Mirror symmetry and the quantum McKay correspondence

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To my parents

Contents

Introduction

Motivation: The Crepant Resolution Conjecture

The subject matter of this thesis lies at the intersection of birational and enumerative algebraic geometry. The questions we will consider fall within the broad overarching problem of understanding the behaviour of curve-counting invariants of varieties (or orbifolds) under birational transformations.

A much-studied setup is that of crepant desingularisations: for $\mathcal{X}$ a complex algebraic orbifold (smooth Deligne–Mumford stack) with coarse moduli space X, a resolution of singularity $\pi : Z \to X$ is **crepant** if $\pi^* K_X = K_Z$. The archetypical example is the classical McKay correspondence for Kleinian singularities, which are coarse quotient spaces $X = \mathbb{C}^2/G$ for finite subgroups $G < \mathrm{SL}(2, \mathbb{C})$. Such a space X has an isolated singularity at the origin, and its minimal resolution $Z \to X$ is crepant. A classical observation of McKay [50] asserts that the intersection graph of the exceptional divisors in Z coincides with the affine Dynkin diagram associated with the representation theory of G. Our thesis will be devoted to upgrading this connection to the study of the respective quantum cohomologies and their underlying theory of curve counting invariants (**Gromov–Witten invariants**).

From a birational perspective, a central question is how to associate invariants to Deligne–Mumford stacks in crepant birational relationship. An early example is the equality between the stringy Euler number of the orbifold and the Euler number of its crepant resolution, motivated by the seminal work of Dixon, Harvey, Vafa, and Witten [29] on orbifold string theory. The formalisation of this equivalence began with Batyrev [5, 4] and was proven for global quotients by Denef–Loeser [28] using motivic integration. Yasuda [64] subsequently refined this framework as an isomorphism of cohomologies as graded vector spaces, proving that orbifolds in crepant birational relationship have matching orbifold Betti numbers.

The *classical* underlying cohomology rings of $\mathcal{X}$ and Z are not, however, isomorphic. Following Chen and Ruan's [21] construction of orbifold cohomology and orbifold Gromov–Witten invariants, Ruan famously proposed [58] that the Chen–Ruan orbifold cohomology ring $H^*_{\mathrm{CR}}(\mathcal{X}; \mathbb{C})$ is isomorphic to a suitable specialisation at roots of unity of the *quantum* cohomology ring $\mathrm{QH}^*(Z; \mathbb{C})$ of the resolution. A set of influential conjectures of Bryan–Graber [17], Coates–Iritani–Tseng [25] and Iritani [42] further refined Ruan's original conjecture with a full prediction on how the genus-zero Gromov–Witten theories of varieties (or orbifolds) in crepant birational equivalence are mutually determined.

The modern framework to understand these conjectures is as an isomorphism of quantum $\mathcal{D}-$modules associated to the respective Dubrovin connections [42, 25], inducing a symplectic isomorphism of the associated Givental spaces and a non-trivial relation between the generating function of genus-zero Gromov–Witten invariants of an orbifold and its resolution. A striking conjecture of Iritani [43] further posits an explicit form for this isomorphism as being induced from a natural isomorphism at the level the corresponding derived categories of coherent sheaves via a suitable "central charge" operator, defined in terms of the $\widehat{\Gamma}$-class of the tangent bundle.

The state-of-the-art in the study of the Crepant Resolution Conjecture (CRC) heavily – and virtually invariably – relies on techniques from mirror symmetry. The quantum cohomology rings of targets in a crepant relationship should ostensibly arise from the expansion of a global quantum $\mathcal{D}-$module at different boundary points of the stringy Kähler moduli space. Relating these explicitly requires a resummation and analytic continuation of the respective Gromov–Witten generating series, with the oscillating integral expressions from Landau–Ginzburg mirror models providing the core method for the proof of the CRC in its various guises. Because of that, the near entirety of our knowledge about the CRC is limited to toric stacks, with non-toric setups being largely *terra incognita*. Even the foundational case of the CRC for Kleinian singularities – the *quantum* McKay correspondence – is largely *terra incognita* aside from the case when G is a cyclic (hence Abelian) group.

Abstract

In this thesis, we take a series of steps towards a proof of the quantum McKay correspondence in full generality. We establish a uniform Lie-theoretic mirror theorem for ADE singularities, also covering the as-of-yet untreated non-toric Dynkin series D_n and E_n, in terms of a suitable one-dimensional Landau–Ginzburg model. We then leverage this to the as-of-yet untreated study of the global Dubrovin connection for the D-series, verifying Iritani's structure conjecture for the CRC in that context. We also push the CRC beyond its original geometric incarnation to encompass the representation-theoretically meaningful case of "foldings".

Outline of the thesis

In Chapter 1, we review some basic notions about the classical McKay correspondence, the Chen–Ruan cohomology of quotient stacks, Gromov–Witten theory and Givental's symplectic formalism, and relevant elements from the theory of Frobenius manifolds.

In Chapter 2, we show that, under Dubrovin's notion of "almost" duality, the Frobenius manifold structure on the orbit spaces of the extended affine Weyl groups of type ADE is dual, for suitable choices of weight markings, to the equivariant quantum cohomology of the minimal resolution of the Kleinian singularity of the same Dynkin type. We also provide a uniform Lie-theoretic construction of Landau–Ginzburg mirrors for the quantum cohomology of ADE resolutions. The mirror B-model is described by a one-dimensional LG superpotential associated to the spectral curve of the $\widehat{\mathrm{ADE}}$ affine relativistic Toda chain. The content of this chapter originally appeared in a published joint work [14] with Andrea Brini and Ian A.B. Strachan.

In Chapter 3, we follow a framework to study the crepant resolution conjecture for the Kleinian singularities of type D, which allows to explicitly compute the symplectic transformation between Givental spaces of the orbifold and resolution. We provide such computation for the $\mathrm{D}_3 = \mathrm{A}_3$ singularity and discuss its relation with Iritani's integral structure. The main tool is mirror symmetry for minimal resolution of Kleinian singularities as developed in Chapter 2.

In Chapter 4, we compute the untwisted part of the $\mathbb{C}^\times$-equivariant orbifold quantum cohomology of certain finite cyclic quotients of the minimal resolutions of Kleinian singularities, which we refer to as their foldings. We then formulate a conjecture for the full $\mathbb{C}^\times$-equivariant orbifold quantum cohomology, motivated by the Crepant Resolution Conjecture, and provide two pieces of supporting evidence. We also identify the resulting Frobenius structure with known Frobenius structures associated with the corresponding non-simply-laced root systems.

In Appendix A, we briefly review Lauricella hypergeometric functions and provide connection formulas for their analytic continuation. These results are used in Chapter 3.

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Notation

$\mathcal{R}$, resp. $\mathcal{R}^+$	An ADE root system, resp. its set of positive roots
$\Pi = \{\alpha_1, \ldots, \alpha_l\}$	The set of simple roots of $\mathcal{R}$
$\Omega = \{\omega_1, \ldots, \omega_l\}$	The set of fundamental weights of $\mathcal{R}$
ρ_i, resp. Γ_i	Irreducible representation, resp. weight system, with highest weight ω_i
$(\mathfrak{h}, \langle,\rangle)$	The Cartan subalgebra of $\mathfrak{g}$, together with its Cartan–Killing form
$\mathfrak{g}$	The complex simple Lie algebra associated to $\mathcal{R}$
$C_{ij} = \langle \alpha_i, \alpha_j \rangle$	The Cartan matrix of $\mathcal{R}$
$\widehat{\omega}$	A marked fundamental weight of $\mathcal{R}$ as in Figure 2.1.
$\mathcal{W}/\widehat{\mathcal{W}}/\widetilde{\mathcal{W}}$	The Weyl/affine Weyl/extended affine Weyl group of type $(\mathcal{R}, \widehat{\omega})$
$G_{\mathcal{R}}$	The Kleinian subgroup of $SL(2;\mathbb{C})$ corresponding to $\mathcal{R}$
$X_{\mathcal{R}}$, resp. $Z_{\mathcal{R}}$	The Kleinian singularity of type $\mathcal{R}$, resp. its minimal resolution
$(t_i)_{i=1}^l$	Linear coordinates for $H^2(Z_{\mathcal{R}})$ in the dual basis of $\{E_i\}_{i=1}^l$
$(x_g)_{g \in \mathrm{Irrep}(G_{\mathcal{R}})}$	Linear coordinates for $H^*_{\mathrm{CR}}(\mathcal{X}_{\mathcal{R}})$ in the basis $\{\mathbf{1}_g\}_{g \in \mathrm{Irrep}(G_{\mathcal{R}})}$
T	The torus action on Z specified by $\widehat{\omega}$
$\mathcal{M}_{\mathrm{AW}}$	The extended affine Weyl Frobenius manifold of type $(\mathcal{R}, \widehat{\omega})$
$\mathcal{M}_{\mathrm{GW}}$	The equivariant quantum cohomology of Z of type $(\mathcal{R}, \widehat{\omega})$
$\mathcal{M}_{\mathrm{LG}}$	The Landau–Ginzburg Frobenius manifold of the affine relativistic Toda chain of type $(\mathcal{R}, \widehat{\omega})$
$(\eta_\bullet, c_\bullet, \eta^\flat_\bullet, c^\flat_\bullet)$	The metric, product tensor, and their duals on $\mathcal{M}_\bullet$ ($\bullet \in \{\mathrm{AW}, \mathrm{GW}, \mathrm{LG}\}$)
$(Y_\alpha)_{\alpha=1}^l$	Basic Weyl-invariant Fourier polynomials of $\mathfrak{h}$
$(W_\alpha)_{\alpha=1}^l$	The fundamental characters of $\exp(\mathfrak{g})$
$(y_\alpha)_{\alpha=1}^{l+1}$	Basic extended affine Weyl-invariant Fourier polynomials of $\mathfrak{h} \oplus \mathbb{C}$
$(q_i)_{i=1}^{l+1}$, resp. $(p_i)_{i=1}^{l+1}$	Linear coordinates for $\mathfrak{h} \oplus \mathbb{C}$ in the basis $\Pi^\vee$, resp. $\Omega^\vee$
$(t_A)_{A=1}^{l+1}$	A flat coordinate chart for $\mathcal{M}_{\mathrm{AW}}$
$\mathfrak{I}$	A set of initial conditions for $\mathcal{M}_{\mathrm{AW}}$

ν	Generator of $H^*_{\mathbb{C}^\times}(\mathrm{pt}) \cong \mathbb{C}[\nu]$ in degree 2; we localise to $\mathbb{C}(\nu)$
$\mathcal{Z}$	A smooth quasi-projective variety or quotient stack, equipped with a $\mathbb{C}^\times-$action
$I\mathcal{Z}$	Inertia stack of $\mathcal{Z}$
$K(\mathcal{Z})$	Topological $K-$group of $\mathcal{Z}$
$H(\mathcal{Z})$	Cohomology $H^*_{\mathbb{C}^\times}(\mathcal{Z})$, if $\mathcal{Z}$ is a smooth variety; Chen–Ruan cohomology $H^*_{\mathrm{CR},\mathbb{C}^\times}(\mathcal{Z})$, if $\mathcal{Z}$ is an orbifold
$\mathcal{H}(\mathcal{Z})$	Givental's space $H(\mathcal{Z}) \otimes \mathbb{C}(\!(z^{-1})\!)$ of $\mathcal{Z}$
$\mathrm{QH}^*_{\mathbb{C}^\times}(\mathcal{Z})$	$\mathbb{C}^\times-$equivariant quantum cohomology of $\mathcal{Z}$
$(\mathcal{R}, \Phi_{\mathcal{R}})$	A pair of root system with a symmetric group $\Phi_{\mathcal{R}} = \langle\zeta\rangle$
$(\mathcal{R}^{\mathrm{fold}}, \mathcal{R}^{\mathrm{ave}})$	Root systems obtained by folding, averaging
$\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}/X_{\mathcal{R}^{\mathrm{fold}}}$	Quotient stack $[Z_{\mathcal{R}}/\Phi_{\mathcal{R}}]$/ coarse quotient space $Z_{\mathcal{R}}/\Phi_{\mathcal{R}}$
$\mathcal{R}^{\mathrm{res}}$	Root systems corresponding to minimal resolutions of $X_{\mathcal{R}^{\mathrm{fold}}}$

Table 1: Notation used throughout the text. When working in local coordinates for $\mathcal{M}_{\mathrm{AW}}$, we will consistently use lower-case Latin indices for a flat chart for the intersection form; upper-case Latin indices for a flat chart for the Saito metric; and Greek indices for a coordinate chart given by basic invariants. When the context is clear and a choice of root system $\mathcal{R}$ is fixed, we will often omit the subscript $\mathcal{R}$ in the relevant expressions.

Chapter 1

Preliminaries

§ 1.1 Kleinian singularities and the McKay correspondence

In this section, we very briefly review the classical theory of Kleinian singularities and the McKay correspondence, which bridges the representation theory of finite subgroups of $\mathrm{SL}(2;\mathbb{C})$ and the geometry of their minimal resolutions.

1.1.1 Kleinian singularities

Let $G < \mathrm{SL}(2;\mathbb{C})$, $|G| < \infty$ be a finite subgroup. The corresponding *Kleinian singularity* (also referred to as du Val singularity, or a surface singularity of ADE type) is defined as the affine scheme

$$X := \operatorname{Spec}\left(\mathbb{C}[x,y]^G\right) ,$$

where the action on $\mathbb{C}^2$ by G is induced by restriction of the fundamental representation of $\mathrm{SL}(2;\mathbb{C})$. The finite subgroups $G < \mathrm{SL}(2;\mathbb{C})$ and their corresponding singularities are classified by root systems of type $\mathcal{R} = \mathrm{ADE}_l$:

$$G \simeq \begin{cases} \mathbb{Z}/(n+1)\mathbb{Z} & \mathcal{R} = \mathrm{A}_n \,, \\ BD_{4l-2} = \langle 2,2,n\rangle & \mathcal{R} = \mathrm{D}_n \,, \\ BT = \langle 2,3,3\rangle & \mathcal{R} = \mathrm{E}_6 \,, \\ BO = \langle 2,3,4\rangle & \mathcal{R} = \mathrm{E}_7 \,, \\ BI = \langle 2,3,5\rangle & \mathcal{R} = \mathrm{E}_8 \,. \end{cases} \tag{1.1}$$

The type of root systems reflects the resolution geometry of Kleinian singularities as follows.

Klein's classical presentation of the ring of invariants,

$$\mathbb{C}[x,y]^G \simeq \frac{\mathbb{C}[u,v,w]}{\langle \mathcal{I}_G \rangle} \,, \tag{1.2}$$

where

$$\mathcal{I}_G(u,v,w) = \begin{cases} w^2 + u^2 + v^{n+1} & \mathcal{R} = \mathrm{A}_n\,, \\ w^2 + v(u^2 + v^{n-2}) & \mathcal{R} = \mathrm{D}_n\,, \\ w^2 + u^3 + v^4 & \mathcal{R} = \mathrm{E}_6\,, \\ w^2 + u^3 + uv^3 & \mathcal{R} = \mathrm{E}_7\,, \\ w^2 + u^3 + v^5 & \mathcal{R} = \mathrm{E}_8\,, \end{cases} \tag{1.3}$$

realises X as a hypersurface in $\mathbb{C}^3$ with an isolated singularity at the origin. There is a well-known canonical minimal resolution

$$Z \xrightarrow{\pi} X \tag{1.4}$$

obtained through a sequence of blowing-ups of the singularity [56]. The exceptional locus $E = \pi^{-1}(0)$ is a union of smooth rational curves $E_1, \ldots, E_n$ intersecting transversally. The dual graph of these curves is the Dynkin diagram of ADE type in the classification.

1.1.2 The McKay correspondence

Let $\mathrm{Irr}(G) = \{\rho_0, \rho_1, \ldots, \rho_n\}$ denote the set of isomorphism classes of irreducible representations of G, where ρ_0 is the trivial representation. Let Q be the defining representation of G given by the inclusion $G < \mathrm{SL}(2;\mathbb{C})$, and let χ_Q denote its character.

The *McKay graph* of G is an undirected graph whose vertices are indexed by $\mathrm{Irr}(G)$. The number of arrows between ρ_i and ρ_j is given by the multiplicity of ρ_j in the tensor product $\rho_i \otimes Q$. Because χ_Q is real, this is equal to the multiplicity of ρ_i in the tensor product $\rho_j \otimes Q$.

McKay [50] observed a striking bijection between the representation theory of G and the geometry of the minimal resolution Z:

Theorem 1.1.1 (McKay correspondence)**.** *The McKay graph of G is the affine ADE Dynkin diagram corresponding to the type of G. Furthermore, by deleting the vertex corresponding to the trivial representation ρ_0, one obtains the classical ADE Dynkin diagram, which is the dual intersection graph of the exceptional curves $E_1, \ldots, E_n$ in the minimal resolution Z.*

Geometrically, this establishes a natural bijection between the non-trivial irreducible representations of G and the irreducible components of the exceptional divisor:

$$\rho_i \longleftrightarrow E_i, \qquad \text{for } 1 \leq i \leq n. \tag{1.5}$$

The irreducible components E_i are -2 curves. Therefore under this correspondence, the intersection matrix of the exceptional curves is exactly the negative of the Cartan matrix of the corresponding root system $\mathcal{R}$:

$$E_i \cdot E_j = -C_{ij}. \tag{1.6}$$

1.1.3 Dynkin diagrams

Below we list the affine Dynkin diagrams associated with the Kleinian singularities.

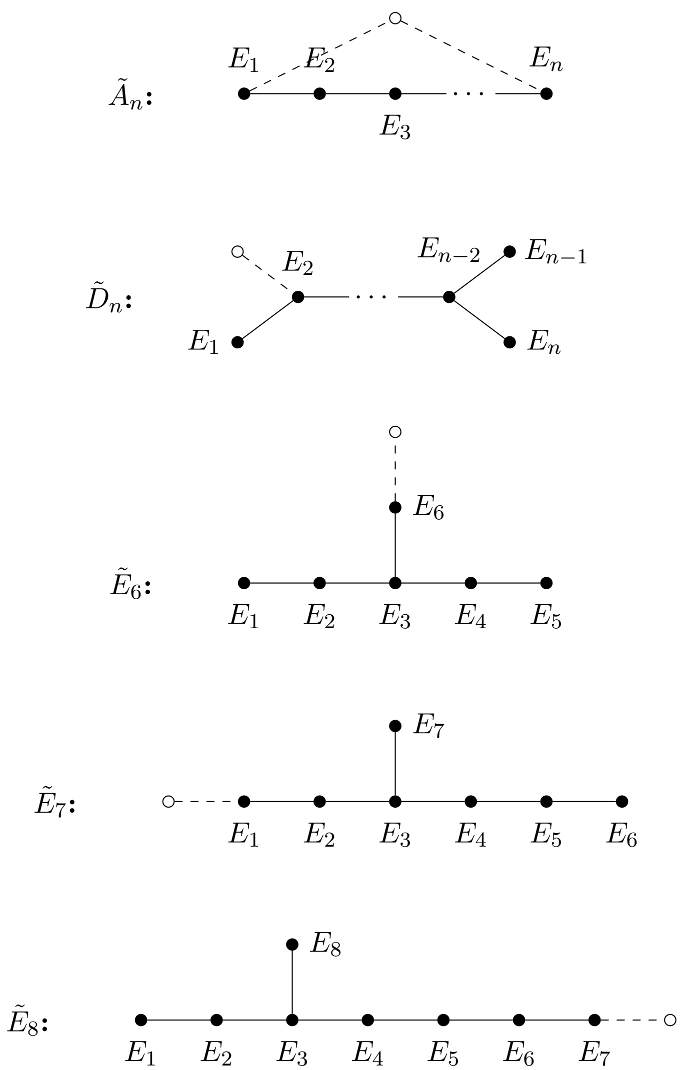


Figure 1.1: Affine Dynkin diagrams of ADE type.

§ 1.2 Chen–Ruan cohomology of global quotient stacks

A *complex orbifold* generalises the notion of a complex manifold by admitting local finite quotient singularities [60]. Locally, it is modeled on quotients U/G, where G is a finite group acting holomorphically and effectively on an open subset $U \subseteq \mathbb{C}^n$. In the algebraic category, the geometric data of a complex orbifold formally corresponds to a smooth, separated Deligne-Mumford stack over $\mathbb{C}$ with generically trivial stabilisers [48]. Throughout this thesis, we treat these two notions as equivalent.

1.2.1 Assumptions and notations

All varieties and stacks are defined over the base field $\mathbb{C}$. Moreover, all stacks considered in this section are global quotient stacks of smooth varieties by finite groups.

Let Y be a smooth quasi-projective variety and let G be a finite group acting on Y. We write

$$\mathcal{X} := [Y/G] \tag{1.7}$$

for the associated global quotient stack, and denote by $\pi\colon \mathcal{X} \to Y/G$ the coarse moduli map.

$\mathbb{C}^\times$-action. Assume that $\mathbb{C}^\times$ acts algebraically on Y and commutes with the G-action, so that it induces an action on $\mathcal{X}$. The induced $\mathbb{C}^\times$-action on the inertia stack $I\mathcal{X}$ is described in (1.10). We assume moreover that the fixed locus $Y^{\mathbb{C}^\times}$ is compact. This hypothesis ensures that the localised $\mathbb{C}^\times$-equivariant pushforwards appearing below are well-defined (see Section 1.2.3).

Coefficients of (co)homology. Unless stated otherwise, all (co)homology groups are taken with $\mathbb{Q}$-coefficients. When a $\mathbb{C}^\times$-action is present, we work $\mathbb{C}^\times$-equivariantly and denote equivariant cohomology by $H^*_{\mathbb{C}^\times}(\,\cdot\,)$. In this setting, all classes are understood as equivariant lifts; moreover, whenever we apply localisation we implicitly extend scalars from $H^*_{\mathbb{C}^\times}(\mathrm{pt}) \cong \mathbb{Q}[\nu]$ to its field of fractions $\mathbb{Q}(\nu)$.

(Co)homology of quotient stack and the Kronecker pairing. Let $q\colon Y \to Y/G$ be the quotient map. With $\mathbb{Q}$-coefficients we identify $H^*(Y/G)$ with the G-invariant subspace $H^*(Y)^G$ via pullback q^*. On homology we use the averaging projector

$$\mathrm{Av}\colon H_2(Y;\mathbb{Q}) \to H_2(Y;\mathbb{Q})^G, \qquad \mathrm{Av}(\gamma) := \frac{1}{|G|}\sum_{g\in G} g_*(\gamma), \tag{1.8}$$

and we regard the degree lattice for the quotient as $\mathrm{Av}\big(H_2(Y;\mathbb{Z})\big) \subset H_2(Y;\mathbb{Q})^G$. The Kronecker pairing on the quotient is defined by evaluation upstairs: for $\alpha \in H^k(Y/G)$ and $\overline{\beta} \in \mathrm{Av}\big(H_2(Y;\mathbb{Z})\big)$,

$$\langle \alpha, \overline{\beta}\rangle_{Y/G} \;:=\; \langle q^*\alpha, \overline{\beta}\rangle_Y. \tag{1.9}$$

1.2.2 Chen–Ruan cohomology

Chen and Ruan [21] introduced a cohomology theory for orbifolds based on the *inertia stack* $I\mathcal{X}$. In the present setup, the inertia stack decomposes as a disjoint union indexed by the conjugacy classes of G:

$$I\mathcal{X} \;\cong\; \bigsqcup_{(g)\in\mathrm{Conj}(G)} \mathcal{X}_{(g)}, \qquad \mathcal{X}_{(g)} := [Y^g/C(g)], \tag{1.10}$$

where Y^g is the fixed locus and $C(g)$ is the centraliser of g.

The connected components of the substacks $\mathcal{X}_{(g)}$ are called *sectors*. Those contained in $\mathcal{X}_{(1)}$ form the **untwisted** sector, and the remaining ones are **twisted** sectors.

There is a canonical involution

$$\iota\colon I\mathcal{X} \to I\mathcal{X}, \qquad (x,g) \mapsto (x,g^{-1}), \tag{1.11}$$

which exchanges $\mathcal{X}_{(g)}$ with $\mathcal{X}_{(g^{-1})}$.

Age. For $x \in Y^g$, the element g acts on T_xY with eigenvalues $\exp(2\pi\mathrm{i}\,\theta_j)$, where $\theta_j \in [0,1)$. Define

$$\operatorname{age}(g)\big|_x := \sum_j \theta_j. \tag{1.12}$$

On each connected component of Y^g this is constant, so we view $\operatorname{age}(g)$ as a locally constant function on $\mathcal{X}_{(g)}$.

State space. The Chen–Ruan cohomology of $\mathcal{X}$ is

$$H^*_{\mathrm{CR}}(\mathcal{X}) := H^*(I\mathcal{X}), \tag{1.13}$$

equipped with the Chen–Ruan grading by declaring that a class $\alpha \in H^k(\mathcal{X}_{(g)})$ has shifted degree

$$\deg_{\mathrm{CR}}(\alpha) := k + 2\ \operatorname{age}(g). \tag{1.14}$$

Equivalently, as a graded vector space one may write

$$H^*_{\mathrm{CR}}(\mathcal{X}) \;\cong\; \bigoplus_{(g)} H^{*-2\ \operatorname{age}(g)}(\mathcal{X}_{(g)}). \tag{1.15}$$

We call the contribution from twisted sectors the *twisted part* of $H^*_{\mathrm{CR}}(\mathcal{X})$, namely

$$H^{*,\mathrm{tw}}_{\mathrm{CR}}(\mathcal{X}) \;:=\; \bigoplus_{(g)\neq(1)} H^{*-2\ \operatorname{age}(g)}(\mathcal{X}_{(g)}). \tag{1.16}$$

Orbifold Poincaré pairing. For the remainder of this subsection, assume that $\mathcal{X}$ is proper. The orbifold Poincaré pairing on $\mathcal{X}$ is defined using the standard pairing and the involution on $I\mathcal{X}$:

$$(\alpha,\beta)_{\mathrm{CR}} \;:=\; \int_{I\mathcal{X}} \alpha \cup \iota^*(\beta), \qquad \alpha,\beta \in H^*_{\mathrm{CR}}(\mathcal{X}), \tag{1.17}$$

where the integration over a global quotient $\mathcal{X}_{(g)} = [Y^g/C(g)]$ is understood as

$$\int_{[Y^g/C(g)]} (\cdot) \;:=\; \frac{1}{|C(g)|} \int_{Y^g} (\cdot) \tag{1.18}$$

whenever this interpretation is convenient.

Double inertia stack and the obstruction bundle. The *double inertia stack* (or 2-fold inertia) of $\mathcal{X}$ is defined by

$$I_2\mathcal{X} \;\coloneqq\; I\mathcal{X} \times_{\mathcal{X}} I\mathcal{X}, \tag{1.19}$$

with the two projections $\mathrm{e}_1, \mathrm{e}_2 \colon I_2\mathcal{X} \to I\mathcal{X}$. There is also a natural morphism $\mu\colon I_2\mathcal{X} \to I\mathcal{X}$ induced by multiplying stabiliser elements.

Set

$$Y^{g,h} \coloneqq Y^g \cap Y^h, \qquad C(g,h) \coloneqq C(g) \cap C(h), \qquad \mathcal{X}^{g,h} \coloneqq [Y^{g,h}/C(g,h)]. \tag{1.20}$$

Then $I_2\mathcal{X}$ admits the presentation

$$I_2\mathcal{X} \;\cong\; \Big[\coprod_{(g,h)\in G\times G} Y^{g,h} \Big/ G\Big] \;\cong\; \coprod_{[(g,h)]\in G\backslash(G\times G)} \mathcal{X}^{g,h}, \tag{1.21}$$

where G acts on $G \times G$ by simultaneous conjugation. On the (g,h)-component, the morphisms

$$\mathrm{e}_1 : \mathcal{X}^{g,h} \to \mathcal{X}_{(g)}, \qquad \mathrm{e}_2 : \mathcal{X}^{g,h} \to \mathcal{X}_{(h)}, \qquad \mu : \mathcal{X}^{g,h} \to \mathcal{X}_{(gh)} \tag{1.22}$$

are induced by the inclusions $Y^{g,h} \hookrightarrow Y^g$, $Y^{g,h} \hookrightarrow Y^h$, and $Y^{g,h} \hookrightarrow Y^{gh}$, together with the corresponding inclusions of centralisers. Equivalently, the multiplication map is induced by $(y,g,h) \mapsto (y,gh)$ on stabilisers.

There is a canonically defined obstruction bundle $\mathrm{Ob}_{g,h}$ on each component $\mathcal{X}^{g,h} \subset I_2\mathcal{X}$. Its Euler class

$$e_{g,h} \coloneqq e(\mathrm{Ob}_{g,h}) \in H^{2r_{g,h}}(\mathcal{X}^{g,h}) \tag{1.23}$$

plays a central role in the definition of the orbifold Chen–Ruan cup product; see [21, 37] for its construction and properties.

Orbifold cup product. For the following componentwise formula only, assume that G is abelian; this includes all groups appearing in the folding constructions of this thesis. Let $\alpha \in H^*(\mathcal{X}_{(g)})$ and $\beta \in H^*(\mathcal{X}_{(h)})$ be homogeneous. Their Chen–Ruan product $\alpha \star_{\mathrm{CR}} \beta$ is the class in the (gh)-sector defined by

$$\alpha \star_{\mathrm{CR}} \beta \;\coloneqq\; \mu_*\Big(\mathrm{e}_1^*\alpha \,\cup\, \mathrm{e}_2^*\beta \,\cup\, e_{g,h}\Big) \;\in\; H^*(\mathcal{X}_{(gh)}). \tag{1.24}$$

Taking the direct sum over $g \in G$ and inserting the degree shifts yields a graded product on $H^*_{\mathrm{CR}}(\mathcal{X})$. For a nonabelian group, the product is obtained by summing the same push–pull construction over the simultaneous-conjugacy classes of pairs (g',h') with $g' \in (g)$ and $h' \in (h)$.

The Chen–Ruan product, together with the orbifold Poincaré pairing, defines the *degree-zero three point function* for any $\alpha, \beta, \gamma \in H^*_{\mathrm{CR}}(\mathcal{X})$:

$$\langle \alpha, \beta, \gamma \rangle_0 \;\coloneqq\; (\alpha \star_{\mathrm{CR}} \beta, \gamma)_{\mathrm{CR}}. \tag{1.25}$$

1.2.3 Equivariant variants

Chen–Ruan cohomology admits a natural $\mathbb{C}^\times$-equivariant refinement. In this thesis we work primarily in this $\mathbb{C}^\times$-equivariant setting.

$\mathbb{C}^\times$-equivariant Chen–Ruan cohomology. We set

$$H^*_{\mathrm{CR},\mathbb{C}^\times}(\mathcal{X}) \coloneqq H^*_{\mathbb{C}^\times}(I\mathcal{X}), \tag{1.26}$$

equipped with the same Chen–Ruan grading convention as in Section 1.2.2. The equivariant orbifold Poincaré pairing is

$$(\alpha, \beta)_{\mathrm{CR},\mathbb{C}^\times} \;\coloneqq\; \int_{I\mathcal{X}}^{\mathbb{C}^\times} \alpha \cup \iota^*(\beta) \in \mathbb{Q}(\nu), \tag{1.27}$$

where $\int^{\mathbb{C}^\times}$ denotes the $\mathbb{C}^\times$-equivariant pushforward to a point.

Remark 1.2.1 (Non-proper targets)**.** Since Y is only quasi-projective, the non-equivariant pushforward $\int_{I\mathcal{X}}$ need not be defined. In this thesis all pairings and correlators are understood $\mathbb{C}^\times$-equivariantly, and their values lie in the localised coefficient ring $\mathbb{Q}(\nu)$. The hypothesis that $Y^{\mathbb{C}^\times}$ is compact ensures that these localised pushforwards are well-defined via localisation.

$\mathbb{C}^\times$-equivariant Chen–Ruan product. The Chen–Ruan product admits a $\mathbb{C}^\times$-equivariant refinement

$$\star_{\mathrm{CR},\mathbb{C}^\times} \;:\; H^*_{\mathrm{CR},\mathbb{C}^\times}(\mathcal{X}) \otimes H^*_{\mathrm{CR},\mathbb{C}^\times}(\mathcal{X}) \longrightarrow H^*_{\mathrm{CR},\mathbb{C}^\times}(\mathcal{X}), \tag{1.28}$$

defined by the same correspondence on the double inertia stack as in Section 1.2.2, with all pullbacks, pushforwards, and Chern classes taken $\mathbb{C}^\times$-equivariantly. Accordingly, the degree–zero three–point function (1.25) admits a tautological $\mathbb{C}^\times$-equivariant extension.

§ 1.3 Gromov–Witten invariants and quantum cohomology

In this section, we briefly recall the necessary background on Gromov–Witten theory for both smooth varieties and orbifolds. Our main references are [21, 1, 20].

1.3.1 Gromov–Witten invariants of smooth varieties

Let Z be a smooth quasi-projective variety equipped with a $\mathbb{C}^\times$-action. Let $\overline{M}_{g,n}(Z, \beta)$ denote the moduli space of genus-g, n-pointed stable maps to Z of class $\beta \in H_2(Z, \mathbb{Z})$. By the standard theory of Gromov–Witten invariants, it admits a perfect obstruction theory and a virtual fundamental class

$$[\overline{M}_{g,n}(Z, \beta)]^{\mathrm{vir}} \in A_{\mathrm{vdim}}\big(\overline{M}_{g,n}(Z, \beta)\big),$$

where the virtual dimension is given by

$$\operatorname{vdim} \overline{M}_{g,n}(Z,\beta) = (1-g)(\dim Z - 3) + n + \int_\beta c_1(T_Z). \tag{1.29}$$

The $\mathbb{C}^\times$-action on Z naturally induces a $\mathbb{C}^\times$-action on the moduli space $\overline{M}_{g,n}(Z,\beta)$, and the virtual fundamental class has a natural equivariant refinement

$$[\overline{M}_{g,n}(Z,\beta)]^{\mathrm{vir},\mathbb{C}^\times} \in A^{\mathbb{C}^\times}_{\mathrm{vdim}}\big(\overline{M}_{g,n}(Z,\beta)\big).$$

Gromov–Witten invariants are intersection numbers against the virtual fundamental class. In the $\mathbb{C}^\times$-equivariant setup, one needs to use Graber–Pandharipande's virtual localisation theorem [40] to define such invariants for a noncompact target. For every (g,n,β) under consideration, assume that the fixed locus of the induced $\mathbb{C}^\times$-action on $\overline{M}_{g,n}(Z,\beta)$ is proper. Virtual localisation then defines, for insertions $\alpha_1,\dots,\alpha_n \in H^*_{\mathbb{C}^\times}(Z)$, the *primary $\mathbb{C}^\times$-equivariant Gromov–Witten invariants*:

$$\big\langle \alpha_1,\dots,\alpha_n \big\rangle^{Z,\mathbb{C}^\times}_{g,n,\beta} := \sum_{F\subset \overline{M}_{g,n}(Z,\beta)^{\mathbb{C}^\times}} \int_{[F]^{\mathrm{vir}}} \frac{i_F^*\Big(\prod_{i=1}^n \mathrm{ev}_i^*(\alpha_i)\Big)}{e_{\mathbb{C}^\times}(N_F^{\mathrm{vir}})} \in \mathbb{Q}(\nu), \tag{1.30}$$

where $\mathrm{ev}_i\colon \overline{M}_{g,n}(Z,\beta) \longrightarrow Z$ is the evaluation map, the sum runs over the connected components F of the fixed locus, $i_F\colon F \hookrightarrow \overline{M}_{g,n}(Z,\beta)$ is the inclusion, and the Euler class of the virtual normal bundle N_F^{vir} is inverted in the localised equivariant cohomology of a point.

Moreover, let $\mathcal{L}_i, 1\le i\le n$ be the universal cotangent line bundle on $\overline{M}_{g,n}(Z,\beta)$ whose fibers are cotangent lines at the ith marked point of the source curve, and $\psi_i = c_1(\mathcal{L}_i)$. Then *descendant $\mathbb{C}^\times$-equivariant Gromov–Witten invariants* are defined as:

$$\big\langle \alpha_1\psi_1^{k_1},\dots,\alpha_n\psi_n^{k_n} \big\rangle^{Z,\mathbb{C}^\times}_{g,n,\beta} := \sum_{F\subset \overline{M}_{g,n}(Z,\beta)^{\mathbb{C}^\times}} \int_{[F]^{\mathrm{vir}}} \frac{i_F^*\Big(\prod_{i=1}^n \mathrm{ev}_i^*(\alpha_i)\psi_i^{k_i}\Big)}{e_{\mathbb{C}^\times}(N_F^{\mathrm{vir}})} \in \mathbb{Q}(\nu). \tag{1.31}$$

1.3.2 Gromov–Witten invariants of orbifolds

Gromov–Witten invariants are also defined for orbifolds. This theory was first developed by Chen–Ruan [20] in the symplectic category and later by Abramovich–Graber–Vistoli [1] in the algebraic category. In this subsection, $\mathcal{X}$ may be any smooth orbifold, not necessarily a global quotient stack. Assume that $\mathcal{X}$ is equipped with a $\mathbb{C}^\times$-action. Following [1], one considers the moduli stack $\mathcal{K}_{g,n}(\mathcal{X},\beta)$ of genus-g, n-pointed twisted stable maps to $\mathcal{X}$ of degree β. It carries a canonical perfect obstruction theory and hence a virtual fundamental class

$$[\mathcal{K}_{g,n}(\mathcal{X},\beta)]^{\mathrm{vir}} \in A_{\mathrm{vdim}}\big(\mathcal{K}_{g,n}(\mathcal{X},\beta)\big).$$

Here, vdim denotes the virtual dimension on a connected component: if the i-th marking lies in the twisted sector of $I\mathcal{X}$ of age age_i $(1 \le i \le n)$, then

$$\operatorname{vdim}\mathcal{K}_{g,n}(\mathcal{X},\beta) \;=\; (1-g)\big(\dim\mathcal{X}-3\big) \;+\; n \;+\; \int_\beta c_1(T_{\mathcal{X}}) \;-\; \sum_{i=1}^{n}\mathrm{age}_i\,. \tag{1.32}$$

If $\mathcal{X}$ is equipped with a $\mathbb{C}^\times$-action, $\mathcal{K}_{g,n}(\mathcal{X},\beta)$ inherits this action, and the canonical obstruction theory admits a natural $\mathbb{C}^\times$-equivariant enhancement. We thus obtain a $\mathbb{C}^\times$-equivariant virtual fundamental class

$$[\mathcal{K}_{g,n}(\mathcal{X},\beta)]^{\mathrm{vir},\mathbb{C}^\times} \in A^{\mathbb{C}^\times}_{\mathrm{vdim}}\big(\mathcal{K}_{g,n}(\mathcal{X},\beta)\big).$$

There are *(unrigidified) evaluation morphisms*

$$\widetilde{\mathrm{ev}}_i\colon \mathcal{K}_{g,n}(\mathcal{X},\beta) \longrightarrow I\mathcal{X}, \qquad i=1,\dots,n, \tag{1.33}$$

recording the twisted sector (monodromy) at the i-th marking. Composing with the rigidification morphism $\rho\colon I\mathcal{X}\to \bar{I}\mathcal{X}$ yields the *rigidified* evaluation morphisms $\mathrm{ev}_i := \rho\circ\widetilde{\mathrm{ev}}_i$ as in [1, §4.4].

$\mathbb{C}^\times$-equivariant orbifold Gromov–Witten invariants. Just as in the smooth case, for every (g,n,β) under consideration, assume that the fixed locus of the induced $\mathbb{C}^\times$-action on $\mathcal{K}_{g,n}(\mathcal{X},\beta)$ is proper. The virtual localisation theorem [40] then defines, for insertions $\alpha_i \in H^*_{\mathrm{CR},\mathbb{C}^\times}(\mathcal{X}) = H^*_{\mathbb{C}^\times}(I\mathcal{X})$, the corresponding $\mathbb{C}^\times$-*equivariant orbifold Gromov–Witten invariants*:

$$\big\langle \alpha_1,\dots,\alpha_n\big\rangle^{\mathcal{X},\mathbb{C}^\times}_{g,n,\beta} \;:=\; \sum_{F\subset\mathcal{K}_{g,n}(\mathcal{X},\beta)^{\mathbb{C}^\times}} \int_{[F]^{\mathrm{vir}}} \frac{i_F^*\Big(\prod_{i=1}^n \widetilde{\mathrm{ev}}_i^*(\alpha_i)\Big)}{e_{\mathbb{C}^\times}(N_F^{\mathrm{vir}})} \;\in\; \mathbb{Q}(\nu). \tag{1.34}$$

Similar to the smooth case, define $\mathcal{L}_i, 1\le i\le n$ as the universal cotangent line bundle on $\overline{M}_{g,n}(Z,\beta)$ whose fibers are cotangent lines at the ith marked point of the *coarse moduli space* of the source curve, and $\psi_i = c_1(\mathcal{L}_i)$. Then the *descendant* $\mathbb{C}^\times$-*equivariant orbifold Gromov–Witten invariants* are defined as:

$$\big\langle \alpha_1\psi_1^{k_1},\dots,\alpha_n\psi_n^{k_n}\big\rangle^{\mathcal{X},\mathbb{C}^\times}_{g,n,\beta} \;:=\; \sum_{F\subset\mathcal{K}_{g,n}(\mathcal{X},\beta)^{\mathbb{C}^\times}} \int_{[F]^{\mathrm{vir}}} \frac{i_F^*\Big(\prod_{i=1}^n \widetilde{\mathrm{ev}}_i^*(\alpha_i)\psi_i^{k_i}\Big)}{e_{\mathbb{C}^\times}(N_F^{\mathrm{vir}})} \;\in\; \mathbb{Q}(\nu). \tag{1.35}$$

Remark 1.3.1. If $\mathcal{X}$ is a Calabi–Yau threefold, i.e. $\dim\mathcal{X}=3$ and $\omega_{\mathcal{X}}\cong\mathcal{O}_{\mathcal{X}}$ (equivalently $c_1(T_{\mathcal{X}})=0$), then (1.32) reduces to

$$\operatorname{vdim}\mathcal{K}_{g,n}(\mathcal{X},\beta) = n - \sum_{i=1}^n \mathrm{age}_i,$$

and in particular $\operatorname{vdim}\mathcal{K}_{g,0}(\mathcal{X},\beta)=0$.

1.3.3 $\mathbb{C}^\times$-equivariant quantum cohomology

To treat the smooth and orbifold cases uniformly, let $\mathcal{Z}$ denote either a smooth quasi-projective variety or an orbifold. We establish the convention that $H(\mathcal{Z})$ refers to the standard equivariant cohomology $H^*_{\mathbb{C}^\times}(\mathcal{Z})$ for the smooth case, and the equivariant Chen–Ruan cohomology $H^*_{\mathrm{CR},\mathbb{C}^\times}(\mathcal{Z})$ for the orbifold case. Correspondingly, $\langle \cdots \rangle^{\mathcal{Z},\mathbb{C}^\times}_{g,n,\beta}$ indicates the equivariant Gromov–Witten invariants defined in (1.30) or (1.34).

Let τ be a formal parameter in $H(\mathcal{Z})$. All $\mathbb{C}^\times$-equivariant products below are taken after localising the equivariant parameter, so the resulting Frobenius structures are defined over $\mathbb{C}(\nu)$. The $\mathbb{C}^\times$*-equivariant big quantum product* $*_\tau$ on $H(\mathcal{Z})$ is defined by

$$(\alpha_1 *_\tau \alpha_2, \alpha_3)_{\mathcal{Z},\mathbb{C}^\times} \;:=\; \sum_\beta \sum_{n\geq 0} \frac{1}{n!} \big\langle \alpha_1, \alpha_2, \alpha_3, \tau, \ldots, \tau \big\rangle^{\mathcal{Z},\mathbb{C}^\times}_{0,n+3,\beta}, \tag{1.36}$$

where the pairing on the left is the standard equivariant Poincaré pairing (or the equivariant orbifold Poincaré pairing, respectively).

The $\mathbb{C}^\times$*-equivariant quantum cohomology* is defined as the family

$$\mathrm{QH}^*_{\mathbb{C}^\times}(\mathcal{Z}) \;:=\; \big(H(\mathcal{Z}), *_\tau, \langle , \rangle, \mathbf{1}\big), \tag{1.37}$$

viewed as a (formal) Frobenius manifold structure over $\mathbb{C}(\nu)$.

Remark 1.3.2 (Fundamental class axiom)**.** The **fundamental class axiom** for Gromov–Witten invariants says that

$$\big\langle \alpha_1, \ldots, \alpha_n, \mathbf{1} \big\rangle^{\mathcal{Z},\mathbb{C}^\times}_{g,n+1,\beta} = 0,$$

except for the case $(g, n+1, \beta) = (0,3,0)$, where

$$\big\langle \alpha, \ldots, \beta, \mathbf{1} \big\rangle^{\mathcal{Z},\mathbb{C}^\times}_{0,3,0} = \int_{\mathcal{Z}}^{\mathbb{C}^\times} \alpha \cup \beta.$$

This implies that $\mathbf{1}$ is the unit of the Frobenius algebra.

Remark 1.3.3 (Divisor equation)**.** For a divisor class $D \in H^2_{\mathbb{C}^\times}(\mathcal{Z})$ and an effective curve class $\beta \neq 0$, the **divisor equation** for Gromov–Witten invariants implies that

$$\big\langle \alpha_1, \ldots, \alpha_n, D \big\rangle^{\mathcal{Z},\mathbb{C}^\times}_{0,n+1,\beta} = \int_\beta D \, \big\langle \alpha_1, \ldots, \alpha_n \big\rangle^{\mathcal{Z},\mathbb{C}^\times}_{0,n,\beta}.$$

Thus repeated divisor insertions in the big quantum product exponentiate to the factor $e^{\int_\beta \tau'}$ below. The degree-zero contribution is the usual classical term and is included separately in the quantum product.

Remark 1.3.4 (Small quantum cohomology and Novikov parameters)**.** The product defined above is the *big* quantum product, since it depends on a general parameter $\tau \in H(\mathcal{Z})$.

The $\mathbb{C}^\times$-equivariant *small* quantum cohomology keeps only the dependence on $H^2_{\mathbb{C}^\times}(\mathcal{Z})$, and the curve classes are encoded by Novikov variables. Concretely, restricting to $\tau' \in H^2_{\mathbb{C}^\times}(\mathcal{Z})$ and applying the divisor equation, (1.36) becomes, after resummation:

$$(\alpha_1 *_{\tau'} \alpha_2, \alpha_3)_{\mathcal{Z},\mathbb{C}^\times} = \sum_\beta e^{\int_\beta \tau'} \langle \alpha_1, \alpha_2, \alpha_3 \rangle^{\mathcal{Z},\mathbb{C}^\times}_{0,3,\beta} .$$

The $e^{\int_\beta \tau'}$ coefficients are often replaced by **Novikov parameters** Q^β. Let $\mathrm{Eff}(\mathcal{Z}) \subset H_2(\mathcal{Z};\mathbb{Z})$ be the semigroup of effective curve classes, choose generators $\beta_1, \dots, \beta_r$ of $\mathrm{Eff}(\mathcal{Z})$, and let

$$\Lambda_{\mathrm{Nov}} \coloneqq \mathbb{Q}[[Q_1, \dots, Q_r]]$$

be the Novikov ring. For $\beta = \sum_{i=1}^r d_i \beta_i$ we write

$$Q^\beta \coloneqq \prod_{i=1}^r Q_i^{d_i} \in \Lambda_{\mathrm{Nov}}.$$

The $\mathbb{C}^\times$-equivariant *small* quantum product $\star_{\mathrm{sm}}$ on $H(\mathcal{Z}) \otimes \Lambda_{\mathrm{Nov}}$ is defined by

$$(\alpha_1 \star_{\mathrm{sm}} \alpha_2, \alpha_3)_{\mathcal{Z},\mathbb{C}^\times} = \sum_{\beta \in \mathrm{Eff}(\mathcal{Z})} Q^\beta \langle \alpha_1, \alpha_2, \alpha_3 \rangle^{\mathcal{Z},\mathbb{C}^\times}_{0,3,\beta} . \tag{1.38}$$

If Gromov–Witten invariants can be defined for $\mathcal{Z}$ without localisation (e.g., if $\mathcal{Z}$ is proper), the same construction applies and defines the non-equivariant small quantum cohomology on the standard (or non-equivariant Chen–Ruan) cohomology spaces.

§ 1.4 Quantum $\mathcal{D}-$module and Givental space

Let $\mathcal{Z}$ be an smooth quasi-projective variety or global quotient stack that admits a $\mathbb{C}^\times$-action with compact support. This allows us to define $\mathbb{C}^\times$-equivariant Gromov–Witten invariants of $\mathcal{Z}$ and $\mathbb{C}^\times$-equivariant orbifold quantum cohomology, as explained in Section 1.3.

In this section, we review the formalism of quantum $\mathcal{D}-$modules and Givental's symplectic vector space, which provide a geometric framework for generating functions of Gromov–Witten invariants. This section closely follows Appendix A of [13]. For more standard references, see [22].

1.4.1 The quantum D-module

The Dubrovin connection. Recall that the state space of the (orbifold) quantum product is given by the $\mathbb{C}^\times$-equivariant (Chen–Ruan) cohomology $H(\mathcal{Z})$.

The quantum $\mathcal{D}-$module is defined on the tangent bundle $TH(\mathcal{Z})$ equipped with the Dubrovin connection. For a cohomology class $\tau \in H(\mathcal{Z})$ and a tangent vector $\alpha \in T_\tau H(\mathcal{Z})$, the Dubrovin connection in the formal parameter z-direction is given by

$$\nabla_\alpha \coloneqq \partial_\alpha + \frac{1}{z}(\alpha *_\tau), \tag{1.39}$$

where $*_\tau$ is the $\mathbb{C}^\times$-equivariant big orbifold quantum product from (1.36). The flatness of the connection ∇ reflects the associativity of the quantum product and the WDVV equations.

The fundamental solution matrix. The fundamental solution to the Dubrovin connection is governed by the $\mathrm{End}(H)$-valued function $S(\tau, z)\colon H(\mathcal{Z}) \to H(\mathcal{Z})[\![z^{-1}]\!]$, defined by

$$S(\tau, z)(\alpha) \coloneqq \alpha - \sum_\beta \sum_{n \geq 0} \frac{1}{n!}\frac{1}{z}\Big\langle \alpha, \tau, \dots, \tau, \frac{\phi_i}{z+\psi}\Big\rangle^{\mathcal{X},\mathbb{C}^\times}_{0,n+2,\beta}\phi^i, \tag{1.40}$$

where $\{\phi_i\}$ is a homogeneous basis of $H(\mathcal{Z})$, $\{\phi^i\}$ is the dual basis with respect to the equivariant orbifold Poincaré pairing $(\cdot,\cdot)_{\mathrm{CR},\mathbb{C}^\times}$, and ψ denotes the first Chern class of the universal cotangent line bundle at the last marked point. The operator $S(\tau, z)$ sends any element $\alpha \in H^*_{\mathrm{CR},\mathbb{C}^\times}(\mathcal{Z})$ to a flat section of the $\mathcal{D}-$module. Furthermore, it satisfies the unitarity property $S(\tau, z)S(\tau, -z)^* = \mathrm{Id}$.

1.4.2 Givental's symplectic space and the Lagrangian cone

The symplectic vector space. Following Givental, we enlarge the state space to the infinite-dimensional loop space

$$\mathcal{H}_\mathcal{Z} \coloneqq H(\mathcal{Z})(\!(z^{-1})\!). \tag{1.41}$$

This loop space is equipped with a natural non-degenerate symplectic pairing Ω given by

$$\Omega(f, g) \coloneqq \mathrm{Res}_{z=0}\big(f(-z), g(z)\big)_{\mathrm{CR},\mathbb{C}^\times} dz, \qquad f, g \in \mathcal{H}_\mathcal{Z}. \tag{1.42}$$

The space $\mathcal{H}_\mathcal{Z}$ comes with a natural polarisation $\mathcal{H}_\mathcal{Z} = \mathcal{H}_{\mathcal{Z},+} \oplus \mathcal{H}_{\mathcal{Z},-}$, defined by

$$\mathcal{H}_{\mathcal{Z},+} \coloneqq H(\mathcal{Z})[z], \qquad \mathcal{H}_{\mathcal{Z},-} \coloneqq z^{-1}H(\mathcal{Z})[\![z^{-1}]\!]. \tag{1.43}$$

The subspaces $\mathcal{H}_{\mathcal{Z},+}$ and $\mathcal{H}_{\mathcal{Z},-}$ are maximally isotropic with respect to Ω. Furthermore, $\mathcal{H}_\mathcal{Z}$ can be identified with the cotangent bundle $T^*\mathcal{H}_{\mathcal{Z},+}$ by viewing $\mathcal{H}_{\mathcal{Z},-}$ as the dual of $\mathcal{H}_{\mathcal{Z},+}$ via the symplectic pairing. If $\{\phi_i\}$ is a homogeneous basis of $H(\mathcal{Z})$ and

$\{\phi^i\}$ is its dual basis, then an element in $\mathcal{H}_{\mathcal{Z}}$ is written in Darboux coordinates as $\sum(q_k^i\phi_i z^k + p_{k,i}\phi^i(-z)^{-k-1})$, where the q_k^i are coordinates on $\mathcal{H}_{\mathcal{Z},+}$ and the $p_{k,i}$ are the corresponding cotangent coordinates from $\mathcal{H}_{\mathcal{Z},-}$.

The J-function and the Lagrangian cone. The genus-zero Gromov–Witten theory of $\mathcal{Z}$ is geometrically encoded by Givental's Lagrangian cone $\mathcal{L}_{\mathcal{Z}} \subset \mathcal{H}_{\mathcal{Z}}$. To define it, we first consider the genus-zero descendent prepotential

$$\mathcal{F}^0_{\mathcal{Z}}(\mathbf{t}) = \sum_{\beta}\sum_{n\geq 0}\frac{1}{n!}\langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi)\rangle^{\mathcal{Z}}_{0,n,\beta}, \tag{1.44}$$

where $\mathbf{t}(z) \in \mathcal{H}_{\mathcal{Z},+}$. Recall from (1.43) that the symplectic space $\mathcal{H}_{\mathcal{Z}}$ can be identified with the cotangent bundle $T^*\mathcal{H}_{\mathcal{Z},+}$. The Lagrangian cone $\mathcal{L}_{\mathcal{Z}}$ is then the formal submanifold of $\mathcal{H}_{\mathcal{Z}}$ defined as the graph of the differential $d\mathcal{F}^0_{\mathcal{Z}}$.

A distinguished family on the Lagrangian cone is given by the J-function:

$$J_{\mathcal{Z}}(\tau, z) := z\mathbf{1} + \tau + \sum_{\beta}\sum_{n\geq 0}\frac{1}{n!}\langle \tau, \ldots, \tau, \frac{\phi_i}{z-\psi}\rangle^{\mathcal{Z},\mathbb{C}^\times}_{0,n+1,\beta}\phi^i. \tag{1.45}$$

The relation

$$J_{\mathcal{Z}}(\tau, z) = zS(\tau, -z)^\dagger\mathbf{1} \tag{1.46}$$

implies that $J_{\mathcal{Z}}(\tau, z)$ solves the quantum differential equation:

$$z\frac{\partial}{\partial t^\alpha}J_{\mathcal{Z}}(\tau, z) = \phi_\alpha *_\tau J_{\mathcal{Z}}(\tau, z). \tag{1.47}$$

In coordinates, expanding $J_{\mathcal{Z}}(\tau, z) = \sum_\mu J^\mu(\tau, z)\phi_\mu$, this reads as

$$z\frac{\partial J^\mu}{\partial t^\alpha} = \sum_\nu C^\mu_{\alpha\nu}(\tau)J^\nu, \tag{1.48}$$

where $C^\mu_{\alpha\nu}(\tau)$ are the structure constants of the big quantum product, defined by $\phi_\alpha *_\tau \phi_\nu = \sum_\mu C^\mu_{\alpha\nu}(\tau)\phi_\mu$.

Remark 1.4.1. Recall Remark 1.3.4 that restricting to $\tau' \in H^2_{\mathbb{C}^\times}(Z)$, the big quantum cohomology becomes the small quantum cohomology. Correspondingly and using Novikov parameters, applying the Divisor equation, the *small $J-$function* is

$$J^{\mathrm{sm}}_{\mathcal{Z}}(\tau', z) := J_{\mathcal{Z}}(\tau', z) = z\mathrm{e}^{\tau'/z}\left(\mathbf{1}_{\mathcal{Z}} + \sum_{\beta,k}Q^\beta\phi^k\langle\frac{\phi_k}{z(z-\psi)}\rangle\right). \tag{1.49}$$

Givental's formalism and crepant resolution conjecture. Let Z be a crepant resolution of the coarse moduli space X of a Gorenstein orbifold $\mathcal{X}$. Coates–Corti–Iritani–Tseng and Ruan made a conjecture that relates Givental's formalism to the crepant resolution conjecture, and its basic statement is:

Conjecture 1.4.2 ([25, 26])**.** *There is a degree-preserving $\mathbb{C}(\!(z^{-1})\!)$-linear symplectic isomorphism $\mathbb{U} : \mathcal{H}_{\mathcal{X}} \to \mathcal{H}_Z$ and a choice of analytic continuation of $\mathcal{L}_{\mathcal{X}}$ and $\mathcal{L}_Z$ such that $\mathbb{U}(\mathcal{L}_{\mathcal{X}}) = \mathcal{L}_Z$.*

1.4.3 Integral structure

The $\widehat{\Gamma}$-class. Following Iritani[42, 43], the natural integral structure on the space of solutions to the quantum differential equation is governed by the topological K-group $K(\mathcal{Z})$, suitably modified by the $\widehat{\Gamma}$-class.

Let $T\mathcal{Z}$ be the tangent bundle of $\mathcal{Z}$, and suppose its Chern roots are δ_i. For a smooth manifold, the $\widehat{\Gamma}$-class of $\mathcal{Z}$ is the characteristic class defined by

$$\widehat{\Gamma}_{\mathcal{Z}} \coloneqq \prod_i \Gamma(1+\delta_i), \tag{1.50}$$

where $\Gamma(x)$ is the classical Gamma function.

For an orbifold $\mathcal{Z}$, this definition naturally extends to the inertia stack $I\mathcal{Z}$ by incorporating the age shifts of the twisted sectors. On a connected component $I\mathcal{Z}_v$ of the inertia stack, the pullback of the tangent bundle decomposes into eigenbundles $T\mathcal{Z}|_{I\mathcal{Z}_v} = \bigoplus_{0\le f<1}(T\mathcal{Z})_{v,f}$ on which the universal stabiliser acts with eigenvalue $e^{2\pi \mathrm{i} f}$. If $\delta_{v,f,i}$ are the Chern roots of the eigenbundle $(T\mathcal{Z})_{v,f}$, the orbifold $\widehat{\Gamma}$-class is defined as

$$\widehat{\Gamma}_{\mathcal{Z}} \coloneqq \bigoplus_v \prod_{0\le f<1} \prod_i \Gamma(1-f+\delta_{v,f,i}). \tag{1.51}$$

Integral structure framing. The integral structure on the Givental space $\mathcal{H}(\mathcal{Z})$ is defined by an explicit framing map from the topological K-group $K(\mathcal{Z})$. To a K-theory class $V \in K(\mathcal{Z})$, the *integral structure framing* map $\widehat{\Psi}_{\mathcal{Z}} : K(\mathcal{Z}) \to \mathcal{H}(\mathcal{Z})$ takes the form:

$$\widehat{\Psi}_{\mathcal{Z}}(V) \coloneqq (2\pi)^{-\frac{\dim \mathcal{Z}}{2}} z^{-\mu} \left(\widehat{\Gamma}_{\mathcal{Z}} \cup (2\pi \mathrm{i})^{\frac{\deg}{2}} \mathrm{inv}^* \mathrm{ch(V)}\right). \tag{1.52}$$

Here, $\mathrm{inv}\colon \mathrm{I}\mathcal{Z} \to \mathrm{I}\mathcal{Z}$ is the canonical involution on the inertia stack, sending a sector with stabiliser element g to the sector with stabiliser element g^{-1}. The operator deg is the real degree operator applied to homogeneous cohomology classes in $H(\mathcal{Z})$, and μ is the grading operator. The image of this map under the fundamental solution map defines a full integral lattice inside the space of multivalued flat sections.

Crepant resolution conjecture with an integral structure. The integral structure provides a bridge between the $K-$group McKay correspondence and the crepant resolution conjecture. Rooted in the language of Givental's space, Iritani proposed the following

Conjecture 1.4.3 ([42]). *The isomorphism in Conjecture 1.4.2 can be lifted to a $K-$group isomorphism $\mathbb{U}_K : K(\mathcal{X}) \to K(Z)$*

$$\begin{tikzcd} K_{\mathbb{C}^\times}(\mathcal{X}) \arrow[r, "\mathbb{U}_K"] \arrow[d, "\widehat{\Psi}_{\mathcal{X}}"] & K_{\mathbb{C}^\times}(Z) \arrow[d, "\widehat{\Psi}_Z"] \\ \mathcal{H}(\mathcal{X}) \arrow[r, "\mathbb{U}"] & \mathcal{H}(Z) \end{tikzcd} \tag{1.53}$$

which preserves the Mukai pairing and commutes with the tensor by a topological line bundle L on the coarse moduli space of $\mathcal{X}$.

§ 1.5 Frobenius manifolds

1.5.1 Generalities on Frobenius manifolds

Let M be an n-dimensional complex manifold. We will write $\mathcal{T}_M$ (resp. Ω_M) for the sheaf of holomorphic sections of the holomorphic tangent (resp. cotangent) bundle $T^{1,0}M$ (resp. $T^*_{1,0}M$), and $\mathfrak{X}(M) \coloneqq H^0(M, \mathcal{T}_M)$ for the space of global holomorphic vector fields on M. A holomorphic Frobenius manifold structure[1] on M is a 4-tuple $\mathcal{M} \coloneqq (M, c, \eta, e)$ satisfying the following axioms:

FM1 the *metric* $\eta \in H^0(M, \mathrm{Sym}^2\Omega_M)$ is a flat perfect symmetric pairing on $\mathcal{T}_M$;

FM2 the *product* $c \in H^0(M, \mathrm{Sym}^3\Omega_M)$ is a totally symmetric $(0,3)$-tensor

$$\eta(X, Y \circ Z) = \eta(X \circ Y, Z) \coloneqq c(X, Y, Z) \in \mathcal{O}_M(M)\,, \quad X, Y, Z \in \mathfrak{X}(M)\,,$$

inducing, holomorphically in $p \in M$, a structure of commutative, unital, associative Frobenius algebra on the tangent fibre $T^{1,0}_p M$;

FM3 $\nabla_W c(X, Y, Z)$ is totally symmetric in $W, X, Y, Z \in \mathfrak{X}(M)$, where ∇ denotes the Levi-Civita connection of η;

FM4 the *identity vector field* $e \in \mathfrak{X}(M)$, defined by its fibrewise restriction to the identity of the algebra, is horizontal with respect to ∇, $\nabla e = 0$.

Two supplementary conditions are often imposed on $\mathcal{M}$, the second of which may or may not be realised in the context of this chapter.

FM5 $\mathcal{M}$ is *semi-simple* if the set

$$\mathrm{Discr}(\mathcal{M}) \coloneqq \{p \in M \mid \exists\, v \in T_p M \text{ with } v \circ v = 0\}$$

has positive complex co-dimension;

FM6 $\mathcal{M}$ is *conformal* if there exists a holomorphic vector field $E \in \mathfrak{X}(M)$ which is covariantly linear, $\nabla\nabla E = 0$, and is such that

$$\mathcal{L}_E \circ = \circ\,, \quad \mathcal{L}_E \eta = (d-2)\eta\,,$$

for some constant $d \in \mathbb{Q}$, known as the *charge* of $\mathcal{M}$.

[1] We call *Frobenius manifold* here what is often elsewhere referred to as a *weak* (or *almost*) *Frobenius manifold*, since we do not require axiom **FM6** (covariant linearity of E) to hold in our definition. The classical notion of Frobenius manifold in [30] is what we call *conformal Frobenius manifold* in this thesis. We will reserve the terminology "almost Frobenius manifold" to indicate a Frobenius manifold where **FM4** (covariant constancy of e) is dropped.

All examples in this chapter will be semi-simple, but not necessarily conformal. We will write

$$M^{\rm ss} := M \setminus \mathrm{Discr}(\mathcal{M})$$

to indicate the open semi-simple locus of $\mathcal{M}$, and the calligraphic notation $\mathcal{M}^{\rm ss}$ for the Frobenius manifold structure induced by restriction of $\mathcal{M}$ to $M^{\rm ss}$.

Since the metric η is flat, the $\mathrm{GL}(n,\mathbb{C})$-equivalence class of flat frames for η equips a Frobenius manifold with a canonical $\mathrm{GL}(n,\mathbb{C}) \ltimes \mathbb{C}^n$ affine-linear equivalence class of flat charts, in which the Gram matrix of the pairing η is constant. The axioms **FM1-FM6** in one such chart $(t_1, \dots, t_n)$ amount to the local existence of a holomorphic function F (the *prepotential*) with the following properties:

(i) the unit vector field is
$$e = \frac{\partial}{\partial t_1} ;$$

(ii) the Gram matrix $\eta_{AB} := \eta\left(\frac{\partial}{\partial t_A}, \frac{\partial}{\partial t_B}\right)$ is constant, non-degenerate, and equal to the Hessian matrix of $\partial_{t_1} F$,
$$\eta_{AB} = \frac{\partial^3 F}{\partial t_1 \partial t_A \partial t_B} ;$$

(iii) the prepotential is weighted quasi-homogeneous in its arguments,
$$\sum_{A=1}^{n} \left(p_A t_A \frac{\partial}{\partial t_A} + r_A \frac{\partial}{\partial t_A} \right) F = (3-d) F ,$$
for some p_A, q_A, $d \in \mathbb{Q}$;

(iv) for all $A, B, M, N \in \{1, \dots, n\}$, writing
$$\eta^{CD} := (\eta^{-1})_{CD} ,$$
the prepotential satisfies the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations,
$$\sum_{C,D=1}^{n} \left(\frac{\partial^3 F}{\partial t_A \partial t_B \partial t_C} \eta^{CD} \frac{\partial^3 F}{\partial t_D \partial t_M \partial t_N} - \frac{\partial^3 F}{\partial t_A \partial t_M \partial t_C} \eta^{CD} \frac{\partial^3 F}{\partial t_D \partial t_B \partial t_N} \right) = 0 . \tag{1.54}$$

1.5.2 Dubrovin duality

Assume now that $\mathcal{M} = (M, c, \eta, e, E)$ is a conformal Frobenius manifold, so that E is the Euler vector field appearing in **FM6**. The intersection form is defined on the open locus where E is invertible in the Frobenius algebra. In the semisimple setting considered below, we restrict to the intersection of this locus with the semisimple locus and continue to denote it by $M^{\rm ss}$. Thus one may define a second metric $\eta^\flat$ (the *intersection form*) by

$$\eta^\flat(X,Y) \coloneqq \eta\left(E^{-1/2}\circ X, E^{-1/2}\circ Y\right) = \eta\left(E^{-1}\circ X, Y\right). \tag{1.55}$$

A consequence of the Frobenius axioms together with **FM6** [30, Lect. 3] is that the metric $\eta^\flat$ is also flat; we will usually denote by $(x_1,\ldots,x_n)$ a choice of a flat coordinate chart for it. Alongside $\eta^\flat$, we can define a second commutative, associative, $\mathcal{O}_{M^{\rm ss}}$-algebra structure on $\mathcal{X}(M^{\rm ss})$ with unit E via

$$c^\flat(X,Y,Z) \coloneqq \eta^\flat(X, Y\star Z), \quad X\star Y \coloneqq E^{-1}\circ X\circ Y. \tag{1.56}$$

The $\star$-product is compatible (in the sense of Frobenius algebras) with the pairing $\eta^\flat$: this is an immediate consequence of the $\circ$-product being compatible with respect to the pairing η.

Proposition 1.5.1 ([32, 54])**.** *Let $\mathcal{M}=(M,c,\eta,e,E)$ be a semi-simple conformal Frobenius manifold of charge $d=1$. Then $\mathcal{M}=(M^{\rm ss},c^\flat,\eta^\flat,E)$ is a semi-simple Frobenius manifold satisfying axioms* **FM1-4***, with flat unit $E\in\mathcal{X}(M^{\rm ss})$.*

In particular, if $d=1$ and in a flat chart $(x_1,\ldots,x_n)$ for the intersection form $\eta^\flat$, there exists a solution $F^\flat$ of the WDVV equations (1.54) (with η, $(t_A)_\alpha$ replaced by $\eta^\flat$, $(x_i)_i$) where furthermore

$$E=\frac{\partial}{\partial x_1}, \qquad \eta^\flat_{ij}=\frac{\partial^3 F^\flat}{\partial x_1\partial x_i\partial x_j}.$$

The Frobenius manifold structure $\mathcal{M}^\flat=(M^{\rm ss},c^\flat,\eta^\flat,E)$ induced by $F^\flat$ will be called the *Dubrovin-dual* structure to $\mathcal{M}$. By Proposition 1.5.1 and (1.56), it will satisfy axioms **FM1-5**, but it will not generally satisfy the additional conformality axiom **FM6**. The two perfect pairings η and $\eta^\flat$ on $\mathcal{T}_M^{\rm ss}$ locally induce isometries between tangent and cotangent fibres,

$$(T_pM^{\rm ss},\eta)\simeq\left(T_p^*M^{\rm ss},\eta^{-1}\right), \qquad \left(T_pM^{\rm ss},\eta^\flat\right)\simeq\left(T_p^*M^{\rm ss},(\eta^\flat)^{-1}\right). \tag{1.57}$$

In Einstein's convention, this would be the familiar operation of "raising the indices" with η^{AB} (resp. $(\eta^\flat)^{ij}$). The isomorphisms (1.57) define two, a priori distinct, $\mathcal{O}_{M^{\rm ss}}$-algebra structures on holomorphic 1-forms $\theta,\chi\in\Omega_{M^{\rm ss}}$,

$$\theta\,\widehat{\circ}\,\chi, \qquad \theta\,\widehat{\star}\,\chi, \tag{1.58}$$

respectively given by the dual of the $\circ$-product under η^{-1}, and the dual of the $\star$-product under $(\eta^\flat)^{-1}$ in (1.57). By (1.55) and (1.56), these two dual products on Ω_M are in fact identically isomorphic:

$$\theta\,\widehat{\circ}\,\chi=\theta\,\widehat{\star}\chi. \tag{1.59}$$

Spelling out (1.59) in the respective flat charts for η and $\eta^\flat$ results in the following non-trivial relation between the prepotentials of $\mathcal{M}$ and $\mathcal{M}^\flat$:

$$\frac{\partial^3 F^\flat}{\partial x_i \partial x_j \partial x_k} = \sum_{a,b,A,B,C,M,N=1}^{n} \eta^\flat_{ia}\eta^\flat_{jb}\frac{\partial t_C}{\partial x_k}\frac{\partial x_a}{\partial t_A}\frac{\partial x_b}{\partial t_B}\eta^{AM}\eta^{BN}\frac{\partial^3 F}{\partial t_C \partial t_M \partial t_N}\,. \tag{1.60}$$

1.5.3 Frobenius structures on Hurwitz spaces

Let $N \in \mathbb{Z}_{>0}$ and $H_{g,\mathsf{m}}$ be the moduli space of smooth genus g-covers of $\mathbb{P}^1$ with ramification profile at infinity specified by a partition $\mathsf{m} \vdash N$. We will write π, λ and Σ_i for, respectively, the universal family, the universal map, and the sections marking $\{\infty_i\}_i \coloneqq \lambda^{-1}([1:0])$, as per the following commutative diagram:

$$\begin{tikzcd} C_g \arrow[r, hook] \arrow[d] & \mathcal{C}_{g,\mathsf{m}} \arrow[r, "\lambda"] \arrow[d, "\pi"] & \mathbb{P}^1 \\ {[\lambda]} \arrow[r, hook, "\mathrm{pt}"] \arrow[u, bend left, "P_i"] & H_{g,\mathsf{m}} \arrow[u, bend left, "\Sigma_i"] & \end{tikzcd} \tag{1.61}$$

We furthermore denote by $\mathrm{d} = \mathrm{d}_\pi$ the relative differential with respect to the universal family and $p_i^{\mathrm{cr}} \in C_g \simeq \pi^{-1}([\lambda])$ the critical locus $\mathrm{d}\lambda = 0$ of the universal map. By the Riemann existence theorem, the critical values of λ,

$$(u_i)_{i=1,\ldots,d_{g;\mathsf{m}}}\,,$$

serve as local coordinates away from the discriminant locus $u_i = u_j$ for $i \neq j$. On its complement, we can construct a family of semi-simple, commutative, $\mathbb{C}$-algebra structures on the tangent fibres at $(u_1, \ldots, u_{d_{g,\mathsf{m}}})$ by stipulating that the coordinate vector fields in the u-chart are the idempotents of the algebra,

$$\partial_{u_i} \cdot \partial_{u_j} = \delta_{ij}\partial_{u_i}. \tag{1.62}$$

Let $\mu : \mathcal{C}_{g,\mathsf{m}} \to \mathbb{P}^1$ be a surjective morphism such that the relative one-form

$$\mathrm{d}\log\mu \in \Omega^1_{\mathcal{C}_{g,\mathsf{m}}/H_{g,\mathsf{m}}}(\infty_0 + \cdots + \infty_m)$$

is an exact third-kind differential[2] on the fibres of the universal curve with simple poles at ∞_i, with residues

$$\operatorname*{Res}_{\infty_i} \mathrm{d}\log\mu = a_i \in \mathbb{Z}\,.$$

If μ does not factor through λ, this defines an Ehresmann connection on $TC_{g,\mathsf{m}}$ where points in nearby fibres of the universal curve are identified if they have the same image under μ. This defines a meromorphic derivation

$$\delta_{\partial_{u_i}} \ : \ H^0(\mathcal{C}_{g,\mathsf{m}}, \mathcal{O}_{\mathcal{C}_{g,\mathsf{m}}}) \longrightarrow H^0(\mathcal{C}_{g,\mathsf{m}}, \mathcal{K}_{\mathcal{C}_{g,\mathsf{m}}})$$

[2] This is a special case of a type III admissible differential, in the classification of [30, Lect. 5].

defined in local coordinates $p = (u_1, \ldots, u_{d_{g;\mathsf{m}}}; \mu)$ for $C_{g,\mathsf{m}}$ as the partial derivative taken with respect to u_i whilst keeping μ constant. A Frobenius manifold structure $\mathcal{H}^{[\mu]}_{g,\mathsf{m}} := (M_{\mathrm{LG}}, c_{\mathrm{LG}}, \eta_{\mathrm{LG}}, e)$ can be defined on the Hurwitz space $M_{\mathrm{LG}} := H_{g,\mathsf{m}}$ by the residue formulas [30]

$$\eta_{\mathrm{LG}}(X,Y) := \sum_i \operatorname*{Res}_{p_i^{\mathrm{cr}}} \frac{\delta_X \ \lambda \ \delta_Y \lambda}{\mathrm{d}\lambda} \phi^2 \,, \tag{1.63}$$

$$c_{\mathrm{LG}}(X,Y,Z) := \sum_i \operatorname*{Res}_{p_i^{\mathrm{cr}}} \frac{\delta_X \lambda \ \delta_Y \lambda \ \delta_Z \lambda}{\mathrm{d}\lambda} \phi^2 \,, \tag{1.64}$$

$$\eta^\flat_{\mathrm{LG}}(X,Y) := \sum_i \operatorname*{Res}_{p_i^{\mathrm{cr}}} \frac{\delta_X \log\lambda \ \delta_Y \log\lambda}{\mathrm{d}\log\lambda} \phi^2 \,, \tag{1.65}$$

where

$$\phi := \frac{\mathrm{d}\mu}{\mu} \,.$$

The universal map λ and the relative differential ϕ are referred to as the *superpotential* and the *primitive form* of $\mathcal{H}_{g,\mathsf{m}}$.

Chapter 2

Dubrovin duality and mirror symmetry for ADE resolutions

§ 2.1 Introduction

Let $l \in \mathbb{Z}_{>0}$ and $\mathcal{R} = \mathrm{ADE}_l$ be a rank$-l$ simply-laced irreducible root system, and fix a choice of fundamental weight $\widehat{\omega}$ for the complex simple Lie algebra associated to $\mathcal{R}$ as follows:

- when $\mathcal{R} = \mathrm{A}_l$, $\widehat{\omega}$ can be any fundamental weight;
- when $\mathcal{R} = \mathrm{D}_l$ or $\mathcal{R} = \mathrm{E}_l$, $\widehat{\omega}$ will be the highest weight of the fundamental representation of highest dimension.

We will call the datum $(\mathcal{R}, \widehat{\omega})$ a *marked ADE pair*. The corresponding Dynkin diagrams, with node marking specified by $\widehat{\omega}$, are shown in Figure 2.1. In this Chapter we will be concerned with three classes of Frobenius manifolds associated to a given pair $(\mathcal{R}, \widehat{\omega})$, arising respectively from representation theory, enumerative algebraic geometry, and integrable systems.

- For $\mathcal{R}$ the root system of any complex simple Lie algebra, and $\widehat{\omega} \in \Omega$ any fundamental weight, Dubrovin and Zhang [35] famously constructed a canonical semi-simple Frobenius manifold structure $\mathrm{EAW}(\mathcal{R}, \widehat{\omega})$ on the orbits of the reflection representation of the $\widehat{\omega}$-extended affine Weyl group of $\mathcal{R}$, generalising the classical construction of polynomial Frobenius manifolds on orbit spaces of Coxeter groups. The specialisation to $(\mathcal{R}, \widehat{\omega})$ being a marked ADE pair will be the setup of sole concern to us in this work, and we will use the shorthand notation

$$\mathcal{M}_{\mathrm{AW}} \coloneqq \mathrm{EAW}(\mathcal{R}, \widehat{\omega}) \,.$$

- Let $G < \mathrm{SL}(2,\mathbb{C})$, $|G| < \infty$ be the Kleinian subgroup of type $\mathcal{R}$, and let $Z = \widetilde{\mathbb{C}^2/G}$ be the minimal resolution of the associated canonical surface singularity. The pair $(\mathcal{R}, \widehat{\omega})$ specifies a $\mathbb{C}^\times$-action on Z, point-wise fixing the irreducible component of the exceptional locus of Z corresponding to the marked fundamental weight $\widehat{\omega}$ under the McKay correspondence [56]. The associated Frobenius manifold is the $\mathbb{C}^\times$-equivariant quantum cohomology of Z,

$$\mathcal{M}_{\mathrm{GW}} \coloneqq \mathrm{QH}_{\mathbb{C}^\times}(Z)\,.$$

- In [31, 30], Dubrovin constructs a Frobenius manifold structure $\mathrm{LG}(\lambda,\phi)$ on the Hurwitz moduli space of ramified covers of the projective line with given genus and ramification profile at infinity. Here, λ (the *Landau–Ginzburg superpotential*) denotes the universal map, and ϕ is the additional datum of a Saito form [59] on the fibres of the family. One can associate to the pair $(\mathcal{R}, \widehat{\omega})$ an algebraically completely integrable system – the type $(\mathcal{R}, \widehat{\omega})$ affine relativistic Toda chain [38] at vanishing Casimir – whose isospectral dynamics is encoded in a special Frobenius submanifold of a certain Hurwitz space [15]. In this context λ is the spectral parameter of the relativistic Toda Lax matrix, and ϕ is the differential of the (logarithm of the) argument of its characteristic polynomial. We will denote this Frobenius manifold as
$$\mathcal{M}_{\mathrm{LG}} \coloneqq \mathrm{LG}(\lambda,\phi)\,.$$

2.1.1 Dubrovin duality and mirror symmetry

Given a Frobenius manifold $\mathcal{M}$ with a linear Euler vector field E and semi-simple product
$$\circ : \Gamma(\mathcal{M}, \mathcal{T}\mathcal{M}) \otimes_{\mathcal{O}_\mathcal{M}} \Gamma(\mathcal{M}, \mathcal{T}\mathcal{M}) \longrightarrow \Gamma(\mathcal{M}, \mathcal{T}\mathcal{M})\,,$$
one can construct [32] a second family $\mathcal{M}^\flat$ of Frobenius rings on the locus where E is invertible in the $\circ$-algebra. This is obtained by pre-composing the flat pairing on $\mathcal{M}$ with multiplication of each of its entries by $E^{-1/2}$:

$$\mathcal{M} \xrightarrow{E^{-1/2}\circ} \mathcal{M}^\flat \tag{2.1}$$

In a canonical coordinate chart for $\mathcal{M}$, (2.1) corresponds to rescaling the coefficients of the (diagonal) Gram matrix of the Frobenius pairing by the inverse diagonal matrix of the canonical coordinates. It is further shown in [32] that the resulting rescaled pairing on $\mathcal{M}^\flat$ is flat, and induced by a *dual* prepotential function to the original prepotential of $\mathcal{M}$. We will call $\mathcal{M}^\flat$ the *Dubrovin-dual* [1] Frobenius manifold of $\mathcal{M}$.

[1] The operation in (2.1) is often referred to as an "almost-duality" of Frobenius manifolds, owing to the fact that $\mathcal{M}^\flat$ is a weak (or *almost*) Frobenius manifold: we review this in Section 1.5. The terminology "duality" is perhaps somewhat improper, as for once the operation in (2.1) is not involutive, but we abide by historical convention and refer to it as such (see also [49, 27]).

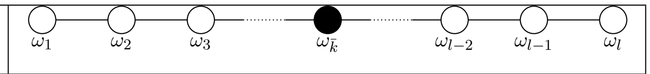
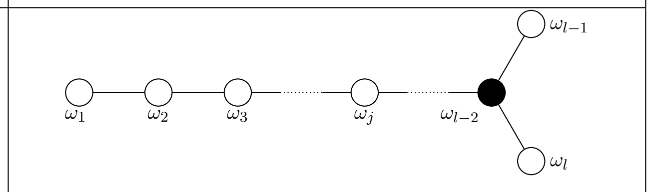
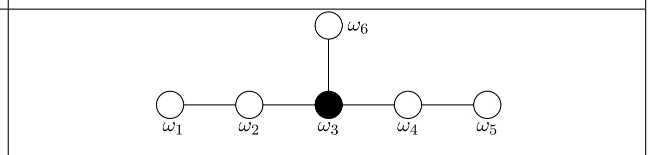
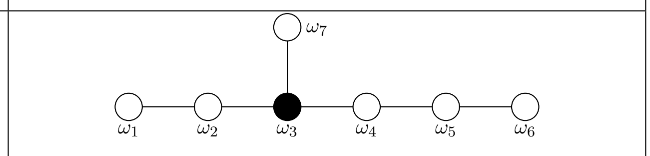
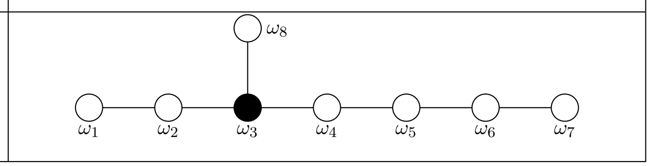
$\mathcal{R}$	$\widehat{\omega}$	Marked ADE Dynkin diagram
A_l	$\omega_{\bar{k}}$	
D_l	ω_{l-2}	
E_6	ω_3	
E_7	ω_3	
E_8	ω_3	

Figure 2.1: Marked Dynkin diagrams of pairs $(\mathcal{R}, \widehat{\omega})$. The marked node corresponding to the weight $\widehat{\omega}$ is indicated in black.

Dubrovin's duality acquires a particularly salient form when $\mathcal{M}$ admits a realisation as a Landau–Ginzburg model on a family of algebraic curves. In this case, (2.1) amounts to replacing the superpotential by its logarithm [30, 32, 57]:

$$\mathcal{M}^\flat \simeq \mathrm{LG}(\log \lambda, \phi)\,.$$

From the point of view of the bihamiltonian quasi-linear integrable hierarchy defined on the loop space of a Frobenius manifold [30], a mirror-symmetry presentation of $\mathcal{M}$ as a Landau–Ginzburg model is equivalent to a dispersionless Lax–Sato formulation of the integrable flows, with λ coinciding with the Lax symbol [62]. Dubrovin's duality (2.1), on the other hand, corresponds to expressing the integrable flows on $\mathcal{M}$ in Darboux coordinates for the second Poisson bracket of the Principal Hierarchy (the limiting value of the Dubrovin–Novikov pencil at infinity), and accordingly to a different ("dual") notion of topological τ-function in the sense of [36, 33].

The main result of this Chapter shows that, for all marked ADE pairs $(\mathcal{R}, \widehat{\omega})$, the Frobenius manifolds $\mathcal{M}_{\mathrm{AW}}$, $\mathcal{M}_{\mathrm{GW}}$ and $\mathcal{M}_{\mathrm{LG}}$ are either non-trivially isomorphic, or Dubrovin-dual to each other.

Theorem (=Theorems 2.3.1 and 2.3.2)**.** *For all marked ADE pairs* $(\mathcal{R}, \widehat{\omega})$*, we have*

$$
\begin{tikzcd}
\mathcal{M}_{\mathrm{AW}} \arrow[rr, "\simeq"] \arrow[d, "E^{-1/2}\circ"'] \arrow[drr, phantom, "\square"] & & \mathcal{M}_{\mathrm{LG}} \arrow[d, "E^{-1/2}\circ"] \\
\mathcal{M}^{\flat}_{\mathrm{AW}} \arrow[dr, "\simeq"] & & \mathcal{M}^{\flat}_{\mathrm{LG}} \arrow[dl, "\simeq"'] \\
& \mathcal{M}_{\mathrm{GW}} &
\end{tikzcd}
$$

The isomorphism in the top row,

$$\mathcal{M}_{\mathrm{AW}} \simeq \mathcal{M}_{\mathrm{LG}}\,,$$

was proved in [15]. Therefore, the statement that the equivariant quantum cohomology of ADE resolutions is Dubrovin-dual to the corresponding extended affine Weyl Frobenius manifold,

$$\mathcal{M}_{\mathrm{GW}} \simeq \mathcal{M}^{\flat}_{\mathrm{AW}}\,, \tag{2.2}$$

is logically equivalent to proving its mirror realisation as an LG model on a family of relativistic Toda spectral curves, upon replacing the spectral parameter as $\lambda \longrightarrow \log\lambda$,

$$\mathcal{M}_{\mathrm{GW}} \simeq \mathcal{M}^{\flat}_{\mathrm{LG}}\,. \tag{2.3}$$

Our strategy will be to prove the mirror theorem (2.3) first, as a means to establish the Dubrovin duality (2.2) as a consequence, for the classical series $\mathcal{R} = \mathrm{A}_l$ and $\mathcal{R} = \mathrm{D}_l$; and vice versa for the exceptional series $\mathcal{R} = \mathrm{E}_l$.

Theorem 2.3.2 answers constructively, in all Dynkin types, a long-standing question about mirror symmetry in the fundamental setup of quantum cohomology of du Val resolutions. When $\mathcal{R} = \mathrm{A}_l$, Z is a smooth 2-dimensional toric Calabi–Yau surface: in this case a Landau–Ginzburg mirror has been known since Givental's work on equivariant toric mirror symmetry [39, 23]. The case where $\mathcal{R} \neq \mathrm{A}_l$ and Z is not toric has been outstanding to-date, as the methods of [39] cannot be directly applied to this more general setup. An important consequence of (2.3) is that it automatically provides a global integral representation of the components of the J-function as univariate Laplace-type integrals: for type $\mathcal{R} = \mathrm{A}_l$, this enhanced control on their analytic continuation was brought to fruit in [13] to prove Iritani's integral K-theoretic and higher genus full-descendent Crepant Resolution Conjectures, using R-matrix quantisation techniques. Theorem 2.3.2 opens the way for a similar analysis for all ADE types, which will be explored in future work. The isomorphism (2.2) further suggests a conjectural integrable hierarchy governing the higher genus Gromov–Witten theory of Z: this should coincide with the one constructed in [53] expressed in a suitable set of dual variables, given by Darboux co-ordinates for a second Hamiltonian structure. For the classical series $\mathcal{R} = \mathrm{A}_l$ and $\mathcal{R} = \mathrm{D}_l$, the hierarchy is a particular rational reduction of the 2-Toda hierarchy [11].

2.1.2 Organisation of the chapter

The background on Frobenius manifolds and Dubrovin's duality used in this chapter is recalled in Section 1.5. The chapter itself is organised as follows. In Section 2.2, we construct the three Frobenius manifolds $\mathcal{M}_{\mathrm{AW}}$, $\mathcal{M}_{\mathrm{LG}}$ and $\mathcal{M}_{\mathrm{GW}}$ specified by the datum of a marked ADE pair $(\mathcal{R}, \widehat{\omega})$. In Section 2.3.1, we will explain how the embedding of the LG model (λ, ϕ) into a genus zero Hurwitz-Frobenius manifold for the classical series $\mathcal{R} = \mathrm{A}_l$ and $\mathcal{R} = \mathrm{D}_l$ allows to systematically determine the structure constants of $\mathcal{M}_{\mathrm{LG}}^\flat$. Armed with this, the mirror theorem in (2.3), and therefore the duality (2.2) with the extended affine Weyl orbit spaces, can be deduced upon comparison with the genus zero Gromov–Witten calculations of [16, 13]. In Section 2.3.2, we give a general representation-theoretic argument, applicable to all extended affine Weyl Frobenius manifolds, showing that the structure constants of the Frobenius product on both sides of (2.2) belong to a certain finite-dimensional vector space of quasi-homogeneous polynomials. As such, the corresponding $(2,1)$-tensors are determined by their values on a (small) finite set of points $\mathfrak{I}$ in the semi-simple locus of $\mathcal{M}_{\mathrm{AW}}$, which we call a set of *initial conditions* for $\mathcal{M}_{\mathrm{AW}}$. We perform this analysis specifically for $\mathcal{R} = \mathrm{E}_l$, and show how the reduction to initial conditions drastically simplifies the verification of (2.2), and therefore the proof of the mirror theorem in (2.3), which we carry out for the entire exceptional series.

§ 2.2 Frobenius manifolds of type $(\mathcal{R}, \widehat{\omega})$

With these preliminaries, we shall consider three classes of semi-simple Frobenius manifolds associated to a marked ADE pair $(\mathcal{R}, \widehat{\omega})$.

2.2.1 Extended affine Weyl Frobenius manifolds

Let $\mathfrak{g} = \mathfrak{ade}_l$ be the rank-l complex, simple, simply-laced Lie algebra with root system $\mathcal{R}$. We shall denote by $\mathfrak{h}$ the associated Cartan subalgebra, and by $\mathcal{W}$ its Weyl group. The action of $\mathcal{W}$ on $\mathfrak{h}$ lifts to an action of the affine Weyl group $\widehat{\mathcal{W}} \cong \mathcal{W} \ltimes \Lambda_r^\vee$, with $\Lambda_r^\vee$ the lattice of co-roots:

$$\begin{aligned} \widehat{\mathcal{W}} \times \mathfrak{h} &\longrightarrow \mathfrak{h}, \\ ((w, \alpha^\vee), h) &\longrightarrow w(h) + 2\pi \mathrm{i} \alpha^\vee. \end{aligned} \tag{2.4}$$

For $(\mathcal{R}, \widehat{\omega})$ a marked ADE pair, the corresponding $\widehat{\omega}$-*extended* affine Weyl group $\widetilde{\mathcal{W}}$ is defined as the semi-direct product $\widetilde{\mathcal{W}} := \widehat{\mathcal{W}} \ltimes \mathbb{Z}$ acting on $\mathfrak{h} \oplus \mathbb{C}$ by

$$\begin{aligned} \widetilde{\mathcal{W}} \times \mathfrak{h} \oplus \mathbb{C} &\longrightarrow \mathfrak{h} \oplus \mathbb{C}, \\ ((w, \alpha^\vee, l), (h, v)) &\longrightarrow (w(h) + 2\pi \mathrm{i} \alpha^\vee + 2\pi \mathrm{i} l \widehat{\omega}, v - 2\pi \mathrm{i} l). \end{aligned} \tag{2.5}$$

Let Σ denote the hyperplane arrangement associated to the root system $\mathcal{R}$, and $\mathfrak{h}^{\mathrm{reg}} := \mathfrak{h} \backslash \Sigma$ be the set of regular elements in $\mathfrak{h}$. The restriction of (2.5) to $\mathfrak{h}^{\mathrm{reg}} \oplus \mathbb{C}$ is a free affine action, whose quotient defines the regular orbit space of the extended affine Weyl group of $\mathcal{R}$ with marked weight $\widehat{\omega}$ as

$$M_{\mathrm{AW}} := (\mathfrak{h}^{\mathrm{reg}} \times \mathbb{C}) / \widetilde{\mathcal{W}} \cong \mathcal{T}^{\mathrm{reg}} / \mathcal{W} \times \mathbb{C}^*, \tag{2.6}$$

where $\mathcal{T}^{\mathrm{reg}} = \exp(\mathfrak{h}^{\mathrm{reg}})$ is the image of the set of regular elements of $\mathfrak{h}^{\mathrm{reg}}$ under the exponential map to the maximal torus $\mathcal{T}$. Let $(q_1, \dots, q_l)$ be linear coordinates on $\mathfrak{h}$ w.r.t. the co-root basis $\{\alpha_1^\vee, \dots, \alpha_l^\vee\}$, and extend these to linear coordinates $(q_1, \dots, q_l; q_{l+1})$ on $\mathfrak{h} \oplus \mathbb{C}$, giving local coordinates on the regular orbit space. Denoting $\langle \alpha, \beta \rangle$ the pairing on $\mathfrak{h}^*$ induced by the restriction of the Killing form on the Cartan subalgebra, and writing

$$C_{ab} = \left\langle \alpha_a, \alpha_b^\vee \right\rangle , \quad d_a := \langle \omega_a, \widehat{\omega} \rangle , \quad \widehat{d} := \langle \widehat{\omega}, \widehat{\omega} \rangle ,$$

where, as throughout, the indices on the Cartan matrix are renamed according to context, we can define a non-degenerate pairing ξ on $\mathfrak{h} \times \mathbb{C}$ by orthogonal extension of minus the Cartan–Killing form on $\mathfrak{h}$ as

$$\eta^\flat_{\mathrm{AW}}(\partial_{q_a}, \partial_{q_b}) := \begin{cases} -C_{ab} & \text{if } a, b < l+1, \\ \widehat{d} & \text{if } a = b = l+1, \\ 0 & \text{otherwise,} \end{cases} \tag{2.7}$$

with q_{l+1} parametrising linearly the right summand in $\mathfrak{h} \oplus \mathbb{C}$. The quotient map $\aleph : \mathcal{T}^{\mathrm{reg}} \times \mathbb{C}^* \to M_{\mathrm{AW}}$ from (2.6) defines a principal $\mathcal{W}$-bundle on M_{AW}: a section $\tilde{\sigma}_i$ lifts a (sufficiently small) open $U \subset M_{\mathrm{AW}}$ to the i^{th} sheet of the cover $V_i \in \widetilde{\sigma}_i^{-1}(U) \equiv V_1 \sqcup \dots \sqcup V_{|\mathcal{W}|}$. The invariant ring $\mathcal{I} := \mathbb{C}[\mathcal{T}]^{\mathcal{W}}$ is, from classical results about exponential Weyl invariants [8], a polynomial ring $\mathcal{I} \simeq \mathbb{C}[Y_1, \dots, Y_l]$, where

$$Y_i := S_{\mathcal{W}}(\mathrm{e}^{\langle \omega_i, h \rangle}) , \tag{2.8}$$

and $S_{\mathcal{W}}$ is the average over the Weyl orbit. Equivalently, we have $\mathcal{I} \simeq \mathbb{C}[W_1, \dots, W_l]$, where

$$W_i := \sum_{\omega \in \Gamma_i} \mathrm{e}^{\langle \omega, h \rangle} . \tag{2.9}$$

Proposition 2.2.1 ([35, Thm. 1.1])**.** *Let $\mathcal{A}$ be the ring of $\widetilde{\mathcal{W}}$-invariant Fourier polynomials in the variables $q_1, \dots, q_l$ and $\frac{q_{l+1}}{f}$ that are bounded in the limit:*

$$q = q^{[0]} + \widehat{\omega} \tau, \quad q_{l+1} = q_{l+1}^{[0]} - \tau, \quad \tau \to +\infty, \tag{2.10}$$

for any $\left(q^{[0]}, q^{[0]}_{l+1}\right)$*, where* f *is the determinant of the Cartan matrix. Then*

$$\mathcal{A} \simeq \mathbb{C}\left[y_1, \ldots, y_l, \mathrm{e}^{y_{l+1}}\right],$$

with

$$y_\alpha = \begin{cases} \mathrm{e}^{d_\alpha q_{l+1}} Y_\alpha\,, & \alpha = 1, \ldots, l\,, \\ q_{l+1}\,, & \alpha = l+1\,. \end{cases} \tag{2.11}$$

and $d_i := \langle \omega_i, \widehat{\omega} \rangle$*.*

In the following, for $k \in \{1, \ldots, l\}$ such that $\omega_{\bar{k}} = \widehat{\omega}$, we will denote $\widehat{y} \coloneqq y_{\bar{k}}$. We will define a grading on $\mathcal{A}$ by

$$\deg\, y_\alpha = d_\alpha\,(\alpha = 1, \ldots, l)\,, \qquad \deg\, \mathrm{e}^{y_{l+1}} = 1\,.$$

The following reconstruction theorem holds [35, Thm 2.1].

Theorem 2.2.2. *There exists a unique, up to isomorphism, semi-simple and conformal Frobenius manifold*

$$\mathcal{M}_{\mathrm{AW}} = (M_{\mathrm{AW}}, c_{\mathrm{AW}}, \eta_{\mathrm{AW}}, e, E)$$

of charge $d = 1$ *satisfying the following properties in flat coordinates* $(t_1, \ldots, t_{l+1})$ *for* η_{AW}*:*

AW-I $\widehat{d}E = \partial_{q_{l+1}} = \sum_{J=1}^{l} d_J t_J \partial_{t_J} + \partial_{t_{l+1}}$*;*

AW-II *the intersection form is* $\eta^\flat = \tilde{\sigma}_i^* \xi$*;*

AW-III *the prepotential is polynomial in* $t_1, \ldots, t_{l+1}$*,* $\mathrm{e}^{t_{l+1}}$*.*

2.2.2 Quantum cohomology of ADE resolutions

Let $G < \mathrm{SU}(2)$, $|G| \leq \infty$ be the Kleinian subgroup of type $\mathcal{R} = \mathrm{ADE}_l$. $X = \mathbb{C}^2/G$ be the corresponding Kleinian singularity (see Section 1.1) and

$$Z \xrightarrow{\pi} X \tag{2.12}$$

be its minimal resolution. The intersection diagram describing the configuration of irreducible rational curves in the exceptional locus is the Dynkin diagram of the corresponding ADE type, and the integral homology of the resolution,

$$H_\bullet(Z, \mathbb{Z}) = H_0(Z, \mathbb{Z}) \oplus H_2(Z, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}^l\,,$$

is isomorphic to the affine root lattice of type $\mathcal{R}$. Writing

$$\Pi \coloneqq \{\alpha_1, \ldots, \alpha_l\}\,, \qquad \Omega \coloneqq \{\omega_1, \ldots, \omega_l\}$$

for, respectively, the set of simple roots and fundamental weights of the complex simple Lie algebra associated to $\mathcal{R}$. we will label the irreducible components $\{e_1, \dots, e_l\}$ of the exceptional locus accordingly, so that

$$e_i \longleftrightarrow \alpha_i \longleftrightarrow \omega_i \,. \tag{2.13}$$

Let $\mathsf{T} \simeq \mathbb{C}^\times$ and consider a T-representation on $\mathbb{C}^2$ commuting with the action of G. This induces T-actions on X and Z, respectively by descent to the quotient and by T-equivariance of the resolution (1.4). When $\mathcal{R} = \mathrm{DE}_l$, we have a unique possible choice for the action of T: this is the scalar torus action on $\mathbb{C}^2$ with characters (t, t) on the affine coordinates (x, y) of $\mathbb{C}^2$. When $\mathcal{R} = \mathrm{A}_l$, since G is abelian, we have more generally that the full Cartan torus

$$\mathsf{T}' \coloneqq (\mathbb{C}^\times)^2 < \mathrm{GL}(2, \mathbb{C}) \,,$$

with characters (t_x, t_y) will act effectively on X and Z. For $\bar{k} \in \{1, \dots, l\}$, we will be specially interested in the one-dimensional subtori T acting with characters $(t_x, t_y) = (t^{l+1-\bar{k}}, t^{\bar{k}})$.

As the singularity is the only torus fixed point in X, the T-fixed locus of Z is fully contained in the exceptional set. We will write $[e_i]^\vee$ for the Poincaré dual classes to $[e_i]$ in the locally compact cohomology, φ_i for their canonical lifts to T-equivariant cohomology, and φ_{l+1} for the identity class in $H_{\mathsf{T}}(Z)$. The T-equivariant cohomology of Z is an $(l+1)$-dimensional vector space over the field of fractions of $H(B\mathsf{T}, \mathbb{C}) \simeq \mathbb{C}[\nu]$, where ν is the first Chern class of the T-representation t. For $\mathcal{R} = \mathrm{A}_l$, we can more generally consider the equivariant cohomology of Z with respect to the full 2-dimensional torus T': this will now be an $(l+1)$-dimensional vector space over $\mathbb{C}(\nu_1, \nu_2)$, with $\nu_i \coloneqq c_1(t_i) \in H_{\mathsf{T}'}(\mathrm{pt})$. We will slightly abuse notation and continue to write $\varphi_1, \dots, \varphi_l, \varphi_{l+1}$ for the lift of $e_i^\vee$ in the T'-equivariant cohomology of Z.

Remark 2.2.3. For $(\mathcal{R}, \widehat{\omega})$ a marked ADE pair, the marked fundamental weight $\widehat{\omega} \in \Omega$ corresponds to the (single) irreducible exceptional divisor $\widehat{e} \in H_2(Z, \mathbb{Z})$ which is pointwise fixed by T

$$\widehat{e} \longleftrightarrow \widehat{\alpha} \longleftrightarrow \widehat{\omega} \,.$$

To see this, notice that each double point in the exceptional locus is T-fixed. For $\mathcal{R} = \mathrm{DE}_l$, this entails that the T-action on the irreducible component $\widehat{e}$ labelled by the trivalent marked node in Figure 2.1 must be trivial, since $\widehat{e}$ is a $\mathbb{P}^1$ with three $\mathsf{T} \simeq \mathbb{C}^\times$-fixed points. When $\mathcal{R} = \mathrm{A}_l$, the weights $w_i^\pm$ at the two T'-fixed points of e_i are [13, App. B]

$$(w_i^+, w_i^-) = (-i\nu_1 + (l+1-i)\nu_2, i\nu_1 - (l+1-i)\nu_2) \,.$$

As discussed above, the restriction to a one-dimensional torus T acting with characters $(t^{l+1-\bar{k}}, t^{\bar{k}})$ sets

$$\nu_1 = (l+1-\bar{k})\nu \,, \qquad \nu_2 = \bar{k}\nu \,, \tag{2.14}$$

so that $w_{\bar{k}}^{\pm} = 0$. Hence, in this case,

$$e_{\bar{k}} \longleftrightarrow \alpha_{\bar{k}} \longleftrightarrow \omega_{\bar{k}} = \widehat{\omega}\,.$$

The torus action on Z induces an action on its moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(Z,\beta)$. Since the T-action on Z is free away from the exceptional locus, its fixed locus in $\overline{\mathcal{M}}_{g,n}(Z,\beta)$ is proper, and it carries a torus-equivariant perfect obstruction theory and virtual fundamental class [40]. Gromov–Witten invariants of Z can then be defined by

$$\langle \varphi_{i_1},\dots,\varphi_{i_n}\rangle^Z_{g,n,d} \coloneqq {}^{\mathsf{T}}\!\!\int_{[\overline{\mathcal{M}}_{g,n}(Z,\beta)]^{\rm vir}_{\mathsf{T}}} \prod_{j=1}^n \mathrm{ev}_j^*(\varphi_{i_j}) \coloneqq \int_{[\overline{\mathcal{M}}^{\mathsf{T}}_{g,n}(Z,\beta)]^{\rm vir}_{\mathsf{T}}} \frac{\prod_{j=1}^n \iota^*\mathrm{ev}_j^*(\varphi_{i_j})}{e_{\mathsf{T}}(N^{\rm vir})} \in \mathbb{Q}(\nu)\,,$$

with $\iota : \overline{\mathcal{M}}^{\mathsf{T}}_{g,n}(Z,\beta) \hookrightarrow \overline{\mathcal{M}}_{g,n}(Z,\beta)$ the immersion of the substack of T-fixed points into the moduli space of stable maps, and $N^{\rm vir}$ its T-equivariant normal bundle. Writing $x = \sum_{i=1}^{l+1} q_i\varphi_i \in H_{\mathsf{T}}(Z)$, the genus-zero invariants of Z define a Frobenius manifold satisfying the axioms **FM1-FM5**:

$$\mathcal{M}_{\rm GW} \coloneqq (H_{\mathsf{T}}(Z), c_{\rm GW}, \eta_{\rm GW}, \varphi_{l+1})$$

via

$$c_{\rm GW}(\varphi,\psi,\theta) \coloneqq {}^{\mathsf{T}}\!\!\int_{[Z]_{\mathsf{T}}} \varphi\cup\psi\cup\theta + \sum_{\beta\in H_2(Z,\mathbb{Z})} \langle\varphi,\psi,\theta\rangle^Z_{0,3,\beta}\mathrm{e}^{\langle\beta,x\rangle}\,, \tag{2.15}$$

$$\eta_{\rm GW}(\varphi,\psi) \coloneqq c_{\rm GW}(\varphi_{l+1},\varphi,\psi) = {}^{\mathsf{T}}\!\!\int_{[Z]_{\mathsf{T}}} \varphi\cup\psi\,.$$

In [35], when the torus acts with characters (t,t), the authors solved the genus zero T-equivariant Gromov–Witten theory of Z using degeneration arguments and the Aspinwall–Morrison formula for the super-rigid local curve. The formal power series (2.15) is the Fourier expansion of a trigonometric rational function given explicitly by

$$c_{\rm GW}(\varphi_{l+1},\varphi_i,\varphi_j) = \eta_{\rm GW}(\varphi_i,\varphi_j) = \begin{cases} \frac{1}{\nu^2}\frac{1}{|G|}\,, & i=j=l+1\,,\\ -C_{ij}\,, & i,j\le l\,,\\ 0\,, & \text{else}\,,\end{cases}$$

$$c_{\rm GW}(\varphi_i,\varphi_j,\varphi_k) = -\nu\sum_{\beta\in\mathcal{R}^+}\langle\alpha_i,\beta\rangle\langle\alpha_j,\beta\rangle\langle\alpha_k,\beta\rangle\coth\left(\frac{\sum_{m=1}^l\langle\beta,\alpha_m\rangle q_m}{2}\right)\,,\quad i,j,k\le l\,. \tag{2.16}$$

The corresponding prepotential is $F_{\rm GW} = F^0_{\rm GW} + F^+_{\rm GW}$, where the zero and positive degrees parts read

$$\begin{aligned} F^0_{\rm GW} =& \frac{1}{6\nu^2}\frac{1}{|G|}(q_{l+1})^3 - \frac{1}{2}q_{l+1}\sum_{i,j=1}^l C_{ij}q_iq_j - \frac{\nu}{6}\sum_{\beta\in\mathcal{R}^+}\sum_{i,j,k=1}^l\langle\alpha_i,\beta\rangle\langle\alpha_j,\beta\rangle\langle\alpha_k,\beta\rangle q_iq_jq_k\,,\\ F^+_{\rm GW} =& 2\nu\sum_{\beta\in\mathcal{R}^+}\mathrm{Li}_3\left(\mathrm{e}^{-\langle\beta,h\rangle}\right)\,. \end{aligned} \tag{2.17}$$

When $\mathcal{R} = \mathrm{A}_l$, the same reasoning applied to the full two-dimensional T'-action on Z yields [23, 12]

$$\begin{aligned} F^0_{\mathrm{GW}} =& \frac{1}{6\nu_1\nu_2}\frac{1}{|G|}q_{l+1}^3 - \frac{1}{2}q_{l+1}\sum_{i,j=1}^{l} C_{ij}q_iq_j - \frac{1}{6}\sum_{i,j,k,i',j',k'=1}^{l} C_{ii'}C_{jj'}C_{kk'}\mathfrak{C}_{i'j'k'}q_iq_jq_k\,, \\ F^+_{\mathrm{GW}} =& (\nu_1+\nu_2)\sum_{\beta\in\mathcal{R}^+}\mathrm{Li}_3\left(\mathrm{e}^{-\langle\beta,h\rangle}\right). \end{aligned} \tag{2.18}$$

where the symmetric trilinear form $\mathfrak{C}_{ijk}$ is determined by its value at $i \le j \le k$ as

$$\mathfrak{C}_{ijk} = \frac{(j\nu_1 + (l+1-j)\nu_2)\, i(l+1-k)}{l+1}\,, \qquad i \le j \le k.$$

2.2.3 Landau–Ginzburg mirrors from ADE spectral curves

A general mirror symmetry construction of Frobenius submanifolds of Hurwitz spaces isomorphic to $\mathcal{M}_{\mathrm{AW}}$, and the associated Dubrovin-dual Frobenius manifolds, was given in [15] (see also [35, 34]), as we review here. For $(\mathcal{R}, \widehat{\omega})$ be a marked ADE pair, let $\bar{k} \in \{1,\dots,l\}$ be such that $\omega_{\bar{k}} = \widehat{\omega}$. Fix $0 \neq \omega \in \Lambda^+_w$ a non-zero dominant weight. Starting from the characteristic polynomial of a regular element of $\mathcal{T}$ in the representation ρ_ω,

$$\mathcal{Q} = \prod_{\omega'\in\mathcal{W}(\omega)}\left(\mathrm{e}^{\langle\omega',x\rangle} - \mu\right) \in \mathbb{Q}[Y_1,\dots,Y_l][\mu]\,, \tag{2.19}$$

define

$$\mathcal{P}\left(y_1,\dots,y_{l+1};\lambda,\mu\right) := \mathcal{Q}\left(Y_i = y_i\mathrm{e}^{-d_iy_{l+1}} - \delta_{i\bar{k}}\lambda\mathrm{e}^{-y_{l+1}};\mu\right). \tag{2.20}$$

For fixed $y \in M^{\mathrm{ss}}_{\mathrm{AW}}$, (2.20) defines a plane algebraic curve $C_y := \mathbb{V}\left(\mathcal{P}\right)$. Let $\overline{C_y}$ denote the normalisation of the projective closure of the fibre at y, $g := h^{1,0}\big(\overline{C_y}\big)$, and let m be the ramification profile over infinity of the Cartesian projection $\lambda : \overline{C_y} \to \mathbb{P}^1$. The corresponding family is the pull-back of the universal curve to $M^{\mathrm{ss}}_{\mathrm{AW}}$, where the pull-back metric, product, and intersection tensors are given by (1.63)–(1.65). with $\{p_i^{\mathrm{cr}}\}_m$ the ramification points of $\lambda : \overline{C_w} \longrightarrow \mathbb{P}^1$. For convenience and later comparison with $\mathcal{M}_{\mathrm{GW}}$, it will be helpful to rescale the primitive differential and the linear coordinate on the second factor of $\mathfrak{h}^{\mathrm{reg}} \oplus \mathbb{C}$ as

$$\phi^2 \longrightarrow \frac{2\nu\dim_{\mathbb{C}}\mathfrak{g}}{\langle\omega,\omega+2\mathsf{w}\rangle\dim_{\mathbb{C}}\rho_\omega}\left(\frac{\mathrm{d}\mu}{\mu}\right)^2, \qquad q_{l+1} \to \frac{1}{2\nu}\,q_{l+1}\,, \tag{2.21}$$

where

$$\mathsf{w} := \frac{1}{2}\sum_{\beta\in\mathcal{R}^+}\beta = \sum_{i=1}^{l}\omega_i \tag{2.22}$$

is the Weyl vector. By (1.63)–(1.65), this just results in an overall rescaling of the prepotential for $\mathcal{M}_{\mathrm{LG}}$.

Theorem 2.2.4 (Mirror symmetry for $\mathcal{M}_{\rm AW}$)**.** *For all marked pairs $(\mathcal{R},\widehat{\omega})$, the Landau–Ginzburg formulas (1.63)–(1.65) define a semi-simple conformal Frobenius manifold*

$$\mathcal{M}_{\rm LG} = (M_{\rm LG}, \eta_{\rm LG}, c_{\rm LG}, e, E)\,,$$

with $e = y_{l+1}^{-1}\partial_{\widehat{y}}$, $E = y_{l+1}\partial_{y_{l+1}}$. Furthermore,

$$\mathcal{M}_{\rm LG} \simeq \mathcal{M}_{\rm AW}\,.$$

The next Proposition [32, 57] shows that, for $\mathcal{M} \hookrightarrow \mathcal{H}_{g,\mathsf{m}}$ a Frobenius submanifold of a Hurwitz space, the Dubrovin-dual Frobenius manifold $\mathcal{M}^\flat$ is obtained by replacing the superpotential by its logarithm

$$\lambda \longrightarrow \log\lambda\,.$$

Proposition 2.2.5. *Let $\iota : \mathcal{M} = (M, c, \eta, e, E) \hookrightarrow \mathcal{H}_{g,\mathsf{m}}$ be a semi-simple conformal Frobenius submanifold of a Hurwitz space defined by a Landau–Ginzburg pair $(\tilde{\lambda}, \tilde{\phi})$. Let $\mathcal{M}^{\log}$ be the Frobenius submanifold structure on $M^{\rm ss}$ defined by the formulas (1.63)–(1.64) with $\lambda = \log\tilde{\lambda}$, $\phi = \tilde{\phi}$, and $X, Y, Z \in \iota_*(\mathcal{X}(M))$. Then,*

$$\mathcal{M}^{\log} \simeq \mathcal{M}^\flat\,.$$

From Proposition 2.2.5, the Frobenius structure of $\mathcal{M}^\flat_{\rm LG} \simeq \mathcal{M}^\flat_{\rm AW}$ is given as

$$\eta^\flat_{\rm AW}(X,Y) = \eta^\flat_{\rm LG}(X,Y) = \sum_m \operatorname*{Res}_{p_m^{\rm cr}} \frac{\delta_X\lambda\;\delta_Y\lambda}{\lambda{\rm d}\lambda}\phi^2\,, \tag{2.23}$$

$$c^\flat_{\rm AW}(X,Y,Z) = c^\flat_{\rm LG}(X,Y,Z) = \sum_m \operatorname*{Res}_{p_m^{\rm cr}} \frac{\delta_X\lambda\;\delta_Y\lambda\delta_Z\lambda}{\lambda^2{\rm d}\lambda}\phi^2\,. \tag{2.24}$$

§ 2.3 Dubrovin duality and mirror symmetry

In this section we will state and prove our two main theorems relating the Dubrovin-dual Frobenius structures of $\mathcal{M}_{\rm AW}$ and $\mathcal{M}_{\rm LG}$ to the T-equivariant quantum cohomology of ADE resolutions.

Theorem 2.3.1 (Dubrovin duality for $\mathcal{M}_{\rm AW}$ and $\mathcal{M}_{\rm GW}$)**.** *For all $(\mathcal{R},\widehat{\omega})$, we have $\mathcal{M}^\flat_{\rm AW} \simeq \mathcal{M}_{\rm GW}$.*

Theorem 2.3.2 (Mirror symmetry for $\mathcal{M}_{\rm GW}$)**.** *For all $(\mathcal{R},\widehat{\omega})$, we have $\mathcal{M}^\flat_{\rm LG} \simeq \mathcal{M}_{\rm GW}$.*

Remark 2.3.3. In Theorem 2.3.1, when comparing the Frobenius structures on $\mathcal{M}^\flat_{\mathrm{AW}}$ and $\mathcal{M}_{\mathrm{GW}}$, we shall need to formally set $\nu = 1$: by the Degree Axiom, the dependence on ν is reinstated on the Gromov–Witten side (or introduced on the extended affine Weyl side) upon rescaling

$$F_{\mathrm{GW}} \to \nu F_{\mathrm{GW}}\,, \quad F^\flat_{\mathrm{AW}} \to \nu F^\flat_{\mathrm{AW}}\,, \quad q_{l+1} \to \frac{q_{l+1}}{\nu}\,.$$

Properties **AW-I** and **AW-II** identify the unit and metric with those of quantum cohomology. Thus it remains only to compare the three-point functions, equivalently the Frobenius products.

In the following we will describe how Theorem 2.3.2 for rational spectral curves translates into a comparison statement between the Gromov–Witten calculation in (2.17) and (2.18) on one hand, and an explicit summation of residues which localise to the fibre over zero of the $\lambda-$projection on the other: we will compute this in closed-form for $\mathcal{R} = \mathrm{A}_l$ and $\mathcal{R} = \mathrm{D}_l$, thereby proving Theorem 2.3.2, and therefore Theorem 2.3.1, for the classical series. We also explain in general how to reduce Theorem 2.3.1 to a comparison statement of the structure constants for the Frobenius algebras restricted to a finite set of points on the complement of the discriminant: computing this explicitly for $\mathcal{R} = \mathrm{E}_l$ will provide a proof of Theorem 2.3.1, and therefore Theorem 2.3.2, for the three exceptional cases.

2.3.1 Landau–Ginzburg mirror symmetry

We start by looking at the argument of the residues in (2.23),

$$\Upsilon_{i,j,k}(p) := \frac{\delta^{(\mu)}_{\partial_{q_i}}\lambda\, \delta^{(\mu)}_{\partial_{q_j}}\lambda \delta^{(\mu)}_{\partial_{q_k}}\lambda}{\lambda^2\mu^2\partial_\mu\lambda}\mathrm{d}\mu(p), \tag{2.25}$$

so that

$$\eta^\flat_{\mathrm{LG}}(\partial_{q_i}, \partial_{q_j}) = \sum_m \operatorname*{Res}_{p=p^{\mathrm{cr}}_m} \Upsilon_{i,j,l+1}(p)\,, \quad c^\flat_{\mathrm{LG}}(\partial_{q_i}, \partial_{q_j}, \partial_{q_k}) = \sum_m \operatorname*{Res}_{p=p^{\mathrm{cr}}_m} \Upsilon_{i,j,k}(p)\,. \tag{2.26}$$

From (2.25), we deduce that the pole structure of $\Upsilon_{i,j,k}(p)$ is as follows:

(i) it has at most simple poles at the critical points $\{p^{\mathrm{cr}}_l\}$, for which $\mathrm{d}\lambda(p^{\mathrm{cr}}_i) = 0$;

(ii) it has a pole of order at most

$$\max\{2 - \delta_{i,l+1} - \delta_{j,l+1} - \delta_{k,l+1}, 0\}$$

at $\lambda(p) = 0$: this follows from

$$\delta^{(\mu)}_{\partial_{q_{l+1}}}\lambda \propto \lambda\,,$$

which offsets a linear power in the vanishing of the denominator;

(iii) it has at most simple poles at $\mu(p) = 0$ (when $\lambda(p) = \infty$) when $i = j = k = l + 1$;

(iv) it has a pole of order at most

$$\max\{1 - \delta_{i,l+1} - \delta_{j,l+1} - \delta_{k,l+1}, 0\}$$

at the critical points $\{q_m^{\rm cr}\}$ of the μ-projection, $\mathrm{d}\mu(q_m^{\rm cr}) = 0$. These are the loci where the Ehresmann connection induced by the μ-foliation is singular and $\delta^{(\mu)}_{\partial_{q_i}}\lambda$ possibly develops a pole. These singularities are partially offset by a vanishing of the same order of $\mathrm{d}\mu/\partial_\mu\lambda$.

The residue sums (2.26) pick up the contributions from the residues of type (i) alone: the difficulty in writing the critical points of the superpotential as algebraic functions of $(q_1, \ldots, q_l)$ makes, however, their individual computation unwieldy. To overcome this problem, we turn the contour around and equate the sum of residues at the critical points in (1.65) to a much more manageable sum of residues at poles and zeros of μ and λ (type (ii) and (iii)) as well as a contribution from residues at critical points of μ (type (iv)).

Computing the individual contributions from type (iv) residues is as difficult, if not more, than computing those arising from the critical points of λ. However, there are two scenarios when they can be shown to vanish identically.

- If any of i, j or k is equal to $l + 1$, we have

$$\mathrm{ord}_{q_m^{\rm cr}} \Upsilon_{i,j,k} = \max\{1 - \delta_{i,l+1} - \delta_{j,l+1} - \delta_{k,l+1}, 0\} = 0$$

hence there is no pole at the ramification points of the μ-projection. The sum was calculated explicitly for all extended affine Weyl Frobenius manifolds with canonically marked node in [15, Thm. 3.5], showing that

$$\eta^\flat_{\rm LG}(\partial_{q_i}, \partial_{q_j}) = \sum_m \operatorname*{Res}_{p=p_m^{\rm cr}} \Upsilon_{i,j,l+1}(p) = \eta_{\rm GW}(\partial_{q_i}, \partial_{q_j})$$

as expected.

- If $\deg_\lambda \mathcal{P} = \deg_{\widehat{Y}} \mathcal{Q} = 1$, implying that the spectral curve is rational, we have $\{q_m^{\rm cr}\} = \emptyset$, so once again the sum over residues only picks up contributions from zeros and poles of μ and λ. For $i, j, k \neq l + 1$, the sole contribution arises from the zeroes of the rational function $\lambda(\mu)$.

When $\mathcal{R} = \mathrm{A}_l$ or $\mathcal{R} = \mathrm{D}_l$, it was shown in [15] that $\deg_\lambda \mathcal{P} = 1$, and for these two cases the summation over residues can be performed explicitly[2].

[2]Using the superpotential derived in [34] the almost dual prepotentials for the BCD-cases with arbitrary marked node may easily be calculated. These are of the same form, corresponding to a BC_l-root system.

2.3.1.1 Proof of Theorem 2.3.2 for the A_l series with arbitrary marked weight

Let ω be the highest weight of the defining representation ρ of $\mathfrak{g} = \mathrm{sl}_{l+1}(\mathbb{C})$, and let $1 \leq \bar{k} \leq l$ be a choice of marked node. From (2.20), the corresponding superpotential reads

$$\lambda = \frac{y_{l+1} \prod_{\omega' \in \mathcal{W}(\omega)} (\mathrm{e}^{\langle \omega', h \rangle} - \mu)}{(-\mu)^{l+1-\bar{k}}} = (-1)^{l+1-\bar{k}} y_{l+1} \prod_{j=1}^{l} \kappa_j^{-\frac{l+1-\bar{k}}{l+1}} \frac{(1-q)\prod_{k=1}^{l}(1-\kappa_k q)}{q^{l+1-\bar{k}}}, \tag{2.27}$$

where

$$\kappa_j := \prod_{i=j}^{l} \mathrm{e}^{-p_i}, \; 1 \leq j \leq l, \quad q := \mathrm{e}^{-\langle \omega_{\min}, h \rangle} \mu, \quad \omega_{\min} := -\frac{1}{l+1} \sum_{i=1}^{l} i \alpha_i .$$

The structure constants of the Frobenius manifold structure on $\mathcal{M}_{\mathrm{LG}}^{\flat}$ were computed using the method described in Section 2.3.1 in [57] and, in a slightly generalised fashion, in [13, Thm. 5.4], where more generally the three-point functions of the full T'-equivariant Gromov–Witten theory of Z were seen to equate[3] those arising from the Landau–Ginzburg pair $(\log \lambda, \phi)$, with

$$\lambda = \mathrm{e}^{\frac{q_{l+1}}{\nu_1+\nu_2}} \prod_{j=1}^{l} \kappa_j^{-\frac{\nu_1}{\nu_1+\nu_2}} \frac{(1-q)\prod_{k=1}^{l}(1-\kappa_k q)}{q^{(l+1)\frac{\nu_1}{\nu_1+\nu_2}}}, \quad \phi^2 = (\nu_1+\nu_2) \left(\frac{\mathrm{d}q}{q}\right)^2 .$$

Taking the restriction to the one-dimensional subtorus T that fixes the irreducible component $E_{\bar{k}}$ as in (2.14), the superpotential λ coincides with (2.27), hence

$$c_{\mathrm{GW}}(\varphi_i, \varphi_j, \varphi_k) = c_{\mathrm{LG}}^{\flat}(\partial_{q_i}, \partial_{q_j}, \partial_{q_k}),$$

which yields the statement of Theorem 2.3.2 for $\mathcal{R} = \mathrm{A}_l$ and $\widehat{\omega} = \omega_{\bar{k}}$.

2.3.1.2 Proof of Theorem 2.3.2 for the D_l series

Let ω be the highest weight of the defining representation of $\mathfrak{g} = \mathrm{so}_{2l}(\mathbb{C})$. From (2.20), the corresponding superpotential reads

$$\lambda = \frac{\kappa_{l+1} \prod_{\omega' \in \mathcal{W}(\omega)} (\mathrm{e}^{\langle \omega', q \rangle} - \mu)}{\mu^{l-2}(\mu^2-1)^2} = \frac{\kappa_{l+1} \prod_{i=1}^{l} (\mu - \kappa_i)(\mu - \kappa_i^{-1})}{\mu^{l-2}(\mu^2-1)^2} \tag{2.28}$$

[3] More precisely, the comparison in [13] was performed for $Z \times \mathbb{C}$ with a 2-torus action acting with opposite weights on the canonical bundle of the two factors. The corresponding genus-zero GW potential differs from that of Z by by an overall weight factor of $-(\nu_1 + \nu_2)$.

where

$$\log \kappa_i = \sum_{j=1}^{l} G_{ij} q_j \,, 1 \le i \le l \,, \tag{2.29}$$

and $G_{ij} := \delta_{ij} - \delta_{i,j+1} + \delta_{i,l-1}\delta_{j,l}$; note that $(G^T G)_{ij} = C_{ij}$. As in (2.21), we shall normalise the coordinate q_{l+1} and the quadratic differential associated to the primitive form such that:

$$\log \kappa_{l+1} = \frac{q_{l+1}}{2\nu} \,, \qquad \phi^2 = \nu \left(\frac{\mathrm{d}\mu}{\mu}\right)^2 . \tag{2.30}$$

Since the spectral curve is rational in this case and the μ-projection is unramified, to perform the sum of residues over critical points in (2.23)–(2.24) we the contour of integration around on the fibres of the mirror family, and instead sum over residues at the other poles of the integrand. From (2.28), these are located at the support of the divisor (λ): the locus of zeroes and poles of λ. We thus need to show that

$$c_{\mathrm{GW}}(\varphi_i, \varphi_j, \varphi_k) = c^\flat_{\mathrm{LG}}(\varphi_i, \varphi_j, \varphi_k) = - \sum_{p \in \mathrm{supp}(\lambda)} \operatorname*{Res}_{p} \frac{\delta_{\partial_{q_i}}(\lambda)\delta_{\partial_{q_j}}(\lambda)\delta_{\partial_{q_k}}(\lambda)}{\lambda^2 \mathrm{d}\lambda} \phi^2. \tag{2.31}$$

Write

$$R_{ijk} := \frac{1}{2\nu} c^\flat_{\mathrm{LG}} \left(\frac{\partial}{\partial \log \kappa_i}, \frac{\partial}{\partial \log \kappa_j}, \frac{\partial}{\partial \log \kappa_k} \right) , \tag{2.32}$$

and $R^{[q]}_{ijk}$ for the contribution of the residue at $\mu = q$ in the sum (2.31). We start with the following

Lemma 2.3.4. *We have*

$$\begin{aligned}
& R^{[1]}_{ijk} = R^{[-1]}_{ijk} = 0 \,, \\
& R^{[0]}_{ijk} = R^{[\infty]}_{ijk} = \frac{\delta_{i,l+1}\delta_{j,l+1}\delta_{k,l+1}}{2(l-2)} \,, \\
& R^{[\kappa_m]}_{ijk} + R^{[1/\kappa_m]}_{ijk} \\
=& (\delta_{i,j,m} + \delta_{j,k,m} + \delta_{i,k,m}) \times \left\{ \frac{\delta_{i,j,k} q_i}{3} + (1 - \delta_{i,j,k}) \Big[- (\delta_{i,j,m}\delta_{k,l+1} + \delta_{j,k,m}\delta_{i,l+1} + \delta_{i,k,m}\delta_{j,l+1}) \right. \\
& \left. + (1 - \delta_{i,j,m}\delta_{k,l+1} - \delta_{j,k,m}\delta_{i,l+1} - \delta_{i,k,m}\delta_{j,l+1})(\delta_{i,j,m} p_{mk} + \delta_{k,i,m} p_{mj} + \delta_{j,k,m} p_{mi}) \Big] \right\} ,
\end{aligned}$$

with

$$p_{ij} = \frac{\kappa_i \left(\kappa_j^2 - 1\right)}{(\kappa_i - \kappa_j)(\kappa_i \kappa_j - 1)} \,, \qquad q_k = \sum_{n \neq k} \frac{\kappa_n (1 - \kappa_k^2)}{(\kappa_k - \kappa_n)(\kappa_k \kappa_n - 1)} \,.$$

Proof. The statement follows from a lengthy, but straightforward calculation of the rational residues in

$$R^{[q]}_{ijk} = -\operatorname*{Res}_{\mu=q} \frac{\prod_{m\in\{i,j,k\}}\left(\frac{-\kappa_m}{\mu-\kappa_m} + \frac{\kappa_m^{-1}}{\mu-\kappa_m^{-1}}\right)^{1-\delta_{m,l+1}}}{\sum_{r=1}^{l}\left(\frac{1}{\mu-\kappa_r} + \frac{1}{\mu-\kappa_r^{-1}}\right) - \frac{l-2}{\mu} - \frac{4\mu}{\mu^2-1}} \frac{d\mu}{2\mu^2}\,, \tag{2.33}$$

for $q = 0,\ \infty,\ \pm 1,\ \kappa_m^{\pm}$; we omit the details here. □

Corollary 2.3.5. *We have, for* $1 \le i, j \le l+1$*,*

$$c^{\flat}_{\mathrm{LG}}\left(\frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}, \frac{\partial}{\partial q_{l+1}}\right) = \begin{cases} \frac{1}{4\nu^2(l-2)}\,, & i=j=l+1\,,\\ -C_{ij}\,, & i,j\neq l+1\,,\\ 0 & \text{else}\,. \end{cases} \tag{2.34}$$

Proof. Since $\log \kappa_{l+1} = q_{l+1}/(2\nu)$, the l.h.s. for $i=j=l+1$ is

$$\frac{1}{4\nu^2} R_{l+1,l+1,l+1} = \frac{R^{[0]}_{l+1,l+1,l+1} + R^{[\infty]}_{l+1,l+1,l+1}}{4\nu^2} = \frac{1}{4\nu^2(l-2)}\,.$$

The vanishing when $i \neq j$ and either $i = l+1$ or $j = l+1$ is immediate from Lemma 2.3.4. For the case $1 \le i, j \le l$, we use (2.29) to get

$$c^{\flat}_{\mathrm{LG}}\left(\frac{\partial}{\partial q_{l+1}}, \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j}\right) = \sum_{k,m,n=1}^{l} G_{ki}\left(R^{[\kappa_n]}_{k,m,l+1} + R^{[1/\kappa_n]}_{k,m,l+1}\right)G_{mj} = -\sum_{k=1}^{l} G_{ki}G_{kj} = -C_{ij}\,.$$

□

Recall that, w.r.t. the orthonormal basis $\{\epsilon_i\}_{i=1}^{l}$ of $\mathbb{R}^l$, the D_l root system is

$$\Pi = \{\epsilon_i - \epsilon_{i+1}\}_{i=1}^{l-1} \cup \{\epsilon_{l-1} + \epsilon_l\}\,, \quad \mathcal{R}^+ = \{\epsilon_i \pm \epsilon_j\}_{i<j}\,, \quad \langle \alpha_i, \epsilon_j\rangle = G_{ji}\,.$$

Definition 2.3.6. Choose a bijection

$$\sigma : \left\{(i,j)\,\middle|\, 1 \le i < j \le l\right\} \longleftrightarrow \left\{1, \dots, \frac{l(l-1)}{2}\right\},$$

and for each pair (i,j) with $1 \le i < j \le l$, define $\Theta^{\pm}$ and Θ by

$$\Theta^{+}_{\sigma(i,j),k} := \begin{cases} 1 & \text{if } k=i \text{ or } k=j\,,\\ 0 & \text{otherwise}\,, \end{cases} \qquad \Theta^{-}_{\sigma(i,j),k} := \begin{cases} 1 & \text{if } k=i\,,\\ -1 & \text{if } k=j\,,\\ 0 & \text{otherwise}\,, \end{cases} \qquad \Theta := \begin{pmatrix}\Theta^+\\ \Theta^-\end{pmatrix}.$$

Corollary 2.3.7. *Let* ϵ *be the column vector* $(\epsilon_1, \dots, \epsilon_l)^T$*. Then the rows of* $\Theta\cdot\epsilon$ *give all the positive roots of* D_l*,*

$$\left(\beta_1, \dots \beta_{l(l-1)}\right) = \Theta\epsilon\,. \tag{2.35}$$

Define now coordinates $(\tau_1, \ldots, \tau_l)$ of $H^2(Z, \mathbb{C})$ via

$$(q_1, \ldots, q_l)^T = C^{-1} G^T (\tau_1, \ldots, \tau_l)^T \,. \tag{2.36}$$

By (2.29), we have:

$$(\log \kappa_1, \ldots, \log \kappa_l)^T = G(q_1, \ldots, q_l)^T = GC^{-1}G^T(\tau_1, \ldots, \tau_l)^T$$

Using $G^T G = C$, we get $GC^{-1}G^T = I$, hence $\log \kappa_i = \tau_i$, $1 \leq i \leq l$.

Lemma 2.3.8. *The positive degree part of the genus Gromov–Witten primary potential of Z is*

$$F^+_{\mathrm{GW}} = 2\nu \sum_{i=1}^{l(l-1)} \mathrm{Li}_3 \left(\exp \left(\sum_{j=1}^{l} -\Theta_{ij} \tau_j \right) \right) \,. \tag{2.37}$$

Proof. In term of the matrix $A_{ij} = \langle \beta_i, \alpha_j \rangle$ of coefficients of the positive roots $(\beta_1, \ldots, \beta_{l(l-1)})$ in the basis of fundamental weights $\Omega = \{\omega_1, \ldots, \omega_l\}$,

$$\mathcal{R}^+ = A\Omega \,, \tag{2.38}$$

we find, using (2.36),

$$F^+_{\mathrm{GW}} = 2\nu \sum_{\sigma=1}^{l(l-1)} \mathrm{Li}_3 \left(\exp \left(\sum_{j=1}^{l} -(AC^{-1}G^T)_{\sigma j} \tau_j \right) \right) \,.$$

On the other hand, expressing fundamental weights in the orthonormal basis,

$$\left(\beta_1, \ldots, \beta_{l(l-1)}\right)^T = AC^{-1} \left(\alpha_1, \ldots, \alpha_l\right)^T = AC^{-1} G^T \left(\epsilon_1, \ldots, \epsilon_l\right)^T \,.$$

From this we deduce that $\Theta = AC^{-1}G^T$ by comparing to (2.35), thereby proving the claim. □

Proof of Theorem 2.3.2 for $\mathcal{R} = \mathrm{D}_l$. By Corollary 2.3.5, we need only consider the case $1 \leq i, j, k \leq l$. From Lemma 2.3.8, we get

$$\frac{1}{2\nu} \frac{\partial^3 F^+_{\mathrm{GW}}}{\partial \tau_i \partial \tau_j \partial \tau_k} = -\sum_{\sigma=1}^{l(l-1)} \Theta_{\sigma i} \Theta_{\sigma j} \Theta_{\sigma k} \frac{\exp(-\sum_{m=1}^{l} \Theta_{\sigma m} \tau_m)}{1 - \exp(-\sum_{m=1}^{l} \Theta_{\sigma m} \tau_m)}, \tag{2.39}$$

where σ is an element of the index set $\sigma \in \{(m,n) | 1 \leq m < n \leq l\}$, and consider the case $k = j = i$ for starters. From Definition 2.3.6, this reads explicitly as

$$\frac{1}{2\nu} \frac{\partial^3 F^+_{\mathrm{GW}}}{\partial \tau_i \partial \tau_i \partial \tau_i} = -\sum_{k \neq i} \frac{\exp(-\tau_i - \tau_k)}{1 - \exp(-\tau_i - \tau_k)} - \sum_{k > i} \frac{\exp(-\tau_i + \tau_k)}{1 - \exp(-\tau_i + \tau_k)} + \sum_{k < i} \frac{\exp(-\tau_k + \tau_i)}{1 - \exp(-\tau_k + \tau_i)} \,.$$

Likewise, for $k = j \neq i$, we spell out (2.39) to be

$$\frac{1}{2\nu}\frac{\partial^3 F^+_{\rm GW}}{\partial\tau_i\partial\tau_i\partial\tau_j} = -\frac{\exp(-\tau_i-\tau_j)}{1-\exp(-\tau_i-\tau_j)} + \frac{\exp(-\tau_i+\tau_j)}{1-\exp(-\tau_i+\tau_j)} + \frac{1}{2}(\mathrm{sgn}(i-j)+1)\,.$$

As, for fixed σ, $\Theta_{\sigma i}$ is only non-zero for exactly two values of i, we finally have that, for i, j, k all distinct,

$$\frac{\partial^3 F^+_{\rm GW}}{\partial\tau_i\partial\tau_j\partial\tau_k} = 0\,.$$

Let us compare the above back to the structure constants of the Landau–Ginzburg product (2.32). From Lemma 2.3.4 and for i, j, k all distinct, we find

$$R_{iii} = \frac{1}{2\nu}\frac{\partial^3 F^+_{\rm GW}}{\partial\tau_i\partial\tau_i\partial\tau_i} - l + i\,, \quad R_{iij} = \frac{1}{2\nu}\frac{\partial^3 F^+_{\rm GW}}{\partial\tau_i\partial\tau_i\partial\tau_j} - \frac{1}{2}(\mathrm{sgn}(i-j)+1)\,, \quad R_{ijk} = \frac{1}{2\nu}\frac{\partial^3 F^+_{\rm GW}}{\partial\tau_i\partial\tau_j\partial\tau_k} = 0\,. \tag{2.40}$$

Hence Theorem 2.3.2 for type D_l reduces to verifying the following identities relating the additive terms in the r.h.s. of (2.40) to the T-equivariant triple intersection numbers of Z:

$$\sum_{\sigma=1}^{l(l-1)} \Theta^2_{\sigma i}\Theta_{\sigma j} = \begin{cases} 2(l-i)\,, & 1 \le i = j \le l\,, \\ \mathrm{sgn}(i-j)+1\,, & 1 \le i \neq j \le l\,. \end{cases}$$

These are easily verified using the definition of Θ, concluding the proof. □

2.3.2 Dubrovin duality via initial conditions

A direct proof of Theorem 2.3.1 would entail showing that the structure constants of $\mathcal{M}^\flat_{\rm AW}$, expressed in terms of those of $\mathcal{M}_{\rm AW}$ through the duality relation (1.60), coincide with the quantum cohomology product (2.16) of Z. Explicitly, in our case (1.60) reads

$$\frac{\partial^3 F_{\rm GW}}{\partial q_i\partial q_j\partial q_k}(x) = \sum_{a,b,A,B,C} (\eta^\flat_{\rm AW})_{ia}(\eta^\flat_{\rm AW})_{jb}\frac{\partial t_C}{\partial q_k}\frac{\partial q_a}{\partial t_A}\frac{\partial q_b}{\partial t_B}\,(c_{\rm AW})^{AB}_C\,(t(x))\,, \tag{2.41}$$

where

$$(c_{\rm AW})^{AB}_C = \sum_{M,N=1}^{l+1} \eta^{AM}_{\rm AW}\eta^{BN}_{\rm AW}\frac{\partial^3 F_{\rm AW}}{\partial t_M\partial t_N\partial t_C}\,. \tag{2.42}$$

For a given marked pair $(\mathcal{R}, \widehat{\omega})$, (2.41) could *a priori* be proved by brute-force, as both sides of the equality are calculable, at least in principle. For the r.h.s., closed-form expressions for the prepotential $F_{\rm AW}(t)$ and the Saito flat coordinates $\{t_A(x)\}$ in linear coordinates on $\mathfrak{h}\times\mathbb{C}$ were found in [35, 10, 15]; for the l.h.s., the Gromov–Witten quantum prepotential is given by (2.16).

In practice, however, this approach is largely unfeasible. The entries of the Jacobian matrix $\partial_{q_k} t_C(x)$ are trigonometric polynomials with e.g. up to billions of terms for $\mathcal{R} = \mathrm{E}_8$; for the same reason, computing the inverse $\partial_{t_A} q_a$ relevant to (2.41) is well out of reach of symbolic computation packages, such as *Mathematica*, even for very low values of l.

As we will show, both sides of (2.41) are uniquely determined by their values at a (small) finite number of points in the semi-simple locus $\mathcal{M}_{\rm GW}^{\rm ss} = \mathfrak{h}^{\rm reg} \oplus \mathbb{C}$: we will refer to this as a *set of initial conditions* for the Dubrovin duality. The functional equality (2.41) reduces then to a *numerical* equality over the set of initial conditions, which can in turn be verified very effectively. As explained in Remark 2.3.3, we will formally set $\nu = 1$ throughout this Section.

2.3.2.1 Reduction to initial conditions

We will start by establishing various homogeneity properties for the tensors appearing in (2.41). Following [35], we define $d_{l+1} \coloneqq \deg(y_{l+1}) = 0$. On the complement of the discriminant, the coefficients $(\eta^\flat_{\rm AW})^{\alpha\beta}$ of (2.6) in the chart parametrised by the coordinates $(y_1, \dots, y_{l+1})$ satisfy [35, Lem. 2.1]

$$\deg(\eta^\flat_{\rm AW})^{\alpha\beta} = d_\alpha + d_\beta \,,$$

while the inverse Gram matrix of the Saito metric is, by definition,

$$\eta^{\alpha\beta}_{\rm AW}(y) = \frac{\partial(\eta^\flat_{\rm AW})^{\alpha\beta}}{\partial\widehat{y}} \,.$$

By [35, Cor. 2.5], the flat coordinates $\{t_A\}_{A=1,\dots,l}$ are polynomials in $y_1, \dots, y_l, \mathrm{e}^{y_{l+1}}$ with $\deg t_A = d_A$. Furthermore, from [8], the determinant of the Jacobian matrix associated to the change-of-variables $Y_i \longrightarrow Y_i(q_1, \dots, q_l)$ is

$$\delta \coloneqq \prod_{\beta\in R^+} \left(e^{\langle\beta,h\rangle/2} - e^{-\langle\beta,h\rangle/2}\right) . \tag{2.43}$$

By definition, δ is anti-invariant under the Weyl group action,

$$\delta(wh) = (-1)^{l(w)}\delta(h).$$

By orthogonal extension to $\mathfrak{h} \times \mathbb{C}$, the determinant of the Jacobian matrix associated to the change-of-variables $y_i \longrightarrow y_i(q_1, \dots, q_{l+1})$ is

$$\Delta \coloneqq c\mathrm{e}^{(d_1+\dots+d_l)q_{l+1}}\delta \,, \tag{2.44}$$

and Δ is therefore, for the same reason as δ, anti-invariant w.r.t. the Weyl group action.

Lemma 2.3.9. $\Delta^2 \in \mathcal{A}$. *Moreover,* Δ^2 *is quasi-homogeneous of degree* $2(d_1 + \dots + d_l)$.

Proof. We first show the invariance of Δ^2 under the extended affine Weyl group $\widetilde{\mathcal{W}}$. It suffices to check it on the generators, given by the Weyl reflections, the co-root lattice generators, and an extra translation

$$(h, q_{l+1}) \longrightarrow (h + 2\pi\mathrm{i}\widehat{\omega}, q_{l+1} - 2\pi\mathrm{i})\,. \tag{2.45}$$

Obviously, since Δ is anti-invariant, Δ^2 is a Weyl group invariant. For the invariance under the other generators of $\widetilde{\mathcal{W}}$, we use

$$\delta = \mathrm{e}^{\langle \mathsf{w}, h\rangle} \prod_{\beta\in\mathcal{R}^+} \left(1 - \mathrm{e}^{-\langle\beta, h\rangle}\right)\,, \tag{2.46}$$

where w is the Weyl vector (2.22). An affine translation by a co-root $\alpha^\vee$,

$$h \longrightarrow h + 2\pi\mathrm{i}\alpha^\vee\,,$$

will leave δ (and therefore Δ and Δ^2) invariant, since $\langle \mathsf{w}, \alpha^\vee\rangle$ and $\langle\beta,\alpha^\vee\rangle$, $\beta\in\mathcal{R}^+$ are integers. It only remains to consider the effect of the $\mathbb{Z}$-action generated by (2.45): under this translation, we have

$$\delta \longrightarrow \mathrm{e}^{2\pi\mathrm{i}\langle\mathsf{w},\widehat{\omega}\rangle}\delta\,, \qquad \mathrm{e}^{(d_1+\dots+d_l)q_{l+1}} \longrightarrow \mathrm{e}^{-2\pi i(d_1+\dots+d_l)}\mathrm{e}^{(d_1+\dots+d_l)q_{l+1}}\,.$$

Hence Δ is invariant, using (2.22) and the fact that

$$\langle\mathsf{w},\widehat{\omega}\rangle = \sum_{i=1}^{l} d_i\,, \tag{2.47}$$

since $d_i = \langle\omega_i,\widehat{\omega}\rangle$. To conclude that $\Delta^2\in\mathcal{A}$, it remains to prove boundedness of Δ in the limit (2.10). From (2.47), the restriction of Δ to the locus in (2.10) is

$$c\mathrm{e}^{(d_1+\dots+d_l)q_{l+1}}\mathrm{e}^{\langle\mathsf{w},h\rangle}\prod_{\beta\in\mathcal{R}^+}\left(1-\mathrm{e}^{-\langle\beta,h\rangle}\mathrm{e}^{-\langle\beta,\widehat{\omega}\rangle\tau}\right).$$

This is bounded when $\tau\to+\infty$, since $\langle\beta,\widehat{\omega}\rangle\in\mathbb{Z}_{>0}$. It is finally immediate to see that $\Delta^2(x)$ is quasi-homogeneous of the claimed degree, since it is monomial in $\mathrm{e}^{q_{l+1}}$ with exponent $2(d_1+\dots+d_l)$. ☐

Define now $(2,1)$-tensors

$$\ell_{\mathrm{GW}} \in H^0\big(M_{\mathrm{GW}}, \mathrm{Sym}^2\mathcal{T}_{M_{\mathrm{GW}}}\otimes\mathcal{T}^*_{M_{\mathrm{GW}}}\big)\,,\quad \ell_{\mathrm{AW}} \in H^0\big(M_{\mathrm{AW}}, \mathrm{Sym}^2\mathcal{T}_{\mathcal{M}_{\mathrm{AW}}}\otimes\mathcal{T}^*_{\mathcal{M}_{\mathrm{AW}}}\big)\,,$$

by the following expressions in the chart $(y_1,\dots,y_{l+1})$:

$$\begin{aligned}
(\ell_{\mathrm{GW}})^{\alpha\beta}_{\epsilon} &\coloneqq \sum_{a,b,i,j,k}\frac{\partial^3 F_{\mathrm{GW}}}{\partial q_i\partial q_j\partial q_k}(\eta^\flat_{\mathrm{AW}})^{ia}\frac{\partial y_\alpha}{\partial q_a}(\eta^\flat_{\mathrm{AW}})^{jb}\frac{\partial y_\beta}{\partial q_b}\frac{\partial q_k}{\partial y_\epsilon}\,,\\
(\ell_{\mathrm{AW}})^{\alpha\beta}_{\epsilon} &\coloneqq \sum_{L,M,N}\frac{\partial y_\alpha}{\partial t_L}\frac{\partial y_\beta}{\partial t_M}\frac{\partial t_N}{\partial y_\epsilon}(c_{\mathrm{AW}})^{LM}_{N}\,.
\end{aligned} \tag{2.48}$$

Proposition 2.3.10. *We have*

$$\Delta^2 \left(\ell_{\mathrm{GW}}\right)^{\alpha\beta}_{\epsilon} \in \mathcal{A}\,, \qquad \left(\ell_{\mathrm{AW}}\right)^{\alpha\beta}_{\epsilon} \in \mathcal{A}\,. \tag{2.49}$$

Proof. By (2.8) and (2.11), the entries of the Jacobian matrix (resp. its inverse),

$$\mathcal{J}_{\alpha i} \coloneqq \frac{\partial y_\alpha}{\partial q_i}\,, \quad \text{resp. } \mathcal{J}^{-1}_{i\alpha} = \frac{\partial q_i}{\partial y_\alpha}\,,$$

are Laurent polynomials (resp. rational functions) in $(\mathrm{e}^{q_1}, \dots, \mathrm{e}^{q_{l+1}})$. Since Δ is the determinant of the Jacobian matrix, the expression

$$\Delta \frac{\partial q_k}{\partial y_\epsilon} \in \mathbb{C}[\mathrm{e}^{\pm q_1}, \dots, \mathrm{e}^{\pm q_{l+1}}]$$

is again a Fourier polynomial in $(q_1, \dots, q_{l+1})$. Moreover, from (2.17), the triple derivatives of the Gromov–Witten prepotential are rational functions in $\mathrm{e}^{\pm q_1}, \dots, \mathrm{e}^{\pm q_{l+1}}$ with at most first order poles at $\delta = 0$. Hence,

$$\Delta^2 \left(\ell_{\mathrm{GW}}\right)^{\alpha\beta}_{\epsilon} \in \mathbb{C}[\mathrm{e}^{\pm q_1}, \dots, \mathrm{e}^{\pm q_{l+1}}]$$

is a Fourier polynomial in $(q_1, \dots, q_{l+1})$, as claimed. By Lemma 2.3.9, it remains to check that the tensor components $\left(\ell_{\mathrm{GW}}\right)^{\alpha\beta}_{\epsilon}$ are $\widetilde{\mathcal{W}}$-invariant and bounded in the limit (2.10). Let

$$g^{\eta\alpha} \coloneqq \sum_{a,b} \frac{\partial y_\eta}{\partial q_a} (\eta^\flat_{\mathrm{AW}})^{ab} \frac{\partial y_\alpha}{\partial q_b}\,. \tag{2.50}$$

By [35], $g^{\eta\alpha}$ are elements of $\mathcal{A}$. Therefore, noticing that

$$\begin{aligned} \left(\ell_{\mathrm{GW}}\right)^{\alpha\beta}_{\epsilon} &= \sum_{i,j,k,a,b} \frac{\partial^3 F_{\mathrm{GW}}}{\partial q_i \partial q_j \partial q_k} (\eta^\flat_{\mathrm{AW}})^{ia} \frac{\partial y_\alpha}{\partial q_a} (\eta^\flat_{\mathrm{AW}})^{jb} \frac{\partial y_\beta}{\partial q_b} \frac{\partial q_k}{\partial y_\epsilon} \\ &= \sum_{i,j,k,\delta,\eta} \left(\frac{\partial^3 F_{\mathrm{GW}}}{\partial q_i \partial q_j \partial q_k} \frac{\partial q_i}{\partial y_\eta} \frac{\partial q_j}{\partial y_\delta} \frac{\partial q_k}{\partial y_\epsilon} \right) g^{\eta\alpha} g^{\delta\beta}\,, \end{aligned} \tag{2.51}$$

it suffices to check that

$$\sum_{i,j,k} \frac{\partial^3 F_{\mathrm{GW}}}{\partial q_i \partial q_j \partial q_k} \frac{\partial q_i}{\partial y_\eta} \frac{\partial q_j}{\partial y_\delta} \frac{\partial q_k}{\partial y_\epsilon}$$

is $\widetilde{\mathcal{W}}$-invariant. Since y_α is $\widetilde{\mathcal{W}}$-invariant for $1 \leqslant \alpha \leqslant l$, $y_{l+1} = q_{l+1}$, and the coordinates q_i which are linearly-acted-upon by $\widetilde{\mathcal{W}}$ are contracted, verifying the $\widetilde{\mathcal{W}}$-invariance of the last expression amounts to checking the $\widetilde{\mathcal{W}}$-invariance of the Gromov–Witten potential F_{GW}, up to an additive quadratic shift in x. Recall that

$$\mathrm{Li}_3(\mathrm{e}^z) = c_3(z) + \mathrm{Li}_3(\mathrm{e}^{-z})\,, \tag{2.52}$$

where $c_3(z)$ is a cubic polynomial in z, as follows from integrating both sides of the geometric series identity $\mathrm{Li}_0(\mathrm{e}^z) = 1 - \mathrm{Li}_0(\mathrm{e}^{-z})$. Expanding the sum over positive roots in (2.17) to a sum over all roots using in (2.52), we have

$$F_{\mathrm{GW}} = D_3(q) + 2\nu \sum_{\beta \in \mathcal{R}} \mathrm{Li}_3(\mathrm{e}^{-\langle \beta, h\rangle})$$

for a cubic polynomial $D_3(q)$. It is straightforward to verify that $D_3(q)$ is Weyl-invariant, hence F_{GW} is Weyl-invariant. The invariance under the extended affine action is trivial, since for each β, $\mathrm{e}^{\langle \beta, h\rangle}$ is invariant under the corresponding translation in $\widetilde{\mathcal{W}}$.
It only remains to verify the boundedness property. Under the limit (2.10), since the entries of the Jacobian matrix $\mathcal{J}_{\alpha i}$ are bounded, so are the coefficients of its inverse $\mathcal{J}^{-1}_{i\alpha}$, and we have already shown that Δ is bounded. As for the triple derivatives of the Gromov–Witten potential, boundedness is trivial when either of $i, j, k = l+1$ by the string equation and (2.16). For $i, j, k \leq l$, we have

$$\frac{\partial^3 F_{\mathrm{GW}}}{\partial q_i \partial q_j \partial q_k} = c_{ijk} + \sum_{\beta \in \mathcal{R}^+} d_{ijk}(\beta) \frac{e^{-\langle \beta, h\rangle}}{1 - e^{-\langle \beta, h\rangle}}\,, \tag{2.53}$$

for constants c_{ijk}, $d_{ijk}(\beta)$, with $\beta \in \mathcal{R}^+$. Since $\langle \beta, \widehat{\omega}\rangle > 0$, we have

$$\lim_{\tau \to +\infty} |\mathrm{e}^{-\langle \beta, q_0 + \widehat{\omega}\tau\rangle}| = 0\,,$$

hence (2.53) is bounded at infinity.

As for ℓ_{AW}, by [35, 15], recall that when $1 \leq A \leq l$, $t_A(y)$ is a polynomial in $y_1, \dots, y_l$ and $\mathrm{e}^{y_{l+1}}$ with polynomial inverse $y_i(t)$, and $t_{l+1} = y_{l+1}$. Moreover, $(c_{\mathrm{AW}})^{lm}_n(t)$ are polynomials in $y_1, \dots, y_l$ and $\mathrm{e}^{y_{l+1}}$. Therefore,

$$(\ell_{\mathrm{AW}})^{\alpha\beta}_{\epsilon}(y) \in \mathbb{C}[y_1, \dots, y_l, \mathrm{e}^{y_{l+1}}]\,.$$

The claim then follows as $y_1(x), \dots, y_l(x)$ and $\mathrm{e}^{y_{l+1}}$ are generators of $\mathcal{A}$. □

The square Jacobian factor Δ^2 in $\Delta^2 \ell_{\mathrm{GW}}$ entered our discussion so far only in order to offset the potential double poles of ℓ_{GW} along the discriminant. However, the equality (2.41) that we need to prove, $\ell_{\mathrm{GW}} = \ell_{\mathrm{AW}}$, would lead to the stronger expectation that ℓ_{GW} is in fact regular at $\delta = 0$. We now show that this is indeed the case. We start by proving the following

Lemma 2.3.11. *Let*

$$\begin{array}{rcl} m : \Lambda_w(\mathcal{R}) & \longrightarrow & \mathbb{C} \\ \omega & \longrightarrow & m_\omega \end{array}$$

be a complex-valued map on the weight lattice having finite support and Weyl-invariant fibres,

$$m_{w(\omega)} = m_\omega \quad \forall\, w \in \mathcal{W}\,, \qquad \big|\{\omega \in \Lambda_w(\mathcal{R}) | m_\omega \neq 0\}\big| < \infty\,,$$

and consider the Weyl-invariant Fourier polynomial

$$\mathfrak{p}(q) := \sum_{\omega \in \Lambda_w(\mathcal{R})} m_\omega \mathrm{e}^{\langle \omega, q\rangle} \in \mathbb{C}[\mathrm{e}^{\pm q_1}, \dots, \mathrm{e}^{\pm q_l}]^{\mathcal{W}}.$$

Then, for all $\beta \in \Lambda_r(\mathcal{R})$, the directional derivative of $\mathfrak{p}(q)$ along β is divisible by $(1-\mathrm{e}^{\pm\langle \beta, h\rangle})$ in the ring of Fourier polynomials,

$$\sum_{a=1}^{l} \langle \beta, \omega_a\rangle \frac{\partial \mathfrak{p}(q)}{\partial q_a} \in (1-\mathrm{e}^{\pm\langle \beta, h\rangle})\, \mathbb{C}[\mathrm{e}^{\pm q_1}, \dots, \mathrm{e}^{\pm q_l}].$$

Proof. Since

$$\frac{\partial \mathfrak{p}(q)}{\partial q_a} = \sum_{\omega \in \Lambda_w(\mathcal{R})} m_\omega \langle \omega, \alpha_a^\vee\rangle \mathrm{e}^{\langle \omega, h\rangle}, \qquad \beta = \sum_a \langle \beta, \omega_a\rangle \alpha_a^\vee,$$

we have that

$$\sum_{a=1}^{l} \langle \beta, \omega_a\rangle \frac{\partial \mathfrak{p}(q)}{\partial q_a} = \sum_{\omega \in \Lambda_w(\mathcal{R})} m_\omega \langle \beta, \omega\rangle \mathrm{e}^{\langle \omega, h\rangle}. \tag{2.54}$$

Let now $\beta \in \Lambda_r(\mathcal{R})$ and

$$\Gamma_\pm(\beta) := \left\{ \omega \in \Lambda_w(\mathcal{R}) \,\middle|\, \mathrm{sgn}\left(\langle \beta, \omega\rangle\right) = \pm 1 \right\}.$$

Since $\langle \beta, s_\beta(\omega)\rangle = -\langle \beta, \omega\rangle$, the Weyl reflection across $\beta^\perp$ gives a bijection

$$s_\beta : \Gamma_\pm(\beta) \to \Gamma_\mp(\beta).$$

Moreover, using $m_{s_\beta(\omega)} = m_\omega$, we can rewrite (2.54) as

$$\sum_{\omega \in \Gamma_+(\beta)} m_\omega \langle \beta, \omega\rangle (\mathrm{e}^{\langle \omega, h\rangle} - \mathrm{e}^{\langle s_\beta(\omega), h\rangle}) = \sum_{\omega \in \Gamma_+(\beta)} m_\omega \langle \beta, \omega\rangle \mathrm{e}^{\langle \omega, h\rangle} (1 - \mathrm{e}^{-\frac{2\langle \omega, \beta\rangle \langle \beta, h\rangle}{\langle \beta, \beta\rangle}}).$$

As $\frac{2\langle \omega, \beta\rangle}{\langle \beta, \beta\rangle}$ is a (positive) integer, $(1-\mathrm{e}^{\pm\langle \beta, h\rangle})$ divides the r.h.s. in $\mathbb{C}[\mathrm{e}^{\pm q_1}, \dots, \mathrm{e}^{\pm q_l}]$. □

Proposition 2.3.12. *$(\ell_{\mathrm{GW}})^{\alpha\beta}_\epsilon \in \mathcal{A}$.*

Proof. From (2.11), (2.17) and (2.55), $(\ell_{\mathrm{GW}})^{\alpha\beta}_\epsilon$ is a polynomial in $\mathrm{e}^{q_{l+1}}$. Let then

$$L^{\alpha\beta}_\epsilon := (\ell_{\mathrm{GW}})^{\alpha\beta}_\epsilon \big|_{q_{l+1}=0}.$$

The statement would follow from showing that $L^{\alpha\beta}_\epsilon$, which *a priori* has double poles along $\delta = 0$, is in fact a Fourier polynomial in $(q_1, \dots, q_l)$. By Proposition 2.3.13, $\delta^2 L^{\alpha\beta}_\epsilon$ is Weyl-invariant, and therefore $\delta L^{\alpha\beta}_\epsilon$ is Weyl anti-invariant. By Bourbaki [8, Ch. VI, Sec.3, Prop. 2(iii)], the multiplication by δ is a bijection from the set of Weyl-invariant Fourier polynomials onto the set of Weyl anti-invariant Fourier polynomials. It then suffices to show that

$$\delta L_\epsilon^{\alpha\beta} \in \mathbb{C}[\mathrm{e}^{\pm q_1}, \dots, \mathrm{e}^{\pm q_l}]\,.$$

From (2.48), we have

$$\delta L_\epsilon^{\alpha\beta} = \sum_{i,j,k,a,b} \frac{\partial^3 F_{\mathrm{GW}}}{\partial q_i \partial q_j \partial q_k} (\eta^\flat_{\mathrm{AW}})^{ia} (\eta^\flat_{\mathrm{AW}})^{jb} \mathcal{J}_{\alpha a} \mathcal{J}_{\beta b} \mathcal{J}^{-1}_{k\epsilon}\, \delta \Big|_{q_{l+1}=0},$$

where the r.h.s. has in principle simple poles at $\delta = 0$. Since $\Delta = \det \mathcal{J}$, $\delta \mathcal{J}^{-1}_{k\epsilon}\Big|_{q_{l+1}=0}$ is regular at $\delta = 0$, therefore it would be sufficient to show that

$$\sum_{i,a} \frac{\partial^3 F_{\mathrm{GW}}}{\partial q_i \partial q_j \partial q_k} (\eta^\flat_{\mathrm{AW}})^{ia} \mathcal{J}_{\alpha a}\Big|_{q_{l+1}=0}$$

is pole-free at $\delta = 0$. By (2.17), the triple derivatives of the prepotential are pole-free along the discriminant unless $i, j, k \le l$, in which case[4]

$$\frac{\partial^3 F_{\mathrm{GW}}}{\partial q_i \partial q_j \partial q_k} = c_{ijk} + 2\nu \sum_{\beta \in \mathcal{R}^+} \langle \beta, \alpha_i^\vee \rangle \langle \beta, \alpha_j^\vee \rangle \langle \beta, \alpha_k^\vee \rangle \frac{1}{1 - \mathrm{e}^{\langle \beta, h \rangle}}\,.$$

By Lemma 2.3.11 with $\mathfrak{p}(q) = y_\alpha(q)$, we have that, for all $\beta \in \mathcal{R}^+$, $\sum_a \langle \beta, \omega_a \rangle \mathcal{J}_{\alpha a}$ has a simple zero at $\langle \beta, h \rangle = 0$. Hence, using $(\eta^\flat_{\mathrm{AW}})_{ab} = -C_{ab}$ for $a, b, \le l$ by (2.7), the expression

$$\frac{1}{1 - \mathrm{e}^{\langle \beta, h \rangle}} \sum_{i,a} \langle \beta, \alpha_i^\vee \rangle (C^{-1})_{ia} \mathcal{J}_{\alpha a}$$

is regular on the discriminant $\delta = 0$, concluding the proof. □

Proposition 2.3.13. *For any α, β, ϵ, $(\ell_{\mathrm{GW}})_\epsilon^{\alpha\beta}$ and $(\ell_{\mathrm{AW}})_\epsilon^{\alpha\beta}$ are quasi-homogeneous polynomials in $y_1, \dots, y_l$ and $\mathrm{e}^{y_{l+1}}$ of degree $d_\alpha + d_\beta - d_\epsilon$.*

Proof. The polynomiality statement follows from Propositions 2.2.1, 2.3.10 and 2.3.12. The homogeneity property for ℓ_{GW} can be read off from (2.51): firstly, from (2.50), $g^{\eta\delta}$ and $g^{\delta\beta}$ are quasi-homogeneous of degree $d_\eta + d_\delta$ and $d_\delta + d_\beta$ respectively. The triple derivatives of F_{GW} are quasi-homogeneous of degree zero, and therefore each nonzero term of $(\ell_{\mathrm{GW}})_\epsilon^{\alpha\beta}$ has the same degree $d_\alpha + d_\beta - d_\epsilon$. As for ℓ_{AW}, by [35], t_A is a quasi-homogeneous polynomial of degree d_A for $1 \le A \le l$, and $t_{l+1} = y_{l+1}$. Thus, recalling that $d_{l+1} = 0$, $\frac{\partial y_\alpha}{\partial t_A}$ is quasi-homogeneous of degree $d_\alpha - d_A$, and likewise $\frac{\partial t_B}{\partial y_\beta}$, has quasi-homogeneous degree $d_B - d_\beta$. Furthermore, by [35], $(c_{\mathrm{AW}})_N^{LM}(t)$ is quasi-homogeneous of degree $d_L + d_M - d_N$. Therefore, each nonzero term of

$$(\ell_{\mathrm{AW}})_\epsilon^{\alpha\beta} = \sum_{L,M,N} \frac{\partial y_\alpha}{\partial t_L} \frac{\partial y_\beta}{\partial t_M} \frac{\partial t_N}{\partial y_\epsilon} c_N^{LM}$$

[4]For A_l with a general 2-torus action, the expression is the same upon relacing $2\nu \to \nu_1 + \nu_2$.

will be of the same degree:

$$(d_\alpha - d_L) + (d_\beta - d_M) + (d_N - d_\epsilon) + (d_L + d_M - d_N) = d_\alpha + d_\beta - d_\epsilon \,,$$

concluding the proof. □

Define a grading on $\mathbb{C}[Y_1, \dots, Y_l]$ by $\deg_Y Y_i = d_i$ for $1 \le i \le l$. Writing

$$D := \max\Big\{ (d_\eta + d_\delta - d_\epsilon) | 1 \le \eta, \delta, \epsilon \le l+1 \Big\} \,, \tag{2.55}$$

we will write

$$\mathfrak{V}_{\mathrm{adm}} := \left\{ F \in \mathbb{C}[Y_1, \dots, Y_l] \,\middle|\, \deg_Y F \le D \right\}, \quad S_{\mathrm{adm}} := \left\{ (n_1, \dots, n_l) \in \mathbb{Z}^l_{\geqslant 0} \,\middle|\, \sum_{i=1}^{l} n_i d_i \le D \right\},$$

for, respectively, the finite-dimensional vector subspace of polynomials in $(Y_1, \dots, Y_l)$ of degree less than or equal to D, and the set of monic monomials in $\mathfrak{V}_{\mathrm{adm}}$. We will refer to S_{adm} as the set of *admissible exponents* of the root system $\mathcal{R}$, and to the corresponding monomial basis of $\mathfrak{V}_{\mathrm{adm}}$ as the set of *admissible monomials.* In particular, for $N = (n_1, \dots, n_l) \in S_{\mathrm{adm}}$, we will use the shorthand multi-index notation

$$Y^N := \prod_{i=1}^{l} Y_i^{n_i} \,.$$

Corollary 2.3.14. *We have*

$$(\ell_{\mathrm{GW}})^{\eta\delta}_\epsilon \big|_{q_{l+1}=0} \in \mathfrak{V}_{\mathrm{adm}} \,, \quad (\ell_{\mathrm{AW}})^{\eta\delta}_\epsilon \big|_{q_{l+1}=0} \in \mathfrak{V}_{\mathrm{adm}} \,.$$

Proof. Immediate from Proposition 2.3.13. □

By Corollary 2.3.14, the Weyl-invariant Fourier polynomials $(\ell_{\mathrm{GW}})^{\eta\delta}_\epsilon|_{q_{l+1}=0}$ and $(\ell_{\mathrm{AW}})^{\eta\delta}_\epsilon|_{q_{l+1}=0}$ are elements of the same finite-dimensional vector space $\mathfrak{V}_{\mathrm{adm}}$, with $\dim_{\mathbb{C}} \mathfrak{V}_{\mathrm{adm}} = |S_{\mathrm{adm}}|$. This reduces the verification of the functional relation (2.41) to checking the *numerical* relation

$$(\ell_{\mathrm{GW}})^{\eta\delta}_\epsilon (q^{(K)}) = (\ell_{\mathrm{AW}})^{\eta\delta}_\epsilon (q^{(K)})$$

for *finitely many points* $\{q^{(1)}, \dots, q^{(|S_{\mathrm{adm}}|)}\}$ in $\mathfrak{h}$, each providing a linear constraint on the coefficients of either side of (2.41) as an element of $\mathfrak{V}_{\mathrm{adm}}$. If these points are chosen generically, the resulting linear system will have maximal rank and determine the $(2,1)$ tensors ℓ_{GW} and ℓ_{AW} uniquely.

Definition 2.3.15. A set of $|S_{\mathrm{adm}}|$ points $\{q^{(1)}, \dots, q^{(|S_{\mathrm{adm}}|)}\} \subset M^{\mathrm{ss}}_{\mathrm{AW}}$ will be called a *set of initial conditions* for $\mathcal{M}^\flat_{\mathrm{AW}}$ if the $|S_{\mathrm{adm}}| \times |S_{\mathrm{adm}}|$ generalised Vandermonde matrix minor

$$(Y^N (q^{(M)}))_{N,M \in S_{\mathrm{adm}}}$$

is non-singular,

$$\det_{M,N} \left(Y^N (q^{(M)}) \right) \ne 0 \,. \tag{2.56}$$

Checking (2.41) on a set of initial conditions is poorly suited to a general proof of Theorem 2.3.1 for all marked pairs $(\mathcal{R}, \widehat{\omega})$. On the other hand, it can be performed highly effectively on a case-by-case basis. As it just remains to prove Theorems 2.3.1 and 2.3.2 for $\mathcal{R} = \mathrm{E}_l$, we will construct sets of initial conditions to verify (2.41) directly in these three exceptional cases.

2.3.2.2 Proof of Theorem 2.3.1 for the E_l series

It is straightforward to compute the degree bound D and the number $|S_{\mathrm{adm}}|$ of admissible monomials for the exceptional series E_l. These are reproduced in Table 2.1.

l	D	$\lvert S_{\mathrm{adm}}\rvert$
6	12	151
7	24	254
8	60	434

Table 2.1: Degree bounds and size of the set of initial conditions for the exceptional series E_l, $l = 6, 7, 8$.

Configurations of $|S_{\mathrm{adm}}|$ points that are not initial have measure zero in $\mathrm{Sym}^{|S_{\mathrm{adm}}|}\mathfrak{h}$, since (2.56) is an open condition. Due to the small size of S_{adm} for $\mathcal{R} = \mathrm{E}_l$, it is straightforward to construct an initial set by picking a configuration of $|S_{\mathrm{adm}}|$ points in the Cartan subalgebra and then checking *a posteriori* that the generalised Vandermonde minor (2.56) is indeed non-zero for them. Having constructed a set of initial conditions $\mathfrak{I}$, we are then just left with verifying directly the numerical identities

$$(\ell_{\mathrm{GW}})_\gamma^{\alpha\beta}(q^{(K)}) = (\ell_{\mathrm{AW}})_\gamma^{\alpha\beta}(q^{(K)}), \quad k = 1, \dots, |\mathfrak{I}|.$$

Proposition 2.3.16. *For all* $(\mathcal{R}, \widehat{\omega})$ *with* $\mathcal{R} = \mathrm{E}_l$*, there exists a set of initial conditions* $\mathfrak{I}$ *such that,* $\forall x \in \mathfrak{I}$*,*

$$(\ell_{\mathrm{GW}})_\gamma^{\alpha\beta}(q) = (\ell_{\mathrm{AW}})_\gamma^{\alpha\beta}(q). \tag{2.57}$$

Proof. Direct calculation. □

Example 2.3.17 ($\mathcal{R} = \mathrm{E}_6$)**.** We will sketch here the main elements entering the verification of (2.57) for $\mathcal{R} = \mathrm{E}_6$. The l.h.s. of (2.41), as a function of $x = (q_1, \dots, q_7)$ is explicitly computed by (2.17). As for the r.h.s., $(\eta^\flat_{\mathrm{AW}})_{ia}$ is given by (2.7), and all we need to compute are the Jacobian matrix of the change-of-variables $t \to t(q)$, its inverse, and the $(2,1)$ multiplication tensor c_{AW} in flat coordinates. For the latter, the prepotential of $\mathcal{M}_{\rm AW}$ was computed in [15] to be

$$
\begin{aligned}
F_{\rm AW} = {} & \frac{\mathrm{e}^{12t_7}}{24} + \frac{1}{4}\mathrm{e}^{8t_7}t_1t_5 + \frac{1}{12}\mathrm{e}^{6t_7}\left(t_1^3+t_5^3+3t_6^2\right) + \frac{1}{2}\mathrm{e}^{5t_7}t_1t_6t_5 \\
& + \frac{1}{72}\mathrm{e}^{4t_7}\left(10t_1^2t_5^2+t_1^2t_4-t_2t_5^2-t_2t_4\right) + \frac{\mathrm{e}^{3t_7}}{12}t_6\left(t_1^3-t_2t_1+t_5^3+2t_6^2+t_4t_5\right) \\
& + \frac{\mathrm{e}^{2t_7}}{144}\left(t_5t_1^4-2t_2t_5t_1^2+\left(t_5^4+2t_4t_5^2+36t_6^2t_5+t_4^2\right)t_1+t_2^2t_5\right) + \frac{1}{72}\mathrm{e}^{t_7}\left(t_1^2-t_2\right)\left(t_5^2+t_4\right)t_6 \\
& - \frac{t_1^6+t_5^6}{38880} - \frac{t_2t_1^4-t_4t_5^4+t_2^3-t_4^3}{7776} - \frac{t_2^2t_1^2+t_4^2t_5^2}{2592} - \frac{(t_2t_1-t_4t_5)t_3}{36} - \frac{t_6^4}{192} + \frac{1}{8}t_3t_6^2 + \frac{1}{4}t_3^2t_7\,,
\end{aligned}
$$

from which $(c_{\rm AW})_C^{AB}$ in (2.42) is immediately computed as an explicit polynomial in $t_1,\dots,t_6$ and e^{t_7}. The flat coordinates are related to the fundamental traces in (2.9) as [15]

$$
\begin{aligned}
& t_1 = W_0^{\frac{1}{3}}W_1\,, \quad t_2 = W_0^{\frac{2}{3}}(W_1^2-6W_2-12W_5)\,, \quad t_3 = W_0(2W_1W_5+W_3+3W_6+3)\,, \\
& t_4 = W_0^{\frac{2}{3}}(-W_5^2+12W_1+6W_4)\,, \quad t_5 = W_0^{\frac{1}{3}}W_5\,, \quad t_6 = W_0^{\frac{1}{2}}(W_6+2)\,, \quad t_7 = \frac{\log(W_0)}{6}\,,
\end{aligned}
\tag{2.58}
$$

where furthermore $W_0 = \mathrm{e}^{1/2q_7}$. The fundamental traces W_1 and W_6 in, respectively, the **27** and **78** (adjoint) representation can be computed from the respective weight and root system, and their expressions are given in the supplementary material available with [14]. The remaining traces can be computed from the following relations in the representation ring of E_6:

$$
W_i(q) = W_{6-i}(-q)\,, \quad i=1,\dots,5
$$

$$
W_2(q) = \frac{1}{2}\left(W_1^2(q)-W_1(2q)\right)\,, \quad W_3(q) = \frac{1}{3}\left(W_2(q)W_1(q)-W_1(q)W_1(2q)+W_1(3q)\right)
$$

expressing, respectively, the fact that $\rho_i = \overline{\rho}_{6-i}$ for $1\le i\le 5$ and $\rho_i = \wedge^i\rho_1$ for $i=2,3$. Plugging the resulting expressions into (2.58) gives the change-of-variables $t\to t(q)$, from which the Jacobian coefficients $\partial_{q_i}t_A$ could then be in principle be computed as explicit, if cumbersome, Fourier polynomials with a few thousand terms. The resulting matrix inversion computing $\partial_{t_A}q_i$ as rational trigonometric functions of q is far out of reach of modern symbolic computation packages. On the other hand, evaluating on a (rational) set of initial conditions $\mathfrak{I}$ dramatically reduces the unwieldy expressions above in the field $\mathbb{Q}(\mathrm{e}^{q_1},\dots,\mathrm{e}^{q_7})$ to eminently manageable manipulations of rational numbers. The exact inversion of the *numerical* 7×7 matrix $\partial_{t_A}q_i|_{q=q^{(K)}}$ over $\mathbb{Q}$ for $q^{(K)}\in\mathfrak{I}$ takes now a fraction of a second in *Mathematica* on an entry-level desktop computer[5], as does the evaluation of all the other quantities entering (2.41). This allows for a straightforward and fast verification of (2.41) over $\mathfrak{I}$, and therefore, by Proposition 2.3.13 and Corollary 2.3.14, on the whole of $M_{\rm AW}$.

[5]Absolute clock-times based on a setup with Intel Core i7-8700 @3.20 GHz processor and 16GB RAM.

The same *Mathematica* calculations take a couple of minutes for $\mathcal{R} = \mathrm{E}_7$, and a few hours for $\mathcal{R} = \mathrm{E}_8$ with the same setup. The Wolfram Language code used to verify Proposition 2.3.16 is also available in the supplementary material for [14].

Corollary 2.3.18. *Theorems 2.3.1 and 2.3.2 hold for* $(\mathcal{R}, \widehat{\omega})$ *with* $\mathcal{R} = \mathrm{E}_l$.

Proof. Theorem 2.3.1 follows from Propositions 2.3.13 and 2.3.16 and Corollary 2.3.14, and it implies Theorem 2.3.2 by Theorem 2.2.4. □

Example 2.3.19 (Landau–Ginzburg mirror symmetry for the E_6 du Val resolution)**.** Let $\mathcal{R} = \mathrm{E}_6$, so that $\widehat{\omega} = \omega_3$ is the highest weight of its 2925-dimensional fundamental representation. By Theorem 2.3.2 and (2.21), the T-equivariant quantum cohomology of the du Val resolution of type E_6 is mirror to a one-dimensional Landau–Ginzburg model with a log-meromorphic superpotential $\log\lambda$ and logarithmic primitive form

$$\phi^2 = \frac{\nu}{3}\left(\frac{\mathrm{d}\mu}{\mu}\right)^2,$$

where λ and μ are the two Cartesian projections on the family of plane algebraic curves over $\mathfrak{h}^{\mathrm{reg}} \oplus \mathbb{C}$ in (2.20) given by

$$\mathcal{P}(y_1, \dots y_7; \lambda, \mu) = 0\,,$$

and we took $\omega \coloneqq \omega_1$ in (2.20) to be the highest weight of the 27-dimensional fundamental representation ρ_1. Expressing the characters of the exterior powers of ρ_1 in terms of fundamental characters $W_1, \dots W_6$ (see [7, Eq. (6.21)]), and further relating the latter to the basic invariants as

$$W_1 = Y_1\,, \quad W_2 = Y_2 + 5Y_5\,, \quad W_3 = Y_3 + 4Y_1Y_5 - 9Y_6 - 63\,,$$
$$W_4 = Y_4 + 5Y_1\,, \quad W_5 = Y_5 \quad W_6 = Y_6 + 6\,,$$

as can be ascertained from a direct inspection of the fundamental weight systems Γ_i $(i = 1 \dots 6)$, we compute from (2.20)–(2.21) that

$$\mathcal{P}(y_1, \dots, y_7; \lambda, \mu) = \mathcal{Q}(Y_1, \dots, Y_6, \mu) + \mathrm{e}^{-q_7/(2\nu)}\mathcal{Q}^{[1]}(Y_1, \dots, Y_6, \mu)\lambda + \mathrm{e}^{-q_7/\nu}\mu^9\left(\mu^3 - 1\right)^3\lambda^2\Big|_{Y_\alpha = y_\alpha \mathrm{e}^{-d_\alpha y_7}}$$

where $\mathcal{Q}$ is the characteristic polynomial in (2.19), and

$$\begin{aligned}
\mathcal{Q}^{[1]} = {} & 1 - 2\mu^2 Y_1 + \mu^3(3Y_6 + 20) + \mu^4\left(Y_1^2 - 2Y_2 - 13Y_5\right) - \mu^5(Y_6Y_1 + 9Y_1 + Y_4) \\
& + \mu^6(2Y_3 + 12Y_1Y_5 - 21Y_6 - 150) + \mu^7\left(5Y_2 - 10Y_1^2 - Y_4Y_1 + 24Y_5 - Y_5Y_6\right) \\
& + \mu^8\left(Y_5Y_1^2 + 3Y_6Y_1 + 26Y_1 - 13Y_5^2 + Y_4 - 2Y_2Y_5\right) \\
& + \mu^9\left(6Y_5Y_1 - Y_1^3 + 3Y_2Y_1 - 6Y_3 + 3Y_4Y_5 + 48Y_6 + 343\right) \\
& + \mu^{10}\left(4Y_5 - 4Y_1^2 - 2Y_5^2Y_1 - Y_4Y_1 - 3Y_2 + 3Y_5Y_6\right) \\
& + \left(\mu^k \longrightarrow (-\mu)^{21-k},\, Y_1 \longleftrightarrow Y_5,\, Y_2 \longleftrightarrow Y_4\right).
\end{aligned}$$

As a check, from (2.23), the dual pairing on $M_{\rm AW} \simeq M_{\rm LG}$ is

$$\eta^\flat_{\rm LG}(\partial_{q_i},\partial_{q_j}) = \frac{1}{6}\sum_m \operatorname*{Res}_{p_m^{\rm cr}} \frac{\delta_{\partial_{q_i}}\lambda\,\delta_{\partial_{q_j}}\lambda}{\lambda\mu\,\partial_\mu\lambda}\frac{\mathrm{d}\mu}{\mu} = -\frac{1}{6}\sum_{\lambda(p),\mu(p)\in\{0,\infty\}} \operatorname*{Res}_{p} \frac{\delta_{\partial_{q_i}}\lambda\,\delta_{\partial_{q_j}}\lambda}{\lambda\mu\,\partial_\mu\lambda}\frac{\mathrm{d}\mu}{\mu}\,.$$

For $1 \leq i,j \leq l$, it is straightforward to check that the only non-vanishing residues arise from the zeroes of λ, i.e. when $\mu = \mathrm{e}^{\langle\omega',q\rangle}$ for $\omega' \in \Gamma_1$. We find

$$\sum_{\omega'\in\Gamma_1} \operatorname*{Res}_{\mu=\mathrm{e}^{\langle\omega',q\rangle}} \frac{\delta_{\partial_{q_i}}\lambda\,\delta_{\partial_{q_j}}\lambda}{\lambda\mu\,\partial_\mu\lambda}\frac{\mathrm{d}\mu}{\mu} = \sum_{\omega'\in\Gamma_1}\left\langle\omega',\alpha_i\right\rangle\left\langle\omega',\alpha_j\right\rangle = \begin{pmatrix} 12 & -6 & 0 & 0 & 0 & 0 \\ -6 & 12 & -6 & 0 & 0 & 0 \\ 0 & -6 & 12 & -6 & 0 & -6 \\ 0 & 0 & -6 & 12 & -6 & 0 \\ 0 & 0 & 0 & -6 & 12 & 0 \\ 0 & 0 & -6 & 0 & 0 & 12 \end{pmatrix}_{ij},$$

and therefore

$$\eta^\flat_{\rm LG}(\partial_{q_i},\partial_{q_j}) = \eta_{\rm GW}(\varphi_i,\varphi_j) = -C_{ij}\,,$$

recovering the expression for the 2-point intersection pairing on Z in (2.16).

Chapter 3

On the crepant resolution conjecture for type $\mathbb{D}$ Kleinian singularities

§ 3.1 Introduction

Let $G = G_{\mathrm{D}_m} < \mathrm{SL}(2;\mathbb{Z})$ be the Kleinian subgroup of type D_m, and $X = \mathbb{C}^2/G$ be the corresponding Kleinian singularity. Let $f : Z \to X$ be the minimal (and crepant) resolution of X, and $\mathcal{X} = [\mathbb{C}^2/G]$ be the underlying orbifold of X.

The setup of the Crepant Resolution Conjecture (CRC) therefore applies. For type D Kleinian singularities, Hu [41] verified the Bryan–Gholampour conjecture for $\mathcal{X}$, thereby matching the quantum cohomology of $\mathcal{X}$ and Z. What remains open is the Coates–Ruan formulation of the CRC stated in Conjecture 1.4.2. It predicts a degree-preserving symplectic isomorphism

$$\mathbb{U}\colon \mathcal{H}_{\mathcal{X}} \longrightarrow \mathcal{H}_Z$$

that, after analytic continuation, maps $\mathcal{L}_{\mathcal{X}}$ to $\mathcal{L}_Z$.

In this chapter, we construct a candidate transformation $\mathbb{U}_\rho$ that matches the two Lagrangian cones after analytic continuation. Recall that Conjecture 1.4.3 predicts that such a transformation is induced from a K-group isomorphism through Iritani's integral structure.

The key input is the mirror symmetry for Z that is developed in Chapter 2, following a framework based on the work of Brini–Cavalieri–Ross[13], where they proved the crepant resolution conjecture for $[\mathbb{C}^2/\mathbb{Z}_{n+1}] \times \mathbb{C}$ using one-dimensional Landau–Ginzburg mirrors. Analogous applications of mirror symmetry to the study of crepant resolutions can be traced back to toric examples [23, 24], where the general theory of toric mirror symmetry is used. We outline the three key steps in the framework as follows:

3.1.1 The framework

Recall that every weak Frobenius manifold $\mathcal{M}^\flat$ is equipped with a pencil of flat connections, known as the Dubrovin connection. In the context of quantum cohomology, the components of the J-function provide the flat coordinates for this connection. For more details, see Section 1.4.

Step 1 Show the following statement:

Claim 3.1.1. *There exists a (weak) Frobenius manifold $\mathcal{M}^\flat$ and two special points* LR, OP $\in \mathcal{M}^\flat$, *such that:*

(1) in a neighbourhood U_Z of LR $\in \mathcal{M}^\flat$, $\mathrm{QH}^*_{\mathbb{C}^\times}(Z) \cong \mathcal{M}^\flat\big|_{U_Z}$,

(2) in a neighbourhood $U_\mathcal{X}$ of OP $\in \mathcal{M}^\flat$, $\mathrm{QH}^*_{\mathbb{C}^\times}(\mathcal{X}) \cong \mathcal{M}^\flat\big|_{U_\mathcal{X}}$.

Consequently, the components J_Z and $J_\mathcal{X}$ are local deformed flat coordinates for the same Frobenius manifold at distinct points, which implies that they are related via analytic continuation or parallel transport. In principle, this reduces the problem to finding the explicit analytic continuation that links the respective J-functions: if there is $\mathbb{U}$ that sends J_Z to $J_\mathcal{X}$, then it induces an isomorphism of Givental's spaces. However, directly computing this continuation on the A-model side is hopeless; because they are defined as formal power series, their global analytic properties remain highly elusive. This problem is translated to the analytic continuation of the mirror B-model periods, which typically have more tractable analytic behaviour.

Step 2 Suppose $\mathcal{M}^\flat$ has a Landau-Ginzburg model $(\log(\lambda), \phi)$, then show that

Claim 3.1.2. *The gradients of oscillatory integrals $\int \lambda^{1/z}\phi$ are flat sections of the Dubrovin connection associated to $\mathcal{M}^\flat$. Moreover, one can choose a family of integral contours γ_i such that $\int_{\gamma_i} \lambda^{1/z}\phi$ give a framing of deformed flat coordinates.*

As a corollary, components of J-functions are linear combinations of these integrals near LR(resp. OP), for Z (resp. $\mathcal{X}$). The components of $J-$functions are therefore linear combinations of oscillatory integrals at LR/OP.

Computing $\mathbb{U}$ is, at this point, equivalent to explicitly comparing $J-$functions with oscillatory integrals at LR/OP, which becomes the final step:

Step 3 Choose a global framing of oscillatory integrals that is defined in particular in a neighbourhood of path ρ connecting LR and OP. Compare the local behaviour of this framing near LR/OP with $J_Z/J_\mathcal{X}$ and obtain the linear map $\mathbb{U}$ that sends J_Z to $J_\mathcal{X}$.

3.1.2 Organisation and main results

The chapter is organised as follows. Section 3.2 describes the Chen–Ruan cohomology of $\mathcal{X}$ and reviews the Bryan–Gholampour conjecture. The analysis of the Coates–Ruan conjecture for $\mathcal{X}$ then proceeds according to the framework explained in Section 3.1.1. Specifically, Section 3.3 uses mirror symmetry to establish Step 1. Following this, Section 3.4 develops the theory of twisted periods for one-dimensional Landau–Ginzburg models, applying it to establish Step 2 for $\mathcal{X}$. Finally, Section 3.5 constructs a candidate transformation $\mathbb{U}_\rho$ for general type D_m by comparing the local expressions of the global period framing, and describes the transformation predicted by Conjecture 1.4.3. The candidate transformation maps the corresponding Lagrangian cones to one another. It remains to prove that $\mathbb{U}_\rho$ satisfies all the remaining properties required by Conjecture 1.4.2. Thus the conjecture is not proved in full. In the D_3 case the candidate transformation is computed explicitly and compared, in suitable limits, with the transformation predicted by the integral structure. Necessary connection formulas of relevant Lauricella functions are given in Appendix A.

We now list the main results of this chapter.

Let $\mathcal{R}$ be the D_m root system, $\mathcal{M}$ be the associated extended affine Weyl Frobenius manifold and $\mathcal{M}^\flat$ be its dual Frobenius manifold. Let (λ, ϕ) be the superpotential of $\mathcal{M}$. These structures are introduced in Chapter 2.

Proposition 3.1.3 (=Proposition 3.3.1)**.** *Claim 3.1.1 holds for* $(\mathcal{X}, Z, \mathcal{M}^\flat)$.

Theorem 3.1.4 (=Propositions 3.4.11 and 3.4.12)**.** *Claim 3.1.2 holds for* $(\mathcal{X}, Z, \mathcal{M}^\flat)$.

Proposition 3.1.5 (= Proposition 3.5.8)**.** $\mathbb{U}_\rho$ *is explicitly computed in* D_3 *case.*

Proposition 3.1.6 (See Propositions 3.5.8, 3.5.9 and 3.5.11)**.** *In the* D_3 *case,* $\mathbb{U}_\rho$ *is induced by an isomorphism*

$$\mathbb{U}_K : K_{\mathbb{C}^\times}(\mathcal{X}) \to K_{\mathbb{C}^\times}(Z) \tag{3.1}$$

via the $\mathbb{C}^\times$*-equivariant integral structure framings* $\widehat{\Psi}_{\mathcal{X}}$ *and* $\widehat{\Psi}_Z$*. Moreover,* $\mathbb{U}_K$ *is linear over* $\mathrm{Frac}(K_{\mathbb{C}^\times}(\mathrm{pt}))$ *and its non-equivariant limit is linear over* $\mathbb{Z}$*.*

§ 3.2 Cohomological and Bryan–Gholampour conjecture for $\mathcal{X}$

We briefly review the classical and quantum McKay correspondence for type D Kleinian singularities in order to fix conventions.

The Kleinian subgroup $G < \mathrm{SL}(2;\mathbb{C})$ of type D_m is the binary dihedral group

$$G = G_{\mathrm{D}_m} = \mathbb{BD}_{4m-8}. \tag{3.2}$$

Explicitly, G is generated by two matrices in $\mathrm{SL}(2;\mathbb{C})$:

$$a = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.3}$$

where $\zeta = \exp\left(\frac{i\pi}{2(m-2)}\right)$. These generators satisfy the relations

$$a^{(m-2)} = b^2 \,, b^4 = I \,, bab^{-1} = a^{-1}. \tag{3.4}$$

Let $\pi : Z \to \mathbb{C}^2/G_{\mathrm{D}_m}$ denote its minimal resolution, and $E_1, \dots, E_m$ be the exceptional curves of π. We express a general element of $H^2(Z, \mathbb{C})$ as

$$\tau = \sum_{j=1}^{m} t_j \, [E_j]. \tag{3.5}$$

By the McKay correspondence (see Section 1.1), the irreducible representations of G are in bijection with the vertices of the affine Dynkin diagram $\tilde{D}_m$. There are four one-dimensional representations $(\rho_0, \rho_1, \rho_{m-1}, \rho_m)$ and $m-3$ two-dimensional representations $(\rho_2, \dots, \rho_{m-2})$, where for $2 \le k \le m-2$

$$\rho_k(a) = \begin{pmatrix} \zeta^k & \\ & \zeta^{-k} \end{pmatrix}, \ \rho_k(b) = \begin{pmatrix} & 1 \\ (-1)^k & \end{pmatrix} \tag{3.6}$$

and for $k \in \{0, 1, m-1, m\}$, ρ_k depends on whether m is even or odd as follows:

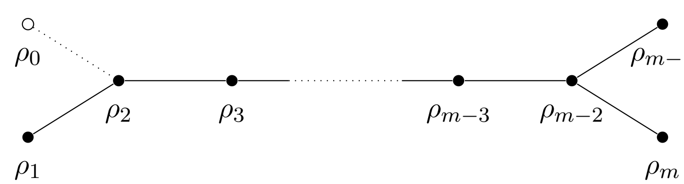
	a	b
ρ_1	1	1
ρ_2	1	-1
ρ_3	-1	-1
ρ_4	-1	1

m even

	a	b
ρ_1	1	1
ρ_2	1	-1
ρ_3	-1	i
ρ_4	-1	$-$i

m odd

It can then be checked that the McKay graph of these representation is the affine D_m Dynkin diagram:



(3.7)

The Chen–Ruan orbifold cohomology of the stack $\mathcal{X} = [\mathbb{C}^2/G]$ is spanned by the twisted sectors in bijection with the $m+1$ conjugacy classes of G:

$$H^*_{\mathrm{orb}}(\mathcal{X}, \mathbb{C}) = \bigoplus_{(g) \in \mathrm{Conj}(G)} H^*(\mathcal{X}^g, \mathbb{C}). \tag{3.8}$$

Let $\mathbf{1}_g$ be the fundamental class of the twisted sector $\mathcal{X}^g$, we write a general element of $H^*_{\mathrm{CR}}(\mathcal{X})$ as

$$\sum_{g \in \mathrm{Conj}(G_{\mathrm{D}_m})} x_g \, \mathbf{1}_g. \tag{3.9}$$

For an irreducible curve E_v, let ρ_v be the corresponding irreducible representation of G under the McKay correspondence, and Q be its defining representation. An immediate consequence of the McKay correspondence is that the following change of variables

$$t_v = \frac{1}{|G|}\sum_{g\in G}\sqrt{\chi_Q(g)-2}\,\chi_{\rho_v}(g)\,x_g, \tag{3.10}$$

where the index ν for ρ_v is consistent with the classical McKay correspondence (3.7), induces an linear isomorphism $\phi : H^2_{\mathrm{orb}}(\mathcal{X},\mathbb{C}) \xrightarrow{\sim} H^2(Z,\mathbb{C})$ and preserves the (Chen–Ruan) Poincaré pairing, as introduced in Bryan–Gholampour [16].

Bryan and Gholampour further conjectured that the genus-zero Gromov-Witten theory of $\mathcal{X}$ is equivalent to that of the resolution Z after analytic continuation. They made the following explicit prediction for the genus zero prepotential of the $\mathcal{X}$:

Conjecture 3.2.1 (Bryan–Gholampour/Bryan–Graber)**.** *Let the coordinates $\{t_v\}$ and $\{x_g\}$ of $H(Z)$ and $H(\mathcal{X})$ be as in (3.10) and (3.9) respectively. Then the genus zero $\mathbb{C}^\times$-equivariant Gromov–Witten prepotentials F_0^Z and $F_0^{\mathcal{X}}$ of $H(Z)$ and $H(\mathcal{X})$ are equal, up to analytic continuation and change of variables*

$$t_v = 2\pi\mathrm{i}\frac{\dim\rho_v}{|G|} + \frac{1}{|G|}\sum_{g\in G}\sqrt{\chi_V(g)-2}\,\chi_{\rho_v}(g)\,x_g, \tag{3.11}$$

Proving this conjecture requires a rigorous computation of the involved Hodge integrals and a structural understanding of the descendant invariants. This long-standing task was finally accomplished by Hu [41]:

Theorem 3.2.2 (Hu)**.** *The Bryan–Gholampour Conjecture 3.2.1 is true for binary dihedral groups.*

Our subsequent study is fundamentally based on this result.

§ 3.3 Global mirror symmetry

In this section, we show Step 1 for type D Kleinian singularities. This means giving the local isomorphism between the quantum cohomology of $\mathcal{X}$ (resp. Z) and the dual Frobenius manifold of the associated extended affine Weyl Frobenius manifolds.

3.3.1 Global mirror symmetry

Let $\mathcal{M}^\flat$ be the dual Frobenius manifold of the extended affine Weyl Frobenius manifold $\mathcal{M}$ associated with the D_m root system, and let $\mathrm{LG}(\log\lambda,\phi)$ be its Landau–Ginzburg model. As in (2.28), its superpotential is

$$\lambda = \frac{\kappa_{l+1}\prod_{i=1}^{l}(\mu-\kappa_i)(\mu-\kappa_i^{-1})}{\mu^{l-2}(\mu^2-1)^2}, \qquad \phi = \frac{d\,\mu}{\mu}, \tag{3.12}$$

where (see (2.29))

$$\log\kappa_i = \sum_{j=1}^{l} G_{ij}q_j\,, 1 \le i \le l\,, \tag{3.13}$$

and $G_{ij} \coloneqq \delta_{ij} - \delta_{i,j+1} + \delta_{i,l-1}\delta_{j,l}$.

Proposition 3.3.1. *Locally identify the base manifold of* $\mathcal{M}^\flat$ *with* $H(Z)$ *and* $H(\mathcal{X})$ *via* (3.11)*. Take two special points* $\mathrm{LR}, \mathrm{OP} \in \mathcal{M}$ *as*

$$\mathrm{LR} = \{\mathrm{Re}(t_i) \to -\infty\}, \tag{3.14}$$

$$\mathrm{OP} = \left\{t_v = 2\pi\mathrm{i}\frac{\dim\rho_v}{|G|}\right\}, \tag{3.15}$$

Here LR *denotes the large radius limit, and* OP *denotes the orbifold point. Then*

(1) near a neighbourhood of $\mathrm{LR} \in \mathcal{M}$*,* $\mathrm{QH}^*_{\mathbb{C}^\times}(Z) \cong \mathcal{M}^\flat \cong \mathrm{LG}(\log\lambda, \phi)$*;*

(2) near a neighbourhood of $\mathrm{OP} \in \mathcal{M}$*,* $\mathrm{QH}^*_{\mathbb{C}^\times}(\mathcal{X}) \cong \mathcal{M}^\flat \cong \mathrm{LG}(\log\lambda, \phi)$*.*

Proof. Part (1) follows from Theorems 2.3.1 and 2.3.2, specialised to the root system D_m. For part (2), Hu's theorem identifies the quantum cohomology of $\mathcal{X}$ with that of Z under the affine change of variables (3.11). The origin in the orbifold coordinates is mapped to OP, so transporting the identification in part (1) gives the required local isomorphism near OP. □

This establishes Step 1.

§ 3.4 Twisted periods of one-dimensional Landau-Ginzburg mirrors

3.4.1 Generalities on deformed flat coordinates

Let $\phi = d(\log\mu)$ and $\lambda(t,\mu)$ be a superpotential that parametrises a Hurwitz space:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\lambda"] \arrow[d, "\pi"] & \mathbb{P}^1 \\ \mathcal{B} & \end{tikzcd} \tag{3.16}$$

Associated to the pair (λ, ϕ) are two weak Frobenius manifolds $\mathcal{M}$ and $\mathcal{M}^\flat$, with the multiplication given by:

$$\eta(X \circ Y, Z) = \sum_{p \in cr(\lambda)} \underset{\mu=p}{\boldsymbol{Res}} \frac{X(\lambda)Y(\lambda)Z(\lambda)}{\partial\lambda/\partial\log\,\mu} d\log\,\mu \tag{3.17}$$

$$g(X \star Y, Z) = \sum_{p \in cr(\lambda)} \underset{\mu=p}{\boldsymbol{Res}} \frac{X(\log\,\lambda)Y(\log\,\lambda)Z(\log\,\lambda)}{\partial\log\,\lambda/\partial\log\,\mu} d\log\,\mu \tag{3.18}$$

An overview of the theories mentioned above can be found in Section 1.5.

Consider the associated pencil of flat coordinates:

$$\nabla_X^{\hbar,I} = \nabla_X^{\boldsymbol{L.C},\eta} + \hbar X \circ \tag{3.19}$$

$$\nabla_X^{\hbar,II} = \nabla_X^{\boldsymbol{L.C},g} + \hbar X \star \tag{3.20}$$

By [30], they are flat pencils of connections on TM. The gradients of a deformed flat coordinate are solutions to the associated system of differential equations. For example, let I be deformed flat coordinates of $\nabla_X^{\hbar,I}$ and t_α are flat coordinate of η, then $\xi_\beta = \partial I/\partial t_\beta$ are solutions to the following system:

$$\partial_\alpha \xi_\beta = \hbar C_{\alpha\beta}^\gamma \xi_\gamma, \tag{3.21}$$

where $C_{\alpha,\beta}^\gamma$ are structure constants of the multiplication $\circ$:

$$\frac{\partial}{\partial_\alpha} \circ \frac{\partial}{\partial_\beta} = C_{\alpha\beta}^\gamma \frac{\partial}{\partial_\gamma}. \tag{3.22}$$

In this chapter, we will always write structure constants in flat coordinates of η, which we denote by$\{t_\alpha\}$. For example:

$$(C_\alpha)_\beta^\gamma = C_{\alpha\beta}^\gamma, \tag{3.23}$$

$$E = E^\epsilon \partial_\epsilon, \tag{3.24}$$

$$\mathcal{U}_\beta^\alpha = E^\epsilon C_{\epsilon\beta}^\alpha, \tag{3.25}$$

$$\mathcal{V} = \frac{2-d}{2} - \nabla E. \tag{3.26}$$

Proposition 3.4.1 ([32])**.** *In flat coordinates of η, the system of differential equations for (gradients of) deformed flat coordinates of the dual-Frobenius manifold is given by*

$$\partial_\alpha \xi \cdot \mathcal{U} = \xi \cdot (\mathcal{V} + \hbar - 1/2) \cdot C_\alpha, \tag{3.27}$$

where $\xi = (\xi_1, \cdots, \xi_m)$ is viewed as a row vector and $\cdot$ is the matrix multiplication.

We will prove the following two statements:

Proposition 3.4.2. *Let* $\lambda, \phi = d\log\mu$ *be a Hurwitz Frobenius manifold, then the integral*

$$\int_\Gamma \mathrm{e}^{\hbar\lambda}\phi$$

is a solution to the system (3.21).

Proposition 3.4.3. *Let* $\lambda, \phi = d\log\mu$ *be a Hurwitz Frobenius manifold. Then, under the assumption that*

(1) $E(\lambda) = \lambda$;

(2) ∇E *is diagonal;*

(3) $d = 1$,

the integral

$$\int_\Gamma \lambda^\hbar\phi$$

is a solution to the system (3.27).

For simplicity, we take $x = \log\mu$ throughout the section. The idea is to follow the theory of primitive form, in particular in the context of Hurwitz Frobenius manifolds as developed by Milanov[51].

3.4.2 Primitive form and flat coordinates to first structure connection

Proposition 3.4.4. *For any Hurwitz Frobenius manifold, there are functions* $K_{\alpha\beta}$, *defined by the first equation below, such that*

$$\partial_\alpha\lambda\partial_\beta\lambda = C^\gamma_{\alpha\beta}\partial_\gamma\lambda + K_{\alpha\beta}\frac{\partial\lambda}{\partial x}, \tag{3.28}$$

$$\partial_\alpha\partial_\beta\lambda = \frac{\partial K_{\alpha\beta}}{\partial x}, \tag{3.29}$$

Proof. The first equation is the same as saying that the **Kodaira-Spencer map**:

$$\mathcal{T}B \to \pi_*\mathcal{O}_C : v \mapsto \tilde{v}(\lambda)\big|_C \tag{3.30}$$

is an isomorphism, where C is the set of critical point of λ. This is explained in [51, 3.1].

The second equation follows from the fact that the primary form $\phi = d\log\mu = dx$ in Dubrovin's classification is a primitive form [51, Proposition6.4] and $\{t^\alpha\}$ are flat coordinates, which we now explain.

Let $\mathcal{H} = \pi_* H^1_{d+\hbar d\lambda\wedge}(\Omega^*_{\mathcal{C}/B})$ be the cohomology bundle of the twisted De Rham differentials, where $\Omega^*_{\mathcal{C}/B} = \Omega^*_{\mathcal{C}}/\left(\pi^*(\Omega^1_B)\wedge\Omega^{*-1}\Omega_C)\right)$ is the sheaf of relative differential forms. On the formal complement $\mathcal{H}((z))$ there is a **Gauss-Manin connection** ($z = 1/\hbar$ in this note):

$$z\nabla^{\text{G.M.}}_v[\omega] = [rel \circ \iota_{\tilde{v}}(zd_{\mathcal{C}} + d_{\mathcal{C}}\lambda\wedge)\tilde{\omega}], \tag{3.31}$$

where $\tilde{v}$ (resp. $\tilde{\omega}$) are lifting of v (resp. ω) to a vector field (resp. a honest differential form) on $\mathcal{C}$.

One of the conditions of primitive form ϕ (see [51, 5.1] or [52, 4.6]) then says that

$$z\nabla^{\text{G.M.}}_\alpha z\nabla^{\text{G.M.}}_\beta \phi = (C^\gamma_{\alpha\beta} + z\Gamma^\gamma_{\alpha\beta})z\nabla^{\text{G.M.}}_\gamma \phi, \tag{3.32}$$

where $C^\gamma_{\alpha\beta}$ and $\Gamma^\gamma_{\alpha\beta}$ are structure constants of the multiplication and Christoffel coefficients of the metric respectively. Since $\{t^\alpha\}$ are flat coordinates, $\Gamma^\gamma_{\alpha\beta} = 0$ and therefore

$$\boldsymbol{R.H.S} = C^\gamma_{\alpha\beta}\nabla^{\text{G.M.}}_\gamma \phi \tag{3.33}$$
$$= C^\gamma_{\alpha\beta}\partial_\gamma\lambda dx. \tag{3.34}$$

Now by definition the L.H.S. is

$$z\nabla^{\text{G.M.}}_\alpha z\nabla^{\text{G.M.}}_\beta dx = z\nabla^{\text{G.M.}}_\alpha(\partial_\beta\lambda dx) \tag{3.35}$$
$$= [(z\partial_\alpha\partial_\beta\lambda + \partial_\alpha\lambda\partial_\beta\lambda)dx]. \tag{3.36}$$

Using the first equation,

$$\boldsymbol{L.H.S} - \boldsymbol{R.H.S} = [(z\partial_\alpha\partial_\beta\lambda + K_{\alpha\beta}\frac{\partial\lambda}{\partial x})dx]. \tag{3.37}$$

Noticing that

$$(d_x + zd_x\lambda\wedge)K_{\alpha\beta} = \left(\frac{\partial K_{\alpha\beta}}{\partial x} + zK_{\alpha\beta}\frac{\partial\lambda}{\partial x}\right)dx, \tag{3.38}$$

so

$$0 = \boldsymbol{L.H.S} - \boldsymbol{R.H.S} = z\left[(\partial_\alpha\partial_\beta\lambda - \frac{\partial K_{\alpha\beta}}{\partial x})dx\right]. \tag{3.39}$$

The last differential is zero in $\mathcal{H}((z))$ if and only if

$$\partial_\alpha\partial_\beta\lambda - \frac{\partial K_{\alpha\beta}}{\partial x} = 0,$$

namely the second equation holds.

□

Proof of Proposition 3.4.2. Let

$$I(t,\hbar)=\int_\Gamma \mathrm{e}^{\hbar\lambda}\,dx \qquad \text{and} \qquad \xi_\beta=\partial_\beta I.$$

Using the two identities in Proposition 3.4.4, a straightforward computation gives

$$\partial_\alpha\xi_\beta=\hbar\int_\Gamma(\partial_\alpha\partial_\beta\lambda+\hbar\partial_\alpha\lambda\partial_\beta\lambda)\,\mathrm{e}^{\hbar\lambda}\,dx \tag{3.40}$$

$$=\hbar C^\gamma_{\alpha\beta}\xi_\gamma+\hbar\int_\Gamma\frac{\partial}{\partial x}\left(K_{\alpha\beta}\mathrm{e}^{\hbar\lambda}\right)\,dx \tag{3.41}$$

$$=\hbar C^\gamma_{\alpha\beta}\xi_\gamma. \tag{3.42}$$

Thus ξ satisfies (3.21). □

3.4.3 Twisted periods of dual Frobenius manifolds

Proof of Proposition 3.4.3. This argument is based on [32, Proposition 5.1]; the difference here is that the Euler vector field can be more general. We use the two identities established in Proposition 3.4.4. Let

$$I(t,\hbar)=\int_\Gamma\lambda^\hbar dx, \tag{3.43}$$

then taking derivatives gives

$$\partial_\alpha\partial_\beta I/\hbar=\partial_\alpha\int_\Gamma\partial_\beta\lambda\lambda^{\hbar-1}dx \tag{3.44}$$

$$=\int_\Gamma\left[(\partial_\alpha\partial_\beta\lambda)\lambda^{\hbar-1}+(\hbar-1)\partial_\alpha\lambda\partial_\beta\lambda\lambda^{\hbar-2}\right]dx \tag{3.45}$$

$$=(\hbar-1)\sum_\gamma C^\delta_{\alpha\beta}\int_\Gamma\partial_\delta\lambda\lambda^{\hbar-2}dx+\int_\Gamma\left[\frac{\partial K_{\alpha\beta}}{\partial x}\lambda^{\hbar-1}+(\hbar-1)K_{\alpha\beta}\frac{\partial\lambda}{\partial x}\lambda^{\hbar-2}\right]dx\,, \tag{3.46}$$

where the two identities from Proposition 3.4.4 are used in the last line. Notice that the second part of the last equation is integrating an exact form, which by Stokes' theorem is zero. This means

$$\partial_\alpha\partial_\beta I/\hbar=(\hbar-1)\sum_\delta C^\delta_{\alpha\beta}\int_\Gamma\partial_\delta\lambda\lambda^{\hbar-2}dx\,. \tag{3.47}$$

Now we use quasi-homogeneous property of λ to show that I satisfy (3.27). Let

$$E=\sum_\alpha(a_\alpha+b_\alpha t_\alpha)\frac{\partial}{\partial t_\alpha}, \tag{3.48}$$

then $E(\lambda)=\lambda$ implies

$$\lambda=\sum_{\alpha}(a_\alpha+b_\alpha t_\alpha)\partial_\alpha\lambda. \tag{3.49}$$

Multiplying by $\partial_\beta\lambda$ and using the first equation, we get

$$\begin{aligned}\lambda\partial_\beta\lambda&=\sum_{\alpha}(a_\alpha+b_\alpha t_\alpha)\partial_\alpha\lambda\partial_\beta\lambda & (3.50)\\ &=\sum_{\alpha}(a_\alpha+b_\alpha t_\alpha)(\sum_{\gamma}C_{\alpha\beta}^{\gamma}\partial_\gamma\lambda+K_{\alpha\beta}\frac{\partial\lambda}{\partial x}). & (3.51)\end{aligned}$$

Recalling the definition of the matrix $\mathcal{U}$, this gives

$$\sum_{\gamma}\mathcal{U}_\beta^\gamma\partial_\gamma\lambda=\lambda\partial_\beta\lambda-\sum_{\alpha}(a_\alpha+b_\alpha t_\alpha)K_{\alpha\beta}\frac{\partial\lambda}{\partial x}. \tag{3.52}$$

Now, take $\xi=(\partial_1 I,\cdots,\partial_m I)$, then the **L.H.S.** of the equation (3.27) reads

$$\begin{aligned}(\partial_\beta\xi\cdot\mathcal{U})_\alpha/\hbar=&\,\mathcal{U}_\alpha^\epsilon\partial_\beta\xi_\epsilon/\hbar\\ &=\mathcal{U}_\alpha^\epsilon(\hbar-1)C_{\epsilon\beta}^{\gamma}\int_\Gamma\partial_\gamma\lambda\lambda^{\hbar-2}dx\,,\end{aligned}$$

using associativity of $E\circ\partial_\alpha\circ\partial_\beta$, this further equals

$$(\hbar-1)C_{\alpha\beta}^{\delta}\int_\Gamma(\mathcal{U}_\delta^\gamma\partial_\gamma\lambda)\lambda^{\hbar-2}dx. \tag{3.53}$$

Now using (3.52) we just obtained, this equals

$$\begin{aligned}&(\hbar-1)C_{\alpha\beta}^{\delta}\int_\Gamma\partial_\delta\lambda\lambda^{\hbar-1}dx-(\hbar-1)C_{\alpha\beta}^{\delta}\int_\Gamma\sum_{\epsilon}(a_\epsilon+b_\epsilon t_\epsilon)K_{\epsilon\delta}\frac{\partial\lambda}{\partial x}\lambda^{\hbar-2}dx & (3.54)\\ =&(\hbar-1)C_{\alpha\beta}^{\delta}\int_\Gamma\partial_\delta\lambda\lambda^{\hbar-1}dx+C_{\alpha\beta}^{\delta}\int_\Gamma\sum_{\epsilon}(a_\epsilon+b_\epsilon t_\epsilon)\frac{\partial K_{\epsilon\delta}}{\partial x}\lambda^{\hbar-1}dx, & (3.55)\end{aligned}$$

where integration by parts is used in the second line.

Now taking ∂_δ to

$$\lambda=\sum_{\epsilon}(a_\epsilon+b_\epsilon t_\epsilon)\partial_\epsilon\lambda \tag{3.56}$$

and using the second equation, we get

$$\partial_\delta\lambda=b_\delta\partial_\delta\lambda+\sum_{\epsilon}(a_\epsilon+b_\epsilon t_\epsilon)\frac{\partial K_{\epsilon\delta}}{\partial x}, \tag{3.57}$$

which means

$$\sum_{\epsilon}(a_\epsilon+b_\epsilon t_\epsilon)\frac{\partial K_{\epsilon\delta}}{\partial x}=(1-b_\delta)\partial_\delta\lambda. \tag{3.58}$$

Using this, (3.55) is just

$$\sum_{\delta}(\hbar-1+1-b_\delta)C^{\delta}_{\alpha\beta}\int_{\Gamma}\partial_\delta\lambda\lambda^{\hbar-1}dx \tag{3.59}$$

$$=1/\hbar\sum_{\delta}(\hbar-b_\delta)C^{\delta}_{\alpha\beta}\xi_\delta. \tag{3.60}$$

Now recalling that

$$\mathcal{V}=\frac{2-d}{2}-\nabla E=1/2-\mathrm{diag}\{b_1,\cdots,b_m\},$$

so

$$\mathcal{V}-1/2=-\nabla E=-\mathrm{diag}\{b_1,\cdots,b_m\}\,, \tag{3.61}$$

therefore

$$(\partial_\beta\xi\cdot\mathcal{U})_\alpha=\sum_{\delta}(\hbar+\mathcal{V}-1/2)^{\delta}_{\delta}C^{\delta}_{\alpha\beta}\xi_\delta=(\xi\cdot(\hbar+\mathcal{V}-1/2)\cdot C_\beta)_\alpha. \tag{3.62}$$

This is exactly (3.27). □

Remark 3.4.5. Every condition is used in this proof, including $d=1$, which guarantees $\mathcal{V}-1/2=-\nabla E$. In particular, the condition $(d=1, E(\lambda)=\lambda)$ can be replaced by other conditions, for example the one in [32, Proposition 5.1].

3.4.4 Restricting to Frobenius submanifolds

Super-potentials of Kleinian resolutions parameterise **sub-**manifolds of Hurwitz spaces. In order to understand the differential equations of its oscillatory integrals, it's necessary to ask how the system restricts to submanifolds.

For our purpose, we restrict to the case where the restriction is given by restricting flat coordinates $t=(s_1,...,s_m)$ to $s=(s_1,...,s_n)$ for some $n<m$.

Proposition 3.4.6. *If* $\mathrm{span}\{\partial/\partial s_{n+1},\cdots,\partial/\partial s_m\}=\mathrm{span}\{\partial/\partial s_1,\cdots,\partial/\partial s_n\}^{\perp}$, *in order that $I(s,z)$ is a solution to the system*

$$\frac{\partial^2}{\partial s_i\partial s_j}I=\hbar\sum_{k=1}^{n}C^{k}_{ij}\frac{\partial}{\partial s_k}I, 1\le i,j\le n, \tag{3.63}$$

a sufficient condition is that

$$C_{ijk}=0,\qquad\forall\,1\le i,j\le n,\quad k>n. \tag{3.64}$$

Proof. This is obvious. □

Proposition 3.4.7. *The rank of $H_1^{\mathrm{lf}}(\mathbb{P}^1 \setminus (\lambda); \mathcal{L})$ is equal to the number of distinct finite poles and zeroes, minus one.*

Proof. See [63, VIII, Proposition 3.1]. □

Proposition 3.4.8. *For any full Hurwitz Frobenius manifold $\mathcal{H}$, taking local basis $\gamma_i : \mathcal{H} \to V, 1 \le i \le N$ of 1-cycles for the homology bundle, then the induced map*

$$\mathcal{H} \to \mathbb{C}^N : t \mapsto I_i(t) = \int_{\gamma_i} \lambda^\hbar \phi$$

is a local homeomorphism at any semisimple point.

Proof. It suffices to take a special local basis. Near a semisimple point, we can use canonical coordinates $\{u_i\}$. Then using path of steepest descent to represent the cycles, each integral has $u_i^\hbar$ as the leading term of the asymptotic expansion, which are algebraically independent because u_i are coordinates. □

3.4.5 Application I: twisted periods satisfy quantum differential equation

Consider the Hurwitz space $\mathcal{H} = \mathcal{H}_{0,\mathfrak{m}}$, where $\mathfrak{m} = (m-3, m-3, 1, 1)$. This is the space of covers of $\mathbb{P}^1$ with ramification profile over ∞ of type $\mathfrak{m}$: there are four preimages of ∞ with λ having order $m_i + 1$ at each preimage. By a Riemann-Hurwitz formula computation, this is a space of dimension:

$$2g + 2m + \sum m_i = 2m + 2. \tag{3.65}$$

A general element of this space is given by

$$\lambda = \mathrm{e}^{x_0} \frac{\prod_{i=1}^m (\mu - \mathrm{e}^{x_i})(\mu - \mathrm{e}^{y_i})}{\mu^{m-2}(\mu+1)^2(\mu - e^{y_0})^2}. \tag{3.66}$$

On $\mathbb{P}^1$ there is an involution:

$$\theta : \mu \mapsto 1/\mu \tag{3.67}$$

which acts on $\mathcal{H}$. Take $\mathcal{M}$ as its invariant subset. It's easy to see that $\mathcal{M}$ is an $m+1$ dimensional space with a general element given by:

$$\widetilde{\lambda} = \mathrm{e}^{x_0} \frac{\prod_{i=1}^m (\mu - \mathrm{e}^{x_i})(\mu - \mathrm{e}^{-x_i})}{\mu^{m-2}(\mu+1)^2(\mu - 1)^2}. \tag{3.68}$$

From this expression it's easy to see the conformal dimension is $d = 1$, with $E = \partial/\partial x_0$.

Take $(x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_m)$ as the coordinates, then $\mathcal{M} = \{y_i = -x_i, 1 \le i \le m, y_0 = 0\} \subset \mathcal{H}$.

We want to use the theories we have discussed so far to study $\int_\Gamma \widetilde{\lambda}^\hbar d\mu/\mu$. Here we take Γ as an element of $H_1^{\mathrm{lf}}(\mathbb{P}^1 \setminus (\lambda); \mathcal{L})$.

Proposition 3.4.9. *Take*

$$e_0 = \partial/\partial x_0, f_0 = \partial/\partial y_0, \tag{3.69}$$
$$e_k = \partial/\partial x_{2k-1} - \partial/\partial x_{2k}, f_k = \partial/\partial x_{2k-1} + \partial/\partial x_{2k}, 1 \leq k \leq m. \tag{3.70}$$

Restricting these vector fields to M, then the following statements hold:

(1) $\mathcal{TM} = \mathrm{span}\{e_0, e_1, \ldots, e_m\}$*;*

(2) $\mathcal{TM}^\perp = \mathrm{span}\{f_0 + e_0, f_k - e_0, 1 \leq k \leq m\}$*.*

Proof. (1) is obvious, (2) follows from straightforward computation. □

Corollary 3.4.10. *For any $\Gamma \in H_1^{\mathrm{lf}}(\mathbb{P}^1 \setminus (\lambda); \mathcal{L})$, the integral $\int_\Gamma \widetilde{\lambda}^\hbar d\mu/\mu$ satisfies the system (3.27) for the twisted periods of the corresponding dual Frobenius manifold.*

Proof. By Proposition 3.4.3, the statement is true for the whole Hurwitz space. Now apply Proposition 3.4.6. □

The next question is to find enough 1-cycles.

Proposition 3.4.11. *There exist local sections $s_k : 0 \leq k \leq n$ of homology bundle, such that the induced map*

$$\mathcal{M} \to \mathbb{C}^{m+1} : t \mapsto I_i(t) = \int_{s_i} \widetilde{\lambda}^\hbar \phi$$

is a local homeomorphism.

Proof. Using Proposition 3.4.8, choosing a basis of $2m+2$ cycles induces a local homeomorphism, at a small neighborhood in $\mathcal{H}$. $\mathcal{M}$ is therefore locally homeomorphic to its image. Now using the involution θ, we can choose these basis elements in pairs with an additional condition $\gamma_{2k} = \gamma_{2k-1} \circ \theta$. Then, because

$$\widetilde{\lambda}(1/\mu) = \widetilde{\lambda}(\mu), \quad d\log(1/\mu) = -d\log(1/\mu), \tag{3.71}$$

the image of $\int_{\gamma_{2k-1}}, 1 \leq k \leq m+1$ is locally homeomorphic to $\mathcal{M}$.

For the last argument: let

$$f_k = \int_{\gamma_{2k-1}} \widetilde{\lambda}^\hbar \phi, \qquad g_k = \int_{\gamma_{2k}} \widetilde{\lambda}^\hbar \phi, 1 \leq k \leq m+1, \tag{3.72}$$

then they give local coordinates of $\mathcal{H}$. By the inverse function theorem, $\{f_k + g_k\}_{k=1}^{n+1}$ cut out a submanifold X of half dimension. On the other side, by (3.71),

$$(f_k + g_k)\big|_{\mathcal{M}} = 0 \tag{3.73}$$

hence $\mathcal{M} \subset X$. Since $\dim \mathcal{M} = \dim X$, $\mathcal{M}$ is locally homeomorphic to X. As a conclusion, $f_1, \ldots, f_{m+1}$ give local coordinates of $\mathcal{M}$. □

Proposition 3.4.12. *$\{s_i\}$ can be chosen as $\{[0, e^{-x_1}], \ldots, [0, e^{-x_m}], [0, 1]\}$.*

Proof. From the proof of Proposition 3.4.11, it suffices to show that the cycles s_i and $s_i \circ \theta$ generate the relative homology group

$$H_1^{\mathrm{lf}}(\mathbb{P}^1 \setminus (\lambda); \mathcal{L}).$$

Taking linear combinations of

$$[0, e^{-x_1}], \ldots, [0, e^{-x_m}], [0, 1]$$

gives

$$[1, e^{-x_1}], \ldots, [1, e^{-x_m}], [1, 0].$$

Applying the involution $\theta(\mu) = 1/\mu$ gives the further cycles

$$[1, e^{x_1}], \ldots, [1, e^{x_m}], [1, \infty].$$

Together these are the standard basis of $2m + 2$ cycles used in the proof of Proposition 3.4.8. Hence the proposed cycles and their images under θ generate the required relative homology group. $\square$

Proof of Theorem 3.1.4. By Propositions 3.4.11 and 3.4.12, the periods over these explicit contours give a framing of local deformed flat coordinates on $\mathcal{M}$. By Corollary 3.4.10, their gradients satisfy the dual quantum differential equation. Hence Claim 3.1.2 holds. $\square$

3.4.6 Application II: hypergeometric form of twisted periods

An immediate application of Section 3.4.5 is the following

Proposition 3.4.13. *Let*

$$\lambda(\kappa_1, \cdots, \kappa_m; \mu) = \kappa_0 \frac{\prod_{i=1}^{m} (\mu - \kappa_i)(\mu - \kappa_i^{-1})}{\mu^{m-2}(\mu^2 - 1)^2}, \qquad \phi = \frac{d\mu}{\mu} \tag{3.74}$$

$$\gamma_k = [0, 1/\kappa_k], k = 1, \cdots, m; \gamma_{m+1} = [0, 1]. \tag{3.75}$$

Further let Z be the minimal resolution of the D_m Kleinian singularity with the $\mathbb{C}^\times$-action induced by the diagonal one on $\mathbb{C}^2$ as in Chapter 2. Then, the associated periods

$$\Pi_i = \int_{\gamma_i} \lambda^{2\nu/z} \phi, \tag{3.76}$$

give a framing of deformed flat coordinates of the Dubrovin connection of the $\mathbb{C}^\times$ equivariant quantum cohomology $\mathrm{QH}^*_{\mathbb{C}^\times}(Z)$, under the identification as in (2.29).

Remark 3.4.14. By the discussions in Section 3.4.5, these integrals give framing of deformed flat coordinates of the Dubrovin connection of the Frobenius structure associated to $(log(\lambda), \phi)$. By Chapter 2, this Frobenius structure is isomorphic to $\mathrm{QH}^*_{\mathbb{C}^\times}(Z)$. The scaling $\hbar = 2\nu$ is in order to match exactly the equivariant parameter in the Dubrovin connection of $\mathrm{QH}^*_{\mathbb{C}^\times}(Z)$.

Pochhammer integrals (3.76) of powers of rational functions are Lauricella functions (see Appendix A.1 for details). Applying to the periods (3.76), we get the following

Proposition 3.4.15. *The integrals* Π_i *can be expressed as:*

$$\begin{aligned}\Pi_i =& \frac{\Gamma(-\frac{2\nu(m-2)}{z})\Gamma(\frac{2\nu}{z}+1)}{\Gamma(1-\frac{2\nu(m-3)}{z})}\kappa_i^{\frac{2\nu(m-2)}{z}} \\ & F_D^{(2m+1)}\Big(-\frac{2\nu(m-2)}{z}; -\frac{2\nu}{z}, \cdots, -\frac{2\nu}{z}, \frac{4\nu}{z}, \frac{4\nu}{z}, -\frac{2\nu}{z}, \cdots, -\frac{2\nu}{z}; 1-\frac{2\nu(m-3)}{z} \\ & \Big| \frac{1}{\kappa_1\kappa_i}, \cdots, \frac{1}{\kappa_m\kappa_i}, \frac{1}{\kappa_i}, -\frac{1}{\kappa_i}, \frac{\kappa_m}{\kappa_i}, \cdots, \frac{\hat{\kappa}_i}{\kappa_i}, \cdots, \frac{\kappa_1}{\kappa_i}\Big),\end{aligned} \tag{3.77}$$

and

$$\begin{aligned}\Pi_{m+1} =& \frac{\Gamma(-\frac{2\nu(m-2)}{z})\Gamma(1-\frac{4\nu}{z})}{\Gamma(1-\frac{2\nu m}{z})} \\ & F_D^{(2m+1)}\Big(-\frac{2\nu(m-2)}{z}; -\frac{2\nu}{z}, \cdots, -\frac{2\nu}{z}, \frac{4\nu}{z}, -\frac{2\nu}{z}, \cdots, -\frac{2\nu}{z}; 1-\frac{2\nu m}{z} \\ & \Big| \frac{1}{\kappa_1}, \cdots, \frac{1}{\kappa_m}, -1, \kappa_m, \cdots, \kappa_1\Big).\end{aligned} \tag{3.78}$$

§ 3.5 Symplectic transformation for type D Kleinian singularities

In Proposition 3.5.1 and Definition 3.5.2, we use the common period framing to define the analytic-continuation transformation $\mathbb{U}_\rho$ and reduce its computation to the two comparison matrices $\mathcal{A}$ and $\mathcal{B}$. The remainder of Section 3.5.2 explains how to compute these matrices from the local behaviour at LR and OP. To compare this transformation with Iritani's integral structure, Proposition 3.5.5 expresses it in terms of the corresponding map on equivariant K-groups. We then specialise to D_3: Proposition 3.5.7 supplies the large-radius asymptotics that determine the resolution-side comparison matrix $\mathcal{A}$, while evaluation of the periods at the orbifold point determines $\mathcal{B}$. Proposition 3.5.8 computes these matrices explicitly, and Propositions 3.5.9 and 3.5.11 compare the result with the equivariant and non-equivariant integral structures.

3.5.1 Symplectic transformation from mirror symmetry

We have shown that twisted periods Π_i give a framing of the basis for the deformed flat coordinates of the Dubrovin connection over $\mathcal{M}$. Moreover, $\mathcal{M}$ is locally isomorphic to quantum cohomologies of Z and $\mathcal{X}$. This in particular means that Π_i give deformed flat coordinates of the quantum $\mathcal{D}-$module on the corresponding neighborhood. Following (3.96), we analytically continue Π_i along a path ρ in the domain

$$\frac{1}{\kappa_k} \ll 1, \frac{\kappa_{i+1}}{\kappa_i} \ll 1, \quad 1 \leqslant k \leq m, 1 \leqslant i \leq m-1. \tag{3.79}$$

to obtain Π_i^σ, then it follows that

Proposition 3.5.1. *There are matrices $\mathcal{A}$ and $\mathcal{B}$ with entries in $\mathbb{C}(\!(1/z)\!)$ such that*

$$\mathcal{A}\{\Pi_1^\rho, \cdots, \Pi_{m+1}^\rho\}^T = \{J_1^Z \cdots, J_{m+1}^Z\}^T \tag{3.80}$$

on $U^Z \subset \mathcal{M}$ and

$$\mathcal{B}\{J_1^{\mathcal{X}} \cdots, J_{m+1}^{\mathcal{X}}\}^T = \{\Pi_1, \cdots, \Pi_{m+1}\}^T \tag{3.81}$$

on $U^{\mathcal{X}} \subset \mathcal{M}$.

Thus Proposition 3.5.1 reduces the construction of $\mathbb{U}_\rho$ to determining the two comparison matrices $\mathcal{A}$ and $\mathcal{B}$ from the local behaviour of the same period framing at LR and OP.

Definition 3.5.2. Let

$$\mathbb{U}_\rho^{\text{mat}} := \mathcal{A}\mathcal{B}. \tag{3.82}$$

In the bases $\{\delta_i\}_{i=1}^{m+1}$ of $\mathcal{H}(\mathcal{X})$ and $\{\phi_i\}_{i=1}^{m+1}$ of $\mathcal{H}(Z)$, this matrix defines a $\mathbb{C}(\!(1/z)\!)$-linear map $\mathbb{U}_\rho \colon \mathcal{H}(\mathcal{X}) \to \mathcal{H}(Z)$ by

$$\{\delta_1, \cdots, \delta_{m+1}\} \mapsto \{\phi_1, \cdots, \phi_{m+1}\}\mathbb{U}_\rho^{\text{mat}}. \tag{3.83}$$

Remark 3.5.3. Varying the choice of path results in a transformation of $\mathcal{A}$ by an associated monodromy matrix. Explicitly, let ρ and ρ' be two paths from OP to LR, then there exists $\mathbb{M}$ such that

$$\{\Pi_1^\rho, \cdots, \Pi_{m+1}^\rho\}^T = \mathbb{M}\{\Pi_1^{\rho'}, \cdots, \Pi_{m+1}^{\rho'}\}^T, \tag{3.84}$$

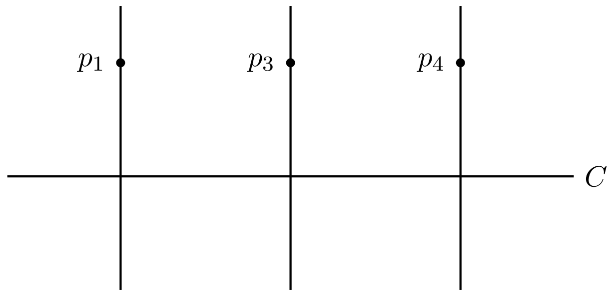
resulting

$$\mathcal{A}_{\rho'} = \mathcal{A}_\rho \mathbb{M}. \tag{3.85}$$

3.5.2 Computing $\mathbb{U}$ by comparing local behaviours

The comparison between twisted periods and J_Z near LR is achieved by comparing monodromies. For this purpose, it's convenient to use the localised basis of $H(Z)$.

The fixed points of $\mathbb{C}^\times$ on Z consist of $m-1$ points and 1 curve. We denote the central curve as C and label p_i from the left to right following the indices in the Dynkin diagram as in Section 1.1.3. For example, in the D_4 case they are labelled as follows:



Let $i : Z^{\mathbb{C}^\times} \subset Z$ be the inclusion, then the localisation morphism

$$i^* : \ H^*_{\mathbb{C}^\times}(Z) \to H^*_{\mathbb{C}^\times}(Z^{\mathbb{C}^\times}) \cong \oplus_{i=1}^{m-3} H^*(p_i) \bigoplus H^*(C) \bigoplus \oplus_{i=m-1}^{m} H^*(p_i) \tag{3.86}$$

is an isomorphism of vector spaces over $\mathrm{Frac}(\mathrm{H}^*_{\mathbb{C}^\times}(\mathrm{pt})) = \mathbb{C}(\nu)$.

Take $\{\psi_i = c_1^{\mathbb{C}^\times}(\mathcal{N}_{E_i \setminus Z})\}_{i=1}^m \cup \{\mathbf{1}_Z\}$ as the basis for $H^*_{\mathbb{C}^\times}(Z)$. Let $P_i, i \neq m-2$ be the fundamental class of p_i, $\mathbf{1}_C$ be the fundamental class of C and $\omega_C = c_1(\mathcal{O}_C(1))$.

The image of $\mathbf{1}_Z$ is just the identity

$$i^*(\mathbf{1}_Z) = \sum_{\substack{1 \le j \le m \\ j \neq m-2}} P_j + \mathbf{1}_C, \tag{3.87}$$

and the image of ψ_i can be computed using weights of the $\mathbb{C}^\times$ on $\mathcal{N}_{E_i \setminus Z}$. Notice that

(W1) The tangent weights at intersection points of exceptional curves add to 2ν;

(W2) If an irreducible curve has two fixed points, then tangent weights along the curve adds to 0,

the images of ψ_i can therefore computed as

$$i^*(\psi_i) = \begin{cases} w_i^L \ P_i + w_i^R \ P_{i+1} & 1 \le i \le m-4 \\ 4\nu \ P_i + 1 \ \omega_C & i = m-3, \ m-1, \ m \ , \\ -2 \ \omega_C + 2\nu \ \mathbf{1}_C & \end{cases} \tag{3.88}$$

where $\{w_i^L, w_i^R\}_{i=1}^{M-4}$ are the unique solutions to

$$2\nu - w_i^L = -2\nu + w_i^R \ , \ w_i^R + w_{i+1}^L = 2\nu \ , \ w_{m-4}^R = -2\nu. \tag{3.89}$$

We now compute the components of J_Z. From (1.49), it's easy to see

$$J^{\mathrm{sm},Z}(\tau', z) \sim z e^{\tau'/z} \mathbf{1}_Z \ , \ \tau' \to \mathrm{LR}. \tag{3.90}$$

Let τ' be a general point of $H^2(Z)$, then writing τ' in the chosen basis yields

$$i^*(\tau') =: \sum_{\substack{1\leq j\leq m\\ j\neq m-2}} L^j(\tau')\, P_j + L^{m-2}(\tau')\, \mathbf{1}_C + L^C(\tau')\, \omega_C/z. \tag{3.91}$$

It then follows that near

$$\mathrm{LR} = \mathrm{Re}\left\{ \int_{E_i} \tau' \mapsto -\infty, 1 \leq i \leq m \right\}, \tag{3.92}$$

the components of it in localised basis

$$J^{\mathrm{sm},Z}(\tau', z) = \sum_{\substack{1\leq j\leq m\\ j\neq m-2}} J_j^{\mathrm{sm},Z}\, P_j + J_{m-2}^{\mathrm{sm},Z}\, \mathbf{1}_C + J_C^{\mathrm{sm},Z}\, \omega_C \tag{3.93}$$

satisfy

$$J_j^{\mathrm{sm},Z} \sim \begin{cases} z\, \mathrm{e}^{L^j(\tau')/z} & j \neq m-2 \\ z\, \mathrm{e}^{L^{m-2}(\tau')/z} & j = m-2 \end{cases}, \tag{3.94}$$

and

$$J_C^{\mathrm{sm},Z} \sim L^C(\tau'). \tag{3.95}$$

On the mirror side, the twisted periods also serve as a framing of deformed flat coordinates, which are given by explicit hypergeometric functions as in Proposition 3.4.15.

Note that the large radius limit sends the parameters $\kappa_1, \cdots, \kappa_m$ to the domain:

$$\frac{1}{\kappa_k} \ll 1, \frac{\kappa_{i+1}}{\kappa_i} \ll 1, \quad 1 \leqslant k \leq m, 1 \leqslant i \leq m-1. \tag{3.96}$$

Thus their leading asymptotic behaviours can be explicitly described, using connecting formulas developed in Appendix A.3.

We now turn to the comparison at OP. Since the components of $J^{\mathcal{X}}$ at OP satisfy

$$J^{\mathcal{X}}(0, z) = z\, \mathbf{1}_0, \tag{3.97}$$

$$\frac{\partial J^{\mathcal{X}}}{\partial x_k}(0, z) = \mathbf{1}_k, \tag{3.98}$$

and $\mathcal{B}$ doesn't depend on x_i, to compute $\mathcal{B}$, it suffices to study the evaluation and derivatives of twisted periods at OP.

Remark 3.5.4. The superpotential, and therefore twisted periods in fact degenerate a lot at OP. Using (2.29) and (3.15), one find

$$\kappa_k\big|_{\mathrm{OP}} = \begin{cases} -1 & k = 1 \\ 1 & k = m \\ \mathrm{e}^{-i\pi(2m-2k-1)/(2m-4)} & 2 \leq k \leq m-1 \end{cases}, \tag{3.99}$$

and therefore

$$\lambda\big|_{\mathrm{OP}} = \mu^{m-2} + \frac{1}{\mu^{m-2}}. \tag{3.100}$$

3.5.3 Symplectic transformation from integral structure

Recall that Iritani proposes the following commuting diagram (1.53):

$$\begin{tikzcd} K_{\mathbb{C}^\times}(\mathcal{X}) \arrow[r, "\mathbb{U}_K"] \arrow[d, "\widehat{\Psi}_{\mathcal{X}}"] & K_{\mathbb{C}^\times}(Z) \arrow[d, "\widehat{\Psi}_Z"] \\ \mathcal{H}(\mathcal{X}) \arrow[r, "\mathbb{U}"] & \mathcal{H}(Z) \end{tikzcd} \tag{3.101}$$

where $\mathbb{U}$ induces an isomorphism of quantum $\mathcal{D}-$modules in the Coates–Ruan conjecture, and is induced by an isometry $\mathbb{U}_K$ of K-groups via the vertical integral-structure framing maps $\widehat{\Psi}_{\mathcal{X}}$ and $\widehat{\Psi}_Z$ defined in (1.52).

Proposition 3.5.5. *Let $\{\mathscr{L}_i^Z\}_{i=1}^{m+1}$ (resp. $\{\mathscr{L}_i^{\mathcal{X}}\}_{i=1}^{m+1}$) be a basis of $K_{\mathbb{C}^\times}(Z)$ (resp. $K_{\mathbb{C}^\times}(\mathcal{X})$), and let $\{\phi_i\}_{i=1}^{m+1}$ (resp. $\{\delta_i\}_{i=1}^{m+1}$) be a basis of $H(Z)$ (resp. $H(\mathcal{X})$). Suppose the map $\mathbb{U}$ is induced by an isomorphism $\mathbb{U}_K$ of $K-$groups. Define their matrices in these bases by*

$$\{\delta_1, \cdots, \delta_{m+1}\} \mapsto \{\phi_1, \cdots, \phi_{m+1}\}\mathbb{U}^{\mathrm{mat}} \tag{3.102}$$

and

$$\{\mathscr{L}_1^{\mathcal{X}}, \cdots, \mathscr{L}_{m+1}^{\mathcal{X}}\} \mapsto \{\mathscr{L}_1^Z, \cdots, \mathscr{L}_{m+1}^Z\}\mathbb{U}_K^{\mathrm{mat}}. \tag{3.103}$$

Then the matrix of $\mathbb{U}$ can be computed as

$$\mathbb{U}^{\mathrm{mat}} = \Gamma_Z \,\mathrm{ch}_Z\, \mathbb{U}_K^{\mathrm{mat}}\, \mathrm{ch}_{\mathcal{X}}^{-1}\, \Gamma_{\mathcal{X}}^{-1}, \tag{3.104}$$

where the matrices $\Gamma_{\mathcal{X}}$, $\mathrm{ch}_{\mathcal{X}}$ and Γ_Z, ch_Z are determined by:

$$(\delta_1, \cdots, \delta_{m+1}) \cup \widehat{\Gamma}(T_{\mathcal{X}}) = (\delta_1, \cdots, \delta_{m+1})\Gamma_{\mathcal{X}}, \tag{3.105}$$
$$(\phi_1, \cdots, \phi_{m+1}) \cup \widehat{\Gamma}(T_Z) = (\phi_1, \cdots, \phi_{m+1})\Gamma_Z, \tag{3.106}$$
$$(\mathrm{ch}(\mathscr{L}_1^{\mathcal{X}}), \cdots, \mathrm{ch}(\mathscr{L}_{m+1}^{\mathcal{X}})) = (\delta_1, \cdots, \delta_{m+1})\mathrm{ch}_{\mathcal{X}}, \tag{3.107}$$
$$(\mathrm{ch}(\mathscr{L}_1^Z), \cdots, \mathrm{ch}(\mathscr{L}_{m+1}^Z)) = (\phi_1, \cdots, \phi_{m+1})\mathrm{ch}_Z. \tag{3.108}$$

Remark 3.5.6. For minimal resolution of Kleinian singularities, they are canonical choices of basis for K-groups. Let $\{\rho\}$ be the set of irreducible representations of the Kleinian subgroup that defines $\mathcal{X}$. There is a canonical vector bundle $L_\rho \in K(Z)$ such that

$$\mathcal{O}_\rho := \mathcal{O}_{\mathbb{C}^2} \otimes \rho \in K(\mathcal{X}) \longleftrightarrow L_\rho \tag{3.109}$$

induces an isomorphism of K-groups.

Moreover, denote by E_ρ the irreducible component of the exceptional E in Z that corresponds to an nontrivial irreducible ρ under the McKay correspondence, then the isomorphism above sends $\mathcal{O}_0 \otimes \rho \in K_0(\mathcal{X})$ to $\mathcal{O}_{E_\rho}(-1) \in K_E(Z)$, and $\mathcal{O}_0$ to $\mathcal{O}_E(-1)$.

3.5.4 Computing $\mathbb{U}$: D_3 case

The D_3 singularity is also the A_3 singularity, where the Kleinian subgroup is $\mathbb{Z}_4$. In this case, much of the computation about the integral structure central charge can be found in [13], with the equivariant parameters α_1 and α_2 there restricted to

$$\alpha_1 = \alpha_2 = 1/2\ \nu. \tag{3.110}$$

For $D_3 = A_3$, we use the following basis for the $K-$groups and compute their integral structure central charges.

On the orbifold side, the irreducible representations $\rho_k(\zeta) := \zeta^k$, with $\zeta = \mathrm{e}^{2\pi\mathrm{i}/4}$ give a basis

$$\{\mathcal{O}_{-k} = \mathcal{O}_{\mathbb{C}^2} \otimes \rho_{-k}\}_{k=0}^{3} \tag{3.111}$$

of $K([\mathbb{C}^2/\mathbb{Z}_4])$. The central charge under this basis is therefore

$$\begin{aligned}
&\left(z^{-\frac{1}{2}\deg}\widehat{\Gamma}(T_{\mathcal{X}})\right) \cup \left((2\pi\mathrm{i})^{\frac{1}{2}\deg}\,\mathrm{inv}^*\mathrm{ch}(\mathcal{O}_j)\right) \\
=&\Big(\sum_{v=0}^{3} \zeta^{-vj}\mathbf{1}_v\Big) \cup \Big(\Gamma\left(1-\frac{1}{2}\frac{\nu}{z}\right)^2 \mathbf{1}_0 \\
&+\sum_{v=1}^{3}\Gamma\left(1-\frac{v}{4}-\frac{1}{2}\frac{\nu}{z}\right)\Gamma\left(\frac{v}{4}-\frac{1}{2}\frac{\nu}{z}\right)\mathbf{1}_v/z\Big) \qquad (3.112)\\
=&\Gamma\left(1-\frac{1}{2}\frac{\nu}{z}\right)^2 \mathbf{1}_0 \\
&+\sum_{v=1}^{3}\Gamma\left(1-\frac{v}{4}-\frac{1}{2}\frac{\nu}{z}\right)\Gamma\left(\frac{v}{4}-\frac{1}{2}\frac{\nu}{z}\right)\zeta^{-vj}\,\mathbf{1}_v/z. \qquad (3.113)
\end{aligned}$$

On the resolution side, a basis is given by

$$\{\mathcal{O}(\phi_k)\}_{k=1}^{3} \cup \{\mathcal{O}_Z\}, \tag{3.114}$$

where $\mathcal{O}(\phi_k)$ restrict to $\mathcal{O}(1)$ on E_k and trivial bundle on the other curves. The localisation of the dual basis $\psi_i := c_1(\mathcal{N}_{E_i\backslash Z})$ reads as

$$i^*(\psi_1) = 2\nu\ \mathbf{1}_C - 2\ \omega_C, \tag{3.115}$$
$$i^*(\psi_2) = \omega_C + 4\nu\ P_2, \tag{3.116}$$
$$i^*(\psi_3) = \omega_C + 4\nu\ P_3. \tag{3.117}$$

Using Iritani's integral-structure framing [43, Definition 2.11],

$$\big(\phi_1, \phi_2, \phi_3\big) = -\big(\psi_1, \psi_2, \psi_3\big)C_{D_3}^{-1}, \tag{3.118}$$

the localisation of the basis ϕ_i is given by

$$i^*(\phi_1) = -2\nu\ \mathbf{1}_C + 1\ \omega_C - 2\nu\ P_2 - 2\nu\ P_3, \tag{3.119}$$
$$i^*(\phi_2) = -\nu\ \mathbf{1}_C - 3\nu\ P_2 - 1\nu\ P_3, \tag{3.120}$$
$$i^*(\phi_3) = -\nu\ \mathbf{1}_C - 1\nu\ P_2 - 3\nu\ P_3. \tag{3.121}$$

It follows that their Chen characters are

$$\mathrm{ch}(\mathcal{O}(\phi_1)) = \mathrm{e}^{2\pi\mathrm{i}(-2\nu)}\ \mathbf{1}_C\ (e^{2\pi\mathrm{i}\omega_C}) + \mathrm{e}^{2\pi\mathrm{i}(-2\nu)}\ P_2 + \mathrm{e}^{2\pi\mathrm{i}(-2\nu)}\ P_3 \tag{3.122}$$
$$= e^{2\pi\mathrm{i}(-2\nu)}\ (\mathbf{1}_C + 2\pi\mathrm{i}\,\omega_C) + \mathrm{e}^{2\pi\mathrm{i}(-2\nu)}\ P_2 + \mathrm{e}^{2\pi\mathrm{i}(-2\nu)}\ P_3, \tag{3.123}$$
$$\mathrm{ch}(\mathcal{O}(\phi_2)) = \mathrm{e}^{2\pi\mathrm{i}(-\nu)}\ \mathbf{1}_C + \mathrm{e}^{2\pi\mathrm{i}(-3\nu)}\ P_2 + \mathrm{e}^{2\pi\mathrm{i}(-1\nu)}\ P_3, \tag{3.124}$$
$$\mathrm{ch}(\mathcal{O}(\phi_3)) = \mathrm{e}^{2\pi\mathrm{i}(-\nu)}\ \mathbf{1}_C + \mathrm{e}^{2\pi\mathrm{i}(-1\nu)}\ P_2 + \mathrm{e}^{2\pi\mathrm{i}(-3\nu)}\ P_3. \tag{3.125}$$

Moreover, the Chen character of the trivial bundle $\mathcal{O}_Z$ is simply

$$\mathrm{ch}(\mathcal{O}_Z) = \mathbf{1}_C + P_2 + P_3. \tag{3.126}$$

Localisation also allows computing Gamma class of Z. Using that the tangent weights at P_i are $\{4\nu, -2\nu\}$ and is 2ν on $\mathcal{N}_{C\backslash Z}$, the Gamma class of Z is given by

$$\begin{aligned}\Big(z^{-\frac{1}{2}\deg}\,\widehat{\Gamma}(T_Z)\Big) =&\Gamma\Big(1+\frac{4\nu}{z}\Big)\Gamma\Big(1-\frac{2\nu}{z}\Big)\ P_2 + \Gamma\Big(1+\frac{4\nu}{z}\Big)\Gamma\Big(1-\frac{2\nu}{z}\Big)\ P_3\\ &+\Gamma\Big(1+\frac{2\omega_c}{z}\Big)\Gamma\Big(1+\frac{2\nu-2\omega_C}{z}\Big)\big|_C,\end{aligned} \tag{3.127}$$

where the last term follows from the decomposition $T_Z\big|_C = T_C \oplus \mathcal{N}_{C\backslash Z}$ and

$$c_1^{\mathbb{C}^\times}(T_C) = 2\omega_C,\ c_1^{\mathbb{C}^\times}(\mathcal{N}_{C\backslash Z}) = -2\omega_C + 2\nu. \tag{3.128}$$

Moreover, using

$$\Gamma(1+y+x) \sim \Gamma(1+y)\left[1+\psi(1+y)x + O(x^2)\right], \tag{3.129}$$
$$\Gamma(1+y) \sim 1 - \gamma\, y + O(y^2), \tag{3.130}$$

where γ is the Euler's constant and $\psi(y) = \Gamma'(y)/\Gamma(y)$ is the digamma function, the last term of the Gamma class can be expanded as

$$\Gamma\Big(1+\frac{2\omega_C}{z}\Big)\Gamma\Big(1+\frac{2\nu-2\omega_C}{z}\Big)\big|_C \tag{3.131}$$
$$=\Big[1-2\gamma\frac{\omega}{z}\Big]\,\Gamma\Big(1+\frac{2\nu}{z}\Big)\,\Big[1+\psi(1+\frac{2\nu}{z})\times(-2\frac{\omega_C}{z})\Big] \tag{3.132}$$
$$=\Gamma\Big(1+\frac{2\nu}{z}\Big)\,\Big[\mathbf{1}_C + \big(-2\gamma - 2\psi(1+\frac{2\nu}{z})\big)\frac{\omega_C}{z}\Big]. \tag{3.133}$$

Combining these, the central charge of the resolution under the given basis is given by

$$\begin{aligned}
&\Big(z^{-\frac{1}{2}\deg}\,\widehat{\Gamma}(T_Z)\Big)\cup\Big((2\pi\mathrm{i})^{\frac{1}{2}\deg}\,\mathrm{ch}(\mathcal{O}(\phi_1))\Big)\\
=&\Gamma\Big(1+\frac{4\nu}{z}\Big)\Gamma\Big(1-\frac{2\nu}{z}\Big)e^{2\pi\mathrm{i}(-2\nu)}\;P_2+\Gamma\Big(1+\frac{4\nu}{z}\Big)\Gamma\Big(1-\frac{2\nu}{z}\Big)e^{2\pi\mathrm{i}(-2\nu)}\;P_3\\
&+\Gamma\Big(1+\frac{2\nu}{z}\Big)\left[\mathbf{1}_C+\big(-2\gamma-2\psi(1+\frac{2\nu}{z})\big)\frac{\omega_C}{z}\right]\left[e^{2\pi\mathrm{i}(-2\nu)}\;(\mathbf{1}_C+2\pi\mathrm{i}\omega_C)\right]\\
=&\Gamma\Big(1+\frac{4\nu}{z}\Big)\Gamma\Big(1-\frac{2\nu}{z}\Big)e^{2\pi\mathrm{i}(-2\nu)}\;P_2+\Gamma\Big(1+\frac{4\nu}{z}\Big)\Gamma\Big(1-\frac{2\nu}{z}\Big)e^{2\pi\mathrm{i}(-2\nu)}\;P_3\\
&+\Gamma\Big(1+\frac{2\nu}{z}\Big)e^{2\pi\mathrm{i}(-2\nu)}\left[\mathbf{1}_C+\big(-2\gamma-2\psi(1+\frac{2\nu}{z})+2\pi\mathrm{i}\big)\frac{\omega_C}{z}\right],
\end{aligned}\tag{3.134}$$

$$\begin{aligned}
&\Big(z^{-\frac{1}{2}\deg}\,\widehat{\Gamma}(T_Z)\Big)\cup\Big((2\pi\mathrm{i})^{\frac{1}{2}\deg}\,\mathrm{ch}(\mathcal{O}(\phi_2))\Big)\\
=&\Gamma\Big(1+\frac{4\nu}{z}\Big)\Gamma\Big(1-\frac{2\nu}{z}\Big)e^{2\pi\mathrm{i}(-3\nu)}\;P_2+\Gamma\Big(1+\frac{4\nu}{z}\Big)\Gamma\Big(1-\frac{2\nu}{z}\Big)e^{2\pi\mathrm{i}(-1\nu)}\;P_3\\
&+\Gamma\Big(1+\frac{2\nu}{z}\Big)e^{2\pi\mathrm{i}(-\nu)}\left[\mathbf{1}_C+\big(-2\gamma-2\psi(1+\frac{2\nu}{z})\big)\frac{\omega_C}{z}\right],
\end{aligned}\tag{3.135}$$

$$\begin{aligned}
&\Big(z^{-\frac{1}{2}\deg}\,\widehat{\Gamma}(T_Z)\Big)\cup\Big((2\pi\mathrm{i})^{\frac{1}{2}\deg}\,\mathrm{ch}(\mathcal{O}(\phi_3))\Big)\\
=&\Gamma\Big(1+\frac{4\nu}{z}\Big)\Gamma\Big(1-\frac{2\nu}{z}\Big)e^{2\pi\mathrm{i}(-1\nu)}\;P_2+\Gamma\Big(1+\frac{4\nu}{z}\Big)\Gamma\Big(1-\frac{2\nu}{z}\Big)e^{2\pi\mathrm{i}(-3\nu)}\;P_3\\
&+\Gamma\Big(1+\frac{2\nu}{z}\Big)e^{2\pi\mathrm{i}(-\nu)}\left[\mathbf{1}_C+\big(-2\gamma-2\psi(1+\frac{2\nu}{z})\big)\frac{\omega_C}{z}\right],
\end{aligned}\tag{3.136}$$

$$\begin{aligned}
&\Big(z^{-\frac{1}{2}\deg}\,\widehat{\Gamma}(T_Z)\Big)\cup\Big((2\pi\mathrm{i})^{\frac{1}{2}\deg}\,\mathrm{ch}(\mathcal{O}_Z)\Big)\\
=&\Gamma\Big(1+\frac{4\nu}{z}\Big)\Gamma\Big(1-\frac{2\nu}{z}\Big)\;P_2+\Gamma\Big(1+\frac{4\nu}{z}\Big)\Gamma\Big(1-\frac{2\nu}{z}\Big)\;P_3\\
&+\Gamma\Big(1+\frac{2\nu}{z}\Big)\left[\mathbf{1}_C+\big(-2\gamma-2\psi(1+\frac{2\nu}{z})\big)\frac{\omega_C}{z}\right].
\end{aligned}\tag{3.137}$$

We now specialise the notation of Proposition 3.5.5 to $\mathbb{U}=\mathbb{U}_\rho$ and compute $\mathbb{U}_\rho^{\mathrm{mat}}$. We fix the basis for $H(Z)$ as the localised basis $\{\mathbf{1}_C,\omega,P_3,P_4\}$ and for $H(\mathcal{X})$ the fundamental class of sectors $\{\mathbf{1}_2,\mathbf{1}_1,\mathbf{1}_3,\mathbf{1}_0\}$ corresponding to the conjugacy classes of $\mathbb{Z}_4=\langle\zeta\rangle$:

$$\mathbf{1}_k\longleftrightarrow\zeta^k.\tag{3.138}$$

We begin by computing the asymptotics of J^Z in the localised basis using (3.94) and (3.95). With the localisation data determined by (3.88), these asymptotics can be computed as

Proposition 3.5.7. *The leading asymptotics of components of*

$$J^{\mathrm{sm},Z}(\tau',z)=J_1^{\mathrm{sm},Z}\;\mathbf{1}_C+J_C^{\mathrm{sm},Z}\;\omega_C+J_3^{\mathrm{sm},Z}\;P_3+J_4^{\mathrm{sm},Z}\;P_4\tag{3.139}$$

are given by

$$J_1^{\mathrm{sm},Z} = z\ \exp\left(\frac{\nu}{z}(2q_1+q_2+q_3)\right), \tag{3.140}$$
$$J_C^{\mathrm{sm},Z} = -q_1\ \exp\left(\frac{\nu}{z}(2q_1+q_2+q_3)\right), \tag{3.141}$$
$$J_3^{\mathrm{sm},Z} = z\ \exp\left(\frac{\nu}{z}(2q_1+3q_2+q_3)\right), \tag{3.142}$$
$$J_4^{\mathrm{sm},Z} = z\ \exp\left(\frac{\nu}{z}(2q_1+q_2+3q_3)\right). \tag{3.143}$$

These asymptotics determine the resolution-side comparison matrix $\mathcal{A}$. The orbifold-side comparison matrix $\mathcal{B}$ is determined below by evaluating the periods and their derivatives at OP.

Since we only need leading asymptotics in the large radius limit (3.96), the integrals in Proposition 3.4.15 can be first reduced to

$$\Pi_i = \frac{\Gamma(-\frac{2\nu(m-2)}{z})\Gamma(\frac{2\nu}{z}+1)}{\Gamma(1-\frac{2\nu(m-3)}{z})}\kappa_i^{\frac{2\nu(l-2)}{z}} F_D^{(i-1)}\left(-\frac{2\nu(m-2)}{z}; -\frac{2\nu}{z},\cdots,-\frac{2\nu}{z}; 1-\frac{2\nu(m-3)}{z}\,\middle|\,\frac{\kappa_{i-1}}{\kappa_i},\cdots,\frac{\kappa_1}{\kappa_i}\right) \tag{3.144}$$

for $1 \le i \le m$ and

$$\Pi_{m+1} = \frac{\Gamma(-\frac{2\nu(m-2)}{z})\Gamma(1-\frac{4\nu}{z})}{\Gamma(1-\frac{2\nu m}{z})} F_D^{(l+1)}\left(-\frac{2\nu(m-2)}{z}; \frac{4\nu}{z}, -\frac{2\nu}{z},\cdots,-\frac{2\nu}{z}; 1-\frac{2\nu m}{z}\,\middle|\,-1,\kappa_m,\cdots,\kappa_1\right). \tag{3.145}$$

The connection formula in Appendix A.3 thus applies, recursively, to these functions. $\mathcal{A}^{-1}$ is then read off from the coefficients.

On the orbifold side, following Section 3.5.2, the matrix $\mathcal{B}$ can be obtained by computing evaluations of periods and their derivatives at

$$q_1 = q_2 = q_3 = -\frac{\pi\mathrm{i}}{2} \tag{3.146}$$

where

$$\kappa_1 = -1,\ \kappa_2 = -\mathrm{i},\ \kappa_3 = 1. \tag{3.147}$$

After a careful computation following strategies above, we get:

Proposition 3.5.8.

$\mathcal{A}^{-1} =$

$$\begin{pmatrix} -\frac{\pi\csc\left(\frac{2\pi\nu}{z}\right)}{z} & 0 & 0 & 0 \\ \frac{e^{\frac{2i\pi\nu}{z}}\left(-H_{\frac{2\nu}{z}}-H_{-\frac{z+2\nu}{z}}+i\pi\right)}{z} & -e^{\frac{2i\pi\nu}{z}} & 0 & 0 \\ -\frac{2e^{\frac{2i\pi\nu}{z}}H_{-\frac{z+2\nu}{z}}}{z} & -e^{\frac{2i\pi\nu}{z}} & \frac{e^{\frac{4i\pi\nu}{z}}\Gamma\left(\frac{2\nu}{z}\right)\Gamma\left(\frac{z+2\nu}{z}\right)}{z\Gamma\left(\frac{4\nu}{z}+1\right)} & 0 \\ -\frac{2e^{\frac{2i\pi\nu}{z}}H_{-\frac{z+2\nu}{z}}}{z} & -e^{\frac{2i\pi\nu}{z}} & \frac{e^{\frac{2i\pi\nu}{z}}\Gamma\left(-\frac{4\nu}{z}\right)\Gamma\left(\frac{2\nu}{z}\right)}{z\Gamma\left(-\frac{2\nu}{z}\right)} & -\frac{\pi^{3/2}2^{-\frac{4\nu}{z}-1}e^{\frac{2i\pi\nu}{z}}\csc\left(\frac{4\pi\nu}{z}\right)}{\nu\Gamma\left(-\frac{2\nu}{z}\right)\Gamma\left(\frac{2\nu}{z}+\frac{1}{2}\right)} \end{pmatrix}, \tag{3.148}$$

where

$$H_x = \psi(1+x)+\gamma, \tag{3.149}$$

and

$\mathcal{B} =$

$$\begin{pmatrix} -\frac{i2^{\frac{2\nu}{z}-\frac{3}{2}}\Gamma\left(\frac{z+2\nu}{z}\right)\Gamma\left(\frac{1}{2}-\frac{2\nu}{z}\right)}{\sqrt{\pi}z} & -\frac{\sqrt{\pi}2^{\frac{2\nu}{z}-1}e^{\frac{2i\pi\nu}{z}}\Gamma\left(\frac{z-2\nu}{2z}\right)}{z\Gamma\left(-\frac{\nu}{z}\right)} & -\frac{i2^{\frac{2\nu}{z}-\frac{3}{2}}\Gamma\left(\frac{z+2\nu}{z}\right)\Gamma\left(\frac{1}{2}-\frac{2\nu}{z}\right)}{\sqrt{\pi}z} & -\frac{\sqrt{\pi}2^{\frac{2\nu}{z}}e^{\frac{2i\pi\nu}{z}}\Gamma\left(1-\frac{\nu}{z}\right)}{\Gamma\left(\frac{z-2\nu}{2z}\right)} \\ 0 & \frac{(-i)^{\frac{2\nu+z}{z}}2^{\frac{2\nu}{z}-1}\Gamma\left(\frac{z-2\nu}{2z}\right)\Gamma\left(\frac{\nu}{z}+1\right)}{\sqrt{\pi}z} & 0 & -\frac{(-2i)^{\frac{2\nu}{z}}\Gamma\left(\frac{z+2\nu}{2z}\right)\Gamma\left(1-\frac{\nu}{z}\right)}{\sqrt{\pi}} \\ -\frac{(-1)^{\frac{2\nu}{z}}2^{\frac{2\nu}{z}-\frac{3}{2}}\Gamma\left(\frac{z+2\nu}{z}\right)\Gamma\left(\frac{1}{2}-\frac{2\nu}{z}\right)}{\sqrt{\pi}z} & \frac{\sqrt{\pi}2^{\frac{2\nu}{z}-1}\Gamma\left(\frac{z-2\nu}{2z}\right)}{z\Gamma\left(-\frac{\nu}{z}\right)} & \frac{(-1)^{\frac{2\nu}{z}}2^{\frac{2\nu}{z}-\frac{3}{2}}\Gamma\left(\frac{z+2\nu}{z}\right)\Gamma\left(\frac{1}{2}-\frac{2\nu}{z}\right)}{\sqrt{\pi}z} & -\frac{\sqrt{\pi}2^{\frac{2\nu}{z}}\Gamma\left(1-\frac{\nu}{z}\right)}{\Gamma\left(\frac{z-2\nu}{2z}\right)} \\ 0 & \frac{\sqrt{\pi}2^{\frac{2\nu}{z}-1}\Gamma\left(\frac{z-2\nu}{2z}\right)}{z\Gamma\left(-\frac{\nu}{z}\right)} & 0 & -\frac{\sqrt{\pi}2^{\frac{2\nu}{z}}\Gamma\left(1-\frac{\nu}{z}\right)}{\Gamma\left(\frac{z-2\nu}{2z}\right)} \end{pmatrix}. \tag{3.150}$$

By Definition 3.5.2, this determines $\mathbb{U}_\rho^{\mathrm{mat}} = \mathcal{A}\mathcal{B}$ in the fixed D_3 bases. This is the matrix of the explicit transformation stated in Proposition 3.1.5.

We finish this section with a few checks in comparison with integral structures. By Proposition 3.5.5, if $\mathbb{U}_\rho$ is induced by a $K-$group isomorphism $\mathbb{U}_K$, their matrices should satisfy

$$\mathbb{U}_\rho^{\mathrm{mat}} = \Gamma_Z\,\mathrm{ch}_Z\,\mathbb{U}_K^{\mathrm{mat}}\,\mathrm{ch}_{\mathcal{X}}^{-1}\,\Gamma_{\mathcal{X}}^{-1}. \tag{3.151}$$

We define $\mathbb{U}_K^{\mathrm{mat}}$ by this relation and let $\mathbb{U}_K$ denote the map represented by this matrix in the fixed K-theory bases.

Proposition 3.5.9.

$$\mathbb{U}_K^{\mathrm{mat}} \in \mathrm{GL}_4\big(\mathrm{Frac}(K_{\mathbb{C}^\times}(\mathrm{pt}))\big), \tag{3.152}$$

where the Chern character of the generator q *of* $K_{\mathbb{C}^\times}(\mathrm{pt}) = \mathbb{Z}[\mathsf{q},\mathsf{q}^{-1}]$ *is mapped to*

$$\mathsf{q} \longrightarrow \left(\frac{\pi i}{z}\right)^{\deg/2}\mathrm{ch}_{\mathbb{C}^\times}(\mathsf{q}) = e^{\frac{\pi i\nu}{z}}. \tag{3.153}$$

Remark 3.5.10. A highly non-trivial result from this computation is that all the transcendental factors in $\mathcal{A}$ and $\mathcal{B}$ (and thus $\mathbb{U}_\rho^{\mathrm{mat}}$), such as Gamma functions and digamma functions of ν/z, delicately cancel out after lifting to $\mathbb{U}_K^{\mathrm{mat}}$. This is a precision check of the (equivariant extension of) Iritani's integral structure conjecture.

Proposition 3.5.11. *Under the non-equivariant limit that sends ν to 0,*

$$\lim_{\nu\to 0} \mathbb{U}_K^{\mathrm{mat}} = \begin{pmatrix} -1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 2 & 0 & 2 & 1 \end{pmatrix}. \tag{3.154}$$

In particular,

$$\lim_{\nu\to 0} \mathbb{U}_K^{\mathrm{mat}} \in \mathrm{GL}_4(\mathbb{Z}). \tag{3.155}$$

This further shows that, under the non-equivariant limit, $\mathbb{U}_K$ identifies the integral lattice of $K(\mathcal{X})$ with that of $K(Z)$, again in keeping with Iritani's integral structure conjecture. Moreover, since $\lim_{\nu\to 0} \mathbb{U}_\rho$ is the Bryan–Gholampour change of variables (which can be seen from either the relation $\mathbb{U}_\rho J_{\mathcal{X}} = J_Z$, or by direct inspection), it is further compatible with the discussion in [43, Remark 3.32] in the non-equivariant setup.

Proof of Proposition 3.1.6. Proposition 3.5.8 determines the transformation $\mathbb{U}_\rho$ explicitly through its matrix $\mathbb{U}_\rho^{\mathrm{mat}} = \mathcal{A}\mathcal{B}$. Using the integral-structure framing diagram, Eq. (3.104) defines the corresponding map

$$\mathbb{U}_K \colon K_{\mathbb{C}^\times}(\mathcal{X}) \longrightarrow K_{\mathbb{C}^\times}(Z).$$

Proposition 3.5.9 shows that its matrix satisfies

$$\mathbb{U}_K^{\mathrm{mat}} \in \mathrm{GL}_4\big(\mathrm{Frac}(K_{\mathbb{C}^\times}(\mathrm{pt}))\big).$$

It is therefore an isomorphism over $\mathrm{Frac}(K_{\mathbb{C}^\times}(\mathrm{pt}))$, and the framing diagram shows that it induces $\mathbb{U}_\rho$. Finally, Proposition 3.5.11 computes its non-equivariant limit and shows that

$$\lim_{\nu\to 0} \mathbb{U}_K^{\mathrm{mat}} \in \mathrm{GL}_4(\mathbb{Z}).$$

Thus the non-equivariant limit is integral, which proves both assertions of Proposition 3.1.6. $\square$

Chapter 4

On orbifold quantum cohomology of foldings of ADE resolutions

§ 4.1 Introduction

Let $\mathcal{R}$ be a root system and let $\Phi_{\mathcal{R}}$ be a finite cyclic group acting on $\mathcal{R}$ by automorphisms induced by symmetries of its Dynkin diagram. The pairs $(\mathcal{R}, \Phi_{\mathcal{R}})$ we consider form four classes listed in the first row of Figure 4.1. For $\mathcal{R} = \mathrm{A}_{2n-1}$ with $n \geq 2$, the group $\Phi_{\mathcal{R}} \cong \mathbb{Z}_2$ fixes the marked central vertex and interchanges the two arms. For $\mathcal{R} = \mathrm{D}_{m+1}$ with $m \geq 3$, the group $\Phi_{\mathcal{R}} \cong \mathbb{Z}_2$ fixes the horizontal chain, including the marked vertex, and interchanges the two remaining vertices. For $\mathcal{R} = \mathrm{E}_6$, the group $\Phi_{\mathcal{R}} \cong \mathbb{Z}_2$ fixes the left two vertices and interchanges the two remaining arms. Finally, for $\mathcal{R} = \mathrm{D}_4$, the group $\Phi_{\mathcal{R}} \cong \mathbb{Z}_3$ fixes the marked central vertex and cyclically permutes the three outer vertices.

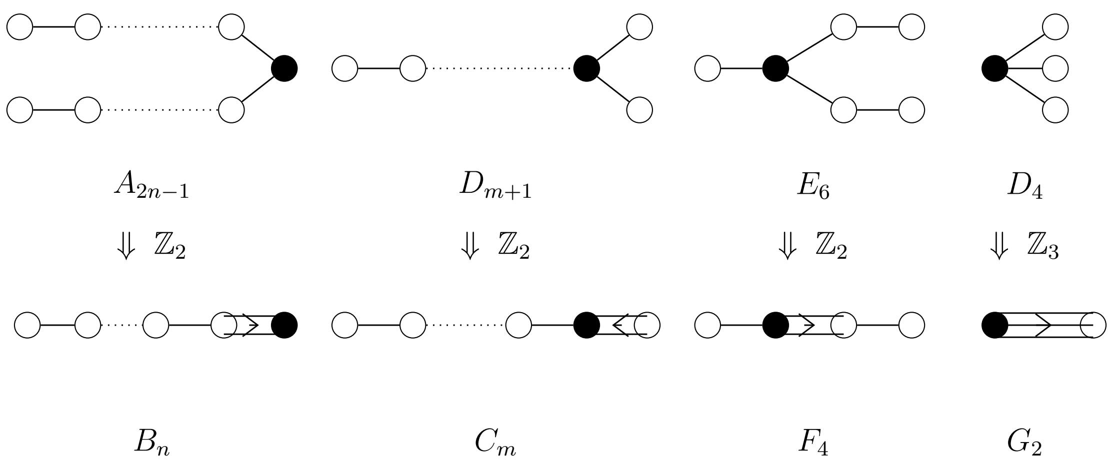


Figure 4.1: Dynkin symmetries and foldings

Associated with such a pair $(\mathcal{R}, \Phi_{\mathcal{R}})$ are two non-simply-laced root systems, one is obtained by **restricting** to the $\Phi_{\mathcal{R}}$-invariant roots, denoted by $\mathcal{R}^{\text{fold}}$ and its Dynkin diagram is shown in the second row of Figure 4.1, another is obtained by **averaging** roots, denoted by $\mathcal{R}^{\text{ave}}$. These root systems can be summarised in the following table:

$\mathcal{R}$	$\Phi_{\mathcal{R}}$	$\mathcal{R}^{\text{fold}}$	$\mathcal{R}^{\text{ave}}$	$N_{\mathcal{R}^{\text{fold}}}$
A_{2n-1}	$\mathbb{Z}_2$	B_n	C_n	2
D_{m+1}	$\mathbb{Z}_2$	C_m	B_m	m
E_6	$\mathbb{Z}_2$	F_4	F_4	3
D_4	$\mathbb{Z}_3$	G_2	G_2	2

Table 4.1: restricting and averaging root systems under a Dynkin symmetry

Here, **averaging** of $\beta \in \mathfrak{h}_{\mathcal{R}}$, denoted by $\overline{\beta}$, is defined by

$$\overline{\beta} = \frac{1}{|\Phi_{\mathcal{R}}|} \sum_{g \in \Phi_{\mathcal{R}}} g \circ \beta. \tag{4.1}$$

Let $G_{\mathcal{R}} < \mathrm{SL}(2, \mathbb{C})$ be the corresponding Kleinian subgroup in the ADE classification, $\mathbb{C}^2/G_{\mathcal{R}}$ be the corresponding Kleinian singularity and $Z_{\mathcal{R}} := \widetilde{\mathbb{C}^2/G_{\mathcal{R}}}$ be its minimal resolution. The geometry of $Z_{\mathcal{R}}$ is closely related to root system $\mathcal{R}$ via the McKay correspondence. For example, there is a canonical identification $H_2(Z_{\mathcal{R}}; \mathbb{Z}) \cong \Lambda_{\mathcal{R}}$, under which the classes of exceptional curves correspond to simple roots, and the intersection form agrees with the negative of the inner product on $\Lambda_{\mathcal{R}}$.

The $\Phi_{\mathcal{R}}$-action on $\mathcal{R}$ also admits a geometric interpretation. In [61], Slodowy shows that there exists a natural $\Phi_{\mathcal{R}}$-action on $Z_{\mathcal{R}}$, such that the exceptional curves are stable under $\Phi_{\mathcal{R}}$ and the action on the curves is compatible with the action on Dynkin diagrams. Moreover, $\Phi_{\mathcal{R}}$ is free on the complement of fixed points, giving the same number of isolated $\mathrm{A}_{|\Phi_{\mathcal{R}}|-1}$ singularities on the quotient space $Z_{\mathcal{R}}/\Phi_{\mathcal{R}}$. The number of fixed points on $Z_{\mathcal{R}}$ is denotes by $N_{\mathcal{R}^{\text{fold}}}$ and listed in Table 4.1.

Let $\mathcal{X}_{\mathcal{R}^{\text{fold}}} = [Z_{\mathcal{R}}/\Phi_{\mathcal{R}}]$ be the corresponding orbifold/quotient stack, which we call the **folding** of $Z_{\mathcal{R}}$. The diagonal $\mathbb{C}^\times$-action on $\mathbb{C}^2$ induces a $\mathbb{C}^\times$-action on $Z_{\mathcal{R}}$, and fixes an irreducible component of the exceptional curves corresponding to the marked node in Figure 4.1. Moreover, the $\mathbb{C}^\times$-action descends to $\mathcal{X}_{\mathcal{R}^{\text{fold}}}$. Associated with these data is the $\mathbb{C}^\times$**-equivariant orbifold quantum cohomology** $\mathrm{QH}^*_{\mathbb{C}^\times}(\mathcal{X}_{\mathcal{R}^{\text{fold}}})$. Relevant theories are reviewed in Section 1.2.

4.1.1 Main results

We first study quantum multiplication on the untwisted sector. To state our result, we describe the relevant cohomology and degree lattices. Consider the quotient map $\pi\colon Z_{\mathcal{R}} \to \mathcal{X}_{\mathcal{R}^{\text{fold}}}$. The pullback

$$\pi^*\colon H^2(\mathcal{X}_{\mathcal{R}^{\text{fold}}}; \mathbb{C}) \longrightarrow H^2(Z_{\mathcal{R}}; \mathbb{C})$$

is naturally identified with the embedding of root systems $\mathcal{R}^{\text{fold}} \subset \mathcal{R}$. Under this identification, the orbifold Poincaré pairing agrees with the negative of the inner product, scaled by $\frac{1}{|\Phi_{\mathcal{R}}|}$:

$$H^2(\mathcal{X}_{\mathcal{R}^{\text{fold}}};\mathbb{C}) \cong \left(\mathfrak{h}_{\mathcal{R}^{\text{fold}}}, -\frac{1}{|\Phi_{\mathcal{R}}|}\langle,\rangle\right). \tag{4.2}$$

The degree lattice, or equivalently the second homology, of $\mathcal{X}_{\mathcal{R}^{\text{fold}}}$ is obtained by averaging curve classes on $Z_{\mathcal{R}}$. Thus,

$$H_2(\mathcal{X}_{\mathcal{R}^{\text{fold}}};\mathbb{Z}) \cong \Lambda_{\mathcal{R}^{\text{ave}}}. \tag{4.3}$$

The root systems $\mathcal{R}^{\text{fold}}$ and $\mathcal{R}^{\text{ave}}$ are dual to each other, consistently with the Kronecker pairing on $\mathcal{X}_{\mathcal{R}^{\text{fold}}}$.

Under these identifications, and writing $\mathcal{R}^{\text{ave},+}$ for the set of positive roots of $\mathcal{R}^{\text{ave}}$, our first main result is the following.

Theorem 4.1.1. *For $\phi_i, \phi_j, \tau \in H^2(\mathcal{X}_{\mathcal{R}^{\text{fold}}}) \cong \mathfrak{h}_{\mathcal{R}^{\text{fold}}}$, quantum multiplication is given by*

$$\phi_i \star_\tau \phi_j = -\nu^2|G_{\mathcal{R}}|\langle\phi_i,\phi_j\rangle + \nu \sum_{\overline{\beta}\in\mathcal{R}^{\text{ave},+}} \langle\overline{\beta},\phi_i\rangle\langle\overline{\beta},\phi_j\rangle \frac{1+\mathrm{e}^{-\langle\overline{\beta},\tau\rangle}}{1-\mathrm{e}^{-\langle\overline{\beta},\tau\rangle}}\overline{\beta}^\vee. \tag{4.4}$$

This theorem gives a closed formula for quantum multiplication on the untwisted sector.

We next turn to the full orbifold quantum cohomology, including products involving twisted sectors. To investigate these products, we use the fact that the coarse moduli space of $\mathcal{X}_{\mathcal{R}^{\text{fold}}}$ admits a crepant resolution. In this setup, Ruan's *Crepant Resolution Conjecture* [58] (CRC) suggests comparing this orbifold quantum cohomology with the quantum cohomology of the crepant resolution.

In Bryan-Graber's later formalism of CRC, an explicit affine change of variables is an essential part of the CRC, see [17]. Although substantial work has been carried out for Kleinian singularities [17, 16, 23, 41], such a change does not appear to have been formulated previously for the quotient stacks considered here. This leads us to propose the following conjecture:

Conjecture 4.1.2 (= Conjecture 4.3.2)**.** *After analytic continuation and an affine change of variables as in Definition 4.3.1, there is an isomorphism*

$$\mathrm{QH}^*_{\mathbb{C}^\times}\left(Z_{\mathcal{R}^{\text{res}}}\right) \;\cong\; \mathrm{QH}^*_{\text{orb},\mathbb{C}^\times}\left(\mathcal{X}_{\mathcal{R}^{\text{fold}}}\right). \tag{4.5}$$

Here $Z_{\mathcal{R}^{\text{res}}}$ denotes the crepant resolution of the coarse moduli space of $\mathcal{X}_{\mathcal{R}^{\text{fold}}}$. This resolution is described in Section 4.3.1.

We establish two results that provide evidence for this conjecture. We state both in simplified form; their precise formulations are given in Propositions 4.3.5 and 4.3.11, respectively.

Proposition 4.1.3 (=Proposition 4.3.5)**.** *After an explicit specialisation of the exceptional quantum parameters, the quantum-corrected cohomology of $Z_{\mathcal{R}^{\mathrm{res}}}$ is isomorphic, as a ring, to the Chen–Ruan cohomology of $\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}$.*

Proposition 4.1.4 (=Proposition 4.3.11)**.** *The affine transformation in Conjecture 4.3.2 is compatible, via Iritani's integral structure, with the Fourier–Mukai transform, in the sense that it identifies the corresponding integral central charges, up to the specified signs in the* $(\mathrm{D}_4, \mathrm{G}_2)$ *case.*

The first proposition verifies the *cohomological crepant resolution conjecture* of Ruan in [58], which is a cohomological limit predicted by Conjecture 4.3.2. The second proposition provides categorical evidence for the proposed affine transformation under Iritani's formalism of *crepant resolution conjecture with an integral structure* developed in [42, 43]. Neither proposition proves the full identification of quantum products after analytic continuation though; establishing such an identification is left for future work.

Finally, we relate the untwisted quantum product to known Frobenius structures associated with non-simply-laced root systems. The precise formulations are given in Corollary 4.4.1 and Theorem 4.4.2. Let

$$\iota\colon H^*(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}) \longrightarrow H^*_{\mathrm{CR}}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}})$$

denote the inclusion of the untwisted sector. As a direct consequence of Theorem 4.1.1, we obtain:

Corollary 4.1.5 (=Corollary 4.4.1)**.** *After the change of variables*

$$Q^{\bar{\beta}} = \mathrm{e}^{-\langle\bar{\beta},\tau\rangle},$$

the untwisted quantum cohomology $\iota^*\big(\mathrm{QH}^*_{\mathbb{C}^\times}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}})\big)$ is isomorphic, as a Frobenius algebra, to the one constructed by Bryan–Gholampour [16, Theorem 6] for $\mathcal{R}^{\mathrm{ave}}$.

Using in addition the Landau–Ginzburg description of the extended affine Weyl Frobenius manifolds, we prove:

Theorem 4.1.6 (=Theorem 4.4.2)**.** *The untwisted quantum cohomology $\iota^*\big(\mathrm{QH}^*_{\mathbb{C}^\times}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}})\big)$ is isomorphic to the Dubrovin dual $\mathcal{M}^\flat_{\mathcal{R}^{\mathrm{ave}}}$ of the extended affine Weyl Frobenius manifold $\mathcal{M}_{\mathcal{R}^{\mathrm{ave}}}$.*

Thus, the folding construction provides a geometric realisation of these Frobenius structures associated with non-simply-laced root systems.

4.1.2 Organisation

The chapter is organised as follows.

In Section 1.2, we recall the necessary background on Chen–Ruan cohomology, equivariant Gromov–Witten theory, and orbifold quantum cohomology for global quotient stacks.

In Section 4.2, we prove Theorem 4.1.1. The proof strategy is outlined in Section 4.2.1 and reduces the theorem to two key ingredients: the construction of two threefolds and a multiple-cover calculation. These are established in Sections 4.2.2 and 4.2.3, respectively.

In Section 4.3, we describe the relevant crepant resolutions and the affine identification of their cohomology groups before formulating Conjecture 4.3.2. The supporting evidence is developed in Sections 4.3.3 and 4.3.4.

Finally, in Section 4.4, we compare Theorem 4.1.1 with the known Frobenius structures associated with non-simply-laced root systems.

§ 4.2 Proof of Theorem 4.1.1

4.2.1 Proof of Theorem 4.1.1 assuming Property 1 and Proposition 4.2.4

We prove Theorem 4.1.1 in this section. The argument relies on property 1 and Proposition 4.2.4, which are established in Section 4.2.2 and Section 4.2.3 respectively.

We first describe the Chen–Ruan cohomology of $\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}$. By Slodowy [61], the Φ_R-action on $Z_{\mathcal{R}}$ has $N_{\mathcal{R}^{\mathrm{fold}}}$ fixed points $\{p_k\}_{k=1}^{N_{\mathcal{R}^{\mathrm{fold}}}}$. Therefore, the inertia stack decomposes as

$$I\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}} = \mathcal{X}_{\mathcal{R}^{\mathrm{fold}}} \bigsqcup_{k=1}^{N_{\mathcal{R}^{\mathrm{fold}}}} \sqcup_{i=1}^{|\Phi_{\mathcal{R}}|-1} [p_k/\Phi_{\mathcal{R}}], \tag{4.6}$$

and

$$H^*_{\mathrm{CR}}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}};\mathbb{C}) \cong H^*(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}};\mathbb{C}) \bigoplus_{k=1}^{N_{\mathcal{R}^{\mathrm{fold}}}} \oplus_{i=1}^{|\Phi_{\mathcal{R}}|-1} H^{*-2}(B\mathbb{Z}_{|\Phi_{\mathcal{R}}|-1};\mathbb{C}). \tag{4.7}$$

Proposition 4.2.1. *$\forall \phi, \phi', \tau \in H^2(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}};\mathbb{C})$ and $\delta \in H^{\mathrm{tw}}_{\mathrm{CR}}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}};\mathbb{C})$,*

$$(\phi \star_\tau \phi', \delta)_{\mathrm{CR},\mathbb{C}^\times} = 0, \tag{4.8}$$

Proof. In orbifold Gromov-Witten theory for global quotients (see for example [45, (10)]), the product of monodromies at marked point need to be 1 in order to have non-empty moduli space, otherwise the GW invariant is zero. As a consequence, each term

$$\langle \phi, \phi', \tau \cdots, \tau, \delta \rangle_{0,n+3,A}, \tag{4.9}$$

in the expansion of quantum product is zero, since the product of monodromies is a nontrivial $|\Phi_{\mathcal{R}}|$-th root of unity. $\square$

As a corollary, the quantum product is closed on the subspace

$$H^*(\mathcal{X}_{\mathcal{R}^{\text{fold}}};\mathbb{C}) \subset H^*_{\text{CR}}(\mathcal{X}_{\mathcal{R}^{\text{fold}}};\mathbb{C}). \tag{4.10}$$

We start with computations of several genus zero Gromov-Witten invariants of $\mathcal{X}_{\mathcal{R}^{\text{fold}}}$.

Proposition 4.2.2. *For any $\phi_i, \phi_j, \phi_k \in H^2(\mathcal{X}_{\mathcal{R}^{\text{fold}}};\mathbb{C}) \cong \mathfrak{h}_{\mathcal{R}^{\text{fold}}}$, the degree 0 invariants are given by*

(1) $\langle 1, 1, 1\rangle_0 = \frac{1}{|\Phi_{\mathcal{R}}|}\frac{1}{t^2|G_{\mathcal{R}}|}$;

(2) $\langle \phi_i, 1, 1\rangle_0 = 0$;

(3) $\langle \phi_i, \phi_j, 1\rangle_0 = \frac{1}{|\Phi_{\mathcal{R}}|}\langle \phi_i, \phi_j\rangle$;

(4) $\langle \phi_i, \phi_j, \phi_k\rangle_0 = \frac{-\nu}{|\Phi_{\mathcal{R}}|}\sum_{\overline{\beta}\in\mathcal{R}^{\text{ave},+}} \langle \phi_i, \overline{\beta}\rangle\langle \phi_j, \overline{\beta}\rangle\langle \phi_k, \overline{\beta}^\vee\rangle$.

Proof. By the comparison (4.2) of Poincaré pairings, the degree 0 of $\mathcal{X}_{\mathcal{R}^{\text{fold}}}$ satisfy

$$\langle \cdot, \cdot, \cdot\rangle_0^{\mathcal{X}_{\mathcal{R}^{\text{fold}}}} = \frac{1}{|\Phi_{\mathcal{R}}|}\langle \cdot, \cdot, \cdot\rangle_0^{Z_{\mathcal{R}}}. \tag{4.11}$$

The later ones are computed in [16] and (1) – (3) immediately follows. By [16],

$$\langle \phi_i, \phi_j, \phi_k\rangle_0^{Z_{\mathcal{R}}} = -\nu \sum_{\beta\in\mathcal{R}^+} \langle \phi_i, \beta\rangle\langle \phi_j, \beta\rangle\langle \phi_k, \beta\rangle, \tag{4.12}$$

(4) then follows from the following properties:

- $\langle \phi, \beta\rangle = \langle \phi, \overline{\beta}\rangle$ for any $\phi \in \mathfrak{h}_{R^{\text{fold}}} = \mathfrak{h}_{\mathcal{R}}^{\Phi_{\mathcal{R}}}$;
- $\sum_{g\in\Phi_{\mathcal{R}}} g\circ\beta = \overline{\beta}^\vee$, shown in [16, Lemma 9].

□

Theorem 4.2.3. *For any $A \in H_2(\mathcal{X}_{\mathcal{R}^{\text{fold}}};\mathbb{Z}) \cong \Lambda_{\mathcal{R}^{\text{ave}}}$, the degree A invariants are given by*

$$\langle\rangle_A^{\mathcal{X}_{\mathcal{R}^{\text{fold}}}} = \begin{cases} 2\nu\frac{1}{d^3}\frac{2}{\langle\beta,\beta\rangle}\frac{1}{|\Phi_{\mathcal{R}}|}, & A = d\bar{\beta}, \\ 0, & \textit{otherwise.}\end{cases}$$

Outline of proof. In Definition 4.2.5, we will define two three-folds W and W' satisfying:

Property 1. *(P1) W and W' admit $\Phi_{\mathcal{R}}$-actions;*

(P2) *W is deformation equivalent to W' ;*

(P3) *The $\Phi_{\mathcal{R}}$-action is compatible with the embedding $Z_{\mathcal{R}} \subset W$ and deformation;*

(P4) *$Z_{\mathcal{R}} \subset W$ and contains all compact curves of W. There is a contracting map $W \to W^{\mathrm{aff}}$ where all the curves get contracted and W^{aff} is affine;*

(P5) *Given any positive root $\beta \in \mathcal{R}$, there exists an isolated $(-1,-1)$ curve $C_\beta \subset W'$. Moreover, $\{C_\beta : \beta \in \mathcal{R}^+\}$ exhaust all the compact curves in W' and $\Phi_{\mathcal{R}}$ acts on $\{C_\beta : \beta \in \mathcal{R}^+\}$ respecting the $\Phi_{\mathcal{R}}$-action on the index β;*

(P6) *On $\mathcal{N}_{Z_{\mathcal{R}}\backslash W}$, $\Phi_{\mathcal{R}}$ acts trivially. On W', the stabliser of $\Phi_{\mathcal{R}}$ is either trivial or $\Phi_{\mathcal{R}}$. At a $\Phi_{\mathcal{R}}$ fixed point on C_β, $\Phi_{\mathcal{R}}$ acts with weights 1 (or -1) on tangent direction on C_β, and $(-1,0)$ (or $(1,0)$) on the normal direction;*

(P7) *W and W' are Calabi–Yau, and the $\Phi_{\mathcal{R}}$-actions preserve the Calabi–Yau form on them.*

These properties above immediately imply that $\langle\rangle_A^{\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}} = 2\nu\langle\rangle_A^{[W/\Phi_{\mathcal{R}}]} = 2\nu\langle\rangle_A^{[W'/\Phi_{\mathcal{R}}]}$. The first identity follows from the fact that $\Phi_{\mathcal{R}}$ acts trivially on $\mathcal{N}_{Z_{\mathcal{R}}\backslash W}$, and $\mathcal{N}_{Z_{\mathcal{R}}\backslash W} \cong \mathcal{O}_{Z_{\mathcal{R}}} \otimes \mathbb{C}_{2\nu}$ as a $\mathbb{C}^\times$ bundle. The second one follows from deformation invariance of orbifold Gromov-Witten invariants.

By *(P4)*, for any genus-zero stable map to $[W'/\Phi_{\mathcal{R}}]$ in class $d\beta$ the image is *equal* to C_β. Take

$$H_\beta \;:=\; \mathrm{Stab}_{\Phi_{\mathcal{R}}}(C_\beta) \;=\; \{\, g \in \Phi_{\mathcal{R}} \mid g(C_\beta) = C_\beta \,\}.$$

Since C_β is a $(-1,-1)$-curve and does not intersect with other C_β, $\beta' \neq \beta$, the degree-$d\beta$ invariant is the local contribution supported on C_β, namely

$$\langle\,\rangle_{0,d\beta}^{[W'/\Phi_{\mathcal{R}}]} \;=\; \int_{[\overline{\mathcal{K}}_{0,0}([C_\beta/H_\beta],d)]^{\mathrm{vir}}} e\Big(R^1\pi_* f^* N_{C_\beta/W'}\Big)\,, \tag{4.13}$$

where $\pi : \mathcal{C} \to \overline{M}_{0,0}([C_\beta/H_\beta], d)$ and $f : \mathcal{C} \to [C_\beta/H_\beta]$ are the universal curve and map.

A direct check shows that

$$|H_\beta| \;=\; \frac{\langle\bar\beta,\bar\beta\rangle}{2}\cdot|\Phi_{\mathcal{R}}|. \tag{4.14}$$

The theorem therefore reduces to the following multicover formula:

Proposition 4.2.4 (=Corollary 4.2.9)**.** *Let x, y be homogeneous coordinates of $\mathbb{P}^1$, and let ζ be a primitive root of unity of order n. Consider the $\mathbb{Z}_n = \langle\zeta\rangle$-action on $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ as follows:*

(1) *On $\mathbb{P}^1$, $\zeta[x : y] = [\zeta x : y]$;*

(2) *View x, y as generators of $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$, then on the first (resp. second) summand $\mathcal{O}_{\mathbb{P}^1}(-1)$, ζ has weight 0 (resp. 1) in x and weight -1(resp. 0) in y.*

Let $[\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1))/\mathbb{Z}_n]$ *be the corresponding quotient stack/orbifold, then*

$$\langle\,\rangle_{0,d}^{[\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1))/\mathbb{Z}_n]} = \frac{1}{n\,d^3}. \tag{4.15}$$

□

Proof of Theorem 4.1.1 assuming Property 1 and Proposition 4.2.4. From Proposition 4.2.1, the orbifold quantum product is closed on the untwisted sector. Therefore, to compute $\phi_i \star_\tau \phi_j$, it suffices to compute its pair with $H^*(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}};\mathbb{C})$. By Remark 1.3.2 the pairing with $\mathbf{1}$ is just the classical pairing , which has been shown in Proposition 4.2.2. Therefore, it remains to show, for any $\phi_i,\phi_j,\phi_k,\tau\in H^2(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}};\mathbb{C})$, it holds

$$(\phi_i \star_\tau \phi_j,\phi_k) = \frac{-\nu}{|\Phi_{\mathcal{R}}|}\sum_{\overline{\beta}\in\mathcal{R}^{\mathrm{ave},+}} \langle\overline{\beta},\phi_i\rangle\langle\overline{\beta},\phi_j\rangle\langle\overline{\beta}^\vee,\phi_j\rangle\frac{1+\mathrm{e}^{-\langle\bar{\beta},\tau\rangle}}{1-\mathrm{e}^{-\langle\bar{\beta},\tau\rangle}}. \tag{4.16}$$

Recall that

$$(\phi_i \star_\tau \phi_j,\phi_k) = \langle\phi_i,\phi_j,\phi_k\rangle_0 + \sum_{A\neq 0}\mathrm{e}^{\int_A\tau}\int_A\phi_i\int_A\phi_j\int_A\phi_k\,\langle\,\rangle_A. \tag{4.17}$$

Here the first term is the degree-zero contribution, which is computed in Proposition 4.2.2. The remaining terms are determined by the positive-degree no-point invariants. By Theorem 4.2.3, they vanish unless $A = d\bar{\beta}$, where $\bar{\beta}\in\mathcal{R}^{\mathrm{ave},+}$ and $d\geq 1$. Under the root-lattice identifications,

$$\int_{d\bar{\beta}}\phi = -d\langle\bar{\beta},\phi\rangle. \tag{4.18}$$

Thus the three divisor pairings contribute a factor $-d^3$. The formula in Theorem 4.2.3 contains a factor d^{-3}, so these powers of d will cancel. For each such root, the remaining coefficient will give

$$-2\nu\frac{2}{\langle\bar{\beta},\bar{\beta}\rangle}\frac{1}{|\Phi_{\mathcal{R}}|}\sum_{d\geq 1}\mathrm{e}^{-d\langle\bar{\beta},\tau\rangle} = -2\nu\frac{2}{\langle\bar{\beta},\bar{\beta}\rangle}\frac{1}{|\Phi_{\mathcal{R}}|}\frac{\mathrm{e}^{-\langle\bar{\beta},\tau\rangle}}{1-\mathrm{e}^{-\langle\bar{\beta},\tau\rangle}}. \tag{4.19}$$

Notice that

$$\bar{\beta}^\vee = \frac{2\bar{\beta}}{\langle\bar{\beta},\bar{\beta}\rangle}, \tag{4.20}$$

combining this series with the degree-zero contribution gives the factor

$$\frac{-\nu}{|\Phi_{\mathcal{R}}|}\frac{1+\mathrm{e}^{-\langle\bar{\beta},\tau\rangle}}{1-\mathrm{e}^{-\langle\bar{\beta},\tau\rangle}} \tag{4.21}$$

in (4.17). This completes the proof. □

4.2.2 Construction of W and W' with Property 1

Restricting Grothendieck's simultaneous resolution

$$\begin{tikzcd} \widetilde{\mathfrak{g}} \arrow[r, "\pi"] \arrow[dr, "\widetilde{\chi}"'] & \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d, "\chi"] \\ & \mathfrak{h} \arrow[r] & \mathfrak{h}/W \end{tikzcd}$$

to the Slodowy slice through a subregular nilpotent element yields

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\pi_S"] \arrow[dr, "\widetilde{\chi}_S"'] & S \times_{\mathfrak{h}/W} \mathfrak{h} \arrow[d, "\chi_S"] \\ & \mathfrak{h} \end{tikzcd} \tag{4.22}$$

Property 2 ([61])**.** *The objects in the diagram (4.22) have the following properties:*

- *over $0 \in \mathfrak{h}$, π_S is the minimal resolution of the corresponding Kleinian singularity;*
- *$\chi^{-1}(\tau)$ is smooth, and $\widetilde{\chi}_S^{-1}(\tau)$ contains a curve only when $\tau \in H_\beta$ for some root β;*
- *Let $\tau \in \cup_\beta H_\beta$, then $\chi_S^{-1}(\tau)$ has the singularity type determined by the root subsystem*
$$\{\alpha \in \mathcal{R}, \tau \in H_\alpha\}; \tag{4.23}$$
- *The symmetric group $\Phi_\mathcal{R}$ acts everywhere on the resolution diagram and commutes with the arrows in (Eq. (4.22)). Any $\sigma \in \Phi_\mathcal{R}$ induces a homeomorphism $\sigma : \chi_S^{-1}(\tau) \mapsto \chi_S(\sigma^{-1}(\tau))$ and stabilises the exceptional curves. In particular, over 0, the $\Phi_\mathcal{R}$–action on curves is compatible with the Dynkin symmetry.*

Let $i : \mathbb{C}^2 \to \mathfrak{h}^{\Phi_\mathcal{R}}$ be any **generic** linear embedding.

Definition 4.2.5. Let W and W' be the total of spaces in the following pull back families:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & i^*\widetilde{S} \arrow[d] & W' \arrow[l] \arrow[d] \\ \{0\} \times \mathbb{C} \arrow[r] & \mathbb{C} \times \mathbb{C} & \mathbb{C} \times \{1\} \arrow[l] \end{tikzcd}$$

Proposition 4.2.6. *W and W' satisfy the properties listed in Property 1.*

Proof. (P1-P3) Since $\mathrm{Im}(i : \mathbb{C}^2 \to \mathfrak{h}_{\mathcal{R}})$ lies in the invariant subspace $\mathfrak{h}_{\mathcal{R}}^{\Phi_{\mathcal{R}}}$, P1-P3 follows from Property 2;

(P4) By construction, $Z_{\mathcal{R}} = \widetilde{\chi_S}^{-1}(0) \subset W$. To show $Z_{\mathcal{R}}$ contains all compact curves of W, it suffices to show $\mathfrak{h}_{\mathcal{R}}^{\Phi_{\mathcal{R}}}$ is not contained in any hypersurface H_β, in another word, $\langle \beta, - \rangle$ is non-zero on $\mathfrak{h}_{\mathcal{R}}^{\Phi_{\mathcal{R}}}$. For this, we use $\langle \beta, v \rangle = \langle \bar{\beta}, v \rangle$ if $v \in \mathfrak{h}_{\mathcal{R}}^{\Phi_{\mathcal{R}}}$ and notice that the averaging $\bar{\beta} \in \mathfrak{h}_{\mathcal{R}}^{\Phi_{\mathcal{R}}}$ is a positive root of $\mathcal{R}^{\mathrm{ave}}$ and hence is nonzero. The affineness follows from the affineness of $S = e + Ker(\mathrm{ad} f)$;

(P5) Since $\mathfrak{h}_{\mathcal{R}}^{\Phi_{\mathcal{R}}}$ is not contained in any hyperplane H_β, for a generic choice of the embedding $i : \mathbb{C} \times \{1\} \hookrightarrow \mathfrak{h}_{\mathcal{R}}^{\Phi_{\mathcal{R}}}$ the curve $i(\mathbb{C} \times \{1\})$ meets each H_β transversely, and avoids all pairwise intersections $H_\beta \cap H_{\beta'}$ unless β and β' lie in the same $\Phi_{\mathcal{R}}$–orbit. Consequently, the exceptional curves in W are contained in the fiber of $\widetilde{\chi_S}$ over the (necessarily unique) point

$$i(\mathbb{C} \times \{1\}) \cap H_\beta.$$

If $i(\mathbb{C} \times \{1\})$ meets a single hyperplane H_β (and no other $H_{\beta'}$), then the corresponding fiber contains a unique (-2)-curve. If instead $i(\mathbb{C} \times \{1\})$ passes through a point lying on several hyperplanes, then by construction this can only happen for hyperplanes $H_{\beta'}$ with $\beta' \in \Phi_{\mathcal{R}} \cdot \beta$. In this case the fiber contains $|\Phi_{\mathcal{R}} \cdot \beta|$ distinct (-2)-curves, naturally indexed by the roots in the orbit: we denote by $C_{\beta'}$ the component which deforms along $H_{\beta'}$ in the family. Moreover, it can be checked that if $\beta' = g\beta$ for $1 \neq g \in \Phi_{\mathcal{R}}$, then β and β' are orthogonal in $\mathcal{R}$, therefore C_β and $C_{\beta'}$ do not intersect in W'. Finally, by transversality of the intersection, each C_β are $(-1,-1)$-curves.

(P6) $\Phi_{\mathcal{R}}$ acts trivially on $\mathcal{N}_{Z_{\mathcal{R}} \backslash W}$ by construction. On W', the fixed points lie on C_β corresponding to a fixed positive root $\beta \in \mathcal{R}^{\mathrm{fold}}$ by property 2. In this case let $\tau_\beta = H_\beta \cap i(\mathbb{C} \times \{1\})$ and

$$\tilde{S}_\beta = \widetilde{\chi}_S^{-1}(\tau_\beta)\ , S_\beta = \chi_S^{-1}(\tau_\beta), \tag{4.24}$$

then $\pi_S : \tilde{S}_\beta \to S_\beta$ is the resolution of an A_1 singularity; let $C_\beta \subset \tilde{S}_\beta$ be the exceptional curve. To check the tangent weights, we use the short exact sequence

$$0 \to \mathcal{N}_{C_\beta \backslash \tilde{S}_\beta} \to \mathcal{N}_{C_\beta \backslash W'} \to \mathcal{N}_{\tilde{S}_\beta \backslash W'} \to 0 \tag{4.25}$$

By construction, $\mathcal{N}_{C_\beta \backslash \tilde{S}_\beta}$ is trivial with trivial Φ_R action. It then suffices to show that $S_\beta/\Phi_{\mathcal{R}}$ is a A_3 singularity $(\mathbb{C}^2/\mathbb{Z}_2)/(\mathbb{Z}_4/\mathbb{Z}_2)$ when $\Phi_R \cong \mathbb{Z}_2$ and an A_5 singularity $(\mathbb{C}^2/\mathbb{Z}_2)/(\mathbb{Z}_6/\mathbb{Z}_2)$ when $\Phi_R \cong \mathbb{Z}_3$, therefore $\Phi_{\mathcal{R}}$ acts on $\tilde{S}_\beta \cong T^*\mathbb{P}^1$ is induced by the action $z \mapsto \zeta z$, where ζ is a primitive root of unity of order $|\Phi_{\mathcal{R}}|$. To see this, we notice that, by [19, Chap3], $S_\beta/\Phi_{\mathcal{R}}$ is a simple singularity and therefore there is a Kleinian subgroup $G < SL(2;\mathbb{C})$ such that $\Phi_{\mathcal{R}} \cong G/\mathbb{Z}_2$. By the classification of Kleinian subgroups, $G \cong \mathbb{Z}_4$ and $G \cong \mathbb{Z}_6$ are the only Kleinian subgroups of order 4 and 6 respectively.

(P7) By construction, W^{aff} is an complete intersections in an affine variety S, hence is Calabi–Yau by adjunction formula. Moreover, since the resolution $W \to W^{\mathrm{aff}}$ is an simultaneous resolution over $\mathbb{C}$ and moreover is a minimal resolution of Kleinian singularities over each $b \in \mathbb{C}$ hence crepant, $W \to W^{\mathrm{aff}}$ is crepant. From this it follows that W is Calabi–Yau. The same argument applies to M'. To check that $\Phi_{\mathcal{R}}$ preserves Calabi–Yau forms, it suffices to check $det\big(g : \omega_x \to \omega_x\big) = 1$ at any fixed point x, since W and W' are connected. For W, a fixed point over $\{0\} \times \{0\}$ satisfy this property, since g acts trivially on the base of the family and gives an $\mathrm{A}_{|\Phi_{\mathcal{R}}|-1}$-singularity to the quotient surface. For W', this has been shown in (P6).

□

4.2.3 Multi-cover formula for $[\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))/\mathbb{Z}_n]$

When $n = 1$, the multicover formula in Proposition 4.2.4 is the famous Aspinwall–Morrison formula [3]. When $n > 1$, it can be obtained from Johnson-Pandripande-Tseng's computation in [46] about local orbifold Gromov-Witten invariants on orbifold $\mathbb{P}^1$, which we explain now.

Lemma 4.2.7. *Consider the action of $\mathbb{Z}_n$ on $\mathbb{P}^1$ and on the total space $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ defined in Proposition 4.2.4. The quotient stack $[\mathbb{P}^1/\mathbb{Z}_n]$ is isomorphic to the root stack $\mathbb{P}^1[n, n]$ of [46]. Moreover, the two line bundles on the quotient stack induced by the summands of $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ coincide with the duals $L_0^\vee$ and $L_\infty^\vee$ defined in [46].*

Proof. We compare the stacks and line bundles respectively.

(1) $[\mathbb{P}^1/\mathbb{Z}_n] \simeq \mathbb{P}^1[n, n]$. The quotient stack $[\mathbb{P}^1/\mathbb{Z}_n]$ has coarse space $\mathbb{P}^1$ with orbifold structure given as: at 0 and ∞, a neighborhood is uniformised by the branched covering map $z \mapsto z^n$. Therefore, it is isomorphic to the complex orbicurve $\big(\mathbb{P}^1, (0, \infty), (n, n)\big)$ defined in [20, Definition 2.2.2], which by [18, Example 2.4.5], coincide with the root stack $\mathbb{P}^1[n, n]$.

(2) *Identification of line bundles.* L_0 and L_∞ in [46] are defined as line bundles associated to the divisors:

$$L_0 = \mathcal{O}_{[\mathbb{P}^1/\mathbb{Z}_n]}([0/\mathbb{Z}_n]),\ L_\infty = \mathcal{O}_{[\mathbb{P}^1/\mathbb{Z}_n]}([\infty/\mathbb{Z}_n]), \tag{4.26}$$

It amounts to showing that their dual bundle coincide, as a $\mathbb{Z}_n$ equivariant bundle, with the summands of $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. To prove it, recall that the $\mathbb{Z}_n$ action is given by

$$\zeta[x : y] \;=\; [\zeta x : y], \tag{4.27}$$

therefore, the relative weight of x/y is 1. Since 0 and ∞ are cut out by x and y respectively, x (resp. y) needs to have weight 0 as global sections of L_0 (resp. L_∞). This coincides with the construction in Proposition 4.2.4.

□

Let $\overline{\mathcal{K}}_{0,0}\big(\mathbb{P}^1[n,n],d\big)$ be the moduli stack of genus-0 twisted stable maps $f\colon C \to \mathbb{P}^1[n,n]$ of degree d/n (in the convention of Section 1.2.1), where the domain C has no stacky points. Viewing local invariants as twisted invariants, we have

$$\langle\rangle_{d/n}^{[\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1))/\mathbb{Z}_n]} = \int_{[\overline{\mathcal{K}}_0(\mathbb{P}^1[n,n],d)]^{\mathrm{vir}}} e\big(R^1\pi_* f^*(L_0^\vee \oplus L_\infty^\vee)\big) \tag{4.28}$$

Therefore the invariant is exactly the one studied by Johnson-Pandripande-Tseng. In fact they give a formula for a generating function of

$$C_{g,\alpha+\beta}(d) := \int_{[\overline{\mathcal{K}}_{g,\alpha+\beta}(\mathbb{P}^1[a,b],d)]^{\mathrm{vir}}} e\big(R^1\pi_* f^*(L_0^\vee \oplus L_\infty^\vee)\big)$$

where $g,a,b \in \mathbb{Z}_{\geq 0}$ and $\alpha = (\alpha_1,\dots,\alpha_r), \beta = (\beta_1,\dots,\beta_s)$ with $1 \leq \alpha_i \leq a-1$ and $1 \leq \beta_j \leq b-1$. It's easy to see that with $a = b = n$, our invariant is their $C_{g,\alpha+\beta}(nd)$ with $\alpha = \beta = \vec{0}$ and can be extracted from their formula:

Theorem 4.2.8 (Johnson–Pandharipande–Tseng [46])**.** *Let $\alpha = (\alpha_1,\dots,\alpha_r)$ and $\beta = (\beta_1,\dots,\beta_s)$ with $1 \leq \alpha_i \leq a-1$ and $1 \leq \beta_j \leq b-1$.*

(1) $C_{g,\alpha+\beta}(d) = 0$ unless

$$d - \sum_{i=1}^{r} \alpha_i \equiv 0 \pmod{a} \qquad \textit{and} \qquad d - \sum_{j=1}^{s} \beta_j \equiv 0 \pmod{b}.$$

(2) If the congruences hold, then

$$\sum_{g\geq 0} C_{g,\alpha+\beta}(d)\, u^{2g} = (-1)^{r - \frac{\sum_{i=1}^r \alpha_i}{a} + \frac{d}{a} + s - \frac{\sum_{j=1}^s \beta_j}{b} + \frac{d}{b}} \frac{d^{r+s-3}}{a^{r-1}b^{s-1}} \left(\frac{ud/2}{\sin(ud/2)}\right)^2. \tag{4.29}$$

Now we can conclude the proof:

Corollary 4.2.9.

$$C_{0,\vec{0}}(nd) = (-1)^{2n}\frac{(nd)^3}{n^2} = \frac{1}{nd^3} \tag{4.30}$$

Proof. Noticing that

$$\left(\frac{ud/2}{\sin(ud/2)}\right)^2 = 1 + O(u^2), \tag{4.31}$$

$C_{0,\vec{0}}(nd)$ can be read from $g = 0$ term of (4.29). □

§ 4.3 A crepant resolution conjecture for $X_{\mathcal{R}^{\mathrm{fold}}}$

We have so far described the contribution from the *untwisted sector* to $\mathrm{QH}^*_{\mathbb{C}^\times}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}})$. In this section we formulate a conjecture for the full (orbifold) quantum cohomology and present supporting evidence. Our strategy is to compare it with the quantum cohomology of a crepant resolution of the coarse moduli space $X_{\mathcal{R}^{\mathrm{fold}}}$.

4.3.1 Crepant resolution of $X_{\mathcal{R}^{\mathrm{fold}}}$

As explained in the introduction, the surface $X_{\mathcal{R}^{\mathrm{fold}}}$ has $N_{\mathcal{R}^{\mathrm{fold}}}$ isolated Kleinian singularities: each is of type A_1 when $\Phi_{\mathcal{R}} \cong \mathbb{Z}_2$, and of type A_2 when $\Phi_{\mathcal{R}} \cong \mathbb{Z}_3$. This means $X_{\mathcal{R}^{\mathrm{fold}}}$ admits a crepant resolution by taking the minimal resolutions of these singularities:

$$\psi\colon Z_{\mathcal{R}^{\mathrm{res}}} \longrightarrow X_{\mathcal{R}^{\mathrm{fold}}}. \tag{4.32}$$

The setup of the *Crepant Resolution Conjecture* [58, 17] therefore applies and predicts that the orbifold quantum cohomology of $\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}$ can be obtained from the quantum cohomology of $Z_{\mathcal{R}^{\mathrm{res}}}$ by analytic continuation together with an explicit affine-linear isomorphism $H^*\big(Z_{\mathcal{R}^{\mathrm{res}}}\big) \cong H^*_{\mathrm{CR}}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}})$. One may therefore obtain the full $\mathrm{QH}^*_{\mathbb{C}^\times}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}})$ from $\mathrm{QH}^*_{\mathbb{C}^\times}(Z_{\mathcal{R}^{\mathrm{res}}})$ by taking such an affine map.

The quantum cohomology of $Z_{\mathcal{R}^{\mathrm{res}}}$ is completely clear. By [61], $Z_{\mathcal{R}^{\mathrm{res}}}$ is in fact the minimal resolution of a Kleinian surface singularity $\mathbb{C}^2/\Gamma_{\mathcal{R}^{\mathrm{res}}}$, where the root system $\mathcal{R}^{\mathrm{res}}$ associated to $\mathcal{R}^{\mathrm{fold}}$ is recorded in Table 4.2:

$\mathcal{R}^{\mathrm{fold}}$	$\mathcal{R}^{\mathrm{res}}$
B_n	D_{n+1}
C_m	D_{2m}
E_6	E_7
D_4	E_6

Table 4.2: minimal resolution types for the folded surfaces.

$\mathrm{QH}^*_{\mathbb{C}^\times}(Z_{\mathcal{R}^{\mathrm{res}}})$ therefore has been studied in detail by Bryan-Gholampour [16]. In this section, we formulate a precise conjecture, in our examples, for the affine-linear transformation relating $\mathrm{QH}^*_{\mathbb{C}^\times}(Z_{\mathcal{R}^{\mathrm{res}}})$ and $\mathrm{QH}^*_{\mathbb{C}^\times}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}})$.

4.3.2 Crepant resolution conjecture for $X_{\mathcal{R}^{\mathrm{fold}}}$

The affine-linear transformation will be written as blocked affine map under natural splitting of cohomologies.

Let $1 \le k \le N_{\mathcal{R}^{\mathrm{fold}}}$. Denote by $p_k \in Z_{\mathcal{R}}$ the fixed points, and $s_k \in X_{\mathcal{R}^{\mathrm{fold}}}$ the images of p_k under the quotient map. Write $E^{(k)} \subset Z_{\mathcal{R}^{\mathrm{res}}}$ for the exceptional configuration over s_k. Decomposing into irreducible components, $E^{(k)} = E_1^{(k)}$ when $\Phi_{\mathcal{R}} \cong \mathbb{Z}_2$ and $E^{(k)} = E_1^{(k)} \sqcup E_2^{(k)}$ when $\Phi_{\mathcal{R}} \cong \mathbb{Z}_3$.

As in (4.7), the definition of Chen–Ruan cohomology yields a natural splitting of graded vector spaces

$$H^*_{\mathrm{CR},\mathbb{C}^\times}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}) = H^*_{\mathbb{C}^\times}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}) \bigoplus_{k=1}^{N_{\mathcal{R}^{\mathrm{fold}}}} V_{s_k}, \tag{4.33}$$

where V_{s_i} is spanned by fundamental classes of components of $I\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}$ that are supported on s_i.

On the other hand, a corresponding splitting respecting ψ can be made as

$$H^*_{\mathbb{C}^\times}\left(Z_{\mathcal{R}^{\mathrm{res}}}\right) = \psi^*\left(H^*_{\mathbb{C}^\times}(X_{\mathcal{R}^{\mathrm{fold}}})\right) \bigoplus_{k=1}^{N_{\mathcal{R}^{\mathrm{fold}}}} W_{s_k}, \tag{4.34}$$

where W_{s_i} is spanned by exceptional classes supported on E_i.

Definition 4.3.1 (Affine identification Φ)**.** Define an affine map

$$\Phi : \; H^*\big(Z_{\mathcal{R}^{\mathrm{res}}}\big) \;\longrightarrow\; H^*_{\mathrm{CR}}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}})$$

as follows. On $\psi^* H^*(Z_{R'})$ we set $\Phi = \mathrm{id}$ via the isomorphism

$$H^*(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}) \cong H^*(X_{\mathcal{R}^{\mathrm{fold}}}) \tag{4.35}$$

For each k, we define $\Phi|_{W_{s_k}}$ to be:

(1) When $(\mathcal{R}, \mathcal{R}^{\mathrm{fold}}) = (A_{2n-1}, B_n), (D_{m+1}, C_m)$ or (E_6, F_4),

$$t_1^{(k)} := -\pi\mathrm{i} + \mathrm{i}\, x_1^{(k)}\,; \tag{4.36}$$

(2) When $(\mathcal{R}, \mathcal{R}^{\mathrm{fold}}) = (D_4, G_2)$,

$$t_i^{(k)} := \frac{2\pi\mathrm{i}}{3} + \frac{2\,\mathrm{i}}{3}\sum_{j=1}^{2} \sin(\frac{j}{3}\pi)\zeta^{-ij} x_j^{(k)}\,,\; i = 1, 2 \tag{4.37}$$

where $\zeta = \mathrm{e}^{2\pi\mathrm{i}/3}$.

Conjecture 4.3.2 (Crepant resolution prediction for $\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}$)**.** *After analytic continuation and affine change of variable as in Definition 4.3.1, there is an isomorphism*

$$\mathrm{QH}^*_{\mathbb{C}^\times}\big(Z_{\mathcal{R}^{\mathrm{res}}}\big) \;\cong\; \mathrm{QH}^*_{\mathrm{orb},\mathbb{C}^\times}\big(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}\big). \tag{4.38}$$

4.3.3 Chen–Ruan cohomology ring of $\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}$

Let $\Delta_\psi^+ = \mathrm{Ker}(\psi_*) \cap \mathrm{Eff}(Z_{\mathcal{R}^{\mathrm{res}}})$ be the semigroup of effective curve classes that is contracted by ψ. Following Ruan and Perroni [58, 55], we define :

Definition 4.3.3 (Quantum corrected ring of $Z_{\mathcal{R}^{\mathrm{res}}}$)**.** Let $Q_i^{(k)}$ be the quantum parameter with respect to $E_i^{(k)}$. For any $\phi_1, \phi_2, \phi_3 \in H^*(Z_{\mathcal{R}^{\mathrm{res}}})$, define the quantum corrected 3−point function $\langle\phi_1, \phi_2, \phi_3\rangle_{\mathrm{qc}}(q)$ as

$$\langle\phi_1, \phi_2, \phi_3\rangle_{\mathrm{qc}}(q) = \sum_{\beta\in\Delta_\psi^+} Q^\beta \langle\phi_1, \phi_2, \phi_3\rangle_{0,3,\beta}^{Z_{\mathcal{R}^{\mathrm{res}}},\mathbb{C}^\times}, \tag{4.39}$$

and quantum corrected product $\cup_\psi$ by

$$(\phi_1 \cup_\psi \phi_2, \phi_3)^{Z_{\mathcal{R}^{\mathrm{res}}}} = (\phi_1 \cup \phi_2, \phi_3)^{Z_{\mathcal{R}^{\mathrm{res}}}} + \langle\phi_1, \phi_2, \phi_3\rangle_{\mathrm{qc}}(q). \tag{4.40}$$

Proposition 4.3.4.

$$(\phi_1 \cup_\psi \phi_2, \phi_3)^{Z_{\mathcal{R}^{\mathrm{res}}}} = \nu \sum_{\beta\in\mathcal{R}^{\phi,+}} \langle\phi_1, \beta\rangle\langle\phi_2, \beta\rangle\langle\phi_3, \beta\rangle \frac{1+Q^\beta}{1-Q^\beta}, \tag{4.41}$$

where

$$\mathcal{R}^{\phi,+} = \mathrm{Ker}(\psi_*) \cap \mathcal{R}^{\mathrm{res},+} \cong \begin{cases} \prod_{i=1}^{N_{\mathcal{R}^{\mathrm{fold}}}} \mathcal{R}_{A_1}^+ & \textit{if } \Phi_{\mathcal{R}} \cong \mathbb{Z}_2, \\ \prod_{i=1}^{N_{\mathcal{R}^{\mathrm{fold}}}} \mathcal{R}_{A_2}^+ & \textit{if } \Phi_{\mathcal{R}} \cong \mathbb{Z}_3. \end{cases} \tag{4.42}$$

under the identification between $H_2(Z_{\mathcal{R}^{\mathrm{res}}}; \mathbb{Z})$ and the root lattice of $\mathcal{R}^{\mathrm{res}}$.

Proof. Notice that the quantum-corrected ring is just a specialisation of the full quantum cohomology ring to the contribution from contracted curve classes, so the result follows from comparison with [16]. By [16], only positive roots contribute to nontrivial Gromov–Witten invariants. It then remains to describe the contracted roots as in (4.42), which can be checked directly. □

For projective surfaces with a single A_1 or A_2 singularity, or in general projective varieties with transversal A_1 or A_2 singularities, Perroni [55] proves that under an explicit change of basis and specialisation of the quantum parameters, there is a ring isomorphism between their orbifold cohomology and certain quantum-corrected cohomology of their crepant resolutions, where the same splitting as (4.33) and (4.34) is used.

In our setup, $X_{\mathcal{R}^{\mathrm{fold}}}$ is non-compact and admits more than one A_1 or A_2 singularity. Moreover, an equivariant variant is used to define (quantum) cohomology. Nevertheless, the same construction yields a ring isomorphism.

Proposition 4.3.5. *Equip $H^*(Z_{\mathcal{R}^{\mathrm{res}}})$ with the quantum-corrected product $\cup_\psi$ defined in Definition 4.3.3. Then the map $H^*(Z_{\mathcal{R}^{\mathrm{res}}})(q) \to H^*_{\mathrm{CR}}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}})$ defined by the following formulas:*

(1) when $(\mathcal{R}, \mathcal{R}^{\mathrm{fold}}) = (A_{2n-1}, B_n), (D_{m+1}, C_m)$ *or* (E_6, F_4)*,*

$$E_1^{(k)} \mapsto 2\mathrm{i}\ e_1^{(k)}\ ,\ Q^{(k)} = -1; \tag{4.43}$$

(2) when $(\mathcal{R}, \mathcal{R}^{\mathrm{fold}}) = (D_4, G_2)$*,*

$$E_i^{(k)} \mapsto 2\mathrm{i} \sum_{j=1}^{2} \sin\left(\frac{j}{3}\pi\right) \zeta^{ij}\, e_j^{(k)}\ ,\ Q_i^{(k)} = \zeta \tag{4.44}$$

where $\zeta = \mathrm{e}^{2\pi \mathrm{i}/3}$*,*

is a ring isomorphism.

Proof. By construction, the linear map in the statement agrees, on each block

$$\psi^*\left(H^*_{\mathbb{C}^\times}(X_{\mathcal{R}^{\mathrm{fold}}})\right) \oplus W_{s_k},$$

with the map appearing in Perroni's formula [55, Proposition 6.2]. Using the same proof as in [55] with the equivariant variant, a ring isomorphism therefore holds on each block:

$$\psi^*\left(H^*_{\mathbb{C}^\times}(X_{\mathcal{R}^{\mathrm{fold}}})\right) \oplus W_{s_k} \cong H^*_{\mathbb{C}^\times}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}) \oplus V_{s_k}, \forall k. \tag{4.45}$$

Consequently, to reduce to Perroni's result, it suffices to verify the following two claims:

Claim 1. For $1 \le k \le N_{\mathcal{R}^{\mathrm{fold}}}$, the space $\psi^*\left(H^*_{\mathbb{C}^\times}(X_{\mathcal{R}^{\mathrm{fold}}})\right) \oplus W_{s_k}$ (resp. $H^*_{\mathbb{C}^\times}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}) \oplus V_{s_k}$) is closed under $\cup_\psi$ (resp. $\star_{\mathrm{CR},\mathbb{C}^\times}$).

Claim 2. $W_{s_{k_1}} \cup_\psi W_{s_{k_2}} = 0$ and $V_{s_{k_1}} \star_{\mathrm{CR}} V_{s_{k_2}} = 0$ if $k_1 \neq k_2$.

The resolution side follows from Proposition 4.2.2: from it we see that $(\phi_1 \cup_\psi \phi_2, \phi_3)^{Z_{\mathcal{R}^{\mathrm{res}}}}$ is zero as long as two of ϕ_1, ϕ_2, ϕ_3 are supported over fibres of distinct singular points.

On the orbifold side, we first show Claim 2. For classes δ_1 and δ_2 supported over distinct singular points, by construction,

$$\delta_1 \star_{\mathrm{CR}} \delta_2 \ \coloneqq\ \mu_*\Big(e_1^*\delta_1\ \cup\ e_2^*\delta_2\ \cup\ e_{g,h}\Big), \tag{4.46}$$

Since $e_1^*(\delta_1)$ and $e_2^*(\delta_2)$ have different support, $e_1^*\delta_1 \ \cup\ e_2^*\delta_2 = 0$. Hence one has $V_{s_{k_1}} \star_{\mathrm{CR}} V_{s_{k_2}} = 0$ if $k_1 \neq k_2$.

It remains to show Claim 1 on the orbifold side. First, the untwisted part is by itself closed, so is each component V_{s_k}. In order to show Claim 1, it suffices to show, for any $\phi \in H^*_{\mathbb{C}^\times}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}})$, and δ_1 and δ_2 supported over distinct singular points,

$$(\phi \star_{\mathrm{CR},\mathbb{C}^\times} \delta_1, \delta_2)^{\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}} = 0 \tag{4.47}$$

By associativity

$$(\phi \star_{\mathrm{CR},\mathbb{C}^\times} \delta_1, \delta_2)^{\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}} = (\phi, \delta_1 \star_{\mathrm{CR},\mathbb{C}^\times} \delta_2)^{\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}}, \tag{4.48}$$

it reduces to Claim 2. □

Remark 4.3.6. This result can be viewed as evidence for Conjecture 4.3.2. Indeed, assuming that Conjecture 4.3.2 holds, this result can be obtained by taking a certain limit, as follows. On the resolution side, the quantum modified ring can be viewed as the Frobenius algebra at the **partial large radius limit**:

$$\mathrm{Re}\langle \tau, E\rangle \to -\infty, \tag{4.49}$$

in the big quantum product, or

$$Q_E \to 0 \tag{4.50}$$

in the small quantum product, for all uncontracted curves $E \subset Z_{\mathcal{R}^{\mathrm{res}}}$.

Correspondingly, on the orbifold side, all curves in $X_{\mathcal{R}^{\mathrm{fold}}}$ are of the form $\psi(E)$, with E an uncontracted curve in $Z_{\mathcal{R}^{\mathrm{res}}}$, under the **large radius limit**

$$\mathrm{Re}\langle \tau, E\rangle \to -\infty, \tag{4.51}$$

in the big quantum product, or

$$Q_{\psi(E)} \to 0, \tag{4.52}$$

in the small quantum product, $\mathrm{QH}^*_{\mathrm{orb}}\big(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}\big)$ degenerates to the ordinary Chen–Ruan cohomology $H^*_{\mathrm{CR}}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}})$.

To finish the remark, one can check that the linear map in Proposition 4.3.5 is the inverse of the linear part of Definition 4.3.1.

4.3.4 Compatibility with a Fourier–Mukai transform

Ruan's conjecture is closely related to the (largely open) DK conjecture [6, 47], which predicts that K-equivalent varieties/orbifolds should be derived equivalent and hence have isomorphic K-groups. A conceptual link is provided by Iritani's integral structure and central charge on K-theory [43]: he explains how a derived equivalence induces a concrete affine transformation on cohomology, compatible with the relevant enumerative structures.

We first fix the integral-structure conventions. For a smooth Deligne–Mumford stack $\mathcal{Z}$ of complex dimension d and $V \in K_c(\mathcal{Z})$, we use Iritani's cohomology-valued class [43, Equation (12)]

$$\Psi_{\mathcal{Z}}(V) := (2\pi)^{-d/2}\widehat{\Gamma}(T\mathcal{Z}) \cup (2\pi \mathrm{i})^{\deg/2}\,\mathrm{inv}^*\,\widetilde{\mathrm{ch}}(V), \qquad d = \dim_{\mathbb{C}} \mathcal{Z}, \tag{4.53}$$

where deg is the ordinary cohomological degree on the inertia stack. Following [43, Proposition 3.28], the integral central charge is then

$$\mathcal{F}_{\mathcal{Z}}(V)(\tau) = (2\pi)^{-d/2}\mathrm{i}^{-d}\left(J_{\mathcal{Z}}(\tau, z)\big|_{z=-1}, \Psi_{\mathcal{Z}}(V)\right)_{\mathrm{orb}}. \tag{4.54}$$

For the noncompact spaces considered here, we take $V \in K_c(\mathcal{Z})$, so that $\Psi_{\mathcal{Z}}(V)$ is compactly supported and the orbifold pairing in (4.54) is well-defined.

Remark 4.3.7. The J-function $J_{\mathcal{Z}}(\tau, z)$ used by Iritani in this setup is normalised with expansion ([43, Equation (55)])

$$J_{\mathcal{Z}}(\tau, z) = 1 + \frac{\tau}{z} + O(z^{-2}). \tag{4.55}$$

Conjecture 4.3.8 (Following Iritani [43, Corollary 3.30] [42, Proposal 5.7])**.** *Let $\mathcal{X}$ be an orbifold with coarse moduli space X, and let $\phi\colon Z \to X$ be a crepant resolution. There exists an isomorphism*

$$\mathbb{U}_K\colon K_c(Z) \longrightarrow K_c(\mathcal{X}) \tag{4.56}$$

induced by a Fourier–Mukai equivalence. After analytic continuation, there exists an affine change of flat coordinates $\mathbb{U}_{\mathrm{coh}}$ with the following properties:

(1) the quantum $\mathcal{D}$-modules of Z and $\mathcal{X}$ are isomorphic in a way that preserves their $\widehat{\Gamma}$-integral structures and is induced by $\mathbb{U}_K$;

(2) $\mathbb{U}_{\mathrm{coh}}$ identifies the corresponding quantum products;

(3) for every $V \in K_c(Z)$,

$$\mathcal{F}_Z(V)(\tau) = \pm\,\mathcal{F}_{\mathcal{X}}\big(\mathbb{U}_K(V)\big)\big(\mathbb{U}_{\mathrm{coh}}(\tau)\big), \tag{4.57}$$

where the sign $\pm$ is independent of V.

For Kleinian singularities, the derived McKay correspondence gives an isomorphism

$$K_c\big(\widetilde{\mathbb{C}^2/G}\big) \;\simeq\; K_0^G(\mathbb{C}^2), \tag{4.58}$$

for any finite subgroup $G < \mathrm{SL}(2;\mathbb{C})$. Iritani explains how the associated integral central charges recover the coordinate changes predicted by the quantum McKay correspondence; see [43, §3.9].

In the remainder of this section, we verify this predicted central-charge compatibility for the Fourier–Mukai transform (4.60) and the affine change of variables in Definition 4.3.1.

By [44, Theorem 1.6], $\Phi_{\mathcal{R}}$−Hilb$(Z_{\mathcal{R}})$ is a crepant resolution of $\mathbb{C}^2/\Gamma_{\mathcal{R}^{res}}$ and therefore isomorphic to the minimal resolution $Z_{\mathcal{R}^{\mathrm{res}}}$. Let $\mathcal{U}_{\mathcal{R}} \subset Z_{\mathcal{R}^{\mathrm{res}}} \times Z_{\mathcal{R}}$ be the universal scheme, with projections

$$p\colon \mathcal{U}_{\mathcal{R}} \longrightarrow Z_{\mathcal{R}^{\mathrm{res}}}, \qquad q\colon \mathcal{U}_{\mathcal{R}} \longrightarrow Z_{\mathcal{R}}. \tag{4.59}$$

The derived McKay correspondence of Bridgeland–King–Reid [9] then implies that the Fourier–Mukai transform

$$Rq_* \circ p^* : K(Z_{\mathcal{R}^{\mathrm{res}}}) \to K^{\Phi_{\mathcal{R}}}(Z_{\mathcal{R}}), \tag{4.60}$$

is a derived equivalence. In particular, coherent sheaves on $Z_{\mathcal{R}^{\mathrm{res}}}$ naturally correspond to $\Phi_{\mathcal{R}}$-equivariant sheaves on $Z_{\mathcal{R}}$.

Given a vertex ρ in the Dynkin diagram of $\mathcal{R}^{res}$, let $C_\rho \subset Z_{\mathcal{R}^{\mathrm{res}}}$ be the corresponding Dynkin curve.

Proposition 4.3.9. *The image of $\mathcal{O}_{C_\rho}(-1)$ under $Rq_* \circ p^*$, as $\Phi_{\mathcal{R}}$-equivariant sheaves $\mathcal{L}_\rho$ on $K(Z_{\mathcal{R}})$, are as follows:*

(1) If C_ρ is contracted to a point x_ρ on the quotient $X_{\mathcal{R}^{\mathrm{fold}}}$ corresponding to a $\Phi_{\mathcal{R}}$–fixed point $y_\rho \in Z_{\mathcal{R}}$, then define $\mathcal{L}_\rho := k(y_\rho) \otimes \omega$, where $\omega : \Phi_{\mathcal{R}} \to \mathbb{C}^\times$ is the irreducible representation of $\Phi_{\mathcal{R}}$ in the local McKay correspondence of ψ at x_ρ;

*(2A) If $\pi^{-1}(\psi(C_\rho))$ is a **free** $\Phi_{\mathcal{R}}$-orbit of irreducible curves $\Phi_{\mathcal{R}} \cdot E_\rho$, where $E_\rho \subset Z_{\mathcal{R}}$ is any representative. In this case define $\mathcal{L}_\rho := \sum_{g \in \Phi_{\mathcal{R}}} \mathcal{O}_{g(E_\rho)}(-1)$;*

(2B) Otherwise, $\pi^{-1}(\psi(C_\rho))$ is a single curve E_ρ, which is stable under the $\Phi_{\mathcal{R}}$ action. In this case define $\mathcal{L}_\rho = \mathcal{O}_{E_\rho}(-|\Phi_{\mathcal{R}}|)$.

Proof. We use the property that the Fourier–Mukai transform of a sheaf can be restricted to a neighbourhood of its support and show (FM) in the two cases respectively:

(1) If C_ρ is contracted to a point x_ρ, then over a $\Phi_{\mathcal{R}}$-invariant neighbourhood of x_ρ, (p, q) can be identified with

$$\begin{tikzcd} & \mathcal{U} \subset \Phi_{\mathcal{R}}\text{–Hilb}(\mathbb{C}^2) \times \mathbb{C}^2 \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \widetilde{\mathbb{C}^2/\Phi_{\mathcal{R}}} & & \mathbb{C}^2 \end{tikzcd} \tag{4.61}$$

which is the same as in the local McKay correspondence;

(2A) If $\pi^{-1}(\psi(C_\rho))$ is a free $\Phi_{\mathcal{R}}$-orbit, then q is an isomorphism and p is a covering map over a neighbourhood of $\psi(C_\rho)$. It follows that $\mathcal{L}_\rho := \sum_{g \in \Phi_{\mathcal{R}}} \mathcal{O}_{g(E_\rho)}(-1)$.

(2B) If $\pi^{-1}(\psi(C_\rho))$ is a single curve E_ρ, then over a neighbourhood of $\psi(C_\rho)$, the pair (p, q) can be identified with

$$\begin{tikzcd} & \mathcal{U} \subset Y \times Tot(\mathcal{O}(-2)) \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Y & & Tot(\mathcal{O}(-2)) \end{tikzcd} , \tag{4.62}$$

where $Y = \Phi_{\mathcal{R}}\text{–Hilb}(Tot(\mathcal{O}(-2)))$ is the minimal resolution of an A_3 (resp. A_5) singularity when $\Phi_{\mathcal{R}} \cong \mathbb{Z}_2$ (resp. $\mathbb{Z}_3$), and C_ρ is the component corresponding to the middle vertex in the corresponding Dynkin diagram. Therefore, p is a degree $|\Phi_{\mathcal{R}}|$ map over C_ρ and q is an isomorphism restricting to $p^{-1}(C_\rho) \cong \mathbb{P}^1$. As a consequence, we get

$$q_* p^*(\mathcal{O}_{C_\rho}(-1)) = \mathcal{O}_{E_\rho}(-|\Phi_{\mathcal{R}}|). \tag{4.63}$$

□

To show compatibility with integral structure, we start with a local computation.

Lemma 4.3.10. *Let $n \geq 2$ and $\zeta = \mathrm{e}^{2\pi\mathrm{i}/n}$. Consider the cotangent lift of the $\mathbb{Z}_n = \langle\zeta\rangle$ action*

$$\zeta^k \circ [z_0, z_1] = [\zeta^k z_0, z_1]$$

on $\mathbb{P}^1$ to $Tot(\mathcal{O}_{\mathbb{P}^1}(-2)) = T^*\mathbb{P}^1$. Let

$$i : [\mathbb{P}^1/\mathbb{Z}_n] \longrightarrow [Tot(\mathcal{O}_{\mathbb{P}^1}(-2))/\mathbb{Z}_n]$$

be the zero-section inclusion, and equip $\mathcal{O}_{\mathbb{P}^1}(-n)$ with the trivial action on the fibers over fixed points p_0 and p_∞.

Let $\mathcal{F}_\tau(V)$ denote the integral central charge (4.54), where Ψ is the cohomology-valued class defined in (4.53). Then

$$\mathcal{F}_\tau\big(i_*\mathcal{O}_{\mathbb{P}^1}(-n)\big) = -\frac{n-1}{n} + \frac{\mathrm{i}\tau_0}{2\pi n} + \mathcal{F}_{p_0,\mathrm{tw}} + \mathcal{F}_{p_\infty,\mathrm{tw}}, \tag{4.64}$$

with

$$\mathcal{F}_{p_0,\mathrm{tw}} = \sum_{k=1}^{n-1} \frac{1-\zeta^k}{4\pi n \sin(\pi k/n)}\tau_{0,k}, \qquad \mathcal{F}_{p_\infty,\mathrm{tw}} = \sum_{k=1}^{n-1} \frac{1-\zeta^{-k}}{4\pi n \sin(\pi k/n)}\tau_{\infty,k}. \tag{4.65}$$

Here the coordinates $\tau_0, \tau_{0,k}, \tau_{\infty,k}$ of

$$\tau = \tau_0 H + \sum_{k=1}^{n-1} \tau_{0,k} e_{0,k} + \sum_{k=1}^{n-1} \tau_{\infty,k} e_{\infty,k} \in H^*_{\mathrm{CR}}([Tot(\mathcal{O}_{\mathbb{P}^1}(-2))/\mathbb{Z}_n]) \tag{4.66}$$

are defined by taking $H = c_1(\mathcal{O}_{\mathbb{P}^1}(1))$, normalized by

$$\int_{[\mathbb{P}^1/\mathbb{Z}_n]} H = \frac{1}{n}, \tag{4.67}$$

while $\mathrm{e}_{0,k}$ and $\mathrm{e}_{\infty,k}$ are the fundamental classes of the twisted sectors over p_0 and p_∞, respectively, labelled by ζ^k.

Proof. Write $\mathcal{X}_n = [Tot(\mathcal{O}_{\mathbb{P}^1}(-2))/\mathbb{Z}_n]$ and let $N = \mathcal{O}_{\mathbb{P}^1}(-2)$ be the normal bundle of the zero section. Grothendieck–Riemann–Roch for the closed immersion i gives

$$\mathrm{ch}(i_*L)\,\mathrm{Td}(T\mathcal{X}_n) = i_*\big(\mathrm{ch}(L)\,\mathrm{Td}(T[\mathbb{P}^1/\mathbb{Z}_n])\big), \tag{4.68}$$

or, equivalently,

$$\operatorname{ch}(i_*L) = i_*\big(\operatorname{ch}(L)\operatorname{Td}(N)^{-1}\big). \tag{4.69}$$

For $L = \mathcal{O}_{\mathbb{P}^1}(-n)$, using $H^2 = 0$, one has

$$\operatorname{ch}(L) = 1 - nH, \qquad \operatorname{Td}(N) = 1 - H, \qquad \operatorname{Td}(N)^{-1} = 1 + H. \tag{4.70}$$

Since $i_*(1) = [\mathbb{P}^1/\mathbb{Z}_n]$ and $i_*(H) = \frac{1}{n}[\mathrm{pt}]$, the untwisted component is

$$\begin{aligned} \widetilde{\operatorname{ch}}\big(i_*\mathcal{O}_{\mathbb{P}^1}(-n)\big)\big|_{(e)} &= i_*\big((1-nH)(1+H)\big) \\ &= [\mathbb{P}^1/\mathbb{Z}_n] - \frac{n-1}{n}[\mathrm{pt}]. \end{aligned} \tag{4.71}$$

Let $q : Tot(\mathcal{O}_{\mathbb{P}^1}(-2)) \to \mathbb{P}^1$ be the projection. The equivariant Koszul resolution of the zero section is

$$0 \longrightarrow q^*(L \otimes N^\vee) \longrightarrow q^*L \longrightarrow i_*L \longrightarrow 0. \tag{4.72}$$

By the definition of the orbifold Chern character [43, 2.4–2.5],

$$\widetilde{\operatorname{ch}}\big(i_*L\big)\big|_{(p,\zeta^k)} = \operatorname{Tr}(\zeta^k|L_p)\left(1 - \operatorname{Tr}(\zeta^k|N_p^\vee)\right). \tag{4.73}$$

At p_0, the tangent and normal characters are (ζ, ζ^{-1}), while at p_∞ they are (ζ^{-1}, ζ). Together with the trivial fiber action on L, this gives

$$\widetilde{\operatorname{ch}}\big(i_*\mathcal{O}_{\mathbb{P}^1}(-n)\big)\big|_{(p_0,\zeta^k)} = 1 - \zeta^k, \qquad \widetilde{\operatorname{ch}}\big(i_*\mathcal{O}_{\mathbb{P}^1}(-n)\big)\big|_{(p_\infty,\zeta^k)} = 1 - \zeta^{-k}. \tag{4.74}$$

The untwisted Gamma class restricts to 1, and on every twisted sector labelled by ζ^k it is

$$\widehat{\Gamma}_k = \Gamma\left(\frac{k}{n}\right)\Gamma\left(1 - \frac{k}{n}\right) = \frac{\pi}{\sin(\pi k/n)}. \tag{4.75}$$

Specializing (4.53) to the surface $\mathcal{X}_n$ gives

$$\Psi(V) = \frac{1}{2\pi}\widehat{\Gamma}(T\mathcal{X}_n) \cup (2\pi \mathrm{i})^{\deg/2} \operatorname{inv}^* \widetilde{\operatorname{ch}}(V), \tag{4.76}$$

where deg is the ordinary cohomological degree on the inertia stack, its untwisted component is

$$\Psi\big(i_*\mathcal{O}_{\mathbb{P}^1}(-n)\big)\big|_{(e)} = \mathrm{i}[\mathbb{P}^1/\mathbb{Z}_n] + \frac{2\pi(n-1)}{n}[\mathrm{pt}], \tag{4.77}$$

and its twisted components are

$$\Psi\big(i_*\mathcal{O}_{\mathbb{P}^1}(-n)\big)\big|_{(p_0,\zeta^k)} = \frac{1-\zeta^{-k}}{2\sin(\pi k/n)}[\mathrm{pt}_{0,k}], \tag{4.78}$$

$$\Psi\big(i_*\mathcal{O}_{\mathbb{P}^1}(-n)\big)\big|_{(p_\infty,\zeta^k)} = \frac{1-\zeta^{k}}{2\sin(\pi k/n)}[\mathrm{pt}_{\infty,k}]. \tag{4.79}$$

To pair these components with the twisted coordinates of τ, by definition of the orbifold Poincaré pairing,

$$\big(e_{j,k}, [\mathrm{pt}_{j,n-k}]\big)_{\mathrm{orb}} = \frac{1}{n}, \qquad j \in \{0, \infty\}. \tag{4.80}$$

Thus the coefficient of $\tau_{j,k}$ is read from the $(j, n-k)$-component of Ψ. By (4.55), the part of $J(\tau, z)|_{z=-1}$ that pairs nontrivially with the compactly supported components of $\Psi(V)$ is $1-\tau$; the remaining $O(z^{-2})$ terms have no component in the dual ordinary degrees. Since $\dim_{\mathbb{C}} \mathcal{X}_n = 2$, the prefactor in (4.54) is $-1/(2\pi)$. Hence

$$\mathcal{F}_\tau(V) = -\frac{1}{2\pi}(1-\tau, \Psi(V))_{\mathrm{orb}} \tag{4.81}$$

for $V = i_*\mathcal{O}_{\mathbb{P}^1}(-n)$. Substituting the preceding components gives the constant term $-(n-1)/n$, the untwisted linear term $\mathrm{i}\tau_0/(2\pi n)$, and the two twisted sums stated above. □

Proposition 4.3.11. *The change of variables in Definition 4.3.1 is compatible with the Fourier–Mukai transform* (4.60) *under the integral structure central charge in the following sense:*

(1) when $(\mathcal{R}, \mathcal{R}^{\mathrm{fold}}) = (A_{2n-1}, B_n), (D_{m+1}, C_m)$ *or* (E_6, F_4)*,*

$$\mathcal{F}(\mathcal{O}_{C_\rho}(-1)) = \mathcal{F}(\mathcal{L}_\rho) \tag{4.82}$$

(2) when $(\mathcal{R}, \mathcal{R}^{\mathrm{fold}}) = (D_4, G_2)$*,*

$$\mathcal{F}(\mathcal{O}_{C_\rho}(-1)) = \pm\mathcal{F}(\mathcal{L}_\rho), \tag{4.83}$$

where the sign $\pm$ *is* -1 *when* C_ρ *is contracted under* ψ *and* $+1$ *in the other cases.*

Proof. We notice that the image of the central charge of a sheaf depends only on a neighbourhood of its support, where the Fourier-Mukai transform has been described in the proof of Proposition 4.3.9. We show the compatibility corresponding to the three cases respectively:

(1) Suppose first that C_ρ is contracted to a fixed point. Restricting to a $\Phi_{\mathcal{R}}$-invariant neighbourhood, (p, q) is the usual local McKay correspondence for the cyclic quotient singularity of type $A_{|\Phi_{\mathcal{R}}|-1}$, hence of type A_1 or A_2. In this local situation, the change of integral central charges has been computed in [43]. Comparing with the affine identification in Definition 4.3.1, we obtain

$$\mathcal{F}(\mathcal{O}_{C_\rho}(-1)) = \mathcal{F}(\mathcal{L}_\rho) \tag{4.84}$$

for the order-two foldings in part (1) of the statement. In the (D_4, G_2) case, the affine identification in Definition 4.3.1 differs from the convention of [43] by an overall negative sign, so for the contracted curves one obtains

$$\mathcal{F}(\mathcal{O}_{C_\rho}(-1)) = -\mathcal{F}(\mathcal{L}_\rho). \tag{4.85}$$

This is the negative sign in part (2) of the statement, and the same sign is used below for the side-block curves in case $(2B)$.

(2A) Suppose that $\pi^{-1}(\psi(C_\rho))$ is a free $\Phi_{\mathcal{R}}$-orbit. From [43], we know that

$$\mathcal{F}_{Z_{\mathcal{R}^{\mathrm{res}}}}(\mathcal{O}_{C_\rho}(-1)) = -\frac{1}{2\pi \mathrm{i}}\, t_\rho. \tag{4.86}$$

Since the support lies in the free locus of the quotient stack, the orbifold central charge is the ordinary central charge upstairs divided by $|\Phi_{\mathcal{R}}|$. Thus

$$\mathcal{F}_{\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}}(\mathcal{L}_\rho) = \frac{1}{|\Phi_{\mathcal{R}}|}\mathcal{F}_{Z_{\mathcal{R}}}\Big(\sum_{g\in\Phi_{\mathcal{R}}} \mathcal{O}_{g(E_\rho)}(-1)\Big) = -\frac{1}{2\pi \mathrm{i}\,|\Phi_{\mathcal{R}}|}\sum_{g\in\Phi_{\mathcal{R}}} t_{g(E_\rho)}. \tag{4.87}$$

On the other hand, by the averaging convention (4.1), (4.3), and the identity on the untwisted summand in Definition 4.3.1, the flat coordinates satisfy

$$t_\rho = \frac{1}{|\Phi_{\mathcal{R}}|}\sum_{g\in\Phi_{\mathcal{R}}} t_{g(E_\rho)}. \tag{4.88}$$

Hence $\mathcal{F}(\mathcal{O}_{C_\rho}(-1)) = \mathcal{F}(\mathcal{L}_\rho)$ in case $(2A)$, which is the $+$ sign in the statement.

(2B) The integral central charge of $\mathcal{L}_\rho$ is computed in Lemma 4.3.10. The resolution side is the minimal resolution of an $A_{2|\Phi_{\mathcal{R}}|-1}$ singularity, with $C_\rho = C_{|\Phi_{\mathcal{R}}|}$ the middle curve, and the integral central charge of $\mathcal{O}_{C_\rho}(-1)$ is computed in [43] as

$$-\frac{1}{2\pi \mathrm{i}}\tau\cap C_\rho. \tag{4.89}$$

Take $C_1,\dots,C_{2|\Phi_{\mathcal{R}}|-1}$ as the $A_{2|\Phi_{\mathcal{R}}|-1}$ chain on the resolution. The basis matching the splitting (4.34) is

$$\{D = \psi^*(\pi_*(E_\rho)), C_1,\dots,C_{|\Phi_{\mathcal{R}}|-1}, C_{|\Phi_{\mathcal{R}}|+1},\dots,C_{2|\Phi_{\mathcal{R}}|-1}\}, \tag{4.90}$$

where

$$D = C_{|\Phi_{\mathcal{R}}|} + \sum_{k=1}^{|\Phi_{\mathcal{R}}|-1}\frac{k}{|\Phi_{\mathcal{R}}|}C_k + \sum_{k=|\Phi_{\mathcal{R}}|+1}^{2|\Phi_{\mathcal{R}}|-1}\frac{2|\Phi_{\mathcal{R}}|-k}{|\Phi_{\mathcal{R}}|}C_k, \tag{4.91}$$

where the coefficients are determined by the relations

$$D\cdot C_k = 0, \qquad k\neq|\Phi_{\mathcal{R}}|. \tag{4.92}$$

Now write

$$\tau = s\,D + \sum_{k=1}^{|\Phi_{\mathcal{R}}|-1} p_k C_k + \sum_{k=|\Phi_{\mathcal{R}}|+1}^{2|\Phi_{\mathcal{R}}|-1} p_k C_k \in H^*(Tot(\widetilde{\mathcal{O}_{\mathbb{P}^1}(-2)})/\mathbb{Z}_{|\Phi_{\mathcal{R}}|}), \tag{4.93}$$

and let

$$t_k = \tau \cap C_k = \begin{cases} p_{k-1} - 2p_k + p_{k+1} & k \neq |\Phi_{\mathcal{R}}|, \\ -\frac{2}{|\Phi_{\mathcal{R}}|}\, s + p_{|\Phi_{\mathcal{R}}|-1} + p_{|\Phi_{\mathcal{R}}|+1} & k = |\Phi_{\mathcal{R}}|, \end{cases} \tag{4.94}$$

where $p_0 = p_{|\Phi_{\mathcal{R}}|} = p_{2|\Phi_{\mathcal{R}}|} = 0$, and $-2/|\Phi_{\mathcal{R}}|$ is the intersection number $D \cdot C_{|\Phi_{\mathcal{R}}|}$. Notice also that $D \cdot C_{|\Phi_{\mathcal{R}}|} = -2H \cdot \psi(C_{|\Phi_{\mathcal{R}}|})$. Since the untwisted degree-two block is one-dimensional and the identification preserves the pairing with the middle curve, this fixes the relation

$$\tau_0 = -2s \tag{4.95}$$

on the untwisted part.

It remains to substitute the affine coordinates in Definition 4.3.1 into the expression for $t_{|\Phi_{\mathcal{R}}|}$. Following the notation of Definition 4.3.1, we relabel the two local blocks by the fixed points p_0 and p_∞; the corresponding affine coordinates are $x_j^{(0)}$ and $x_j^{(\infty)}$. When $|\Phi_{\mathcal{R}}| = 2$, C_1 and C_3 are -2 curves, so

$$p_1 = -\frac{1}{2} t_1, \qquad p_3 = -\frac{1}{2} t_3, \tag{4.96}$$

and the affine substitution is the one in Definition 4.3.1, with $x_1^{(0)} = \tau_{0,1}$ and $x_1^{(\infty)} = \tau_{\infty,1}$. No additional sign is inserted in this case. Thus

$$p_1 + p_3 = \pi \mathrm{i} - \frac{\mathrm{i}}{2} \left(\tau_{0,1} + \tau_{\infty,1} \right). \tag{4.97}$$

When $|\Phi_{\mathcal{R}}| = 3$, the left-hand A_2 block satisfies

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = -\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \qquad p_2 = -\frac{t_1 + 2t_2}{3}. \tag{4.98}$$

On this block the sector labels are $x_j^{(0)} = \tau_{0,j}$. At p_∞ the tangent character is inverse to the tangent character at p_0. Therefore the local A_2 ordering used in Definition 4.3.1 is opposite to the global ordering of the right-hand side of the chain $C_1 - \cdots - C_5$: the ordered global pair is (t_5, t_4), and the sector labels satisfy

$$(x_1^{(\infty)}, x_2^{(\infty)}) = (\tau_{\infty,2}, \tau_{\infty,1}). \tag{4.99}$$

With this ordering, the right-hand block satisfies

$$\begin{pmatrix} t_5 \\ t_4 \end{pmatrix} = -\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} p_5 \\ p_4 \end{pmatrix}, \qquad p_4 = -\frac{t_5 + 2t_4}{3}. \tag{4.100}$$

We then substitute the affine coordinates from Definition 4.3.1. In the (D_4, G_2) case, the side-block curves belong to the contracted-to-a-fixed-point case treated in (1) above, so the negative sign there gives $(t_1, t_2) = -(t_1^{(0)}, t_2^{(0)})$ and $(t_5, t_4) = -(t_1^{(\infty)}, t_2^{(\infty)})$. Consequently,

$$
\begin{aligned}
p_2 &= \frac{2\pi \mathrm{i}}{3} - \frac{\mathrm{i}\sqrt{3}}{9}\left((1-\zeta)\tau_{0,1} + (1-\zeta^2)\tau_{0,2}\right), \\
p_4 &= \frac{2\pi \mathrm{i}}{3} - \frac{\mathrm{i}\sqrt{3}}{9}\left((1-\zeta^2)\tau_{\infty,1} + (1-\zeta)\tau_{\infty,2}\right).
\end{aligned}
\tag{4.101}
$$

Substituting the preceding expressions into $-\frac{1}{2\pi\mathrm{i}} t_{|\Phi_{\mathcal{R}}|}$, together with $\tau_0 = -2s$, gives exactly the central charge in Lemma 4.3.10. Hence $\mathcal{F}(\mathcal{O}_{C_\rho}(-1)) = \mathcal{F}(\mathcal{L}_\rho)$ in case $(2B)$, which is the $+$ sign in the statement since C_ρ is not contracted by ψ in this case.

□

Remark 4.3.12. The ρ-dependent sign in the (D_4, G_2) case, equivalently in the local $|\Phi_{\mathcal{R}}| = 3$ comparison, is not part of Iritani's original proposal. It appears here because the affine change of variables matches the central charges only after inserting this sign. A conceptual explanation of this sign is still missing, and we leave it for future work.

§ 4.4 Frobenius structures associated with affine $BCFG$ root systems

Given any irreducible root system Φ, including the non-simply-laced ones, Dubrovin–Zhang [35] construct a natural Frobenius structure $\mathcal{M}_\Phi$ on the orbit space of the corresponding extended affine Weyl group. Later work of Brini–van Gemst [15] provides Lie-theoretic Landau–Ginzburg superpotentials for all these Frobenius manifolds. When Φ is simply-laced, Brini–Ma–Strachan [14] relate the quantum cohomology of the minimal resolution of the corresponding Kleinian singularity to the Dubrovin dual of $\mathcal{M}_\Phi$. When Φ is non-simply-laced, a similar Frobenius algebra has been constructed by Bryan–Gholampour [16], but the geometric counterpart is not yet known. In this section, we explain how the untwisted part of the orbifold quantum cohomology $\mathrm{QH}^*_{\mathbb{C}^\times}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}})$ provides a geometric counterpart of these Frobenius structures.

Let $\mathcal{R}^{\mathrm{fold}}$ be one of the non-simply-laced root systems in Table 4.1, and let $\iota : H^*(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}) \to H^*_{\mathrm{CR}}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}})$ be the restriction.

The first observation is the comparison with Bryan–Gholampour's construction.

Corollary 4.4.1. *After applying*

$$
Q^{\bar{\beta}} = \mathrm{e}^{-\langle \bar{\beta}, \tau \rangle}, \tag{4.102}
$$

$\iota^*\big(\mathrm{QH}^*_{\mathbb{C}^\times}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}})\big)$ *is isomorphic, as a Frobenius algebra, to the one constructed in [16, Theorem 6] for* $\mathcal{R}^{\mathrm{ave}}$.

Proof. Notice that Theorem 4.2.3 coincides with [16, Theorem 6] exactly. $\square$

The second observation extends the simply-laced result of Brini–Ma–Strachan [14] to the non-simply-laced cases.

Theorem 4.4.2. *$\iota^*(\mathrm{QH}^*_{\mathbb{C}^\times}(\mathcal{X}_{\mathcal{R}^{\mathrm{fold}}}))$ is isomorphic to the Dubrovin dual $\mathcal{M}^\flat_{\mathcal{R}^{\mathrm{ave}}}$ of the extended affine Weyl Frobenius manifold $\mathcal{M}_{\mathcal{R}^{\mathrm{ave}}}$ corresponding to $\mathcal{R}^{\mathrm{ave}}$.*

Proof. By Brini–van Gemst [15], $\mathcal{M}_{\mathcal{R}}$ is isomorphic to a Landau-Ginzburg(LG) model $(\lambda_{\mathcal{R}}, \phi)$. For a LG model (λ, ϕ), its Dubrovin dual is another LG model$(\log(\lambda), \phi)$. Having this, the result follows from proving $\lambda_{\mathcal{R}}|_{\mathfrak{h}_{\mathcal{R}^{\mathrm{ave}}}} = \lambda_{\mathcal{R}^{\mathrm{ave}}}$. This fact is briefly mentioned in [15], and we explain it here in more details.

We recall the construction of the LG superpotential given any irreducible root system Φ. First, Φ is associated with a Lie algebra $\mathfrak{g}_\Phi$. λ_Φ is then constructed by solving zeroes of

$$\mathcal{P}(w_0, \ldots, w_{l_\Phi}; \lambda_\Phi, \mu) = \mathcal{Q}^{\mathrm{red}}(\chi_i = w_i - \delta_{i,\bar{k}}\frac{\lambda_\Phi}{w_0}; \mu), \tag{4.103}$$

where $\mathcal{Q}^{\mathrm{red}}$ is the $(1-\mu)$-free part of the characteristic polynomial

$$(1-\mu)^{z_0}\mathcal{Q}^{\mathrm{red}} = \mathcal{Q} = \det_\rho(g - \mu 1) \in \mathbb{Z}[\chi_1, \cdots, \chi_{l_\Phi}][\mu] \tag{4.104}$$

of a fundamental representation ρ of $\mathfrak{g}_\Phi$ with minimal dimension, and $\chi_i(g) := Tr_{\rho_i}(g)$ is the i^{th} fundamental character. Moreover, $\bar{k}$ is a marked node in correspondence (see [14]) with the index of the $\mathbb{C}^\times$ fixed curve in the exceptional curves, shown in Figure 4.1. We denote the fundamental weight corresponding to the marked node by ω^{tw}. In order to prove $\lambda_{\mathcal{R}}|_{\mathfrak{h}_{\mathcal{R}^{\mathrm{ave}}}} = \lambda_{\mathcal{R}^{\mathrm{ave}}}$, it's enough to show

1 $\rho_{\mathcal{R}}\big|_{\mathfrak{h}_{\mathcal{R}^{\mathrm{ave}}}}$ contains $\rho_{\mathcal{R}^{\mathrm{ave}}}$ with multiplicity exactly 1;

2 $V(\omega^{\mathrm{tw}}_{\mathcal{R}})\big|_{\mathfrak{h}_{\mathcal{R}^{\mathrm{ave}}}}$ contains $V(\omega^{\mathrm{tw}}_{\mathcal{R}^{\mathrm{ave}}})$ with multiplicity exactly 1.

This is checked directly. $\square$

Appendix A

Lauricella functions and their analytic continuation

§ A.1 Lauricella functions

The Lauricella function $F_D^{(N)}$ is the analytic continuation of the generalised hypergeometric series defined by:

$$F_D^{(N)}(a; b_1, \cdots, b_N; c \big| x_1, \cdots, x_N) \\ := \sum_{i_1, \cdots, i_N \geqslant 0} \frac{(a)_{\sum i_k}}{(c)_{\sum i_k}} \prod_{k=1}^{N} \frac{(b_k)_{i_k}}{(i_k)!} x_k^{i_k}. \tag{A.1}$$

An important property of $F_D^{(n)}$ is its one-dimensional Euler-type integral representation:

$$F_D^{(n)}(a, b_1, \cdots, b_N, c, x_1, \cdots, x_N) \\ = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 dt \;\; t^{a-1}(1-t)^{c-a-1} \prod_{k=1}^{N} (1 - x_k t)^{-b_k}. \tag{A.2}$$

The Lauricella series reduce to the Gauss hypergeometric series when $N = 1$ and to the Appell hypergeometric series when $N = 2$.

Fixing the variable $x_1, \cdots, x_{N-1}$ and re-expand the series w.r.t. x_N, one get the following useful formula:

$$F_D^{(N)}(a; b_1, \cdots, b_N; c \big| x_1, \cdots, x_N) \\ = \sum_{i_1, \cdots, i_{N-1}} \frac{(a)_{\sum_{j=1}^{N-1} i_j}}{(c)_{\sum_{j=1}^{N-1} i_j}} \prod_{j=1}^{N-1} \frac{(b_j)_{i_j} x_j^{i_j}}{i_j!} {}_2F_1 \left(a + \sum_{j=1}^{N-1} i_j, b_N, c + \sum_{j=1}^{N-1} i_j \big| x_N \right) \tag{A.3}$$

§ A.2 Analytic continuation of Gauss hypergeometric functions

When $N = 1$, a Lauricella function becomes an Euler hupergeometric function: $F_D^{(1)}(a, b, c|z) = {}_2F_1(a, b, c|z)$. In this case the following connection formula holds:

If $a - b \notin \mathbb{Z}$,

$$\begin{aligned} {}_2F_1(a, b, c|z) =& \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1(a, a-c+1; a-b+1|\frac{1}{z}) \\ &+ \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1(b, b-c+1; -a+b+1|\frac{1}{z}) \end{aligned} \tag{A.4}$$

If $a = b$,

$$\begin{aligned} {}_2F_1(a, a, c|z) =& (-z)^{-a}\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \\ &\times \sum_{n\geqslant 0} \frac{(a)_n(1-c+a)_n}{(n!)^2} z^{-n}[\log(-z) + 2\psi(n+1) - \psi(a+n) - \psi(c-a-n)]. \end{aligned} \tag{A.5}$$

$a = b$ is the special case of $b = a + m, m \in \mathbb{Z}_+$,

$$\begin{aligned} &{}_2F_1(a, a+m, c|z) = {}_2F_1(a+m, a, c|z) \\ =&(-z)^{-a-m}\frac{\Gamma(c)}{\Gamma(a+m)\Gamma(c-a)} \times \sum_{n\geqslant 0}\frac{(a)_{n+m}(1-c+a)_{n+m}}{n!(n+m)!}z^{-n} \\ &\times [\log(-z) + \psi(1+m+n) - \psi(1+n) - \psi(a+m+n) - \psi(c-a-m-n)] \\ &+ (-z)^{(-a)}\frac{\Gamma(c)}{\Gamma(a+m)}\sum_{n=0}^{m-1}\frac{\Gamma(m-n)(a)_n}{n!\Gamma(c-a-n)}z^{-n}. \end{aligned} \tag{A.6}$$

§ A.3 Analytic continuation of multi-variable Lauricella functions

The multi-variable case is less well understood. When $a; b_1, \cdots, b_N; c$ are generic, using connection formula in generic case and restricting to the asymptotically leading terms, the following connection formula holds:

Proposition A.3.1 ([13, (243)]).

$$
\begin{aligned}
& F_D^{(N)}(a; b_1, \cdots, b_N; c \big| x_1, \cdots, x_N) \\
\sim & (-x_N)^{-a} \Gamma\left(\begin{matrix} c, & b_N - a \\ b_N, & c-a \end{matrix} \right) \\
& \times F_D^{(N)}(a; b_1, \cdots, b_{N-1}, 1-c+a; 1-b_N+a \big| \frac{x_1}{x_N}, \cdots, \frac{1}{x_N}) \\
+ & (-x_N)^{-b_N} \Gamma\left(\begin{matrix} c, & a - b_N \\ a, & c - b_N \end{matrix} \right) \\
& \times F_D^{(N-1)}(a - b_N; b_1, \cdots, b_{N-1}; c - b_N \big| x_1, \cdots, x_{N-1}).
\end{aligned} \tag{A.7}
$$

When $a; b_1, \cdots, b_N; c$ are not generic, the analytic continuation is more complicated, as we have seen in $N = 1$ case. A key result we need is the following

Proposition A.3.2 ([2, (A.11)]).

$$
\begin{aligned}
& F_D^{(n)}(a, b_1, \cdots, b_{n-1}, a, c | x_1, \cdots, x_n) \\
= & \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} (-x_n)^{-a} \sum_M \sum_{m_n=0}^{\infty} \frac{\Gamma(c-a-|M|)}{\Gamma(c-a+|M|)} \frac{(a)_{|M|+m_n} (1-c+a)_{2|M|+m_n}}{(|M|+m_n)! m_n!} \prod_{i=1}^{n-1} \frac{(b_i)_{m_i}}{m_i!} \times \\
& (\log(-x_n) + h_{m_n}) \left(\frac{x_1}{x_n}\right)^{m_1} \cdots \left(\frac{x_{n-1}}{x_n}\right)^{m_{n-1}} \left(\frac{1}{x_n}\right)^{m_n} \\
& + \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} (-x_n)^{-a} \sum_M \sum_{m_n=0}^{|M|-1} \frac{(a)_{m_n} \Gamma(|M| - m_n)}{m_n! (c-a)_{|M|-m_n}} \prod_{i=1}^{n-1} \frac{(b_i)_{m_i}}{m_i!} x_1^{m_1} \cdots x_{n-1}^{m_{n-1}} \left(\frac{1}{x_n}\right)^{m_n},
\end{aligned} \tag{A.8}
$$

here

$$
h_{m_n} = \psi(1 + |M| + m_n) + \psi(1 + m_n) - \psi(a + |M| + m_n) - \psi(c - a - m_n).
$$

Corollary A.3.3.

$$
\begin{aligned}
& F_D^{(N)}(a; b_1, \cdots, b_{N-1}, a; c \big| x_1, \cdots, x_N) \\
\sim & \Gamma\left(\begin{matrix} c \\ a, c-a \end{matrix} \right) (-x_N)^{-a} (\log(-x_N) + h_0) \\
+ & \Gamma\left(\begin{matrix} c \\ a, c-a \end{matrix} \right) (-x_N)^{-a} \sum_{\mathcal{M}, |\mathcal{M}| \geqslant 1} \frac{\Gamma(|\mathcal{M}|)}{(c-a)_{|\mathcal{M}|}} \prod_{j=1}^{N-1} \frac{(b_j)_{m_j}}{(m_j)!} x_j^{m_j}
\end{aligned} \tag{A.9}
$$

Lemma A.3.4.

$$\begin{aligned}&\sum_{\mathcal{M},|\mathcal{M}|\geqslant 1}\frac{\Gamma(|\mathcal{M}|)}{(c-a)_{|\mathcal{M}|}}\prod_{j=1}^{N-1}\frac{(b_j)_{m_j}}{(m_j)!}x_k^{m_j}\\=&\frac{d}{d\epsilon}\bigg|_{\epsilon=0}F_D^{N-1}(\epsilon;b_1,\cdots,b_{n-1};c-a\big|x_1,\cdots,x_{n-1}).\end{aligned}$$

Proof. Notice that

$$\begin{aligned}&\sum_{\mathcal{M},|\mathcal{M}|\geqslant 1}\frac{\Gamma(|\mathcal{M}|)}{(c-a)_{|\mathcal{M}|}}\prod_{j=1}^{N-1}\frac{(b_j)_{m_j}}{(m_j)!}x_k^{m_j}\\=&\lim_{\epsilon\to 0}\frac{F_D^{N-1}(\epsilon;b_1,\cdots,b_{n-1};c-a\big|x_1,\cdots,x_{n-1})-1}{1/\Gamma(\epsilon)}.\end{aligned}$$

The statement is therefore an application of L'Hôpital's rule.

□

Taking derivatives in Proposition A.3.1, where we take $a=\epsilon$, we arrive at

Lemma A.3.5.

$$\begin{aligned}&\frac{d}{d\epsilon}\bigg|_{\epsilon=0}F_D^{N-1}(\epsilon;b_1,\cdots,b_{N-1};c-a\big|x_1,\cdots,x_{N-1})\\&\sim-\log(-x_{N-1})+(-\psi(b_{N-1})+\psi(-a+c))\\&+(-x_{N-1})^{-b_{N-1}}\Gamma\left(\begin{matrix}c-a,-b_{N-1}\\c-a-b_{N-1}\end{matrix}\right)\\&\times F_D^{(N-2)}(-b_{N-1},b_1,\cdots,b_{N-2},c-a-b_{N-1}\big|x_1,\cdots,x_{N-2}),\end{aligned}\tag{A.10}$$

here if $N=2$, then the $F_D^{(N-2)}$ term is 1.

Proof. We follow the proof of Proposition A.3.1. Using the identity for $F_D^{(N)}$, we get a linear combination of Lauricella function and a power series with Gamma function coefficients. We then take derivatives of these two terms.

□

Corollary A.3.6.

$$
\begin{aligned}
&F_D^{(N)}(a;b_1,\cdots,b_{N-1},a;c\big|x_1,\cdots,x_N)\\
\sim&\Gamma\left(\begin{matrix} c \\ a, c-a \end{matrix}\right)(-x_N)^{-a}(\log(-x_N)-\log(-x_{N-1})+h_0')\\
+&\Gamma\left(\begin{matrix} c & -b_{N-1} \\ a & c-a-b_{N-1} \end{matrix}\right)(-x_N)^{-a}(-x_{N-1})^{-b_{N-1}}\\
&\times F_D^{(N-2)}(-b_{N-1};b_1,\cdots,b_{N-2};c-a-b_{N-1}\big|x_1,\cdots,x_{N-2})
\end{aligned}
\tag{A.11}
$$

here

$$
\begin{aligned}
h_0' &= h_0-\psi(b_{n-1})+\psi(-a+c)\\
&=2\psi(1)-\psi(a)-\psi(c-a)-\psi(b_{n-1})+\psi(-a+c)\\
&=2\psi(1)-\psi(a)-\psi(b_{n-1}),
\end{aligned}
$$

and if $N=2$, then the $F_D^{(N-2)}$ term is 1.

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