Studies in Automorphic Representations and the Fargues-Fontaine Curve

University of Sheffield

A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy

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June 1, 2026

Abstract

This thesis consists of two parts.

In the first part, we study the relationship between special values of L-functions and congruences of automorphic representations. We relate $L(1, \pi, \mathrm{Ad}^\circ)$ to the congruence ideals for cohomological cuspidal automorphic representations π of GL_n over any number field. We then use this result to deduce relationships between the congruences of automorphic forms and adjoint L-functions. For CM fields, we apply the result, together with local-global compatibility for Galois representations and Galois deformation theory, to obtain a lower bound on the cardinality of certain Selmer groups in terms of $L(1, \pi, \mathrm{Ad}^\circ)$, giving partial progress toward the Bloch-Kato conjecture. The main method is to relate automorphic representations to the cohomology of locally symmetric spaces for GL_n. To implement this strategy, we analyze the relations between several cohomology theories. Cohomology plays a central role because cup products correspond to Rankin-Selberg convolutions and hence to adjoint L-functions. This part of the essay has been published in [Fon26].

In the second part, we introduce a notion of Higgs bundles on the Fargues-Fontaine curve, motivated by the trace formula and the geometric ideas underlying the fundamental lemma. We show that there is an Artin v-stack parametrising Higgs bundles on the Fargues-Fontaine curve. Then we prove three abelianisation theorems, such as a version of the BNR correspondence, which relates Higgs bundles to line bundles on suitable spectral curves. We further describe an action of a Picard stack on the moduli stack of Higgs bundles and show that, modulo this action, there is a natural injective map of étale stacks from the product of B_{dR}^+-affine Springer fibers to the Hitchin fiber that induces an equivalence of categories on every geometric point. Finally, we discuss connections with number-theoretic objects.

Declaration

I, the author, confirm that the Thesis is my own work. I am aware of the University's Guidance on Academic Integrity (https://www.sheffield.ac.uk/study-skills/assessment/academic-integrity/academic-integrity). This work has not previously been presented for an award at this, or any other, university.

Contents

Chapter 1

Adjoint L-functions, congruence ideals, and Selmer groups over GL_n

In this part, we relate $L(1, \pi, \mathrm{Ad}^\circ)$ to the congruence ideals for cohomological cuspidal automorphic representations π of GL_n over any number field. We then use this result to deduce relationships between the congruences of automorphic forms and adjoint L-functions. For CM fields, we apply the result to obtain a lower bound on the cardinality of certain Selmer groups in terms of $L(1, \pi, \mathrm{Ad}^\circ)$. This part of the thesis has been published in [Fon26].

1.1 Introduction

In number theory, we study many different types of L-functions, such as L-functions of elliptic curves, Artin L-functions, and Dirichlet L-functions. A recurring theme is that the special values of these L-functions often encode deep arithmetic information. Some notable examples include the analytic class number formula, the Herbrand-Ribet theorem, and the Birch-Swinnerton-Dyer conjecture.

The focus of this part of the thesis is the adjoint L-function of an automorphic representation. Special values of adjoint L-functions frequently reflect congruences between automorphic forms, and these congruences can in turn be measured by the so-called congruence ideal. The relationship between adjoint L-values and congruence ideals has by now become a central theme in modern number theory; for a historical overview, we refer the reader to the introduction of [BR17]. Here we offer a complementary motivation.

In 1994, Andrew Wiles and Richard Taylor proved Fermat's last theorem by showing that every semistable elliptic curve is modular. The key is to establish a modularity lifting theorem, which in their setup boils down to showing that the universal deformation ring R of some residual Galois representation is isomorphic to a suitable Hecke algebra T. To do so, they used the Taylor-Wiles method to prove a modularity lifting theorem in the minimally ramified case [TW95]. To extend the theorem to the non-minimal case, Wiles [Wil95] used a numerical criterion for ring isomorphism. To apply the criterion involves studying the tangent space of R on the one hand and examining an invariant called the congruence ideal η_T of T on the other hand.

It has since proved fruitful to study congruence ideals not only for Hecke algebras but also for various cohomology modules. In [TU21], the congruence ideals associated with the cohomology of certain locally symmetric spaces are related to the value at $s = 1$ of the adjoint L-function for GL_2 over specific number fields. Using the work of [BR17], we generalize these results to GL_n over an arbitrary number field.

Let π be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, where F is a number field. Our first main result is Theorem 1.4.14, which establishes that, under suitable assumptions, the (cohomological) congruence ideal of π is generated by a special value of the normalised adjoint L-function $L^{alg}(1, \pi, \mathrm{Ad}^0, \epsilon)$.

Building on Theorem 1.4.14, we derive consequences for congruences between automorphic forms. In Corollaries 1.4.17 and 1.4.20, we show that under suitable assumptions, if the p-adic valuation of $L(1, \pi, \mathrm{Ad}^\circ, \epsilon)$ is positive, then there is an automorphic representation $\pi' \ncong \pi$ whose Hecke eigensystem is congruent to that of π.

In the last subsection, we provide in Theorem 1.4.26 a lower bound for the cardinality of a certain Selmer group in terms of special values of L-functions when F is a CM field. This can be viewed as partial progress on the Bloch-Kato conjecture.

1.1.1 Strategy

We briefly summarise the strategy of this thesis, omitting certain technical details in order to emphasise the main ideas.

The starting point is to realise automorphic representations in cohomology and to work at the level of cohomology. By results of Borel and Franke [Bor83, Fra98], any regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ contributes to the cohomology of the locally symmetric space X_U associated with GL_n. Poincaré duality furnishes a perfect pairing between the singular cohomology and the compactly supported cohomology of X_U with $\mathbb{Z}_p$-coefficients. Under the additional assumption that the cohomology of the boundary of X_U is p-torsion free, this pairing induces a perfect pairing on the inner cohomology of X_U.

By Lemma 1.3.5, any such perfect pairing that is suitably compatible with the action of the Hecke algebra yields an explicit formula for the corresponding congruence ideal: the ideal is generated by the value of the pairing on appropriate $\mathbb{Z}_p$-generators. In our setting, the relevant pairing is the cup product on cohomology, which, on the automorphic side, corresponds essentially to the Petersson inner product of two automorphic forms. Via the Rankin-Selberg method, this inner product can be expressed in terms of the adjoint L-function. This leads directly to a formula for the congruence ideal in terms of the value $L(1, \pi, \mathrm{Ad}^0, \epsilon)$ in Theorem 1.4.14.

Using the relationship between congruence ideals and congruences of automorphic representations (Lemma 1.3.6), we then obtain criteria for the existence of congruent automorphic representations in terms of adjoint L-values (Corollary 1.4.17). Finally, invoking the local-global compatibility results for Galois representations established in [CN23], together with standard techniques from Galois deformation theory, we derive in section 1.4.5 (when F is a CM or totally real field) a lower bound for the order of a certain Selmer group in terms of special values of adjoint L-functions.

1.1.2 Notation

For a number field F, we write $\mathbb{A}_F$ for the ring of adeles, $F_\infty := F \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{v|\infty} F_v$,

$$\mathbb{A}_F^\infty$$

the ring of finite adeles, and $\mathbb{A}_F^{\infty,S}$ for the ring of finite adeles without the components at S. The contragredient of an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ is denoted by $\widetilde{\pi}$.

If G is a locally profinite group and U is an open compact subgroup of G, then we let

$$\begin{aligned}\mathcal{H}(G,U) &:= \{U\text{-biinvariant compactly supported functions } G \to \mathbb{Z}\}\\ &\cong \mathbb{Z}[U\backslash G/U],\end{aligned}$$

where multiplication is given by convolution with respect to the Haar measure on G which gives U volume 1. If F is a number field and $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_p$, then

$$\mathbb{T}^S := \mathcal{H}(\mathrm{GL}_n(\mathbb{A}_F^{\infty,S}), \prod_{v\notin S\cup\{\infty\}} \mathrm{GL}_n(\mathcal{O}_{F_v})) \otimes_{\mathbb{Z}} \mathcal{O}.$$

(This of course depends on $F, \mathcal{O}$.)

We write Ad^0 for the (trace zero) adjoint representation, which is the representation of $\mathrm{GL}_n(\mathbb{C})$ on the $n \times n$ trace zero matrices by conjugation.

All L-functions include the L-factors at infinity unless otherwise stated.

If F is a number field and $G = \mathrm{GL}_{n/F}$, then K_∞ will mean the product of $A_G := \mathbb{R}_{>0}$ [1]and the standard maximal compact subgroup of $G(F \otimes_{\mathbb{Q}} \mathbb{R})$, so

$$K_\infty \cong \mathbb{R}_{>0} \cdot (O_n(\mathbb{R})^{r_1} \times U_n(\mathbb{R})^{r_2}),$$

where r_1, r_2 are the number of real and complex places of F respectively and $U_n(\mathbb{R}) := \{g \in \mathrm{GL}_n(\mathbb{C}) : \bar{g}^T g = 1\}$ is the unitary group. Also, $\mathfrak{g}$ will be the Lie algebra of $\mathrm{GL}_n(F_\infty)$.

Acknowledgement

I would like to thank my supervisor Tobias Berger for his help with this project, James Newton and Robert Kurinczuk for lots of constructive feedback which greatly enhances the quality of this thesis, Anantharam Raghuram for sketching the proof of Lemma 1.2.16

[1] $A_G = \mathbb{R}_{>0}$ is embedded diagonally into the centre of $G(F_\infty)$.

part (b) in an email, Haluk Sengun for suggesting nice references for learning about cohomology of arithmetic groups, and an anonymous referee for giving many useful comments on the previous version of this part of the thesis. This work was completed while in receipt of the EPSRC Doctoral Training Partnership (DTP) Studentship with grant number EP/W524360/1.

1.2 Cohomology

Fix a number field F with r_1 real places and r_2 complex places for the entire section. Readers not very familiar with the relationship between automorphic representations and cohomology are recommended to read the excellent papers [Sch09, Clo90].

1.2.1 Cohomology and Hecke operators

We shall mostly follow [KT17, section 6] and [Han17, section 2.1] to define the cohomology group and Hecke operators. Most of the material is standard, but there are many different variations, so we think it is necessary to state clearly our conventions.

Let $G = \mathrm{GL}_n$. For an open compact subgroup $U \subset G(\mathbb{A}_F^\infty)$, we define[2]

$$X_U := G(F)\backslash G(\mathbb{A}_F)/K_\infty^\circ U.$$

and

$$X := G(F_\infty)/K_\infty^\circ.$$

Note that $X_U = G(F)\backslash(X \times G(\mathbb{A}_F^\infty)/U)$.

We call an element $g = (g_v)_v \in G(\mathbb{A}_F^\infty)$ ***neat*** if $\cap_v \Gamma_v = \{1\}$, where $\Gamma_v \subset \bar{\mathbb{Q}}^\times$ is the torsion subgroup of the subgroup generated by the eigenvalues of g_v. We call an open compact subgroup $K \subset G(\mathbb{A}_F^\infty)$ ***neat*** if all of its elements are neat.

Definition 1.2.1. *We call $U \subset G(\mathbb{A}_F^\infty)$ a* ***good subgroup*** *if it is a neat subgroup of the form $U = \prod_v U_v \subset \prod \mathrm{GL}_n(\mathcal{O}_{F_v})$.*

Let U be a good subgroup and M be a $\mathbb{Z}[U]$-module. We define a locally constant sheaf $\mathcal{L}_M$ on X_U as the sheaf of continuous sections of the map

$$G(F)\backslash(X \times G(\mathbb{A}_F^\infty) \times M)/U \to X_U$$

where $G(F) \times U$ acts on $X \times G(\mathbb{A}_F^\infty) \times M$ by $(\gamma, u)(x, g, m) = (\gamma x, \gamma g u^{-1}, um)$ and M is equipped with the discrete topology. We define

$$H^*(X_U, M) := H^*(X_U, \mathcal{L}_M)$$

to be the sheaf cohomology.

[2] We quotient out by K_∞° rather than K_∞ because we would like to realise cohomological automorphic representations in cohomology. If we instead quotient out by K_∞, then some of these representations only appear in the relative Lie algebra after twisting by a character ϵ of $\widehat{K_\infty/K_\infty^\circ}$, but the fact that ϵ is not a character of $U \subset G(\mathbb{A}_F^\infty)$ makes it unclear how to define the inner cohomology with coefficient in $M_\lambda \otimes \epsilon$. See however [Jan24, Proposition 3.2.5, Theorem 4.3.5] for a possible way to do this using locally algebraic representations.

Proposition 1.2.2. *[KT17, Proposition 6.2] If U is a neat*[3] *subgroup, then*

$$H^*(X_U, M) \cong H^*(C^\bullet_{\mathbb{A}}(U, M)),$$

where

$$C^\bullet_{\mathbb{A}}(U, M) := \mathrm{Hom}_{G(F)\times U}(C_{\mathbb{A},\bullet}, M)$$

and $C_{\mathbb{A},\bullet}$ is the group of singular chains on $X \times G(\mathbb{A}_F^\infty)$ with $\mathbb{Z}$ coefficients.

To define the action of the Hecke algebra, we suppose M is actually a left $\mathbb{Z}[\Delta]$-module, where $\Delta \subset G(\mathbb{A}_F^\infty)$ is a submonoid containing U. Note that the compactness of U implies that $U \subset \Delta$ is a Hecke pair. Let $\mathcal{H}(\Delta, U)$ be the set of locally constant, compactly supported functions $f : \Delta \to \mathbb{Z}$ which are U-biinvariant. We can (and will) regard it as a subalgebra of $\mathcal{H}(G(\mathbb{A}_F^\infty), U)$.

For $\delta \in \Delta$, let the characteristic function $[U\delta U]$ acts on the complex $C^\bullet_{\mathbb{A}}(U, M)$ by

$$([U\delta U]^*\phi)(\sigma) = \sum \delta_i \phi(\delta_i^{-1}\sigma)$$

where $U\delta U = \bigsqcup_i \delta_i U$, $\phi \in C^\bullet_{\mathbb{A}}(U, M)$ and $\sigma \in C_{\mathbb{A},\bullet}$. This is independent of the choices of δ_i. By taking cohomology, we get an action of $\mathcal{H}(\Delta, U)$ on $H^*(X_U, M)$.

Let T_n and B_n be the standard diagonal torus and Borel subgroup of $\mathrm{GL}_{n/\mathbb{Z}}$. Let w_0 be the longest element of the Weyl group[4].

Definition 1.2.3. *[Ger19, Definition 2.1] Let A be a commutative ring. If $\lambda \in \mathbb{Z}^n$ is a dominant weight for GL_n, then we define the algebraic representation $Ind_{B_n}^{\mathrm{GL}_n}(w_0\lambda)_{/A}$ of $\mathrm{GL}_{n/A}$ to be*

$$\{f \in A[\mathrm{GL}_n] : f(bg) = (w_0\lambda)(b)f(g) \text{ for all } A\text{-algebras } B, g \in \mathrm{GL}_n(B), b \in B_n(B)\}$$

where $A[\mathrm{GL}_n] := Mor_{\mathrm{Spec}\,A}(\mathrm{GL}_{n/A}, \mathbb{A}^1_A)$ [5] *and $\mathrm{GL}_{n/A}$ acts by right translation. We let*

$$M_{\lambda,A} := Ind_{B_n}^{\mathrm{GL}_n}(w_0\lambda)_{/A}(A),$$

which is a representation of $\mathrm{GL}_n(A)$.

If E is a p-adic field with ring of integers $\mathcal{O}$, then from [Ger19, page 1349], $M_{\lambda,\mathcal{O}}$ is finite free over $\mathcal{O}$. Also, $M_{\lambda,\mathcal{O}} \otimes_{\mathcal{O}} E = M_{\lambda,E}$ is the algebraic representation of $\mathrm{GL}_n(E)$ of highest weight λ. By [NT16, page 19], for all $\mathcal{O}$-algebras R, the natural map $M_{\lambda,\mathcal{O}} \otimes_{\mathcal{O}} R \to M_{\lambda,R}$ is an isomorphism.

We write $\mathbb{Z}^n_+ := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n : \lambda_1 \geq \cdots \geq \lambda_n\}$. Let E be a finite extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$ which contains all embeddings of F to $\overline{\mathbb{Q}}_p$, $\mathcal{O}$ be the ring of integers of E, and $\mu \in (\mathbb{Z}^n_+)^{\mathrm{Hom}(F,E)}$.

[3] In [KT17], this is stated for good subgroups, but the proof actually works for all neat subgroups.

[4] If we write the characters of T_n as $\lambda = (\lambda_1, \dots, \lambda_n)$, then $w_0\lambda = (\lambda_n, \dots, \lambda_1)$

[5] Here, $\mathbb{A}^1_A = \mathrm{Spec}\,A[T]$ is the affine line over A.

We define the $\mathcal{O}$-module

$$M_\mu := M_{\mu,\mathcal{O}} := \bigotimes_{\tau \in \mathrm{Hom}(F,E)} M_{\mu_\tau,\mathcal{O}}$$

which receives an action of $\prod_{v|p} \mathrm{GL}_n(\mathcal{O}_{F_v})$ by $(g_v)_v \cdot \otimes m_\tau = \otimes g_{v(\tau)} m_\tau$, where $v(\tau)$ is the place of F induced by τ. Then $\mathrm{GL}_n(\mathbb{A}_F^{\infty,p}) \times U_p$ acts on $M_{\mu,\mathcal{O}}$ by projection to $U_p := \prod_{v|p} U_v$. By the formalism above[6], $\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_F^{\infty,p}), U^p)$ acts on $H^*(X_U, M_{\mu,\mathcal{O}})$.

We define

$$M_{\mu,E} := \otimes_{\tau \in \mathrm{Hom}(F,E)} M_{\mu_\tau,E}$$

as an $E[\prod_{v|p} \mathrm{GL}_n(F_v)]$-module. Then $\mathrm{GL}_n(\mathbb{A}_F^\infty)$ acts on $M_{\mu,E}$ by projection to $\mathrm{GL}_n(F_p) := \mathrm{GL}_n(\prod_{v|p} F_v)$. By the formalism above, $\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_F^\infty), U)$ acts on $H^*(X_U, M_{\mu,E})$. This is compatible with the construction above, i.e. we have an isomorphism $H^*(X_U, M_{\mu,\mathcal{O}}) \otimes_\mathcal{O} E \cong H^*(X_U, M_{\mu,E})$ that is Hecke-equivariant for the restriction map $\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_F^{\infty,p}), U^p) \to \mathcal{H}(\mathrm{GL}_n(\mathbb{A}_F^\infty), U)$.

Fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ for the rest of this section. We define $M_{\mu,\mathbb{C}} := \otimes_{\tau \in \mathrm{Hom}(F,E)} M_{\mu_\tau,\mathbb{C}}$. This is acted on by $\prod_{v|p} \mathrm{GL}_n(F_v)$ via ι and also by $G(F_\infty)$, where $F_\infty := F \otimes_\mathbb{Q} \mathbb{R}$, by

$$G(F_\infty) \hookrightarrow G(F \otimes_\mathbb{Q} \mathbb{C}) = \prod_{\tau \in \mathrm{Hom}(F,\mathbb{C})} G(\mathbb{C}) \curvearrowright \otimes_{\tau \in \mathrm{Hom}(F,E)} M_{\mu_\tau,\mathbb{C}}$$

using the identification $\mathrm{Hom}(F,E) = \mathrm{Hom}(F,\mathbb{C})$ given by ι. Note that the 2 actions of $G(F)$ on $M_{\mu,\mathbb{C}}$ agree. As $M_{\mu,\mathbb{C}}$ is an irreducible representation of $\mathrm{GL}_n(F \otimes_\mathbb{Q} \mathbb{C})$, it has a central character. In particular, A_G acts[7] by a character

$$\chi^{-1} : A_G \to \mathbb{C}^\times$$

on $M_{\mu,\mathbb{C}}$. One can show that for all good subgroup $U \le G(\mathbb{A}_F^\infty)$,

$$H^*(X_U, M_{\mu,\mathbb{C}}) \cong H^*(\mathfrak{g}, K_\infty^\circ, C^\infty(G(F)\backslash G(\mathbb{A}_F)/U, \chi) \otimes_\mathbb{C} M_{\mu,\mathbb{C}}),$$

where $\mathfrak{g} = Lie(G(F_\infty))$ and

$$C^\infty(G(F)\backslash G(\mathbb{A}_F)/U, \chi)$$

is the set of smooth functions $f : G(F)\backslash G(\mathbb{A}_F)/U \to \mathbb{C}$ with $f(ag) = \chi(a)f(g)$ for all $a \in A_G$. The algebra $\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_F^\infty), U)$ acts on $C^\infty(G(F)\backslash G(\mathbb{A}_F)/U, \chi)$ and this induces the Hecke action on the relative Lie algebra cohomology. Then this isomorphism of cohomology is compatible with the action of $\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_F^\infty), U)$ on both sides.

[6]Strictly speaking, the formalism gives us an action of $\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_F^{\infty,p}) \times U_p, U)$. Yet, this algebra is canonically isomorphic to $\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_F^{\infty,p}), U^p)$

[7]Recall that $A_G = \mathbb{R}_{>0}$ is embedded diagonally into the centre of $G(F_\infty)$.

1.2.2 Regular algebraic automorphic representations

Let $G = \mathrm{GL}_n$ as above.

Definition 1.2.4. *Let $\chi : A_G \to \mathbb{C}^\times$ be a continuous homomorphism. We write*

$$L^2(G(F)\backslash G(\mathbb{A}_F), \chi)$$

for the space of measurable functions $f : G(F)\backslash G(\mathbb{A}_F) \to \mathbb{C}$ such that $f(ag) = \chi(a)f(g)$ for all $a \in A_G$ and

$$\int_{G(F)\backslash G(\mathbb{A}_F)^1} |f(g)|^2 dg < \infty,$$

where

$$G(\mathbb{A}_F)^1 := \{g \in G(\mathbb{A}_F) : |\det(g)|_{\mathbb{A}_F} = 1\}$$

and functions which agree almost everywhere are identified. Let

$$L^2_0(G(F)\backslash G(\mathbb{A}_F), \chi)$$

be the subspace of cuspidal functions in $L^2(G(F)\backslash G(\mathbb{A}_F), \chi)$, i.e. elements f of $L^2(G(F)\backslash G(\mathbb{A}_F), \chi)$ such that for the unipotent radical U_P of every proper parabolic subgroup P,

$$\int_{U_P(F)\backslash U_P(\mathbb{A}_F)} f(ug)du = 0$$

*for almost all $g \in G(\mathbb{A}_F)^1$. We define a **cuspidal automorphic representation of character** χ to be an irreducible subrepresentation in $L^2_0(G(F)\backslash G(\mathbb{A}_F), \chi)$ of $G(\mathbb{A}_F)$. A **cuspidal automorphic representation** is one such representation for some χ.*

Definition 1.2.5. *Let $\chi : A_G \to \mathbb{C}^\times$ be a continuous homomorphism. We let*

$$L^2_d(G(F)\backslash G(\mathbb{A}_F), \chi)$$

*be the discrete spectrum, i.e. the closure of the sum of all irreducible subrepresentations of $L^2(G(F)\backslash G(\mathbb{A}_F), \chi)$. We define a **discrete automorphic representation of character** χ to be an irreducible subrepresentation in $L^2_d(G(F)\backslash G(\mathbb{A}_F), \chi)$ of $G(\mathbb{A}_F)$. A **discrete automorphic representation** is one such representation for some χ.*

Remark 1.2.6. Note that every cuspidal automorphic representation is discrete. Also, by our definition, every discrete automorphic representation π is a unitary Hilbert space representation of $G(\mathbb{A}_F)$ after twisting by a suitable character.[8] It follows from the irreducibility of π that π has a central character. Moreover, π_∞ is admissible by a result of Harish-Chandra [GH24, Theorem 4.4.5]

[8] For instance, we can always twist π by a Hecke character such that the action of A_G is trivial, i.e. $\chi = 1$. This reduces us to the case in [GH24, section 3.7].

Definition 1.2.7. *Let $\lambda \in (\mathbb{Z}_+^n)^{\mathrm{Hom}(F,\mathbb{C})}$. We say that a cuspidal automorphic representation π is **regular algebraic**[9]**/cohomological of weight** λ if π_∞ has the same infinitesimal character as $M_{\lambda,\mathbb{C}}^\vee$.*

Lemma 1.2.8. *Let v be an infinite place of F. Let (ρ, V) be an irreducible admissible representation of $G(F_v)$ that has central character ω. Then the infinitesimal character of ρ determines $\omega|_{F_v^{\times,\circ}}$, where $F_v^{\times,\circ}$ is the identity component of $F_v^\times$.*

Proof. First assume v is a real place. Let us use the corresponding real embedding to identify F_v with $\mathbb{R}$. Then there is $s \in \mathbb{C}$ such that $\omega(y) = y^s$ for all $y \in F_v^{\times,\circ} = \mathbb{R}_{>0}$. Let $X = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ as an element in the centre of the complexified universal enveloping algebra of $G(F_v)$. By definition of the Lie algebra action, for all $u \in V_{sm}$,

$$X \cdot u = \frac{d}{dt}\Big|_{t=0} \rho(e^{Xt})u = \frac{d}{dt}\Big|_{t=0} e^{st}u = su.$$

Thus, the infinitesimal character determines s and hence $\omega|_{F_v^{\times,\circ}}$.

Now, assume v is a complex place. Let $\sigma_1, \sigma_2 : F \to \mathbb{C}$ be the two complex embeddings corresponding to v. Use the same notation for the induced isomorphisms $F_v \xrightarrow{\sim} \mathbb{C}$. Then there are $s_1, s_2 \in \mathbb{C}$ with $s_1 - s_2 \in \mathbb{Z}$ such that $\omega(y) = \sigma_1(y)^{s_1}\sigma_2(y)^{s_2}$ for all $y \in F_v^\times$ ($\cong \mathbb{C}^\times$). For $x \in F_v$, let $X = \begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix}$ as an element in the centre of the complexified universal enveloping algebra of $G(F_v)$. By definition of the Lie algebra action, for all $u \in V_{sm}$,

$$X \cdot u = \frac{d}{dt}\Big|_{t=0} \rho(e^{Xt})u = \frac{d}{dt}\Big|_{t=0} e^{s_1\sigma_1(x)t + s_2\sigma_2(x)t}u = (s_1\sigma_1(x) + s_2\sigma_2(x))u.$$

Taking $x = 1$ and $x = \sigma_1^{-1}(i)$ gives us two equations that allows us to solve for s_1, s_2. Thus, the infinitesimal character determines s_1, s_2 and hence $\omega|_{F_v^\times}$. $\square$

Lemma 1.2.9. *If π is a regular algebraic automorphic representation of weight λ, then the restriction of its central character to $F_\infty^{\times\circ}$ is the inverse of that of $M_{\lambda,\mathbb{C}}$. In particular, A_G acts trivially on $\pi_\infty \otimes M_\lambda$.*

Proof. Let $M_{\lambda_v} := \begin{cases} M_{\lambda_\tau,\mathbb{C}} & \text{if } v \text{ is real} \\ M_{\lambda_\tau,\mathbb{C}} \otimes_{\mathbb{C}} M_{\lambda_{\bar\tau},\mathbb{C}} & \text{if } v \text{ is complex} \end{cases}$, where τ is an embedding $F \to \mathbb{C}$ whose associated place is v. Then the infinitesimal character of π_v is the same as that of $M_{\lambda_v}^\vee$. We want to show that the restriction of the central character of π_v to $F_v^{\times,\circ}$ is that of $M_{\lambda_v}^\vee = M_{\lambda_v^\vee}$. This follows from Lemma 1.2.8. $\square$

[9] This is C-algebraic in the sense of Buzzard-Gee.

Definition 1.2.10. *We define*

$$b_n := r_1 \left\lfloor n^2/4 \right\rfloor + r_2 n(n-1)/2,\ t_n := r_1 \left\lfloor (n+1)^2/4 \right\rfloor + r_2 n(n+1)/2 - 1.$$

If n is clear, we may just write b and t instead.[10]

Definition 1.2.11. *Let π be a regular algebraic cuspidal automorphic representation of weight λ. Let $\epsilon \in (K_\infty/K_\infty^\circ)\widehat{} = \{1, sgn\}^{r_1}$. We say that ϵ is a* ***permissible signature*** *if n is even, or n is odd and ϵ_v is the central character of $\pi_v \otimes M_{\lambda_v}$ restricted to $\{\pm 1\}$ for all real places v.*

Lemma 1.2.12. *Let π be a regular algebraic cuspidal automorphic representation of weight λ. Let ϵ be a permissible signature. Then for $i \in \{b_n, t_n\}$, the space*

$$H^i(\mathfrak{g}, K_\infty^0, \pi_\infty \otimes_{\mathbb{C}} M_{\lambda,\mathbb{C}})[\epsilon]$$

is 1 dimensional, where $[\epsilon]$ denotes the ϵ-isotypic component.

The strategy is to use Künneth formula and Clozel's result for Lie algebra cohomology of $\pi_v \otimes M_{\lambda_v}$. A slight complication is caused by the fact that our K_∞ (which contains A_G) does not factor as a product over the infinite places.

Proof. Let

$$G^1(F_\infty) := \left\{ (g_v) \in G(F_\infty) : \prod_{v|\infty} |\det g_v|_{F_v} = 1 \right\},$$
$$G^1(F_v) := \{g_v \in G(F_v) : |\det g_v|_v = 1\},$$
$$H := \left\{ (a_v) \in \prod_{v|\infty} \mathbb{R}_{>0} : \prod_v |a_v|_{F_v} = 1 \right\} \subset G(F_\infty),$$

where we view $\prod_{v|\infty} \mathbb{R}_{>0}$ as a subgroup of the centre of $G(F_\infty)$ via the diagonal embedding. Let $\mathfrak{g}_\infty^1, \mathfrak{g}_v^1, \mathfrak{h}$ be the corresponding Lie algebras. Let

$$K_v^1 := \begin{cases} O_n(\mathbb{R}) \text{ if } v \text{ is real} \\ U_n(\mathbb{R}) \text{ if } v \text{ is complex} \end{cases},$$

which is a subgroup of $G^1(F_v)$, so $\mathfrak{k}_v^1 := Lie(K_v^1) \subset \mathfrak{g}_v^1$.

From the Lie group decompositions $G(F_\infty) = G^1(F_\infty) \times A_G$ and $K_\infty = \prod_{v|\infty} K_v^1 \times A_G$, we get corresponding decompositions of Lie algebras, which in turns give

$$\mathfrak{g}/\mathfrak{k}_\infty = \mathfrak{g}_\infty^1 / \prod_{v|\infty} \mathfrak{k}_v^1 \tag{1.1}$$

[10]It is also common in the literature to write q_0 for b_n and $q_0 + \ell_0$ for t_n.

as $\mathbb{C}[K_\infty]$-modules, where K_∞ acts by conjugation.

Similarly, from the Lie group decomposition $G^1(F_\infty) = \prod_{v|\infty} G^1(F_v) \times H$, we get a corresponding Lie algebra decomposition for $\mathfrak{g}^1_\infty$, which we can substitute to equation (1.1) to get

$$\mathfrak{g}/\mathfrak{k}_\infty = \left(\prod_{v|\infty} \frac{\mathfrak{g}^1_v}{\mathfrak{k}^1_v}\right) \times \mathfrak{h} \tag{1.2}$$

as $\mathbb{C}[K_\infty]$-modules. Note that the action of K_∞ on $\mathfrak{h}$ is trivial.

The relative Lie algebra complex computing $H^i(\mathfrak{g}, K^0_\infty, \pi_\infty \otimes_\mathbb{C} M_{\lambda,\mathbb{C}})$ is by definition the i-th cohomology of

$$\begin{aligned}&(\wedge^*(\mathfrak{g}/\mathfrak{k}_\infty)^\vee \otimes_\mathbb{R} (\pi_\infty)_{K_\infty\text{-fin}} \otimes_\mathbb{C} M_{\lambda,\mathbb{C}})^{K^\circ_\infty}\\ &= \left(\wedge^*\left(\prod_{v|\infty} \frac{\mathfrak{g}^1_v}{\mathfrak{k}^1_v}\right)^\vee \otimes_\mathbb{R} \wedge^*\mathfrak{h}^\vee \otimes_\mathbb{R} (\pi_\infty)_{K_\infty\text{-fin}} \otimes_\mathbb{C} M_{\lambda,\mathbb{C}}\right)^{K^\circ_\infty}\end{aligned}$$

where $(\pi_\infty)_{K\text{-fin}}$ means the K_∞-finite vectors of π_∞. As the action of K_∞ on $\mathfrak{h}$ is trivial, we can pull out that factor from the cohomology and get

$$H^i(\mathfrak{g}, K^0_\infty, \pi_\infty \otimes_\mathbb{C} M_{\lambda,\mathbb{C}}) = \bigoplus_{a+b=i} H^a\left(\left(\wedge^*\left(\prod_{v|\infty} \frac{\mathfrak{g}^1_v}{\mathfrak{k}^1_v}\right)^\vee \otimes_\mathbb{R} (\pi_\infty)_{K_\infty\text{-fin}} \otimes_\mathbb{C} M_{\lambda,\mathbb{C}}\right)^{K^\circ_\infty}\right) \otimes_\mathbb{R} \wedge^b\mathfrak{h}^\vee.$$

Since A_G acts trivially on $\wedge^*\left(\prod_{v|\infty} \frac{\mathfrak{g}^1_v}{\mathfrak{k}^1_v}\right)^\vee$ and $\pi_\infty \otimes_\mathbb{C} M_{\lambda,\mathbb{C}}$ by Lemma 1.2.9, we can replace K°_∞ on the right hand side by $\prod_{v|\infty} K^1_v$. Then by definition

$$H^i(\mathfrak{g}, K^0_\infty, \pi_\infty \otimes_\mathbb{C} M_{\lambda,\mathbb{C}}) = \bigoplus_{a+b=i} H^a\left(\prod_{v|\infty} \mathfrak{g}^1_v, \prod_{v|\infty} \mathfrak{k}^1_v, \pi_\infty \otimes_\mathbb{C} M_{\lambda,\mathbb{C}}\right) \otimes_\mathbb{R} \wedge^b\mathfrak{h}^\vee,$$

which by Künneth formula [BW00, section 1.3 equation (2)] equals

$$\bigoplus_{a_1+\cdots+a_m+b=i} H^{a_1}\left(\mathfrak{g}^1_v, \mathfrak{k}^1_v, \pi_v \otimes_\mathbb{C} M_{\lambda_v,\mathbb{C}}\right) \otimes_\mathbb{C} \cdots \otimes_\mathbb{C} H^{a_m}\left(\mathfrak{g}^1_v, \mathfrak{k}^1_v, \pi_v \otimes_\mathbb{C} M_{\lambda_v,\mathbb{C}}\right) \otimes_\mathbb{R} \wedge^b\mathfrak{h}^\vee,$$

where $m := r_1+r_2$, $M_{\lambda_v,\mathbb{C}} := \begin{cases} M_{\lambda_\tau,\mathbb{C}} & \text{if } v \text{ is real} \\ M_{\lambda_\tau,\mathbb{C}} \otimes_\mathbb{C} M_{\lambda_{\bar\tau},\mathbb{C}} & \text{if } v \text{ is complex}\end{cases}$, and τ is an embedding $F \to \mathbb{C}$ whose associated place is v. The result now follows from [Clo90, Lemma 3.14], Remark 1.2.13, and the fact that $\mathfrak{h} \cong \mathbb{R}^{r_1+r_2-1}$. □

Remark 1.2.13. For even n, [Clo90, Lemma 3.14] only worked with trivial ϵ_v. To deduce the result for general ϵ, one can twist π by a suitable Hecke character $F^\times\backslash\mathbb{A}^\times_F \to \{\pm1\}$, which can for instance be constructed from a suitable character of $\overline{F^\times F^{\times,\circ}_\infty}\backslash\mathbb{A}^\times_F \cong \mathrm{Gal}(F^{ab}/F)$.

1.2.3 More on cohomology

Definition 1.2.14. *We define the* ***cuspidal cohomology*** *as*

$$H^*_{cusp}(X_U, M_{\mu,\mathbb{C}}) := H^*(\mathfrak{g}, K^\circ_\infty, M_{\mu,\mathbb{C}} \otimes_\mathbb{C} L^2_0(G(F)\backslash G(\mathbb{A}_F), \chi)^U),$$

where χ^{-1} is the restriction to A_G of the central character of $G(F \otimes \mathbb{C})$ on $M_{\mu,\mathbb{C}}$.

The cuspidal cohomology is also acted on by $\mathcal{H}(G(\mathbb{A}_F^\infty), U)$ and there is a $\mathcal{H}(G(\mathbb{A}_F^\infty), U)$-equivariant injection

$$H^*_{cusp}(X_U, M_{\mu,\mathbb{C}}) \hookrightarrow H^*(X_U, M_{\mu,\mathbb{C}}).$$

By multiplicity one and semisimplicity of $L^2_0(G(F)\backslash G(\mathbb{A}_F), \chi)$ [GH24, Corollary 9.1.3, Theorem 11.4.3], we know

$$L^2_0(G(F)\backslash G(\mathbb{A}_F), \chi) = \widehat{\bigoplus}_\pi \pi,$$

where the sum ranges over all cuspidal automorphic representations of character χ. We have a corresponding decomposition of the cuspidal cohomology into finite algebraic sum [Sch09, Theorem 4.1], [Clo90, Lemma 3.15]:

$$H^*_{cusp}(X_U, M_{\mu,\mathbb{C}}) = \oplus_\pi H^*(\mathfrak{g}, K^\circ_\infty, M_{\mu,\mathbb{C}} \otimes_\mathbb{C} \pi_\infty) \otimes_\mathbb{C} (\pi^\infty)^U. \tag{1.3}$$

By strong multiplicity one, for every cuspidal automorphic representation of character χ, the $(\pi^{\infty,S})^{U^S}$-isotypic component is

$$H^*_{cusp}(X_U, M_{\mu,\mathbb{C}})[(\pi^{\infty,S})^{U^S}] = H^*(\mathfrak{g}, K^\circ_\infty, \pi_\infty \otimes_\mathbb{C} M_{\mu,\mathbb{C}}) \otimes_\mathbb{C} (\pi^\infty)^U. \tag{1.4}$$

By Lemma 1.2.12, if π is regular algebraic cuspidal of weight λ with $\pi^U \neq 0$, then there is a $\mathcal{H}(\mathrm{GL}_n(\mathbb{A}_F^\infty), U)$-equivariant injection $(\pi^\infty)^U \hookrightarrow H^{b_n}_{cusp}(X_U, M_{\lambda,\mathbb{C}})$. The same holds for the top degree t_n.

Definition 1.2.15. *We define the* ***inner cohomology*** *by $H^*_! = im(H^*_c \to H^*)$, where H^*_c is the compactly supported cohomology.*

Then $H^*_!(X_U, M_{\mu,A})$ for $A \in \{\mathcal{O}, E, \mathbb{C}\}$ is also acted on by the Hecke algebras by restriction of the action on $H^*(X_U, M_{\mu,A})$. Moreover, there is a Hecke-equivariant injection

$$H^*_{cusp}(X_U, M_{\mu,\mathbb{C}}) \hookrightarrow H^*_!(X_U, M_{\mu,\mathbb{C}}).$$

Lemma 1.2.16. *(a) $\mathcal{H}_\mathbb{C} := \mathcal{H}(G^S, U^S) \otimes_\mathbb{Z} \mathbb{C}$ acts semisimply on $H^*_!(X_U, M_{\mu,\mathbb{C}})$ and there is a decomposition*

$$H^*_!(X_U, M_{\mu,\mathbb{C}}) \cong \bigoplus_{\Pi \in \Pi_d} m(\Pi)(\Pi^{\infty,S})^{U^S},$$

where Π_d is the set of isomorphism classes of all discrete automorphic representations occurring as a subrepresentation of $L^2_d(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F), \chi)$ and $m(\Pi) \in \mathbb{Z}_{\geq 0}$.

(b) *If π is a cuspidal automorphic representation of character χ, then the $(\pi^{\infty,S})^{U^S}$-isotypic component is*

$$H^*_!(X_U, M_{\mu,\mathbb{C}})[(\pi^{\infty,S})^{U^S}] = H^*(\mathfrak{g}, K^\circ_\infty, \pi_\infty \otimes_{\mathbb{C}} M_{\mu,\mathbb{C}}) \otimes_{\mathbb{C}} (\pi^\infty)^U.$$

(c) *If $U^S := \mathrm{GL}_n(\prod_{v \notin S \cup \{\infty\}} \mathcal{O}_{F_v})$, then the ring*

$$\mathbb{T}^S_{\mathbb{C}}(H^*_!(X_U, M_{\mu,\mathbb{C}})) := im(\mathcal{H}_{\mathbb{C}} \to \mathrm{End}_{\mathbb{C}}(H^*_!(X_U, M_{\mu,\mathbb{C}})))$$

is reduced.

Proof. Let $H^*_2 = H^*(\mathfrak{g}, K^\circ_\infty, L^2_d(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F), \chi)_{sm} \otimes_{\mathbb{C}} M_{\mu,\mathbb{C}})$, where $()_{sm}$ stands for the smooth vectors. We have a Hilbert space decomposition

$$L^2_d(\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A}_F), \chi) \cong \widehat{\bigoplus}_{\Pi \in \Pi_d} m_d(\Pi)\Pi.$$

By the multiplicity one theorem for the discrete spectrum proved by Moeglin and Waldspurger [MW89], we have $m_d(\Pi) = 1$ for all such Π. We have a corresponding decomposition for the cohomology [BG83, Theorem 5.3]:

$$H^*_2 \cong \bigoplus_{\Pi \in \Pi_d} H^*(\mathfrak{g}, K^\circ_\infty, \Pi_\infty \otimes_{\mathbb{C}} M_{\mu,\mathbb{C}}) \otimes_{\mathbb{C}} (\Pi^\infty)^U. \tag{1.5}$$

Each Π^∞ is an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}^\infty_F)$, so it is factorisable, so $(\Pi^\infty)^U = (\Pi_S)^{U_S} \otimes_{\mathbb{C}} (\Pi^{\infty,S})^{U^S}$ is isomorphic to a direct sum of simple modules of $\mathcal{H}_{\mathbb{C}}$. It follows that H^*_2 is a semisimple $\mathcal{H}_{\mathbb{C}}$-module.

By a result of Borel [Clo90, Proposition 3.18], we have injections

$$H^*_{cusp}(X_U, M_{\mu,\mathbb{C}}) \hookrightarrow H^*_!(X_U, M_{\mu,\mathbb{C}}) \hookrightarrow \widetilde{H}^*_2 \tag{1.6}$$

where $\widetilde{H}^*_2$ is the image of $H^*_2 \to H^*(X_U, M_{\mu,\mathbb{C}})$. Submodules of semisimple modules are semisimple, so $H^*_!(X_U, M_{\mu,\mathbb{C}})$ is a semisimple $\mathcal{H}_{\mathbb{C}}$-module and part (a) follows.

For part (b), recall that discrete automorphic representations are isobaric (as they are Speh representations), so they satisfy strong multiplicity one, so $H^*_2[(\pi^{\infty,S})^{U^S}] = H^*(\mathfrak{g}, K^\circ_\infty, \pi_\infty \otimes_{\mathbb{C}} M_{\mu,\mathbb{C}}) \otimes_{\mathbb{C}} (\pi^\infty)^U$ by (1.5). But this is also the $(\pi^{\infty,S})^{U^S}$-isotypic component of the cuspidal cohomology by equation (1.4). Part (b) now follows from (1.6).

For part (c), assume $U^S := \mathrm{GL}_n(\prod_{v \notin S \cup \{\infty\}} \mathcal{O}_{F_v})$. By Satake isomorphism, $\mathcal{H}_{\mathbb{C}}$ is commutative. Thus, $(\Pi^{\infty,S})^{U^S}$ is 0-dimensional or 1-dimensional. Hence each element of $\mathcal{H}_{\mathbb{C}}$ acts on $(\Pi^{\infty,S})^{U^S}$ by a scalar. The result now follows from part (a). $\square$

1.3 Abstract congruence ideals

In this section, we will define congruence ideals as in [TU21, section 2.1]. We will also establish some of their properties in the abstract algebraic setting. These will be applied to the Hecke algebras and cohomology of locally symmetric space in the next section.

Let $\mathcal{O}$ be a complete discrete valuation ring with uniformizer ϖ and field of fractions E. Let T be a reduced finite flat local $\mathcal{O}$-algebra, $\lambda : T \to \mathcal{O}$ be an $\mathcal{O}$-algebra homomorphism. This, being a section of the structure map $\mathcal{O} \to T$, is necessarily surjective. We first recall a standard concept, which already appeared in [Wil95]:

Definition 1.3.1. $\eta_\lambda := \lambda(Ann_T(\ker\lambda))$.

It turns out that to study η_λ, it is useful to generalise this concept to modules over T.

Let M be a finitely generated T-module which is free over $\mathcal{O}$. Write $M_E = M \otimes_{\mathcal{O}} E$ and $T_E = T \otimes_{\mathcal{O}} E$. Note that $M \hookrightarrow M_E$ and $T \hookrightarrow T_E$ by $\mathcal{O}$-flatness of T and M. Also, λ induces an E-algebra map $\lambda_E : T_E \to E$. Note that T_E is a finite dimensional E-vector space, so it is Artinian. It follows that

$$\begin{aligned} T_E &\cong \prod_{\mathfrak{p}\in SpecT_E} (T_E)_{\mathfrak{p}} && (1.7)\\ &\cong \prod_{\mathfrak{p}\in SpecT_E} T_E/\mathfrak{p} && (1.8)\\ &\cong E \times \prod_{\mathfrak{p}\neq ker\lambda_E} T_E/\mathfrak{p} && (1.9)\end{aligned}$$

Here, (1.7) is given by the diagonal map; (1.8) is true as T_E is reduced; (1.9) is true by the first isomorphism theorem. The upshot is that we have a canonical decomposition

$$T_E \cong E \times T_E^c$$

of E-algebras, where the first projection is given by λ_E.

Let $e_\lambda = e \in T_E$ be the element corresponding to $(1, 0) \in E \times T_E^c$. In other words, e is the unique element of T_E such that $\lambda_E(e) = 1$ and $e \in \bigcap_{\mathfrak{p}\neq\ker\lambda_E} \mathfrak{p}$. Define two T-submodules of M_E by

$$M^\lambda := eM$$

and

$$M_\lambda := eM \cap M = M[\ker\lambda]$$

where the last equality is proved in lemma 1.3.3. In [TU21, section 2.1], M_λ is defined as $eM_E \cap M$, but it is equivalent to our definition, because if $m \in eM_E \cap M$, then $m = em \in M \cap eM$.

Definition 1.3.2. *Define the* ***congruence module*** $C_0^\lambda(M)$ *by*

$$C_0^\lambda(M) := \frac{M^\lambda}{M_\lambda}$$

and the ***congruence ideal*** *to be its Fitting ideal*

$$\eta_\lambda(M) := Fitt_{\mathcal{O}}(C_0^\lambda(M)).$$

Note that $C_0^\lambda(M)$ is a finite torsion $\mathcal{O}$-module, so $\eta^\lambda(M)$ is completely determined by the cardinality of $C_0^\lambda(M)$. More precisely, if $C_0^\lambda(M)$ has cardinality $|\mathcal{O}/\varpi|^a$, then $\eta_\lambda(M) = (\varpi^a)$.

To see how definitions 1.3.1, 1.3.2 are related, note:

Lemma 1.3.3. *Let M and T be as above.*

1. $M_\lambda = M[\ker \lambda] := \{m \in M : (\ker \lambda)m = 0\}$.
2. $\eta_\lambda(T) = \eta_\lambda$.

Proof. For (1), we first consider a slightly more general setup. Suppose we have a product of commutative rings $A \times B$ acting on N. We can decompose N into $N_1 \times N_2$ accordingly. It is then clear that $N_1 = N[\ker \pi_A]$, where $\pi_A : A \times B \to A$ is the first projection.

In our case, taking $A \times B = E \times T_E^c$ and $N = M_E$ shows $eM_E = M_E[\ker \lambda_E]$. Hence $M_\lambda = eM_E \cap M = \{m \in M : m \otimes 1 \in M_E \text{ is annihilated by } (\ker \lambda) \otimes_{\mathcal{O}} E\} = M[\ker \lambda]$ since M is a free $\mathcal{O}$-module.

For (2), note that under the $\mathcal{O}$-module isomorphism $T_E \cong E \times T_E^c$, the $\mathcal{O}$-module $T^\lambda = eT$ corresponds to $\lambda(T) \times 0 = \mathcal{O} \times 0$ while $T_\lambda = T[\ker \lambda]$ corresponds to $\lambda(T[\ker \lambda]) \times 0$. Hence, $C_0^\lambda(T) = \mathcal{O}/\lambda(T[\ker \lambda])$ and $\eta_\lambda(M) = \lambda(T[\ker \lambda]) = \eta_\lambda$. $\square$

Observe that $M^\lambda = eM \subset M_E$ is torsion free and finitely generated over $\mathcal{O}$, so it is a free $\mathcal{O}$-module. The same is true for $M_\lambda = M[\ker \lambda]$.

Lemma 1.3.4. *(cf. [TU21, equation (2.2)]) If $rk_{\mathcal{O}} M_\lambda = 1$, then $C_0^\lambda(M) = \mathcal{O}/\eta_\lambda(M)$ and $\eta_\lambda(M) \supset \eta_\lambda$.*

Proof. We know that M^λ/M_λ is a finite torsion $\mathcal{O}$-module, so $rk_{\mathcal{O}}(M^\lambda) = rk_{\mathcal{O}}(M_\lambda) = 1$. Thus, there exists $m \in M$ such that $eM = \mathcal{O}em$ as $\mathcal{O}$-module. Then we have a surjection of $\mathcal{O}$-modules

$$C_0^\lambda(T) = \frac{eT}{T[\ker \lambda]} \twoheadrightarrow C_0^\lambda(M) = \frac{eM}{M[\ker \lambda]}$$
$$x \mapsto xm.$$

The lemma now follows from the observation that $C_0^\lambda(T) = \mathcal{O}/\eta_\lambda$. $\square$

Lemma 1.3.5. *Let T, λ, e be as above. Let $\widetilde{T}$ be a finite flat local $\mathcal{O}$-algebra, $\widetilde{\lambda} : \widetilde{T} \to \mathcal{O}$ be an $\mathcal{O}$-algebra homomorphism, and $\widetilde{e}$ be the corresponding idempotent in $\widetilde{T}$. Let M_1, M_2 be T-module and $\widetilde{T}$-module respectively that are finite free over $\mathcal{O}$. Suppose that there is an $\mathcal{O}$-bilinear perfect pairing*[11]

$$[\,,\,] : M_1 \times M_2 \to \mathcal{O}$$

such that[12] $[eM_1, (1-\widetilde{e})M_2] = 0$ *and* $[(1-e)M_1, \widetilde{e}M_2] = 0$.

[11]This means the two induced maps $M_1 \to \mathrm{Hom}_{\mathcal{O}}(M_2, \mathcal{O})$ and $M_1 \to \mathrm{Hom}_{\mathcal{O}}(M_2, \mathcal{O})$ are isomorphisms.

[12]We also write $[\,,\,]$ for its extension $(M_1)_E \times (M_2)_E \to E$.

(a) *Then* $[\,,\,]$ *induces an* $\mathcal{O}$*-bilinear perfect pairing*

$$C_0^{\lambda}(M_1) \times C_0^{\widetilde{\lambda}}(M_2) \to E/\mathcal{O}$$

and $\eta_\lambda(M_1) = \eta_{\widetilde{\lambda}}(M_2)$.

(b) *If* $M_1[\ker \lambda]$ *and* $M_2[\ker \widetilde{\lambda}]$ *are both free* $\mathcal{O}$*-modules of rank 1 with respective bases* δ_1, δ_2*, then*

$$\eta_\lambda(M_1) = \eta_{\widetilde{\lambda}}(M_2) = ([\delta_1, \delta_2]).$$

The key observation is that for every finite torsion $\mathcal{O}$-module N, we have a (non-canonical) isomorphism $N \cong \mathrm{Hom}_{\mathcal{O}}(N, E/\mathcal{O})$. Hence, by considering the cardinalities of $C_0^{\lambda}(M_1)$ and $C_0^{\widetilde{\lambda}}(M_2)$, we know the pairing in (a) is perfect if and only if it is non-degenerate.

This lemma appears similar to [TU21, Proposition 2.3], but a key difference is that we do not assume $[tx, y] = [x, ty]$ for all $t \in T$. Instead, our analogous conditions are $[eM_1, (1 - \widetilde{e})M_2] = 0$ and $[(1 - e)M_1, \widetilde{e}M_2] = 0$. This distinction is important because we will later apply this lemma to the cup product, which satisfies our conditions but not theirs. In the GL_2 setting, they twisted the pairing by the Atkin-Lehner involution to make their conditions hold, but we are unaware of any such involution for GL_n.

Proof. It is easy to see that $[\,,\,]$ extends to an E-bilinear perfect pairing $[\,,\,] : (M_1)_E \times (M_2)_E \to E$. Let $em_1 \in eM_1$ (with $m_1 \in M_1$) and $\widetilde{e}m_2 \in M_2 \cap \widetilde{e}M_2$ (with $m_2 \in M_2$). Then $[em_1, \widetilde{e}m_2] = [em_1 + (1 - e)m_1, \widetilde{e}m_2] = [m_1, \widetilde{e}m_2] \in \mathcal{O}$ because $(m_1, \widetilde{e}m_2) \in M_1 \times M_2$. A symmetric consideration shows $[\,,\,]$ induces a map

$$\langle\,,\,\rangle : C_0^{\lambda}(M_1) \times C_0^{\widetilde{\lambda}}(M_2) \to E/\mathcal{O}.$$

To show $\langle\,,\,\rangle$ is non-degenerate, we let $m_1 \in M_1$. Suppose that

$$\langle em_1, -\rangle \in \mathrm{Hom}_{\mathcal{O}}(C_0^{\widetilde{\lambda}}(M_2), E/\mathcal{O})$$

is zero. This means that for all $m_2 \in M_2$,

$$[em_1, \widetilde{e}m_2] = [em_1, \widetilde{e}m_2 + (1 - \widetilde{e})m_2] = [em_1, m_2] \in \mathcal{O}$$

so $[em_1, -] \in \mathrm{Hom}_{\mathcal{O}}(M_2, \mathcal{O})$. By perfectness, there exists $n \in M_1$ such that $[em_1, -] = [n, -]$, so $n = em_1 \in eM_1 \cap M_1 = (M_1)_\lambda$ by perfectness again, so $em_1 = 0 \in C_0^{\lambda}(M_1)$. By symmetry, $\langle\,,\,\rangle$ is non-degenerate and hence perfect. Also,

$$C_0^{\lambda}(M_1) \cong \mathrm{Hom}_{\mathcal{O}}(C_0^{\widetilde{\lambda}}(M_2), E/\mathcal{O}) \cong C_0^{\widetilde{\lambda}}(M_2).$$

For part (b), let $(\varpi^a) := \eta_\lambda(M_1) = \eta_{\widetilde{\lambda}}(M_2)$. By lemma 1.3.4, $C_0^{\lambda}(M_1)$ and $C_0^{\widetilde{\lambda}}(M_2)$ are free $\mathcal{O}/(\varpi^a)$-modules of rank 1, with respective bases b_1, b_2 say. By part (a), we have isomorphism

$$\frac{\mathcal{O}}{(\varpi^a)} \cong \frac{\mathcal{O}}{(\varpi^a)} b_1 \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}\left(\frac{\mathcal{O}}{(\varpi^a)} b_2, E/\mathcal{O}\right) \xrightarrow{\sim} \frac{\varpi^{-a}\mathcal{O}}{\mathcal{O}} \xrightarrow{\cdot \varpi^a} \frac{\mathcal{O}}{(\varpi^a)}$$

$$f \mapsto f(b_2).$$

This isomorphism maps 1 to $[b_1, b_2]\varpi^a \pmod{\varpi^a}$, so $[b_1, b_2]\varpi^a \in \mathcal{O}^\times$. As $\mathcal{O}b_i/\mathcal{O}\delta_i \cong \mathcal{O}/(\varpi^a)$, we know $\delta_i \in \varpi^a b_i \mathcal{O}^\times$. We deduce that $[\delta_1, \delta_2] \in \varpi^a \mathcal{O}^\times$, as desired. $\square$

The following lemma explains why η_λ is called the congruence ideal:

Lemma 1.3.6. *Let E be a non-Archimedean local field with ring of integers $\mathcal{O}$, T be a reduced finite flat local $\mathcal{O}$-algebra, $\lambda : T \to \mathcal{O}$ be an $\mathcal{O}$-algebra homomorphism. Then $\eta_\lambda \neq \mathcal{O}$ if and only if there is a finite field extension L of E and an $\mathcal{O}$-algebra homomorphism $\lambda' : T \to \mathcal{O}_L$ such that (viewing λ as a homomorphism to $\mathcal{O}_L$) we have $\lambda \neq \lambda'$ and*

$$\lambda \equiv \lambda' \pmod{\varpi_L}.$$

Here $\mathcal{O}_L$ is the ring of integers of L and ϖ_L is a uniformizer of $\mathcal{O}_L$.

Remark 1.3.7. Recall that T is finite over $\mathcal{O}$, so any $\mathcal{O}$-algebra homomorphism $T \to \overline{E}$ has image in $\mathcal{O}_L$ for some finite field extension L of E. Thus, we can rephrase the lemma: $\eta_\lambda \neq \mathcal{O}$ if and only if there is an $\overline{E}$-algebra homomorphism $\lambda' : T \otimes_{\mathcal{O}} \overline{E} \to \overline{E}$ such that $\lambda' \neq \lambda \otimes_{\mathcal{O}} \overline{E}$ and $|\lambda(t) - \lambda'(t)| < 1$ for all $t \in T$, where $|\cdot|$ is an absolute value on $\overline{E}$ extending that of E.

Proof. Given λ, we can decompose $T \otimes_{\mathcal{O}} E \cong E \times T_E^c$ and get an idempotent $e \in T \otimes_{\mathcal{O}} E$ as before. Define $T^c := im(T \to T \otimes_{\mathcal{O}} E \to T_E^c)$.

We assume that λ' is as in the statement of the lemma. We claim that λ' factors through $T \to T^c$. As $\mathcal{O}_L \subset \mathcal{O}_L \otimes_{\mathcal{O}} E$, it suffices to show

$$\lambda'_E := \lambda' \otimes E : T \otimes_{\mathcal{O}} E \to \mathcal{O}_L \otimes_{\mathcal{O}} E$$

factors through T_E^c. Note that $\mathcal{O}_L \otimes_{\mathcal{O}} E = L$ is an integral domain while $T \otimes_{\mathcal{O}} E \cong E \times T_E^c$, so λ'_E must factor through E or T_E^c. Since λ'_E is an E-algebra homomorphism and $\lambda'_E \neq \lambda_E$, λ'_E cannot factor through E. Hence λ'_E must factor through T_E^c, as claimed. Suppose $\eta_\lambda = \mathcal{O}$. This means $eT \cap T = eT$, i.e. $T \supset eT$. In particular, $(1, 0) = e \cdot (1, 1) \in T$. Then

$$\begin{aligned} 1 &= \lambda(1, 0) \\ &\equiv \lambda'(1, 0) \pmod{\varpi_L} \\ &\equiv 0 \pmod{\varpi} \end{aligned}$$

where the last equality holds since λ' factors through T^c. We get a contradiction, so $\eta_\lambda \neq \mathcal{O}$.

Conversely, suppose $\eta_\lambda \neq \mathcal{O}$. The key observation is that $\mathcal{O}/\eta_\lambda = \mathcal{O} \otimes_T T^c$, because $\ker(T \twoheadrightarrow T^c) = T \cap eT$. Thus we have an $\mathcal{O}$-algebra homomorphism

$$f : T^c \twoheadrightarrow \mathcal{O} \otimes_T T^c = \mathcal{O}/\eta_\lambda \twoheadrightarrow \mathcal{O}/\varpi.$$

We get a commutative diagram

$$\begin{tikzcd}
& \mathcal{O} \arrow[dr, two heads] & \\
T \arrow[ur, two heads] \arrow[dr, two heads] & & \mathcal{O}/\varpi \\
& T^c \arrow[ur, two heads, "f"'] &
\end{tikzcd}$$

We want to lift f. There is a classical argument due to Deligne and Serre. Let $\mathfrak{m}^c := \ker f$. Consider the structure map $\mathcal{O} \to T^c$. As $\varpi \in \mathfrak{m}^c$, the prime ideal $\mathfrak{m}^c$ lies above $\varpi\mathcal{O}$. Note that multiplication by ϖ is invertible in T_E, so T_E and hence T^c is $\mathcal{O}$-torsion free. Thus, T^c is a finite free $\mathcal{O}$-module. By flatness, the going down property holds, so there is a prime ideal $\mathfrak{p}^c \subset \mathfrak{m}^c$ lying above (0), so $\mathcal{O} \hookrightarrow T^c/\mathfrak{p}^c$. This is a finite extension, so

$$L := Frac(T^c/\mathfrak{p}^c)$$

is a finite extension of E. We know $T^c/\mathfrak{p}^c$ is finite over $\mathcal{O}$ and hence integral over $\mathcal{O}$, so

$$T^c/\mathfrak{p}^c \subset \mathcal{O}_L.$$

We want to show that this inclusion is a local homomorphism of local rings. As $\mathcal{O}$ is a Henselian ring and $T^c/\mathfrak{p}^c$ is an integral domain, $T^c/\mathfrak{p}^c$ is a local ring. Note $\mathfrak{m}' := \varpi_L\mathcal{O}_L \cap T^c/\mathfrak{p}^c$ is a prime ideal of $T^c/\mathfrak{p}^c$ lying above $\varpi\mathcal{O}$ (since its pullback to $\mathcal{O}$ is the preimage of $\varpi_L\mathcal{O}_L$ under the $\mathcal{O}$-algebra map $\mathcal{O} \to T^c/\mathfrak{p}^c \to \mathcal{O}_L$). By the incomparability theorem for injective integral ring extensions, $\mathfrak{m}'$ is a maximal ideal of $T^c/\mathfrak{p}^c$. Hence, the fact that $T^c/\mathfrak{p}^c$ is a local ring implies $\mathfrak{m}' = \mathfrak{m}^c/\mathfrak{p}^c$. This gives us another commutative diagram

$$\begin{tikzcd}
& T^c/\mathfrak{m}^c \arrow[r] & \mathcal{O}_L/\varpi_L \\
T^c \arrow[ur, two heads, "f"] \arrow[r, two heads] & T^c/\mathfrak{p}^c \arrow[u, two heads] \arrow[r, hook] & \mathcal{O}_L \arrow[u]
\end{tikzcd}$$

Combining the two diagrams gives us

$$\begin{tikzcd}
& \mathcal{O} \arrow[drr] & & \\
T \arrow[ur, "\lambda"] \arrow[dr] & & & \mathcal{O}_L/\varpi_L \\
& T^c \arrow[r] & \mathcal{O}_L \arrow[ur] &
\end{tikzcd}$$

Denote the bottom map $T \to \mathcal{O}_L$ by λ'. It remains to show $\lambda \neq \lambda'$ as maps to $\mathcal{O}_L$. If this is false, then for all $(x, y) \in T \subset \mathcal{O} \times T^c$, we have $x - y \in \mathfrak{p}^c$ by construction of λ'. In particular, if $(x, y) \in T \cap eT$, then $y = 0$ and $x \in \mathcal{O} \cap \mathfrak{p}^c = (0)$ as $\mathfrak{p}^c$ lies above (0). This means $T \cap eT = 0$, which is false as $\varpi^a e \in T \cap eT$ for $a \in \mathbb{Z}$ sufficiently large. $\square$

1.4 $L(1,\pi,\mathrm{Ad}^\circ)$ and congruences for automorphic representations

In this section, we will apply the results from the last section to study congruence ideals of Hecke algebras and cohomology of locally symmetric spaces. Then we will relate them to Selmer groups.

We shall need some notation and assumptions. Let

- F be a number field
- p be a prime. Starting from section 1.4.2, we will also assume

$$p > 2.$$

- $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ be a fixed isomorphism. We will often use it implicitly.
- π be a cuspidal, regular algebraic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight $\iota\mu$, where $\mu \in (\mathbb{Z}^n)^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_p)}$
- $U_{v,m} = \{g \in \mathrm{GL}_n(\mathcal{O}_{F_v}) : \text{the last row of } g \pmod{\varpi_v^m} \text{ is } \begin{pmatrix} 0 & \dots & 0 & 1\end{pmatrix}\}$ for $m \in \mathbb{Z}_{\geq 0}$, where ϖ_v is a uniformiser of F_v. Let $f_v \in \mathbb{Z}_{\geq 0}$ be the unique integer such that[13]

$$\dim_{\mathbb{C}}(\pi_v)^{U_{v,f_v}} = 1.$$

  Let $U = \prod_{v\nmid\infty} U_v$, where $U_v = U_{v,f_v}$. Assume that U is **neat**.
- S be a finite set of finite places of F containing all v such that π_v is ramified.
- $G^S = \mathrm{GL}_n(\mathbb{A}_F^{S,\infty})$ and $U^S := \mathrm{GL}_n(\prod_{v\notin S\cup\{\infty\}} \mathcal{O}_{F_v})$.
- $E \subset \overline{\mathbb{Q}}_p$ be a local field containing $\iota^{-1}(\mathbb{Q}(\pi))$ [14]and the image of every embedding $F \hookrightarrow \overline{\mathbb{Q}}_p$
- $\mathcal{O}$ be the ring of integers of E, ϖ a uniformizer of $\mathcal{O}$
- $\epsilon \in \widehat{K_\infty/K_\infty^\circ}$ a permissible signature (Definition 1.2.11)
- X_U the locally symmetric space of $\mathrm{GL}_{n,F}$ (Section 1.2.1), ∂X_U the boundary of its Borel-Serre compactification.
- $\mathbb{T}^S := \mathcal{H}(G^S, U^S) \otimes_{\mathbb{Z}} \mathcal{O}$.

Note that $\mathbb{T}^S$ is a commutative $\mathcal{O}$-algebra. If M is an $\mathcal{O}$-module equipped with an $\mathcal{O}$-algebra homomorphism $\mathbb{T}^S \to \mathrm{End}_{\mathcal{O}}(M)$, then we define

$$\mathbb{T}^S(M) := im(\mathbb{T}^S \to \mathrm{End}_{\mathcal{O}}(M)).$$

[13]Such a f_v always exists by the uniqueness of local new vector [GH24, Theorem 11.5.6].

[14]$\mathbb{Q}(\pi)$ is the field of rationality of π [Clo90, section 3 page 101].

For completeness, let us also remark that in Lemma 1.4.1 below, we shall show that there is a Hecke eigensystem $\Lambda : \mathbb{T}^S \to \mathcal{O}$ attached to π. We will let

$$\mathfrak{m} := \ker(\Lambda \pmod{\varpi}).$$

1.4.1 Hecke eigensystems

We first show that we have a Hecke eigensystem attached to π.

Lemma 1.4.1. *We have an $\mathcal{O}$-algebra homomorphism*

$$\Lambda : \mathbb{T}^S \to \mathcal{O}$$

sending $t \in \mathbb{T}^S$ to its eigenvalue on $(\iota^{-1}\pi^\infty)^U$. This homomorphism factors through $\mathbb{T}^S(H_!^*(X_U, M_{\mu,\mathcal{O}}))$, where $H_!^* = im(H_c^* \to H^*)$ is the inner cohomology and $H^* = \oplus_{i\geq 0} H^i$.

Proof. In this proof, we may sometime abuse notation and regard a $\mathbb{C}$-vector space as an $\mathcal{O}$-module via the map $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. Let $\mathcal{H} := \mathcal{H}(G^S, U^S)$.

As $\mathrm{GL}_n(\mathbb{A}_F^S)$ acts on the $\overline{\mathbb{Q}}_p$-vector space $\iota^{-1}\pi^\infty$, we know $\mathbb{T}^S$ acts on the U^S-invariant $(\iota^{-1}\pi^\infty)^{U^S}$ and hence also on $(\iota^{-1}\pi^\infty)^U$. Moreover, for all finite $v \notin S$, π_v is unramified so $\pi_v^{U_v}$ is a one dimensional $\overline{\mathbb{Q}}_p$-vector space. It follows that each element of $\mathbb{T}^S$ acts by a scalar on $(\iota^{-1}\pi^\infty)^U$, so we get an $\mathcal{O}$-algebra homomorphism

$$\mathbb{T}^S \to \overline{\mathbb{Q}}_p \tag{1.10}$$

sending an element to its eigenvalue.

Note that $\mathbb{T}^S(H_!^*(X_U, M_{\mu,\mathcal{O}}))$ acts on $H_!^*(X_U, M_{\mu,\mathcal{O}}) \otimes_{\mathcal{O}} \mathbb{C} \cong H_!^*(X_U, M_{\mu,\mathbb{C}})$. Since π is cohomological of weight μ, we have $\mathcal{H}$-equivariant injections

$$(\pi^\infty)^U \hookrightarrow H_{cusp}^*(X_U, M_{\mu,\mathbb{C}}) \hookrightarrow H_!^*(X_U, M_{\mu,\mathbb{C}}).$$

Pick any non-zero $x \in (\pi^\infty)^U$ and let y be its image under this injection. For all $t \in \mathbb{T}^U$, its eigenvalue on $(\iota^{-1}\pi^\infty)^U$ only depends on how it acts on y and this is determined by the image of t in $\mathbb{T}^S(H_!^*(X_U, M_{\mu,\mathcal{O}}))$. It follows that (1.10) factors through $\mathbb{T}^S(H_!^*(X_U, M_{\mu,\mathcal{O}}))$. It remains to show that the image of (1.10) lies in $\mathcal{O}$.

We first show that its image lies in E. By [Clo90, Proposition 3.1], there is a $\mathrm{GL}_n(\mathbb{A}_F^\infty)$-stable $\mathbb{Q}(\pi)$-vector subspace W of π^∞ such that $\pi^\infty = W \otimes_{\mathbb{Q}(\pi)} \mathbb{C}$. Then $(\pi^\infty)^U = W^U \otimes_{\mathbb{Q}(\pi)} \mathbb{C}$. Let $h \in \mathcal{H}$. We have already seen that it acts by a scalar on $(\pi^\infty)^U$, so the same is true for W^U. As W^U is a $\mathbb{Q}(\pi)$-vector space, the scalar must lies in $\mathbb{Q}(\pi)$. Hence the image of (1.10) lies in $\iota^{-1}(\mathbb{Q}(\pi)) \subset E$.

Finally, by the existence of Borel-Serre compactification of X_U, we know that $H_!^*(X_U, M_{\mu,\mathcal{O}})$ is a finite $\mathcal{O}$-module, so

$$H_!^*(X_U, M_{\mu,\mathcal{O}})/\mathcal{O}\text{-torsion}$$

is a finite free $\mathcal{O}$-module stable under $\mathcal{H}$. Pick an $\mathcal{O}$-basis $\mathcal{B}$ for this module. We can then express the action of $\mathcal{H}$ on $H_!^*(X_U, M_{\mu,\mathcal{O}})/\mathcal{O}$-torsion by matrices with entries in $\mathcal{O}$.

We can view $\mathcal{B}$ as a $\mathbb{C}$-basis for $H_!^*(X_U, M_{\mu,\mathbb{C}})$. Then the action of $\mathcal{H}$ on $H_!^*(X_U, M_{\mu,\mathbb{C}})$ is given by the same matrices. Hence each eigenvalue of $\mathcal{H}$ on this space is a root of a monic polynomial over $\mathcal{O}$. We know they lie in E by the previous paragraph, so they lie in $\mathcal{O}$ as $\mathcal{O}$ is integrally closed. $\square$

We let $\mathfrak{m} = \ker(\Lambda \mod \varpi) \subset \mathbb{T}^S$, which is a maximal ideal. We define

$$T := \mathbb{T}^S(H_!^*(X_U, M_{\mu,\mathcal{O}}))_{\mathfrak{m}}/\mathcal{O}\text{-torsion}.$$

Lemma 1.4.2. *T is a reduced finite flat*[15] *local $\mathcal{O}$-algebra. Also, Λ induces a local $\mathcal{O}$-algebra map*

$$\lambda : T \to \mathcal{O}.$$

Proof. Let $H := H_!^*(X_U, M_{\mu,\mathcal{O}})$. By the existence of Borel-Serre compactification of X_U, H is a finite $\mathcal{O}$-module, so $\mathbb{T}^S(H)$ is a finite $\mathcal{O}$-algebra. We let

$$\Lambda' : \mathbb{T}^S(H) \to \mathcal{O}$$

be the map induced by Λ, which exists by lemma 1.4.1, and let $q : \mathbb{T}^S \to \mathbb{T}^S(H)$ be the quotient map, so $\Lambda = \Lambda' \circ q$. Then $\mathfrak{m} \supset \ker q$, so $q(\mathfrak{m})$ is a maximal ideal of $\mathbb{T}^S(H)$ and hence

$$\mathbb{T}^S(H)_{\mathfrak{m}} = \mathbb{T}^S(H)_{q(\mathfrak{m})}$$

is a finite local $\mathcal{O}$-algebra. The same is thus true for T. As T is a finitely generated torsion-free $\mathcal{O}$-module and $\mathcal{O}$ is a PID, T is flat over $\mathcal{O}$. Hence $T \hookrightarrow T \otimes_{\mathcal{O}} \mathbb{C}$. As $\mathbb{C}$ is $\mathcal{O}$-flat, $T \otimes_{\mathcal{O}} \mathbb{C} = \mathbb{T}^S_{\mathbb{C}}(H_!^*(X_U, M_{\mu,\mathbb{C}}))_{\mathfrak{m}}$, which is reduced by Lemma 1.2.16. Thus, T, which injects into $T \otimes_{\mathcal{O}} \mathbb{C}$, is also reduced.

Note that $q(\mathfrak{m}) = \ker(\Lambda' \mod \varpi)$. It follows that Λ' induces an $\mathcal{O}$-algebra map $\mathbb{T}^S(H_!^*(X_U, M_{\mu,\mathcal{O}}))_{q(\mathfrak{m})} \to \mathcal{O}$ and hence an $\mathcal{O}$-algebra map $\lambda : T \to \mathcal{O}$, i.e. a commutative diagram

$$\begin{tikzcd} T \arrow[r, "\lambda"] & \mathcal{O} \\ \mathcal{O} \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

It follows that $\lambda^{-1}(\varpi\mathcal{O})$ is a prime ideal of T lying above $\varpi\mathcal{O}$ (i.e. the preimage of $\lambda^{-1}(\varpi\mathcal{O})$ along the structure map $\mathcal{O} \to T$ is $\varpi\mathcal{O}$). Since T is a local $\mathcal{O}$-algebra, the maximal ideal of T also lies above $\varpi\mathcal{O}$. By the incomparability theorem for injective integral ring extensions, we deduce that $\lambda^{-1}(\varpi\mathcal{O})$ is the maximal ideal of T. Thus, λ is a local homomorphism. $\square$

Remark 1.4.3. Let us show that

$$T \cong \mathbb{T}^S(\overline{H}_!^*(X_U, M_\mu)_{\mathfrak{m}}),$$

[15] Note that since $\mathcal{O}$ is a PID, an $\mathcal{O}$-module is finite flat if and only if it is finite free.

where $\overline{H}_!^*(X_U, M_\mu) := H_!^*(X_U, M_\mu)/\mathcal{O}$-torsion. (In the following, a bar on top of an $\mathcal{O}$-module will usually mean the module modulo its $\mathcal{O}$-torsion.) Write $H = H_!^*(X_U, M_{\mu,\mathcal{O}})$. There is an obvious surjection

$$\mathbb{T}^S(H) \to \mathbb{T}^S((\overline{H})_\mathfrak{m}). \tag{1.11}$$

Let $t \in \mathbb{T}^S(H)$. Since H is a finitely generated $\mathcal{O}$-module and $(\overline{H})_\mathfrak{m}$ is a quotient of it, we know that t is in the kernel of (1.11) if and only if there exists $a \in \mathbb{N}, b \in \mathbb{T}^S - \mathfrak{m}$ such that $\varpi^a bt = 0$, which is equivalent to $t \in \ker(\mathbb{T}^S(H) \twoheadrightarrow \mathbb{T}^S(H)_\mathfrak{m}/\mathcal{O}\text{-tors})$. Thus (1.11) induces the desired isomorphism.

1.4.2 Congruence ideals for automorphic representations

The previous lemma means we are now in the situation of section 1.3.

Definition 1.4.4. *With the setup above, define the* ***congruence ideal***

$$\eta_\pi := \eta_\lambda = \lambda(Ann_T(\ker\lambda)) = Fitt_\mathcal{O}\left(\frac{eT}{eT \cap T}\right)$$

as in definitions 1.3.1, 1.3.2, where e is the idempotent in $T \otimes_\mathcal{O} E$ corresponding to $(1,0)$ in the decomposition $T \otimes_\mathcal{O} E \cong E \times T_E^c$ induced by λ. For $i \in \{b,t\}$ and a permissible $\epsilon \in \widehat{K_\infty/K_\infty^\circ}$, define the ***cohomological congruence ideal***

$$\eta_{\pi,i,\epsilon} := \eta_\lambda(H_!^i(X_U, M_{\mu,\mathcal{O}})_\mathfrak{m}[\epsilon]/\mathcal{O}\textit{-torsion}) = Fitt_\mathcal{O}\left(\frac{eH}{eH \cap H}\right)$$

where $H = H_!^i(X_U, M_{\mu,\mathcal{O}})_\mathfrak{m}[\epsilon]/\mathcal{O}$-torsion.

Let $\mathcal{O}_{\overline{\mathbb{Q}}_p}$ be the ring of integers of $\overline{\mathbb{Q}}_p$ and $\mathfrak{m}_{\overline{\mathbb{Q}}_p}$ be its maximal ideal.

Lemma 1.4.5. *Fix $i \in \{b,t\}$ and a permissible $\epsilon \in \widehat{K_\infty/K_\infty^\circ}$. Let $p > 2$ and $\mathbb{T}^S_{\overline{\mathbb{Q}}_p} := \mathbb{T}^S \otimes_\mathcal{O} \overline{\mathbb{Q}}_p$.*

(a) Let $\mathfrak{n}$ be a maximal ideal of $\mathbb{T}^S$ and $V = H^i(X_U, M_{\mu,\overline{\mathbb{Q}}_p})_\mathfrak{n}$. Then

$$V = \oplus_{\alpha:\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V)\to\overline{\mathbb{Q}}_p} V[(\ker\alpha)^\infty]$$

where α runs over $\overline{\mathbb{Q}}_p$-algebra homomorphisms $\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V) \to \overline{\mathbb{Q}}_p$ and $[(\ker\alpha)^\infty]$ is the set of elements annihilated by some power of $\ker\alpha$. Any such α satisfies $(\alpha|_{\mathbb{T}^S})^{-1}(\mathfrak{m}_{\overline{\mathbb{Q}}_p}) = \mathfrak{n}$.

(b) Let $\mathfrak{n}$ be a maximal ideal of $\mathbb{T}^S$ and $W = H_!^i(X_U, M_{\mu,\overline{\mathbb{Q}}_p})_\mathfrak{n}$. Then

$$W = \oplus_{\alpha:\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(W)\to\overline{\mathbb{Q}}_p} W[\ker\alpha]$$

where α runs over $\overline{\mathbb{Q}}_p$-algebra homomorphisms $\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(W) \to \overline{\mathbb{Q}}_p$. Any such α satisfies $(\alpha|_{\mathbb{T}^S})^{-1}(\mathfrak{m}_{\overline{\mathbb{Q}}_p}) = \mathfrak{n}$.

(c) *Let $H = H^i_!(X_U, M_{\mu,\mathcal{O}})_{\mathfrak{m}}[\epsilon]/\mathcal{O}$-torsion. Then $H[\ker\lambda]$ is a free $\mathcal{O}$-module of rank one whose base change to $\mathbb{C}$ is canonically isomorphic to*

$$H^i_!(X_U, M_{\mu,\mathbb{C}})[(\pi^{\infty,S})^{U^S} \times \epsilon]$$

as a $\mathbb{C}$-vector space.

(d) *$\eta_{\pi,i,\epsilon} \supset \eta_\pi$. Equality holds if H is a free T-module of rank 1.*

Proof. Note that $\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V)$ is finite dimensional as a $\overline{\mathbb{Q}}_p$-vector space, so it is an Artinian ring. In particular, it is the (finite) product of its localisations at maximal ideas. Hence we get a corresponding descomposition

$$V = \oplus_{\mathfrak{p}\in \mathrm{mSpec}\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V)} V_{\mathfrak{p}}.$$

By [Bel21, Section 2.5.1], $V_{\mathfrak{p}} = V[\mathfrak{p}^\infty]$. By Zariski's lemma, $\mathrm{mSpec}\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V) = \{\ker\alpha \mid \alpha : \mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V) \to \overline{\mathbb{Q}}_p$ is a $\overline{\mathbb{Q}}_p$-algebra homomorphism$\}$. We thus get the desired decomposition of V.

To see that $(\alpha|_{\mathbb{T}^S})^{-1}(\mathfrak{m}_{\overline{\mathbb{Q}}_p}) = \mathfrak{n}$, note that $\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V) = \mathbb{T}^S(H^i(X_U, M_\mu)_{\mathfrak{n}}) \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p$ by $\mathcal{O}$-flatness of $\overline{\mathbb{Q}}_p$. Thus, α gives rise to an $\mathcal{O}$-algebra homomorphism $\mathbb{T}^S(H^i(X_U, M_\mu)_{\mathfrak{n}}) \to \overline{\mathbb{Q}}_p$. The image of the this homomorphism necessarily lies in $\mathcal{O}_{\overline{\mathbb{Q}}_p}$ because $\mathbb{T}^S(H^i(X_U, M_\mu)_{\mathfrak{n}})$ is finitely generated as an $\mathcal{O}$-module. By the same reasoning as Remark 1.4.3,

$$\mathbb{T}^S(H^i(X_U, M_\mu)_{\mathfrak{n}}) = \mathbb{T}^S(H^i(X_U, M_\mu))_{\mathfrak{n}}.$$

Thus, we get an $\mathcal{O}$-algebra homomorphism $a : \mathbb{T}^S(H^i(X_U, M_\mu))_{\mathfrak{n}} \to \mathcal{O}_{\overline{\mathbb{Q}}_p}$. We have $a^{-1}(\mathfrak{m}_{\overline{\mathbb{Q}}_p}) = \mathfrak{n}\mathbb{T}^S(H^i(X_U, M_\mu))_{\mathfrak{n}}$ by the going up theorem [Sta25, Lemma 00GU]. This proves $(\alpha|_{\mathbb{T}^S})^{-1}(\mathfrak{m}_{\overline{\mathbb{Q}}_p}) = \mathfrak{n}$.

Part (b) can be proved in the same way using Lemma 1.2.16(a). One can also prove it in a more straightforward manner.

For part (c), we know $T[\ker\lambda]$ is a finite free $\mathcal{O}$-module since it is torsion-free and finitely generated. To find its rank, note that $\ker\lambda$ is the image of $\ker\Lambda = (t - \Lambda(t) : t \in \mathbb{T}^S)$ under the projection $q : \mathbb{T}^S \twoheadrightarrow T$. As T is Noetherian, we can pick $t_1, \dots, t_n \in \mathbb{T}^S$ such that $q((t_i - \Lambda(t_i) : 1 \le i \le n)) = \ker\lambda$. Then

$$H[\ker\lambda] = H[\{t_i - \Lambda(t_i) : 1 \le i \le n\}] = \ker(H \xrightarrow{m \mapsto (t_i - \Lambda(t_i)m)} H^n).$$

Taking kernel commutes with flat base change, so

$$\begin{aligned} H[\ker\lambda] \otimes_{\mathcal{O}} \mathbb{C} &= (H^i_!(X_U, M_{\mu,\mathcal{O}})_{\mathfrak{m}}[\epsilon] \otimes_{\mathcal{O}} \mathbb{C})[\ker\lambda] \\ &= H^i_!(X_U, M_{\mu,\mathbb{C}})_{\mathfrak{m}}[\epsilon][\ker\lambda] \\ &= H^i_!(X_U, M_{\mu,\mathbb{C}})[\epsilon][\ker\lambda] \qquad (1.12) \\ &= H^i_!(X_U, M_{\mu,\mathbb{C}})[(\pi^{\infty,S})^{U^S}][\epsilon] \\ &= H^i(\mathfrak{g}, K^\circ_\infty, \pi_\infty \otimes_{\mathbb{C}} M_{\mu,\mathbb{C}})[\epsilon] \otimes_{\mathbb{C}} (\pi^\infty)^U. \qquad (1.13) \end{aligned}$$

To see that equation (1.12) holds, note that $\mathcal{O}$ is a Henselian local ring, so

$$\mathbb{T}^S(H^i_!(X_U, M_{\mu,\mathcal{O}})) = \oplus_{\mathfrak{n}} \mathbb{T}^S(H^i_!(X_U, M_{\mu,\mathcal{O}}))_{\mathfrak{n}},$$

where $\mathfrak{n}$ runs over the maximal ideal of $\mathbb{T}^S$ that contains $\ker(\mathbb{T}^S \twoheadrightarrow \mathbb{T}^S(H^i_!(X_U, M_{\mu,\mathcal{O}})))$. Thus, $H^i_!(X_U, M_{\mu,\mathbb{C}}) = \oplus_{\mathfrak{n}} H^i_!(X_U, M_{\mu,\mathbb{C}})_{\mathfrak{n}}$. By part (b), if $\mathfrak{n} \neq \mathfrak{m}$, then

$$H^i_!(X_U, M_{\mu,\mathbb{C}})_{\mathfrak{n}}[\ker \lambda] = 0.$$

Equation (1.13) holds by lemma 1.2.16. By Lemma 1.2.12 and the choice of U, (1.13) is a one dimensional $\mathbb{C}$-vector space. This proves part (c).

The first part of (d) now follows from lemma 1.3.4. The remaining part is easy. $\square$

1.4.3 Betti-Whittaker periods

We shall mostly follow [BR17] to define the Betti-Whittaker period for this subsection. Fix all Haar measures as in that paper. Let $F, \pi, \ldots$ be defined as before.

Definition 1.4.6. *Let $i \in \{b_n, t_n\}$. Pick the generator w_∞ (depends on i) of the 1-dimensional space (Lemma 1.2.12)*

$$H^i(\mathfrak{g}, K^0_\infty, W(\pi_\infty) \otimes M_{\mu,\mathbb{C}})[\epsilon]$$

as in [BR17, Section 3.2.2]. Fix a continuous unitary homomorphism $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}^\times$ such that ψ_v is non-trivial[16] for all v. We shall abuse notation and let ψ also denote the corresponding standard character on the unipotent radical of G, i.e. $\psi(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n})$. Let $W(\pi^\infty)$ be the Whittaker model of π^∞ with respect to ψ^∞. Let V_π be the subspace[17] of $L^2_0(G(F)\backslash G(\mathbb{A}_F), \chi)$ realizing π. Define $\mathcal{F}_{\pi^\infty,\epsilon,w_\infty,i}$ as the composition of the isomorphisms

$$\begin{aligned} W(\pi^\infty)^U &\xrightarrow{\sim} W(\pi^\infty)^U \otimes_{\mathbb{C}} H^i(\mathfrak{g}, K^0_\infty, W(\pi_\infty) \otimes M_{\mu,\mathbb{C}})[\epsilon] \\ &\xrightarrow{\sim} H^i(\mathfrak{g}, K^0_\infty, W(\pi)^U \otimes M_{\mu,\mathbb{C}})[\epsilon] \\ &\xrightarrow{\sim} H^i(\mathfrak{g}, K^0_\infty, V^U_\pi \otimes M_{\mu,\mathbb{C}})[\epsilon] \\ &= H^i_{cusp}(X_U, M_{\mu,\mathbb{C}})[(\pi^\infty)^U \times \epsilon] \\ &\xrightarrow{\sim} H^i_!(X_U, M_{\mu,\mathbb{C}})[(\pi^{\infty,S})^{U^S} \times \epsilon]. \end{aligned}$$

The first map is $w^\infty \mapsto w^\infty \otimes w_\infty$; the second map is trivial; the third map is the inverse of the map

$$\begin{aligned} V_\pi &\xrightarrow{\sim} W(\pi) \\ f &\mapsto \left(g \mapsto \int_{U_n(F)\backslash U_n(\mathbb{A}_F)} f(ug)\psi^{-1}(u)du \right) \end{aligned}$$

[16] Such a ψ exists, e.g. take $\psi_v(x) = e^{-2\pi i x}$ for real v, $\psi_v(x) = e^{-2\pi i(x+\bar{x})}$ for complex v, and $\psi_v(x) = e^{2\pi i Tr_{F_v/\mathbb{Q}_p}(x)}$ for all $v \mid p$ and all rational primes p, and $\psi = \prod_v \psi_v$.

[17] Well-defined by the multiplicity one theorem.

and U_n is the unipotent radical of the standard Borel subgroup B_n. The last isomorphism is by Lemma 1.2.16.

Definition 1.4.7. *For each finite place v, let w_v be the essential vector*[18] *of π_v in its ψ_v-Whittaker model. Let $w^\infty = \otimes_{v<\infty} w_v$. Then $w^\infty \in W(\pi^\infty)^U$ by our choice of U and the definition of essential vector. Let $H = H^i_!(X_U, M_{\mu,\mathcal{O}})_\mathfrak{m}[\epsilon]/\mathcal{O}$-torsion. By Lemma 1.4.5, $H[\ker\lambda] \otimes_\mathcal{O} \mathbb{C} = H^i_!(X_U, M_{\mu,\mathbb{C}})[(\pi^{\infty,S})^{U^S} \times \epsilon]$. We define the* ***period***

$$\mathfrak{p}_{\pi,i,\epsilon}$$

to be the number in $\mathbb{C}^\times$ such that $\mathcal{F}_{\pi^\infty,\epsilon,w_\infty,i}(w^\infty)/\mathfrak{p}_{\pi,i,\epsilon}$ is an $\mathcal{O}$-generator of $H[\ker\lambda]$. This is well defined up to multiplication by $\mathcal{O}^\times$ by Lemma 1.4.5(c).

1.4.4 $L(1, \pi, \mathrm{Ad}^\circ)$ and congruence ideals

Recall that $U_v = \mathrm{GL}_n(\mathcal{O}_v)$ for all finite $v \notin S$. Let

$$\begin{aligned}\theta : \mathcal{H}^S(G^S, U^S) &\to \mathcal{H}^S(G^S, U^S)\\ [U^S g U^S] &\mapsto [U^S g^{-1} U^S].\end{aligned}$$

This is an $\mathcal{O}$-algebra isomorphism.[19]

Lemma 1.4.8. *The Hecke eigensystem $\widetilde{\Lambda} : \mathbb{T}^S \to \mathcal{O}$ attached to the contragredient $\widetilde{\pi}$ is given by $\Lambda \circ \theta$.*

Proof. It is enough to show the analogous fact after base changing from $\mathcal{O}$ to $\mathbb{C}$ and working at a single place. Let $v \notin S$ be a finite place, $G = \mathrm{GL}_n(F_v)$, $K = \mathrm{GL}_n(\mathcal{O}_{F_v})$, $V = \pi_v$.

It is well known that the restriction map induces an isomorphism

$$(\widetilde{V})^K \xrightarrow{\sim} \widetilde{V^K} \tag{1.14}$$

so in particular $(\widetilde{V})^K$ is 1-dimensional.

Let $g \in G$. By the same proof as [DS05, Lemma 5.5.1 (c)], there exist $g_1, \dots, g_m \in G$ such that $KgK = \sqcup_i g_i K = \sqcup K g_i$. For all $f \in (\widetilde{V})^K$ and $v \in V^K$,

$$([KgK]f)(v) = (\sum g_i \cdot f)(v) = f(\sum g_i^{-1} v) = f([Kg^{-1}K]v) = \lambda([Kg^{-1}K])f(v).$$

By (1.14), $[KgK]f = \lambda([Kg^{-1}K])f$, as desired. $\square$

Proposition 1.4.9 (Poincaré duality)**.** *Let $d = dim X_U$. The cup product induces a perfect pairing*

$$[\,,\,] : H^i_c(X_U, M_\mu)/(\mathcal{O} - tors) \times H^{d-i}(X_U, M_\mu^\vee)/(\mathcal{O} - tors) \to \mathcal{O}$$

[18]If π_v is unramified, then w_v is the unique element in $W(\pi_v, \psi_v)^{\mathrm{GL}_n(\mathcal{O}_{F_v})}$ with $w_v(I_n) = 1$.
[19]This is a homomorphism because $\mathcal{H}^S(G^S, U^S)$ is commutative.

where[20] $M_\mu^\vee = \mathrm{Hom}_{\mathcal{O}}(M_\mu, \mathcal{O})$. *If S is a finite set of finite places such that $U_v = \mathrm{GL}_n(\mathcal{O}_v)$ for all finite $v \notin S$, then*

$$[tx, y] = [x, \theta(t)y]$$

for all $t \in \mathcal{H}^S(G^S, U^S)$, $x \in H_c^i(X_U, M_\mu)/(\mathcal{O} - tors)$, $y \in H^{d-i}(X_U, M_\mu^\vee)/(\mathcal{O} - tors)$.

A version of this is proved in [Har08, Theorem 4.8.9], but the proof is not that easy. We shall deduce this from Verdier duality instead.

Proof. As stated in [ACC+23, Proposition 2.2.20], we have by Verdier duality an isomorphism

$$RHom_{\mathcal{O}}(R\Gamma_c(X_U, M_\mu), \mathcal{O}) \cong R\Gamma(X_U, M_\mu^\vee)[d]$$

in the derived category of $\mathcal{O}$-modules $D(\mathcal{O})$. It follows from one of the spectral sequences for Ext that we have a spectral sequence

$$E_2^{i,j} = Ext_{\mathcal{O}}^i(H_c^{-j}(X_U, M_\mu), \mathcal{O}) \Rightarrow E^{i+j} = H^{i+j+d}(X_U, M_\mu^\vee).$$

Since $\mathcal{O}$ is a PID, the only non-zeros terms lie in $\{(i,j) : 0 \leq i \leq 1, -d \leq j \leq 0\}$. For all $j \in \mathbb{Z}$, we have an exact sequence

$$0 \to E_2^{1,-j-1} \to E^{-j} \to E_2^{0,-j} \to 0$$

i.e.

$$0 \to Ext_{\mathcal{O}}^1(H_c^{j+1}(X_U, M_\mu), \mathcal{O}) \to H^{d-j}(X_U, M_\mu^\vee) \xrightarrow{f} \mathrm{Hom}_{\mathcal{O}}(H_c^j(X_U, M_\mu), \mathcal{O}) \to 0.$$

Since $H_c^{j+1}(X_U, M_\mu)$ is finitely generated over $\mathcal{O}$ by the existence of Borel-Serre compactification, the second term is $\mathcal{O}$-torsion. On the other hand, the 4th term is $\mathcal{O}$-torsion free. It follows that $\ker f$ is precisely the $\mathcal{O}$-torsion of $H^{d-j}(X_U, M_\mu^\vee)$, so f induces an isomorphism

$$\widetilde{f} : H^{d-j}(X_U, M_\mu^\vee)/(\mathcal{O} - \mathrm{tors}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(H_c^j(X_U, M_\mu)/(\mathcal{O} - \mathrm{tors}), \mathcal{O}).$$

We have a pairing

$$H^{d-j}(X_U, M_\mu^\vee)/(\mathcal{O} - \mathrm{tors}) \times H_c^j(X_U, M_\mu)/(\mathcal{O} - \mathrm{tors}) \to \mathcal{O}$$
$$(a, b) \mapsto \widetilde{f}(a)(b).$$

This is a perfect pairing because both $H^{d-j}(X_U, M_\mu^\vee)/(\mathcal{O} - \mathrm{tors})$ and $H_c^j(X_U, M_\mu)/(\mathcal{O} - \mathrm{tors})$ are finite free $\mathcal{O}$-modules and $\widetilde{f}$ is an isomorphism. It is well-known that this is given by the cup product. The last assertion about the action of the Hecke algebra follows from [ACC+23, Proposition 2.2.20]. □

[20] In general, $M_\mu^\vee$ and $M_{\mu^\vee}$ are ***not*** isomorphic.

The cup product induces a pairing

$$[\,,\,] : H_!^i(X_U, M_\mu)/(\mathcal{O} - \text{tors}) \times H_!^{d-i}(X_U, M_\mu^\vee)/(\mathcal{O} - \text{tors}) \to \mathcal{O}.$$

To see this, it is enough to show that the cup product induces a pairing

$$H_!^i(X_U, M_\mu) \times H_!^{d-i}(X_U, M_\mu^\vee) \to \mathcal{O}.$$

We thus need to show that the map

$$\begin{aligned} H_!^i(X_U, M_\mu) \times H_!^{d-i}(X_U, M_\mu^\vee) &\to H_c^d(X_U, M_\mu \otimes M_\mu^\vee) \\ (x, y) &\mapsto x_c \cup y_c \end{aligned}$$

is well-defined, where x_c is a lift of x to $H_c^i(X_U, M_\mu)$, y_c is a lift of y to $H_c^{d-i}(X_U, M_\mu^\vee)$, and $\cup$ is the cup product on $H_c^i(X_U, M_\mu) \times H_c^{d-i}(X_U, M_\mu^\vee)$. If x'_c (resp. y'_c) is another lift of x (resp. y) to $H_c^i(X_U, M_\mu)$ (resp. $H_c^{d-i}(X_U, M_\mu^\vee)$), then $x'_c = x_c + \epsilon$ and $y'_c = y_c + \delta$, where $\epsilon \in \ker(H_c^i(X_U, M_\mu) \to H^i(X_U, M_\mu))$ and $\delta \in \ker(H_c^{d-i}(X_U, M_\mu^\vee) \to H^{d-i}(X_U, M_\mu^\vee))$. Recall that to form the cup product of two elements in the compactly supported cohomology, we can first map one of the elements to the usual cohomoology and then take the cup product between usual and compactly supported cohomology. It follows that

$$x'_c \cup y'_c = x_c \cup y_c + \epsilon \cup y_c + x_c \cup \delta + \epsilon \cup \delta = x_c \cup y_c.$$

For convenience, let

$$\overline{H}_!^i(X_U, M_\mu) := H_!^i(X_U, M_\mu)/(\mathcal{O} - \text{tors})$$

and $\overline{H}_!^{d-i}(X_U, M_\mu^\vee) := H^{d-i}(X_U, M_\mu^\vee)/(\mathcal{O} - \text{tors})$.

Let $q_1 : \mathbb{T}^S \twoheadrightarrow \mathbb{T}^S(\overline{H}_!^b(X_U, M_\mu))$ be the quotient map. Note that $\mathbb{T}^S(\overline{H}_!^b(X_U, M_\mu))$ is a finite $\mathcal{O}$-algebra, so it is the (finite) product of its localisations at maximal ideals. Thus,

$$\overline{H}_!^b(X_U, M_\mu) = \bigoplus_{\substack{\mathfrak{m}_1 \in \mathrm{mSpec}\mathbb{T}^S \\ \mathfrak{m}_1 \supset \ker q_1}} \overline{H}_!^b(X_U, M_\mu)_{\mathfrak{m}_1}.$$

Note also that $supp_{\mathbb{T}^S}(\overline{H}_!^b(X_U, M_\mu)) = V(Ann_{\mathbb{T}^S}(\overline{H}_!^b(X_U, M_\mu))) = V(\ker q_1)$, so $\overline{H}_!^b(X_U, M_\mu)_{\mathfrak{m}_1} = 0$ if $\mathfrak{m}_1 \not\supset \ker q_1$. It follows that

$$\overline{H}_!^b(X_U, M_\mu) = \bigoplus_{\mathfrak{m}_1 \in \mathrm{mSpec}\mathbb{T}^S} \overline{H}_!^b(X_U, M_\mu)_{\mathfrak{m}_1}. \tag{1.15}$$

We have a similar decomposition for $\overline{H}_!^t(X_U, M_\mu^\vee)$. Thus, we can restrict $[\,,\,]$ to these direct summands to get pairings

$$\overline{H}_!^b(X_U, M_\mu)_{\mathfrak{m}_1} \times \overline{H}_!^t(X_U, M_\mu^\vee)_{\mathfrak{m}_2} \to \mathcal{O}$$

for any $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{mSpec}\mathbb{T}^S$. Similarly, we have decompositions for $\overline{H}^b(X_U, M_\mu)$ and $\overline{H}_!^t(X_U, M_\mu^\vee)$ into direct sums of their localisations at maximal ideals of $\mathbb{T}^S$. Thus, we can restrict $[\,,\,]$ to these direct summands to get pairings

$$\overline{H}^b(X_U, M_\mu)_{\mathfrak{m}_1} \times \overline{H}_c^t(X_U, M_\mu^\vee)_{\mathfrak{m}_2} \to \mathcal{O}$$

for any $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{mSpec}\mathbb{T}^S$.

Let ∂X_U denote the boundary of the Borel-Serre compactification of X_U. Let $\widetilde{\mathfrak{m}} = \theta(\mathfrak{m}) \subset \mathbb{T}^S$, which equals $\ker(\widetilde{\Lambda} \mod \varpi)$ by Lemma 1.4.8, where $\widetilde{\Lambda} : \mathbb{T}^S \to \mathcal{O}$ is the Hecke eigensystem attached to the contragredient $\widetilde{\pi}$. Let $\widetilde{\epsilon} : K_\infty/K_\infty^\circ \to \{\pm 1\}$ be the character such that for every real place v, if $x_v \in K_v/K_v^\circ$ is non-trivial, then $\widetilde{\epsilon}(x_v) = (-1)^{n-1}\epsilon(x_v)$.

Corollary 1.4.10. *Assume $H^b(\partial X_U, M_\mu)_\mathfrak{m}$ is $\mathcal{O}$-torsion free and $p > 2$.*

(a) Then

$$[\,,\,] : \overline{H}_!^b(X_U, M_\mu)_\mathfrak{m} \times \overline{H}_!^t(X_U, M_\mu^\vee)_{\widetilde{\mathfrak{m}}} \to \mathcal{O} \tag{1.16}$$

and

$$[\,,\,] : \overline{H}_!^b(X_U, M_\mu)_\mathfrak{m}[\epsilon] \times \overline{H}_!^t(X_U, M_\mu^\vee)_{\widetilde{\mathfrak{m}}}[\widetilde{\epsilon}] \to \mathcal{O}$$

are both perfect pairings.

(b) θ induces[21] an isomorphism $\mathbb{T}^S(\overline{H}_!^b(X_U, M_\mu)_\mathfrak{m}) \cong \mathbb{T}^S(\overline{H}_!^t(X_U, M_\mu^\vee)_{\widetilde{\mathfrak{m}}})$.

Proof.

Claim. Let $m_1, m_2 \subset \mathbb{T}^S$ be maximal ideals of $\mathbb{T}^S$. Then

$$[\overline{H}^b(X_U, M_\mu)_{\mathfrak{m}_1}, \overline{H}_c^t(X_U, M_\mu^\vee)_{\mathfrak{m}_2}] = 0$$

unless $\mathfrak{m}_2 = \theta(\mathfrak{m}_1)$. Similarly,

$$[\overline{H}_!^b(X_U, M_\mu)_{\mathfrak{m}_1}, \overline{H}_!^t(X_U, M_\mu^\vee)_{\mathfrak{m}_2}] = 0$$

unless $\mathfrak{m}_2 = \theta(\mathfrak{m}_1)$.

We shall show the first one. The second one is similar and easier. To show this, we let $A := H^b(X_U, M_{\mu,\overline{\mathbb{Q}}_p})_{\mathfrak{m}_1}, B := H_c^t(X_U, M^\vee_{\mu,\overline{\mathbb{Q}}_p})_{\mathfrak{m}_2}$. Suppose $[A, B] \neq 0$. By Lemma 1.4.5(a) and its analogue for the compactly supported cohomology, there exist $a \in A[(\ker\alpha)^\infty], b \in B[(\ker\beta)^\infty]$ such that $[a, b] \neq 0$, where $\alpha, \beta : \mathbb{T}^S_{\overline{\mathbb{Q}}_p} \to \overline{\mathbb{Q}}_p$ are $\overline{\mathbb{Q}}_p$-algebra homomorphisms with $(\alpha|_{\mathbb{T}^S})^{-1}(\mathfrak{m}_{\overline{\mathbb{Q}}_p}) = \mathfrak{m}_1$ and $(\beta|_{\mathbb{T}^S})^{-1}(\mathfrak{m}_{\overline{\mathbb{Q}}_p}) = \mathfrak{m}_2$. Let $t \in \mathbb{T}^S$. Then $t - \alpha(t) \in \ker\alpha$, so $(t - \alpha(t))^r a = 0$ for some $r \geq 1$. For all $y \in B[(\ker\beta)^\infty]$, we have

$$0 = [(t - \alpha(t))^r a, y] = [x, (\theta(t) - \alpha(t))^r y]$$

[21]By 'induces', we mean the map given by lifting an element of $\mathbb{T}^S(\overline{H}_!^b(X_U, M_\mu)_\mathfrak{m})$ to $\mathbb{T}^S$, applying θ to it, and then projecting it to $\mathbb{T}^S(\overline{H}_!^t(X_U, M_\mu^\vee)_{\widetilde{\mathfrak{m}}})$.

Thus, $(\theta(t)-\alpha(t))^r$ is not surjective as an $\overline{\mathbb{Q}}_p$-linear endomorphism of $B[(\ker\beta)^\infty]$, so it is not injective, so $(\theta(t)-\alpha(t))^r c = 0$ for some $c \in B[(\ker\beta)^\infty] \setminus \{0\}$. By the definition of B, there exists $s \geq 1$ such that $(\theta(t)-\beta(\theta(t)))^s c = 0$. Since $c \neq 0$, we must have $\alpha(t) = \beta(\theta(t))$. Thus, $\alpha|_{\mathbb{T}^S} = \beta|_{\mathbb{T}^S} \circ \theta$. It follows that $\mathfrak{m}_1 = (\alpha|_{\mathbb{T}^S})^{-1}(\mathfrak{m}_{\overline{\mathbb{Q}}_p}) = (\beta|_{\mathbb{T}^S} \circ \theta)^{-1}(\mathfrak{m}_{\overline{\mathbb{Q}}_p}) = \theta(\mathfrak{m}_2)$.

Using this claim, we shall now deduce the first part of (a) by a similar argument to [BR17, section 4.2.4] under our assumption that $H^b(\partial X_U, M_\mu)_\mathfrak{m}$ is $\mathcal{O}$-torsion free.

We first note that the base change of (1.16) to E

$$[\,,\,] : H^b_!(X_U, M_{\mu,E})_\mathfrak{m} \times H^t_!(X_U, M^\vee_{\mu,E})_{\widetilde{\mathfrak{m}}} \to E \tag{1.17}$$

is a perfect pairing, because this is non-degenerate by Poincaré duality and the claim proved above. Similarly,

$$[\,,\,] : H^b(X_U, M_{\mu,E})_\mathfrak{m} \times H^t_c(X_U, M^\vee_{\mu,E})_{\widetilde{\mathfrak{m}}} \to E \tag{1.18}$$

is a perfect pairing.

Next, note that since $\overline{H}^b_!(X_U, M_\mu)_\mathfrak{m}$ and $\overline{H}^t_!(X_U, M^\vee_\mu)_{\widetilde{\mathfrak{m}}}$ are both finite free $\mathcal{O}$-modules, to show that (1.16) is a perfect pairing, it is enough to show that the induced map

$$\overline{H}^b_!(X_U, M_\mu)_\mathfrak{m} \to \mathrm{Hom}_\mathcal{O}(\overline{H}^t_!(X_U, M^\vee_\mu)_{\widetilde{\mathfrak{m}}}, \mathcal{O}) \tag{1.19}$$

is an isomorphism. To see injectivity, note that $\overline{H}^b_!(X_U, M_\mu)_\mathfrak{m} \hookrightarrow \overline{H}^b_!(X_U, M_\mu)_\mathfrak{m} \otimes_\mathcal{O} E = H^b_!(X_U, M_{\mu,E})_\mathfrak{m}$ and similarly for the right hand side of (1.19). Thus, we deduce the injectivity of (1.19) from the non-degeneracy of (1.17).

To check surjectivity of (1.19), let $f \in \mathrm{Hom}_\mathcal{O}(\overline{H}^t_!(X_U, M^\vee_\mu)_{\widetilde{\mathfrak{m}}}, \mathcal{O})$. Analogous to (1.15), $\overline{H}^t_!(X_U, M^\vee_\mu)$ is the direct sum of its localisations at maximal ideals of $\mathbb{T}^S$. We extend f to $f' : \overline{H}^t_!(X_U, M^\vee_\mu) \to \mathcal{O}$ by setting it to be 0 on other summands. Let $q : \overline{H}^t_c(X_U, M^\vee_\mu) \twoheadrightarrow \overline{H}^t_!(X_U, M^\vee_\mu)$ be the natural map. By Proposition 1.4.9, there exists $x \in \overline{H}^b(X_U, M_\mu)$ such that

$$f' \circ q = [x, -].$$

By the claim, we can assume that x lies in the direct summand $\overline{H}^b(X_U, M_\mu)_\mathfrak{m}$ of $\overline{H}^b(X_U, M_\mu)$. On the other hand, by the perfectness of (1.19), there exists $z \in H^b_!(X_U, M_{\mu,E})_\mathfrak{m}$ such that $f_E = [z, -]$, where f_E is the base change of f along $\mathcal{O} \to E$. Then

$$f'_E \circ q_E = [z, -]$$

as maps $H^t_c(X_U, M^\vee_{\mu,E}) \to E$. By the perfectness of (1.18), $x_E = z$, where x_E is the image of x in $H^b(X_U, M_{\mu,E})_\mathfrak{m}$. In particular,

$$x_E \in H^b_!(X_U, M_{\mu,E})_\mathfrak{m}.$$

Recall that there is a long exact sequence

$$\cdots \to H^i_c(X_U, M_\mu) \to H^i(X_U, M_\mu) \to H^i(\partial X_U, M_\mu) \to \ldots$$

We lift x to $H^b(X_U, M_\mu)_\mathfrak{m}$ and consider its image y in $H^b(X_U, M_\mu)_\mathfrak{m}/H^b_!(X_U, M_\mu)_\mathfrak{m} \subset H^b(\partial X_U, M_\mu)_\mathfrak{m}$. By what we have just shown, $y_E \in H^b(\partial X_U, M_{\mu,E})_\mathfrak{m}$ is 0. Since $H^b(\partial X_U, M_\mu)_\mathfrak{m}$ is $\mathcal{O}$-torsion-free, $y = 0$. Thus, $x \in \overline{H}^b_!(X_U, M_\mu)_\mathfrak{m}$.

For the second part of (a), note that we have a decomposition

$$\overline{H}^b_!(X_U, M_\mu)_\mathfrak{m} = \bigoplus_{\epsilon_1 \in \widehat{K_\infty/K^\circ_\infty}} \overline{H}^b_!(X_U, M_\mu)_\mathfrak{m}[\epsilon_1]$$

since $p > 2$. The perfectness then follows from the first part and the proof of [BR17, Proposition 3.3.1]. (The proof there works here in view of the decomposition of $H^i(\mathfrak{g}, K^0_\infty, \pi_\infty \otimes_\mathbb{C} M_{\mu,\mathbb{C}})$ in the last part of the proof of 1.2.12.)

Part (b) follows from part (a) and the same argument as [ACC+23, Corollary 2.2.21], namely the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{T}^S \arrow[r] \arrow[d, "\theta"'] & End_\mathcal{O}(\overline{H}^b_!(X_U, M_\mu)_\mathfrak{m}) \arrow[d, "\text{transpose}"] \\
\mathbb{T}^S \arrow[r] & End_\mathcal{O}(\overline{H}^t_!(X_U, M^\vee_\mu)_{\widetilde{\mathfrak{m}}})
\end{tikzcd}$$

□

Lemma 1.4.11. *Let $\phi, \widetilde{\phi}$ be the specific cusp forms in the space of cusp forms affording $\pi, \widetilde{\pi}$ respectively which are defined in [BR17, Section 2.2].*[22] *Define*

$$\langle \phi, \widetilde{\phi} \rangle = \int_{A_G G(F)\backslash G(\mathbb{A}_F)} \phi(g)\widetilde{\phi}(g)dg.$$

Then

$$\langle \phi, \widetilde{\phi} \rangle = \frac{\prod_{v|\infty} \mathfrak{c}^\sharp_v(w_v, \widetilde{w}_v) \cdot L(1, \pi, \mathrm{Ad}^\circ)}{\alpha_F\ \mathfrak{p}_{ram}(\pi)},$$

where $\alpha_F := \frac{\widehat{\Phi}_f(0)}{n \operatorname{Res}_{s=1} \widetilde{\zeta}_F(s)}$ with Φ_f the characteristic function of $\prod_{v\nmid\infty} \mathcal{O}^n_{F_v}$, $\widehat{\Phi}_f$ its Fourier transform, and $\widetilde{\zeta}_F$ the completed zeta function. The measures, $\mathfrak{c}^\sharp_v(w_v, \widetilde{w}_v)$, and $\mathfrak{p}_{ram}(\pi)$ are defined as in [BR17, section 2]. Also, $L(1, \pi, \mathrm{Ad}^\circ)$ is the value at 1 of the Langlands L-function[23].

Proof. Note that $\int_{A_G G(F)\backslash G(\mathbb{A}_F)} = \int_{Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} \int_{A_G G(F)\backslash Z(\mathbb{A}_F)G(F)}$, where Z is the centre of G. Also, $A_G G(F)\backslash Z(\mathbb{A}_F)G(F) = F^\times\backslash \mathbb{A}^1_F$, where $\mathbb{A}^1_F := \{x \in \mathbb{A}_F : |x|_{\mathbb{A}_F} = 1\}$.

[22]The corresponding Whittaker vectors of $\phi, \widetilde{\phi}$ are tensor products of local Whittaker vectors. At finite places, the local Whittaker vectors are the essential vectors. At infinite places, the local Whittaker vectors are the 'cohomological vectors' as defined in [BR17].

[23]The Langlands L-function is defined as the product over all places of F of the local L-factors. The local L-factor can be defined as the local L-factor of the corresponding Weil(-Deligne) representation under the local Langlands correspondence at a (non-)Archimedean place.

It follows that

$$\langle \phi, \widetilde{\phi} \rangle = vol(F^\times \backslash \mathbb{A}_F^1) \int_{Z(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} \phi(g)\widetilde{\phi}(g)dg.$$

The result now follows from [BR17, equation (2.2.11)]. They obtained their result by relating the Petersson inner product with $L(1, \pi, \mathrm{Ad}^0)$ by the Rankin-Selberg method and using the fact that $L(s, \pi \times \widetilde{\pi}) = \widetilde{\zeta}_F(s)L(s, \pi, \mathrm{Ad}^0)$. □

Lemma 1.4.12. *We have*

$$[\vartheta^\circ_{b,\epsilon}, \widetilde{\vartheta}^\circ_{t,\widetilde{\epsilon}}] = L^{alg}(1, \pi, \mathrm{Ad}^0, \epsilon),$$

where [,] *is the pairing induced by cup product as before,* $\vartheta^\circ_{b,\epsilon}$ *is an* $\mathcal{O}$*-basis of* $\overline{H}^b_!(X_U, M_\mu)_{\mathfrak{m}}[\epsilon][\ker \lambda]$, $\widetilde{\vartheta}^\circ_{t,\widetilde{\epsilon}}$ *is an* $\mathcal{O}$*-basis of* $\overline{H}^t_!(X_U, M^\vee_\mu)_{\widetilde{\mathfrak{m}}}[\widetilde{\epsilon}][\ker \widetilde{\lambda}]$*, and*

$$L^{alg}(1, \pi, \mathrm{Ad}^0, \epsilon) := \frac{L(1, \pi, \mathrm{Ad}^0)}{\alpha_F\, \mathfrak{p}_{ram}(\pi)\mathfrak{p}_\infty(\pi)\mathfrak{p}_{\pi,b,\epsilon}\mathfrak{p}_{\widetilde{\pi},t,\widetilde{\epsilon}}}.$$

Here, $\mathfrak{p}_\infty(\pi)$ *is defined as in [BR17, equation (3.3.9)] and* $\mathfrak{p}_{\pi,b,\epsilon}, \mathfrak{p}_{\widetilde{\pi},t,\widetilde{\epsilon}}$ *are defined in Definition 1.4.7.*[24]

Proof. This is more or less what [BR17, section 3.3.3] obtained, except that our space X_U is different from the locally symmetric spaces they used. As in [BR17, p.658], we have

$$\begin{aligned}
[\mathfrak{p}_{\pi,b,\epsilon}\vartheta^\circ_{b,\epsilon}, \mathfrak{p}_{\widetilde{\pi},t,\widetilde{\epsilon}}\widetilde{\vartheta}^\circ_{t,\widetilde{\epsilon}}] &= \frac{1}{vol(U)} \int_{G(F)\backslash G(\mathbb{A}_F)/K^\circ_\infty U} \varsigma \\
&= \frac{1}{vol(U)} \int_{G(F)\backslash G(\mathbb{A}_F)/A_G U} \varsigma \\
&= \int_{G(F)\backslash G(\mathbb{A}_F)/A_G} \varsigma \\
&= \frac{L(1, \pi, \mathrm{Ad}^0)}{\alpha_F\, \mathfrak{p}_{ram}(\pi)\mathfrak{p}_\infty(\pi)}
\end{aligned}$$

where ς has the same meaning as that in [BR17, p.658] and in the last equality we used Lemma 1.4.11 instead of [BR17, equation (2.2.11)]. Dividing both sides by $\mathfrak{p}_{\pi,b,\epsilon}\mathfrak{p}_{\widetilde{\pi},t,\widetilde{\epsilon}}$ gives the result. □

Remark 1.4.13. α_F depends only on F, $\mathfrak{p}_{ram}(\pi)$ depends only on the ramified components of π, and $\mathfrak{p}_\infty(\pi)$ depends only on π_∞.

We can now prove our first main theorem.

[24] However, our $L^{alg}(1, \pi, \mathrm{Ad}^0, \epsilon)$ and periods are slightly different from that in [BR17] due to our different choice of X_U.

Theorem 1.4.14. *Let the assumptions be as at the start of Section 1.4. Suppose that* $H^b(\partial X_U, M_\mu)_\mathfrak{m}$ *is* $\mathcal{O}$*-torsion free. Then*

$$\eta_{\pi,b,\epsilon} = \eta_{\widetilde{\pi},t,\widetilde{\epsilon}} = (L^{alg}(1, \pi, \mathrm{Ad}^0, \epsilon)).$$

Proof. Since $H^b(\partial X_U, M_\mu)_\mathfrak{m}$ is $\mathcal{O}$-torsion free, the cup product gives a perfect pairing

$$[\,,\,] : \overline{H}^b_!(X_U, M_\mu)_\mathfrak{m}[\epsilon] \times \overline{H}^t_!(X_U, M_\mu^\vee)_{\widetilde{\mathfrak{m}}}[\widetilde{\epsilon}] \to \mathcal{O}$$

by Corollary 1.4.10.[25] We would like to apply Lemma 1.3.5. The conditions in part (b) of that lemma is satisfied by Lemma 1.4.5 and the fact that $(\pi^\infty)^U$ and $(\widetilde{\pi}^\infty)^U$ are one dimensional.

Recall that $\lambda : T \to \mathcal{O}$ is the Hecke eigensystem for π. This factors through $\mathbb{T}^S(\overline{H}^b_!(X_U, M_\mu)_\mathfrak{m})$ because the action of the Hecke algebra on $\overline{H}^*_!(X_U, M_\mu)_\mathfrak{m}$ preserves degree and π is isomorphic to a submodule of $\overline{H}^b_!(X_U, M_\mu)_\mathfrak{m} \otimes_\mathcal{O} \mathbb{C}$.

Similar to how we defined the idempotent $e \in T_E$ using $\lambda : T \to \mathcal{O}$, we can define an idempotent $e' \in \mathbb{T}^S(\overline{H}^b_!(X_U, M_\mu)_\mathfrak{m})_E$ using the induced map on the quotient $\mathbb{T}^S(\overline{H}^b_!(X_U, M_\mu)_\mathfrak{m}) \to \mathcal{O}$. It is clear from the definitions of e and e' that e' is the image of e under $T_E \twoheadrightarrow \mathbb{T}^S(\overline{H}^b_!(X_U, M_\mu)_\mathfrak{m})_E$.

The same argument (with $H^b_!(X_U, M_\mu)_\mathfrak{m}$ replaced by $H^t_!(X_U, M_\mu^\vee)_{\widetilde{\mathfrak{m}}}$) works for the contragredient $\widetilde{\pi}$, $\widetilde{\lambda}$, $\widetilde{e}, \widetilde{e}'$ and we know $\widetilde{e}'$ is the image of $\widetilde{e}$ under $\widetilde{T}_E \twoheadrightarrow \mathbb{T}^S(\overline{H}^t_!(X_U, M_\mu^\vee)_{\widetilde{\mathfrak{m}}})_E$.

By Lemma 1.4.8, we know $\widetilde{\lambda} = \lambda \circ \theta$. It follows from definition and part (b) of Corollary 1.4.10 that $\theta(e') = \widetilde{e}'$. Thus, for all $x \in \overline{H}^b_!(X_U, M_\mu)_\mathfrak{m}[\epsilon]$, $y \in \overline{H}^t_!(X_U, M_\mu^\vee)_{\widetilde{\mathfrak{m}}}[\widetilde{\epsilon}]$, we have

$$[ex, y] = [e'x, y] = [x, \theta(e')y] = [x, \widetilde{e}'y] = [x, \widetilde{e}y].$$

From this[26] and the fact that e, e' are idempotents, we deduce that all the conditions of Lemma 1.3.5 are satisfied.

By Lemma 1.3.5 and Lemma 1.4.12,

$$\eta_{\pi,b,\epsilon} = \eta_{\widetilde{\pi},t,\widetilde{\epsilon}} = [\vartheta^\circ_{b,\epsilon}, \widetilde{\vartheta}^\circ_{t,\widetilde{\epsilon}}] = (L^{alg}(1, \pi, \mathrm{Ad}^0, \epsilon)).$$

□

Remark 1.4.15. The theorem is a generalisation of [TU21, Proposition 4.12 first part and Lemma 5.6(iv)], where analogous results were obtained for GL_2 over a totally real field and over an imaginary quadratic field.[27]

[25]This is the only place in this proof where we use the assumption that $H^b(\partial X_U, M_\mu)_\mathfrak{m}$ is $\mathcal{O}$-torsion free.

[26]The reason that we need to argue via $e', \widetilde{e}'$ rather than $e, \widetilde{e}$ directly is that we do not have the analogue of part (b) of Corollary 1.4.10 for the entire inner cohomology. We only have it for the bottom and top degrees.

[27]At least in the totally real case, although it was not explicitly stated, they implicitly assumed that the ideal $\mathfrak{m}$ is non-Eisenstein, as they applied the Poincaré duality result from [TU21, Proposition 4.10 part 3], which was established only under this assumption in their paper.

An analogous result is stated in [BR17, section 4]. It differs from our result in the following ways: the result there is for the product of $L^{alg}(1, \pi, \mathrm{Ad}^0, \epsilon)$ over all permissible ϵ while our result is for each individual permissible ϵ. Additionally, we localize at a maximal ideal throughout, which is necessary for relating congruence ideals to Selmer groups (see Theorem 1.4.26 below). Localisation also makes some results slightly harder to prove, but requires a weaker hypothesis that $H^b(\partial X_U, M_\mu)_\mathfrak{m}$, rather than $H^b(\partial X_U, M_\mu)$, is $\mathcal{O}$-torsion free (see Lemma 1.4.19 below for a case where this holds). Although this weaker hypothesis was mentioned in [BR17, page 669], the necessary (small) adjustments to their proof were not worked out. We have provided more detailed arguments for certain parts. Furthermore, to facilitate the use of some results in [ACC+23] and [NT16], we work with a different locally symmetric space, preventing us from applying some results from [BR17] directly. For the same reason, our $L^{alg}(1, \pi, \mathrm{Ad}^0, \epsilon)$ is slightly different from theirs.

Remark 1.4.16. It may be possible to determine $\mathfrak{p}_\infty(\pi)$ explicitly as a power of $2\pi i$ using techniques of [GL21], but we have not attempted this. In that paper, they precisely determined some archimedean zeta-integrals by replacing π with simpler automorphic representations π' with $\pi_\infty \cong \pi'_\infty$. Here, 'simpler' means π' is automorphically induced from a Hecke character or is an isobaric sum of Hecke characters. This approach allows them to relate the L-function of π to those of Hecke characters, which, in turn, are related to CM periods by results of Blasius.

Now, we will explain why this is related to congruences of automorphic representations. Roughly speaking, if $L^{alg}(1, \pi, \mathrm{Ad}^0, \epsilon)$ is not a p-adic unit, then π is congruent to another automorphic representation. The converse holds if the maximal ideal $\mathfrak{m}$ is non-Eisenstein.

Corollary 1.4.17. *Let the assumptions be as at the start of Section 1.4. Suppose that $H^b(\partial X_U, M_\mu)_\mathfrak{m}$ is $\mathcal{O}$-torsion free. Consider the following two statements:*

(a) $\varpi \mid L^{alg}(1, \pi, \mathrm{Ad}^0, \epsilon)$.

(b) There is a discrete automorphic representation $\pi' \not\cong \pi$ of $\mathrm{GL}_n(\mathbb{A}_F)$ with $H^*_!(X_U, M_{\mu,\mathbb{C}})[(\pi'^{\infty,S})^{U^S}] \neq 0$ whose Hecke eigensystem[28] *$\Lambda' : \mathbb{T}^S \to \overline{\mathbb{Q}}_p$ satisfies $|\Lambda(t) - \Lambda'(t)|_p < 1$ for all $t \in \mathbb{T}^S$.*

Then (a) implies (b). If $H^b_!(X_U, M_{\mu,\mathcal{O}})_\mathfrak{m}[\epsilon]/\mathcal{O}$-torsion is a free T-module, then (b) implies (a).

Note that π' needs not be cuspidal even though we start with a cuspidal π. See Corollary 1.4.20 below however.

[28] This is defined using the fixed isomorphism $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ as in Lemma 1.4.1. The only differences are that $(\pi'^{\infty,S})^{U^S}$ only appears in the inner cohomology but not the cuspidal cohomology, and the image of Λ' needs not lie in $\mathcal{O}$, but only an integral extension.

Proof. Abusing notation, we shall identify $\overline{\mathbb{Q}}_p$ with $\mathbb{C}$ using our fixed isomorphism ι. Note that $\pi' \cong \pi$ if and only if $\Lambda' = \Lambda$ by the strong multiplicity one theorem (where we regard Λ as having codomain in $\overline{\mathbb{Q}}_p$).

Suppose $\varpi \mid L^{alg}(1, \pi, \mathrm{Ad}^0, \epsilon)$. By Theorem 1.4.14 and Lemma 1.3.4, $\eta_\pi \neq \mathcal{O}$. By Remark 1.3.7 there exists a $\overline{\mathbb{Q}}_p$-algebra homomorphism $\lambda' : T \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_p$ such that $\lambda' \neq \lambda \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p$ and $|\lambda(t) - \lambda'(t)| < 1$ for all $t \in T$. We have a natural map

$$\mathbb{T}^S(H^*_!(X_U, M_\mu)) \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p = \mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V) \to T \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p,$$

where $V := H^*_!(X_U, M_{\mu,\overline{\mathbb{Q}}_p})$ and $\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V) := im(\mathbb{T}^S \otimes_{\mathcal{O}} \bar{\mathbb{Q}}_p \to \mathrm{End}_{\bar{\mathbb{Q}}_p}(V))$. Composing this with λ', we get a $\overline{\mathbb{Q}}_p$-algebra homomorphism

$$f : \mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V) \to \overline{\mathbb{Q}}_p.$$

Let $\mathfrak{n} := \ker f$. Then

$$V_{\mathfrak{n}} \neq 0$$

because $Supp_{\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V)}(V) = \{\mathfrak{p} \in Spec(\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V)) : \mathfrak{p} \supset Ann_{\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V)}(V)\} = Spec(\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V))$. By [Bel21, section 2.5.1], $V[\mathfrak{n}] \neq 0$. By Lemma 1.2.16 part (a), there is a discrete automorphic representation π' such that $(\pi'^{\infty,S})^{U^S}$ is isomorphic to a sub-$\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V)$-module of V and $(\pi'^{\infty,S})^{U^S}[\mathfrak{n}] \neq 0$. Note that $(\pi'^{\infty,S})^{U^S} = \otimes'_v(\pi'_v)^{U_v}$ is one dimensional over $\overline{\mathbb{Q}}_p$, so $(\pi'^{\infty,S})^{U^S}[\mathfrak{n}] = (\pi'^{\infty,S})^{U^S}$. The Hecke eigensystem attached to π' has the desired property.

Suppose $H^b_!(X_U, M_{\mu,\mathcal{O}})_{\mathfrak{m}}[\epsilon]/\mathcal{O}$-torsion is a free T-module, say, of rank d, and there is a π' satisfying the statement of the corollary. Note that $d \geq 1$ by Lemma 1.4.5(c). By freeness, $\eta_{\pi,b,\epsilon} = \eta_\pi^d$, so by Theorem 1.4.14, it suffices to show that $\eta_\pi \neq \mathcal{O}$. By Lemma 1.3.6, it suffices to show that there is an $\mathcal{O}$-algebra homomorphism $\lambda' : T \to \overline{\mathbb{Q}}_p$ with $\lambda \neq \lambda'$ and $|\lambda(t) - \lambda'(t)| < 1$ for all $t \in T$. Equivalently, we need to show that there is an $\mathcal{O}$-algebra homomorphism $\Lambda'' : \mathbb{T}^S \to \overline{\mathbb{Q}}_p$ that factors through $\mathbb{T}^S(H^*_!(X_U, M_\mu))$ with $\Lambda \neq \Lambda''$ and $|\Lambda(t) - \Lambda''(t)| < 1$ for all $t \in \mathbb{T}^S$, because any such Λ'' necessarily factors through T. For this, we can take Λ'' to be the Hecke eigensystem attached to π'. $\square$

Remark 1.4.18. A version of Corollary 1.4.17 appears in [BR17, Theorem 4.3.1], but the conditional converse is not stated explicitly and is not proved. An analogue of Corollary 1.4.17 in the case of GL_2 is also proved in [Nam15, Theorem 5.25] under certain conditions for minimal and ordinary eigenforms.

We think the condition for the converse, namely that $H^b_!(X_U, M_{\mu,\mathcal{O}})_{\mathfrak{m}}[\epsilon]/\mathcal{O}$-torsion is a free T-module, is not unreasonable. When F is a CM field, it should be possible to verify this using the Taylor-Wiles method if $\mathfrak{m}$ is non-Eisenstein and the Galois representation attached to π is a minimally ramified deformation of its residual representation. However, under our current knowledge of Galois representations, such an approach will require a lot of extra assumptions and conjectures (such as the vanishing of $H^i(X_U, k)_{\mathfrak{m}}$

outside the cuspidal range, existence of Hecke algebra valued Galois representations (without nilpotent ideals), local-global compatibility of such representations). Therefore, we do not pursue this approach here. See, however, [Han13] for the GL_2 case.[29] It may also be possible to extend the freeness to the non-minimal case using the methods of [IKM22a, IKM22b].

Now, assume in addition that F is a CM field or a totally real number field, and S contains $\{\text{places } w : w|_{\mathbb{Q}} \text{ is ramified in } F \text{ or } w|_{\mathbb{Q}} = p\}$. (Recall that we also make the assumption that S contains all v such that π_v is ramified.) By [HLTT16, Theorem A], there is a Galois representation

$$\rho_\pi : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$$

attached to π such that for all $v \notin S$ of F, the characteristic polynomial of the image of Frob_v is

$$X^n - T_{v,1}X^{n-1} + \cdots + (-1)^i q_v^{i(i-1)/2} T_{v,i} X^{n-i} + \cdots + (-1)^n q_v^{n(n-1)/2} T_{v,n},$$

where $T_{v,i} = [\mathrm{GL}_n(\mathcal{O}_{F_v}) diag(\varpi_v, \ldots, \varpi_v, 1, \ldots, 1) \,\mathrm{GL}_n(\mathcal{O}_{F_v})]$ with ϖ_v appearing i times. From now on, we assume that $\mathfrak{m}$ is ***non-Eisenstein***, i.e. the residual Galois representation $\overline{\rho}_\pi : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is (absolutely) irreducible.

Lemma 1.4.19. *Let the assumptions be as above (so in particular $\mathfrak{m}$ is non-Eisenstein). Then*

(a) $H^*(\partial X_U, M_\mu)_\mathfrak{m} = 0$,

(b) $H^*(X_U, M_\mu)_\mathfrak{m} = H^*_!(X_U, M_\mu)_\mathfrak{m}$,

(c) $T = \mathbb{T}^S(H^*(X_U, M_\mu))_\mathfrak{m}/\mathcal{O}\text{-}tors$,

(d) $H^*(X_U, M_{\mu,\mathbb{C}})_\mathfrak{m} = H^*_!(X_U, M_{\mu,\mathbb{C}})_\mathfrak{m} = H^*_{cusp}(X_U, M_{\mu,\mathbb{C}})_\mathfrak{m}$.

Proof. The key input for the first part is [NT16, Theorem 4.2], which states that for every smooth $\mathcal{O}[U_S]$-module A that is finite as an $\mathcal{O}$-module, $H^*(\partial X_U, A)_\mathfrak{m} = 0$. They proved this using the fact that ∂X_U admits a stratification with strata indexed by conjugacy classes of proper parabolic subgroups of GL_n and by an in depth analysis of the cohomology of each stratum.

Recall that by [NT16, page 19], $M_\mu \otimes_\mathcal{O} \mathcal{O}/\varpi = M_{\mu,\mathcal{O}/\varpi}$, which receives an action of $\mathrm{GL}_n(\mathcal{O}/\varpi)$, compatible with that of $\mathrm{GL}_n(\mathcal{O})$. Thus $M_{\mu,\mathcal{O}/\varpi}$ is a smooth $\mathcal{O}[U_S]$-module that is finite as $\mathcal{O}$-module, so

$$H^*(\partial X_U, M_{\mu,\mathcal{O}/\varpi})_\mathfrak{m} = 0.$$

[29] We think that in that (well-written) paper, it is necessary for $p \geq 7$ instead of $p \geq 3$ as stated. This ensures the image of the residual Galois representation is enormous, guaranteeing the existence of Taylor-Wiles primes. Also, it seems that for the first equation on page 8 to hold, one should patch the (derived) dual of $C_n^\bullet$ rather than $C_n^\bullet$ itself.

We get the desired result by considering the long exact sequence associated to

$$0 \to M_\mu \xrightarrow{\varpi} M_\mu \to M_{\mu,\mathcal{O}/\varpi} \to 0$$

and applying the Nakayama lemma to the finite $\mathcal{O}$-module $H^*(\partial X_U, M_\mu)_\mathfrak{m} = 0$.

Part (b) now follows from the long exact sequence

$$\cdots \to H^i_c(X_U, M_\mu) \to H^i(X_U, M_\mu) \to H^i(\partial X_U, M_\mu) \to \dots$$

For part (c), note that $T^S(H)_\mathfrak{m}/\mathcal{O}\text{-tors} \cong \mathbb{T}^S(\overline{H_\mathfrak{m}})$ for $H \in \{H^*(X_U, M_\mu), H^*_!(X_U, M_\mu)\}$ by the same proof as Remark 1.4.3

For part (d), it is proved in [ACC+23, Theorem 2.4.10] using Franke's decomposition of $H^*(X_U, M_{\mu,\mathbb{C}})$ via automorphic forms that $H^*(X_U, M_{\mu,\mathbb{C}})_\mathfrak{m} = H^*_{cusp}(X_U, M_{\mu,\mathbb{C}})_\mathfrak{m}$. As $H^*_!(X_U, M_{\mu,\mathbb{C}})_\mathfrak{m}$ is always sandwiched between these two groups, these groups are all equal. (The first equality also follows from part (b).) □

For readers' convenience, we restate our running assumptions.

Corollary 1.4.20. *Let the assumptions be as at the start of Section 1.4. Suppose in addition that F is a CM or totally real number field, and S contains $\{$places w : $w|_\mathbb{Q}$ is ramified in F or $w|_\mathbb{Q} = p\}$ (and all v such that π_v is ramified).*

Assume that $\mathfrak{m}$ is non-Eisenstein. Consider the following two statements:

(a)

$$\varpi \mid L^{alg}(1, \pi, \mathrm{Ad}^0, \epsilon).$$

(b) There is a cohomological cuspidal automorphic representation $\pi' \ncong \pi$ of weight $\iota\mu$ of $\mathrm{GL}_n(\mathbb{A}_F)$ with $(\pi')^U \neq 0$ whose Hecke eigensystem $\Lambda' : \mathbb{T}^S \to \overline{\mathbb{Q}}_p$ satisfies $|\Lambda(t) - \Lambda'(t)|_p < 1$ for all $t \in \mathbb{T}^S$.

Then (a) implies (b). If $H^b_!(X_U, M_{\mu,\mathcal{O}})_\mathfrak{m}[\epsilon]/\mathcal{O}$-torsion is a free T-module, then (b) implies (a).

Proof. The proof is just a slight variation of that of Corollary 1.4.17. Abusing notation, we shall identify $\overline{\mathbb{Q}}_p$ with $\mathbb{C}$ using our fixed isomorphism ι. Suppose $\varpi \mid L^{alg}(1, \pi, \mathrm{Ad}^0, \epsilon)$. By Theorem 1.4.14 and Lemma 1.3.4, $\eta_\pi \neq \mathcal{O}$. By Remark 1.3.7 there exists a $\overline{\mathbb{Q}}_p$-algebra homomorphism $\lambda' : T \otimes_\mathcal{O} \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_p$ such that $\lambda' \neq \lambda \otimes_\mathcal{O} \overline{\mathbb{Q}}_p$ and $|\lambda(t) - \lambda'(t)| < 1$ for all $t \in T$. We have a natural map

$$\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V) \to T \otimes_\mathcal{O} \overline{\mathbb{Q}}_p = \mathbb{T}^S_{\bar{\mathbb{Q}}_p}(H^*_{cusp}(X_U, M_{\mu,\bar{\mathbb{Q}}_p})_\mathfrak{m}),$$

where the last equality is by Lemma 1.4.19, $V := H^*_{cusp}(X_U, M_{\mu,\overline{\mathbb{Q}}_p})$, and $\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V) := im(\mathbb{T}^S \otimes_\mathcal{O} \bar{\mathbb{Q}}_p \to \mathrm{End}_{\bar{\mathbb{Q}}_p}(V))$. Composing this with λ', we get a $\overline{\mathbb{Q}}_p$-algebra homomorphism

$$f : \mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V) \to \overline{\mathbb{Q}}_p.$$

Let $\mathfrak{n} := \ker f$. Then

$$V_{\mathfrak{n}} \neq 0$$

because $Supp_{\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V)}(V) = \{\mathfrak{p} \in Spec(\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V)) : \mathfrak{p} \supset Ann_{\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V)}(V)\} = Spec(\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V))$. By [Bel21, section 2.5.1], $V[\mathfrak{n}] \neq 0$. Thus, there is a cohomological cuspidal automorphic representation π' of weight $\iota\mu$ such that $(\pi'^{\infty,S})^{U^S}$ is isomorphic to a sub-$\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(V)$-module of V and $(\pi'^{\infty,S})^{U^S}[\mathfrak{n}] \neq 0$. Note that $(\pi'^{\infty,S})^{U^S} = \otimes'_v(\pi'_v)^{U_v}$ is one dimensional over $\overline{\mathbb{Q}}_p$, so $(\pi'^{\infty,S})^{U^S}[\mathfrak{n}] = (\pi'^{\infty,S})^{U^S}$. The Hecke eigensystem attached to π' has the desired property.

When $H^b_!(X_U, M_{\mu,\mathcal{O}})_{\mathfrak{m}}[\epsilon]/\mathcal{O}$-torsion is a free T-module, the converse follows from Corollary 1.4.17 because if π' is a cohomological cuspidal automorphic representation π' of weight $\iota\mu$ of $\mathrm{GL}_n(\mathbb{A}_F)$ with $(\pi')^U \neq 0$, then π' is a discrete automorphic representation and $0 \neq H^b_{cusp}(X_U, M_{\mu,\mathbb{C}})[(\pi'^{\infty,S})^{U^S}] \subset H^b_!(X_U, M_{\mu,\mathbb{C}})[(\pi'^{\infty,S})^{U^S}]$. $\square$

1.4.5 Selmer groups

We now illustrate how to combine the results above with deformation theory to obtain some Bloch-Kato type results relating Selmer groups and L-functions. We use the same definitions and notation of local and global deformation problems as in [ACC$^+$23, section 6.2.1] and we will always take $\Lambda_v = \mathcal{O}$ for all $v \in S$. In particular, $\bar{\rho} : G_{F,S} \to \mathrm{GL}_n(k)$ is absolutely irreducible, $\mathcal{D}_v$ is a local deformation problem for each $v \in S$,

$$\mathcal{S} = (\bar{\rho}, S, \{\mathcal{O}\}_{v\in S}, \{\mathcal{D}_v\}_{v\in S})$$

is a global deformation problem, and $R_{\mathcal{S}}$ is the ring representing the deformation functor of type $\mathcal{S}$.

Fix $\rho : G_{F,S} \to \mathrm{GL}_n(\mathcal{O})$ a lifting of $\bar{\rho}$ of type $\mathcal{S}$. For each $m \geq 1$, let

$$\mathcal{O}_m := \mathcal{O} \oplus \frac{\pi^{-m}\mathcal{O}}{\mathcal{O}}\epsilon$$

with multiplication given by $(a, b\epsilon)(c, d\epsilon) = (ac, (bc + ad)\epsilon)$. This is a local $\mathcal{O}$-algebra and there is a natural map $\mathcal{O}_m \twoheadrightarrow \mathcal{O}$ given by projection to the first factor.

Definition 1.4.21. *We let*

$$\mathcal{L}^1_v(\pi^{-m}\mathcal{O}/\mathcal{O})$$

be the preimage of $\mathcal{D}_v(\mathcal{O}_m)$ under the isomorphism

$$Z^1\left(G_{F_v}, \mathrm{Ad}\,\rho \otimes_{\mathcal{O}} \frac{\pi^{-m}\mathcal{O}}{\mathcal{O}}\right) \xrightarrow{\sim} \{\textit{liftings } G_{F_v} \to \mathrm{GL}_n(\mathcal{O}_m) \textit{ of } \rho|_{G_{F_v}}\}$$

given by $c \mapsto (1 + c\epsilon)\rho|_{G_{F_v}}$, where Z^1 means the group of continuous 1-cocycles.

We know

$$Z^1(G_{F_v}, \mathrm{Ad}\,\rho \otimes_{\mathcal{O}} E/\mathcal{O}) \xrightarrow{\sim} \{\text{liftings } G_{F_v} \to \mathrm{GL}_n(\mathcal{O} \oplus \frac{E}{\mathcal{O}}\epsilon) \text{ of } \rho|_{G_{F_v}}\} \tag{1.20}$$

Since[30] $Z^1(G_{F_v}, \mathrm{Ad}\,\rho \otimes_{\mathcal{O}} E/\mathcal{O}) = \varinjlim_m Z^1(G_{F_v}, \mathrm{Ad}\,\rho \otimes_{\mathcal{O}} \frac{\pi^{-m}\mathcal{O}}{\mathcal{O}})$, any lifting $G_{F_v} \to$

[30] Note that $\mathrm{Ad}\,\rho \otimes_{\mathcal{O}} E/\mathcal{O}$ is a discrete G_{F_v}-module.

$\mathrm{GL}_n(\mathcal{O} \oplus \dfrac{E}{\mathcal{O}}\epsilon)$ of $\rho|_{G_{F_v}}$ necessarily has image in $\mathrm{GL}_n(\mathcal{O}_m)$ for some $m \geq 1$.

Definition 1.4.22. [31] *A lifting $G_{F_v} \to \mathrm{GL}_n(\mathcal{O} \oplus \dfrac{E}{\mathcal{O}}\epsilon)$ of $\rho|_{G_{F_v}}$ is of type $\mathcal{D}_v$ if it is of type $\mathcal{D}_v$ when it is regarded as a lift with codomain in $\mathrm{GL}_n(\mathcal{O}_m)$ for some m (or, equivalently, for all m for which $\mathrm{GL}_n(\mathcal{O}_m)$ contains the image of the lift.)*

The following is immediate from the definitions.

Definition/Lemma 1.4.23. The following subgroups of $Z^1(G_{F_v}, \mathrm{Ad}\,\rho \otimes_{\mathcal{O}} E/\mathcal{O})$ are equal. We denote them by $\mathcal{L}_v^1(E/\mathcal{O})$.

(i) $\varinjlim_m \mathcal{L}_v^1(\pi^{-m}\mathcal{O}/\mathcal{O})$

(ii) preimage of {liftings $G_{F_v} \to \mathrm{GL}_n(\mathcal{O} \oplus \dfrac{E}{\mathcal{O}}\epsilon)$ of $\rho|_{G_{F_v}}$ of type $\mathcal{D}_v$} under the isomorphism (1.20).

Since $\rho|_{G_{F_v}}$ is of type $\mathcal{D}_v$, $a\rho|_{G_{F_v}}a^{-1}$ is also of type $\mathcal{D}_v$ for all $a \in \ker(\mathrm{GL}_n(\mathcal{O}_m) \to \mathrm{GL}_n(\mathcal{O}))$ for all m by definition of local deformation problem. It follows that $\mathcal{L}_v^1(\pi^{-m}\mathcal{O}/\mathcal{O})$ and $\mathcal{L}_v^1(E/\mathcal{O})$ both contain the group of 1-boundaries.

Definition 1.4.24. *We define $\mathcal{L}_v(\pi^{-m}\mathcal{O}/\mathcal{O})$ to be the image of $\mathcal{L}_v^1(\pi^{-m}\mathcal{O}/\mathcal{O})$ under $Z^1 \to H^1$. Similarly, we define $\mathcal{L}_v(E/\mathcal{O})$ to be the image of $\mathcal{L}_v^1(E/\mathcal{O})$ under $Z^1 \to H^1$. Equivalently, by exactness of direct limits, $\mathcal{L}_v(E/\mathcal{O}) = \varinjlim_m \mathcal{L}_v(\pi^{-m}\mathcal{O}/\mathcal{O})$.*

We also define the Selmer groups

$$H^1_{\mathcal{S}}\left(\mathrm{Ad}\,\rho \otimes_{\mathcal{O}} \frac{\pi^{-m}\mathcal{O}}{\mathcal{O}}\right) := \left\{c \in H^1\left(G_{F,S}, \mathrm{Ad}\,\rho \otimes_{\mathcal{O}} \frac{\pi^{-m}\mathcal{O}}{\mathcal{O}}\right) : c_v \in \mathcal{L}_v(\pi^{-m}\mathcal{O}/\mathcal{O})\ \forall v \in S\right\}$$

and

$$H^1_{\mathcal{S}}\left(\mathrm{Ad}\,\rho \otimes_{\mathcal{O}} \frac{E}{\mathcal{O}}\right) := \left\{c \in H^1\left(G_{F,S}, \mathrm{Ad}\,\rho \otimes_{\mathcal{O}} \frac{E}{\mathcal{O}}\right) : c_v \in \mathcal{L}_v(E/\mathcal{O})\ \forall v \in S\right\},$$

where c_v is the restriction of c to G_{F_v}. It is easy to verify that

$$H^1_{\mathcal{S}}\left(\mathrm{Ad}\,\rho \otimes_{\mathcal{O}} \frac{E}{\mathcal{O}}\right) = \varinjlim_m H^1_{\mathcal{S}}\left(\mathrm{Ad}\,\rho \otimes_{\mathcal{O}} \frac{\pi^{-m}\mathcal{O}}{\mathcal{O}}\right).$$

Lemma 1.4.25. *The strict equivalence class $[\rho]$ of ρ gives rise to a local $\mathcal{O}$-algebra homomorphism $R_{\mathcal{S}} \xrightarrow{\theta} \mathcal{O}$. Let $\mathfrak{p} := \ker\theta$. Then*

$$\mathrm{Hom}_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2, E/O) \cong H^1_{\mathcal{S}}(\mathrm{Ad}\,\rho \otimes_{\mathcal{O}} E/\mathcal{O}),$$

[31]Since $\mathcal{O} \oplus \frac{E}{\mathcal{O}}\epsilon$ is not a complete local ring, the term 'of type $\mathcal{D}_v$' is not defined for liftings to $\mathrm{GL}_n(\mathcal{O} \oplus \frac{E}{\mathcal{O}}\epsilon)$ a priori.

Proof. This is well-known so we will just sketch a proof. Note that $\mathfrak{p}/\mathfrak{p}^2$ is a finitely generated $R_{\mathcal{S}}/\mathfrak{p} = \mathcal{O}$-module, so any $\mathcal{O}$-algebra homomorphism $\mathfrak{p}/\mathfrak{p}^2 \to E/\mathcal{O}$ has image contained in $\varpi^{-m}\mathcal{O}/\mathcal{O}$ for some $m \geq 1$. Thus, if we know

$$\mathrm{Hom}_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2, \varpi^{-m}\mathcal{O}/\mathcal{O}) \cong H^1_{\mathcal{S}}(\mathrm{Ad}\,\rho \otimes_{\mathcal{O}} \varpi^{-m}\mathcal{O}/\mathcal{O})$$

for all $m \geq 1$, then taking colimit will give the desired result. This follows from the chain of isomorphisms

$$\begin{aligned}
&H^1_{\mathcal{S}}(\mathrm{Ad}\,\rho \otimes_{\mathcal{O}} \varpi^{-m}\mathcal{O}/\mathcal{O}) \\
\cong\ &\{\text{liftings } G_{F,S} \to \mathrm{GL}_n(\mathcal{O}_m) \text{ of } \rho \text{ of type } \mathcal{S}\}/\ker(\mathrm{GL}_n(\mathcal{O}_m) \to \mathrm{GL}_n(\mathcal{O})) && (1.21)\\
\cong\ &\{\text{deformations } G_{F,S} \to \mathrm{GL}_n(\mathcal{O}_m) \text{ of } [\rho] \text{ of type } \mathcal{S}\} && (1.22)\\
\cong\ &\{f \in \mathrm{Hom}_{\mathcal{O}}(R_{\mathcal{S}}, \mathcal{O}_m) : f \pmod{\epsilon} = \theta\}\\
\cong\ &\mathrm{Hom}_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2, \varpi^{-m}\mathcal{O}/\mathcal{O}).
\end{aligned}$$

In (1.21), $\ker(\mathrm{GL}_n(\mathcal{O}_m) \to \mathrm{GL}_n(\mathcal{O}))$ acts on $\{\text{liftings } G_{F,S} \to \mathrm{GL}_n(\mathcal{O}_m) \text{ of } \rho \text{ of type } \mathcal{S}\}$ by conjugation. In (1.22), by deformations of $[\rho]$, we mean deformations of $\bar{\rho}$ whose pushforward to $\mathrm{GL}_n(\mathcal{O})$ is $[\rho]$. To show the bijectivity of (1.21) and (1.22), note that the centralizer of the image of ρ in $\mathrm{GL}_n(\mathcal{O})$ is $\mathcal{O}^\times$ by [CHT08, lemma 2.1.8] and $\bar{\rho}$ is absolutely irreducible. The other steps are easy. □

Theorem 1.4.26. *Let the assumptions be as at the start of Section 1.4. Suppose in addition that F is a CM or totally real number field, and S contains $\{$places w : $w|_{\mathbb{Q}}$ is ramified in F or $w|_{\mathbb{Q}} = p\}$ (and all v such that π_v is ramified).*

Assume $\mathfrak{m}$ is non-Eisenstein. Then there is a continuous Galois representation

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(T)$$

such that for all $v \notin S$ of F, the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ is

$$X^n - T_{v,1}X^{n-1} + \cdots + (-1)^i q_v^{i(i-1)/2} T_{v,i} X^{n-i} + \cdots + (-1)^n q_v^{n(n-1)/2} T_{v,n},$$

where $T_{v,i} = [\mathrm{GL}_n(\mathcal{O}_{F_v}) diag(\varpi_v, \ldots, \varpi_v, 1, \ldots, 1)\,\mathrm{GL}_n(\mathcal{O}_{F_v})]$ with ϖ_v appearing i times.

Assume that $\rho_{\mathfrak{m}}$ is a lifting of $\bar{\rho}_{\mathfrak{m}}$ of type $\mathcal{S}$, where $\mathcal{S} = (\bar{\rho}_{\mathfrak{m}}, S, \{\mathcal{O}\}_{v \in S}, \{\mathcal{D}_v\}_{v \in S})$ is some global deformation problem. Let $\rho := \lambda \circ \rho_{\mathfrak{m}}$, where $\lambda : T \to \mathcal{O}$ is induced from Λ as in Lemma 1.4.2. Then

$$\#H^1_{\mathcal{S}}(\mathrm{Ad}\,\rho \otimes_{\mathcal{O}} E/\mathcal{O}) \geq \#(\mathcal{O}/L^{alg}(1, \pi, \mathrm{Ad}^\circ, \epsilon)) \tag{1.23}$$

where # denotes the order of a group.

Proof. The key to proving the existence[32] of $\rho_{\mathfrak{m}}$ is to use [HLTT16, Theorem A] and Carayol's lemma.

[32] If one makes extra assumptions on F and S, then the existence of $\rho_{\mathfrak{m}}$ follows immediately from [ACC+23, theorem 2.3.7].

Since $\mathfrak{m}$ is non-Eisenstein, $H^*_!(X_U, M_{\mu,\overline{\mathbb{Q}}_p})_{\mathfrak{m}} = H^*_{cusp}(X_U, M_{\mu,\overline{\mathbb{Q}}_p})_{\mathfrak{m}}$ by Lemma 1.4.19. Let $W = H^*_{cusp}(X_U, M_{\mu,\overline{\mathbb{Q}}_p})_{\mathfrak{m}}$. By (the proof of) Lemma 1.4.5(b),

$$W = \oplus_{\alpha:\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(W)\to\overline{\mathbb{Q}}_p} W[\ker\alpha] \tag{1.24}$$

where α runs over $\overline{\mathbb{Q}}_p$-algebra homomorphisms $\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(W) \to \overline{\mathbb{Q}}_p$. Any such α satisfies $(\alpha|_{\mathbb{T}^S})^{-1}(\mathfrak{m}_{\overline{\mathbb{Q}}_p}) = \mathfrak{m}$. By equation (1.3) and [BW00, Ch.II, Prop. 3.1],

$$H^*_{cusp}(X_U, M_{\mu,\mathbb{C}}) = \oplus_{\pi'} H^*(\mathfrak{g}, K^\circ_\infty, M_{\mu,\mathbb{C}} \otimes_{\mathbb{C}} \pi'_\infty) \otimes_{\mathbb{C}} (\pi'^\infty)^U,$$

where π' runs over all cohomological cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight μ with $\pi'^U \neq 0$. It follows that any α as above gives rise to a cohomological cuspidal automorphic representation π_α of $\mathrm{GL}_n(\mathbb{A}_F)$ of weight μ such that $\pi_\alpha^U \neq 0$ and $t \in \mathbb{T}^S$ acts on π^U by $\alpha(t)$.

We have a $\overline{\mathbb{Q}}_p$-algebra homomorphism

$$\begin{aligned} h : T \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p = \mathbb{T}^S_{\overline{\mathbb{Q}}_p}(W) \to & \prod_{\alpha:\mathbb{T}^S_{\overline{\mathbb{Q}}_p}(W)\to\overline{\mathbb{Q}}_p} \overline{\mathbb{Q}}_p \\ t \mapsto & (\alpha(t))_\alpha. \end{aligned}$$

By (1.24), h is injective. If h is not surjective, then its image is a proper $\overline{\mathbb{Q}}_p$-subalgebra of $\prod_\alpha \overline{\mathbb{Q}}_p$, so, by a simple arguement, there exist $\beta, \gamma : \mathbb{T}^S_{\overline{\mathbb{Q}}_p}(W) \to \overline{\mathbb{Q}}_p$ such that $\beta \neq \gamma$ and the image is contained in $\{z \in \prod_\alpha \overline{\mathbb{Q}}_p : z_\beta = z_\gamma\}$. This is impossible by the strong multiplicity one theorem. Hence, h is an isomorphism.

By [HLTT16, Theorem A], for every α, there is a unique continuous semisimple representation

$$\rho_{\pi_\alpha} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$$

such that for all $v \notin S$, ρ_{π_α} is unramified at v and $\rho_{\pi_\alpha}(\mathrm{Frob}_v)$ has characteristic polynomial $\alpha(P_v(X))$, where $\alpha(P_v(X))$ is the polynomial obtained by applying α to every coefficient of $P_v(X) := X^n - T_{v,1}X^{n-1} + \cdots + (-1)^i q_v^{i(i-1)/2} T_{v,i} X^{n-i} + \cdots + (-1)^n q_v^{n(n-1)/2} T_{v,n}$. By the Baire category theorem and the compactness of $\mathrm{Gal}(\overline{F}/F)$, after conjugating ρ_{π_α} by elements of $\mathrm{GL}_n(\overline{\mathbb{Q}}_p)$, we can assume that $\rho_{\pi_\alpha}(\mathrm{Gal}(\overline{F}/F)) \subset \mathrm{GL}_n(\mathcal{O}_L)$ for every α, where L is a finite extension of E. See [CHT08, Lemma 2.1.5] for the proof of an analogous result. By the Brauer-Nesbitt theorem, every residual Galois representation $\overline{\rho}_{\pi_\alpha} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(k_L)$, where k_L is the residue field of L, is isomorphic to $\overline{\rho}_\pi$. We can conjugate ρ_{π_α} by elements of $\mathrm{GL}_n(\mathcal{O}_L)$ to ensure that all these residual Galois representations are equal to $\overline{\rho}_\pi$, not just conjugate to it. By enlarging L if necessary, we can also assume $h(T) \subset \prod_\alpha \mathcal{O}_L$.

Let

$$\rho = (\rho_{\pi_\alpha}) : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(T \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p) = \prod_\alpha \mathrm{GL}_n(\overline{\mathbb{Q}}_p).$$

For every $v \notin S$, ρ is unramified at v and $\rho(\mathrm{Frob}_v)$ has characteristic polynomial $P_v(X)$.

Let $A := \{(x_\alpha) \in \prod_\alpha \mathcal{O}_L : \bar{x}_\alpha$ are all equal and lies in $\mathcal{O}/\varpi\}$, where $\bar{x}_\alpha$ is the reduction mod $\mathfrak{m}_L$ of x. This is a local ring with maxial ideal $\mathfrak{m}_A := \{(x_\alpha) \in \prod_\alpha \mathcal{O}_L : \bar{x}_\alpha = 0$ for all $\alpha\}$ and residue field $\mathcal{O}/\varpi$. This is also finitely generated as an $\mathcal{O}$-module, so it is a complete Noetherian ring.[33] Thus, $A \in CNL_{\mathcal{O}}$ (the category of complete Noetherian local $\mathcal{O}$-algebra with residue field $\mathcal{O}/\varpi$).

In the remaining part, let us denote the image of $\mathfrak{m}$ in T also as $\mathfrak{m}$. Note that $T \subset A$ and $T \in CNL_{\mathcal{O}}$. Also, $T \cap \mathfrak{m}_A = \mathfrak{m}$ by, so $T \hookrightarrow A$ is continuous. This is a closed map of topological spaces, because T is compact[34] and A is Hausdorff. In particular, T is a closed subset of A.

Note that $\rho(\mathrm{Gal}(\overline{F}/F)) \subset \mathrm{GL}_n(A)$, so we can view ρ as a representation

$$\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(A).$$

The residual Galois representation $\bar{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(A/\mathfrak{m}_A)$ is absolutely irreducible because $\mathfrak{m}$ is non-Eisenstein. We know that $\mathrm{tr}(\rho(\mathrm{Frob}_v)) = \pm T_{v,1} \in T$ for all $v \notin S$. By Chebotarev density theorem and the fact that $T \subset A$ is a closed subset, $\mathrm{tr}\rho(\mathrm{Gal}(\overline{F}/F)) \subset T$. By Carayol's lemma ([CHT08, Lemma 2.1.10]), there exists $g \in \mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathcal{O}/\varpi)$ such that $g\rho(\mathrm{Gal}(\overline{F}/F))g^{-1} \subset \mathrm{GL}_n(T)$. Then $g\rho g^{-1}$ is unramified outside S, so it gives rise to

$$\rho_{\mathfrak{m}} : G_{F,S} \to \mathrm{GL}_n(T)$$

such that for all $v \notin S$ of F, the characteristic polynomial of $\rho_{\mathfrak{m}}(\mathrm{Frob}_v)$ is $P_v(X)$. This proves the first part.

For the second part, assume that $\rho_{\mathfrak{m}}$ is of type $\mathcal{S}$. Its strict equivalence class induces an $\mathcal{O}$-algebra homomorphism $f : R_{\mathcal{S}} \to T$. We know that $\mathbb{T}^S$ is generated by

$$\{T_v^i, (T_v^n)^{-1} : v \notin S, 1 \leq i \leq n\}$$

as an $\mathcal{O}$-algebra[35]. For all $g \in G_{F,S}$, every coefficient of the characteristic polynomial of $\rho_{\mathfrak{m}}(g)$ is in the image of $R_{\mathcal{S}} \to T$. Taking $\mathrm{Frob}_v^{\pm 1}$, we know f is surjective.

If $H^1_{\mathcal{S}}(\mathrm{Ad}\,\rho \otimes_{\mathcal{O}} E/\mathcal{O})$ is infinite, then equation (1.23) is trivial. Suppose $H^1_{\mathcal{S}}(\mathrm{Ad}\,\rho \otimes_{\mathcal{O}} E/\mathcal{O})$ is finite. Let $\theta = \lambda \circ f$ and $\mathfrak{p} := \ker\theta$. It follows from Lemma 1.4.25 that $\mathfrak{p}/\mathfrak{p}^2$ is also finite and

$$\#H^1_{\mathcal{S}}(\mathrm{Ad}\,\rho \otimes_{\mathcal{O}} E/\mathcal{O}) = \#(\mathfrak{p}/\mathfrak{p}^2).$$

By [DDT97, page 141 equation (5.2.3)],

$$\#(\mathfrak{p}/\mathfrak{p}^2) \geq \#(\mathcal{O}/\eta_{R_{\mathcal{S}}})$$

where $\eta_{R_{\mathcal{S}}} := \theta(Ann_{R_{\mathcal{S}}}(\mathfrak{p}))$. By [DDT97, page 140 equation (5.2.2)],

$$\#(\mathcal{O}/\eta_{R_{\mathcal{S}}}) \geq \#(\mathcal{O}/\eta_T),$$

[33]It is clear that A is ϖ-adically complete. By the going up and incomparability theorem, $\sqrt{\varpi A} = \cap_{\mathfrak{p} \subset \varpi A} \mathfrak{p} = \mathfrak{m}_A$. It follows that A is also $\mathfrak{m}_A$-adically complete.

[34]To see this, note that $T = \lim_i T/m^i$ is profinite.

[35]This can be easily deduced by applying the Satake isomorphism to $\mathcal{H}(\mathrm{GL}_n(F_v), \mathrm{GL}_n(\mathcal{O}_{F_v})) \otimes_{\mathbb{Z}} \mathbb{Z}[q_v^{1/2}]$.

where $\eta_T = \lambda(Ann_T(\ker\lambda))$, which is the same as η_π in the previous subsections. By Lemma 1.3.4, Lemma 1.4.19, and Theorem 1.4.14,

$$\#(\mathcal{O}/\eta_\pi) \geq \#(\mathcal{O}/\eta_{\pi,b,\epsilon}) \geq \#(\mathcal{O}/L^{alg}(1,\pi,\mathrm{Ad}^\circ,\epsilon)).$$

We get the desired inequality. $\square$

As an illustration of Theorem 1.4.26, let us give an example. Let $\mathcal{D}_v^\square$ be the functor on $CNL_\mathcal{O}$ (category of complete Noetherian local $\mathcal{O}$-algebras with residue fields k) that sends A to the set of all lifts of $\bar{\rho}_\mathfrak{m}|_{G_{F_v}}$ to A. In the Fontaine-Laffaille case, if v is a p-adic place, we let $\mathcal{D}_v^{FL}$ be the local deformation problem that sends any $A \in CNL_\mathcal{O}$ that is finite over $\mathcal{O}$ to all liftings of $\bar{\rho}_\mathfrak{m}|_{G_{F_v}}$ to A that are Fontaine-Laffaille of type $(\mu_\tau)_{\tau\in Hom(F_v,E)}$. See [ACC+23, sections 4.1, 6.2.14] for more detail.

Corollary 1.4.27. *Let the assumptions be as at the start of Section 1.4. Suppose in addition that*

- *F is a CM or totally real number field.*
- *S contains $\{$places $w : w|_\mathbb{Q}$ is ramified in F or $w|_\mathbb{Q} = p\}$ (and all v such that π_v is ramified).*
- *$U_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$ for all $v \mid p$.*
- *the residual Galois representation $\bar{\rho}_\pi : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{F}_p)$ is irreducible and decomposed generic [ACC+23, Definition 4.3.1].*

Then for all $v \mid p$, $\rho_m|_{G_{F_v}}$ is Fontaine-Laffaille of type $(\mu_\tau)_{\tau\in Hom(F_v,E)}$, where $\rho_\mathfrak{m}$ is the Galois representation constructed in Theorem 1.4.26. In particular, ρ_m is a lifting of $\bar{\rho}_\mathfrak{m}$ of type $\mathcal{S} = (\bar{\rho}_\mathfrak{m}, S, \{\mathcal{O}\}_{v\in S}, \{\mathcal{D}_v^{FL}\}_{v|p} \cup \{\mathcal{D}_v^\square\}_{v\in S-\{v|p\}})$ and

$$\#H^1_\mathcal{S}(\mathrm{Ad}\,\rho \otimes_\mathcal{O} E/\mathcal{O}) \geq \#(\mathcal{O}/L^{alg}(1,\pi,\mathrm{Ad}^\circ)).$$

Proof. This follows from the proof of Theorem 1.4.26 and by replacing the use of [HLTT16, Theorem A] by [CN23, Theorem 4.3.1]. $\square$

Remark 1.4.28. To relate this Selmer group to the Bloch-Kato Selmer group, see [DFG04, lemma 2.1] and [Dim09, section 7.3]. In their setup, the $\mathcal{O}$-Fitting ideal of their Selmer group was equal to that of the Bloch-Kato one multiplied by $\prod_{v\in\Sigma} Fitt_\mathcal{O}(H^1_f(F_v,(\mathrm{Ad}^\circ\rho_f)^*(1)))$ for some finite set of places Σ. Each term in the product was then shown to be equal to the local Tamagawa number divided by the local L-factor at that place. Combining this with a suitable $R = T$ theorem, they were able to deduce a form of the Bloch-Kato conjecture using similar argument to the proof of Theorem 1.4.26.

Chapter 2

Higgs bundles on the Fargues-Fontaine curve

In this part, we introduce a notion of Higgs bundles on the Fargues-Fontaine curve. We establish a version of the BNR correspondence, which relates Higgs bundles to line bundles on suitable curves. We then describe an action of a Picard stack on the moduli stack of Higgs bundles and show that, modulo this action, there is a natural injective map of étale-stacks from the product of B_{dR}^{+}-affine Springer fibers to the Hitchin fiber that induces an equivalence of categories on every geometric point. Finally, we discuss connections with number-theoretic objects.

2.1 Introduction

Higgs bundles on algebraic varieties play an important role in several areas of mathematics. To motivate this thesis, let us give two motivations from the Langlands program for studying Higgs bundles. We will be slightly imprecise to streamline the discussion.

1. **Langlands duality**
   Let X be a smooth projective curve over $\mathbb{C}$. It has been conjectured that (cf. [DP12, Conjecture 2.5]) there is an equivalence of categories[1]

   $$\mathrm{QCoh}(\mathrm{Higgs}_G) \simeq \mathrm{QCoh}(\mathrm{Higgs}_{\check{G}}),$$

   where G is a connected reductive group over $\mathbb{C}$, $\check{G}$ is its dual group, and Higgs_G is the moduli stack of G-Higgs bundles on X. This is highly interesting because it is an instance of Langlands duality: it involves both G and $\check{G}$. Even for $G = \mathrm{GL}_n$, this is a non-trivial assertion, because this equivalence is expected to be compatible with certain Fourier-Mukai equivalence, and in particular is not satisfied by the identity functor. To be less vague, there is a Hitchin fibration $\mathrm{Higgs}_G \to \mathcal{A}_G$. By the BNR theorem [BNR89, Proposition 3.6], the fibres of this map over a suitable

[1] This is just an approximation. It is unknown precisely what the two categories should be.

locus on $\mathcal{A}_G$ are given by the Jacobian of the spectral curves (up to some stacky parts). In particular, there is a Fourier-Mukai equivalence [Muk81] on the derived category of quasi-coherent sheaves on the fibres. The conjectured equivalence above should restrict to this Fourier-Mukai equivalence.

2. **Proof of the fundamental lemma**
   One of the central tools in the study of the Langlands conjectures is the trace formula. Most of its applications rely on a deep and highly nontrivial input known as the fundamental lemma, which relates local orbital integrals attached to two different reductive groups. By [Wal97], to prove the fundamental lemma for all non-Archimedean local fields, it is enough to prove it for non-Archimedean local fields with equal characteristic. In [Ngô10], Ngô achieved this using the geometry of Higgs bundles.

   The guiding principle is a global-to-local philosophy. Locally, orbital integrals can be interpreted in terms of point counts on affine Springer fibers, certain closed subsheaves of the affine Grassmannians. Globally, the Hitchin fiber in the moduli stack Higgs_G serves as a global avatar of affine Springer fibers. The reason is that there are certain Picard stacks acting on Higgs_G and the affine Springer fibers. Ngo proves in [Ngô10, Proposition 4.15.1] that modulo the action of these Picard stacks, the Hitchin fiber factorises as a product of affine Springer fibers over all places. This product formula makes precise the global-local relationship and allows one to deduce identities between orbital integrals from global geometric properties of the Hitchin fibration.

   It is worth emphasizing that Ngô works with a variant of the usual notion of Higgs bundles: instead of the canonical line bundle Ω^1_X, one allows an arbitrary line bundle on X. This additional flexibility plays an essential role in the geometric arguments.

In this thesis, we introduce a notion of Higgs bundles on the fundamental curve of p-adic Hodge theory, namely the Fargues-Fontaine curve. Introduced by Laurent Fargues and Jean-Marc Fontaine, this curve provides a geometric framework for studying p-adic Hodge theory and the Local Langlands Program. In particular, the study of G-bundles on the Fargues-Fontaine curve and ℓ-adic sheaves on their moduli stacks plays a central role in the geometric approach to local Langlands developed in [FS21].

From this perspective, the local Langlands program can be viewed as a form of geometric Langlands theory on the Fargues-Fontaine curve. It is therefore natural to extend geometric constructions from the classical setting of smooth projective curves to this p-adic context. In this thesis, we carry out such an extension for the notion of Higgs bundles.

In contrast to the classical setting, the Fargues-Fontaine curve is an adic space[2], rather than an algebraic variety. As a consequence, certain basic tools from algebraic

[2]There is also a schematic version of the Fargues-Fontaine curve, but that is also not an algebraic variety.

geometry are not available. In particular, the notion of Kähler differentials does not exist for the Fargues-Fontaine curve, which prevents one from defining Higgs bundles in the usual way. We thus adapt Ngo's definition of Higgs bundles, i.e. we fix a line bundle $\mathcal{L}$ on the Fargues-Fontaine curve X_S and define a ($\mathcal{L}$-twisted G-)Higgs bundle on X_S to be a pair $(\mathcal{E}, \phi)$, where

- $\mathcal{E}$ is a G-bundle on X_S,
- $\phi \in H^0(X_S, \mathrm{Ad}(\mathcal{E}) \otimes_{\mathcal{O}_{X_S}} \mathcal{L})$.

We let Higgs_G denote the moduli stack of $\mathcal{L}$-twisted G-Higgs bundles. We also introduce in Definition 2.6.1 a local analogue of Higgs_G, which we call B^+_{dR}-affine Springer fiber.

We prove three abelianisation theorems in this thesis. As we noted in motivation 1, to formulate motivation 1 for GL_n, we need to know that the Hitchin fibers are the Jacobians of certain curves (up to some stacky parts), so that we have Fourier-Mukai equivalence on them. In Theorem 2.4.9, we prove a version of the BNR correspondence, which relates GL_n-Higgs bundles to line bundles on suitable curves.[3] This can be viewed as progress towards formulating an analogue of motivation 1 for the Fargues-Fontaine curve. In Proposition 2.5.19, we prove that

$$\mathrm{Higgs}_G^{reg} \cong \mathcal{P},$$

where Higgs_G^{reg} is a certain sub-prestack of Higgs_G and $\mathcal{P}$ is a certain Picard prestack which classifies $J_{\mathcal{L}}$-bundles on the Fargues-Fontaine curve, for some commutative group scheme $J_{\mathcal{L}}$ over the Hitchin base. Similarly, in Proposition 2.7.7, we prove that

$$\mathrm{Gr}^{reg}_{G,a,v} \cong \mathcal{P}^{loc}_{a_v},$$

where $\mathrm{Gr}^{reg}_{G,a,v}$ is a certain sub-v-sheaf of the B^+_{dR}-affine Springer fiber $\mathrm{Gr}_{G,a,v}$ and $\mathcal{P}^{loc}_{a_v}$ is the B^+_{dR}-affine Grassmannian of a commutative group scheme J_{a_v}.

In Theorem 2.7.22, we exhibit the close relationship between the Hitchin fibers Higgs_a and the affine Springer fibers $\mathrm{Gr}_{G,a,v}$ by showing that for generically regular semisimple a, there is a natural injective morphism of étale-stacks

$$\prod_{\text{closed } v \in |X_C^{sch}|} [\mathrm{Gr}_{G,a,v} / \mathcal{P}^{loc}_{a,v}]_{\text{ét}} \to [\mathrm{Higgs}_a / \mathcal{P}_a]_{\text{ét}},$$

which is an equivalence on every geometric point. This can be viewed as progress towards motivation 2 for arbitrary non-Archimedean local fields. Finally, we will discuss connections with number-theoretic objects.

[3]These will be the spectral curves. In the same way that the Fargues-Fontaine curve is not a curve in the usual sense, the spectral curve is also not a curve in the usual sense. However, the spectral curve will be a finite covering of the Fargues-Fontaine curve.

Convention

If $\mathcal{C}$ is a category, then by a prestack on $\mathcal{C}$, we mean a 2-functor $\mathcal{C}^{op} \to \mathrm{Grpd}$, where Grpd is the (2,1)-category of (small) groupoids. For a scheme or adic space X, we denote by $|X|$ its underlying topological space, not just the closed points. Unless otherwise stated, by a stack, we mean a stack in groupoids. By a vector bundle on X, we mean a locally finite free $\mathcal{O}_X$-module on X. Unless otherwise stated, our G-torsors will be ***right*** G-torsors and all other (group) actions will be left actions.

If $\mathcal{C}$ is a site, $X \in \mathcal{C}$ and G is a group sheaf on $\mathcal{C}$, then a G-torsor/G-bundle on X is a sheaf $\mathcal{E}$ on $\mathcal{C}_{/X}$ equipped with a right $G|_X$-action on it for which there is a cover $\{U_i \to X\}$ and $G|_{U_i}$-equivariant isomorphisms $\mathcal{E}|_{U_i} \cong G|_{U_i}$, where $G|_{U_i}$ acts on itself via right multiplication. If $\mathcal{F}$ is a sheaf on $\mathcal{C}$ with a left G-action, then we define $\mathcal{E} \times^G \mathcal{F}$ to be the sheafification of $U \mapsto (\mathcal{E}(U) \times \mathcal{F}(U))/G(U)$ where $G(U)$ acts as $g \cdot (x, y) = (xg^{-1}, gy)$. We will denote the trivial G-bundle by $\mathcal{E}_0$ or G.

We let $\mathrm{Perf}, \mathrm{Perf}_{\overline{\mathbb{F}}_q}, \mathrm{Perf}_S$ be the category of perfectoid spaces over $\mathbb{F}_p$, $\overline{\mathbb{F}_q}$, S respectively. We let X_S denote the Fargues-Fontaine curve. If S is affinoid perfectoid, then we let X_S^{sch} denote the schematic Fargues-Fontaine curve. If $\mathcal{V}$ is a vector bundle on the Fargues-Fontaine curve X_S, we let $\mathcal{H}^0(\mathcal{V})$ be the functor on Perf_S given by $T \mapsto H^0(X_T, \mathcal{V})$. We let $\mathbb{D}_T$ denote $\operatorname{Spec} B^+_{Div^1}(T)$ and $\mathbb{D}^*_T$ denote $\operatorname{Spec} B_{Div^1}(T)$. Note that this depends on the map $T \to Div^1_X$, not just on T.

If A is a Huber ring, then $\operatorname{Spa} A := \operatorname{Spa}(A, A^\circ)$.

If X is an affine scheme, then $\mathcal{O}(X)$ is the ring of global sections of its structure sheaf $\mathcal{O}_X$.

Unless otherwise stated, by 'ring', we mean a commutative ring.

Acknowledgements

I would like to thank my supervisor Tobias Berger for providing feedback of drafts of this thesis, James Newton and Robert Kurinczuk for pointing out various errors in a previous version of this thesis, and Ken Lee for providing the proof of Lemma 2.2.16. This work was completed while in receipt of the EPSRC Doctoral Training Partnership (DTP) Studentship with grant number EP/W524360/1.

2.2 Background

In this section, we provide some background useful for this thesis. While most of these materials are standard, there are some new results which do not seem to exist in the literature, including Proposition 2.2.11, Proposition 2.2.15, Lemma 2.2.16, the computations in Section 2.2.13, Lemma 2.2.89, Lemma 2.2.91, Lemma 2.2.95.

2.2.1 Recollections on stacks

Let $\mathcal{C}$ be a site. We shall mainly apply this to the fppf site of schemes and the étale site of sousperfectoid spaces as in the appendix.

If $\mathcal{F}$ is a sheaf on $\mathcal{C}$ and $X \in \mathcal{C}$, then we abbreviate $\mathcal{F}|_{\mathcal{C}_{/X}}$ as $\mathcal{F}|_X$.

Let G be a group sheaf on $\mathcal{C}$.

Definition 2.2.1. *For $X \in \mathcal{C}$, a **G-torsor/G-bundle** on X is a sheaf $\mathcal{E}$ on $\mathcal{C}_{/X}$ equipped with a right $G|_X$-action on it for which there is a cover $\{U_i \to X\}$ and $G|_{U_i}$-equivariant isomorphisms $\mathcal{E}|_{U_i} \cong G|_{U_i}$, where $G|_{U_i}$ acts on itself via right multiplication.*

We recommend the readers to read [SW20, Appendix to Lecture XIX] for some more discussions on G-torsors in the settings of schemes and adic spaces.

Definition 2.2.2. *Let $\mathcal{F}$ be a sheaf on $\mathcal{C}$ acted on by G. We define $[\mathcal{F}/G]$ to be the stack*

$$\begin{aligned} \mathcal{C}^{op} &\to \mathrm{Grpd} \\ S &\mapsto (\mathcal{E} \text{ a } G\text{-torsor on } S, f : \mathcal{E} \to \mathcal{F}|_S \text{ a } G\text{-equivariant map}). \end{aligned}$$

Since by our convention $\mathcal{E}$ and $\mathcal{F}$ receive right and left G-actions respectively, by 'G-equivariant', we mean $f(eg) = g^{-1}f(e)$.

Let us recall a well-known lemma.

Lemma 2.2.3. *$[\mathcal{F}/G]$ is the stackification of the prestack $[\mathcal{F}/_{pre}\, G] : \mathcal{C}^{op} \to \mathrm{Grpd}$ sending S to the quotient groupoid[4] $\mathcal{F}(S)/G(S)$.*

Proof. By [Sta25, 04TQ, 04TR], $[\mathcal{F}/G]$ is indeed a stack. We have a map of prestacks $[\mathcal{F}/_{pre}\, G] \to [\mathcal{F}/G]$ given on S-valued points by

$$\begin{aligned} \mathcal{F}(S)/G(S) &\to [\mathcal{F}/G](S) \\ x \in \mathcal{F}(S) &\mapsto (\text{trivial } G\text{-bundle on } S,\ g \mapsto g^{-1}x) \\ g \in G(S) &\mapsto g\cdot \end{aligned}$$

It is easy to verify using [Sta25, Lemma 02ZN] that this map exhibits $[\mathcal{F}/G]$ as the stackification of $[\mathcal{F}/_{pre}\, G]$. $\square$

Let us also recall a well-known description of the sections of associated bundles via G-equivariant maps.

Lemma 2.2.4. *Let $\mathcal{E}$ be a G-torsor on $X \in \mathcal{C}$. We have a bijection between*

1. *G-equivariant maps $\mathcal{E} \to \mathcal{F}|_X$ and*
2. *elements of $(\mathcal{E} \times^G \mathcal{F}|_X)(X)$.*

In particular, we can identify $[\mathcal{F}/G]$ with the stack

$$S \mapsto (\mathcal{E} \text{ a } G\text{-torsor on } S, s \text{ an element of } (\mathcal{E} \times^G \mathcal{F}|_X)(X)).$$

[4]The groupoid whose objects are elements of $X(S)$ with morphisms given by $\mathrm{Mor}(x, y) = \{g \in G(S) : gx = y\}$.

Proof. By replacing $\mathcal{C}$ by $\mathcal{C}|_X$, we can assume X is the final object of $\mathcal{C}$. Let $f : \mathcal{E} \to \mathcal{F}$ be a G-equivariant map. Then we get a G-equivariant map $(\mathrm{id}, f) : \mathcal{E} \to \mathcal{E} \times \mathcal{F}$, and hence a map

$$X = \mathcal{E}/G \to (\mathcal{E} \times \mathcal{F})/G = \mathcal{E} \times^G \mathcal{F},$$

[5]i.e. an element of $(\mathcal{E} \times^G \mathcal{F})(X)$.

Conversely, let $s \in (\mathcal{E} \times^G \mathcal{F})(X)$. By abuse of notation, we let s also denote the composite $\mathcal{E} \to X \to \mathcal{E} \times^G \mathcal{F}$, where the second map corresponds to the original s. We get a G-equivariant map

$$\mathcal{E} \xrightarrow{(id,s)} \mathcal{E} \times (\mathcal{E} \times^G \mathcal{F}) \xrightarrow{r} \mathcal{F}$$

where r is the sheafification of the map (as $U \in \mathcal{C}$ vary)

$$\begin{aligned} \mathcal{E}(U) \times (\mathcal{E}(U) \times \mathcal{F}(U))/G(U) &\to \mathcal{F}(U) \\ (e, [e, g] &\mapsto g). \end{aligned}$$

When defining r, we have used the fact that $G(U)$ acts on $\mathcal{E}(U)$ freely. This follows from the isomorphism $G \times \mathcal{E} \xrightarrow{(pr_2,\text{action})} \mathcal{E} \times \mathcal{E}$, which holds because by [Sta25, 04TQ], we can check this étale locally on X and hence assume $\mathcal{E}$ is the trivial G-torsor, for which the isomorphism is clear.

It is straightforward to check that these two maps are inverses of each other. □

We will also use the following lemma, whose proof is evident, without explicitly mentioning it.

Lemma 2.2.5. *Suppose we have*

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

where X, Y, Z are groupoids and f, g are functors. Assume that for each $x \in X$ and map $f(x) \xrightarrow{\beta} z$, there is a map $x \xrightarrow{\alpha} x'$ such that $f(\alpha) = \beta$. (I.e. f satisfies the 'path lifting property'.) Then the 2-fiber product[6] *and the 1-fiber product*[7] *of the diagram are canonically equivalent.*

[5]The canonical map $\mathcal{E}/G \to X$ is an isomorphism locally, and hence is an isomorphism by [Sta25, 04TQ].

[6]The groupoid whose objects are $(x, y, f(x) \xrightarrow{\sim} g(y))$ and morphisms $(x, y, f(x) \xrightarrow{\sim} g(y)) \to (x', y', f(x') \xrightarrow{\sim} g(y'))$ are pairs $(x \xrightarrow{\alpha} x', y \xrightarrow{\alpha'} y')$ such that the obvious diagram commutes.

[7]The groupoid whose objects are (x, y) such that $f(x) = g(y)$ and morphisms $(x, y) \to (x', y')$ are pairs $(x \xrightarrow{\alpha} x', y \xrightarrow{\alpha'} y')$ such that $f(\alpha) = g(\alpha')$.

2.2.2 Picard stack

Recall that a ***Picard groupoid*** is by definition a symmetric monoidal groupoid G for which every element $x \in G$ admits an inverse $x^{-1} \in G$ such that $x \otimes x^{-1} \cong 1$, where 1 is the monoidal unit. For example, if X is a scheme, then the groupoid of line bundles on X equipped with the tensor product structure is a Picard groupoid. One can think of Picard groupoids as categorifications of abelian groups.

Let us recall the following definition in [EGNO15, Definition 7.1.2][8]:

Definition 2.2.6. *Let X be a groupoid and G be a Picard groupoid. An **action** of G on X is a functor $\otimes : G \times X \to X$ together with two natural isomorphisms $m_{g,h,x} : (g \otimes h) \otimes x \xrightarrow{\sim} g \otimes (h \otimes x)$ and $l_x : 1 \otimes x \xrightarrow{\sim} x$ such that the obvious diagrams*

$$\begin{tikzcd}
 & ((g \otimes h) \otimes k) \otimes x \arrow[dl] \arrow[dr] & \\
(g \otimes (h \otimes k)) \otimes x \arrow[d] & & (g \otimes h) \otimes (k \otimes x) \arrow[d] \\
g \otimes ((h \otimes k) \otimes x) \arrow[rr] & & g \otimes (h \otimes (k \otimes x))
\end{tikzcd}$$

and

$$\begin{tikzcd}
(g \otimes 1) \otimes x \arrow[rr] \arrow[dr] & & g \otimes (1 \otimes x) \arrow[dl] \\
 & g \otimes x &
\end{tikzcd}$$

commute for all $g, h, k \in G$ and $x \in X$.

In this case, we can form the 2-categorical quotient of $[X/G]$. It is a 2-category which can be explicitly described as follows (cf. [Ngô06, end of section 4]):

- The objects are objects of X.
- The 1-morphisms from x_1 to x_2 are pairs (g, α), where $g \in G$ and $\alpha \in Mor_X(gx_1, x_2)$.
- The 2-morphisms from (g, α) to (g', α') are $\phi \in Mor_G(g, g')$ such that

$$\begin{tikzcd}
gx_1 \arrow[r, "\alpha"] \arrow[d, "\phi \otimes id_{x_1}"'] & x_2 \\
g'x_1 \arrow[ur, "\alpha'"'] &
\end{tikzcd}$$

  commutes.

It turns out that in the case of interest in this thesis, every 2-quotient is equivalent to a 1-groupoid.

[8] There, X is called a left module category over G.

Lemma 2.2.7. *[Ngô06, Lemme 4.7] Let X be a groupoid and G be a Picard groupoid acting on X. Then the 2-quotient $[X/G]$ is equivalent to a 1-groupoid if and only if the natural map induced by the action*

$$\mathrm{Aut}_G(1_G) \to \mathrm{Aut}_X(x)$$

is an injection for every $x \in X$.

Proof. It is evident from the explicit description above that the 2-quotient $[X/G]$ is equivalent to a 1-groupoid if and only if for every $g \in G, x \in X$, the natural map

$$\mathrm{Aut}_G(g) \to \mathrm{Aut}_X(gx)$$

is an injection. This is equivalent to the condition in the statement of the lemma, because we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Aut}_G(g) \arrow[r] & \mathrm{Aut}_X(gx) \\ \mathrm{Aut}_G(1_G) \arrow[u, "g\otimes -"] \arrow[ur] & \end{tikzcd}$$

where the vertical map is an isomorphism. □

Now let $\mathcal{C}$ be a site. Let $*$ be the terminal object in the topos. Note that for any stack $\mathcal{P}$ on $\mathcal{C}$, an element in $\mathcal{P}(*) := \mathrm{Mor}(*, \mathcal{P})$ is the same as a compatible collection of objects in $\mathcal{P}(U)$ as $U \in \mathcal{C}$ varies.

Definition 2.2.8. *A* ***Picard prestack*** *on $\mathcal{C}$ is a prestack $\mathcal{P}$ on $\mathcal{C}$ together with*

- *a morphism $\otimes : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$,*
- *a 2-morphism*[9] $\otimes \circ (\otimes \times \mathrm{id}) \Rightarrow \otimes \circ (\mathrm{id} \times \otimes)$,
- *a 2-morphism*[10] $\otimes \Rightarrow \otimes \circ \mathrm{flip}$,
- *an element $1 \in \mathcal{P}(*)$,*
- *a 2-morphism $1 \otimes \mathrm{id} \Rightarrow \mathrm{id}$,*
- *a 2-morphism $\mathrm{id} \otimes 1 \Rightarrow \mathrm{id}$,*

such that for every $U \in \mathcal{C}$, these data make $\mathcal{P}(U)$ a Picard groupoid.

Remark 2.2.9. One should think of a Picard prestack as a 'commutative group prestack', i.e. a prestack $\mathcal{P}$ with a group operation $\otimes$ such that $x \otimes y = y \otimes x$. However, it is unnatural to assume equality holds. One should instead have an isomorphism $x \otimes y \xrightarrow{\sim} y \otimes x$ for every x, y. Moreover, these isomorphisms should be compatible. The definition above formalises this idea.

[9] Here, both $\otimes \circ (\otimes \times \mathrm{id})$ and $\otimes \circ (\mathrm{id} \times \otimes)$ are morphisms $\mathcal{P} \times \mathcal{P} \times \mathcal{P} \to \mathcal{P}$. A map between these morphisms is thus a 2-morphism in the category of stacks.

[10] Here, flip is the morphism $\mathcal{P} \times \mathcal{P} \to \mathcal{P} \times \mathcal{P}$ given by $(x, y) \mapsto (y, x)$.

Definition 2.2.10. *Let X be a prestack and $\mathcal{P}$ be a Picard prestack. An **action** of $\mathcal{P}$ on X is*

- *a morphism $\otimes : \mathcal{P} \times X \to X$,*
- *a 2-morphism*[11] $\otimes \circ (\otimes \times \mathrm{id}_X) \Rightarrow \otimes \circ (\mathrm{id}_{\mathcal{P}} \times \otimes)$
- *a 2-morphism $1_{\mathcal{P}} \otimes \mathrm{id}_X \Rightarrow \mathrm{id}_X$*

such that for every $U \in \mathcal{C}$, these data define an action of $\mathcal{P}(U)$ on $X(U)$ in the sense of Definition 2.2.6.

Let J be a commutative group sheaf on $\mathcal{C}$. Since J is commutative, we can identify left J-torsors and right J-torsors in the naive way.[12] In particular, we can form the contracted product $Q_1 \times^J Q_2$ of two J-torsors Q_1 and Q_2. The contracted product $Q_1 \times^J Q_2$ is still a J-torsor by the commutativity of J. Moreover, if Q is a J-torsor, then Q' defined to be the same sheaf Q with J-action given by

$$\begin{gathered} Q \times J \to Q \times J \to Q \\ (q, j) \mapsto (q, j^{-1}) \end{gathered}$$

satisfies $Q \times^J Q' \cong J_a$. Thus $[*/J]$ is a Picard stack.

Recall that for a prestack X on a site $\mathcal{C}$, its inertia prestack $\mathcal{I}_X$ is the prestack whose U-valued points are the groupoid of pairs (x, θ), where $x \in X(U), \theta \in \mathrm{Aut}(x)$. If $x \in X(U)$, then we let $\underline{\mathrm{Aut}}(x)$ denote the presheaf on $\mathcal{C}_{/U}$ given by $V \mapsto \mathrm{Aut}(x|_V)$.

Proposition 2.2.11. *Let $\mathcal{C}$ be a site, X be a stack over B, and J be a commutative group sheaf on $\mathcal{C}$. Suppose we have a morphism*

$$f : J \times X \to \mathcal{I}_X$$

over X. Equivalently, for every $U \in \mathcal{C}$ and $x \in X(U)$, we have a morphism

$$J|_U \to \underline{\mathrm{Aut}}(x) \tag{2.1}$$

such that for every $U \in \mathcal{C}$ and every morphism $x \xrightarrow{\sim} y$ in $X(U)$, the following commutes

$$\begin{tikzcd} & J|_U \arrow[dl] \arrow[dr] & \\ \underline{\mathrm{Aut}}(x) \arrow[rr] & & \underline{\mathrm{Aut}}(y) \end{tikzcd}$$

where the horizontal map is given by conjugation. If (2.1) is a group homomorphism for every $U \in \mathcal{C}$ and $x \in X(U)$, then there is a canonical action

$$[*/J] \times X \to X. \tag{2.2}$$

[11] Both $\otimes \circ (\otimes \times \mathrm{id}_X)$ and $\otimes \circ (\mathrm{id}_{\mathcal{P}} \times \otimes)$ are morphisms $\mathcal{P} \times \mathcal{P} \times X \to X$.

[12] I.e. if Q is a right J-torsor, then we view it as a left J-torsor with the same J-action, without taking inverse.

Remark 2.2.12. Informally, the action is given by twisting objects and morphisms in X by J-bundles.

Remark 2.2.13. The assumption that J is commutative ensures that $[*/J]$ is a Picard stack, so its action on another stack makes sense.

Proof. For $U \in \mathcal{C}$ and $x \in X(U)$, we denote the image of j under $J|_U \to \underline{\mathrm{Aut}}(x)$ as j_x.

We define a morphism of prestacks $[*/_{pre}\ J] \times X \to X$, which on U-valued points (for $U \in \mathcal{C}$) is given by

$$(*, x) \mapsto x$$

on objects and is given by

$$((*, x) \xrightarrow{(j,\theta)} (*, y)) \mapsto (x \xrightarrow{\theta \circ j_x = j_y \circ \theta} y)$$

on morphisms. Taking the stackification gives the desired action on stacks. $\square$

Remark 2.2.14. Let us describe the action explicitly on objects. To avoid confusing with index, let us write $j(x)$ for what we denote as j_x above. Let Q be a J-bundle on U and $x \in X(U)$. Fix a covering $\{U_a \to U\}$ of U and trivialisations $\iota_a : Q|_{U_a} \xrightarrow{\sim} J|_{U_a}$. Write $U_{ab} = U_a \times_U U_b$. Let

$$\iota_b|_{U_{ab}} \circ \iota_a|_{U_{ab}}^{-1} =: j_{ab} \in H^0(U_{ab}, Q).$$

Then

$$(x|_{U_a}, j_{ab}(x|_{U_{ab}})),$$

where $j_{ab}(x|_{U_{ab}})$ is viewed as an isomorphism $(x|_{U_a})|_{U_{ab}} \xrightarrow{\sim} (x|_{U_b})|_{U_{ab}}$, is a descent datum on X, so it glues to an object on $X(U)$. It is straightforward to check that this is the image of (j, x) under the map (2.2).

We will need the following slight generalisation of Proposition 2.2.11.

Proposition 2.2.15. *Let $\mathcal{C}$ be a site, B be a sheaf, X be a stack on $\mathcal{C}$, and $h : J \to B$ be a commutative group sheaf over B. Suppose we have a morphism*

$$f : J \times_B X \to \mathcal{I}_X$$

over X. In particular, for all $U \in \mathcal{C}$, $x \in X(U)$ having image b in $B(U)$, we get a map

$$\{j \in J(U) : h(j) = b\} \to \mathrm{Aut}(x)$$

Suppose that this is a group homomorphism. Then there is a canonical action

$$[B/J] \times_B X \to X$$

of stacks on $\mathcal{C}$ over B.

Proof. This follows from Proposition 2.2.11 by replacing $\mathcal{C}$ by $\mathcal{C}_{/B}$ and identifying stacks over B with stacks on $\mathcal{C}_{/B}$. $\square$

2.2.3 An exactness criterion

We will often prove results about G-bundles by first proving them for vector bundles, and then generalise the results using the Tannakian formalism, namely by interpreting G-bundles on X as *exact* tensor functors $\mathrm{Rep}(G) \to \{\text{vector bundles on } X\}$. The following result is useful for proving exactness of functors in various situations.

Lemma 2.2.16. *Let* $\left\{X_i \xrightarrow{f_i} X\right\}_{i\in I}$ *be morphisms of locally ringed spaces such that* $\cup f_i(X_i) = X$. *Then a sequence of vector bundles on* X

$$0 \to \mathcal{E} \xrightarrow{a} \mathcal{E}' \xrightarrow{b} \mathcal{E}'' \to 0 \tag{2.3}$$

with $b \circ a = 0$ *is exact if and only if*

$$0 \to f_i^*\mathcal{E} \to f_i^*\mathcal{E}' \to f_i^*\mathcal{E}'' \to 0 \tag{2.4}$$

is exact for every i.

The proof is given to us by Ken Lee in the context of part (2) of Lemma 2.2.89, namely in the case where X and X_i are the Fargues-Fontaine curves. We sincerely thank him for it. The generalisation to locally ringed spaces is proved in the same way.

To simplify notation, in the proof, whenever Y is a locally ringed space and $y \in Y$, then we let $\kappa(y) = \mathcal{O}_{Y,y}/m_y$ denote the residue field of Y at y. If $\mathcal{F}$ is a sheaf of Y, then $\mathcal{F}(y) := \mathcal{F}_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y)$.

Proof. The 'only if' direction is clear by considering stalks. Let us prove the 'if' direction. By [Sta25, Lemma 00O0], to check that (2.3) is exact, it suffices to show that for every $x \in X$,

$$0 \to \mathcal{E}(x) \to \mathcal{E}'(x) \to \mathcal{E}''(x) \to 0 \tag{2.5}$$

is exact. By assumption, there exists $i \in I$ and $x' \in X_i$ such that $f_i(x') = x$. We can check the exactness of (2.5) after base changing along $\kappa(x) \to \kappa(x')$. Note that $\mathcal{E}(x) \otimes_{\kappa(x)} \kappa(x') = (f_i^*\mathcal{E})(x')$, so we would like to show the exactness of

$$0 \to (f_i^*\mathcal{E})(x') \to (f_i^*\mathcal{E}')(x') \to (f_i^*\mathcal{E}'')(x') \to 0.$$

This follows from the exactness of (2.4). □

2.2.4 Huber rings

Following [SW20, Lecture 2], we briefly review the the theory of Huber rings. Unless otherwise stated, everything in this subsection is from this source.

Definition 2.2.17. *A* ***Huber ring*** *is a topological ring* R *for which there is an open subring* R_0 *whose subspace topology is* I*-adic for some finitely generated ideal* I *of* R_0. *Any such* R_0 *is called a* ***ring of definition***. *Any such* I *is called an* ***ideal of definition***.

Remark 2.2.18. The R_0, I are not part of the data of a Huber ring. A Huber ring can have many different rings/ideals of definition. Also, R_0 needs not be I-adically complete.

Example 2.2.19.

1. Any ring equipped with the discrete topology is a Huber ring.
2. $\mathbb{Q}_p$ is a Huber ring with ring of definition $\mathbb{Z}_p$.
3. If R is a ring with a finitely generated ideal, then R equipped with the I-adic topology is a Huber ring. For example, $\mathbb{Z}_p[\![T]\!]$ with the (p, T)-adic topology is a Huber ring.
4. (Non-example) [Wei19, Section 1.4] One may think that $\mathbb{Q}_p[\![T]\!]$ is a Huber ring with ring of definition $\mathbb{Z}_p[\![T]\!]$ and ideal of definition $(p, T)\mathbb{Z}_p[\![T]\!]$. This is false. In fact, $\mathbb{Q}_p[\![T]\!]$ with this topology is not a topological ring, because as $n \to \infty$, $T^n \to 0$ but $p^{-1}T^n \not\to 0$.

Let R be a Huber ring.

Definition 2.2.20. *An element $r \in R$ is called **topologically nilpotent** if $\lim_{n\to\infty} r^n = 0$. We let $R^{\circ\circ}$ denote the set of topologically nilpotent elements in R. An element r is a **pseudouniformizer** if it is topologically nilpotent and is a unit. A Huber ring is **Tate** if it has a pseudouniformizer.*

For example, $\mathbb{Q}_p$ is Tate with p as a pseudouniformizer. On the other hand, $\mathbb{Z}_p$ is not Tate.

Definition 2.2.21. *A subset $S \subset R$ is called **bounded** if for every open subgroup $U \subset R$, there is an open neighborhood V of 0 such that $S \cdot V \subset U$.*

Example 2.2.22.

- Any finite subset is bounded. The union of any two bounded subsets is bounded.
- Suppose that R is Tate with a pseudouniformizer ϖ and R_0 is a ring of definition. Then a subset $S \subset R$ is bounded if and only if $S \subset \varpi^{-n} R_0$ for some $n \in \mathbb{Z}$.

Definition 2.2.23. *An element $r \in R$ is called **powerbounded** if the set $\{r^n : n \in \mathbb{Z}_{\geq 1}\}$ is bounded. We let R° denote the set of powerbounded elements in R.*

It can be shown that R° is an open and integrally closed subring of R. However, it needs not be a ring of definition of R. In fact, R° is a ring of definition of R if and only if R is uniform.

Definition 2.2.24. *R is called **uniform** if R° is a bounded subset of R.*

Example 2.2.25. Let K be a non-Archimedean valued field with absolute value $|\cdot|$. Then a subset $S \subset K$ is bounded if and only if $S \subset \{x \in K : |x| \leq r\}$ for some $r > 0$. Also $K^\circ = \{x \in K : |x| \leq 1\}$. This is bounded so K is uniform.

Example 2.2.26. Consider the Huber ring $R = \mathbb{Q}_p[T]/(T^2)$ with ring of integers $R_0 = \mathbb{Z}_p[T]/(T^2)$ and ideal of definition pR_0. Then p is a pseudouniformizer of R, so R is Tate. It is easy to see that

- $a \in R^\circ$ for all $a \in \mathbb{Z}_p$,
- $bT \in R^\circ$ for all $b \in \mathbb{Q}_p$,
- $c \notin R^\circ$ if $c \in \mathbb{Q}_p \setminus \mathbb{Z}_p$.

Since R° is a ring, it follows that $R^\circ = \{a + bT : a \in \mathbb{Z}_p, b \in \mathbb{Q}_p\}$. This is not bounded, so R is not uniform.

Definition 2.2.27. *R is called **complete** if every Cauchy sequence in R has a unique limit.*

Equivalently, some (equivalently, every) ring of definition R_0 is I-adically complete, where I is an ideal of definition of R_0. The inclusion functor {complete Huber rings} $\hookrightarrow$ {Huber rings} has a left adjoint, called ***completion***. Explicitly, if we let R_0 be a ring of definition of R and $I \subset R_0$ be an ideal of definition, then the completion $\widehat{R}$ can be constructed as

$$\widehat{R_0} \otimes_{R_0} R$$

with ring of definition $\widehat{R_0} := \lim_n R_0/I^n$ and ideal of definition $I\widehat{R_0}$. See [Hub93, Lemma 1.6]. As a group, $\widehat{R}$ is isomorphic to $\lim_n R/I^n$. Note however that I^n is not an ideal of R, so R/I^n is not a ring.

2.2.5 Perfectoid rings

Fix a prime p. In this section, we give a brief summary of the theory of perfectoid spaces, with an emphasis on examples.

Definition 2.2.28. *A **perfectoid ring**[13] R is a complete uniform Tate ring such that*

- *there is a pseudo-uniformizer $\pi \in R$ such that $\pi^p \mid p$ in R° and*
- *the Frobenius map $R^\circ/\pi \to R^\circ/\pi^p$ is an isomorphism.*

By [SW20, Remark 6.1.3], it is equivalent to require R to be a complete uniform Tate ring such that

- there is a pseudo-uniformizer $\pi \in R$ such that $\pi^p \mid p$ in R° and
- the Frobenius map $R^\circ/p \to R^\circ/p$ is surjective.

[13]These are also sometimes called perfectoid Tate rings so as to distinguish them from integral perfectoid rings.

For instance, $\mathbb{C}_p$ and $\overline{\mathbb{F}_p((t))}^\wedge$ are perfectoid rings. More generally, if K is an algebraically closed, complete, non-Archimedean valued field (whose absolute value is non-trivial), then K is perfectoid. The product of any two perfectoid rings is a perfectoid ring. To construct more examples, it is useful to note the following lemma, whose proof is easy.

Lemma 2.2.29. *Let K be a field equipped with a non-trivial non-Archimedean absolute value $|\cdot|$. Denote its completion with respect to the metric by $\widehat{K}$. Let $t \in K$ be any pseudo-uniformizer. Then*

$$\begin{aligned}\mathcal{O}_{\widehat{K}} &= \text{completion of } \mathcal{O}_K \text{ w.r.t. } |\cdot| \\ &= t\text{-adic completion of } \mathcal{O}_K\end{aligned}$$

and

$$\mathcal{O}_{\widehat{K}}/(t) = \mathcal{O}_K/(t).$$

Example 2.2.30. $\mathbb{Q}_p((p^{1/p^\infty})) := \mathbb{Q}_p(p^{1/p^\infty})^\wedge$ is a perfectoid ring, where $\mathbb{Q}_p(p^{1/p^\infty})$ is the field obtained from $\mathbb{Q}_p$ by adjoining a compatible sequence of p-th power roots of unity. To see this, let $K = \mathbb{Q}_p(p^{1/p^\infty})$. Clearly, $\widehat{K}$ is non-Archimedean, not discretely valued, and $|p| < 1$. It remains to show that the Frobenius is surjective on $\mathcal{O}_{\widehat{K}}/(p) = \mathcal{O}_K/(p)$. Note that for all $n \geq 1$, $\mathcal{O}_{\mathbb{Q}_p(p^{1/p^n})} = \mathbb{Z}_p[p^{1/p^n}]$, so

$$\mathcal{O}_K = \mathbb{Z}_p[p^{1/p^\infty}].$$

Every element in $\mathcal{O}_K/(p)$ has a representative of the form $\sum_{i=0}^{p^n} a_i p^{i/p^n}$ for some $a_i \in \mathbb{Z}_p, n \geq 1$. This is mapped by $\sum_{i=0}^{p^n} a_i p^{i/p^{n+1}}$ under the Frobenius on $\mathcal{O}_K/(p)$. This proves the surjectivity of the Frobenius.

If we view $\mathbb{Q}_p((p^{1/p^\infty}))$ as a subring of $\mathbb{C}_p$, then every element of $\mathbb{Q}_p((p^{1/p^\infty}))$ can be uniquely written as

$$\sum_{i \in \mathbb{Z}[\frac{1}{p}]} a_i p^i$$

where every $a_i \in \{0, 1, \ldots, p-1\}$ and for every $n \in \mathbb{N}$, $a_i = 0$ for almost all $i \leq n$. Conversely, every such sum gives an element of $\mathbb{Q}_p((p^{1/p^\infty}))$.

Similarly, we can show that $\mathbb{Q}_p^{cycl} := \mathbb{Q}_p(\mu_{p^\infty})^\wedge$ and $\mathbb{F}_p((t^{1/p^\infty})) := \mathbb{F}_p((t))(t^{1/p^\infty})^\wedge$, where $\mu_{p^\infty} := \{x \in \overline{\mathbb{Q}}_p : x^{p^n} = 1 \text{ for some } n \geq 1\}$, are both perfectoid rings. Explicitly, we can write elements of $\mathbb{F}_p((t^{1/p^\infty}))$ as formal power series:

$$\mathbb{F}_p((t^{1/p^\infty})) = \left\{ \sum_{i \in \mathbb{Z}[\frac{1}{p}]} a_i t^i : \begin{array}{l} \text{every } a_i \in \mathbb{F}_p \\ \text{and for every } n \in \mathbb{N}, \\ a_i = 0 \text{ for almost all } i \leq n. \end{array} \right\}.$$

Definition 2.2.31. *A* ***perfectoid field*** *is a perfectoid ring whose underlying ring is a field. Equivalently [Ked18, Theorem 4.2], it is a complete non-Archimedean field K (i.e. topology is induced by a non-Archimedean absolute value) such that*

- *K is not discretely valued,*
- $|p| < 1$,
- *the Frobenius is surjective on $\mathcal{O}_K/p$.*

The topology of any complete Tate ring is induced by a (sub-multiplicative) norm. The point of [Ked18, Theorem 4.2] is that, for a perfectoid field, one can construct a *multiplicative* norm inducing the topology.

Example 2.2.32. [Zhu, Example 3.1.9] If K is a perfectoid field, then $K\langle T^{1/p^\infty}\rangle$ is a perfectoid ring. Here, $K\langle T^{1/p^\infty}\rangle = \mathcal{O}_K\langle T^{1/p^\infty}\rangle[\pi^{-1}]$ and $\mathcal{O}_K\langle T^{1/p^\infty}\rangle = \mathcal{O}_K[T^{1/p^\infty}]^\wedge$, where the completion is with respect to the π-adic topology and π any pseudouniformizer of K. We equip it with the topology such that $\pi^n\mathcal{O}_K\langle T^{1/p^\infty}\rangle$ for $n \in \mathbb{N}$ forms an open neighborhood basis of 0. Explicitly, we can write elements of the ring as formal power series:

$$K\langle T^{1/p^\infty}\rangle = \left\{ \sum_{i\in\mathbb{Z}[\frac{1}{p}]_{\geq 0}} a_i T^i : \begin{array}{l} \text{every } a_i \in K \\ \text{and for every neighborhood } U \text{ of } 0 \text{ in } K, \\ a_i \in U \text{ for almost all } i. \end{array} \right\}.$$

Fix an absolute value $|\cdot|$ that induces the topology on K. We define the Gauss norm $||\cdot||$ on $K\langle T^{1/p^\infty}\rangle$ by

$$\left|\left| \sum_{i\in\mathbb{Z}[\frac{1}{p}]_{\geq 0}} a_i T^i \right|\right| := \sup_i |a_i|.$$

This induces the topology on $K\langle T^{1/p^\infty}\rangle$. Moreover, this norm is multiplicative (see [Zhu, Example 3.1.9]). It follows that the set of powerbounded elements of $K\langle T^{1/p^\infty}\rangle$ is precisely $\mathcal{O}_K\langle T^{1/p^\infty}\rangle$. It is now clear that $K\langle T^{1/p^\infty}\rangle$ is indeed perfectoid.

Similarly, $K\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle$ and $K\langle T_1^{\pm 1/p^\infty}, \ldots, T_n^{\pm 1/p^\infty}\rangle$ are perfectoid rings.

The following proposition shows that for a ring of characteristic p, being perfectoid is essentially the same as being perfect.

Proposition 2.2.33. *[SW20, Proposition 6.1.6] Let R be a complete Tate ring with $pR = 0$. Then R is perfectoid if and only if R is perfect (i.e. the Frobenius on R is a bijection).*

The hard part of the proof is that, if R is perfect (and complete Tate), then it is uniform. This is proved by the Banach open mapping theorem.

We also get lots of examples of perfectoid rings by the following important theorem.

Theorem 2.2.34. *[SW20, Theorem 7.4.5] Any finite étale extension of a perfectoid ring is perfectoid.*

In particular, any finite field extension of a perfectoid field is perfectoid.[14]

[14]Note that every finite field extension of a perfectoid field K is separable, because either $\mathrm{char} K = 0$ or $\mathrm{char} K = p$ is perfect.

2.2.6 Relation with integral perfectoid rings

Definition 2.2.35. *A commutative ring A is called* ***special integral perfectoid*** *if there is a non-zero divisor $\pi \in A$ such that*

- *A is π-adically complete,*
- *$\pi^p \mid p$ in A, and*
- *the Frobenius map $A/\pi \to A/\pi^p$ is an isomorphism.*

While this concept is quite standard, the terminology is not standard. Every special integral perfectoid ring is integral perfectoid[15] in the sense of [BMS18, Definition 3.5] by [BMS18, Lemmas 3.9, 3.10].[16]

There is a tight relationship between perfectoid rings and special integral perfectoid rings.

Proposition 2.2.36. *If A, π are as in Definition 2.2.35, then $R := A[\frac{1}{\pi}]$, equipped with the topology such that $\pi^n A$ ($n \in \mathbb{N}$) is an open neighborhood basis of 0, is a perfectoid ring. Also, every perfectoid ring arises in this way.*

Proof. The first part is stated in [BMS18, Lemma 3.21] and proved in [Sch12, Lemma 5.6]. One can also consult [Heu, Proposition 1.2.14] for the same proof without explicitly mentioning almost mathematics.

The second part is true because if R is any perfectoid ring, then R° is special integral perfectoid, so taking $A := R^\circ$ works. $\square$

Remark 2.2.37. Let A, π, R be as in the first part of Proposition 2.2.36. Up to multiplying π by a unit in A, π admits compatible p-power roots in A. Then it is shown in the proof of [Heu, Proposition 1.2.14] that the ring of powerbounded elements R° in R is

$$\left\{x \in R : \pi^{1/p^n} x \in A \text{ for all } n \in \mathbb{N}\right\}.$$

In particular, $A \subset R^\circ$ is an almost isomorphism in the sense of almost mathematics.

Among other things, Proposition 2.2.36 is useful for proving perfectoidness. Let us give an example, generalising Example 2.2.32.

Example 2.2.38. Let R be a perfectoid ring. Then $R\langle T^{1/p^\infty}\rangle$ is also perfectoid. Here, $R\langle T^{1/p^\infty}\rangle := R^\circ\langle T^{1/p^\infty}\rangle[\pi^{-1}]$ and $R^\circ\langle T^{1/p^\infty}\rangle := R^\circ[T^{1/p^\infty}]^\wedge$, where the completion is with respect to the π-adic topology and π any pseudouniformizer of R. We equip it with the topology such that $\pi^n R^\circ\langle T^{1/p^\infty}\rangle$ for $n \in \mathbb{N}$ forms an open neighborhood basis of 0. Note that none of these depends on the choice of the pseudouniformizer.

Similarly, $R\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^n}\rangle$ is also perfectoid.

[15] In [BMS18], integral perfectoid rings are simply called perfectoid rings. What we called perfectoid are called 'perfectoid in Fontaine's sense' there.

[16] In fact, by those two lemmas, special integral perfectoid rings are precisely those integral perfectoid rings which are π-adically complete for some non-zero-divisor π such that $\pi^p \mid p$.

2.2.7 Tilting

One of the key properties of perfectoid rings is that they have strong relation with characteristic p rings.

Let R be a perfectoid ring.

Definition 2.2.39. *The **tilt** of R is*

$$R^\flat := \varprojlim_{x \mapsto x^p} R,$$

equipped with the inverse limit topology.

This can be equipped with the structure of a topological ring. See [SW20, Section 6.2] for the detail. Then it is a characteristic p perfectoid ring. Note that if R is of characteristic p, then $R^\flat = R$.

Example 2.2.40. We have $\mathbb{Q}_p((p^{1/p^\infty}))^\flat = \mathbb{F}_p((t^{1/p^\infty}))$, $(\mathbb{Q}_p^{cycl})^\flat = \mathbb{F}_p((t^{1/p^\infty}))$, $\mathbb{C}_p^\flat = \overline{\mathbb{F}_p((t))}^\wedge$. If K is a perfectoid field, then $K\langle T^{1/p^\infty}\rangle^\flat = K^\flat\langle T^{1/p^\infty}\rangle$.

A perfectoid R-algebra is a perfectoid ring S equipped with a continuous ring homomorphism $R \to S$.

Theorem 2.2.41. *[SW20, Theorems 6.2.7, 7.4.5]*

1. *We have an equivalence of categories*

$$\{\text{perfectoid } R\text{-algebras}\} \xrightarrow{\sim} \{\text{perfectoid } R^\flat\text{-algebras}\}$$
$$S \mapsto S^\flat.$$

2. *Every finite étale R-algebra, equipped with the induced topology[17], is a perfectoid ring. Moreover, the above equivalence restricts to give an equivalence of categories*

$$\{\text{finite étale } R\text{-algebras}\} \xrightarrow{\sim} \{\text{finite étale } R^\flat\text{-algebras}\}$$
$$S \mapsto S^\flat.$$

2.2.8 Adic spaces

Following [SW20, Lectures 2, 3], we will briefly review the theory of adic spaces.

If Γ is a totally ordered abelian group, whose group operation is written multiplicatively, then we let $\Gamma \cup \{0\}$ denote the totally ordered monoid $\Gamma \sqcup \{0\}$ with 0 the minimum element and $0 \cdot a = a \cdot 0 = 0$.

Definition 2.2.42. *A **valuation** of a topological ring A is a function $|\cdot| : A \to \Gamma \cup \{0\}$, where Γ is a totally ordered abelian group, such that*

[17]If A is a topological ring, then any finitely generated A-module has a canonical topology, given by picking a surjection $A^n \twoheadrightarrow M$ and equipping M with the quotient topology. This is independent of the choice of the surjection. See [GR18, Lemma 8.3.34].

- $|0| = 0, |1| = 1$,
- $|ab| = |a||b|$,
- $|a + b| \leq \max(|a|, |b|)$.

*If moreover for every $a \in A$, the subset $\{x \in A : |x| < |a|\}$ is open, then we say that $|\cdot|$ is a **continuous** valuation.*

Definition 2.2.43. *Two valuations $|\cdot|$ and $|\cdot|'$ on A are **equivalent** when it is the case that for $a, b \in A$, $|a| \leq |b|$ if and only if $|a|' \leq |b|'$.*

Definition 2.2.44.

- *A **ring of integral elements** R^+ of a Huber ring R is an open and integrally closed subring of R^0.*
- *A **Huber pair** is a pair (R, R^+) where R is a Huber ring and R^+ is a ring of integral elements in R.*
- *A **morphism** of Huber pairs $(R, R^+) \to (S, S^+)$ is a continuous homomorphism*

$$\phi : R \to S$$

such that $\phi(R^+) \subset S^+$.

- *A Huber pair (R, R^+) is called **complete** if R is complete.*[18]

The inclusion functor {complete Huber pairs} $\hookrightarrow$ {Huber pairs} has a left adjoint, called ***completion***. Explicitly, if (R, R^+) is a Huber pair, then its completion is $(\widehat{R}, \widehat{R^+})$, where $\widehat{R}$ is the completion of Huber ring and $\widehat{R^+}$ is the topological closure of R^+ in $\widehat{R}$.

Let (R, R^+) be a Huber pair. Note that we always have $R^{\circ\circ} \subset R^+ \subset R^{\circ} \subset R$.

Definition 2.2.45. *The **adic spectrum** $\mathrm{Spa}(R, R^+)$ of (R, R^+) is the set of equivalence classes of continuous valuations $|\cdot|$ on A such that $|s| \leq 1$ for every $s \in R^+$. We equip it with the topology generated by open subsets of the form $U(\frac{f}{g}) = \{|\cdot| \in \mathrm{Spa}(A, A^+) : |f| \leq |g| \neq 0\}$ as f, g varies over A, i.e. the finite intersections of subsets of this form forms a basis for the topology.*

Example 2.2.46.

1. Let $R = R^+ = \mathbb{F}_q$ with the discrete topology. Let $|\cdot|$ be a continuous valuation on $\mathbb{F}_q$. Then for all $x \in \mathbb{F}_q \setminus \{0\}$, $|x|^q = |x^q| = 1$. As ordered abelian groups have no non-trivial torsion element, $|x| = 1$. Thus, $\mathrm{Spa}(\mathbb{F}_q, \mathbb{F}_q)$ is a singleton, consisting of the trivial valuation.
2. Let $(K, |\cdot|)$ be a valued field, where $|\cdot|$ is a non-trivial absolute value on K. Let K^+ be an open and bounded valuation subring of K. Then we have a homeomorphism between $\mathrm{Spa}(K, K^+)$ and $\mathrm{Spec}(K^+/K^{\circ\circ})$. See [Zhu, Example 2.4.7.(3)]. In particular, $|\mathrm{Spa}(K, K^+)|$ is a totally ordered chain of specialisations.

[18] Note that R^+ is closed in R. Thus, if R is complete, then so is R^+.

Proposition 2.2.47. *[SW20, Proposition 2.3.9] The natural map*

$$\mathrm{Spa}(\widehat{R}, \widehat{R}^+) \xrightarrow{\sim} \mathrm{Spa}(R, R^+)$$

is a homeomorphism.

Definition 2.2.48. *A **rational subset** of* $\mathrm{Spa}(R, R^+)$ *is a subset of the form*

$$U\left(\frac{f_1, \ldots, f_n}{g}\right) = \left\{ |\cdot| \in \mathrm{Spa}(A, A^+) : |f_1|, \ldots, |f_n| \leq |g| \neq 0 \right\}$$

where $(f_1, \ldots, f_n)R$ *is an open ideal of* R.

It is easy to see that any rational subset is an open subset of $\mathrm{Spa}(R, R^+)$. In fact, the rational subsets form a basis for the topology of $\mathrm{Spa}(R, R^+)$.

We would like to endow $\mathrm{Spa}(R, R^+)$ with a sheaf of topological rings. We shall do so by specifying the values of the sheaf on the basis of rational subsets.

Lemma 2.2.49. *[SW20, Theorem 3.1.3] Let* U *be a rational subset of* $\mathrm{Spa}(R, R^+)$. *Then there is a complete Huber pair* (R_U, R_U^+) *and a morphism* $(R, R^+) \to (R_U, R_U^+)$ *such that the map*

$$\mathrm{Spa}(R_U, R_U^+) \to \mathrm{Spa}(R, R^+) \tag{2.6}$$

factors through U *and* (R_U, R_U^+) *is universal in such complete Huber pairs. Moreover,* (2.6) *is a homeomorphism onto* U.

Definition 2.2.50. *Let* $X = \mathrm{Spa}(R, R^+)$. *For an open subset* $W \subset X$, *define*

$$\mathcal{O}_X(W) := \lim_{W \subset U \text{ rational}} R_W$$

equipped with the inverse limit topology and

$$\mathcal{O}_X^+(W) := \lim_{W \subset U \text{ rational}} R_W^+$$

equipped with the inverse limit topology.

Then $\mathcal{O}_X, \mathcal{O}_X^+$ are both presheaves of topological rings. However, they are not necessarily sheaves of topological rings.

Remark 2.2.51. We have $\mathcal{O}_X(X) = \widehat{R}$ (not R in general).

Definition 2.2.52. *A Huber pair* (R, R^+) *is **sheafy** if* $\mathcal{O}_{\mathrm{Spa}(R,R^+)}$ *is a sheaf of topological rings. (Then* $\mathcal{O}^+_{\mathrm{Spa}(R,R^+)}$ *is also a sheaf of topological rings.)*

Theorem 2.2.53. *[SW20, Theorem 3.1.8][BV15, Theorem 7] Let* (R, R^+) *be a complete Huber pair and let* $X = \mathrm{Spa}(R, R^+)$. *Then* $\mathcal{O}_X$ *is a sheaf of topological rings in the following situations.*

1. *(Schemes) The topological ring* R *is discrete.*

2. *(Formal schemes) The ring R has a Noetherian ring of definition.*

3. *(Rigid spaces) The topological ring R is Tate and strongly Noetherian, i.e. the rings $R\langle T_1, \dots, T_n\rangle$ is Noetherian for all $n \geq 0$.*

4. *(Stably uniform) $\mathcal{O}_X(U)$ is uniform for every rational subset $U \subset X$.*

Example 2.2.54. If $X = \mathrm{Spa}(R, R^+)$, where R is a perfectoid ring, then $\mathcal{O}_X(U)$ is a perfectoid ring for every rational subset $U \subset X$ by [SW20, Theorem 6.1.10]. Thus, (R, R^+) is stably uniform and hence sheafy.

Definition 2.2.55. *Define a category (V) as follows:*

- *Objects: triples $(X, \mathcal{O}_X, (|\cdot(x)|)_{x\in X})$, where X is a topological space, $\mathcal{O}_X$ is a sheaf of topological rings on X , and each $|\cdot(x)|$ is an equivalence class of continuous valuations on $\mathcal{O}_{X,x}$.*

- *Morphisms: maps of topologically ringed spaces $f : X \to Y$ (so that the map $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ is continuous for each open $V \subset Y$) such that the pullback of $|\cdot(x)|$ along $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ gives $|\cdot(f(x))|$.*

If (A, A^+) is a sheafy Huber pair, then $X = \mathrm{Spa}(A, A^+)$ is naturally an object in (V). Indeed, we have already defined its topology and structure sheaf. For all $x \in X$ and any rational subset $x \in U \subset X$, the equivalence class of continuous valuations $|\cdot(x)|$ extends uniquely to an equivalence class of continuous valuations on $\mathcal{O}_X(U)$, because $\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to X$ is a bijection onto U. The valuations are compatible as U varies. Thus, the valuations extend uniquely to a class of continuous valuations on the stalk $\mathcal{O}_{X,x}$, as desired. We call such an X an ***affinoid adic space***.

Definition 2.2.56. *An* ***adic space*** *is an object $(X, \mathcal{O}_X, (|\cdot(x)|)_{x\in X})$ of (V) that admits a covering by spaces U_i such that for each i,*

$$(U_i, \mathcal{O}_X|_{U_i}, (|\cdot(x)|)_{x\in U_i}) \cong \mathrm{Spa}(R_i, R_i^+)$$

for some sheafy Huber pair (R_i, R_i^+).

Remark 2.2.57.

1. Any adic space is automatically a *locally* ringed space and any morphism between them is automatically a morphism of *locally* ringed space [Wed, pages 76, 77].

2. For every sheafy Huber pair (R, R^+), the natural map $\mathrm{Spa}(\widehat{R}, \widehat{R}^+) \to \mathrm{Spa}(R, R^+)$ is an isomorphism.

3. The contravariant functor $(R, R^+) \to \mathrm{Spa}(R, R^+)$ from sheafy *complete* Huber pairs to adic spaces is fully faithful.

2.2.9 Perfectoid spaces

Definition 2.2.58. *An* ***affinoid perfectoid space*** *is an adic space of the form* $\mathrm{Spa}(R, R^+)$*, where* R *is a perfectoid ring and* R^+ *is an open and integrally closed subring of* R°*.*

Definition 2.2.59. *A* ***perfectoid space*** *is an adic space* X *for which there is an open cover* $X = \cup U_i$ *where each* U_i *is affinoid perfectoid.*

Remark 2.2.60. Let (R, R^+) be a complete Huber pair. Suppose that $\mathrm{Spa}(R, R^+)$ is a perfectoid space. Then we know that it has an open cover $\mathrm{Spa}(R, R^+) = \cup_i \mathrm{Spa}(S_i, S_i^+)$ where each S_i is a perfectoid ring. It is unknown whether R itself is a perfectoid ring. Thus, one needs to be careful that 'affinoid perfectoid' is different from 'affinoid and perfectoid.' In most literature and this thesis, 'affinoid perfectoid' is used in the sense of Definition 2.2.58.

We can extend the operation of tilting to perfectoid spaces.

Theorem 2.2.61 (Tilting equivalence)**.** *Let* X *be a perfectoid space. Then tilting induces an equivalence of categories*

$$\{\textit{perfectoid spaces over } X\} \xrightarrow{\sim} \left\{\textit{perfectoid spaces over } X^\flat\right\}.$$

Moreover, we have a homeomorphism $|X| \cong |X^\flat|$*.*

One can define étale morphisms and étale sites of perfectoid spaces [Sch17, Definition 6.2]. It turns out that tilting preserves the étale sites:

Theorem 2.2.62. *[Sch17, Theorem 6.3] Let* X *be a perfectoid space. Then tilting induces an equivalence of étale sites*

$$X_{\text{ét}} \xrightarrow{\sim} X^\flat_{\text{ét}}.$$

This equivalence restricts to an equivalence on the finite étale sites.

Definition 2.2.63. *If* Y *is a perfectoid space over* $\mathrm{Spa}\,\mathbb{F}_p$*, then an* ***untilt*** *of* Y *is a pair* (X, ι)*, where*

- X *is a perfectoid space,*
- ι *is an isomorphism* $X^\flat \xrightarrow{\sim} Y$*.*

Note that there is always a trivial untilt, namely $(Y, Y^\flat = Y \xrightarrow{\mathrm{id}} Y)$.

For later uses, let us also recall the following definition.

Definition 2.2.64. *A perfectoid space* X *is called* ***strictly totally disconnected*** *if it is qcqs and every étale cover of it splits.*

By [Sch17, Proposition 1.15], a perfectoid space is strictly totally disconnected if and only if it is affinoid and every connected component of X is of the form $\mathrm{Spa}(K, K^+)$ for K an algebraically closed perfectoid field with an open and bounded valuation subring K^+.

There is a useful generalisation of perfectoid spaces, called sousperfectoid spaces, which were introduced in [HK25].

Definition 2.2.65. *[SW20, Definition 6.3.1] Let R be a complete Tate-$\mathbb{Z}_p$-algebra. Then R is called **sousperfectoid** if there is a perfectoid ring S with an injection $R \hookrightarrow S$ that splits as topological R-modules.*

Definition 2.2.66. *An **affinoid sousperfectoid space** is an adic space of the form $\mathrm{Spa}(R, R^+)$, where R is a sousperfectoid ring and R^+ is an open and integrally closed subring of R°. A **sousperfectoid space** is an adic space X for which there is an open cover $X = \cup U_i$ where each U_i is affinoid perfectoid.*

For example, the Fargues-Fontaine curve is a sousperfectoid space. Some properties of sousperfectoid rings are summarised in [SW20, Section 6.3].

2.2.10 Diamonds

We now introduce diamonds, which are the analogue of algebraic spaces in the perfectoid settings.

We can generalise the concept of étale morphisms to pro-étale morphisms. See [Sch17, Definition 7.8].

Definition 2.2.67. *[Sch17, Definition 11.1] Let $X \in \mathrm{Perf}$. A **representable equivalence relation** on X is a perfectoid space $R \subset X \times X$ such that for every $Y \in \mathrm{Perf}$, $R(Y) \subset X(Y) \times X(Y)$ is an equivalence relation. A **pro-étale equivalence relation** on X is a representable equivalence relation $R \subset X \times X$ such that the two projections $R \rightrightarrows X$ are both pro-étale.*

Definition 2.2.68. *[SW20, Definition 8.3.1] A **diamond** is a pro-étale sheaf $\mathcal{D}$ on* Perf *for which $\mathcal{D} = X/R$, where X is a perfectoid space over $\mathbb{F}_p$ and R is a pro-étale equivalence relation on X.*

Here, the quotient X/R is taken in the sense of pro-étale sheaves.

There is a particularly nice class of diamonds, called locally spatial diamonds. See [Sch17, Definition 11.17].

Definition 2.2.69. *Let X be an adic space over $\mathbb{Z}_p$. Define $X^\diamondsuit$ as the presheaf on* Perf *given by $S \mapsto \left\{(S^\sharp, \iota, S^\sharp \to X) : (S^\sharp, \iota) \text{ is an untilt of } S\right\} / \cong$, where two elements are equivalent if and only if there is an isomorphism between the untilts that commute with the maps to X.*

Then $X^\diamondsuit$ is a v-sheaf (see next section) for every adic space X over $\mathbb{Z}_p$ by [SW20, Lemma 18.1.1]. If X is an analytic adic space over $\mathbb{Z}_p$, then $X^\diamondsuit$ is a locally spatial diamond with $|X^\diamondsuit| = |X|$ by [Sch17, Lemma 15.6]. If X is a perfectoid space, then $X^\diamondsuit = X^\flat$. Thus, one can view $(-)^\diamondsuit$ as a generalisation of tilting to general adic spaces. One can then generalise the tilting equivalence, see [Sch17, Lemma 15.6].

We define $\operatorname{Spd}(A, A^+) := (\operatorname{Spa}(A, A^+))^\diamondsuit$.

Example 2.2.70. [SW20, Section 8.4] Take $X = \operatorname{Spa} \mathbb{Q}_p$. Then

$$X^\diamondsuit = (\operatorname{Spa} \mathbb{F}_p((t^{1/p^\infty})))/\underline{\mathbb{Z}_p^\times},$$

where the action is given by $\gamma \cdot t := (1+t)^\gamma - 1$ for $\gamma \in \mathbb{Z}_p^\times$.

2.2.11 v-stacks

There is a notion of v-site of perfectoid spaces, see [Sch17, Definition 8.1]. This is an analogue of the fpqc sites on schemes for perfectoid spaces. We can then define v-sheaf and v-stacks:

Definition 2.2.71. *A* ***v-sheaf*** *is a sheaf on the v-site* Perf*. A* ***v-stack*** *is a stack on the v-site* Perf*.*

It turns out that every diamond is a v-sheaf [Sch17, Proposition 11.9]. This is an analogue of the fact that every algebraic space is a fpqc sheaf [Sta25, 0APL].

Below is a useful lemma.

Lemma 2.2.72. *[SW20, Proposition 11.10] Any sub-v-sheaf of a diamond is a diamond.*

Definition 2.2.73. *A v-sheaf $\mathcal{F}$ is* ***small*** *if there is a surjective map of v-sheaves $X \to \mathcal{F}$ from a perfectoid space X over $\mathbb{F}_p$. A v-stack Y is* ***small*** *if there is a surjective map of v-stacks $X \to Y$ from a perfectoid space X over $\mathbb{F}_p$ such that $X \times_Y X$ is a small v-sheaf.*

Definition 2.2.74. *A v-stack X is* ***quasi-compact*** *if there is an affinoid perfectoid space S and a surjection $S \to X$ of v-stacks.*

Definition 2.2.75. *Let $f : X \to Y$ be a morphism of v-stacks. Then f is*

- *a* ***closed immersion*** *if for every strictly totally disconnected perfectoid space S with a map $S \to Y$, the pullback $X \times_Y S \to S$ is representable by a closed immersion of perfectoid spaces (see [Sch17, Definition 5.6]);*
- ***representable in locally spatial diamonds*** *if for every morphism $Z \to Y$ from a locally spatial diamond Z, the fibre product $X \times_Y Z$ is a locally spatial diamond;*
- ***quasi-compact*** *if for every affinoid perfectoid space S with a map $S \to Y$, the fibre product $X \times_Y S$ is quasi-compact;*
- ***quasi-separated*** *if the diagonal $\Delta : X \to X \times_Y X$ and the iterated diagonal $\Delta_\Delta : X \to X \times_{X \times_Y X} X$ are quasi-compact;*
- *0-**truncated** if $X(S) \to Y(S)$ is faithful for every $S \in$ Perf;*

- ***partially proper*** *if it is quasiseparated,* 0*-truncated, and for every perfectoid ring* R *with an open and integrally closed subring* R^+*, and every diagram*

$$\begin{tikzcd} \mathrm{Spa}(R, R^\circ) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spa}(R, R^+) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

there is a unique dotted arrow making the diagram commutes.

Let us be more precise about the definition for partially properness, following [Sta25, 0CL9]. The diagram above (without the dotted arrow) is equivalent to the data of a triple $(y \in Y(R, R^+),\ x \in X(R, R^\circ),\ y|_{(R,R^\circ)} \xrightarrow{\gamma} f(x))$. There is a category Dot, called the category of dotted arrows, defined as follows:

- objects are triples $(x^+ \in X(R, R^+),\ x^+|_{(R,R^\circ)} \xrightarrow{\alpha} x,\ y \xrightarrow{\beta} f(x^+))$ such that $f(\alpha) \circ \beta|_{(R,R^\circ)} = \gamma$
- morphisms $(x_1^+, \alpha_1, \beta_1) \to (x_2^+, \alpha_2, \beta_2)$ are given by morphisms $x_1^+ \to x_2^+$ that are compatible with the α_i and β_i.

If f is 0-truncated, then there are no non-trivial automorphisms in Dot, so Dot is equivalent to a set. By 'there is a unique dotted arrow making the diagram commutes', we mean Dot is equivalent to a set with one element. In other words, the tripe (x^+, α, β) is unique up to (necessarily unique by 0-truncatedness) isomorphism.

Definition 2.2.76. *[FS21, Definition IV.1.1] An* ***Artin v-stack*** *is a small v-stack* X *for which the diagonal* $X \to X \times X$ *is representable in locally spatial diamonds, and there is a surjective morphism* $f : U \to X$ *from a locally spatial diamond* U *such that* f *is separated and cohomologically smooth.*

2.2.12 Fargues-Fontaine curve

Let E be a non-Archimedean local field with residue field $\mathbb{F}_q$. Let π be a uniformizer of E.

Definition 2.2.77. *For an affinoid perfectoid* $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_q}$*, define*

$$\mathcal{Y}_{E,S} := \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus V([\varpi])$$

and

$$Y_{E,S} := \mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) \setminus V(\pi[\varpi]),$$

where ϖ *is a pseudouniformizer of* R *and* $W_{\mathcal{O}_E}(R^+)$ *is equipped with the* $(\pi, [\varpi])$*-adic topology. We define the* ***(relative) Fargues-Fontaine curve*** *to be*

$$X_{E,S} := Y_{E,S}/\phi_S^{\mathbb{Z}},$$

where[19] ϕ_S *is the automorphism on* $Y_{E,S}$ *induced by the q-Frobenius*[20] *on* R^+*. For*

[19] The quotient makes sense as an adic space because the action of ϕ on on Y_S is free and totally discontinuous by [FS21, Proposition II.1.16].

[20] The morphism $x \mapsto x^q$.

general $S \in \mathrm{Perf}_{\mathbb{F}_q}$, we define $X_{E,S}$ to be the adic space obtained by gluing.[21]

When there is no ambiguity, we will write $\mathcal{Y}_S$ for $\mathcal{Y}_{E,S}$, Y_S for $Y_{E,S}$, and X_S for $X_{E,S}$.
Here,

$$W_{\mathcal{O}_E}(R^+)$$

is the unique π-adically complete, flat $\mathcal{O}_E$-algebra such that $W_{\mathcal{O}_E}(R^+)/\pi = R^+$ over $\mathbb{F}_q$. Its existence and uniqueness can be proved using the vanishing of cotangent complex on perfect rings and the relationship between cotangent complex and lift of algebras, as summarised in [Sch12, Theorem 5.11]. We also have $W_{\mathcal{O}_E}(R^+) = W(R^+)\widehat{\otimes}_{W(\mathbb{F}_q)}\mathcal{O}_E$, where the completion is the π-adic completion. Explicitly,

$$\begin{aligned} W_{\mathcal{O}_E}(R^+) &= W(R^+) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E && \text{if } \mathrm{char} E = 0, \\ W_{\mathcal{O}_E}(R^+) &= R^+[[t]] && \text{if } E = \mathbb{F}_q((t)). \end{aligned}$$

In any case, every element in $W_{\mathcal{O}_E}(R^+)$ can be written uniquely as $\sum_{i=0}^{\infty}[a_i]\pi^i$, where $a_i \in R^+$ and $[-]$ is the Teichmuller lift, and conversely every such element defines an element of $W_{\mathcal{O}_E}(R^+)$.

Proposition 2.2.78. *[FS21, before Definition I.2.2] Let $E = \mathbb{F}_q((t))$ and $S = \mathrm{Spa}(R, R^+)$, where R has pseudouniformizer ϖ. Then*

1. $\mathrm{Spa}\, W_{\mathcal{O}_E}(R^+) = \mathrm{Spa}\, R^+[[t]]$ *is the open unit disc over* $\mathrm{Spa}\, R^+$*, namely*

$$\mathrm{Spa}\, \mathbb{F}_q[[t]] \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}\, R^+.$$

2. $\mathcal{Y}_S = \mathrm{Spa}\, R^+[[t]] \setminus V(\varpi)$ *is the open unit disc over* $\mathrm{Spa}(R, R^+)$*, namely*

$$\mathrm{Spa}\, \mathbb{F}_q[[t]] \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}(R, R^+).$$

3. $Y_S = \mathrm{Spa}\, R^+[[t]] \setminus V(t\varpi)$ *is the punctured open unit disc over* $\mathrm{Spa}(R, R^+)$*, namely*

$$\mathrm{Spa}\, \mathbb{F}_q((t)) \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}(R, R^+).$$

Let us justify why $\mathrm{Spa}\, \mathbb{F}_q[[t]] \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}\, R^+$ is called the open unit disc over $\mathrm{Spa}\, R^+$. Let (A, A^+) be a complete Huber pair over $\mathbb{F}_q$. Then

$$\begin{aligned} (\mathrm{Spa}\, \mathbb{F}_q[[t]] \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}\, R^+)(A, A^+) &= \mathrm{Hom}_{\mathbb{F}_q, cts}(\mathbb{F}_q[[t]], A^+) \times (\mathrm{Spa}\, R^+)(A, A^+) \\ &= A^{\circ\circ} \times (\mathrm{Spa}\, R^+)(A, A^+), \end{aligned}$$

where $\mathrm{Hom}_{cts, \mathbb{F}_q}$ stands for the set of continuous $\mathbb{F}_q$-algebra homomorphisms. This is what the (A, A^+)-valued points of the open unit disc over $\mathrm{Spa}\, R^+$ should be. The same argument justifies why $\mathrm{Spa}\, \mathbb{F}_q[[t]] \times_{\mathrm{Spa}\, \mathbb{F}_q} \mathrm{Spa}(R, R^+)$ is called the open unit disc over $\mathrm{Spa}(R, R^+)$.

[21] This gluing makes sense because if $S \to T$ is an open immersion of affinoid perfectoid spaces, then $X_S \to X_T$ is an open immersion by [FS21, Proposition II.1.3].

Similarly, if (A, A^+) be a complete Huber pair over (R, R^+), then

$$\begin{aligned}(\operatorname{Spa}\mathbb{F}_q((t)) \times_{\operatorname{Spa}\mathbb{F}_q} \operatorname{Spa}(R, R^+))(A, A^+) &= \operatorname{Hom}_{cts,\mathbb{F}_q}((\mathbb{F}_q((t)), \mathbb{F}_q[[t]]), (A, A^+)) \times \operatorname{Spa}(R, R^+)(A, A^+) \\ &= (A^{\circ\circ} \cap A^\times) \times \operatorname{Spa}(R, R^+)(A, A^+).\end{aligned}$$

This justifies why $\operatorname{Spa}\mathbb{F}_q((t)) \times_{\operatorname{Spa}\mathbb{F}_q} \operatorname{Spa}(R, R^+)$ is called the punctured open unit disc over $\operatorname{Spa}(R, R^+)$.

Proof of Proposition 2.2.78. Let us show (3). The others are similar. Let (A, A^+) be a complete Huber pair over $\mathbb{F}_q$. Then $Y_S(A, A^+)$ is the subset of

$$(\operatorname{Spa} R^+[[t]])(A, A^+) = \operatorname{Hom}_{\mathbb{F}_q,cts}(R^+[[t]], A^+)$$

such that the induced map $\operatorname{Spa}(A, A^+) \to \operatorname{Spa} R^+[[t]]$ lies in the open subspace $\operatorname{Spa} R^+[[t]] \setminus V(t\varpi)$. In other words, it is those $f \in \operatorname{Hom}_{\mathbb{F}_q,cts}(R^+[[t]], A^+)$ such that for all $|\cdot| \in \operatorname{Spa}(A, A^+)$, $|f(t\varpi)| \neq 0$. By [SW20, Proposition 2.3.10], it is those $f \in \operatorname{Hom}_{\mathbb{F}_q,cts}(R^+[[t]], A^+)$ such that $f(t\varpi) \in A^\times$. We can identify this with $(A^{\circ\circ} \cap A^\times) \times \operatorname{Spa}(R, R^+)(A, A^+)$. □

While the definition of Y_S and X_S may look a bit complicated, it turns out that the diamonds associated with them have simple and intuitive descriptions.

Proposition 2.2.79. *[FS21, Proposition II.1.17] For every $S \in \operatorname{Perf}_{\mathbb{F}_q}$, $Y_S^\diamondsuit = \operatorname{Spd} E \times S$ and $X_S^\diamondsuit = \operatorname{Spd} E \times S/\phi_S^{\mathbb{Z}}$.*

Definition 2.2.80. *For $\lambda = \frac{d}{r}$ with $d, r \in \mathbb{Z}$ and $r > 0$, we have the vector bundle $\mathcal{O}_{Y_S}^r$ with a ϕ_S-linear automorphism* $\begin{pmatrix} 0 & 1 & 0 & \dots & \\ & 0 & 1 & 0 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & 0 & 1 \\ \pi^{-d} & 0 & & & 0 \end{pmatrix}\phi_S$*, where the matrix is of size $r \times r$. This descends to a vector bundle on X_S, which we denote by $\mathcal{O}(\lambda)$.*

Note that the power of π in the definition is $-d$, not d.

When S is an affinoid perfectoid space over $\mathbb{F}_q$, there is a schematic version of the Fargues-Fontaine curve.

Definition 2.2.81. $X_S^{sch} := \operatorname{Proj}\left(\bigoplus_{m\geq 0} H^0(X_S, \mathcal{O}_{X_S}(m))\right)$.

We have the following GAGA result:

Proposition 2.2.82. *[FS21, after Remark II.2.8] Suppose S is affinoid perfectoid. Then we have a natural map of locally ringed spaces*

$$f : X_S \to X_S^{sch}$$

and pullback along this map induces an equivalence of categories

$$f^* : \{\textit{vector bundles on } X_S^{sch}\} \xrightarrow{\simeq} \{\textit{vector bundles on } X_S\}.$$

In addition, for every vector bundle $\mathcal{E}$ on X_S^{sch}, we have

$$H^i(X_S^{sch}, \mathcal{E}) = H^i(X_S, f^*(\mathcal{E}))$$

for all $i \geq 0$.

Remark 2.2.83. By [FS21, proof of Proposition II.2.7], the equivalence above is an equivalence of exact categories, i.e. a sequence $0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$ of vector bundles on X_S^{sch} is exact if and only if $0 \to f^*\mathcal{E}_1 \to f^*\mathcal{E}_2 \to f^*\mathcal{E}_3 \to 0$ is exact. It follows from the Tannakian interpretation of G-bundles that we have an equivalence of categories

$$f^* : \{G\text{-bundles on } X_S^{sch}\} \xrightarrow{\simeq} \{G\text{-bundles on } X_S\}$$

for any linear algebraic group G over E.

2.2.13 Global sections of the Fargues-Fontaine curve

As an attempt to understand the Fargues-Fontaine curve, let us compute the global sections of Y_S when $E = \mathbb{F}_q((t))$.

The following lemma is a slight variant of [Ber10, Proposition 32.3].

Lemma 2.2.84. *Let A be a commutative ring and $f, g \in A$ be non-zero-divisors such that $A[\frac{1}{f}] \cap A[\frac{1}{g}] = A$, where the intersection is taken in $A[\frac{1}{fg}]$. Then we have an isomorphism*

$$A[X]/(gX - f) \xrightarrow{\sim} A\left[\frac{f}{g}\right]$$

given by $X \mapsto \frac{f}{g}$, where $A[\frac{f}{g}]$ is the subring of $A[\frac{1}{g}]$ generated by A and $\frac{f}{g}$.

Proof. The proof in [Ber10, Proposition 32.3] works. □

Let $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_q}$ and ϖ be a pseudouniformizer of R admitting a compatible sequence $(\varpi^{1/p^m})_{m \geq 1}$ of p-power roots of ϖ. Take $E = \mathbb{F}_q((t))$.

Let $a = \frac{1}{n}$ for some $n \geq 1$ which is a power of p. Let us compute the rational subspace $U_n := \{|t| \leq |\varpi^a| \neq 0\}$ of $\mathrm{Spa}\, R^+[\![t]\!]$. By definition, this rational subspace is $\mathrm{Spa}(B_n, B_n^+)$, where $B_n = R^+[\![t]\!]\langle \frac{t}{\varpi^a} \rangle [\frac{1}{\varpi}]$ and B_n^+ is the integral closure of $R^+[\![t]\!]\langle \frac{t}{\varpi^a} \rangle$ in B_n. Here, $R^+[\![t]\!]\langle \frac{t}{\varpi^a} \rangle$ is the $(\varpi, t) = (\varpi)$-adic completion of $R^+[\![t]\!][\frac{t}{\varpi^a}]$. For $i \geq 1$, we have

$$\begin{aligned} R^+[\![t]\!]\left[\frac{t}{\varpi^a}\right]/(\varpi^i) &\cong R^+[\![t]\!][X]/(\varpi^a X - t, \varpi^i) \\ &= (R^+/\varpi^i)[t, X]/(\varpi^a X - t) \\ &= (R^+/\varpi^i)[X], \end{aligned}$$

where we use Lemma 2.2.84 in the first line. The inverse limit of these rings is $R^+\langle X \rangle$. Thus, $(B_n, B_n^+) = (R\langle X \rangle, R^+\langle X \rangle)$. We can identify B_n with

$$R\langle \frac{t}{\varpi^a} \rangle = \left\{ \sum_{i=0}^{\infty} r_i t^i | r_i \in R, \lim_{i \to \infty} r_i \varpi^{i/n} = 0 \right\}$$

and B_n^+ with $R^+\langle\frac{t}{\varpi^a}\rangle = \left\{\sum_{i=0}^{\infty} r_i t^i | r_i \in R, \lim_{i\to\infty} r_i\varpi^{i/n} = 0, r_i\varpi^{i/n} \in R^+ \forall i\right\}$.

Next, we want to compute the rational subspace $V_n = \{|\varpi^n| \le |t| \le |\varpi^a| \ne 0\}$ of $\operatorname{Spa} R^+[\![t]\!]$. This is also the rational subspace $\{|\varpi^n| \le |t| \ne 0\} = \{|\varpi^{n-a}| \le |X| \ne 0\}$ of U_n, where $X = \frac{t}{\varpi^a}$. Thus, $V_n = \operatorname{Spa}(B_n', B_n'^+)$, where $B_n' = R^+\langle X\rangle\langle\frac{\varpi^{n-a}}{X}\rangle[\frac{1}{X}]$ and $B_n'^+$ is the integral closure of $R^+\langle X\rangle\langle\frac{\varpi^{n-a}}{X}\rangle$ in B_n'. For $i \ge 1$, we have

$$\begin{aligned} R^+\langle X\rangle\left[\frac{\varpi^{n-a}}{X}\right]/(\varpi^i) &\cong R^+\langle X\rangle[Y]/(XY - \varpi^{n-a}, \varpi^i) \\ &= (R^+/\varpi^i)[X,Y]/(XY - \varpi^{n-a}) \end{aligned}$$

The inverse limit of these rings is $R^+\langle X, Y\rangle/(XY - \varpi^{n-a})$, because in both the inverse limit and this ring, every element can be written uniquely as $\sum_{j\ge 0} r_j X^j + \sum_{k\ge 1} r_k' Y^k$, where $r_j, r_k' \in R^+$ and $\lim_{j\to\infty} r_j = \lim_{k\to\infty} r_k' = 0$. We can identify B_n' with

$$\left\{\sum_{i\in\mathbb{Z}} r_i t^i | r_i \in R, \lim_{i\to\infty} r_i\varpi^{i/n} = 0, \lim_{i\to-\infty} r_i\varpi^{in} = 0\right\}.$$

Since $Y_S = \cup_{n\ge 1} V_n$, we know that

$$H^0(Y_S, \mathcal{O}_{Y_S}) = \cap_{n\ge 1} B_n'.$$

In other words, it is those series $\sum_{i\in\mathbb{Z}} r_i t^i$ with suitable convergence conditions on r_i as $i \to \pm\infty$. Note in particular that $r_i \in R^{\circ\circ}$ for all i sufficiently negative.

Example 2.2.85. Recall that ϕ_S acts on Y_S via the morphism induced by $f(t) \mapsto f(t^q)$ on $R^+[\![t]\!]$. Using the analysis above, it is easy to check that

$$\begin{aligned} H^0(X_S, \mathcal{O}_{X_S}(1)) &= H^0(Y_S, \mathcal{O}_{Y_S})^{\phi_S = t} \\ &= \left\{\sum_{i\in\mathbb{Z}} r_i t^i \in H^0(Y_S, \mathcal{O}_{Y_S}) : r_{i+1}^q = r_i \ \forall i \in \mathbb{Z}\right\} \\ &= \left\{\sum_{i\in\mathbb{Z}} r_0^{q^{-i}} t^i : r_0 \in R^{\circ\circ}\right\} \\ &\cong R^{\circ\circ}. \end{aligned}$$

Similarly, for every $n \in \mathbb{Z}_{\ge 1}$, $H^0(X_S, \mathcal{O}_{X_S}(n)) = H^0(Y_S, \mathcal{O}_{Y_S})^{\phi_S = t^n} \cong (R^{\circ\circ})^n$.

Example 2.2.86. Note that

$$H^0(X_S, \mathcal{O}_{X_S}) = H^0(Y_S, \mathcal{O}_{Y_S})^{\phi_S = \mathrm{id}} = \left\{\sum_{i\in\mathbb{Z}} r_i t^i \in H^0(Y_S, \mathcal{O}_{Y_S}) : r_i^q = r_i \ \forall i \in \mathbb{Z}\right\}.$$

If $S = \operatorname{Spa}(R, R^+)$, where R is a field, then one can easily check that $H^0(X_S, \mathcal{O}_{X_S}) = \mathbb{F}_q((t))$.

There is a more general result, valid for any non-Archimedean local field:

Proposition 2.2.87. *[FS21, Proposition II.2.5] For any non-Archimedean local field E with residue field $\mathbb{F}_q$ and $S \in \mathrm{Perf}_{\mathbb{F}_q}$, we have*

$$H^0(X_S, \mathcal{O}) = \mathrm{Cont}(|S|, E),$$

where Cont *denotes the set of continuous functions. Also, for any $\lambda = \frac{r}{s}$ with coprime integers $r, s > 0$, we have*

$$H^0(X_S, \mathcal{O}(\lambda)) \cong (\mathcal{O}_S(S)^{\circ\circ})^r \text{ if } \begin{cases} \mathrm{char} E = 0 \text{ and } 0 < \lambda \leq [E : \mathbb{Q}_p], \text{ or} \\ \mathrm{char} E = p. \end{cases}$$

2.2.14 Moduli stack of G-bundles

Definition 2.2.88. *For a linear algebraic group G over E, we define Bun_G to be the prestack on[22] $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ taking $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ to the groupoid of G-bundles on X_S.*

This is a v-stack [FS21, before Definition III.1.2]. This is proved as follows: One first proves that $S \mapsto \{\text{vector bundles on } X_S\}$ is a v-stack [FS21, Proposition II.2.1]. One can view a G-bundle on X_S as an exact tensor functor $\mathrm{Rep}_E(G) \to \{\text{vector bundles on } X_S\}$. Using these two facts and part (2) of the following lemma, one can deduce that Bun_G is a v-stack.

Lemma 2.2.89. *Let $S' \to S$ be a v-cover of perfectoid spaces over $\mathrm{Spa}\, \mathbb{F}_q$. Let $c : X_{S'} \to X_S$ be the associated map on the Fargues-Fontaine curves.*

1. *The map on the underlying topological spaces $|c| : |X_{S'}| \to |X_S|$ is surjective.*
2. *A sequence*

$$0 \to \mathcal{E}_1 \xrightarrow{f} \mathcal{E}_2 \xrightarrow{g} \mathcal{E}_3 \to 0$$

of vector bundles on X_S with $g \circ f = 0$ is exact if and only if its pullback

$$0 \to c^*\mathcal{E}_1 \to c^*\mathcal{E}_2 \to c^*\mathcal{E}_3 \to 0$$

is exact.

Proof. For part (1), note that as in [FS21, Section II.1.2],

$$|X_S| = |X_S^{\diamondsuit}| = |S/\phi^{\mathbb{Z}} \times \mathrm{Spd}\, E| \cong |S \times (\mathrm{Spd}\, E)/\phi^{\mathbb{Z}}|$$

because the absolute Frobenius $\phi \times \phi$ of $S \times \mathrm{Spd}\, E$ acts trivially on the topological space. Since $S' \to S$ is a v-cover, it is a surjection of v-sheaves. It follows that its base change $S' \times (\mathrm{Spd}\, E)/\phi^{\mathbb{Z}} \to S \times (\mathrm{Spd}\, E)/\phi^{\mathbb{Z}}$ is also a surjection of v-sheaves. The map on the underlying topological spaces is thus surjective.

Part (2) follows from part (1) and Lemma 2.2.16. $\square$

[22] We switch to $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ because this is the setup where Bun_G is studied in [FS21].

2.2.15 B_{dR}^{+}-Affine Grassmannian

Lemma 2.2.90 (Beauville-Laszlo lemma)**.** *[BL95] Let A be a ring and $f \in R$ be a non-zero-divisor. Let $\widehat{A} = \lim_n A/f^n$. There is an equivalence of categories between {finite projective A-modules} and the category of triples (F, G, φ), where*

- *F is a finite projective $A[\frac{1}{f}]$-module*
- *G is a finite projective $\widehat{A}$-module*
- *$\phi : F \otimes_A \widehat{A} \xrightarrow{\sim} G[\frac{1}{f}]$ is an $\widehat{A}[\frac{1}{f}]$-module isomorphism.*

Lemma 2.2.91. *Let A be a commutative ring and $f \in R$ be a non-zero-divisor. A sequence*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

of finite projective A-modules is exact if and only if its pullbacks

$$0 \to M_1 \otimes_A A\left[\frac{1}{f}\right] \to M_2 \otimes_A A\left[\frac{1}{f}\right] \to M_3 \otimes_A A\left[\frac{1}{f}\right] \to 0$$

and

$$0 \to M_1 \otimes_A \widehat{A} \to M_2 \otimes_A \widehat{A} \to M_3 \otimes_A \widehat{A} \to 0$$

are exact.

Proof. By Lemma 2.2.16, it is enough to show that the natural map $\operatorname{Spec} A\left[\frac{1}{f}\right] \sqcup \operatorname{Spec} \widehat{A} \to \operatorname{Spec} A$ is surjective (on the underlying topological spaces). This follows from [BL95, Lemme 1] by taking M to be the residue field of A at every prime ideal of A. $\square$

The following follows immediately from Lemmas 2.2.90 and 2.2.91.

Corollary 2.2.92. *Let A be an E-algebra and $f \in R$ be a non-zero-divisor. Let $\widehat{A} = \lim_n A/f^n$. Let G be a linear algebraic group over E. There is an equivalence of categories between {G-bundles on $\operatorname{Spec} A$} and the category of triples $(\mathcal{F}, \mathcal{G}, \varphi)$, where*

- *$\mathcal{F}$ is a G-bundle on $\operatorname{Spec} A[\frac{1}{f}]$*
- *$\mathcal{G}$ is a G-bundle on $\operatorname{Spec} \widehat{A}$*
- *$\phi : \mathcal{F}|_{\operatorname{Spec} \widehat{A}[\frac{1}{f}]} \xrightarrow{\sim} \mathcal{G}|_{\operatorname{Spec} \widehat{A}[\frac{1}{f}]}$ is an isomorphism.*

Let us recall the definitions of $B_{dR}^{+}(R)$ and $B_{dR}(R)$. Let R be a perfectoid E-algebra with a pseudo-uniformizer ϖ. Then we have Fontaine's map

$$\theta : W_{\mathcal{O}_E}(R^{\circ\flat}) \twoheadrightarrow R^{\circ}.$$

Its kernel is generated by an element ξ of the form $\pi + [\varpi^\flat]\alpha$ for some $\alpha \in W_{\mathcal{O}_E}(R^{\circ\flat})$ and pseudouniformizer $\varpi^\flat$ of $R^\flat$. We define

$$B_{dR}^+(R) := \varprojlim_i W_{\mathcal{O}_E}(R^{\circ\flat})\left[\frac{1}{[\varpi^\flat]}\right]/(\xi^i)$$

and

$$B_{dR}(R) := B_{dR}^+(R)\left[\frac{1}{\xi}\right].$$

One can show that $B_{dR}^+(R)$ is ξ torsion-free, ξ-adically complete, and $B_{dR}^+(R)/(\xi) \cong R$.

Let G be a connected reductive group over E.

Definition 2.2.93. *The B_{dR}^+-**affine Grassmannian** Gr_G is the v-sheaf on* Perf *given by sending an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ to the equivalence classes of triples $\{(S^\sharp, \mathcal{E}, \iota)\}$, where*

- *$S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ is an untilt of S over $\mathrm{Spa}\, E$*
- *$\mathcal{E}$ is a G-bundle on $\mathrm{Spec}\, B_{dR}^+(R^\sharp)$*
- *$\iota : \mathcal{E}|_{\mathrm{Spec}\, B_{dR}(R^\sharp)} \xrightarrow{\sim} \mathcal{E}_0|_{\mathrm{Spec}\, B_{dR}(R^\sharp)}$ is an isomorphism to the trivial G-bundle.*

One way to prove that this is a v-sheaf is to first show that

$$S \mapsto \left\{\text{vector bundles on } \mathrm{Spec}\, B_{dR}^+(R^\sharp)\right\}$$

is a v-stack [SW20, Corollary 17.1.9] and then use the Tannakian formalism together with Lemma 2.2.95 below.

One can show using the Beauville-Laszlo lemma that this is isomorphic to the v-sheaf sending an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ to the equivalence classes of triples $\{(S^\sharp, \mathcal{E}, \iota)\}$, where

- $S^\sharp = \mathrm{Spa}(R^\sharp, R^{\sharp+})$ is an untilt of S over $\mathrm{Spa}\, E$
- $\mathcal{E}$ is a G-bundle on X_S^{sch}
- $\iota : \mathcal{E}|_{X_S^{sch} \setminus \mathrm{Spec}\, R^\sharp} \xrightarrow{\sim} \mathcal{E}_0|_{X_S^{sch} \setminus \mathrm{Spec}\, R^\sharp}$ is an isomorphism to the trivial G-bundle.

In particular, we have a morphism $\mathrm{Gr}_G \to \mathrm{Bun}_G$, called the ***Beauville-Laszlo morphism***. This is surjective as a morphism of pro-étale stacks [FS21, Proposition III.3.1].

Let us state a handy lemma.

Lemma 2.2.94. *Let R be a commutative ring and $\xi \in R$. Let*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \tag{2.7}$$

be morphisms of ξ-complete R-modules. Assume that $g \circ f = 0$ and ξ is neither a zero divisor in B nor C. If

$$0 \to A/\xi \to B/\xi \to C/\xi \tag{2.8}$$

is exact, then so is (2.7). If $B/\xi \to C/\xi$ is surjective, then $B \to C$ is surjective.

Proof. Assume (2.8) holds. It is straightforward to check by induction that for every $n \geq 1$,

$$0 \to A/\xi^n \to B/\xi^n \to C/\xi^n$$

is exact. It is also clear that

$$0 \to \varprojlim A/\xi^n \to \varprojlim B/\xi^n \to \varprojlim C/\xi^n$$

is exact. The exactness of (2.7) now follows by the ξ-completeness of A, B, C.

Assume $B/\xi \to C/\xi$ is surjective. Let $c \in C$. Then there exists $b_0 \in B$ such that $c = g(b_0) + \xi c_1$ for some $c_1 \in C$. There exists $b_1 \in B$ such that $c_1 = g(b_1) + \xi c_2$ for some $c_2 \in C$, and hence $c = g(b_0 + \xi b_1) + \xi^2 c_2$. We continue this process. Then $b_0 + \xi b_1 + \xi^2 b_2 + \dots$ defines an element in B whose image in C is c by ξ-completeness of B and C. Thus $B \to C$ is surjective. $\square$

Lemma 2.2.95. *Let $h : S' = \mathrm{Spa}(R', R'^+) \to S = \mathrm{Spa}(R, R^+)$ be a v-cover of affinoid perfectoid spaces over $\mathrm{Spa}\, E$.*

1. *A sequence*

$$0 \to P_1 \xrightarrow{f} P_2 \xrightarrow{g} P_3 \to 0 \tag{2.9}$$

of finite projective R-modules with $g \circ f = 0$ is exact if and only if its pullback

$$0 \to P_1 \otimes_R R' \to P_2 \otimes_R R' \to P_3 \otimes_R R' \to 0 \tag{2.10}$$

is exact.

2. *A sequence*

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0 \tag{2.11}$$

of finite projective $B_{dR}^+(R)$-modules with $g \circ f = 0$ is exact if and only if its pullback

$$0 \to M_1 \otimes_{B_{dR}^+(R)} B_{dR}^+(R') \to M_2 \otimes_{B_{dR}^+(R)} B_{dR}^+(R') \to M_3 \otimes_{B_{dR}^+(R)} B_{dR}^+(R') \to 0 \tag{2.12}$$

is exact.

This does not follow immediately from Lemma 2.2.16, because neither $\mathrm{Spec}(R') \to \mathrm{Spec}\, R$ nor $\mathrm{Spec}\, B_{dR}^+(R') \to \mathrm{Spec}\, B_{dR}^+(R)$ seem to be surjective.

Proof. For part (1), the 'only if' direction is clear. Let us now prove the 'if' direction. Note that $\widetilde{P_i} := P_i \otimes_R \mathcal{O}_S$ are vector bundles on S. By [Ked19, Theorem 1.3.4], $H^1(S, \widetilde{P_i}) = 0$. Thus, to prove the exactness of (2.9), it suffices to show that the sequence of vector bundles on S

$$0 \to \widetilde{P_1} \to \widetilde{P_2} \to \widetilde{P_3} \to 0$$

is exact. Since $S' \to S$ is a v-cover, $|S'| \twoheadrightarrow |S|$. By Lemma 2.2.16, it suffices to show that

$$0 \to h^*\widetilde{P_1} \to h^*\widetilde{P_2} \to h^*\widetilde{P_3} \to 0$$

is exact. Note that $h^*\widetilde{P_i} = \widetilde{P_i \otimes_R R'} := (P_i \otimes_R R') \otimes_{R'} \mathcal{O}_{S'}$, so the desired result follows from the exactness of (2.10).

For part (2), the 'only if' direction is clear. Let us now prove the 'if' direction. Note that M_i is ξ-adically complete: There is a $B^+_{dR}(R)$-module N_i such that

$$M_i \oplus N_i \cong B^+_{dR}(R)^{n_i}$$

for some n_i. Since $B^+_{dR}(R)$ is ξ-adically complete, $B^+_{dR}(R)^{n_i} \xrightarrow{\sim} \lim_m B^+_{dR}(R)^{n_i}/\xi^m$ is an isomorphism, which implies that $M_i \xrightarrow{\sim} \lim_{m\geq 1} M_i/\xi^m$. By Lemma 2.2.94, to show the exactness of (2.11), it suffices to show that

$$0 \to M_1/\xi \to M_2/\xi \to M_3/\xi \to 0$$

is exact. Since (2.12) consists of finite projective $B^+_{dR}(R')$-modules, its base change along $B^+_{dR}(R') \to B^+_{dR}(R')/\xi$ is still exact, i.e.

$$0 \to M_1/\xi \otimes_R R' \to M_2/\xi \otimes_R R' \to M_3/\xi \otimes_R R' \to 0$$

is exact. Now the result follows from part (1). $\square$

2.2.16 Moduli stack of L-parameters

We will briefly state the definition and some properties of the moduli stack of L-parameters, which are studied in [FS21, DHKM25, Zhu25]. We shall follow the presentation in [FS21]. Fix a prime $\ell \neq p$. We have the dual group $\widehat{G}$ of G over $\mathbb{Z}$.

The idea is that there is a scheme $Z^1(W_E, \widehat{G})$ whose Λ-valued points, for a $\mathbb{Z}_\ell$-algebra Λ, are the continuous 1-cocycles $\phi : W_E \to \widehat{G}(\Lambda)$. However, it is not clear which topology we should put on $\widehat{G}(\Lambda)$. We can solve this problem using condensed mathematics.

Let ProFin be the category of profinite sets whose morphisms are continuous maps. We say that $\{f_i : X_i \to X\}_{i\in I}$ is a cover in ProFin if I is finite and $X = \cup f_i(X_i)$. Then ProFin is a site.[23] A condensed set/group/ring is a sheaf of sets/groups/rings on ProFin.

[23] This is not literally true, because ProFin is not a small category. To be rigorous, one should restrict to κ-small profinite sets for a suitable cardinality κ and then take the union over these κ. Alternatively, one can also consider the category of light profinite sets instead. We shall ignore these set-theoretic difficulties in this subsection.

For example, if T is a T1-topological space, then we have a condensed set

$$\underline{T} : \mathrm{ProFin}^{op} \to \mathrm{Set}\, S \mapsto \{\text{continuous maps } S \to T\}.$$

If Λ is any $\mathbb{Z}_\ell$-algebra, then we have a condensed ring $\underline{\Lambda} := \underline{\Lambda_{\mathrm{disc}}} \otimes_{\underline{\mathbb{Z}_{\ell,\mathrm{disc}}}} \underline{\mathbb{Z}_\ell}$, where Λ_{disc} and $\mathbb{Z}_{\ell,\mathrm{disc}}$ are equipped with the discrete topology. If we let $\Lambda_{\mathrm{ind}-\ell}$ denote Λ equipped with the ind-ℓ-adic topology[24], then in fact $\underline{\Lambda_{\mathrm{ind}-\ell}} = \underline{\Lambda}$.

Note that $\widehat{G}(\underline{\Lambda})$ is a condensed group, given by $S \mapsto \widehat{G}(\underline{\Lambda}(S))$.

Recall that the action of W_E on $\widehat{G}$ factors through a finite quotient of W_E. It follows that[25] $\underline{W_E}$ acts on $\widehat{G}(\underline{\Lambda})$.

Theorem 2.2.96. *[FS21, Theorem VIII.0.1] There is a scheme $Z^1(W_E, \widehat{G})$ whose Λ-valued points, for a $\mathbb{Z}_\ell$-algebra Λ, are the condensed 1-cocycles $\phi : \underline{W_E} \to \widehat{G}(\underline{\Lambda})$. The scheme is a disjoint union of finite type $\mathbb{Z}_\ell$-schemes.*

Note that $\widehat{G}$ acts on $Z^1(W_E, \widehat{G})$ via twisted conjugation: $(g * \phi)(w) = g\phi(w)({}^w g)^{-1}$. Define

$$\mathrm{Par}_{\widehat{G}} := [Z^1(W_E, \widehat{G})/\widehat{G}].$$

2.2.17 Categorical conjecture

Let G be a quasi-split reductive group over E. Let ℓ be a prime different from p. In this subsection, we shall view $\mathrm{Par}_{\widehat{G}}$ as a stack on $\overline{\mathbb{Q}_\ell}$-algebras, i.e. we base change $\mathrm{Par}_{\widehat{G}}$ from $\mathbb{Z}_\ell$ to $\overline{\mathbb{Q}_\ell}$. Pick a Whittaker datum, i.e. a Borel $B \subset G$ and a generic character $\psi : U(E) \to \overline{\mathbb{Q}_\ell}^\times$, where U is the unipotent radical of B. Let $\mathcal{W}_\psi = i_{1!}\mathrm{cInd}_{U(E)}^{G(E)}\psi \in \mathcal{D}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})$, where $i_1 : \mathrm{Bun}_G^1 \to \mathrm{Bun}_G$ is the usual open immersion. Fix $\sqrt{q} \in \overline{\mathbb{Q}_\ell}^\times$. Then we have a spectral action of $\mathrm{Perf}(\mathrm{Par}_{\widehat{G}})$ on $\mathcal{D}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})$, where $\mathrm{Perf}(-)$ denotes the ∞-category of perfect complexes. The action on the Whittaker sheaf gives rise to a map

$$\begin{aligned} \mathrm{Perf}^{qc}(\mathrm{Par}_{\widehat{G}}) &\to \mathcal{D}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell}) \\ M &\mapsto M * \mathcal{W}_\psi \end{aligned}$$

and hence a map

$$\mathrm{QCoh}(\mathrm{Par}_{\widehat{G}}) = \mathrm{Ind}\,\mathrm{Perf}^{qc}(\mathrm{Par}_{\widehat{G}}) \to \mathcal{D}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})$$

by the universal property of ind-completion, where QCoh is the derived ∞-category of quasi-coherent sheaves. By the adjoint functor theorem, this map has a right adjoint

$$\mathcal{D}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell}) \to \mathrm{QCoh}(\mathrm{Par}_{\widehat{G}}). \tag{2.13}$$

[24] I.e. the colimit topology $M = \mathrm{colim}_{N \subset M}(N, \ell\text{-adic topology})$, where N runs through the finitely generated sub-$\mathbb{Z}_\ell$-module of M. More explicitly, a subset $S \subset M$ is open if and only if $S \cap N$ is open in $(N, \ell\text{-adic topology})$ for every finitely generated sub-$\mathbb{Z}_\ell$-module of M.

[25] To see this, let $S \in \mathrm{ProFin}$, $w \in \underline{W_E}(S)$, $g \in \widehat{G}(\underline{\Lambda}(S))$. Say the action of W_E on $\widehat{G}$ factors through a finite quotient Q. Let S_q be the fibre of $q \in Q$ in the composite map the composite $S \xrightarrow{w} W_E \to Q$. Then the action is defined as ${}^w g := ({}^q(g|_{S_q}))_{q \in Q} \in \prod_{q \in Q} \widehat{G}(\underline{\Lambda})(S_q) \cong \widehat{G}(\underline{\Lambda})(S)$. This is independent of the choice of Q.

Conjecture 2.2.97. *[FS21, Conjecture X.1.4] The restriction of* (2.13) *to* $\mathcal{D}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})^\omega$ *is fully faithful with essential image* $\mathrm{Coh}(\mathrm{Par}_{\widehat{G}})$*. Thus,* $\mathcal{D}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})^\omega \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Par}_{\widehat{G}})$.

Here, $\mathrm{Coh}(\mathrm{Par}_{\widehat{G}})$ is the ∞-category of bounded complexes with quasicompact support and coherent cohomology sheaves. By [FS21, Theorem VII.7.4], $\mathcal{D}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})$ is compactly generated. In particular, if Conjecture 2.2.97 is true, then taking ind-completion gives an equivalence

$$\mathbb{L}_G : \mathcal{D}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell}) \xrightarrow{\sim} \mathrm{IndCoh}(\mathrm{Par}_{\widehat{G}}).$$

2.3 Moduli stack of Higgs bundles

In this section, we define the moduli stack of Higgs bundles and establish some basic properties of it.

Let E be a non-Archimedean local field with residue field $\mathbb{F}_q$, where q is a power of p. Let G be a connected reductive group over E. Let $\mathfrak{g}(E)$ be its Lie algebra. Fix $S \in \mathrm{Perf}_{\mathbb{F}_q}$ and a line bundle $\mathcal{L}$ on the Fargues-Fontaine curve X_S. Inspired by [Ngô10, definition 4.2.1], we make the following definition:

Definition 2.3.1. *For* $T \in \mathrm{Perf}_S$*, a* $\mathcal{L}$*-twisted* G*-Higgs bundle on* X_T *is a pair* $(\mathcal{E}, \phi)$*, where*

- $\mathcal{E}$ *is a* G*-bundle on* X_T*,*
- $\phi \in H^0(X_T, \mathrm{Ad}(\mathcal{E}) \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_T)$.

Here, $\mathcal{L}_T$ is the pullback of $\mathcal{L}$ to X_T and $\mathrm{Ad}(\mathcal{E})$ is the adjoint bundle.[26] We will usually just call them G-Higgs bundles or Higgs bundles. We will let $\mathrm{Ad}(\mathcal{E})_{\mathcal{L}}$ denote $\mathrm{Ad}(\mathcal{E}) \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_T$.

Definition 2.3.2. *Let* $\mathrm{Higgs}_{G,\mathcal{L}}$ *be the prestack on* Perf_S *taking* $T \in \mathrm{Perf}_S$ *to the groupoid of Higgs bundles on* X_T*, where an isomorphism* $(\mathcal{E}_1, \phi_1) \xrightarrow{\sim} (\mathcal{E}_2, \phi_2)$ *is an isomorphism* $\mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2$ *mapping* ϕ_1 *to* ϕ_2.

When there is no ambiguity, we will write $\mathrm{Higgs}_{G,\mathcal{L}}$ as Higgs_G.

Example 2.3.3.

- For $G = \mathbb{G}_m$, we have $\mathrm{Higgs}_G = \mathrm{Bun}_{\mathbb{G}_m} \times \mathcal{H}^0(\mathcal{L})$.
- For $G = \mathrm{GL}_n$, there is a more concrete description of G-Higgs bundles. We have an equivalence of groupoids

$$\begin{aligned} \{\text{rank } n \text{ vector bundles on } X_S\} &\simeq \{\mathrm{GL}_n\text{-bundles on } X_S\} \\ \mathcal{V} &\mapsto \underline{\mathrm{Isom}}_{\mathcal{O}_{X_S}}(\mathcal{O}_{X_S}^n, \mathcal{V}) \\ \mathcal{E} \times^{\mathrm{GL}_n} \mathcal{O}_X^n &\leftarrow\!\shortmid \mathcal{E}. \end{aligned}$$

[26] If one view a G-bundle as an exact tensor functor $\mathrm{Rep}(G) \to$ (vector bundles on X_T), then this is the image of $\mathfrak{g}(E)$ with the adjoint action. If one view a G-bundle as a sheaf on X_T, then $\mathrm{Ad}(\mathcal{E}) = \mathcal{E} \times^G (\mathfrak{g}(E) \otimes_E \mathcal{O}_{X_T})$.

If $\mathcal{V}$ corresponds to $\mathcal{E}$, then $\mathrm{Ad}(\mathcal{E})_{\mathcal{L}} = \underline{\mathrm{Hom}}(\mathcal{V}, \mathcal{V} \otimes \mathcal{L})$, where the right hand side is the sheaf on X_S sending U to $\mathrm{Hom}_{\mathcal{O}_U}(\mathcal{V}|_U, (\mathcal{V} \otimes \mathcal{L})|_U)$. Thus, we can identify a GL_n-Higgs bundles with a pair $(V, \phi : V \to V \otimes_{\mathcal{O}_{X_S}} \mathcal{L})$, where V is a rank n vector bundle on X_S and ϕ is $\mathcal{O}_{X_S}$-linear.

Lemma 2.3.4. *The prestack* Higgs_G *is a v-stack.*

Proof. This follows easily from the fact that Bun_G is a v-stack and for any G-bundle $\mathcal{E}$, the functor $T \mapsto H^0(X_T, \mathrm{Ad}(\mathcal{E}) \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_T)$, being the Banach-Colmez space associated with $\mathrm{Ad}(\mathcal{E}) \otimes_{\mathcal{O}_{X_S}} \mathcal{L}$, is a v-sheaf by [FS21, Proposition II.2.1]. $\square$

Proposition 2.3.5. *The natural map* $\mathrm{Higgs}_G \to \mathrm{Bun}_G$ *is representable in locally spatial diamonds and is partially proper (cf. Definition 2.2.75.)*

Proof. Let us denote this map by f. The proof of [FS21, Proposition III.1.3] shows that there is a perfectoid space T such that $T \twoheadrightarrow \mathrm{Bun}_G$ as pro-étale stacks. This map corresponds to a G-bundle $\mathcal{E}$ on X_T. The base change of f along this map is

$$\mathrm{Higgs}_G \times_{\mathrm{Bun}_G} T = \mathcal{H}^0(\mathrm{Ad}(\mathcal{E}) \otimes \mathcal{L}_T) \to T.$$

As $\mathcal{H}^0(\mathrm{Ad}(\mathcal{E}) \otimes \mathcal{L}_T)$ is a locally spatial diamond, this map is representable in locally spatial diamonds. Moreover, by [FS21, Proposition II.2.6], this map is quasi-separated. It follows from [Sch17, Proposition 10.11] that f is quasi-separated. By [Sch17, Proposition 13.4(iv)], f is representable in locally spatial diamonds.

It is clear that f is 0-truncated and we have just shown that f is quasi-separated. We can deduce that f is partially proper by applying the following fact to the adjoint bundle: if $\mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space over S and $\mathcal{V}$ is a vector bundle on $X_{(R,R^+)}$, then we have an isomorphism $H^0(X_{(R,R^+)}, \mathcal{V}) \xrightarrow{\sim} H^0(X_{(R,R^\circ)}, \mathcal{V})$. This fact is immediate from the partial properness of the map $\mathcal{H}^0(\mathcal{V}) \to S$, proved in [FS21, Proposition II.2.16]. $\square$

Corollary 2.3.6. Higgs_G *is an Artin v-stack if*[27] $S \in \mathrm{Perf}_{\overline{F}_q}$.

Proof. This follows from Proposition 2.3.5 and [FS21, Proposition IV.1.8(iii)]. $\square$

Proposition 2.3.7. *[TL26, Proposition 2.20] If* $S \in \mathrm{Perf}_{\overline{F}_q}$ *and* $\mathcal{L} = \mathcal{O}_{X_S}$*, then* $\mathrm{Higgs}_G \times_{\mathrm{Bun}_G} \mathrm{Bun}_G^1 \cong [\underline{\mathfrak{g}(E)}/\underline{G(E)}]$.

Proof. This is true because Higgs_G is the same as the stack Bun_G^X in [TL26], where $X = \mathfrak{g}$ with the adjoint action of G. $\square$

[27]We make the assumption that $S \in \mathrm{Perf}_{\overline{F}_q}$ because in [FS21], Bun_G is defined on $\mathrm{Perf}_{\overline{F}_q}$.

2.4 GL_n case

In this section, we take $G = \mathrm{GL}_n$. We will usually view Higgs_G as the stack that sends T to the groupoid of pairs $(V, \phi : V \to V \otimes_{\mathcal{O}_{X_T}} \mathcal{L}_T)$, where V is a vector bundle on X_T and ϕ is $\mathcal{O}_{X_T}$-linear.

The aim of this section is to study the Hitchin fibration $\mathrm{Higgs}_G \to \mathcal{A}_G$ (cf. Definition 2.4.1), which essentially maps a pair (V, ϕ) to the characteristic polynomial of ϕ. We prove an analogue of the BNR theorem, which relates fibres of the Hitchin fibration to line bundles on the spectral curves. This is an interesting result, because it is an instance of abelianisation: we reduce the study of (non-abelian) GL_n objects to (abelian) GL_1 objects. We also prove a GAGA type result for the spectral curve, which gives an equivalence between the category of vector bundles on the spectral curve and that on a schematic version of the spectral curve.

2.4.1 Hitchin fibration

Definition 2.4.1. *We define the* ***Hitchin base*** $\mathcal{A}_{G,\mathcal{L}} := \prod_{i=1}^n \mathcal{H}^0(\mathcal{L}^{\otimes i})$*, which is a locally spatial diamond. When there is no ambiguity, we will write* $\mathcal{A}_{G,\mathcal{L}}$ *as* $\mathcal{A}_G$*.*

We define the Hitchin fibration

$$\mathrm{Higgs}_G \to \mathcal{A}_G$$

to be the map which on T*-valued points sends* $(V, \phi : V \to V \otimes \mathcal{L}_T)$ *to* $((-1)^i \operatorname{Tr} \Lambda^i \phi)_i$*.*

We call the image of $(V, \phi) \in \mathrm{Higgs}_G(T)$ *under this map its* ***characteristic polynomial****. We will also call* $Z^n - \mathrm{Tr}(\phi) Z^{n-1} + \mathrm{Tr}(\Lambda^2 \phi) Z^{n-2} + \cdots + (-1)^n \mathrm{Tr}(\Lambda^n \phi) \in \prod_{i=1}^n H^0(X_T, \mathcal{L}^{\otimes i}) Z^i$ *the characteristic polynomial of* (V, ϕ)*.*

Here, $\operatorname{Tr} \Lambda^i \phi$ is defined as follows: We apply Λ^i to ϕ to get a map

$$\Lambda^i \phi : \Lambda^i V \to \Lambda^i (V \otimes \mathcal{L}) \cong \Lambda^i V \otimes \mathcal{L}^{\otimes i},$$

which corresponds to an element in $H^0(X_T, (\Lambda^i V)^\vee \otimes \Lambda^i V \otimes \mathcal{L}_T^{\otimes i})$. We can apply the evaluation map to get an element in $H^0(X_T, \mathcal{L}_T^{\otimes i})$, which we call $\operatorname{Tr} \Lambda^i \phi$.

2.4.2 Spectral curve

There is a way to describe the fibre of the Hitchin fibration in terms of line bundles on the spectral curve, which we will define now. This is desirable because it allows us to relate GL_n objects with GL_1 objects.

Now, fix $S \in \mathrm{Perf}_{\mathbb{F}_q}$, $\mathcal{L}$ a line bundle on X_S, and $(a_i) \in \prod_{i=1}^n \mathcal{L}^{\otimes i}(X_S)$. To construct the spectral curve, we shall construct adic spaces $\mathrm{Spa}(B_U, B_U^+)$ over U for open subsets $U \subset X_S$ and then glue these spaces.

Let $U \subset X_S$ be an affinoid sousperfectoid space such that $\mathcal{L}|_U$ is isomorphic to $\mathcal{O}_U$. Let $A = \mathcal{O}_U(U), A^+ = \mathcal{O}_U^+(U)$. If $\iota : \mathcal{L}|_U \xrightarrow{\sim} \mathcal{O}_U$ is a trivialisation, then

$$B_{U,\iota} := A[Z]/(Z^n + \iota(a_1) Z^{n-1} + \cdots + \iota(a_n))$$

equipped with the induced topology in footnote 17 is a sheafy Huber ring by [HK25, Corollary 4.7]. If $\iota' : \mathcal{L}|_U \xrightarrow{\sim} \mathcal{O}_U$ is another trivialisation. Then $\iota' = a\iota$ for some $a \in A^\times$, so

$$\begin{aligned} B_{U,\iota} &\to B_{U,\iota'} \\ Z &\mapsto a^{-1}Z \end{aligned}$$

is an isomorphism. Define[28]

$$B_U := \operatorname{colim}_{\iota:\mathcal{L}|_U \xrightarrow{\sim} \mathcal{O}_U} B_{U,\iota}$$

and

$$B_U^+ = \text{integral closure of } A^+ \text{ in } B_U.$$

Lemma 2.4.2. *B_U^+ is a ring of integral elements of B_U, i.e. B_U^+ is an open and integrally closed subring of $(B_U)^\circ$.*

Proof. We identify B_U with some $B_{U,\iota}$ by fixing a trivialisation ι. The map $A \hookrightarrow B_U$ is an adic morphism by [SW20, Lemma 5.1.2], so it sends powerbounded elements to powerbounded elements. Thus, $A^+ \subset (B_U)^\circ$. Thus

$$\begin{aligned} B_U^+ &\subset \text{integral closure of } (B_U)^\circ \text{ in } B_U \\ &= (B_U)^\circ. \end{aligned}$$

It remains to show that B_U^+ is open in B_U. Let ϖ be a pseudouniformizer of A. Then $\varpi^N Z$ is integral over A^+ for some $N \geq 1$, so $B_U^+ \supset A^+[\varpi^N Z]$. It is direct from the definition of the topology on B_U that $A^+[\varpi^N Z]$ is open in B_U, so B_U^+ is also open. $\square$

Hence, we have an adic space $\operatorname{Spa}(B_U, B_U^+)$ with a canonical map

$$\pi_U : \operatorname{Spa}(B_U, B_U^+) \to \operatorname{Spa}(A, A^+) = U.$$

This is compatible with passing to open subspace of U:

Lemma 2.4.3. *Let $U' \subset U$ be an affinoid sousperfectoid open subspace of U. Then the canonical U-morphism $\operatorname{Spa}(B_{U'}, B_{U'}^+) \to \pi_U^{-1}(U')$ is an isomorphism. In other words, we have a Cartesian diagram*

$$\begin{tikzcd} \operatorname{Spa}(B_{U'}, B_{U'}^+) \arrow[r] \arrow[d, "\pi_{U'}"'] & \operatorname{Spa}(B_U, B_U^+) \arrow[d, "\pi_U"] \\ U' \arrow[r] & U. \end{tikzcd}$$

[28] Of course $B_U \cong B_{U,\iota}$ for every ι, but B_U is more canonical in the sense that it is independent of the choice of ι.

Proof. We shall verify that they have the same functor of points. Fix a trivialisation $\iota : \mathcal{L}_U \xrightarrow{\sim} \mathcal{O}_U$ to make the identifications $B_U \cong B_{U,\iota}$ and $B_{U'} \cong B_{U',\iota|_{U'}}$. Let $U' = \mathrm{Spa}(A', A'^+)$. Let (R, R^+) be a complete Huber pair over (A, A^+). Recall that $B_{U'}^+$ is the integral closure of A'^+ in $B_{U'} = \frac{A'[Z]}{(Z^n+\iota(a_1)Z^{n-1}+\cdots+\iota(a_n))}$. Thus

$$\begin{aligned}
&(\mathrm{Spa}(B_{U'}, B_{U'}^+))(R, R^+) \\
&= \mathrm{Hom}_{(A,A^+)}\left((\frac{A'[Z]}{(Z^n + \iota(a_1)Z^{n-1} + \cdots + \iota(a_n))}, B_{U'}^+), (R, R^+) \right) \\
&= \mathrm{Hom}_{(A,A^+)}((A', A'^+), (R, R^+)) \times \left\{ r \in R : r^n + \iota(a_1)r^{n-1} + \cdots + \iota(a_n) = 0 \right\}
\end{aligned}$$

This is $\left\{ r \in R : r^n + \iota(a_1)r^{n-1} + \cdots + \iota(a_n) = 0 \right\}$ if the structure map $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(A, A^+)$ factors through U' and empty otherwise.

Similarly, if the structure map $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(A, A^+)$ factors through U', then

$$\begin{aligned}
&\pi_U^{-1}(U')(R, R^+) \\
&= \mathrm{Hom}_U(\mathrm{Spa}(R, R^+), \mathrm{Spa}(B_U, B_U^+)) \\
&= \mathrm{Hom}_{(A,A^+)}\left((\frac{A[Z]}{(Z^n + \iota(a_1)Z^{n-1} + \cdots + \iota(a_n))}, B_U^+), (R, R^+) \right) \\
&= \left\{ r \in R : r^n + \iota(a_1)r^{n-1} + \cdots + \iota(a_n) = 0 \right\}.
\end{aligned}$$

If the structure map $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(A, A^+)$ does not factor through U', then $\pi_U^{-1}(U')(R, R^+) = \varnothing$. This finishes the proof. $\square$

Definition 2.4.4. *Define the* ***spectral curve*** $C_{a,S}$ *to be the adic space obtained by gluing* $\mathrm{Spa}(B_U, B_U^+)$ *as* U *runs through all the affinoid sousperfectoid open subsets of* X_S *such that* $\mathcal{L}|_U \cong \mathcal{O}_U$.

This gluing is valid by Lemma 2.4.3. By gluing π_U, we get a morphism $\pi : C_{a,S} \to X_S$.

Let $U = \mathrm{Spa}(A, A^+)$ be an affinoid sousperfectoid open subset of X_S such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Then $(\pi^*\mathcal{L})|_{\pi^{-1}(U)}$ is the vector bundle associated with the B_U-module[29]

$$\mathcal{L}(U) \otimes B_U = \mathcal{L}(U) \otimes_A \mathrm{colim}_\iota\, A[Z_\iota]/(Z_\iota^n + \iota(a_1)Z_\iota^{n-1} + \cdots + \iota(a_n)).$$

The element

$$\iota^{-1}(1) \otimes Z_\iota$$

is independent of the choice of ι, because if $\iota' = a\iota$ is another trivialisation of $\mathcal{L}|_U$, then $\iota^{-1}(1) \otimes Z_\iota = \iota^{-1}(1) \otimes a^{-1}Z_{\iota'} = \iota'^{-1}(1) \otimes Z_{\iota'}$. Thus, we can glue them as U varies to obtain an element

$$\tau \in H^0(C_{a,S}, \pi^*\mathcal{L}).$$

We call this the ***tautological section*** of $\pi^*\mathcal{L}$.

Informally, locally we have $U = \mathrm{Spa}(A, A^+)$, $\pi^{-1}(U) \cong \mathrm{Spa}(A[Z]/(Z^n + \iota(a_1)Z^{n-1} + \cdots + \iota(a_n)), \text{integral closure of } A^+)$, $\tau = Z$.

[29]We rename Z by Z_ι so as to distinguish the different copies of Z.

Lemma 2.4.5. *Let A be a commutative ring, M be a finite free A-module, $\phi \in \mathrm{End}_A(M)$ with characteristic polynomial $f(Z) = (Z - \lambda_1) \dots (Z - \lambda_n)$, where $\lambda_i \in A$ and $\lambda_i - \lambda_j \in A^\times$ for all $i \neq j$. If we regard M as the $A[Z]/(f)$-module with Z acting as ϕ, then M is a finite projective $A[Z]/(f)$-module of rank 1.*

Proof. As $\lambda_i - \lambda_j \in A^\times$ for all $i \neq j$,

$$A[Z]/(f) \cong A[Z]/(Z - \lambda_1) \times \dots \times A[Z]/(Z - \lambda_n) \cong A^n.$$

Let e_i be the idempotent in $A[Z]/(f)$ corresponding to the i-th factor, i.e. $e_i = \prod_{j \neq i} \frac{Z - \lambda_j}{\lambda_i - \lambda_j}$. Note that $1 = e_1 + \dots + e_n$ and $e_i e_j = \delta_{ij}$. Then

$$M = e_1 M \oplus \dots \oplus e_n M \tag{2.14}$$

as $A[Z]/(f) = A^n$-module. It suffices to show that each $e_i M$ is a finite projective A-module of rank 1.

Each $e_i M$ is a direct summand of the free A-module M, so each $e_i M$ is a finite projective A-module. By (2.14), it suffices to show $M_i \neq 0$ for all i. Let K be a field over A. If we view ϕ as a matrix over K, then it has characteristic polynomial f with eigenspace decomposition

$$M \otimes_A K = ((e_1 M) \otimes_A K) \oplus \dots \oplus ((e_n M) \otimes_A K).$$

Note that $(e_i M) \otimes_A K$ is the λ_i-eigenspace, so it is non-zero. It follows that each $e_i M$ is non-zero and the claim follows. ☐

Lemma 2.4.6. *Let $A \to B$ be a ring homomorphism such that B is free of rank n as an A-module. Let M be a finite projective B-module of rank 1. Then M is a finite projective A-module of rank n.*

Proof. Let $h : \operatorname{Spec} B \to \operatorname{Spec} A$ be the corresponding map on affine schemes. This is flat and of finite presentation, so it is open. Let $\widetilde{M}$ be the quasi-coherent sheaf on $\operatorname{Spec} B$ corresponding to M. We can pick an open cover $\{U_i\}$ of $\operatorname{Spec} B$ such that $\widetilde{M}|_{U_i} \cong \mathcal{O}_{U_i}$ for all i. Then $\{h(U_i)\}$ is an open cover of $\operatorname{Spec} A$ and $(h_* \widetilde{M})|_{h(U_i)} \cong (h_* \mathcal{O}_{U_i})|_{h(U_i)} \cong \mathcal{O}^n_{h(U_i)}$. ☐

We have the analogue of the BNR theorem [BNR89, Proposition 3.6] in our setting:

Proposition 2.4.7. *For $1 \leq i \leq n$, let $a_i \in \mathcal{L}^{\otimes i}(X_S)$. Let $f(Z) = Z^n + a_1 Z^{n-1} + \dots + a_n \in \oplus_{i=1}^n \mathcal{L}^{\otimes i}(X_S) Z^i$. Assume that for all $x \in X_S$, there is an open neighborhood U of x such that $\mathcal{L}|_U$ is trivial and $f(Z)$ factorizes as $(Z - \lambda_1) \dots (Z - \lambda_n)$, where $\lambda_i \in \mathcal{L}(U)$ and $\lambda_i - \lambda_j \in \mathcal{L}(U)^\times$ for[30] all $i \neq j$. Then we have an equivalence of categories*

$$\{\textit{line bundles on } C_{a,S}\} \to \{\textit{rank } n \textit{ Higgs bundles on } X_S \textit{ with characteristic polynomial } f\}$$
$$\mathcal{F} \mapsto (\pi_* \mathcal{F}, \pi_*(\tau \otimes \mathcal{F})),$$

[30] Here is the definition of $\mathcal{L}|_U(U)^\times$: Pick an isomorphism $\iota : \mathcal{L}|_U \xrightarrow{\sim} \mathcal{O}_U$; $\mathcal{L}(U)^\times$ is the subset of $\mathcal{L}(U)$ which corresponds to $\mathcal{O}(U)^\times$ under the induced isomorphism $\mathcal{L}(U) \cong \mathcal{O}_U(U)$. This is independent of the choice of ι.

where $\pi : C_{a,S} \to X_S$ is the canonical map and $\pi_*(\tau \otimes \mathcal{F})$ is defined as follows: τ corresponds to a map $\mathcal{O} \to \pi^*\mathcal{L}$; we can tensor it with $\mathcal{F}$ to get $\mathcal{F} \to (\pi^*\mathcal{L}) \otimes \mathcal{F}$; then we apply π_* to get the desired map $\pi_*\mathcal{F} \to \pi_*((\pi^*\mathcal{L}) \otimes \mathcal{F}) = \mathcal{L} \otimes \pi_*\mathcal{F}$.

Proof. For $x \in X_S$, let us call an open neighborhood $x \in U \subset X_S$ ***good*** if U is affinoid sousperfectoid and $\mathcal{L}|_U \cong \mathcal{O}_U$. Note that on such a U, we can identify f with a polynomial with coefficients in $\mathcal{O}_U(U)$.

Let $\mathcal{F}$ be a line bundle on $C_{a,S}$. Let us first show that $\pi_*\mathcal{F}$ is a rank n vector bundle on X_S. Let $x \in X$ and $U = \mathrm{Spa}(A, A^+) \subset X$ be a good neighborhood. We know that $\pi^{-1}(U) \cong \mathrm{Spa}(A[Z]/(f), \widetilde{A^+})$, where $\widetilde{A^+}$ is the integral closure of A^+ in $A[Z]/(f)$. Then $\mathcal{F}|_{\pi^{-1}(U)}$ corresponds to a finite projective $A[Z]/(f)$-module of rank 1, which, by Lemma 2.4.6, is a finite projective A-module of rank n. Thus, $(\pi_*\mathcal{F})|_U$ is a rank n vector bundle. It follows that the same holds for $\pi_*\mathcal{F}$.

To see that $\varphi := \pi_*(\tau \otimes \mathcal{F})$ has characteristic polynomial f, i.e. $a_i = (-1)^i \operatorname{Tr} \Lambda^i \phi$ for all $1 \leq i \leq n$, note that the formation of $\operatorname{Tr} \Lambda^i \varphi$ is compatible with restriction to open subsets. We can therefore check them locally, so we are reduced to showing the following: if M is a finite projective $A[Z]/(f)$-module of rank 1, then $Z \in \mathrm{End}_A(M)$, viewed as an A-linear endomorphism of the finite projective rank n A-module M, has characteristic polynomial f. We can localise at A to assume that M is free of rank n as an A-module. Then this is standard.

Conversely, let $(\mathcal{V}, \phi)$ be a rank n Higgs bundles on X_S with characteristic polynomial f. Let $x \in X_S$ and $U = \mathrm{Spa}(A, A^+) \subset X_S$ be a good neighborhood such that $\mathcal{V}|_U \cong \mathcal{O}_U^n$ and $f(Z)$ factorises as in the statement of the proposition. We can view the finite free A-module $\mathcal{V}(U)$ as a $B_U = \mathrm{colim}_\iota\, A[Z_\iota]/(Z_\iota^n + \iota(a_1)Z_\iota^{n-1} + \cdots + \iota(a_n))$-module with Z_ι acting as the endomorphism

$$\mathcal{V}(U) \xrightarrow{\phi} (\mathcal{V} \otimes \mathcal{L})(U) \xrightarrow{1 \otimes \iota} \mathcal{V}(U).$$

This is well-defined, because if $\iota' = a\iota$ is another trivialisation of ι, then $a^{-1}Z_{\iota'}$ ($= Z_\iota$ in B_U) acts as a^{-1} times $\mathcal{V}(U) \xrightarrow{\phi} (\mathcal{V} \otimes \mathcal{L})(U) \xrightarrow{1 \otimes \iota'} \mathcal{V}(U)$, which is the same as the way that Z_ι acts. By Lemma 2.4.5, $\mathcal{V}(U)$ is a finite projective B_U-module of rank 1. Thus, it gives rise to a line bundle W on $\pi^{-1}(U)$. We can glue W as U varies to get a line bundle on $C_{a,S}$.

It is not hard to verify that these two constructions are inverses of each other. □

Remark 2.4.8. In the classical case, where X is a smooth projective curve with genus at least 2 over an algebraically closed field k and $\mathcal{L} = \Omega_{X/k}$, then one can show that the assumption in the proposition is never satisfied by considering the degree of the discriminant divisor. However, the assumption in the proposition is possible in our setup. If $\mathcal{L} = \mathcal{O}_{X_S}$, then the assumption is roughly saying that the characteristic polynomial has distinct roots, which seems reasonable.

As an immediate corollary of Proposition 2.4.7, we get the following relation between the Hitchin fibers and line bundles on the spectral curve.

Theorem 2.4.9. *In the situation of Proposition 2.4.7, we have a canonical isomorphism of v-stacks*

$$\mathrm{Higgs}_{n,a} \cong \mathrm{Pic}_{C_a}.$$

Here, $\mathrm{Higgs}_{n,a}$ *is the v-stack on* Perf_S *sending* T *to the groupoid of rank* n *Higgs bundles on* X_S *with characteristic polynomial* f*, i.e. the Hitchin fiber* $\mathrm{Higgs}_{\mathrm{GL}_n} \times_{\mathcal{A}_{\mathrm{GL}_n}} S$*;* Pic_{C_a} *is the v-stack on* Perf_S *sending* T *to the groupoid of line bundles on* $C_{a,T}$*.*

Remark 2.4.10. In the setting of Proposition 2.4.7, the spectral cover $C_{a,S}$ is a degree n covering space of X_S, i.e. for all $x \in X_S$, there is an open neighborhood U of x such that $\pi^{-1}(U) \cong \coprod_{i=1}^n U$ over U. Moreover, for all $T \in \mathrm{Perf}_S$,

$$C_{a,T} = C_{a,S} \times_{X_S} X_T.$$

Indeed, there is a natural map $C_{a,T} \to C_{a,S} \times_{X_S} X_T$. Locally, X_S is given by $\mathrm{Spa}(A, A^+)$, X_T is given by $\mathrm{Spa}(B, B^+)$, $C_{a,S}$ is given by $\mathrm{Spa}(A[Z]/(f), \widetilde{A}^+)$, and both $C_{a,T}$ and $C_{a,S} \times_{X_S} X_T$ are locally given by $\mathrm{Spa}(B[Z]/(f), \widetilde{B}^+)$, where $\sim$ denotes integral closure. The claim follows. This motivates the setup in the next subsection.

2.4.3 Covering space of the Fargues-Fontaine curve

Fix $S \in \mathrm{Perf}_{\mathbb{F}_q}, n \geq 1$. Let $\pi : C_S \to X_S$ be a degree n covering of X_S, i.e. C_S is an adic space and for all $x \in X_S$, there is an open neighborhood U of x such that $\pi^{-1}(U) \cong \coprod_{i=1}^n U$ over U.

Let us start with an easy lemma.

Lemma 2.4.11. *If* $\mathcal{E}$ *is a vector bundle of rank* m *on* C_S*, then* $\pi_*\mathcal{E}$ *is a vector bundle of rank* mn *on* X_S*. Also,* $\pi_* : Ab(C_S) \to Ab(X_S)$ *is an exact functor, where* $Ab(-)$ *stands for the category of abelian sheaves. In particular, for all* $\mathcal{F} \in Ab(C_S)$*,*

$$H^i(C_S, \mathcal{F}) = H^i(X_S, \pi_*\mathcal{F}) \tag{2.15}$$

for all $i \geq 0$*.*

Proof. The first statement follows immediately from the fact that π is a covering of degree n. For the second statement, note that for all $\mathcal{F} \in Ab(C_S)$ and $x \in X_S$, we have $(\pi_*\mathcal{F})_x = \oplus_{y \in \pi^{-1}(x)} \mathcal{F}_y$ because π is a covering. □

Let $\mathcal{O}_{C_S}(1) := \pi^*\mathcal{O}_{X_S}(1)$. Let us prove an ampleness result for $\mathcal{O}_{C_S}(1)$.

Proposition 2.4.12. *Assume* S *is affinoid perfectoid. Let* $\mathcal{E}$ *be a vector bundle on* C_S*. Then there exists* $N \geq 1$ *such that for all* $i \geq N$*,* $\mathcal{E}(i) := \mathcal{E} \otimes_{\mathcal{O}_{C_S}} \mathcal{O}_{C_S}(1)^{\otimes i}$ *is globally generated*[31] *and* $H^1(C_S, \mathcal{E}(i)) = 0$*. Moreover,* $H^j(C_S, \mathcal{E}) = 0$ *for all* $j \geq 2$*.*

[31] An $\mathcal{O}_{C_S}$-module $\mathcal{F}$ is called globally generated if there is an $m \geq 1$ and a surjection $\mathcal{O}_{C_S}^m \twoheadrightarrow \mathcal{F}$.

Proof. By Lemma 2.4.11, $\pi_*\mathcal{E}$ is a vector bundle on X_S. By equation (2.15) and [FS21, beginning of section II.2], $H^j(C_S, \mathcal{E}) = 0$ for all $j \geq 2$. By [FS21, Theorem II.2.6], there is an $N \geq 1$ such that for all $i \geq N$, $(\pi_*\mathcal{E})(i) := (\pi_*\mathcal{E}) \otimes_{\mathcal{O}_{X_S}} \mathcal{O}_{X_S}(1)^{\otimes i}$ is globally generated and $H^1(X_S, (\pi_*\mathcal{E})(i)) = 0$. Note that

$$(\pi_*\mathcal{E})(i) = \pi_*(\mathcal{E} \otimes_{\mathcal{O}_{C_S}} \pi^*\mathcal{O}_{X_S}(i)) = \pi_*(\mathcal{E}(i))$$

by the projection formula. By equation (2.15), $H^1(C_S, \mathcal{E}(i)) = 0$ for all $i \geq N$. Finally, as $(\pi_*\mathcal{E})(i)$ is globally generated, there is an $m \geq 1$ and a surjection

$$\mathcal{O}_{X_S}^m \twoheadrightarrow (\pi_*\mathcal{E})(i) = \pi_*(E(i)).$$

Applying π^* gives a surjection

$$g : \mathcal{O}_{C_S}^m \twoheadrightarrow \pi^*\pi_*(E(i)).$$

Now, observe that the natural map

$$h : \pi^*\pi_*(E(i)) \to E(i)$$

is surjective, as can be checked on stalks. Hence we get a surjection $h \circ g : \mathcal{O}_{X_S}^m \twoheadrightarrow E(i)$, showing that $E(i)$ is globally generated. $\square$

We shall also need a general topological lemma. Let us write qc as a shorthand for 'quasi-compact'. Recall that a topological space is called quasi-separated if the intersection of any two qc open subsets is qc. We refer the reader to [Sch17, section 2] for the definitions of spectral and locally spectral spaces. Recall that every adic space is a locally spectral space. Also, a locally spectral space is spectral if and only if it is quasi-compact and quasi-separated (qcqs).

Lemma 2.4.13. *Let Z be a locally spectral space. Assume that Z admits a finite open cover $\{U_i\}_{i=1}^m$ such that U_i and $U_i \cap U_j$ are qc for all i, j. Then Z is qcqs.*

Proof. Quasi-compactness of Z is clear. The proof for quasi-separatedness is almost identical to that in [Sil]. Let V be a qc open subset of Z. Let us show that $V \cap U_1$ is qc. Since Z is locally spectral and V is qc, we can write

$$V = \bigcup_{i=1}^{m} \bigcup_{j=1}^{M} V_{ij}$$

for some qc open $V_{ij} \subset U_i$ and $M \in \mathbb{N}$. Then

$$\begin{aligned} V \cap U_1 &= \bigcup_{i=1}^{m} \bigcup_{j=1}^{M} (V_{ij} \cap U_1) \\ &= \bigcup_{i=1}^{m} \bigcup_{j=1}^{M} (V_{ij} \cap U_1 \cap U_j). \end{aligned}$$

By assumption, each $U_1 \cap U_j$ is qc. Since V_{ij} and $U_1 \cap U_j$ are both qc open subsets of U_j, we know that $(V_{ij} \cap U_1 \cap U_j)$ is also qc by the quasi-separatedness of U_j. Hence $V \cap U_1$ is also qc.

Now, suppose W is also a qc open subset of Z. Then $W \cap U_1$ is qc by the above. By quasi-separatedness of U_1, $V \cap W \cap U_1$ is also qc. By symmetry, the same holds with U_1 replaced by any of the U_i. Hence, $V \cap W = \cup_{i=1}^m (V \cap W \cap U_i)$, is also qc. □

We have the following GAGA result:

Theorem 2.4.14. *Suppose S is affinoid perfectoid. Let*

$$C_S^{sch} := \mathrm{Proj}\left(\bigoplus_{m \geq 0} H^0(C_S, \mathcal{O}_{C_S}(m))\right).$$

Then we have a natural map of locally ringed spaces

$$f : C_S \to C_S^{sch}$$

and pullback along this map induces an equivalence of categories

$$f^* : \{\textit{vector bundles on } C_S^{sch}\} \xrightarrow{\simeq} \{\textit{vector bundles on } C_S\}.$$

In addition, for every vector bundle $\mathcal{E}$ on C_S^{sch}, we have

$$H^i(C_S^{sch}, \mathcal{E}) = H^i(C_S, f^*(\mathcal{E}))$$

for all $i \geq 0$.

Proof. For affinoid perfectoid S, X_S is qcqs. Thus, there is a finite open cover $\{U_i\}_{i=1}^m$ such that $U_i \cap U_j$ is qc for all i, j. Since $\pi : C_S \to X_S$ is a covering of degree n, π is quasi-compact, i.e. the preimage of any qc open subset is qc. It follows that $\{\pi^{-1}(U_i)\}_{i=1}^m$ is an open cover of C_S satisfying the condition of Lemma 2.4.13. Thus, C_S is qcqs, and hence is a spectral space. The theorem now follows from Proposition 2.4.12 and the general GAGA theorem in [FS21, Proposition II.2.7]. □

Remark 2.4.15. It is conceivable that for $S = \mathrm{Spa}\, K$, where K is an algebraically closed perfectoid field over $\mathbb{F}_q$, one can establish an analogue of [FS21, Proposition II.2.9] about the properties of C_S^{sch}. Once this is established, one should be able to get an analogue of [FS21, Proposition II.2.10] on the classification of line bundles on C_S^{sch} and hence a concrete description of the Hitchin fiber via Proposition 2.4.7.

2.5 Hitchin fibration and action of Picard stack

The aim of this section is to study the Hitchin fibration for general G and study the action of a certain Picard stack on Higgs_G (cf. Section 2.2.2.) Starting from this section, we will mostly use the schematic version of the Fargues-Fontaine curve instead of the adic space version.

Let E be a non-Archimedean local field with residue field $\mathbb{F}_q$, where q is a power of p. Let G be a split connected reductive group over E, B be a Borel subgroup over E, T be a split maximal torus in B. Assume $char E \nmid |W|$, where W is the Weyl group of G. We let $\mathfrak{g}$ be the Lie algebra of G, thought of as a scheme over E.[32] Let $\mathfrak{t}$ be the Lie algebra of T.

Fix an affinoid perfectoid space S over $\mathbb{F}_q$ and a line bundle $\mathcal{L}$ on X_S^{sch}. As a rule of thumb, in this section, every stack is either a stack on Perf_S with the v-topology or a stack on $\mathrm{Sch}_{X_S^{sch}}$ with the fppf topology.

2.5.1 Hitchin fibration

Let $\mathfrak{c} := \mathfrak{g} /\!\!/ G$ be the GIT quotient of $\mathfrak{g}$ by the adjoint action of G, i.e.

$$\mathfrak{c} := \mathfrak{g} /\!\!/ G := \operatorname{Spec} \mathcal{O}(\mathfrak{g})^G$$

where $\mathcal{O}(\mathfrak{g})$ is the ring of global sections of the structure sheaf of $\mathfrak{g}$. Let us recall the meaning of $\mathcal{O}(\mathfrak{g})^G$: if we view elements of $\mathcal{O}(\mathfrak{g})$ as morphisms $\mathfrak{g} \to \mathbb{A}^1$, then $\mathcal{O}(\mathfrak{g})^G$ are the morphisms such that the two compositions

$$G \times \mathfrak{g} \underset{\text{action}}{\overset{\pi_2}{\rightrightarrows}} \mathfrak{g} \to \mathbb{A}^1$$

agree. In other words,

$$\mathcal{O}(\mathfrak{g})^G = eq\left(\mathcal{O}(\mathfrak{g}) \underset{\text{action}^*}{\overset{1\otimes -}{\rightrightarrows}} \mathcal{O}(G) \otimes_E \mathcal{O}(\mathfrak{g}) \right). \tag{2.16}$$

Let us recall the Chevalley restriction theorem.

Proposition 2.5.1. *[Ngô10, Theorem 2.1] The inclusion $\mathfrak{t} \to \mathfrak{g}$ induces an isomorphism*

$$\mathcal{O}(\mathfrak{g})^G \xrightarrow{\sim} \mathcal{O}(\mathfrak{t})^W.$$

Moreover, $\mathcal{O}(\mathfrak{g})^G$ is isomorphic to the polynomial algebra $E[T_1, \dots, T_r]$ over E, where r is the rank of G, and we can pick the isomorphism such that $T_1, \dots, T_r$ correspond to homogeneous elements of $\mathcal{O}(\mathfrak{g}) = Sym_E(Lie(G)^\vee)$. The degrees of these elements are independent of their choices.

The proposition implies that $\mathfrak{c} \cong \mathbb{A}_E^r$ over E. The inclusion $\mathcal{O}(\mathfrak{g})^G \subset \mathcal{O}(\mathfrak{g})$ induces a map

$$\mathfrak{g} \to \mathfrak{c}.$$

This map is invariant under the adjoint action of G on $\mathfrak{g}$ by equation (2.16). We base change both sides along $X_S^{sch} \to \operatorname{Spec} E$ to get

$$\mathfrak{g}_{X_S^{sch}} \to \mathfrak{c}_{X_S^{sch}}.$$

[32] I.e. $\mathfrak{g} = \operatorname{Spec} Sym_E(Lie(G)^\vee)$, where $Lie(G) = \ker(G(E[\epsilon]/(\epsilon^2)) \to G(E))$.

We can view both sides as vector bundles on X_S^{sch} and tensor both sides with $\mathcal{L}$ to get a G-equivariant map

$$\mathfrak{g}_{\mathcal{L}} \to \mathfrak{c}_{\mathcal{L}} \tag{2.17}$$

where $\mathfrak{g}_{\mathcal{L}} := \mathfrak{g}_{X_S^{sch}} \otimes_{\mathcal{O}_{X_S^{sch}}} \mathcal{L}$ with the adjoint G-action and $\mathfrak{c}_{\mathcal{L}} := \mathfrak{c}_{X_S^{sch}} \otimes_{\mathcal{O}_{X_S^{sch}}} \mathcal{L}$ with the trivial G-action.

Definition 2.5.2. *We call $\mathcal{A}_{G,\mathcal{L}} := \mathcal{H}^0(\mathfrak{c}_{\mathcal{L}})$ the Hitchin base.*

Note that this is a Banach-Colmez space. In particular, it is a locally spatial diamond. When there is no ambiguity, we will write $\mathcal{A}_{G,\mathcal{L}}$ as $\mathcal{A}_G$.

Remark 2.5.3. We can be more concrete about the definition of $\mathcal{A}_G$. Pick homogeneous $f_1, \dots, f_r \in \mathcal{O}(\mathfrak{g})^G$ such that the map $E[T_1, \dots, T_r] \to \mathcal{O}(\mathfrak{g})^G$, $T_i \to f_i$ is an isomorphism. Let $\deg(f_i) = d_i$. Then we get an isomorphism of schemes

$$\begin{aligned} \mathfrak{c} &\xrightarrow{\sim} \mathbb{A}_E^r \\ f_i &\leftarrow\!\shortmid T_i. \end{aligned}$$

This induces a $\mathbb{G}_m$-equivariant isomorphism of sheaves on X_S^{sch}

$$\mathfrak{c}_{X_S^{sch}} \cong \mathcal{O}^r_{X_S^{sch}},$$

where $\mathbb{G}_m$ acts on the i-th component of $\mathcal{O}^r_{X_S^{sch}}$ via $t \mapsto t^{d_i}$. Twisting both sides by the $\mathbb{G}_m$-torsor $\widetilde{\mathcal{L}} = \underline{Isom}_{\mathcal{O}_{X_S}^{sch}}(\mathcal{O}_{X_S}^{sch}, \mathcal{L})$, we obtain

$$\mathfrak{c}_{\mathcal{L}} \cong \mathcal{L}^{\otimes d_1} \oplus \cdots \oplus \mathcal{L}^{\otimes d_r}$$

because on the i-th component, we have an isomorphism[33] $\mathcal{O}_{X_S}^{sch} \times^{\mathbb{G}_m} \widetilde{\mathcal{L}} \xrightarrow{\sim} \mathcal{L}^{\otimes d_i}$ given by sheafifiying $(z, f) \mapsto f(z) \otimes f(1) \otimes \cdots \otimes f(1)$. It follows that

$$\mathcal{A}_G \cong \prod_{i=1}^r \mathcal{H}^0(\mathcal{L}^{\otimes d_i}).$$

For example, for $G = \mathrm{GL}_n$, if we pick $f_i = (-1)^i \operatorname{Tr}(\Lambda^i(\ ))$ for $1 \le i \le n$, then we get back Definition 2.4.1.

Remark 2.5.4. From the analogue of Lemma A.0.8 for the fppf site on schemes, we can deduce that if $\mathcal{E}$ is a G-bundle on X_S^{sch}, then

$$\begin{aligned} \mathcal{E} \times^G \mathfrak{g}_{\mathcal{L}} &\cong \mathcal{E} \times^G (\widetilde{\mathcal{L}} \times^{\mathbb{G}_m} \mathfrak{g}_{X_S^{sch}}) \\ &= \widetilde{\mathcal{L}} \times^{\mathbb{G}_m} (\mathcal{E} \times^G \mathfrak{g}_{X_S^{sch}}) \\ &= \widetilde{\mathcal{L}} \times^{\mathbb{G}_m} \mathrm{Ad}(\mathcal{E}) \\ &\cong \mathcal{L} \otimes_{\mathcal{O}_{X_S^{sch}}} \mathrm{Ad}(\mathcal{E}) \end{aligned}$$

where we use the analogue of Lemma A.0.8 for the first and last isomorphisms.

[33] Note that the action of $\mathbb{G}_m$ on $\mathcal{O}_{X_S}^{sch}$ is different on each component.

Definition 2.5.5. *We define the* ***Hitchin fibration*** *to be the map*

$$\mathrm{Higgs}_G \to \mathcal{A}_G \tag{2.18}$$

which on T-valued points (for affinoid perfectoid T) is given by sending $(\mathcal{E}, \phi) \in \mathrm{Higgs}_G(T)$ to its image under

$$\mathrm{Ad}(\mathcal{E})_{\mathcal{L}}(X_T^{sch}) = (\mathcal{E} \times^G \mathfrak{g}_{\mathcal{L}})(X_T^{sch}) \to (\mathcal{E} \times^G \mathfrak{c}_{\mathcal{L}})(X_T^{sch}) = \mathfrak{c}_{\mathcal{L}}(X_T^{sch}),$$

where the middle map is induced from equation (2.17)*. We call the image of $(\mathcal{E}, \phi)$ under* (2.18) *its* ***characteristic polynomial****.*

Remark 2.5.6. Note that by Lemma 2.2.4, we have

$$\mathrm{Higgs}_G = \underline{\mathrm{Map}}(FF, [\mathfrak{g}_{\mathcal{L}}/G]).$$

Here, $\underline{\mathrm{Map}}(FF, [\mathfrak{g}_{\mathcal{L}}/G])$ denotes [34] the prestack on Perf_S sending T to the groupoid $[\mathfrak{g}_{\mathcal{L}}/G](X_T^{sch})$, where G acts on $\mathfrak{g}_{\mathcal{L}}$ via the adjoint action. Similarly, we have

$$\mathcal{A}_G := \underline{\mathrm{Map}}(FF, \mathfrak{c}_{\mathcal{L}}).$$

The Hitchin fibration is obtained by applying $\underline{\mathrm{Map}}(FF, \ )$ to

$$[\mathfrak{g}_{\mathcal{L}}/G] \to \mathfrak{c}_{\mathcal{L}}. \tag{2.19}$$

Example 2.5.7. Assume $G = \mathrm{GL}_n$. Let us show that we recover Definition 2.4.1. To identify $\mathcal{A}_G$ with $\prod_{i=1}^n \mathcal{H}^0(\mathcal{L}^{\otimes i})$ as in Remark 2.5.3, we pick $f_i = (-1)^i \mathrm{Tr}(\Lambda^i(\ ))$ for $1 \le i \le n$ as a basis of $\mathcal{O}(\mathfrak{g})^G$.

Let $\mathcal{V}$ be a rank n vector bundle on X_S^{sch}. Its corresponding GL_n-bundle is $\mathcal{E} := \underline{\mathrm{Isom}}_{\mathcal{O}_{X_S^{sch}}}(\mathcal{O}^n_{X_S^{sch}}, \mathcal{V})$. Let $\phi \in H^0(X_S^{sch}, \mathrm{Ad}(\mathcal{E})_{\mathcal{L}}) = \mathrm{Hom}_{\mathcal{O}_{X_S^{sch}}}(\mathcal{V}, \mathcal{V} \otimes \mathcal{L})$. We want to show that the image of $(\mathcal{E}, \phi)$ (via Definition 2.5.5) and the image of $(\mathcal{V}, \phi)$ (via Definition 2.4.1) in $\mathfrak{c}_{\mathcal{L}}$ agree. It suffices to check this on an open cover of X_S^{sch}. Let $U = \mathrm{Spec}\, B$ be an open affine subset of X_S^{sch} such that $\mathcal{L}$, $\mathcal{E}$, $\mathcal{V}$ are all trivial. Then $\phi|_U$ corresponds to an element $M \in \mathfrak{g}(U) = \mathfrak{gl}_{n\times n}(B)$. The restriction of the map in Definition 2.5.5 to U is given by

$$\begin{aligned} \mathfrak{g}(A) &\to \mathfrak{c}(A) = (\mathrm{Spec}\, E[f_1, \dots, f_n])(A) \cong A^n \\ M &\mapsto (f_i(M))_{1\le i\le n} = ((-1)^i \mathrm{Tr}(\Lambda^i M))_{1\le i \le n}. \end{aligned}$$

This agrees with Definition 2.4.1.

[34] Note that by 2-Yoneda lemma, this is also the prestack sending T to the groupoid of morphisms $X_T^{sch} \to [\mathfrak{g}_{\mathcal{L}}/G]$ of stacks on $\mathrm{Sch}_{X_S^{sch}, fppf}$. This justifies the notation $\underline{\mathrm{Map}}(FF, -)$.

2.5.2 $\mathfrak{g}^{reg}$

Lemma 2.5.8. *The functor given by*

$$\begin{aligned} I : \{\textit{schemes over } E\}^{op} &\to \mathrm{Set} \\ U &\mapsto \{(g,v) \in G(U) \times \mathfrak{g}(U) : \mathrm{Ad}_g(v) = v\} \end{aligned}$$

is representable by an affine scheme of finite type over E.

Proof. Fix an embedding $G \hookrightarrow \mathrm{GL}_n$ of group schemes. Let

$$\begin{aligned} I' : \{\text{schemes over } E\}^{op} &\to \mathrm{Set} \\ U &\mapsto \{(g,v) \in \mathrm{GL}_n(U) \times \mathfrak{gl}_n(U) : \mathrm{Ad}_g(v) = v\}. \end{aligned}$$

Then $I = I' \times_{\mathrm{GL}_n \times \mathfrak{gl}_n} (G \times \mathfrak{g})$, so it suffices to prove the lemma for I replaced by I'. Write $\begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix} \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}^{-1} - \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$. Then I' is represented by

$$\operatorname{Spec} E[X_{11}, \dots, X_{nn}, Y_1, \cdots, Y_n][1/det(X_{ij})]/(f_1, \dots, f_n).$$

□

We shall identify I with the scheme that represents it. Note that there is a projection map $I \to \mathfrak{g}$, $(g,v) \mapsto v$. Also, I is a group scheme over $\mathfrak{g}$.

Lemma 2.5.9. *[Ryd] Let H be a group scheme locally of finite type over a scheme T. Then the function $f : T \to \mathbb{Z}$, $t \mapsto dim(H_t)$ is upper semi-continuous, i.e. for all $n \in \mathbb{Z}$, $\{t \in T : \dim(H_t) \le n\}$ is open.*

Proof. This is the proof in [Ryd]. We include it here as [Ryd] is subject to change.

Let $s : H \to T$ be the structure map. Let $T \xrightarrow{e} H$ be the unit map. By [Sta25, lemma 045X], f is the composite of the maps e and

$$\begin{aligned} H &\to \mathbb{Z} \\ h &\mapsto \dim_h(H_{s(h)}). \end{aligned} \tag{2.20}$$

As e is continuous and (2.20) is upper semi-continuous by [Sta25, lemma 02FZ], their composite is also upper semi-continuous. □

Applying this lemma to $I \to \mathfrak{g}$, we know that $\{v \in \mathfrak{g} : \dim I_v \le r\}$ is open in $\mathfrak{g}$.

Definition 2.5.10. *Define $\mathfrak{g}^{reg}$ to be the open subscheme of $\mathfrak{g}$ whose underlying subset is $\{v \in \mathfrak{g} : \dim I_v \le r\}$.*

In fact, for every $v \in \mathfrak{g}$, we have $\dim I_v = r$.

Note that $G \times \mathbb{G}_m$ acts on I, where G acts via $h \cdot (g, v) := (hgh^{-1}, \mathrm{Ad}_h(v))$ and $\mathbb{G}_m$ acts via $t \cdot (g, v) := (g, tv)$.

Let us recall the construction of the Kostant section, as in [CZ17, section 2.3]. For each simple root α_i, pick $f_i \in \mathfrak{g}_{-\alpha_i}(E) \setminus \{0\}$. Let $f = \sum f_i$. Complete f into an $\mathfrak{sl}_2$-triple e, f, h. Let $\mathfrak{g}^e$ be the centraliser of e in $\mathfrak{g}$.

Proposition 2.5.11 (Kostant)**.** *[Ngô06, Theorem 2.1] $f + \mathfrak{g}^e \subset \mathfrak{g}^{reg}$ and the restriction of $\mathfrak{g} \to \mathfrak{c} = g//G$ to $f + \mathfrak{g}^e$ gives an isomorphism*

$$f + \mathfrak{g}^e \xrightarrow{\sim} \mathfrak{c}. \tag{2.21}$$

The ***Kostant section*** $\mathrm{Kos} : \mathfrak{c} \to \mathfrak{g}$ is the composition of the inverse of (2.21) with the inclusion $f + \mathfrak{g}^e \hookrightarrow \mathfrak{g}$. Then Kos is a section of the natural map $\mathfrak{g} \to \mathfrak{c}$. Note that the Kostant section actually lands in $\mathfrak{g}^{reg}$.

We define $\mathfrak{g}^{reg}_{\mathcal{L}} := \widetilde{\mathcal{L}} \times^{\mathbb{G}_m} (\mathfrak{g}^{reg} \times_{\mathrm{Spec}\, E} X^{sch}_S)$ and $\widetilde{\mathcal{L}} = \underline{\mathrm{Isom}}(\mathcal{O}_X, \mathcal{L})$. A priori, this is a fppf sheaf on $\mathrm{Sch}_{X^{sch}_S}$, but this is representable by a scheme by fppf descent of quasi-affine schemes [Sta25, 0247], so we will also regard it as a scheme over X^{sch}_S.

Lemma 2.5.12. *Let U be a scheme over X^{sch}_S. Assume $v_1, v_2 \in \mathfrak{g}^{reg}_{\mathcal{L}}(U)$ have the same image in $\mathfrak{c}_{\mathcal{L}}(U)$. Then there is a fppf cover $W \to U$ such that $v_1|_W, v_2|_W$ are $G(W)$-conjugate.*

Proof. Let c be the common image of v_1, v_2 in $\mathfrak{c}_{\mathcal{L}}(U)$. Fix a Kostant section $\mathrm{Kos} : \mathfrak{c} \to \mathfrak{g}$. The morphism $G \times \mathfrak{c} \to \mathfrak{g}^{reg}$, given on Z-valued points (for $Z \in \mathrm{Sch}_E$) by $(g, c) \mapsto \mathrm{Ad}_g(\mathrm{Kos}(c))$, is a smooth and surjective morphism of schemes over $\mathfrak{c}$, as stated in [Ngô06, proof of Proposition 3.4]. Note that this map is $\mathbb{G}_m$-equivariant. Thus, $G \times \mathfrak{c}_{\mathcal{L}} \to \mathfrak{g}^{reg}_{\mathcal{L}}$ is a smooth and surjective morphism of schemes over $\mathfrak{c}_{\mathcal{L}}$. It follows that

$$G \times \mathfrak{c}_{\mathcal{L}} \to \mathfrak{g}^{reg}_{\mathcal{L}} \tag{2.22}$$

is surjective as morphism of fppf sheaves on $\mathrm{Sch}_{X^{sch}_S}$.

Thus, there is a fppf cover $W \to U$ and $(g_1, c_1) \in (G \times \mathfrak{c}_{\mathcal{L}})(W)$ such that

$$\mathrm{Ad}_{g_1}(\mathrm{Kos}(c_1)) = v_1.$$

As (2.22) is a morphism over $\mathfrak{c}_{\mathcal{L}}$, $c_1 = c|_W$. Hence $\mathrm{Ad}_{g_1}(\mathrm{Kos}(c)) = v_1$. Similarly, up to replacing W by a fppf cover, there is $g_2 \in G(W)$ such that $\mathrm{Ad}_{g_2}(\mathrm{Kos}(c)) = v_2$. Thus, $v_1|_W, v_2|_W$ are $G(W)$-conjugate. □

2.5.3 Regular centraliser

Let $I|_{\mathfrak{g}^{reg}} := I \times_{\mathfrak{g}} \mathfrak{g}^{reg}$. It turns out that the group scheme $I|_{\mathfrak{g}^{reg}} \to \mathfrak{g}^{reg}$ descends to a group scheme over $\mathfrak{c}$:

Lemma 2.5.13. *[Ngô10, Lemme 2.1.1] There is a unique smooth commutative group scheme J over $\mathfrak{c}$ equipped with a group scheme isomorphism over $\mathfrak{g}^{reg}$*

$$J \times_{\mathfrak{c}} \mathfrak{g}^{reg} \xrightarrow{\sim} I|_{\mathfrak{g}^{reg}}. \tag{2.23}$$

that is compatible with the $G \times \mathbb{G}_m$-action. This extends uniquely to a homomorphism over $\mathfrak{g}$

$$J \times_{\mathfrak{c}} \mathfrak{g} \to I. \tag{2.24}$$

that is compatible with the $G \times \mathbb{G}_m$-action.

Remark 2.5.14. [Ngô10, after Lemme 2.1.1] It we pick a Kostant section $Kos : \mathfrak{c} \to \mathfrak{g}$, then $J \cong I \times_{\mathfrak{g},Kos} \mathfrak{c}$.

As the projection map $I \to \mathfrak{g}$ is $G \times \mathbb{G}_m$-equivariant, we can base change both sides to X_S^{sch} and twist the map by the $\mathbb{G}_m$-torsor $\widetilde{\mathcal{L}} = \underline{Isom}(\mathcal{O}, \mathcal{L})$ to get a G-equivariant map

$$I_{\mathcal{L}} \to \mathfrak{g}_{\mathcal{L}}$$

of fppf sheaves on $\mathrm{Sch}_{X_S^{sch}}$, where $I_{\mathcal{L}} := I_{X_S^{sch}} \otimes_{\mathcal{O}_{X_S^{sch}}} \mathcal{L}$. Similarly, we can twist $J \to \mathfrak{c}$ to get

$$J_{\mathcal{L}} \to \mathfrak{c}_{\mathcal{L}}.$$

Note that for all $U \in \mathrm{Sch}_{X_S^{sch}}$, we have a canonical isomorphism

$$I_{\mathcal{L}}(U) \cong \{(g, v) \in G(U) \times \mathfrak{g}_{\mathcal{L}}(U) : \mathrm{Ad}_g(v) = v\}. \tag{2.25}$$

To see this, note that the right hand side defines a sheaf and there is an obvious map from $I_{\mathcal{L}}$ to it, which is an isomorphism locally (and hence globally). Similarly, if we fix a Kostant section Kos, then for all $U \in \mathrm{Sch}_{X_S^{sch}}$, we have a canonical isomorphism

$$J_{\mathcal{L}}(U) \cong \{(g, c) \in G(U) \times \mathfrak{c}_{\mathcal{L}}(U) : \mathrm{Ad}_g(\mathrm{Kos}(c)) = \mathrm{Kos}(c)\}. \tag{2.26}$$

2.5.4 Action of Picard prestack on Higgs bundles

Our aim in this subsection is to define the action of a certain Picard prestack $\mathcal{P}$ on Higgs_G.

We can twist (2.24) by $\mathcal{L}$ to get a group homomorphism over $\mathfrak{g}_{\mathcal{L}}$

$$J_{\mathcal{L}} \times_{\mathfrak{c}_{\mathcal{L}}} \mathfrak{g}_{\mathcal{L}} \to I_{\mathcal{L}} \tag{2.27}$$

that is compatible with the G-action. This induces a map

$$[(J_{\mathcal{L}} \times_{\mathfrak{c}_{\mathcal{L}}} \mathfrak{g}_{\mathcal{L}})/G] \cong J_{\mathcal{L}} \times_{\mathfrak{c}_{\mathcal{L}}} [\mathfrak{g}_{\mathcal{L}}/G] \to [I_{\mathcal{L}}/G]. \tag{2.28}$$

Note that $[I_{\mathcal{L}}/G] = \mathcal{I}_{[\mathfrak{g}_{\mathcal{L}}/G]}$, the inertia stack of $[\mathfrak{g}_{\mathcal{L}}/G]$. Indeed, it is clear from the definition and equation (2.25) that the inertia prestack of the quotient prestack[35]

[35] We mean the prestack $S \mapsto [\mathfrak{g}_{\mathcal{L}}(U)/G(U)]$.

$[\mathfrak{g}_{\mathcal{L}}/_{pre}\, G]$ is $[I_{\mathcal{L}}/_{pre}\, G]$. Recall that on prestacks, the operations of taking stackification and taking inertia prestack commutes [Sta25, Lemma 06NS]. It follows that $[I_{\mathcal{L}}/G]$ is the inertia stack of $[\mathfrak{g}_{\mathcal{L}}/G]$.

Thus, we can write (2.28) as

$$J_{\mathcal{L}} \times_{\mathfrak{c}_{\mathcal{L}}} [\mathfrak{g}_{\mathcal{L}}/G] \to \mathcal{I}_{[\mathfrak{g}_{\mathcal{L}}/G]} \tag{2.29}$$

which is a morphism over $[\mathfrak{g}_{\mathcal{L}}/G]$ satisfying the condition of Proposition 2.2.15 because (2.27) is a group homomorphism over $\mathfrak{g}_{\mathcal{L}}$. By Proposition 2.2.15, we get an action

$$[\mathfrak{c}_{\mathcal{L}}/J_{\mathcal{L}}] \times_{\mathfrak{c}_{\mathcal{L}}} [\mathfrak{g}_{\mathcal{L}}/G] \to [\mathfrak{g}_{\mathcal{L}}/G] \tag{2.30}$$

over $\mathfrak{c}_{\mathcal{L}}$.

Example 2.5.15. Take $G = \mathbb{G}_m$. Let $\mathrm{Tot}(\mathcal{L}) \to X_S^{sch}$ be the geometric vector bundles corresponding to $\mathcal{L}$. Then $\mathfrak{g} = \mathfrak{c} = \mathbb{A}^1_E$, $\mathfrak{g}_{\mathcal{L}} = \mathrm{Tot}(\mathcal{L})$ with trivial G-action, and $I_{\mathcal{L}} = J_{\mathcal{L}} = \mathbb{G}_m \times_E \mathrm{Tot}(\mathcal{L})$. By definition, (2.30) is the morphism $[\mathcal{L}/G] \times_{\mathcal{L}} [\mathcal{L}/G] \to [\mathcal{L}/G]$ given by taking the stackification of the map

$$\begin{aligned} &[\mathcal{L}/_{pre}\, G] \times_{\mathcal{L}} [\mathcal{L}/_{pre}\, G] \to [\mathcal{L}/_{pre}\, G] \\ &\qquad\qquad (v, v) \mapsto v \qquad\qquad\qquad \text{on objects} \\ &\qquad\qquad (g_1, g_2) \mapsto g_1 g_2 \qquad\qquad \text{on morphisms.} \end{aligned}$$

Thus, if we identify $[\mathcal{L}/G]$ as the stack taking a scheme X to the groupoid of pairs $(\mathcal{F}, v)$, where $\mathcal{F}$ is a line bundle on X and $v \in H^0(X, \mathcal{L})$ is a global section of the pullback of $\mathcal{L}$ to X, then (2.30) is the morphism

$$\begin{aligned} &[\mathcal{L}/G] \times_{\mathcal{L}} [\mathcal{L}/G] \to [\mathcal{L}/G] \\ &((\mathcal{F}_1, v), (\mathcal{F}_2, v)) \mapsto (\mathcal{F}_1 \otimes \mathcal{F}_2, v). \end{aligned}$$

Let us return to the general case.

Definition 2.5.16. *Define* $\mathcal{P} := \underline{\mathrm{Map}}(FF, [\mathfrak{c}_{\mathcal{L}}/J_{\mathcal{L}}])$, *i.e. the prestack*

$$\begin{aligned} &\mathrm{Perf}_S^{op} \to \mathrm{Grpd} \\ &\quad T \mapsto \left\{(a, Q) : a \in \mathfrak{c}_{\mathcal{L}}(X_T^{sch}),\ Q \textit{ is a } J_{\mathcal{L}} \times_{\mathfrak{c}_{\mathcal{L}}, a} X_T^{sch}\textit{-torsors on } X_T^{sch}\right\}. \end{aligned}$$

We also define $J_a := J_{\mathcal{L}} \times_{\mathfrak{c}_{\mathcal{L}}, a} X_T^{sch}$.

Remark 2.5.17. If $S = \mathrm{Spa}\, C$ for an algebraically closed perfectoid field C over $\mathbb{F}_q$, then for $T \to S$, we can by Lemma 2.7.1 identify J_a-bundles on X_T^{sch} with exact $\mathcal{O}_{\mathfrak{c}_{\mathcal{L}}}$-linear symmetric monoidal functors $\mathrm{Rep}_{\mathfrak{c}_{\mathcal{L}}}(J_{\mathcal{L}}) \to \mathrm{VB}(X_T^{sch}) \simeq \mathrm{VB}(X_T)$. Since $T \mapsto \mathrm{VB}(X_T)$ is a v-stack, $\mathcal{P}$ is also a v-stack by Lemma 2.2.89.

Taking $\underline{\mathrm{Map}}(FF, -)$ of (2.30), we get an action

$$\mathcal{P} \times_{\mathcal{A}_G} \mathrm{Higgs}_G \to \mathrm{Higgs}_G \tag{2.31}$$

of $\mathcal{P}$ on Higgs_G over $\mathcal{A}_G$. Let us denote the image of (Q, x) by $Q * x$.

Lemma 2.5.18. *Let $(\mathcal{E}, \phi) \in \mathrm{Higgs}_G(T)$, which corresponds to a map $X_T^{sch} \to [\mathfrak{g}_{\mathcal{L}}/G]$. Let a be the composite map $X_T^{sch} \to [\mathfrak{g}_{\mathcal{L}}/G] \to \mathfrak{c}$. Let $Z \to X_T^{sch}$ be a scheme over X_T^{sch}. If the image of $Z \to X_T^{sch} \to [\mathfrak{g}_{\mathcal{L}}/G]$ lies in $[\mathfrak{g}_{\mathcal{L}}^{reg}/G]$, then $J_a|_Z \xrightarrow{\sim} \underline{Aut}(\mathcal{E}, \phi)|_Z$.*

Proof. The base change of (2.29) along $[\mathfrak{g}_{\mathcal{L}}^{reg}/G] \to [\mathfrak{g}_{\mathcal{L}}/G]$, i.e. the map

$$J_{\mathcal{L}} \times_{\mathfrak{c}_{\mathcal{L}}} [\mathfrak{g}_{\mathcal{L}}^{reg}/G] \xrightarrow{\sim} \mathcal{I}_{[\mathfrak{g}_{\mathcal{L}}^{reg}/G]}, \tag{2.32}$$

is an isomorphism, because it is induced from isomorphism (2.23) (in the same way that (2.29) is induced from (2.24)). Base changing (2.32) along the map $Z \to X_T^{sch} \to [\mathfrak{g}_{\mathcal{L}}^{reg}/G]$ gives an isomorphism

$$J_a|_Z \xrightarrow{\sim} Z \times_{[\mathfrak{g}_{\mathcal{L}}^{reg}/G]} \mathcal{I}_{[\mathfrak{g}_{\mathcal{L}}^{reg}/G]} \cong \underline{Aut}(\mathcal{E}, \phi)|_Z.$$

□

We have the following analogue of [Ngô10, Proposition 2.2.1].

Proposition 2.5.19. *Let $\mathrm{Higgs}_G^{reg} := \underline{\mathrm{Map}}(FF, [\mathfrak{g}_{\mathcal{L}}^{reg}/G])$. Then*

$$\mathrm{Higgs}_G^{reg} \cong \mathcal{P}$$

as prestacks. This isomorphism is canonical if we choose a Kostant section.

Proof. We first prove that $[\mathfrak{g}_{\mathcal{L}}^{reg}/G]$ is a neutral gerbe over $\mathfrak{c}_{\mathcal{L}}$ as stacks on $(\mathrm{Sch}_{X_S^{sch}})_{fppf}$. Fix a Kostant section $\mathrm{Kos} : \mathfrak{c} \to \mathfrak{g}^{reg}$. Then the map $\mathfrak{c}_{\mathcal{L}} \to \mathfrak{g}_{\mathcal{L}}^{reg} \to [\mathfrak{g}_{\mathcal{L}}^{reg}/G]$ provides an object in $x = [\mathfrak{g}_{\mathcal{L}}^{reg}/G](\mathfrak{c}_{\mathcal{L}})$. Let $U \in \mathrm{Sch}_{X_S^{sch}}$. Suppose $(\mathcal{E}_1, \phi_1), (\mathcal{E}_2, \phi_2) \in [\mathfrak{g}_{\mathcal{L}}^{reg}/G](U)$ have the same image in $\mathfrak{c}_{\mathcal{L}}(U)$. We would like to show that locally in U, $(\mathcal{E}_1, \phi_1)$ and $(\mathcal{E}_2, \phi_2)$ are isomorphic. By passing to a fppf cover of U, we may assume $\mathcal{E}_1, \mathcal{E}_2$ are trivial G-bundles and $\mathcal{L} \cong \mathcal{O}$. Then $\phi_1, \phi_2 \in \mathfrak{g}(U)$ have the same image in $\mathfrak{c}(U)$. Since ϕ_1, ϕ_2 are regular, they are $G(U)$-conjugate after passing to a fppf cover V of U by Lemma 2.5.12. Thus, $(\mathcal{E}_1, \phi_1)|_V \cong (\mathcal{E}_2, \phi_2)|_V$. Hence, $[\mathfrak{g}_{\mathcal{L}}^{reg}/G]$ is a neutral gerbe over $\mathfrak{c}_{\mathcal{L}}$ by [Sta25, Lemma 06P1].

Let us calculate $\underline{\mathrm{Aut}}(x)$. Let $U \to \mathfrak{c}_{\mathcal{L}}$, which corresponds to $c \in \mathfrak{c}_{\mathcal{L}}(U)$. Then we can identify $x|_U$ with $(\mathcal{E}_0, \mathrm{Kos}(c)) \in [\mathfrak{g}_{\mathcal{L}}^{reg}/G](U)$, where $\mathcal{E}_0$ is the trivial G-bundle on U and $\mathrm{Kos}(c)$ is viewed as a global section of $\mathrm{Ad}(\mathcal{E}_0)_{\mathcal{L}}$. Thus,

$$\underline{\mathrm{Aut}}(x)(U) = \{g \in G(U) : \mathrm{Ad}_g(\mathrm{Kos}(c)) = \mathrm{Kos}(c)\}.$$

By (2.26), $\underline{\mathrm{Aut}}(x) = J_{\mathcal{L}}$.

By a general property of neutral gerbe, we have an isomorphism $[\mathfrak{c}_{\mathcal{L}}/J_{\mathcal{L}}] \xrightarrow{\sim} [\mathfrak{g}_{\mathcal{L}}^{reg}/G]$ given by the stackification of the map

$$\mathfrak{c}_{\mathcal{L}}(U)/J_{\mathcal{L}}(U) \to [\mathfrak{g}_{\mathcal{L}}^{reg}/G](U) \tag{2.33}$$
$$c \mapsto (\mathcal{E}_0, \mathrm{Kos}(c)) \tag{2.34}$$
$$j \mapsto j. \tag{2.35}$$

Applying $\underline{\mathrm{Map}}(FF, -)$ to $[\mathfrak{c}_{\mathcal{L}}/J_{\mathcal{L}}] \xrightarrow{\sim} [\mathfrak{g}_{\mathcal{L}}^{reg}/G]$, we obtain $\mathcal{P} \xrightarrow{\sim} \mathrm{Higgs}_G^{reg}$. □

Remark 2.5.20. By Remark 2.5.17 and Proposition 2.5.19, we know that if $S = \operatorname{Spa} C$, then Higgs_G^{reg} is a v-stack.

We have two actions of $\mathcal{P}$ on Higgs_G^{reg} over $\mathcal{A}_G$:

1. the one in (2.31) restricted[36] to Higgs_G^{reg}
2. the one obtained by identifying $\mathrm{Higgs}_G^{reg} \cong \mathcal{P}$ and using the contracted product on $\mathcal{P}$.

These two actions are equivalent.

Lemma 2.5.21. *The two actions above are equivalent. More precisely, the two* 1*-morphisms* $\mathcal{P} \times_{\mathcal{A}} \mathrm{Higgs}_G^{reg} \to \mathrm{Higgs}_G^{reg}$ *are isomorphic in the* 2*-category of prestacks on* Perf_S*.*

Proof. It suffices to show that the two actions of $[\mathfrak{c}_{\mathcal{L}}/J_{\mathcal{L}}]$ on $[\mathfrak{g}_{\mathcal{L}}^{reg}/G]$ over $\mathfrak{c}_{\mathcal{L}}$ are isomorphic. In other words, we want to show the two 1-morphisms $[\mathfrak{c}_{\mathcal{L}}/J_{\mathcal{L}}] \times_{\mathfrak{c}_{\mathcal{L}}} [\mathfrak{g}_{\mathcal{L}}^{reg}/G] \to [\mathfrak{g}_{\mathcal{L}}^{reg}/G]$ are isomorphic. Equivalently, by identifying $[\mathfrak{c}_{\mathcal{L}}/J]$ with $[\mathfrak{g}_{\mathcal{L}}^{reg}/G]$ via the map (2.33), we need to show the two 1-morphisms

$$[\mathfrak{c}_{\mathcal{L}}/J_{\mathcal{L}}] \times_{\mathfrak{c}_{\mathcal{L}}} [\mathfrak{c}_{\mathcal{L}}/J] \xrightarrow{\sim} [\mathfrak{c}_{\mathcal{L}}/J_{\mathcal{L}}] \times_{\mathfrak{c}_{\mathcal{L}}} [\mathfrak{g}_{\mathcal{L}}^{reg}/G] \rightrightarrows [\mathfrak{g}_{\mathcal{L}}^{reg}/G]$$

are isomorphic. By property of stackification [Sta25, Lemma 04W9], it suffices to see the two 1-morphisms $[\mathfrak{c}_{\mathcal{L}}/_{pre}\ J] \times_{\mathfrak{c}_{\mathcal{L}}} [\mathfrak{c}_{\mathcal{L}}/_{pre}\ J_{\mathcal{L}}] \to [\mathfrak{g}_{\mathcal{L}}^{reg}/G]$ are isomorphic. This can be checked by chasing through the definitions: on objects, they both send (c, c) to $(\mathcal{E}_0, \mathrm{Kos}(c))$; on morphisms, they both send (j_1, j_2) to $j_1 j_2$. $\square$

2.6 Affine Springer fiber

In this section, we define and study the B_{dR}-affine Springer fibres. One can view them as the local analogues of Hitchin fibers.

Recall that [FS21, section VI.1] if S is a perfectoid space over $\mathbb{F}_q$, then $Div_X^1(S)$ is the set of closed Cartier divisors of degree 1 of X_S. Locally in the analytic topology of S, any such Cartier divisor arises from an untilt of S over E. If S is affinoid and $S \to Div_X^1$ is a map whose corresponding Cartier divisor D_S arises from an untilt of S over E, then we define[37]

$$B^+_{\mathrm{Div}^1_X}(S)$$

to be the global sections of the completion of $\mathcal{O}_{X_S}$ along $\mathcal{I}_S$, where $I_S \subset \mathcal{O}_{X_S}$ is the ideal sheaf corresponding to D_S, and

$$B_{\mathrm{Div}^1_X}(S)$$

[36]The action preserves the regularity condition, because the map $J \times_{\mathfrak{c}} \mathfrak{g} \to I$ restricts to a map $J \times_{\mathfrak{c}} \mathfrak{g}^{reg} \to I|_{\mathfrak{g}^{reg}}$.

[37]Note that $B^+_{\mathrm{Div}^1_X}(S)$ depends on the map $S \to Div_X^1$ and not just S.

to be $B^+_{\mathrm{Div}^1_X}(S)[\frac{1}{I_S}]$ where I_S is the ideal of $B^+_{\mathrm{Div}^1_X}(S)$ determined by $\mathcal{I}_S$. If we pick an untilt $S^\sharp = \mathrm{Spa}(R, R^+)$ of S which gives rise to D_S, then we can identify $B^+_{\mathrm{Div}^1_X}(S)$ with $B^+_{dR}(R)$ and $B_{\mathrm{Div}^1_X}(S)$ with $B_{dR}(R)$.

In the remainder of this thesis, let $\mathbb{D}_T$ denote $\mathrm{Spec}\, B^+_{Div^1}(T)$ and $\mathbb{D}^*_T$ denote $\mathrm{Spec}\, B_{Div^1}(T)$. Note that this depends on the map $T \to Div^1_X$, not just on T. Also let $\mathcal{E}_0$ denote the trivial G-bundle.

Motivated by [Ngô10, Definition 3.2.2], we make the following definition.

Definition 2.6.1. *Let S be a perfectoid space over $\mathbb{F}_q$, $D_S \in Div^1_X(S)$, $\mathcal{L}$ be a line bundle on $\mathbb{D}_S = \mathrm{Spec}\, B_{Div^1}(S)$, and $\gamma \in \mathfrak{g}_{\mathcal{L}}(\mathbb{D}_T)$. Define the* ***($B_{dR}$-)affine Springer fibre*** *$\mathrm{Gr}^\gamma_{G,S/Div^1}$ as the v-sheaf $\mathrm{Perf}^{op}_S \to \mathrm{Set}$ given by sending (an affinoid perfectoid $T \to S$ for which the pullback D_T of D_S arises from an untilt of T over E)*[38] *to the isomorphism classes of triples*

$$(\mathcal{E}, \phi, \iota),$$

where

- *$\mathcal{E}$ is a G-bundle on $\mathbb{D}_T$,*
- *$\phi \in H^0(\mathbb{D}_T, \mathrm{Ad}(\mathcal{E})_{\mathcal{L}})$,*
- *$\iota : \mathcal{E}|_{\mathbb{D}^*_T} \xrightarrow{\sim} \mathcal{E}_0|_{\mathbb{D}^*_T}$ is an isomorphism whose induced map $\mathrm{Ad}(\mathcal{E})_{\mathcal{L}}|_{\mathbb{D}^*_T} \xrightarrow{\sim} \mathrm{Ad}(\mathcal{E}_0)_{\mathcal{L}}|_{\mathbb{D}^*_T}$ sends $\phi|_{\mathbb{D}^*_T}$ to γ.*

Proposition 2.6.2. *$\mathrm{Gr}^\gamma_{G,S/Div^1}$ is indeed a v-sheaf (on the basis {affinoid perfectoid $T \to S$ for which the pullback D_T of D_S arises from an untilt of T over E} of Perf_S, and hence extends uniquely to a v-sheaf on the entire Perf_S).*

Proof. The lemma follows from the fact that the B^+_{dR}-affine Grassmannian Gr_G is a v-sheaf and the following: If $T' \to T$ is a v-cover of affinoid perfectoid spaces over S for which D_T arises from an untilt and $\mathcal{V}$ is a vector bundle on $\mathrm{Spec}\, B^+_{\mathrm{Div}^1_X}(T)$, then

$$H^0(\mathrm{Spec}\, B^+_{\mathrm{Div}^1_X}(T), \mathcal{V}) \to H^0(\mathrm{Spec}\, B^+_{\mathrm{Div}^1_X}(T'), \mathcal{V}) \rightrightarrows H^0(\mathrm{Spec}\, B^+_{\mathrm{Div}^1_X}(T''), \mathcal{V})$$

is an equaliser diagram, where $T'' = T' \times_T T'$. This is proved in a similar way to [CY23, end of the proof of proposition 2.1]. Let us give more detail for their proof. Let $Q = \mathrm{Spec}\,\mathrm{Sym}\,\mathcal{V}^\vee$, so $\mathcal{V} = \mathrm{Hom}_{\mathrm{Spec}\, B^+_{\mathrm{Div}^1_X}(T)}(-, Q)$. We need to show

$$\mathrm{Hom}_{B^+_{\mathrm{Div}^1_X}(T)}(\mathcal{O}_Q(Q), B^+_{\mathrm{Div}^1_X}(T)) \to \mathrm{Hom}_{B^+_{\mathrm{Div}^1_X}(T)}(\mathcal{O}_Q(Q), B^+_{\mathrm{Div}^1_X}(T')) \rightrightarrows \mathrm{Hom}_{B^+_{\mathrm{Div}^1_X}(T)}(\mathcal{O}_Q(Q), B^+_{\mathrm{Div}^1_X}(T''))$$

[38] Recall that $\mathrm{Div}^1_X = \mathrm{Spd}\, E/\varphi^{\mathbb{Z}}$, where the quotient is taken in the category of sheaves on $\mathrm{Perf}_{\mathbb{F}_q}$ for the topology of open covers [FS21, after definition II.1.19]. It follows that {affinoid perfectoid $T \to S$ for which the pullback D_T of D_S arises from an untilt of T over E} contains every totally disconnected perfectoid space over S.

is an equaliser diagram. By the definition of limit, it suffices to show

$$B^+_{\mathrm{Div}^1_X}(T) \to B^+_{\mathrm{Div}^1_X}(T') \rightrightarrows B^+_{\mathrm{Div}^1_X}(T'') \tag{2.36}$$

is an equaliser diagram in the category of $B^+_{\mathrm{Div}^1_X}(T)$-algebras. Fix an untilt $T^\sharp = \mathrm{Spa}(R, R^+)$ that gives rise to D_T. This induces untilts $T'^\sharp = \mathrm{Spa}(R', R'^+)$ and $T''^\sharp = \mathrm{Spa}(R'', R''^+)$ of T' and T''. We need to see

$$B^+_{dR}(R) \to B^+_{dR}(R') \rightrightarrows B^+_{dR}(R'')$$

is an equaliser diagram. The kernel of $W_{\mathcal{O}_E}(R^{\circ,\flat}) \twoheadrightarrow R^\circ$ is generated by a primitive element ξ of degree 1. The images of ξ in $B^+_{dR}(R')$ and $B^+_{dR}(R'')$ are also primitive elements of degree 1. By Lemma 2.2.94 (with g taken to be the difference of the two maps $B^+_{dR}(R') \rightrightarrows B^+_{dR}(R'')$), it suffices to show

$$R \to R' \rightrightarrows R''$$

is an equaliser diagram. This is true because $\mathrm{Spa}(R', R'^+) \to \mathrm{Spa}(R, R^+)$ is a v-cover and $\mathcal{O}$ is a v-sheaf. □

Lemma 2.6.3. *Let R be a perfectoid E-algebra and $\mathcal{V}$ be a vector bundle on $\mathrm{Spec}\, B^+_{dR}(R)$. Then the restriction map $H^0(\mathrm{Spec}\, B^+_{dR}(R), \mathcal{V}) \to H^0(\mathrm{Spec}\, B_{dR}(R), \mathcal{V})$ is injective.*

Proof. Let $W = \mathrm{Spec}(\mathrm{Sym}\, \mathcal{V}^\vee)$ be the total space of $\mathcal{V}$. We need to show that

$$\mathrm{Hom}_{B^+_{dR}(R)}(\mathcal{O}_W(W), B^+_{dR}(R)) \to \mathrm{Hom}_{B^+_{dR}(R)}(\mathcal{O}_W(W), B_{dR}(R))$$

is injective. This is true because $B^+_{dR}(R) \to B_{dR}(R)$ is injective. □

Proposition 2.6.4. *The natural map*

$$\mathrm{Gr}^\gamma_{G,S/Div^1} \to \mathrm{Gr}_{G,S/\mathrm{Div}^1}$$

is a closed immersion. Also, $\mathrm{Gr}^\gamma_{G,S/Div^1}$ is a small v-sheaf (cf. Definition 2.2.73).

Proof. By Lemma 2.6.3, the natural map

$$\mathrm{Gr}^\gamma_{G,S/Div^1} \to \mathrm{Gr}_{G,S/\mathrm{Div}^1}$$

is an injection of v-sheaves. Since $\mathrm{Gr}_{G,S/\mathrm{Div}^1}$ is a small v-sheaf, the same is true for $\mathrm{Gr}^\gamma_{G,S/Div^1}$.

The proof that $\mathrm{Gr}^\gamma_{G,S/Div^1} \to \mathrm{Gr}_{G,S/\mathrm{Div}^1}$ is a closed immersion is similar to that of [SW20, Lemma 19.1.4]. Let $Z = \mathrm{Spa}(A, A^+)$ be a strictly totally disconnected perfectoid space. Let $Z \to \mathrm{Gr}_{G,S/\mathrm{Div}^1}$ be a map, which corresponds to a pair $(\mathcal{E}, \iota : \mathcal{E}|_{\mathbb{D}_Z^*} \to G|_{\mathbb{D}_Z^*}) \in \mathrm{Gr}_{G,S/\mathrm{Div}^1}(Z)$. Let $\mathcal{F} = \mathrm{Gr}^{\gamma}_{G,S/Div^1} \times_{\mathrm{Gr}_{G,S/\mathrm{Div}^1}} Z$, so we have a Cartesian diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & Z \arrow[d] \\ \mathrm{Gr}^{\gamma}_{G,S/Div^1} \arrow[r] & \mathrm{Gr}_{G,S/\mathrm{Div}^1} . \end{tikzcd}$$

We need to show that $\mathcal{F} \to Z$ is representable by a closed immersion of perfectoid spaces.

Since Z is strictly totally disconnected, every étale cover of it splits. By the same proof as Lemma 2.7.3, $\mathcal{E}$ and $\mathcal{L}$ are both trivial. Without loss of generality, $\mathcal{E} = G$ and $\mathcal{L} = \mathcal{O}$, so $(\mathcal{E}, \iota) = (G, \iota : G|_{\mathbb{D}_Z^*} \xrightarrow{g\cdot} G|_{\mathbb{D}_Z^*})$ for some $g \in G(\mathbb{D}_Z)$. Let $T = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space. Then

$$\mathcal{F}(T) = \left\{ f : T \to Z | f^*(\mathrm{Ad}_{g^{-1}}(\gamma)) \in Lie(G) \otimes_E B^+_{dR}(R^\sharp) \right\}.$$

Upon choosing basis for $Lie(G)$, $Lie(G) \otimes_E B^+_{dR}(R^\sharp) \cong B^+_{dR}(R^\sharp)^m$ functorial in $R^\sharp$. Without loss of generality, $m = 1$. We have thus reduced to showing that for any $\delta \in B_{dR}(A^\sharp)$, the subfunctor

$$T \mapsto \left\{ f : T \to Z | f^*(\delta) \in B^+_{dR}(R^\sharp) \right\}$$

of Z is representable by a closed immersion.

We have $\delta \in \xi^{-i} B^+_{dR}(A^\sharp)$ for some $i \geq 1$. Recall that a closed subspace of a strictly totally disconnected perfectoid space is also strictly totally disconnected. It suffices to show that the subfunctor

$$T \mapsto \left\{ f : T \to Z | f^*(\delta) \in \xi^{-i+1} B^+_{dR}(R^\sharp) \right\}$$

of Z is representable by a closed immersion $Z_i = \mathrm{Spa}(A_i, A_i^+) \subset Z$, because once we have shown this, we can replace Z by Z_i, pullback δ to $B_{dR}(A_i^\sharp)$, and repeat the process with i replaced by $i-1$ until we reach $i = 1$. Recall that $\xi^{-i} B^+_{dR}(R^\sharp)/\xi^{-i+1} B^+_{dR}(R^\sharp) \cong R^\sharp$. We have finally reduced to showing that for $a \in A^\sharp$, the subfunctor

$$T = \mathrm{Spa}(R, R^+) \mapsto \left\{ f : T \to Z | f^*(a) = 0 \in R^\sharp \right\}$$

of Z is representable by a closed immersion. This can be shown as in the proof of [SW20, Lemma 19.1.4]. $\square$

Proposition 2.6.5. *Consider the presheaf* $\mathrm{Perf}_S^{op} \to \mathrm{Set}$ *given by sending (an affinoid perfectoid* $T \to S$ *for which the pullback* D_T *of* D_S *arises from an untilt of* T *over* E*) to*

$$\{g \in G(B_{\mathrm{Div}^1_X}(T))/G(B^+_{\mathrm{Div}^1_X}(T)) : \mathrm{Ad}_{g^{-1}}(\gamma) \in \mathfrak{g}_{\mathcal{L}}(B^+_{\mathrm{Div}^1_X}(T))\}.$$

Its sheafification with respect to the analytic topology on Perf_S *is* $\mathrm{Gr}^{\gamma}_{G,S/Div^1}$.

Proof. Let us denote the presheaf in the statement as $\mathcal{F}$. We have a map of presheaves $\mathcal{F} \to \mathrm{Gr}^{\gamma}_{G,S/Div^1}$ given by

$$\begin{aligned} \mathcal{F}(T) &\to \mathrm{Gr}^{\gamma}_{G,S/Div^1}(T) \\ g &\mapsto (\mathcal{E}_0, \mathrm{Ad}_{g^{-1}}(\gamma), \mathcal{E}_0|_{\mathbb{D}^*_T} \xrightarrow{g\cdot} \mathcal{E}_0|_{\mathbb{D}^*_T}). \end{aligned}$$

This map is injective, so its sheafification is also injective. The surjectivity of the sheafified map follows from [CY23, Theorem 3.1]. □

Remark 2.6.6. Suppose that C is an algebraically closed perfectoid field. Then every line bundle on $\mathbb{D}_C$ is trivial by the same arguement as the proof of Lemma 2.7.3. By Proposition 2.6.5,

$$\mathrm{Gr}^{\gamma}_{G,S/Div^1}(\mathrm{Spa}\, C) \cong \left\{ g \in G(C((\xi)))/G(C[\![\xi]\!]) : \mathrm{Ad}_{g^{-1}}(\gamma) \in \mathfrak{g}(C[\![\xi]\!]) \right\},$$

as in the case for the classical affine Springer fiber. This expression also suggests that $\mathrm{Gr}^{\gamma}_{G,S/Div^1}$ may be related to orbital integrals.

2.7 Product formula

In this section, we shall show that modulo the actions of certain Picard stacks, there is a natural injective map of étale-stacks from the product of affine Springer fibers to the Hitchin fibre that induces an equivalence of categories on every geometric point. This is a geometric manifestation of the fact that global orbital integral factorises as the product of local orbital integrals. This also partly justifies the claim that affine Springer fibers are local analogues of Hitchin fibers.

Let E be a non-Archimedean local field with residue field $\mathbb{F}_q$, where q is a power of p. Let G be a split connected reductive group over E, B be a Borel subgroup over E, T be a split maximal torus in B. Assume $char E \nmid |W|$, where W is the Weyl group of G. Fix an algebraically closed perfectoid field C over $\mathbb{F}_q$. Let $\mathcal{L}$ be a line bundle on X^{sch}_C. Fix a Kostant section $\mathrm{Kos} : \mathfrak{c} \to \mathfrak{g}$.

Lemma 2.7.1. *[Wed23, Lemma A.25, Example A.28, Proposition A.30] Let S be a separated regular noetherian scheme of dimension ≤ 1 and $G \to S$ be an affine[39] flat group scheme of finite type. Then for any prestack $\mathcal{X}$ over S, the groupoids of cohomological G-bundles over $\mathcal{X}$ and of Tannakian G-bundles over $\mathcal{X}$ are equivalent.*

Here,

- *A cohomological G-bundle on $\mathcal{X}$ is an fpqc-sheaf on* $\mathrm{Aff}/\mathcal{X}$ *endowed with a $G \times_S \mathcal{X}$-action over $\mathcal{X}$ which is locally for the fpqc-topology on $\mathcal{X}$ isomorphic to $G_{\mathcal{X}}$.*
- *A Tannakian G-bundle on $\mathcal{X}$ is an exact $\mathcal{O}_S$-linear symmetric monidal functor* $\mathrm{Rep}_S(G) \to \mathrm{VB}(\mathcal{X})$, *where* $\mathrm{Rep}_S(G)$ *is the category of vector bundles $\mathcal{V}$ on T with a morphism of group schemes $G \to \underline{Aut}_{\mathcal{O}_S}(\mathcal{V})$ and* $\mathrm{VB}(\mathcal{X})$ *is the category of vector bundles on $\mathcal{X}$.*

[39] We mean the morphism $G \to S$ is affine. The schemes G and S need not be affine.

Definition 2.7.2. *Let $a \in \mathcal{A}_G(\operatorname{Spa} C)$. Let v be a closed point of X_C^{sch} and $a_v := a|_{\operatorname{Spec} B^+_{\operatorname{Div}^1_X}(\operatorname{Spa} C)}$.*

1. *Define $\operatorname{Gr}_{G,a,v}$ to be $\operatorname{Gr}^{\operatorname{Kos}(a_v)}_{G,\operatorname{Spa} C/\operatorname{Div}^1}$. Here, the map $\operatorname{Spa} C \to \operatorname{Div}^1_X$ is the map corresponding to v, a_v is the restriction of $a : X_C^{sch} \to \mathfrak{c}_{\mathcal{L}}$ to $\operatorname{Spec} B^+_{Div^1}(\operatorname{Spa} C)$. In other words, $\operatorname{Gr}_{G,a,v}$ is the v-sheaf $\operatorname{Perf}^{op}_{\operatorname{Spa} C} \to \operatorname{Set}$ given by sending (an affinoid perfectoid $T \to S$ for which the pullback D_T of D_S arises from an untilt of T over E) to the isomorphism classes of triples*

$$(\mathcal{E}, \phi, \iota),$$

where

- *$\mathcal{E}$ is a G-bundle on $\mathbb{D}_T$,*
- *$\phi \in H^0(\mathbb{D}_T, \operatorname{Ad}(\mathcal{E})_{\mathcal{L}})$,*
- *$\iota : \mathcal{E}|_{\mathbb{D}_T^*} \xrightarrow{\sim} \mathcal{E}_0|_{\mathbb{D}_T^*}$ is an isomorphism whose induced map $\operatorname{Ad}(\mathcal{E})_{\mathcal{L}}|_{\mathbb{D}_T^*} \xrightarrow{\sim} \operatorname{Ad}(\mathcal{E}_0)_{\mathcal{L}}|_{\mathbb{D}_T^*}$ sends $\phi|_{\mathbb{D}_T^*}$ to $\operatorname{Kos}(a_v)$.*

2. *Let $\operatorname{Gr}^{reg}_{G,a,v}$ be the sub-presheaf of $\operatorname{Gr}_{G,a,v}$ consisting of those $(\mathcal{E}, \phi, \iota)$ with $\phi \in H^0(\mathbb{D}_T, \mathcal{E} \times^G \mathfrak{g}^{reg}_{\mathcal{L}})$.*

3. *Define $\mathcal{P}^{loc}_{a,v} = \operatorname{Gr}_{J_{\mathcal{L}} \times_{\operatorname{Div}^1, a} \operatorname{Spa} C}$ to be the presheaf over $\operatorname{Perf}_{\operatorname{Spa} C}$ sending $T = \operatorname{Spa}(R, R^+)$ to*

$$\left\{ J_{\mathcal{L}} \times_{\mathfrak{c}_{\mathcal{L}}, a_v} \mathbb{D}_T\text{-torsors } Q \text{ on } \mathbb{D}_T \text{ with a trivialisation of } Q|_{\mathbb{D}_T^*} \right\} / \cong .$$

   We also define $J_{a_v} := J_{\mathcal{L}} \times_{\mathfrak{c}_{\mathcal{L}}, a_v} \mathbb{D}_T$.

Recall that $\mathbb{D}_T$ denotes $\operatorname{Spec} B^+_{Div^1}(T)$ and $\mathbb{D}_T^*$ denotes $\operatorname{Spec} B_{Div^1}(T)$, and this depends on the closed point v. In case of ambiguity, we will instead denote them by

$$\mathbb{D}_{T,v}$$

and

$$\mathbb{D}^*_{T,v}.$$

If we do not take isomorphism classes in the definition of $\mathcal{P}^{loc}_{a,v}$, we will get a prestack that is equivalent to $\mathcal{P}^{loc}_{a,v}$ by an argument similar to Lemma 2.6.3. We can by Lemma 2.7.1 identify J_{a_v}-bundles on $\mathbb{D}_T$ with exact $\mathcal{O}_{\mathfrak{c}_{\mathcal{L}}}$-linear symmetric monoidal functors

$$\operatorname{Rep}_{\mathfrak{c}_{\mathcal{L}}}(J_{\mathcal{L}}) \to \operatorname{VB}(\mathbb{D}_T).$$

Since $T \mapsto \operatorname{VB}(\mathbb{D}_T)$ is a v-stack by [SW20, Corollary 17.1.9], $\mathcal{P}^{loc}_{a,v}$ is a v-sheaf by Lemma 2.2.95.

Let $\phi_0 := \operatorname{Kos}(a_v)$.

Note that for all $(\mathcal{E}, \phi, \iota) \in \mathrm{Gr}_{G,v,a}(T)$, the image of $(\mathcal{E}, \phi)$ under the Hitchin fibration is a_v, because by Lemma 2.6.3, the image is determined by its restriction to $\mathbb{D}_T^*$ and we have an isomorphism $\iota : \mathcal{E}|_{\mathbb{D}_T^*} \xrightarrow{\sim} \mathcal{E}_0|_{\mathbb{D}_T^*}$ which sends $\phi|_{\mathbb{D}_T^*}$ to $\mathrm{Kos}(a_v)|_{\mathbb{D}_T^*}$.

Similar to Section 2.5.4, we can define an action

$$\mathcal{P}_{a_v}^{loc} \times_{\mathrm{Spa}\, C} \mathrm{Gr}_{G,a,v} \to \mathrm{Gr}_{G,a,v} \tag{2.37}$$

of $\mathcal{P}_{a,v}^{loc}$ on $\mathrm{Gr}_{G,a,v}$ over $\mathrm{Spa}\, C$. Indeed, if $(Q, t : Q|_{\mathbb{D}_T^*} \xrightarrow{\sim} J_{a_v}|_{\mathbb{D}_T^*}) \in \mathcal{P}_{a_v}^{loc}(T)$ and $(\mathcal{E}, \phi, \iota) \in \mathrm{Gr}_{G,a,v}(T)$, then as in Section 2.5.4, we get $Q * (\mathcal{E}, \phi)$. The trivialisations t and ι induce a trivialisation $Q * (\mathcal{E}, \phi)|_{\mathbb{D}_T^*} \xrightarrow{\sim} J_{a_v} * (\mathcal{E}_0, \phi_0)|_{\mathbb{D}_T^*} \cong (\mathcal{E}_0, \phi_0)$.

Lemma 2.7.3. *Let Q be a J_{a_v}-bundle on $\mathbb{D}_T$, where $T = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space over C. Then there exists an étale cover $T' \to T$ such that $Q|_{\mathbb{D}_{T'}}$ is a trivial $J_{a_v}|_{\mathbb{D}_{T'}}$-bundle.*

Proof. The usual proof for affine Grassmannian works here. Let us record it here to show that it works in this setup.

We have a map $B_{dR}^+(R^\sharp) \to R^\sharp$, i.e. a map $f : \mathrm{Spec}\, R^\sharp \to \mathbb{D}_T$. The base change f^*Q is a J_{a_v}-bundle on $\mathrm{Spec}\, R^\sharp$. As $J_{a_v} \to \mathbb{D}_T$ is smooth, by passing to an étale cover of $\mathrm{Spa}(R, R^+)$, we may assume f^*Q is the trivial J_{a_v}-bundle. We claim that Q is also trivial.

Since J_{a_v} is affine, Q is representable by an affine scheme $\mathfrak{Q}$ over $\mathbb{D}_T$ by descent of affine morphisms. As f^*Q is trivial, there is an element $s_1 \in \mathrm{Mor}_{\mathbb{D}_T}(\mathrm{Spec}\, R^\sharp, \mathfrak{Q})$. As $\mathfrak{Q} \to \mathbb{D}_T$ is smooth (because smoothness can be checked fpqc locally), there is a morphism s_2 such that the following commutes:

$$\begin{tikzcd}
\mathrm{Spec}\, R^\sharp \arrow[r, "s_1"] \arrow[d] & \mathfrak{Q} \arrow[d] \\
\mathrm{Spec}\, B_{dR}^+(R^\sharp)/\xi^2 \arrow[r] \arrow[ur, dotted, "s_2"] & \mathbb{D}_T
\end{tikzcd}$$

Similarly, there is a morphism s_3 such that the following commutes:

$$\begin{tikzcd}
\mathrm{Spec}\, B_{dR}^+(R^\sharp)/\xi^2 \arrow[r, "s_2"] \arrow[d] & \mathfrak{Q} \arrow[d] \\
\mathrm{Spec}\, B_{dR}^+(R^\sharp)/\xi^3 \arrow[r] \arrow[ur, dotted, "s_3"] & \mathbb{D}_T
\end{tikzcd}$$

We can continue this process and get compatible sequence of maps $s_1, s_2, \ldots$ i.e. compatible sequence of maps $H^0(\mathfrak{Q}, \mathcal{O}_{\mathfrak{Q}}) \to B_{dR}^+(R^\sharp)/\xi^n$ over $B_{dR}^+(R^\sharp)$. This gives a map $H^0(\mathfrak{Q}, \mathcal{O}_{\mathfrak{Q}}) \to \lim_n B_{dR}^+(R^\sharp)/\xi^n = B_{dR}^+(R^\sharp)$ over $B_{dR}^+(R^\sharp)$, i.e. an element of $\mathrm{Mor}_{\mathbb{D}_T}(\mathbb{D}_T, \mathfrak{Q})$, so Q is trivial. $\square$

Lemma 2.7.4. *Let $(\mathcal{E}, \phi, \iota) \in \mathrm{Gr}^{reg}_{G,v,a}(T)$. Then there exists an étale cover $T' \to T$ such that*[40]

$$(\mathcal{E}, \phi, \iota) = (\mathcal{E}_0, \phi_0, \mathcal{E}_0|_{\mathbb{D}^*_{T'}} \xrightarrow{j\cdot} \mathcal{E}_0|_{\mathbb{D}^*_{T'}})$$

in $\mathrm{Gr}^{reg}_{G,v,a}(T')$ for some $j \in J_{a_v}(\mathbb{D}^*_{T'})$. Moreover, j is unique up to right multiplication by elements of $J_{a_v}(\mathbb{D}_{T'})$.

Proof. Let $Q_{\mathcal{E},\phi} := \underline{\mathrm{Isom}}((\mathcal{E}_0, \phi_0), (\mathcal{E}, \phi))$. Let us show that $Q_{\mathcal{E},\phi}$ is a J_{a_v}-bundle on $\mathbb{D}_T$. There is a right J_{a_v}-action on $Q_{\mathcal{E},\phi}$ given by $f * j := f(j\cdot\ )$. There is a fppf cover U of $\mathbb{D}_T$ such that $\mathcal{E}|_U \cong \mathcal{E}_0|_U$. Let us fix such an isomorphism and identify $\mathcal{E}|_U \cong \mathcal{E}_0|_U$. Then for all $T \to U$, we can identify $Q_{\mathcal{E},\phi}(T)$ with

$$\{g \in G(T) : \mathrm{Ad}_g(\phi_0) = \phi\}.$$

We know $\phi|_U, \phi_0|_U$ have the same image under the map $\mathfrak{g}^{reg}_{\mathcal{L}}(U) \to \mathfrak{c}_{\mathcal{L}}(U)$. By regularity and Lemma 2.5.12, there is a fppf cover V of U and $g \in G(V)$ such that $\mathrm{Ad}_g(\phi_0|_V) = \phi|_V$. Then we have a J_{a_v}-equivariant isomorphism $J_{a_v}|_V \to Q_{\mathcal{E},\phi}|_V$ given by $j \mapsto gj$. Thus, $Q_{\mathcal{E},\phi}$ is a J_{a_v}-bundle on $\mathbb{D}_T$.

By Lemma 2.7.3, there exists an étale cover $T' \to T$ such that $Q_{\mathcal{E},\phi}|_{\mathbb{D}'_T}$ is a trivial J_{a_v}-bundle. In particular, it has a global section, i.e. $(\mathcal{E}_0, \phi_0) \cong (\mathcal{E}, \phi)$. The lemma is now clear. $\square$

Proposition 2.7.5. *Let $LJ_{a_v}/L^+J_{a_v} : \mathrm{Perf}^{op}_{\mathrm{Spa}\, C} \to \mathrm{Set}$ be the v-sheafification of the functor sending an affinoid perfectoid T to $J_{a_v}(\mathbb{D}_T)/J_{a_v}(\mathbb{D}^*_T)$. We have an isomorphism*

$$\begin{aligned} LJ_{a_v}/L^+J_{a_v} &\xrightarrow{\sim} \mathcal{P}^{loc}_{a,v} \\ j \in J_{a_v}(\mathbb{D}_T) &\mapsto (J_{a_v}|_{\mathbb{D}_T}, J_{a_v}|_{\mathbb{D}^*_T} \xrightarrow{j\cdot} J_{a_v}|_{\mathbb{D}^*_T}). \end{aligned}$$

We also have an isomorphism

$$\begin{aligned} LJ_{a_v}/L^+J_{a_v} &\xrightarrow{\sim} \mathrm{Gr}^{reg}_{G,a,v} \\ j \in J_{a_v}(\mathbb{D}_T) &\mapsto (\mathcal{E}_0|_{\mathbb{D}_T}, \phi_0|_{\mathbb{D}_T}, \mathcal{E}_0|_{\mathbb{D}^*_T} \xrightarrow{j\cdot} \mathcal{E}_0|_{\mathbb{D}^*_T}). \end{aligned} \tag{2.38}$$

In particular, $\mathrm{Gr}^{reg}_{G,a,v}$ is a v-sheaf.

Proof. The first isomorphism follows from Lemma 2.7.3. The injectivity of (2.38) is clear. By Lemma 2.7.4, this map is étale locally surjective as a map of presheaves. Since $\mathrm{Gr}^{reg}_{G,a,v}$ is a sub-presheaf of $\mathrm{Gr}_{G,a,v}$, we know that $\mathrm{Gr}^{reg}_{G,a,v}$ is a separated presheaf, i.e. for all v-cover $T' \to T$, the map $\mathrm{Gr}^{reg}_{G,a,v}(T) \to \mathrm{Gr}^{reg}_{G,a,v}(T')$ is injective. Combining these information, we can deduce that (2.38) is surjective as a map of presheaves. Hence it is an isomorphism. $\square$

[40] Recall that elements in $\mathrm{Gr}^{reg}_{G,v,a}(T')$ are *isomorphism classes*, so equality means equality of isomorphism classes, not of the representatives.

Remark 2.7.6. By Lemma 2.7.3, we also have

$$LJ_{a_v}/_{\text{ét}}\, L^+J_{a_v} \xrightarrow{\sim} \mathcal{P}^{loc}_{a,v},$$

where $LJ_{a_v}/_{\text{ét}}\, L^+J_{a_v}$ is the étale sheafification of $T \mapsto J_{a_v}(\mathbb{D}_T)/J_{a_v}(\mathbb{D}_T^*)$. It follows that $LJ_{a_v}/L^+J_{a_v} = LJ_{a_v}/_{\text{ét}}\, L^+J_{a_v}$.

It follows from Proposition 2.7.5 that $\mathrm{Gr}^{reg}_{G,a,v} \cong LJ_{a_v}/L^+J_{a_v} \cong \mathcal{P}^{loc}_{a,v}$. We can give a more direct description of this isomorphism.

Proposition 2.7.7. *We have an isomorphism*

$$F : \mathrm{Gr}^{reg}_{G,a,v} \xrightarrow{\sim} \mathcal{P}^{loc}_{a_v}$$

which on T-valued points (for affinoid $T \in \mathrm{Perf}_{\mathrm{Spa}\, C}$) are given by

$$(\mathcal{E}, \phi, \iota) \mapsto (Q_{\mathcal{E},\phi}, \mathrm{triv})$$

where $Q_{\mathcal{E},\phi} := \underline{\mathrm{Isom}}((\mathcal{E}_0, \phi_0), (\mathcal{E}, \phi))$ and triv *is the trivialisation on $Q_{\mathcal{E},\phi}|_{\mathbb{D}_T^*}$ given by*

$$\underline{\mathrm{Isom}}((\mathcal{E}_0, \phi_0), (\mathcal{E}, \phi))|_{\mathbb{D}_T^*} \xrightarrow{\iota_*} \underline{\mathrm{Isom}}((\mathcal{E}_0, \phi_0), (\mathcal{E}_0, \phi_0))|_{\mathbb{D}_T^*} \cong J_{a_v}|_{\mathbb{D}_T^*}.$$

Moreover, F agrees with the isomorphism induced by Proposition 2.7.5, i.e. the following commutes

$$\begin{tikzcd}
 & LJ_{a_v}/L^+J_{a_v} \arrow[dl] \arrow[dr] & \\
\mathrm{Gr}^{reg}_{G,a,v} \arrow[rr, "F"'] & & \mathcal{P}^{loc}_{a,v}
\end{tikzcd}$$

Proof. Note that $Q_{\mathcal{E},\phi}$ is indeed a J_{a_v}-bundle on $\mathbb{D}_T$ by the proof of Lemma 2.7.4. It is straightforward to verify the commutativity of the diagram. We deduce that F is an isomorphism from the fact that the other two morphisms of the diagram are isomorphisms. $\square$

We have three actions of $\mathcal{P}^{loc}_{a_v}$ on $\mathrm{Gr}^{reg}_{G,a,v}$ over $\mathrm{Spa}\, C$:

1. the one in (2.37) restricted[41] to $\mathrm{Gr}^{reg}_{G,a,v}$
2. the one obtained by identifying $\mathrm{Gr}^{reg}_{G,a,v} \cong \mathcal{P}^{loc}_{a_v}$ and using the contracted product on $\mathcal{P}^{loc}_{a_v}$.
3. the one obtained by identifying $\mathrm{Gr}^{reg}_{G,a,v} \cong LJ_{a_v}/L^+J_{a_v}$, $\mathcal{P}^{loc}_{a_v} \cong LJ_{a_v}/L^+J_{a_v}$ and using the multiplication map $LJ_{a_v}/L^+J_{a_v}$ given by $(j_1, j_2) \mapsto j_1 j_2$.

These three actions are equivalent.

[41]The action preserves the regularity condition, because the map $J \times_{\mathfrak{c}} \mathfrak{g} \to I$ restricts to a map $J \times_{\mathfrak{c}} \mathfrak{g}^{reg} \to I|_{\mathfrak{g}^{reg}}$.

Lemma 2.7.8. *The three actions above are equivalent over* $\mathrm{Spa}\, C$. *More precisely, the three* 1*-morphisms* $\mathcal{P}^{loc}_{a_v} \times_{\mathrm{Spa}\, C} \mathrm{Gr}^{reg}_{G,a,v} \to \mathrm{Gr}^{reg}_{G,a,v}$ *are isomorphic in the* 2*-category of v-stacks over* $\mathrm{Spa}\, C$.

Proof. It is straightforward to verify that the first two actions both agree with the third one, so they all agree. $\square$

Lemma 2.7.9. *Let*

$$LJ_{a_v} : \mathrm{Perf}^{op}_{\mathrm{Spa}\, C} \to \mathrm{Set}$$

be the functor sending an affinoid perfectoid T *(for which the pullback* D_T *of* $D_{\mathrm{Spa}\, C}$ *arises from an untilt of* T *over* E*) to* $J_{a_v}(\mathbb{D}^*_T)$. *Let*

$$L^+ J_{a_v} : \mathrm{Perf}^{op}_{\mathrm{Spa}\, C} \to \mathrm{Set}$$

be the functor sending an affinoid perfectoid T *(for which the pullback* D_T *of* $D_{\mathrm{Spa}\, C}$ *arises from an untilt of* T *over* E*) to* $J_{a_v}(\mathbb{D}_T)$. *Then both* LJ_{a_v} *and* $L^+J_{a_v}$ *define small v-sheaves on* $\mathrm{Perf}_{\mathrm{Spa}\, C}$.

Proof. That LJ_{a_v} and $L^+J_{a_v}$ are both v-sheaves can be proved by the same argument as Proposition 2.6.2. By the proof of [FS21, Proposition III.1.3], to show that LJ_{a_v} and $L^+J_{a_v}$ are small, it is enough to show that if $T_i = \mathrm{Spa}(R_i, R_i^+)$, $i \in I$ is an ω_1-cofiltered inverse system of affinoid perfectoid spaces with limit $T = \mathrm{Spa}(R, R^+)$, then

$$\mathrm{colim}_i (LJ_{a_v})(T_i) \xrightarrow{\sim} (LJ_{a_v})(T),\ \mathrm{colim}_i (L^+J_{a_v})(T_i) \xrightarrow{\sim} (L^+J_{a_v})(T).$$

By [Sch17, Proposition 6.5],

$$\mathrm{colim}_i\, R_i \xrightarrow{\sim} R,\ \mathrm{colim}_i\, R_i^+ \xrightarrow{\sim} R^+.$$

By the ω_1-cofilterness of I and the construction of the Witt ring by Witt polynomials,

$$\mathrm{colim}_i\, W_{\mathcal{O}_E}(R_i^+) \xrightarrow{\sim} W_{\mathcal{O}_E}(R^+).$$

Fix $i_0 \in I$, ϖ be a pseudouniformizer of R_0 and $\xi \in \ker(W_{\mathcal{O}_E}(R_0^+) \to R_0^{+\sharp})$ a primitive element of degree 1. Then $(\xi) = \ker(W_{\mathcal{O}_E}(R^+) \to R^{+\sharp})$. Then

$$\begin{aligned} B^+_{dR}(R^\sharp) &= \lim_{m \geq 1} W_{\mathcal{O}_E}(R^+)\left[\frac{1}{[\varpi]}\right]/(\xi^m) \\ &= \lim_{m\geq 1} \mathrm{colim}_i \left(W_{\mathcal{O}_E}(R_i^+)\left[\frac{1}{[\varpi]}\right]/(\xi^m)\right) \\ &= \mathrm{colim}_i \lim_{m \geq 1} \left(W_{\mathcal{O}_E}(R_i^+)\left[\frac{1}{[\varpi]}\right]/(\xi^m)\right) \\ &= \mathrm{colim}_i\, B^+_{dR}(R_i^\sharp), \end{aligned}$$

where the second to last equality holds because in the category of rings, ω_1-filtered colimits commute with sequential limits. Inverting ξ shows that we also have $B_{dR}(R^\sharp) = \mathrm{colim}_i\, B_{dR}(R_i^\sharp)$.

Note that $J_{\mathcal{L}} \times_{\mathfrak{c}_{\mathcal{L}}} \mathbb{D}_{\operatorname{Spa} C}$ is an affine scheme of finite presentation over $\mathbb{D}_{\operatorname{Spa} C}$. Let A be the ring of global section of $J_{\mathcal{L}} \times_{\mathfrak{c}_{\mathcal{L}}} \mathbb{D}_{\operatorname{Spa} C}$. Then

$$\begin{aligned}
(L^+ J_{a_v})(T) &= J_{a_v}(\mathbb{D}_T) \\
&= \operatorname{Hom}_{D_{\operatorname{Spa} C}}(\mathbb{D}_T, J_{\mathcal{L}} \times_{\mathfrak{c}_{\mathcal{L}}} \mathbb{D}_{\operatorname{Spa} C}) \\
&= \operatorname{Hom}_{B^+_{dR}(C^\sharp)}(A, B^+_{dR}(R^\sharp)) \\
&= \operatorname{colim}_i \operatorname{Hom}_{B^+_{dR}(C^\sharp)}(A, B^+_{dR}(R^\sharp_i)) \\
&= \operatorname{colim}_i (L^+ J_{a_v})(T_i)
\end{aligned}$$

where the second to last equality holds because A is of finite presentation over $B^+_{dR}(C^\sharp)$. Similarly, $(LJ_{a_v})(T) = \operatorname{colim}_i (LJ_{a_v})(T_i)$. □

Lemma 2.7.10. *If $\mathcal{F}$ is a small v-sheaf with a group action of a small group v-sheaf*[42] *$\mathcal{G}$, then $[\mathcal{F}/\mathcal{G}]$ is a small v-stack.*

Proof. As $\mathcal{F}$ is small, there is a perfectoid space X with a v-surjection $\alpha : X \twoheadrightarrow \mathcal{F}$. We have the following commutative diagram, where the two squares are Cartesian.

$$\begin{tikzcd}
 & & X \arrow[d, "\alpha"] \\
\mathcal{G} \times X \arrow[r, "\mathrm{id}\times\alpha"] \arrow[d, "pr_2"'] & \mathcal{G} \times \mathcal{F} \arrow[r, "act"] \arrow[d, "pr_2"'] & \mathcal{F} \arrow[d] \\
X \arrow[r, "\alpha"'] & \mathcal{F} \arrow[r] & {[\mathcal{F}/\mathcal{G}]}
\end{tikzcd}$$

It follows that $X \times_{[\mathcal{F}/\mathcal{G}]} X = (\mathcal{G} \times X) \times_{\mathcal{F}} X$. We would like to show that this is small.

We have an injection of v-sheaves $(\mathcal{G} \times X) \times_{\mathcal{F}} X \hookrightarrow \mathcal{G} \times X \times X$. As $\mathcal{G} \times X \times X$ is small, there is a perfectoid space Y with a v-surjection $Y \twoheadrightarrow \mathcal{G} \times X \times X$. We form the fibre product diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[r, hook] \arrow[d, two heads] & Y \arrow[d, two heads] \\
(\mathcal{G} \times X) \times_{\mathcal{F}} X \arrow[r, hook] & \mathcal{G} \times X \times X.
\end{tikzcd}$$

As $\mathcal{H}$ is sub-v-sheaf of Y and Y is a diamond, $\mathcal{H}$ is also a diamond by [SW20, Proposition 11.10]. In particular, $\mathcal{H}$ admits a v-surjection from a perfectoid space. Thus, the same holds for $(\mathcal{G} \times X) \times_{\mathcal{F}} X$. □

Corollary 2.7.11. *The v-stack $[\operatorname{Gr}_{G,a,v} / \mathcal{P}^{loc}_{a,v}]$ is small.*

Proof. By Proposition 2.6.4 and Lemma 2.7.9, $\operatorname{Gr}_{G,a,v}$ and $\mathcal{P}^{loc}_{a,v} = LJ_{a_v}/L^+J_{a_v}$ are both small v-sheaves. The result now follows from Lemma 2.7.10. □

[42] I.e. a group object in the category of v-sheaves.

Note that each root $\alpha : T \to \mathbb{G}_m$ of G induces a map $d\alpha : \mathfrak{t} \to \mathbb{A}^1_E$ on Lie algebras. Their product gives a map $\prod_{\alpha\in\Phi} d\alpha : \mathfrak{t} \to \mathbb{A}^1_E$, which is invariant under the Weyl group W, where Φ is the set of roots of G. Hence, it induces a morphism

$$disc := \prod_{\alpha\in\Phi} d\alpha : \mathfrak{c} \to \mathbb{A}^1_E.$$

Define $\mathfrak{c}^{rs}$ to be the affine scheme given by the locus where $disc$ does not vanish. Let

$$\mathfrak{c}^{rs}_{\mathcal{L}} = \widetilde{\mathcal{L}} \times^{\mathbb{G}_m} \mathfrak{c}^{rs}_{X^{sch}_C}.$$

The preimage of $\mathfrak{c}^{rs}$ in $\mathfrak{g} \to \mathfrak{c}$ is contained in $\mathfrak{g}^{reg}$. Thus, the preimage of $\mathfrak{c}^{rs}_{\mathcal{L}}$ in $\mathfrak{g}_{\mathcal{L}} \to \mathfrak{c}_{\mathcal{L}}$ is contained in $\mathfrak{g}^{reg}_{\mathcal{L}}$.

For $a \in \mathcal{A}_G(\operatorname{Spa} C)$, we define

$$\mathrm{Higgs}_a := \mathrm{Higgs}_G \times_{\mathcal{A}_G} \operatorname{Spa} C,$$

where the map $\operatorname{Spa} C \to \mathcal{A}_G$ is the one corresponding to a. We call it the ***Hitchin fibre***. We also define

$$\mathcal{P}_a := \mathcal{P} \times_{\mathcal{A}_G} \operatorname{Spa} C.$$

In analogy with [Ngô06, Definition 4.4], we make the following definition.

Definition 2.7.12. *We say that $a \in \mathcal{A}_G(\operatorname{Spa} C)$ is **generically regular semisimple** if a, as a map $X^{sch}_C \to \mathfrak{c}_{\mathcal{L}}$, sends the generic point η of X^{sch}_C to $\mathfrak{c}^{rs}_{\mathcal{L}}$.*

Lemma 2.7.13. *The group scheme $J_{\mathcal{L}}|_{\mathfrak{c}^{rs}_{\mathcal{L}}} \to \mathfrak{c}^{rs}_{\mathcal{L}}$ is a torus.*

Proof. By [Ngô10, Proposition 2.4.2] and [CZ17, Proposition 2.5.1], $J|_{\mathfrak{c}^{rs}} \to \mathfrak{c}^{rs}$ is a torus. Thus, the twist $J_{\mathcal{L}}|_{\mathfrak{c}^{rs}_{\mathcal{L}}} \to \mathfrak{c}^{rs}_{\mathcal{L}}$ is also a torus. □

Lemma 2.7.14. *Suppose that $a \in \mathcal{A}_G(\operatorname{Spa} C)$ is generically regular semisimple.*

1. *For every closed point $v \in |X^{sch}_C|$, $J_{a_v}|_{\mathbb{D}^*_{C,v}} = \mathbb{D}^*_{C,v} \times_{\mathfrak{c}_{\mathcal{L}}} J_{\mathcal{L}}$ is a torus over $\mathbb{D}^*_{C,v}$.*

2. *Let $T = \operatorname{Spa}(K, K^+) \in \mathrm{Perf}_{\operatorname{Spa} C}$, where K is algebraically closed. Then every J_{a_v}-torsor on $\mathbb{D}^*_{T,v}$ is trivial.*

Proof. The element $a \in \mathcal{A}_G(\operatorname{Spa} C)$ corresponds to a map $f : X^{sch}_C \to \mathfrak{c}_{\mathcal{L}}$. Since a is generically regular semisimple, $f(X^{sch}_C) \cap \mathfrak{c}^{rs}_{\mathcal{L}} \neq \varnothing$. As $\mathfrak{c}^{rs}_{\mathcal{L}}$ is open in $\mathfrak{c}_{\mathcal{L}}$, $f^{-1}(\mathfrak{c}^{rs}_{\mathcal{L}})$ is a non-empty open subset in X^{sch}_C. Since X^{sch}_C is Noetherian, 1-dimensional, and irreducible, the complement of $f^{-1}(\mathfrak{c}^{rs}_{\mathcal{L}})$ in X^{sch}_C is a finite set of closed points.

For every closed point $v \in |X^{sch}_C|$, the map

$$\mathbb{D}^*_{C,v} \to X^{sch}_C \tag{2.39}$$

factors through the open subscheme $f^{-1}(\mathfrak{c}^{rs}_{\mathcal{L}}) \subset X^{sch}_C$. To see this, recall that $\mathbb{D}_{C,v} = \operatorname{Spec} \widehat{\mathcal{O}_{X^{sch}_C,v}}$. The map

$$\mathbb{D}_{C,v} \to X^{sch}_C \tag{2.40}$$

is given by $\operatorname{Spec} \widehat{\mathcal{O}_{X_C^{sch},v}} \to \operatorname{Spec} \mathcal{O}_{X_C^{sch},v} \to X$. Pick an open neighborhood $U = \operatorname{Spec} B$ of $v \in X_C^{sch}$ such that $U \setminus \{v\} \subset f^{-1}(\mathfrak{c}_{\mathcal{L}}^{rs})$ and an element $\xi \in B$ such that $\{v\}$ is the vanishing locus of ξ. Then (2.40) factors through U. The map $\mathbb{D}_{C,v} \to U$ corresponds to the ring homomorphism $B \to \widehat{\mathcal{O}_{X_C^{sch},v}}$. Inverting ξ gives $B[\frac{1}{\xi}] \to \widehat{\mathcal{O}_{X_C^{sch},v}}[\frac{1}{\xi}]$. Taking Spec gives $\mathbb{D}_{C,v}^* \to U \setminus \{v\}$. Thus, (2.39) factors as $\mathbb{D}_{C,v}^* \to U \setminus \{v\} \hookrightarrow f^{-1}(\mathfrak{c}_{\mathcal{L}}^{rs}) \to X_C^{sch}$.

It follows from this and Lemma 2.7.13 that J_{a_v} is a torus over $\mathbb{D}_{C,v}$. Recall that

$$\mathbb{D}_{T,v}^* = \operatorname{Spec} B_{dR}(K^\sharp) \cong \operatorname{Spec} K^\sharp((\xi)).$$

By [Gre66, Section 2 Theorem 2], $K^\sharp((T))$ is a C_1 field. Since $K^\sharp((\xi))$ is a C_1-field and $J_{a_v}|_{\mathbb{D}_{T,v}^*}$ is a connected reductive group over $\operatorname{Spec} K^\sharp((T))$,

$$H^1_{\text{ét}}(\mathbb{D}_{T,v}^*, J_{a_v}) = H^1(K^\sharp((\xi)), J_{a_v}) = 0$$

[43]by [BS68, Section 8.6]. This is the desired result. ☐

Remark 2.7.15. Suppose that $a \in \mathcal{A}_G(\operatorname{Spa} C)$ is generically regular semisimple. The element a corresponds to a map $f : X_C^{sch} \to \mathfrak{c}_{\mathcal{L}}$. For each closed point $v \in |X_C^{sch}|$ that lies in $f^{-1}(\mathfrak{c}_{\mathcal{L}}^{rs})$, $J_{a_v} = \mathbb{D}_{C,v} \times_{\mathfrak{c}_{\mathcal{L}}} J_{\mathcal{L}}$ is a torus over $\mathbb{D}_{C,v}$ by Lemma 2.7.13 and the proof of Lemma 2.7.14. Also, $\operatorname{Gr}_{G,a,v}^{reg} = \operatorname{Gr}_{G,a,v}$, because if $(\mathcal{E}, \phi, \iota) \in \operatorname{Gr}_{G,a,v}(T)$, then the image of $(\mathcal{E}, \phi)$ under the Hitchin fibration is a_v, which lies in $\mathfrak{c}_{\mathcal{L}}^{rs}(\mathbb{D}_{T,v})$, implying that $\phi \in H^0(\mathbb{D}_T, \mathcal{E} \times^G \mathfrak{g}_{\mathcal{L}}^{reg})$. Thus, by Proposition 2.7.7,

$$\operatorname{Gr}_{G,a,v} \cong \mathcal{P}_{a_v}^{loc}. \tag{2.41}$$

In other words, in this case (which is the case for all but finitely many v), the affine Springer fiber is the affine Grassmannian of a torus. We can view this as an instance of abelianisation. Let us also note that (2.41) respects the $P_{a_v}^{loc}$-actions on both sides by Lemma 2.7.8. In particular, the quotient $[\operatorname{Gr}_{G,a,v} / \mathcal{P}_{a,v}^{loc}]$ is trivial.

Lemma 2.7.16. *Let T be a perfectoid space and $t \in H^0(T, \mathcal{O}_T)$. Suppose that $f^*(t) = 0$ for every algebraically closed perfectoid field K and every map $f : \operatorname{Spa}(K, K^\circ) \to T$. Then $t = 0$.*

Proof. Without loss of generality, we assume that $T = \operatorname{Spa}(A, A^+)$ is affinoid perfectoid. Let $x \in T$. Let $K(x)$ be the completed residue field at x. As T is analytic, the absolute value on $K(x)$ is non-trivial. Since $K(x)$ is a complete valued field, its absolute value extends uniquely to its algebraic closure $\overline{K(x)}$ and then to the completion $\widehat{\overline{K(x)}} =: C(x)$. By standard argument, $C(x)$ is algebraically closed. It follows that $C(x)$ is perfectoid. Since A is uniform, the map

$$A \to \prod_{x \in T} C(x)$$

[43]The latter cohomology is Galois cohomology.

is injective by a result of Berkovich [SW20, Theorem 5.2.1]. In other words, the map

$$H^0(T, \mathcal{O}_T) \to \prod_{x \in T} H^0(\mathrm{Spa}(C(x), C(x)^\circ), \mathcal{O})$$

is injective. The desired result follows. $\square$

Lemma 2.7.17. *Let S be a perfectoid space over $\mathbb{F}_q$, $\mathcal{E}$ be a vector bundle on X_S.*

1. *Let $t \in H^0(X_S, \mathcal{E})$. Suppose that for every algebraically closed perfectoid field K with open and integrally closed subring $K^+ \subset K^\circ$ and every map $\mathrm{Spa}(K, K^+) \to S$, we have $t|_{X_{\mathrm{Spa}(K,K^+)}} = 0$. Then $t = 0$.*
2. *Let $\alpha : \mathcal{E} \to \mathcal{E}$ be a morphism. Suppose that for every algebraically closed perfectoid field K with open and integrally closed subring $K^+ \subset K^\circ$ and every map $\mathrm{Spa}(K, K^+) \to S$, we have $\alpha|_{X_{\mathrm{Spa}(K,K^+)}} = \mathrm{id}$. Then $\alpha = \mathrm{id}$.*

By Proposition 2.2.82, the same also holds with X_S replaced by X_S^{sch}.

Proof. We first prove (1). Without loss of generality, assume that S is affinoid perfectoid. Since S is affinoid, there exist integers $n, m \geq 1$ such that $\mathcal{O}_{X_S}^m \twoheadrightarrow \mathcal{E}^\vee(n)$ by [FS21, Theorem II.2.6]. Thus, $\mathcal{E} \hookrightarrow \mathcal{O}_{X_S}^m(n)$. Hence, we may without loss of generality assume $\mathcal{E} = \mathcal{O}_{X_S}(n)$. Let $S^\sharp$ be an untilt of S over E. By [FS21, Proposition II.2.3], for every integer k, there is a short exact sequence

$$0 \to \mathcal{O}_{X_S}(k-1) \to \mathcal{O}_{X_S}(k) \to \mathcal{O}_{S^\sharp} \to 0.$$

Taking global section gives us an exact sequence

$$0 \to H^0(X_S, \mathcal{O}_{X_S}(k-1)) \xrightarrow{f_k} H^0(X_S, \mathcal{O}_{X_S}(k)) \xrightarrow{g_k} H^0(S^\sharp, \mathcal{O}_{S^\sharp})$$

which is compatible with change of S. By Lemma 2.7.16 and the assumption on t, $g_n(t) = 0$. Thus, t lies in $H^0(X_S, \mathcal{O}_{X_S}(n-1))$. Repeating this process, we reduce to the case where $t \in H^0(X_S, \mathcal{O}_{X_S})$.

By [FS21, Proposition II.2.5(ii)], $\mathcal{H}^0(\mathcal{O}) = \underline{E}$. Thus, t corresponds to a continuous map $T : |S| \to E$ whose composite with every map $|\mathrm{Spa}(K, K^+)| \to |S|$ is the zero map. Since the images of the maps $|\mathrm{Spa}(K, K^+)| \to |S|$ cover $|S|$ as (K, K^+) varies, T must be the zero map. Therefore $t = 0$.

For (2), note that α can be identified with a global section of the vector bundle $\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathcal{E}, \mathcal{E})$, so the result follows from (1). $\square$

We have the following analogue of [Ngô06, Lemme 4.5].

Lemma 2.7.18. *Suppose $a \in \mathcal{A}_G(\mathrm{Spa}\, C)$ is generically regular semisimple. Let S be an affinoid perfectoid space over $\mathrm{Spa}\, C$. Then the 2-quotient $[\mathrm{Higgs}_a(S)/\mathcal{P}_a(S)]$ is equivalent to a 1-groupoid.*

Proof. By Lemma 2.2.7, we need to show that for all $(\mathcal{E}, \phi) \in \mathrm{Higgs}_a(S)$, the natural map

$$F : \mathrm{Aut}(1_{J_a}) \to \mathrm{Aut}((\mathcal{E}, \phi))$$

is injective, where 1_{J_a} is the trivial J_a-torsor on X_S^{sch}.

We first prove it for $S = \mathrm{Spa}\, C$. We employ the same strategy as [Ngô06, Lemme 4.5] for this case. We shall focus on the omitted detail there. We have a commutative diagram

$$\begin{tikzcd} \mathrm{Aut}(1_{J_a}) \arrow[r] \arrow[d] & \mathrm{Aut}((\mathcal{E}, \phi)) \arrow[d] \\ \mathrm{Aut}(1_{J_a}|_\eta) \arrow[r] & \mathrm{Aut}((\mathcal{E}, \phi)|_\eta). \end{tikzcd}$$

Note that $\mathrm{Aut}(1_{J_a}) = H^0(X_S^{sch}, J_a)$ and $\mathrm{Aut}(1_{J_a}|_\eta) = H^0(\eta, J_a|_\eta)$. By Lemma 2.5.18 and the assumption that a is generically regular semisimple, the bottom map is an isomorphism. The desired result will follow if we show that the left vertical map is injective. Since $I_{\mathcal{L}} \to \mathfrak{g}_{\mathcal{L}}$ is an affine morphism, the same is true for its base change $\pi : J_a \to X_C^{sch}$. We would like to show that the restriction map

$$\mathrm{Mor}_{X_C^{sch}}(X_C^{sch}, J_a) \to \mathrm{Mor}_{X_C^{sch}}(\eta, J_a)$$

is injective. Let $f, g \in \mathrm{Mor}_{X_C^{sch}}(X_C^{sch}, J_a)$ with $f|_\eta = g|_\eta$. Let $U = \mathrm{Spec}\, B$ be a non-empty open subscheme of X_C^{sch}. We can view the restrictions of f, g to U as maps $f|_U, g|_U : U \to \pi^{-1}(U)$. We know these two maps agree after restriction along $\mathrm{Spec}\,\mathrm{Frac}(B) \to \mathrm{Spec}\, B$. As U and $f^{-1}(U)$ are both affine schemes, we know $f|_U = g|_U$. As U is arbitrary, $f = g$.

Let us return to the case of general affinoid perfectoid S over $\mathrm{Spa}\, C$. Let $f, g \in \mathrm{Aut}(1_{J_a})$ with $F(f) = F(g)$. Then $F(f)|_{X_K^{sch}} = F(g)|_{X_K^{sch}}$ for all $\mathrm{Spa}(K, K^+) \to S$ with K algebraically closed perfectoid field K and open integrally closed subring $K^+ \subset K^\circ$. The same proof as above proves that $f|_{X_K^{sch}} = g|_{X_K^{sch}}$.

By Lemma 2.7.1, we can identify J_a-bundles on X_S with exact $\mathcal{O}_{\mathfrak{c}_{\mathcal{L}}}$-linear symmetric monoidal functors $\mathrm{Rep}_{\mathfrak{c}_{\mathcal{L}}}(J_{\mathcal{L}}) \to \mathrm{VB}(X_S^{sch})$. Then the result follows from part (2) of Lemma 2.7.17. □

Due to this lemma, we will henceforth regard $[\mathrm{Higgs}_a(S)/\mathcal{P}_a(S)]$ as a 1-groupoid.

Definition 2.7.19. *Define*

$$[\mathrm{Higgs}_a/\mathcal{P}_a]_{\text{ét}}$$

to be the étale-stackification of $T \mapsto [\mathrm{Higgs}_a(T)/\mathcal{P}_a(T)]$. *Define* $[\mathrm{Gr}_{G,a,v}/\mathcal{P}_{a,v}^{loc}]_{\text{ét}}$ *to be the étale-stackification of* $S \mapsto [\mathrm{Gr}_{G,a,v}(T)/\mathcal{P}_{a,v}^{loc}(T)]$.

Remark 2.7.20. Let $T \to \mathrm{Spa}\, C$ be an affinoid perfectoid. Using the fact that $\mathcal{P}_a$ and Higgs_a are both v-stacks, one can easily verify that for any $x, y \in \mathrm{Higgs}_a(T)$, the presheaf of isomorphisms from x to y in $[\mathrm{Higgs}_a(T)/\mathcal{P}_a(T)]$ is a v-sheaf. It follows from [Sta25, Lemma 02ZN] that this is also the sheaf of isomorphisms from x to y in

$[\mathrm{Higgs}_a/\mathcal{P}_a](T)$. In particular, the isomorphisms from x to y in $[\mathrm{Higgs}_a(T)/\mathcal{P}_a(T)]$ and in $[\mathrm{Higgs}_a/\mathcal{P}_a]_{\text{ét}}(T)$ agree.

Remark 2.7.21. We take étale stackification to ensure that for every geometric point[44] $\mathrm{Spa}(K, K^+)$, the value at $\mathrm{Spa}(K, K^+)$ of a presheaf $\mathcal{F}$ agrees with the value at $\mathrm{Spa}(K, K^+)$ of the stackification of $\mathrm{Spa}(K, K^+)$. This is used in the proof below.

We have the following analogue of [Ngô06, Theoreme 4.6], which roughly says that modulo the actions of the Picard stacks, the Hitchin fiber factorises as a product of affine Springer fibers.

Theorem 2.7.22. *Suppose that $a \in \mathcal{A}_G(\mathrm{Spa}\, C)$ is generically regular semisimple. Then there is a canonical morphism*

$$F : \prod_{\textit{closed } v \in |X_C^{sch}|} [\mathrm{Gr}_{G,a,v} / \mathcal{P}_{a,v}^{loc}]_{\text{ét}} \to [\mathrm{Higgs}_a/\mathcal{P}_a]_{\text{ét}}$$

of étale-stacks over $\mathrm{Spa}\, C$ and all but finitely many terms on the left are trivial (i.e. isomorphic to $\mathrm{Spa}\, C$). This is injective, i.e. the map on the $\mathrm{Spa}(R, R^+)$-valued points is fully faithful for every $\mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathrm{Spa}\, C}$. Moreover, F is an equivalence on the $\mathrm{Spa}(K, K^+)$-valued points for every geometric point $\mathrm{Spa}(K, K^+) \in \mathrm{Perf}_{\mathrm{Spa}\, C}$.

Proof. The element $a \in \mathcal{A}_G(\mathrm{Spa}\, C)$ corresponds to a map $f : X_C^{sch} \to \mathfrak{c}_{\mathcal{L}}$. We have already seen in the proof of Lemma 2.7.14 that $U := f^{-1}(\mathfrak{c}_{\mathcal{L}}^{rs})$ is a non-empty open subset in X_C^{sch} whose complement in X_C^{sch} is a finite set of closed points. For each closed point $v \in |X_C^{sch}|$ that lies in U, the quotient $[\mathrm{Gr}_{G,a,v} / \mathcal{P}_{a,v}^{loc}]$ is trivial by Remark 2.7.15.

Using Propositions 2.5.19 and 2.7.7 and Lemma 2.7.14, we can prove the remaining assertion in the same way as [Ngô06, Theoreme 4.6]. Let us sketch the argument.

Let $T \to \mathrm{Spa}\, C$ be an affinoid perfectoid (for which the pullback D_T of $D_{\mathrm{Spa}\, C}$ arises from an untilt of T over E). Let $(\mathcal{E}_v, \phi_v, \iota_v)_v \in \prod_{\text{closed } v \in |X_C^{sch}|} \mathrm{Gr}_{G,a,v}(T)$. Let U' be the pullback of U along $X_T^{sch} \to X_C^{sch}$. By the Beauville-Laszlo lemma, we can glue $(\mathcal{E}_v)_{v \in |X_C^{sch}| \setminus |U|}$ with the trivial G-bundle on U' via the trivialisations (ι_v) to get a G-bundle $\mathcal{E}$ on X_T^{sch}. Note that global sections of $\mathrm{Ad}(\mathcal{E})_{\mathcal{L}}$ correspond to morphisms $\mathcal{O} \to \mathrm{Ad}(\mathcal{E})_{\mathcal{L}}$, so by the fully faithfulness part of the Beauville-Laszlo lemma, we can glue $(\phi_v)_{v \in |X_C^{sch}| \setminus |U|}$ with $\phi_0|_{U'} \in H^0(U', \mathrm{Ad}(\mathcal{E})_{\mathcal{L}})$ to get a section $\phi \in H^0(X_T^{sch}, \mathrm{Ad}(\mathcal{E})_{\mathcal{L}})$. Then $(\mathcal{E}, \phi) \in \mathrm{Higgs}_a(T)$. This defines a map $\prod_{\text{closed } v \in |X_C^{sch}|} \mathrm{Gr}_{G,a,v} \to \mathrm{Higgs}_a$, which is $\mathcal{P}_{a,v}^{loc}$ and $\mathcal{P}_a$ equivariant, so it induces a map

$$F_0 : \prod_{v \in |X_C^{sch}| \setminus |U|} [\mathrm{Gr}_{G,a,v} /_{pre}\, \mathcal{P}_{a,v}^{loc}] \to [\mathrm{Higgs}_a /_{pre}\, \mathcal{P}_a].$$

Taking étale stackification gives

$$F : \prod_{v \in |X_C^{sch}| \setminus |U|} [\mathrm{Gr}_{G,a,v} / \mathcal{P}_{a,v}^{loc}]_{\text{ét}} \to [\mathrm{Higgs}_a/\mathcal{P}_a]_{\text{ét}}.$$

[44] A geometric point is an adic space of the form $\mathrm{Spa}(K, K^+)$, where K is an algebraically closed perfectoid field and K^+ is an open and bounded valuation subring of K.

Since F is the stackification of F_0 and $[\mathrm{Higgs}_a(T)/\mathcal{P}_a(T)] \to [\mathrm{Higgs}_a/\mathcal{P}_a]_{\text{ét}}(T)$ is fully faithful for every affinoid $T \in \mathrm{Perf}_{\mathrm{Spa}\, C}$ by Remark 2.7.20, to show that F is injective, it is enough to prove that $F_0(T)$ is fully faithful for every T in a basis of Perf_C. Let $T \to \mathrm{Spa}\, C$ be an affinoid perfectoid (for which the pullback D_T of $D_{\mathrm{Spa}\, C}$ arises from an untilt of T over E). Let $m = (m_v), n = (n_v) \in \prod_{v \in |X_C^{sch}| \setminus |U|} \mathrm{Gr}_{G,a,v}(T)$. Then a morphism from $F(m)$ to $F(n)$ in $[\mathrm{Higgs}_a(T)/\mathcal{P}_a(T)]$ is given by a pair (P, θ), where $P \in \mathcal{P}_a(T)$ and $\theta : P * F(m) \xrightarrow{\sim} F(n)$ is an isomorphism. We know that $F(m)|_{U'} = (\mathcal{E}_0|_U, \phi_0|_U)$ and $F(n)|_{U'} = (\mathcal{E}_0|_U, \phi_0|_U)$. By Proposition 2.5.19 and Lemma 2.5.21, we necessarily have $P|_{U'} \cong J_a|_{U'}$. We also know that for each $v \in |X_C^{sch}| \setminus |U|$, $F(m)|_{\mathbb{D}_{T,v}} = m_v$ and $F(n)|_{\mathbb{D}_{T,v}} = n_v$. By the Beauville-Laszlo lemma, the data of such a pair (P, θ) is thus the same as $(P_v) \in \prod_{v \in |X_C^{sch}| \setminus |U|} \mathcal{P}_{a,v}^{loc}(T)$.

Finally, let $T = \mathrm{Spa}(K, K^+) \in \mathrm{Perf}_{\mathrm{Spa}\, C}$ be a geometric point. To show that

$$F(T) : \prod_{v \in |X_C^{sch}| \setminus |U|} [\mathrm{Gr}_{G,a,v}(T)/\mathcal{P}_{a,v}^{loc}(T)] \to [\mathrm{Higgs}_a(T)/\mathcal{P}_a(T)]$$

is an equivalence of categories, it remains to show that this is essentially surjective. Let $(\mathcal{E}, \phi) \in \mathrm{Higgs}_a(T)$. By Proposition 2.5.19 and Lemma 2.5.21, there is a J_a-torsor P' on U' and an isomorphism $P' * (\mathcal{E}, \phi)|_{U'} \xrightarrow{\iota} (\mathcal{E}_0, \phi_0)|_{U'}$. By Lemma 2.7.14, for every $v \in |X_C^{sch}| \setminus |U|$, $P'|_{\mathbb{D}_{T,v}^*}$ is a trivial J_{a_v}-torsor on $\mathbb{D}_{T,v}^*$. In particular, it can be extended to a J_{a_v}-torsor on $\mathbb{D}_{T,v}$. By the Beauville-Laszlo lemma, we can thus extend P' to a J_a-torsor on X_T^{sch}, which we still denote as P'. By replacing $(\mathcal{E}, \phi)$ by $P' * (\mathcal{E}, \phi)$, we can assume we have an isomorphism $(\mathcal{E}, \phi)|_{U'} \xrightarrow{\iota} (\mathcal{E}_0, \phi_0)|_{U'}$. Then $((\mathcal{E}, \phi)|_{\mathbb{D}_{T,v}}, \iota|_{\mathbb{D}_{T,v}^*})_v \in \prod_{v \in |X_C^{sch}| \setminus |U|} \mathrm{Gr}_{G,a,v}(T)$ and its image under F is isomorphic to $(\mathcal{E}, \phi)$. $\square$

2.8 Connections with number-theoretic objects

2.8.1 Adelic description of $\mathrm{Higgs}_G(C)$

Fix C an algebraically closed perfectoid field. For simplicity, we take $\mathcal{L} = \mathcal{O}_{X_C^{sch}}$ (in the definition of Higgs bundles). Throughout the section, we let η be the generic point of $X_C^{sch} = X_{\mathrm{Spa}\, C}^{sch}$ and

$$F = \mathcal{O}_{X_C^{sch}, \eta}$$

be the function field of X_C^{sch}. If $E = \mathbb{Q}_p$, we have $F = \mathrm{Frac}(B_{crys}^{\phi=1})$, where B_{crys} is the crystalline period ring [SW20, Theorem 13.5.3]. For each closed point v of X_C^{sch}, we let $\mathcal{O}_v := \widehat{\mathcal{O}_{X_C^{sch}, v}} = B^+_{\mathrm{Div}^1_X}(\mathrm{Spa}\, C)$ and $\mathbb{D}_v := \mathrm{Spec}\, \mathcal{O}_v$ $(= \mathbb{D}_{C,v}$ in previous sections), where the implicit map $\mathrm{Spa}\, C \to \mathrm{Div}^1_X$ corresponds to the untilt corresponding to v. Also, let $F_v := \mathrm{Frac}(\mathcal{O}_v) = B_{\mathrm{Div}^1_X}(\mathrm{Spa}\, C)$ and $\mathbb{D}_v^* := \mathrm{Spec}\, F_v (= \mathbb{D}_{C,v}^*$ in previous sections). Note that $F \subset F_v$. Following [Ngô06, Section 1], we shall give in this section an adelic description of $\mathrm{Higgs}_G(C)$.

Proposition 2.8.1. *Every G-torsor on* $\operatorname{Spec} F$ *can be extended to $X^{sch}_{\operatorname{Spa} C}$. In particular, we have a surjection $B(G) \twoheadrightarrow H^1(F,G)$, where $B(G)$ is the Kottwitz set of G and the cohomology group on the right is the Galois cohomology.*

Proof. For every untilt $C^\sharp$ of C, $B_{dR}(C^\sharp) \cong C^\sharp((t))$ is a C_1-field by [Gre66, Section 2 Theorem 2], so every G-bundle on it is trivial by [BS68, Section 8.6]. The proof in [Ngô06, Lemma 1.1] then gives the result. $\square$

Remark 2.8.2. In the case where E is a finite extension of $\mathbb{Q}_p$, Fargues conjectured in [Far20, Conjecture 3.10] that F is a C_1-field. If this is true, then $H^1(F,G) = 1$.

For each $c \in H^1(F,G)$, we can by Proposition 2.8.1 pick a G-torsor $\mathcal{E}_c$ on X^{sch}_C such that the cohomology class corresponding to $(\mathcal{E}_c)_\eta$ is c. Let

$$G_c := \underline{\operatorname{Aut}}(\mathcal{E}_c)$$

be the sheaf on $(Sch/X^{sch}_C)_{\mathrm{fppf}}$ given by $(U \to X^{sch}_C) \mapsto \operatorname{Aut}_G(\mathcal{E}_c|_U)$. Let

$$\mathfrak{g}_c$$

be the sheaf on $(Sch/X^{sch}_C)_{\mathrm{fppf}}$ given by $\operatorname{Spec} R \mapsto \ker(G_c(R[\epsilon]) \to G_c(R))$, where $\epsilon^2 = 0$.

Lemma 2.8.3. *If $\gamma \in \mathfrak{g}_c(F)$, then there is an open subscheme $U \subset X^{sch}_C$ such that γ extends to an element of $\mathfrak{g}_c(U)$, and this extension is unique for any fixed U. The same holds for $\gamma' \in G_c(F)$.*

The idea is very simple: objects finitely presented over $\operatorname{Frac}(B)$ are defined over $B[\frac{1}{b}]$ for some b.

Proof. Let us prove the lemma for $\mathfrak{g}_c(F)$. The proof for $G_c(F)$ is similar.

Pick a non-empty affine open subscheme $U' = \operatorname{Spec} B \subset X^{sch}_C$. Then $\mathcal{E}_c|_{U'}$ is representable by a flat, finitely presented affine scheme $U = \operatorname{Spec} A$ over U'. Then $\gamma : \mathcal{E}_c|_{\operatorname{Spec} F[\epsilon]} \to \mathcal{E}_c|_{\operatorname{Spec} F[\epsilon]}$ corresponds to a $\operatorname{Frac}(B)[\epsilon]$-algebra homomorphism $A \otimes_B \operatorname{Frac}(B)[\epsilon] \to A \otimes_B \operatorname{Frac}(B)[\epsilon]$. As A is finitely presented over B, there exists $b \in B \setminus \{0\}$ such that the ring homomorphism arises from a $B[\frac{1}{b}][\epsilon]$-algebra homomorphism

$$A \otimes_B B[\frac{1}{b}][\epsilon] \to A \otimes_B B[\frac{1}{b}][\epsilon]. \tag{2.42}$$

This is G-equivariant. By the B-flatness of A, $A \otimes_B B[\frac{1}{b}] \hookrightarrow A \otimes_B \operatorname{Frac} B$ is injective. It follows that (2.42) is G-equivariant, is in $\mathfrak{g}_c(\operatorname{Spec} B[\frac{1}{b}]) = \ker(\operatorname{Aut}_G(\mathcal{E}_c|_{B[\frac{1}{b}][\epsilon]}) \to \operatorname{Aut}_G(\mathcal{E}_c|_{B[\frac{1}{b}]}))$, and is the unique extension of γ to $\operatorname{Spec} B[\frac{1}{b}]$. The uniqueness of extension to general U (such that γ extends to an element of $\mathfrak{g}_c(U)$) can be deduced by applying the above to an affine open $U' \subset U$ and take $b = 1$. $\square$

Lemma 2.8.4. *We have* $\operatorname{Ad}(\mathcal{E}_c) \cong \mathfrak{g}_c$.

Proof. Let U be an affine scheme over X_C^{sch}. Pick an affine fppf cover $V \to U$ and an isomorphism $\mathcal{E}_c|_V \xrightarrow{\alpha} G|_V$. Let $\pi_1, \pi_2 : V \times_U V \to V$ be the projections to the left and right factors respectively. Let $t = \pi_2^*(\alpha) \circ \pi_1^*(\alpha)$. To give an element of $\mathrm{Ad}(\mathcal{E}_c)(U)$ is the same as giving an element of $\mathrm{eq}(\mathrm{Ad}(\mathcal{E}_c)(V) \rightrightarrows \mathrm{Ad}(\mathcal{E}_c)(V \times_U V))$, i.e. an element $v \in \mathfrak{g}(V)$ such that $\mathrm{Ad}_t(\pi_1^* v) = \pi_2^* v$. The same description holds for $\mathfrak{g}_c(U)$. This gives an isomorphism

$$\mathrm{Ad}(\mathcal{E}_c)(U) \cong \mathfrak{g}_c(U) \tag{2.43}$$

By tracing through the definitions, we see that this is given by descending

$$\begin{aligned}\mathrm{Ad}(\mathcal{E}_c)(V) = (\mathcal{E}_c \times^G \mathfrak{g})(V) &\xrightarrow{\sim} \mathfrak{g}_c(V)\\ (e, v) &\mapsto \alpha^{-1} \circ \mathrm{Ad}_{\alpha(e)}(v) \circ \alpha,\end{aligned}$$

where $\mathfrak{g}_c(V)$ is viewed as an element of $\mathrm{Aut}_G(\mathcal{E}_c|_{\mathcal{O}(V)[\epsilon]})$ and $\mathrm{Ad}_{\alpha(e)}(v)$ is viewed as an element of $\mathrm{Aut}_G(G|_{\mathcal{O}(V)[\epsilon]})$. It is clear from this expression that (2.43) is independent of the choice of α and is compatible with passing to affine fppf cover $V' \to V$ of V, so (2.43) is also independent of the choice of V. As U varies, we get an isomorphism of sheaves. □

For each closed point v of X_C^{sch}, we pick a trivialisation

$$\iota_v : \mathcal{E}_c|_{\mathbb{D}_v} \xrightarrow{\sim} G|_{\mathbb{D}_v}$$

which exists by the proof of Lemma 2.7.3. Then ι_v induces isomorphisms $G_c|_{\mathbb{D}_v} \xrightarrow{\sim} G|_{\mathbb{D}_v}$ and $\mathfrak{g}_c|_{\mathbb{D}_v} \xrightarrow{\sim} \mathfrak{g}|_{\mathbb{D}_v}$. We shall use these isomorphisms to view elements of $G_c(F_v)$ as elements of $G(F_v)$ and to view elements of $\mathfrak{g}_c(F_v)$ as elements of $\mathfrak{g}(F_v)$.

Note that $\mathfrak{g}_c(F) \subset \mathfrak{g}_c(F_v)$. To see this, note that by definition $\mathfrak{g}_c(F) \subset \mathrm{Aut}(\mathcal{E}_c|_{\mathrm{Spec}\, F[\epsilon]})$. We know that $\mathcal{E}_c|_{\mathrm{Spec}\, F[\epsilon]}$ is representable by an affine flat $\mathrm{Spec}\, F[\epsilon]$-scheme $\mathrm{Spec}\, A$. In particular, $A \hookrightarrow A \otimes_{F[\epsilon]} F_v[\epsilon]$, so $\mathrm{Aut}(A) \hookrightarrow \mathrm{Aut}(A \otimes_{F[\epsilon]} F_v[\epsilon])$. This implies $\mathfrak{g}_c(F) \subset \mathfrak{g}_c(F_v)$. In particular, elements of $\mathfrak{g}_c(F)$ can also be viewed as elements of $\mathfrak{g}(F_v)$.

We have the following analogue of [Ngô06, Corollaire 1.3], whose proof is omitted there. The relevant [Ngô06, Proposition 1.2] is proved there however.

Proposition 2.8.5. *We have an equivalence of categories between* $\mathrm{Higgs}_G(C)$ *and the groupoid of triples*

$$(c, \gamma, (g_v)_{\textit{closed } v \in |X_C^{sch}|})$$

where

- $c \in H^1(F, G)$
- $\gamma \in \mathfrak{g}_c(F)$
- $g_v \in G(F_v)/G(\mathcal{O}_v)$ *such that*[45] $\mathrm{Ad}_{g_v^{-1}}(\gamma) \in \mathfrak{g}(\mathcal{O}_v)$ *for every* v *and* g_v *is trivial for all but finitely many* v.

The morphisms are given by

$$\begin{aligned}&\mathrm{Mor}((c,\gamma,(g_v)),(c',\gamma',(g'_v)))\\ =&\begin{cases}\{h\in G_c(F): hg_v\in g'_vG(\mathcal{O}_v)\ \textit{for}^{46}\textit{all } v,\gamma'=\mathrm{Ad}_h(\gamma)\} & \textit{if } c=c'\\ \varnothing & \textit{if } c\neq c'.\end{cases}\end{aligned}$$

Proof. The proof is similar to how one expresses $\mathrm{Bun}_G(\mathbb{F}_q)$ as a double adelic quotient. Let us denote the groupoid of triples in the proposition as $\mathcal{T}$.

We first construct a functor $f:\mathcal{T}\to \mathrm{Higgs}_G(C)$. Let $(c,\gamma,(g_v)_{\text{closed } v\in|X_C^{sch}|})\in\mathcal{T}$. By Lemma 2.8.3, we can pick a non-empty open subscheme U of X_C^{sch} such that g_v is trivial for all $v\in U$ and γ extends to an element of $\mathfrak{g}_c(U)$, which we still denote as γ. By the Beauville-Laszlo lemma, we can glue $\mathcal{E}_c|_U$ with the trivial G-bundles $G|_{\mathbb{D}_v}$ for $v\in|X_C^{sch}|\setminus|U|$ via the isomorphism

$$(\mathcal{E}_c|_U)|_{\mathbb{D}_v^*}\xrightarrow{g_v^{-1}\iota_v}(G|_{\mathbb{D}_v})|_{\mathbb{D}_v^*}$$

to get a G-bundle $\mathcal{E}$ on X_C^{sch}. This is well-defined because changing the representatives of $g_v\in G(F_v)/G(\mathcal{O}_v)$ gives another G-bundle on X_C^{sch} with a canonical isomorphism to $\mathcal{E}$. (Alternatively, one can pick a representative for each object of $\mathcal{T}$.) By the Beauville-Laszlo lemma, the sections $\gamma\in\mathfrak{g}_c(U)\cong\mathrm{Ad}(\mathcal{E}_c)(U)$ and $\mathrm{Ad}_{g_v^{-1}}(\iota_v(\gamma))\in\mathfrak{g}(\mathbb{D}_v)=\mathrm{Ad}(G|_{\mathbb{D}_v})(\mathbb{D}_v)$ glue to give a section $\phi\in H^0(X_C^{sch},\mathrm{Ad}(\mathcal{E}))$. Up to canonical isomorphism, $(\mathcal{E},\phi)$ is independent of the choice of U. This defines f on objects.

Let $h\in\mathrm{Mor}((c,\gamma,(g_v)),(c',\gamma',(g'_v)))$. Then $c=c'$. Fix representative for g_v,g'_v in $G(\mathcal{O}_v)$, which we still denote as g_v,g'_v. By Lemma 2.8.3, we can pick a non-empty open subscheme W of X_C^{sch} such that g_v,g'_v are trivial for all $v\in W$ and γ,γ' extend to elements of $\mathfrak{g}_c(W)$ and h extends to an element in $G_c(W)$. By the Beauville-Laszlo lemma, the isomorphisms $h:(\mathcal{E}_c|_W,\gamma)\to(\mathcal{E}_c|_W,\gamma')$ and $g_v'^{-1}hg_v:(G|_{\mathbb{D}_v},\mathrm{Ad}_{g_v^{-1}}(\gamma))\to(G|_{\mathbb{D}_v},\mathrm{Ad}_{g_v'^{-1}}(\gamma'))$ glues to an isomorphism $(\mathcal{E},\phi)\to(\mathcal{E}',\phi')$. This defines the functor F.

It is easy to verify using the Beauville-Laszlo lemma that F is fully faithful. It remains to show that F is essentially surjective. Let $(\mathcal{E},\phi)\in\mathrm{Higgs}_G(C)$. Pick an isomorphism $t_\eta:\mathcal{E}_\eta\xrightarrow{\sim}(\mathcal{E}_c)_\eta$ for some $c\in H^1(F,G)$. This extends to $t_\eta:\mathcal{E}|_U\xrightarrow{\sim}\mathcal{E}_c|_U$ for some non-empty open subscheme $U\subset X_C^{sch}$. For each $v\in|X_C^{sch}|\setminus|U|$, pick an isomorphism $t_v:\mathcal{E}|_{\mathbb{D}_v}\xrightarrow{\sim}G|_{\mathbb{D}_v}$. Then

$$G|_{\mathbb{D}_v^*}\xrightarrow{t_v^{-1}}\mathcal{E}|_{\mathbb{D}_v^*}\xrightarrow{t_\eta}(\mathcal{E}_c)|_{\mathbb{D}_v^*}\xrightarrow{\iota_v}G|_{\mathbb{D}_v^*}$$

corresponds to an element $g_v\in G(F_v)$. For $v\in U$, set $g_v=1$. Via t_η and Lemma 2.8.4, we get an isomorphism $\mathrm{Ad}(\mathcal{E})|_U\xrightarrow{\sim}\mathrm{Ad}(\mathcal{E}_c)|_U\cong\mathfrak{g}_c|_U$. The image of $\phi|_U$ gives an element in $\mathfrak{g}_c(U)$, which can be restricted to η to give $\gamma\in\mathfrak{g}_c(F)$. Then $F(c,\gamma,(g_v))\cong(\mathcal{E},\phi)$. $\square$

[45]More precisely, $\mathrm{Ad}_{g_v^{-1}}(\iota_v\gamma\iota_v^{-1})\in\mathfrak{g}(\mathcal{O}_v)$.

[46]More precisely, $\iota_vh\iota_v^{-1}g_v\in g'_vG(\mathcal{O}_v)$.

Remark 2.8.6. In analogy with [Ngô06, after Corollaire 1.3], this proposition roughly suggests that the cardinality of $\mathrm{Higgs}_G(C)$ as a groupoid is given by the orbital integrals

$$\sum_{c\in H^1(F,G)} \sum_{\gamma\in\mathfrak{g}_c(F)/\mathrm{conj.}} \int_{G_\gamma(F)\backslash G(\mathbb{A}_F)} 1_{\prod_v \mathfrak{g}(\mathcal{O}_v)}(\mathrm{Ad}_{g^{-1}}\gamma)dg$$

where

- $\mathbb{A}_F = \prod_v'(F_v, \mathcal{O}_v) \cong \prod_v'(C_v^\sharp((t)), C_v^\sharp[\![t]\!])$ is a restricted direct product and $C_v^\sharp$ is the untilt of C corresponding to v
- G_γ is the centraliser of γ in G
- $1_{\prod_v \mathfrak{g}(\mathcal{O}_v)}$ is the indicator function of $\prod_v \mathfrak{g}(\mathcal{O}_v)$.

However, as stated, this equation does not make much sense, since no Haar measures on the groups have been specified; even after fixing such measures, both the cardinality of $\mathrm{Higgs}_G(C)$ and the sums involved should be infinite. We do not attempt here to make the connection with orbital integrals fully rigorous.

Appendix A

Étale topology on sousperfectoid spaces

In Remark 2.5.6, we showed that $\mathrm{Higgs}_G = \underline{\mathrm{Map}}(FF, [\mathfrak{g}_{\mathcal{L}}/G])$, where $[\mathfrak{g}_{\mathcal{L}}/G]$ is viewed as an fppf stack on the category of schemes over X_S^{sch}. We also use various other fppf stacks in the main text. In the appendix, we show that, we can instead use adic spaces. More precisely, we use stacks on the big étale site of sousperfectoid spaces. The reason for this choice of site is that, we would like to consider G-bundles on the adic Fargues-Fontaine curve X_S. We thus want to have a site that includes X_S as an object. To get the correct notion of G-bundles on adic spaces, which are by definition étale locally trivial, we use the étale topology. We restrict to sousperfectoid spaces, rather than using all adic spaces, because the notion of G-bundle is particularly well-behaved on them [SW20, Theorem 19.5.2] and the fibre product of sousperfectoid spaces $X \times_Z Y$ is still a sousperfectoid space if $X \to Z$ is étale by Lemma A.0.1.

One advantage of using this formalism is that, we do not have to restrict to affinoid S in defining v-stack whose S-valued points involve the Fargues-Fontaine curve.

Let sPerfd be the category of sousperfectoid spaces, which is introduced by Hansen-Kedlaya, cf. [SW20, section 6.3], [HK25, section 7]. In this section, we will study some basic properties of the étale topology on sPerfd.

Recall that étale maps of (pre)adic spaces are defined in [KL15, Definition 8.2.16].

Lemma A.0.1. *Let $f : X \to Z, g : Y \to Z$ be two maps of sousperfectoid spaces with f étale. Then the fibre product $X \times_Z Y$ exists in the category of adic spaces and is a sousperfectoid space. Moreover, we get a surjection of topological spaces*

$$|X \times_Z Y| \twoheadrightarrow |X| \times_{|Z|} |Y|.$$

Proof. By the usual argument, this reduces to the case where X, Y, Z are affinoid sousperfectoid and $f = f_1 \circ f_2 \circ f_3$, where f_1, f_3 are open immersions and f_2 is finite étale. Then the claim that $X \times_Z Y$ exists and is sousperfectoid follows from [SW20, Proposition 6.3.3].

For the last claim, one can use [KL15, Lemma 9.1.3]. Alternatively, one can easily reduce the claim to the case where X, Y, Z are affinoids and then apply [Joh]. $\square$

By this lemma and [KL15, Lemma 8.2.17(c)], we get the étale site on sPerfd:

Definition A.0.2. *We denote by*

- $\mathrm{sPerfd}_{\text{ét}}$ *the big étale site, whose objects are sousperfectoid spaces and coverings are étale covers;*
- $\mathrm{sPerfd}_{X,\text{ét}}$ *the big étale site over a sousperfectoid space X, whose objects are sousperfectoid spaces over X and coverings are étale covers;*
- $X_{\text{ét}}$ *the small étale site over a sousperfectoid space X, whose objects are sousperfectoid spaces étale over X and coverings are étale covers.*

By an étale sheaf on X, we mean a sheaf on $X_{\text{ét}}$.

We shall establish some basic properties of these sites, following the strategy of [Sch17, sections 8, 9].

Lemma A.0.3. *The presheaves $\mathcal{O}, \mathcal{O}^+ : \mathrm{sPerfd}_{\text{ét}}^{op} \to \mathrm{Set}$ which send X to $\mathcal{O}_X(X), \mathcal{O}_X^+(X)$ are sheaves.*

Proof. They are sheaves for the analytic topology, so it is enough to check the sheaf axioms for an étale cover of the form $\{Y \to X\}$ with X affinoid sousperfectoid. This follows from [KL15, Theorem 8.2.22(c)]. $\square$

Proposition A.0.4. *The site $\mathrm{sPerfd}_{\text{ét}}$ is subcanonical, i.e. the presheaf represented by any $X \in \mathrm{sPerfd}$ is a sheaf.*

Proof. Let $X \in \mathrm{sPerfd}$ and $\mathcal{F}$ be the presheaf represented by X. Let $q : Z \twoheadrightarrow Y$ be an étale cover in sPerfd. We want to show that

$$\mathcal{F}(Y) \xrightarrow{\sim} eq(\mathcal{F}(Z) \rightrightarrows \mathcal{F}(Z \times_Y Z)).$$

By [KL15, Lemma 8.2.17(b)], every étale morphism is open.[1] Being surjective and open, the map $|q| : |Z| \to |Y|$ on the underlying topological spaces is a quotient map.

Let $f \in eq(\mathcal{F}(Z) \rightrightarrows \mathcal{F}(Z \times_Y Z))$. By surjectivity of $|Z \times_Y Z| \to |Z| \times_{|Y|} |Z|$, the two composite maps $|Z| \times_{|Y|} |Z| \rightrightarrows |Z| \xrightarrow{|f|} |X|$ agree. As $|q|$ is a quotient map, $|f|$ factors as

$$\begin{tikzcd} {|Z|} \arrow[rr, "|f|"] \arrow[dr, "|q|"'] & & {|X|} \\ & {|Y|} \arrow[ur] & \end{tikzcd}.$$

The problem can then be considered locally on X, so we can assume X is affinoid. The corollary now follows from the fact that $\mathcal{O}$, $\mathcal{O}^+$ are étale sheaves by Lemma A.0.3. $\square$

[1]The lemma there directly implies that every étale morphism whose domain is quasi-compact is an open mapping. We can deduce the same result for a general étale morphism by writing the domain as a union of quasi-compact open subsets.

Remark A.0.5. In [SW20, Appendix to Lecture XIX], a cohomological G-torsor $\mathcal{E}$ on a sousperfectoid space X is viewed as a sheaf on the small étale site of X. By [SW20, Theorem 19.5.2], $\mathcal{E}$ is the étale sheaf of sections of a geometric G-torsor $\mathcal{P}$, which is sousperfectoid. By Proposition A.0.4, $\mathrm{Hom}_X(\ ,\mathcal{P})$ is a sheaf on the big étale site of X, whose restriction to $X_{\text{ét}}$ is $\mathcal{E}$. It follows that we can equivalently regard a cohomological G-torsor as a sheaf on the big étale site.

We have étale descent of vector bundles on sousperfectoid spaces:

Lemma A.0.6. *The prestack sending $X \in \mathrm{sPerfd}$ to the groupoid of vector bundles on X is an étale stack.*

Proof. This is a stack for the analytic topology, so it is enough to prove the axiom of stack for étale covering between affinoid sousperfectoid spaces. The lemma thus follows from the fact that for $X = \mathrm{Spa}(A, A^+)$ affinoid sousperfectoid, we have $\{$finite locally free $\mathcal{O}_X$-modules$\} \simeq \{$finite projective A-modules$\} \simeq \{$finite locally free $\mathcal{O}_{X_{\text{ét}}}$-modules$\}$ by [KL15, Theorem 8.22]. □

Lemma A.0.7. *Let $X \in \mathrm{sPerfd}$. We have equivalences of symmetric monoidal categories*

$$\{\textit{vector bundles on } \mathrm{sPerfd}_{X,\text{ét}}\} \xrightarrow{\sim} \{\textit{vector bundles on } X_{\text{ét}}\} \xrightarrow{\sim} \{\textit{vector bundles on } X\}.$$

By a vector bundle on $\mathrm{sPerfd}_{X,\text{ét}}$, we mean a sheaf $\mathcal{F}$ of $\mathcal{O}_{\mathrm{sPerfd}_{X,\text{ét}}}$-module on $\mathrm{sPerfd}_{X,\text{ét}}$ which is locally finite free, i.e. there is an étale cover $\{Y_i \to X\}$ of X by sousperfectoid Y_i such that $\mathcal{F}|_{\mathrm{sPerfd}_{,Y_i\text{ét}}} \cong \mathcal{O}^{n_i}_{\mathrm{sPerfd}_{,Y_i\text{ét}}}$ for some $n_i \in \mathbb{Z}_{\geq 0}$.

Proof. Given a vector bundle on $\mathrm{sPerfd}_{X,\text{ét}}$, we can restrict it to get a vector bundle on $X_{\text{ét}}$. This, together with the obvious maps on morphisms, define a functor which we call a.

Given a vector bundle on $X_{\text{ét}}$, we can restrict it to get a sheaf on X, which is a vector bundle by Lemma A.0.6. We can also restrict morphisms. We call this functor b.

Given a vector bundle $\mathcal{V}$ on X, we can form a sousperfectoid geometric vector bundle $\mathrm{Tot}(\mathcal{V}) \to X$ such that $\mathrm{Hom}_X(\ ,\mathrm{Tot}(\mathcal{V})) = \mathcal{V}$ as sheaves on X. Then $\mathrm{Hom}_X(\ ,\mathrm{Tot}(\mathcal{V}))$, regarded as a functor on $\mathrm{sPerfd}^{op}_{X,\text{ét}}$, is a sheaf on $\mathrm{sPerfd}_{X,\text{ét}}$ as $\mathrm{sPerfd}_{X,\text{ét}}$ is subcanonical by Proposition A.0.4. To endow it with an $\mathcal{O}_{\mathrm{sPerfd}_{X,\text{ét}}}$-module structure, we can work locally and assume $\mathcal{V} \cong \mathcal{O}^n_X$, so $\mathrm{Tot}(\mathcal{V}) = \mathbb{A}^n_X$ and $\mathrm{Hom}_X(\ ,\mathrm{Tot}(\mathcal{V})) = \mathcal{O}^n_{\mathrm{sPerfd}_{X,\text{ét}}}$, which has a canonical $\mathcal{O}_{\mathrm{sPerfd}_{X,\text{ét}}}$-module structure. It is easy to see this is a vector bundle on $\mathrm{sPerfd}_{X,\text{ét}}$. This assignment, together with the obvious maps on morphisms, define a functor which we call c.

It is not hard to show that $c\circ b\circ a, a\circ c\circ b, b\circ a\circ c$ are all equivalent to the identities. To enhance a (and similarly for b) to a symmetric monoidal functor, we can use the universal property of sheafification to define for vector bundles $\mathcal{E}_1, \mathcal{E}_2$ on $\mathrm{sPerfd}_{X,\text{ét}}$ a map $a(\mathcal{E}_1) \otimes a(\mathcal{E}_2) \to a(\mathcal{E}_1 \otimes E_2)$, which can be easily shown to be symmetric monoidal by working locally. As c is a quasi-inverse of $b\circ a$, it can also be enhanced to a symmetric monoidal functor. □

Let us state the following easy lemma, which relates two ways to twist a sheaf by a line bundle.

Lemma A.0.8. *Let X be a sousperfectoid space and $\mathcal{L}$ be a line bundle. Let $\widetilde{\mathcal{L}} = \underline{Isom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{L})$ be the associated $\mathbb{G}_m$-bundle, where $\mathbb{G}_m$ acts via $f * a := af$. Suppose $\mathcal{F}$ is an $\mathcal{O}_X$-module. Then we have a canonical isomorphism of $\mathcal{O}_X$-modules*

$$\widetilde{\mathcal{L}} \times^{\mathbb{G}_m} \mathcal{F} \xrightarrow{\sim} \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}$$

where the $\mathcal{O}_X$-module structure on $\widetilde{\mathcal{L}} \times^{\mathbb{G}_m} \mathcal{F}$ is induced[2] *from $\mathcal{F}$.*

Proof. Define the map as the sheafification of $(h, v) \mapsto h(1) \otimes v$. To check that this is an isomorphism, we can work locally on X, so we may without loss of generality assume $\mathcal{L} = \mathcal{O}_X$. Then $\widetilde{\mathcal{L}} \times^{\mathbb{G}_m} \mathcal{F} = \mathcal{F} \xrightarrow{\sim} \mathcal{F} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{F}$. □

As an application, we show that we can replace some uses of fppf stacks by étale stacks on sousperfectoid spaces. There are some clash of notations with the main text. We hope this is not too confusing.

We just give one example. We can view G as an étale sheaf on sPerfd_{X_S}, whose $\mathrm{Spa}(R, R^+)$-valued points are $G(R)$. To see that this is indeed an étale sheaf, note that G is smooth over $\operatorname{Spec} E$, so its analytification[3] over $\operatorname{Spa} E$ exists and is a sousperfectoid adic space. Thus, the presheaf it represents is an étale sheaf on $\mathrm{sPerfd}_{\operatorname{Spa} E}$ (and hence also on sPerfd_{X_S}) by Proposition A.0.4.

Similarly, we can view $\mathfrak{g}$ as an étale sheaf on sPerfd_{X_S}, whose $\mathrm{Spa}(R, R^+)$-valued points are $\mathfrak{g}(R)$. We can then form $\mathfrak{g}_{\mathcal{L}} := \mathfrak{g} \otimes_{\mathcal{O}_{X_S}} \mathcal{L}$, which is a vector bundle on X_S. We can define the étale stack $[\mathfrak{g}_{\mathcal{L}}/G]$ on sPerfd_{X_S}. Then $\mathrm{Higgs}_G = \underline{\mathrm{Map}}(FF, [\mathfrak{g}_{\mathcal{L}}/G])$, where $\underline{\mathrm{Map}}(FF, [\mathfrak{g}_{\mathcal{L}}/G])$ denotes the prestack on Perf_S sending T to the groupoid $[\mathfrak{g}_{\mathcal{L}}/G](X_T)$. Note that by 2-Yoneda lemma, this is also the prestack sending T to the groupoid of morphisms $X_T \to [\mathfrak{g}_{\mathcal{L}}/G]$ of stacks on sPerfd_{X_S}.

[2] The addition on $\widetilde{\mathcal{L}} \times^{\mathbb{G}_m} \mathcal{F}$ is obtained by sheafifying $(a, x) + (a, y) = (a, x + y)$.

[3] See [Zav24, Section 6] for the notion of analytification, which is a process that turns schemes into adic spaces.

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