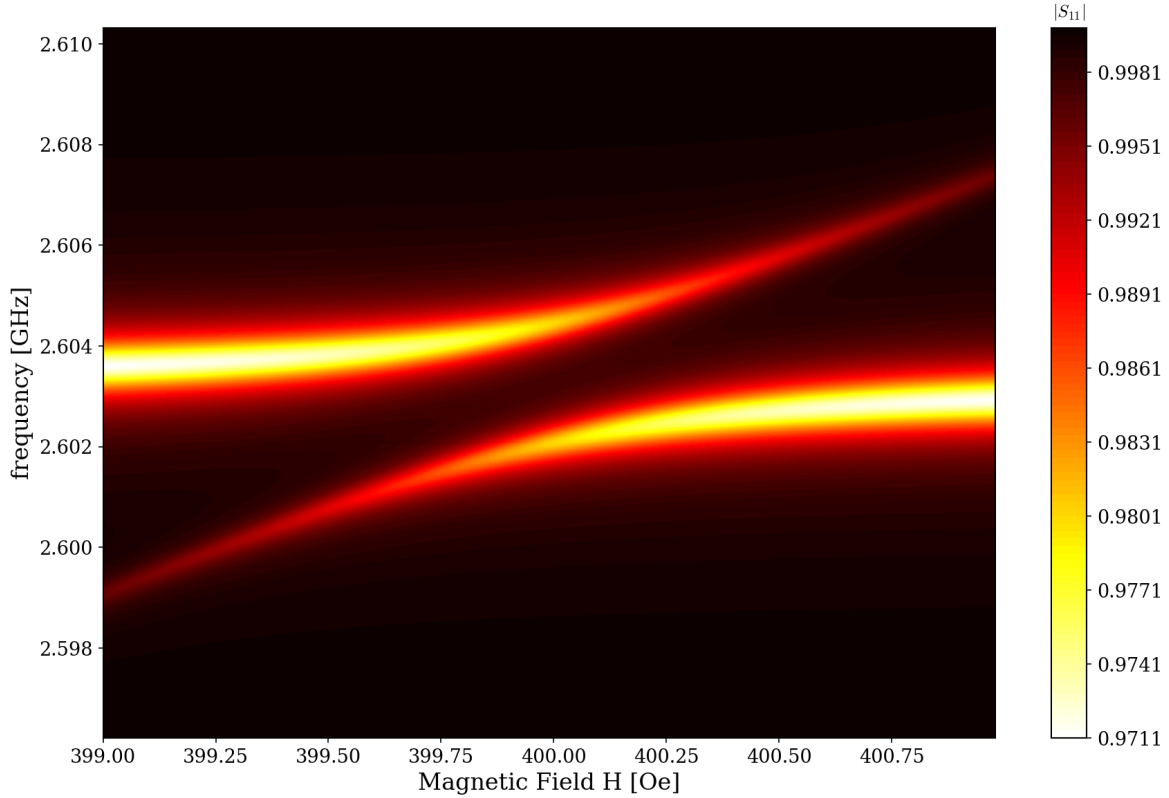


Figure 5.4: Reflection spectrum $|S_{11}|$ of the coupled cavity-magnon system as a function of magnetic field H and probe frequency ω . The system is driven at a detuning $\Delta = \omega_d - 1.5\omega_b$ with an input power of $P = 0.2$ nW. The colour map highlights the hybridised polariton modes arising from coherent magnon–photon coupling, where the avoided crossing structure reflects the strong-coupling regime. The narrow magnetic-field window is chosen to ensure operation within the weakly nonlinear regime, where the nonlinear frequency shift remains below half of the magnon linewidth.

An avoided crossing is observed at this point, indicating the formation of hybridised polaritonic modes. The resulting splitting $\Delta\omega = 2g_{ma}$ confirms that the magnon–photon coupling rate g_{ma} exceeds both the cavity and magnon decay rates, thereby placing

the system in the strong-coupling regime. High reflection amplitudes correspond to efficient energy exchange between the coupled modes. Away from resonance, the cavity mode remains largely unaffected by the magnetic field, and the magnon mode follows its expected Kittel-like dispersion.

A key aspect of this result is the intentional restriction of the magnetic field to a narrow range around $H = 400$ Oe. This constraint is essential to ensure that the system remains within the weakly nonlinear regime. In this regime, the variation of the magnon frequency $\omega_m(H)$ is sufficiently small so that the nonlinear frequency shift remains well below half of the magnon linewidth. As a result, the condition $\delta\omega_{\text{NL}} < \kappa_m/2$ is satisfied, which ensures the validity of the linearised treatment of the system.

Within this controlled regime, there are no signatures of bistability, spectral collapse, or dynamical instability. Instead, the spectrum reflects stable coherent hybridisation, with contrast governed by the pump-induced magnon population. This population enhances the effective magnomechanical coupling $G_{mb} = g_{mb}\sqrt{n_m}$, which modifies the polaritonic dispersion while preserving linear-response stability.

Overall, these results demonstrate that operating within a carefully chosen narrow magnetic-field window and red-detuned driving configuration enhances coherent magnon–photon–phonon interactions while strictly maintaining linear-response stability. This balance is crucial for observing controlled hybridisation effects without triggering nonlinear dynamical instabilities.

Fig. 5.5 displays the reflection spectrum $|S_{11}|$ of a hybrid cavity–magnon system as a function of probe frequency for various values of the bias magnetic field H , with a fixed microwave drive at frequency $\omega_d = \omega_a - 1.5\omega_b$. The left panels illustrate the complete spectral response, while the right panels provide an enlarged view in terms of the two-photon detuning $\Delta_{sd} = \omega_s - \omega_d$. In this regime, the cavity photon mode is strongly coupled to a magnon mode, resulting in field-dependent hybrid polariton excitations. Variation of the magnetic field shifts the magnon frequency $\omega_m(H)$, which continuously tunes the hybridisation condition and modifies the reflection spectrum.

The full spectra exhibit a broad Lorentzian-like dip corresponding to the lower hybrid magnon–photon polariton mode. This feature results from the dressed cavity response due to strong coherent coupling between the cavity field and the collective magnon excitation. The position and linewidth of this resonance are highly sensitive to the bias magnetic field, reflecting the evolution of the hybrid mode composition from photon-like to magnon-like character as the detuning between the bare modes is varied.

Examination of the zoomed-in panels reveals a narrow spectral feature embedded within the broader polaritonic response. This structure emerges near the two-photon detuning condition $\Delta_{sd} \approx \omega_b$, indicating the onset of coherent magnon–phonon interaction mediated by the driven cavity field. In contrast to the broad hybrid resonance, this feature arises from additional sideband processes introduced by the mechanical (phononic) degree of freedom, which contributes a frequency-dependent self-energy correction to

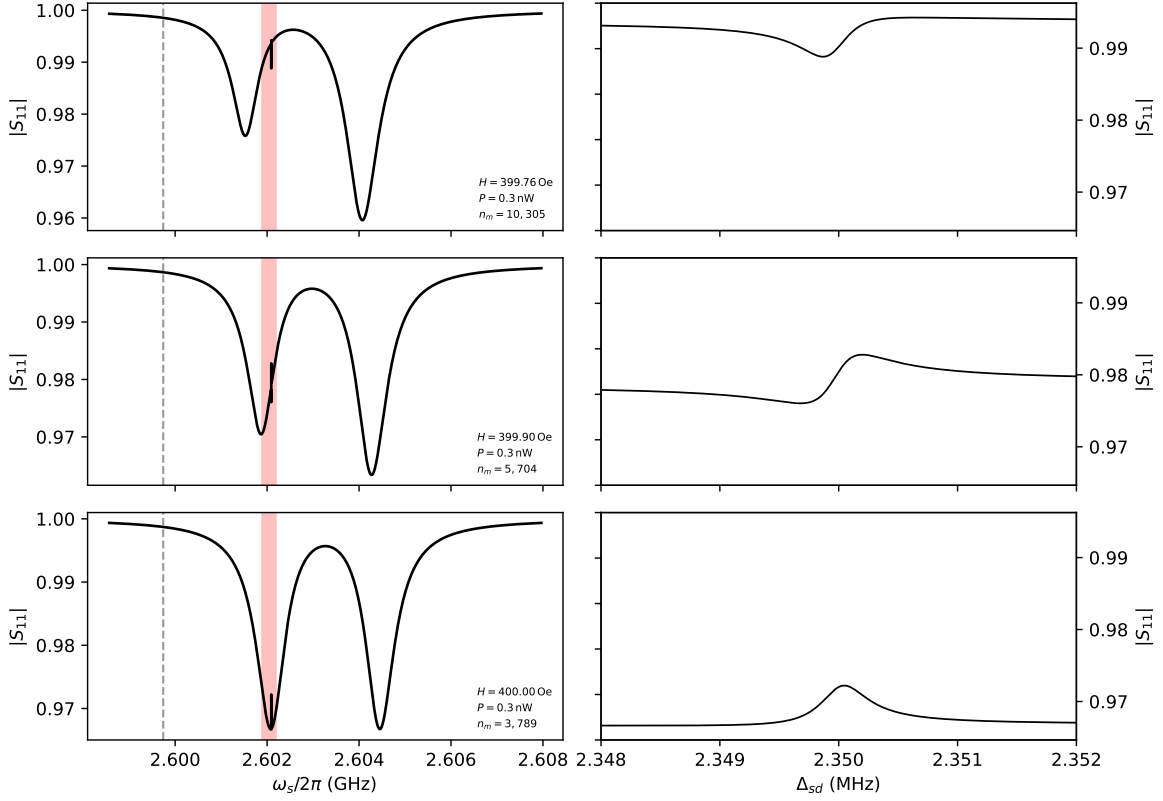


Figure 5.5: Reflection spectra $|S_{11}|$ of a hybrid cavity-magnon system as a function of probe frequency for different bias magnetic fields H , under a fixed microwave drive at frequency ω_d . The left panels show the full spectral response, where a broad Lorentzian-like dip corresponds to the lower hybrid magnon-photon polariton mode arising from strong photon-magnon coupling. The field dependence reflects the tuning of the magnon frequency $\omega_m(H)$ and the resulting evolution of the hybridised modes. The right panels show a zoomed-in view in terms of the two-photon detuning $\Delta_{sd} = \omega_s - \omega_d$, revealing a sharp spectral feature near $\Delta_{sd} \approx \omega_b$. This structure originates from coherent magnon-phonon interaction mediated by the cavity field and appears as a Fano-like interference profile superimposed on the hybrid polariton background. The line shape varies with the bias field due to changes in the relative strengths and phases of the phonon-assisted scattering pathways. At specific detunings, the asymmetric profile evolves toward a symmetric transparency window, indicating a crossover to magnomechanically induced transparency (MMIT).

the effective cavity susceptibility.

The resulting narrow resonance displays a characteristic Fano-like asymmetric line shape, which arises from interference between the broad background response of the hybrid polariton and the narrow resonant scattering pathway associated with phonon-assisted processes. The relative strength and phase of these two channels vary with the applied

magnetic field, resulting in a continuous evolution of the spectral profile. Specifically, the interference may be predominantly destructive or constructive on opposite sides of the resonance, producing asymmetric dip–peak structures in the reflection spectrum.

With further tuning of the bias magnetic field, the system enters a regime where the drive detuning becomes resonant with the lower hybrid mode. This transition alters the line shape from an asymmetric Fano profile to a more symmetric transparency-like feature. Such behaviour is consistent with magnomechanically induced transparency (MMIT), in which coherent interference between excitation pathways suppresses absorption in the hybrid mode. Overall, the figure illustrates the coexistence of strong magnon–photon hybridisation and coherent magnon–phonon coupling, resulting in a hierarchy of spectral features from broad polaritonic resonances to narrow interference-induced transparency windows.

5.6 Phonon Self-Energy and Magnon Spring Effect

Magnons exert an effective radiation-pressure-like force on the mechanical mode due to magnetoelastic coupling. This leads to a feedback of the magnetic dynamics onto the mechanical vibrations. This feedback gives rise to a dynamical modification of the mechanical response, commonly referred to as dynamical backaction [14]. In the absence of magnomechanical coupling, the mechanical oscillator is fully described by its bare susceptibility, determined solely by its resonance frequency and intrinsic damping. When coupled to the magnon mode, however, the mechanical dynamics are changed through this interaction-induced feedback. As a consequence, the mechanical susceptibility is modified, resulting in shifts of the mechanical resonance frequency and changes in its effective damping rate. These modifications are the signature of the magnon-induced phonon self-energy on the mechanical mode.

The linearised equations of fluctuation can be rewritten as:

$$\chi_a^{-1}[\omega] \delta \hat{a}[\omega] = -ig_{ma} \delta \hat{m}[\omega], \quad (5.65)$$

$$\chi_m^{-1}[\omega] \delta \hat{m}[\omega] = -ig_{ma} \delta \hat{a}[\omega] - iG_{mb} \left(\delta \hat{b}[\omega] + \delta \hat{b}^\dagger[-\omega] \right), \quad (5.66)$$

$$\chi_b^{-1}[\omega] \delta \hat{b}[\omega] = -i \left(G_{mb}^* \delta \hat{m}[\omega] + G_{mb} \delta \hat{m}^\dagger[-\omega] \right), \quad (5.67)$$

where the linearised (enhanced) magnomechanical coupling is

$$G_{mb} \equiv g_{mb} \bar{m}, \quad \bar{m} \equiv \langle \hat{m} \rangle, \quad (5.68)$$

which may in general be complex (its phase is set by the drive).

From Eq. (5.65),

$$\delta \hat{a}[\omega] = -ig_{ma} \chi_a[\omega] \delta \hat{m}[\omega]. \quad (5.69)$$

Substituting Eq. (5.69) into Eq. (5.66) yields

$$\left(\chi_m^{-1}[\omega] + g_{ma}^2 \chi_a[\omega]\right) \delta \hat{m}[\omega] = -i G_{mb} \left(\delta \hat{b}[\omega] + \delta \hat{b}^\dagger[-\omega]\right). \quad (5.70)$$

Define the dressed magnon susceptibility

$$\Xi[\omega] \equiv \frac{1}{\chi_m^{-1}[\omega] + g_{ma}^2 \chi_a[\omega]}. \quad (5.71)$$

Then

$$\delta \hat{m}[\omega] = -i G_{mb} \Xi[\omega] \left(\delta \hat{b}[\omega] + \delta \hat{b}^\dagger[-\omega]\right). \quad (5.72)$$

Taking the Hermitian conjugate at $-\omega$ gives

$$\delta \hat{m}^\dagger[-\omega] = +i G_{mb}^* \Xi^*[-\omega] \left(\delta \hat{b}[\omega] + \delta \hat{b}^\dagger[-\omega]\right). \quad (5.73)$$

Inserting Eqs. (5.72)–(5.73) into Eq. (5.67) yields

$$\begin{aligned} \chi_b^{-1}[\omega] \delta \hat{b}[\omega] &= -i \left[G_{mb}^* \left(-i G_{mb} \Xi[\omega] \right) + G_{mb} \left(+i G_{mb}^* \Xi^*[-\omega] \right) \right] \left(\delta \hat{b}[\omega] + \delta \hat{b}^\dagger[-\omega] \right) \\ &= -|G_{mb}|^2 \left(\Xi[\omega] - \Xi^*[-\omega] \right) \left(\delta \hat{b}[\omega] + \delta \hat{b}^\dagger[-\omega] \right). \end{aligned} \quad (5.74)$$

We therefore identify the (magnon-induced) phonon self-energy

$$\Sigma_0[\omega] \equiv |G_{mb}|^2 \left(\Xi[\omega] - \Xi^*[-\omega] \right), \quad (5.75)$$

so that the phonon equation becomes the coupled $(b, b^\dagger)$ relation

$$\chi_b^{-1}[\omega] \delta \hat{b}[\omega] = -\Sigma_0[\omega] \left(\delta \hat{b}[\omega] + \delta \hat{b}^\dagger[-\omega] \right). \quad (5.76)$$

Taking the Hermitian conjugate of Eq. (5.76) at $-\omega$ and using $(\delta \hat{b}[\omega])^\dagger = \delta \hat{b}^\dagger[-\omega]$ gives

$$\left(\chi_b^*[-\omega]\right)^{-1} \delta \hat{b}^\dagger[-\omega] = -\Sigma_0^*[-\omega] \left(\delta \hat{b}[\omega] + \delta \hat{b}^\dagger[-\omega] \right). \quad (5.77)$$

From Eq. (5.75) one has $\Sigma_0^*[-\omega] = -\Sigma_0[\omega]$, hence

$$\left(\chi_b^*[-\omega]\right)^{-1} \delta \hat{b}^\dagger[-\omega] = +\Sigma_0[\omega] \left(\delta \hat{b}[\omega] + \delta \hat{b}^\dagger[-\omega] \right). \quad (5.78)$$

Solving Eq. (5.78) for $\delta \hat{b}^\dagger[-\omega]$ yields

$$\delta \hat{b}^\dagger[-\omega] = \frac{\Sigma_0[\omega]}{\left(\chi_b^*[-\omega]\right)^{-1} - \Sigma_0[\omega]} \delta \hat{b}[\omega]. \quad (5.79)$$

Therefore,

$$\delta\hat{b}[\omega] + \delta\hat{b}^\dagger[-\omega] = \left[1 + \frac{\Sigma_0[\omega]}{(\chi_b^*[-\omega])^{-1} - \Sigma_0[\omega]} \right] \delta\hat{b}[\omega]. \quad (5.80)$$

Substituting Eq. (5.80) into Eq. (5.76) gives the closed form

$$\chi_b^{-1}[\omega] \delta\hat{b}[\omega] = -\Sigma_0[\omega] \left[1 + \frac{\Sigma_0[\omega]}{(\chi_b^*[-\omega])^{-1} - \Sigma_0[\omega]} \right] \delta\hat{b}[\omega] \quad (5.81)$$

with $\Sigma_0[\omega]$ and $\Xi[\omega]$ defined in Eqs. (5.75) and (5.71).

Equivalently, inserting Eq. (5.75) into Eq. (5.81) yields

$$\chi_b^{-1}[\omega] \delta\hat{b}[\omega] = -|G_{mb}|^2 \left(\Xi[\omega] - \Xi^*[-\omega] \right) \left[1 + \frac{|G_{mb}|^2 \left(\Xi[\omega] - \Xi^*[-\omega] \right)}{(\chi_b^*[-\omega])^{-1} - |G_{mb}|^2 \left(\Xi[\omega] - \Xi^*[-\omega] \right)} \right] \delta\hat{b}[\omega] \quad (5.82)$$

Bringing the interaction-induced term to the left-hand side, the dressed mechanical response can be written as,

$$\left(\chi_b^{-1}[\omega] - i\Sigma[\omega] \right) \delta\hat{b}[\omega] = 0, \quad (5.83)$$

which identifies the phonon self-energy as

$$\Sigma[\omega] = i|G_{mb}|^2 \left(\Xi[\omega] - \Xi^*[-\omega] \right) \left[1 + \frac{|G_{mb}|^2 \left(\Xi[\omega] - \Xi^*[-\omega] \right)}{(\chi_b^*[-\omega])^{-1} - |G_{mb}|^2 \left(\Xi[\omega] - \Xi^*[-\omega] \right)} \right]. \quad (5.84)$$

If the antiresonant ($b, b^\dagger$) mixing is negligible, i.e. $| |G_{mb}|^2 (\Xi - \Xi^*) \chi_b^*[-\omega] | \ll 1$, the bracketed correction in Eq. (5.84) can be approximated by unity, yielding

$$\Sigma[\omega] \simeq i|G_{mb}|^2 \left(\Xi[\omega] - \Xi^*[-\omega] \right). \quad (5.85)$$

which is the final expression for the self-energy of the phonon. In the weak-coupling limit, the magnon-phonon interaction modifies the mechanical mode, giving rise to a frequency shift $\delta\omega_b = -\text{Re} \Sigma[\omega]$, referred to as the magnon spring effect, and an additional damping rate $\Gamma_{\text{mag}} = 2\text{Im} \Sigma[\omega]$ called magnomechanical damping. The resulting mechanical frequency and linewidth are therefore

$$\tilde{\omega}_b = \omega_b + \delta\omega_b, \quad \Gamma_{\text{tot}} = \Gamma_b + \Gamma_{\text{mag}}. \quad (5.86)$$

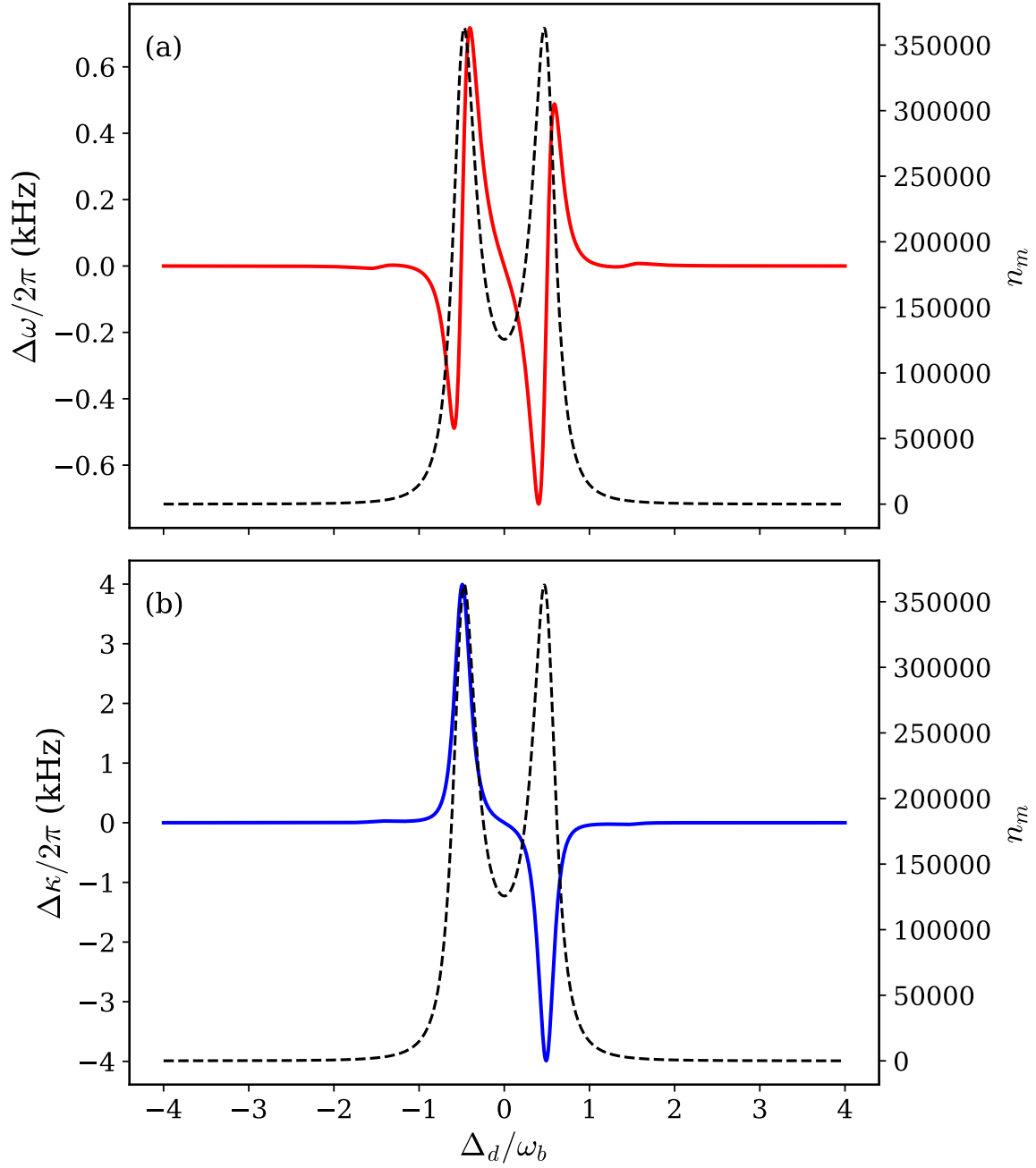


Figure 5.6: The magnon spring effect showing the nonlinear modification of (a) the effective magnon frequency shift $\Delta\omega$ and (b) the linewidth change $\Delta\kappa$ as a function of the normalized drive-cavity detuning $\Delta_d = (\omega_d - \omega_a)/\omega_b$. The dashed black curve shows the corresponding steady-state magnon population n_m . Calculations are performed in the linear regime at an input power of 0.15 nW.

Figure 5.3 presents the nonlinear response of the coupled cavity–magnon system in vertically stacked plots. The upper panel illustrates the effective frequency renormalisation $\Delta\omega$, while the lower panel depicts the corresponding change in the damping rate $\Delta\kappa$ as a function of the normalized drive-cavity detuning $\Delta_d = (\omega_d - \omega_a)/\omega_b$. In both panels, the dashed curve denotes the steady-state magnon population n_m , which mediates the nonlinear backaction within the hybrid system. There is a clear correlation between increased magnon occupation and the emergence of nonlinear frequency and damping corrections, confirming that the magnon mode serves as the primary channel for nonlinear feedback.

Although resonant enhancement occurs in n_m , both $\Delta\omega$ and $\Delta\kappa$ remain bounded across the entire detuning range examined. Notably, even at the detuning corresponding to maximum magnon population, the induced frequency shift is substantially smaller than the intrinsic magnon linewidth. This result demonstrates that the condition $\delta\omega_{\text{NL}} < \kappa_m/2$ is maintained throughout the parameter regime, thereby ensuring that the system operates within the linear and dynamically stable regime.

A direct comparison of the frequency and damping responses reveals that the nonlinear modification of the linewidth exhibits a similar qualitative trend to the frequency shift, but with reduced amplitude. This observation indicates that dissipative renormalisation is weaker than coherent frequency pulling under the present parameter conditions. This asymmetry originates from the structure of the hybrid Green’s function, in which reactive (real part) and dissipative (imaginary part) contributions are weighted differently.

In summary, the results confirm that although strong magnon-mediated coupling produces measurable nonlinear corrections, the system remains well within the linear-response limit for the selected drive power. This finding supports the validity of employing a linearised theoretical framework and demonstrates that the nonlinear frequency shift consistently remains below half of the magnon linewidth across all detuning conditions examined.

5.7 Cooperativity

Cooperativity C is a key figure of merit in cavity magnomechanical systems, quantifying the relative strength of coherent coupling compared to dissipative losses within the hybrid system. More fundamentally, cooperativity serves as a dimensionless parameter that measures the efficiency of excitation exchange between subsystems: microwave cavity photons, magnons, and mechanical phonons prior to irreversible loss to their respective environments.

In open quantum systems, high cooperativity indicates a regime in which coherent energy or information exchange between subsystems occurs multiple times during the lifetime of an excitation. In contrast, when $C \ll 1$, dissipative processes dominate, resulting in the loss of transferred excitations to the environment before significant hybrid dynamics

can develop. This interpretation is widely adopted in both cavity optomechanics and cavity magnomechanics [1] where cooperativity delineates the transition between weak and strong dynamical backaction regimes. By analogy with cavity optomechanics and following established treatments of cavity magnomechanical systems, the cooperativity under red-detuned driving is expressed as

$$C = \frac{P_d \kappa_{ex} g_{mb}^2}{\hbar \omega_d \omega_b^2 \kappa_b} \frac{2 \sin^4 \theta \cos^2 \theta}{\kappa_a \cos^2 \theta + \kappa_m \sin^2 \theta}, \quad (5.87)$$

Here, P_d denotes the microwave drive power, κ_{ex} is the external coupling rate of the cavity, g_{mb} represents the intrinsic magnon–mechanical coupling strength, and κ_b is the intrinsic mechanical damping rate. The hybrid character of the system is introduced via the mixing angle θ , which determines the photon–magnon composition of the dressed eigenmodes.

The hybridisation angle is defined by

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{2g_{ma}}{\omega_m(H) - \omega_a} \right), \quad (5.88)$$

In this expression, g_{ma} is the magnon–photon coupling rate, $\omega_m(H)$ denotes the field-dependent magnon frequency and ω_a is the bare cavity resonance frequency. This mixing is essential for redistributing damping and interaction strengths among subsystems, resulting in the pronounced magnetic-field dependence of cooperativity observed in Fig. 5.7.

The thermal occupation of the mechanical mode is determined by the Bose-Einstein distribution.

$$n_b^i = \frac{1}{\exp\left(\frac{\hbar \omega_b}{k_B T_{\text{bath}}}\right) - 1}, \quad (5.89)$$

This equation describes the equilibrium phonon population determined by the cryogenic environment. In the high-temperature, or classical, limit, it simplifies to $n_b^i \approx k_B T_{\text{bath}} / \hbar \omega_b$, which demonstrates the linear dependence of thermal occupation on temperature.

Cooperativity determines the efficiency of dynamical backaction cooling in the mechanical subsystem. In the resolved-sideband and weak-noise limit, the steady-state phonon occupation is reduced as follows:

$$n_b = \frac{n_b^i}{1 + C}, \quad (5.90)$$

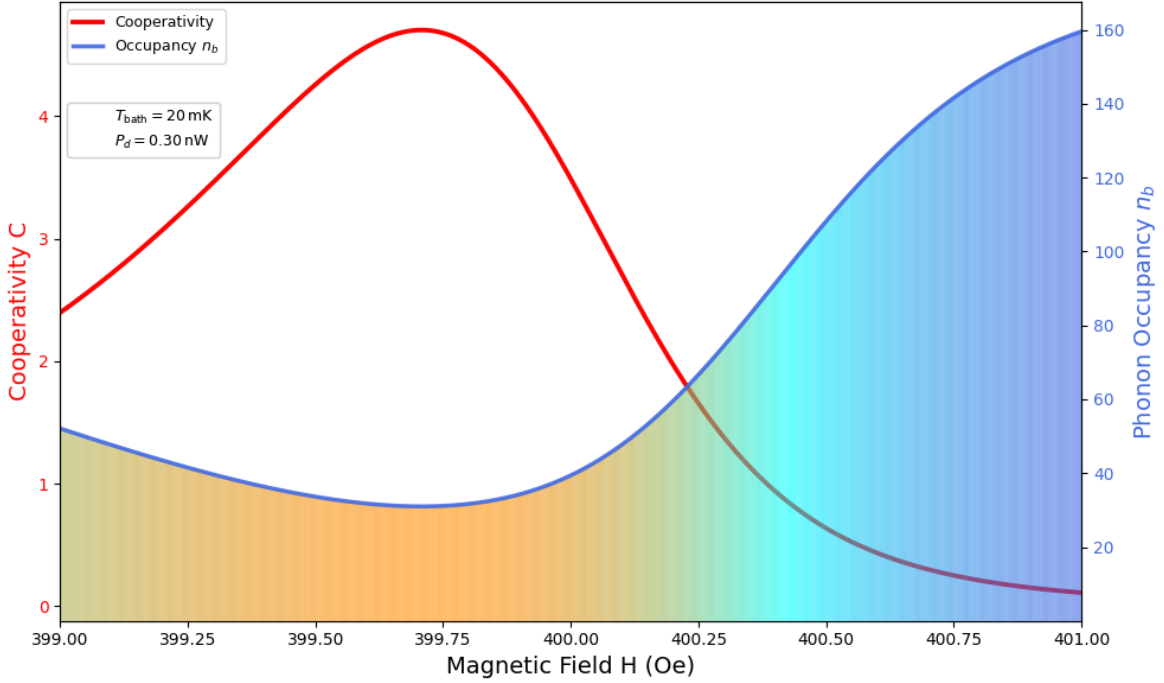


Figure 5.7: Magnetic-field dependence of the cooperativity C and the resulting phonon occupation n_b for a red-detuned drive at $P_d = 0.3$ nW. The bath occupation is determined by the Bose–Einstein distribution at $T_{\text{bath}} = 20$ mK for $\omega_b/2\pi = 2.35$ MHz, yielding $n_b^i \approx 177$. Cooperativity is significantly enhanced near the magnon–cavity resonance as a result of hybridisation, which enables dynamical backaction cooling of the mechanical mode.

This relationship explicitly demonstrates that increasing cooperativity suppresses thermal fluctuations by enhancing the effective damping of the mechanical mode.

This behaviour can also be interpreted in terms of an effective mechanical linewidth:

$$\tilde{\kappa}_b = \kappa_b(1 + C), \quad (5.91)$$

This linewidth increases as a result of radiation-pressure-like backaction. The corresponding effective temperature of the mechanical resonator is then reduced according to

$$T_{\text{eff},b} \approx \frac{\kappa_b}{\tilde{\kappa}_b} T_{\text{bath}} = \frac{T_{\text{bath}}}{1 + C}. \quad (5.92)$$

This relation offers a direct thermodynamic interpretation of cooperativity as a factor that enhances cooling.

For the experimental parameters considered, with a mechanical frequency of $\omega_b/2\pi = 2.35$ MHz and a bath temperature of $T_{\text{bath}} \approx 20$ mK, the equilibrium phonon occupation is $n_b^i \approx 177$, consistent with Fig. 5.7. Adjusting the external magnetic field allows the cooperativity to be varied by more than an order of magnitude due to magnon–photon hybridisation effects, which enables significant suppression of the mechanical occupation.

In the present device, however, the achievable cooperativity is ultimately limited by the onset of Kerr nonlinearity in the magnetic subsystem at higher drive powers (approximately 0.3 nW). This nonlinearity restricts the maximum stable intracavity field and consequently bounds the achievable cooling performance within the linear regime. As a result, ground-state cooling is not attained under the current operating conditions.

Future advancements in device engineering, such as optimising mode volume, enhancing photon coupling strength, and reducing decay rates, are expected to extend the linear operating regime. These improvements would enable higher cooperativity scaling with input power and may provide access to the strong backaction cooling regime necessary for ground-state preparation of the mechanical mode.

Chapter 6

Nano Magnomechanical Interaction Under Blue-Detuned Drive

This chapter analyses the same tripartite cavity–magnon–phonon system introduced previously, now operated under a blue-detuned microwave drive. The system consists of a microwave cavity mode $\hat{a}$, a collective spin (magnon) mode $\hat{m}$, and a mechanical (phonon) mode $\hat{b}$, coupled through the magnon–photon dipole interaction $g_{ma}(\hat{a} + \hat{a}^\dagger)(\hat{m} + \hat{m}^\dagger)$ and the magnetostrictive radiation-pressure-type interaction $g_{mb}\hat{m}^\dagger\hat{m}(\hat{b} + \hat{b}^\dagger)$.

The entire analytical machinery developed in the red-detuned chapter carries over unchanged. In particular, the laboratory-frame Hamiltonian, the transformation to the drive-rotating frame and the rotating-wave approximation, the Heisenberg–Langevin equations of motion, the steady-state mean-field solution for the cavity and magnon amplitudes, the linearised fluctuation equations, the cavity reflection coefficient obtained from input–output theory, and the magnon-induced phonon self-energies are all identical in form. Therefore, the derivations are not reproduced here, and the reader is referred to the preceding chapter. The system parameters also remain unchanged and are those listed in Table 1.1.

The sole modification is the sign of the drive detuning. Whereas the red-detuned chapter employed a pump placed below the cavity resonance,

$$\omega_d = \omega_a - 1.5\omega_b \quad \text{and} \quad \omega_d = \omega_a - 0.5\omega_b, \quad (6.1)$$

the present chapter considers the blue-detuned configurations

$$\omega_d = \omega_a + 1.5\omega_b \quad \text{and} \quad \omega_d = \omega_a + 0.5\omega_b, \quad (6.2)$$

so that the cavity detuning $\Delta_a = \omega_a - \omega_d$ becomes *negative*. Every steady-state relation, susceptibility, and self-energy retains its previous functional form; only the value and sign of Δ_a entering those expressions change.

The physical consequence of this single sign reversal is decisive. Under red detuning the pump preferentially drives the anti-Stokes (phonon-absorbing) sideband and extracts energy from the mechanical mode, resulting in cooling and a positive dynamical-backaction damping. Under blue detuning the pump instead favours the Stokes (phonon-emitting) sideband and deposits energy into the mechanical mode. The magnon-induced contribution to the mechanical damping therefore reverses sign and becomes negative, corresponding to anti-damping and mechanical amplification rather than cooling. The remainder of this chapter establishes this picture quantitatively through the steady-state magnon population, the linearity of the driven response, the reflection spectrum, and the magnon-induced spring/anti-damping effect, before turning to the cooperativity analysis.

6.1 Steady-State Magnon Population

Figure ?? shows the steady-state magnon population n_m as a function of the normalised cavity detuning Δ_a/ω_b and the bias field H , obtained from the same mean-field steady-state solution used in the red-detuned analysis. The population is governed by the magnon–photon hybridisation and is essentially symmetric about $\Delta_a/\omega_b = 0$: two bright ridges trace the upper and lower polariton branches and form an avoided crossing near $H \simeq 400$ Oe, where the bare cavity and Kittel-like magnon frequencies become degenerate and the cavity feeds the magnon mode most efficiently.

For the blue-detuned operating points of Eq. (6.2) the relevant region is the left half of the map, $\Delta_a/\omega_b < 0$ (i.e. $\omega_d > \omega_a$). The accessible magnon population there is comparable to the red-detuned case, confirming that blue driving establishes a substantial coherent magnon amplitude and, consequently, a substantial enhancement of the magnomechanical coupling $G_{mb} = g_{mb}\sqrt{n_m}$. As before, the bias field is restricted to a narrow window around $H = 400$ Oe so that the magnon-frequency excursion remains small and the system stays weakly nonlinear; no bistability or abrupt spectral distortion is observed.

6.2 Linearity of the Driven Magnon Mode for Both Detunings

To confirm that the blue-detuned operating points remain within the linearised regime, we evaluate the magnon Kerr ratio $K_m n_m / \kappa_m$ as a function of the bias field for both drive conditions of Eq. (6.2). The linearised treatment is valid provided the Kerr-induced frequency shift stays below half the magnon linewidth, $K_m n_m < \kappa_m/2$, i.e. $K_m n_m / \kappa_m < 0.5$.

Figure 6.1 shows the case $\omega_d = \omega_a + 0.5\omega_b$ at $P = 0.15$ nW. The steady-state magnon number (top panel) peaks at $n_m \simeq 3.8 \times 10^5$ near $H \simeq 400.05$ Oe, and the corresponding

6.2. LINEARITY OF THE DRIVEN MAGNON MODE FOR BOTH DETUNINGS

Kerr ratio (bottom panel) reaches a maximum of $\simeq 0.29$, remaining below the linearity threshold by a factor of approximately 1.7.

Figure 6.2 shows the more strongly detuned case $\omega_d = \omega_a + 1.5\omega_b$ at the higher power $P = 0.30$ nW. Here the population peaks at $n_m \simeq 4.6 \times 10^5$ near $H \simeq 400.8$ Oe, while the Kerr ratio peaks at approximately 0.35, remaining below threshold by a factor of about 1.4 and reaches as low as 3×10^{-3} at $H = 400$ Oe.

In both cases, the resonance maximum occurs above $H = 400$ Oe, which is the mirror image of the red-detuned chapter (where the maxima lie below 400 Oe), because the blue-detuned drive becomes resonant with a hybrid mode at a field where $\omega_m(H) > \omega_a$. Importantly, the Kerr ratio remains below 0.5 across the entire field window for both detunings, exactly as found for the red-detuned drives. This confirms that the linearised description is valid for both configurations and that the larger detuning tolerates the higher drive power while preserving linear-response stability.

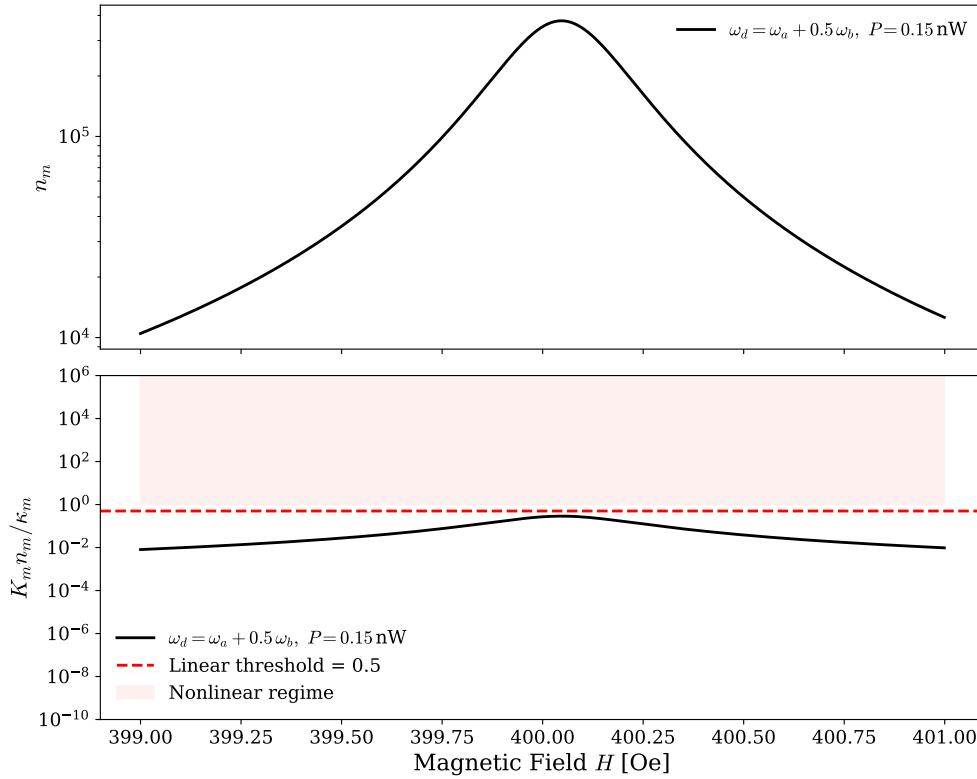


Figure 6.1: Linearity analysis for the blue-detuned drive $\omega_d = \omega_a + 0.5\omega_b$ at $P = 0.15$ nW. Top: steady-state magnon number n_m versus H , peaking near $H \simeq 400.05$ Oe. Bottom: Kerr ratio $K_m n_m / \kappa_m$ versus H ; the dashed line is the linearity threshold 0.5 and the shaded region marks the nonlinear regime. The ratio peaks at $\simeq 0.29$, remaining below threshold throughout.

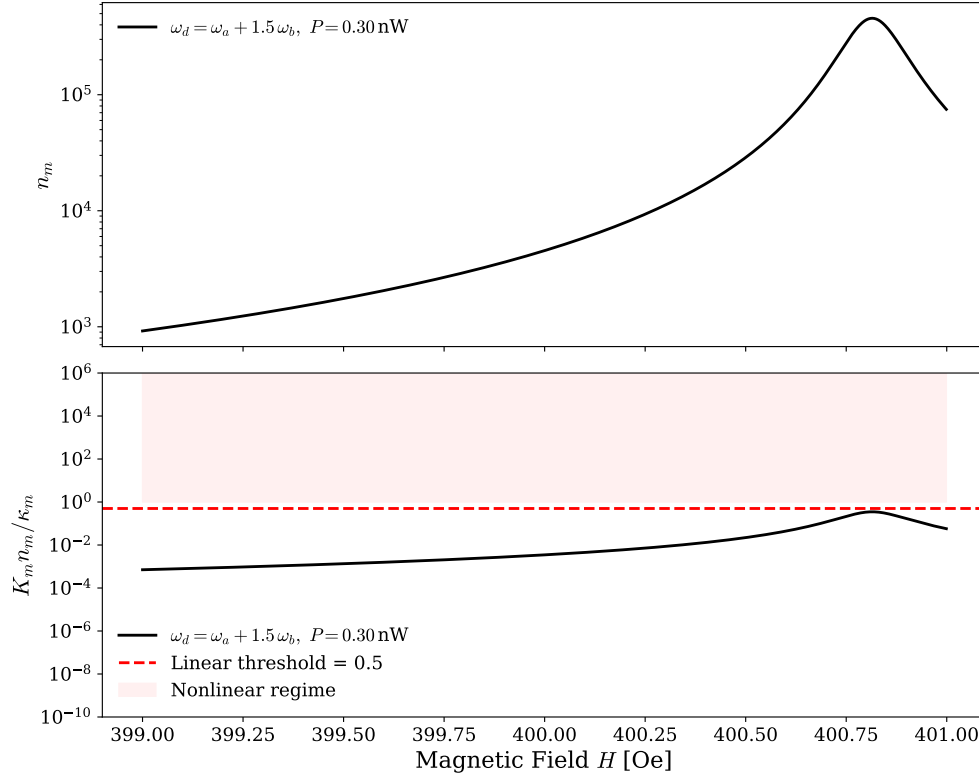


Figure 6.2: Linearity analysis for the blue-detuned drive $\omega_d = \omega_a + 1.5 \omega_b$ at $P = 0.30$ nW. Top: steady-state magnon number n_m versus H , peaking near $H \simeq 400.8$ Oe. Bottom: Kerr ratio $K_m n_m / \kappa_m$ versus H ; the ratio peaks at $\simeq 0.35$, remaining below the linearity threshold of 0.5 over the considered field range.

6.3 Reflection Spectrum

The cavity reflection spectrum is computed from the same input-output expression and dressed susceptibilities as in the red-detuned chapter (5.47)–(5.64), with the drive set to the blue-detuned value $\omega_d = \omega_a + 1.5 \omega_b$. Figure 6.3 shows $|S_{11}|$ as a function of bias field H and probe frequency. A field-independent cavity band and a field-dependent Kittel-like magnon band form an avoided crossing near $H \simeq 400$ Oe, producing upper and lower hybrid polariton branches split by $2g_{ma}$. The splitting confirms operation in the strong-coupling regime, $g_{ma} > \kappa_a, \kappa_m$.

The overall structure does not depend on the sign of the drive detuning; the role of the blue-detuned pump is to set the steady magnon population n_m and hence the enhanced coupling $G_{mb} = g_{mb} \sqrt{n_m}$, which determines the dynamical backaction analysed below. Within the narrow field window, the spectrum is smooth and stable, with no signatures of bistability or spectral collapse.

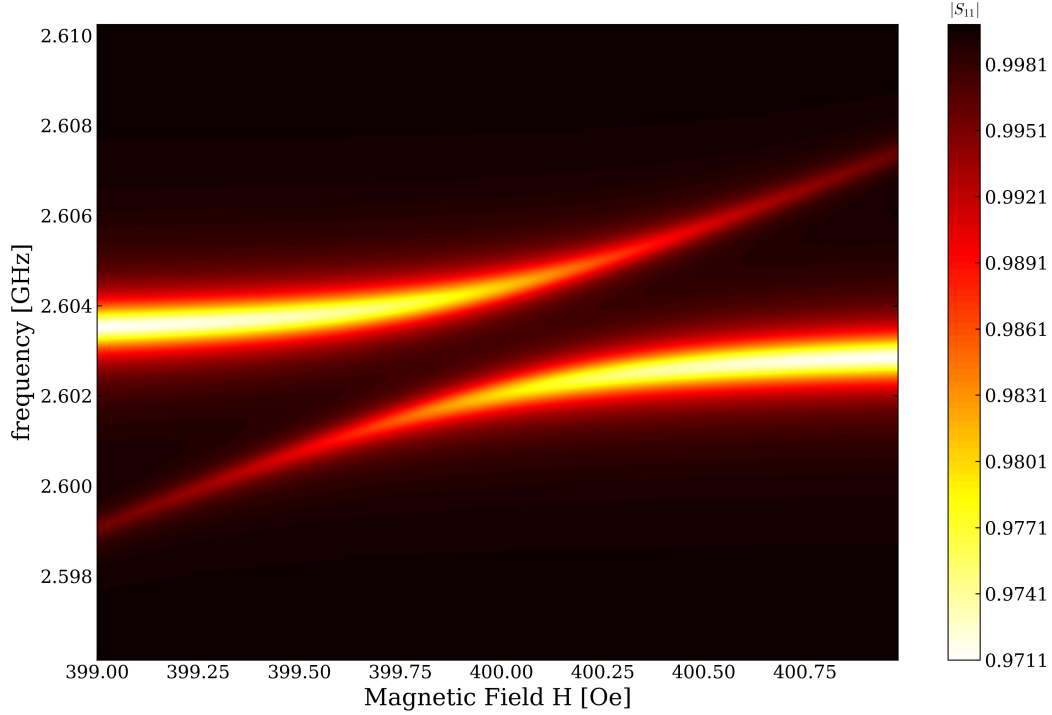


Figure 6.3: Reflection spectrum $|S_{11}|$ of the coupled cavity-magnon system under blue-detuned driving ($\omega_d = \omega_a + 1.5\omega_b$, $P = 0.3$ nW) as a function of magnetic field H and probe frequency. The avoided crossing near $H \simeq 400$ Oe reflects strong magnon–photon coupling, with the two hybridised polariton branches split by $2g_{ma}$.

6.4 Magnomechanically Induced Spectral Features

Figure 6.4 presents the reflection spectrum $|S_{11}|$ as a function of probe frequency for several values of the bias field H , with a fixed blue-detuned drive at $\omega_d = \omega_a + 1.5\omega_b$ and $P = 0.3$ nW. The left panels show the full spectral response; the right panels provide an enlarged view in terms of the two-photon detuning $\Delta_{sd} = \omega_s - \omega_d$.

The full spectra (left) display the two broad hybrid magnon–photon polariton dips arising from strong photon–magnon coupling. The defining feature of the blue-detuned configuration is the position of the drive (grey dashed line): ω_d now lies *above* both polariton branches, in contrast to the red-detuned case, where it is positioned below them. The shaded band marks the mechanical sideband, which for blue driving is located *below* the drive, at $\omega_d - \omega_b$.

The enlarged panels (right) reveal a narrow spectral feature near the two-photon detuning $\Delta_{sd} \simeq -\omega_b$. This is the mirror image of the red-detuned result, where the analogous feature appeared at $\Delta_{sd} \simeq +\omega_b$: the sideband flips to the opposite side of the drive because the blue-detuned pump favours the Stokes process at $\omega_d - \omega_b$ rather than the anti-Stokes ($\omega_d + \omega_b$) one. The feature originates from coherent magnon–phonon

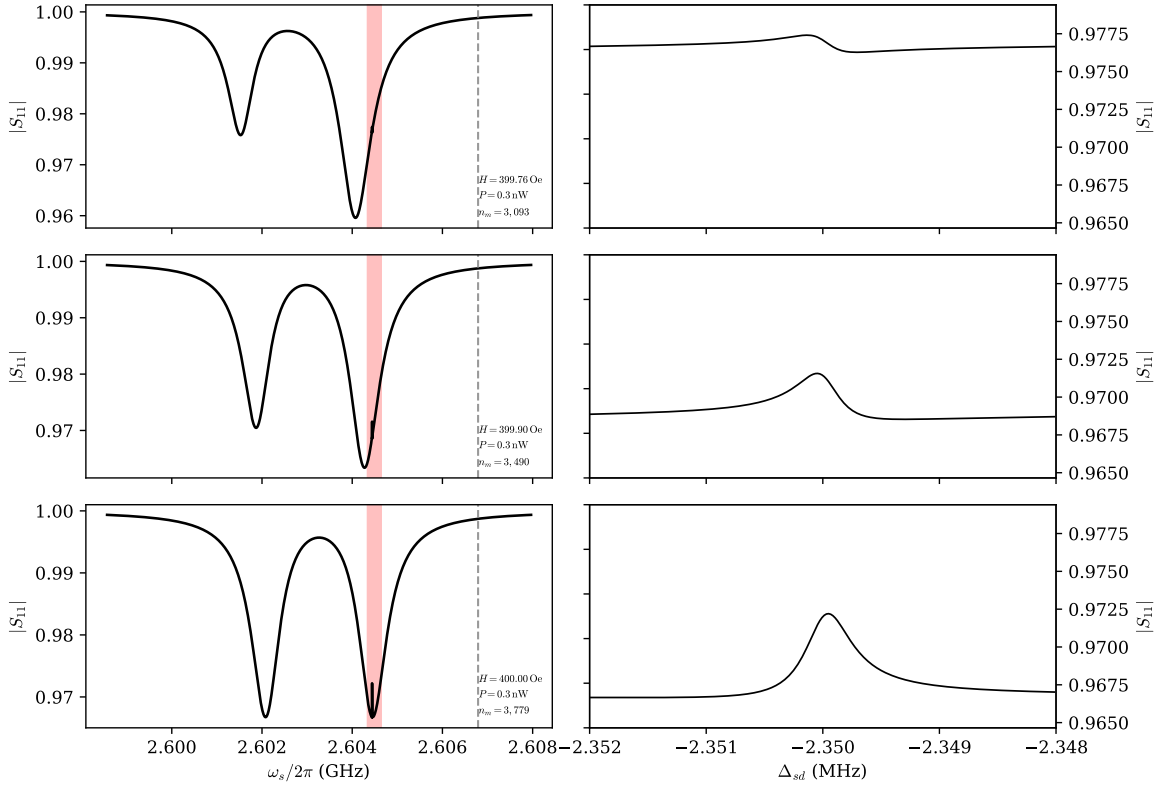


Figure 6.4: Reflection spectra $|S_{11}|$ under blue-detuned driving ($\omega_d = \omega_a + 1.5\omega_b$, $P = 0.3$ nW) for different bias fields H . Left: full spectral response; the grey dashed line marks the drive frequency ω_d , which lies above both polariton branches, and the shaded band marks the mechanical sideband at $\omega_d - \omega_b$. Right: enlarged view in terms of the two-photon detuning $\Delta_{sd} = \omega_s - \omega_d$, revealing a narrow magnomechanical feature near $\Delta_{sd} \simeq -\omega_b$. The line shape evolves from a Fano-like asymmetric profile toward a symmetric transparency window with bias field, signalling the crossover to magnomechanically induced transparency (MMIT).

interaction mediated by the cavity field and appears as a Fano-like asymmetric profile superimposed on the hybrid polariton background. As the bias field is tuned, the relative strengths and phases of the phonon-assisted scattering pathways change, and the line shape evolves continuously from an asymmetric dip-peak structure toward a more symmetric transparency-like window, indicating a transition to magnomechanically induced transparency (MMIT). The coexistence of the broad polaritonic resonances and this narrow interference feature mirrors the hierarchy of spectral features found in the red-detuned case, transposed to the opposite mechanical sideband.

6.5 Cooperativity and the Absence of Amplification

The cooperativity under blue-detuned drive is defined as:

$$C = \frac{P_d \kappa_{ex} g_{mb}^2}{\hbar \omega_d \omega_b^2 \kappa_b} \frac{2 \cos^4 \theta \sin^2 \theta}{\kappa_a \sin^2 \theta + \kappa_m \cos^2 \theta}, \quad (6.3)$$

with the drive set to the blue-detuned value $\omega_d = \omega_a + 1.5 \omega_b$. It quantifies the rate of coherent magnon–phonon excitation exchange relative to the dissipative losses and, under blue driving, governs the strength of the mechanical amplification. Because blue driving favours Stokes scattering, the dynamical backaction damping is negative, $\Delta \kappa_b = -\kappa_b C < 0$. Thus, although the cooperativity C remains positive, it enters the total mechanical linewidth with a negative sign, $\tilde{\kappa}_b = \kappa_b(1 - C)$, leading to linewidth narrowing and phonon anti-damping (heating) rather than cooling. As C grows, the mode is anti-damped and reaches the parametric-instability (phonon-lasing) threshold at $C = 1$ where the total mechanical linewidth vanishes, $\tilde{\kappa}_b = 0$). Observable amplification, therefore, requires the cooperativity to approach unity from below.

Figure 6.5 shows the field dependence of C for a blue-detuned drive at $P_d = 0.3$ nW and $T_{\text{bath}} = 20$ mK. Hybridisation produces a single broad maximum near $H \simeq 400.3$ Oe, where the photon–magnon mixing most efficiently funnels the coherent excitation into the magnomechanical channel. Crucially, the cooperativity peaks at only $C \simeq 0.47$ and remains *below unity* across the entire accessible field window, never reaching the threshold marked by the dotted line.

The physical origin of this limitation is the restriction on drive power. The cooperativity scales linearly with P_d , so reaching the amplification threshold $C = 1$ would require roughly a two-fold increase in power. However, as established in the linearity analysis of Section 3, increasing the drive power raises the steady-state magnon population n_m and hence the Kerr ratio $K_m n_m / \kappa_m$. At the field of maximum cooperativity the system already operates within a factor of order unity of the linearity threshold $K_m n_m / \kappa_m = 0.5$; pushing the power high enough to bring C to unity would drive the magnon mode into the nonlinear regime, invalidating the linearised description and introducing Kerr-induced bistability and frequency pulling. The device is therefore confined to $P_d \lesssim 0.3$ nW, where the achievable cooperativity remains capped at $C < 1$.

Consequently, although the dynamical backaction unambiguously exhibits the signature of amplification, namely the negative magnomechanical damping $\Delta \kappa < 0$ of Fig. 5.6, the system cannot be driven into a regime of net mechanical amplification. The anti-damping reduces but never cancels the intrinsic mechanical loss, since $C < 1$ implies $\tilde{\kappa}_b = \kappa_b(1 - C) > 0$ everywhere. Net amplification, phonon lasing, and the associated parametric instability all lie beyond the Kerr-limited power window of the present device.

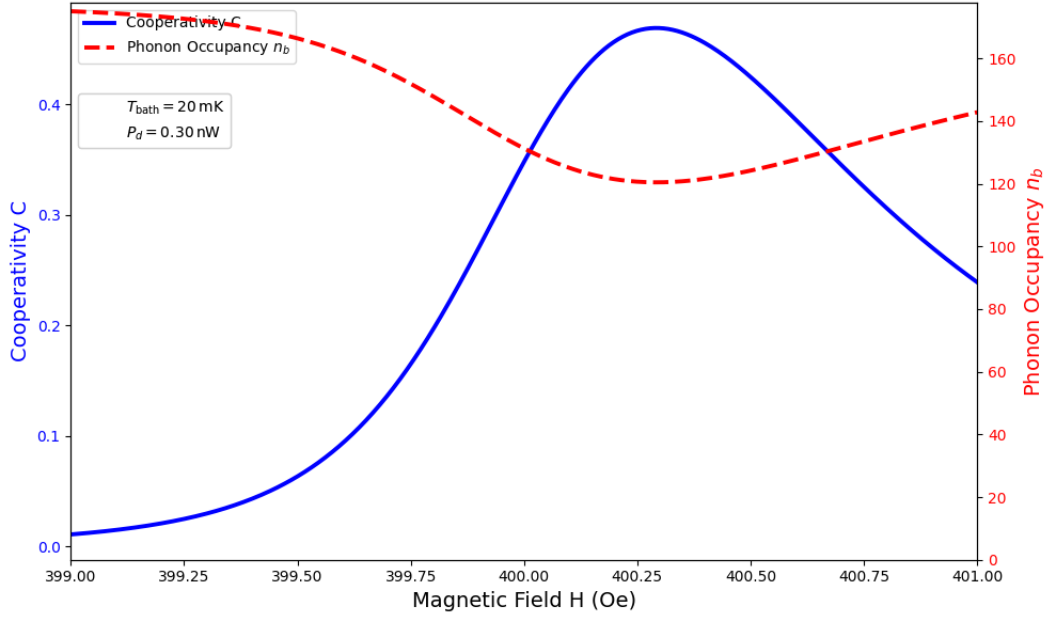


Figure 6.5: Magnetic-field dependence of the cooperativity C for a blue-detuned drive $\omega_d = \omega_a + 1.5\omega_b$ at $P_d = 0.3$ nW and $T_{\text{bath}} = 20$ mK. The cooperativity peaks at $C \simeq 0.47$ near $H \simeq 400.3$ Oe and remains below unity over the entire field window, so the system does not reach the amplification (phonon-lasing) regime; the drive power cannot be raised further without violating the magnon linearity condition.

6.6 Conclusion

In this chapter, we analysed the nano-magnomechanical system under a blue-detuned microwave drive, $\omega_d = \omega_a + 1.5\omega_b$ and $\omega_d = \omega_a + 0.5\omega_b$. Because the Hamiltonian, equations of motion, linearisation, reflection spectrum, phonon self-energy, and cooperativity are formally identical to the red-detuned case, the entire change of physics follows from the single sign reversal of the drive detuning, which renders $\Delta_a = \omega_a - \omega_d$ negative.

The steady-state magnon population (Fig. 5.7) retains the hybridised polariton structure with an avoided crossing near $H \simeq 400$ Oe, and the linearity analysis for both detunings (Figs. 6.1 and 6.2) confirms that the Kerr ratio $K_m n_m / \kappa_m$ stays below the threshold of 0.5, with the resonance maxima now shifted to $H > 400$ Oe, the mirror image of the red-detuned case. The reflection spectrum (Fig. 6.3) displays the same strong-coupling avoided crossing, while the zoomed spectra (Fig. 6.4) reveal the narrow magnomechanically-induced-transparency feature transposed to the opposite mechanical sideband, $\Delta_{sd} \simeq -\omega_b$, reflecting the Stokes-dominant character of blue driving.

The central physical result is the reversal of the dynamical backaction. The magnon-induced phonon self-energy yields a *negative* magnomechanical damping (Fig. 5.6), the hallmark of mechanical amplification and the mirror image of the red-detuned

cooling. However, the cooperativity analysis (Fig. 6.5) shows that, at the drive powers compatible with linear, stable operation, the cooperativity peaks at only $C \simeq 0.47$ and never reaches the amplification threshold $C = 1$. Because blue driving yields an effective linewidth $\tilde{\kappa}_b = \kappa_b(1 - C)$, $C < 1$ ensures that total mechanical dissipation remains strictly positive ($\tilde{\kappa}_b > 0$). Because the cooperativity grows linearly with power while the Kerr nonlinearity simultaneously sets in, the power cannot be raised sufficiently to access the amplification regime without leaving the linearised, dynamically stable window. The system therefore exhibits the clear backaction signature of amplification and phonon heating without permitting net mechanical gain. Accessing genuine amplification and phonon lasing would require device improvements, a higher mechanical quality factor, stronger photon coupling, or a suppressed magnon Kerr coefficient that extend the linear operating range to higher drive powers, thereby pushing the cooperativity above unity.

Chapter 7

Kerr Non Linear Effect

7.1 Nonlinear System Hamiltonian

In the last chapter, a theoretical framework for a hybrid cavity magnomechanical system with three coupled excitations was developed. These excitations include a microwave cavity photon mode, a magnon mode representing collective spin dynamics in a magnetic material, and a mechanical phonon mode. Within this framework, the magnon excitation is treated as a simple harmonic oscillator. This allows us to treat the entire system using a linearised Hamiltonian. This framework allows us to develop equations of motion for the system, analyse the steady-state behaviour, and compute observable properties such as the cavity reflection spectrum using the input-output formalism [23].

Although this linear approximation is both mathematically useful and physically intuitive, it is only accurate in the weak driving regime, where the magnon population is small. However, as the driving strength is increased, the accuracy of this approximation starts to degrade. This is because any magnetic material is nonlinear by its very nature, owing to the spin interactions and crystal structure of the material itself [28]. One of the nonlinear effects is the energy dependence of the magnetisation direction relative to the crystal axes, which is known as magnetocrystalline anisotropy. This gives rise to an anharmonic correction of the magnon modes.

A good way of describing this phenomenon is by using a nonlinearity of the Kerr type for the magnon mode [125, 36, 1]. This can be understood as a physical interaction between magnons, where the frequency of the magnons is affected by the number of magnons already present. This is similar to saying that the frequency of the magnon mode is affected by the intensity of the magnon mode. When the system is driven by a strong microwave signal, a large population of magnons can be created, and the nonlinear effects become relevant. This gives rise to phenomena that cannot be described by a linear model, for example, frequency shifts, bistability, and changes in spectral response

[36].

In order to account for these effects, we modify the Hamiltonian provided in the previous chapter by adding the nonlinear term corresponding to the leading nonlinear contribution of the magnon mode. For the vast majority of experimentally relevant scenarios, this nonlinear contribution may be approximated by the self-Kerr term, which involves a quartic term in the magnon operators. The cross-Kerr term, which describes a two-mode nonlinear interaction where the frequency of one mode shifts proportionally to the excitation number of a second mode (typically governed by $\hbar K_{mb} m^\dagger m b^\dagger b$) is neglected because the phonon population is lower than the magnon population by several orders of magnitude. Other nonlinear contributions among different modes are also possible but are expected to be negligible and may thus be safely neglected for the purposes of this analysis.

In the following sections, we will use the same notation and theoretical framework as previously established and focus on the specific impact of the inclusion of the nonlinear term corresponding to the magnon Kerr nonlinearity on the Hamiltonian of the system.

We consider a tripartite hybrid quantum system composed of three interacting bosonic modes: microwave cavity photons, collective spin excitations (magnons), and mechanical vibrations (phonons). The annihilation (creation) operators for the cavity, magnon, and mechanical modes are denoted by $\hat{a}$ ($\hat{a}^\dagger$), $\hat{m}$ ($\hat{m}^\dagger$), and $\hat{b}$ ($\hat{b}^\dagger$), respectively. In the laboratory frame, the total Hamiltonian of the system is given by

$$\begin{aligned} \hat{H} = & \omega_a \hat{a}^\dagger \hat{a} + \omega_m \hat{m}^\dagger \hat{m} + \omega_b \hat{b}^\dagger \hat{b} + g_{ma} (\hat{a} \hat{m}^\dagger + \hat{a}^\dagger \hat{m}) + g_{mb} \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger) \\ & + K_m (\hat{m}^\dagger \hat{m})^2 + \sqrt{\kappa_{ex}} \varepsilon_d (\hat{a} e^{i\omega_d t} + \hat{a}^\dagger e^{-i\omega_d t}) + \sqrt{\kappa_{ex}} \varepsilon_s (\hat{a} e^{i\omega_s t} + \hat{a}^\dagger e^{-i\omega_s t}). \end{aligned} \quad (7.1)$$

The first three terms describe the free Hamiltonians of the uncoupled cavity, mechanical, and magnon modes, modeled as harmonic oscillators with resonance frequencies ω_a , ω_b , and ω_m , respectively. These terms represent the energy stored in the corresponding bosonic degrees of freedom.

The coherent magnon–photon interaction is described by the beam-splitter term $g_{ma} (\hat{a} \hat{m}^\dagger + \hat{a}^\dagger \hat{m})$, where g_{ma} denotes the coupling strength arising from the magnetic dipole interaction between the cavity microwave field and the collective spin excitations.

The magnon–phonon interaction is captured by the term $g_{mb} \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger)$, which originates from magnetostrictive coupling and has the form of a radiation-pressure interaction [2].

The nonlinear contribution included in this model is the nonlinear term arising from magnon self-Kerr interaction

$$K_m (\hat{m}^\dagger \hat{m})^2,$$

with

$$K_m = \frac{13 \hbar K_{\text{an}} \gamma_c^2}{16 M^2 V_m}, \quad (7.2)$$

where K_m is the magnon self-Kerr coefficient, $\hbar$ is the reduced Planck constant, K_{an} is the magnetocrystalline anisotropy constant, γ_c is the gyromagnetic ratio, M is the saturation magnetization, and V_m is the volume of the magnetic material. This nonlinear term describes the intrinsic anharmonicity of the magnon mode. This term arises primarily from magnetocrystalline anisotropy and leads to a magnon frequency that depends on the magnon population.

The remaining terms describe coherent microwave drives applied to the cavity mode. A strong drive at frequency ω_d and amplitude ε_d populates the cavity, while a weaker probe at frequency ω_s and amplitude ε_s is used to probe the system response. The parameter κ_{ex} denotes the external cavity coupling rate.

The dynamics are simplified by eliminating the explicit time dependence associated with the coherent driving field through a transformation to a frame rotating at the drive frequency ω_d . This transformation is implemented using the unitary operator.

$$\hat{U}(t) = \exp[i\omega_d t (\hat{a}^\dagger \hat{a} + \hat{m}^\dagger \hat{m})]. \quad (7.3)$$

Under this transformation, the Hamiltonian in the rotating frame becomes

$$\hat{H}' = \hat{U} \hat{H} \hat{U}^\dagger + i \frac{d\hat{U}}{dt} \hat{U}^\dagger.$$

After applying the unitary transformation and neglecting the weak probe field whose amplitude is much smaller than that of the strong drive field, the Hamiltonian takes the form

$$\begin{aligned} \hat{H}' = & \Delta_a \hat{a}^\dagger \hat{a} + \Delta_m \hat{m}^\dagger \hat{m} + \omega_b \hat{b}^\dagger \hat{b} + g_{ma} (\hat{a} \hat{m}^\dagger + \hat{a}^\dagger \hat{m}) + g_{mb} \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger) \\ & + K_m (\hat{m}^\dagger \hat{m})^2 + \sqrt{2\kappa_{\text{ex}}} \varepsilon_d (\hat{a} + \hat{a}^\dagger) \end{aligned} \quad (7.4)$$

where $\Delta_a = \omega_a - \omega_d$ and $\Delta_m = \omega_m - \omega_d$ denote the detunings of the cavity and magnon modes from the drive frequency, respectively. The nonlinear contribution retained in this model is the magnon self-Kerr interaction $K_m (\hat{m}^\dagger \hat{m})^2$, which accounts for the intrinsic anharmonicity of the magnon mode.

7.2 Derivation of the Nonlinear Equations of Motion

The time evolution of each mode is determined using the Heisenberg equation of motion

$$\dot{\hat{\mathcal{O}}} = \frac{1}{i\hbar} [\hat{\mathcal{O}}, \hat{H}'] - \kappa_{\mathcal{O}} \hat{\mathcal{O}}, \quad (7.5)$$

where $\hat{\mathcal{O}} \in \{\hat{a}, \hat{m}, \hat{b}\}$. Here $\kappa_a = \kappa_{\text{ex}} + \kappa_i$ denotes the total cavity decay rate, including both external coupling and intrinsic loss. The parameters κ_m and κ_b represent the magnon and mechanical damping rates, respectively. As in the previous chapter, input noise terms for the magnon and mechanical modes are neglected since they are much smaller than the coherent drive.

Since the nonlinear extension introduced in this chapter only affects the magnon mode through the self-Kerr interaction, the equations of motion for the cavity and mechanical modes remain unchanged from the linear theory presented in Chapter 5. For completeness we briefly state them below.

The cavity dynamics are identical to those obtained in Chapter 5 and are given by

$$\dot{\hat{a}} = -(i\Delta_a + \kappa_a) \hat{a} - ig_{ma} \hat{m} - i\sqrt{2\kappa_{\text{ex}}}\varepsilon_d. \quad (7.6)$$

Similarly, the mechanical mode equation remains unchanged and takes the form

$$\dot{\hat{b}} = -(i\omega_b + \kappa_b) \hat{b} - ig_{mb} \hat{m}^\dagger \hat{m}. \quad (7.7)$$

The equation of motion for the magnon mode is modified due to the presence of the self-Kerr interaction. Starting from Eq. (7.5), we obtain

$$\begin{aligned} \dot{\hat{m}} &= \frac{1}{i\hbar} [\hat{m}, \hat{H}'] - \kappa_m \hat{m} \\ &= -i\Delta_m [\hat{m}, \hat{m}^\dagger \hat{m}] - ig_{ma} [\hat{m}, (\hat{a} \hat{m}^\dagger + \hat{a}^\dagger \hat{m})] - ig_{mb} [\hat{m}, \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger)] \\ &\quad - iK_m [\hat{m}, (\hat{m}^\dagger \hat{m})^2] - \kappa_m \hat{m}. \end{aligned} \quad (7.8)$$

The required commutators are

$$[\hat{m}, \hat{m}^\dagger \hat{m}] = \hat{m}, \quad (7.9)$$

$$[\hat{m}, \hat{a} \hat{m}^\dagger + \hat{a}^\dagger \hat{m}] = \hat{a}, \quad (7.10)$$

$$[\hat{m}, \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger)] = \hat{m} (\hat{b} + \hat{b}^\dagger), \quad (7.11)$$

$$[\hat{m}, (\hat{m}^\dagger \hat{m})^2] = 2\hat{m}^\dagger \hat{m} \hat{m} + \hat{m}. \quad (7.12)$$

Substituting these results yields

$$\begin{aligned}\dot{\hat{m}} &= -i\Delta_m \hat{m} - ig_{ma} \hat{a} - ig_{mb} \hat{m}(\hat{b} + \hat{b}^\dagger) - 2iK_m \hat{m}^\dagger \hat{m} \hat{m} - \kappa_m \hat{m} \\ &= -\left[i\Delta_m + ig_{mb}(\hat{b} + \hat{b}^\dagger) + 2iK_m \hat{m}^\dagger \hat{m} + \kappa_m\right] \hat{m} - ig_{ma} \hat{a}.\end{aligned}\quad (7.13)$$

where the term $K_m \hat{m} \ll \Delta_m \hat{m}$, hence neglected. The additional nonlinear term $2K_m \hat{m}^\dagger \hat{m}$ represents the Kerr-induced shift of the magnon frequency, which depends on the magnon population.

7.3 Steady-State Solutions and Mean-Field Magnon

Under the strong coherent drive, the system is linearized by expanding the mode operators around their classical averages:

$$\hat{a} = \langle a \rangle + \delta a = a + \delta a, \quad (7.14)$$

$$\hat{m} = \langle m \rangle + \delta m = m + \delta m, \quad (7.15)$$

$$\hat{b} = \langle b \rangle + \delta b = b + \delta b, \quad (7.16)$$

where $a = \langle a \rangle$, $m = \langle m \rangle$, and $b = \langle b \rangle$ denote the classical steady-state amplitudes of the cavity, magnon, and mechanical modes, respectively, while δa , δm , and δb represent the corresponding quantum fluctuations.

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The steady-state equations for the classical amplitudes are obtained by taking expectation values of the equations of motion and factorizing operator products (for example, $\langle \hat{m}^\dagger \hat{m} \rangle \approx |m|^2$). This yields

$$0 = (-i\Delta_a - \kappa_a)a - ig_{ma}m - i\sqrt{2\kappa_{ex}\varepsilon_d}, \quad (7.17)$$

$$0 = (-i\omega_b - \kappa_b)b - ig_{mb}|m|^2, \quad (7.18)$$

$$0 = [-i\Delta_m - ig_{mb}(b + b^*) - 2iK_m|m|^2 - \kappa_m]m - ig_{ma}a. \quad (7.19)$$

The nonlinear contribution enters through the term $2K_m|m|^2$, which represents a Kerr-induced shift of the magnon frequency that depends on the magnon population. As a result, the steady-state magnon amplitude is determined by a nonlinear equation, reflecting the intrinsic anharmonicity of the magnon mode.

From Eq. (7.18), solving for the steady-state mechanical amplitude gives

$$b = \frac{-ig_{mb}|m|^2}{i\omega_b + \kappa_b}, \quad (7.20)$$

$$b^* = \frac{ig_{mb}|m|^2}{-i\omega_b + \kappa_b}. \quad (7.21)$$

The combination $b + b^*$ appearing in the magnon equation becomes

$$\begin{aligned} b + b^* &= g_{mb}|m|^2 \left(\frac{-i}{i\omega_b + \kappa_b} + \frac{i}{-i\omega_b + \kappa_b} \right) \\ &= \frac{2g_{mb}\omega_b|m|^2}{\omega_b^2 + \kappa_b^2}. \end{aligned} \quad (7.22)$$

Substituting this expression into the magnon steady-state equation, Eq. (7.19), we obtain

$$0 = [-i\Delta_m - ig_{mb}(b + b^*) - 2iK_m|m|^2 - \kappa_m] m - ig_{ma}a. \quad (7.23)$$

Solving for the magnon amplitude yields

$$m = \frac{-ig_{ma}a}{i\Delta_m + ig_{mb}(b + b^*) + 2iK_m|m|^2 + \kappa_m}. \quad (7.24)$$

From Eq. (7.17), the cavity steady-state amplitude is

$$a = \frac{-ig_{ma}m - i\sqrt{2\kappa_{ex}\varepsilon_d}}{i\Delta_a + \kappa_a}. \quad (7.25)$$

Substituting Eq. (7.25) into Eq. (7.24) gives a self-consistency equation for m :

$$m = \frac{-ig_{ma}}{i\Delta_m + ig_{mb}(b + b^*) + 2iK_m|m|^2 + \kappa_m} \cdot \frac{-ig_{ma}m - i\sqrt{2\kappa_{ex}\varepsilon_d}}{i\Delta_a + \kappa_a}. \quad (7.26)$$

Rearranging, we obtain

$$[(i\Delta_a + \kappa_a)(i\Delta_m + ig_{mb}(b + b^*) + 2iK_m|m|^2 + \kappa_m) + g_{ma}^2] m = -g_{ma}\sqrt{2\kappa_{ex}\varepsilon_d}. \quad (7.27)$$

Taking the modulus squared of both sides yields an implicit equation for the magnon population $n_m = |m|^2$:

$$n_m |(i\Delta_a + \kappa_a)(i\Delta_m + ig_{mb}(b + b^*) + 2iK_m n_m + \kappa_m) + g_{ma}^2|^2 = g_{ma}^2 \cdot 2\kappa_{ex} |\varepsilon_d|^2. \quad (7.28)$$

Substituting Eq. (7.22) into Eq. (7.28) eliminates b and b^* . Defining the total effective Kerr shift parameter

$$K_{\text{eff}} = 2K_m + \frac{2g_{mb}^2 \omega_b}{\omega_b^2 + \kappa_b^2}, \quad (7.29)$$

Eq. (7.28) can be expanded directly into a closed cubic equation in n_m :

$$c_3 n_m^3 + c_2 n_m^2 + c_1 n_m + c_0 = 0, \quad (7.30)$$

where the real coefficients are expressed directly in terms of system parameters:

$$c_3 = (\Delta_a^2 + \kappa_a^2) K_{\text{eff}}^2, \quad (7.31)$$

$$c_2 = 2K_{\text{eff}} [\Delta_m (\Delta_a^2 + \kappa_a^2) - \Delta_a g_{ma}^2], \quad (7.32)$$

$$c_1 = (\kappa_a \kappa_m - \Delta_a \Delta_m + g_{ma}^2)^2 + (\Delta_a \kappa_m + \Delta_m \kappa_a)^2, \quad (7.33)$$

$$c_0 = -2g_{ma}^2 \kappa_{ex} |\varepsilon_d|^2. \quad (7.34)$$

Eq. (7.30) allows direct numerical or analytical evaluation of the steady-state magnon magnitude $|m|$ as a function of drive strength $|\varepsilon_d|^2$ or detuning.

However, when the Kerr nonlinearity becomes significant, Eq. (7.28) must be solved self-consistently. In this regime the nonlinear dependence of the magnon frequency on $|m|^2$ can lead to multiple steady-state solutions and bistable behaviour.

7.4 Linearisation About the Steady State and Fluctuation Dynamics

We consider the coupled Heisenberg equations

$$\dot{\hat{a}} = -(i\Delta_a + \kappa_a) \hat{a} - ig_{ma} \hat{m} - i\sqrt{2\kappa_{ex}} \varepsilon_d, \quad (7.35)$$

$$\dot{\hat{b}} = -(i\omega_b + \kappa_b) \hat{b} - ig_{mb} \hat{m}^\dagger \hat{m}, \quad (7.36)$$

$$\dot{\hat{m}} = - \left[i\Delta_m + ig_{mb}(\hat{b} + \hat{b}^\dagger) + 2iK_m \hat{m}^\dagger \hat{m} + \kappa_m \right] \hat{m} - ig_{ma} \hat{a}. \quad (7.37)$$

The nonlinearities arise from the operator products $\hat{m}^\dagger \hat{m}$, $\hat{m}(\hat{b} + \hat{b}^\dagger)$, and the Kerr term $(\hat{m}^\dagger \hat{m}) \hat{m}$. We linearise these equations about the steady state by decomposing each operator into a steady-state mean field and a fluctuation operator as in Eqs. (7.14)–(7.16).

The nonlinear operator products are expanded to first order in fluctuations by writing $\hat{m} = m + \delta\hat{m}$ and $\hat{b} = b + \delta\hat{b}$, and neglecting $\mathcal{O}(\delta^2)$ terms.

$$\begin{aligned}\hat{m}^\dagger\hat{m} &= (m^* + \delta\hat{m}^\dagger)(m + \delta\hat{m}) \\ &= |m|^2 + m^* \delta\hat{m} + m \delta\hat{m}^\dagger + \mathcal{O}(\delta^2).\end{aligned}\tag{7.38}$$

$$\begin{aligned}\hat{m}(\hat{b} + \hat{b}^\dagger) &= (m + \delta\hat{m}) \left[(b + b^*) + (\delta\hat{b} + \delta\hat{b}^\dagger) \right] \\ &= m(b + b^*) + m(\delta\hat{b} + \delta\hat{b}^\dagger) + (b + b^*) \delta\hat{m} + \mathcal{O}(\delta^2).\end{aligned}\tag{7.39}$$

$$\begin{aligned}(\hat{m}^\dagger\hat{m})\hat{m} &= (|m|^2 + m^* \delta\hat{m} + m \delta\hat{m}^\dagger)(m + \delta\hat{m}) \\ &= |m|^2 m + 2|m|^2 \delta\hat{m} + m^2 \delta\hat{m}^\dagger + \mathcal{O}(\delta^2).\end{aligned}\tag{7.40}$$

7.4.1 Linearized Equations of Motion

Substituting these expansions into Eqs. (7.35)–(7.37) and subtracting the steady-state equations yields the linearized fluctuation equations:

$$\dot{\delta\hat{a}} = -(i\Delta_a + \kappa_a) \delta\hat{a} - ig_{ma} \delta\hat{m},\tag{7.41}$$

$$\dot{\delta\hat{b}} = -(i\omega_b + \kappa_b) \delta\hat{b} - ig_{mb} (m^* \delta\hat{m} + m \delta\hat{m}^\dagger),\tag{7.42}$$

$$\begin{aligned}\dot{\delta\hat{m}} &= -(i\Delta'_m + \kappa_m) \delta\hat{m} - 2iK_m m^2 \delta\hat{m}^\dagger - ig_{ma} \delta\hat{a} \\ &\quad - ig_{mb} m \delta\hat{b} - ig_{mb} m \delta\hat{b}^\dagger,\end{aligned}\tag{7.43}$$

where the effective magnon detuning is

$$\Delta'_m = \Delta_m + g_{mb}(b + b^*) + 4K_m |m|^2.\tag{7.44}$$

7.5 Nonlinear Reflection Spectrum with Magnon Self-Kerr

Let the cavity be probed at frequency ω_s with amplitude ϵ_s , in the presence of a strong drive at ω_d which sets the steady-state magnon amplitude $m \equiv \langle \hat{m} \rangle$. Define the probe-drive detuning

$$\Omega \equiv \omega_s - \omega_d.\tag{7.45}$$

The linearized fluctuation equations including the magnon self-Kerr term are

$$\dot{\delta\hat{a}} = -(i\Delta_a + \kappa_a) \delta\hat{a} - ig_{ma} \delta\hat{m} + \sqrt{2\kappa_{ex}} \delta\hat{a}_{in}, \quad (7.46)$$

$$\dot{\delta\hat{b}} = -(i\omega_b + \kappa_b) \delta\hat{b} - ig_{mb} (m^* \delta\hat{m} + m \delta\hat{m}^\dagger), \quad (7.47)$$

$$\dot{\delta\hat{m}} = -(i\Delta'_m + \kappa_m) \delta\hat{m} - 2iK_m m^2 \delta\hat{m}^\dagger - ig_{ma} \delta\hat{a} - ig_{mb} m (\delta\hat{b} + \delta\hat{b}^\dagger), \quad (7.48)$$

with the effective magnon detuning including Kerr,

$$\Delta'_m = \Delta_m + g_{mb}(b + b^*) + 4K_m |m|^2. \quad (7.49)$$

Following the same derivation as in the linear case, the mechanics-induced magnon self-energy remains,

$$\Sigma_{mb}(\Omega) = |G_{mb}|^2 [\chi_b(\Omega) + \chi_b^*(-\Omega)], \quad G_{mb} = g_{mb}m, \quad (7.50)$$

and the magnon-dressed susceptibility, including Kerr, is

$$\Xi[\Omega] = \frac{1}{\chi_m^{-1}(\Omega) + \Sigma_{mb}(\Omega) + 2iK_m |m|^2}, \quad (7.51)$$

where $\chi_m^{-1}(\Omega) = \kappa_m + i(\Delta'_m - \Omega)$.

Finally, eliminating the magnon yields the cavity's effective susceptibility

$$\chi_a^{\text{eff}}(\Omega) = \frac{1}{\chi_a^{-1}(\Omega) + g_{ma}^2 \Xi[\Omega]} = \frac{1}{\kappa_a + i(\Delta_a - \Omega) + \frac{g_{ma}^2}{\kappa_m + i(\Delta'_m - \Omega) + \Sigma_{mb}(\Omega) + 2iK_m |m|^2}}. \quad (7.52)$$

The reflection spectra is then

$$S_{11} = 1 - 2\kappa_{ex} \chi_a^{\text{eff}}(\Omega) = 1 - \frac{2\kappa_{ex}}{\kappa_a + i(\Delta_a - \Omega) + \frac{g_{ma}^2}{\kappa_m + i(\Delta'_m - \Omega) + \Sigma_{mb}(\Omega) + 2iK_m |m|^2}}. \quad (7.53)$$

where the nonlinear contribution comes from the term

$$\frac{g_{ma}^2}{\kappa_m + i(\Delta'_m - \Omega) + \Sigma_{mb}(\Omega) + 2iK_m |m|^2}$$

7.6 Phonon Self-Energy and Non-Linear magnon spring effect Magnon

Magnons exert an effective radiation-pressure-like force on the mechanical mode due to magnetoelastic coupling. This feedback modifies the mechanical response, giving rise to

dynamical backaction [14]. In the absence of magnomechanical coupling, the mechanical oscillator is fully described by its bare susceptibility, determined solely by its resonance frequency and intrinsic damping. When coupled to the magnon mode, however, the mechanical dynamics are changed through this interaction-induced feedback. These modifications are the signature of the magnon-induced phonon self-energy on the mechanical mode.

The linearized equations of fluctuation, including the magnon self-Kerr term, are

$$\chi_a^{-1}[\omega] \delta \hat{a}[\omega] = -ig_{ma} \delta \hat{m}[\omega], \quad (7.54)$$

$$\chi_m^{-1}[\omega] \delta \hat{m}[\omega] = -ig_{ma} \delta \hat{a}[\omega] - iG_{mb} \left(\delta \hat{b}[\omega] + \delta \hat{b}^\dagger[-\omega] \right) - 2iK_m \bar{m}^2 \delta \hat{m}^\dagger[-\omega], \quad (7.55)$$

$$\chi_b^{-1}[\omega] \delta \hat{b}[\omega] = -i \left(G_{mb}^* \delta \hat{m}[\omega] + G_{mb} \delta \hat{m}^\dagger[-\omega] \right), \quad (7.56)$$

where

$$G_{mb} \equiv g_{mb} \bar{m}, \quad \bar{m} \equiv \langle \hat{m} \rangle, \quad (7.57)$$

is the linearized (enhanced) magnomechanical coupling .

From the cavity equation,

$$\delta \hat{a}[\omega] = -ig_{ma} \chi_a[\omega] \delta \hat{m}[\omega]. \quad (7.58)$$

substituting this into the magnon equation gives

$$\left(\chi_m^{-1}[\omega] + g_{ma}^2 \chi_a[\omega] \right) \delta \hat{m}[\omega] - 2iK_m \bar{m}^2 \delta \hat{m}^\dagger[-\omega] = -iG_{mb} \left(\delta \hat{b}[\omega] + \delta \hat{b}^\dagger[-\omega] \right). \quad (7.59)$$

Because the magnon self-Kerr nonlinearity couples the annihilation fluctuation $\delta \hat{m}[\omega]$ directly to its creation conjugate $\delta \hat{m}^\dagger[-\omega]$, a scalar susceptibility treatment fails. To accurately capture the frequency-dependent squeezing and phase correlation, we must set up the linear system in a 2×2 Nambu matrix basis.

Defining the cavity-modified inverse magnon susceptibility as $\tilde{\chi}_m^{-1}[\omega] \equiv \chi_m^{-1}[\omega] + g_{ma}^2 \chi_a[\omega]$, pairing the equation for $\delta \hat{m}[\omega]$ with its Hermitian conjugate at frequency $-\omega$ forms the coupled matrix equation:

$$\begin{pmatrix} \tilde{\chi}_m^{-1}[\omega] & 2iK_m \bar{m}^2 \\ -2iK_m (\bar{m}^*)^2 & (\tilde{\chi}_m^{-1}[-\omega])^* \end{pmatrix} \begin{pmatrix} \delta \hat{m}[\omega] \\ \delta \hat{m}^\dagger[-\omega] \end{pmatrix} = \begin{pmatrix} -iG_{mb} \\ +iG_{mb}^* \end{pmatrix} \begin{pmatrix} \delta \hat{b}[\omega] + \delta \hat{b}^\dagger[-\omega] \end{pmatrix}. \quad (7.60)$$

Inverting this matrix gives the determinant $D[\omega] \equiv \tilde{\chi}_m^{-1}[\omega] (\tilde{\chi}_m^{-1}[-\omega])^* - 4K_m^2 |\bar{m}|^4$. Solving for the magnon fluctuation operators yields:

$$\delta \hat{m}[\omega] = -iG_{mb} \Xi_1[\omega] \left(\delta \hat{b}[\omega] + \delta \hat{b}^\dagger[-\omega] \right), \quad (7.61)$$

$$\delta \hat{m}^\dagger[-\omega] = +iG_{mb}^* \Xi_2[\omega] \left(\delta \hat{b}[\omega] + \delta \hat{b}^\dagger[-\omega] \right), \quad (7.62)$$

where the correct dressed Kerr-magnon response functions are defined as

$$\Xi_1[\omega] \equiv \frac{(\tilde{\chi}_m^{-1}[-\omega])^* + 2K_m|\bar{m}|^2}{D[\omega]}, \quad \Xi_2[\omega] \equiv \frac{\tilde{\chi}_m^{-1}[\omega] - 2K_m|\bar{m}|^2}{D[\omega]}. \quad (7.63)$$

The distinct response functions $\Xi_1[\omega]$ and $\Xi_2[\omega]$ reflect the phase-sensitive amplification and frequency asymmetry introduced by the Kerr medium, which directly renormalises the dynamic feedback exerted on the phonons.

Inserting into the phonon equation yields

$$\chi_b^{-1}[\omega] \delta\hat{b}[\omega] = -|G_{mb}|^2 (\Xi_1[\omega] - \Xi_2[\omega]) (\delta\hat{b}[\omega] + \delta\hat{b}^\dagger[-\omega]) \quad (7.64)$$

so the magnon-induced phonon self-energy is

$$\Sigma_0[\omega] \equiv |G_{mb}|^2 (\Xi_1[\omega] - \Xi_2[\omega]). \quad (7.65)$$

Proceeding as before, we eliminate $\delta\hat{b}^\dagger[-\omega]$ to get

$$\delta\hat{b}[\omega] + \delta\hat{b}^\dagger[-\omega] = \left[1 + \frac{\Sigma_0[\omega]}{(\chi_b^*[-\omega])^{-1} - \Sigma_0[\omega]} \right] \delta\hat{b}[\omega]. \quad (7.66)$$

The closed phonon equation becomes

$$\boxed{\chi_b^{-1}[\omega] \delta\hat{b}[\omega] = -\Sigma_0[\omega] \left[1 + \frac{\Sigma_0[\omega]}{(\chi_b^*[-\omega])^{-1} - \Sigma_0[\omega]} \right] \delta\hat{b}[\omega]}. \quad (7.67)$$

Bringing the interaction-induced term to the LHS, the dressed mechanical equation reads

$$\boxed{(\chi_b^{-1}[\omega] - i\Sigma[\omega]) \delta\hat{b}[\omega] = 0}. \quad (7.68)$$

The full phonon self-energy, including antiresonant corrections, is

$$\boxed{\Sigma[\omega] = i|G_{mb}|^2 (\Xi_1[\omega] - \Xi_2[\omega]) \left[1 + \frac{|G_{mb}|^2 (\Xi_1[\omega] - \Xi_2[\omega])}{(\chi_b^*[-\omega])^{-1} - |G_{mb}|^2 (\Xi_1[\omega] - \Xi_2[\omega])} \right]}. \quad (7.69)$$

If the antiresonant term is small, i.e. $||G_{mb}|^2(\Xi_1[\omega] - \Xi_2[\omega]) \chi_b^*[-\omega]| \ll 1$, we can neglect the bracketed correction:

$$\boxed{\Sigma[\omega] \simeq i|G_{mb}|^2 (\Xi_1[\omega] - \Xi_2[\omega])}. \quad (7.70)$$

The resulting mechanical frequency shift and damping are

$$\tilde{\omega}_b = \omega_b + \delta\omega_b, \quad \Gamma_{\text{tot}} = \Gamma_b + \Gamma_{\text{mag}}, \quad \delta\omega_b = -\text{Re} \Sigma[\omega], \quad \Gamma_{\text{mag}} = 2 \text{Im} \Sigma[\omega]. \quad (7.71)$$

7.7 Results and Discussions

Having established the linear coupling between magnon and phonon modes in the previous section, we now investigate how the intrinsic Kerr nonlinearity of the magnon mode fundamentally alters the phonon response under strong driving. Rather than inducing a simple scalar frequency shift, the Kerr interaction couples the magnon fluctuation operator $\delta\hat{m}[\omega]$ directly to its conjugate $\delta\hat{m}^\dagger[-\omega]$. Evaluated via a 2×2 Nambu matrix formulation, this interaction introduces frequency-dependent squeezing, phase correlations, and asymmetric dressed response functions $\Xi_1[\omega]$ and $\Xi_2[\omega]$ that govern the phonon self-energy $\Sigma[\omega]$. In this section, we systematically explore this nonlinear phonon response as a function of drive detuning and power.

Figure 7.1 illustrates the dependence of the frequency shift ($\Delta\omega/2\pi$) and linewidth shift ($\Delta\kappa/2\pi$) on the normalized drive detuning (Δ_d/ω_b) across three drive powers (0.15 nW, 0.5 nW, and 1 nW) in the linear regime where the Kerr coefficient vanishes ($K = 0$). As shown in Panel (a), the frequency shift exhibits characteristic dispersive, antisymmetric features centered around the sidebands at $\Delta_d/\omega_b = \pm 0.5$. Near zero detuning ($\Delta_d/\omega_b = 0$), a distinct zero-crossing emerges, marking a Fano-type interference that arises from quantum wave interference between the discrete mode and the continuous background drive continuum. As the input power increases, the amplitude of this frequency shift scales proportionally without altering its fundamental lineshape, confirming a purely linear response.

In Panel (b), the linewidth shift displays symmetric, Lorentzian-like profiles corresponding to absorption and amplification at the sideband locations. At the red sideband ($\Delta_d/\omega_b = -0.5$), a positive shift ($\Delta\kappa > 0$) indicates enhanced dissipation through induced damping, whereas at the blue sideband ($\Delta_d/\omega_b = +0.5$), a negative shift ($\Delta\kappa < 0$) reflects anti-damping and linewidth narrowing. Because the Kerr nonlinearity is absent ($K = 0$), the overall response functions maintain strict mirror and anti-symmetry around $\Delta_d = 0$. The peak and dip amplitudes scale strictly linearly with drive power, increasing from 0.15 nW up to 1 nW, without exhibiting any of the asymmetric frequency-pulling or bistability typically introduced by a non-zero Kerr term.

Figure 7.2 shows the phonon frequency shift $\Delta\omega_b/2\pi$ and phonon damping rate $\Delta\kappa_b/2\pi$ as functions of the normalised drive detuning $(\omega_d - \omega_a)/\omega_b$ for three drive powers $P \in \{0.15, 0.5, 1\}$ nW. Both panels display a pronounced spectral asymmetry between the red- and blue-detuned sidebands resulting from the Kerr-modified Nambu response. In panel (a), the frequency shift $\Delta\omega_b/2\pi$ exhibits a dominant positive peak near the red sideband $(\omega_d - \omega_a)/\omega_b \approx -0.5$ (reaching ~ 11.6 kHz for 1 nW) and a secondary positive peak near the blue sideband $(\omega_d - \omega_a)/\omega_b \approx 0.5$ (~ 5.3 kHz). In panel (b), the damping rate $\Delta\kappa_b/2\pi$ exhibits a positive cooling peak at the red sideband (~ 20 kHz for 1 nW) and a larger negative amplification trough at the blue sideband (~ -29 kHz for 1 nW).

This pronounced sideband asymmetry reflects the parametric interference between

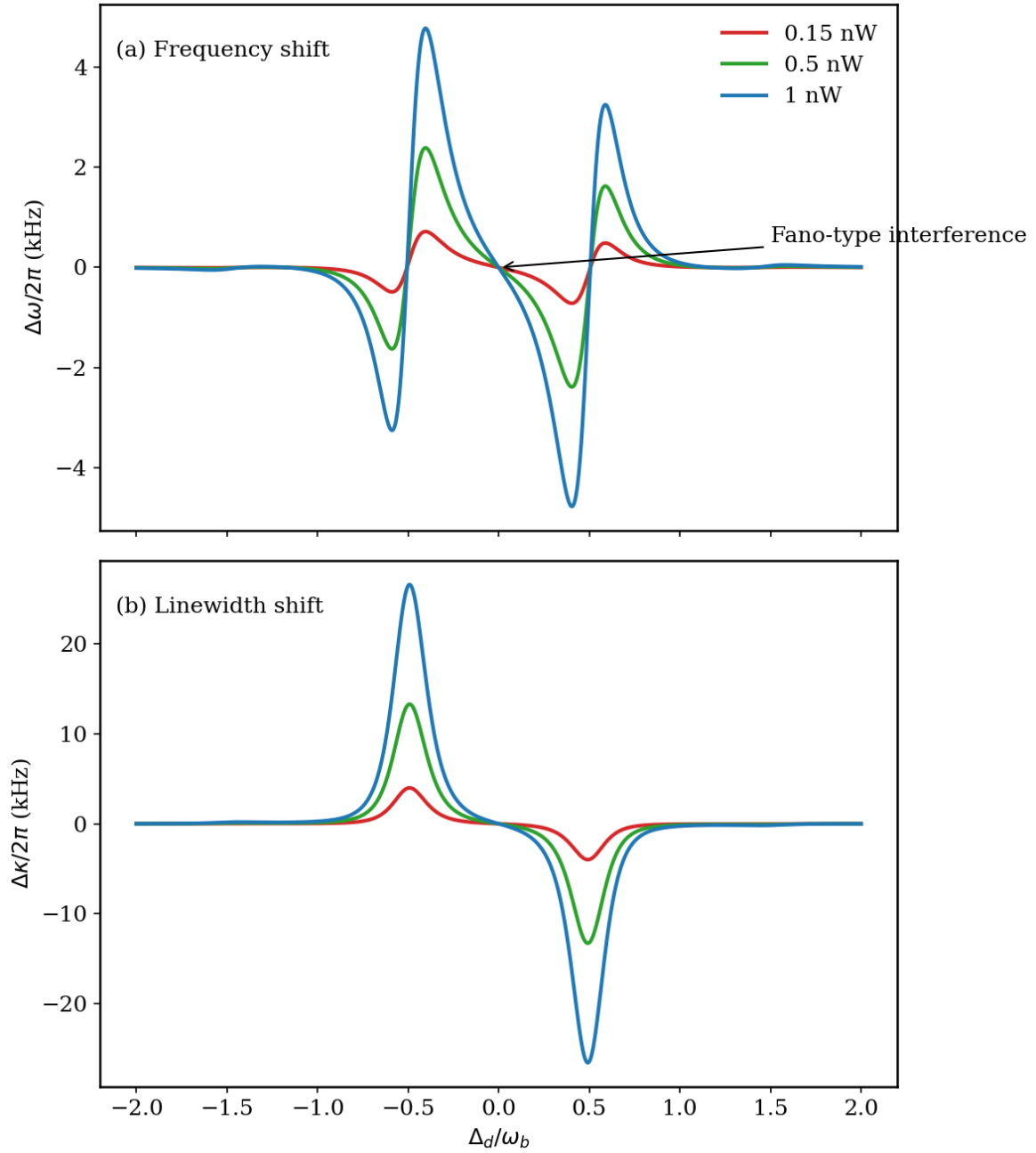


Figure 7.1: (a) Phonon frequency shift $\Delta\omega_b/2\pi$ as a function of the normalised drive detuning $(\omega_d - \omega_a)/\omega_b$ for varying drive powers P when $K_m = 0.2\text{Hertz}$. (b) Phonon damping $\Delta\kappa_b/2\pi$ as a function of the normalised drive detuning $(\omega_d - \omega_a)/\omega_b$ for varying drive powers P .

Stokes and anti-Stokes sidebands under Kerr-induced squeezing: while anti-Stokes scattering extracts phonons on the red sideband, the matrix cross-coupling between

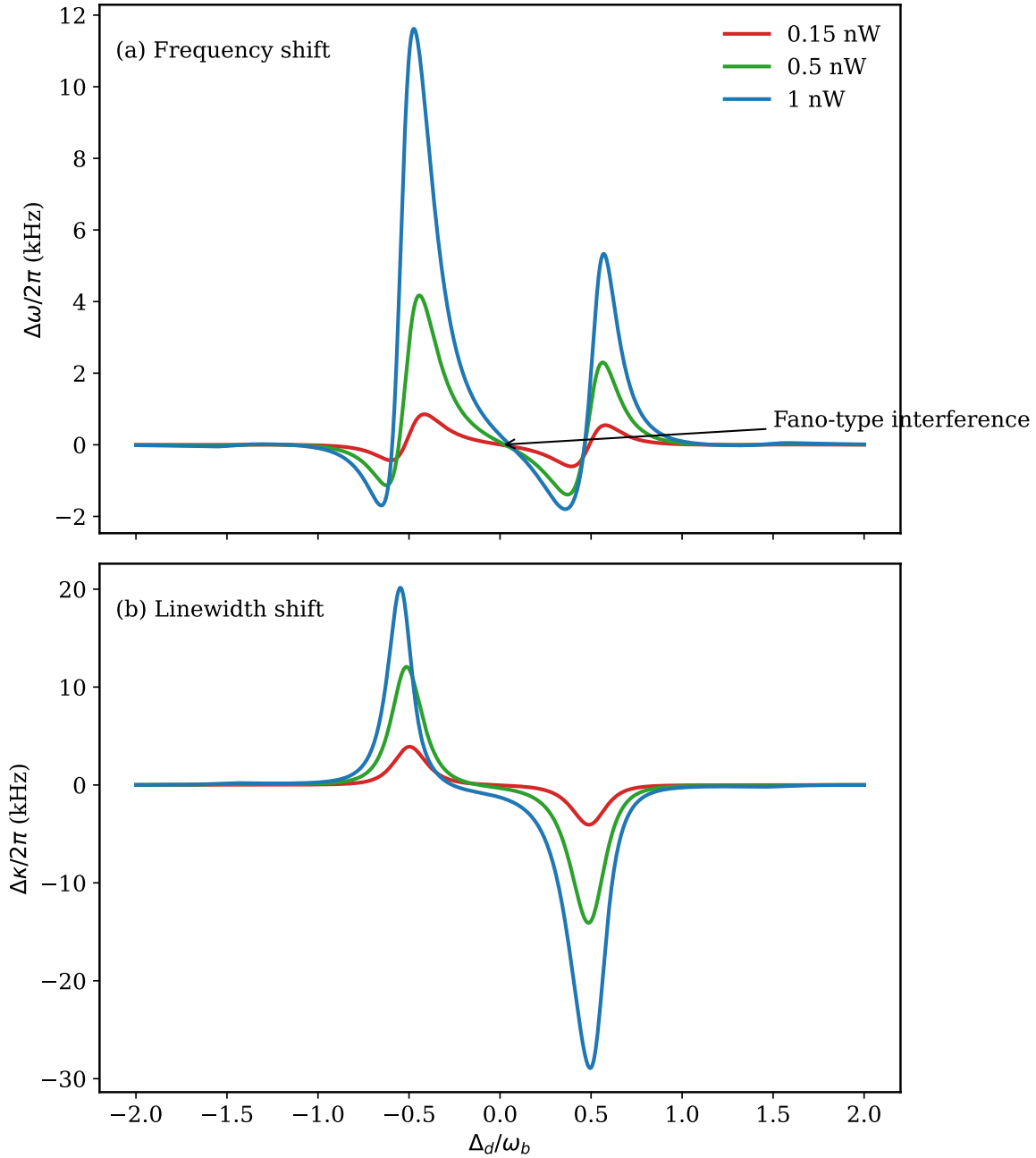


Figure 7.2: (a) Phonon frequency shift $\Delta\omega_b/2\pi$ as a function of the normalised drive detuning $(\omega_d - \omega_a)/\omega_b$ for varying drive powers P when $K_m = 0.2 \text{ Hertz}$. (b) Phonon damping $\Delta\kappa_b/2\pi$ as a function of the normalised drive detuning $(\omega_d - \omega_a)/\omega_b$ for varying drive powers P .

$\delta\hat{m}[\omega]$ and $\delta\hat{m}^\dagger[-\omega]$ breaks standard scalar symmetry, enhancing the red-sideband optical spring while pushing the blue-sideband damping into strong negative values.

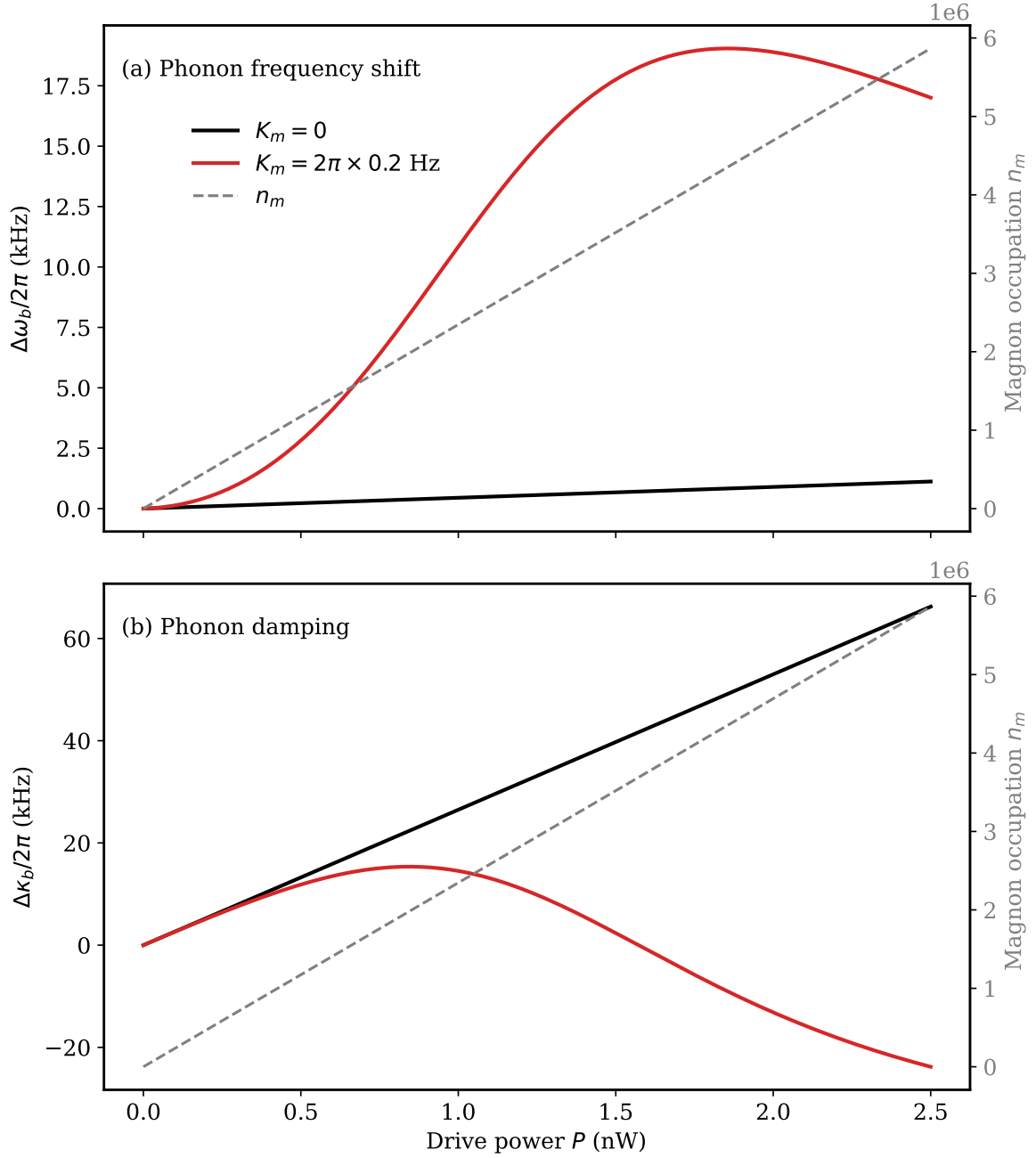


Figure 7.3: (a) Phonon frequency shift $\Delta\omega_b/2\pi$ and steady-state magnon occupation n_m (grey dashed curve, right axis) as a function of drive power P at fixed detuning $\omega_d = \omega_a - 0.5\omega_b$. (b) Phonon damping $\Delta\kappa_b/2\pi$ and magnon occupation n_m as a function of P for $K_m = 0$ (linear) and $K_m = 2\pi \times 0.2$ Hz (nonlinear).

The most prominent features occur near $(\omega_d - \omega_a)/\omega_b \approx \pm 0.5$, where the cavity-magnon dressed modes satisfy the effective sideband resonance condition, causing $|\Delta\omega_b|$ and

$|\Delta\kappa_b|$ reaching their extrema. The sharpness of these peaks confirms that the system operates in the *resolved-sideband regime* ($\omega_b \gg \kappa_a$). A secondary structure is visible near $|(\omega_d - \omega_a)/\omega_b| \approx 1.2\text{--}1.4$ arises from the hybridisation of the magnon and microwave cavity into polariton branches separated by $\sim 2g_{ma}$.

At a given detuning, the amplitude of both $\Delta\omega_b$ and $\Delta\kappa_b$ grows with drive power P via $G_{mb}^2 \propto \bar{m} \propto P$. For $P = 1$ nW, the peak extra damping reaches several tens of kilohertz, demonstrating a large many-magnon enhancement of the effective magnetostrictive interaction.

Figure 7.3 presents the same observables alongside the steady-state magnon population n_m (grey dashed line, right axis) as functions of drive power P at a fixed drive detuning $\omega_d = \omega_a - 0.5\omega_b$, comparing the linear magnon case ($K_m = 0$) with the Kerr-nonlinear case ($K_m = 2\pi \times 0.2$ Hz).

The secondary y-axis confirms that the magnon population grows strictly linearly with drive power, rising from $n_m = 0$ at $P = 0$ nW to $n_m \approx 5.8 \times 10^6$ at $P = 2.5$ nW. This linear growth establishes that all non-monotonic features and threshold behaviours observed in the nonlinear observables stem entirely from the Kerr-induced off-diagonal Nambu cross-coupling ($2K_m n_m$), rather than any trivial saturation of the steady-state magnon population.

In the linear case ($K_m = 0$, black curves), both $\Delta\omega_b/2\pi$ and $\Delta\kappa_b/2\pi$ scale linearly with P (and thus with n_m), reaching $\Delta\omega_b/2\pi \approx 1.1$ kHz and $\Delta\kappa_b/2\pi \approx 66$ kHz at $P = 2.5$ nW.

In stark contrast, the Kerr-nonlinear case ($K_m = 2\pi \times 0.2$ Hz, red curves) demonstrates a clear decoupling between frequency shift and damping behaviour:

- **Phonon Frequency Shift [Panel (a)]:** At low powers ($P \lesssim 0.3$ nW), $\Delta\omega_b/2\pi$ grows superlinearly as the off-diagonal coupling $2K_m n_m$ dresses the dispersive optical spring. It reaches a high maximum of $\Delta\omega_b/2\pi \approx 19.0$ kHz near $P \approx 1.85$ nW ($n_m \approx 4.3 \times 10^6$) more than 20 times larger than the linear baseline before turning slightly downward to ~ 17.0 kHz at $P = 2.5$ nW as matrix denominator saturation sets in.
- **Phonon Damping [Panel (b)]:** The damping rate $\Delta\kappa_b/2\pi$ tracks the linear cooling response up to $P \approx 0.3$ nW, reaches a broad cooling maximum of $\Delta\kappa_b/2\pi \approx 15.2$ kHz around $P \approx 0.85$ nW ($n_m \approx 2.0 \times 10^6$), and then steadily declines. It crosses zero ($\Delta\kappa_b = 0$) at an instability threshold of $P \approx 1.60$ nW ($n_m \approx 3.7 \times 10^6$). Beyond this point, the system transitions into parametric phonon amplification (negative linewidth shift), reaching $\Delta\kappa_b/2\pi \approx -23.8$ kHz at $P = 2.5$ nW.

This fundamental divergence where the frequency shift continues climbing toward its peak while the damping rolls over and drops into amplification highlights the dual role of Kerr-induced Bogoliubov squeezing: it strongly amplifies reactive dispersive coupling

(optical spring) while suppressing anti-Stokes cooling in favour of Stokes parametric amplification.

Conclusions

We have analysed the phonon frequency shift and damping rate of a magnomechanical system in the resolved-sideband regime as functions of both drive detuning and drive power, incorporating the full 2×2 Nambu matrix description of the magnon Kerr nonlinearity. The detuning-dependent spectra (Fig. 7.2) reveal a strong sideband asymmetry in both the optical spring shift and phonon damping, driven by Kerr-induced squeezing and cavity-magnon polariton hybridisation.

By tracking the steady-state magnon population n_m explicitly (Fig. 7.3), we have demonstrated that while n_m grows linearly with power up to $\sim 5.8 \times 10^6$, the Kerr cross-coupling $2K_m n_m$ induces a profound functional divergence between reactive dispersion and dissipative cooling. The optical spring shift is strongly enhanced (peaking at ~ 19.0 kHz near $n_m \approx 4.3 \times 10^6$), whereas the phonon damping peaks early (15.2 kHz at $n_m \approx 2.0 \times 10^6$) before crossing into parametric amplification at $n_m \approx 3.7 \times 10^6$. These findings establish clear operational bounds for magnomechanical devices: moderate drive powers leverage Kerr cross-coupling for enhanced optical spring tuning, whereas higher drive powers can be deliberately exploited to drive phonon self-oscillation and parametric amplification.

Chapter 8

Magnon-Phonon Entanglement via Sideband-Selective Coupling in Cavity Magnomechanics

8.1 Introduction

Entanglement, described as the “characteristic trait of quantum mechanics” [126], is central to understanding the foundations of physical reality and advancing quantum information science [127, 128]. In recent decades, entanglement has been experimentally generated and controlled in microscopic systems, including photons, atoms, and ions [129]. However, a fundamental question remains: to what extent can quantum entanglement persist as systems become more macroscopic and approach the classical regime [130]? Addressing this question is crucial for testing the limits of quantum mechanics and for developing scalable quantum technologies based on solid-state and mechanical platforms [2].

Experiments investigating macroscopic quantum phenomena have demonstrated single-particle interference in large molecules [131], entanglement between collective atomic spins [132], and coherent quantum behaviour in Josephson-junction qubits [133]. Mechanical oscillators are especially compelling candidates for studying the quantum-to-classical transition because they closely resemble classical objects while remaining experimentally controllable. Advances in micro- and nanofabrication now enable the preparation and manipulation of mechanical resonators with high precision [134]. Theoretical proposals have identified viable methods for cooling these systems to the quantum regime and observing nonclassical behaviour [135]. Quantum-limited measurements have advanced to the point where ground-state detection of mechanical motion is experimentally feasible [18]. However, the unambiguous observation of robust quantum entanglement involving mechanical oscillators remains a significant challenge.

Optomechanical coupling via radiation pressure has become a leading approach for preparing and manipulating quantum states of mechanical systems [15]. Applications include quantum state transfer between light and mechanics, coherent control of macroscopic motion, and the generation of entanglement between distinct mechanical resonators [136, 137]. Recently, hybrid quantum systems have further expanded these possibilities by enabling coherent interactions among photons, spins, and phonons. These systems offer a versatile platform for investigating quantum coherence and information transfer across multiple physical degrees of freedom.

Cavity magnomechanical systems have attracted significant attention because they integrate microwave cavity photons, collective spin excitations (magnons), and mechanical vibrations within a single hybrid device. These systems are especially appealing due to the exceptional magnetic and mechanical properties of yttrium iron garnet (YIG), which supports low-loss magnon modes and magnetostrictive interactions that couple magnons to mechanical motion. Recent experiments have demonstrated strong and tunable coupling between cavity microwave photons and magnons in YIG spheres, as well as coherent magnon–phonon interactions resulting from magnetostriction [24, 125]. These advances have stimulated substantial theoretical interest in determining whether robust quantum correlations and entangled states can be generated among the interacting modes of cavity magnomechanical systems.

Achieving stable entanglement between magnonic and mechanical degrees of freedom would mark a significant step toward realising macroscopic quantum states in solid-state environments. In addition to its foundational importance, such entanglement could facilitate coherent quantum transduction, long-lived quantum memories, and scalable interfaces connecting spin, mechanical, and microwave platforms. Hybridised magnon–photon polaritons, in particular, serve as effective intermediaries that enable mechanical motion to inherit quantum coherence from electromagnetic and spin excitations.

This chapter presents a theoretical framework for generating steady-state magnon–phonon entanglement in a cavity magnomechanical system operating in the resolved-sideband regime. Application of a red-detuned microwave drive induces an effective magnomechanical interaction with a beam-splitter-like form, enhancing coherent mode conversion and enabling dynamical backaction cooling to suppress thermal fluctuations. Under these conditions, quantum correlations arise between the mechanical phonon mode and a hybridised magnon–photon polariton mode.

Logarithmic negativity is employed to quantify the resulting nonclassical correlations as a measure of bipartite continuous-variable entanglement. The analysis demonstrates that significant steady-state entanglement can persist under experimentally realistic conditions, including finite thermal occupation and moderate dissipation rates. These findings underscore the robustness of quantum correlations in cavity magnomechanical systems and reinforce their potential as a platform for exploring macroscopic quantum physics and developing scalable hybrid quantum technologies.

8.2 Mode Entanglement in Hybrid Bosonic Systems

Quantum entanglement is frequently described in terms of distinguishable particles, such as two electrons or two photons. In these cases, correlations arise between identifiable components, which is suitable for localised particles or separate qubits. However, the present system exhibits fundamentally different characteristics.

The elementary excitations in this context, magnons and phonons, do not constitute particles in the conventional sense. Instead, they are collective bosonic quasiparticles. A magnon represents a coherent spin-wave excitation within a magnetic lattice, whereas a phonon corresponds to a quantised vibration of a mechanical resonator [28]. Neither excitation can be individually labelled or tracked over time. Therefore, referring to "entanglement between this magnon and that phonon" lacks operational meaning [138].

It is more appropriate to consider the collective bosonic modes themselves as the relevant subsystems. In this device, strong coupling between the microwave cavity field and the magnon mode generates hybridised normal modes, referred to as cavity magnon polaritons. Simultaneously, the cantilever supports a distinct flexural phonon mode, which can couple coherently to the magnetic excitation via magnetostriction [2].

Accordingly, the framework of mode entanglement is employed [139]. Each subsystem is modelled as a bosonic harmonic oscillator characterised by its own creation and annihilation operators. In this context, entanglement denotes a non-separable quantum state of these modes, encompassing correlations in their quadratures or occupations that cannot be reproduced by any classical probabilistic mixture [127]. This chapter examines the quantum correlations between the mechanical phonon mode and the lower hybridised magnon-photon polariton mode.

8.3 System Hamiltonian

The Hamiltonian for the complete system, consisting of microwave photons, magnons, and phonon (flexure) modes, is given by

$$\begin{aligned} \hat{H} = & \omega_a \hat{a}^\dagger \hat{a} + \omega_b \hat{b}^\dagger \hat{b} + \omega_m \hat{m}^\dagger \hat{m} + g_{ma} (\hat{a} + \hat{a}^\dagger) (\hat{m} + \hat{m}^\dagger) \\ & + g_{mb} \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger) + \epsilon_d \sqrt{2\kappa_{ex}} (\hat{a} e^{i\omega_d t} + \hat{a}^\dagger e^{-i\omega_d t}) \\ & + \epsilon_s \sqrt{2\kappa_{ex}} (\hat{a} e^{i\omega_s t} + \hat{a}^\dagger e^{-i\omega_s t}) \end{aligned} \quad (8.1)$$

where ϵ_d and ϵ_s are the drive and weak probe fields, respectively. g_{ma} is the coupling strength between the cavity mode ($\hat{a}$) and magnon mode ($\hat{m}$), and g_{mb} is the coupling between magnon ($\hat{m}$) and phonon modes ($\hat{b}$) via magnetostrictive forces. Omitting the weak probe and applying the rotating wave approximation in a frame rotating at the drive frequency, the Hamiltonian becomes:

$$\hat{H} = \Delta_a \hat{a}^\dagger \hat{a} + \omega_b \hat{b}^\dagger \hat{b} + \Delta_m \hat{m}^\dagger \hat{m} + g_{ma} (\hat{a} \hat{m}^\dagger + \hat{a}^\dagger \hat{m}) + g_{mb} \hat{m}^\dagger \hat{m} (\hat{b} + \hat{b}^\dagger) + \epsilon_d \sqrt{2\kappa_{ex}} (\hat{a} + \hat{a}^\dagger) \quad (8.2)$$

where, $\Delta_a = \omega_d - \omega_a$ and $\Delta_m = \omega_d - \omega_m$. For clarity and to ensure this chapter remains self-contained, the cavity and magnon detunings are defined here as $\Delta_a = \omega_d - \omega_a$ and $\Delta_m = \omega_d - \omega_m$. Under this convention, a positive detuning ($\Delta_{a,m} > 0$) directly represents the blue-detuned regime ($\omega_d > \omega_{a,m}$). This convention is specifically chosen so that the blue-detuned regime appears naturally on the right-hand side ($\Delta > 0$) of zero detuning in all subsequent figures, thereby clearly illustrating the onset of inter-mode entanglement where the two-mode squeezing interaction terms become resonant.

Due to the high spin density of YIG, the magnon-photon coupling rate g_{ma} exceeds both the cavity and magnon decay rates κ and κ_m , respectively. This leads to hybridised modes separated by $\Delta\omega = 2g_{ma}$, with $g_{ma} \approx 10 \times 10^6 \text{ rads}^{-1}$. The drive frequency is set so that $\omega_d = \omega_a - 1.5\omega_b$, and $2g_{ma} = \omega_b$. Our model assumes $\langle \hat{m} \rangle \ll 5N$, with $N = \rho V_m \approx 1.2 \times 10^9$ and $\rho = 4.22 \times 10^{27} \text{ m}^{-3}$.

The time evolution of each mode is given by the nonlinear Langevin equations:

$$\dot{\hat{a}} = (-i\Delta_a - \kappa_a) \hat{a} - ig_{ma} \hat{m} - i\epsilon_d \sqrt{2\kappa_{ex}} + \hat{a}_{in} \quad (8.3a)$$

$$\dot{\hat{m}} = (-i\Delta_m - \kappa_m) \hat{m} - ig_{ma} \hat{a} - ig_{mb} \hat{m} q + \hat{m}_{in} \quad (8.3b)$$

$$\dot{\hat{p}} = -\omega_b q - \kappa_b p - g_{mb} \hat{m}^\dagger \hat{m} + \sigma \quad (8.3c)$$

$$\dot{\hat{q}} = \omega_b p \quad (8.3d)$$

where q and p are the dimensionless phonon position and momentum operators defined as

$$\hat{q} = \frac{1}{\sqrt{2}} (\hat{b} + \hat{b}^\dagger)$$

and

$$\hat{p} = \frac{1}{i\sqrt{2}} (\hat{b} - \hat{b}^\dagger)$$

The vacuum input noise operators $\hat{a}_{in}, \hat{m}_{in}$ satisfy:

$$\langle \hat{a}^{in}(t) \hat{a}^{in\dagger}(t') \rangle = \delta(t - t') \quad (8.4a)$$

$$\langle \hat{m}^{in}(t) \hat{m}^{in\dagger}(t') \rangle = \delta(t - t') \quad (8.4b)$$

The Hermitian Brownian noise operator σ has correlation:

$$\langle \sigma(t) \sigma(t') \rangle = \frac{\kappa_b}{\omega_b} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \omega \left[\coth \left(\frac{\hbar\omega}{2k_B T} \right) + 1 \right] \quad (8.5)$$

The dynamics can be linearised by expanding each operator as a steady-state mean plus fluctuation:

$$\hat{O}(t) = \langle O \rangle + \delta O(t)$$

Substituting into the equations and keeping only zeroth-order terms (i.e., ignoring fluctuations and noise) gives the steady-state equations:

After linearisation, the steady-state solutions can be written as

$$0 = (-i\Delta_a - \kappa_a)\langle a \rangle - ig_{ma}\langle m \rangle - i\epsilon_d\sqrt{2\kappa_{\text{ex}}} \quad (8.6a)$$

$$0 = (-i\Delta_m - \kappa_m)\langle m \rangle - ig_{ma}\langle a \rangle - ig_{mb}\langle m \rangle\langle q \rangle \quad (8.6b)$$

$$0 = -\omega_b\langle q \rangle - g_{mb}\langle \hat{m}^\dagger \hat{m} \rangle \quad (8.6c)$$

$$0 = \langle p \rangle \quad (\text{at steady state}) \quad (8.6d)$$

We approximate:

$$\langle \hat{m}^\dagger \hat{m} \rangle \approx |\langle m \rangle|^2$$

From equation (8.6c), we solve for $\langle q \rangle$:

$$\langle q \rangle = -\frac{g_{mb}}{\omega_b} |\langle m \rangle|^2$$

Substituting into equation (8.6b):

$$(-i\Delta_m - \kappa_m)\langle m \rangle - ig_{ma}\langle a \rangle + \frac{ig_{mb}^2}{\omega_b} |\langle m \rangle|^2 \langle m \rangle = 0$$

Neglecting the nonlinear term $|\langle m \rangle|^2 \langle m \rangle$ under first-order linearisation:

$$(-i\Delta_m - \kappa_m)\langle m \rangle - ig_{ma}\langle a \rangle = 0$$

$$\Rightarrow \langle m \rangle = \frac{-g_{ma}\langle a \rangle}{\Delta_m - i\kappa_m}$$

Substituting into equation (8.6a):

$$(-i\Delta_a - \kappa_a)\langle a \rangle + \frac{ig_{ma}^2\langle a \rangle}{\Delta_m - i\kappa_m} = i\epsilon_d\sqrt{2\kappa_{\text{ex}}}$$

Multiplying both sides by $\Delta_m - i\kappa_m$:

$$\langle a \rangle [(-i\Delta_a - \kappa_a)(\Delta_m - i\kappa_m) + ig_{ma}^2] = i\epsilon_d\sqrt{2\kappa_{\text{ex}}}(\Delta_m - i\kappa_m)$$

Thus:

$$\langle a \rangle = \frac{i\epsilon_d \sqrt{2\kappa_{\text{ex}}}(\Delta_m - i\kappa_m)}{(-i\Delta_a - \kappa_a)(\Delta_m - i\kappa_m) + ig_{ma}^2}$$

Multiplying numerator and denominator by -1 , we obtain

$$\langle a \rangle = \frac{\epsilon_d \sqrt{2\kappa_{\text{ex}}}(i\Delta_m + \kappa_m)}{(i\Delta_a + \kappa_a)(i\Delta_m + \kappa_m) + ig_{ma}^2}$$

The steady-state solutions can be summarised as

$$\langle a \rangle = \frac{\epsilon_d \sqrt{2\kappa_{\text{ex}}}(i\Delta_m + \kappa_m)}{(i\Delta_a + \kappa_a)(i\Delta_m + \kappa_m) + ig_{ma}^2} \quad (8.7a)$$

$$\langle m \rangle = \frac{-g_{ma}\langle a \rangle}{\Delta_m - i\kappa_m} \quad (8.7b)$$

$$\langle p \rangle = 0 \quad (8.7c)$$

$$\langle q \rangle = -\frac{g_{mb}}{\omega_b} |\langle m \rangle|^2 \quad (8.7d)$$

We proceed with the linearized fluctuation equations for operators $\delta a, \delta m, \delta q, \delta p$ around steady-state values:

$$\delta \dot{a} = (-i\Delta_a - \kappa_a)\delta a - ig_{ma}\delta m + \sqrt{2\kappa_a}\delta a^{\text{in}} \quad (8.8a)$$

$$\delta \dot{m} = (-i\Delta_m - \kappa_m)\delta m - ig_{ma}\delta a - iG_{mb}\delta q + \sqrt{2\kappa_m}\delta m^{\text{in}} \quad (8.8b)$$

$$\delta \dot{q} = \omega_b \delta p \quad (8.8c)$$

$$\delta \dot{p} = -\omega_b \delta q - \kappa_b \delta p - g_{mb}(\hat{m}^\dagger \hat{m} - \langle \hat{m}^\dagger \hat{m} \rangle) + \sigma \quad (8.8d)$$

The nonlinear term involving the product of operators can be linearized by expanding around the steady state $\langle m \rangle$:

$$\hat{m}^\dagger \hat{m} = |\langle m \rangle|^2 + \langle m \rangle \delta m^\dagger + \langle m \rangle^* \delta m + \delta m^\dagger \delta m.$$

Assuming fluctuations are small, we neglect the second-order term $\delta m^\dagger \delta m$, so

$$\hat{m}^\dagger \hat{m} - \langle \hat{m}^\dagger \hat{m} \rangle \approx \langle m \rangle \delta m^\dagger + \langle m \rangle^* \delta m.$$

The enhanced coupling is then defined by

$$G_{mb} = g_{mb}\langle m \rangle.$$

Substituting this into Eq. (8.8d) yields

$$\delta\dot{p} = -\omega_b\delta q - \kappa_b\delta p - G_{mb}\delta m^\dagger - G_{mb}^*\delta m + \sigma.$$

We define the Hermitian quadratures (position- and momentum-like) operators for the cavity and magnon modes as

$$\begin{cases} \delta x_a = \frac{\delta a + \delta a^\dagger}{\sqrt{2}}, & \delta p_a = \frac{\delta a^\dagger - \delta a}{i\sqrt{2}}, \\ \delta x_m = \frac{\delta m + \delta m^\dagger}{\sqrt{2}}, & \delta p_m = \frac{\delta m^\dagger - \delta m}{i\sqrt{2}}. \end{cases}$$

Similarly, for the input noise operators,

$$\begin{cases} \delta x_a^{\text{in}} = \frac{\delta a^{\text{in}} + \delta a^{\text{in}\dagger}}{\sqrt{2}}, & \delta p_a^{\text{in}} = \frac{\delta a^{\text{in}\dagger} - \delta a^{\text{in}}}{i\sqrt{2}}, \\ \delta x_m^{\text{in}} = \frac{\delta m^{\text{in}} + \delta m^{\text{in}\dagger}}{\sqrt{2}}, & \delta p_m^{\text{in}} = \frac{\delta m^{\text{in}\dagger} - \delta m^{\text{in}}}{i\sqrt{2}}. \end{cases}$$

We rewrite Eq. (8.8a) and its conjugate:

$$\delta\dot{a} = (-i\Delta_a - \kappa_a)\delta a - ig_{ma}\delta m + \sqrt{2\kappa_a}\delta a^{\text{in}},$$

$$\delta\dot{a}^\dagger = (i\Delta_a - \kappa_a)\delta a^\dagger + ig_{ma}\delta m^\dagger + \sqrt{2\kappa_a}\delta a^{\text{in}\dagger}.$$

Adding these and dividing by $\sqrt{2}$,

$$\delta\dot{x}_a = -\kappa_a\delta x_a - \Delta_a\delta p_a - g_{ma}\delta p_m + \sqrt{2\kappa_a}\delta x_a^{\text{in}}.$$

Subtracting and dividing by $i\sqrt{2}$,

$$\delta\dot{p}_a = \Delta_a\delta x_a - \kappa_a\delta p_a + g_{ma}\delta x_m + \sqrt{2\kappa_a}\delta p_a^{\text{in}}.$$

Similarly, from Eqs. (8.8b) and its conjugate,

$$\delta\dot{m} = (-i\Delta_m - \kappa_m)\delta m - ig_{ma}\delta a - iG_{mb}\delta q + \sqrt{2\kappa_m}\delta m^{\text{in}},$$

$$\delta \dot{m}^\dagger = (i\Delta_m - \kappa_m)\delta m^\dagger + ig_{ma}\delta a^\dagger + iG_{mb}^*\delta q + \sqrt{2\kappa_m}\delta m^{\text{in}\dagger}.$$

Adding,

$$\delta \dot{x}_m = -\kappa_m\delta x_m - \Delta_m\delta p_m - g_{ma}\delta p_a + \sqrt{2}\text{Im}(G_{mb})\delta q + \sqrt{2\kappa_m}\delta x_m^{\text{in}}.$$

Subtracting,

$$\delta \dot{p}_m = \Delta_m\delta x_m - \kappa_m\delta p_m + g_{ma}\delta x_a + \sqrt{2}\text{Re}(G_{mb})\delta q + \sqrt{2\kappa_m}\delta p_m^{\text{in}}.$$

Finally, the linearized equations in quadratures can be written compactly as

$$\begin{cases} \delta \dot{x}_a = -\kappa_a\delta x_a - \Delta_a\delta p_a - g_{ma}\delta p_m + \sqrt{2\kappa_a}\delta x_a^{\text{in}} \\ \delta \dot{p}_a = \Delta_a\delta x_a - \kappa_a\delta p_a + g_{ma}\delta x_m + \sqrt{2\kappa_a}\delta p_a^{\text{in}} \\ \delta \dot{x}_m = -\kappa_m\delta x_m - \Delta_m\delta p_m - g_{ma}\delta p_a + \sqrt{2}\text{Im}(G_{mb})\delta q + \sqrt{2\kappa_m}\delta x_m^{\text{in}} \\ \delta \dot{p}_m = \Delta_m\delta x_m - \kappa_m\delta p_m + g_{ma}\delta x_a + \sqrt{2}\text{Re}(G_{mb})\delta q + \sqrt{2\kappa_m}\delta p_m^{\text{in}} \\ \delta \dot{q} = \omega_b\delta p \\ \delta \dot{p} = -\omega_b\delta q - \kappa_b\delta p - \sqrt{2}\text{Re}(G_{mb})\delta x_m - \sqrt{2}\text{Im}(G_{mb})\delta p_m + \sigma \end{cases}$$

where

$$G_{mb} = g_{mb}\langle m \rangle.$$

The system of equations can then be written compactly as

$$\dot{\mathbf{u}}(t) = \mathbf{A} \mathbf{u}(t) + \mathbf{n}(t)$$

where we choose the phase of the steady-state magnon amplitude such that G_{mb} is real.

Let

$$\hat{u} = (\delta x_a, \delta p_a, \delta x_m, \delta p_m, \delta q, \delta p)^\top$$

be the vector of linearized fluctuation operators. The dynamics can then be written as

$$\dot{\hat{u}}(t) = A\hat{u}(t) + \hat{n}(t),$$

where A is the 6×6 drift matrix governing the deterministic evolution of the fluctuations, and $\hat{n}(t)$ is a vector of δ -correlated noise operators representing the coupling to the thermal baths and input fields.

The drift matrix A for our system, expressed in the quadrature basis, is given by

$$\mathbf{A} = \begin{bmatrix} -\kappa_a & -\Delta_a & 0 & -g_{ma} & 0 & 0 \\ \Delta_a & -\kappa_a & g_{ma} & 0 & 0 & 0 \\ 0 & -g_{ma} & -\kappa_m & -\Delta_m & 0 & 0 \\ g_{ma} & 0 & \Delta_m & -\kappa_m & \sqrt{2}G_{mb} & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega_b \\ 0 & 0 & -\sqrt{2}G_{mb} & 0 & -\omega_b & -\kappa_b \end{bmatrix}.$$

The corresponding noise vector is

$$\hat{n}(t) = \begin{pmatrix} \sqrt{2\kappa_a} \delta x_a^{\text{in}} \\ \sqrt{2\kappa_a} \delta p_a^{\text{in}} \\ \sqrt{2\kappa_m} \delta x_m^{\text{in}} \\ \sqrt{2\kappa_m} \delta p_m^{\text{in}} \\ 0 \\ \sigma \end{pmatrix}.$$

The formal solution to the linearized Langevin equation is

$$\hat{u}(t) = e^{At}\hat{u}(0) + \int_0^t e^{A(t-\tau)}\hat{n}(\tau) d\tau,$$

where $\hat{u}(0)$ is the initial state vector.

Steady State and Stability

The system attains a steady state as $t \rightarrow \infty$. For stability, the influence of the initial state must diminish, which requires that all eigenvalues of the drift matrix A possess negative real parts. This condition guarantees that the system's fluctuations remain bounded over time. Stability can be assessed by analyzing the real parts of the eigenvalues of A or by applying established criteria such as the Routh-Hurwitz criterion [140]. In our numerical analysis, we confirm that this stability condition is satisfied.

Covariance Matrix and Entanglement

Since the quantum noise sources are assumed to be Gaussian with zero mean, and we are analyzing linearized quantum fluctuation dynamics, the steady state of the system corresponds to a continuous-variable tripartite Gaussian state. This state is completely characterized by a 6×6 covariance matrix (CM), denoted by $\mathcal{V}$. The elements of $\mathcal{V}$ are defined as the symmetrized correlations of the fluctuation operators in the steady state:

$$\mathcal{V}_{ij} = \frac{1}{2} \langle \{\hat{u}_i, \hat{u}_j\} \rangle_{\text{steady state}}, \quad i, j = 1, \dots, 6$$

where $\{, \}$ denotes the anticommutator.

For a stable system, the steady-state covariance matrix $\mathcal{V}$ satisfies the continuous-variable Lyapunov equation:

$$A\mathcal{V} + \mathcal{V}A^\top = -D$$

where A is the drift matrix, and D is the diffusion matrix. The diffusion matrix D accounts for the strength of the noise sources coupled to each mode and is given by: $D_{ij} = \delta_{ij} \cdot 2\Gamma_i (N_i + \frac{1}{2})$ with $\Gamma_i = (\kappa_a, \kappa_m, \kappa_b)$, $\omega_i = (\omega_a, \omega_m, \omega_b, \omega_b)$ and $N_i = \left(\exp\left(\frac{\hbar\omega_i}{k_B T}\right) - 1 \right)^{-1}$ is the mean thermal excitation number for mode i (cavity a , magnon m , mechanical b) at temperature T and frequency ω_j . The term $\frac{1}{2}$ in the diffusion matrix arises from the quantum vacuum fluctuations.

Entanglement in a continuous-variable Gaussian state can be quantified using the logarithmic negativity (E_N). For a bipartite system, E_N is calculated from the partial transpose of the covariance matrix. Specifically, for a 4×4 covariance matrix of a two-mode subsystem, the logarithmic negativity is given by $E_N = \max(0, -\log_2(2\nu_{\min}))$ [141], where $\nu_{\min}$ is the minimum symplectic eigenvalue of the partial transpose of the covariance matrix. Entanglement is present when $E_N > 0$, which corresponds to $\nu_{\min} < 0.5$, which is equivalent to Simon's necessary and sufficient entanglement non-positive partial transpose criterion for Gaussian states[142]. The quantum effect of entanglement can be observed when the system is sufficiently cooled and the coupling strengths are large enough to overcome the thermal noise.

8.4 Results

Steady-state quantum correlations in a hybrid cavity magnomechanical system are analyzed by calculating the logarithmic negativity from the full Gaussian covariance matrix of the linearized dynamics. This entanglement landscape provides a detailed perspective on how coherent hybridization, magnetic tuning, and dissipative processes collectively determine the emergence of macroscopic quantum correlations between the magnon and phonon subsystems.

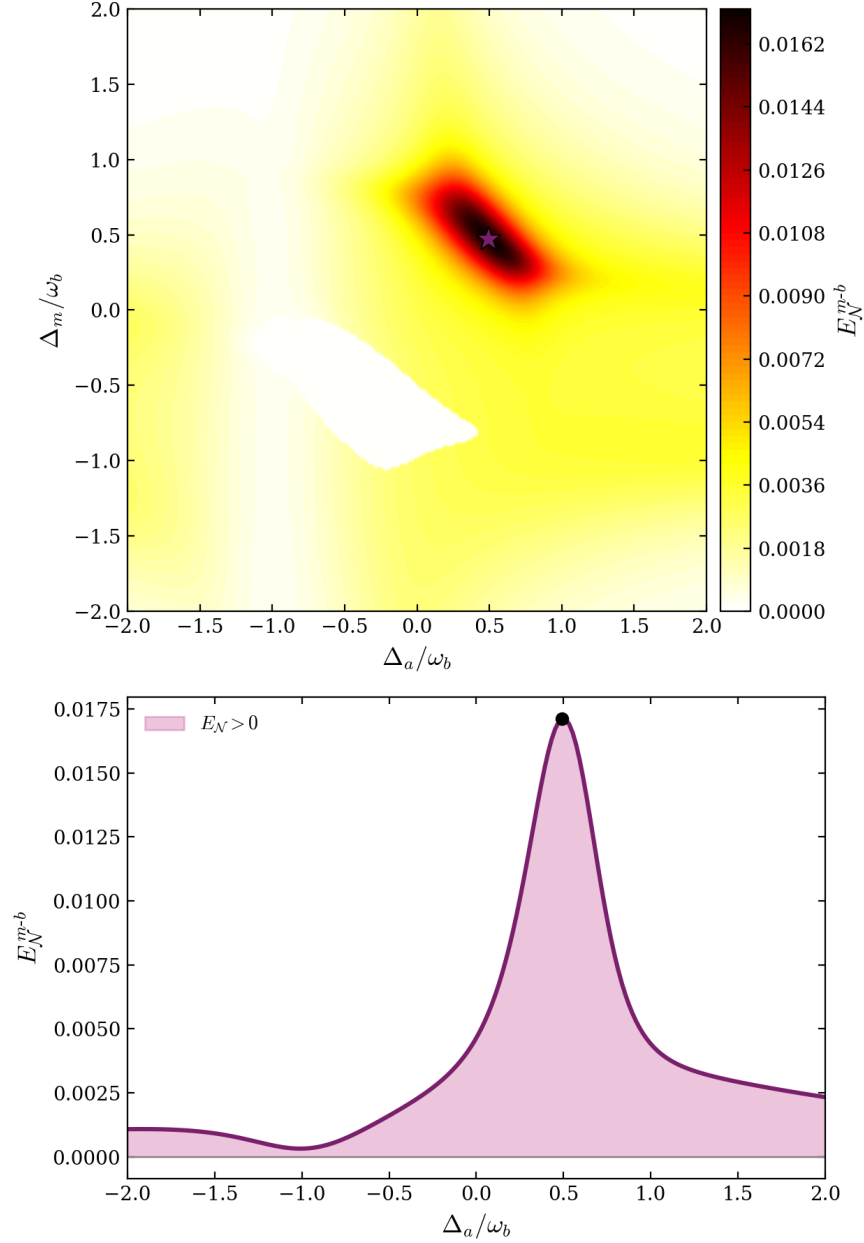


Figure 8.1: Steady-state magnon–phonon entanglement at a drive power $P = 0.3$ nW and bias field $H \simeq 400$ Oe. **(Top)** Logarithmic negativity E_N^{m-b} as a function of the normalized cavity detuning Δ_a/ω_b and magnon detuning Δ_m/ω_b ; the star marks the global maximum $E_N^{m-b} \simeq 0.017$ at $(\Delta_a/\omega_b, \Delta_m/\omega_b) \simeq (0.49, 0.47)$. **(Bottom)** Line cut at the optimal magnon detuning $\Delta_m/\omega_b \simeq 0.47$, showing E_N^{m-b} versus Δ_a/ω_b . The shaded region indicates the parameter window where $E_N^{m-b} > 0$.

Magnon-phonon entanglement

Figure 8.1 presents the steady-state magnon-phonon logarithmic negativity E_N^{m-b} as a function of the normalised cavity detuning Δ_a/ω_b and magnon detuning Δ_m/ω_b

(top panel), together with a line cut taken at the optimal magnon detuning $\Delta_m/\omega_b \simeq 0.47$ (bottom panel). The entanglement forms a single compact lobe centred near $(\Delta_a/\omega_b, \Delta_m/\omega_b) \simeq (0.49, 0.47)$, where it reaches its maximum value $E_{\mathcal{N}}^{m-b} \simeq 0.017$. This optimal point coincides with the parametric (two-mode-squeezing) resonance: at red detuning the anti-Stokes scattering of the drive resonantly enhances the effective magnomechanical coupling $G_{mb} = g_{mb}\sqrt{n_m}$, generating quantum correlations between the magnon and phonon quadratures. The line cut exhibits a single, well-defined maximum, and the shaded region marks the parameter window where $E_{\mathcal{N}}^{m-b} > 0$. Away from this window the coupling detunes off resonance, and the correlations are washed out by dissipation, recovering a separable state.

Photon-magnon entanglement

Figure 8.2 displays the corresponding photon-magnon logarithmic negativity $E_{\mathcal{N}}^{a-m}$. The qualitative structure resembles the magnon-phonon case, with a single entangled lobe in the red-detuned regime and a clear optimum. However, the peak value, $E_{\mathcal{N}}^{a-m} \simeq 1.7 \times 10^{-3}$, is roughly an order of magnitude smaller and is shifted to $\Delta_a/\omega_b \simeq 0.83$. This suppression results directly from the dissipation hierarchy of the system. The cavity field is the driven, strongly damped mode ($\kappa_a/\omega_b \simeq 0.2$), and therefore decoheres on the fastest timescale, sharing the least entanglement. The magnon mode thus acts as the central node of the tripartite system, remaining strongly entangled with the long-lived phonon and only weakly with the lossy photon. The line cut again reveals a single maximum bounded by a narrow $E_{\mathcal{N}}^{a-m} > 0$ window, consistent with entanglement transfer mediated by the beam-splitter-type photon-magnon interaction.

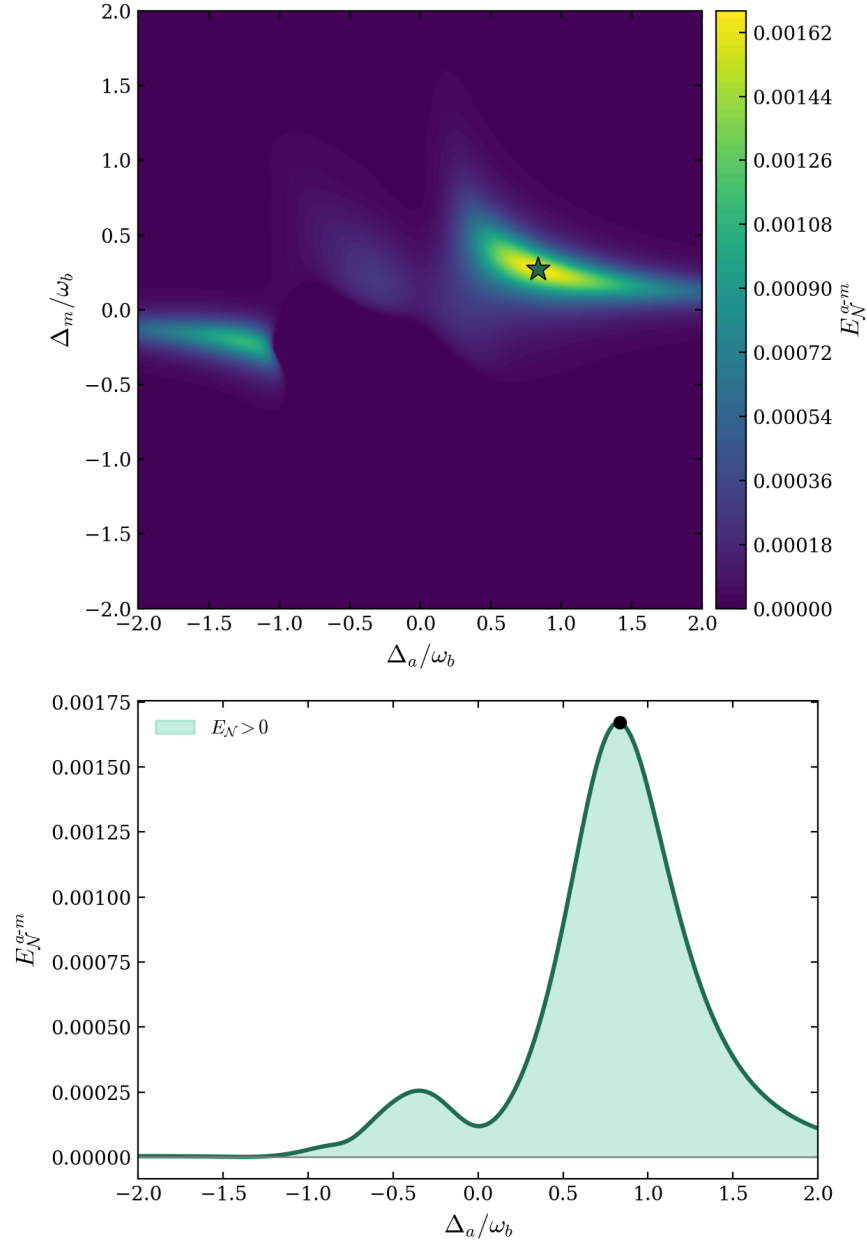


Figure 8.2: Steady-state photon-magnon entanglement at a drive power $P = 0.3$ nW and bias field $H = 400$ Oe. **(Top)** Logarithmic negativity E_N^{a-m} as a function of the normalized cavity detuning Δ_a/ω_b and magnon detuning Δ_m/ω_b ; the star marks the global maximum $E_N^{a-m} \simeq 1.7 \times 10^{-3}$ at $(\Delta_a/\omega_b, \Delta_m/\omega_b) \simeq (0.83, 0.27)$. **(Bottom)** Line cut at the optimal magnon detuning $\Delta_m/\omega_b \simeq 0.27$, showing E_N^{a-m} versus Δ_a/ω_b . The shaded region indicates the parameter window where $E_N^{a-m} > 0$.

8.5 Conclusion

This chapter investigated the steady-state quantum entanglement generated in a driven optomagnomechanical system, in which a microwave cavity mode is coupled to a magnon mode of a YIG film that, in turn, couples to a mechanical (phonon) mode through magnetostriction. Working in the linearized regime, we constructed the quantum Langevin equations for the quadrature fluctuations, derived the corresponding drift and diffusion matrices, and obtained the stationary covariance matrix by solving the Lyapunov equation $AV + VA^\top = -D$. The bipartite entanglement of each of the two-mode subsystem was then quantified through the logarithmic negativity $E_{\mathcal{N}}$ evaluated on the relevant 4×4 blocks of the covariance matrix, with dynamical stability enforced via the Routh-Hurwitz condition on the eigenvalues of the drift matrix.

For a fixed bias field $H \simeq 400$ Oe and drive power $P = 0.3$ nW, we mapped the entanglement as a function of the normalised cavity and magnon detunings, Δ_a/ω_b and Δ_m/ω_b . The magnon-phonon subsystem was found to be the most strongly correlated pair, reaching a maximum logarithmic negativity $E_{\mathcal{N}}^{m-b} \simeq 0.017$ in the red-detuned regime near $(\Delta_a/\omega_b, \Delta_m/\omega_b) \simeq (0.49, 0.47)$. This optimum coincides with the parametric resonance at which the drive's anti-Stokes sideband maximally enhances the effective magnomechanical coupling $G_{mb} = g_{mb}\sqrt{n_m}$, confirming that the entanglement originates from the radiation-pressure-like magnetostrictive interaction. The photon-magnon subsystem displayed the same qualitative behaviour, but an entanglement roughly an order of magnitude weaker, $E_{\mathcal{N}}^{a-m} \simeq 1.7 \times 10^{-3}$, peaked at a distinct detuning $\Delta_a/\omega_b \simeq 0.83$. This hierarchy reflects the dissipative structure of the system: the strongly damped, externally driven cavity field decoheres on the fastest timescale, so the magnon mode acts as the central node that mediates correlations, sharing the bulk of its entanglement with the long-lived phonon mode rather than with the lossy photon.

Collectively, these results demonstrate that a single red-detuned microwave drive is sufficient to establish stationary, robust entanglement between disparate degrees of freedom: spin, light, and mechanical motion within a compact solid-state platform. The detuning dependence provides a clear experimental prescription: operating in the red-sideband regime at the identified optimal working point maximises the achievable entanglement, while the well-defined $E_{\mathcal{N}} > 0$ windows delimit the parameter range over which nonclassical correlations survive. The magnitude of the entanglement is ultimately limited by the thermal occupation of the mechanical bath; reaching the strongly entangled regime requires the system cooperativity to exceed the thermal phonon number, which favours low temperatures, high mechanical quality factors, and increased drive power. These findings establish optomagnomechanical systems as a promising resource for hybrid quantum networks and provide a basis for further studies of tripartite entanglement, entanglement transfer, and their experimental verification.

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