

Analysis and Control of Large-Scale Multi-agent System using Partial Differential Equations

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by

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Dedicated to my beloved parents.

Acknowledgements

I would like to begin this acknowledgement with a Sanskrit *shlok* that reflects the timeless reverence accorded to teachers:

Om Ganeshaya Namah!

*Guru Brahma, Guru Vishnu, Guru Devo Maheshwara,
Guru Sakshat Param Brahma, Tasmai Shri Gurave Namah.*

This verse conveys that the *Guru* (teacher) is revered as Brahma—the creator of knowledge, Vishnu—the nurturer and preserver of wisdom, and Maheshwara—the remover of ignorance. The Guru is regarded as the direct embodiment of the Supreme Truth. To such a teacher, one offers the highest form of respect and gratitude.

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Abstract

Multi-agent system (MAS) consist of multiple interacting agents that cooperate to accomplish tasks that cannot be efficiently achieved by a single agent. Such systems arise in numerous applications across engineering, biology, and the social sciences, motivating their systematic study. Traditional approaches study the dynamics of each agent individually, typically modelling them using ordinary differential equations (ODEs) with interactions represented through graph-theoretic structures. As the number of agents increases, this results in a large system of ODEs whose dimension grows with the population size, making analysis and control design increasingly challenging. To address this limitation, several studies adopt a continuum perspective in which the collective behaviour of a large population of agents is approximated using partial differential equations (PDEs). Building upon this framework, this thesis investigates the problem of deploying a large-scale multi-agent system onto a predefined target. A leader–follower architecture is considered in which a subset of agents, referred to as leaders, can assess their deviation from the target, while follower agents rely only on local interactions with their neighbours. The resulting collective behaviour can be described by a semilinear parabolic PDE. Based on this model, global control strategies are designed that utilize information available to the leaders. In particular, communication between leaders is introduced to improve the estimation of the global error state, which enhances the effectiveness of the control strategy. Furthermore, a Wentzell-type boundary control is proposed to regulate the behaviour of boundary agents, allowing them to balance the deployment objective with the need to preserve swarm cohesion. Sufficient conditions for successful deployment are derived in the form of Linear Matrix Inequalities (LMIs). Numerical simulations confirm that the proposed control strategies achieve faster convergence and improved swarm cohesion compared with approaches that assume non-communicating leaders.

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Abbreviations

MAS	Multi-Agent System
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
LMI	Linear Matrix Inequality

Symbols

Sets and Functional Spaces

$\mathbb{R}, \mathbb{N}$	Sets of real and natural numbers, respectively.
$\mathbb{R}^n$	n -dimensional Euclidean space with norm $ x _2 = (\sum_{i=1}^n x_i ^2)^{1/2}$.
$\mathbb{R}^{n \times m}$	Set of real matrices with the induced norm $ \cdot _2$.
$C^k[a, b]$	Space of k -times continuously differentiable functions.
$C_c^\infty(\Omega)$	Space of infinitely differentiable functions with compact support in Ω .
$L^2(0, 1)$	Square-integrable functions with norm $\ f\ _{L^2} = \left(\int_0^1 f(x) _2^2 dx\right)^{1/2}$.
$H^1(0, 1)$	Sobolev space with norm $\ f\ _{H^1}^2 = \ f\ _{L^2}^2 + \ f_x\ _{L^2}^2$.
$\langle f, g \rangle_{L^2}$	Inner product $\int_0^1 f(x)^\top g(x) dx$.

MAS States and Error Variables

$z_i(t)$	State of agent i in $\mathbb{R}^n$ at time t .
$z(t, x)$	Continuum state representation at time t and position x .
$\gamma(x)$	Target deployment curve.
$e(t, x)$	Error state defined as $z(t, x) - \gamma(x)$.
$\nu(t, x)$	Approximation error between the error state and its estimate.

Control Gains and Parameters

a	Interaction strength (diffusion coefficient).
κ, σ	Boundary tracking and coupling gains.
K	Leader state propagation gain.
h, Δ	Nominal agent spacing and maximum inter-leader spacing.

Matrix Notation

$\text{col}\{\cdot\}$	Column stack operator.
$M^\top$	Transpose of matrix M .
$P > 0$	Positive definite symmetric matrix P .
$*$	Symmetric block entry in a matrix.
I_n	Identity matrix of dimension n .

Introduction

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1.1 Introduction to Multi-agent System

Consider a scenario in which a fleet of autonomous aerial robots collaboratively surveys a large agricultural field. Maintaining continuous coordination through a central controller may be impractical in this scenario due to communication limitations and the large spatial coverage of the environment. Consequently, the robots rely on local sensing and interactions with neighbouring agents to distribute themselves across the field, avoid collisions, respond to obstacles, and collectively gather environmental data such as temperature, humidity, and crop health indices. The success of the mission depends not on any individual robot but on their coordinated collective behaviour. This scenario illustrates a typical *multi-agent system (MAS)* in action [Davoodi et al. \(2021\)](#). To better understand such systems, [Durfee et al. \(1989\)](#) defines a MAS as a loosely coupled structure composed of multiple agents that interact to solve problems which cannot be effectively addressed by a single agent due to limited capability, incomplete information, resource constraints, or inefficiency.

MAS is a key area within *Distributed Artificial Intelligence (DAI)*, along with *Parallel AI* and *Distributed Problem Solving (DPS)*. Parallel AI focuses on developing algorithms, programming languages, and system architectures that run tasks in parallel to improve the speed and performance of traditional AI methods [Bond and Gasser \(2014\)](#). DPS, on the other hand, breaks down a large problem into smaller parts, assigning each subtask to a separate computing node within a cooperative network [Wooldridge \(2009\)](#). In a similar way, agents in MAS work together to solve problems. However, they provide more flexibility than the fixed computing units used in DPS. MAS agents can adapt, make decisions, and coordinate with one another, making them better suited for solving complex and dynamic tasks. To better understand how MAS handle such challenges, we now outline the main features that define their structure and behaviour.

First, *heterogeneity* distinguishes systems of identical agents from those with varied capabilities, allowing task-specific specialisation [Vrancken and Soares \(2009\)](#); [Khodaverdian \(2014\)](#); [Kim and Matson \(2016\)](#); second, *mobility* separates static agents from mobile ones, with the latter providing

wider coverage but requiring more intricate coordination Wooldridge (2009); Wang et al. (2013, 2014). Third, the *interaction topology* dictates how information spreads: it can be *static*, with fixed links, or *dynamic*, where links form and break as agents move, crucial for aerial, underwater, and vehicular swarms Olfati-Saber and Murray (2004); Guo et al. (2014); Liu et al. (2015). Fourth, agents' *decision-making logic* may be *linear*, allowing straightforward analysis, or *nonlinear*, capturing richer behaviours typical of biological and socially inspired systems Zhao et al. (2013); Li et al. (2012); Zhang et al. (2015). Fifth, the *leadership structure* may be *leader–follower*, where designated agents set the objective, or *leaderless*, where consensus emerges without central authority González-Pardo et al. (2012); Angeli and Bliman (2005). Sixth, *data-transmission frequency* can be *time-triggered*, with periodic updates, or *event-triggered*, transmitting only when significant changes occur, thus saving bandwidth Guo et al. (2014); Li et al. (2014). Seventh, real systems must tolerate *delays* from communication, sensing, or computation; delay-aware control is essential for stability Wen et al. (2013); Liu et al. (2014). Eighth, the required *agreement complexity* ranges from single-parameter consensus (e.g., position) to multi- or higher-order consensus (e.g., velocity, acceleration) Wen et al. (2013).

This thesis focuses on the modelling and control of large-scale MAS, where a large population of agents interacts through local communication and coordination mechanisms. Such systems typically involve mobile agents, dynamic interaction topologies, and coordination structures such as leader–follower architectures, which enable collective behaviour to emerge from local interactions. Examples of large-scale MAS arise in many domains, including flocking behaviour in biological swarms, vehicular traffic systems in transportation engineering, coordinated drone formations in robotics, opinion dynamics in social networks, goods movement in manufacturing, and pedestrian flows in civil and architectural engineering. Figure 1.1 illustrates several representative examples, including flocking birds, traffic streams, a social interaction network, and coordinated robotic agents.

To ground this discussion, Section 1.2 provides a brief overview of classical large-scale MAS models, such as the *Hegselmann–Krause* opinion model, the *social force* model for pedestrians, and the *Vicsek* and *Cucker–Smale* flocking models, each formulated as a large number of ordinary differential equations (ODEs). While these agent-level ODE approaches capture individual behaviour with high fidelity, their computational cost rises sharply with the number of agents, making them impractical for truly large-scale systems. This *scalability problem* is examined in Section 1.3, with



Figure 1.1: Examples of large-scale MAS (clockwise from top left): bird flocking, vehicular traffic, social network, and robot formation. The images are generated using AI-based tools (ChatGPT and Gemini) for illustrative purposes.

illustrative examples ranging from traffic congestion and power-grid instabilities to orbital debris accumulation in satellite constellations.

To overcome this limitation, Section 1.4 introduces *continuum modelling* as a scalable alternative. The first argument for this approach stems from observations in biology, where collective motion often resembles fluid-like behaviour, motivating the use of partial differential equations (PDEs) to describe the swarm's dynamics [Toner and Tu \(1995\)](#); [Flierl et al. \(1999\)](#); [Vicsek and Zafeiris \(2012\)](#); [Rastgoftar and Jayasuriya \(2014\)](#). Second, the graph-theoretic tools traditionally used to model agent interactions reveal a structural similarity between graph Laplacians and differential operators, establishing a rigorous correspondence between discrete and continuous representations.

Section 1.5 provides background on PDE-based Lyapunov stability analysis, and Section 1.6 reviews relevant literature on PDE control techniques, including backstepping, sliding mode control, flatness-based design, and spatial decomposition methods. A detailed review of PDE-based MAS control literature appears later in Section 2.3 of Chapter 2. The chapter concludes with a summary of this thesis's novel contributions in Section 1.8, followed by an outline of the thesis structure in the next section.

1.2 Classical Agent-Based Models

This section gives a short look at well-known agent-based models in multi-agent research. In fields such as social science, biology, and robotics, self-moving agents are often described by sets of ODEs that capture simple rules of attraction, repulsion, and alignment motivated from graph theory [Cordier et al. \(2005\)](#); [Herty and Pareschi \(2010\)](#); [Degond et al. \(2011\)](#); [Bellomo and Soler \(2012\)](#). While these discrete formulations are intuitive, they face a significant analytical challenge: as the number of agents N increases, the system becomes a high-dimensional set of coupled equations. Performing stability analysis for such a large system under a discrete scenario is mathematically difficult, as the complexity of the analysis scales directly with the population size. This motivates the transition to the continuum (PDE) framework, where the swarm is described by a single equation, allowing us to utilise tools from functional analysis and operator theory.

1.2.1 The Vicsek Model for Swarming and Schooling Behavior

The *Vicsek model* [Vicsek et al. \(1995\)](#) describes how self-propelled agents, like schooling fish, locally adjust their headings to align with nearby neighbours. The motion is confined to the plane $\mathbb{R}^2$. At time t , agent i has position $z_i(t) \in \mathbb{R}^2$ and moves at constant speed $v > 0$ with heading angle $\theta_i(t)$. Its velocity is $v_i(t) = v(\cos \theta_i(t), \sin \theta_i(t))$. With sampling step $\Delta t > 0$, the update rules are:

$$\mathbf{z}_i(t + \Delta t) = \mathbf{z}_i(t) + \mathbf{v}_i(t) \Delta t, \quad (1.1)$$

$$\theta_i(t + \Delta t) = \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} \theta_j(t) + \eta_i(t), \quad (1.2)$$

where $\mathcal{N}_i$ denotes the set of agents within a fixed interaction radius $r > 0$ of agent i . Analysing this model under the discrete scenario is challenging because the coupling between N individual trajectories makes system stability analysis cumbersome as N grows. In contrast, the PDE approach collapses these individual headings into a continuous distribution, where alignment can be studied as the convergence of a single state variable.

1.2.2 The Cucker-Smale Alignment Model

The *Cucker-Smale (CS) model* [Cucker and Smale \(2007\)](#) captures how agents gradually align their velocities. Let $z_i(t) \in \mathbb{R}^n$ be the position and $v_i(t) \in \mathbb{R}^n$ the velocity of agent i . The influence of agent j on agent i is weighted by $a_{ij} = \eta(\|z_j(t) - z_i(t)\|^2)$, with $\eta(y) = \frac{K}{(y + \sigma^2)^\beta}$, where $K, \sigma, \beta > 0$. For N agents the CS dynamics are:

$$\dot{z}_i(t) = v_i(t), \quad (1.3)$$

$$\dot{v}_i(t) = \frac{1}{N} \sum_{j=1}^N a_{ij} (v_j(t) - v_i(t)), \quad (1.4)$$

Stability analysis for the discrete CS model requires managing an $N \times N$ matrix of interactions. As N becomes sufficiently large, this high-dimensional analysis becomes complicated. By reformulating this as a PDE, the problem is simplified into an abstract evolution equation in a Hilbert space, where operator theory provides the necessary tools to perform stability analysis which is independent of the number of agents.

Comparison with the Vicsek Model. The Vicsek and CS models both achieve alignment, yet they weight neighbours in very different ways. In the Vicsek model, an agent averages the headings of all neighbours inside a fixed radius r , using the indicator weight

$$\eta_V(y) = \begin{cases} 1, & y \leq r^2, \\ 0, & y > r^2, \end{cases}$$

The CS model replaces this step weight with a smooth, distance-dependent kernel

$$\eta_{CS}(y) = \frac{K}{(y + \sigma^2)^\beta}, \quad K, \sigma, \beta > 0.$$

Despite their differences, both models suffer from the same scalability limitation in a discrete setting. Analysing stability becomes increasingly complex as number of agents increases and the interaction graph evolves, whereas a PDE-based framework provides a more systematic and scalable approach to stability analysis.

1.2.3 Vehicular Traffic Models

Traffic models track each vehicle's longitudinal position $\mathbf{z}_i(t)$ and speed $v_i(t)$ as time-varying state variables.

First, In the *Intelligent Driver Model* (IDM) [Treiber et al. \(2000\)](#), drivers aim for a desired cruising speed v_0 . The dynamic spacing is $s^*(v, \Delta v) = s_0 + vT + \frac{v \Delta v}{2\sqrt{ab}}$. The IDM dynamics are:

$$\dot{z}_i = v_i, \quad \dot{v}_i = a \left[1 - \left(\frac{v_i}{v_0} \right)^\delta - \left(\frac{s^*(v_i, \Delta v_i)}{s_i} \right)^2 \right].$$

Second, in the *Optimal Velocity (Bando) model* [Bando et al. \(1995\)](#), drivers adapt their speed toward an *optimal velocity* $V(s_i)$. The dynamics are:

$$\dot{z}_i = v_i, \quad \dot{v}_i = \alpha [V(s_i) - v_i].$$

Finally, the *Follow-the-Leader (FtL) model* [Gazis et al. \(1961\)](#) assumes that the acceleration of vehicle i depends on the speed difference to the vehicle ahead. Stability analysis for long chains of vehicles is traditionally carried out by analysing N coupled ODEs, a process that becomes increasingly difficult to generalise as N grows. In contrast, a PDE-based description enables the analysis of the collective flow as a single dynamical system, allowing the use of energy methods and operator-theoretic tools to assess the stability of the entire formation.

1.2.4 The Hegselmann-Krause Opinion Model

In the *HK model* [Rainer and Krause \(2002\)](#), each agent i holds an opinion $z_i(t) \in \mathbb{R}^n$. Interactions occur only within a confidence radius $r > 0$. The dynamics are:

$$\dot{z}_i = \frac{1}{|\mathcal{N}_i(t)|} \sum_{j \in \mathcal{N}_i(t)} (z_j - z_i).$$

The discrete HK model is notoriously difficult to analyse when the population is large, as the interaction rules create complex, state-dependent couplings. This highlights the fundamental advantage of the PDE-based approach developed in this thesis: by treating the agents as a continuum, we can apply rigorous scale-invariant stability criteria that remain mathematically tractable regardless of whether the swarm consists of dozens or thousands of agents.

1.3 Scalability Problem

This section highlights why analysing large-scale systems is essential by drawing on three illustrative examples: urban traffic, space traffic and collision risk (Kessler syndrome), and power grids. These examples demonstrate that poor coordination at scale can result in substantial economic, environmental, or operational consequences. In the second part of the section, we discuss the limitations of classical agent-based models when applied to such large-scale settings and highlight the pressing need for scalable alternatives.

1. Urban traffic: Traffic congestion is a major problem in cities. It causes long delays, wastes fuel, increases air pollution, and leads to big economic losses. In 2024, London was the most congested city in Europe. On average, drivers lost 101 hours sitting in traffic, costing the UK around 7.7 billion [INRIX \(2024\)](#). As cities grow and more vehicles appear on the roads, it becomes very important to understand traffic flow in order to plan better roads and reduce delays.

2. Space debris and the Kessler syndrome: The number of satellites in low Earth orbit is growing quickly, which raises the risk of collisions and space debris. This situation is known as the *Kessler syndrome*, where one collision can trigger many more. Even in 1984, Earth's orbit was considered crowded with just 160 satellites [EoS \(1984\)](#). By 2021, there were over 4,500 active satellites. In 2022, a solar storm caused the failure of 40 out of 49 newly launched *Starlink* satellites [BBC \(2022\)](#). As satellite numbers continue to rise, we need better ways to study and manage space traffic to avoid dangerous chain reactions.

3. Large-scale power grids: Power grids are massive systems that must operate continuously without interruption. A well-known example of failure occurred during the 2003 blackout across the U.S. and Canada, which left over 50 million people without electricity and caused estimated economic losses of \$4–6 billion [White et al. \(2003\)](#). One reason was the failure of computer tools that were supposed to check the grid's health in real time [U.S.-Canada Power System Outage Task Force \(2004\)](#). As power networks grow larger, it becomes harder to monitor and control them. This shows why we need better ways to study such big systems.

Taken together, these examples demonstrate the importance of analysing such systems at scale. However, existing classical models are not well-suited for this purpose. For instance, in the case of traffic congestion, models such as the Intelligent Driver Model (IDM), Optimal Velocity Model

(OVM), and Follow-the-Leader (FtL) model (see Section 1.2.3) describe the motion of each vehicle using separate differential equations. As the number of vehicles increases, the system becomes increasingly complex, making it difficult to analyse or design effective control strategies. Moreover, since traffic density changes in real time, the number of vehicles is not fixed. This means that any controller designed for a specific population size may no longer perform well as the system evolves, requiring the entire analysis to be repeated and control gains to be recalibrated.

A similar issue arose during the 2003 U.S.–Canada power blackout. The control system relied on two software tools: the *State Estimator (SE)* and the *Real-Time Contingency Analysis (RTCA)* engine. The SE computed bus voltages and power flows by solving a large nonlinear algebraic system, and the RTCA used these results to simulate thousands of possible line failures. On the day of the blackout, the network included over 43,000 buses and 57,000 transmission lines. Each analysis cycle required solving massive, sparse nonlinear systems, which scaled poorly with system size. Although these solvers were deterministic, they were also fragile: a small modelling error—such as incorrectly marking a transmission line as active—caused the SE to fail, which in turn disabled the RTCA engine. This left operators blind during the critical period of cascading failures ([U.S.-Canada Power System Outage Task Force, 2004](#), pp. 47–48).

These scalability challenges highlight the need for alternative modelling frameworks that are independent of the number of agents. A promising approach, adopted in this thesis, is to approximate the swarm as a spatial continuum, leading to a partial differential equation (PDE) representation. Unlike classical agent-based models, the PDE model is independent of the number of agents, and it is easy to perform analysis and design control for large-scale settings.

1.4 PDEs for Swarms

To overcome the scalability limits of agent-based models, we adopt *continuum modelling* as a functional alternative for large-scale MAS. This transition from discrete individuals to continuous representations is driven by two primary motivations.

1.4.1 Motivation 1: Fluid-Like Behaviour

The first motivation arises from the natural emergence of collective behaviour in biological systems. In small groups, coordination often requires explicit external intervention. For instance, a small number of sheep typically require guidance from a shepherd to maintain cohesion and direction. In contrast, as the population size increases, coordination emerges spontaneously through local interactions alone. Large herds of sheep are able to move cohesively without continuous external control, maintaining structure and direction through simple neighbour-based interactions.

As the number of agents becomes very large, such systems undergo a qualitative transition: they no longer behave as a collection of discrete individuals, but rather as a single, cohesive entity exhibiting fluid-like behaviour. As seen in Figure 1.2, a dense herd of sheep or a school of fish forms a smooth, cohesive mass characterised by its overall shape and flow. These large-scale, inherently smooth structures motivate the use of continuum models rather than discrete agent representations, as they allow us to capture the fluid-like nature of collective dynamics in a mathematically tractable manner.



Figure 1.2: Examples of large-scale collective motion in biological systems: (left) a herd of sheep and (right) a school of fish, both exhibiting fluid-like behaviour. The images are generated using AI-based tools (ChatGPT) for illustrative purposes.

1.4.2 Motivation 2: Graph Laplacian as a Discrete Differential Operator

Another motivation for adopting a continuum framework is that the graph Laplacian, which governs interactions in MAS, can be interpreted as a discrete analogue of the Laplacian operator arising in

PDEs. To better understand this connection, we first introduce the basic graph-theoretic framework and then consider a representative example.

A graph is defined as an ordered pair $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_N\}$ is the set of nodes (agents), and $E \subseteq V \times V$ is the set of edges encoding the interaction topology. An edge $(v_i, v_j) \in E$ indicates that agent v_i interacts with agent v_j , and for undirected graphs this interaction is symmetric. The set of neighbours of agent v_i is denoted by $\mathcal{N}_i = \{v_j \in V \mid (v_i, v_j) \in E\}$.

To analyse the structure of inter-agent coupling, we use matrix-based representations. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ is defined as

$$a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E, \\ 0, & \text{otherwise,} \end{cases}$$

and the degree matrix $D = \text{diag}(d_1, \dots, d_N)$ has entries $d_i = \sum_{j=1}^N a_{ij}$. The graph Laplacian is then defined as $L = D - A$.

Example 1: Consider a multi-agent system consisting of five agents with the undirected topology illustrated in Figure 1.3.

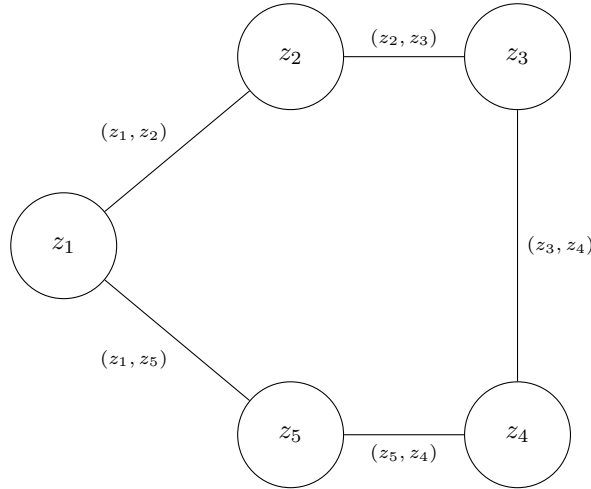


Figure 1.3: Undirected graph representing a five-agent interaction network.

For the state vector $\mathbf{z} = [z_1, \dots, z_5]^T \in \mathbb{R}^5$, the resulting adjacency matrix A , degree matrix D , and the graph Laplacian matrix L is:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Under the dynamics $\dot{\mathbf{z}} = -L\mathbf{z}$, the evolution of each agent is governed by the local difference with its neighbours. For an internal node such as z_3 , the coupling term is derived from the third row of $L\mathbf{z}$, yielding $(L\mathbf{z})_3 = -z_2 + 2z_3 - z_4$.

The structural relationship to the continuous Laplacian operator ∇^2 is established by treating these agents as discrete spatial samples of a continuous distribution $z(t, x)$ over the unit interval $x \in [0, 1]$. In one dimension, the Laplacian, ∇^2 , is defined as the second-order spatial derivative $\frac{\partial^2 z}{\partial x^2}$. By utilizing the Taylor series expansion of $z(t, x)$ around a point x_i for a fixed time t with uniform spacing h , we have:

$$z(t, x_i + h) = z(t, x_i) + h \frac{\partial z}{\partial x} \Big|_{(t, x_i)} + \frac{h^2}{2} \frac{\partial^2 z}{\partial x^2} \Big|_{(t, x_i)} + \mathcal{O}(h^3) \quad (1.5)$$

$$z(t, x_i - h) = z(t, x_i) - h \frac{\partial z}{\partial x} \Big|_{(t, x_i)} + \frac{h^2}{2} \frac{\partial^2 z}{\partial x^2} \Big|_{(t, x_i)} - \mathcal{O}(h^3) \quad (1.6)$$

Summing these expressions leads to the finite-difference approximation of the operator at x_i :

$$\frac{\partial^2 z}{\partial x^2} \Big|_{(t, x_i)} = \frac{z(t, x_i + h) - 2z(t, x_i) + z(t, x_i - h)}{h^2} + \mathcal{O}(h^2). \quad (1.7)$$

By comparing the interaction term from Example 1, $-z_2 + 2z_3 - z_4$, with the numerator of Equation (1.7), it is evident that the graph Laplacian L is a discrete discretisation of the negative continuous operator $-\nabla^2$, scaled by the spatial resolution h^2 . This structural equivalence motivates the modeling of discrete, ODE-based graph-theoretic MAS as continuous PDEs.

Analytical Advantages of the Continuum Limit

The primary advantage of the continuum formulation lies in providing a tractable framework for the stability analysis of large-scale MAS. In the discrete setting, the swarm state is defined by a vector

$\mathbf{z} = [z_1, \dots, z_N]^T \in \mathbb{R}^N$, where each $z_i \in \mathbb{R}$ denotes the state of an individual agent. Conventional stability analysis utilizes quadratic Lyapunov functions in the Euclidean space $(\mathbb{R}^N, \|\cdot\|_2)$. However, the resulting Lyapunov analysis is closely tied to the algebraic properties of the graph Laplacian L , which depend on the underlying interaction topology and may become difficult to characterise analytically as N increases.

In contrast, the continuum limit represents the collective state as a function $z(t, x)$ over a spatial domain Ω . This allows the analysis to be conducted in the Hilbert space $L^2(\Omega)$, where the system's energy is described using integral functionals that do not explicitly depend on the number of agents. A natural Lyapunov functional in this setting is the squared L^2 norm: $\|z(t, \cdot)\|_{L^2}^2$. This formulation replaces high-dimensional matrix-based expressions arising in Lyapunov stability analysis with integral inequalities, leading to a more transparent interpretation of the system's energy. By shifting the analysis to the L^2 function space, stability properties can be studied through the structure of the governing PDE, such as boundary conditions and domain geometry, rather than the explicit dimensionality of the discrete agent model.

1.4.3 Precedents in Continuum Modelling: Fokker-Planck Equation

The transition from *individual agent dynamics* to continuous partial differential equations is a well-established framework in statistical mechanics and stochastic control. A primary example of this relationship is the *Fokker-Planck equation* (FPE), first introduced to describe the time evolution of the probability density function (PDF) of particles under the influence of drag forces and random thermal motions [Fokker \(1914\)](#); [Planck \(1917\)](#). In recent decades, the FPE has been successfully adapted to model the *collective spatial distribution* of large-scale robotic swarms [Hamann \(2010\)](#); [Prorok et al. \(2011\)](#). The mathematical link between the individual and collective scales is rigorous; for a single agent whose motion is governed by a stochastic differential equation (SDE)

$$\dot{R}(t) = A(R(t), t) + B(R(t), t)F(t), \quad (1.8)$$

where A represents a deterministic drift and B scales a stochastic perturbation $F(t)$, it is a fundamental result that for a sufficiently large population of independent agents, the spatial density

$\rho(r, t)$ satisfies the Fokker-Planck PDE Risken (1989):

$$\frac{\partial \rho(r, t)}{\partial t} = -\nabla \cdot (A(r, t)\rho(r, t)) + \frac{1}{2}Q\nabla^2 (B^2(r, t)\rho(r, t)). \quad (1.9)$$

This equation decomposes the collective evolution into two physically intuitive components: a *drift term* that models deterministic behaviour by directing the density along the vector field $A(r, t)$, and a *diffusion term* that captures the spreading of the density due to random perturbations, governed by the Laplacian operator ∇^2 .

The relevance of the FPE to the present work lies in its role as a conceptual and mathematical precursor. Although the agents considered in this thesis are modelled using *deterministic ODEs* rather than SDEs, the underlying principle remains analogous: as the number of agents increases, the collective dynamics can be described by a PDE.

1.5 Background on PDE-Based Lyapunov Stability

This thesis investigates the stability of PDE models using Lyapunov theory as a primary analytical tool. A comprehensive exposition of Lyapunov methods for finite-dimensional systems, both linear and nonlinear, is provided in Khalil and Grizzle (2002). In contrast, the extension of these techniques to infinite-dimensional systems remains a more intricate task. Foundational contributions in this direction can be found in Henry (2006); Walker (2013).

The analysis of stability in infinite-dimensional settings relies heavily on functional spaces and their associated norms. These spaces, such as $L^2([0, 1])$ and $H^1([0, 1])$, provide the structure needed to define well-posedness and stability. However, unlike the finite-dimensional case, no universal framework exists for assessing the stability of PDEs, even when the system is linear. Stability analysis in this context typically requires the development of problem-specific norm estimates, making the appropriate choice of norm a critical aspect of the methodology. A detailed introduction to functional analysis and related spaces can be found in Kreyszig (1991).

Let X denote a suitable function space over the interval $[0, 1]$, equipped with a norm $\|\cdot\|_X$. Typical choices for X include $L^2([0, 1])$ and $H^1([0, 1])$, depending on the regularity requirements of the system under study. We denote by $z(t) \in X$ the system state at time $t \geq 0$, where $z(t) : (0, 1) \rightarrow \mathbb{R}$

and $\|z(t, \cdot)\|_X < \infty$. We consider the abstract evolution equation

$$\frac{dz}{dt} = \mathcal{A}z, \quad z \in X, \quad t > 0, \quad (1.10)$$

where $\mathcal{A}$ is a linear operator, typically of elliptic type, such as the Laplacian Δ or a more general sectorial operator (for a formal definition, see [Henry \(2006\)](#)). We assume that the zero function $z(t, x) \equiv 0$ is a steady-state solution of (1.10). This equilibrium, $z = 0$, will be the focus of our stability analysis.

Definition 1.1. *The zero solution $z(t) \equiv 0$ of (1.10) is said to be stable if, for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that, for all initial conditions satisfying $\|z(0, \cdot)\|_X < \delta$, the corresponding solution obeys $\|z(t, \cdot)\|_X < \varepsilon$ for all $t \geq 0$. If, in addition, $\lim_{t \rightarrow \infty} \|z(t, \cdot)\|_X = 0$ for all such initial conditions, then the zero solution is called asymptotically stable. The equilibrium is said to be exponentially stable if there exist constants $M \geq 1$ and $\delta > 0$ such that*

$$\|z(t, \cdot)\|_X \leq M e^{-\delta t} \|z(0, \cdot)\|_X, \quad \text{for all } t \geq 0.$$

Here, M is referred to as the overshoot coefficient, and δ denotes the decay rate.

Energy-based methods have been extensively employed in PDE control [Coron \(2007\)](#); [Mazenc and Prieur \(2011\)](#). The key idea is to construct a functional that captures the system's stability or energy characteristics, and then design control strategies that enforce its decrease over time. While this provides a systematic framework, identifying suitable functionals and dealing with boundary effects remains technically challenging, especially in complex PDE settings.

Example. Consider the following reaction-diffusion equation:

$$z_t = a z_{xx} + \lambda(x)z, \quad z(t, 0) = 0, \quad z(t, 1) = 0, \quad (1.11)$$

where $z(t, x)$ denotes the system state for $x \in [0, 1]$ and $t > 0$. Let $\bar{\lambda}$ denote the supremum of $\lambda(x)$ over the interval $[0, 1]$, i.e., $\bar{\lambda} = \sup_{x \in [0, 1]} \lambda(x)$. If $\bar{\lambda} < \frac{\varepsilon}{4}$, the zero equilibrium $z(t, x) = 0$ of (1.11) tends asymptotically to zero. This property will be justified in the subsequent analysis.

Consider the Lyapunov functional: $V(t) = \int_0^1 z^2(x, t) dx$. For simplicity, the time and spatial dependence of z is omitted in the following derivation. Differentiating $V(t)$ with respect to time yields:

$$\frac{dV(t)}{dt} = 2 \int_0^1 z z_t dx = 2 \int_0^1 z (a z_{xx} + \lambda(x) z) dx = 2a \int_0^1 z z_{xx} dx + 2 \int_0^1 \lambda(x) z^2 dx.$$

Using integration by parts and applying the boundary conditions, we have: $\int_0^1 z z_{xx} dx = - \int_0^1 z_x^2 dx$.

Applying Wirtinger's inequality, $\int_0^1 z_x^2 dx \geq \pi^2 \int_0^1 z^2 dx$, and combining the results, we obtain:

$$\frac{dV(t)}{dt} \leq -2a \int_0^1 z_x^2 dx + 2\bar{\lambda} \int_0^1 z^2 dx \leq 2(-a\pi^2 + \bar{\lambda}) \int_0^1 z^2 dx = -2(a\pi^2 - \bar{\lambda}) V(t).$$

This implies decay of $V(t)$:

$$V(t) \leq e^{-2\left(\frac{a\pi^2}{2} - \frac{\bar{\lambda}}{2}\right)t} V(0).$$

Hence, the zero solution $z(t, x) \equiv 0$ converges to zero. In particular, if $\bar{\lambda} \leq 0$, the origin converges to zero for any positive diffusion coefficient a .

1.6 Literature Review of PDE Control Methods

This review does not aim to provide an exhaustive survey of all control strategies for PDE systems, as the literature spans broad areas including optimal control, estimation, and other advanced theoretical developments beyond the scope of this work. Instead, we focus on model-based feedback control methods relevant to large-scale MAS modelled by PDEs. To provide a consistent basis for comparing these strategies, we define a general class of one-dimensional (1D) non-autonomous PDEs for a state $z(t, x)$ defined on the spatial domain $x \in [0, L]$:

$$\frac{\partial z(t, x)}{\partial t} = \mathcal{F}\left(t, x, z, \frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2}\right) + \mathcal{B}u(t, x) \quad (1.12)$$

subject to boundary conditions:

$$\mathcal{G}_0\left(z, \frac{\partial z}{\partial x}\right)\Big|_{x=0} = \phi_0(t), \quad \mathcal{G}_L\left(z, \frac{\partial z}{\partial x}\right)\Big|_{x=L} = \phi_L(t) \quad (1.13)$$

where $\mathcal{F}$ is a potentially nonlinear spatial operator, $\mathcal{B}$ is the control input operator, and $u(t, x)$ is the control signal. Framing the literature in terms of (1.12) clarifies the structural limitations of different methods, which are often restricted by the specific form of the operator $\mathcal{F}$ or the spatial distribution of the actuation. This review is structured by first classifying PDE control methods into *early* and *late lumping*, with focus on the latter as it aligns with the thesis framework. The late lumping methods are then organised based on the spatial location of actuation: in-domain and boundary control.

When designing controllers for distributed-parameter systems, two major modelling strategies are typically employed: *early lumping* and *late lumping*. *Early lumping* involves discretising the spatial operator $\mathcal{F}$ in (1.12) to obtain a finite-dimensional ODE system, typically using methods like finite difference, finite element Meirovitch (1991); Preumont (2002), modal and spectral techniques Georgakis et al. (1977); Balas (1978b); Meirovitch (1991); Fuller et al. (1996); Gawronski (2004), model reduction strategies such as balanced truncation and proper orthogonal decomposition Atwell and King (2001); Banks et al. (2002); Singler and Batten (2009), or Galerkin and inertial manifold approaches Ray (1981); Aamo and Krstic (2013); Foias et al. (1989); Christofides and Daoutidis (1997); Christofides and Chow (2002). With the ODE model in hand, standard finite-dimensional feedback laws can be used, yet these finite dimensional controllers may miss crucial PDE dynamics, causing spillover-related performance loss or instability Balas (1978a); for nonlinear systems the reduced model is valid only locally, so driving the state outside this range can erode robustness or even destabilise the loop, as fluid-flow studies show King et al. (2005).

Late lumping, in contrast, designs the controller directly on the original infinite-dimensional PDE model (1.12), without first reducing the operator $\mathcal{F}$ to a finite-dimensional approximation. This approach enables a rigorous extension of classical control techniques to distributed systems using semigroup theory Salamon (1987); Curtain and Zwart (2020); Luo et al. (2012); Tucsnak and Weiss (2009), which provides the mathematical foundation for applying frequency-domain methods Callier and Desoer (2003); Desoer and Wang (1980); Bontsema et al. (1988); Morris and Vidyasagar (1990); Lenz et al. (1991); Meurer et al. (2008), eigenvalue and structural assignment based on spectral properties Lasiecka and Triggiani (1983); Curtain (1985); Rebarber (1989); Xu and Sallet (1996), and optimal state-feedback via operator-based LQR design Curtain and Zwart (2012); Banks et al. (1996). Stability analysis in this setting typically relies on semigroup methods, spectral analysis, and contraction theory Dafermos and Slemrod (1973); Triggiani (1975); Pritchard and

Zabczyk (1981); Pazy (2012); Curtain and Zwart (2012); Cazenave and Haraux (1998); Luo et al. (2012). Since this thesis adopts a PDE-based modelling framework for large-scale MAS, in contrast to the more traditional ODE-based approach, further discussion of early lumping methods is not pursued. The remainder of the review focuses exclusively on late lumping approaches, categorised by the spatial influence of the input operator $\mathcal{B}$. We further note that Control strategies for PDEs typically fall into two categories: in-domain actuation and boundary actuation.

1.6.1 In-Domain Control

In-domain control addresses scenarios where the actuation operator $\mathcal{B}$ in (1.12) is active within the spatial interval $(0, L)$. Regulating these interior points is challenging because fully distributed actuation is seldom practical in real applications (in this thesis, an in-domain scheme is nevertheless adopted, as justified in Chapter 2). Moreover, when $\mathcal{B}$ represents point-wise actuation (e.g., leaders in a swarm), the influence of the control signal is highly localised, which necessitates a design capable of ensuring stability despite sparse spatial measurements. A seemingly natural strategy for in-domain actuation is to reformulate the problem as an equivalent boundary control task, allowing the use of existing boundary control techniques to construct the corresponding in-domain controllers Rudolph and Woittennek (2002); Yang et al. (2016); Zheng and Zhu (2015). This reinterpretation, however, fails for point actuators: their highly localised influence cannot be captured by boundary feedback alone. In the linear setting, the superposition principle offers a further simplification: a PDE with multiple in-domain actuators can be decomposed into a set of parallel sub-systems, each containing a single in-domain actuator. Applying the Laplace transform to each sub-problem yields explicit in-domain control laws. The effectiveness of this method has been demonstrated in the context of the heat equation Zheng and Zhu (2015) and the Euler-Bernoulli beam equation Yang et al. (2016). For non-linear PDEs, the superposition principle breaks down, so this approach can not be used. Geometric control theory has been introduced to address controllability and observability in such systems Henry (2006); Jurdjevic (1997), with the zero-dynamics inverse approach offering a structured design tool. A simpler static in-domain control method based on zero dynamics has been proposed in Aulisa and Gilliam (2013, 2014, 2015).

A notable contribution to the in-domain control of non-linear PDE systems is the spatial decomposition approach proposed in [Fridman and Blighovsky \(2012\)](#), which is adopted in this thesis to develop coordination strategies for MAS in Section 2.8. The authors design sampled-data controllers for uncertain semi-linear parabolic PDEs with distributed control on a finite interval. Although such systems can, in principle, be stabilised using linear infinite-dimensional state-feedback, the full PDE state is generally not measurable in practice, making direct implementation of such feedback infeasible. To overcome this, the authors propose a sampled-data static output feedback strategy, where the state is measured at a finite number of spatially fixed sampling points at discrete time instants. The control input is constructed using a finite set of spatially localised shape functions, each scaled by the corresponding sampled measurement, and is piecewise-constant in time. Stability of the closed-loop system is analysed using a Lyapunov-based method that employs Wirtinger’s and Halanay’s inequalities, resulting in sufficient conditions for exponential stabilisation expressed as Linear Matrix Inequalities (LMIs). These LMIs further yield bounds on the maximum allowable spatial and temporal sampling intervals, as well as on the exponential decay rate of the system. Sampled-data observers for the damped semilinear wave equation with point measurements over a spatial interval were introduced in [Terushkin and Fridman \(2019\)](#). For the non-delay case, continuous-time distributed control of the one-dimensional heat equation using pointlike measurements—interpreted as spatial averages over small subdomains was proposed in [Wang and Wu \(2014\)](#) and later extended to the sampled-data setting in [Wang and Wang \(2019\)](#). In the two-dimensional case, [Selivanov and Fridman \(2019\)](#) developed distributed sampled-data observers and controllers for semilinear heat equations under similar pointlike measurement assumptions. Other studies on distributed sampled-data and delayed control of parabolic PDEs has been studied in [Am and Fridman \(2014\)](#); [Kang and Fridman \(2018\)](#); [Selivanov and Fridman \(2016a, 2017\)](#).

1.6.2 Boundary Control

Boundary control assumes $\mathcal{B} = 0$ inside the domain, with the system driven solely by the boundary operators $\mathcal{G}_0$ and $\mathcal{G}_L$ in (1.13). This offers a practical alternative to in-domain actuation because sensors and actuators are placed only at the spatial boundary, simplifying implementation and motivating extensive study for both linear and nonlinear PDEs [Balogh and Krstic \(2002\)](#); [Fursikov \(2001\)](#); [Krstic and Smyshlyaev \(2008\)](#); [Triggiani \(1980\)](#). Within this setting, the *backstepping* method has become the cornerstone technique for stabilization: originating from Volterra-type

transformations proposed in the 1970s–1980s [Colton \(1977\)](#); [Seidman \(1984\)](#), it was rigorously formalized by Smyshlyaev and Krstic, who developed a late-lumping boundary-feedback framework for one-dimensional reaction–diffusion–advection equations and its observer counterpart [Smyshlyaev and Krstic \(2004, 2005\)](#). Structurally, backstepping relies on a transformation that maps the original system (1.12) into a stable target system. While this method has been successfully extended to various nonlinear and higher-dimensional systems [de Andrade et al. \(2024\)](#); [Wang et al. \(2025\)](#), its application is typically limited by the complexity of solving the associated kernel PDEs as the operator $\mathcal{F}$ becomes more complex. Subsequent extensions include a transformation that relocates interior actuation to the boundary under specific conditions [Tsubakino et al. \(2012\)](#), comprehensive treatments for higher-dimensional models via backstepping and flatness [Meurer \(2012\)](#), exponential H^1 -stabilization of hyperbolic–parabolic couplings with interior interactions [Chen et al. \(2023\)](#), boundary regulation of a heat equation on a time-varying domain for crystal-growth applications [Izadi et al. \(2015\)](#), and traffic-flow stabilization using Aw–Rascle–Zhang and LWR models augmented by adaptive and learning strategies [Yu and Krstic \(2022\)](#). Ongoing advances in boundary backstepping design are reflected in recent contributions such as [de Andrade et al. \(2024\)](#); [Kerschbaum and Deutscher \(2019\)](#); [Wang et al. \(2025\)](#); [Lamarque et al. \(2025\)](#).

Complementary to backstepping-based designs, *optimal control* and *state-estimation* techniques have also been extended to boundary-controlled PDEs. The linear–quadratic regulator (LQR) framework has been generalised to PDE systems [Bernard and Krstic \(2014\)](#); [Krener \(2021\)](#), producing feedback laws that optimise a quadratic cost function while respecting PDE constraints. However, implementation remains difficult because it involves solving an operator-valued Riccati equation. Another widely used approach is the Kalman filter, applied to obtain estimates of system variables. Backstepping observers are typically derived via systematic design, whereas filtering-based methods aim to reduce estimation errors caused by noise and uncertainty. A recent contribution [Afshar et al. \(2023\)](#) adapts the Extended Kalman Filter to semilinear infinite-dimensional systems, delivering state estimates for nonlinear PDEs under suitable detectability requirements.

The concept of *differential flatness* has emerged as a powerful framework in the control of finite-dimensional nonlinear systems and has found widespread application [Rigatos et al. \(2016\)](#); [Zhang et al. \(2017\)](#); [Rothfuss et al. \(1996\)](#). Fundamentally, differential flatness implies that all state variables and control inputs of a system can be parameterised by a suitably chosen set of flat outputs and their time derivatives. Beyond finite-dimensional systems, differential flatness has

also been extended to infinite-dimensional systems [Meurer \(2011, 2012\)](#); [Meurer and Kugi \(2009\)](#); [Schörkhuber et al. \(2013\)](#). In the context of (1.12), flatness requires that the state $z(t, x)$ and input $u(t, x)$ be representable as an infinite power series of a chosen flat output. A structural limitation of this methodology is the requirement for Gevrey-class regularity, which restricts the method to specific semi-linear or quasi-linear forms of $\mathcal{F}$ that ensure the convergence of these series representations [Schörkhuber et al. \(2013\)](#). Consequently, in practical implementations, flatness-based controllers are constructed using a finite number of terms from the series expansion, i.e., through partial sums that approximate the original infinite series [Laroche et al. \(2000\)](#). It has been effectively applied to PDEs for trajectory tracking and stabilisation by enabling systematic motion planning via feedforward input design [Meurer \(2011, 2012\)](#); [Meurer and Kugi \(2009\)](#); [Rudolph and Woittennek \(2008\)](#). Moreover, this approach has been extended for broader classes of PDEs, including semi and quasi-linear parabolic systems [Lynch and Rudolph \(2007, 2002\)](#); [Martin et al. \(2014\)](#); [Meurer and Zeitz \(2004\)](#); [Woittennek \(2011\)](#).

The port-Hamiltonian (pH) systems framework offers a rigorous and energy-consistent methodology for the modelling, analysis, and control of complex dynamical systems. The pH framework exploits the energy-consistent structure of the interconnection within the spatial operator $\mathcal{F}$, separating the interconnection structure from the components' constitutive relations. Initially developed for lumped systems, the pH framework was later extended to encompass distributed parameter systems in the foundational work of van der Schaft and Maschke [van der Schaft and Maschke \(2002\)](#). This has been further studied in [Duindam et al. \(2009\)](#); [Jacob and Zwart \(2012\)](#); [Wu and Swei \(2017\)](#); [Macchelli et al. \(2016\)](#).

Utkin's early 1980s work extended sliding mode control (SMC) to distributed parameter systems (DPS) governed by PDEs, marking a foundational shift toward infinite-dimensional control. [In terms of the general structure \(1.12\)](#), a fundamental challenge in extending SMC to distributed parameter systems is that spatial derivatives in PDEs may grow rapidly over time, even if the state itself remains small. As a result, the influence of spatial dynamics cannot be effectively attenuated using bounded, high-frequency switching inputs, which are typically sufficient in finite-dimensional SMC design. The first extension of sliding mode control (SMC) to infinite-dimensional systems was presented in [Orlov and Utkin \(1983\)](#) for specific parabolic and hyperbolic PDEs, demonstrating its effectiveness in this setting. The method used modal decomposition to separate the original system into a finite-dimensional (possibly unstable) part and an infinite-dimensional stable part.

A stabilising SMC law was then designed for the finite-dimensional subsystem. This approach was later extended to observer and output feedback designs [Orlov \(2002\)](#); [Orlov and Dochain \(2002\)](#); [Orlov et al. \(2004\)](#). Subsequently, [Orlov and Utkin \(1987\)](#) provided a rigorous justification for the equivalent control method in infinite-dimensional Hilbert spaces. This theoretical development, combined with Lyapunov–Krasovskii and Lyapunov–Razumikhin methods, led to various SMC schemes for distributed and time-delay systems, including first- and second-order controllers [Orlov \(2020\)](#); [Yan et al. \(2017\)](#).

1.7 Analytic Exposition of Selected PDE Control Approaches

In this section, we provide an analytical exposition of four influential PDE control frameworks: backstepping, sliding mode control, the flatness-based approach, and the spatial decomposition method. These approaches have been widely studied in the context of MAS, particularly in problems related to consensus, formation control, and coordinated deployment (see [Section 2.3](#) for an overview of PDE-based approaches for MAS). Rather than presenting a purely descriptive survey, the objective here is to critically examine these methods in order to assess their suitability for deployment problems in large-scale MAS. In particular, we consider practical aspects of implementation, including the location of actuation, the availability of state measurements, and the ability of the control strategy to operate under limited sensing and communication capabilities.

Many classical PDE control techniques are formulated in the context of boundary actuation, where control inputs are applied only at the spatial boundaries of the domain. In the setting of swarm deployment, this corresponds to a scenario in which only the boundary agents, one located at the left end of the swarm and one at the right end, act as leaders that influence the remaining agents. The information generated by these boundary leaders must then propagate through the swarm via chains of local interactions between neighbouring agents. For large swarms, this sequential propagation can introduce significant delays, meaning that agents located deeper inside the swarm may react only after considerable time has elapsed. Consequently, the information received by interior agents may already be outdated, which can slow down the overall deployment process.

In contrast, the spatial decomposition method is inherently an in-domain control approach, in which control action is applied at multiple locations within the spatial domain. In the context of MAS, these locations correspond to leader agents distributed throughout the interior of the swarm,

whose measurements guide the behaviour of nearby followers. This distributed influence structure enables faster propagation of deployment information across the swarm and avoids the need for highly aggressive control actions, such as the high-frequency switching commonly associated with sliding mode control. The present thesis provides a recent application of the spatial decomposition framework to the design of control strategies for swarm deployment problems (see Section 2.8).

1.7.1 Backstepping

Backstepping is a systematic boundary control technique. It was first proposed for finite-dimensional systems Koditschek (1987); Tsiniias (1989); Sontag and Sussmann (1989); Byrnes and Isidori (1989) and was later extended to infinite-dimensional problems Krstic and Smyshlyaev (2008). In the PDE setting, it builds an invertible Volterra-type transformation, an idea already used in earlier studies of PDE controllability Colton (1977); Seidman (1984), that converts the open-loop plant towards a stabilised reference model. The accompanying boundary feedback law then guarantees closed-loop stability. To illustrate the approach, consider the one-dimensional reaction-diffusion equation with an unstable reaction term:

$$\begin{cases} z_t(x, t) = z_{xx}(x, t) + \lambda z(x, t), & x \in (0, 1), t > 0, \\ z(0, t) = 0, \quad z(1, t) = U(t), \end{cases} \quad (1.14)$$

where $\lambda > 0$ is a constant and $U(t)$ is the boundary input. The term λz makes the system unstable for large enough λ . The backstepping method constructs a Volterra transformation that maps the original state $z(x, t)$ into a target state $w(x, t)$, governed by a stable PDE:

$$w(x, t) = z(x, t) - \int_0^x k(x, y) z(y, t) dy, \quad (1.15)$$

where $k(x, y)$ is the kernel. The target system is the heat equation with homogeneous Dirichlet boundary conditions:

$$\begin{cases} w_t(x, t) = w_{xx}(x, t), & x \in (0, 1), t > 0, \\ w(0, t) = 0, \quad w(1, t) = 0. \end{cases} \quad (1.16)$$

To determine the kernel $k(x, y)$, we differentiate the transformation (1.15) and substitute the plant dynamics $z_t = z_{xx} + \lambda z$. After integration by parts, we obtain:

$$w_t(x, t) = z_{xx}(x, t) - k(x, x)z_x(x, t) + k(x, 0)z_x(0, t) + \int_0^x (k_{yy}(x, y) - \lambda k(x, y)) z(y, t) dy, \quad (1.17)$$

$$w_{xx}(x, t) = z_{xx}(x, t) - \frac{d}{dx}k(x, x)z(x, t) - k(x, x)z_x(x, t) - \int_0^x k_{xx}(x, y)z(y, t) dy. \quad (1.18)$$

Subtracting (1.18) from (1.17) gives:

$$\begin{aligned} w_t(x, t) - w_{xx}(x, t) &= \left[\lambda + 2 \frac{d}{dx}k(x, x) \right] z(x, t) - k_y(x, 0)z(0, t) \\ &\quad + \int_0^x (k_{xx}(x, y) - k_{yy}(x, y) - \lambda k(x, y)) z(y, t) dy. \end{aligned}$$

To ensure $w_t = w_{xx}$ for all admissible $z(x, t)$, the kernel must satisfy $k_{xx} - k_{yy} - \lambda k = 0$, with $k(x, 0) = -\frac{\lambda}{2}x$ and $k_y(x, x) = 0$. The kernel PDE is defined on the triangular domain $\{(x, y) \in [0, 1]^2 : 0 \leq y \leq x\}$. Its solution, obtained using the method of successive approximations [Krstic and Smyshlyaev \(2008\)](#), is given by:

$$k(x, y) = -\lambda x \frac{I_1 \left(\sqrt{\lambda(x^2 - y^2)} \right)}{\sqrt{\lambda(x^2 - y^2)}}, \quad (1.19)$$

where I_1 is the modified Bessel function of the first kind [Watson \(1922\)](#). The resulting boundary feedback law is:

$$U(t) = z(1, t) = - \int_0^1 \frac{\lambda I_1 \left(\sqrt{\lambda(1 - y^2)} \right)}{\sqrt{\lambda(1 - y^2)}} z(y, t) dy. \quad (1.20)$$

This control law guarantees exponential stability of the closed-loop system in the L^2 -norm. While backstepping provides a mathematically rigorous framework for stabilising PDE systems through boundary actuation, several considerations arise when applying this approach to large-scale multi-agent systems. First, the method relies on constructing an invertible Volterra transformation, such as (1.15), which maps the original unstable PDE to a stable target system. For many nonlinear or more complex PDE models, identifying a suitable transformation that achieves this mapping can be analytically challenging. Second, once the transformation is defined, the associated kernel PDE must be solved in order to determine the kernel function $k(x, y)$. As illustrated in (1.19), the resulting expressions may involve special functions, and in more general settings the kernel equations may not admit closed-form solutions and must instead be solved numerically. Finally,

because the control input $U(t)$ in (1.20) is applied only at the boundary of the spatial domain, the influence of the control action propagates through the system from the boundary toward the interior. In the context of swarm deployment, this corresponds to a situation where boundary agents influence the rest of the swarm through chains of local interactions. For large swarms, such sequential propagation may introduce delays before interior agents respond to the control signal, which can slow down the overall coordination process.

1.7.2 Sliding Mode Control

The first publication on sliding mode control (SMC) is attributed to Emelyanov [Emelyanov \(1957\)](#), with further developments for finite-dimensional systems carried out in collaboration with his team, including V. Utkin, V. Taran, N. Kostyleva, A. Shubladze, V. Ezerov, and E. Dubrovsky [Emelyanov and Taran \(1963\)](#); [Emelyanov et al. \(1968\)](#); [Emelyanov \(1967, 1970\)](#). A substantial body of literature on SMC has since been published, including books, special issues, and research articles [Utkin \(2013\)](#); [Edwards and Spurgeon \(1998\)](#); [Zinober \(1994\)](#); [Young et al. \(1999\)](#); [Sabanovic et al. \(2004\)](#); [Edwards et al. \(2006\)](#); [Bartolini et al. \(2008, 1997\)](#); [MISAWA and UTKIN \(2000\)](#); [Bartoszewicz et al. \(2008\)](#). The first extension to infinite-dimensional systems appeared in [Orlov and Utkin \(1983\)](#), addressing certain parabolic and hyperbolic PDEs.

Consider the PDE model describing thermal fields governed by:

$$\begin{cases} Q_t = Q_{xx} + DQ + Fu(x, t), & x \in (0, 1), t > 0, \\ Q_x(0, t) = Q_x(1, t) = 0, & t > 0, \\ Q(x, 0) = Q_0(x), & x \in [0, 1], \end{cases} \quad (1.21)$$

where $Q(x, t) \in \mathbb{R}^n$ denotes the distributed state and $u(x, t) \in \mathbb{R}^m$ is the distributed input. From a physical perspective, this model represents the heating of n thermally coupled plants using m distributed actuators. The matrix $D \in \mathbb{R}^{n \times n}$ characterises internal and environmental heat exchange, while $F \in \mathbb{R}^{n \times m}$ models the spatial distribution of input effects. Neumann boundary conditions enforce zero flux at the spatial endpoints.

To design the sliding mode controller, we partition the state and system matrices as

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \quad F = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \quad D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix},$$

with $Q_1 \in \mathbb{R}^{n-m}$, $Q_2 \in \mathbb{R}^m$, $F_1 \in \mathbb{R}^{(n-m) \times m}$, and $F_2 \in \mathbb{R}^{m \times m}$. The sliding variable is introduced as

$$S(Q) := c_1 Q_1 + c_2 Q_2 \in \mathbb{R}^m,$$

where $c_1 \in \mathbb{R}^{m \times (n-m)}$, $c_2 \in \mathbb{R}^{m \times m}$ with $\det(c_2) \neq 0$. This defines the sliding manifold

$$\mathcal{M} := \{Q \in \mathbb{R}^n : S(Q) = 0\}.$$

The control objective is to drive the state $Q(\cdot, t)$ to the origin. This is achieved in two phases. In the first phase, a discontinuous *unit control* drives the sliding variable $S(Q)$ to zero in finite time, thereby forcing the system onto the manifold $\mathcal{M}$. In the second phase, the *equivalent control* maintains the trajectory on $\mathcal{M}$, and the reduced dynamics in terms of Q_1 are shown to be exponentially stable.

Differentiating $S(Q) = c_1 Q_1 + c_2 Q_2$ and using the dynamics of Q in (1.21), we obtain

$$\begin{aligned} S_t &= c_1(Q_1)_t + c_2(Q_2)_t \\ &= c_1((Q_1)_{xx} + D_{11}Q_1 + D_{12}Q_2 + F_1u) + c_2((Q_2)_{xx} + D_{21}Q_1 + D_{22}Q_2 + F_2u) \\ &= S_{xx} + \underbrace{(c_1D_{11} + c_2D_{21})}_{=: C_{11}} Q_1 + \underbrace{(c_1D_{12} + c_2D_{22})}_{=: C_{12}} Q_2 + \underbrace{(c_1F_1 + c_2F_2)}_{=: cF} u. \end{aligned} \quad (1.22)$$

To enforce finite-time convergence to the manifold, we let $\det(cF) \neq 0$, and define the scaled sliding variable $S_1(Q) := (cF)^{-1}S(Q)$. The unit control is chosen as

$$u(Q) = -\frac{M(Q) S_1(Q)}{\|S_1(Q, \cdot, t)\|_{L^2(0,1)}}, \quad (1.23)$$

where $M(Q) := M_0\|Q(\cdot, t)\|_{L^2(0,1)} + M_1$, $M_0 := \|(cF)^{-1}cD\|$, $M_1 > 0$. Consider the Lyapunov functional $V_1(t) := \|S_1(\cdot, t)\|_{L^2(0,1)}^2$. Differentiating along the solutions, integrating by parts, and

substituting (1.23) yields

$$\begin{aligned}
\frac{1}{2}\dot{V}_1(t) &= \int_0^1 S_1^\top S_{1t} dx = \int_0^1 S_1^\top (S_{1xx} + (cF)^{-1}cDQ + u) dx \\
&= -\|S_1\|^2 + \langle S_1, (cF)^{-1}cDQ \rangle + \langle S_1, u \rangle \\
&\leq -\|S_1\|^2 + M_0\|Q\|_{L^2}\|S_1\|_{L^2} - (M_0\|Q\|_{L^2} + M_1)\|S_1\|_{L^2} \\
&= -M_1\|S_1\|_{L^2} = -(M_1 + 1)\sqrt{V_1(t)}.
\end{aligned}$$

This implies that $V_1(t) \rightarrow 0$ in finite time; more precisely, $S(\cdot, t) \equiv 0$ for all $t \geq T$, where $T \leq V_1(0)/M_1$. Since c_2 is invertible, on the manifold $\mathcal{M}$, we have, $Q_2 = -c_2^{-1}c_1Q_1$, then (1.22) becomes $0 = \underbrace{(C_{11} - C_{12}c_2^{-1}c_1)}_{=: A_S}Q_1 + cF u_{\text{eq}}$. Substituting this expressions into the dynamics of Q_1 , we obtain

$$\begin{aligned}
(Q_1)_t &= (Q_1)_{xx} + D_{11}Q_1 + D_{12}Q_2 + F_1u_{\text{eq}} = (Q_1)_{xx} + (D_{11} - D_{12}c_2^{-1}c_1 - F_1(cF)^{-1}A_S)Q_1 \\
&= (Q_1)_{xx} + RQ_1,
\end{aligned} \tag{1.24}$$

where $R := D_{11} - D_{12}c_2^{-1}c_1 - F_1(cF)^{-1}A_S$. To establish $Q \rightarrow 0$ in L^2 -norm, let $W > 0$ be the solution of the Lyapunov equation $R^\top W + WR = -I$, and define the Lyapunov functional $V(t) := \int_0^1 Q_1^\top W Q_1 dx$. Differentiating along (1.24)

$$\begin{aligned}
\dot{V}(t) &= 2 \int_0^1 Q_1^\top W (Q_1)_t dx = 2 \int_0^1 Q_1^\top W (Q_1)_{xx} dx + 2 \int_0^1 Q_1^\top W R Q_1 dx \\
&= -2 \int_0^1 (Q_1)_x^\top W (Q_1)_x dx + \int_0^1 Q_1^\top (WR + R^\top W) Q_1 dx \\
&= -2 \int_0^1 (Q_1)_x^\top W (Q_1)_x dx - \int_0^1 Q_1^\top Q_1 dx.
\end{aligned}$$

The matrix $WR - R^\top W$ is skew-symmetric, and can be verified directly. Consequently, for any vector Q_1 , the scalar quantity $Q_1^\top (WR - R^\top W) Q_1$ satisfies $Q_1^\top (WR - R^\top W) Q_1 = -Q_1^\top (WR - R^\top W) Q_1$, which implies that it must be zero. Furthermore, using the Lyapunov identity $R^\top W + WR = -I$, and denoting by $\lambda_{\max}$ the largest eigenvalue of the positive definite matrix W , we have $Q_1^\top W Q_1 \leq \lambda_{\max} Q_1^\top Q_1$ to conclude the desired upper bound on the Lyapunov derivative. Using the upper bound $Q_1^\top W Q_1 \leq \lambda_{\max} Q_1^\top Q_1$, we conclude

$$\dot{V}(t) \leq -\lambda_{\max}^{-1} V(t),$$

which implies exponential decay. Since on the manifold $Q_2 = -c_2^{-1}c_1Q_1$, we conclude $Q \rightarrow 0$ in L^2 -norm.

While sliding mode control is an effective control strategy, its standard formulation relies on discontinuous control inputs. In particular, the control law given in (1.23) introduces a switching action that forces the sliding variable $S(Q)$ to reach and remain on the sliding manifold. In practice, this switching may occur at very high frequency, leading to the well-known phenomenon of *chattering*. In the context of large-scale multi-agent systems, where each agent is equipped with physical motors and limited power resources, such aggressive control actions may place unnecessary demands on the hardware. For the purposes of swarm deployment considered in this thesis, it is therefore desirable to employ control strategies that achieve stability while maintaining smoother control inputs that are more compatible with the physical limitations of real agents.

1.7.3 Flatness-Based Control Design

The concept of *differential flatness* was originally introduced in the context of finite-dimensional systems by Fliess et al. (1995), and later extended to infinite-dimensional systems in a series of works Fliess (1998); Fliess et al. (1998); Fliess and Mounier (1999), laying the foundation for flatness-based control of PDEs. Subsequent contributions, including Laroche et al. (2000); Rudolph (2003); Freudenthaler and Meurer (2020), expanded the theoretical framework and demonstrated its relevance in boundary control problems for distributed parameter systems.

We now consider the one-dimensional heat equation defined on the spatial domain $x \in (0, 1)$ and evolving over time $t \in \mathbb{R}$, given by

$$z_t(t, x) - \gamma z_{xx}(t, x) = 0, \quad z_x(t, 0) = 0, \quad z(t, 1) = u(t),$$

where $z(t, x)$ denotes the temperature profile along the domain and $u(t)$ represents the boundary control input, and γ is the thermal diffusivity of the material. The control objective is to regulate the temperature distribution to a spatially uniform, constant value Z after a finite time interval $T > 0$. Specifically, we aim to achieve $z(t, x) = Z$ for all $t > T$ and all $x \in [0, 1]$. To proceed, we adopt the methodology introduced by Fliess (1998), wherein the solution $z(t, x)$ is assumed to admit a formal power series expansion in the spatial variable x . Accordingly, we express the state

as

$$z(t, x) = \sum_{i=0}^{\infty} a_i(t) \frac{x^i}{i!},$$

where $a_i(t)$ are time-varying coefficients to be determined. Substituting this series representation into the heat equation requires the computation of the relevant partial derivatives:

$$\frac{\partial z}{\partial t}(t, x) = \sum_{k=0}^{\infty} \frac{da_k(t)}{dt} \frac{x^k}{k!}, \quad \frac{\partial z}{\partial x}(t, x) = \sum_{k=0}^{\infty} a_{k+1}(t) \frac{x^k}{k!}, \quad \frac{\partial^2 z}{\partial x^2}(t, x) = \sum_{k=0}^{\infty} a_{k+2}(t) \frac{x^k}{k!}.$$

Of particular interest is the temperature at the boundary $x = 0$, which from the series expansion evaluates to $z(t, 0) = a_0(t)$. We denote this quantity by $y(t)$ and identify it as the flat output of the system. Furthermore, the Neumann boundary condition $z_x(0, t) = 0$ implies that $a_1(t) = 0$. Substituting the series expressions into the governing PDE and equating terms of like powers of x , we arrive at the following recursive relation:

$$\frac{d}{dt} a_k(t) = \gamma a_{k+2}(t).$$

This relation enables the computation of all coefficients $a_k(t)$ in terms of the flat output $y(t)$ and its time derivatives. In particular, one obtains

$$a_k = \begin{cases} 0 & \text{for } k \text{ odd,} \\ \frac{1}{\sqrt{\gamma^k}} \frac{d^k y(t)}{dt^k} & \text{for } k \text{ even.} \end{cases}$$

Hence, the full temperature distribution can be written explicitly in terms of the flat output as

$$z(t, x) = \sum_{k=0}^{\infty} \left(\frac{x^2}{\gamma} \right)^k \frac{1}{(2k)!} \frac{d^{2k} y(t)}{dt^{2k}}.$$

This expression clearly demonstrates that the state $z(t, x)$ is fully determined by the flat output $y(t)$ and an infinite number of its time derivatives. Moreover, evaluating this series at $x = 1$, we obtain a closed-form expression for the boundary control input:

$$u(t) = z(t, 1) = \sum_{k=0}^{\infty} \left(\frac{1}{\gamma} \right)^k \frac{1}{(2k)!} \frac{d^{2k} y(t)}{dt^{2k}}.$$

While the flatness-based approach is very useful for trajectory planning, it also has some limitations

when applied to multi-agent systems. First, the state and control input are expressed as infinite series expansions, as shown in the expressions for $z(t, x)$ and the control law $u(t)$. In practice, these infinite series must be truncated in order to implement the controller, which introduces approximation errors and may affect the accuracy of the resulting motion. Second, flatness-based control is primarily a feedforward design approach, where the control input $u(t)$ is constructed from the flat output and its derivatives. This design assumes that the system model is known accurately and that disturbances are small. However, in large-scale swarm systems, agents may experience disturbances such as wind, sensor noise, or local hardware faults. In such situations, additional feedback mechanisms may be required to compensate for these real-time deviations.

1.7.4 Spatial Decomposition Method

The *Spatial Decomposition Method* is an effective in-domain control approach for partial differential equations (PDEs). As the term suggests, the spatial domain is divided into a finite number of subintervals, with the PDE state known or estimated at the endpoints of each. These values serve as a basis for reconstructing the state over the full domain. The motivation for this approach, i.e., estimating the PDE state using partial spatial information, can be explained through the following example. Consider the semilinear parabolic PDE:

$$z_t(t, x) = az_{xx}(t, x) + F(t, z) + v(t, x), \quad x \in (0, 1), \quad t \in [0, \infty),$$

where $a > 0$, and the nonlinearity F satisfies a Lipschitz condition: $\|F(t, z)\| \leq L\|z\|$. We assume zero Dirichlet boundary conditions: $z(t, 0) = 0$, $z(t, 1) = 0$. The control objective is to design $v(t, x)$ such that the solution $z(t, x)$ becomes exponentially stable in the L^2 -norm. Define the Lyapunov functional: $V(t) = \int_0^1 z^2(t, x) dx$. Differentiating V and using integration by parts along with the Lipschitz property of F , we obtain:

$$\begin{aligned} \frac{dV(t)}{dt} &\leq -2a\|z_x(t, \cdot)\|^2 + 2L\|z(t, \cdot)\|^2 + \int_0^1 z(t, x) v(t, x) dx \\ &\leq -2(a\pi^2 - L)\|z(t, \cdot)\|^2 + \int_0^1 z(t, x) v(t, x) dx. \end{aligned}$$

To ensure that $\frac{dV(t)}{dt} < 0$, one can choose a feedback law $v(t, x) = -Kz(t, x)$ with $K > L$. However, in a practical MAS, the full state $z(t, x)$ is not measurable. This limitation is addressed by the

spatial decomposition method proposed by [Fridman and Blighovsky \(2012\)](#), which partitions the domain into sub-intervals and reconstructs the control law using only pointwise measurements at discrete locations. This framework is uniquely suited for the large-scale MAS coordination explored in this thesis for several reasons. First, unlike backstepping, which relies on a few leaders at the boundary and suffers from propagation delays, the approach in [Fridman and Blighovsky \(2012\)](#) allows for 'distributed leaders' throughout the swarm. This ensures that agents in the center of the group respond to control signals just as quickly as those at the edges. Second, this method provides a smooth feedback response, avoiding the excessive physical demand on motors caused by the switching in sliding mode control or the mathematical approximations required in flatness-based designs. By aligning the mathematical partitioning of the PDE with the physical distribution of the agents, we achieve a system that is both faster to stabilise and more robust to real-world sensing constraints.

1.8 Thesis Contribution

1.8.1 Problem Statement

In this PhD thesis, we study the problem of *swarm deployment*, where a large number of agents must be deployed onto a predefined *target curve* in the plane. A key difficulty is that the agents do not have access to the location of the curve. Each agent operates independently and is governed by its own non-linear dynamics, with only limited local sensing and communication capabilities.

As a result, the main challenge is to design *distributed control strategies* that enable the swarm to collectively achieve the desired deployment. The control design must satisfy the following core objectives:

1. *Rapid convergence* of each agent's position to the desired target curve.
2. *Swarm cohesion*, ensuring that the collective structure remains intact throughout the deployment process.
3. Robust performance across a wide range of initial conditions and swarm sizes.

A formal definition of the deployment problem is given in Section 2.4, along with a precise mathematical characterisation in Definition 2.4.

1.8.2 Related Research

The works of Wei et al. (2019a); Terushkin and Fridman (2021) introduced scalable control strategies for swarm deployment that remain effective regardless of the number of agents. These strategies are based on a *distance-based interaction topology* Oh et al. (2015), wherein each agent measures its relative position with respect to its immediate neighbours and adjusts its state accordingly. This local interaction structure promotes swarm cohesion, enforces uniform inter-agent spacing, and enables distributed coordination. Crucially, it also ensures that if one agent converges to its target, its immediate neighbours also converge.

In the limit of large swarm sizes, this interaction framework admits a *continuum approximation*, resulting in a PDE-based model for the swarm’s collective error dynamics. Specifically, the model leads to a *linear parabolic PDE* in the work of Wei et al. (2019a), and a *damped wave equation* in the work of Terushkin and Fridman (2021), both governing the evolution of the agents’ positional errors relative to the target.

However, the resulting PDE-based error system is inherently unstable. While local interactions are sufficient to align neighbouring agents, they cannot guarantee deployment since none of the agents has access to the location of the target curve. Thus, the system alone cannot stabilise the swarm towards the target.

To overcome this limitation, a global control strategy was proposed in Wei et al. (2019a); Terushkin and Fridman (2021) by adapting the *spatial decomposition method*, an in-domain stabilisation technique originally developed for semilinear parabolic PDEs in Fridman and Blighovsky (2012). In this approach, a subset of agents, referred to as *leaders*, is used to guide the swarm by providing direct measurements of their positional error relative to the target. While both Wei et al. (2019a) and Terushkin and Fridman (2021) adopt this overall control strategy, they apply it to different PDE-based models of multi-agent systems: the linear parabolic PDE in Wei et al. (2019a), and the damped wave equation in Terushkin and Fridman (2021). The key components of the control strategy are:

1. *Leader-Based Target Tracking*: Leaders are sparsely distributed throughout the swarm (including at the boundaries) and do *not* interact with other leaders. Each leader can compute its positional error relative to the target curve and shares this information with its neighbouring agents. Since the followers cannot measure their own error, they adopt the positional error of the *nearest leader* as their own estimate. As a result, the swarm collectively constructs a piecewise constant approximation of the true PDE error state.
2. *Boundary Leader Decoupling*: Internal leaders remain locally coupled to the swarm, whereas boundary leaders are decoupled from swarm dynamics and act independently. This design choice enables faster convergence at the swarm boundaries.

To quantify the performance of their strategy, the authors used an L^2 -norm bound on the difference between the PDE error state and its piecewise-constant approximation constructed by the swarm using the leaders' positional errors. This bound, originally developed in [Fridman and Blighovsky \(2012\)](#) in the context of robust control of semilinear parabolic PDEs, depends linearly on the spacing between consecutive leaders. By incorporating this result into a Lyapunov-based stability framework, they derived *Linear Matrix Inequalities (LMIs)* whose feasibility determines the number of leaders *sufficient* to guarantee successful deployment of the swarm onto the target curve.

Despite its elegance and scalability, this approach reveals *three critical limitations*, which motivate the core research questions of this dissertation.

1.8.3 Motivating Insights and Questions

While the framework proposed in [Wei et al. \(2019a\)](#); [Terushkin and Fridman \(2021\)](#) provides a scalable and distributed approach to swarm deployment, a closer examination reveals certain design trade-offs that motivate the present work. From a system-design perspective, these frameworks offer clear advantages, as they do not require communication between leaders. This reduces communication overhead, energy consumption, and implementation complexity. However, this design also introduces a trade-off that becomes more pronounced when faster convergence of the swarm to the target curve is desired. Since, the swarm constructs only a piecewise constant approximation of the PDE error state. This approximation is reasonably accurate within a small neighbourhood around the sampling point, i.e. the nearest leader, but its quality deteriorates as one moves farther away. Consequently, agents that are not close to any leader receive a poor approximation of their

positional error, leading to less effective control updates and slower convergence. To achieve faster convergence under these conditions, a high leader density may be required.

This observation raises an important question: can the introduction of communication between leaders improve the propagation of information across the swarm and reduce the number of leaders required? By allowing leaders to communicate, it may be possible to obtain a more accurate and globally consistent approximation of the PDE error. This leads to the hypothesis that enabling communication between leaders can allow the swarm to achieve faster convergence using fewer leader agents, thereby improving the economic viability of the deployment strategy by reducing the need for high leader density.

A related issue concerns the behaviour of boundary leaders. In existing strategies, boundary leaders evolve independently under dynamic Dirichlet-type conditions and move directly toward the target curve. While this simplifies the control design, it may reduce coordination with the interior swarm. In particular, boundary leaders may reach the target earlier than interior agents, which can affect the overall cohesion of the swarm during deployment. This motivates the investigation of alternative strategies in which boundary leaders remain coupled to the swarm dynamics.

These observations lead to the following research questions:

Question 1: *Can enabling communication between leaders improve convergence speed while reducing the number of leaders required for successful swarm deployment, thereby improving economic viability?*

Question 2: *How can the control laws for boundary leaders be modified so that they remain coupled to the swarm dynamics, moving in coordination with the group and contributing to overall cohesion even when they are on the target curve initially?*

These questions guide the investigations in this dissertation, with the aim of improving the efficiency, convergence speed, and robustness of swarm deployment strategies.

1.8.4 Contributions

The following contributions address the research questions raised above by introducing novel control design techniques and theoretical insights.

Contribution 1: (*Improved Error Estimation via Leader Communication*) We enhance the existing deployment framework by enabling inter-leader communication, allowing leaders to exchange their positional errors relative to the target curve. This communication enables the swarm to construct a piecewise linear approximation of the PDE error state. As a result, each follower estimates its positional error as a linear interpolation of the errors from its *left and right leaders*, rather than relying solely on the nearest one. This yields a more accurate error estimate, which improves the quality of the control input and enables more effective correction of agent positions. The resulting refinement significantly accelerates convergence and enhances overall deployment performance.

Contribution 2: (*Enhanced Boundary Control via Wentzell Conditions*) To prevent boundary leaders from either advancing toward the target curve ahead of the swarm or remaining fixed on it when initially placed there, we propose a boundary control strategy that dynamically couples each boundary leader with its immediate swarm neighbour. This coupling leads to the formulation of *Wentzell boundary conditions*, which keep boundary leaders integrated with the swarm throughout the deployment process. In particular, if a boundary leader is initially on the target curve, the proposed dynamics cause it to move back toward the swarm and then advance in coordination with it, thereby maintaining global cohesion.

Table 1.1: Feature-by-Feature Comparison with the Approach of [Wei et al. \(2019a\)](#)

Feature	Model in Wei et al. (2019a)	Present Work
<i>Distance-based topology</i>	✓	✓
<i>Semi-linear PDE</i>	✗	✓
<i>Wentzell boundary condition</i>	✗	✓
<i>Dynamic Dirichlet-type boundary</i>	✓	✓
<i>Inter-leader communication</i>	✗	✓
<i>Swarm cohesion guarantee</i>	✗	✓
<i>Deployment with fewer leaders</i>	✗	✓
<i>Rapid agent convergence to target</i>	✗	✓

Contribution 3: To support the proposed leader-communication strategy, we derive two novel L^2 -norm bounds on the difference between the PDE error state and its piecewise linear approximation.

The first bound is analogous in form to the estimate established in [Fridman and Blighovsky \(2012\)](#), but is newly derived for the piecewise linear setting and follows a distinct proof. This guarantees that the proposed approach is at least as effective as existing methods, since both rely on the same core inequality.

The second bound provides a significant refinement: it involves the second-order spatial derivative of the error and depends *quadratically* on the spacing between consecutive leaders. This sharper estimate demonstrates that the approximation error decays much faster than in the piecewise constant case considered in [Wei et al. \(2019a\)](#); [Terushkin and Fridman \(2021\)](#), where the bound scales only *linearly* with spacing. The quadratic dependence means that even small reductions in leader spacing lead to significantly more accurate error estimates, allowing agents to correct their positions with higher precision.

By incorporating these bounds into a Lyapunov-based stability analysis, we derive LMIs that characterise sufficient conditions for deployment. Numerical results show that the number of leaders required to guarantee successful deployment is strictly lower than in the original strategy of [Wei et al. \(2019a\)](#); [Terushkin and Fridman \(2021\)](#).

1.9 Thesis Organisation

Chapter 2 reviews the consensus problem in MAS, considering both ODE and PDE models. The chapter sets out the assumptions on the ODE-based formulation that allow the design of local controllers, ensuring cohesion within the formation and enabling the derivation of a PDE-based error system in the large-population limit. It also introduces the modelling of boundary agents, leading to Dirichlet, Neumann, and Wentzell boundary conditions. Their physical interpretations are discussed in the context of the deployment problem. To address the instability of the PDE system with Wentzell boundaries, an in-domain control approach is developed by decomposing the spatial domain into intervals, with leaders placed at the interval boundaries. These leaders are assumed to communicate with one another, and their interaction forms the basis for a global control strategy that improves deployment performance.

Chapter 3 provides the mathematical background required to analyse the PDE-based system. It begins with an introduction to semigroup theory, which is used to establish the well-posedness of the closed-loop dynamics. Lyapunov-based stability analysis is then carried out for both nonlinear and linear systems, leading to stability conditions expressed as LMIs. The feasibility of these LMIs is discussed, showing that the proposed strategies are implementable and robust.

Chapter 4 validates the theoretical results through numerical simulations. MATLAB-based simulations are used to test the effectiveness of the proposed deployment strategies. The chapter examines

the effect of inter-leader communication, compares local and global strategies, and evaluates the influence of different boundary conditions on convergence and cohesion. The results show that the proposed controllers outperform existing approaches that exclude inter-leader communication and Wentzell boundary control, leading to faster convergence and better cohesion.

Chapter 5 revisits the core problem of swarm deployment and presents a structured summary of the control framework developed to address it. A concise overview of the main results is then provided. Beyond these results, the chapter outlines two natural extensions, namely time-delay systems and multi-agent deployment in two-dimensional domains, along with preliminary findings: a stability theorem under time delays and novel L^2 -approximation bounds for the 2D setting. The chapter closes with a brief reflection on the broader significance and potential impact of the work.

PDE-based MAS Deployment

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2.1 MAS Coordination Control

Formation vs Consensus vs Deployment

Formation control concerns the distributed coordination of multiple agents such that a prescribed geometric configuration, for example, a line, circle, or lattice, is maintained through local interactions, without necessitating convergence to a single location [Oh et al. \(2015\)](#). Consensus, by

contrast, entails the asymptotic agreement of agent states (e.g., position, heading, velocity, or measured variables) via iterative information exchange with neighbours; it can be viewed as a limiting case of formation control wherein all inter-agent distances converge to zero [Olfati-Saber et al. \(2007\)](#). Deployment addresses the spatial allocation of agents (e.g., mobile sensors, autonomous drones, robotic swarms) across a domain to achieve coverage goals such as maximising area surveillance or minimising redundancy. For a foundational treatment of these ideas, the reader is referred to [Olfati-Saber and Murray \(2004\)](#); [Ren et al. \(2005\)](#); [Olfati-Saber et al. \(2007\)](#); [Mesbahi and Egerstedt \(2010\)](#); [Oh et al. \(2015\)](#).

Although Chapter 1 surveyed the principal mathematical models for MAS coordination, we now offer a concise historical overview of their evolution. The origins of MAS coordination can be traced to biological observations of collective behaviours such as flocking in birds, schooling in fish, and swarming in insects. A pivotal milestone in this direction was Reynolds' Boid model [Reynolds \(1987\)](#), which demonstrated that realistic group movement could emerge from simple local interaction rules. Inspired by this, [Vicsek et al. \(1995\)](#) introduced a statistical mechanics framework to capture *phase transitions* in self-propelled particles, providing early insights into how local alignment rules lead to global coordination.

Transitioning from these biologically inspired and simulation-driven models to formal mathematical theory, a rigorous framework for consensus in dynamic networks was developed by [Saber and Murray \(2003\)](#); [Olfati-Saber and Murray \(2004\)](#), building on foundational work by [Fax \(2002\)](#); [Fax and Murray \(2004\)](#). Their approach employed *graph theory* and *matrix analysis*, particularly the use of the *Laplacian matrix*, to characterise conditions under which agents can achieve consensus. This work introduced key ideas such as *consensus protocols*, *algebraic connectivity*, and *solvability conditions*, and demonstrated how these properties affect convergence speed and robustness to time delays.

Parallel to this, [Jadbabaie et al. \(2003\)](#) addressed the *alignment problem*, focusing on how agents reach agreement through distributed interactions without relying on explicit optimisation. Their analysis formalised the behaviour observed in the Vicsek model. Further theoretical advancements by Moreau [Moreau \(2005\)](#) and Ren and Beard [Ren and Beard \(2005\)](#) extended the framework to account for *directed communication graphs* and more general agent dynamics, such as *second-order integrator systems*. These works established that consensus is achievable in directed networks under certain topological conditions, such as the presence of a *directed spanning tree*.

The coordination literature also draws heavily from earlier contributions in *distributed computing and control*, including work by [Tsitsiklis \(1984\)](#), [Borkar and Varaiya \(2003\)](#), and foundational texts such as [Lynch \(1996\)](#). In fact, the formulation of consensus in [Olfati-Saber and Murray \(2004\)](#) was originally motivated by the vision of *amorphous computing*, as proposed in [Abelson et al. \(2000\)](#); [Nagpal \(2002\)](#), where large populations of agents collectively perform computation through local interaction. This foundation later supported the development of *flocking algorithms* with provable convergence guarantees, including capabilities for *obstacle avoidance* and resilience to *adversarial agents* [Olfati-Saber \(2006\)](#).

2.2 Characterisation of MAS Coordination Techniques

This section characterises MAS coordination in terms of what each agent *senses*, what the system *controls*, and how agents interact. The sensed variables define the sensing capabilities required per agent, while the controlled variables determine the necessary inter-agent coupling. Controlling absolute *positions* requires minimal coordination, whereas controlling *displacements* or *distances* imposes sustained coupling to maintain rigid-body-like behaviour. Thus, the choice of controlled variable directly shapes the coordination goal and corresponding interaction topology. Based on this, we classify strategies as *position-based*, *displacement-based*, or *distance-based*, and the overall architecture as either *leader-based* or *leaderless*, depending on sensing availability. We now briefly describe the three coordination schemes.

- **Position-based coordination:** Each agent measures its position with respect to a global coordinate system and controls it directly to achieve a desired configuration. The agent doesn't need to sense or communicate with its neighbours to accomplish the overall objective, because its goal is defined absolutely, not relative to others. This means agents operate independently, and no relative distance or direction measurements between agents are required to achieve the desired formation, consensus, or deployment. Note that the control architecture imposes a strong structural requirement: a shared global reference frame and a predefined global target for each agent. This makes the approach inherently centralised in its information structure, even if the control computation itself is decentralised.

- **Displacement-based coordination:** In this approach, agents do not need to know their own absolute positions. Instead, they control how far and in which direction their neighbours are, based on a desired set of relative displacements. These displacements are defined with respect to a global coordinate system, and each agent is assumed to sense the relative positions of its neighbours in that frame. While agents do not need to know their own absolute positions, they must share a common orientation so that directions are interpreted consistently. The global coordinate system itself does not need to be fully known.
- **Distance-based coordination:** Agents control only the distances to their neighbours to realise the desired formation, consensus, or deployment, which is defined by the prescribed inter-agent distances. Each agent senses relative positions in its own local coordinate frame, and these local frames do not need to be aligned. No absolute positions or shared orientation are required. The overall configuration emerges from local distance constraints. Compared with displacement-based schemes, agents agree only on how far apart to be, not on direction, making this the most decentralised of the three approaches.

Fig. 2.1 illustrates the three strategies. In position-based control (a), agents operate in a global frame with known positions (indicated by maroon arrows) and move directly toward the target curve. In displacement-based control (b), agents are still in a global frame but use relative position vectors to neighbours (shown by double arrows). In distance-based control (c), agents rely only on local coordinates and inter-agent distances (dashed lines), without needing a global frame.

Having characterised coordination strategies by the types of controlled variables, we now turn to the role of *leaders* in structuring MAS behaviour.

Definition 2.1 (Ajwad (2020)). *A leader is an agent which produces a reference state representing a control objective or feature of common interest for the other agents in the MAS. A leader may be either real, meaning a physical agent with its own dynamics, or virtual, serving only as a prescribed trajectory or signal.*

Consider a MAS composed of $N + 1$ agents indexed by $i \in \{0, 1, \dots, N\}$. Each agent evolves according to the nonlinear dynamics:

$$\dot{z}_i(t) = F_i(z_i(t), u_i(t)), \quad z_i(t) \in \mathbb{R}^n, \quad u_i(t) \in \mathbb{R}^m \quad (2.1)$$

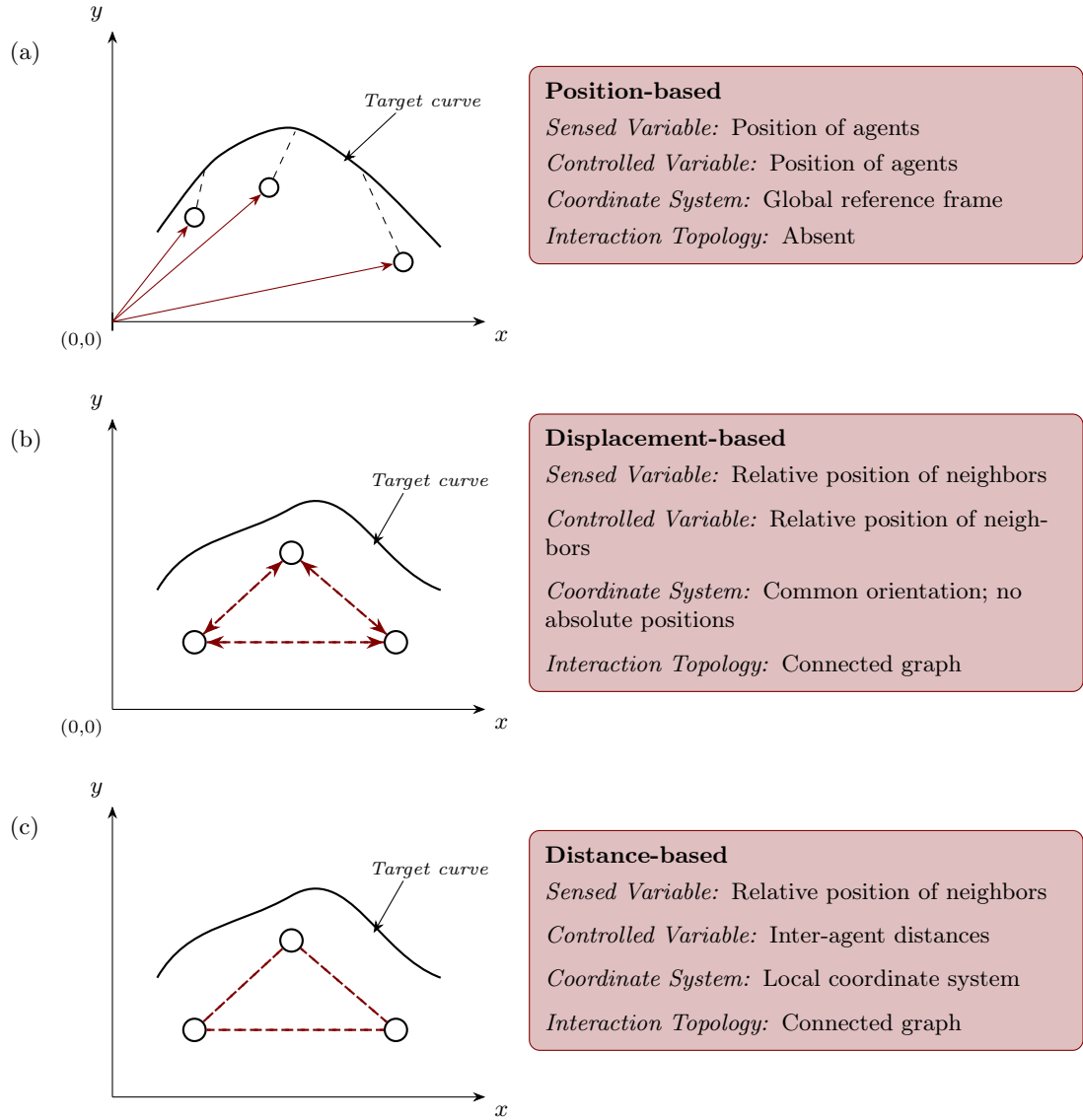


Figure 2.1: Position-, displacement-, and distance-based deployment strategies, each with a schematic (left) and summary (right).

where $z_i(t)$ denotes the state of agent i , $u_i(t)$ its control input, and $F_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a smooth vector field describing its local dynamics. Here, we give formal mathematical definitions for MAS to achieve consensus, formation and deployment.

Definition 2.2 (Ajwad (2020)). *The MAS (2.1) achieves leaderless consensus if for all $z_i(0)$, and for all $i, j = 0, 1, \dots, N$, $\|z_i(t) - z_j(t)\|_2 \rightarrow 0$ as $t \rightarrow \infty$, where $\|\cdot\|_2$ denotes the standard Euclidean norm on $\mathbb{R}^n$. In the leader-following consensus, the objective is not only to ensure that each agent of MAS reaches consensus on a common state, but that common state must also converge to the*

reference trajectory produced by the leader, i.e. the MAS (2.1) achieves leader-following consensus if for all $z_i(0)$, $\|z_i(t) - z_0(t)\|_2 \rightarrow 0$ as $t \rightarrow \infty$ for all $i = 1, 2, \dots, N$.

Definition 2.3 (Dong and Hu (2017); Yuhang et al. (2022)). Let $\bar{z}_i(t) \in \mathbb{R}^n$ denote the desired relative position of agent i within a formation, and let $b(t) \in \mathbb{R}^n$ represent the trajectory of the formation centre. The MAS (2.1) achieves leaderless formation if

$$\lim_{t \rightarrow \infty} \|z_i(t) - \bar{z}_i(t) - b(t)\|_2 = 0, \quad \forall i \in \{1, \dots, N\}.$$

In a multi-leader setting, suppose M agents are followers and $N - M + 1$ are leaders indexed by $\{M, \dots, N\}$. The system achieves time-varying formation tracking if there exist weights $\alpha_k \geq 0$ with $\sum_{k=M+1}^N \alpha_k = 1$ such that

$$\lim_{t \rightarrow \infty} \left\| z_i(t) - h_i(t) - \sum_{k=M}^N \alpha_k z_k(t) \right\|_2 = 0, \quad \forall i \in \{0, \dots, N\},$$

where $h_i(t) \in \mathbb{R}^n$ defines the desired offset of agent i relative to the moving formation reference.

Definition 2.4 (Deployment). Let $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ be a target curve and define $\gamma_i := \gamma(i/N)$. The MAS (2.1) achieves deployment if

$$\lim_{t \rightarrow \infty} \|z_i(t) - \gamma_i\|_2 = 0, \quad \forall i \in \{0, \dots, N\}.$$

This can be realised under both leaderless and leader-based protocols, though leader guidance may accelerate convergence.

2.3 Overview of PDE-Based Approaches for MAS

Early foundational work by Ferrari-Trecate et al. (2006) introduced a partial difference equation (PdE) framework applied to graphs to analyse collective dynamics, marking a critical shift toward continuous approximations of discrete MAS models. A significant advancement appeared in Frihauf and Krstic (2010), where a PDE-based methodology was introduced for distributed organisation of agent populations along geometric paths. Treating the collective evolution as a spatially distributed field, the authors applied *backstepping* to develop a boundary control strategy that ensured the

spatial configuration evolved toward a predefined path, while preserving relative distances among agents. This idea was extended in [Meurer and Krstic \(2011\)](#), where finite-time deployment was achieved using a distributed trajectory design method formulated through a first-order hyperbolic model, demonstrating rapid formation acquisition in time-critical settings. Complementary to these developments, [Hao and Barooah \(2012\)](#) analysed the approximation accuracy between PDE models and discretised agent-based systems. Their work confirmed that PDEs can serve as accurate representations of coupled ODE-based MAS dynamics. Distributed *in-domain control* remains relatively less explored. An early effort in this direction is [Demetriou \(2013\)](#), which studied the synchronisation of parabolic PDE-modelled agents with actuation distributed within the domain. In a related advancement, [Qi et al. \(2014\)](#) developed a multidimensional backstepping method for deployment in 3D environments based on a *reaction-diffusion* model. [Pilloni et al. \(2015\)](#) proposed a local interaction protocol enabling MASs to achieve *average consensus* under spatial disturbances affecting boundary conditions. Building on this, [Xiong and Li \(2017\)](#) introduced a *nonfragile feedback boundary control* strategy using (LMIs) for robust stability verification. Further extending the PDE modelling, [Qi et al. \(2017\)](#) explored a second-order hyperbolic PDE in the context of adaptive multi-agent arrangements, taking advantage of propagation effects inherent in the model to dynamically reconfigure formations. Leaderless consensus problems were tackled in both heat and wave PDE frameworks in [Fu et al. \(2018\)](#), while [He \(2018\)](#) employed *adaptive sliding-mode control* to manage disturbances in perturbed parabolic PDEs. The wave-based leaderless consensus problem was further analysed in [Chen et al. \(2020\)](#). Communication delay, an important practical concern, was addressed in [Qi et al. \(2019\)](#), where the authors modelled delay dynamics using a transport equation and designed a backstepping controller to ensure robust formation control in the presence of latency. In another notable contribution, [Wei et al. \(2019a\)](#) analysed agent deployment on open and closed curves using *linear parabolic PDEs*, employing dynamic or periodic boundary conditions. Their *in-domain leader-follower control* approach ensured stability via LMI-based methods. Experimental validation was presented by [Freudenthaler and Meurer \(2020\)](#), who implemented a PDE-based coordination strategy combining trajectory linearisation with boundary feedback techniques, and validated the method on a physical testbed, confirming its practical applicability. Other theoretical advances include the use of *Lyapunov-based* boundary control in wave PDEs [Aguilar et al. \(2020\)](#), and synchronisation of *semilinear hyperbolic PDEs* under boundary actuation using LMIs [Ferrante et al. \(2021\)](#). [Terushkin and Fridman \(2021\)](#) addressed the deployment of nonlinear MASs using semi-linear parabolic PDEs on open curves.

Table 2.1: Overview of PDE-Based MAS Control Strategies

References	PDE Type	Problem Addressed	Control Type	Key Methods / Features
Demetriou (2013) ; Wei et al. (2019a) ; Terushkin and Fridman (2021) ; Hasanzadeh and Tang (2024)	Parabolic, Linear, Semi-linear	Synchronisation, deployment, dynamic average consensus	In-domain	Leader-follower networks, LMIs, Neumann boundaries
Fu et al. (2018) ; Wei et al. (2019a)	Heat, Wave, Parabolic	Distributed consensus and formation	In-domain	Relative position control, LMIs, stability analysis
Frihauf and Krstic (2010) ; Meurer and Krstic (2011) ; Qi et al. (2014) ; Zhang et al. (2024) ; Freudenthaler and Meurer (2016)	Parabolic, Transport, Reaction-Diffusion	Leader-based deployment, formation, time-critical control	Boundary	Backstepping, flatness-based planning, experimental validation
Qi et al. (2017, 2019)	Wave, Transport with Delay	Dynamic formation, delay compensation	Boundary	Wave-based design, transport modelling, backstepping
Orlov et al. (2016) ; Pilloni et al. (2015) ; He (2018)	Perturbed Diffusion, Parabolic	Leaderless and leader-following consensus	Boundary	Nonlinear sliding-mode control, adaptive strategies
Aguilar et al. (2020) ; Chen et al. (2020)	Wave	Synchronisation, bipartite consensus	Boundary	Lyapunov-based protocols, Neumann actuation
Ferrante et al. (2021) ; Selivanov and Fridman (2022)	Semilinear, First-order Hyperbolic	Output synchronization, single-leader control	Boundary	LMIs with nonlinearities, leader via boundary actuation

The work in [Hasanzadeh and Tang \(2024\)](#) studied *dynamic average consensus* with in-domain actuation and *Neumann boundary conditions*. [Selivanov and Fridman \(2022\)](#) proposed a novel *single boundary leader* approach using first-order hyperbolic PDEs. Building on earlier work of [Qi et al. \(2014\)](#), [Zhang et al. \(2024\)](#) extended the framework to radially varying domains, thereby expanding the range of attainable formations. An approach known as *continuification* reformulates large systems of ODEs or SDEs into a smaller set of PDEs that describe emergent system-level behaviour. This method has been successfully applied to swarm robotics [Maffettone et al. \(2024\)](#) and the synchronisation of spin-torque oscillators [Nikitin et al. \(2023\)](#). A recent decentralised implementation of this strategy was proposed in [Di Lorenzo et al. \(2025\)](#), which combines *kernel density estimation* with a *PI-consensus* algorithm to estimate densities and implement control in a fully distributed manner.

2.4 Problem Setup and Control Objective

We address the problem of *swarm deployment*, in which a large number of agents must be *deployed* to distinct, pre-assigned *positions along a smooth target curve*. The central challenge arises from the fact that agents possess only limited local sensing and communication capabilities, while also evolving under unstable, nonlinear open-loop dynamics. Consequently, achieving the desired deployment requires a careful balance between stabilising each agent’s individual trajectory and preserving sufficient inter-agent connectivity to propagate the deployment objective across the swarm. The control law must therefore resolve the inherent tension between local reactive behaviour and the overall deployment goal, ensuring convergence of the swarm without compromising its structural cohesion.

Mathematical Formulation

We model the multi-agent system (MAS) as a collection of $N + 1$ agents operating in n -dimensional Euclidean space $\mathbb{R}^n$ (with $n = 3$ in the simulations of Chapter 4). Each agent evolves according to:

$$\dot{z}_i(t) = f(t, z_i(t)) + U_i(t), \quad i \in \mathcal{I} := \{0, 1, \dots, N\}, \quad (2.2)$$

where $z_i(t)$ is the state of agent i and $f : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ represents the unstable open-loop dynamics. Let $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ denote the smooth target curve. We define the target coordinate for

each agent i as:

$$\gamma_i := \gamma(i/N), \quad \forall i \in \mathcal{I}. \quad (2.3)$$

The objective is to construct distributed control laws $U_i(t) : [0, \infty) \rightarrow \mathbb{R}^n$ such that the swarm achieves deployment onto the curve, i.e.,

$$\lim_{t \rightarrow \infty} z_i(t) = \gamma_i. \quad (2.4)$$

Performance Requirements

The proposed control law $U_i(t)$ must satisfy the following three core requirements:

1. **Rapid Convergence:** Agents must achieve their specific target coordinates significantly faster than the approaches presented in [Wei et al. \(2019a\)](#); [Terushkin and Fridman \(2021\)](#).
2. **Swarm Cohesion:** The agents must maintain structural integrity and local connectivity throughout the deployment process.
3. **Robustness:** The system must remain resilient with respect to a wide range of initial conditions and variations in swarm size (N).

2.4.1 Overview of the Approach

To construct the distributed control laws $U_i(t)$ in (2.2), we first decompose it into two parts as:

$$U_i(t) = \underbrace{u_i(t)}_{\text{local controller}} + \underbrace{v_i(t)}_{\text{global controller}}.$$

This decomposition allows us to design local interaction rules $u_i(t)$ that preserve cohesion and enable continuum approximations, as well as global strategies $v_i(t)$ that guide the entire swarm towards the target curve. We next describe them.

- **Local Coordination Strategy (Section 2.5):** The local controller $u_i(t)$ (Section 2.5) is designed within a leader-follower framework (Section 2.2) for the MAS (2.2) with a decentralized, distance-based topology. Using only inter-agent distances, $u_i(t)$ maintains cohesion while keeping agents sufficiently close to enable a PDE approximation independent of agent

count. However, $u_i(t)$ alone does not guarantee convergence to the target curve, as explained in Section 2.8; hence we introduce a global control $v_i(t)$.

- **Global Coordination Strategy (Section 2.8):** The global controller $v_i(t)$ (Section 2.8), advances the leader-based coordination strategy developed in Wei et al. (2019a); Terushkin and Fridman (2021), enabling agents to estimate and correct their positional error relative to the target curve. We deploy sparsely distributed leaders that communicate with neighbouring leaders, yielding a piecewise *linear* approximation of the error PDE state. Agents between two leaders interpolate their error from both (rather than using the nearest alone as in Wei et al. (2019a); Terushkin and Fridman (2021)), which improves estimation accuracy and speeds convergence.

2.5 Local Coordination Strategy and System Assumptions

This section presents the design of the local control input $u_i(t)$ for the interior agents in the MAS. Control strategies for the boundary agents are discussed separately in Section 2.7. The objective of the local controller is to enforce spatial coordination within the swarm, ensuring that agents evolve cohesively while remaining consistent with the configuration induced by the target curve.

To isolate the role of local interactions, we first consider the MAS dynamics without drift or global input:

$$\dot{z}_i(t) = u_i(t), \quad i \in \mathcal{I} := \{0, 1, \dots, N\}. \quad (2.5)$$

A classical approach in swarm coordination is to drive each agent toward the average of its immediate neighbours, thereby promoting cohesion through diffusive coupling. This behaviour can be enforced via the local control law

$$u_i(t) = \frac{2a}{h^2} \left(\frac{z_{i+1}(t) + z_{i-1}(t)}{2} - z_i(t) \right), \quad (2.6)$$

where $h = 1/N$ denotes the nominal inter-agent spacing and $a > 0$ is a scalar gain. Under this law, each agent is continuously pulled toward the midpoint of its neighbours, ensuring that deviations are dissipated over time and the swarm remains cohesive.

However, such averaging dynamics do not encode any information about the desired target configuration. In particular, they drive the system toward consensus rather than toward a prescribed spatial arrangement. To address this limitation, we modify the interaction law so that it not only preserves connectivity but also embeds the target coordinates into the local interactions.

The key idea is based on a *relative spacing principle*: each interior agent i seeks to match its relative positional differences with its neighbours $i \pm 1$ to the corresponding desired differences defined by the target configuration. That is, the agent regulates the quantities

$$(z_{i\pm 1}(t) - z_i(t)) \quad \text{toward} \quad (\gamma_{i\pm 1} - \gamma_i),$$

thereby driving the mismatch between the current and desired inter-agent displacements to zero. Accordingly, for each interior agent $i \in \{1, \dots, N-1\}$, we propose the following distributed local control law:

$$u_i(t) = \frac{a}{h^2} \times \underbrace{\left([z_{i+1}(t) - z_i(t)] - [\gamma_{i+1} - \gamma_i] + [z_{i-1}(t) - z_i(t)] - [\gamma_{i-1} - \gamma_i] \right)}_{\text{Agent } i \text{ aligns its relative position to neighbours with the desired relative spacing.}} \quad (2.7)$$

where $z_i(t)$ denotes the current position of agent i , γ_i is its assigned point on the target curve, $a > 0$ is a local control gain, and the parameter $h = \frac{1}{N}$ is a design parameter representing a theoretical reference spacing between adjacent agents.

Remark 2.5. Had we adopted the control law (2.7) without the h^2 term in the denominator, then as N becomes large (i.e., $h \rightarrow 0$), the agents become densely packed. Consequently, the relative differences $(z_{i\pm 1} - z_i)$ and $(\gamma_{i\pm 1} - \gamma_i)$ between neighbouring agents $i-1$, i , and $i+1$ become very small, resulting in a control input at the i -th agent that is nearly zero, even when its neighbouring agents have updated their positions. In this case, the swarm behaves like a highly viscous medium, responding very slowly to local interactions. By adopting the control law (2.7), the sensitivity of the i -th agent to the minute positional changes of its neighbouring agents $i+1$ and $i-1$ is amplified. This ensures that, even as the number of agents increases and the swarm becomes denser, the control law produces a meaningful response.

The following lemma formalises a key consequence of this design: if a local neighborhood reaches its target configuration, the intermediate agent is naturally driven to its specific assignment.

Lemma 2.6. *Consider an interior agent i with the control law (2.7). If its neighbours satisfy $z_{i-1}(t) = \gamma_{i-1}$ and $z_{i+1}(t) = \gamma_{i+1}$, then agent i converges exponentially to its target position γ_i with rate $\frac{2a}{h^2}$, i.e., $|z_i(t) - \gamma_i| \leq |z_i(0) - \gamma_i|e^{-\frac{2a}{h^2}t}$ for all $t \geq 0$.*

Proof. Define the tracking error $e_i(t) := z_i(t) - \gamma_i$. Substituting $z_{i\pm 1}(t) = \gamma_{i\pm 1}$ into (2.7), the equation (2.5), then changes to:

$$\dot{z}_i(t) = \frac{a}{h^2} \left([\gamma_{i+1} - z_i(t)] - [\gamma_{i+1} - \gamma_i] + [\gamma_{i-1} - z_i(t)] - [\gamma_{i-1} - \gamma_i] \right) = -\frac{2a}{h^2} (z_i(t) - \gamma_i).$$

This implies $\dot{e}_i(t) = -\frac{2a}{h^2} e_i(t)$, from which the exponential convergence follows. $\square$

Finally, to compensate for the nonlinear dynamics $f(t, z_i)$ in the original system (2.2), we incorporate a feedforward term evaluated along the desired trajectory. The complete local controller is given by:

$$u_i(t) = \underbrace{\frac{a}{h^2} [(z_{i+1}(t) - z_i(t)) + (z_{i-1}(t) - z_i(t))]}_{\text{alignment term}} - \underbrace{\frac{a}{h^2} [(\gamma_{i+1} - \gamma_i) + (\gamma_{i-1} - \gamma_i)]}_{\text{spacing offset}} - \underbrace{f(t, \gamma_i)}_{\text{nonlinear compensation}}. \quad (2.8)$$

Assumptions

For the local controller (2.8) to be implemented in a distributed manner, we assume the following local information is available to each interior agent $i \in \{1, \dots, N-1\}$

- A1.** Each interior agent $i \in \{1, \dots, N-1\}$ can measure its relative position with its immediate neighbors, i.e., it has access to $z_i(t) - z_{i-1}(t)$ and $z_i(t) - z_{i+1}(t)$. The boundary agents $i = 0$ and $i = N$ can sense only their respective interior neighbours.
- A2.** Each agent i knows the desired relative positions, i.e., $\gamma_i - \gamma_{i-1}$ and $\gamma_i - \gamma_{i+1}$.
- A3.** Each agent has access to its local dynamics evaluated along the target curve, $f(t, \gamma_i)$.
- A4.** The nonlinear function f satisfies Lipschitz condition in z with Lipschitz constant $L > 0$:

$$\|f(t, z_1) - f(t, z_2)\| \leq L\|z_1 - z_2\|, \quad \forall z_1, z_2 \in \mathbb{R}^n. \quad (2.9)$$

Assumption **A1** supports the alignment term, Assumption **A2** the formation offset term, and Assumption **A3** the nonlinear compensation term, all in the local controller (2.8). Assumption **A4** ensures that the nonlinear term does not grow too rapidly with respect to the state, which is essential for establishing well-posedness and stability of the closed-loop PDE system.

2.6 Semilinear PDE for Swarm Deployment

We now derive a continuum model for the multi-agent system (MAS) using the local control law (2.8). Substituting the expression for $u_i(t)$ into the agent dynamics (2.2), we obtain the following closed-loop system for each interior agent $i \in \mathcal{I} \setminus \{0, N\}$:

$$\dot{z}_i(t) = \frac{a}{h^2} \left[(z_{i+1} - 2z_i + z_{i-1}) - (\gamma_{i+1} - 2\gamma_i + \gamma_{i-1}) \right] + f(t, z_i) - f(t, \gamma_i) + v_i(t). \quad (2.10)$$

To study the large-scale behaviour of the swarm, we introduce continuous spatial and temporal variables by defining distributed state and control fields $z(t, x), v(t, x): [0, \infty) \times [0, 1] \rightarrow \mathbb{R}^n$ such that

$$z(t, ih) := z_i(t), \quad v(t, ih) := v_i(t), \quad \text{where } h = 1/N.$$

Using second-order central difference approximations, the spatial finite difference term becomes

$$\frac{z_{i+1}(t) - 2z_i(t) + z_{i-1}(t)}{h^2} \approx z_{xx}(t, x),$$

and similarly, the discrete second difference of γ_i approximates $\gamma_{xx}(x)$. As the number of agents $N \rightarrow \infty$ (i.e., $h \rightarrow 0$), equation (2.10) converges to the following semilinear parabolic PDE (cf. Ferrari-Trecate et al. (2006)):

$$z_t(t, x) = a [z_{xx}(t, x) - \gamma_{xx}(x)] + f(t, z(t, x)) - f(t, \gamma(x)) + v(t, x), \quad x \in [0, 1], \quad t \geq 0. \quad (2.11)$$

Defining the deployment error as $e(t, x) := z(t, x) - \gamma(x)$, the error dynamics are governed by the following PDE:

$$e_t(t, x) = a e_{xx}(t, x) + F(t, x, e) + v(t, x), \quad (2.12)$$

where F collects the nonlinear terms arising from the substitution $z = e + \gamma$ into the original dynamics, and is defined as

$$F(t, x, e) := f(t, \gamma(x) + e) - f(t, \gamma(x)). \quad (2.13)$$

Assumption **A3** (see Section 2.5) guarantees that f is Lipschitz continuous in its second argument. Therefore, the nonlinearity $F(t, x, e)$ satisfies the bound

$$|F(t, x, e)| \leq L|e|, \quad \forall x \in [0, 1], t \geq 0, e \in \mathbb{R}^n, \quad (2.14)$$

for some constant $L > 0$. This PDE will serve as the basis for the control design and stability analysis presented in the subsequent sections.

2.7 Modelling the Behaviour of the Boundary Agents

The behaviour of boundary agents strongly influences the overall deployment of the multi-agent system, in particular the boundary conditions of the PDE model derived in Section 2.6. This section classifies boundary-agent configurations and explains how each configuration maps to a specific boundary condition in the continuum limit.

We extend the assumptions in Section 2.5 to include the sensing and feedback capabilities of leaders:

A5. Certain agents, designated as *leaders*, have access to absolute position information. Specifically, a leader at index $i \in \mathcal{I}$ can measure its absolute deviation from the target curve, $z_i(t) - \gamma_i$.

We classify boundary-agent behaviour along three attributes: (i) dynamic vs. static, (ii) leader vs. follower, and (iii) cohesion-preserving vs. not. The diagram below summarises this taxonomy.

2.7.1 Dirichlet Condition

If boundary agents are static but not placed at their assigned positions on the target curve, their error remains nonzero for all time and the deployment objective cannot be achieved. The only

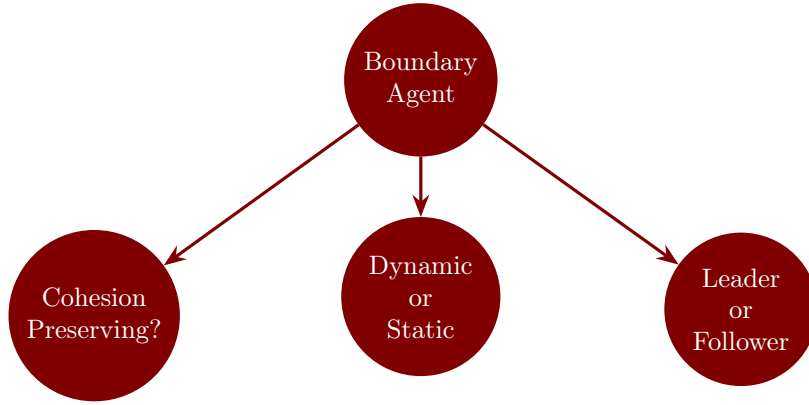


Figure 2.2: Boundary-agent attributes: dynamic/static, leader/follower, cohesion-preserving/not.

meaningful static configuration is thus to fix the boundary agents at their desired target locations.

In that case, they act as ideal leaders and enforce

$$e(t, 0) = 0, \quad e(t, 1) = 0, \quad (2.15)$$

which corresponds to homogeneous Dirichlet boundary conditions for the PDE model (2.12). *Note that swarm cohesion is lost near the boundaries, because static agents do not adapt to interior swarm dynamics.*

2.7.2 Neumann Condition

We now treat the boundary agents exactly like all other agents in the swarm; that is, they rely on relative measurements and move with the swarm. The left boundary agent can measure $z_0(t) - z_1(t)$ and $\gamma_0 - \gamma_1$, and the right boundary agent can measure $z_N(t) - z_{N-1}(t)$ and $\gamma_N - \gamma_{N-1}$. They enforce that their relative distance to the immediate interior neighbor equals the desired relative position on the target curve. With inter-agent spacing h , this yields

$$\frac{-(z_0(t) - z_1(t))}{h} + \frac{\gamma_0 - \gamma_1}{h} = 0, \quad \frac{z_N(t) - z_{N-1}(t)}{h} - \frac{\gamma_N - \gamma_{N-1}}{h} = 0.$$

Passing to the continuum limit $h \rightarrow 0$ gives

$$e_x(t, 0) = 0, \quad e_x(t, 1) = 0, \quad (2.16)$$

which are homogeneous *Neumann boundary conditions* for the PDE model (2.12).

2.7.3 Wentzell Boundary Conditions

We now consider a realistic setting in which the boundary agents act as *dynamic leaders*. In addition to tracking their assigned target positions and compensating for the local drift $f(t, z_i)$ in (2.2), they remain cohesively coupled to the swarm through feedback from their immediate neighbors. This behavior is implemented by the boundary control laws

$$\begin{aligned} u_0(t) &= \frac{\sigma}{h} \left[(z_1(t) - z_0(t)) - (\gamma_1 - \gamma_0) \right] - \kappa (z_0(t) - \gamma_0) - f(t, \gamma_0) \\ &\approx \sigma e_x(t, 0) - \kappa e(t, 0) - f(t, \gamma_0), \\ u_N(t) &= -\frac{\sigma}{h} \left[(z_N(t) - z_{N-1}(t)) - (\gamma_N - \gamma_{N-1}) \right] - \kappa (z_N(t) - \gamma_N) - f(t, \gamma_N) \\ &\approx -\sigma e_x(t, 1) - \kappa e(t, 1) - f(t, \gamma_N), \end{aligned} \tag{2.17}$$

where $\kappa > 0$ is the boundary tracking gain, $\sigma > 0$ is the boundary coupling gain, and $h = 1/N$ denotes the inter-agent spacing. Substituting (2.17) into the boundary-agent dynamics (2.2) and expressing the result in error coordinates yields the Wentzell (dynamic) boundary conditions for the error PDE (2.12):

$$\begin{aligned} e_t(t, 0) + \kappa e(t, 0) - \sigma e_x(t, 0) - F(t, 0, e(t, 0)) - v(t, 0) &= 0, \\ e_t(t, 1) + \kappa e(t, 1) + \sigma e_x(t, 1) - F(t, 1, e(t, 1)) - v(t, 1) &= 0, \end{aligned} \tag{2.18}$$

where F is defined in (2.13), and $v(t, x)$ is the global coordination strategy to be introduced in Section 2.8.

Remark. The coupling term σe_x appears with opposite signs at the two boundaries in (2.18) due to the orientation of the outward unit normal. At $x = 0$, the outward normal points in the negative x -direction, while at $x = 1$ it points in the positive x -direction. As a result, the diffusive coupling across the boundary carries opposite signs, consistent with the physical geometry of the domain.

2.7.4 Dirichlet-Type Dynamic Boundary Conditions

We next consider dynamic boundary agents that act as leaders but are *not* cohesion-preserving: they measure their absolute errors $(z_0(t) - \gamma_0)$ and $(z_N(t) - \gamma_N)$, but they do not use relative measurements to neighbors $(z_0(t) - z_1(t))$ and $(z_N(t) - z_{N-1}(t))$. Setting $\sigma = 0$ in (2.18) removes the neighbour coupling and yields the following dynamic (Dirichlet-type) boundary conditions:

$$\begin{aligned} e_t(t, 0) &= -\kappa e(t, 0) + F(t, 0, e(t, 0)) + v(t, 0), \\ e_t(t, 1) &= -\kappa e(t, 1) + F(t, 1, e(t, 1)) + v(t, 1). \end{aligned} \tag{2.19}$$

In the limit of fast boundary dynamics (large κ), these conditions drive the boundary error toward zero, recovering the classical Dirichlet behaviour at steady state.

	Wentzell	Dirichlet (dynamic)	Neumann	Dirichlet
Dynamic	✓	✓	✓	✗
Leader	✓	✓	✗	✓
Cohesion-preserving	✓	✗	✓	✗

Table 2.2: Boundary-agent attributes across boundary-condition types.

2.8 Motivation and Global Coordination Design

This section motivates the design of global control strategies for the error PDE system (2.12). We begin by examining how the system behaves under the various boundary conditions discussed in the previous section.

Under Neumann boundary conditions (2.7.2), there are no leaders to guide the swarm, so deployment cannot be achieved. In contrast, the Dirichlet, Dynamic Dirichlet, and Wentzell cases allow boundary agents to act as leaders by measuring their absolute error, enabling deployment under certain conditions. To illustrate, consider the Dirichlet case where boundary agents are fixed at their target positions. Due to the chain topology and the consensus-based control law (2.8), the adjacent agent (e.g., $i = 1$ or $i = N - 1$) adjusts its position to align with both the fixed boundary and its neighbour. Since the boundary agent does not move, agent $i = 1$ covers a larger distance to maintain the alignment. This displacement propagates inward, with each agent responding to the previous one, driving the entire swarm toward the target curve. This process succeeds only when

the interaction strength a in (2.8) dominates the local nonlinear dynamics. Specifically, global convergence is ensured if $a > L$, where L is the Lipschitz constant from Assumption A4.

Remark. With Wentzell boundaries, the boundary agents move with the swarm. The first interior agents ($i = 1$ and $i = N - 1$) make only small moves, so alignment spreads slowly and deployment takes longer. Under Dynamic Dirichlet, the boundary agents move toward their targets aggressively; agents $i = 1$ and $i = N - 1$ travel more distance than in Wentzell, so convergence is faster, but still slower than static Dirichlet. With static Dirichlet, the boundaries are fixed; agents $i = 1$ and $i = N - 1$ travel the most, and alignment spreads the fastest. Figure 2.3 illustrates this observation: the left one shows that for strong interaction strength ($a = 1 > 0.3 = L$), Dirichlet control converges fastest; the right one shows that for low interaction strength ($a = 0.01 < 0.3 = L$) the system is unstable.

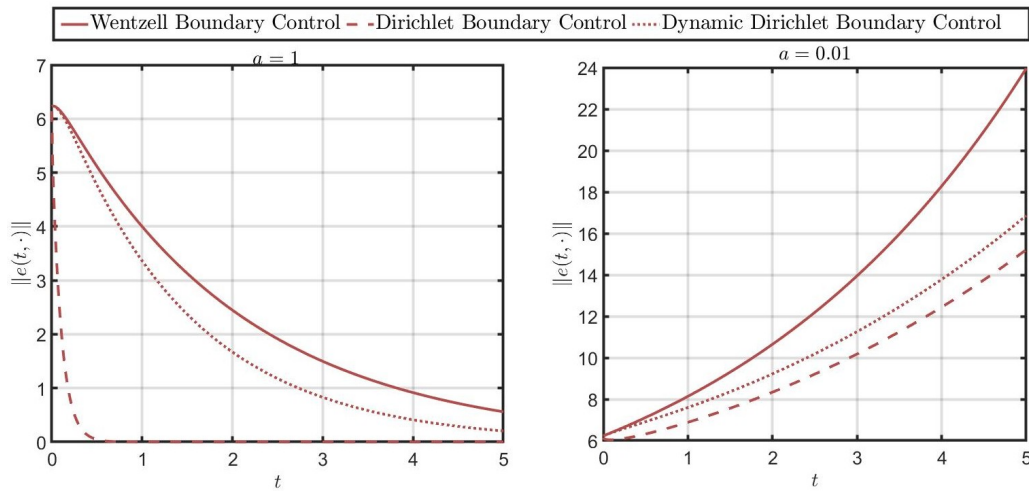


Figure 2.3: Convergence of $\|e(t, \cdot)\|^2$ of the PDE (2.12) with $v(t, x) = 0$ under Wentzell, Dirichlet, and dynamic Dirichlet boundary conditions, for strong (left) and weak (right) interaction strength a .

In practice, large interaction strength is rarely feasible, and for small a , the open-loop error PDE (2.12) becomes unstable. To address this, Wei et al. (2019a); Terushkin and Fridman (2021) designed a global coordination strategy for MAS by adopting the spatial decomposition framework from Fridman and Blighovsky (2012), where leader agents construct a piecewise constant approximation of the error PDE. Followers then adopt the *nearest leader's* error as their own and adjust their positions accordingly. This strategy assumes no communication between leaders. In this work, we extend it by enabling leader-to-leader communication, which improves performance in two ways:

First, when boundary agents reach the target early, their zero error is adopted by nearby agents as their own error estimate. These agents may then falsely believe they are correctly positioned and stop adjusting. While deployment will still occur eventually (as neighbouring agents continue correcting their errors and force these agents to realign), the convergence time increases. Enabling communication between leaders mitigates this delay.

Second, enabling communication between leaders allows the construction of a piecewise linear approximation of the PDE state, rather than a piecewise constant one. This yields more accurate error estimates and improves convergence speed.

We now present our global coordination design. Specifically, we construct a piecewise linear approximation of $e(t, x)$ using the leaders' measurements. We place $M + 1$ ($\leq N + 1$) leaders at $x_j \in \{ih\}_{i \in \mathcal{I}} \subset [0, 1]$, $j = 0, \dots, M$, so that $x_0 = 0$ and $x_M = 1$. That is, $x_j = \sum_{k=1}^j \Delta_k$ for $j \geq 1$, where Δ_k is a multiple of h and is the spacing between two consecutive leaders. The maximum space between the leaders is $\Delta := \max_k \Delta_k$. The leaders provide the measurements

$$y_j(t) := e(t, x_j) = z(t, x_j) - \gamma(x_j), \quad j \in \{0, 1, \dots, M\}.$$

By communicating with each other, the leaders can construct the piecewise linear state approximation: For $x \in [x_j, x_{j+1}]$,

$$\begin{aligned} e(t, x) &\approx y_j(t) + \frac{y_{j+1}(t) - y_j(t)}{\Delta_{j+1}}(x - x_j) \\ &= y_j(t) \left(1 - \frac{x - x_j}{\Delta_{j+1}}\right) + y_{j+1}(t) \frac{x - x_j}{\Delta_{j+1}}. \end{aligned}$$

Introducing

$$b_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}}, & x \in [x_{j-1}, x_j], \\ \frac{x_{j+1} - x}{x_{j+1} - x_j}, & x \in [x_j, x_{j+1}], \\ 0, & x \notin [x_{j-1}, x_{j+1}] \end{cases} \quad (2.20)$$

with $x_{-1} = 0$ and $x_{M+1} = 1$, we obtain

$$e(t, x) \approx \sum_{j=0}^M y_j(t) b_j(x). \quad (2.21)$$

This motivates the following global controller:

$$v(t, x) = -K \sum_{j=0}^M b_j(x) y_j(t) = -K e(t, x) + K \nu(t, x) \quad (2.22)$$

with a leader state propagation gain $K > 0$ and approximation error

$$\nu(t, x) = e(t, x) - \sum_{j=0}^M y_j(t) b_j(x).$$

Substituting (2.22) into (2.12) and (2.18), we obtained the closed-loop system

$$e_t(t, x) = a e_{xx}(t, x) + F(t, x, e) - K e(t, x) + K \nu(t, x), \quad (2.23)$$

accompanied by the boundary conditions

$$\begin{aligned} e_t(t, 0) &= -(\kappa + K) e(t, 0) + \sigma e_x(t, 0) + F(t, 0, e(t, 0)), \\ e_t(t, 1) &= -(\kappa + K) e(t, 1) - \sigma e_x(t, 1) + F(t, 1, e(t, 1)). \end{aligned} \quad (2.24)$$

To analyse the effect of the approximation error $\nu(t, x)$ on the closed-loop system, we require bounds that relate the L^2 norm of a function to its spatial derivatives. Such estimates are essential for quantifying how the approximation error depends on the inter-leader spacing Δ . Classical inequalities, such as Poincaré and Wirtinger, provide precisely this connection under appropriate boundary or integral constraints. These tools enable us to convert spatial regularity of the PDE error state into explicit bounds on the approximation error, which will be used to derive the LMI-based stability conditions in Chapter 3, and solving these LMIs in Chapter 4 provides a sufficient condition on the number of leaders needed for successful deployment.

Lemma 2.7 (Poincaré's inequality, Payne and Weinberger (1960)). *If $g \in H^1(0, l)$ is such that $\int_0^l g(x) dx = 0$, then $\|g\| \leq \frac{l}{\pi} \|g'\|$.*

Lemma 2.8 (Wirtinger's inequality, Hardy et al. (1952)). *If $g \in H_0^1(0, l)$, then $\|g\| \leq \frac{l}{\pi} \|g'\|$.*

Lemmas 2.7 and 2.8 have been used in Fridman and Am (2012); Fridman and Blighovsky (2012) to bound the error of piecewise constant state approximations constructed using averaged and point measurements. In contrast, we employ the following lemma to derive bounds for a piecewise linear approximation.

Lemma 2.9. *If $g \in H^2(0, l)$ with $g(0) = g(l) = 0$, then $\|g\| \leq \frac{l}{\pi} \|g'\| \leq \frac{l^2}{\pi^2} \|g''\|$.*

Proof. Since $g(0) = 0 = g(l)$, Wirtinger's inequality implies $\|g\| \leq \frac{l}{\pi} \|g'\|$. Since $\int_0^l g'(x) dx = g(l) - g(0) = 0$, Poincaré's inequality gives

$$\|g\| \leq \frac{l}{\pi} \|g'\| \leq \frac{l^2}{\pi^2} \|g''\|.$$

□

We now observe, for $x \in [x_{j-1}, x_j]$,

$$\nu(t, x) = e(t, x) - y_{j-1}(t)b_{j-1}(x) - y_j(t)b_j(x)$$

since $b_i(x) = 0$ when $i \notin \{j-1, j\}$. Then, recalling that $y_j(t) = e(t, x_j)$, we obtain

$$\begin{aligned} \nu_x(t, x) &= e_x(t, x) - e(t, x_{j-1})b'_{j-1}(x) - e(t, x_j)b'_j(x) = e_x(t, x) - \frac{e(t, x_j) - e(t, x_{j-1})}{\Delta_j} \\ &= e_x(t, x) - \frac{1}{\Delta_j} \int_{x_{j-1}}^{x_j} e_s(t, s) ds. \end{aligned}$$

Since $\nu(t, x_j) = 0$, Lemma 2.8 gives

$$\begin{aligned} \|\nu(t, \cdot)\|_{[x_{j-1}, x_j]}^2 &\leq \frac{\Delta_j^2}{\pi^2} \|\nu_x(t, \cdot)\|_{[x_{j-1}, x_j]}^2 = \frac{\Delta_j^2}{\pi^2} \|e_x(t, \cdot) - \frac{1}{\Delta_j} \int_{x_{j-1}}^{x_j} e_s(t, s) ds\|_{[x_{j-1}, x_j]}^2 \\ &= \frac{\Delta_j^2}{\pi^2} \|e_x(t, \cdot)\|_{[x_{j-1}, x_j]}^2 - \frac{2\Delta_j}{\pi^2} \int_{x_{j-1}}^{x_j} e_x(t, x) dx \int_{x_{j-1}}^{x_j} e_s(t, s) ds \\ &\quad + \frac{1}{\pi^2} \int_{x_{j-1}}^{x_j} [\int_{x_{j-1}}^{x_j} e_s(t, s) ds]^2 dx \\ &= \frac{\Delta_j^2}{\pi^2} \|e_x(t, \cdot)\|_{[x_{j-1}, x_j]}^2 - \frac{\Delta_j}{\pi^2} [\int_{x_{j-1}}^{x_j} e_x(t, x) dx]^2 \\ &\leq \frac{\Delta_j^2}{\pi^2} \|e_x(t, \cdot)\|_{[x_{j-1}, x_j]}^2 \leq \frac{\Delta_j^2}{\pi^2} \|e_x(t, \cdot)\|_{[x_{j-1}, x_j]}^2. \end{aligned}$$

Since $\|\nu(t, \cdot)\|_{[0, 1]}^2 = \sum_{j=1}^M \|\nu(t, \cdot)\|_{[x_{j-1}, x_j]}^2$, we have

$$\|\nu(t, \cdot)\|_{[0, 1]}^2 \leq \frac{\Delta_j^2}{\pi^2} \|e_x(t, \cdot)\|_{[0, 1]}^2. \quad (2.25)$$

Using Lemma 2.9 and the relation $b_j'' = 0$, we also have

$$\|\nu(t, \cdot)\|_{[x_{j-1}, x_j]}^2 \leq \frac{\Delta_j^4}{\pi^4} \|\nu_{xx}(t, \cdot)\|_{[x_{j-1}, x_j]}^2 = \frac{\Delta_j^4}{\pi^4} \|e_{xx}(t, \cdot)\|_{[x_{j-1}, x_j]}^2.$$

Therefore,

$$\|\nu(t, \cdot)\|_{[0,1]}^2 \leq \frac{\Delta^4}{\pi^4} \|e_{xx}(t, \cdot)\|_{[0,1]}^2. \quad (2.26)$$

Remark 2.10. If the leaders do not communicate, the piecewise linear approximation (2.21) can be replaced by a piecewise constant one where $b_j(x)$ are the characteristic functions of $[(x_j + x_{j-1})/2, (x_j + x_{j+1})/2]$. Then (2.25) still holds (with a simpler proof, see [Fridman and Blighovsky \(2012\)](#); [Selivanov and Fridman \(2016b\)](#)), but (2.26) does not. That is, communications between leaders allow for the bound (2.26), which improves the stability analysis as illustrated by the numerical simulations in Chapter 4.

This chapter has developed a PDE model of swarm deployment that captures the essential roles of local interaction, leader influence, and boundary behaviour. The use of Wentzell-type boundary conditions ensures that boundary leaders remain coupled to the swarm, while leader-to-leader communication improves the spread of information across the domain. In the next chapter, we build on this model to establish well-posedness using semigroup theory and perform a Lyapunov-based analysis to derive stability conditions in terms of linear matrix inequalities.

Stability Analysis

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In this chapter, we study the closed-loop dynamics of the PDE-based MAS introduced in Chapter 2. Section 3.1 recalls semigroup-theoretic background, Section 3.2 addresses well-posedness of the nonlinear closed-loop system, and Section 3.3 presents stability analysis via Lyapunov functionals that incorporate boundary terms.

3.1 Semigroup-Theoretic Background

Semigroup theory provides the natural framework for studying PDE dynamics in abstract Banach spaces. It allows us to reformulate the closed-loop PDE as an abstract Cauchy problem, where strongly continuous semigroups (C_0 -semigroups) describe the system's time evolution and their generators capture the infinitesimal dynamics. We begin by recalling the definitions of a C_0 -semigroup and its generator.

Definition 3.1 (Pazy (2012)). *Let X be a Banach space. A one-parameter family of operators $\{T(t)\}_{t \geq 0} \subset \mathcal{L}(X)$ is a strongly continuous (i.e. C_0) semigroup on X if the following hold:*

1. $T(0) = I$.
2. $T(t)T(s) = T(t + s)$ for all $s, t \geq 0$.

3. For all $f \in X$, $T(h)f \rightarrow f$ strongly in X as $h \rightarrow 0^+$.

Further, such a C_0 semigroup is said to be ω -contractive for some $\omega \in \mathbb{R}$ if, additionally,

$$\|T(t)\|_{\mathcal{L}(X)} \leq e^{\omega t} \quad \text{for all } t \geq 0.$$

Definition 3.2 (Pazy (2012)). Let $\{T(t)\}_{t \geq 0}$ be a C_0 -semigroup on a Banach space X . Define an operator A as

$$Au := \lim_{t \rightarrow 0^+} \frac{T(t)u - u}{t} = \left. \frac{d}{dt} T(t)u \right|_{t=0},$$

with domain $D(A) = \{u \in X : \text{above limit exists in } X\}$. The operator A is called the (infinitesimal) generator of the semigroup.

The following theorem characterises the relationship between a C_0 -semigroup and its generator, and explains why one may intuitively view $T(t)$ as e^{At} for some operator A .

Theorem 3.3 (Pazy (2012), Corollary 1.2.3 and Theorem 1.2.4). Suppose A is the generator of a C_0 -semigroup $\{T(t)\}_{t \geq 0}$ on a Banach space X . Then for all $u \in D(A)$, we have the following:

1. $T(t)u \in D(A)$ for all $t \geq 0$, i.e. $D(A)$ is an invariant set for the semigroup.
2. $AT(t)u = T(t)Au$ for each $t > 0$, i.e. generators and semigroups commute.
3. The mapping $t \mapsto T(t)u \in X$ is $C^1(0, \infty; X)$.
4. We have $\frac{d}{dt} T(t)u = AT(t)u$ for all $t > 0$.

We now turn our attention to the abstract initial value problem (IVP) posed on a Banach space X , given by

$$\begin{cases} \frac{du}{dt} = Au, & t > 0, \\ u(0) = g, \end{cases} \quad (3.1)$$

where A is a (possibly unbounded) linear operator on X and $g \in X$ is the initial condition. A function $u : [0, \infty) \rightarrow X$ is called a *classical solution* of (3.1) if it satisfies the regularity condition

$$u \in C([0, \infty); D(A)) \cap C^1((0, \infty); X),$$

with $u(0) = g$, $u(t) \in D(A)$ for all $t > 0$, and the differential equation is satisfied pointwise on $(0, \infty)$; see (Pazy, 2012, Chapter 4). Here, the intersection of these spaces implies that u is continuous on $[0, \infty)$, continuously differentiable on $(0, \infty)$, and $u(t)$ remains in the domain of A for all $t > 0$ with $Au(\cdot)$ being continuous.

The existence of such classical solutions can be guaranteed by the above technical results. Specifically, if the operator A is the generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0} \subset \mathcal{L}(X)$, and if the initial condition $g \in D(A)$, then the function

$$u(t) = T(t)g$$

constitutes a classical solution to the problem (3.1). This follows directly from Theorem 3.3, which asserts that $T(t)$ preserves the domain of the generator and satisfies the pointwise differential relation $\frac{d}{dt}T(t)g = AT(t)g$ for all $t > 0$.

Now, our goal is to answer the question: what are the necessary and sufficient conditions for a given linear operator A on a Banach space X to be the generator of a C_0 -semigroup on X ? Before addressing this question, we introduce some definitions that are essential for stating these conditions.

Definition 3.4. Let X and Y be Banach spaces, and let $A : D(A) \subset X \rightarrow Y$ be a linear operator with domain $D(A)$. Then A is said to be closed if, for every sequence $\{x_n\} \subset D(A)$ such that $x_n \rightarrow x$ in X and $Ax_n \rightarrow y$ in Y , we have $x \in D(A)$ and $Ax = y$.

Said more abstractly, an operator A is said to be closed if its graph

$$\text{gra}(A) := \{x \oplus Ax : x \in D(A)\}$$

is a closed subset of $X \oplus Y$.

Definition 3.5. Let $A : D(A) \subset X \rightarrow X$ be a closed linear operator on a Banach space X with dense domain $D(A)$. We define the resolvent set of A to be

$$\rho(A) := \{\lambda \in \mathbb{C} : (\lambda I - A) : D(A) \subset X \rightarrow X \text{ is a bijection}\}.$$

and $(\lambda I - A)^{-1}$ is bounded in X . Furthermore, for each $\lambda \in \rho(A)$, we define the resolvent operator $R_\lambda(A) := (\lambda I - A)^{-1} : X \rightarrow X$.

With the above, we are now able to give a necessary and sufficient condition for a given operator A to generate a C_0 -semigroup on X . This is provided by the famous Hille–Yosida Theorem.

Theorem 3.6 (Pazy (2012), Hille–Yosida Theorem). *Let A be a linear operator on a Banach space X . Then A is the generator of a C_0 -semigroup on X if and only if the following conditions hold:*

1. $D(A)$ is dense in X , and A is a closed operator.
2. There exists a constant $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$, and

$$\|R_\lambda(A)\| \leq \frac{1}{\lambda - \omega}, \quad \text{for all } \lambda > \omega.$$

In this case, the associated semigroup $\{T(t)\}_{t \geq 0}$ is ω -contractive, i.e. it satisfies $\|T(t)\| \leq e^{\omega t}$ for all $t \geq 0$.

3.2 Well-Posedness

Proving well-posedness for the closed-loop system (2.23)–(2.24) is non-trivial due to the Wentzell-type boundary conditions, which involve the second-order spatial derivative in the boundary operator itself. Unlike standard Dirichlet or Robin conditions, Wentzell conditions imply that the boundary dynamics are as regular as the interior dynamics, effectively coupling the boundary values to the domain through the governing PDE. Establishing well-posedness is a critical prerequisite for the stability analysis in Section 3.3; it ensures that the solutions we analyse via Lyapunov functionals actually exist, are unique, and possess the H^1 regularity required for the norm estimates in Theorem 3.10.

To this end, we consider the nonlinear parabolic PDE subject to generalized Wentzell boundary conditions of the form:

$$\frac{\partial e}{\partial t} = Ae + F(e), \quad \text{in } \Omega, \quad Ae + \sigma \frac{\partial e}{\partial n} + \kappa e = 0 \quad \text{on } \partial\Omega, \quad t \geq 0, \quad (3.2)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary. The differential operator A is defined as $Ae = \nabla \cdot (a \nabla e)$, where the diffusion coefficient $a \in C^1(\Omega)$ is strictly positive throughout the domain. The boundary functions σ and κ belong to $C^1(\partial\Omega)$ and are nonnegative,

with $\sigma(x) > 0$ on $\partial\Omega$. The vector $n(x)$ denotes the unit outward normal at each boundary point $x \in \partial\Omega$. Under these conditions, it was established in Favini et al. (2002) that the corresponding Cauchy problem for (3.2) is governed by a strongly continuous contraction semigroup on a suitable $L^p(\Omega)$ space for all $1 \leq p < \infty$, and on $C(\overline{\Omega})$. Furthermore, this semigroup is analytic when $1 < p < \infty$. By grounding our MAS model in this framework, we ensure that the stability conditions derived via LMIs describe a system that is both mathematically sound and physically realisable.

In the one-dimensional case, semigroup generation for problems involving Wentzell-type boundary conditions on the space $C[0, 1]$ has been analysed in both linear and nonlinear settings; see Favini et al. (2000). While classical Robin boundary conditions—of the form $\sigma \frac{\partial e}{\partial n} + \kappa e = 0$ on $\partial\Omega$ —are typically studied in standard $L^p(\Omega)$ spaces Agmon et al. (1959); Lunardi (2012); Daners (2000), the generalised Wentzell conditions require a more sophisticated functional framework. A key observation in Favini et al. (2002) is that the presence of the boundary flux term $\sigma \frac{\partial u}{\partial n}$ naturally leads to the use of weighted function spaces to maintain the operator's self-adjointness and dissipativity.

To account for this coupling between the domain and its boundary, we define the measure $d\mu := dx \oplus \frac{a}{\sigma} dS$, where dx is the Lebesgue measure on Ω , and dS is the surface measure on the boundary subset Σ . This induces the product space

$$L^p(\overline{\Omega}, d\mu) := L^p(\Omega, dx) \times L^p(\Sigma, \frac{a}{\sigma} dS), \quad 1 \leq p < \infty,$$

consisting of pairs $U = (u, v)$ that represent the state in the interior and on the boundary, respectively.

In our specific one-dimensional setting with $\Omega = (0, 1)$, this framework results in the Hilbert space

$$X = L^2(0, 1) \times \mathbb{R}^2, \tag{3.3}$$

equipped with the weighted inner product:

$$\langle v_1, v_2 \rangle_X = \langle f_1, f_2 \rangle_{L^2} + \frac{a}{\sigma} \alpha_1^\top \alpha_2, \tag{3.4}$$

where $v_i = (f_i, \alpha_i) \in X$. This space is essential to our main contribution, as it allows us to treat the distributed state $e(t, \cdot)$ and the discrete boundary values $e(t, 0), e(t, 1)$ jointly. Within

this Hilbert space, the boundary conditions (2.24) for $\sigma > 0$ can be rigorously expressed in the following Wentzell form:

$$\begin{aligned} ae_{xx}(t, 0) - \sigma e_x(t, 0) + \kappa e(t, 0) &= 0, \\ ae_{xx}(t, 1) + \sigma e_x(t, 1) + \kappa e(t, 1) &= 0. \end{aligned}$$

By reformulating the boundary dynamics this way, we transform the original PDE into an abstract evolution equation that is amenable to the stability analysis presented in the subsequent sections.

We now introduce the operator $\mathcal{G}$ on the Hilbert space X , which will be used to formulate the closed-loop system in an abstract evolution framework. A key aspect in defining the domain $D(\mathcal{G})$ is the regularity required to satisfy the boundary conditions in a pointwise sense. If we consider the standard Sobolev space $H^2(0, 1)$, the second-order derivative f'' belongs to $L^2(0, 1)$. Since functions in $L^2(0, 1)$ are defined only almost everywhere, the pointwise evaluations $f''(0)$ and $f''(1)$ are not well-defined. Consequently, the Wentzell boundary conditions cannot be imposed rigorously at this level of regularity. By the *Sobolev Embedding Theorem* Evans (2022), specifically the embedding $H^k(0, 1) \hookrightarrow C^{k-1}[0, 1]$ for $k > 1/2$, it follows that for $f \in H^3(0, 1)$, the second derivative f'' belongs to $H^1(0, 1)$. Since $H^1(0, 1) \subset C[0, 1]$, it follows that $f'' \in C[0, 1]$, which ensures that the boundary values $f''(0)$ and $f''(1)$ are well-defined in the classical sense.

Accordingly, the operator $\mathcal{G}$ is defined as:

$$\mathcal{G} \begin{bmatrix} f(\cdot) \\ f(0) \\ f(1) \end{bmatrix} = \begin{bmatrix} af''(\cdot) \\ \sigma f'(0) - \kappa f(0) \\ -\sigma f'(1) - \kappa f(1) \end{bmatrix}, \quad (3.5)$$

with the domain $D(\mathcal{G}) \subset X$ specified as:

$$D(\mathcal{G}) = \left\{ \begin{bmatrix} f \\ f(0) \\ f(1) \end{bmatrix} \in X \left| \begin{array}{l} f \in H^3(0, 1), \\ af''(0) - \sigma f'(0) + \kappa f(0) = 0, \\ af''(1) + \sigma f'(1) + \kappa f(1) = 0 \end{array} \right. \right\}. \quad (3.6)$$

The PDE system given by (2.23) and (2.24) can now be expressed in the following abstract form:

$$\frac{dE}{dt} = \mathcal{G}E + \mathcal{F}(t, E), \quad (3.7)$$

where

$$E(t) = \begin{bmatrix} e(t, \cdot) \\ e(t, 0) \\ e(t, 1) \end{bmatrix}, \quad \mathcal{F}(t, E) = \begin{bmatrix} F(t, \cdot, e(t, \cdot)) - K \sum_{j=0}^M b_j(x) e(t, x_j) \\ F(t, 0, e(t, 0)) - K e(t, 0) \\ F(t, 1, e(t, 1)) - K e(t, 1) \end{bmatrix}. \quad (3.8)$$

in (3.3), with an appropriate inner product specified in (3.4).

Now that the abstract problem has been formulated in the appropriate space, we shall establish its well-posedness by applying results from Pazy (2012) (see Theorem 3.7 below). Before presenting this theorem, we recall the definition of a fractional Banach space associated with the operator $-A$. In particular, we define $X_{\frac{1}{2}} := D((-A)^{\frac{1}{2}})$, that is, the domain of the square root of the closed linear positive operator $-A$. It is equipped with the graph norm: $\|x\|_{X_{\frac{1}{2}}} := \|(-A)^{\frac{1}{2}}x\|_X$, for $x \in X_{\frac{1}{2}}$.

Theorem 3.7 ((Pazy, 2012, Theorem 6.3.1)). *Let $\mathcal{G}$ be the infinitesimal generator of an analytic semigroup $T(t)$ satisfying $\|T(t)\| \leq M$. Assume further that 0 belongs to the resolvent set of $\mathcal{G}$. Let $\mathcal{F} : \mathcal{U} \rightarrow X$ be as in (3.8), where $\mathcal{U}$ is an open subset of $\mathbb{R} \times X_{\frac{1}{2}}$ with $X_{\frac{1}{2}} = D((- \mathcal{G})^{\frac{1}{2}})$. Assume that for every $(t, E) \in \mathcal{U}$ there exists a neighborhood $\mathcal{V} \subset \mathcal{U}$ and constant $L \geq 0$, such that*

$$\|\mathcal{F}(t_1, E_1) - \mathcal{F}(t_2, E_2)\|_X \leq L(|t_1 - t_2| + \|x_1 - x_2\|_{X_{\frac{1}{2}}}) \quad (3.9)$$

for all $(t_i, E_i) \in \mathcal{V}$. Then, for every initial data $(t_0, E_0) \in \mathcal{U}$, the initial value problem

$$\frac{dE}{dt} = \mathcal{G}E + \mathcal{F}(t, E),$$

admits a unique local solution $E \in C([t_0, t_1]; X) \cap C^1((t_0, t_1); X)$, where $t_1 = t_1(t_0, x_0) > t_0$.

We now verify the conditions of the above theorem and recall the main generation result from Favini et al. (2002), which shows that an elliptic operator with Wentzell boundary conditions generates a C_0 -semigroup.

Theorem 3.8 (Favini et al. (2002), Theorem 3.1). *Let $a, \sigma > 0$ and $\kappa \geq 0$, then the operator $\mathcal{G}$ in (3.5) with domain $D(\mathcal{G})$ defined in (3.6) generates a C_0 -contraction semigroup on $X = L^2(0, 1) \times \mathbb{R}^{2n}$. Moreover, this semigroup is analytic.*

From the proof of Proposition 4.3 in [Marta \(2024\)](#), the spectrum of $\mathcal{G}$ is negative, and therefore $[0, \infty)$ is contained in the resolvent set of $\mathcal{G}$. Moreover, by Theorem 3.6, the associated semigroup $\{T(t)\}_{t \geq 0}$ satisfies $\|T(t)\| \leq 1$ for all $t \geq 0$. We now show that the operator $-\mathcal{G}$ is positive, so that the space $X_{\frac{1}{2}} = D\left((-\mathcal{G})^{\frac{1}{2}}\right)$ is well defined.

Lemma 3.9. *Let $\mathcal{G}$ be the operator defined in equation (3.7). Then the operator $-\mathcal{G} : D(\mathcal{G}) \rightarrow X$ is positive definite with respect to the inner product defined in (3.4).*

Proof. Let $E(t) = \begin{bmatrix} e(t, \cdot) \\ e(t, 0) \\ e(t, 1) \end{bmatrix} \in D(\mathcal{G})$. Then

$$\langle -\mathcal{G}E(t), E(t) \rangle_X = \left\langle \begin{bmatrix} -ae''(\cdot) \\ -\sigma e_x(t, 0) + \kappa e(t, 0) \\ \sigma e_x(t, 1) + \kappa e(t, 1) \end{bmatrix}, \begin{bmatrix} e(t, \cdot) \\ e(t, 0) \\ e(t, 1) \end{bmatrix} \right\rangle_X.$$

Using the inner product in X , we compute

$$\begin{aligned} \langle -\mathcal{G}E(t), E(t) \rangle_X &= \left\langle \frac{a}{\sigma} [(-\sigma e_x(t, 0) + \kappa e(t, 0)) e(t, 0) + (\sigma e_x(t, 1) + \kappa e(t, 1)) e(t, 1)] \right. \\ &\quad \left. + -ae_{xx}(t, \cdot), e(t, \cdot) \right\rangle_{L^2} \\ &= -a \int_0^1 e_{xx}(t, x) e(t, x) dx + ae_x(t, x) e(t, x) \Big|_{x=0}^1 + \frac{a}{\sigma} \kappa [e^2(t, 0) + e^2(t, 1)] \\ &= a \|e_x(t, \cdot)\|_{L^2}^2 + \frac{a}{\sigma} \kappa [e^2(t, 0) + e^2(t, 1)] > 0. \end{aligned}$$

□

Finally, we show that $\mathcal{F} : X_{1/2} \rightarrow X$ is Lipschitz continuous. Since f is Lipschitz (see Assumption A4), the last two components of $\mathcal{F}$ are also Lipschitz, and

$$\|F(t, \cdot, e_1(t, \cdot)) - F(t, \cdot, e_2(t, \cdot))\| \leq L \|e_1(t, \cdot) - e_2(t, \cdot)\|.$$

By the Sobolev embedding theorem (see, e.g., Lemma 3 of [Kang and Fridman \(2018\)](#)), there exists C such that $\max_{x \in [0, 1]} |e_1(t, x) - e_2(t, x)| \leq C \|e_1(t, \cdot) - e_2(t, \cdot)\|_{H^1}$. Since $K > 0$ and $\sum_{j=0}^M b_j(x) = 1$,

this yields

$$\begin{aligned} \left\| K \sum_{j=0}^M b_j(\cdot) [e_1(t, x_j) - e_2(t, x_j)] \right\| &\leq K \max_{x \in [0,1]} |e_1(t, x) - e_2(t, x)| \left\| \sum_{j=0}^M b_j(\cdot) \right\| \\ &\leq KC \|e_1(t, \cdot) - e_2(t, \cdot)\|_{H^1} \leq M \|E_1(t) - E_2(t)\|_{1/2} \end{aligned}$$

for some $M > 0$. Therefore, $\mathcal{F}$ is Lipschitz continuous. Then, Theorem 3.7 guarantees that (3.7) has a unique solution

$$e \in C([0, T], L^2(0, 1)) \cap C^1((0, T), L^2(0, 1))$$

for $e(0, \cdot) \in H^1(0, 1)$. Since $\mathcal{F}(t, 0) = 0$, (Pazy, 2012, Theorem 6.3.1) guarantees that $T = \infty$.

3.3 LMI-Based Lyapunov Stability Analysis

In this section, we derive stability conditions for the closed-loop error system (2.23), (2.24) in Theorem 3.3, valid for $\sigma > 0$ in (2.24). When $\sigma = 0$, the conditions of Theorem 3.10 are not feasible. However, in the linear case, these conditions can be modified to accommodate $\sigma = 0$, as established in Theorem 3.13 and Corollary 3.14.

Stability of the nonlinear system

Theorem 3.10. *Consider the closed-loop system (2.23), (2.24) with agent interaction strength $a > 0$, positive controller gains K , σ , and κ , a maximum inter-leader spacing Δ , and a nonlinearity F subject to (2.14). For a given decay rate $\delta \in (0, K)$, if there exist positive scalars p, μ , and λ_i , $i \in \{0, 1, 2, 3\}$, such that $\mu(K - \delta) + a \geq \lambda_0 \frac{\Delta^2}{2\pi^2}$, and*

$$\mathcal{N} = \begin{bmatrix} \mathcal{N}_{11} & 0 & 1 & K \\ * & \mathcal{N}_{22} & -\mu & -\mu K \\ * & * & -\lambda_2 & 0 \\ * & * & * & -\lambda_1 - \lambda_0 \end{bmatrix} \leq 0, \quad \mathcal{M} = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & p \\ * & -2\sigma\mu & -\mu \\ * & * & -\lambda_3 \end{bmatrix} \leq 0 \quad (3.10)$$

where

$$\begin{aligned}\mathcal{N}_{11} &= -2(K - \delta) + \lambda_2 L^2, & \mathcal{M}_{11} &= -2p(\kappa + K - \delta) + \lambda_3 L^2, \\ \mathcal{N}_{22} &= -2\mu a + \lambda_1 \Delta^4 / \pi^4, & \mathcal{M}_{12} &= \sigma p + \mu \kappa - a.\end{aligned}$$

Then the closed-loop system (2.23), (2.24) is exponentially stable in the $\mathcal{H}^1$ norm with the decay rate δ , i.e.,

$$\exists C > 0: \|e(t, \cdot)\|_{\mathcal{H}^1} \leq C e^{-\delta t} \|e(0, \cdot)\|_{\mathcal{H}^1}. \quad (3.11)$$

Proof. Consider the Lyapunov functional

$$\begin{aligned}V &= V_1 + V_B, \\ V_1 &= \|e(t, \cdot)\|^2 + \mu \|e_x(t, \cdot)\|^2, \\ V_B &= p|\eta(t)|^2, \quad \eta(t) = \text{col}\{e(t, 0), e(t, 1)\}.\end{aligned} \quad (3.12)$$

Differentiating V_1 along the trajectories of (2.23), (2.24) and integrating by parts, we obtain

$$\begin{aligned}\dot{V}_1 &= 2\langle e(t, \cdot), e_t(t, \cdot) \rangle + 2\mu \langle e_x(t, \cdot), e_{xt}(t, \cdot) \rangle = 2\langle e(t, \cdot), e_t(t, \cdot) \rangle + 2\mu \left(e_x^T e_t \Big|_0^1 - \langle e_{xx}(t, \cdot), e_t(t, \cdot) \rangle \right) \\ &= 2 \int_0^1 (e - \mu e_{xx})^T \left(a e_{xx} + F - K e + K \nu \right) - 2\sigma \mu |\eta_x|^2 - 2\mu (\kappa + K) e_x^T e \Big|_0^1 + 2\mu e_x^T F \Big|_0^1\end{aligned} \quad (3.13)$$

with $\eta_x(t) = \text{col}\{e_x(t, 0), -e_x(t, 1)\}$. Integrating by parts, we obtain

$$2(a + \mu K) \int_0^1 e^T e_{xx} = 2(a + \mu K) \left(e^T e_x \Big|_0^1 - \|e_x(t, \cdot)\|^2 \right). \quad (3.14)$$

From (2.14), (2.25), and (2.26), we have

$$0 \leq \lambda_0 \left(\frac{\Delta^2}{\pi^2} \|e_x(t, \cdot)\|^2 - \|\nu(t, \cdot)\|^2 \right), \quad \lambda_0 > 0, \quad (3.15a)$$

$$0 \leq \lambda_1 \left(\frac{\Delta^4}{\pi^4} \|e_{xx}(t, \cdot)\|^2 - \|\nu(t, \cdot)\|^2 \right), \quad (3.15b)$$

$$0 \leq \lambda_2 \left(L^2 \|e(t, \cdot)\|^2 - \|F(t, \cdot, e(t, \cdot))\|^2 \right), \quad (3.15c)$$

$$0 \leq \lambda_3 \left(L^2 |\eta(t)|^2 - |\eta_F(t)|^2 \right), \quad (3.15d)$$

where $\eta_F(t) = \text{col}\{F(t, e(t, 0)), F(t, e(t, 1))\}$. Using (3.14) in (3.13), and adding the right-hand sides of (3.15a)–(3.15c) to $\dot{V}_1$, we obtain

$$\begin{aligned} \dot{V}_1 + 2\delta V_1 \leq & \int_0^1 \zeta^T(t, x)(\mathcal{N} \otimes I_n)\zeta(t, x) dx - 2\sigma\mu|\eta_x(t)|^2 + 2(\mu\kappa - a)\eta^T(t)\eta_x(t) - 2\mu\eta_x^T(t)\eta_F(t) \\ & - 2\left([\mu(K - \delta) + a] - \lambda_0 \frac{\Delta^2}{2\pi^2}\right)\|e_x(t, \cdot)\|^2, \end{aligned}$$

where $\zeta(t, x) = \text{col}\{e(t, x), e_{xx}(t, x), F(t, e), \nu(t, x)\}$. Differentiating V_B along the trajectories of (2.24), we find

$$\begin{aligned} \dot{V}_B + 2\delta V_B \leq & 2p\eta^T(t)\eta_t(t) + 2p\delta|\eta(t)|^2 + (3.15d) \\ = & (-2p(\kappa + K - \delta) + \lambda_3 L^2)|\eta(t)|^2 + 2p\sigma\eta^T(t)\eta_x(t) + 2p\eta^T(t)\eta_F(t) - \lambda_3|\eta_F(t)|^2. \end{aligned}$$

Setting $\zeta_B(t) = \text{col}\{\eta(t), \eta_x(t), \eta_F(t)\}$, we have

$$\dot{V} + 2\delta V \leq \int_0^1 \zeta^T(t, x)(\mathcal{N} \otimes I_n)\zeta(t, x) dx + \zeta_B^T(t)(\mathcal{M} \otimes I_{2n})\zeta_B(t) - 2\left([\mu(K - \delta) + a] - \lambda_0 \frac{\Delta^2}{2\pi^2}\right)\|e_x(t, \cdot)\|^2.$$

Since $\mu(K - \delta) + a \geq \lambda_0 \frac{\Delta^2}{2\pi^2}$, $\mathcal{M} \leq 0$ and $\mathcal{N} \leq 0$, we have $\dot{V}(t) \leq -2\delta V(t)$, which implies $V(t) \leq e^{-2\delta t}V(0)$. By Lemma 3 of Kang and Fridman (2018), $V_B(t) \leq c_B\|e(t, \cdot)\|_{H^1}^2$ for some c_B . Then,

$$c_1\|e(t, \cdot)\|_{H^1}^2 \leq V(t) \leq c_2\|e(t, \cdot)\|_{H^1}^2,$$

where $c_1 = \min\{1, \mu\}$ and $c_2 = \max\{1, \mu\} + c_B$. Therefore,

$$\|e(t, \cdot)\|_{H^1}^2 \leq \frac{V(t)}{c_1} \leq \frac{e^{-2\delta t}}{c_1} V(0) \leq \frac{c_2}{c_1} e^{-2\delta t} \|e(t, \cdot)\|_{H^1}^2,$$

which implies the exponential stability in the $\mathcal{H}^1$ norm. $\square$

Remark 3.11. The PDE model (2.23)–(2.24) analysed in Theorem 3.10 is derived as the continuum limit of the discrete MAS governed by the ODEs (2.2). Accordingly, the stability conditions derived for the PDE-based MAS, expressed in the form of LMIs in Theorem 3.10, also hold for the corresponding discrete MAS when the number of agents is sufficiently large. This is validated numerically by solving the LMIs and performing phase portrait simulations for the discrete MAS under feasible parameter settings, considering different initial conditions and large swarm sizes (see Chapter 4 and Figs. 4.10–4.12). In all cases, successful deployment of agents onto the target curve is observed.

Remark 3.12. The conditions of Theorem 3.10 are feasible when L and Δ are sufficiently small, and K is sufficiently large.

Proof. Since $\sigma > 0$, the relation

$$\mathcal{M}_{2:3} = \begin{bmatrix} -2\sigma\mu & -\mu \\ -\mu & -\lambda_3 \end{bmatrix} < 0$$

holds for a large enough $\lambda_3 > 0$. Then, by the Schur complement lemma, $\mathcal{M} < 0$ follows from

$$-2p(\kappa + K - \delta) + \lambda_3 L^2 + \begin{bmatrix} \mathcal{M}_{12} & p \end{bmatrix} \mathcal{M}_{2:3}^{-1} \begin{bmatrix} \mathcal{M}_{12} & p \end{bmatrix}^T < 0,$$

which holds for a large enough $K > 0$. Furthermore, the Schur complement lemma implies that $\mathcal{N} < 0$ if

$$\begin{bmatrix} \mathcal{N}_{11} & 0 \\ 0 & \mathcal{N}_{22} \end{bmatrix} - \begin{bmatrix} 1 & -K \\ -\mu & -\mu K \end{bmatrix} \begin{bmatrix} \lambda_2^{-1} & 0 \\ 0 & \mathcal{N}_{44}^{-1} \end{bmatrix} \begin{bmatrix} 1 & -\mu \\ K & -\mu K \end{bmatrix} \leq 0.$$

Since $|\mathcal{N}_{44} = -(\lambda_0 + \lambda_1)| \rightarrow \infty$ (for large enough λ_0, λ_1), the above holds for $L = 0$, large enough $\lambda_2 > 0$, and small enough $\Delta > 0$. By continuity, it remains true for a small enough $L > 0$. $\square$

Stability of the linear system

Theorem 3.10 cannot be used if $\sigma = 0$ because $\mathcal{M}$ in (3.10) becomes infeasible. Specifically, when $\sigma = 0$, the matrix $\mathcal{M}$ takes the form:

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & p \\ * & 0 & -\mu \\ * & * & -\lambda_3 \end{bmatrix}.$$

A necessary condition for $\mathcal{M} \leq 0$ is that if any diagonal entry is zero, the corresponding row and column must also be zero. Since $\mathcal{M}_{22} = 0$ but the off-diagonal entry $\mathcal{M}_{23} = -\mu$ is strictly non-zero (as $\mu > 0$ is required for the positivity of V_1 in (3.12)), $\mathcal{M}$ can never be negative semi-definite. However, the conditions can be adjusted to allow for $\sigma = 0$ if the system is linear, i.e., $f(t, z_i) = Lz_i$

in (2.2). In this case, the closed-loop error system (2.23), (2.24) with $F(e) = Le(t, x)$ simplifies to

$$\begin{aligned} e_t(t, x) &= ae_{xx}(t, x) - (K - L)e(t, x) + K\nu(t, x), \\ e_t(t, 0) &= -(\kappa + K - L)e(t, 0) + \sigma e_x(t, 0), \\ e_t(t, 1) &= -(\kappa + K - L)e(t, 1) - \sigma e_x(t, 1). \end{aligned} \quad (3.16)$$

The following theorem provides simplified stability conditions that can be used with $\sigma = 0$, in which case they can be simplified further (see Corollary 3.14).

Theorem 3.13. *Consider the closed-loop system (3.16) with agent interaction strength $a > 0$, reaction coefficient (or linear gain) $L > 0$, and maximum gap between leaders Δ . For a given decay rate $\delta > 0$ and controller gains $K > L + \delta$, $\kappa > 0$, and $\sigma \geq 0$, if there exist positive scalars p , μ , λ_0 , and λ_1 such that*

$$\mu(K - L - \delta) + a \geq \lambda_0 \frac{\Delta^2}{2\pi^2}, \quad (3.17)$$

$$\bar{\mathcal{N}} = \begin{bmatrix} \bar{\mathcal{N}}_{11} & 0 & K \\ * & \bar{\mathcal{N}}_{22} & -\mu K \\ * & * & -\lambda_0 - \lambda_1 \end{bmatrix} \leq 0, \quad (3.18)$$

and

$$\bar{\mathcal{M}} = \begin{bmatrix} -2p(\kappa + K - L - \delta) & \sigma p + \mu\kappa - a \\ * & -2\sigma\mu \end{bmatrix} \leq 0 \quad (3.19)$$

where

$$\begin{aligned} \bar{\mathcal{N}}_{11} &= -2(K - L - \delta), \\ \bar{\mathcal{N}}_{22} &= -2\mu a + \lambda_1 \Delta^4 / \pi^4. \end{aligned}$$

Then the closed-loop system (3.16) is exponentially stable in the $\mathcal{H}^1$ norm with the decay rate δ in the sense of (3.11).

Proof. Consider the Lyapunov functional V defined in (3.12). Calculations similar to (3.13) give

$$\dot{V}_1 = 2 \int_0^1 (e - \mu e_{xx})^T \left(ae_{xx} - (K - L)e + K\nu \right) - 2\sigma\mu |\eta_x|^2 - 2\mu(\kappa + K - L) e_x^T e \Big|_0^1$$

with $\eta_x(t) = \text{col}\{e_x(t, 0), -e_x(t, 1)\}$. Integrating by parts, and adding the right-hand sides of (3.15a) and (3.15b), we obtain

$$\dot{V}_1 + 2\delta V_1 \leq \int_0^1 \bar{\zeta}^T (\bar{\mathcal{N}} \otimes I_n) \bar{\zeta} + 2(a - \mu\kappa) e_x^T e \Big|_0^1 - 2\sigma\mu|\eta_x|^2 - 2\left([\mu(K - L - \delta) + a] - \lambda_0 \frac{\Delta^2}{2\pi^2}\right) \|e_x(t, \cdot)\|^2,$$

where $\bar{\zeta}(t, x) = \text{col}\{e(t, x), e_{xx}(t, x), \nu(t, x)\}$. Differentiating V_B along the trajectories of (3.16), we find

$$\dot{V}_B + 2\delta V_B = 2p\eta^T(t)\eta_t(t) + 2p\delta|\eta(t)|^2 = [-2p(\kappa + K - L - \delta)]|\eta(t)|^2 + 2p\sigma\eta^T(t)\eta_x(t).$$

Setting $\bar{\zeta}_B(t) = \text{col}\{\eta(t), \eta_x(t)\}$, we have

$$\begin{aligned} \dot{V} + 2\delta V &\leq \int_0^1 \bar{\zeta}^T(t, x) (\bar{\mathcal{N}} \otimes I_n) \bar{\zeta}(t, x) dx + \bar{\zeta}_B^T(t) (\mathcal{M} \otimes I_{2n}) \bar{\zeta}_B(t) \\ &\quad - 2\left([\mu(K - L - \delta) + a] - \lambda_0 \frac{\Delta^2}{2\pi^2}\right) \|e_x(t, \cdot)\|^2. \end{aligned}$$

Therefore, (3.18) and (3.19) guarantee $\dot{V} \leq -2\delta V$. The rest of the proof is similar to that of Theorem 3.10. $\square$

Corollary 3.14. *Consider the closed-loop system (3.16) with agent interaction strength $a > 0$, reaction coefficient $L > 0$, and maximum gap between leaders Δ . Given controller gains $K > L + \delta$, $\kappa > 0$, and $\sigma = 0$, let there exist a positive λ_0, λ_1 such that (3.17)-(3.18) holds with $\mu = a/\kappa$. Then the closed-loop system (3.16) is exponentially stable in the $\mathcal{H}^1$ norm with the decay rate δ in the sense of (3.11).*

Proof. If $\sigma = 0$, then (3.19) holds for $\mu = a/\kappa$. If (3.18) holds for this μ , then the conditions of Theorem 3.13 are satisfied, implying the exponential stability of (3.16). $\square$

Remark 3.15. The conditions of Theorem 3.13 and Corollary 3.14 are feasible when Δ is sufficiently small and K is sufficiently large. The proof is similar to Remark 3.12.

With this, we conclude the chapter. In the following chapter, we turn to simulation studies that illustrate the theoretical results and examine in detail the effects of boundary control and leader communication on swarm deployment.

Simulations

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In the preceding chapters, we developed a theoretical framework for modelling and controlling large-scale MAS using a PDE-based method. Specifically, we considered MAS composed of $N + 1$ agents in $\mathbb{R}^n$, with agent interactions and local dynamics leading to a semilinear parabolic PDE with Wentzell boundary conditions in Chapter 2, while Chapter 3 established the well-posedness and stability conditions in the form of LMIs for the resulting closed-loop PDE system.

This chapter focuses on validating and extending these theoretical insights through comprehensive numerical simulations. A key part of this process involves solving the LMIs introduced in Chapter

3 to identify feasible system parameter sets that guarantee stability. This is achieved using YALMIP, a MATLAB-based modelling language for convex optimisation problems, in conjunction with the solver SeDuMi, which is specifically designed to handle semidefinite programming tasks.

All simulations are implemented in MATLAB, and the full source code is available on the author's GitHub repository:

https://github.com/ShubhamKhansili1/OJAG_PDE_solution

4.1 Simulation Setup and Research Questions

Consider the MAS defined in (2.2), consisting of 141 agents ($N = 140$), each with a three-dimensional state ($n = 3$). The objective is to deploy the agents from the initial position, $(0, 0, 0)$, onto a smooth spatial curve given by

$$\gamma(s) = (1.5 \sin s, 1.5 \cos s, 6), \quad s \in [0, 1].$$

Two types of local agent dynamics are studied: (i) linear, $f(t, z_i) = Lz_i$, and (ii) nonlinear, $f(t, z_i) = L \sin(z_i)$. In both cases, local controllers u_i are designed according to (2.8) for interior agents and (2.17) for boundary agents. The error dynamics PDE is defined in (2.12) with boundary conditions (2.18). Building on this model, global controllers $v_i = v(t, i/N)$ are designed in (2.22).

Coordination Strategies Under Comparison

To evaluate the effectiveness of various coordination mechanisms for distributed swarm deployment, we consider four coordination strategies summarised in Table 4.1. Each strategy is characterised by the presence or absence of two coordination features: boundary coupling via Wentzell control and communication among leaders. We begin with **Strategy 0**, which follows the approach proposed in Wei et al. (2019a); Terushkin and Fridman (2021). In this strategy, the swarm constructs a piecewise-constant approximation of the PDE error state $e(t, x)$ using a finite set of non-communicating leaders distributed across the domain. Agents between two leaders estimate their error solely from the nearest leader. Thus, if boundary leaders are initialised on the target, i.e. $e(t, 0) = e(t, 1) = 0$, neighbouring agents incorrectly infer zero error even when far from the

target. This slows convergence and increases deployment time. Crucially, **Strategy 0** employs dynamic Dirichlet boundary conditions, under which boundary agents do not care about the swarm. For example, if the boundary agents are already on the target while the swarm is not, they remain fixed, breaking cohesion.

Strategy	Boundary Coupling? (Wentzell)	Leader-Leader Communication?
0	✗	✗
1	✗	✓
2	✓	✗
3	✓	✓

Table 4.1: Summary of strategies indicating presence of boundary coupling and leader connection.

To overcome the limitations of **Strategy 0**, we consider three alternative strategies that progressively enhance coordination. **Strategy 1** introduces leader-to-leader communication, allowing leaders to share their positional error information and enabling the swarm to construct a piecewise-linear approximation of the error state. Followers then estimate their positional error by linearly interpolating the errors of the neighbouring leaders to the left and right. **Strategy 2** introduces boundary coupling via Wentzell control, enabling boundary agents to remain dynamically linked with the swarm, thereby improving swarm cohesion. **Strategy 3**, proposed in this work, integrates both mechanisms, leader communication and boundary coupling. These four strategies serve as the foundation for all simulation-based comparisons in this chapter.

This chapter addresses the following key questions through simulation studies:

Question 1: *What is the sufficient number of leaders required to guarantee deployment under Strategy 3 compared to Strategy 0?*

Section 4.2 investigates this by solving the LMIs derived in Chapter 3, which incorporate the approximation bounds discussed earlier.

Question 2: *Does the combination of Wentzell boundary control and leader-to-leader communication improve the deployment speed compared to strategies without them?*

This is explored in Section 4.3 by comparing the transient responses under each coordination strategy.

Question 3: *When boundary agents are initially placed on the target, can Wentzell boundary feedback improve cohesion by causing them to first move towards the swarm and then return to the target curve?*

This question is also addressed in Section 4.3. These questions are central to evaluating the effectiveness of the proposed strategy and are answered through detailed simulation results presented in the remainder of this chapter.

4.2 Simulation of LMI Feasibility and Leader Requirements

In this section, we address **Question 1** from Section 4.1: *What is the sufficient number of leaders required to guarantee deployment under Strategy 3 compared to Strategy 0?* To answer this, we solve the LMIs derived in Chapter 3. Given specific values of the system parameters $K, \kappa, \sigma, L, \delta$, and a , the LMI feasibility problem yields an admissible bound on the leader spacing Δ . Feasible values of Δ determine the sufficient number of leaders required for stability, since

$$N_{\text{leaders}} = 1 + \text{ceil} \left(\frac{1}{\Delta} \right).$$

Setup for LMI Feasibility

We consider the linear error system (3.16) with

$$K = 1.4, \quad \kappa = 1, \quad \sigma = 6, \quad L = 0.3, \quad \delta = 1, \quad (4.1)$$

where K, κ, σ , and δ are the leader propagation, boundary agents tracking, boundary coupling, and decay gains, respectively. Since the system is linear, the nonlinear term in (2.2) is written as $f(t, z_i) = Lz_i$, with L acting as a linear gain in this setting.

The values presented in the subsequent tables are generated by numerically verifying the feasibility of the conditions in Theorem 3.13. Since the maximum allowable spacing Δ is a scalar parameter appearing in the LMIs, we determine its upper bound using a linear search as described in Algorithm 1. The subsequent analysis systematically examines the influence of key control parameters on the sufficient number of leaders required for deployment, considering first the interaction

strength, then five representative parameter sets, and finally the effect of the global gain K and the linear gain L . The same procedure is then extended to the nonlinear error system governed by Theorem 3.10.

Algorithm 1 Computation of Maximum Admissible Leader Spacing Δ

Input: Fixed system parameters $\{K, \kappa, \sigma, L, \delta, a\}$ and Strategy $S \in \{0, 3\}$

```

1: Set search resolution  $\epsilon > 0$ 
2: Initialise  $\Delta = \epsilon$ 
3: while  $\Delta \leq 1.0$  do
4:   if LMI conditions in Theorem 3.13 are feasible for current  $\Delta$  then
5:      $\Delta_{\max} \leftarrow \Delta$  {Update the largest feasible value found}
6:      $\Delta \leftarrow \Delta + \epsilon$ 
7:   else
8:     break {Exit loop at the first point of infeasibility}
9:   end if
10: end while
11: return  $\Delta_{\max}$ 

```

4.2.1 Impact of Interaction Strength a

Table 4.2 presents the maximum values of Δ for different choices of interaction strength a such that the LMIs of Theorem 3.13 are feasible. In all cases, **Strategy 3** i.e. Wentzell boundary control with inter-leader communication allows for larger Δ , implying that fewer leaders are required to successfully deploy the MAS. Specifically, for $a = 4 \times 10^{-3}$, we obtain $\Delta \leq 0.17$, meaning that seven leaders are needed: $1 + 1/0.17 \approx 6.8$. For **Strategy 0** i.e. with dynamic Dirichlet boundary control and without inter-leader communication, the feasible value decreases to $\Delta \leq 0.08$, requiring twice as many leaders: $1 + 1/0.08 \approx 13.5$.

As a increases, the feasible Δ also increases, indicating that fewer leaders are needed for stability. For instance, when $a = 0.14$, then $\Delta = 1$, meaning the system can be stabilised using only two boundary leaders. A higher value of a allows agents to correct their positional errors more quickly, which reduces the need for frequent leader feedback.

Table 4.2: Allowable leader spacing Δ from Theorem 3.13 for different values of interaction strength a , under Strategy 0 and Strategy 3 for parameter values in (4.1).

Control Strategy	Interaction Strength (a)							
	0.001	0.0025	0.004	0.01	0.14	0.2	0.45	0.5
Strategy 3 (Δ)	0.08	0.13	0.17	0.27	1.00	1.00	1.00	1.00
Strategy 0 (Δ)	0.04	0.07	0.08	0.14	0.53	0.63	0.95	1.00

4.2.2 Leader Requirements for Five Different Parameter Sets

We next fix $a = 0.01$ and vary $(K, \kappa, \sigma, \delta, L)$ across five representative cases, listed in Table 4.3. In every case, the sufficient leader requirement under Strategy 3 is reduced compared with Strategy 0, in some instances by as much as 50%. This demonstrates the robustness of Strategy 3 across different controller choices.

Table 4.3: Sufficient number of leaders required under different configurations, with and without communication, for various system parameters and $a = 0.01$. Values are obtained from the LMIs in Theorem 3.13.

System Parameters	Case 1	Case 2	Case 3	Case 4	Case 5
Leader propagation gain, K	5	2.1	2	1.5	1
Boundary tracking gain, κ	4	1	2	3	2
Boundary coupling gain, σ	1	2	1	1	4
Decay rate, δ	2	1	0.8	1	0.4
Linear gain, L	2	1	1	0.4	0.5
Strategy					
Strategy 3 (Δ)	7	6	6	5	5
Strategy 0 (Δ)	10	13	9	9	10

4.2.3 Effect of Global Gain K

Table 4.4 shows how the maximum allowable leader spacing Δ changes with the global gain K . When K is small, the impact of the approximation error term $K\nu(t, x)$ in (2.23) is limited. In this case, the improved accuracy of the piecewise-linear approximation used in Strategy 3 allows for a larger value of Δ , meaning fewer leaders are needed. As K increases, even small approximation errors become more significant. At this point, both Strategy 0 and Strategy 3 give similar values of Δ . However, this does not mean both strategies converge at the same rate. A higher K improves the speed at which leader information spreads across the swarm. This will be demonstrated in

Figure 4.7 in the next section. For very large values of K , the LMIs become infeasible due to error amplification.

Table 4.4: Allowable leader spacing Δ for different values of global control gain K , under Strategy 0 and Strategy 3 with $a = 0.001, \kappa = 1, \sigma = 6, L = 0.3, \delta = 1$.

Global Control Gain K	1	1.31	1.4	3	5	8	10^4
Strategy 3 (Δ)	Infeas.	0.08	0.08	0.08	0.07	0.06	Infeas.
Strategy 0 (Δ)	Infeas.	0.01	0.04	0.08	0.07	0.06	Infeas.

4.2.4 Effect of Linear Gain L

Table 4.5 investigates the effect of the linear gain L . As L increases, the influence of each agent's local dynamics becomes stronger, which destabilises the open-loop MAS. Consequently, the admissible leader spacing Δ decreases, and more leaders are required to maintain stability.

Table 4.5: Allowable leader spacing Δ for different values of the linear gain L , with $a = 0.001, K = 1, \kappa = 1, \sigma = 1, \delta = 0.1$.

Linear Gain L	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Strategy 3 (Δ)	0.15	0.14	0.12	0.11	0.11	0.10	Infeas.
Strategy 0 (Δ)	0.15	0.14	0.12	0.10	0.08	0.06	Infeas.

4.2.5 Nonlinear Case

Finally, Table 4.6 presents admissible Δ values for the nonlinear system governed by Theorem 3.10. As in the linear case, Strategy 3 consistently permits larger Δ , confirming that inter-leader communication and Wentzell coupling reduce leader requirements even under nonlinear dynamics.

Table 4.6: Allowable leader spacing Δ from Theorem 3.10 for different values of interaction strength a , under Strategy 0 and Strategy 3 for $K = 1, \kappa = 2, \sigma = 6, L = 0.1, \delta = 0.8$.

Interaction Strength a	0.001	0.01	0.1	0.25
Strategy 3 (Δ)	0.13	0.24	0.78	1
Strategy 0 (Δ)	0.10	0.19	0.62	0.99

Figure 4.1 shows the maximum feasible values of Δ from Theorems 3.10 (top) and 3.13 (bottom) for interacting and non-interacting leader cases, plotted against the local control gain a . As a increases, the feasible Δ also increases, indicating that fewer leaders are required for stability. For example, when $a = 0.14$, we obtain $\Delta = 1$, which means the system can be stabilised using only

two boundary leaders (see the lower plot of Figure 4.1 and Table 4.2). Since the spatial domain is normalised to $[0, 1]$, Δ must satisfy $0 < \Delta \leq 1$. Notably, $\Delta = 1$ remains feasible for all $a \geq 0.14$.

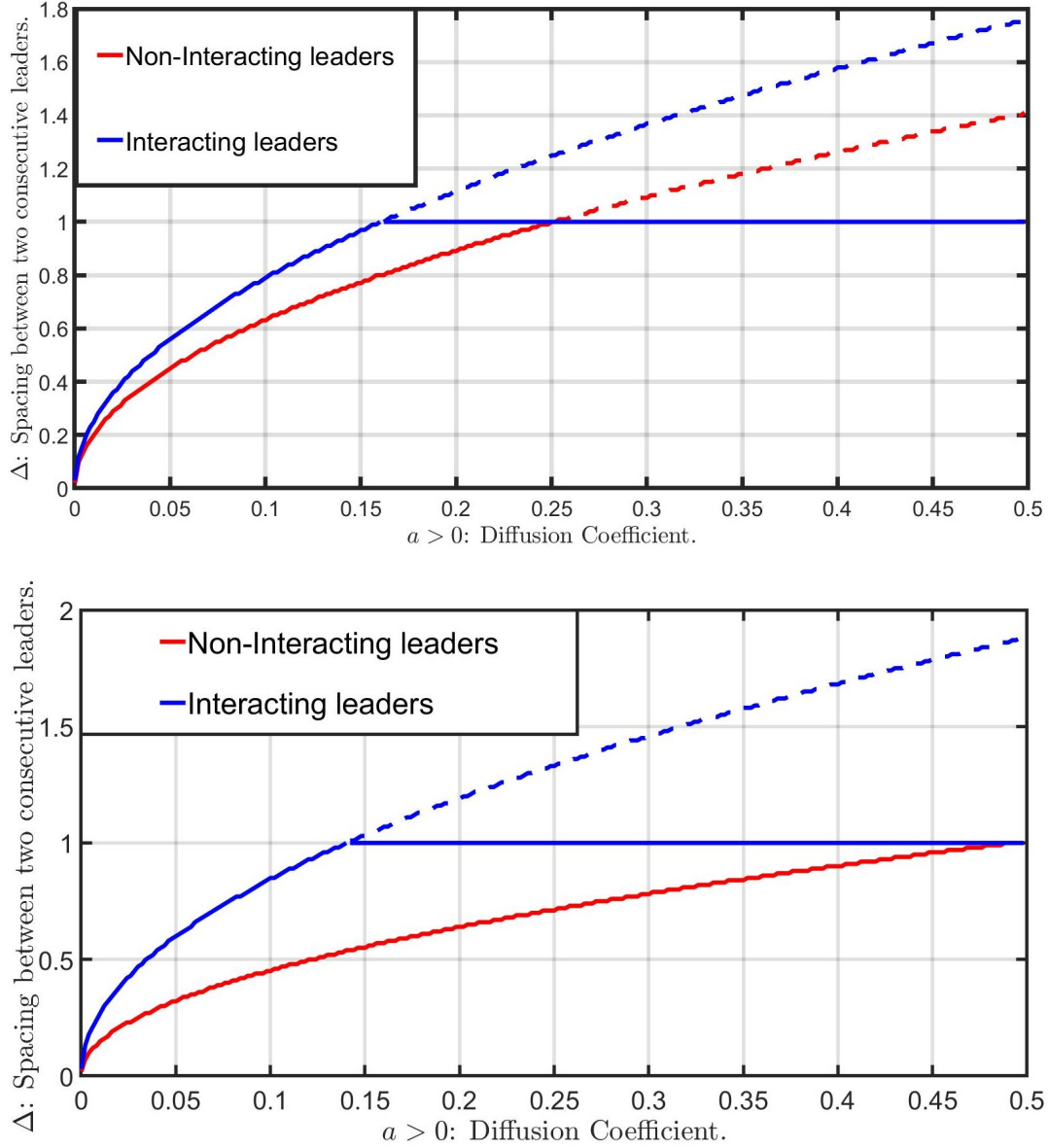


Figure 4.1: Comparison of admissible leader spacing Δ versus interaction strength a . (Top) Results from Theorem 3.10 for the nonlinear PDE system. (Bottom) Results from Theorem 3.13 for the linearised case.

4.3 Improved Deployment via Wentzell and Leader Communication

This section examines the convergence performance of the four strategies introduced earlier, with a particular focus on the advantages offered by **Strategy 3**, which combines Wentzell boundary feedback with leader-to-leader communication. By synthesizing these two distinct mechanisms, we aim to uncover how their interplay facilitates more robust swarm behavior than either component could achieve in isolation. The analysis is based on simulations using the linear error system (3.16) under varying interaction strengths, propagation gains, and initial configurations. These diverse simulation parameters allow for a comprehensive sensitivity analysis, ensuring that our findings regarding Strategy 3 are consistent across a broad range of operational conditions. Our goal is to demonstrate how specific design features, namely, boundary control and inter-leader communication, affect convergence speed and swarm cohesion.

Note: In all simulations presented in this thesis, the error evolution is shown on a logarithmic scale. This representation makes it easier to observe the convergence behaviour of the system over time, particularly when the swarm gets closer to the target curve and the error becomes very small. Since the convergence is exponential, the logarithmic scale helps highlight and compare the decay rates of different strategies more clearly. It also allows both large initial errors and small residual errors to be visualised effectively within the same plot.

4.3.1 Comparing Strategies: Equal Leaders, Unequal Performance

We now address **Question 2** by evaluating how combining Wentzell boundary control with inter-leader communication impacts deployment speed. Essentially, we want to see if this specific combination, defined as **Strategy 3**, performs better than the simpler strategies of **Strategy 0**, **Strategy 1**, and **Strategy 2** which were discussed earlier.

Using the parameter values specified in equation (4.1) with $a = 0.0025$, a feasibility check of the LMI-based stability conditions indicates that a sufficient number of leaders for a stable system is nine (see the orange column in Table 4.2). The top portion of Figure 4.2 illustrates the evolution of the boundary errors $\log \|e(t, 0)\|$ and $\log \|e(t, 1)\|$ for both $\sigma = 0$ and $\sigma > 0$, in the presence

of leader-to-leader communication. The bottom portion of the same figure shows the evolution of the norm $\log \|e(t, \cdot)\|$ for four distinct strategies: **Strategy 3** (solid orange), **Strategy 2** (dashed orange), **Strategy 1** (solid red), and **Strategy 0** (dashed red). This simulation is performed for the scenario in which all agents are initially positioned at the origin. This choice provides a clearer visualisation of how the proposed control laws guide the swarm during deployment. In particular, it highlights the action of the local controller, which is designed to maintain the relative spacing between neighbouring agents equal to the desired spacing on the target curve.

Figure 4.2 (Top) shows that with $\sigma = 0$ and leader communication, as in **Strategy 1**, the boundary agents reach the target more quickly, indicating a loss of swarm cohesion, compared to the Wentzell-controlled case, i.e., **Strategy 3**. However, the bottom plot in Figure 4.2 clearly shows that this faster boundary motion does not lead to improved overall performance: the global error decays more slowly for both communicating and non-communicating leaders under $\sigma = 0$, i.e., **Strategy 1** and **Strategy 0**, than for their Wentzell-controlled counterparts, i.e., **Strategy 3** and **Strategy 2**. This observation confirms that prioritising boundary speed without proper feedback damping, provided here by the Wentzell condition, actually hinders the collective convergence of the entire MAS.

A similar observation to that in Figure 4.2 (Bottom) is obtained when the boundary agents are initially placed on the target curve while the rest of the swarm is at the origin. Testing this alternative configuration ensures that the superiority of **Strategy 3** is not merely a byproduct of the initial error magnitude but is inherent to the control architecture. This case, illustrated in Figure 4.3, corresponds to the parameter values in Table 4.4 with $K = 8$. The consistency across these different initial conditions suggests that the proposed feedback mechanism is robust to spatial perturbations within the swarm. Thus, regardless of whether leader communication is present, Wentzell boundary control ($\sigma > 0$) results in lower error and stronger swarm cohesion. The presence of $\sigma > 0$ acts as a regulatory mechanism that prevents the boundary agents from moving significantly faster than the diffusive dynamics of the interior agents. Furthermore, communication among leaders consistently improves convergence, as evidenced by the solid curves (with communication) outperforming their dashed counterparts (without communication). The combination of leader communication and Wentzell control, i.e. **Strategy 3**, therefore yields the best overall convergence performance.

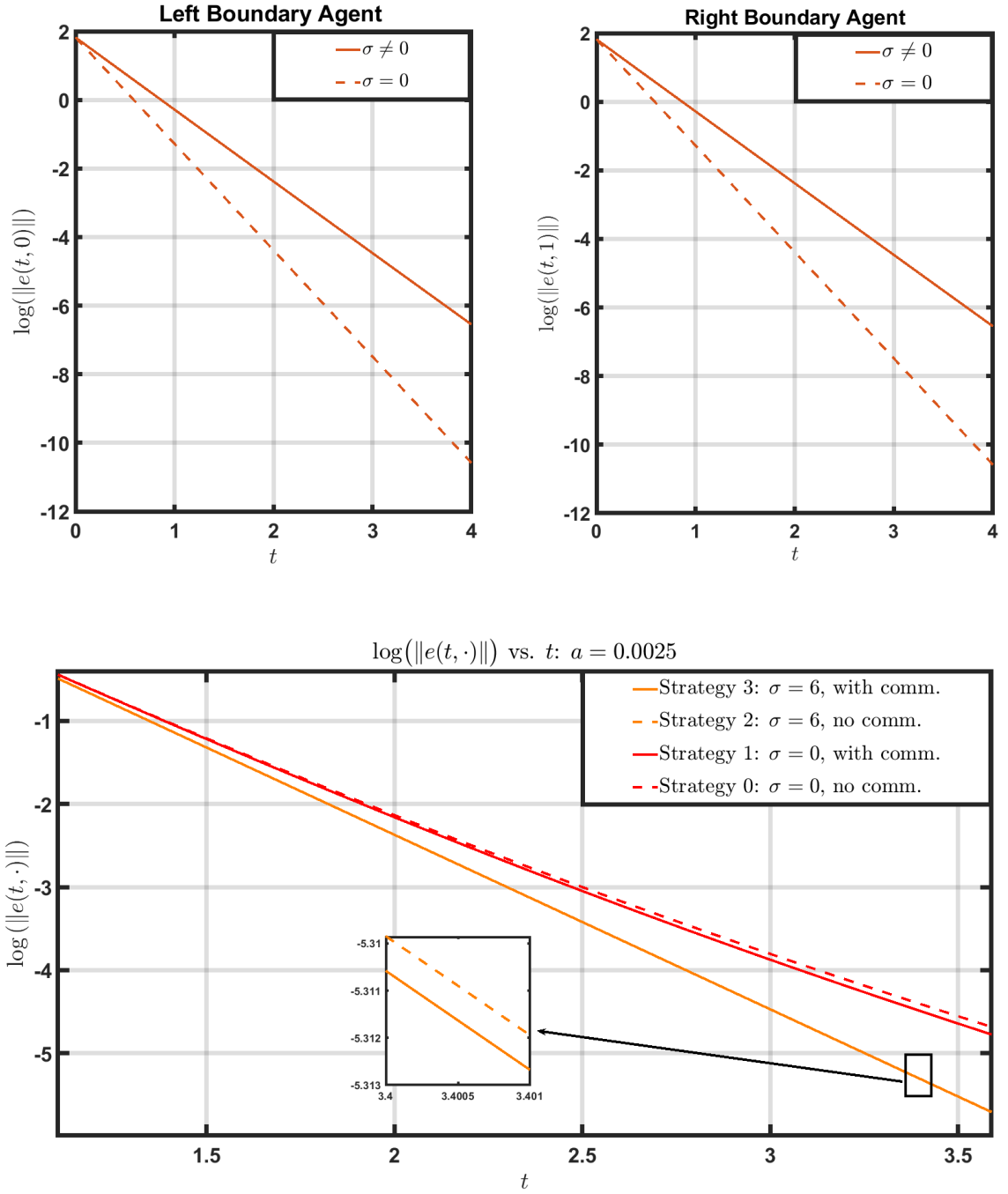


Figure 4.2: Time evolution of deployment error from equation (3.16) for $a = 0.0025$ and parameters as given in (4.1). (Top) Evolution of boundary errors $\log \|e(t, 0)\|$ and $\log \|e(t, 1)\|$ for $\sigma = 6$ (Wentzell) and $\sigma = 0$ (Dirichlet) with leader communication. (Bottom) Evolution of global error $\log \|e(t, \cdot)\|$ under all four strategies: Strategy 0 and Strategy 1 (Dirichlet), Strategy 2 and Strategy 3 (Wentzell), each with and without leader communication.

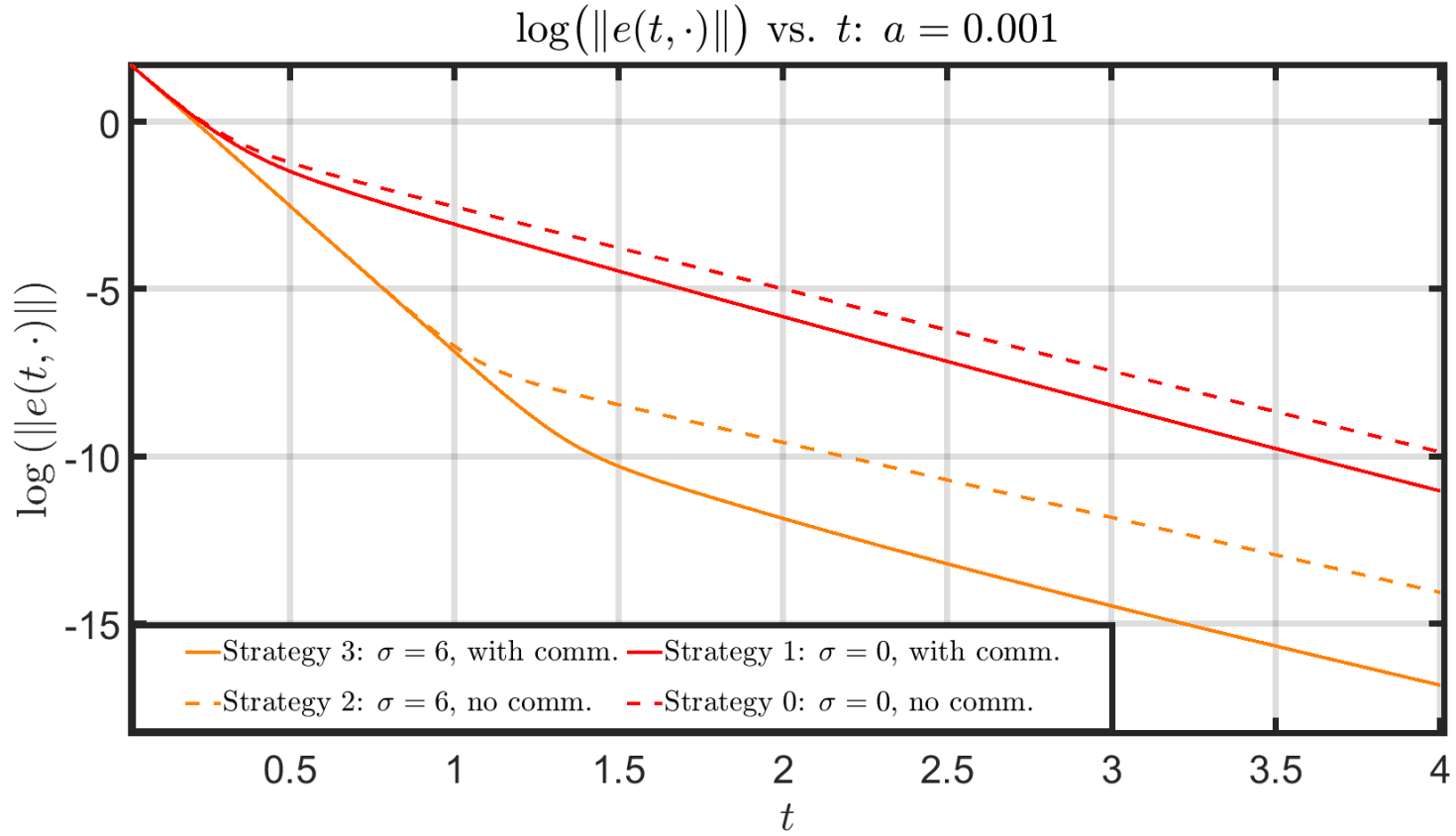


Figure 4.3: Time evolution of $\log(\|e(t, \cdot)\|)$ from equation (3.16), with boundary agents initially on the target curve and all other agents at the origin, for $\sigma = 6$ (Wentzell boundary conditions) and $\sigma = 0$ (dynamic Dirichlet boundary conditions), each with and without communication, for $K = 8$ and system parameters as specified in Table 4.4.

4.3.2 Effect of Weak Interaction Strength a

For the weak interaction strength $a = 0.001 < 0.0025$, the sufficient number of leaders required to ensure stability under **Strategy 3** increases to 14 (see the maroon column in Table 4.2). This increase indicates that when the interaction strength between agents becomes weaker, the corrective action generated by the local controller (2.8) is reduced, causing the agents to adjust their positions more slowly. Consequently, a greater burden is placed on the global controller (2.22) to propagate positional information throughout the swarm, and therefore a larger number of leaders is required to maintain stable deployment behaviour. To assess how reduced interaction strength affects the evolution of $\log \|e(t, \cdot)\|$ under different coordination strategies, the number of leaders was fixed at 9 to match the $a = 0.0025$ case. The system was simulated under all four strategies, and the results are presented in Figure 4.4. Since 9 leaders are sufficient for $a = 0.0025$ but below the threshold of 14 required for $a = 0.001$, this comparison illustrates how the evolution of $\log \|e(t, \cdot)\|$ changes for each strategy when the local coupling gain weakens while the number of leaders remains unchanged.

Under **Strategy 0** and **Strategy 1**, the boundary leaders move rapidly towards the target, leaving the swarm behind; the agents near the boundary leaders require more time to align, which leads to a significant slowdown in convergence. In contrast, **Strategy 2** and **Strategy 3** apply Wentzell boundary conditions that keep the boundary agents moving in coordination with the swarm, so even with weak interaction strength the agents remain aligned with the boundary leaders.

The performance for $a = 0.0025$ is shown in Figure 4.2, and the same qualitative behavior is observed in Figure 4.4 for $a = 0.001$. The Wentzell boundary condition cases exhibit similar performance in both scenarios, whereas the dynamic Dirichlet-type boundary conditions lead to slower convergence when the interaction strength is weak. A comparison of the two figures shows that the performance gap between the communicating and non-communicating cases under Wentzell boundary control becomes more pronounced in the weak interaction case. This suggests that leader-to-leader communication becomes increasingly vital as local agent interactions diminish, as it provides a global coordination signal that helps synchronize the swarm's overall trajectory. Figures 4.5 and 4.6 show 3D surface plots of the deployment error $e(t, x)$ for 9 leaders under **Strategy 3** and **Strategy 0**, respectively (with parameter values in (4.1) and $a = 0.001$).

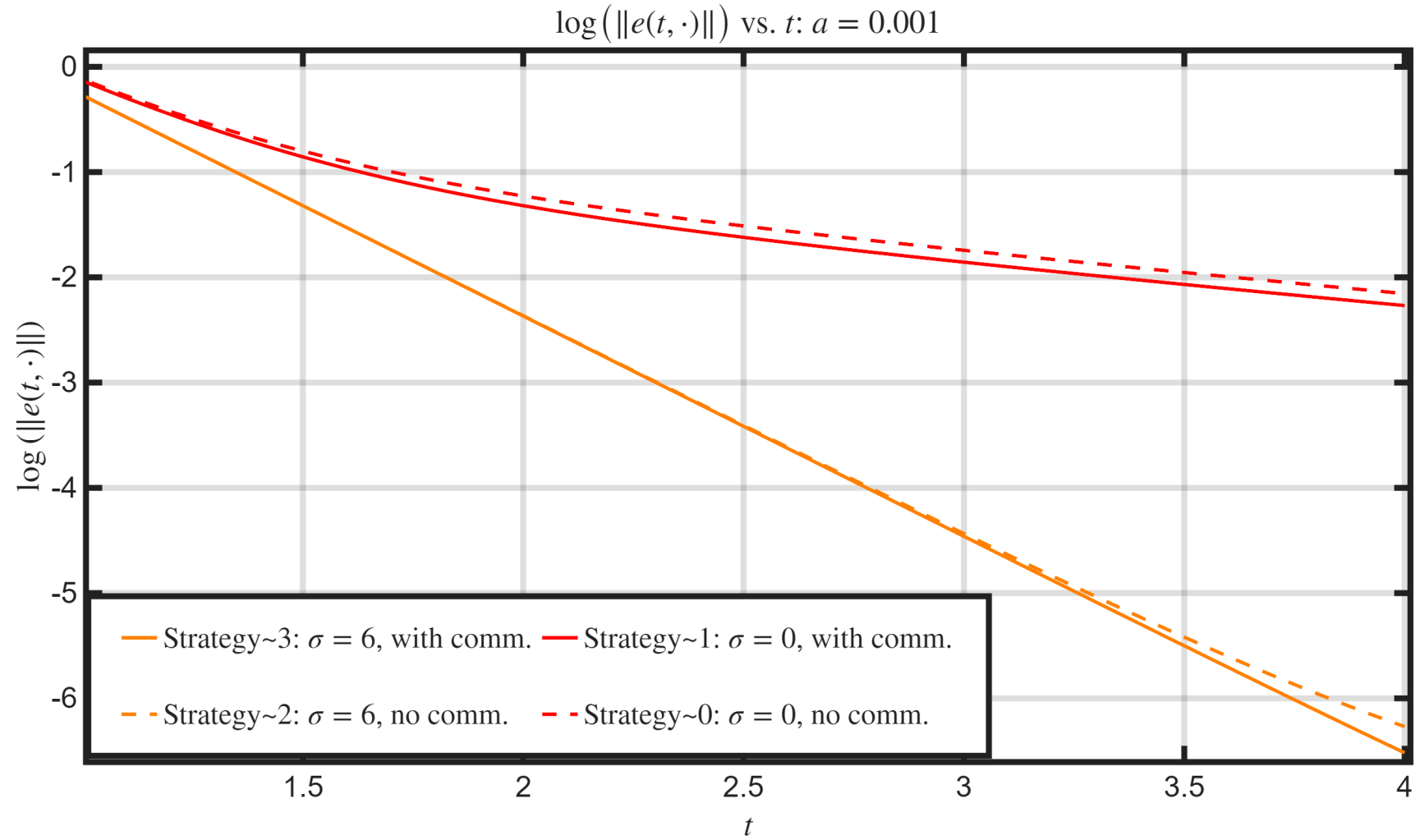


Figure 4.4: Time evolution of $\log(\|e(t, \cdot)\|)$ from equation (3.16) for $\sigma = 6$ (Wentzell boundary conditions) and $\sigma = 0$ (dynamic Dirichlet boundary conditions), each with and without communication among nine leaders, for $a = 0.001$ and system parameters as specified in (4.1).

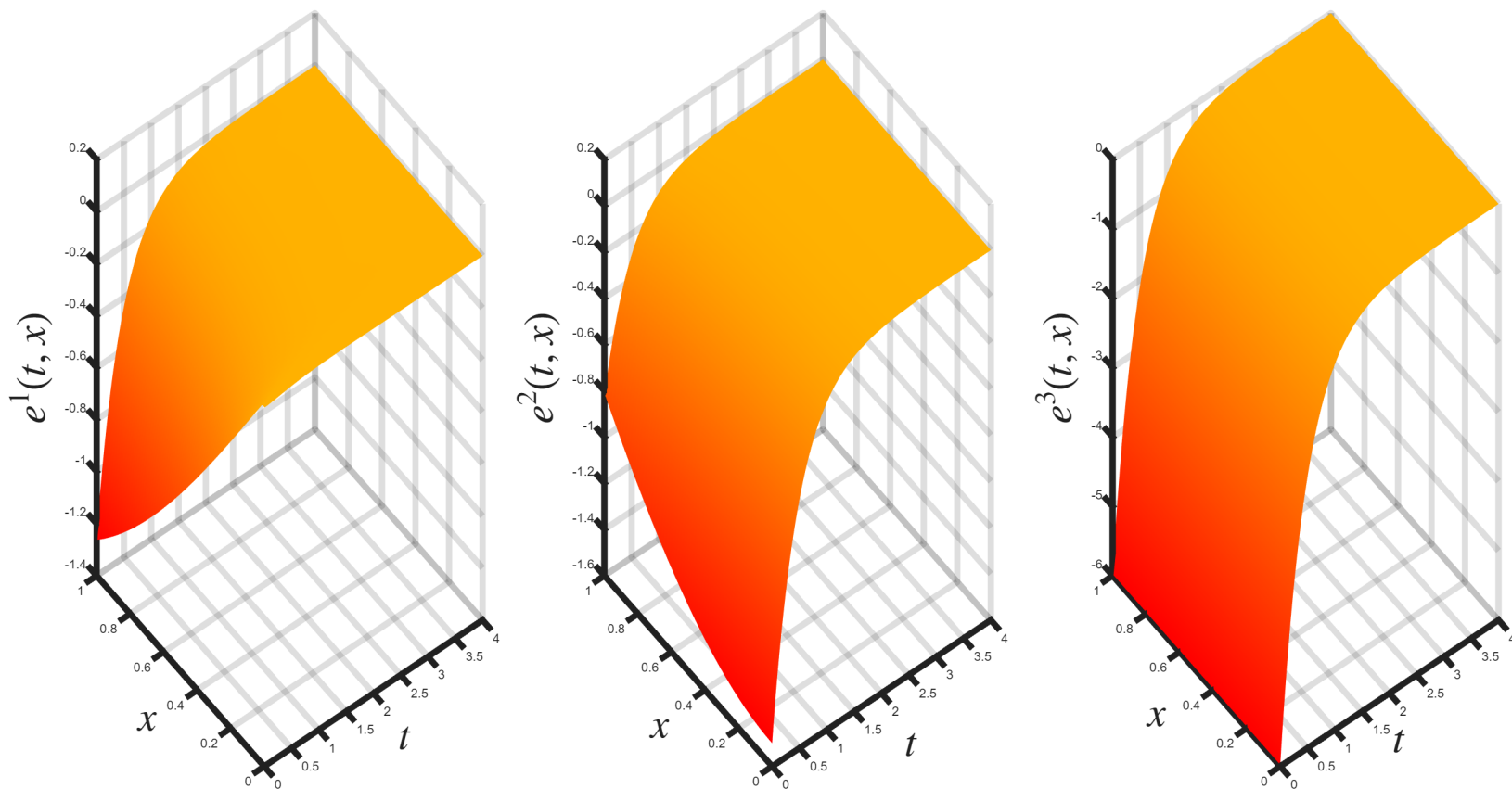


Figure 4.5: 3D surface plot of the deployment error $e(t, x)$ for $a = 0.001$ with 9 leaders under Strategy 3.

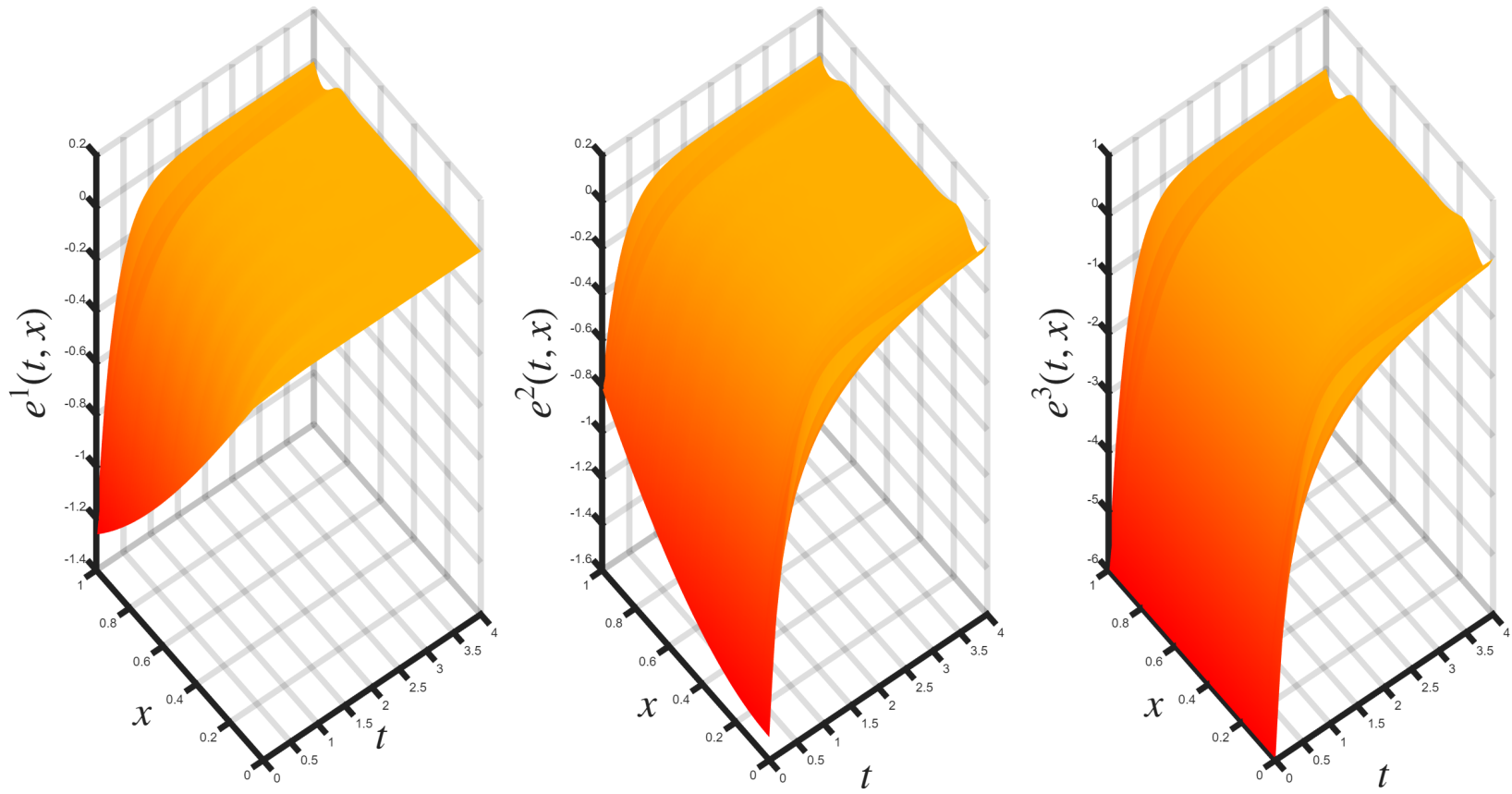


Figure 4.6: 3D surface plot of the deployment error $e(t, x)$ for $a = 0.001$ with 9 leaders under Strategy 0.

4.3.3 Impact of Global Gain K on Convergence Speed

For $a = 0.0025$, as shown in Figure 4.2, the performance curves for the leader-leader communication and no-communication strategies under Wentzell boundary conditions are almost identical, making the benefit of leader communication difficult to discern from the convergence profiles. To better highlight this difference, the leader propagation gain K defined in (2.22) was increased. A higher value of K accelerates the diffusion of positional error information from the leaders to the agents, resulting in faster convergence to the desired deployment curve. Table 4.4 lists the sufficient number of leaders required for stable deployment for $K = 1.4, 3, 5, 8$, which are 14, 14, 16, and 18 leaders respectively, for $a = 0.001$ and system parameters given in (4.1). Figure 4.7 shows that with increasing K , the performance gap between the communicating and non-communicating strategies under Wentzell boundary conditions becomes more evident. In contrast, this gap remains less pronounced under dynamic Dirichlet boundary conditions, again demonstrating the advantages of Wentzell boundary control. Overall, the combination of Wentzell boundary conditions and leader-to-leader communication delivers the best performance.

4.3.4 Extra Leaders Needed to Match Wentzell + Communication

Figure 4.8 shows that Strategy 3, i.e., Wentzell boundary control with leader-to-leader interaction, consistently outperforms Strategy 0, i.e., dynamic Dirichlet boundary control without communication, when both use the same number of leaders. For $K = 1.4, 3, 5$, and 8 , Strategy 3 achieves faster convergence with 14, 14, 16, and 18 leaders respectively. To match this performance, Strategy 0 requires approximately 5, 5, 5, and 7 additional leaders in each case, as evident from the figure and supported by the theoretical stability thresholds in Table 4.4. This demonstrates that combining Wentzell boundary conditions with leader-to-leader communication not only reduces the number of leaders needed for stability but also delivers superior convergence speed compared to the non-communicating Dirichlet case.

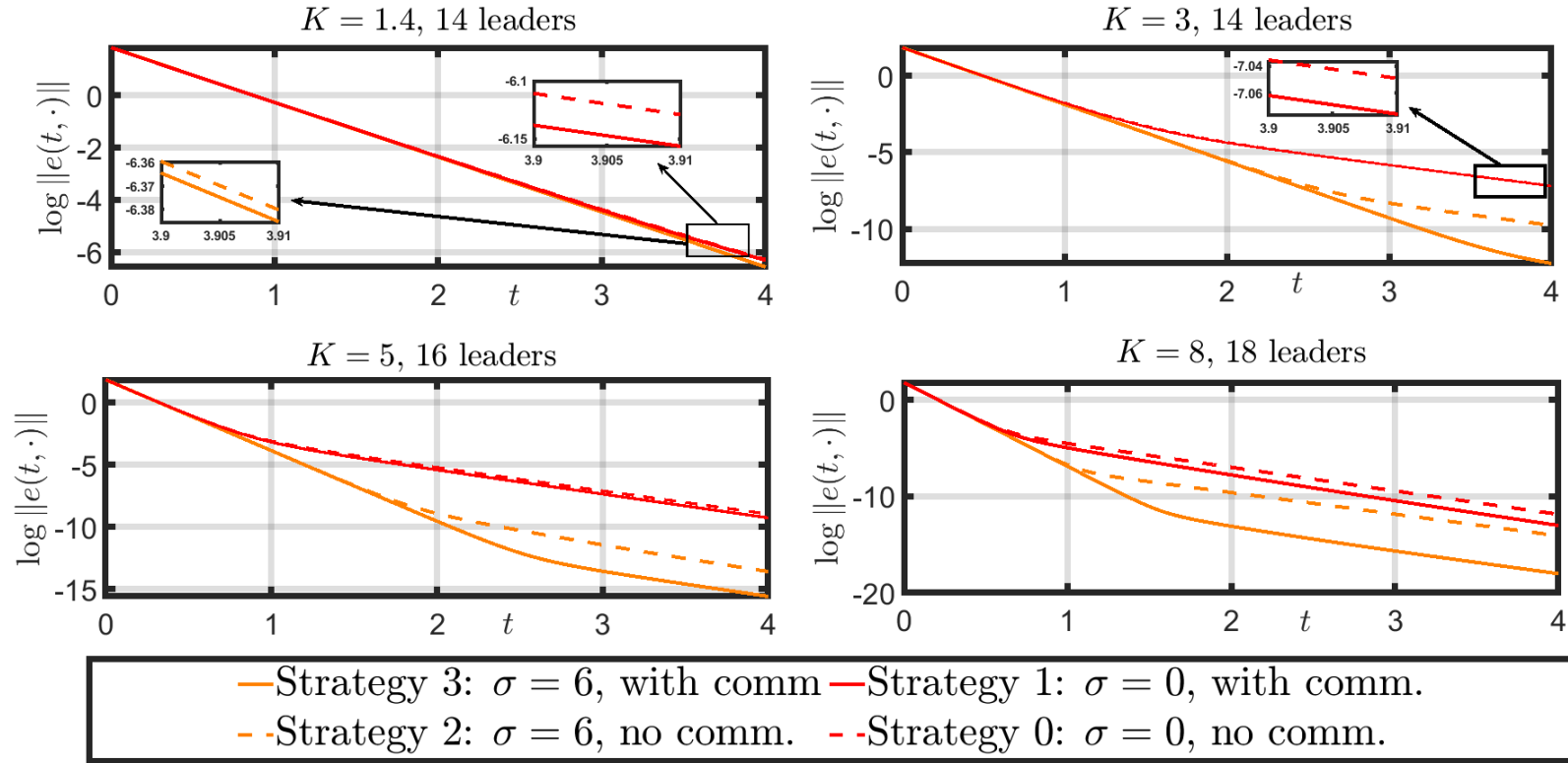


Figure 4.7: Time evolution of $\log(\|e(t, \cdot)\|)$ from equation (3.16) for $\sigma = 6$ (Wentzell boundary conditions) and $\sigma = 0$ (dynamic Dirichlet boundary conditions), each with and without leader-to-leader communication. Results are shown for $K = 1.4, 3, 5, 8$, corresponding to sufficient leader counts of 14, 14, 16, and 18 respectively, for $a = 0.001$ and system parameters as given in (4.1).

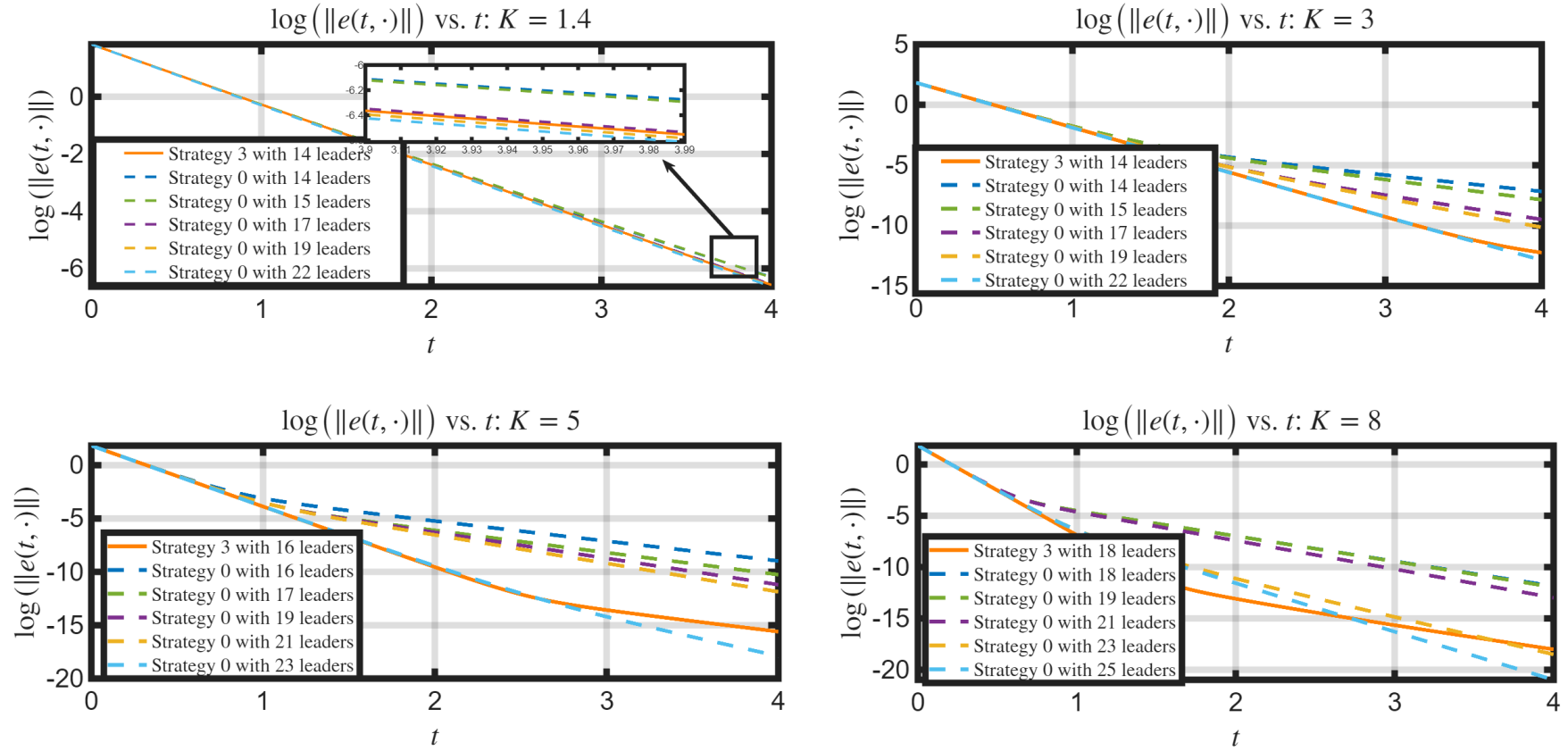


Figure 4.8: Time evolution of $\log(\|e(t, \cdot)\|)$ from (3.16) under Strategy 3 and Strategy 0. In all four cases, Strategy 0 requires more leaders to match the performance of Strategy 3. Results are shown for $K = 1.4, 3, 5, 8$, with sufficient leader counts of 14, 14, 16, and 18 respectively, for $a = 0.001$ and system parameters given in (4.1).

4.3.5 Role of Boundary Control Gain σ

Figure 4.9 shows the effect of the boundary control gain σ under **Strategy 3**, with parameters given in (4.1) and diffusion gain $a = 1 \times 10^{-3}$. The parameter σ is the control gain associated with the Wentzell boundary condition and determines how strongly the boundary leaders respond to their neighbouring agents. Intuitively, the boundary leaders are required to move towards the target curve while simultaneously remaining integrated with the swarm through interaction with their immediate neighbours. For smaller values of σ , the boundary leaders place greater emphasis on moving towards the target curve, whereas increasing σ increases the influence of neighbouring agents on the boundary dynamics and causes the boundary leaders to move more consistently with the swarm. The effect of changing σ is now examined through the simulations shown in Figure 4.9.

The left plot shows that for small and moderate values, increasing σ improves convergence, since the boundary agents move more consistently with the swarm and help the group reach the target curve faster. Increasing σ strengthens the coupling between the boundary and interior agents, which improves cohesion and reduces the tendency of the boundary leaders to separate rapidly from the swarm. However, when σ becomes very large (right plot, $\sigma = 80$), the system destabilises. This happens because the effect of σ dominates the target-tracking gain κ , so the boundary agents lose their role of guiding the swarm to the desired curve. In this case, the boundary leaders prioritise maintaining coordination with neighbouring agents more strongly than progressing towards the target curve.

At the same time, a large σ amplifies any noise or small mismatch when measuring the distance to the nearest neighbor, and this error is fed back into the swarm. Consequently, even small neighbour-measurement errors generate large corrective actions at the boundary, which destabilises the swarm dynamics. Together, these effects prevent convergence and destabilise the formation. Thus, σ must be kept moderate: large enough to aid coordination, but not so large that it overwhelms κ or magnifies neighbor-measurement noise.

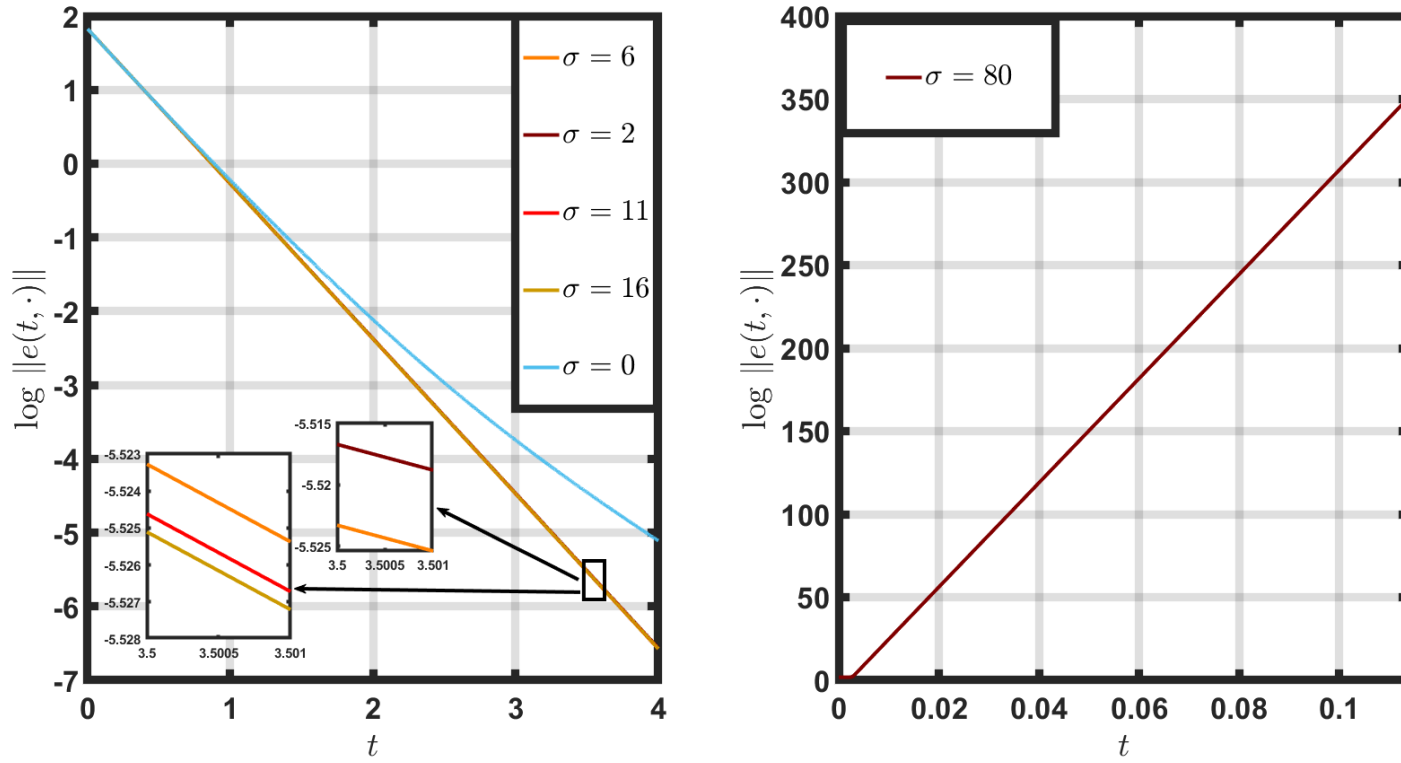


Figure 4.9: Effect of the boundary control gain σ on swarm convergence under Strategy 3. (Left) For small and moderate σ , increasing its value improves convergence as the boundary agents move more consistently with the swarm. (Right) For very large σ , the system destabilises because σ dominates the target-tracking effect of κ and amplifies neighbour-measurement noise.

4.3.6 Deployment Trajectories and Scalability

4.3.6.1 All Agents Initialised at the Origin

Figure 4.10 (left) illustrates the phase portrait of PDE-based MAS (3.16) with system parameters as specified in (4.1), and interaction strength $a = 1 \times 10^{-3}$ (see Table 4.2) in which all 141 agents are initially located at the origin. The maroon curve represents the predefined target curve, while the black lines indicate the trajectories of individual agents. Among these agents, a subset of 14 leaders, identified by indices $i = 0, 11, 22, 32, 43, 54, 65, 75, 86, 97, 108, 118, 129, 140$, are highlighted in orange. The corresponding 2D projection (right) shows the trajectory of a boundary agent, with black arrows indicating the direction of motion. The leaders effectively guide the swarm to converge and track the desired trajectory.

4.3.6.2 Boundary Agents on Target, Swarm at Origin

Figure 4.11 (left) presents a modified scenario in which only the boundary agents are initialised directly on the target curve, while all other agents start from the origin. The maroon target curve and black agent trajectories are consistent with the previous case, with leaders again shown in orange. The 2D projection (right) reveals the motion of the boundary agents, who initially approach the swarm, perform a U-turn, and then align with the group. This behaviour emphasises the boundary agent's role in enhancing swarm cohesion and transmitting more accurate positional error feedback to the swarm.

4.3.6.3 Scalability with Respect to Agent Count

Finally, we illustrate the successful deployment of agents through the phase portraits shown in Figure 4.12 for $N = 40$ and $N = 70$. These results confirm that, regardless of the number of agents, the swarm converges to the target curve under the same set of system parameters identified earlier. This highlights a key advantage of the PDE modelling framework: stability conditions and controller design remain independent of the agent count. In contrast, ODE-based formulations require repeating the stability analysis whenever N changes, making them less scalable. Thus, the PDE approach offers a robust and efficient control framework for large-scale MAS.

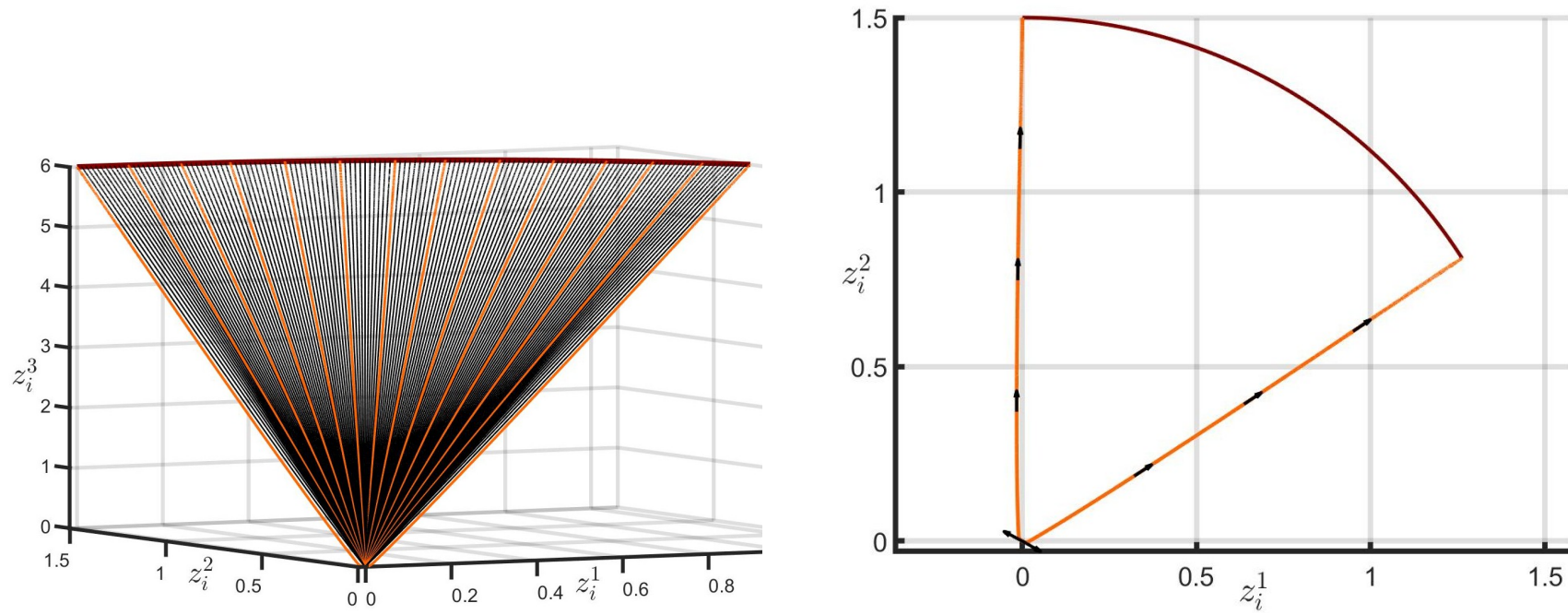


Figure 4.10: Phase portrait when all agents are initialised at the origin. The maroon curve denotes the target, and orange lines denote the leaders. The right figure shows a 2D projection of boundary agent motion with directional arrows.

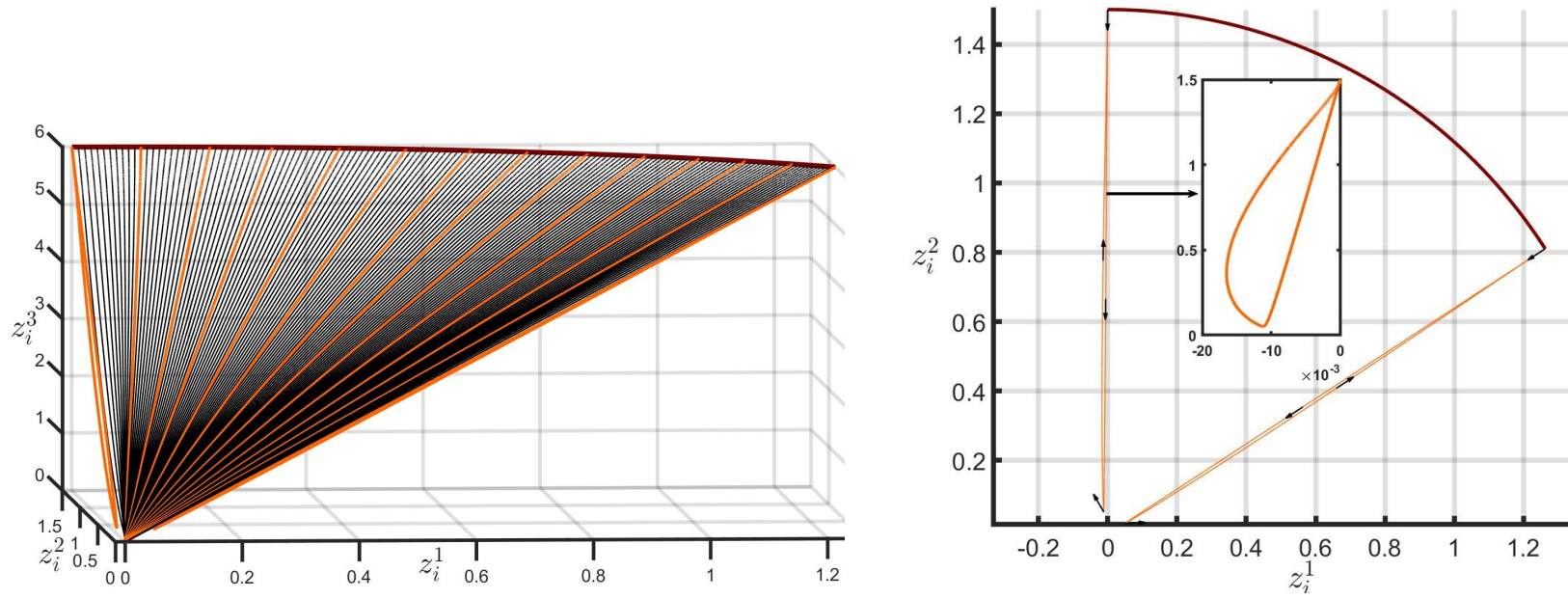


Figure 4.11: Phase portrait when only the boundary agents are initialised on the target curve and all other agents start at the origin. The boundary agents first move toward the swarm and subsequently align with it. For better visualisation, the 2D projection of the boundary agents' behaviour is shown on the right.

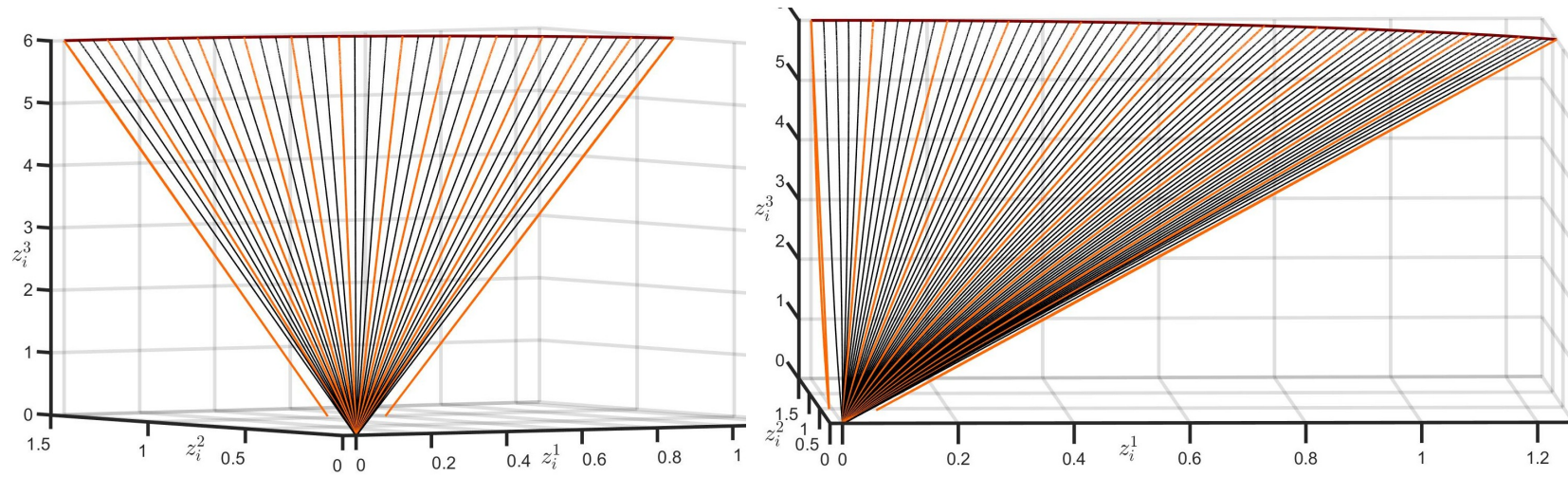


Figure 4.12: Phase portraits showing successful deployment of agents to the target curve for different swarm sizes: (left) $N = 40$ agents and (right) $N = 70$ agents, with boundary agents initially on the target curve. Stability is achieved under the same PDE-based parameter settings, demonstrating the scalability of the proposed control strategy.

4.4 Summary of Simulation Results

The simulation study provides several consistent insights into the role of boundary control and leader communication in PDE-based swarm deployment.

First, the best performance is obtained under **Strategy 3**, in which boundary leaders are governed by Wentzell conditions and inter-leader communication is enabled. Across all scenarios, this strategy yields faster convergence and requires fewer leaders than any of the alternatives.

Second, swarm cohesion is preserved under Wentzell boundary feedback. This is most clearly demonstrated in the scenario where both boundary agents are initially placed on the target curve while the remaining swarm starts at the origin. In this case, the boundary leaders first move towards the swarm, perform a U-turn, and then proceed in coordination with the group towards the target curve. This behaviour ensures that the swarm remains cohesive.

Third, the introduction of leader-to-leader communication substantially reduces the sufficient number of leaders required for stability. In several cases, the required number of leaders is almost halved compared with earlier approaches that did not employ communication.

Table 4.7: Summary of control gains in the PDE (2.23) with boundary conditions (2.24), and their qualitative effects on swarm behaviour.

Control gain	Interpretation and qualitative effect
<i>Interaction strength a</i>	Governs alignment between neighbouring agents. For small a , convergence is slow and more leaders are required. As a increases, coupling strengthens and fewer leaders are sufficient.
<i>Boundary coupling gain σ</i>	Couples boundary agents to swarm. Moderate values strengthen cohesion, but if σ is too small, cohesion is lost, while very large values destabilise the system by overwhelming target-tracking and amplifying neighbour noise.
<i>Leader propagation gain K</i>	Determines how strongly leader information spreads through the swarm. Small K makes leader accuracy critical, with linear approximations outperforming constant ones. Large K accelerates convergence but magnifies approximation errors.

Finally, the role of the various control gains can be summarised as shown in Table 4.7. Each gain has a distinct influence on the stability and collective behaviour of the swarm. The interaction strength a determines the effectiveness of local alignment; the boundary tracking gain κ anchors the swarm to the target; the boundary coupling gain σ ensures coordinated motion at the boundaries

but must remain moderate; and the leader propagation gain K controls the speed at which leader information diffuses through the swarm.

Furthermore, the PDE-based framework ensures scalability, since the same set of system parameters can be applied consistently as the swarm size increases. With these findings, we close this chapter, and in the next chapter, we turn to possible extensions of this work.

Conclusion

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This concluding chapter revisits the core problem of swarm deployment and presents a structured summary of the control framework developed to address it. A concise overview of the main results is then provided. Beyond these results, the chapter outlines two natural extensions, time-delay systems and multi-agent deployment in two-dimensional domains, along with preliminary findings: a stability theorem under time delays and novel L^2 -approximation bounds for the 2D setting. The chapter closes with a brief reflection on the broader significance and potential impact of the work.

5.1 Summary of Main Results

This section summarises the proposed control design, which forms the core contribution of the thesis, and presents the resulting system performance, highlighting improvements over existing strategies.

5.1.1 Problem Statement Revisited

In conclusion, this thesis addressed the problem of *swarm deployment*, in which a large population of agents was required to position themselves along a prescribed *target curve* without having direct access to its location. Each agent evolved according to nonlinear dynamics and relied solely on local sensing and limited communication. The objective was to design distributed control laws that guided the swarm to the desired configuration, while maintaining cohesion, ensuring robustness across a wide range of initial conditions and population sizes, and achieving faster convergence compared to existing strategies in [Wei et al. \(2019b\)](#); [Terushkin and Fridman \(2021\)](#).

5.1.2 Control Design Summary

More specifically, the control strategy developed in this thesis consists of two design stages. The first stage focuses on local coordination, under the assumptions formalised in Section 2.5. The strategy was based on a distance-based interaction topology [Oh et al. \(2015\)](#), wherein each agent measured its relative position with respect to immediate neighbours and adjusted its state accordingly. This local interaction promoted swarm cohesion, enforced uniform inter-agent spacing, enabled distributed coordination, and compensated for the agents' nonlinear dynamics. Importantly, it ensured that the convergence of any agent to its target position influenced neighbouring agents to do the same. For agents located at the swarm boundaries, dedicated coordination rules were designed (see Section 2.7) that exploited both their access to positional error and relative measurements with adjacent interior agents. In the limit of large swarm sizes, this local coordination framework admitted a *continuum approximation*, giving rise to a PDE model for the collective error dynamics subject to Wentzell boundary conditions.

The second design aspect addressed global coordination, as presented in Section 2.8. This was achieved by adapting the spatial decomposition method, originally developed for in-domain stabilisation of semilinear parabolic PDEs [Fridman and Blighovsky \(2012\)](#). A sparse subset of agents, termed *leaders*, was assigned the role of estimating their own positional error relative to the target and propagating this information throughout the swarm. These leaders, distributed across the spatial domain including the boundaries, were allowed to communicate with one another. Followers, which could not access their own positional error, approximated it using a linear interpolation of the error information received from adjacent leaders. This mechanism enabled the swarm to

construct a piecewise-linear approximation of the true PDE error state, resulting in faster convergence and more accurate positional error correction compared to earlier strategies based on non-communicating leaders [Wei et al. \(2019a\)](#); [Terushkin and Fridman \(2021\)](#).

5.1.3 Result 1

This thesis demonstrated that enabling communication between leaders leads to significantly faster convergence of the swarm to the target curve compared to earlier strategies in which leaders operate independently [Wei et al. \(2019a\)](#); [Terushkin and Fridman \(2021\)](#). This improvement is supported by numerical simulations, as illustrated in Figures 4.2 and 4.3, which show the evolution of the logarithmic error norm $\|\log(e(t, \cdot))\|$ for the PDE system (3.16) under two scenarios: (i) when all agents are initially at the origin, and (ii) when only the boundary leaders are placed on the target while the rest of the swarm starts at the origin. In both cases, the solid red curve (leader communication enabled) exhibits substantially faster decay than the dashed red curve (non-communicating leaders), confirming the benefit of the proposed approach.

This performance arises from the swarm's ability to construct a piecewise-linear approximation of the PDE error state when leaders are allowed to communicate, as opposed to the piecewise-constant approximation used in previous methods. With a more accurate error representation, followers receive improved positional error estimates, resulting in more effective control actions and quicker convergence.

5.1.4 Result 2

This thesis demonstrated that replacing dynamic Dirichlet conditions with Wentzell boundary control significantly improved swarm cohesion and convergence. This improvement is supported by simulation results shown in Figures 4.2, 4.3, and 4.11. Under Dirichlet boundary conditions, the boundary leaders moved rapidly towards the target curve (see the top panel of Figure 4.2), often detaching from the swarm. This undermined cohesion, especially when boundary leaders were initially placed on the target curve, where they remained fixed and offered zero guidance to neighbouring agents in the absence of leader communication. By contrast, the Wentzell boundary formulation introduced in this thesis ensured that boundary leaders remained dynamically coupled to the swarm. When initially positioned on the target, these agents first move toward the swarm

to maintain cohesion and then return to the curve in coordination with the rest of the swarm, as shown in Figure 4.11.

Furthermore, the Wentzell boundary control resolved a key limitation of earlier designs in the absence of leader communication. Under Dirichlet conditions, boundary leaders initially placed on the target remained stationary, causing nearby followers to adopt a zero-error estimate, which slowed down the deployment. In contrast, the Wentzell formulation caused these boundary agents to first move towards the swarm, enabling nearby followers to receive a non-zero and more accurate estimate of their positional error. This effect is confirmed in Figure 4.3, where the dashed orange curve (Wentzell control without leader communication) shows faster convergence than the dashed red curve (Dirichlet control without leader communication).

5.1.5 Result 3

This thesis further established two novel L^2 -norm bounds on the difference between the PDE error state and its piecewise-linear approximation (Section 2.8). By embedding these bounds into a Lyapunov-based stability framework, a set of LMIs was derived that characterises sufficient conditions for successful deployment. The numerical results demonstrated that the number of leaders required to satisfy these conditions is significantly lower under the proposed strategy than in the original methods of Wei et al. (2019a); Terushkin and Fridman (2021). For instance, as highlighted in maroon in Table 4.2, deployment was guaranteed with only 14 leaders under the proposed design, whereas 25 leaders were required under the earlier non-communicating Dirichlet-based strategy. It is important to note that these values correspond to sufficient, not minimal, leader configurations that ensure convergence within the derived theoretical guarantees.

5.2 Preliminary Results Towards Future Work

5.2.1 Future Work 1: Time Delays

An important direction explored during this PhD concerns the role of time delays in the global coordination strategy proposed in this thesis. In practice, it is natural to expect that the information propagated from leaders to followers, specifically, the leaders' positional error, may be received with

communication or processing delays. This motivates the study of the PDE system (2.23)–(2.24) under delay. In particular, when leader communication is subject to a time-varying delay $\tau(t)$ with

$$0 \leq \tau(t) \leq \tau_M, \quad \dot{\tau}(t) \leq \vartheta < 1, \quad (5.1)$$

where $\tau_M > 0$ is the maximum allowable delay, and $\vartheta \in (0, 1)$ is a constant that bounds the rate of change of the delay, the construction of the piecewise-linear approximation of the PDE error state is modified accordingly:

$$e(t - \tau(t), x) \approx \sum_{j=0}^M y_j(t - \tau(t)) b_j(x), \quad (5.2)$$

where $b_j(x)$ are the shape functions defined in (2.20) for $x \in [0, 1]$, and $y_j(t - \tau(t))$ denotes the delayed positional error of the j -th leader. Based on this, the global coordination controller is expressed as

$$v(t - \tau(t), x) = -K \sum_{j=0}^M b_j(x) y_j(t - \tau(t)) = -K e(t - \tau(t), x) + K \nu(t - \tau(t), x), \quad (5.3)$$

where $K > 0$ is the leader information gain, and the approximation error is defined as

$$\nu(t - \tau(t), x) := e(t - \tau(t), x) - \sum_{j=0}^M y_j(t - \tau(t)) b_j(x).$$

The resulting closed-loop error dynamics of the PDE system with delay are given by

$$e_t(t, x) = a e_{xx}(t, x) + F(t, x, e) - K e(t, x) + K e_d(t, x) + K \nu(t - \tau(t), x), \quad (5.4)$$

where

$$e_d(t, x) := e(t, x) - e(t - \tau(t), x),$$

subject to the Wentzell boundary conditions

$$\begin{aligned} e_t(t, 0) &= -(\kappa + K) e(t, 0) + \sigma e_x(t, 0) + F(t, 0, e(t, 0)), \\ e_t(t, 1) &= -(\kappa + K) e(t, 1) - \sigma e_x(t, 1) + F(t, 1, e(t, 1)). \end{aligned} \quad (5.5)$$

In the delayed setting, the approximation error $\nu(t - \tau(t), x)$ satisfies the inequalities

$$\|\nu(t - \tau(t), \cdot)\|_{[0,1]}^2 \leq \frac{\Delta^2}{\pi^2} \|e_x(t - \tau(t), \cdot)\|_{[0,1]}^2, \quad (5.6)$$

$$\|\nu(t - \tau(t), \cdot)\|_{[0,1]}^2 \leq \frac{\Delta^4}{\pi^4} \|e_{xx}(t - \tau(t), \cdot)\|_{[0,1]}^2. \quad (5.7)$$

5.2.1.1 Preliminary Theorem: Stability Conditions

Before we present results on time delays, we recall Halanay's inequality. This classical result will be instrumental in compensating certain troublesome terms that arise in the Lyapunov analysis of time-delay systems, and will subsequently allow us to establish exponential stability.

Lemma 5.1 (Halanay (1966) Halanay's Inequality). *Let $0 < \delta_1 < 2\delta$ and let $V : [t_0 - h, \infty) \rightarrow [0, \infty)$ be an absolutely continuous function that satisfies*

$$\dot{V}(t) \leq -2\delta V(t) + \delta_1 \sup_{-h \leq \theta \leq 0} V(t + \theta), \quad t \geq t_0. \quad (5.8)$$

Then

$$V(t) \leq e^{-2\alpha(t-t_0)} \sup_{-h \leq \theta \leq 0} V(t_0 + \theta), \quad t \geq t_0, \quad (5.9)$$

where $\alpha > 0$ is a unique positive solution of

$$\alpha = \delta - \frac{\delta_1 e^{2\alpha h}}{2}. \quad (5.10)$$

We now apply this result to the closed-loop system under consideration.

Theorem 5.2. *Consider the time-delayed closed-loop system (5.4), (5.5), with interaction strength $a > 0$, controller gains $K > 0$, $\sigma > 0$, $\kappa > 0$, and maximum inter-leader spacing $\Delta > 0$. Assume that the nonlinearity $F(t, x, e)$ satisfies the Lipschitz condition (2.14) with Lipschitz constant $L > 0$, and that the time-varying delay $\tau(t)$ is bounded as $0 \leq \tau(t) \leq \tau_M$ with $\dot{\tau}(t) \leq \vartheta < 1$, for some constants $\tau_M > 0$ and $\vartheta \in (0, 1)$. Fix a desired exponential decay rate $\delta > 0$, and let $\delta_1 > 0$ satisfy $2\delta > \delta_1$. If there exist positive scalars $p, \bar{p}, \mu, \bar{q}, r, g, \bar{g}$, and λ_i for $i \in \{0, 1, 2, 3\}$, such that the*

following LMIs hold:

$$\lambda_0 \frac{\Delta^2}{\pi^2} - \delta_1 \mu a \leq 0, \quad \lambda_1 \frac{\Delta^4}{\pi^4} - \bar{g}(1 - \vartheta) e^{-2\delta\tau_M} \leq 0, \quad \mathcal{M} = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & p \\ * & -2\sigma\mu a & -\mu a \\ * & * & -\lambda_3 \end{bmatrix} \leq 0, \quad (5.11)$$

and

$$\mathcal{N} = \begin{bmatrix} -2\mu + \tau_M^2 r & 0 & 0 & \bar{p} & \mu K & \mu K & \mu \\ * & \mathcal{N}_{22} & 0 & 0 & \bar{q} K & \bar{q} K & \bar{q} \\ * & * & \mathcal{N}_{33} & 0 & 0 & 0 & 0 \\ * & * & * & -2a\bar{p} + \bar{g} & -\bar{p} K & -\bar{p} K & -\bar{p} \\ * & * & * & * & -\tau_M r e^{-2\delta\tau_M} & 0 & 0 \\ * & * & * & * & * & -\lambda_0 - \lambda_1 & 0 \\ * & * & * & * & * & * & -\lambda_2 \end{bmatrix} \leq 0, \quad (5.12)$$

where

$$\mathcal{M}_{11} = -2p(\kappa + K - \delta) + \lambda_3 L^2,$$

$$\mathcal{M}_{12} = \mu a(\kappa + K) - a\bar{q} - K\bar{p} + p\sigma,$$

$$\mathcal{N}_{22} = -2((K - \delta)\bar{q} - \delta K\mu) + g + \lambda_2 L^2,$$

$$\mathcal{N}_{33} = -2(a\bar{q} + K\bar{p} - \delta\mu a),$$

then a unique strong solution to the PDE (5.4) with boundary conditions (5.5), initialized with

$$e(t_0, \cdot) \in H^1(0, 1), \quad e(t, x) \equiv 0 \quad \text{for } t < t_0, \quad (5.13)$$

satisfies the following exponential decay estimate:

$$q \|e(t, \cdot)\|^2 + \mu a \|e_x(t, \cdot)\|^2 \leq e^{-2\alpha(t-t_0)} [q \|e(t_0, \cdot)\|^2 + \mu a \|e_x(t_0, \cdot)\|^2 + p (e^2(t_0, 0) + e^2(t_0, 1))],$$

for all $t \geq t_0$, where $\alpha > 0$ is the unique positive solution to (5.10).

Proof. For notational simplicity, we suppress explicit dependence on the spatial variable in inner products and norms. For instance, we write $\langle e, e_t \rangle := \langle e(t, \cdot), e_t(t, \cdot) \rangle$ and $\|e\|^2 := \|e(t, \cdot)\|^2$, with all expressions understood as functions over the spatial domain $[0, 1]$ at fixed time t .

To analyse the stability of the delayed PDE system (5.4) with boundary conditions (5.5), we construct the following *Lyapunov–Krasovskii functional*, which accounts for both the spatial dynamics and the time-varying delay:

$$\begin{aligned}
 V(t) = & \underbrace{q\|e(t, \cdot)\|^2 + \mu a\|e_x(t, \cdot)\|^2}_{V_1(t)} + \underbrace{p(e^2(t, 0) + e^2(t, 1))}_{V_B(t)} \\
 & + \underbrace{\tau_M r \int_0^1 \int_{-\tau_M}^0 \int_{t+\theta}^t e^{2\delta(s-t)} e_s^2(s, x) ds d\theta dx}_{V_{\text{del},1}(t)} \\
 & + \underbrace{g \int_0^1 \int_{t-\tau_M}^t e^{2\delta(s-t)} e^2(s, x) ds dx}_{V_{\text{del},2}(t)} + \underbrace{\bar{g} \int_0^1 \int_{t-\tau(t)}^t e^{2\delta(s-t)} e_{xx}^2(s, x) ds dx}_{V_{\text{del},3}(t)}.
 \end{aligned}$$

We begin by computing the time derivative of $V_1(t)$, which yields $\dot{V}_1(t) = 2q \langle e(t, \cdot), e_t(t, \cdot) \rangle + 2\mu a \langle e_x(t, \cdot), e_{xt}(t, \cdot) \rangle$. Next, observe that the following holds:

$$2 \langle \bar{q}e + \mu e_t - \bar{p}e_{xx}, -e_t + ae_{xx} - Ke + F(t, x, e) + Ke_d + K\nu(t - \tau(t), x) \rangle = 0. \quad (5.14)$$

We now add (5.14) to the expression for $\dot{V}_1(t)$, thereby obtaining:

$$\begin{aligned}
 \dot{V}_1(t) &= 2q \langle e(t, \cdot), e_t(t, \cdot) \rangle + 2\mu a \langle e_x(t, \cdot), e_{xt}(t, \cdot) \rangle + (5.14) \\
 &= 2q \langle e, e_t \rangle + 2\mu a \langle e_x, e_{xt} \rangle - 2 \langle \bar{q}e + \mu e_t - \bar{p}e_{xx}, e_t \rangle \\
 &\quad + 2 \langle \bar{q}e + \mu e_t - \bar{p}e_{xx}, ae_{xx} - Ke \rangle \\
 &\quad + 2 \langle \bar{q}e + \mu e_t - \bar{p}e_{xx}, F(t, x, e) + Ke_d + K\nu(t - \tau(t), x) \rangle.
 \end{aligned}$$

Rearranging terms yields:

$$\begin{aligned}
 \dot{V}_1(t) &= 2(q - \bar{q} - K\mu) \langle e, e_t \rangle + 2\mu a \langle e_x, e_{xt} \rangle - 2\mu \|e_t\|^2 + 2(\mu a + \bar{p}) \langle e_{xx}, e_t \rangle \\
 &\quad - 2K\bar{q} \|e\|^2 - 2a\bar{p} \|e_{xx}\|^2 + 2(a\bar{q} + K\bar{p}) \langle e_{xx}, e \rangle \\
 &\quad + 2 \langle \bar{q}e + \mu e_t - \bar{p}e_{xx}, F(t, x, e) + Ke_d + K\nu(t - \tau(t), x) \rangle.
 \end{aligned}$$

Upon substituting $q = \bar{q} + K\mu$, and applying integration by parts, we observe that $2\mu a \langle e_{xx}, e_t \rangle = 2\mu a e_x e_t|_0^1 - 2\mu a \langle e_x, e_{xt} \rangle$, and $2(a\bar{q} + K\bar{p}) \langle e_{xx}, e \rangle = 2(a\bar{q} + K\bar{p}) e_x e|_0^1 - 2(a\bar{q} + K\bar{p}) \|e_x\|^2$.

Substituting the boundary conditions (5.5) and the above relations into the expression for $\dot{V}_1(t)$, and grouping terms with an exponential weighting, we obtain the following differential inequality:

$$\begin{aligned}
\dot{V}_1(t) + 2\delta V_1(t) = & -2\mu \|e_t\|^2 - 2((K - \delta)\bar{q} - \delta K\mu) \|e\|^2 - 2(a\bar{q} + K\bar{p} - \delta\mu a) \|e_x\|^2 \\
& - 2a\bar{p} \|e_{xx}\|^2 + 2\bar{p} \langle e_{xx}, e_t \rangle \\
& + 2 \langle \bar{q}e + \mu e_t - \bar{p}e_{xx}, F + Ke_d + K\nu(t - \tau(t), \cdot) \rangle \\
& + 2(\mu a(\kappa + K) - a\bar{q} - K\bar{p}) \langle \eta, \eta_x \rangle - 2\sigma\mu a \|\eta_x\|^2 - 2\mu a \langle \eta_x, \eta_F \rangle,
\end{aligned} \tag{5.15}$$

where $\eta(t) := \begin{bmatrix} e(t, 0) \\ e(t, 1) \end{bmatrix}$, $\eta_x(t) := \begin{bmatrix} e_x(t, 0) \\ -e_x(t, 1) \end{bmatrix}$, $\eta_F(t) := \begin{bmatrix} F(t, e(t, 0)) \\ F(t, e(t, 1)) \end{bmatrix}$.

Differentiating $V_{\text{del},1}$ we obtain

$$\dot{V}_{\text{del},1}(t) = \tau_m r \int_0^1 \int_{-\tau_m}^0 [e_t^2(t, x) - e^{2\delta\theta} e_t^2(t + \theta, x)] d\theta dx - 2\delta V_{\text{del},1}(t). \tag{5.16}$$

Changing variables $s = t + \theta$ in the second integral ($\theta \in [-\tau_m, 0] \iff s \in [t - \tau_m, t]$) gives

$$\dot{V}_{\text{del},1}(t) + 2\delta V_{\text{del},1}(t) = \tau_m^2 r \|e_t(t, \cdot)\|^2 - \tau_m r \int_0^1 \int_{t-\tau_m}^t e^{2\delta(s-t)} e_s^2(s, x) ds dx. \tag{5.17}$$

By Jensen's inequality [Gu et al. \(2004\)](#) we have

$$\begin{aligned}
\dot{V}_{\text{del},1}(t) + 2\delta V_{\text{del},1}(t) &= \tau_m^2 r \|e_t\|^2 - \tau_m r \int_0^1 \int_{t-\tau_m}^t e^{2\delta(s-t)} e_s^2(s, x) ds dx \\
&= \tau_m^2 r \|e_t\|^2 - \tau_m r e^{-2\delta\tau_m} \int_0^1 \left(\int_{t-\tau_m}^{t-\tau(t)} + \int_{t-\tau(t)}^t \right) e_s^2(s, x) ds dx \\
&\leq \tau_m^2 r \|e_t\|^2 - r e^{-2\delta\tau_m} \int_0^1 \left(\int_{t-\tau_m}^{t-\tau(t)} e_s(s, x) ds \right)^2 dx \\
&\quad - r e^{-2\delta\tau_m} \int_0^1 \left(\int_{t-\tau(t)}^t e_s(s, x) ds \right)^2 dx \\
&= \tau_m^2 r \|e_t\|^2 - r e^{-2\delta\tau_m} \|e(t - \tau(t), \cdot) - e(t - \tau_m, \cdot)\|^2 \\
&\quad - r e^{-2\delta\tau_m} \|e(t, \cdot) - e(t - \tau(t), \cdot)\|^2 \\
&\leq \tau_m^2 r \|e_t\|^2 - r e^{-2\delta\tau_m} \|e_d(t, \cdot)\|^2.
\end{aligned} \tag{5.18}$$

Differentiating $V_{\text{del},2}$, and $V_{\text{del},3}$ we obtain

$$\begin{aligned} \dot{V}_{\text{del},2}(t) + \dot{V}_{\text{del},3}(t) &= g \int_0^1 \left[e^2(t, x) - e^{-2\delta\tau_m} e^2(t - \tau_m, x) \right] dx - 2\delta V_{\text{del},2} + \bar{g} \|e_{xx}(t, \cdot)\|^2 \\ &\quad - \bar{g}(1 - \dot{\tau}(t)) e^{-2\delta\tau(t)} \|e_{xx}(t - \tau(t), \cdot)\|^2 - 2\delta V_{\text{del},3}(t). \end{aligned}$$

Using the assumption $\dot{\tau}(t) \leq \vartheta$, for some constant $\vartheta < 1$, and neglecting $-ge^{-2\delta\tau_m} \|e(t - \tau_m, \cdot)\|^2$, we obtain

$$\begin{aligned} \dot{V}_{\text{del},2}(t) + \dot{V}_{\text{del},3}(t) + 2\delta(V_{\text{del},2}(t) + V_{\text{del},3}(t)) &\leq g \|e(t, \cdot)\|^2 + \bar{g} \|e_{xx}(t, \cdot)\|^2 \\ &\quad - \bar{g}(1 - \vartheta) e^{-2\delta\tau_m} \|e_{xx}(t - \tau(t), \cdot)\|^2. \end{aligned} \quad (5.19)$$

From (2.14), (5.6), and (5.7), we have

$$0 \leq \lambda_0 \left(\frac{\Delta^2}{\pi^2} \|e_x(t - \tau(t), \cdot)\|^2 - \|\nu(t - \tau(t), \cdot)\|^2 \right), \quad \lambda_0 > 0, \quad (5.20a)$$

$$0 \leq \lambda_1 \left(\frac{\Delta^4}{\pi^4} \|e_{xx}(t - \tau(t), \cdot)\|^2 - \|\nu(t - \tau(t), \cdot)\|^2 \right), \quad (5.20b)$$

$$0 \leq \lambda_2 (L^2 \|e(t, \cdot)\|^2 - \|F(t, \cdot, e(t, \cdot))\|^2), \quad (5.20c)$$

$$0 \leq \lambda_3 (L^2 |\eta(t)|^2 - |\eta_F(t)|^2), \quad (5.20d)$$

Differentiating V_B along the trajectories of (2.24), we find

$$\begin{aligned} \dot{V}_B + 2\delta V_B &\leq 2p\eta^T(t)\eta_t(t) + 2p\delta|\eta(t)|^2 + (5.20d) \\ &= (-2p(\kappa + K - \delta) + \lambda_3 L^2) |\eta(t)|^2 + 2p\sigma\eta^T(t)\eta_x(t) + 2p\eta^T(t)\eta_F(t) \\ &\quad - \lambda_3 |\eta_F(t)|^2. \end{aligned} \quad (5.21)$$

Taking into account the estimates (5.15), (5.18), (5.19), and (5.21), and setting

$$\begin{aligned} \zeta(t, x) &:= \text{col}\{e_t(t, x), e(t, x), e_x(t, x), e_{xx}(t, x), e_d(t, x), \nu(t - \tau(t), x)\}, \\ \zeta_B(t) &:= \text{col}\{\eta(t), \eta_x(t), \eta_F(t)\}, \end{aligned}$$

we obtained the following inequality:

$$\begin{aligned} \dot{V}(t) + 2\delta V(t) &\leq \dot{V}(t) + 2\delta V(t) + (5.20a) + (5.20b) + (5.20c) + (5.20d) \\ &= \int_0^1 \zeta^T(t, x)(\mathcal{N} \otimes I_n) \zeta(t, x) dx + \zeta_B^T(t)(\mathcal{M} \otimes I_{2n})\zeta_B(t) \\ &\quad - \left(\bar{g}(1 - \vartheta)e^{-2\delta\tau_m} - \lambda_1 \frac{\Delta^4}{\pi^4} \right) \|e_{xx}(t - \tau(t), \cdot)\|^2 + \lambda_0 \frac{\Delta^2}{\pi^2} \|e_x(t - \tau(t), \cdot)\|^2. \end{aligned}$$

To compensate for the term $\lambda_0 \frac{\Delta^2}{\pi^2} \|e_x(t - \tau(t), \cdot)\|^2$, note that

$$\sup_{\theta \in [-\tau_M, 0]} V(t + \theta) = \sup_{s \in [t - \tau_M, t]} V(s).$$

Since $t - \tau(t) \in [t - \tau_M, t]$, it follows that

$$V(t - \tau(t)) \leq \sup_{\theta \in [-\tau_M, 0]} V(t + \theta).$$

Hence, for any $\delta_1 > 0$,

$$-\delta_1 \sup_{\theta \in [-\tau_M, 0]} V(t + \theta) \leq -\delta_1 V(t - \tau(t)) \leq -\delta_1 \mu a \|e_x(t - \tau(t), \cdot)\|^2.$$

Therefore, we obtain

$$\begin{aligned} \dot{V}(t) + 2\delta V(t) - \delta_1 \sup_{\theta \in [-\tau_M, 0]} V(t + \theta) &\leq \int_0^1 \zeta^T(t, x)(\mathcal{N} \otimes I_n) \zeta(t, x) dx \\ &\quad + \zeta_B^T(t)(\mathcal{M} \otimes I_{2n})\zeta_B(t) \\ &\quad - \left(\bar{g}(1 - \vartheta)e^{-2\delta\tau_m} - \lambda_1 \frac{\Delta^4}{\pi^4} \right) \|e_{xx}(t - \tau(t), \cdot)\|^2 \\ &\quad - \left(\delta_1 \mu a - \lambda_0 \frac{\Delta^2}{\pi^2} \right) \|e_x(t - \tau(t), \cdot)\|^2. \end{aligned} \tag{5.22}$$

Application of Halanay's inequality, together with $V(t_0) = \sup_{\theta \in [-\tau_M, 0]} V(t_0 + \theta)$ (cf. (5.13)), completes the proof.

□

5.2.2 Future Work 2: MAS in 2D Domain

Extending the control strategies developed in this thesis to two-dimensional domains presents a promising direction for future research. While the deployment framework introduced in [Wei et al. \(2019a\)](#); [Terushkin and Fridman \(2021\)](#) has not yet been generalised to MAS operating in 2D spatial domains, related work in [Selivanov and Fridman \(2019\)](#) has considered 2D diffusion-based control for PDE systems.

The 2D extension may be formulated by considering a MAS composed of $(N + 1)^2$ agents, indexed over a two-dimensional grid, where each agent evolves according to nonlinear dynamics:

$$\dot{z}_{i,j}(t) = f(t, z_{i,j}(t)) + U_{i,j}(t), \quad i, j \in \mathcal{I} := \{0, 1, \dots, N\}, \quad (5.23)$$

with $z_{i,j}(t) \in \mathbb{R}^n$, and $f : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. The control objective is to design distributed control law $U_{i,j}(t)$ that steer the agents to a prescribed 2D surface $\gamma : [0, 1]^2 \rightarrow \mathbb{R}^n$, such that $\lim_{t \rightarrow \infty} z_{i,j}(t) = \gamma(i/N, j/N)$, $\forall i, j \in \mathcal{I}$.

We envision extending the control framework developed in Chapter 2 to the 2D case by following a similar decomposition of the distributed control law: $U_{i,j}(t) = \underbrace{u_{i,j}(t)}_{\text{local coordination}} + \underbrace{v_{i,j}(t)}_{\text{global coordination}}$. To facilitate this, the original assumptions in Chapter 2 must be suitably adapted for the two-dimensional setting:

- B1.** The nonlinear function f is differentiable with respect to z and satisfies a global Lipschitz condition: $\|f_z(t, z)\| \leq L$ for some $L > 0$.
- B2.** Each agent $(i, j) \in \{1, \dots, N - 1\}^2$ measures its relative positions with respect to its four nearest neighbours. That is, agent (i, j) can measure

$$z_{i,j}(t) - z_{i-1,j}(t), \quad z_{i+1,j}(t) - z_{i,j}(t), \quad z_{i,j+1}(t) - z_{i,j}(t), \quad z_{i,j}(t) - z_{i,j-1}(t).$$

It also knows the corresponding desired relative positions,

$$\gamma_{i,j} - \gamma_{i-1,j}, \quad \gamma_{i+1,j} - \gamma_{i,j}, \quad \gamma_{i,j} - \gamma_{i,j-1}, \quad \gamma_{i,j+1} - \gamma_{i,j},$$

as well as the local target dynamics $f(t, \gamma_{i,j})$.

B3. Leaders can measure the difference between their own position and their prescribed target position, i.e. $z_{i,j}(t) - \gamma_{i,j}$, and know $f(t, \gamma_{i,j})$.

Then, similar to the local control design in equation (2.8) for the 1D case, the local controller for agents $(i, j) \in \{1, \dots, N-1\}^2$ in the 2D domain is given by:

$$u_{i,j}(t) = \frac{a}{h^2} [(z_{i+1,j}(t) - z_{i,j}(t)) + (z_{i-1,j}(t) - z_{i,j}(t)) + (z_{i,j+1}(t) - z_{i,j}(t)) + (z_{i,j-1}(t) - z_{i,j}(t))] \\ - \frac{a}{h^2} [(\gamma_{i+1,j} - \gamma_{i,j}) + (\gamma_{i-1,j} - \gamma_{i,j}) + (\gamma_{i,j+1} - \gamma_{i,j}) + (\gamma_{i,j-1} - \gamma_{i,j})] - f(t, \gamma_{i,j}).$$

We now derive a continuum model for MAS in the 2D domain. Substituting the 2D local control law into the agent dynamics yields the following closed-loop update for each interior agent $(i, j) \in \{1, \dots, N-1\}^2$:

$$\dot{z}_{i,j}(t) = \frac{a}{h^2} [(z_{i+1,j} + z_{i-1,j} + z_{i,j+1} + z_{i,j-1} - 4z_{i,j}) \\ - (\gamma_{i+1,j} + \gamma_{i-1,j} + \gamma_{i,j+1} + \gamma_{i,j-1} - 4\gamma_{i,j})] + f(t, z_{i,j}) - f(t, \gamma_{i,j}) + v_{i,j}(t).$$

To study the large-scale behavior of the swarm in two dimensions, we introduce continuous spatial and temporal variables by defining distributed state and control fields $z(t, x, y), v(t, x, y): [0, \infty) \times [0, 1]^2 \rightarrow \mathbb{R}^n$ such that

$$z(t, ih, jh) := z_{i,j}(t), \quad v(t, ih, jh) := v_{i,j}(t), \quad \text{where } h = 1/N.$$

Passing to the continuum limit as $h \rightarrow 0$, and setting $z(t, x, y) \approx z_{i,j}(t)$, $v(t, x, y) \approx v_{i,j}(t)$, we obtain the following semilinear parabolic PDE:

$$z_t(t, x, y) = a [z_{xx} + z_{yy} - \gamma_{xx} - \gamma_{yy}] + f(t, z(t, x, y)) - f(t, \gamma(x, y)) + v(t, x, y),$$

for $(x, y) \in [0, 1]^2$, $t \geq 0$. Letting $e(t, x, y) := z(t, x, y) - \gamma(x, y)$, the resulting PDE for the error dynamics is:

$$e_t(t, x, y) = a \Delta e(t, x, y) + F(t, x, y, e) + v(t, x, y), \quad (5.24)$$

where $\Delta := \partial_{xx} + \partial_{yy}$ is the Laplacian in 2D, and $F(t, x, y, e) := f(t, \gamma(x, y) + e) - f(t, \gamma(x, y))$.

We now extend the dynamic boundary leader behaviour to the two-dimensional setting. As in the 1D case, boundary agents track their assigned target positions while remaining coupled to the

swarm through feedback from neighbouring agents. This leads to Wentzell boundary conditions on each edge of the spatial domain $[0, 1]^2$. The deployment error then satisfies the following dynamic boundary conditions:

$$\begin{aligned}
e_t(t, 0, y) + \kappa e(t, 0, y) - \sigma e_x(t, 0, y) - F(t, 0, y, e(t, 0, y)) - v(t, 0, y) &= 0, \\
e_t(t, 1, y) + \kappa e(t, 1, y) + \sigma e_x(t, 1, y) - F(t, 1, y, e(t, 1, y)) - v(t, 1, y) &= 0, \\
e_t(t, x, 0) + \kappa e(t, x, 0) - \sigma e_y(t, x, 0) - F(t, x, 0, e(t, x, 0)) - v(t, x, 0) &= 0, \\
e_t(t, x, 1) + \kappa e(t, x, 1) + \sigma e_y(t, x, 1) - F(t, x, 1, e(t, x, 1)) - v(t, x, 1) &= 0,
\end{aligned} \tag{5.25}$$

where $\kappa > 0$ and $\sigma > 0$ denote the tracking and coupling gains, respectively. The sign of each spatial derivative term reflects the outward unit normal direction on each edge, preserving the physical interpretation of diffusive coupling.

Global Coordination Strategy (2D)

By arguments analogous to the 1D case in Chapter 2, the PDE system (5.24)–(5.25) is unstable in open loop. We therefore adopt the spatial decomposition approach used in Chapter 2 for 1D, and divide the square domain $D = [0, 1]^2$ into a finite collection of non-overlapping triangular regions $\{T_k\}_{k=1}^M$. Specifically, for any two distinct triangles T_i and T_j , we require

$$\text{int}(T_i) \cap \text{int}(T_j) = \emptyset, \quad \text{for all } i \neq j,$$

where $\text{int}(T_k)$ denotes the interior of triangle $T_k \subset D$. Adjacent triangles may share a common vertex or edge, but no interior points. Leaders are placed at the vertices of the triangular regions and can measure their positional error:

$$y_k(t) := e(t, x_k, y_k) = z(t, x_k, y_k) - \gamma(x_k, y_k), \quad (x_k, y_k) \in \mathcal{V},$$

where $\mathcal{V}$ denotes the set of all triangle vertices. Within each triangle $T \subset D$ with vertices $(x_{k_1}, y_{k_1}), (x_{k_2}, y_{k_2}), (x_{k_3}, y_{k_3}) \in \mathcal{V}$, the corresponding leaders exchange their measurements and construct a linear approximation of the error state:

$$\bar{e}_T(t, x, y) := \lambda_1(x, y) y_{k_1}(t) + \lambda_2(x, y) y_{k_2}(t) + \lambda_3(x, y) y_{k_3}(t), \quad (x, y) \in T, \tag{5.26}$$

where the weights $\lambda_1(x, y), \lambda_2(x, y), \lambda_3(x, y) \in [0, 1]$ satisfy $\lambda_1(x, y) + \lambda_2(x, y) + \lambda_3(x, y) = 1$. The global controller is then defined over the entire domain D as:

$$v(t, x, y) := -K \sum_{k=1}^M \chi_{T_k}(x, y) \bar{e}_{T_k}(t, x, y),$$

where $K > 0$ is a global control gain and $\chi_{T_k}(x, y)$ is the indicator function of triangle T_k :

$$\chi_{T_k}(x, y) := \begin{cases} 1, & (x, y) \in T_k, \\ 0, & \text{otherwise.} \end{cases}$$

Remark. The principal analytical challenge at this stage is to derive L^2 -norm bound on the interpolation error $e - \bar{e}$ over the domain D , where e denotes the true deployment error and $\bar{e}$ its piecewise-linear approximation. In the case where leaders do not communicate and only construct a piecewise-constant approximation based on local measurements, such a bound has been established in Theorem 1 of [Selivanov and Fridman \(2019\)](#). Specifically, for any function $e \in H^2((0, l_1) \times (0, l_2))$, the following inequality holds:

$$\|e(t, \cdot, \cdot) - e(t, 0, 0)\|^2 \leq \frac{1}{\lambda_{(1,0)}} \left(\frac{2l_1}{\pi} \|e_x\| \right)^2 + \frac{1}{\lambda_{(0,1)}} \left(\frac{2l_2}{\pi} \|e_y\| \right)^2 + \frac{1}{\lambda_{(1,1)}} \left(\frac{4l_1l_2}{\pi^2} \|e_{xy}\| \right)^2,$$

for any weights $\lambda_{(1,0)}, \lambda_{(0,1)}, \lambda_{(1,1)} \in \mathbb{R}_{>0}$ satisfying the condition $\lambda_{(1,0)} + \lambda_{(0,1)} + \lambda_{(1,1)} = 1$.

We now prove the bounds on the approximation error when leaders can communicate.

5.2.2.1 Preliminary Result: L^2 -Bound on Approximation Error

Lemma 5.3. *Let $T \subset D$ be a triangular region with vertices $(0, 0)$, $(l_1, 0)$, and $(0, l_2)$, where $l_1, l_2 > 0$. Suppose $e \in \mathcal{H}^3(T, \mathbb{R})$, and the values of e at the three vertices are known. Define the linear approximation $\bar{e}_T(t, x, y)$ as in (5.26), using the weights: $\lambda_1(x, y) = 1 - \frac{x}{l_1} - \frac{y}{l_2}$, $\lambda_2(x, y) = \frac{x}{l_1}$, $\lambda_3(x, y) = \frac{y}{l_2}$. The corresponding linear interpolant is then given by*

$$\bar{e}_T(t, x, y) := \lambda_1(x, y) y_{k_1}(t) + \lambda_2(x, y) y_{k_2}(t) + \lambda_3(x, y) y_{k_3}(t), \quad (x, y) \in T.$$

Then, for any parameters $0 < \beta_1, \beta_2, \theta_1, \theta_2 < 1$, the approximation error satisfies:

$$\begin{aligned} \|e - \bar{e}_T\|_{L^2(T)}^2 &\leq \frac{1}{\beta_1\beta_2(1-\theta_1)} \left(\frac{l_1}{\pi}\right)^4 \|e_{xx}\|_{L^2(T)}^2 + \frac{1}{\beta_1(1-\beta_2)(1-\theta_2)} \left(\frac{l_2}{\pi}\right)^4 \|e_{yy}\|_{L^2(T)}^2 \\ &\quad + \frac{1}{1-\beta_1} \left(\frac{4l_1l_2}{\pi^2}\right)^2 \|e_{xy}\|_{L^2(T)}^2 + \frac{1}{\beta_1\beta_2\theta_1} \left(\frac{2l_1^2l_2}{\pi^3}\right)^2 \|e_{xxy}\|_{L^2(T)}^2 \\ &\quad + \frac{1}{\beta_1(1-\beta_2)\theta_2} \left(\frac{2l_1l_2^2}{\pi^3}\right)^2 \|e_{yyx}\|_{L^2(T)}^2. \end{aligned}$$

Proof. Let $E(t, x, y) := e(t, x, y) - \bar{e}_T(t, x, y)$ denote the approximation error. We decompose this error into three parts:

The deviation along the x -axis: $E(t, x, 0) = e(t, x, 0) - \left(1 - \frac{x}{l_1}\right) e(t, 0, 0) - \frac{x}{l_1} e(t, l_1, 0)$,

The deviation along the y -axis: $E(t, 0, y) = e(t, 0, y) - \left(1 - \frac{y}{l_2}\right) e(t, 0, 0) - \frac{y}{l_2} e(t, 0, l_2)$,

The mixed residual term: $R(t, x, y) = e(t, x, y) - e(t, x, 0) - e(t, 0, y) + e(t, 0, 0)$, with $R(t, 0, y) = 0$.

This decomposition facilitates the application of Lemmas 2.8 and 2.9. Observe that $E(t, \cdot, 0)$ vanishes at $x = 0$ and $x = l_1$, and similarly $E(t, 0, \cdot)$ vanishes at $y = 0$ and $y = l_2$. Therefore, using Lemma 2.9, and for any $0 < \theta_1 < 1$ (See Lemma 3 in [Selivanov and Fridman \(2019\)](#)), we obtain:

$$\begin{aligned} \|E(t, \cdot, 0)\|_{L^2(T)}^2 &\leq \frac{l_1^4}{\pi^4} \|e_{xx}(t, \cdot, 0)\|_{L^2(T)}^2 = \frac{l_1^4}{\pi^4} \|e_{xx}(t, \cdot, 0) - e_{xx}(t, \cdot, \cdot) + e_{xx}(t, \cdot, \cdot)\|_{L^2(T)}^2 \\ &\leq \frac{4l_1^4}{\pi^4\theta_1} \|e_{xx}(t, \cdot, 0) - e_{xx}(t, \cdot, \cdot)\|_{L^2(T)}^2 + \frac{l_1^4}{\pi^4(1-\theta_1)} \|e_{xx}(t, \cdot, \cdot)\|_{L^2(T)}^2 \\ &\leq \frac{l_1^4}{\pi^4} \left(\frac{4l_2^2}{\pi^2\theta_1} \|e_{xxy}(t, \cdot, \cdot)\|_{L^2(T)}^2 + \frac{1}{1-\theta_1} \|e_{xx}(t, \cdot, \cdot)\|_{L^2(T)}^2 \right). \end{aligned}$$

Similarly, for $0 < \theta_2 < 1$, we obtain:

$$\|E(t, 0, \cdot)\|_{L^2(T)}^2 \leq \frac{l_2^4}{\pi^4} \left(\frac{4l_1^2}{\pi^2\theta_2} \|e_{yyx}(t, \cdot, \cdot)\|_{L^2(T)}^2 + \frac{1}{1-\theta_2} \|e_{yy}(t, \cdot, \cdot)\|_{L^2(T)}^2 \right).$$

Applying Wirtinger's inequality in the x -direction yields:

$$\begin{aligned} \|R(t, \cdot, \cdot)\|_{L^2(T)}^2 &\leq \frac{4l_1^2}{\pi^2} \int_0^{l_2} \int_0^{l_1(1-\frac{y}{l_2})} \left(\frac{\partial R}{\partial x}(t, x, y) \right)^2 dx dy \\ &= \frac{4l_1^2}{\pi^2} \int_0^{l_2} \int_0^{l_1(1-\frac{y}{l_2})} (e_x(t, x, y) - e_x(t, x, 0))^2 dx dy \\ &\leq \frac{16l_1^2l_2^2}{\pi^4} \|e_{xy}(t, \cdot, \cdot)\|_{L^2(T)}^2. \end{aligned}$$

Combining all the above and applying Lemma 3 in [Selivanov and Fridman \(2019\)](#), we obtain the following bound for any $0 < \beta_1, \beta_2 < 1$:

$$\begin{aligned}
\|E(t, \cdot, \cdot)\|_{L^2(T)}^2 &= \|E(t, x, 0) + E(t, 0, y) + R(t, x, y)\|_{L^2(T)}^2 \\
&\leq \frac{1}{\beta_1 \beta_2} \|E(t, x, 0)\|_{L^2(T)}^2 + \frac{1}{\beta_1 (1 - \beta_2)} \|E(t, 0, y)\|_{L^2(T)}^2 \\
&\quad + \frac{1}{1 - \beta_1} \|R(t, x, y)\|_{L^2(T)}^2 \\
&\leq \frac{1}{\beta_1 \beta_2 (1 - \theta_1)} \left(\frac{l_1}{\pi}\right)^4 \|e_{xx}\|_{L^2(T)}^2 \\
&\quad + \frac{1}{\beta_1 (1 - \beta_2) (1 - \theta_2)} \left(\frac{l_2}{\pi}\right)^4 \|e_{yy}\|_{L^2(T)}^2 \\
&\quad + \frac{1}{1 - \beta_1} \left(\frac{4l_1 l_2}{\pi^2}\right)^2 \|e_{xy}\|_{L^2(T)}^2 + \frac{1}{\beta_1 \beta_2 \theta_1} \left(\frac{2l_1^2 l_2}{\pi^3}\right)^2 \|e_{xxy}\|_{L^2(T)}^2 \\
&\quad + \frac{1}{\beta_1 (1 - \beta_2) \theta_2} \left(\frac{2l_1 l_2^2}{\pi^3}\right)^2 \|e_{yyx}\|_{L^2(T)}^2.
\end{aligned}$$

□

In this section, we extended the PDE-based swarm deployment framework to two-dimensional domains by formulating a distributed control strategy based on spatial decomposition. A piecewise-linear approximation of the positional error was constructed over triangular regions, and a bound on the resulting approximation error was derived in the L^2 norm.

However, the stability analysis of the full closed-loop system remains an open problem. In particular, the Lyapunov-based techniques used in the 1D setting must now account for higher-order mixed derivatives such as e_{xxy} and e_{yyx} , which pose significant analytical challenges. Future work will focus on developing suitable Lyapunov functionals capable of handling these third-order terms, thereby enabling a complete theoretical guarantee of convergence for the 2D deployment strategy.

5.3 Closing Remarks

Controlling large-scale MAS is a challenging problem, especially when agents have access only to local information and limited communication. Traditional agent-based models become harder to analyse as the number of agents increases, which limits their usefulness for scalable system design. This thesis has shown that using PDEs to model MAS offers a scalable and effective

solution. Once the system is described by a PDE, the control design and stability analysis do not depend on how many agents are in the system. This makes the approach more general and easier to apply to large and changing swarms. The main contribution of this work is a new control strategy that combines inter-leader communication with Wentzell-type boundary conditions. This improves convergence speed, maintains swarm cohesion, and performs better than existing PDE-based methods. Theoretical results were confirmed through detailed simulations. Although the main focus was on the deployment problem, the methods developed here can be applied to other types of distributed systems. It is hoped that this work will help guide future research on scalable and reliable control of MAS using PDEs.

Appendix

Preliminaries on Partial Differential Equations

A partial differential equation (PDE) is defined as a mathematical relationship involving an unknown function of multiple independent variables, $u(x_1, \dots, x_n)$, and its partial derivatives. The fundamental properties and solution techniques for such equations are primarily dictated by their order and structural classification. The classification of a PDE of order m depends on the algebraic manner in which the unknown function u and its derivatives are composed.

- **Linear PDEs:** An equation is categorized as linear if it can be expressed in the form:

$$\sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u = f(x),$$

where the coefficients a_α are functions solely of the independent variables x . Linear PDEs satisfy the *superposition principle*, a fundamental property in functional analysis and control theory.

- **Semilinear PDEs:** A PDE is semilinear if the highest-order derivatives appear linearly with coefficients depending only on the independent variables, while nonlinearities are restricted to lower-order terms:

$$\sum_{|\alpha|=m} a_\alpha(x) D^\alpha u + F(x, u, Du, \dots, D^{m-1}u) = 0.$$

- **Quasilinear PDEs:** In quasilinear equations, the highest-order derivatives appear linearly, but their associated coefficients may depend on the function u and its lower-order derivatives:

$$\sum_{|\alpha|=m} a_\alpha(x, u, Du, \dots, D^{m-1}u) D^\alpha u + F(x, u, Du, \dots, D^{m-1}u) = 0.$$

- **Fully Nonlinear PDEs:** These equations exhibit a nonlinear dependence on the highest-order derivatives and cannot be partitioned into the aforementioned linear structures.

The qualitative behavior of second-order linear PDEs in two independent variables is determined by their principal part. Consider the equation:

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + \text{l.o.t.} = G(x, y), \quad (\text{A.1})$$

where A, B , and C are functions of (x, y) , and l.o.t. denotes lower-order terms. The classification is governed by the sign of the discriminant:

$$\mathcal{D}(x, y) = B^2 - 4AC.$$

The PDE is classified at a point (x, y) as follows:

- **Elliptic** if $\mathcal{D} < 0$: These equations typically describe steady-state phenomena. A canonical example is the *Laplace equation*, $\Delta u = 0$.
- **Parabolic** if $\mathcal{D} = 0$: These equations generally model diffusion processes and irreversible time evolution. The *heat equation*, $u_t = \kappa u_{xx}$, serves as the standard prototype.
- **Hyperbolic** if $\mathcal{D} > 0$: These equations characterize wave propagation and vibrations, where information travels at finite speeds. The *wave equation*, $u_{tt} = c^2 u_{xx}$, is the representative model.

This classification is fundamental as it dictates the well-posedness of the problem, the required regularity of solutions, and the appropriate choice of initial and boundary conditions.

Functional Analysis and Abstract Spaces

The study of complex dynamical systems, particularly those described by partial differential equations, necessitates a departure from finite-dimensional Euclidean spaces toward abstract function spaces. This section provides a rigorous overview of the mathematical foundations required for the analysis of functions and operators in infinite-dimensional settings, establishing the framework for convergence, stability, and the existence of solutions.

Vector Spaces and Linear Structure

A **Vector Space** $\mathcal{V}$ over a scalar field $\mathbb{F}$ (typically $\mathbb{R}$ or $\mathbb{C}$) is the fundamental algebraic structure used to define linear operations. It is a set closed under vector addition and scalar multiplication, satisfying the following axioms for all $u, v, w \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{F}$:

1. Additive Axioms:

- Associativity: $u + (v + w) = (u + v) + w$.
- Commutativity: $u + v = v + u$.
- Identity: There exists a unique zero vector $\mathbf{0} \in \mathcal{V}$ such that $u + \mathbf{0} = u$.
- Inverse: For every $u \in \mathcal{V}$, there exists an element $-u \in \mathcal{V}$ such that $u + (-u) = \mathbf{0}$.

2. Multiplicative and Distributive Axioms:

- Compatibility: $\alpha(\beta u) = (\alpha\beta)u$.
- Identity: $1u = u$, where 1 is the multiplicative identity of $\mathbb{F}$.
- Distributivity: $\alpha(u + v) = \alpha u + \alpha v$ and $(\alpha + \beta)u = \alpha u + \beta u$.

Normed Vector Spaces $(\mathcal{V}, \|\cdot\|)$

To move from pure algebra to analysis, it is necessary to quantify the magnitude of vectors. A **Normed Vector Space** is a vector space equipped with a mapping $\|\cdot\| : \mathcal{V} \rightarrow [0, \infty)$ that satisfies:

1. **Positive Definiteness:** $\|u\| = 0$ if and only if $u = \mathbf{0}$.

2. **Absolute Homogeneity:** $\|\alpha u\| = |\alpha| \|u\|$ for all scalars $\alpha \in \mathbb{F}$.

3. **Triangle Inequality:** $\|u + v\| \leq \|u\| + \|v\|$.

A norm naturally induces a distance metric $d(u, v) = \|u - v\|$, which allows for the rigorous definition of convergence and continuity within the space.

Banach Spaces $(\mathcal{X}, \|\cdot\|)$

A **Banach Space** $\mathcal{X}$ is a normed vector space that is **complete** with respect to the metric induced by its norm. Completeness is a topological property ensuring that every **Cauchy sequence** in $\mathcal{X}$ converges to a limit that is also contained within $\mathcal{X}$. A sequence $\{u_n\}_{n=1}^{\infty} \subset \mathcal{X}$ is said to be Cauchy if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that:

$$\|u_n - u_m\| < \epsilon \quad \forall n, m > N. \quad (\text{A.2})$$

Typical examples of Banach spaces include:

- **Lebesgue Spaces:** The space $L^p(\Omega)$ for $1 \leq p < \infty$, with the norm $\|f\|_p = (\int_{\Omega} |f|^p dx)^{1/p}$.
- **Continuous Spaces:** The space $C(\Omega)$ of continuous functions with the supremum norm $\|f\|_{\infty} = \sup_{x \in \Omega} |f(x)|$.

Hilbert Spaces $(\mathcal{H}, \langle \cdot, \cdot \rangle)$

A **Hilbert Space** $\mathcal{H}$ is a specific class of Banach space where the norm is induced by an **inner product** $\langle \cdot, \cdot \rangle$, such that $\|u\| = \sqrt{\langle u, u \rangle}$. An inner product is a mapping $\mathcal{V} \times \mathcal{V} \rightarrow \mathbb{F}$ that is conjugate symmetric, linear in the first argument, and positive definite.

A Banach space is a Hilbert space if and only if the norm satisfies the **Parallelogram Law**:

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2). \quad (\text{A.3})$$

Hilbert spaces, most notably $L^2(\Omega)$, provide essential geometric tools such as **orthogonality** and **orthogonal projections**.

Sobolev Spaces and Weak Derivatives

In the study of differential equations, the classical requirement of pointwise differentiability is often too restrictive. To address this, **Sobolev Spaces** generalize the notion of differentiability by employing **weak derivatives**.

Let u be a function in the local Lebesgue space $L^1_{\text{loc}}(\Omega)$. A function v is said to be the α -th weak derivative of u , denoted by $D^\alpha u = v$, if for all test functions $\phi \in C_c^\infty(\Omega)$, the following integration-by-parts identity holds:

$$\int_{\Omega} u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_{\Omega} v \phi \, dx. \quad (\text{A.4})$$

A Sobolev space $W^{k,p}(\Omega)$ is defined as the set of functions $u \in L^p(\Omega)$ such that for every multi-index α with $|\alpha| \leq k$, the weak derivative $D^\alpha u$ exists and belongs to $L^p(\Omega)$. This is a Banach space equipped with the norm:

$$\|u\|_{W^{k,p}} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p}^p \right)^{1/p}. \quad (\text{A.5})$$

For $p = 2$, these spaces possess a Hilbert space structure and are conventionally denoted as $H^k(\Omega) = W^{k,2}(\Omega)$.

The Hilbert Space $H^1(\Omega)$ and $H_0^1(\Omega)$

The space $H^1(\Omega)$ is characterized by functions with square-integrable magnitudes and square-integrable gradients:

$$H^1(\Omega) = \{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)\}. \quad (\text{A.6})$$

It is equipped with the inner product $\langle u, v \rangle_{H^1} = \int_{\Omega} (uv + \nabla u \cdot \nabla v) \, dx$. For problems subject to homogeneous Dirichlet conditions, the subspace $H_0^1(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in the H^1 norm.

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List of Publications

Articles Published in Journals

1. Shubham Khansili and Anton Selivanov, “PDE-based deployment of multi-agent systems with enhanced leader communication and Wentzell boundary control,” *Systems & Control Letters*, vol. 204, p. 106201, 2025, Elsevier.

Conference Presentations and Proceedings

1. Shubham Khansili and Anton Selivanov, “PDE-based deployment with communicating leaders for large-scale multi-agent systems,” in *Proceedings of the 2023 European Control Conference (ECC)*, pp. 1–6, IEEE, 2023.

