Cohomology of Diagram Algebras and Generalised Tate Cohomology of Hopf Algebras

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Abstract

This thesis is split into two parts. In Part I, we will prove that the composition product on generalised Tate cohomology of finite-dimensional Hopf algebras over a field is equivalent to the cup product. We then deduce that there exists an A_∞-structure on generalised Tate cohomology of such Hopf algebras using Kadeishvili's Theorem. We also show, by way of example, that Steenrod operations on the ordinary cohomology of such Hopf algebras detect the underlying coalgebra structure.

In Part II, which is based on joint work of the author and Daniel Graves, we prove cohomological versions of homological results in recent literature and apply these to families of diagram algebras. We apply frameworks developed by Boyde and Sroka for studying homology of such algebras to families of diagram algebras not yet appearing in homological algebra literature. These algebras include Tanabe algebras, walled Brauer algebras, dilute Temperley–Lieb algebras and blob algebras. We deduce a number of (co)homological stability results for diagram algebras. We also define a notion of Tate cohomology for the diagram algebras considered in this thesis.

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Contents

Chapter 1

Introduction

1.1 Generalised Tate Cohomology of Hopf Algebras

In homological algebra, one of the most important invariants is the Ext functor. If we work over the group ring $\mathbb{Z}G$ for G a finite group, then the Ext functor recovers the group cohomology of G which is a non-negatively graded abelian group. This was extended in [Tat52] to an integer-graded cohomology theory now known as *Tate cohomology*, which is in some sense "group homology and group cohomology spliced together into a single invariant". Since then, Tate cohomology has been generalised from finite groups to other classes of groups, such as groups with finite virtual cohomological dimension [Far78] and condensed groups [Ghe24a; Ghe24b]. It has also been generalised to various classes of rings [Goi92; BC92; Mis94; CK97; AM02; Hed20], and there have been important generalisations and applications of Tate cohomology in stable homotopy theory [GM95; NS18].

It is well known that group cohomology is in fact a non-negatively graded ring where the multiplicative structure comes from the cup product [Bro82, Section V.3]. If we take trivial coefficients, then in fact one can prove that we can equivalently use the composition product for the multiplicative structure on group cohomology ([Bro82, Section V.4], [Bou07, Section 7.2]). This composition product can be easier to calculate in practice, but more importantly one can also use this to show that group cohomology in fact carries a non-trivial higher multiplicative structure called an A_∞-*structure* [Kel01; Kad80].

Similarly, there is a cup product giving Tate cohomology the structure of an integer-graded ring [Bro82, Section VI.5], and [Ngu13, Section 6] notes without proof that this is also true for generalised Tate cohomology of finite-dimensional Hopf algebras over a field. In this thesis, we give the details of the proof of this claim. We additionally prove that one can also define the composition product for this version of generalised Tate cohomology and that this is equivalent to the cup product, and we then deduce that there exists an A_∞-structure on this version of generalised Tate cohomology. We remark that both of these results appear in the literature in [BKS04, Section 6.4] but we are not aware of any proofs. Along the way, we prove a version of the well-known comparison of projective resolutions [Ben91, Theorem 2.4.2] for the analogous chain complexes used to compute generalised Tate cohomology, complete resolutions.

Finally, certain operations on the cohomology of cocommutative Hopf algebras called *Steenrod operations* were proven to exist in [Liu60] and expanded upon in [May70, Section 11]. These are closely related to the Steenrod operations for the cohomology of spaces developed in [Ste47; Ste53]. We will calculate Steenrod operations on cohomology of two examples of cocommutative Hopf algebras and we will consequently see that, although the additive and multiplicative structure of cohomology of a finite-dimensional Hopf algebra over a field is independent of the underlying coalgebra structure, Steenrod operations do detect the underlying coalgebra structure.

1.2 Cohomology of Diagram Algebras

There are a number of results in algebraic topology on whether a sequence of groups satisfies *homological stability*. This is satisfied if, given a sequence of groups and group homomorphisms

$$G_0 \to G_1 \to G_2 \to \dots,$$

the induced maps on homology $H_d(G_n) \to H_d(G_{n+1})$ are isomorphisms when n is sufficiently large relative to d. Homological stability results have been proven for numerous families of groups, the first of which was proved in [Nak60] for the symmetric groups with other examples including braid groups [Arn70], mapping class groups of surfaces [Har85; Wah07] and Coxeter groups [Hep16]. There has also been a lot of recent work in establishing general frameworks for proving homological stability results [RW17; GKR25; Ran25].

In recent years, there have been many developments in the study of homological stability for algebras, specifically for *diagram algebras*. These come in families indexed by natural numbers n defined on certain bases of graphs of two columns of n vertices. These algebras over a base commutative ring have been studied in knot theory, representation theory and statistical mechanics for several decades. Important examples of such algebras whose homology have been studied include the Temperley–Lieb algebras [BH24] and partition algebras [BHP23]. Both of these examples depend on a parameter in the base ring, and it is shown in the above papers that if the parameter is invertible, then the homology of the algebra in question is globally isomorphic to the homology of a particular group. However, if the parameter in the base ring is not necessarily invertible, then there is instead a stable isomorphism on homology to that of a particular group. In the case of the partition algebras, for example, the homology is stably isomorphic to that of the symmetric groups, and hence by combining this with the stability result for symmetric groups in [Nak60], Boyd, Hepworth and Patzt obtain a homological stability result for the partition algebras. In this thesis, we obtain similar (co)homological stability results for several families of diagram algebras.

Further work in this new field has been done by Boyde [Boy24; Boy25] and Sroka [Sro24]. [Boy25] provides a framework for proving invertible parameter results on homology by showing that certain ideals are generated by idempotent elements, and [Boy24] provides a framework for proving parameter-independent results on homology via a partial projective resolution that Boyde calls the *Mayer–Vietoris complex* (which generalises the *cellular Davis complex* definition of [Sro24, Definition 8]). In addition, [Sro24] proves a global isomorphism result on the homology of the Temperley–Lieb algebra whose basis diagrams have an odd number of vertices in each column. These results fit into the framework of [Boy24] but in those cases, the Mayer–Vietoris complex turns out to be a genuine projective resolution. In this thesis, we will prove cohomological analogues of results on homology in [Boy24; Boy25] and apply them to various diagram algebras.

In particular, we will use these methods to prove cohomological versions of both global and stable isomorphism results first proved in [BHP23] on the homology of partition algebras. We will also prove cohomological versions of Sroka's 'odd n' result for the homology of Temperley–Lieb algebras [Sro24, Theorem A], the global homology of Brauer algebras with invertible parameter [BHP21, Theorem A] or 'odd n' [Boy25, Theorem 1.3], and the global homology of rook algebras with invertible parameter [Boy25, Theorem 5.4].

We also apply results to several diagram algebras which have not yet been studied in homological algebra. We calculate the homology and cohomology of certain subalgebras of the partition algebras, namely the Tanabe algebras, totally propagating partition algebras and uniform block permutation algebras. The Tanabe algebras have a parameter in the base ring, and our result on their (co)homology is independent of both this parameter and the parity of the index n. We believe that this is the first result of this type in the literature.

We calculate the (co)homology of the rook-Brauer algebras. These algebras take two parameters in the base ring, and subalgebras of the rook-Brauer algebras were studied in [Boy25]. [Boy25, Theorem 7.3] claims that the homology of these algebras with one parameter being invertible is isomorphic to the homology of the Brauer algebras with invertible parameter. However, we will see in Subsection 6.3.2 that this claimed isomorphism does not hold for even n and we will prove a corrected version of the theorem. We also prove results for subalgebras of the rook-Brauer algebras not yet studied in (co)homology, namely the walled Brauer algebras and the Motzkin algebras.

Finally, we prove global and stable isomorphism results on the (co)homology of certain variants of the Temperley–Lieb algebras, namely the dilute Temperley–Lieb algebras and the blob algebras. Boyd and Hepworth suggest that these two algebras are candidates for satisfying homological stability [BH24, Section 1.7], and in this thesis we prove that (co)homological stability does indeed hold for these two families of diagram algebras.

We end by defining a notion of Tate cohomology for the diagram algebras considered in this thesis.

1.3 Structure of the Thesis

Part I, which comprises Chapters 2 and 3, is individual work of the author. Chapter 2 is background material on Hopf algebras, homological algebra and higher structures on cohomology (A_∞-algebras and Steenrod operations).

In Chapter 3, we provide a complete proof that generalised Tate cohomology of finite-dimensional Hopf algebras over a field is isomorphic as a graded ring (with the cup product) to the cohomology of the endomorphism dga of a complete resolution (with the composition product). These results appear in the text as Theorem 3.1.0.4 and Proposition 3.2.0.10. We therefore deduce that generalised Tate cohomology carries an A_∞-structure (Corollary 3.2.0.11). These results are known (see, for example, [BKS04, Section 6.4]) but the author is not aware of any proofs in the literature. We prove a corrected version of a result in [Ngu13, Section 8.1] on the additive structure of cohomology of the Sweedler Hopf algebra and then calculate the multiplicative structure using the composition product (Subsection 3.3.1). Finally, we show that there are cohomology operations on usual Ext of a Hopf algebra which detect the comultiplication (Section 3.4). We do this by calculating the Bockstein homomorphism and the Steenrod operation P^0 on the cohomology of two non-isomorphic Hopf algebras which are isomorphic as algebras.

Part II, which comprises Chapters 4–7, and Section 1.2 above are based on joint work of the author and Daniel Graves. The bulk of this material is modified from the content of three papers: 'Cohomology of Tanabe algebras' (published in Extracta Mathematicae in

December 2025) [FG25b]; 'Cohomology of dilute Temperley–Lieb algebras' (accepted for publication in Canadian Mathematical Bulletin in February 2026) [FG26]; and 'Cohomology of rook-Brauer algebras and their subalgebras' (submitted March 2025) [FG25a]. These papers have all been adapted from [FG24a] and [FG24b] (available on the arXiv e-print system) and expanded upon. Results on the (co)homology of blob algebras in Chapter 7 were proved by the author and Daniel Graves following conversations with Guy Boyde during the 'Equivariant homotopy theory in context' programme at the Isaac Newton Institute for Mathematical Sciences, Cambridge in 2025, and that work is part of an ongoing collaboration between Guy Boyde, the author and Daniel Graves [BFG25]. There is also additional material on Tate cohomology of diagram algebras adapted from [FG24a].

Guy Boyde and Daniel Graves have agreed for the joint work described above to be included in this thesis. As is usual in algebraic topology and homological algebra, authors are listed alphabetically by surname and joint authorship is equal.

In Chapter 4, we recall from the literature definitions of various diagram algebras considered in this thesis. We also recall the notion of link states and associated ideals.

In Chapter 5 we recall the notion of augmented algebras and show that the diagram algebras considered in this thesis can be equipped with augmentations, thereby allowing us to define homology and cohomology of these diagram algebras (with coefficients in a trivial module). We then recall results on the homology of algebras in [Boy24; Boy25] and prove cohomological versions of these results. The main original results in this chapter are Theorem 5.2.3.1 and Theorem 5.2.3.4.

In Chapters 6 and 7, we apply results from Chapter 5 to diagram algebras defined in Chapter 4. We obtain a number of global isomorphisms on cohomology by adapting arguments in [Boy25] and we additionally prove stable isomorphisms on cohomology by adapting arguments in [Boy24; Sro24]. The main original results in Chapter 6 are Theorem 6.1.2.1, Theorem 6.2.0.3, Theorem 6.3.2.5 and Theorem 6.4.4.1. The main original results in Chapter 7 are Theorem 7.1.4.1, Theorem 7.1.4.3, Theorem 7.2.1.4 and Corollary 7.2.4.3.

In Appendix A, we discuss questions and ideas for future work on generalised Tate cohomology of Hopf algebras and on cohomology of diagram algebras.

At the end of the thesis, after the bibliography, there is a table of notation for common terms and symbols used herein.

No artificial intelligence has been used in any way, shape or form in the creation of this thesis.

1.4 Conventions

Unless otherwise stated, then throughout this thesis R will be a commutative unital ring, k will be a field and R-modules are understood to be left R-modules. When the ring R is clear from context, we will sometimes use the term "module" instead of "R-module". We will often use the terms "map" and "homomorphism" interchangeably when referring to morphisms of R-modules. We denote the category of R-modules by R-**mod**. When working with an algebra (possibly with extra structure, for example a Hopf algebra) A over a ring R, an unadorned tensor product $\otimes$ will denote the tensor product of R-modules. Similarly, an unadorned Hom is taken over R.

Chain complexes will be written (C_*, d_*^C), where $d_i^C\colon C_i \to C_{i-1}$, cochain complexes will be written (C^*, d_C^*), where $d_C^i\colon C^i \to C^{i+1}$. We sometimes use the term "a complex" to refer to either a chain complex or a cochain complex. We will often suppress the differentials from the notation and simply write C_* and C^*, and where the objects of the complex are understood from context, we will sometimes write the i^{th} differential as just d_i or d^i as appropriate. We fix the convention that a degree r chain map $f\colon C_* \to C'_*$ will be a sequence of maps $f_i\colon C_{i+r} \to C'_i$ commuting with the boundaries on C_* and C'_*.

For $\mathcal{A}$ a category and A, A' objects of $\mathcal{A}$, the covariant and contravariant hom functors are respectively denoted by $\mathrm{Hom}_{\mathcal{A}}(A, -)$ and $\mathrm{Hom}_{\mathcal{A}}(-, A')$. If the source category $\mathcal{A}$ is clear from context, the subscript in the hom functor will sometimes be omitted.

We say that the natural numbers $\mathbb{N}$ includes 0. We write C_n for the cyclic group of order n and Σ_n for the symmetric group of degree n.

There will be some additional conventions for Part II which we will outline in Section 4.1.

Part I

Generalised Tate Cohomology

Chapter 2

Background

In this chapter, we recall various preliminary notions from homological algebra. We will assume the reader is familiar with basic category theory, modules over a commutative ring and their homomorphisms, and (co)chain complexes and (co)chain maps as well as the (co)homology of (co)chain complexes.

We will outline several useful categories, including the category of unbounded chain complexes of left modules over a commutative ring (and its bounded, bounded above, and bounded below subcategories). We also define cohomological functors.

Projective and injective modules will be defined, as well as projective and injective resolutions, allowing us to define the notions of group cohomology and the Ext functor. We also define complete resolutions which will allow us to define one version of Tate cohomology and the complete Ext functor which we will use throughout Part I.

Finally, we recall some basic notions for A_∞-algebras and Steenrod operations.

2.1 Algebras, Coalgebras and Hopf Algebras

We have assumed knowledge of modules over commutative rings but in this thesis we will also often work with modules over more structured algebraic objects such as Hopf algebras. To define Hopf algebras, we first need to define several preliminary notions. The material in this section can be found in many introductory sources to Hopf algebras such as [Mon93; Rad11].

Throughout this section, an unadorned tensor product $\otimes$ is taken to be over R (or k if we work over a field).

Definition 2.1.0.1. A monoid in the (symmetric) monoidal category $(R\text{-}\mathbf{mod}, \otimes, R)$ is called an *R-algebra* or simply an *algebra*. That is, an algebra (A, ∇, η) is an R-module A together with an associative multiplication map ∇ and a unit map $\eta\colon R \to A$ such that $\nabla(\eta \otimes \mathrm{id}) = \nabla(\mathrm{id} \otimes \eta) = \mathrm{id}$. If the context is clear, then we often write $\nabla(a \otimes a') = aa'$. An algebra is *commutative* if $\nabla\tau = \nabla$ where $\tau\colon A \otimes A \to A \otimes A$ is the *twist map of R-modules* which sends $a \otimes a'$ to $a' \otimes a$.

Dually, a comonoid in $(R\text{-}\mathbf{mod}, \otimes, R)$ is called an *R-coalgebra* or simply a *coalgebra*. That is, a coalgebra (C, Δ, ε) is an R-module C together with maps $\Delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to R$ such that the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\mathrm{id}\otimes\Delta} \\ C \otimes C & \xrightarrow{\Delta\otimes\mathrm{id}} & C \otimes C \otimes C, \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow{\scriptstyle\Delta} & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle\mathrm{id}\otimes\varepsilon} \\ C \otimes C & \xrightarrow{\varepsilon\otimes\mathrm{id}} & C. \end{array}$$

The first diagram expresses *coassociativity* of the *comultiplication* Δ and the second shows that ε is a *counit* where we have also used the canonical isomorphism of $C \otimes R$ with C.

A coalgebra is *cocommutative* if $\tau\Delta = \Delta$.

Notation 2.1.0.2. Let (C, Δ, ε) be a coalgebra. We will sometimes use *sumless Sweedler notation* for the comultiplication of an element $c \in C$:

$$\Delta(c) = c_{(1)} \otimes c_{(2)}.$$

We note that other authors may instead include a summation symbol in their notation for the comultiplication of an element $c \in C$ and write $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$.

Using this notation, coassociativity says that $c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}$ and we can in fact write this element unambiguously as $c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$. Similarly, ε being a counit means that $c = \varepsilon(c_{(1)})c_{(2)} = c_{(1)}\varepsilon(c_{(2)})$.

Definition 2.1.0.3. An *R-algebra homomorphism* $A_1 \to A_2$ is an R-linear ring homomorphism (equivalently, an R-algebra homomorphism is a monoid map in R-**mod**). Likewise, an *R-coalgebra homomorphism* $\varphi\colon C_1 \to C_2$ is an R-linear map that respects the comultiplication and the counit of C_1 and C_2.

Definition 2.1.0.4. An *augmented algebra* is an R-algebra A together with a surjective R-algebra homomorphism $\varepsilon\colon A \to R$ called the *augmentation map*.

Examples 2.1.0.5.

1. (R, ∇, id) is the *trivial R-algebra* with multiplication $\nabla(r \otimes r') = rr'$. Likewise, (R, Δ, id) is the trivial R-coalgebra with comultiplication $\Delta(r) = r \otimes 1 (= 1 \otimes r)$.
2. For any two algebras (A_1, ∇_1, η_1) and (A_2, ∇_2, η_2), the tensor product $A_1 \otimes A_2$ is an algebra with multiplication determined by $\nabla(a_1 \otimes a_2 \otimes a_1' \otimes a_2') = a_1a_1' \otimes a_2a_2'$ and unit $\eta(r) = \eta_1(r) \otimes \eta_2(r)$.
3. For any two coalgebras $(C_1, \Delta_1, \varepsilon_1)$ and $(C_2, \Delta_2, \varepsilon_2)$, the tensor product $C_1 \otimes C_2$ is a coalgebra with comultiplication (using Sweedler notation) $\Delta(x \otimes y) = x_{(1)} \otimes y_{(1)} \otimes x_{(2)} \otimes y_{(2)}$ and counit $\varepsilon(x \otimes y) = \varepsilon_1(x)\varepsilon_2(y)$ for $x \in C_1$ and $y \in C_2$.
4. Given an algebra A, the *opposite algebra* A^{op} has multiplication $\nabla^{op} = \nabla\tau$ and same unit as A. Similarly, given a coalgebra C, the *opposite coalgebra* C has comultiplication $\Delta^{op} = \tau\Delta$ and same counit as C.

Notation 2.1.0.6 ([Bak25, Theorems 2.4 and 2.6])**.** We can form a symmetric monoidal category of algebras (respectively coalgebras) $(\mathbf{Alg}_R, \otimes, R)$ (respectively $(\mathbf{Coalg}_R, \otimes, R)$), noting that the tensor product is over R.

Definition 2.1.0.7. An *R-bialgebra* (or simply a *bialgebra*) is a monoid in $\mathbf{Coalg}_R$ or equivalently, a comonoid in $\mathbf{Alg}_R$. That is, a bialgebra $(A, \nabla, \eta, \Delta, \varepsilon)$ is an algebra and a coalgebra such that the comultiplication and counit are algebra homomorphisms (or equivalently, such that the multiplication and unit are coalgebra homomorphisms). This is expressed by the commutativity of the following diagrams:

$$
\begin{array}{ccccc}
A \otimes A & \xrightarrow{\nabla} & A & \xrightarrow{\Delta} & A \otimes A \\
\big\downarrow{\scriptstyle \Delta\otimes\Delta} & & & & \big\uparrow{\scriptstyle \nabla\otimes\nabla} \\
A \otimes A \otimes A \otimes A & & \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} & & A \otimes A \otimes A \otimes A,
\end{array}
\qquad
\begin{array}{ccc}
R & \xrightarrow{\cong} & R \otimes R \\
\big\downarrow{\scriptstyle \eta} & & \big\downarrow{\scriptstyle \eta\otimes\eta} \\
A & \xrightarrow{\Delta} & A \otimes A,
\end{array}
$$

$$
\begin{array}{ccc}
R & \xleftarrow{\cong} & R \otimes R \\
\big\uparrow{\scriptstyle \varepsilon} & & \big\uparrow{\scriptstyle \varepsilon\otimes\varepsilon} \\
A & \xleftarrow{\nabla} & A \otimes A,
\end{array}
\qquad
\begin{array}{ccccc}
 & & A & & \\
 & \nearrow^{\eta} & & \searrow^{\varepsilon} & \\
R & & \xrightarrow{\mathrm{id}} & & R.
\end{array}
$$

Definition 2.1.0.8. An R-bialgebra $(A, \nabla, \eta, \Delta, \varepsilon)$ is called an *R-Hopf algebra* (or a *Hopf algebra over R*, or simply a *Hopf algebra*) if there exists an R-linear map $S\colon A \to A$ such that the following diagram commutes:

$$
\begin{array}{ccccccc}
 & & A \otimes A & \xrightarrow{S\otimes\mathrm{id}} & A \otimes A & & \\
 & \nearrow^{\Delta} & & & & \searrow^{\nabla} & \\
A & & \xrightarrow{\varepsilon} & R & \xrightarrow{\eta} & & A \\
 & \searrow_{\Delta} & & & & \nearrow_{\nabla} & \\
 & & A \otimes A & \xrightarrow{\mathrm{id}\otimes S} & A \otimes A. & &
\end{array}
$$

The map S is called the *antipode*. We often suppress the structure maps in the notation $(A, \nabla, \eta, \Delta, \varepsilon, S)$ and simply write A for a Hopf algebra.

If k is a field, then a *finite-dimensional Hopf algebra* (over k) is a Hopf algebra which is finite-dimensional as a k-vector space. The examples of Hopf algebras we examine in this thesis will generally be finite-dimensional.

We note that S is a ring antihomomorphism, that is $S(ab) = S(b)S(a)$ for $a, b \in A$ [Swe69, Proposition 4.0.1].

Examples 2.1.0.9.

1. If A is a finite-dimensional Hopf algebra, then we can define the *opposite Hopf algebra* A^{op} to comprise the opposite algebra and coalgebra structures, with antipode $S^{op} = S$.

2. For R a ring, G a finite group, we can endow the group algebra RG with a Hopf algebra structure where $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$ and $S(g) = g^{-1}$ for all $g \in G$ (and extended R-linearly). An element g of a general Hopf algebra satisfying $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ is called *grouplike*.

3. For k a field with characteristic p, and n a power of p, the *truncated polynomial algebra* $k[x]/(x^n)$ is a Hopf algebra with $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$ and $S(x) = -x$ [Bak25, Example 3.1]. We require n to be a power of p so that the comultiplication Δ is an algebra homomorphism. An element x of a general Hopf algebra satisfying $\Delta(x) = x \otimes 1 + 1 \otimes x$ is called *primitive*. We will consider the Hopf algebra $\mathbb{F}_3[x]/(x^3)$ later in Chapter 3.

4. Given a field k and a finite-dimensional Hopf algebra $(A, \nabla, \eta, \Delta, \varepsilon, S)$ over k, there is a Hopf algebra structure on the k-linear dual $A^\vee = \mathrm{Hom}_k(A, k)$ which we call the *dual Hopf algebra* $(A^\vee, \nabla^\vee, \eta^\vee, \Delta^\vee, \varepsilon^\vee, S^\vee)$. Its structure maps are dual to the structure maps on A, in the sense that, for example, $\Delta^\vee$ is dual to ∇ on A: $\Delta^\vee(f)(a \otimes a') = f(\nabla(a \otimes a'))$ (see [Mon93, Example 1.3.6 and Theorem 9.1.3]). We will use this in Section 2.3 and Chapter 3 to look at higher structure on cohomology.

We now examine the notion of modules over algebras, and then two important examples of modules over Hopf algebras.

Definition 2.1.0.10. Let (A, ∇, η) be an R-algebra. Then a *left A-module* (M, μ) is an R-module M together with an R-linear map $\mu\colon A \otimes M \to M$ such that the following diagrams commute:

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{\nabla \otimes \mathrm{id}} & A \otimes M \\ \big\downarrow{\scriptstyle \mathrm{id} \otimes \mu} & & \big\downarrow{\scriptstyle \mu} \\ A \otimes M & \xrightarrow{\quad\mu\quad} & M, \end{array} \qquad \begin{array}{ccc} R \otimes M & \xrightarrow{\eta \otimes \mathrm{id}} & A \otimes M \\ & \searrow^{\cong} & \big\downarrow{\scriptstyle \mu} \\ & & M. \end{array}$$

As for R-modules, we will often say 'A-module' to mean 'left A-module', and we often use just M as the notation. There is also an analogous definition of right A-module.

We remark that there is a dual notion of comodules over coalgebras which we will not deal with in this thesis, but the interested reader can refer to, for example, [Mon93, Section 1.6] for further details.

We now look at two important examples of modules over Hopf algebras.

Examples 2.1.0.11. Let A be a Hopf algebra over R.

1. Let $(M_1, \mu_1), (M_2, \mu_2)$ be two left A-modules. Then $(M_1 \otimes M_2, \tilde{\mu})$ is a left A-module where $\tilde{\mu}(a \otimes m_1 \otimes m_2) = \mu_1(a_{(1)} \otimes m_1) \otimes \mu_2(a_{(2)} \otimes m_2)$ (using sumless Sweedler notation of Notation 2.1.0.2 for the comultiplication of a).

2. The *trivial A-module* R has module action given by the counit, that is $\mu(a \otimes r) = \varepsilon(a)r$.

Remark 2.1.0.12 ([Bak25, Section 4.4]). For a Hopf algebra A, given a left A-module M, the dual module $M^\vee = \mathrm{Hom}_R(M, R)$ can also be viewed as a left A-module by using the antipode to twist the natural *right* module action into a *left* module action. In particular, for $a \in A$ and $f \in M^\vee$, we define

$$(a \cdot f)(m) = f(S(a)m)$$

for any $m \in M$.

In addition to Hopf algebras, we define another example of algebras equipped with extra structure. We will not work directly with such algebras, but we will see in Subsection 2.2.2 how it will allow us to define generalised Tate cohomology for Hopf algebras.

Definition 2.1.0.13. A *Frobenius algebra* is a finitely generated R-algebra A equipped with a nondegenerate R-bilinear form $\beta\colon A \times A \to R$ satisfying $\beta(ab, c) = \beta(a, bc)$.

Remark 2.1.0.14. An important fact we will use throughout this thesis is that finite-dimensional Hopf algebras are automatically Frobenius algebras via [LS69, Section 5 "Remark"] and in particular, as noted in [LS69, Section 3], a finite-dimensional Hopf algebra A is isomorphic as an A-module to its linear dual $A^\vee$. Here, the nondegenerate bilinear form is given by composing the product in A with the image of $1 \in A$ under an A-module isomorphism $A \xrightarrow{\cong} A^\vee$ [Bak25, Theorems 5.10 and 5.11].

2.2 Homological Algebra

The material in this section can be found in many textbooks which cover homological algebra such as [Wei94; CE56; Ben91; Bro82]. We recall from Section 1.4 that a degree r chain map $f\colon C_* \to C'_*$ is a sequence of maps $f_i\colon C_{i+r} \to C'_i$ commuting with the boundaries on C_* and C'_*. We first define several useful categories.

Notation 2.2.0.1. The *category of (unbounded) chain complexes of R-modules* has unbounded ($\mathbb{Z}$-graded) chain complexes of R-modules as objects and chain maps between them as morphisms. We denote this category by $\mathbf{Ch}(R)$.

This category has a (full) subcategory formed by bounded above (respectively bounded below, bounded, non-negative) chain complexes which is denoted by $\mathbf{Ch}_-(R)$ (respectively $\mathbf{Ch}_+(R)$, $\mathbf{Ch}_b(R)$ and $\mathbf{Ch}_{\geqslant 0}(R)$). For a chain complex C_*, we will use the notation $C_{\geqslant 0}$ for the non-negative truncation where

$$(C_{\geqslant 0})_i = \begin{cases} C_i, & i \geqslant 0 \\ 0, & i < 0. \end{cases}$$

More generally, we can define the category of chain complexes over any abelian category $\mathcal{A}$ which we write $\mathbf{Ch}(\mathcal{A})$. If the base abelian category is understood, we may write this as $\mathbf{Ch}$. This category is again abelian.

In the following material, an unadorned tensor product $\otimes$ is taken to be over R.

Definition 2.2.0.2. Let C_*, D_* be chain complexes of R-modules. Then for all $i \in \mathbb{Z}$, the *tensor product of chain complexes* $(C \otimes D)_*$ is defined by

$$(C \otimes D)_i = \bigoplus_{p+q=i} C_p \otimes D_q,$$

with differential determined by

$$\partial_i(x \otimes y) = d_p^C(x) \otimes y + (-1)^p x \otimes d_q^D(y)$$

for $x \in C_p, y \in D_q$. We note that a general element of $C_p \otimes D_q$ is a sum of the form $\sum_i x_i \otimes y_i$ and we refer to the $x_i \otimes y_i$ as *elementary tensors*.

We recall from Section 1.4 that our convention is that degree r chain maps *decrease* the degree by r. We have chosen this grading to match [BKS04, Sections 6 and 7] which will be a key reference for us in Chapter 3, however we note that other authors may choose a different convention.

Given chain complexes C_*, C'_*, D_*, D'_*, a degree r chain map $u\colon C_* \to D_*$ and a degree s chain map $v\colon C'_* \to D'_*$, there is a degree $r+s$ chain map $u \otimes v$ defined by

$$(u \otimes v)_i = \bigoplus_{p+q=i} u_p \otimes v_q \colon (C \otimes C')_{i+r+s} \to (D \otimes D')_i$$

for all $i \in \mathbb{Z}$. Note that when applied to an elementary tensor $a \otimes b$ this will introduce a sign of $(-1)^{|a|s}$ coming from the twist map of graded R-modules (see [Hed20, page 60] or [Law13, Section 15]).

Remarks 2.2.0.3.
- Equivalently, this is the direct sum total complex of the bicomplex of the degreewise tensor product.
- This tensor product of chain complexes endows $\mathbf{Ch}(R)$ with a symmetric monoidal structure.

There is another monoidal product on $\mathbf{Ch}(R)$ which will be used in this thesis, namely the complete tensor product of chain complexes.

Definition 2.2.0.4. Let C_*, D_* be chain complexes of R-modules. Then for all $i \in \mathbb{Z}$, the *complete tensor product of chain complexes* $(C \hat{\otimes} D)_*$ is defined by

$$(C \hat{\otimes} D)_i = \prod_{p+q=i} C_p \otimes D_q.$$

Because an arbitrary element of $(C \hat{\otimes} D)_i$ is *not* a finite sum of elements of the form $a \otimes b$, we need to define the differential on $(C \hat{\otimes} D)_*$ in a different way to Definition 2.2.0.2. For an element $y \in (C \hat{\otimes} D)_i$, we denote the projection of y to $C_{p+i} \otimes D_{-p}$ by $y_{p+i,-p}$. Then for $x \in (C \hat{\otimes} D)_i$ we define the differential by

$$\partial_i(x)_{p+i-1,-p} = (d^C_{p+i} \otimes \mathrm{id}_{D_{-p}})(x_{p+i,-p}) + (-1)^{p+i-1}(\mathrm{id}_{C_{p+i-1}} \otimes d^D_{-p+1})(x_{p+i-1,-p+1}).$$

Given chain complexes C_*, C'_*, D_*, D'_*, a degree r chain map $u\colon C_* \to D_*$ and a degree s chain map $v\colon C'_* \to D'_*$, there is a degree $r+s$ chain map $u \hat{\otimes} v$ defined by

$$(u \hat{\otimes} v)_i = \prod_{p+q=i} u_p \otimes v_q \colon (C \hat{\otimes} C')_{i+r+s} \to (D \hat{\otimes} D')_i$$

for all $i \in \mathbb{Z}$. As for the ordinary tensor product of chain complexes, when $u \hat{\otimes} v$ is applied to an elementary tensor $a \otimes b$ this will introduce a sign of $(-1)^{|a|s}$ coming from the twist map of graded R-modules.

Definition 2.2.0.5. For chain complexes C_*, D_* of R-modules, we define the *hom cochain complex* (or *hom complex*) $\underline{\mathrm{Hom}}^{\bullet}_R(C_*, D_*)$ by

$$\underline{\mathrm{Hom}}^n_R(C_*, D_*) = \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(C_i, D_{i-n}),$$

for all $n \in \mathbb{Z}$, where the differential is given by

$$\partial(f) = d^D \circ f - (-1)^n f \circ d^C$$

for all $f \in \underline{\mathrm{Hom}}^n_R(C_*, D_*)$. That is, the degree n components are precisely the degree n maps between the underlying graded modules.

Remark 2.2.0.6. By changing the grading, it is possible to define instead the *hom chain complex*, and this would then define the internal hom of chain complexes. However, we will not need the closed structure of the monoidal category of chain complexes in this thesis, and we have chosen this grading convention for notational ease later on.

As a direct consequence of the definition, for n even (respectively n odd) we see that the n-cocycles in $\underline{\mathrm{Hom}}^\bullet(C_*, D_*)$ are precisely degree n chain maps (respectively degree n anti-chain maps) from C_* to D_* and two such n-cocycles differ by a coboundary if and only if the degree n chain maps (respectively anti-chain maps) are chain homotopic.

Remark 2.2.0.7. For a chain complex C_* and an R-module M, we will sometimes look at the hom complex $\underline{\mathrm{Hom}}^\bullet(C_*, M)$ by viewing M as a chain complex concentrated in degree 0.

2.2.1 Cohomology

We follow [Wei94, Chapter 2] to define cohomology of R-modules.

Definition 2.2.1.1 ([Kro95, Section 4.1]). Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. A *cohomological functor* T^* is a collection of additive functors $T^n\colon \mathcal{A} \to \mathcal{B}$ for $n \in \mathbb{Z}$ such that for every short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

in $\mathcal{A}$, there exists a natural *connecting homomorphism* $\delta^n\colon T^n(C) \to T^{n+1}(A)$ such that the following sequence of maps is exact:

$$\ldots \xrightarrow{p_*} T^{n-1}(C) \xrightarrow{\delta^{n-1}} T^n(A) \xrightarrow{i_*} T^n(B) \xrightarrow{p_*} T^n(C) \xrightarrow{\delta^n} T^{n+1}(A) \xrightarrow{i_*} \ldots.$$

For any two cohomological functors $T^*, U^*\colon \mathcal{A} \to \mathcal{B}$ with respective connecting homomorphisms δ^*, δ'^*, a *morphism of cohomological functors* is a collection of natural transformations $(\tau^n\colon T^n \to U^n)_{n\in\mathbb{Z}}$ such that for every short exact sequence

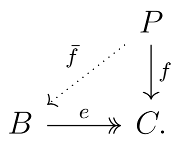
$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$, the following square commutes:

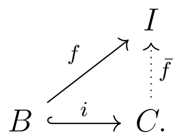
$$\begin{array}{ccc} T^n(C) & \xrightarrow{\delta^n} & T^{n+1}(A) \\ \downarrow{\scriptstyle \tau^n} & & \downarrow{\scriptstyle \tau^{n+1}} \\ U^n(C) & \xrightarrow{\delta'^n} & U^{n+1}(A). \end{array}$$

We now review the key notions of *projective resolutions* and *injective resolutions* of R-modules.

Definition 2.2.1.2. An R-module P is *projective* if it satisfies the following universal lifting property: given a surjection $e\colon B \to C$ and a map $f\colon P \to C$, there exists a map (not necessarily unique) $\bar{f}\colon P \to B$ such that $f = e \circ \bar{f}$. This is expressed by the commutativity of the following diagram



Dually, an R-module I is *injective* if it satisfies the following universal extension property: given an injection $i\colon B \to C$ and a map $f\colon B \to I$, there exists a map (not necessarily unique) $\bar{f}\colon C \to I$ such that $f = \bar{f} \circ i$. This is expressed by the commutativity of the following diagram



In an abelian category such as R-**mod**, however, the following lemma holds, which in practice can be an easier way of considering projective and injective objects.

Lemma 2.2.1.3 ([Wei94, Lemmas 2.2.3 and 2.3.4])**.** *1. An R-module M is projective iff* $\mathrm{Hom}_{R\text{-}\boldsymbol{mod}}(M, -)$ *is an exact functor. That is, M is projective iff the sequence of modules*

$$0 \to \mathrm{Hom}_{R\text{-}\boldsymbol{mod}}(M, A) \to \mathrm{Hom}_{R\text{-}\boldsymbol{mod}}(M, B) \to \mathrm{Hom}_{R\text{-}\boldsymbol{mod}}(M, C) \to 0$$

is exact for every exact sequence of modules $0 \to A \to B \to C \to 0$.

2. A module N is injective iff $\mathrm{Hom}_{R\text{-}\boldsymbol{mod}}(-, N)$ *is an exact functor.* □

Remark 2.2.1.4 ([Ben91, Section 1.6]). For a finite-dimensional k-algebra A, a finitely generated left A-module M is projective if and only if its linear dual $\mathrm{Hom}_k(M, k)$ is an injective right A-module, and vice versa. This can be seen by applying the functor $\mathrm{Hom}_k(-, k)$ to the objects and morphisms in the commutative diagram for projective modules in Definition 2.2.1.2 and noting that this yields the commutative diagram for injective modules.

We will use the following result to see that when we work over sufficiently nice Hopf algebras, projective and injective modules coincide.

Proposition 2.2.1.5 ([Ben91, Proposition 1.6.2])**.** *Let A be a Frobenius algebra. Then an A-module is projective if and only if it is injective.* □

We recall from Remark 2.1.0.14 that finite-dimensional Hopf algebras are Frobenius algebras. We now show that there is in fact a larger class of Hopf algebras which are Frobenius algebras.

Proposition 2.2.1.6. *Let A be a finitely generated Hopf algebra which is projective over the base ring R. Then A is a Frobenius algebra.*

Proof. Let A be a Hopf algebra meeting the conditions of the statement of the proposition and whose dual $A^\vee$ is finitely generated and projective over A. Then Pareigis [Par71] showed that A is a Frobenius algebra. However, Hedenlund [Hed20, Corollary I.2.8] proves that when A is a finitely generated Hopf algebra which is projective over the base ring R, then in fact the dual $A^\vee$ is necessarily finitely generated and projective over A. Hence we can drop Pareigis's conditions on $A^\vee$ and the proposition follows. □

It now follows immediately from Propositions 2.2.1.5 and 2.2.1.6 that for A a finitely generated Hopf algebra which is projective over the base ring R, an A-module is projective if and only if it is injective. This will be crucial for us to construct complete resolutions in Subsection 2.2.2.

Definition 2.2.1.7. Let M be an R-module. A *resolution* of M is an $\mathbb{N}$-graded chain complex of R-modules C_* along with an *augmentation map* $\varepsilon\colon C_0 \to M$ such that

$$\dots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} \dots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} M \to 0$$

is an exact sequence.

A resolution of M where each C_i is projective is called a *projective resolution* of M. We will often denote projective resolutions by P_*.

We will mostly work with resolutions and projective resolutions in this thesis, however we will occasionally make use of the dual notions of coresolutions and injective resolutions.

Definition 2.2.1.8. A *coresolution* of M is an $\mathbb{N}$-graded cochain complex of modules C^* along with a *coaugmentation map* $\eta\colon M \to C^0$ such that

$$0 \to M \xrightarrow{\eta} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \dots \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} \dots$$

is an exact sequence.

A coresolution of M where each C^i is injective is called an *injective resolution* of M. Injective resolutions are often denoted by I^*.

The following result and its dual [Wei94, Comparison Theorem 2.3.7] will be important in Chapter 3 to prove equivalence of two definitions of generalised Tate cohomology.

Theorem 2.2.1.9 (Comparison Theorem of Projective Resolutions [Ben91, Theorem 2.4.2])**.** *Let R be a ring and $f'\colon M \to N$ be a map of R-modules. Let P_* be a projective resolution of M and Q_* be a projective resolution of N. Then f' can be extended to a chain map f*

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_3^P} & P_2 & \xrightarrow{d_2^P} & P_1 & \xrightarrow{d_1^P} & P_0 & \xrightarrow{\varepsilon} & M \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f'} \\
\cdots & \xrightarrow{d_3^Q} & Q_2 & \xrightarrow{d_2^Q} & Q_1 & \xrightarrow{d_1^Q} & Q_0 & \xrightarrow{\varepsilon} & N,
\end{array}$$

and any two such chain maps extending f' are chain homotopic. □

Remark 2.2.1.10. As noted in [Ben91, Remark 2.4.2], Theorem 2.2.1.9 actually holds in greater generality: the top row of the diagram does not need to be acyclic, and the modules Q_i do not need to be projective. We will sometimes refer to this version of the theorem where required.

Definition 2.2.1.11. Let M and N be R-modules and let P_* be a projective resolution of M. Then the i^{th} *cohomology of M with coefficients in N* is

$$\mathrm{Ext}^i_R(M, N) := H^i(\mathrm{Hom}_R(P_*, N)).$$

By Theorem 2.2.1.9, this is independent of the choice of projective resolution of M.

For G a group, if R is the group ring $\mathbb{Z}G$, then $\mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z}, N)$ is the group cohomology of G with coefficients in N [Bro82, Section III.2].

2.2.2 Generalised Tate Cohomology

We define complete resolutions for finitely generated modules over suitable Hopf algebras and note that they always exist. Moreover, we give a convenient way of constructing one for the trivial module from a projective resolution. Throughout this subsection, A is a finitely generated Hopf algebra which is projective over its base ring R.

Definition 2.2.2.1. Let M be a finitely generated A-module. Then a *complete resolution of M* is an acyclic complex $\hat{P}_*$ of finitely generated projective A-modules such that the non-negative truncation $\hat{P}_{\geqslant 0}$ is isomorphic (in $\mathbf{Ch}_{\geqslant 0}(A)$) to some projective resolution P'_* of M, and such that $\mathrm{Hom}_A(\hat{P}_*, A)$ is also acyclic. An acyclic complex $\hat{P}_*$ meeting this latter condition is called *totally acyclic*.

Remark 2.2.2.2. This is a less general definition than is often found in the literature, for example in [AM02, Section 3] where the complete resolution agrees with a projective resolution only in degrees greater than or equal to some non-negative n. However, [AM02, Theorem 3.1] shows that such an n is equal to the *Gorenstein dimension* of A [AB69], and since we have already noted that Hopf algebras are self-injective, then Hopf algebras have Gorenstein dimension 0 (as an equivalent definition for self-injectivity). Hence our definition here makes sense for Hopf algebras, and in fact by the above cited result they always exist for Hopf algebras.

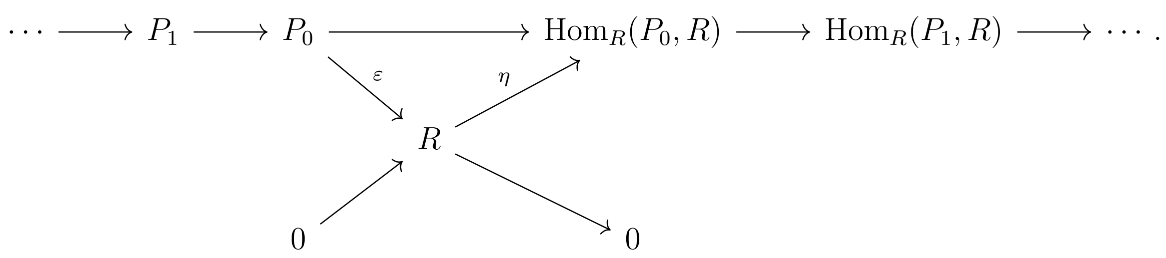
Construction 2.2.2.3. We now construct a complete resolution of the trivial A-module R as follows. Let (P_*, d) be a projective resolution of the trivial A-module R with augmentation ε where each P_i is finitely generated. It is shown in [Hed20, Proposition I.2.25] that under the assumption that A is finitely generated and projected over R then you can always construct such a projective resolution where each P_i is finitely generated. Then the linear dual complex $(\mathrm{Hom}_R(P_*, R), d^*)$ is an injective resolution of $\mathrm{Hom}_R(R, R) \cong R$ with coaugmentation η given by precomposition with ε. We "splice the resolution and coresolution together along R", and define the complex $(\hat{P}_*, d^{\hat{P}})$ by

$$\hat{P}_i = \begin{cases} P_i, & i \geqslant 0 \\ \mathrm{Hom}_R(P_{-i-1}, R), & i < 0 \end{cases}$$

with differential

$$d_i^{\hat{P}} = \begin{cases} d_i, & i > 0 \\ \eta\varepsilon, & i = 0 \\ d^*_{-i-1}, & i < 0, \end{cases}$$

which we can depict as the following diagram



Then by [Hed20, Proposition I.2.21 and Remark I.2.27] this is totally acyclic, and we can see that $\hat{P}_{i\geqslant 0}$ agrees with the projective resolution P_*. Hence this is indeed a complete resolution of R.

We now define generalised Tate cohomology.

Definition 2.2.2.4 ([CK97])**.** Let $\hat{P}_*$ be a complete resolution of the trivial A-module R and M an A-module. The i^{th} *generalised Tate cohomology of* R *with coefficients in* M is

$$\widehat{\mathrm{Ext}}^i_A(R, M) = H^i(\mathrm{Hom}_A(\hat{P}_*, M)).$$

Throughout this thesis, we will often refer to generalised Tate cohomology as *complete Ext*. We will also henceforth often refer to cohomology as defined in Definition 2.2.1.11 as "usual Ext" to differentiate it from complete Ext.

Remarks 2.2.2.5.

1. It is immediate from the definitions of a complete resolution and complete Ext that $\widehat{\mathrm{Ext}}^i_A(R, M) = \mathrm{Ext}^i_A(R, M)$ for all $i \geqslant 1$ and that the map $\mathrm{Hom}_A(P_0, M) \to \mathrm{Hom}_A(\hat{P}_0, M)$ induces a surjection $\mathrm{Ext}^0_A(R, M) \to \widehat{\mathrm{Ext}}^0_A(R, M)$.
2. Generalised Tate cohomology is independent of the choice of complete resolution [AM02, Lemma 5.3].
3. For every short exact sequence of A-modules, there is a doubly-infinite long exact sequence of generalised Tate cohomology modules [AM02, Proposition 5.4]. In particular, $\widehat{\mathrm{Ext}}^*_A(R, -)$ is a cohomological functor.
4. As for usual Ext, complete Ext for the Hopf algebra $\mathbb{Z}G$ recovers the classical definition of Tate cohomology [Bro82, Section VI.4].

There are various alternative definitions of complete Ext over different classes of rings due to [BC92; CK97; Hed20; Mis94; Goi92]. The interested reader can refer to these papers and the survey [Pag16]. We also note that it is proved in [Ghe24a] that complete Ext due to [BC92; CK97; Mis94; Goi92] are isomorphic as cohomological functors, and that [Hed20, Remark I.2.27] shows that the definition of complete Ext due to Hedenlund is (additively) isomorphic to that of Cornick–Kropholler. In particular, Gheorghiu's and Hedenlund's results apply for modules over a finitely generated Hopf algebra which is projective over its base ring, and hence in our context all of these definitions of complete Ext are equivalent.

2.2.3 Multiplicative Structures on Cohomology

Let A now be an R-algebra. Before looking at multiplicative structure on usual Ext, we first consider a different way of viewing $\mathrm{Ext}^i_A(M, N)$. For P_* a projective resolution of M and Q_* a projective resolution of N, we can view elements of this as homotopy classes of degree i chain maps $P_* \to Q_*$. If we have a cocycle $\gamma\colon P_i \to N$, then we can use projectivity of P_i and surjectivity of the augmentation $Q_0 \to N$ to lift this to a map $P_i \to Q_0$, which then extends to a full degree i chain map $P_* \to Q_*$ by Theorem 2.2.1.9 and Remark 2.2.1.10. On the other hand, starting with a degree i chain map $f\colon P_* \to Q_*$, then εf_i is a cocycle $P_i \to N$. One can check that this descends to a well-defined isomorphism on cohomology, and hence we can indeed view $\mathrm{Ext}^i_A(M, N)$ as homotopy classes of degree i chain maps $P_* \to Q_*$ [Bou07, Section 7.2].

Moreover, consider the cohomology of the endomorphism dg algebra $\underline{\mathrm{Hom}}^\bullet_A(P_*, P_*)$ where (P_*, d) is a projective resolution of R and the multiplication is given by composition of maps. An element of the degree i component is a graded degree i map $f\colon P_* \to P_*$ and its differential is defined by $\partial(f) = df - (-1)^i fd$. As above, this is isomorphic as a graded module to $\mathrm{Ext}^*_A(R, R)$, and moreover the multiplication on $\underline{\mathrm{Hom}}^\bullet_A(P_*, P_*)$ descends to cohomology, and so $\mathrm{Ext}^*_A(R, R)$ is in fact a graded algebra (we will retain the use of * for the grading on Ext even though we use $^\bullet$ for the grading on $\underline{\mathrm{Hom}}$). We call the product on $\underline{\mathrm{Hom}}^\bullet_A(P_*, P_*)$ and the induced product on $\mathrm{Ext}^*_A(R, R)$ the *composition product*.

Now, let A be a Hopf algebra over R. We now examine another product on usual Ext and for this we need to use the tensor product of two projective resolutions. Using Definition 2.2.0.2 and the Künneth Theorem [Ben91, Corollary 2.7.2], it is immediate that for P_* a projective resolution of R, $(P \otimes P)_*$ is a projective resolution of $R \otimes R \cong R$, where each component $P_i \otimes P_j$ has an A-module structure as in Examples 2.1.0.11 (1). We can then use Theorem 2.2.1.9 to construct a chain map $\Delta\colon P_* \to (P \otimes P)_*$ such that $(\varepsilon \otimes \varepsilon)\Delta_0 = \varepsilon$. Such a map is called a *diagonal approximation*. This allows us to make the following definition.

Definition 2.2.3.1. Let P_* be a projective resolution of R and M and N be A-modules. Then for $f \in \mathrm{Hom}_A(P_i, M), g \in \mathrm{Hom}_A(P_j, N)$ we have a *cup product*

$$\smile\colon \mathrm{Hom}_A(P_i, M) \otimes \mathrm{Hom}_A(P_j, N) \to \mathrm{Hom}_A(P_{i+j}, M \otimes N)$$

defined by $f \smile g = (f \otimes g)\Delta$ where $\Delta\colon P_* \to (P \otimes P)_*$ is a diagonal approximation.

We note that the Δ used in this definition is *not* the comultiplication on the Hopf algebra A.

One checks that this induces a well defined product on usual Ext

$$\smile\colon \mathrm{Ext}^i_A(R, M) \otimes \mathrm{Ext}^j_A(R, N) \to \mathrm{Ext}^{i+j}_A(R, M \otimes N),$$

and also that it is associative at the cochain level and independent of choice of Δ.

Proposition 2.2.3.2 ([Bou07, Section 7.2]). *The cup product and composition product on* $\mathrm{Ext}^*_A(R,R)$ *coincide.* □

In Chapter 3 we will generalise this to complete Ext for a finite-dimensional Hopf algebra over a field and calculate some examples.

2.3 Higher Structures on Ordinary Ext

In this section, we define the notions of A_∞*-algebras* and *Steenrod operations* on the cohomology of a Hopf algebra. The former can be considered as dgas with associativity only up to homotopy, and the latter encode the failure of the cup product to be commutative on the nose.

2.3.1 A_∞-algebras

In this subsection, we will follow the survey paper [Kel01] to introduce the definition of A_∞-algebras and then state an important result on the structure of their cohomology.

Definition 2.3.1.1. Let k be a field. An A_∞*-algebra over* k is a $\mathbb{Z}$-graded vector space A together with, for $n \geqslant 1$, degree $2-n$ k-linear maps $m_n \colon A^{\otimes n} \to A$ satisfying

$$\sum_{n=r+s+t} (-1)^{r+st} m_{r+1+t}(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = 0$$

for all $n \geqslant 1$.

Setting $n=1$ shows that m_1 is just a differential on A, and setting $n=2$ shows that m_1 is a graded derivation with respect to the multiplication m_2. If we set $n=3$, we get

$$m_2(\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) = m_1 m_3 + m_3(m_1 \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes m_1 \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes m_1).$$

The left hand side is the associator for m_2 and the right hand side is the boundary of m_3 in the hom complex $\underline{\mathrm{Hom}}^\bullet_k(A^{\otimes 3}, A)$. Therefore we can consider m_2 to be a multiplication which is "associative up to the homotopy m_3".

If $m_3 = 0$ then A is a dga (i.e. it is associative on the nose). If we have a non-trivial dga (that is, $m_1 \neq 0$) A, then it is an A_∞-algebra with $m_n = 0$ for all $n \geqslant 3$. In addition, if we take an A_∞-algebra A with $m_1 = 0$ then the boundary of m_3 in the hom complex $\underline{\mathrm{Hom}}^\bullet_k(A^{\otimes 3}, A)$ will vanish, so in that case A will be a dga but with m_3 not necessarily trivial.

If $m_1 = 0$ then A is said to be a *minimal* A_∞-algebra. The following theorem says that every A_∞-algebra admits a minimal model.

Theorem 2.3.1.2 (Kadeishvili's Theorem [Kad80]). *Let* A *be an* A_∞*-algebra. Then its cohomology algebra* $H^*(A)$ *is an* A_∞*-algebra with vanishing differential and multiplication induced by the* m_2 *on* A. □

Remark 2.3.1.3. Theorem 2.3.1.2 is in fact a special case of a more general theorem over certain operads called the *homotopy transfer theorem* [LV12, Section 10.3].

By Proposition 2.2.3.2, for an algebra A over a ring R and P_* a projective resolution of R, we can view $\mathrm{Ext}^*_A(R,R)$ as the cohomology of the dga $\underline{\mathrm{Hom}}^\bullet_A(P_*,P_*)$ where the multiplication is given by the composition product. We can then restrict to taking an algebra A over a field k and apply Theorem 2.3.1.2 to deduce that we have an A_∞-structure on $\mathrm{Ext}^*_A(k,k)$ with m_2 induced by the composition product. We will prove an analogous result for complete Ext of a Hopf algebra over a field in Chapter 3.

The "higher multiplications" m_n for $n \geqslant 3$ are closely related to *Massey products*. We will not define these, but the interested reader can find a definition and examples in [LPWZ09] as well as a proof that the m_n operations on the cohomology of a dga are, up to signs, elements of Massey products [LPWZ09, Theorem 3.1].

2.3.2 Steenrod Operations

In this subsection, we will work over cocommutative Hopf algebras. Throughout this subsection, p will be prime and k will denote the finite field $\mathbb{F}_p$. The content of this section was developed in [Liu60] and then generalised in [May70, Section 11]. We also recommend the unpublished work in progress [Bru09, Chapters 3 and 4].

Theorem 2.3.2.1 ([May70, Theorem 11.8])**.** *Let A be a cocommutative Hopf algebra over $k = \mathbb{F}_p$. Then there exist maps P^i defined on $\mathrm{Ext}^*_A(k,k)$ with*

$$\begin{cases} P^i\colon \mathrm{Ext}^s_A(k,k) \to \mathrm{Ext}^{s+i}_A(k,k) & \text{if } p=2,\\ P^i\colon \mathrm{Ext}^s_A(k,k) \to \mathrm{Ext}^{s+2i(p-1)}_A(k,k) & \text{if } p>2 \end{cases}$$

and if $p > 2$ then there also exist maps βP^i with

$$\beta P^i\colon \mathrm{Ext}^s_A(k,k) \to \mathrm{Ext}^{s+1+2i(p-1)}_A(k,k).$$

These maps satisfy the following properties:

1. *If $p = 2$, then $P^i = 0$ if $i < 0$ or $i > s$;*
2. *If $p > 2$, then $P^i = 0$ if $i < 0$ or $2i > s$;*
3. *If $p > 2$, then $\beta P^i = 0$ if $i < 0$ or $2i \geqslant s$;*
4. *$P^i(x) = x^p$ (where x^p is the p^{th} cup power) if $p = 2$ and $i = s$ or if $p > 2$ and $2i = s$;*
5. *The* Cartan formulas *hold: $P^i(x \smile y) = \sum_{n+m=i} P^n(x) \smile P^m(y)$ and $\beta P^i(x \smile y) = \sum_{n+m=i} \beta P^n(x) \smile P^m(y) + (-1)^s P^n(x) \smile \beta P^m(y)$;*
6. *The* Adem relations *hold [Ade53, Theorem 1.1].*

Definition 2.3.2.2. The maps P^i and, for $p > 2$, the maps βP^i are called *Steenrod operations* or *power operations*. When $p = 2$, the notation $P^i = \mathrm{Sq}^i$ is often used and in this case the maps are called *Steenrod squares*.

Remarks 2.3.2.3. 1. The map βP^i should be considered a single symbol; the notation comes from the fact that in the cohomology of spaces, this is the composite of P^i and the Bockstein homomorphism β, but this is not true in general for cohomology of Hopf algebras. It is true, however, for Hopf algebras A which are *reduced modulo* p, meaning that $A = B \otimes_{\mathbb{Z}} k$ where B is a $\mathbb{Z}$-free Hopf algebra [May70, Theorem 11.8 iii)]. In Section 3.4, we will look at Steenrod operations on the cohomology of an example of a reduced mod 3 Hopf algebra and on the cohomology of a non-reduced mod 3 Hopf algebra.

2. Classical Steenrod operations for cohomology of spaces also include $P^0 = \mathrm{id}$ as an axiom. However, this does not always hold for cohomology of Hopf algebras, as we will see by way of example in Section 3.4.

3. The mod p Steenrod operations defined above generate the *mod* p *extended Steenrod algebra* $\overline{\mathcal{A}}_p$ ([Man01, Section 1], [Bru09, Section 4.3]).

In this thesis, we will only calculate P^0 and βP^0 in examples. We refer the reader to [May70, Section 11] and [BMMS86, Section IV.2] for details on how to construct the other P^i and βP^i.

In order to give the definition of P^0, we need to define a particular cochain complex whose cohomology is $\mathrm{Ext}^*_A(k,k)$, and to do that we need to how a particular induced comultiplication on the augmentation ideal $I(A^\vee) = \ker(\varepsilon_{A^\vee})$ is defined, where we recall that $A^\vee$ is the dual Hopf algebra as defined in Examples 2.1.0.9 (4). We first note that the short exact sequence of k-vector spaces

$$0 \to I(A^\vee) \overset{\iota}{\hookrightarrow} A^\vee \twoheadrightarrow k \to 0$$

splits via the map $p\colon A^\vee \to I(A^\vee)$ defined by

$$p(f) = f - \varepsilon_{A^\vee}(f)\eta_{A^\vee}(1).$$

Definition 2.3.2.4. The *reduced comultiplication on* $I(A^\vee)$ is the map $\tilde{\Delta}\colon I(A^\vee) \to I(A^\vee) \otimes I(A^\vee)$ defined by

$$\tilde{\Delta} = (p \otimes p)\Delta\iota,$$

where Δ is the comultiplication in $A^\vee$ and p, ι are as above.

Definition 2.3.2.5 ([Liu60, Section II.4])**.** Let A be a finite-dimensional Hopf algebra over $k = \mathbb{F}_p$ and let $A^\vee$ be its k-linear dual algebra. Let $I(A^\vee)$ be the augmentation ideal. The *cobar construction of* $I(A^\vee)$ is the cochain complex $B^\bullet$ where $B^m = I(A^\vee)^{\otimes m}$. We will denote elementary tensor elements of B^m by $[x_1|x_2|\dots|x_m]$. The differential $d^m\colon I(A^\vee)^{\otimes m} \to I(A^\vee)^{\otimes(m+1)}$ is defined by

$$d^m([([x_1|\dots|x_m])]) = \sum_{i=1}^{m} (-1)^i [x_1|\dots|\tilde{\Delta}(x_i)|\dots|x_m],$$

where $\tilde{\Delta}$ is the reduced comultiplication as defined in Definition 2.3.2.4.

As A is finite-dimensional, this cochain complex is the A-linear dual of the *bar construction* (see again [Liu60, Section II.4] or [May70, Definitions 11.1], for example) which is a particular projective resolution of k. Hence the cohomology of $B^\bullet$ is indeed equal to $\mathrm{Ext}^*_A(k,k)$, and we can represent elements of $\mathrm{Ext}^n_A(k,k)$ by n-cocycles in $B^\bullet$. We can now state the definition of P^0.

Proposition 2.3.2.6 ([May70, Definition 11.9 and Proposition 11.10])**.** *Let A be a finite-dimensional Hopf algebra over $k = \mathbb{F}_p$, let $B^\bullet$ be the cobar construction of $I(A^\vee)$, and let $[x_1|\dots|x_n]$ be an n-cocycle in $B^\bullet$ representing $x \in \mathrm{Ext}^n_A(k,k)$. Then*

$$P^0(x) = P^0([[x_1|\dots|x_n]]) = [[x_1^p|\dots|x_n^p]].$$

We will apply this to examples in Chapter 3 and we will see that Steenrod operations on the cohomology of a Hopf algebra detect the comultiplication.

Chapter 3

Multiplicative and Higher Structures on Cohomology

In this chapter, we prove a comparison theorem for complete resolutions and use it to show that complete Ext over a finitely generated Hopf algebra which is projective over the base ring can be calculated via homotopy classes of chain maps on a complete resolution. We generalise a result of Brown [Bro82, Section VI.5] that complete diagonal approximations always exist for complete resolutions, and given a certain initial choice in the construction, the construction of a complete diagonal approximation is unique up to chain homotopy. This is used to define the cup product for complete Ext over a finite-dimensional Hopf algebra over a field, and we show that this agrees with the composition product on complete Ext, proving a result stated without proof in [BKS04, Section 6.4]. As a consequence, we see that the multiplicative structure of complete Ext does not depend on the coalgebra structure of the Hopf algebra. However, by way of example we will see that for usual Ext, Steenrod operations (if they exist) do depend on the coalgebra structure.

3.1 Comparison of Complete Resolutions

Recall Theorem 2.2.1.9, the comparison theorem for projective resolutions. In this subsection, we prove an analogue for complete resolutions (see Definition 2.2.2.1). We show the general construction, prove a key property and then state the result applied for complete resolutions.

Construction 3.1.0.1. Let n be an integer, let R be a ring, and let N be an R-module. Let $\hat{P}_*$ be a $\mathbb{Z}$-graded acyclic chain complex of R-modules such that the modules $\hat{P}_i$ are projective for all $i \geqslant n-1$, and let $\hat{Q}_*$ be an acyclic complex of R-modules such that $\hat{Q}_{\geqslant 0}$ is a resolution (not necessarily projective) of N and the modules $\hat{Q}_i$ are injective for all $i \leqslant -1$.

Then given any map $\gamma\colon \hat{P}_n \to N$ which is a cocycle in the complex $\underline{\mathrm{Hom}}^\bullet_R(\hat{P}_*, N)$ (by considering N as a chain complex concentrated in degree 0 as in Remark 2.2.0.7), we construct a degree n chain map $f\colon \hat{P}_* \to \hat{Q}_*$ as follows.

First, given a map $\gamma\colon \hat{P}_n \to N$, we can lift this to a map $f_0\colon \hat{P}_n \to \hat{Q}_0$ such that $\varepsilon \circ f_0 = \gamma$. We can do this because $\hat{P}_n$ is projective and the augmentation $\varepsilon\colon \hat{Q}_0 \to N$ is surjective.

In the diagram below, we now have that the rightmost square commutes, and we wish to show that we can construct the dotted maps f_i for $i \geqslant 1$ such that all the squares commute. The bottom row is exact and the top row is a chain complex (since we took γ to be a cocycle i.e. $\gamma \circ d^{\hat{P}}_{n+1} = 0$).

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d^{\hat{P}}_{n+3}} & \hat{P}_{n+2} & \xrightarrow{d^{\hat{P}}_{n+2}} & \hat{P}_{n+1} & \xrightarrow{d^{\hat{P}}_{n+1}} & \hat{P}_n & \xrightarrow{\gamma} & N \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \| \\
\cdots & \xrightarrow{d^{\hat{Q}}_{3}} & \hat{Q}_2 & \xrightarrow{d^{\hat{Q}}_{2}} & \hat{Q}_1 & \xrightarrow{d^{\hat{Q}}_{1}} & \hat{Q}_0 & \xrightarrow{\varepsilon} & N
\end{array}$$

We follow the procedure in Benson [Ben91, Theorem 2.4.2] to build these maps inductively, including the details for completeness. The base case f_0 has been constructed, and suppose that f_k and f_{k-1} have been constructed for some $k \geqslant 0$ such that $f_{k-1} \circ d^{\hat{P}}_{n+k} =$

$d_k^{\hat{Q}} \circ f_k$ (setting $f_{-1} = \mathrm{id}_N$, $d_n^{\hat{P}} = \gamma$ and $d_0^{\hat{Q}} = \varepsilon$). We have

$$d_k^{\hat{Q}} \circ f_k \circ d_{n+k+1}^{\hat{P}} = f_{k-1} \circ d_{n+k}^{\hat{P}} \circ d_{n+k+1}^{\hat{P}} = 0,$$

so $f_k \circ d_{n+k+1}^{\hat{P}}$ lands in $\mathrm{Im}(d_{k+1}^{\hat{Q}})$ by exactness of the bottom row of the above diagram. Since the corestriction of $d_{k+1}^{\hat{Q}}$ onto $\mathrm{Im}(d_{k+1}^{\hat{Q}})$ is clearly surjective, then by projectivity of $\hat{P}_{n+k+1}$ we have a map $f_{k+1} \colon \hat{P}_{n+k+1} \to \hat{Q}_{k+1}$ such that $f_k \circ d_{n+k+1}^{\hat{P}} = d_{k+1}^{\hat{Q}} \circ f_{k+1}$. Hence the above diagram commutes for the constructed maps f_k for $k \geqslant 0$ as required.

In the diagram below, we have now shown the existence of the maps f_i for all $i \geqslant 0$ and all of the squares consisting solely of solid arrows commute. We now wish to construct the remaining dotted maps f_j for all $j \leqslant -1$ such that all remaining squares commute, which shows that f is a chain map.

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{d_{n+2}^{\hat{P}}} & \hat{P}_{n+1} & \xrightarrow{d_{n+1}^{\hat{P}}} & \hat{P}_n & \xrightarrow{d_n^{\hat{P}}} & \hat{P}_{n-1} & \xrightarrow{d_{n-1}^{\hat{P}}} & \hat{P}_{n-2} & \xrightarrow{d_{n-2}^{\hat{P}}} & \hat{P}_{n-3} & \xrightarrow{d_{n-3}^{\hat{P}}} & \cdots \\
 & & \downarrow f_1 & & \downarrow f_0 & & \vdots f_{-1} & & \vdots f_{-2} & & \vdots f_{-3} & & \\
\cdots & \xrightarrow{d_2^{\hat{Q}}} & \hat{Q}_1 & \xrightarrow{d_1^{\hat{Q}}} & \hat{Q}_0 & \xrightarrow{d_0^{\hat{Q}}} & \hat{Q}_{-1} & \xrightarrow{d_{-1}^{\hat{Q}}} & \hat{Q}_{-2} & \xrightarrow{d_{-2}^{\hat{Q}}} & \hat{Q}_{-2} & \xrightarrow{d_{-3}^{\hat{Q}}} & \cdots
\end{array}$$

We prove this via downward induction on j. We take our base cases to be f_1 and f_0 which we have already shown commute with the differentials on $\hat{P}_*$ and $\hat{Q}_*$. Suppose that maps f_{j+1} and f_j have been constructed for some $j \leqslant 0$ such that $f_j \circ d_{n+j+1}^{\hat{P}} = d_{j+1}^{\hat{Q}} \circ f_{j+1}$. We wish to show that we can construct a map f_{j-1} such that $f_{j-1} \circ d_{n+j}^{\hat{P}} = d_j^{\hat{Q}} \circ f_j$. We have

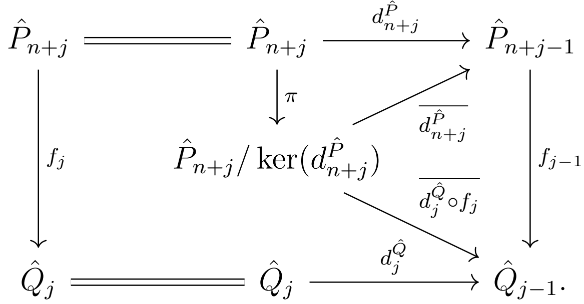
$$d_j^{\hat{Q}} \circ f_j \circ d_{n+j+1}^{\hat{P}} = d_j^{\hat{Q}} \circ d_{j+1}^{\hat{Q}} \circ f_{j+1} = 0,$$

so $d_j^{\hat{Q}} \circ f_j$ restricted to $\mathrm{Im}(d_{n+j+1}^{\hat{P}})$ is 0. But by exactness of $\hat{P}_*$, $\mathrm{Im}(d_{n+j+1}^{\hat{P}}) = \ker(d_{n+j}^{\hat{P}})$. Hence we get an induced map

$$\overline{d_j^{\hat{Q}} \circ f_j} \colon \hat{P}_{n+j}/\ker(d_{n+j}^{\hat{P}}) \to \hat{Q}_{j-1}.$$

Moreover, the induced map $\overline{d_{n+j}^{\hat{P}}}$ on $\hat{P}_{n+j}/\ker(d_{n+j}^{\hat{P}})$ is clearly injective, and hence by injectivity of $\hat{Q}_{j-1}$, there exists a map $f_{j-1} \colon \hat{P}_{n+j-1} \to \hat{Q}_{j-1}$ such that $f_{j-1} \circ \overline{d_{n+j}^{\hat{P}}} = \overline{d_j^{\hat{Q}} \circ f_j}$.

We have $d_j^{\hat{Q}} \circ f_j = \overline{d_j^{\hat{Q}} \circ f_j} \circ \pi$, where $\pi \colon \hat{P}_{n+j} \to \hat{P}_{n+j}/\ker(d_{n+j}^{\hat{P}})$ is the projection map. By construction of the map f_{j-1}, we have shown that the following diagram commutes:



Hence $f_{j-1} \circ d^{\hat{P}}_{n+j} = d^{\hat{Q}}_j \circ f_j$, as required, and we have constructed a degree n chain map $f \colon \hat{P}_* \to \hat{Q}_*$.

Proposition 3.1.0.2. *Let $\gamma, \gamma' \colon \hat{P}_n \to N$ be two cocycles in the complex $\underline{\operatorname{Hom}}^\bullet_R(\hat{P}_*, N)$ which differ by a coboundary, and let $f, f' \colon \hat{P}_* \to \hat{Q}_*$ be degree n chain maps such that $\varepsilon \circ f_0 = \gamma$ and $\varepsilon \circ f'_0 = \gamma'$. Then f is chain homotopic to f'.*

Proof. If γ and γ' are such cocycles, this means there exists a map $\beta \colon \hat{P}_{n-1} \to N$ such that $\gamma = \gamma' + \beta \circ d^{\hat{P}}_n$. We let $f, f' \colon \hat{P}_* \to \hat{Q}_*$ be degree n chain maps such that $\varepsilon \circ f_0 = \gamma$ and $\varepsilon \circ f'_0 = \gamma'$, and let $f'' \colon \hat{P}_* \to \hat{Q}_*$ be a degree n chain map built using Construction 3.1.0.1 from the map $\beta \circ d^{\hat{P}}_n$. By construction, we have $\varepsilon \circ f''_0 = \beta \circ d^{\hat{P}}_n$. We aim to show that there is a chain homotopy h from f to f', i.e. a sequence of maps $h_i \colon \hat{P}_{i+n} \to \hat{Q}_{i+1}$ such that $f_i - f'_i = d^{\hat{Q}}_{i+1} \circ h_i + h_{i-1} \circ d^{\hat{P}}_{i+n}$ for all $i \in \mathbb{Z}$.

As in Construction 3.1.0.1, we follow the procedure in Benson [Ben91, Theorem 2.4.2] to build the maps h_i for $i \geqslant -1$ inductively. First, we can lift β to a map $h_{-1} \colon \hat{P}_{n-1} \to \hat{Q}_0$ such that $\varepsilon \circ h_{-1} = \beta$. We can do this because $\hat{P}_{n-1}$ is projective and the augmentation $\varepsilon \colon \hat{Q}_0 \to N$ is surjective. Hence $\varepsilon \circ f''_0 = \varepsilon \circ h_{-1} \circ d^{\hat{P}}_n$. We therefore have

$$\begin{aligned} \varepsilon \circ (f_0 - f'_0 - f''_0) &= \varepsilon \circ (f_0 - f'_0 - h_{-1} \circ d^{\hat{P}}_n) \\ &= \gamma - \gamma' - \beta \circ d^{\hat{P}}_n \\ &= 0. \end{aligned}$$

Since $\operatorname{Im}(d^{\hat{Q}}_1) = \ker(\varepsilon)$, $f_0 - f'_0 - h_{-1} \circ d^{\hat{P}}_n$ lands in $\operatorname{Im}(d^{\hat{Q}}_1)$. Since the corestriction of $d^{\hat{Q}}_1$ onto $\operatorname{Im}(d^{\hat{Q}}_1)$ is surjective, then by projectivity of $\hat{P}_n$ we have a map $h_0 \colon \hat{P}_n \to \hat{Q}_1$ such that $f_0 - f'_0 - h_{-1} \circ d^{\hat{P}}_n = d^{\hat{Q}}_1 \circ h_0$. We now have the base case for our induction.

Suppose now that h_k and h_{k-1} have been constructed for some $k \geqslant 0$ such that $f_k - f'_k = d^{\hat{Q}}_{k+1} \circ h_k + h_{k-1} \circ d^{\hat{P}}_{k+n}$. We have

$$\begin{aligned} d^{\hat{Q}}_{k+1} \circ (f_{k+1} - f'_{k+1} - h_k \circ d^{\hat{P}}_{n+k+1}) &= (f_k - f'_k - d^{\hat{Q}}_{k+1} \circ h_k) \circ d^{\hat{P}}_{n+k+1} \\ &= h_{k-1} \circ d^{\hat{P}}_{k+n} \circ d^{\hat{P}}_{k+n+1} \\ &= 0. \end{aligned}$$

Hence by exactness of $\hat{Q}_*$ and by projectivity of $\hat{P}_{k+n+1}$, we can find a map

$$h_{k+1} \colon \hat{P}_{k+n+1} \to \hat{Q}_{k+2}$$

such that $f_{k+1} - f'_{k+1} = d^{\hat{Q}}_{k+2} \circ h_{k+1} + h_k \circ d^{\hat{P}}_{k+n+1}$.

We now construct the remaining components h_j for all $j \leqslant -2$ via downward induction on j. Suppose now that h_{j+1} and h_j have been constructed for some $j \leqslant -1$ such that $f_{j+1} - f'_{j+1} = d^{\hat{Q}}_{j+2} \circ h_{j+1} + h_j \circ d^{\hat{P}}_{j+n+1}$. We note that we have already constructed the maps h_0 and h_{-1} for the base case of this downward induction. We have

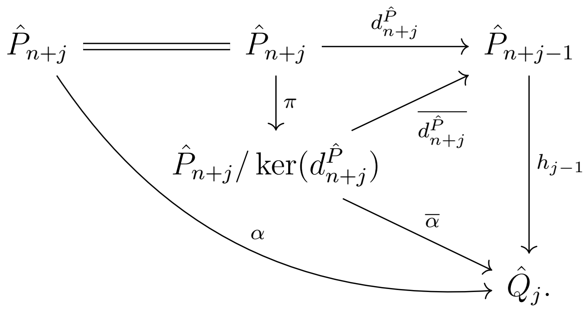
$$\begin{aligned}(f_j - f_j' - d^{\hat{Q}}_{j+1} \circ h_j) \circ d^{\hat{P}}_{j+n+1} &= d^{\hat{Q}}_{j+1} \circ (f_{j+1} - f_{j+1}') - d^{\hat{Q}}_{j+1} \circ (f_{j+1} - f_{j+1}' - d^{\hat{Q}}_{j+2} \circ h_{j+1}) \\ &= d^{\hat{Q}}_{j+1} \circ d^{\hat{Q}}_{j+2} \circ h_{j+1} \\ &= 0.\end{aligned}$$

So denoting $f_j - f_j' - d^{\hat{Q}}_{j+1} \circ h_j$ by α, we see that α restricted to $\mathrm{Im}(d^{\hat{P}}_{j+n+1})$ is 0. But by exactness of $\hat{P}_*$, $\mathrm{Im}(d^{\hat{P}}_{j+n+1}) = \ker(d^{\hat{P}}_{j+n})$. Hence we get an induced map

$$\overline{\alpha}\colon \hat{P}_{j+n}/\ker(d^{\hat{P}}_{j+n}) \to \hat{Q}_j.$$

Moreover, the induced map $\overline{d^{\hat{P}}_{j+n}}$ on this cokernel is clearly injective, and hence by injectivity of $\hat{Q}_j$, we get a map $h_{j-1}\colon \hat{P}_{j+n-1} \to \hat{Q}_j$ such that $h_{j-1} \circ \overline{d^{\hat{P}}_{j+n}} = \overline{\alpha}$.

We have $\alpha \circ h_j = \overline{\alpha} \circ \pi$, where $\pi\colon \hat{P}_{j+n} \to \hat{P}_{j+n}/\ker(d^{\hat{P}}_{j+n})$ is the projection map. By construction of the map h_{j-1}, we have shown that the following diagram commutes.



Hence $\alpha = h_{j-1} \circ d^{\hat{P}}_{j+n}$, i.e. $f_j - f_j' = d^{\hat{Q}}_{j+1} \circ h_j + h_{j-1} \circ d^{\hat{P}}_{j+n}$, as required.

Hence we have constructed a sequence of maps $h_i\colon \hat{P}_{i+n} \to \hat{Q}_{i+1}$ such that $f_i - f_i' = d^{\hat{Q}}_{i+1} \circ h_i + h_{i-1} \circ d^{\hat{P}}_{i+n}$ for all $i \in \mathbb{Z}$, so this defines a chain homotopy h from f to f'. □

Remark 3.1.0.3. This result shows that if we let f, f' be chain maps built as in Construction 3.1.0.1 from γ and γ' respectively, then f and f' are chain homotopic. Moreover, setting $\gamma = \gamma'$ shows that different choices of chain map in Construction 3.1.0.1 are chain homotopic.

As an immediate consequence, we have a well-defined map Ψ from cohomology classes of maps $\hat{P}_n \to N$ to chain homotopy classes of degree n chain maps $\hat{P}_* \to \hat{Q}_*$. We now wish to show that in fact this map is an isomorphism when we work over a finitely generated Hopf algebra which is projective over its base ring and when $\hat{P}_*$ and $\hat{Q}_*$ are complete resolutions of M and N respectively. We recall from Definition 2.2.2.1 that a complete resolution of an A-module is a totally acyclic complex of finitely generated projective A-modules such that the non-negative truncation is isomorphic to a projective resolution of the A-module.

Theorem 3.1.0.4. *Let A be a finitely generated Hopf algebra which is projective over its base ring R, and let N be a finitely generated A-module. Let $\hat{P}_*$ be a complete resolution of the trivial A-module R, and let $\hat{Q}_*$ be a complete resolution of N. Then for all $n \in \mathbb{Z}$, $\widehat{\operatorname{Ext}}^n_A(R, N)$ is isomorphic to the R-module of homotopy equivalence classes of degree n chain maps $\hat{P}_* \to \hat{Q}_*$.*

Proof. We note that the complete resolutions $\hat{P}_*$ and $\hat{Q}_*$ are acyclic by definition, and also recall from Propositions 2.2.1.5 and 2.2.1.6 that A is a Frobenius algebra so injective modules and projective modules coincide. In particular, the objects $\hat{P}_i$ and $\hat{Q}_i$ are injective as well as projective for all $i \in \mathbb{Z}$, and we can apply Construction 3.1.0.1 and Proposition 3.1.0.2.

Let $n \in \mathbb{Z}$. As noted in Section 2.2, the elements of the n^{th} cohomology module $H^n(\underline{\operatorname{Hom}}^\bullet_A(\hat{P}_*, \hat{Q}_*))$ are homotopy equivalence classes of degree n chain maps $\hat{P}_* \to \hat{Q}_*$. The R-module $\widehat{\operatorname{Ext}}^n_A(R, N)$ consists of equivalence classes of cocycles $\hat{P}_* \to N$. We write $\overline{f}$ for the homotopy equivalence class of a degree n chain map $f\colon \hat{P}_* \to \hat{Q}_*$, and we write $[\gamma]$ for the equivalence class of a cocycle $\gamma\colon \hat{P}_* \to N$. We wish to show that there is an R-module isomorphism

$$H^n(\underline{\operatorname{Hom}}^\bullet_A(\hat{P}_*, \hat{Q}_*)) \underset{\Psi}{\overset{\Phi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} \widehat{\operatorname{Ext}}^n_A(R, N).$$

We define the maps Φ, Ψ by

$$\Phi(\overline{f}) = [\varepsilon \circ f_0]$$
$$\Psi([\gamma]) = \overline{g},$$

where $g\colon \hat{P}_* \to \hat{Q}_*$ is the unique up to chain homotopy degree n chain map such that $\varepsilon \circ g_0 = \gamma$ as in Construction 3.1.0.1 and Proposition 3.1.0.2. We want to show that Φ and Ψ are well defined maps.

Let $f, f'\colon \hat{P}_* \to \hat{Q}_*$ be two chain homotopic degree n chain maps, i.e. there exists a sequence of maps $h_i\colon \hat{P}_{i+n} \to \hat{Q}_{i+1}$ such that $f_i - f'_i = d^{\hat{Q}}_{i+1} \circ h_i + h_{i-1} \circ d^{\hat{P}}_{i+n}$ for all $i \in \mathbb{Z}$. Then

$$\begin{aligned}
\Phi(\overline{f}) = [\varepsilon \circ f_0] &= [\varepsilon \circ (f'_0 + d^{\hat{Q}}_1 \circ h_0 + h_{-1} \circ d^{\hat{P}}_n)] \\
&= [\varepsilon \circ f'_0] + [\varepsilon \circ d^{\hat{Q}}_1 \circ h_0] + [\varepsilon \circ h_{-1} \circ d^{\hat{P}}_n] \\
&= \Phi(\overline{f'}) + [0] + [\varepsilon \circ h_{-1} \circ d^{\hat{P}}_n] \\
&= \Phi(\overline{f'}) + [0] + [0]
\end{aligned}$$

since $\varepsilon \circ h_{-1} \circ d^{\hat{P}}_n$ is the boundary of $\varepsilon \circ h_{-1}$ in $\operatorname{Hom}(\hat{P}_*, N)$. Hence Φ is well defined.

Now let $\gamma, \gamma'\colon \hat{P}_n \to N$ be two cocycles representing the same equivalence class in $\widehat{\operatorname{Ext}}^n_A(R, N)$, i.e. there exists a map $\beta\colon \hat{P}_{n-1} \to N$ such that $\gamma = \gamma' + \beta \circ d^{\hat{P}}_n$. Then by Proposition 3.1.0.2, $\Psi([\gamma]) = \Psi([\gamma'])$, hence Ψ is well defined.

We see that Φ is surjective since $\Phi(\Psi([\gamma])) = [\gamma]$. We now show that Φ is also injective so that Φ is in fact bijective with inverse Ψ.

Let $f\colon \hat{P}_* \to \hat{Q}_*$ be a degree n chain map with $\overline{f}$ in the kernel of Φ. Then $\varepsilon \circ f_0$ is the boundary of some map $\beta\colon \hat{P}_{n-1} \to N$, i.e. $\varepsilon \circ f_0 = \beta \circ d_n^{\hat{P}}$. Then we apply Proposition 3.1.0.2 to $\gamma = \beta \circ d_n^{\hat{P}}$ and $\gamma' = 0$ to see that f is null-homotopic. Hence Φ is injective, so it is bijective. Since $\Phi(\Psi([\gamma])) = [\gamma]$, Ψ is the right inverse of Φ and because Φ is bijective it must also be a left inverse of Φ. So Ψ is indeed the inverse of Φ.

Finally, Φ is an R-module homomorphism because ε is an R-module homomorphism and since Ψ is inverse to Φ, Ψ is also an R-module homomorphism. Therefore Φ and Ψ are R-module isomorphisms. □

3.2 Multiplicative Structure

We now wish to compare different products on complete Ext. We will prove that when we work over finite-dimensional Hopf algebras over a field, the isomorphisms Φ and Ψ from Theorem 3.1.0.4 preserve the composition product of chain maps and cup product to be defined below in Definition 3.2.0.9, hence showing that they can be extended to isomorphisms of graded rings. This result first implicitly appeared for Frobenius algebras over a field in [BKS04, Section 6.4], however the result was not proved in that paper, and we are not aware of any proofs of this in the literature for either Frobenius algebras or Hopf algebras. We note that although Theorem 3.1.0.4 was proved over finitely generated Hopf algebras over a ring, we will see that our definition of the cup product uses a construction which requires us to work over finite-dimensional Hopf algebras over a field. We will then show that we get an A_∞-structure on complete Ext.

Following a discussion in [Ngu13, Section 6], we note that constructing the cup product on complete Ext is more challenging than for ordinary Ext. Let A be a finite-dimensional Hopf algebra over a field k, and let $\hat{P}_*$ be a complete resolution of the trivial A-module k. As $\hat{P}_*$ is unbounded, we need to use the complete tensor product $(\hat{P} \mathbin{\hat{\otimes}} \hat{P})_*$ rather than the ordinary (direct sum) tensor product that we used for projective resolutions in Subsection 2.2.3.

Recall from Definition 2.2.0.4 that the complete tensor product $(\hat{P} \mathbin{\hat{\otimes}} \hat{P})_*$ is defined by

$$(\hat{P} \mathbin{\hat{\otimes}} \hat{P})_i = \prod_{p+q=i} \hat{P}_p \otimes \hat{P}_q,$$

with differential defined by

$$\partial_i(x)_{p+i-1,-p} = (d_{p+i}^{\hat{P}} \otimes \mathrm{id}_{\hat{P}_{-p}})(x_{p+i,-p}) + (-1)^{p+i-1}(\mathrm{id}_{\hat{P}_{p+i-1}} \otimes d_{-p+1}^{\hat{P}})(x_{p+i-1,-p+1})$$

for $x \in (\hat{P} \mathbin{\hat{\otimes}} \hat{P})_i$ where $y_{p+i,-p}$ denotes the projection of $y \in (\hat{P} \mathbin{\hat{\otimes}} \hat{P})_i$ to $\hat{P}_{p+i} \otimes \hat{P}_{-p}$. In addition, given chain maps $u, v\colon \hat{P}_* \to \hat{P}_*$ of degrees r and s respectively, there is a degree $r+s$ chain map $u \mathbin{\hat{\otimes}} v$ defined by

$$(u \mathbin{\hat{\otimes}} v)_i = \prod_{p+q=i} u_p \otimes v_q \colon (\hat{P} \mathbin{\hat{\otimes}} \hat{P})_{i+r+s} \to (\hat{P} \mathbin{\hat{\otimes}} \hat{P})_i$$

for all $i \in \mathbb{Z}$, and when $u \mathbin{\hat{\otimes}} v$ is applied to an elementary tensor $a \otimes b$ this will introduce a sign of $(-1)^{|a|s}$ coming from the twist map of graded A-modules.

We can see from this that, where $\varepsilon\colon \hat{P}_0 \to k$ is the augmentation of $\hat{P}_*$ (considered as a degree 0 chain map from $\hat{P}_*$ to the chain complex with k concentrated in degree 0),

$$(\varepsilon \mathbin{\hat{\otimes}} \varepsilon)_i = \begin{cases} \varepsilon \otimes \varepsilon, & i = 0 \\ 0, & \text{otherwise.} \end{cases}$$

In order to define the cup product on complete Ext, we need to define an analogue of a diagonal approximation (see Subsection 2.2.3) for complete resolutions, and then show that such a map always exists.

Definition 3.2.0.1. Let A be a finite-dimensional Hopf algebra over a field k, and let $\hat{P}_*$ be a complete resolution of the trivial A-module k with augmentation ε. A *complete diagonal approximation* is a (degree 0) chain map

$$\hat{\Delta}\colon \hat{P}_* \to (\hat{P} \mathbin{\hat{\otimes}} \hat{P})_*$$

such that $(\varepsilon \mathbin{\hat{\otimes}} \varepsilon) \circ \hat{\Delta}_0 = \varepsilon$, i.e. $\hat{\Delta}$ is an augmentation-preserving chain map.

Remark 3.2.0.2. In general, a direct product of projective modules is not projective. However, recall from Remark 2.2.1.4 that finite-dimensional Hopf algebras are Frobenius algebras, in particular this means that injective and projective modules coincide over a finite-dimensional Hopf algebra by Proposition 2.2.1.5. Since a direct product of injective modules *is* injective (and therefore projective, as above), we see that in this case all the components of $(\hat{P} \mathbin{\hat{\otimes}} \hat{P})_*$ are both projective and injective.

We now show that a complete diagonal approximation always exists for a complete resolution of the trivial A-module k and it is unique up to chain homotopy. We would like to use Construction 3.1.0.1 and Proposition 3.1.0.2 for this, however since $(\hat{P} \mathbin{\hat{\otimes}} \hat{P})_*$ is *not* a complete resolution of k, we do not meet the conditions to apply the results. Instead, we follow [Bro82, Section VI.5] (filling in details not written up over Hopf algebras) to inductively build the degree 0 component of a complete diagonal approximation, and then use a comparison of resolutions type result to extend it to a chain map which will be unique up to chain homotopy.

Let A be a finite-dimensional Hopf algebra over a field k and let $\hat{P}_*$ be a complete resolution of the trivial A-module k. We also denote the underlying k-vector space of $\hat{P}_i$ by $U(\hat{P}_i)$, and likewise the complex of underlying k-vector spaces of $\hat{P}_*$ by $U(\hat{P}_*)$.

The proof relies on showing that the complex $\hat{P}_* \otimes \hat{P}_j$ is contractible as a complex of A-modules for any given integer j. One would like to do this by using a contracting homotopy for $\hat{P}_*$. However, this does not always exist. For example, let $A = \mathbb{F}_2[x]/(x^2)$ be the algebra of dual numbers over $\mathbb{F}_2$. This can be given a Hopf algebra structure with primitive comultiplication, $\varepsilon(x) = 0$ and antipode equal to the identity. There is an A-module complete resolution of $\mathbb{F}_2$ where every module is A and $d_i(1) = x$ for all integers i (with augmentation given by the counit). As noted in [Pos24, Introduction], this complex is acyclic but not contractible. Indeed, any choice of contracting homotopy will not be an A-module map at any given degree. But it is possible to find a contracting homotopy of the complex of underlying $\mathbb{F}_2$-vector spaces.

By [Wei94, Section 1.4], every acyclic complex of vector spaces is contractible. Therefore we can choose a contracting homotopy h for $U(\hat{P}_*)$. We will use this to build a contracting homotopy h' for $\hat{P}_* \otimes \hat{P}_i$.

Proposition 3.2.0.3 ([Bak25, Proposition 4.30]). *Let A be a finite-dimensional Hopf algebra over a field k and let M be an A-module and $U(M)$ be its underlying k-vector space. Let $A \otimes U(M)$ be the extended module for $U(M)$, where the A-module action is given by $a' \cdot (a \otimes m) = a'a \otimes m$. Recall also from Examples 2.1.0.11 (1) that $M \otimes A$ is an A-module with action given by $a' \cdot (m \otimes a) = a'_{(1)} m \otimes a'_{(2)} a$. There is an isomorphism of A-modules (natural in M)*

$$A \otimes U(M) \underset{\Phi^{-1}}{\overset{\Phi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} M \otimes A,$$

where

$$\begin{aligned} \Phi(a \otimes x) &= a_{(1)} x \otimes a_{(2)} \\ \Phi^{-1}(x \otimes a) &= a_{(2)} \otimes S(a_{(1)}) x, \end{aligned}$$

where we use sumless Sweedler notation and S denotes the antipode.

Proof. We note that Φ and Φ^{-1} are both k-linear. We will show that Φ is an A-module map and that Φ^{-1} is indeed the inverse for Φ, and hence automatically an A-module map.

Let $a, a' \in A$ and let $x \in U(M)$. We will show that Φ is a map of left A-modules:

$$\begin{aligned} \Phi(a'(a \otimes x)) = \Phi(a'a \otimes x) &= (a'a)_{(1)} x \otimes (a'a)_{(2)} \\ &= a'_{(1)} a_{(1)} x \otimes a'_{(2)} a_{(2)} && (A \text{ is a bialgebra}) \\ &= a'(a_{(1)} x \otimes a_{(2)}) && (\text{diagonal } A\text{-action on } M \otimes A) \\ &= a' \Phi(a \otimes x). \end{aligned}$$

Hence Φ is indeed A-linear. So it remains to show that Φ is bijective.

$$\begin{aligned} \Phi^{-1}(\Phi(a \otimes x)) = \Phi^{-1}(a_{(1)} x \otimes a_{(2)}) &= a_{(2)(2)} \otimes S(a_{(2)(1)}) a_{(1)} x \\ &= a_{(2)} \otimes S(a_{(1)(2)}) a_{(1)(1)} x && (\text{coassociativity}) \\ &= a_{(2)} \otimes \eta(\varepsilon(a_{(1)})) x && (S \text{ antipode}) \\ &= a_{(2)} \otimes \varepsilon(a_{(1)}) x && (A\text{-module action on } M) \\ &= \varepsilon(a_{(1)}) a_{(2)} \otimes x \\ &= a \otimes x && (A \text{ coalgebra}). \end{aligned}$$

Similarly, we find

$$\begin{aligned} \Phi(\Phi^{-1}(x \otimes a)) = \Phi(a_{(2)} \otimes S(a_{(1)}) x) &= a_{(2)(1)} S(a_{(1)}) x \otimes a_{(2)(2)} \\ &= a_{(1)(2)} S(a_{(1)(1)}) x \otimes a_{(2)} && (\text{coassociativity}) \\ &= \eta(\varepsilon(a_{(1)})) x \otimes a_{(2)} && (S \text{ antipode}) \\ &= x \otimes \varepsilon(a_{(1)}) a_{(2)} && (A\text{-module action on } M) \\ &= x \otimes a && (A \text{ coalgebra}). \end{aligned}$$

Hence Φ and Φ^{-1} are mutually inverse A-module maps. □

From now on, we will always apply Proposition 3.2.0.3 to $M = \hat{P}_i$.

Lemma 3.2.0.4. *Let h be a contracting homotopy for $U(\hat{P}_*)$. Then $h' = \Phi(\mathrm{id} \otimes h)\Phi^{-1}$ is a contracting homotopy for the A-module complex $\hat{P}_* \otimes A$.*

Proof. We note that the differential on the A-module complex $\hat{P}_* \otimes A$ is given by $d \otimes \mathrm{id}_A$, where d is the differential on $\hat{P}_*$. The result follows from

$$\begin{aligned}(d_{i+1} \otimes \mathrm{id}_A)h'_i + h'_{i-1}(d_i \otimes \mathrm{id}_A) &= (d_{i+1} \otimes \mathrm{id}_A)\Phi(\mathrm{id}_A \otimes h)\Phi^{-1} + \Phi(\mathrm{id}_A \otimes h)\Phi^{-1}(d_i \otimes \mathrm{id}_A) \\ &= \Phi(\mathrm{id}_A \otimes d_{i+1})(\mathrm{id}_A \otimes h)\Phi^{-1} + \Phi(\mathrm{id}_A \otimes h)(\mathrm{id}_A \otimes d_i)\Phi^{-1} \\ &= \Phi(\mathrm{id}_A \otimes \mathrm{id}_{\hat{P}_*})\Phi^{-1} \\ &= \mathrm{id}_{\hat{P}_*} \otimes \mathrm{id}_A,\end{aligned}$$

where Φ and Φ^{-1} commute with the differential on $\hat{P}_i \otimes A$ by d being an A-module map. □

We now extend this result to the A-module complex $\hat{P}_* \otimes \hat{P}_j$ for any integer j.

Lemma 3.2.0.5. *Let Q be a projective A-module. Then the A-module complex $\hat{P}_* \otimes Q$ is contractible.*

Proof. Since Q is projective, it is the retract of a free A-module F, and if we take any A-module M, then the tensor product $M \otimes F$ is a direct sum of copies of M. In addition, as $\hat{P}_* \otimes -$ is a functor, it preserves retracts. It follows from these facts that $\hat{P}_* \otimes Q$ is a retract of a direct sum of copies of $\hat{P}_*$, and since retracts and direct sums preserve contractibility then $\hat{P}_* \otimes Q$ is contractible. □

Remark 3.2.0.6. From Lemma 3.2.0.5, we now know that $\hat{P}_* \otimes \hat{P}_j$ is contractible for any integer j. However, the majority of examples of complete resolutions in this thesis (and indeed, over many Hopf algebras) are comprised of complexes of *free* modules rather than projective modules. Hence in practice, we will not need to directly use the retraction and inclusion maps.

We now have the requisite setup to construct a complete diagonal approximation as in [Bro82, Section VI.5, pages 140–141]. The existence of a complete diagonal approximation for finite-dimensional Hopf algebras over a field was stated without proof in [Ngu13, Section 6.1]. However, Nguyen does not make any comment on uniqueness up to chain homotopy of complete diagonal approximations.

Theorem 3.2.0.7. *Let A be a finite-dimensional Hopf algebra over a field k. For any A-module complete resolution $\hat{P}_*$ of k, there exists a complete diagonal approximation $\hat{\Delta}\colon \hat{P}_* \to (\hat{P} \hat{\otimes} \hat{P})_*$. Moreover, given an initial choice of contracting homotopy for $U(\hat{P}_*)$, the complete diagonal approximation is unique up to chain homotopy.*

Proof. Let ε be the augmentation of $\hat{P}_*$. By the first step in the ordinary comparison theorem of projective resolutions, there exists a map $\alpha_0\colon \hat{P}_0 \to \hat{P}_0 \otimes \hat{P}_0$ such that $(\varepsilon \otimes$

$\varepsilon)\alpha_0 = \varepsilon$, noting that $(\hat{P}_{\geqslant 0} \otimes \hat{P}_{\geqslant 0})_*$ is a projective resolution of $k \otimes k \cong k$. Let α_0 be such a map. Then we want to inductively define maps $\alpha_r \colon \hat{P}_0 \to \hat{P}_r \otimes \hat{P}_{-r}$ such that

$$(d_r \otimes \mathrm{id}_{\hat{P}_{-r}})\alpha_r d_1 = (-1)^r (\mathrm{id}_{\hat{P}_{r-1}} \otimes d_{-r+1})\alpha_{r-1} d_1,$$

so that $\alpha = \prod_{r\in\mathbb{Z}} \alpha_r \colon \hat{P}_0 \to \prod_{r\in\mathbb{Z}} \hat{P}_r \otimes \hat{P}_{-r}$ satisfies $\partial_0^{\hat{P}\hat{\otimes}\hat{P}} \alpha d_1 = 0$ and $(\varepsilon \hat{\otimes} \varepsilon)\alpha = \varepsilon$. Then we will use induction and downward induction to extend α to a complete diagonal approximation.

Let $(h^r)^{r\in\mathbb{Z}}$ be a sequence of contracting homotopies on $(\hat{P}_* \otimes \hat{P}_{-r})_{r\in\mathbb{Z}}$, as constructed in Lemma 3.2.0.5. We define

$$\alpha_r := (-1)^r h^r_{r-1} (\mathrm{id}_{\hat{P}_{r-1}} \otimes d_{-r+1})\alpha_{r-1},$$

for $r \geqslant 1$ noting that h^r_{r-1} is the $(r-1)^{\text{th}}$ component of the contracting homotopy h^r on $\hat{P}_* \otimes \hat{P}_{-r}$.

Now, using the definition of α_r and the contracting homotopy property of h^r_{r-1}, we have

$$\begin{aligned}(d_r \otimes \mathrm{id}_{\hat{P}_{-r}})\alpha_r d_1 &= (-1)^r (d_r \otimes \mathrm{id}_{\hat{P}_{-r}}) h^r_{r-1} (\mathrm{id}_{\hat{P}_{r-1}} \otimes d_{-r+1})\alpha_{r-1} d_1 \\ &= (-1)^r (\mathrm{id}_{\hat{P}_{r-1}} \otimes d_{-r+1})\alpha_{r-1} d_1 + \\ &\quad (-1)^{r+1} h^r_{r-2} (d_{r-1} \otimes \mathrm{id}_{\hat{P}_{-r}})(\mathrm{id}_{\hat{P}_{r-1}} \otimes d_{-r+1})\alpha_{r-1} d_1.\end{aligned}$$

We will show, by induction on r, that

$$(d_r \otimes \mathrm{id}_{\hat{P}_{-r}})\alpha_r d_1 = (-1)^r (\mathrm{id}_{\hat{P}_{r-1}} \otimes d_{-r+1})\alpha_{r-1} d_1$$

holds for all positive integers r. We recall that η and ε are respectively the coaugmentation and augmentation of the complete resolution $\hat{P}_*$. Let $r = 1$. Then

$$\begin{aligned}(d_0 \otimes \mathrm{id}_{\hat{P}_0})(\mathrm{id}_{\hat{P}_0} \otimes d_0)\alpha_0 d_1 &= (d_0 \otimes d_0)\alpha_0 d_1 \\ &= (\eta \otimes \eta)(\varepsilon \otimes \varepsilon)\alpha_0 d_1,\end{aligned}$$

using $d_0 = \eta\varepsilon$. But then α_0 is defined such that $(\varepsilon \otimes \varepsilon)\alpha_0 = \varepsilon$, and since $\hat{P}_*$ is a complete resolution then $\varepsilon d_1 = 0$. Therefore $(d_0 \otimes \mathrm{id}_{\hat{P}_0})(\mathrm{id}_{\hat{P}_0} \otimes d_0)\alpha_0 d_1 = 0$. Hence

$$\begin{aligned}(d_1 \otimes \mathrm{id}_{\hat{P}_{-1}})\alpha_1 d_1 &= -(\mathrm{id}_{\hat{P}_0} \otimes d_0)\alpha_0 d_1 + \\ &\quad h^1_{-1}(d_0 \otimes \mathrm{id}_{\hat{P}_{-1}})(\mathrm{id}_{\hat{P}_0} \otimes d_0)\alpha_0 d_1 \\ &= -(\mathrm{id}_{\hat{P}_0} \otimes d_0)\alpha_0 d_1,\end{aligned}$$

so the base case $r = 1$ for the induction holds.

Now assume that the inductive statement holds for some $r \geqslant 1$. Then

$$\begin{aligned}(d_r \otimes \mathrm{id}_{\hat{P}_{-r-1}})(\mathrm{id}_{\hat{P}_r} \otimes d_{-r})\alpha_r d_1 &= -(\mathrm{id}_{\hat{P}_{r-1}} \otimes d_{-r})(d_r \otimes \mathrm{id}_{\hat{P}_{-r}})\alpha_r d_1 \\ &= (-1)^{r+1} (\mathrm{id}_{\hat{P}_{r-1}} \otimes d_{-r})(\mathrm{id}_{\hat{P}_{r-1}} \otimes d_{-r+1})\alpha_{r-1} d_1 \\ &= 0.\end{aligned}$$

Hence

$$\begin{aligned}(d_{r+1} \otimes \mathrm{id}_{\hat{P}_{-r-1}})\alpha_{r+1}d_1 &= (-1)^{r+1}(\mathrm{id}_{\hat{P}_r} \otimes d_{-r})\alpha_r d_1 + \\ &\quad (-1)^{r+2}h_{r-1}^{r+1}(d_r \otimes \mathrm{id}_{\hat{P}_{-r-1}})(\mathrm{id}_{\hat{P}_r} \otimes d_{-r})\alpha_r d_1 \\ &= (-1)^{r+1}(\mathrm{id}_{\hat{P}_r} \otimes d_{-r})\alpha_r d_1\end{aligned}$$

is satisfied which completes the inductive step, hence the statement holds for all positive integers r. We similarly inductively define maps

$$\alpha_s := (-1)^{-s-1}g_{-s-1}^s(d_{s+1} \otimes \mathrm{id}_{\hat{P}_{-s-1}})\alpha_{s+1},$$

for $s \leqslant -1$ where $(g^s)^{s\in\mathbb{Z}}$ is a sequence of contracting homotopies on $(\hat{P}_s \otimes \hat{P}_*)_{s\in\mathbb{Z}}$. These are constructed in the same way as Lemmas 3.2.0.4 and 3.2.0.5 using the isomorphism of A-modules

$$A \otimes U(\hat{P}_i) \cong A \otimes \hat{P}_i$$

from [Bak25, Proposition 4.30] (note that this is slightly different to the isomorphism in Proposition 3.2.0.3). Then we prove by downward induction that

$$(\mathrm{id}_{\hat{P}_s} \otimes d_{-s})\alpha_s d_1 = (-1)^{-s-1}(d_{s+1} \otimes \mathrm{id}_{\hat{P}_{-s-1}})\alpha_{s+1}d_1$$

holds for all $s \leqslant -1$. We omit the details as the downward induction proceeds in the same way as the ordinary induction argument for positive r. Therefore we have constructed a map $\hat{\Delta} = \alpha = \prod_{i\in\mathbb{Z}} \alpha_i \colon \hat{P}_0 \to \prod_{i\in\mathbb{Z}} \hat{P}_i \otimes \hat{P}_{-i}$ such that $\partial_0^{\hat{P}\hat{\otimes}\hat{P}}\alpha d_1 = 0$ and $(\varepsilon \mathbin{\hat{\otimes}} \varepsilon)\alpha = \varepsilon$. Then by the same induction and downward induction argument utilised in Construction 3.1.0.1 and Proposition 3.1.0.2, we can extend this to a complete diagonal approximation $\hat{\Delta} \colon \hat{P}_* \to (\hat{P} \mathbin{\hat{\otimes}} \hat{P})_*$, and given an initial choice of contracting homotopy for $U(\hat{P}_*)$, this extension is unique up to chain homotopy. □

Remark 3.2.0.8. We conjecture that it should be possible to remove the condition of choosing an initial contracting homotopy for $U(\hat{P}_*)$. However, we will see in Proposition 3.2.0.10 that the construction of the cup product does not depend on this choice.

With the existence of this complete diagonal approximation, we can now define a cup product on complete Ext.

Definition 3.2.0.9. Let $i, j \in \mathbb{Z}$ and let f be a map $\hat{P}_i \to k$ and g a map $\hat{P}_j \to k$. Let $\hat{\Delta} \colon \hat{P}_* \to (\hat{P} \mathbin{\hat{\otimes}} \hat{P})_*$ be a complete diagonal approximation. The *cup product*

$$\smile \colon \mathrm{Hom}(\hat{P}_i, k) \otimes \mathrm{Hom}(\hat{P}_j, k) \to \mathrm{Hom}(\hat{P}_{i+j}, k \otimes k)$$

is defined on cochains by

$$f \smile g = (f \mathbin{\hat{\otimes}} g) \circ \hat{\Delta}.$$

This gives a well-defined product on complete Ext

$$\smile \colon \widehat{\mathrm{Ext}}_A^i(k,k) \otimes \widehat{\mathrm{Ext}}_A^j(k,k) \to \widehat{\mathrm{Ext}}_A^{i+j}(k,k)$$

since the cup product of two cocycles is again a cocycle (and using the fact that $k \otimes k \cong k$), and we will show in Proposition 3.2.0.10 that it is independent of choices involved in the construction of $\hat{\Delta}$.

Recall from Theorem 3.1.0.4 (specialised to the case where A is a finite-dimensional Hopf algebra over a field k, and where $N = k$) the following isomorphism of k-vector spaces:

$$H^n(\underline{\mathrm{Hom}}_A^*(\hat{P}_*, \hat{P}_*)) \underset{\Psi}{\overset{\Phi}{\underset{\longleftarrow}{\overset{\longrightarrow}{\cong}}}} \widehat{\mathrm{Ext}}_A^n(k, k).$$

with

$$\Phi(\overline{f}) = [\varepsilon \circ f_0]$$
$$\Psi([\gamma]) = \overline{g},$$

where $g\colon \hat{P}_* \to \hat{P}_*$ is the unique up to chain homotopy degree n chain map such that $\varepsilon \circ g_0 = \gamma$ as in Construction 3.1.0.1 and Proposition 3.1.0.2.

We now prove that Φ preserves the multiplicative structure on both sides, generalising [Bro82, Theorem VI.6.2].

Proposition 3.2.0.10. *Let A be a finite-dimensional Hopf algebra over a field k and let $\hat{P}_*$ be an A-module complete resolution of k, and let Φ be as defined above. Writing $\widehat{\underline{\mathrm{Ext}}}_A^i(k, k)$ for $H^i(\underline{\mathrm{Hom}}_A^*(\hat{P}_*, \hat{P}_*))$, the following diagram commutes for all $n, m \in \mathbb{Z}$:*

$$\begin{array}{ccc} \widehat{\underline{\mathrm{Ext}}}_A^n(k,k) \otimes \widehat{\underline{\mathrm{Ext}}}_A^m(k,k) & \xrightarrow{\ \circ\ } & \widehat{\underline{\mathrm{Ext}}}_A^{n+m}(k,k) \\ {\scriptstyle \Phi\otimes\Phi}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Phi} \\ \widehat{\mathrm{Ext}}_A^n(k,k) \otimes \widehat{\mathrm{Ext}}^m(k,k) & \xrightarrow{\ \smile\ } & \widehat{\mathrm{Ext}}_A^{n+m}(k,k). \end{array}$$

Proof. We follow the line of proof for usual Ext in [Bro82, Theorem V.4.6]. Let f, g be chain maps $\hat{P}_* \to \hat{P}_*$ of degrees n and m respectively, and let $\gamma = \varepsilon \circ f_0$ and $\gamma' = \varepsilon \circ g_0$ (so $\Phi(\overline{f}) = [\gamma], \Phi(\overline{g}) = [\gamma']$). Then

$$\Phi(\overline{f \circ g}) = [\varepsilon(fg)_0] = [(\varepsilon \circ f_0)g_n] = [\gamma \circ g_n].$$

We claim that $\overline{g} = \overline{(\mathrm{id}_{\hat{P}_*} \otimes \gamma')\hat{\Delta}}$, where $\hat{\Delta}\colon \hat{P}_* \to (\hat{P} \hat{\otimes} \hat{P})_*$ is a complete diagonal approximation constructed as in Theorem 3.2.0.7. By Theorem 3.1.0.4, we only need to show that

$$\Phi(\overline{g}) = \Phi(\overline{(\mathrm{id}_{\hat{P}_*} \otimes \gamma')\hat{\Delta}}) = [\gamma'].$$

We have

$$\begin{aligned} \Phi(\overline{(\mathrm{id}_{\hat{P}_*} \otimes \gamma')\hat{\Delta}}) = [\varepsilon \circ ((\mathrm{id}_{\hat{P}_*} \otimes \gamma')\hat{\Delta})_0] &= [(\varepsilon \otimes \mathrm{id}_k)(\mathrm{id}_{\hat{P}_0} \otimes \gamma')\hat{\Delta}_m] \\ &= [(\varepsilon \otimes \gamma')\hat{\Delta}_m] \\ &= [\varepsilon \smile \gamma']. \end{aligned}$$

Now, $(\varepsilon \otimes \mathrm{id}_{\hat{P}_*})\hat{\Delta}\colon \hat{P}_* \to \hat{P}_*$ is a degree 0 map of complete resolutions lifting the identity on k (using the canonical isomorphism $k \otimes k \cong k$ and the fact that $\hat{\Delta}$ is a complete diagonal approximation). Hence the m-cocycle γ' is homotopic to $(\mathrm{id}_k \otimes \gamma')(\varepsilon \otimes \mathrm{id}_{\hat{P}_*})\hat{\Delta} = \varepsilon \smile \gamma'$. Therefore

$$\begin{aligned}\Phi(\overline{(\mathrm{id}_{\hat{P}_*} \otimes \gamma')\hat{\Delta}}) &= [\varepsilon \smile \gamma'] \\ &= [\gamma'].\end{aligned}$$

We also have $\Phi(\overline{g}) = [\gamma']$ by definition, so $\overline{g} = \overline{(\mathrm{id}_{\hat{P}_*} \otimes \gamma')\hat{\Delta}}$. Hence

$$[\gamma \circ g_n] = [(\gamma \otimes \mathrm{id}_k)(\mathrm{id}_{\hat{P}_n} \otimes \gamma')\hat{\Delta}_{n+m}] = [\gamma \smile \gamma'],$$

and $[\gamma \smile \gamma'] = \Phi(\overline{f}) \smile \Phi(\overline{g})$ which shows that the diagram commutes. □

Proposition 3.2.0.10 now allows us to recover the result that the cohomology of the endomorphism algebra of the complete resolution $\hat{P}_*$ is isomorphic to the graded complete Ext algebra of k. As noted in Section 1.1, this result has been stated in [BKS04, Section 6.4] without proof for Frobenius algebras over a field, and we are not aware of any proofs of this in the literature.

Corollary 3.2.0.11. *Let A be a finite-dimensional Hopf algebra over a field k, and let $\hat{P}_*$ be a complete resolution of the trivial A-module k. Then the cohomology of the endomorphism algebra of the complete resolution $\hat{P}_*$ is isomorphic to the graded complete Ext algebra of k. That is,*

$$\widehat{\mathrm{Ext}}^*_A(k,k) \cong H^*(\underline{\mathrm{Hom}}^*_A(\hat{P}_*, \hat{P}_*)).$$

Moreover, as $\underline{\mathrm{Hom}}^*_A(\hat{P}_*, \hat{P}_*)$ is a dg algebra, we get an A_∞-structure on $\widehat{\mathrm{Ext}}^*_A(k,k)$ with m_2 induced by the multiplication on $\underline{\mathrm{Hom}}^*_A(\hat{P}_*, \hat{P}_*)$.

Proof. The first statement follows immediately from Proposition 3.2.0.10. We briefly prove the second result by using Theorem 2.3.1.2 (Kadeishvili's Theorem [Kad80]). Since $\underline{\mathrm{Hom}}^*_A(\hat{P}_*, \hat{P}_*)$ is a chain complex with an associative and unital pairing

$$\mu\colon\ \underline{\mathrm{Hom}}^*_A(\hat{P}_*, \hat{P}_*) \otimes \underline{\mathrm{Hom}}^*_A(\hat{P}_*, \hat{P}_*) \to \underline{\mathrm{Hom}}^*_A(\hat{P}_*, \hat{P}_*)$$

given by composition of chain homomorphisms, then it is in fact a dg algebra.

Now, by [LV12, Lemma 9.4.4], any chain complex of k-vector spaces (and thus any dg algebra) admits its homology as a deformation retract. We can therefore apply Kadeishvili's Theorem to show that there is an A_∞-structure on the homology algebra of any dg algebra, and this A_∞-structure has vanishing differential and m_2 induced by the multiplication on the dg algebra.

In particular, we apply this result to our dg algebra $\underline{\mathrm{Hom}}^*_A(\hat{P}_*, \hat{P}_*)$ and by the isomorphism $\widehat{\mathrm{Ext}}^*_A(k,k) \cong H^*(\underline{\mathrm{Hom}}^*_A(\hat{P}_*, \hat{P}_*))$, we have an A_∞-structure on $\widehat{\mathrm{Ext}}^*_A(k,k)$ with m_2 induced by μ. □

3.3 Examples

3.3.1 The Sweedler Hopf Algebra

We now calculate the additive and multiplicative structure of complete Ext of the Sweedler Hopf algebra [Swe69, pages 89–90].

Definition 3.3.1.1. Let k be a field where $\operatorname{char} k \neq 2$. The *Sweedler Hopf algebra* H_4 is generated as a k-algebra by g, x subject to the relations $g^2 = 1, x^2 = 0$ and $xg = -gx$. We define the comultiplication by

$$\Delta(g) = g \otimes g, \quad \Delta(x) = 1 \otimes x + x \otimes g,$$

the counit by

$$\varepsilon(g) = 1, \quad \varepsilon(x) = 0,$$

and the antipode by

$$S(g) = g, \quad S(x) = gx,$$

and this makes H_4 a 4-dimensional Hopf algebra (with vector space basis $\{1, g, x, gx\}$).

H_4 is often used in the literature as a low-dimensional example of a Hopf algebra that is neither cocommutative nor commutative. We note that H_4 over a field of characteristic 2 would, however, be commutative (since $-1 = 1$, therefore $xg = gx$).

We start by following the calculations of Nguyen [Ngu13, Section 8.1] to compute the additive structure of $\widehat{\operatorname{Ext}}^n_{H_4}(k, k)$. However, we will show that the claimed complete resolution of k used in the paper to compute this is not, in fact, a complete resolution. We will show that a minor modification of this does compute complete Ext.

As an algebra, H_4 can be considered as a skew group algebra (see [Mon93, Example 4.1.6], for example) $B\#kC_2$ where C_2 is the cyclic group of order two generated by g, and $B = k[x]/(x^2)$ are the dual numbers. We take the following B-module resolution of k:

$$\cdots \xrightarrow{d} B \xrightarrow{d} B \xrightarrow{d} B \xrightarrow{d} B \xrightarrow{\varepsilon} k \to 0$$

where $d(b) = bx$ for all $b \in B$. We want to extend this to be an H_4-module projective resolution of k, and we can do this by giving $1, x \in B$ appropriate C_2-actions depending on the degree of the resolution. Let B_0 be the H_4-module which has a basis $\{1_0, x_0\}$ and C_2-action defined by $g \cdot 1_0 = 1_0$ and $g \cdot x_0 = -x_0$. One checks that, together with $x \cdot 1_0 = x_0$, $x \cdot x_0 = 0$, $gx \cdot 1_0 = g \cdot x_0 = -x_0$ and $gx \cdot x_0 = 0$ this does indeed give B_0 an H_4-module structure. Similarly, one checks that B_1 is the H_4-module which has a basis $\{1_1, x_1\}$ and C_2-action defined by $g \cdot 1_1 = -1_1$ and $g \cdot x_1 = x_1$. We can now extend the above resolution to be an H_4-module resolution of the trivial H_4-module k (where H_4 acts via the counit) by putting B_0 in every even degree and B_1 in every odd degree.

Lemma 3.3.1.2. *The following is an H_4-module projective resolution of k:*

$$\cdots \xrightarrow{d} B_1 \xrightarrow{d} B_0 \xrightarrow{d} B_1 \xrightarrow{d} B_0 \xrightarrow{\varepsilon} k \to 0,$$

where $d(1_0) = x_1, d(x_0) = d(x_1) = 0$ and $d(1_1) = x_0$ (that is, d is "multiplication on the right by x"). We denote the resolution by P_*.

Proof. We note that the maps d can easily be shown to be maps of H_4-modules since, for example

$$d(g \cdot 1_0) = d(1_0) = x_1 = g \cdot x_1 = g \cdot d(1_0).$$

We will prove that each H_4-module in P_* is projective by showing that $B_0 \oplus B_1 \cong H_4$ as H_4-modules.

Let A_0 be the k-submodule of H_4 generated by $\{1+g, x-gx\}$. Then $g \cdot (1+g) = 1+g$ and $g \cdot (x-gx) = -(x-gx)$. We also have $x \cdot (1+g) = x - gx$ and $x \cdot (x-gx) = 0$. Hence we can identify $1+g \leftrightarrow 1_0$ and $x - gx \leftrightarrow x_0$ and show that $A_0 \cong B_0$ as H_4-modules.

Let A_1 be the k-module generated by $\{1-g, x+gx\}$. By calculating the C_2-actions and the action of x we can similarly show that $A_1 \cong B_1$ as H_4-modules. Finally, we can show that $\{1+g, x-gx, 1-g, x+gx\}$ is a basis for H_4 because, for example,

$$1 = \frac{1}{2}((1+g) + (1-g))$$

since 2 is invertible in k as we assume that $\operatorname{char} k \neq 2$. The other basis elements are similar, and we conclude that $B_0 \oplus B_1 \cong H_4$ as H_4-modules hence B_0 and B_1 are projective as H_4-modules so P_* is indeed a projective resolution of k. □

When we later compute the multiplicative structure of $\widehat{\operatorname{Ext}}^*_{H_4}(k,k)$, we will need to know what an arbitrary H_4-module homomorphism $B_i \to B_j$ looks like for all choices of (i,j).

Lemma 3.3.1.3. *Let $\chi_{i,j} \colon B_i \to B_j$ be an H_4-module homomorphism for all $i \in \{0,1\}$ and $j \in \{0,1\}$. Then*

$$\chi_{i,j}(b_i) = \begin{cases} \alpha b_j, & i = j = 0 \text{ or } i = j = 1 \\ \beta x \cdot b_j, & \text{otherwise,} \end{cases}$$

for some $\alpha, \beta \in k$. That is, any H_4-module homomorphism $B_0 \to B_0$ or $B_1 \to B_1$ is "multiplication by α" for some $\alpha \in k$, and any H_4-module homomorphism $B_0 \to B_1$ or $B_1 \to B_0$ is "multiplication by βx" for some $\beta \in k$.

Proof. We first look at $\chi_{0,0}$. The image of 1_0 under $\chi_{0,0} \colon B_0 \to B_0$ must be of the form $\alpha 1_0 + \beta x_0$ for some $\alpha, \beta \in k$, and the image of x_0 under $\chi_{0,0}$ must be of the form $\gamma 1_0 + \delta x_0$ for some $\gamma, \delta \in k$. Then

$$g \cdot \chi_{0,0}(1_0) = \alpha g \cdot 1_0 + \beta g \cdot x_0 = \alpha 1_0 - \beta x_0,$$

and

$$\chi_{0,0}(g \cdot 1_0) = \chi_{0,0}(1_0) = \alpha 1_0 + \beta x_0.$$

Since we want $\chi_{0,0}$ to be an H_4-module map, then we must have $\beta = 0$ since $\operatorname{char} k \neq 2$. Similar calculations for the action of other elements of H_4 show that we also have $\gamma = 0$ and $\alpha = \delta$. Hence $\chi_{0,0}(b_0) = \alpha b_0$ for some $\alpha \in k$, for all $b_0 \in B_0$.

The analogous calculations for $\chi_{1,1} \colon B_1 \to B_1$ give $\chi_{1,1}(b_1) = \alpha b_1$ for some $\alpha \in k$, for all $b_1 \in B_1$.

We now look at $\chi_{0,1}$. The image of 1_0 under $\chi_{0,1}\colon B_0 \to B_1$ must be of the form $\alpha 1_1 + \beta x_1$ for some $\alpha, \beta \in k$, and the image of x_0 under $\chi_{0,1}$ must be of the form $\gamma 1_1 + \delta x_1$ for some $\gamma, \delta \in k$. Then

$$g \cdot \chi_{0,1}(1_0) = \alpha g \cdot 1_1 + \beta g \cdot x_1 = -\alpha 1_1 + \beta x_1,$$

and

$$\chi_{0,1}(g \cdot 1_0) = \chi_{0,1}(1_0) = \alpha 1_1 + \beta x_1.$$

Since we want $\chi_{0,1}$ to be an H_4-module map, then we must have $\alpha = 0$ since $\operatorname{char} k \neq 2$. Similar calculations for the action of other elements of H_4 show that we also have $\gamma = \delta = 0$. Hence $\chi_{0,1}(b_0) = \beta x \cdot b_1$ for some $\beta \in k$, for all $b_0 \in B_0$.

The analogous calculations for $\chi_{1,0}\colon B_1 \to B_0$ give $\chi_{1,0}(b_1) = \beta x \cdot b_0$ for some $\beta \in k$, for all $b_1 \in B_1$. □

We now want to splice the projective resolution P_* from Lemma 3.3.1.2 with its linear dual $\operatorname{Hom}_k(P_*, k)$ to obtain a complete resolution of k. The coresolution $\operatorname{Hom}_k(P_*, k)$ is

$$0 \to k \xrightarrow{\varepsilon^*} \operatorname{Hom}_k(B_0, k) \xrightarrow{d^*} \operatorname{Hom}_k(B_1, k) \xrightarrow{d^*} \operatorname{Hom}_k(B_0, k) \xrightarrow{d^*} \operatorname{Hom}_k(B_1, k) \xrightarrow{d^*} \cdots$$

where d^* and ε^* denote precomposition by d and ε respectively. We first note that this really is a coresolution comprising *left* H_4-modules since the linear dual of a left H_4-module is again a left module via the antipode (see Remark 2.1.0.12). In this case, for $h \in H_4$ and $f \in \operatorname{Hom}_k(P_i, k)$ for any $i \in \mathbb{N}$, the left module structure is given by

$$(h \cdot f)(x) = f(S(h)x).$$

In addition, all of the modules are injective as they are the duals of projective modules (see Remark 2.2.1.4) and because H_4 is a finite-dimensional Hopf algebra and hence Frobenius, injective modules and projective modules coincide so all the modules in the coresolution are projective.

Lemma 3.3.1.4. *We can extend the projective resolution in Lemma 3.3.1.2 to the following H_4-module complete resolution of k, which is denoted by $\hat{P}_*$:*

$$\cdots \xrightarrow{d} B_1 \xrightarrow{d} B_0 \xrightarrow{d} B_1 \xrightarrow{d} B_0 \xrightarrow{d} B_1 \xrightarrow{d} B_0 \xrightarrow{d} B_1 \xrightarrow{d} B_0 \xrightarrow{d} \cdots.$$

Proof. We write $\{\delta_1^0, \delta_x^0\}$ and $\{\delta_1^1, \delta_x^1\}$ for the dual bases of $\operatorname{Hom}_k(B_0, k)$ and $\operatorname{Hom}_k(B_1, k)$ respectively. By looking at the action of g and of x on the dual basis elements, we can make the following identifications:

$$\delta_1^0 \leftrightarrow -x_1, \quad \delta_x^0 \leftrightarrow 1_1$$
$$\delta_1^1 \leftrightarrow x_0, \quad \delta_x^1 \leftrightarrow 1_0.$$

For example, we have

$$\begin{aligned}(g \cdot \delta_1^0)(1_0) = \delta_1^0(S(g) \cdot 1_0) &= \delta_1^0(g \cdot 1_0) \\ &= \delta_1^0(1_0) \\ &= 1,\end{aligned}$$

and similarly $(g \cdot \delta_1^0)(x_0) = 0$, so we have $g \cdot \delta_1^0 = \delta_1^0$. We can likewise calculate that $g \cdot \delta_x^0 = -\delta_x^0$, $g \cdot \delta_1^1 = -\delta_1^1$ and $g \cdot \delta_x^1 = \delta_x^1$.

We remark that these identifications are in contradiction to Nguyen, who claims that you identify $\delta_1^0 \leftrightarrow x_1$ because of the action of g. However, this does not agree with the action of x:

$$\begin{aligned}(x \cdot \delta_x^0)(1_0) = \delta_x^0(S(x) \cdot 1_0) &= \delta_x^0((-xg) \cdot 1_0) \\ &= \delta_x^0((-x) \cdot 1_0) \\ &= \delta_x^0(-x_0) \\ &= -1,\end{aligned}$$

and similarly $(x \cdot \delta_x^0)(x_0) = 0$. Hence we get $x \cdot \delta_x^0 = -\delta_1^0$, whereas $x \cdot 1_1 = x_1$. So we instead identify $\delta_1^0 \leftrightarrow -x_1$, and together with the other identifications this provides H_4-module isomorphisms $\mathrm{Hom}_k(B_0, k) \cong B_1$ and $\mathrm{Hom}_k(B_1, k) \cong B_0$.

To splice together the resolution P_* with the coresolution $\mathrm{Hom}_k(P_*, k)$, we need to know what the map $\varepsilon^* \circ \varepsilon \colon B_0 \to B_1$ is. By using the above isomorphisms, we have

$$\varepsilon^*(\varepsilon(1_0)) = \varepsilon^*(1) = -x_1, \quad \varepsilon^*(\varepsilon(x_0)) = \varepsilon^*(0) = 0.$$

Therefore $\varepsilon^* \circ \varepsilon = -d$, and using this map to splice together the resolution P_* with the coresolution $\mathrm{Hom}_k(P_*, k)$ gives us a complete resolution. However, we can instead take the differential in degree 0 of the complete resolution to be $d = (-\varepsilon^*) \circ \varepsilon$: we will still have a totally acyclic complex of projective H_4-modules, and there is a degree 0 chain map isomorphism $\hat{P}_{\geqslant 0} \to P_*$. □

Remark 3.3.1.5. Nguyen claims that the target of the map $\varepsilon^* \circ \varepsilon$ is B_0 rather than B_1, and hence does not get the same complete resolution. However, this does not work because the source of ε is B_0 therefore the target of ε^* is $\mathrm{Hom}_k(B_0, k)$. But as related above, this is isomorphic to B_1. In addition, if we define the map $d \colon B_0 \to B_0$ to be $d(b) = bx$ as before, then this is not a map of H_4-modules. Indeed, $d(g \cdot 1_0) = d(1_0) = x_0$, but $g \cdot d(1_0) = g \cdot x_0 = -x_0$.

Proposition 3.3.1.6. *The additive structure of complete Ext of the Sweedler Hopf algebra H_4 over a field k where $\mathrm{char}\, k \neq 2$ is*

$$\widehat{\mathrm{Ext}}^i_{H_4}(k, k) \cong \begin{cases} k, & i \text{ even} \\ 0, & i \text{ odd}, \end{cases}$$

for all $i \in \mathbb{Z}$.

Proof. To calculate complete Ext, we apply $\mathrm{Hom}_{H_4}(-, k)$ to $\hat{P}_*$ and take cohomology of the resulting complex. We recall from Examples 2.1.0.11 (2) that H_4 acts on the trivial H_4-module k via the counit, so the H_4-action for the trivial module k is given by $g \cdot 1 = 1$ and $x \cdot 1 = 0$.

We write $\overline{B_0}$ and $\overline{B_1}$ for $\operatorname{Hom}_{H_4}(B_0, k)$ and $\operatorname{Hom}_{H_4}(B_1, k)$ respectively. Let $f \in \overline{B_0}$ with $f(1_0) = \alpha$ and $f(x_0) = \beta$. Then

$$\beta = f(x \cdot 1_0) = x \cdot f(1_0) = 0.$$

Checking the actions of other elements of H_4 shows that there are no limitations on α, hence $\overline{B_0} \cong k$. Similarly, if we let $f' \in \overline{B_1}$ with $f'(1_1) = \alpha'$ and $f'(x_1) = \beta'$ one can check the H_4-action to show that $\alpha' = \beta' = 0$ (by again using $\operatorname{char} k \neq 2$) so $\overline{B_1} = 0$.

Therefore we need to take cohomology of the complex

$$\cdots \xrightarrow{0} 0 \xrightarrow{0} k \xrightarrow{0} 0 \xrightarrow{0} k \xrightarrow{0} 0 \xrightarrow{0} k \xrightarrow{0} 0 \xrightarrow{0} k \xrightarrow{0} \cdots,$$

which gives the claimed additive structure of complete Ext as

$$\widehat{\operatorname{Ext}}^i_{H_4}(k, k) \cong \begin{cases} k, & i \text{ even} \\ 0, & i \text{ odd}, \end{cases}$$

for all $i \in \mathbb{Z}$. □

Nguyen's work only examined the additive structure of complete Ext of the Sweedler Hopf algebra. We now wish to go further and calculate the multiplicative structure, for which we will use the composition product. To do this, we first need to see what degree n chain maps $\hat{P}_* \to \hat{P}_*$ we have (which correspond to n-cocycles in $\underline{\operatorname{Hom}}^\bullet_{H_4}(\hat{P}_*, \hat{P}_*)$), and this will allow us to recover the additive structure calculated above. We can already see that if n is even then the components of such chain maps must be H_4-module maps $B_0 \to B_0$ and H_4-module maps $B_1 \to B_1$ and if n is odd then the components are H_4-module maps $B_0 \to B_1$ and $B_1 \to B_0$.

Lemma 3.3.1.7. *Let Φ be a degree $2i$ chain map $\hat{P}_* \to \hat{P}_*$. Then for some $\alpha \in k$, all even components of Φ are defined by $\Phi_0(b_0) = \alpha b_0$ for all $b_0 \in B_0$ and all odd components of Φ are defined by $\Phi_1(b_1) = \alpha b_1$ for all $b_1 \in B_1$. In addition, no two such chain maps which are not equal are chain homotopic, and therefore we recover $\widehat{\operatorname{Ext}}^{2i}_{H_4}(k, k) \cong k$ for all $i \in \mathbb{Z}$, agreeing with our result in Proposition 3.3.1.6.*

Proof. We recall from Lemma 3.3.1.3 that an H_4-module homomorphism $B_0 \to B_0$ is "multiplication by α" for some $\alpha \in k$ and likewise an H_4-module homomorphism $B_1 \to B_1$ is "multiplication by α'" for some $\alpha' \in k$. Hence for such α, α', all even components of Φ are defined by $\Phi_0(b_0) = \alpha b_0$ for all $b_0 \in B_0$ and all odd components of Φ are defined by $\Phi_1(b_1) = \alpha' b_1$ for all $b_1 \in B_1$. Now, Φ being a chain map shows that

$$\Phi_0(d(1_1)) = \Phi_0(x_0) = \alpha x_0 = d(\Phi_1(1_1)) = \alpha' x_0,$$

hence $\alpha = \alpha'$. So for some $\alpha \in k$, then the even degree chain map Φ on every component is "multiplication by α".

Now consider an even degree chain map Φ which is multiplication by α and another even degree chain map Φ' which is multiplication by α' and where $\alpha \neq \alpha'$. Assume, for a contradiction, that $\Phi \simeq \Phi'$ i.e. there exist maps $h_n \colon \hat{P}_{2i-1+n} \to \hat{P}_n$ such that

$\Phi - \Phi' = d \circ h_n + h_{n-1} \circ d$ for all $n \in \mathbb{Z}$. Since h_n is a map $B_0 \to B_1$ if n is odd or a map $B_1 \to B_0$ if n is even, then from Lemma 3.3.1.3 we can write $h_n(b) = c_n bx$ for some $c_n \in k$, for all $n \in \mathbb{Z}$ as these are the only possible H_4-module maps. But then

$$(dh_n + h_{n-1}d)(1_1) = d(c_n x_1) + h_{n-1}(x_0) = 0,$$

and similar calculations on other basis elements show that in fact $d \circ h_n + h_{n-1} \circ d = 0$ for all $n \in \mathbb{Z}$. However, we have $(\Phi - \Phi')(1_0) = \alpha - \alpha'$ and $(\Phi - \Phi')(x_0) = (\alpha - \alpha')x_0$, which are both non-zero by assumption. Hence we get a contradiction and so there is no chain homotopy $\Phi \Rightarrow \Phi'$. Therefore $\widehat{\operatorname{Ext}}^{2i}_{H_4}(k, k) \cong k$ for all $i \in \mathbb{Z}$. □

Lemma 3.3.1.8. *Any degree $2i+1$ chain map $\Psi\colon \hat{P}_* \to \hat{P}_*$ is nullhomotopic, and therefore we recover $\widehat{\operatorname{Ext}}^{2i+1}_{H_4}(k, k) = 0$ for all $i \in \mathbb{Z}$, agreeing with our result in Proposition 3.3.1.6.*

Proof. Similarly to the proof of Lemma 3.3.1.7, we can use Lemma 3.3.1.3 to look at the H_4-action on an odd degree graded module map $\Psi\colon \hat{P}_* \to \hat{P}_*$ to see that every component of such a map is "multiplication by βx for *some* $\beta \in k$". Every odd degree graded module map $\hat{P}_* \to \hat{P}_*$ is in fact a chain map since these components commute with the differential on $\hat{P}_*$, for instance

$$\Psi_0(d(1_0)) = \Psi_0(x_1) = 0 = d(\beta x_1) = d(\Psi_1(1_0)).$$

Hence the components of the chain map Ψ are not necessarily all the same. However, it is a direct consequence of the isomorphism of Theorem 3.1.0.4 and the fact that for any odd degree chain map $\Psi\colon \hat{P}_* \to \hat{P}_*$ we have $\varepsilon \circ \Psi_0 = 0$ (since $\varepsilon(\Psi_0(1_1)) = \varepsilon(\beta x_0) = 0$) to show that any such Ψ is nullhomotopic.

Therefore $\widehat{\operatorname{Ext}}^{2i+1}_{H_4}(k, k) = 0$ for all $i \in \mathbb{Z}$, and along with the result in Lemma 3.3.1.7 we have now completely recovered the additive structure of complete Ext of the Sweedler Hopf algebra. □

We now determine the multiplicative structure. We will use the term (from [BKS04, Proof of Theorem 7.3]) "periodicity isomorphism" for the map $f \in \underline{\operatorname{Hom}}^2_{H_4}(\hat{P}_*, \hat{P}_*)$ which is the identity in every degree as in [BKS04, Proof of Theorem 7.3]. This corresponds to $\alpha = 1 \in k$ in the notation of the proof of Lemma 3.3.1.7.

Proposition 3.3.1.9. *The complete Ext algebra over the Sweedler Hopf algebra H_4 over a field k where* $\operatorname{char} k \neq 2$ *is given by*

$$\widehat{\operatorname{Ext}}^*_{H_4}(k, k) = k[t^{\pm 1}]$$

where $|t| = 2$ and t is the class of the periodicity isomorphism f.

Proof. We see that the periodicity isomorphism f has an inverse $f^{-1} \in \underline{\operatorname{Hom}}^{-2}_{H_4}(\hat{P}_*, \hat{P}_*)$ and f generates every map in $\underline{\operatorname{Hom}}^{2i}_{H_4}(\hat{P}_*, \hat{P}_*)$ for all $i \in \mathbb{Z}$. The proposition is an immediate consequence. □

Remark 3.3.1.10. Let $i \in \mathbb{Z}$ and let $f_i \in \underline{\mathrm{Hom}}^{2i}_{H_4}(\hat{P}_*, \hat{P}_*)$ be the periodicity isomorphism. The cycle selection homomorphism

$$\begin{aligned} k[t^{\pm 1}] &\to \underline{\mathrm{Hom}}^{\bullet}_{H_4}(\hat{P}_*, \hat{P}_*) \\ t^i &\mapsto f_i \end{aligned}$$

is multiplicative so this map is a quasi-isomorphism of dgas. Hence the endomorphism dga $\underline{\mathrm{Hom}}^{\bullet}_{H_4}(\hat{P}_*, \hat{P}_*)$ is actually formal in this case, and there are no non-trivial higher multiplications m_n for $n \geqslant 3$ on $\widehat{\mathrm{Ext}}^*_{H_4}(k, k)$.

3.3.2 A Truncated Polynomial Algebra

We now look to compute complete Ext over the Hopf algebra $A = k[x]/(x^3)$ where $k = \mathbb{F}_3$, with the primitive comultiplication $\varphi(x) = x \otimes 1 + 1 \otimes x$, counit $\varepsilon(x) = 0, \varepsilon(1) = 1$ and antipode $S(x) = -x$. We will also illustrate how to construct a complete diagonal approximation in this example.

We follow [BKS04, Proof of Theorem 7.3] for calculating the additive and multiplicative structure. We start by defining a complete resolution (of A-modules) $\hat{P}_*$ of the trivial A-module k by $\hat{P}_i = A$, with $d_i \colon \hat{P}_i \to \hat{P}_{i-1}$ given by

$$d_i(1) = \begin{cases} -x^2, & i \text{ even} \\ x, & i \text{ odd}, \end{cases}$$

and augmentation given by $\varepsilon(1) = 1$. Henceforth, for clarity, we will write elements of the module $\hat{P}_i$ with the subscript i (e.g. $x_i + 1_i$). 1 written without a subscript indicates the multiplicative identity $1 \in k$.

The complex $\mathrm{Hom}_A(\hat{P}_*, k)$ has zero differential, so when we take cohomology we have $\widehat{\mathrm{Ext}}^i_A(k, k) \cong k$ for all $i \in k$. We now use the chain map formulation of complete Ext (i.e. chain homotopy classes of maps in $\underline{\mathrm{Hom}}^{\bullet}_A(\hat{P}_*, \hat{P}_*)$) to deduce the multiplicative structure.

Let $\overline{s} \in \underline{\mathrm{Hom}}^1_A(\hat{P}_*, \hat{P}_*)$ be the degree 1 anti-chain map defined by

$$s_i(1_{i+1}) = \begin{cases} 1_i, & i \text{ even} \\ x_i, & i \text{ odd}, \end{cases}$$

and let $\overline{y} \in \underline{\mathrm{Hom}}^2_A(\hat{P}_*, \hat{P}_*)$ be the degree 2 chain map defined by

$$y_i(1_{i+2}) = 1_i$$

for all $i \in \mathbb{Z}$. As shown in [BKS04, Proof of Theorem 7.3], if we let s and y be the classes of $\overline{s}$ and $\overline{y}$ respectively, then the complete Ext algebra of A is given by

$$\widehat{\mathrm{Ext}}^*_A(k, k) = \Lambda(s) \otimes k[y^{\pm 1}]$$

where $|s| = 1, |y| = 2$, and $\Lambda(s)$ is the exterior algebra on s (since $\overline{s}^2$ is nullhomotopic via the chain homotopy which is the identity in odd degrees and 0 in even degrees).

Remark 3.3.2.1. As the complete Ext algebra structure is given by the composition product, it is clear that the multiplicative structure (including the higher structure of Massey products) does not depend on the comultiplication on the Hopf algebra A; at no point in the above calculations is it necessary to form the tensor product of two A-modules.

We now use Theorem 3.2.0.7 to build the complete diagonal approximation. In particular, we choose a contracting homotopy for $U(\hat{P}_*)$ and use it to define $\hat{\Delta}_0$, and then inductively construct the other components of the chain map. We recall from Definition 2.2.0.4 that the differential on $(\hat{P} \hat{\otimes} \hat{P})_*$ is given by

$$\partial_i(a)_{p+i-1,-p} = (d_{p+i} \otimes \mathrm{id}_{\hat{P}_{-p}})(a_{p+i,-p}) + (-1)^{p+i-1}(\mathrm{id}_{\hat{P}_{p+i-1}} \otimes d_{-p+1})(a_{p+i-1,-p+1})$$

for $a \in (\hat{P} \hat{\otimes} \hat{P})_i$ where $b_{p+i,-p}$ denotes the projection of $b \in (\hat{P} \hat{\otimes} \hat{P})_i$ to $\hat{P}_{p+i} \otimes \hat{P}_{-p}$, and where d denotes the differential on $\hat{P}_*$ defined above.

Lemma 3.3.2.2. *We have a complete diagonal approximation $\hat{\Delta}\colon \hat{P}_* \to (\hat{P} \hat{\otimes} \hat{P})_*$ defined by*

$$\hat{\Delta}_{n,i}(1_n) = \begin{cases} 1_{i+n} \otimes 1_{-i}, & n \text{ odd} \\ 1_{i+n} \otimes 1_{-i}, & n \text{ even, } i \text{ even} \\ -x_{i+n} \otimes 1_{-i} + 1_{i+n} \otimes x_{-i}, & n \text{ even, } i \text{ odd,} \end{cases}$$

where $\hat{\Delta}_{n,i}$ denotes the composition of $\hat{\Delta}_n$ and the i^{th} projection to $\hat{P}_{i+n} \otimes \hat{P}_{-i}$.

Proof. Let $h_i\colon U(\hat{P}_i) \to U(\hat{P}_{i+1})$ be a map of graded k-vector spaces where

$$h_i\left(\begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix}\right) = \begin{cases} \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix}, & i \text{ even} \\ \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, & i \text{ odd.} \end{cases}$$

Then h is a contracting homotopy for $U(\hat{P}_*)$. Recalling the definition of $\Phi\colon \hat{P}_j \otimes U(\hat{P}_i) \to \hat{P}_i \otimes \hat{P}_j$ and Φ^{-1} from Proposition 3.2.0.3, here we have

y	$\Phi(y)$	$\Phi^{-1}(y)$
$1 \otimes 1$	$1 \otimes 1$	$1 \otimes 1$
$1 \otimes x$	$x \otimes 1$	$x \otimes 1 - 1 \otimes x$
$1 \otimes x^2$	$x^2 \otimes 1$	$x^2 \otimes 1 + x \otimes x + 1 \otimes x^2$
$x \otimes 1$	$x \otimes 1 + 1 \otimes x$	$1 \otimes x$
$x \otimes x$	$x^2 \otimes 1 + x \otimes x$	$x \otimes x - 1 \otimes x^2$
$x \otimes x^2$	$x^2 \otimes x$	$x^2 \otimes x + x \otimes x^2$
$x^2 \otimes 1$	$x^2 \otimes 1 - x \otimes x + 1 \otimes x^2$	$1 \otimes x^2$
$x^2 \otimes x$	$-x^2 \otimes x + x \otimes x^2$	$x \otimes x^2$
$x^2 \otimes x^2$	$x^2 \otimes x^2$	$x^2 \otimes x^2$

We then have a contracting homotopy h' for $\hat{P}_* \otimes \hat{P}_j$ with $h'_i := \Phi(\mathrm{id}_{\hat{P}_j} \otimes h_i)\Phi^{-1}$, i.e.

y	$h'_i(y)$, i even	$h'_i(y)$, i odd
$1 \otimes 1$	0	0
$1 \otimes x$	$-1 \otimes 1$	0
$1 \otimes x^2$	$-x \otimes 1 + 1 \otimes x$	$-1 \otimes 1$
$x \otimes 1$	$1 \otimes 1$	0
$x \otimes x$	$1 \otimes x$	$1 \otimes 1$
$x \otimes x^2$	$-x^2 \otimes 1 + 1 \otimes x^2$	$-x \otimes 1 - 1 \otimes x$
$x^2 \otimes 1$	$x \otimes 1$	$-1 \otimes 1$
$x^2 \otimes x$	$x^2 \otimes 1 + x \otimes x$	$-x \otimes 1 - 1 \otimes x$
$x^2 \otimes x^2$	$-x^2 \otimes x + x \otimes x^2$	$-x^2 \otimes 1 + x \otimes x - 1 \otimes x^2$

Recall that we construct $\hat{\Delta}_0$ inductively, with

$$\hat{\Delta}_{0,i} = (-1)^i h'_{i-1}(\mathrm{id}_{\hat{P}_{i-1}} \otimes d_{i-1})\hat{\Delta}_{0,i-1}$$

for $i \geqslant 1$, where $\hat{\Delta}_{0,0} = \Delta_0$ the degree 0 component of the ordinary diagonal approximation, i.e. $\hat{\Delta}_{0,0}(1) = 1 \otimes 1$. Then one can calculate that $\hat{\Delta}_{0,1}(1) = -x \otimes 1 + 1 \otimes x$ and $\hat{\Delta}_{0,2}(1) = 1 \otimes 1$ and see that this extends 2-periodically to $\hat{\Delta}_{0,i}$ agreeing with the claim of the lemma.

We then use comparison of resolutions and the fact that $\partial_0^{\hat{P}\hat{\otimes}\hat{P}}\hat{\Delta}_0 d_1 = 0$ (by construction, as shown in Theorem 3.2.0.7) to inductively construct the other components in the chain map as claimed.

For example, we know that there exists $\hat{\Delta}_1 \colon \hat{P}_1 \to (\hat{P} \mathbin{\hat{\otimes}} \hat{P})_1$ such that $\partial_1 \hat{\Delta}_1 = \hat{\Delta}_0 d_1$. Therefore we want

$$\begin{aligned}\partial_1(\hat{\Delta}_{1,2j}(1)) = \hat{\Delta}_{0,2j}(d_1(1)) &= \hat{\Delta}_{0,2j}(x) \\ &= x \cdot (\hat{\Delta}_{0,2j}(1)) \\ &= x \cdot (1 \otimes 1) \\ &= x \otimes 1 + 1 \otimes x,\end{aligned}$$

and similarly

$$\begin{aligned}\partial_1(\hat{\Delta}_{1,2j+1}(1)) = x \cdot (\hat{\Delta}_{0,2j+1}(1)) &= x \cdot (-x \otimes 1 + 1 \otimes x) \\ &= -x^2 \otimes 1 + 1 \otimes x^2.\end{aligned}$$

Let $a \in (\hat{P} \mathbin{\hat{\otimes}} \hat{P})_1$ be such that every component of a is $1 \otimes 1$, that is $a_{p+1,-p} = 1_{p+1} \otimes 1_{-p}$ for all $p \in \mathbb{Z}$. Then we calculate that

$$\begin{aligned}\partial_1(a)_{2j+1,-2j-1} &= (d_{2j+2} \otimes \mathrm{id}_{\hat{P}_{-2j-1}})(a_{2j+2,-2j-1}) + (-1)^{2j+1}(\mathrm{id}_{\hat{P}_{2j+1}} \otimes d_{-2j})(a_{2j+1,-2j}) \\ &= d_{2j+2}(1_{2j+2}) \otimes 1_{-2j-1} - 1_{2j+1} \otimes d_{-2j}(1_{-2j}) \\ &= -x^2_{2j+1} \otimes 1_{-2j-1} + 1_{2j+1} \otimes x^2_{-2j-1}\end{aligned}$$

and

$$\begin{aligned}\partial_1(a)_{2j,-2j} &= (d_{2j+1} \otimes \mathrm{id}_{\hat{P}_{-2j}})(a_{2j+1,-2j}) + (-1)^{2j}(\mathrm{id}_{\hat{P}_{2j}} \otimes d_{-2j+1})(a_{2j,-2j+1}) \\ &= d_{2j+1}(1_{2j+1}) \otimes 1_{-2j} + 1_{2j} \otimes d_{-2j+1}(1_{-2j+1}) \\ &= x_{2j} \otimes 1_{-2j} + 1_{2j} \otimes x_{-2j}.\end{aligned}$$

Therefore we see that taking $\hat{\Delta}_1(1) = a$ (that is, taking $\hat{\Delta}_{1,i}(1) = 1 \otimes 1$ for all $i \in \mathbb{Z}$) satisfies $\partial_1 \hat{\Delta}_1 = \hat{\Delta}_0 d_1$, and we can apply the same argument to construct $\hat{\Delta}_2$. Then because $\hat{P}_*$ is 2-periodic, so is the chain map $\hat{\Delta}$. Hence the other components are also as claimed.

Finally, it is clear that $\hat{\Delta}$ is augmentation-preserving since $(\varepsilon \mathbin{\hat{\otimes}} \varepsilon)(\hat{\Delta}_0)(1_0) = 1 \otimes 1 = 1 = \varepsilon(1_0)$. □

3.4 Steenrod Operations on Cohomology

We show that on usual Ext of a cocommutative Hopf algebra, there are cohomology operations which detect the comultiplication. That is, we give an example of an algebra which can be given two different coalgebra structures such that we get two different Hopf algebra structures so that even though their cohomology rings are isomorphic, certain operations on generators of the cohomology depend on the coproduct on the Hopf algebra. We use the example of the truncated polynomial algebra $A = k[x]/(x^3)$ where $k = \mathbb{F}_3$ and the cyclic group algebra $B = kC_3 = k[\alpha]/(\alpha^3 - 1)$ and look at mod 3 power operations $P^i, \beta P^i$.

There is an isomorphism of algebras $A \cong B$ via the map $x \mapsto \alpha - 1$. Hence the cohomology rings of A and B are also isomorphic. Recall the (cocommutative) Hopf algebra structure on A from Subsection 3.3.2 and the (cocommutative) Hopf algebra structure on B from Examples 2.1.0.9 (2).

Recall from Subsection 3.3.2 that the usual cohomology ring of A (and B) is $\mathrm{Ext}^*_A(k,k) = \Lambda(s) \otimes k[y]$ where $|s| = 1, |y| = 2$, and $\Lambda(s)$ is the exterior algebra on s.

Remark 3.4.0.1. On the cohomology of the Hopf algebra B, the Bockstein homomorphism applied to s is $\beta(s) = y$ which can be calculated as follows. We have a short exact sequence of trivial $\mathbb{Z}C_3$-modules

$$0 \to k \to \mathbb{Z}/9\mathbb{Z} \to k \to 0,$$

which gives rise to the short exact sequence of cochain complexes (of $\mathbb{Z}$-modules)

$$0 \to \mathrm{Hom}_{\mathbb{Z}C_3}(P_*, k) \to \mathrm{Hom}_{\mathbb{Z}C_3}(P_*, \mathbb{Z}/9\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}C_3}(P_*, k) \to 0.$$

We can do this because $\mathbb{Z}C_3$ is a Hopf algebra over $\mathbb{Z}$ with structure as in Examples 2.1.0.9 (2).

The Bockstein homomorphism $\beta\colon \mathrm{Ext}^i_{\mathbb{Z}C_3}(\mathbb{Z}, k) \to \mathrm{Ext}^{i+1}_{\mathbb{Z}C_3}(\mathbb{Z}, k)$ is the connecting homomorphism in the long exact sequence on cohomology induced from the above short exact sequence, and we then identify $\mathrm{Ext}^i_{\mathbb{Z}C_3}(\mathbb{Z}, k)$ and $\mathrm{Ext}^{i+1}_{\mathbb{Z}C_3}(\mathbb{Z}, k)$ with $\mathrm{Ext}^i_B(k,k)$ and $\mathrm{Ext}^{i+1}_B(k,k)$ respectively to get the Bockstein homomorphism on the cohomology of the Hopf algebra B.

However, we note that we *cannot* calculate the Bockstein homomorphism on the cohomology of the Hopf algebra A because $\mathbb{Z}[x]/(x^3)$ is *not* a Hopf algebra with primitive comultiplication since Δ is not an algebra map. For example, $\Delta(x^2)\Delta(x) = 3x^2 \otimes x + 3x \otimes x^2$ but $\Delta(x^3) = \Delta(0) = 0$.

We denote the k-linear dual algebras of A and B by $A^\vee$ and $B^\vee$ respectively. The augmentation ideals $I(A^\vee)$ and $I(B^\vee)$ are respectively generated by the dual basis elements $x^\vee, (x^2)^\vee$ and $\alpha^\vee, (\alpha^2)^\vee$. Recall the cobar construction from Definition 2.3.2.5 and that the differential on the cobar construction is determined by

$$d([([x_1|\dots|x_m])]) = \sum_{i=1}^{m} (-1)^i [x_1|\dots|\tilde{\Delta}(x_i)|\dots|x_m],$$

where $\tilde{\Delta}$ is the reduced comultiplication (see Definition 2.3.2.4). Recall also from Proposition 2.3.2.6 that P^0 is defined by cubing (since we work over $\mathbb{F}_3$ here) elements of the cobar construction of the augmentation ideal of the dual algebra, i.e.

$$P^0([[x_1|\dots|x_n]]) = [[x_1^3|\dots|x_n^3]].$$

Lemma 3.4.0.2. *On the cohomology of the Hopf algebra A, we have $P^0 = 0$ and on the cohomology of the Hopf algebra B, we have $P^0 = \mathrm{id}$.*

Proof. The algebra structures of $A^\vee$ and $B^\vee$ are dual to the coalgebra structures of A and B. Since $\varepsilon(x) = 0$ and $\varepsilon(\alpha) = 1$, this means that the units in the dual algebras are given by $\eta_{A^\vee}(1) = 1^\vee$ and $\eta_{B^*}(1) = 1^\vee + \alpha^\vee + (\alpha^2)^\vee$.

The multiplication on the dual Hopf algebra $A^\vee$ is defined via

$$\begin{array}{ccc} A & \longrightarrow & k \\ \downarrow{\scriptstyle\Delta} & & \uparrow{\scriptstyle\cong} \\ A \otimes A & \xrightarrow{f \otimes g} & k \otimes k, \end{array}$$

for $f, g \in A^\vee$, and similarly for $B^\vee$. Hence we can calculate that $(x^\vee)^2 = -(x^2)^\vee$ and $(x^\vee)((x^2)^\vee) = ((x^2)^\vee)(x^\vee) = ((x^2)^\vee)((x^2)^\vee) = 0$, and similarly $(\alpha^\vee)^2 = \alpha^\vee, ((\alpha^2)^\vee)^2 = (\alpha^2)^\vee$ and $(\alpha^\vee)((\alpha^2)^\vee) = ((\alpha^2)^\vee)(\alpha^\vee) = 0$.

Finally, to calculate the differential in the cobar resolution of $I(A^\vee)$ we need to use the reduced comultiplication $\tilde{\Delta}\colon I(A^\vee) \to I(A^\vee) \otimes I(A^\vee)$. Recall from Definition 2.3.2.4 that the short exact sequence of k-vector spaces

$$0 \to I(A^\vee) \overset{\iota}{\hookrightarrow} A^\vee \twoheadrightarrow k \to 0$$

splits via the map $p\colon A^\vee \to I(A^\vee)$ defined by

$$p(f) = f - \varepsilon_{A^\vee}(f)\eta_{A^\vee}(1).$$

We can see from this that $p(1^\vee) = 0$, $p(x^\vee) = x^\vee$ and $p((x^2)^\vee) = (x^2)^\vee$.

The reduced comultiplication on $I(A^\vee)$ is then given by

$$\tilde{\Delta} = (p \otimes p)\Delta\iota,$$

where Δ is the comultiplication in $A^\vee$. As above, Δ is dual to the multiplication in A (using an identification of $(A \otimes A)^\vee$ with $A^\vee \otimes A^\vee$), so we calculate that $\Delta(1^\vee) = 1^\vee \otimes 1^\vee$, $\Delta(x^\vee) = x^\vee \otimes 1^\vee + 1^\vee \otimes x^\vee$ and $\Delta((x^2)^\vee) = (x^2)^\vee \otimes 1^\vee + 1^\vee \otimes (x^2)^\vee + x^\vee \otimes x^\vee$. Hence we have $\tilde{\Delta}(x^\vee) = 0$ and $\tilde{\Delta}((x^2)^\vee) = x^\vee \otimes x^\vee$.

We can now find representing cocycles for s, y in the cobar resolution and use these to calculate P^0. We have $d([x^\vee]) = 0$ so $[x^\vee]$ is a representing 1-cocycle for s, hence

$$P^0(s) = [[(x^\vee)^3]] = 0.$$

Similarly, we see that $d([(x^2)^\vee|x^\vee]+[x^\vee|(x^2)^\vee]) = [x^\vee|x^\vee|x^\vee]-[x^\vee|x^\vee|x^\vee] = 0$ so $[(x^2)^\vee|x^\vee]+[x^\vee|(x^2)^\vee]$ is a representing 2-cocycle for y. Hence

$$P^0(y) = [[((x^2)^\vee)^3|(x^\vee)^3] + [(x^\vee)^3|((x^2)^\vee)^3]] = 0.$$

Therefore $P^0 = 0$.

We follow the same method to calculate the reduced comultiplication on $I(B^\vee)$ where we find that $\tilde{\Delta}(\alpha^\vee) = \tilde{\Delta}((\alpha^2)^\vee) = \alpha^\vee \otimes \alpha^\vee - \alpha^\vee \otimes (\alpha^2)^\vee - (\alpha^2)^\vee \otimes \alpha^\vee + (\alpha^2)^\vee \otimes (\alpha^2)^\vee$.

We then find representing cocycles for s, y in the cobar resolution of $I(B^\vee)$ and find that $[\alpha^\vee]+[(\alpha^2)^\vee]$ is a representing 1-cocycle for s and $[\alpha^\vee|\alpha^\vee]-[(\alpha^2)^\vee|(\alpha^2)^\vee]$ is a representing 2-cocycle for y. By our calculations for the multiplication on the dual Hopf algebra $B^\vee$, we then see that $P^0 = \mathrm{id}$ on the cohomology of B, as claimed. □

Recall from Remarks 2.3.2.3 (1) that a Hopf algebra H is reduced mod 3 if $H = H' \otimes_{\mathbb{Z}} \mathbb{F}_3$ where H' is a $\mathbb{Z}$-free Hopf algebra, and that if H is reduced mod 3 then the Steenrod operation βP^0 on the cohomology of H is given by the composition of β with P^0.

Since $B = \mathbb{Z}C_3 \otimes_{\mathbb{Z}} \mathbb{F}_3$ and $\mathbb{Z}C_3$ is a Hopf algebra (with structure as in Examples 2.1.0.9 (2), as noted in Remark 3.4.0.1), then B is reduced mod 3. However, $A = \mathbb{Z}[x]/(x^3) \otimes_{\mathbb{Z}} \mathbb{F}_3$ but we have seen in Remark 3.4.0.1 that $\mathbb{Z}[x]/(x^3)$ is *not* a Hopf algebra with primitive comultiplication because Δ is not an algebra map. Hence A is *not* reduced mod 3.

This means that we can use Lemma 3.4.0.2 to deduce that $\beta P^0 = \beta$ on the cohomology of B but we cannot use Lemma 3.4.0.2 to deduce βP^0 on the cohomology of A. Based on [May70, Remarks 11.11], we make the following conjecture.

Conjecture 3.4.0.3. The Steenrod operation βP^0 on a degree 1 element of the cohomology of a Hopf algebra does *not* detect the comultiplication.

This conjecture is motivated by the fact that for H a cocommutative Hopf algebra over a field in characteristic p and for $z \in \mathrm{Ext}^1_H(k, k)$, [May70, Remarks 11.11] appears to describe $\beta P^0(z)$ as (up to a sign) an element in the p-fold Massey product $\langle z, \ldots, z\rangle$. If this is true, then as noted in Remark 3.3.2.1 the higher multiplications on the cohomology of H do not depend on the coalgebra structure of H and so we would expect this conjecture to hold. However, this is not explicitly stated in this form by May, only implied by the article's references to [Kra66], and it also references the unpublished work [May] which the author does not believe is available in any form.

In addition, Remark 3.4.0.1 and Lemma 3.4.0.2 tell us that $\beta P^0(s) = y$ on the cohomology of B, and [BKS04, Remark 5.10 and Theorem 7.3] tells us that the 3-fold Massey product $\langle s, s, s\rangle = y$ on the cohomology of A and B.

Part II

Cohomology of Diagram Algebras

Chapter 4

Diagram Algebras

4.1 Background and Conventions

In this chapter, we will recall algebras defined on certain bases of diagrams which we will study throughout Part II of this thesis. We will refer to such algebras as *diagram algebras*.

We will also define certain "half-diagrams" called (left and right) link states of diagrams in some of our diagram algebras. We will then define ideals on bases of diagrams obtained from these link states, and we will use these ideals in Chapters 6 and 7 to prove isomorphisms on (co)homology.

Our conventions in Part II (in addition to those for the overall thesis noted in Section 1.4) will be as follows. Throughout, n will be a positive integer unless otherwise stated. We will denote the set $\{1, 2, \ldots, n\}$ by $\underline{n}$. All n-diagrams in this thesis (which we sometimes call simply diagrams if n is understood) will be graphs on two identical columns of n equally spaced vertices where the edges must be drawn within the rectangle formed by the vertices. The vertices down the left-hand column of an n-diagram will be labelled by $1, \ldots, n$ in ascending order from top to bottom and the vertices down the right-hand column will be labelled by $\overline{1}, \ldots, \overline{n}$ in ascending order from top to bottom. We also have some terminology that will be used throughout the paper.

Definition 4.1.0.1. We collect some important terminology for graphs that will be used throughout the rest of this thesis.

1. An edge that connects the left-hand column of vertices to the right-hand column of vertices will be called a *propagating edge*.
2. A connected component of a diagram which contains vertices in both columns (that is, a connected component containing a propagating edge) will be called a *propagating component*.
3. An edge that connects two vertices in the same column will be called a *non-propagating edge*.
4. A vertex not connected to any other by an edge will be called an *isolated vertex*.
5. If the edges of a diagram can be drawn without crossing within the rectangle formed by the vertices, then we refer to such a diagram as being *planar*. We identify planar diagrams up to isotopy.
6. Any diagram having precisely n propagating components will be called a *permutation diagram*. All other diagrams will be referred to as *non-permutation diagrams*.

Remark 4.1.0.2. We will see that each diagram algebra studied in this thesis will have some subalgebra generated by all permitted permutation diagrams. Since permutation n-diagrams are in bijection with elements of the symmetric group Σ_n, for many diagram algebras this subalgebra will be the symmetric group algebra $R[\Sigma_n]$. However, for the algebras defined on bases of planar diagrams, this subalgebra will be a copy of R.

The multiplication in all of our different diagram algebras will follow the same general process. We glue two diagrams together along a column of vertices and replace a certain feature (such as a loop) in the middle column with a parameter in the ground ring. We will make precise definitions for all diagram algebras considered in this thesis.

As noted in Section 1.3, the contents of this chapter are adapted from joint work of the author with Daniel Graves.

4.2 Partition Algebras and Their Subalgebras

In this section we collect the definitions of certain subalgebras of the partition algebras that we will consider in this thesis, namely the partition algebras themselves, the Tanabe algebras, the totally propagating partition algebras and the uniform block permutation algebras. The partition algebras, $P_n(\delta)$ (where n is a positive integer and δ is a parameter in a unital, commutative ground ring), were introduced independently by Martin [Mar94] and Jones [Jon94] to study the Potts model in statistical mechanics. Jones showed that the partition algebras exhibit a *Schur–Weyl* duality with the symmetric groups, so called because it takes a similar form to the classical statement of Schur–Weyl duality between the symmetric groups and the general linear groups.

In Table 4.2.4.4, we will give a representative diagram of each algebra defined in this section.

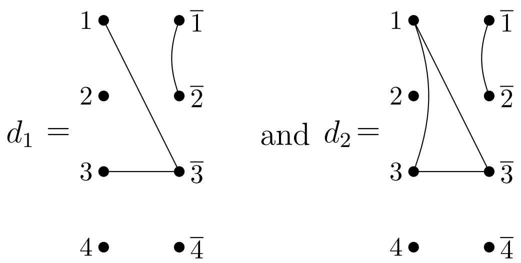
4.2.1 Partition Algebras

Definition 4.2.1.1. A *partition n-diagram* is a graph on two columns of n vertices where each edge is incident to two distinct vertices and there is at most one edge between any two vertices.

We will refer to the number of vertices in a connected component of a partition n-diagram as the *cardinality* of the connected component.

These diagrams are called partition n-diagrams because the connected components of the graph determine a partition of the set $\{1, \overline{1}, \dots, n, \overline{n}\}$. We say two partition n-diagrams are *equivalent* if they determine the same partition of the set $\{1, \overline{1}, \dots, n, \overline{n}\}$. Henceforth, when referring to a partition diagram, we will mean its equivalence class.

Example 4.2.1.2. The partition 4-diagrams



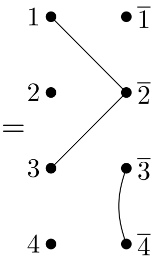
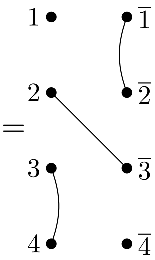
are equivalent.

Definition 4.2.1.3. Let $\delta \in R$. The *partition algebra*, $P_n(\delta)$, is the R-algebra with basis consisting of all partition n-diagrams with the multiplication defined by the R-linear extension of the following product of diagrams. Let d_1 and d_2 be partition n-diagrams. The product $d_1 d_2$ is obtained by the following procedure:

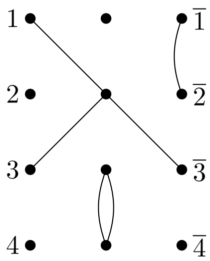
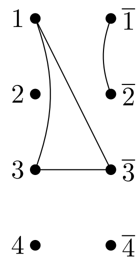
- Place the diagram d_2 to the right of the diagram d_1 and identify the vertices $\overline{1}, \ldots, \overline{n}$ in d_1 with the vertices $1, \ldots, n$ in d_2. Call this diagram with three columns of vertices $d_1 * d_2$. We drop the labels of the vertices in the middle column and we preserve the labels of the left-hand column and right-hand column.
- Count the number of connected components that lie entirely within the middle column. Call this number α.
- Make a new partition n-diagram, d_3, as follows. Given distinct vertices x and y in the set $\{1, \overline{1}, \ldots, n, \overline{n}\}$, d_3 has an edge between x and y if there is a path from x to y in $d_1 * d_2$.
- We define the composite $d_1 d_2 = \delta^\alpha d_3$.

We note that this product is associative and well-defined up to equivalence of partition n-diagrams. The identity element consists of the diagram with n horizontal edges.

Example 4.2.1.4. Here is an example of the composition defined above. Suppose we have diagrams

$d_1 =$  $d_2 =$ 

in $P_n(\delta)$. In this case we have

$d_1 * d_2 =$  and $d_1 d_2 = \delta^2$ 

We refer the reader to [Jon94; Blo03; BHP23] for some more examples of composing partition diagrams. We note that some authors work with two rows of vertices rather than two columns and compose diagrams vertically rather than horizontally.

4.2.2 Tanabe Algebras

In this subsection we define the Tanabe algebras, a family of subalgebras $\mathcal{T}_n(\delta, r)$ of the partition algebra $P_n(\delta)$, one for each positive integer r, introduced by Tanabe in [Tan97]. Tanabe demonstrated that these subalgebras exhibit a Schur–Weyl duality with certain complex reflection groups. Our treatment of the Tanabe algebras will follow that of Orellana [Ore07]. If we fix positive integers r and n, the Tanabe algebra is defined as the subalgebra of the partition algebra spanned R-linearly by those partition n-diagrams such that, for each connected component, the difference between the number of vertices in the left and right columns is congruent to zero modulo r.

Definition 4.2.2.1. Let d be a partition n-diagram. Let C be a connected component of d, let $L(C)$ denote the number of vertices in C in the left-hand column of d and let $R(C)$ denote the number of vertices in C in the right-hand column of d. We define

$$\kappa(C) = \left|L(C) - R(C)\right|.$$

Definition 4.2.2.2. Let $\delta \in R$. Let $r \geqslant 1$. By checking that partition n-diagrams such that $\kappa(C) \equiv 0 \pmod r$ for each connected component C are closed under composition, we define the *Tanabe algebra* $\mathcal{T}_n(\delta, r)$ to be the subalgebra of $P_n(\delta)$ spanned R-linearly by such diagrams.

Remark 4.2.2.3. If we take $r = 1$, we recover the partition algebra, $P_n(\delta)$. When $r = 2$ the family of algebras are sometimes called the *even partition algebras* or the *parity matching algebras* (see [Scr24] for instance).

4.2.3 Totally Propagating Partition Algebras

The totally propagating partition algebras have been studied in [KM08], where it was shown that they exhibit a Schur–Weyl duality with the rook algebras (see Definition 4.3.0.4). The underlying monoid had appeared in earlier papers: [FL98; Mal07; EEF08] (under the name *dual symmetric inverse monoid*). The totally propagating partition algebras have been further studied in [MS21].

Definition 4.2.3.1. Recall that a connected component of a partition n-diagram is called propagating if it contains vertices in both the left-hand column and the right-hand column. A partition n-diagram is called *totally propagating* if all of its connected components are propagating.

Definition 4.2.3.2. The *totally propagating partition algebra*, TPP_n, is the subalgebra of $P_n(\delta)$ spanned R-linearly by the totally propagating partition n-diagrams. We note that in the procedure for composing two totally propagating partition n-diagrams d_1 and d_2, the diagram $d_1 * d_2$ can have no connected components that lie entirely within the middle column so we drop the parameter δ from the notation.

4.2.4 Uniform Block Permutation Algebras

The uniform block permutation algebras were first introduced by Kosuda, under the name *party algebras* [Kos00] (see also [Kos06]). The name *uniform block permutation algebra* was coined by FitzGerald [Fit03]. See [AO08; OSSZ22; OSSZ24] for more recent results about the uniform block permutation algebras.

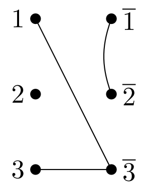
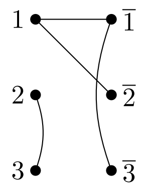
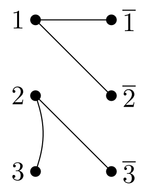
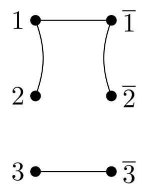
Definition 4.2.4.1. Recall the notation of Definition 4.2.2.1. Let d be a partition n-diagram. A connected component C of d is called *uniform* if $L(C) = R(C)$, that is, if C

has an equal number of vertices in both the left-hand and right-hand column of d. The diagram d is called *uniform* if every connected component of d is uniform. Some authors refer to such a partition as being *balanced* (see [Har18] for instance).

Definition 4.2.4.2. The *uniform block permutation algebra*, U_n, is the subalgebra of $P_n(\delta)$ spanned R-linearly by the uniform partition n-diagrams. We note that in the procedure for composing two uniform partition n-diagrams d_1 and d_2, the diagram $d_1 * d_2$ can have no connected components that lie entirely within the middle column so we drop the parameter δ from the notation.

Remark 4.2.4.3. We note that $R[\Sigma_n]$ is a subalgebra of the partition algebra $P_n(\delta)$, the Tanabe algebra $\mathcal{T}_n(\delta, r)$, the totally-propagating partition algebra TPP_n and the uniform partition algebra U_n. In each case, it is the subalgebra spanned R-linearly by diagrams having precisely n propagating edges.

Table 4.2.4.4. We have the following examples of diagrams for each algebra defined in this section. We have taken $r = 2$ in our representative diagram of the Tanabe algebra.

Diagram algebra name	Diagram algebra symbol	Representative diagram ($n = 3$)
Partition algebra	$P_n(\delta)$	
Tanabe algebra	$\mathcal{T}_n(\delta, r)$	
Totally propagating partition algebra	TPP_n	
Uniform block permutation algebra	U_n	

4.3 Rook-Brauer Algebras and Their Subalgebras

In this section, we recall the definition of the rook-Brauer algebras, the Motzkin algebras, the (planar) rook algebras, the (walled) Brauer algebras and the Temperley–Lieb algebras.

These algebras have been well studied in representation theory, knot theory and statistical mechanics – see, for instance, [Hd14; MM14; DG24] for rook-Brauer, [BH14; JY21] for Motzkin, [HR01; FHH09] for (planar) rook, [Hal96; BSH24] for walled Brauer, and [TL71; Jon83; Jon85] for Temperley–Lieb.

In Table 4.3.0.10, we will give a representative diagram for each algebra defined in this section.

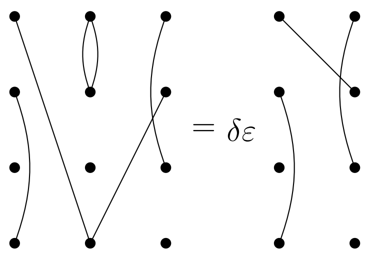
Definition 4.3.0.1. A *rook-Brauer n-diagram* is a graph on two columns of n vertices where each edge is incident to two distinct vertices, there is at most one edge between any two vertices and any vertex is connected to at most one other vertex.

Definition 4.3.0.2. Let δ, $\varepsilon \in R$. The *rook-Brauer algebra*, $\mathcal{RB}_n(\delta, \varepsilon)$, is the R-algebra with basis consisting of all rook-Brauer n-diagrams with the multiplication defined by the R-linear extension of the following product of diagrams. Let d_1 and d_2 be rook-Brauer n-diagrams. The product $d_1 d_2$ is obtained by the following procedure:

- Place the diagram d_2 to the right of the diagram d_1 and identify the vertices $\overline{1}, \ldots, \overline{n}$ in d_1 with the vertices $1, \ldots, n$ in d_2. Call this diagram $d_1 * d_2$. We drop the labels of the vertices in the middle column and we preserve the labels of the left-hand column and right-hand column.
- Count the number of loops that lie entirely within the middle column. Call this number α. Count the number of contractible connected components that lie entirely within the middle column. Call this number β.
- Make a new rook-Brauer n-diagram, d_3, as follows. Given distinct vertices x and y in the set $\{1, \overline{1}, \ldots, n, \overline{n}\}$, d_3 has an edge between x and y if there is a path from x to y in $d_1 * d_2$.
- We define $d_1 d_2 = \delta^\alpha \varepsilon^\beta d_3$.

The product is associative and the identity element consists of the diagram with n horizontal edges.

Example 4.3.0.3. Here is an example of composition in $\mathcal{RB}_4(\delta, \varepsilon)$. We drop the labels on the vertices for a clearer picture. In the diagram below, we form one loop in the middle column and we have one isolated vertex in the middle column and so we obtain a factor of $\delta\varepsilon$.



We refer the reader to the papers [Hd14] and [Boy25] for further examples of composing rook-Brauer diagrams.

Definition 4.3.0.4. Fix $n \geqslant 1$.

1. A *Motzkin n-diagram* is a planar rook-Brauer n-diagram, that is, a rook-Brauer n-diagram such that no edge crosses another. The *Motzkin algebra*, $\mathcal{M}_n(\delta, \varepsilon)$, is the subalgebra of $\mathcal{RB}_n(\delta, \varepsilon)$ spanned R-linearly by the Motzkin n-diagrams.

2. A *rook n-diagram* is a rook-Brauer n-diagram such that each connected component is either an isolated vertex, or it consists of exactly one vertex from the left-hand column and one vertex from the right-hand column. A *planar rook n-diagram* is a rook n-diagram which is planar. The *rook algebra*, $\mathcal{R}_n(\varepsilon)$, is the subalgebra of $\mathcal{RB}_n(\delta, \varepsilon)$ spanned R-linearly by the rook n-diagrams. The *planar rook algebra*, $\mathcal{PR}_n(\varepsilon)$, is the subalgebra of $\mathcal{RB}_n(\delta, \varepsilon)$ spanned R-linearly by the planar rook n-diagrams.

3. A *Brauer n-diagram* is a rook-Brauer n-diagram with no isolated vertices. Let $\delta \in R$. The *Brauer algebra*, $\mathcal{B}_n(\delta)$, is the subalgebra of $\mathcal{RB}_n(\delta, \varepsilon)$ spanned R-linearly by the Brauer n-diagrams.

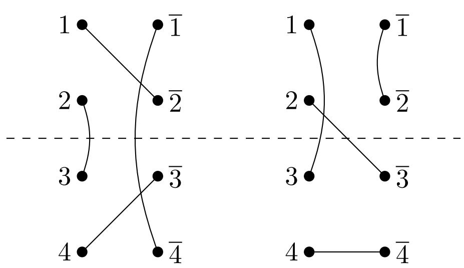
4. A *Temperley–Lieb n-diagram* is a planar rook-Brauer n-diagram with no isolated vertices. Let $\delta \in R$. The *Temperley–Lieb algebra*, $\mathcal{TL}_n(\delta)$, is the subalgebra of $\mathcal{RB}_n(\delta, \varepsilon)$ spanned R-linearly by the Temperley–Lieb n-diagrams.

Remark 4.3.0.5. We note that $R[\Sigma_n]$ is a subalgebra of the rook-Brauer algebra $\mathcal{RB}_n(\delta, \varepsilon)$ and the rook algebra $\mathcal{R}_n(\varepsilon)$. In both cases, it is the subalgebra spanned R-linearly by diagrams having precisely n propagating edges.

Definition 4.3.0.6. Let r and s be non-negative integers with $r + s \geqslant 1$. A *walled Brauer* $(r+s)$*-diagram* is a Brauer $(r+s)$-diagram such that

- a propagating edge must either join one of the first r vertices of the left-hand column with one of the first r vertices of the right-hand column or join one of the last s vertices of the left-hand column with one of the last s vertices of the right-hand column,
- a non-propagating edge in either column must join one of the first r vertices with one of the last s vertices.

Example 4.3.0.7. Consider the two diagrams below. We have drawn in a dashed horizontal line to indicate the division of the vertices into two parts.

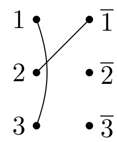
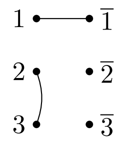
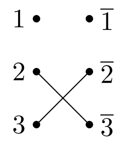
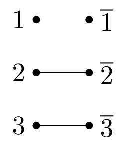
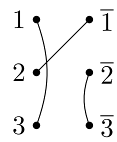
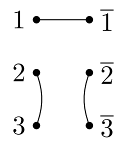
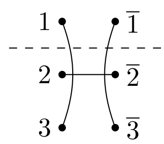


The diagram on the left is a walled Brauer $(2+2)$-diagram. The diagram on the right is not: there is a propagating edge that crosses the dividing line and a non-propagating edge in the right-hand column which does not. Both of these prevent this diagram from being a walled Brauer $(2+2)$-diagram.

Definition 4.3.0.8. Let r and s be non-negative integers. The *walled Brauer algebra* $\mathcal{B}_{r,s}(\delta)$ is the subalgebra of the Brauer algebra $\mathcal{B}_{r+s}(\delta)$ (and therefore, a subalgebra of $\mathcal{RB}_{r+s}(\delta, \varepsilon)$) spanned R-linearly by the walled Brauer $(r + s)$-diagrams. We note that $\mathcal{B}_{r,0} \cong \mathcal{B}_{0,r} \cong R[\Sigma_r]$.

Remark 4.3.0.9. We note that $R[\Sigma_r \times \Sigma_s]$ is a subalgebra of the walled Brauer algebra $\mathcal{B}_{r,s}(\delta)$. It is the subalgebra spanned R-linearly by diagrams having precisely r propagating edges starting at the first r vertices of the left-hand column and s propagating edges starting at the last s vertices of the left-hand column.

Table 4.3.0.10. We have the following examples of diagrams for each algebra defined in this section. We have taken $r = 1$ and $s = 2$ in our representative diagram of the walled Brauer algebra.

Diagram algebra name	Diagram algebra symbol	Representative diagram ($n = 3$)
Rook-Brauer algebra	$\mathcal{RB}_n(\delta, \varepsilon)$	
Motzkin algebra	$\mathcal{M}_n(\delta, \varepsilon)$	
Rook algebra	$\mathcal{R}_n(\varepsilon)$	
Planar rook algebra	$\mathcal{PR}_n(\varepsilon)$	
Brauer algebra	$\mathcal{B}_n(\delta)$	
Temperley–Lieb algebra	$\mathcal{TL}_n(\delta)$	
Walled Brauer algebra	$\mathcal{B}_{r+s}(\delta)$	

4.4 Dilute Temperley–Lieb Algebras

We recall the definition of the dilute Temperley–Lieb algebras. They arise in the study of solvable lattice models, in particular, the dilute A-D-E lattice models (see [Nie90; Roc92; WNS92; WNS93; WPSN94; Gri96; BP97; GN17a; GN17b] for instance). The representation theory of the dilute Temperley–Lieb algebras has been studied in [BS14; Bel15]. These algebras will be defined on bases of diagrams that are defined similarly to Temperley–Lieb diagrams (see Definition 4.3.0.4) except we now allow isolated vertices. There are also additional conditions on the multiplication in these algebras, and the identity elements are not the same as in the Temperley–Lieb algebras.

Remark 4.4.0.1. The dilute Temperley–Lieb algebra, $d\mathcal{TL}_n(\delta)$, defined below, has the same underlying free R-module of diagrams as the Motzkin algebra, $\mathcal{M}_n(\delta, \varepsilon)$ (see Definition 4.3.0.4). However, the product and identity element in the dilute Temperley–Lieb algebra are very different from those in the Motzkin algebra. For example, the Temperley–Lieb algebra is a unital subalgebra of the Motzkin algebra but not of the dilute Temperley–Lieb algebra. In order to distinguish between the two cases, we will refer to a planar rook-Brauer diagram as a Motzkin diagram when thought of as an element of the Motzkin algebra and as a *dilute Temperley–Lieb diagram* when thought of as an element of the dilute Temperley–Lieb algebra.

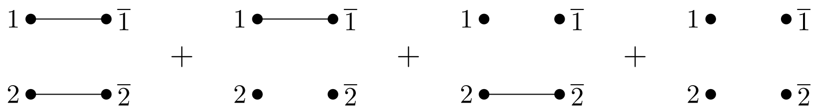
Definition 4.4.0.2. Let $\delta \in R$. The *dilute Temperley–Lieb algebra*, $d\mathcal{TL}_n(\delta)$, is the R-algebra with basis consisting of all dilute Temperley–Lieb n-diagrams with the multiplication defined by the R-linear extension of the following product of diagrams. Let d_1 and d_2 be dilute Temperley–Lieb n-diagrams. The product $d_1 d_2$ is obtained by the following procedure:

- Place the diagram d_2 to the right of the diagram d_1 and identify the vertices $\overline{1}, \ldots, \overline{n}$ in d_1 with the vertices $1, \ldots, n$ in d_2. Call this diagram with three columns of vertices $d_1 * d_2$. We drop the labels of the vertices in the middle column and we preserve the labels of the left-hand column and right-hand column.
- If we have an edge which is connected to neither the left-hand column nor the right-hand column and is not part of a loop, then the product is zero. Similarly, if we have an edge that connects the middle column to either the left-hand column or the right-hand column, but not both, then the product is zero. Such an edge is sometimes called a *floating edge*. We note, however, that if we have an isolated vertex in the middle column this does *not* mean that the product is zero.
- Otherwise, count the number of loops that lie entirely within the middle column. Call this number α.
- Make a new dilute Temperley–Lieb n-diagram, d_3, as follows. Given distinct vertices x and y in the set $\{1, \overline{1}, \ldots, n, \overline{n}\}$, d_3 has an edge between x and y if there is a path from x to y in $d_1 * d_2$. One can view the diagram d_3 as the result of discarding the vertices in the middle column, whilst maintaining the connection between any edges incident to these vertices.
- We define $d_1 d_2 = \delta^\alpha d_3$.

The identity element is the formal sum of all diagrams that can be formed from the diagram with n horizontal edges by omitting k edges ($0 \leqslant k \leqslant n$).

As noted in [BS14, Subsection 2.1], this product is associative since the formation of propagating edges, closed loops and floating edges in a composite of three or more diagrams is independent of the order of multiplication.

Example 4.4.0.3. The identity element in $d\mathcal{TL}_2(\delta)$ is



Example 4.4.0.4. Here are some examples of composition in the dilute Temperley–Lieb algebra $d\mathcal{TL}_2(\delta)$. We drop the labels on the vertices for a clearer picture. The first composite below is zero because we have a floating edge connecting the left-hand column to the middle column and it is not part of a loop.

The first composite below is zero because we have an edge which is contained entirely within the middle column. In the second composite below we form a loop in the middle column and so we obtain a factor of $\delta \in R$.

We refer the reader to [BS14] for further examples of composition in dilute Temperley–Lieb algebras.

4.5 Blob Algebras

The blob algebras are sometimes known as the *Temperley–Lieb algebras of type B*. They have their origins in statistical mechanics, namely the two-dimensional Potts model with toroidal boundary conditions. They were first introduced in the paper [MS94]. Since their introduction the blob algebras and their applications have been well studied (see [MW00; CGM03; GJSV13; EP19; LP20; HMP21] for instance).

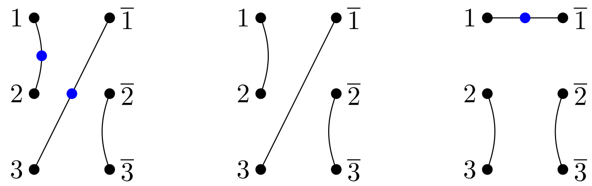
Definition 4.5.0.1. A *blob n-diagram* is a Temperley–Lieb n-diagram in which edges may be decorated with a single blob such that:

- no edge below the top-most propagating edge can be decorated with a blob;
- in any formation of nested non-propagating edges in the diagram, only the outermost may be decorated with a blob.

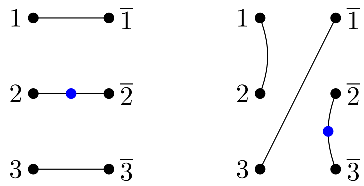
Informally, one can interpret this as saying that any edge decorated with a blob can "see" the top of the rectangle enclosing the vertices without crossing any other edges.

We will often call an edge that has been decorated with a blob a *blobbed edge*, and similarly an edge that is not decorated with a blob an *unblobbed edge*.

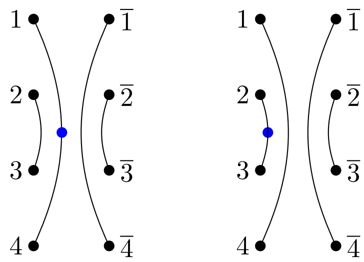
Example 4.5.0.2. The following are examples of blob 3-diagrams (note that the second is an example of blob 3-diagram which is also a Temperley–Lieb 3-diagram):



However, the following are not examples as the blobs occur *below* the top-most propagating edge:



Example 4.5.0.3. Consider the following two diagrams:



The left-hand diagram is a blob 4-diagram since the blobbed non-propagating edge occurs as the outer-most of the nested non-propagating edges. The right-hand diagram is not a blob 4-diagram since the blob occurs on a non-propagating edge that is nested inside another.

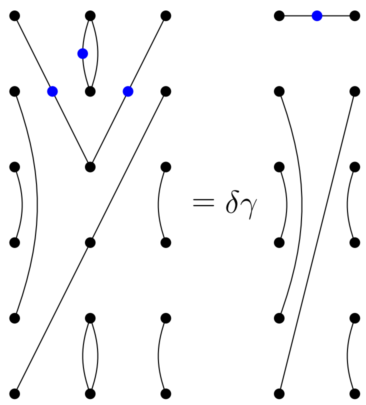
Definition 4.5.0.4. Let δ, $\gamma \in R$. The *blob algebra*, $\mathbf{Bl}_n(\delta, \gamma)$, is the R-algebra with basis consisting of all blob n-diagrams with the multiplication defined by the R-linear extension of the following product of diagrams. Let d_1 and d_2 be blob n-diagrams. The product $d_1 d_2$ is obtained by the following procedure:

- Place the diagram d_2 to the right the diagram d_1 and identify the vertices $\overline{1}, \ldots, \overline{n}$ in d_1 with the vertices $1, \ldots, n$ in d_2. Call this diagram with three columns of vertices $d_1 * d_2$. We drop the labels of the vertices in the middle column and we preserve the labels of the left-hand column and right-hand column.
- Count the number of unblobbed loops that lie entirely within the middle column. Call this number α. Count the number of blobbed loops that lie entirely within the middle column. Call this number β.

- Make a new blob n-diagram, d_3, as follows. Given distinct vertices x and y in the set $\{1, \bar{1}, \ldots, n, \bar{n}\}$, d_3 has an edge between x and y if there is a path from x to y in $d_1 * d_2$. This edge is decorated with a blob if any edge in the path is decorated by a blob (that is, blobs are idempotent).
- We define the composite $d_1 d_2 = \delta^\alpha \gamma^\beta d_3$.

The identity element consists of the diagram with n horizontal edges with no blobs.

Example 4.5.0.5. Here is an example of composition in $\mathbf{Bl}_6(\delta, \gamma)$. We drop the labels on the vertices and just illustrate the composite of the underlying graphs together with factors from the ground ring. In the composite, we have one blobbed loop in the middle column and one unblobbed loop in the middle column. Therefore we obtain a factor of $\delta\gamma$. We also compose two propagating edges that are decorated with blobs. Blobs are idempotent therefore the resulting edge in the composite is also labelled with a single blob.



4.6 Link States and Ideals

In this section we recall the notion of a *link state*. Link states were first introduced in the paper [RS14]. They in turn point out that the notion is related to *parenthesis structures* [Kau90], *arch configurations* [DGG97] and *cellular structure* [GL96]. We refer the reader to [BS14, Section 3] for a more extensive discussion of link states for dilute Temperley–Lieb diagrams and to [MS94, Section 2.2] for the original definition of link states for blob diagrams. We will use link states to define certain left ideals which will be crucial for us to prove global isomorphisms on (co)homology of diagram algebras with invertible parameters in Chapters 6 and 7. Our definitions and notation for link states and related ideals follow [Boy25, Section 1].

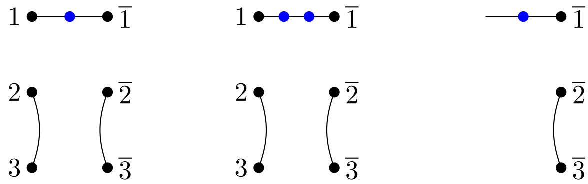
4.6.1 Link States for Diagram Algebras

We begin by recalling link states for the diagram algebras defined in Sections 4.3, 4.4 and 4.5. We will not require link states for subalgebras of the partition algebras in this thesis. Unless otherwise stated, then for the remainder of this section we will use n-diagram to refer to an n-diagram from any of the algebras defined in Sections 4.3, 4.4 and 4.5.

Definition 4.6.1.1. By slicing vertically down the middle of an n-diagram we obtain its *left link state* and *right link state*. Explicitly, we split all propagating edges at their midpoint and preserve all non-propagating edges. A propagating edge that has been split is called a *defect* in its link state.

To obtain the left or right link state of a blob n-diagram, then for a propagating edge decorated with a blob, we double the blob (which is allowed since blobs are idempotent) and split the propagating edge between the two blobs (so the defect in each link state is decorated with a blob).

Example 4.6.1.2. We show an example of the right link state of a blob 3-diagram with a blobbed propagating edge. The second diagram below is equal to the first diagram (since blobs are idempotent) and the third diagram is its right link state.



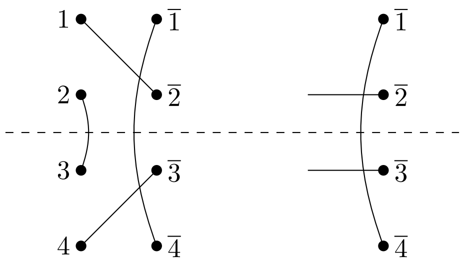
Remark 4.6.1.3. The right link state of an n-diagram consists of a column of n vertices, labelled $\overline{1}, \ldots \overline{n}$, such that at each vertex we have one of the following three situations:

- the vertex has a hanging edge, called a *defect*;
- the vertex is connected to precisely one other vertex by a non-propagating edge;
- the vertex is an isolated vertex.

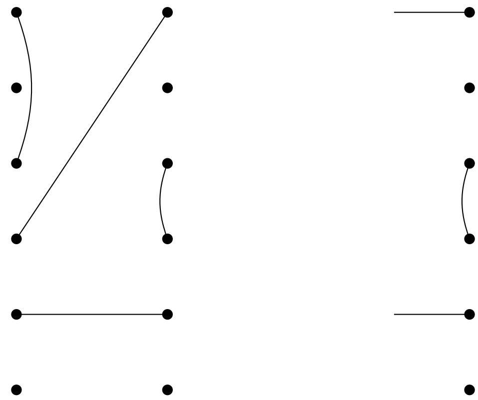
We note that link states for Brauer n-diagrams, Temperley–Lieb n-diagrams and walled Brauer $(r+s)$-diagrams will not have isolated vertices. Link states for Temperley–Lieb n-diagrams, Motzkin n-diagrams and dilute Temperley–Lieb n-diagrams must be planar.

For blob n-diagrams, a blob can only occur on the top-most defect of a link state or on any non-propagating edge above the top-most defect (so long as it is not nested within another non-propagating edge).

Example 4.6.1.4. In the diagram below we see the walled Brauer $(2+2)$-diagram from Example 4.3.0.7, together with its right link state:



Example 4.6.1.5. Here is an example of a dilute Temperley–Lieb 6-diagram (on the left) and its right link state (on the right).



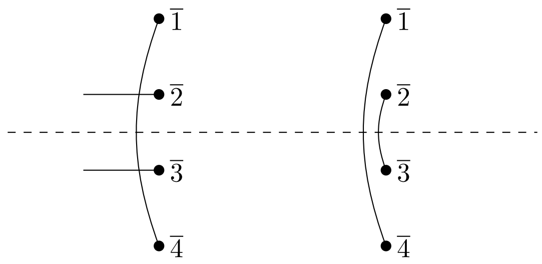
Definition 4.6.1.6. Suppose we have a right link state of an n-diagram. We can remove two defects and replace them with a non-propagating edge joining the two vertices. This operation is called a *splice*. We can also remove a defect, leaving an isolated vertex. This operation is called a *deletion*.

For a right link state of a blob n-diagram, if we perform a splice with a blobbed defect then the resulting non-propagating edge will also be blobbed.

Remark 4.6.1.7. These operations are only valid for some diagram algebras when certain conditions are met:

- We do not allow the deletion of defects when considering the (walled) Brauer algebra or the Temperley–Lieb algebra as this would result in isolated vertices;
- We also do not allow deletion of defects for the dilute Temperley–Lieb algebra. This is because, if we have a dilute Temperley–Lieb diagram d with a propagating edge from a to $\overline{b}$, then the deletion of the corresponding defect in this diagram's right link state is akin to multiplying d on the left by a dilute Temperley–Lieb diagram with an isolated vertex at $\overline{a}$. However, this would result in a floating edge and so the product here is 0 rather than the diagram obtained from d by deleting the propagating edge from a to $\overline{b}$;
- When considering a Temperley–Lieb diagram (respectively Motzkin diagram, dilute Temperley–Lieb diagram), we do not allow splices if the resulting diagram is not also a right link state of a Temperley–Lieb diagram (respectively Motzkin diagram, dilute Temperley–Lieb diagram). In particular, we can splice defects in these diagrams at vertices $\overline{i}$ and $\overline{k}$ if and only if there is not a defect at $\overline{j}$ with $i < j < k$;
- In the case of the walled Brauer algebra, we only allow splices that connect a defect at one of the first r vertices with a defect at one of the last s vertices;
- We do not allow splices that violate the conditions on blobs from Definition 4.5.0.1. In addition, as for the Temperley–Lieb, Motzkin and dilute Temperley–Lieb algebras, we cannot splice defects at vertices $\overline{i}$ and $\overline{k}$ if there is a defect at $\overline{j}$ with $i < j < k$.

Example 4.6.1.8. The diagrams below show the right link state of the walled Brauer (2+2)-diagram from Example 4.6.1.4, together with the result of splicing the two defects.



We now define left ideals of each of our families of diagram algebras which we will use in Chapters 6 and 7 to prove isomorphisms on (co)homology. Along the way, we define certain sets of right link states for rook-Brauer and blob diagrams which will be important to prove more cohomological results in Chapters 5 and 7.

Definition 4.6.1.9. For $0 \leqslant i \leqslant n$, we let P_i denote the set of right link states of rook-Brauer n-diagrams with precisely i defects.

Definition 4.6.1.10. Consider a right link state $p \in P_i$. Let J_p denote the left ideal of $\mathcal{RB}_n(\delta, \varepsilon)$ with basis given by the diagrams having right link state obtained from p by (possibly empty) sequences of splices and deletions.

Definition 4.6.1.11. For a right link state p of a dilute Temperley–Lieb n-diagram, let J_p denote the left ideal of $d\mathcal{TL}_n(\delta)$ with basis given by the diagrams having right link state obtained from p by a (possibly empty) sequence of splices.

Due to the extra conditions on blobs for blob n-diagrams, we need to be careful in defining our left ideals in $\mathbf{Bl}_n(\delta, \gamma)$.

Definition 4.6.1.12. Fix a positive integer n.

- Let Q_0 denote the set of right link states of blob n-diagrams with precisely 0 defects. For $1 \leqslant i \leqslant n-1$, let Q_i denote the set of right link states of blob n-diagrams with precisely i defects, none of them blobbed. For $1 \leqslant i \leqslant n$, let $Q_i^\bullet$ denote the set of right link states of blob n-diagrams with precisely i defects with a blob on the top-most.
- Given a right link state in Q_i (for $2 \leqslant i \leqslant n-1$) or $Q_i^\bullet$ (for $2 \leqslant i \leqslant n$), we can splice two defects.
- Given a right link state in Q_i (for $1 \leqslant i \leqslant n-1$), we can add a blob to the top-most defect. We call this operation *blobbing*.
- Let $R_0 = Q_0$, $R_1 = Q_1^\bullet$ and $R_i = Q_{i-1} \amalg Q_i^\bullet$ for $2 \leqslant i \leqslant n$.
- For a given right link state, q, of a blob n-diagram, let $\mathcal{J}_q$ denote the left ideal of $\mathbf{Bl}_n(\delta, \gamma)$ with basis given by the diagrams having right link state obtained from q by a valid (possibly empty) sequence of blobbing and splice operations.

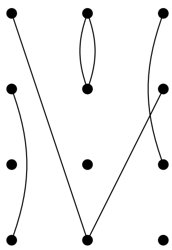
One can see that $J_p \subseteq \mathcal{RB}_n(\delta, \varepsilon)$ is a left ideal as follows: given rook-Brauer n-diagrams x and y, the right link state of the composite xy is obtained from the right link state of y by means of a valid sequence of splicings and deletions. Similar arguments show that $\mathsf{J}_p \subseteq d\mathcal{TL}_n(\delta)$ (by means of a valid sequence of splicings) and $\mathcal{J}_q \subseteq \mathbf{Bl}_n(\delta, \gamma)$ (by means of a valid sequence of splicings and blobbings) are also left ideals.

4.6.2 Sesqui-diagrams and Double Diagrams

In order to prove a number of our results on the cohomology of diagram algebras such as Theorem 6.4.4.1 for walled Brauer algebras and Theorem 7.1.4.1 for dilute Temperley–Lieb algebras, it will be useful to consider the diagrams formed in composition, together with the composite of a diagram and a right link state. This leads to the notions of *double diagram* and *sesqui-diagram* as introduced in [Boy25, Subsection 8.1].

Definition 4.6.2.1. For any of the algebras defined in Sections 4.3, 4.4 and 4.5, the composition of two diagrams d_1 and d_2 involved forming a diagram $d_1 * d_2$ with three columns of vertices, formed by identifying the right-hand column of vertices of d_1 with the left-hand column of vertices in d_2. We call such a diagram a *double diagram*. Henceforth, when using double diagrams we will label the left-hand column of vertices by $1, \dots, n$, the middle column of vertices with $1', \dots, n'$ and the right-hand column of vertices with $\overline{1}, \dots, \overline{n}$.

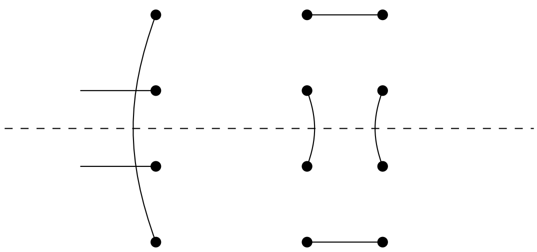
Example 4.6.2.2. The diagram



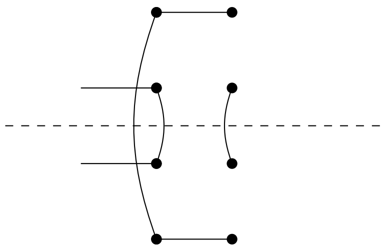
formed in composition of two rook-Brauer diagrams from Example 4.3.0.3 is a double diagram. We omit the labels here for a clearer picture.

Definition 4.6.2.3. For any of the algebras defined in Sections 4.3, 4.4 and 4.5, let p be a right link state of an n-diagram. Let d be an n-diagram. We define the *sesqui-diagram* (p, d) to be the diagram formed by identifying the vertices $\overline{1}, \dots, \overline{n}$ in p with the vertices $1, \dots, n$ in d. After performing this identification, we relabel the vertices. The left-hand column of vertices will be labelled by $1', \dots, n'$ and we preserve the labels $\overline{1}, \dots, \overline{n}$ on the right-hand column.

Example 4.6.2.4. Consider the right link state p and the diagram d (as walled Brauer $(2 + 2)$-diagrams) below:



The sesqui-diagram (p, d) is drawn below (where we have omitted the labelling for a clearer picture):



Chapter 5

Cohomology of Augmented Algebras

In this chapter, we recall the definition of an augmented algebra and show that all the diagram algebras defined in Chapter 4 are augmented algebras. We therefore show that each diagram algebra A has a trivial module via the augmentation and define the homology and cohomology of A accordingly.

We prove cohomological versions of a number of isomorphism results on the homology of algebras in [Boy24; Boy25].

As noted in Section 1.3, the contents of this chapter are adapted from joint work of the author with Daniel Graves. The original results of this chapter first appeared in [FG24b, Section 4] and [FG24a, Sections 6 and 7].

5.1 Augmented Diagram Algebras

Recall from Definition 2.1.0.4 that an R-algebra A is said to be *augmented* if it comes equipped with an R-algebra map $\tau\colon A \to R$, which is called the *augmentation* (we note that an augmentation is usually denoted by ε but we are already using this as a parameter in some of our algebras). We define augmentations and trivial modules for all algebras defined in Chapter 4 and hence define (co)homology of all of these algebras. For each family of diagram algebras we will then define certain ideals which we will make use of in Chapters 6 and 7.

Definition 5.1.0.1. We equip the blob algebra $\mathbf{Bl}_n(\delta, \gamma)$ (see Definition 4.5.0.4) with the augmentation

$$\tau\colon \mathbf{Bl}_n(\delta, \gamma) \to R$$

that sends every non-identity blob n-diagram to $0 \in R$ and the identity diagram to $1 \in R$. That is, the augmentation sends any *unblobbed* permutation diagram to $1 \in R$ and all other diagrams to $0 \in R$.

Let A be any algebra defined in Sections 4.2, 4.3 and 4.4 (that is, A is any diagram algebra considered in this thesis *except* the blob algebra). We equip A with the augmentation $\tau\colon A \to R$ that sends the permutation diagrams to $1 \in R$ and all non-permutation diagrams to $0 \in R$.

Remark 5.1.0.2. The augmentations in Definition 5.1.0.1 are indeed algebra maps because one can check that the composite of two diagrams with i propagating edges and j propagating edges respectively is a scalar multiple of a diagram with at most $\min(i, j)$ propagating edges. For instance, see [Hd14, Section 2.2] for the rook-Brauer algebra case or [BS14, Subsection 2.1] for the dilute Temperley–Lieb case, but the argument is similar for the other algebras.

We now define the trivial module and (co)homology for all of our diagram algebras following [Ben91, Section 2.4].

Definition 5.1.0.3. Let A be any algebra defined in Chapter 4 equipped with its augmentation τ defined in Definition 5.1.0.1. The *trivial A-module* (or just *trivial module*) $\mathbb{1}$ consists of a single copy of the ground ring R, where A acts on R via the augmentation τ.

Definition 5.1.0.4. Let A be any algebra defined in Chapter 4 which we equip with the augmentation τ, and let $\mathbb{1}$ denote the trivial A-module. We define the *homology* and *cohomology* of A to be

$$\mathrm{Tor}_\star^A(\mathbb{1}, \mathbb{1}) \quad \text{and} \quad \mathrm{Ext}_A^\star(\mathbb{1}, \mathbb{1})$$

respectively, and we note that other authors may denote these by $H_*(A)$ or $H^*(A)$ respectively.

We now define a collection of two-sided ideals inside each of the algebras defined in Chapter 4. Each of the definitions made below is seen to be valid by Remark 5.1.0.2.

Definition 5.1.0.5. Let I_{n-1} denote the two-sided ideal of $P_n(\delta)$ spanned R-linearly by all n-diagrams having having at most $n-1$ propagating components.

Let A denote any of the subalgebras of $P_n(\delta)$ defined in Section 4.2 which is equipped with an augmentation by restriction. Since I_{n-1} is a two-sided ideal in $P_n(\delta)$ (spanned R-linearly by all non-permutation diagrams), we see that $A \cap \mathrm{I}_{n-1}$ is a two-sided ideal in A spanned R-linearly by all n-diagrams in A having having at most $n-1$ propagating components.

Definition 5.1.0.6. For $0 \leqslant i \leqslant n-1$, let I_i denote the two-sided ideal of $\mathcal{RB}_n(\delta, \varepsilon)$ spanned R-linearly by rook-Brauer n-diagrams having at most i propagating edges. We will use the fact that $I_{-1} = 0$.

Remark 5.1.0.7. Recall from Definition 4.6.1.9 that P_i is the set of right link states of rook-Brauer n-diagrams with precisely i defects. Then this set consists of the right link states of the diagrams which span the quotient ideal I_i/I_{i-1}. This is an important part of the proof of [Boy25, Theorem 1.1] and Theorem 5.2.3.4.

Let A denote any of the subalgebras of $\mathcal{RB}_n(\delta, \varepsilon)$ defined in Section 4.3 which is equipped with an augmentation by restriction. Since I_{n-1} is a two-sided ideal in $\mathcal{RB}_n(\delta, \varepsilon)$ (spanned R-linearly by all non-permutation diagrams), we see that $A \cap I_{n-1}$ is a two-sided ideal in A spanned R-linearly by all non-permutation diagrams in A.

Definition 5.1.0.8. Let I_{n-1} denote the two-sided ideal of $d\mathcal{TL}_n(\delta)$ spanned R-linearly by dilute Temperley–Lieb n-diagrams having at most $n-1$ propagating edges.

Definition 5.1.0.9. Let $\mathcal{I}_0$ denote the two-sided ideal of $\mathbf{Bl}_n(\delta, \gamma)$ spanned R-linearly by all blob n-diagrams having no propagating edges. For $1 \leqslant i \leqslant n$, let $\mathcal{I}_i$ denote the two-sided ideal spanned R-linearly by diagrams, d, such that either

1. d has at most $i-1$ propagating edges or
2. d has precisely i propagating edges, has at least one blob and has a blob on the top-most propagating edge.

We will use the fact that $\mathcal{I}_{-1} = 0$.

In particular, $\mathcal{I}_n$ is a two-sided ideal in $\mathbf{Bl}_n(\delta, \gamma)$ spanned R-linearly by all non-identity diagrams.

Remark 5.1.0.10. Recall the definition of R_i from Definition 4.6.1.12. Then R_i consists of the right link states of the diagrams which span the quotient ideal $\mathcal{I}_i/\mathcal{I}_{i-1}$. This plays an important role in the proof of Theorem 7.2.1.1.

5.2 Results on the Cohomology of Algebras

We recall the definitions of R-free idempotent left cover and the Mayer–Vietoris complex from [Boy24] (which is a generalisation of Sroka's *cellular Davis complex* [Sro24, Definition 8]) and prove the cohomological analogues of [Boy24, Theorem 1.7] and [Boy25, Theorem 1.11].

5.2.1 Idempotent Left Covers and the Mayer–Vietoris Complex

The material in this subsection comes from [Boy24, Sections 1 and 2].

Definition 5.2.1.1. Let A be an R-algebra. Let I be a two-sided ideal of A. Let $w \geqslant h \geqslant 1$. An *idempotent left cover of I of height h and width w* is a collection of left ideals $J_1, \dots, J_w$ in A such that

- $J_1 + \cdots + J_w = I$;
- for $S \subset \underline{w}$ with $|S| \leqslant h$, the intersection
$$\bigcap_{i \in S} J_i$$
is either zero or is a principal left ideal generated by an idempotent.

If I is free as an R-module, then an idempotent left cover is said to be *R-free* if there is a choice of R-basis for I such that each J_i is free on a subset of this basis.

Definition 5.2.1.2. Let A be an R-algebra. Let $I \subset A$ be a two-sided ideal. Let $J_1, \dots, J_w$ be an idempotent left cover of I. The *Mayer–Vietoris complex associated to the idempotent left cover*, $C_\star$, is the chain complex of left A-modules defined as follows. We set
$$C_p = \bigoplus_{\substack{S \subset \underline{w} \\ |S| = p}} \bigcap_{i \in S} J_i$$
for $1 \leqslant p \leqslant w$. We set $C_0 = A$, $C_{-1} = A/I$ and $C_n = 0$ for $n > w$ and $n < -1$.

The differential $C_0 \to C_{-1}$ is the projection map $A \to A/I$. The differential $C_1 \to C_0$ is the direct sum of the inclusion of the left ideals $J_i \to A$. For $p \geqslant 2$, the differential $C_p \to C_{p-1}$ is defined on the summand $\bigcap_{i \in S} J_i$ by
$$x \mapsto \sum_{j \in S} (-1)^{\#(S,j)} i_{(S,j)}(x)$$
where $\#(S, j)$ is the number of elements of S that are less than j and $i_{(S,j)}$ is the inclusion
$$\bigcap_{i \in S} J_i \to \bigcap_{i \in S \setminus \{j\}} J_i.$$

Recall that for a left A-module M, a *partial projective resolution of length h* of M by left A-modules is an exact sequence

$$P_h \to P_{h-1} \to \cdots \to P_0 \to M \to 0$$

where each P_i is a projective left A-module.

The Mayer–Vietoris complex satisfies the following important properties, as proved in [Boy24, Section 2].

Lemma 5.2.1.3 ([Boy24, Lemma 2.2])**.** *If I is free as an R-module on some basis such that the J_i are free R-modules on subsets of that basis, then the Mayer–Vietoris complex is acyclic.* □

Proposition 5.2.1.4 ([Boy24, Proposition 2.5])**.** *Let A be an R-algebra. Let I be a two-sided ideal of A. Let $J_1, \dots, J_w$ be an R-free idempotent left cover of I of height h.*

The truncation, $C_\star^{\leqslant h}$ of the Mayer–Vietoris complex associated to the idempotent left cover is a length h partial projective resolution of A/I by left A-modules with $C_0 = A$. The partial projective resolution has the additional property that $X \otimes_A C_p^{\leqslant h} = 0$ for $p \geqslant 1$ for any right A-module X on which I acts trivially. If $h = w$ then $C_\star^{\leqslant h} = C_\star$ is a projective resolution of A/I by left A-modules. □

The Mayer–Vietoris complex also satisfies the dual condition to the vanishing of the tensor products.

Lemma 5.2.1.5. *Let A be an R-algebra. Let I be a two-sided ideal of A. Let $J_1, \dots, J_w$ be an R-free idempotent left cover of I of height h. The truncation, $C_\star^{\leqslant h}$ of the Mayer–Vietoris complex associated to the idempotent left cover has the property that*

$$\mathrm{Hom}_A(C_p^{\leqslant h}, Y) = 0$$

for any left A-module Y on which I acts trivially.

Proof. The proof is similar to [Boy24, Lemma 2.3]. It suffices to consider an idempotent generator of $C_p^{\leqslant h}$. Let $\alpha \in A$ and let e be the idempotent generator of one of the summands of $C_p^{\leqslant h}$. For $f \in \mathrm{Hom}_A(C_p^{\leqslant h}, Y)$, we have $f(\alpha e) = f(\alpha e^2) = \alpha e f(e) = 0$, where the first equality uses idempotence of e, the second uses A-linearity and the final equality uses the fact that I (and therefore each J_i) acts trivially on Y. □

5.2.2 A Cohomology Isomorphism in a Range of Degrees

Using the Mayer–Vietoris complex, Boyde proved the following result.

Proposition 5.2.2.1 ([Boy24, Theorem 1.7])**.** *Let A be an augmented R-algebra with trivial module $\mathbb{1}$. Let I be a two-sided ideal of A which is free as an R-module and which acts as multiplication by $0 \in R$ on $\mathbb{1}$. Suppose that there exists an R-free idempotent left cover of I of height h and width w. There is a natural isomorphism of R-modules*

$$\mathrm{Tor}_q^A(\mathbb{1}, \mathbb{1}) \cong \mathrm{Tor}_q^{A/I}(\mathbb{1}, \mathbb{1})$$

for $q \leqslant h$. Furthermore, if $h = w$, then the isomorphism holds for all q. □

In order to show that the cohomological analogue of Proposition 5.2.2.1 holds, we will require the following result also proved by Boyde.

Proposition 5.2.2.2 ([Boy24, Proposition 2.6])**.** *Let M be a left A-module, let N be a right A-module and let $h > 0$. Suppose that there exists a length h partial projective resolution $C_\star^{\leqslant h}$ of M with the property that $N \otimes_A C_\star^{\leqslant h} = 0$ for $p \geqslant 1$. Then*

$$\operatorname{Tor}_q^A(N, M) = 0$$

for $0 < q \leqslant h$, and $N \otimes_A M \cong N \otimes_A C_\star^{\leqslant h}$. □

We now show that the cohomological analogue Proposition 5.2.2.4 also holds.

Proposition 5.2.2.3. *Let M and N be right A-modules. Let I be a two-sided ideal that acts trivially on M and N.*

Suppose that there exists a partial projective resolution of length h, $C_\star^{\leqslant h}$, of A/I by left A-modules such that $C_0^{\leqslant h} = A$ and such that $X \otimes_A C_p^{\leqslant h} = 0$ for $p \geqslant 1$ for any right A-module X on which I acts trivially.

There is a natural isomorphism of R-modules

$$\operatorname{Ext}_A^q(M, N) \cong \operatorname{Ext}_{A/I}^q(M, N)$$

for $q \leqslant h$.

Proof. When $q \leqslant h$, we will show that $\operatorname{Ext}_A^q(M, N)$ and $\operatorname{Ext}_{A/I}^q(M, N)$ are the cohomology of the same cochain complex.

Let $F_\star$ be a free resolution of M by right A-modules. We know that $\operatorname{Ext}_A^\star(M, N)$ is the cohomology of the cochain complex $\operatorname{Hom}_A(F_\star, N)$.

Since I acts trivially on N, we have an isomorphism of cochain complexes

$$\operatorname{Hom}_A(F_\star, N) \cong \operatorname{Hom}_{A/I}\big(F_\star \otimes_A (A/I), N\big)$$

by extension and restriction of scalars.

We observe that since each F_i is free as an A-module, each $F_i \otimes_A (A/I)$ is free as an A/I-module.

In order to deduce the isomorphisms for $q \leqslant h$, it suffices to show that the homology of $F_\star \otimes_A (A/I)$ is isomorphic to M in degree zero and 0 in degrees $0 < q \leqslant h$.

By assumption, $C_\star^{\leqslant h}$ is a partial projective resolution of A/I by left A-modules with the property that $X \otimes_A C_p^{\leqslant h} = 0$ for $p \geqslant 1$ for any right A-module X on which I acts as multiplication by $0 \in k$. Furthermore, $C_0^{\leqslant h} = A$.

Therefore, by applying Proposition 5.2.2.2 ([Boy24, Proposition 2.6]), we have

$$H_q(F_\star \otimes_A (A/I)) = \operatorname{Tor}_q^A(M, A/I) = 0$$

for $0 < q \leqslant h$ and

$$H_0(F_\star \otimes_A (A/I)) = \operatorname{Tor}_0^A(M, A/I) = M \otimes_A (A/I) = M \otimes_A C_0^{\leqslant h} = M \otimes_A A \cong M.$$

This yields the necessary isomorphisms for $q \leqslant h$. □

Proposition 5.2.2.4. *Let A be an augmented R-algebra with trivial module $\mathbb{1}$. Let I be a two-sided ideal of A which is free as an R-module and which acts as multiplication by $0 \in R$ on $\mathbb{1}$. Suppose that there exists an R-free idempotent left cover of I of height h and width w. There is a natural isomorphism of R-modules*

$$\mathrm{Ext}^q_A(\mathbb{1}, \mathbb{1}) \cong \mathrm{Ext}^q_{A/I}(\mathbb{1}, \mathbb{1})$$

for $q \leqslant h$. Furthermore, if $h = w$, then the isomorphism holds for all q.

Proof. This follows from Proposition 5.2.2.3 and Proposition 5.2.1.4 ([Boy24, Proposition 2.4]), taking $M = N = \mathbb{1}$. □

5.2.3 Ideals Generated by Idempotents

In this subsection we prove the cohomological analogues of [Boy25, Theorem 4.3] and [Boy25, Corollary 4.4] and provide an application in the case of subalgebras of the rook-Brauer algebras.

Theorem 5.2.3.1. *Let A be an associative R-algebra. Let M and N be right A-modules. Suppose that I is a two-sided ideal of A such that*

- *I acts trivially on M and N and*
- *as a left ideal I is isomorphic to a direct sum of left ideals $J_1 \oplus \cdots \oplus J_p$ where each J_i is generated as a left ideal by finitely many commuting idempotents.*

Then there is an isomorphism of graded R-modules

$$\mathrm{Ext}^\star_A(M, N) \cong \mathrm{Ext}^\star_{A/I}(M, N).$$

Proof. We will show that $\mathrm{Ext}^\star_A(M, N)$ and $\mathrm{Ext}^\star_{A/I}(M, N)$ are the cohomology of the same cochain complex.

Let $F_\star$ be a free resolution of M by right A-modules. We know that $\mathrm{Ext}^\star_A(M, N)$ is the cohomology of the cochain complex $\mathrm{Hom}_A(F_\star, N)$.

On the other hand, since I acts trivially on N, we have an isomorphism of cochain complexes

$$\mathrm{Hom}_A(F_\star, N) \cong \mathrm{Hom}_{A/I}(F_\star \otimes_A (A/I), N),$$

by extension and restriction of scalars.

We now wish to show that $F_\star \otimes_A (A/I)$ is a resolution of M by free right (A/I)-modules. This is demonstrated in [Boy25, Proof of Theorem 4.3], and we summarise Boyde's argument here which uses the commuting idempotents in the statement and the fact that I acts trivially on M.

The commuting idempotents ensure that I is a projective left A-module, yielding a projective resolution $0 \to I \hookrightarrow A \to 0$ of A/I by projective right A-modules. The fact that I acts trivially on M allows one to deduce that when we tensor over A with M we obtain $M \otimes_A I = 0$. Combining these two things we see that $\mathrm{Tor}^A_\star(M, A/I) = 0$ for $\star \geqslant 1$

and $\mathrm{Tor}_0^A(M, A/I) = M$, that is $F_\star \otimes_A (A/I)$ is a resolution of M by free right (A/I)-modules, as required. Therefore the cohomology of the right-hand side, $\mathrm{Ext}^\star_{A/I}(M, N)$, is isomorphic to $\mathrm{Ext}^\star_A(M, N)$ as required. □

The theorem has two important corollaries. The first covers the case where I itself is generated as a left ideal by finitely many commuting idempotents. This follows immediately from the theorem and we state it without proof. The second covers the case where we have a chain of inclusions of two-sided ideals and each successive quotient of ideals satisfies the conditions of the theorem.

Corollary 5.2.3.2. *Let A be an associative R-algebra. Let M and N be right A-modules. Suppose that I is a two-sided ideal of A such that I acts trivially on M and N and that I is generated as a left ideal by finitely many commuting idempotents. Then there is an isomorphism of graded R-modules*

$$\mathrm{Ext}^\star_A(M, N) \cong \mathrm{Ext}^\star_{A/I}(M, N).$$

□

Corollary 5.2.3.3. *Let A be a unital, associative R-algebra. Let M and N be right A-modules. Let $0 \leqslant l \leqslant m$. Suppose that we have a chain of two-sided ideals*

$$0 = I_{-1} \leqslant I_0 \leqslant I_1 \leqslant \cdots \leqslant I_m \leqslant A,$$

such that

- *each I_j acts trivially on M and N;*
- *for $i \geqslant l$ there is an isomorphism of left A-modules*

$$\frac{I_i}{I_{i-1}} \cong \frac{J_{i,1}}{I_{i-1}} \oplus \cdots \oplus \frac{J_{i,p_i}}{I_{i-1}},$$

  for left ideals $J_{i,j}$ generated by finitely many commuting idempotents.

Then there exist isomorphisms of graded R-modules

$$\mathrm{Ext}^\star_{A/I_{l-1}}(M, N) \cong \mathrm{Ext}^\star_{A/I_l}(M, N) \cong \cdots \cong \mathrm{Ext}^\star_{A/I_m}(M, N).$$

Proof. Starting with the given chain of two-sided ideals and $l \geqslant 0$, the conditions in the statement tell us that for each $i \geqslant l$, the conditions of Theorem 5.2.3.1 are satisfied for the image of each two-sided ideal I_i inside the algebra A/I_{i-1}. Theorem 5.2.3.1 therefore gives the required chain of isomorphisms. □

We now apply Corollary 5.2.3.3 to subalgebras of the rook-Brauer algebras. We will see in Section 6.3 and Section 6.4 that we can apply the below result to all the different families of subalgebras of the rook-Brauer algebras considered in this thesis and defined in Section 4.3 including the rook-Brauer algebras themselves.

We recall from Definition 5.1.0.6 that for $0 \leqslant i \leqslant n-1$, I_i is the two-sided ideal of $\mathcal{RB}_n(\delta, \varepsilon)$ spanned by rook-Brauer n-diagrams having at most i propagating edges. We also recall from Definitions 4.6.1.9 and 4.6.1.10 that for $0 \leqslant i \leqslant n$ and $p \in P_i$ a right link state of a rook-Brauer n-diagram with precisely i defects, J_p is the left ideal of $\mathcal{RB}_n(\delta, \varepsilon)$ spanned by rook-Brauer n-diagrams having right link state obtained from p by (possibly empty) sequences of splices and deletions.

Theorem 5.2.3.4. *Let $0 \leqslant l \leqslant m \leqslant n-1$. Let A be a subalgebra of $\mathcal{RB}_n(\delta, \varepsilon)$ such that*

- *as an R-module, A is free on a subset of the rook-Brauer diagrams and*
- *for i in the range $l \leqslant i \leqslant m$, for each right link state $p \in P_i$, if A contains at least one diagram with right link state p, then A contains an idempotent e_p such that in A we have an equality of left ideals $A \cdot e_p = A \cap J_p$.*

Then we have a chain of isomorphisms

$$\operatorname{Ext}^{\star}_{A/\left(A\cap I_{l-1}\right)}(\mathbb{1}, \mathbb{1}) \cong \operatorname{Ext}^{\star}_{A/(A\cap I_l)}(\mathbb{1}, \mathbb{1}) \cong \cdots \cong \operatorname{Ext}^{\star}_{A/(A\cap I_m)}(\mathbb{1}, \mathbb{1}).$$

Proof. The proof of this theorem follows the same method as [Boy25, Theorem 1.11], given in Section 6 of that paper. We will apply Corollary 5.2.3.3 to the chain of two-sided ideals

$$0 = A \cap I_{-1} \leqslant A \cap I_0 \leqslant \cdots \leqslant A \cap I_m \leqslant A.$$

We note that each of these two-sided ideals acts as multiplication by $0 \in R$ on $\mathbb{1}$ since the diagrams spanning these ideals have fewer than n propagating edges. Therefore the first condition of Corollary 5.2.3.3 is satisfied.

For the second condition, since each I_i is free as an R-module on some basis of diagrams and since every diagram has some right link state, we can write

$$\frac{A \cap I_i}{A \cap I_{i-1}} = \sum_{p \in P_i} \frac{A \cap J_p}{A \cap I_{i-1}},$$

where we recall that the set P_i consists of the right link states of the diagrams that span the quotient I_i/I_{i-1}. The fact that the sum is direct follows by noting that, if we have two distinct elements in P_i, say p and q, then $J_p \cap J_q \subset I_{i-1}$. Since p and q are distinct, an element in $J_p \cap J_q$ must be obtained from both p and q by at least one splice or deletion (depending on which operations are allowed in A). In either case, this yields a diagram in I_{i-1} since the number of defects is reduced. □

Remark 5.2.3.5. In Theorem 7.2.1.1, we will prove an analogous result to Theorem 5.2.3.4 for the blob algebras.

Chapter 6

Cohomology of Diagram Algebras I

In this chapter and Chapter 7, we will apply results from Chapter 5 to the diagram algebras defined in Chapter 4. Here, we will apply results to subalgebras of the partition algebras and subalgebras of the rook-Brauer algebras (the remaining diagram algebras will be considered in Chapter 7). If we work over a subalgebra of the rook-Brauer algebra with invertible parameter(s), then we prove global isomorphisms on (co)homology by adapting arguments first used in [Boy25] to show that certain left ideals defined in Section 4.6 are generated by idempotents and then applying Theorem 5.2.3.4. In the case of partition algebras, our method of proof uses cohomological versions of results in [BHP23].

If we instead work over a diagram algebra with arbitrary parameter(s) (that is, not necessarily invertible in the base ring), then we prove stable isomorphisms on (co)homology by adapting arguments in [Boy24; Sro24] to construct free idempotent left covers (see Subsection 5.2.1) of certain two-sided ideals defined in Section 5.1 and then applying Proposition 5.2.2.4. If the height of the idempotent left cover is equal to the width, then in fact we get a global isomorphism on (co)homology.

As noted in Section 1.3, the contents of this chapter are adapted from joint work of the author with Daniel Graves. The original results of this chapter first appeared in [FG24b, Sections 5 and 6], [FG24a, Sections 7 and 8] and [FG25a, Section 8].

6.1 Cohomology of Partition Algebras

The homology of partition algebras has been studied by Boyd, Hepworth and Patzt [BHP23]. Their results split into two cases, which depend on the parameter δ. They show that if the parameter δ is invertible in the ground ring, then the homology of the partition algebras is globally isomorphic to the homology of the symmetric groups. However, if δ is not invertible then the homology of the partition algebras is only isomorphic to the homology of the symmetric groups in a range. In this section we prove cohomological analogues of these results. We show that if δ is invertible, then the cohomology of the partition algebra $P_n(\delta)$ is globally isomorphic to the cohomology of the symmetric group Σ_n, using the methods introduced in [BHP23]. We also show that for any $\delta \in R$, the cohomology of the partition algebras is stably isomorphic to the group cohomology of the symmetric groups using Proposition 5.2.2.4. As a corollary, we deduce that the partition algebras satisfy cohomological stability.

6.1.1 Partition Algebras with Invertible Parameter

In this subsection we will prove that if the parameter $\delta \in R$ is invertible then the cohomology of $P_n(\delta)$ is globally isomorphic to the cohomology of the symmetric group Σ_n. We do this by finding a replacement of the cohomological version of Shapiro's lemma for the partition algebras. This should be seen as the cohomological companion to [BHP23, Theorem 5.1].

We first recall the cohomological version of Shapiro's lemma (see, for instance, [Ben91, Corollary 2.8.4] or [Bro82, Proposition III.6.2]).

Lemma 6.1.1.1 (Shapiro's lemma (cohomological version))**.** *Let G be a group and let H*

be a subgroup of G. The natural map

$$H^*(G, \mathrm{Hom}_{RH}(RG, \mathbb{1})) \to H^*(H, \mathbb{1})$$

induced by the inclusion $H \hookrightarrow G$ and the RH-module map $\mathrm{Hom}_{RH}(RG, \mathbb{1}) \to \mathbb{1}$ which sends f to $f(1)$ is an isomorphism. □

Definition 6.1.1.2. For $m \geqslant 1$, let J_m denote the left ideal of $P_n(\delta)$ consisting of all diagrams in which, amongst the vertices labelled $\{\overline{1}, \ldots, \overline{m}\}$, there is at least one singleton, or one pair of vertices in the same connected component.

We note that, for $m \leqslant n$, we can consider $P_m(\delta)$ as a subalgebra of $P_n(\delta)$ where we extend a $P_m(\delta)$-diagram to a $P_n(\delta)$-diagram by adding $n - m$ vertices to the left and right columns below the existing ones and adding horizontal propagating edges between the new vertices. We can therefore consider $P_n(\delta)$ as a right $P_m(\delta)$-module under the action of this subalgebra.

Remark 6.1.1.3. As noted in [BHP23, Lemma 5.2], for $m \leqslant n$, $P_n(\delta)$ is a right $R[\Sigma_m]$-module (where Σ_m acts by permuting the vertices $\{\overline{1}, \ldots, \overline{m}\}$) and $P_n(\delta)/\mathrm{J}_m$ is a free right $R[\Sigma_m]$-module (where the above Σ_m action does not change whether there exists a singleton or a pair of vertices in the same connected component in $\{\overline{1}, \ldots, \overline{m}\}$).

For the remainder of this subsection, we will drop the δ and denote the partition algebra simply by P_n in order to ease notation.

As P_n/J_m is a right $R[\Sigma_m]$-module and there is the trivial right $R[\Sigma_m]$-module $\mathbb{1}$, we can consider the module $\mathrm{Hom}_{R[\Sigma_m]}\left(P_n/\mathrm{J}_m, \mathbb{1}\right)$ of $R[\Sigma_m]$-linear homomorphisms of right modules. We can in fact make this into a right P_n-module since P_n acts on the right of both of P_n/J_m and $\mathbb{1}$ where $(f \cdot d)(x) = f(x \cdot d)$. In addition, we have already noted that P_n is a right P_m-module (since P_m is a subalgebra of P_n) and we have the trivial right P_m-module $\mathbb{1}$, and so we can consider the module $\mathrm{Hom}_{P_m}(P_n, \mathbb{1})$ of P_m-linear homomorphisms of right modules. We can in fact make this into a right P_n-module since P_n acts on the right of both of P_n and $\mathbb{1}$.

Lemma 6.1.1.4. *For $m \leqslant n$, there is an isomorphism of right P_n-modules*

$$\mathrm{Hom}_{R[\Sigma_m]}\left(P_n/\mathrm{J}_m, \mathbb{1}\right) \cong \mathrm{Hom}_{P_m}(P_n, \mathbb{1})$$

given by the maps

$$\varphi\colon \mathrm{Hom}_{R[\Sigma_m]}\left(P_n/\mathrm{J}_m, \mathbb{1}\right) \to \mathrm{Hom}_{P_m}(P_n, \mathbb{1})$$

determined by $\varphi(f)(x) = f([x])$ and

$$\psi\colon \mathrm{Hom}_{P_m}(P_n, \mathbb{1}) \to \mathrm{Hom}_{R[\Sigma_m]}\left(P_n/\mathrm{J}_m, \mathbb{1}\right)$$

determined by $\psi(g)([x]) = g(x)$.

Proof. It suffices to show that the maps are well-defined as it is then clear that they are mutually inverse. In particular, we will show that each $\varphi(f)$ is a homomorphism of right P_m-modules, each $\psi(g)$ is a homomorphism of right $R[\Sigma_m]$-modules, and that each $\psi(g)([x])$ depends only on the class $[x]$ and not the element x itself.

We follow the conventions of [BHP23, Lemma 5.3]. We regard J_m as an ideal of $P_n(\delta)$ and write $\mathrm{J}_m \cap P_m$ for the corresponding ideal in $P_m(\delta)$. As noted in [BHP23, Lemma 5.3], $P_m \cong R[\Sigma_m] \oplus (\mathrm{J}_m \cap P_m)$.

To show that the map φ is well-defined, we must show that $\big(\varphi(f)(x)\big)\,\sigma = \varphi(f)(x\sigma)$ for $\sigma \in P_m$. Since $P_m \cong R[\Sigma_m] \oplus (\mathrm{J}_m \cap P_m)$, it suffices to show this for $\sigma \in \Sigma_m$ and for $\sigma \in \mathrm{J}_m \cap P_m$.

For $\sigma \in \Sigma_m$, we have $\big(\varphi(f)(x)\big)\,\sigma = \varphi(f)(x) = f([x])$ since σ acts as multiplication by $1 \in R$ on $\mathbb{1}$. On the other hand, $\varphi(f)(x\sigma) = f([x\sigma]) = f([x])\sigma = f([x])$, where the penultimate equality holds by the $R[\Sigma_m]$-linearity of f and the final equality holds since σ acts as multiplication by $1 \in R$ on $\mathbb{1}$.

For $\sigma \in \mathrm{J}_m \cap P_m$, $\big(\varphi(f)(x)\big)\,\sigma = 0$ since σ acts as multiplication by $0 \in R$ on $\mathbb{1}$. On the other hand, $\varphi(f)(x\sigma) = f([x\sigma]) = f([0]) = 0$ since $\sigma \in \mathrm{J}_m$ and so $[x\sigma] = [0] \in P_n/\mathrm{J}_m$. Therefore we have $\big(\varphi(f)(x)\big)\,\sigma = \varphi(f)(x\sigma)$ for any $\sigma \in P_m$, so each $\varphi(f)$ is a homomorphism of right P_m-modules and so the map φ is well-defined.

To show that the map ψ is well-defined, we must show that for $\sigma \in \Sigma_m$, we have $\psi(g)([x\sigma]) = \psi(g)([x])$ and that $\psi(g)([x]) = g(x) = 0$ if $x \in \mathrm{J}_m$.

If $\sigma \in \Sigma_m$ then $\psi(g)([x\sigma]) = g(x\sigma) = g(x)\sigma = \psi(g)([x])$ by P_m-linearity of g, and since $\sigma \in \Sigma_m \subset P_m$ acts as multiplication by $1 \in R$ on $\mathbb{1}$.

Now, if $x \in \mathrm{J}_m$, then as noted in [BHP23, Proof of Lemma 5.3] we can write x as a sum of terms of the form dx' with $d \in P_n$ and $x' \in \mathrm{J}_m \cap P_m$. For each such summand we have $\psi(g)([dx']) = g(dx') = g(d)x' = 0$ where the penultimate equality holds by P_m-linearity of g and the final equality holds since $x' \in \mathrm{J}_m$ acts as multiplication by $0 \in R$ on $\mathbb{1}$. Therefore ψ is well-defined. □

Theorem 6.1.1.5. *Let $n \geqslant m \geqslant 0$. Suppose that $\delta \in R$ is invertible or that $m < n$. The maps*

$$\pi^*\colon\ \mathrm{Ext}^{\star}_{R[\Sigma_n]}\Big(\mathbb{1}, \mathrm{Hom}_{R[\Sigma_m]}\big(R[\Sigma_n], \mathbb{1}\big)\Big) \to \mathrm{Ext}^{\star}_{P_n}\big(\mathbb{1}, \mathrm{Hom}_{P_m}(P_n, \mathbb{1})\big)$$

and

$$i^*\colon\ \mathrm{Ext}^{\star}_{P_n}\big(\mathbb{1}, \mathrm{Hom}_{P_m}(P_n, \mathbb{1})\big) \to \mathrm{Ext}^{\star}_{R[\Sigma_n]}\Big(\mathbb{1}, \mathrm{Hom}_{R[\Sigma_m]}\big(R[\Sigma_n], \mathbb{1}\big)\Big)$$

induced by the inclusion $R[\Sigma_n] \to P_n$ and projection $\pi\colon P_n \to R[\Sigma_n]$ are mutually inverse isomorphisms.

Proof. Let Q_* be a free resolution of $\mathbb{1}$ by right $R[\Sigma_n]$-modules. We observe that this is also a free resolution of $\mathbb{1}$ by right $R[\Sigma_m]$-modules. The map

$$\mathrm{Hom}_{R[\Sigma_n]}\big(Q_i, \mathrm{Hom}_{R[\Sigma_m]}(R[\Sigma_n], \mathbb{1})\big) \to \mathrm{Hom}_{R[\Sigma_m]}(Q_i, \mathbb{1})$$

which sends $f(x)$ to $f(x)(1)$ induces an isomorphism on cohomology

$$\sigma\colon\ \mathrm{Ext}^{\star}_{R[\Sigma_n]}\big(\mathbb{1}, \mathrm{Hom}_{R[\Sigma_m]}(R[\Sigma_n], \mathbb{1})\big) \xrightarrow{\cong} \mathrm{Ext}^{\star}_{R[\Sigma_m]}(\mathbb{1}, \mathbb{1})$$

by Lemma 6.1.1.1, the cohomological version of Shapiro's lemma.

We have the induced map

$$i^*\colon \operatorname{Ext}^\star_{P_n}\left(\mathbb{1}, \operatorname{Hom}_{P_m}(P_n, \mathbb{1})\right) \to \operatorname{Ext}^\star_{R[\Sigma_n]}\Big(\mathbb{1}, \operatorname{Hom}_{R[\Sigma_m]}\left(R[\Sigma_n], \mathbb{1}\right)\Big).$$

We will construct an isomorphism of graded R-modules

$$\theta\colon \operatorname{Ext}^\star_{R[\Sigma_m]}(\mathbb{1}, \mathbb{1}) \to \operatorname{Ext}^\star_{P_n}\left(\mathbb{1}, \operatorname{Hom}_{P_m}(P_n, \mathbb{1})\right)$$

and show that the composite $\sigma \circ i^* \circ \theta$ is the identity on $\operatorname{Ext}^\star_{R[\Sigma_m]}(\mathbb{1}, \mathbb{1})$.

We begin by constructing θ. Let $F_\star$ be a free resolution of $\mathbb{1}$ by right P_n-modules.

The graded R-module $\operatorname{Ext}^\star_{P_n}\left(\mathbb{1}, \operatorname{Hom}_{P_m}(P_n, \mathbb{1})\right)$ is the cohomology of the cochain complex

$$\operatorname{Hom}_{P_n}\left(F_\star, \operatorname{Hom}_{P_m}(P_n, \mathbb{1})\right).$$

By Lemma 6.1.1.4, there is an isomorphism of cochain complexes

$$\operatorname{Hom}_{P_n}\left(F_\star, \operatorname{Hom}_{P_m}(P_n, \mathbb{1})\right) \cong \operatorname{Hom}_{P_n}\left(F_\star, \operatorname{Hom}_{R[\Sigma_m]}(P_n/\mathrm{J}_m, \mathbb{1})\right).$$

By the tensor-hom adjunction, we have an isomorphism of chain complexes

$$\operatorname{Hom}_{P_n}\left(F_\star, \operatorname{Hom}_{R[\Sigma_m]}(P_n/\mathrm{J}_m, \mathbb{1})\right) \cong \operatorname{Hom}_{R[\Sigma_m]}\left(F_\star \otimes_{P_n} (P_n/\mathrm{J}_m), \mathbb{1}\right).$$

As noted in [BHP23, Proof of Theorem 5.1] (which follows the structure of [BHP21, Proof of Theorem 4.1]), $F_\star \otimes_{P_n} (P_n/\mathrm{J}_m)$ computes $\operatorname{Tor}^{P_n}_*(\mathbb{1}, P_n/\mathrm{J}_m)$ and [BHP23, Theorem 4.2] shows that this vanishes in positive degrees under the assumption that either δ is invertible or $m < n$. Moreover, since $F_\star$ is a free resolution of $\mathbb{1}$ of right P_n-modules and P_n/J_m is a free right $R[\Sigma_m]$-module (see Remark 6.1.1.3), then $F_\star \otimes_{P_n} (P_n/\mathrm{J}_m)$ is in fact a free resolution of $\mathbb{1}$ by right $R[\Sigma_m]$-modules.

Let θ be the induced map on cohomology under the isomorphism of chain complexes

$$\operatorname{Hom}_{R[\Sigma_m]}\left(F_\star \otimes_{P_n} (P_n/\mathrm{J}_m), \mathbb{1}\right) \to \operatorname{Hom}_{P_n}\left(F_\star, \operatorname{Hom}_{P_m}(P_n, \mathbb{1})\right).$$

It remains to show that the composite $\sigma \circ i^* \circ \theta$ is the identity on $\operatorname{Ext}^\star_{R[\Sigma_m]}(\mathbb{1}, \mathbb{1})$.

Following the argument in [BHP21, Proof of Theorem 4.1], we take a projective resolution, $Q_\star$, of $\mathbb{1}$ by right $R[\Sigma_n]$-modules and choose a chain map $\tilde{i}\colon Q_\star \to F_\star$ that lies over the identity map on $\mathbb{1}$ and respects the inclusion $R[\Sigma_n] \to P_n$. The map

$$Q_\star \to F_\star \otimes_{P_n} (P_n/\mathrm{J}_m)$$

determined by $q \mapsto \tilde{i}(q) \otimes (1 + \mathrm{J}_m)$ is a map of projective $R[\Sigma_m]$-resolutions of $\mathbb{1}$, lying above the identity map on $\mathbb{1}$ and respecting the module structure. It is therefore a chain homotopy equivalence.

The composite $\sigma \circ i^* \circ \theta$ is obtained by applying the functor $\operatorname{Hom}_{P_m}(-, \mathbb{1})$ to this chain homotopy equivalence and taking cohomology. □

Corollary 6.1.1.6. *Suppose $\delta \in R$ is invertible. There is an isomorphism of graded R-modules*

$$\operatorname{Ext}^{\star}_{P_n(\delta)}(\mathbb{1}, \mathbb{1}) \cong H^{\star}(\Sigma_n, \mathbb{1}).$$

Proof. Since δ is invertible, we can apply Theorem 6.1.1.5 with $m = n$ and use the identifications $\operatorname{Hom}_{P_n}(P_n, \mathbb{1}) \cong \mathbb{1}$ and $\operatorname{Hom}_{R[\Sigma_n]}\left(R[\Sigma_n], \mathbb{1}\right) \cong \mathbb{1}$. □

6.1.2 Partition Algebras with Any Parameter

We show that the cohomology of the partition algebras is stably isomorphic to the cohomology of the symmetric groups.

Theorem 6.1.2.1. *There is a natural isomorphism of R-modules*

$$\operatorname{Ext}^{q}_{P_n(\delta)}(\mathbb{1}, \mathbb{1}) \cong H^{q}(\Sigma_n, \mathbb{1})$$

for $q \leqslant n - 1$.

Proof. Let $A = P_n(\delta)$. Recall from Definition 5.1.0.5 that I_{n-1} is the two-sided ideal of A spanned R-linearly by the non-permutation diagrams. In particular, there is an isomorphism of R-algebras $P_n(\delta)/\mathrm{I}_{n-1} \cong R[\Sigma_n]$. In order to prove the theorem we make use of Boyde's R-free idempotent left cover of I_{n-1}.

For $i \in \underline{n}$, let K_i denote the left ideal in $P_n(\delta)$ spanned R-linearly by the diagrams where the vertex $\bar{i}$ is an isolated vertex. For distinct i and j in $\underline{n}$ with $i < j$, we let $\mathrm{L}_{i,j}$ denote the left ideal in $P_n(\delta)$ spanned R-linearly by the diagrams where $\bar{i}$ and $\bar{j}$ are in the same connected component.

Boyde ([Boy24, Section 3]) shows that the left ideals K_i and $\mathrm{L}_{i,j}$ form an R-free idempotent left cover of I_{n-1} of height $n-1$. The theorem now follows from Proposition 5.2.2.4. □

As a corollary, we can deduce the cohomological analogue of [BHP23, Theorem B].

Corollary 6.1.2.2. *The inclusion map $P_{n-1}(\delta) \to P_n(\delta)$ induces a map on cohomology*

$$\operatorname{Ext}^{q}_{P_n(\delta)}(\mathbb{1}, \mathbb{1}) \to \operatorname{Ext}^{q}_{P_{n-1}(\delta)}(\mathbb{1}, \mathbb{1})$$

which is an isomorphism in degrees $n \geqslant 2q + 1$. Furthermore,

$$\lim_{n\to\infty} \operatorname{Ext}^{\star}_{P_n(\delta)}(\mathbb{1}, \mathbb{1}) \cong \lim_{n\to\infty} H^{\star}(\Sigma_n, \mathbb{1}).$$

Proof. This follows from Theorem 6.1.2.1 and Nakaoka's result on the cohomology stability of the symmetric groups [Nak60, Corollary 6.7] that the maps $H^q(\Sigma_n, \mathbb{1}) \to H^q(\Sigma_{n-1}, \mathbb{1})$ induced by the inclusions $\Sigma_{n-1} \hookrightarrow \Sigma_n$ are isomorphisms for $q < n/2$. □

6.2 Cohomology of Tanabe Algebras, Totally Propagating Partition Algebras and Uniform Block Algebras

In this section, we will show that for any parameter δ and any $r \geqslant 2$, the (co)homology of the Tanabe algebra $\mathcal{T}_n(\delta, r)$ (see Subsection 4.2.2) is isomorphic to the (co)homology of the symmetric group Σ_n and that this is *independent* of both the parameter δ and the index n. To the best of our knowledge this is the first example of a result of this type to appear in the literature. We also prove that the (co)homology of the totally propagating partition algebras and the (co)homology of the uniform block permutation algebras are isomorphic to the (co)homology of the symmetric groups. The argument is similar in all three cases. We show that the left ideals $\mathrm{L}_{i,j}$ of Definition 6.2.0.1 form an R-free idempotent left cover of $\mathcal{T}_n(\delta, r) \cap \mathrm{I}_{n-1}$ in $\mathcal{T}_n(\delta, r)$ such that the height is equal to the width by adapting the methods introduced in [Boy24, Section 3].

Definition 6.2.0.1. Let $\delta \in R$. Let $r \geqslant 2$. For distinct i and j in $\underline{n}$ with $i < j$, we let $\mathrm{L}_{i,j}$ denote the left ideal in $\mathcal{T}_n(\delta, r)$ (resp. TPP_n, U_n) spanned R-linearly by the diagrams where $\bar{i}$ and $\bar{j}$ are in the same connected component.

Lemma 6.2.0.2. *Let $r \geqslant 2$. Recall that I_{n-1} is the two-sided ideal in $\mathcal{T}_n(\delta, r)$ (resp. TPP_n, U_n) spanned R-linearly by n-diagrams having fewer than n propagating components. The left ideals $\mathrm{L}_{i,j}$ of Definition 6.2.0.1 form an R-free idempotent left cover of I_{n-1} such that the height is equal to the width. In other words, for each algebra*

1. *the $\mathrm{L}_{i,j}$ cover I_{n-1} and*
2. *any intersection of the $\mathrm{L}_{i,j}$ is either zero or a principal left ideal generated by an idempotent.*

Proof. Recall from Definition 4.2.2.1 that $\kappa(C)$ is the absolute value of the difference between the number of vertices in the left-hand column and the number of vertices in the right-hand column of a connected component C in a partition n-diagram.

1. Firstly, we must show that the left ideals $\mathrm{L}_{i,j}$ cover I_{n-1} in each case.

   If a partition lies in $\mathrm{L}_{i,j}$, it contains a connected component with at least two vertices in the right-hand column and so can have at most $n-1$ propagating components. Therefore, the partition lies in I_{n-1}. This is true for all three cases.

   A diagram in $\mathrm{I}_{n-1} \subset \mathcal{T}_n(\delta, r)$ contains at most $n-1$ propagating components and each connected component must satisfy $\kappa(C) \equiv 0 \pmod{r}$. This means the diagram cannot be a permutation diagram and, since $r \geqslant 2$, we cannot have isolated vertices. It follows that at least one connected component contains (at least) two vertices in the right-hand column and so the diagram lies in some $\mathrm{L}_{i,j}$. Hence the $\mathrm{L}_{i,j}$ cover I_{n-1} for the Tanabe algebra $\mathcal{T}_n(\delta, r)$.

   A diagram in $\mathrm{I}_{n-1} \subset U_n$ contains at most $n-1$ propagating components and each connected component must have an equal number of left and right vertices. It follows that at least one connected component contains (at least) two vertices in the right-hand column and so the diagram lies in some $\mathrm{L}_{i,j}$. Hence the $\mathrm{L}_{i,j}$ cover I_{n-1} in U_n.

A diagram in $\mathrm{I}_{n-1} \subset TPP_n$ contains at most $n-1$ propagating components and each connected component must contain both left and right vertices. It follows that at least one connected component contains two vertices in the right-hand column (if each vertex in the right-hand column was in a different connected component, we would have n connected components, by the totally propagating condition). Therefore, the diagram lies in some $\mathrm{L}_{i,j}$ and the $\mathrm{L}_{i,j}$ cover I_{n-1} in TPP_n.

2. We break up the proof of the second part of lemma into three steps.
   (a) Let $\underline{n}^2_<$ be the set of indices (i,j) with $1 \leqslant i < j \leqslant n$. Let $T \subset \underline{n}^2_<$. Let
   $$\mathrm{J} = \bigcap_{(i,j)\in T} \mathrm{L}_{i,j}.$$
   Take $(a,b) \in \underline{n}^2_< \setminus T$. Let $\nu_{a,b}$ be the diagram whose connected components are $\left\{a, b, \overline{a}, \overline{b}\right\}$ and $\left\{i, \overline{i}\right\}$ for $i \in \underline{n} \setminus \{a,b\}$. We note that $\nu_{a,b}$ satisfies $\kappa(C) \equiv 0 \pmod r$ for each $r \geqslant 2$ and that $\nu_{a,b}$ is a totally propagating partition diagram and a uniform block permutation diagram, so the subsequent applies equally well to all three cases. We give the argument for $\mathcal{T}_n(\delta, r)$.

   We claim that $\mathrm{J} \cdot \nu_{a,b} \subset \mathrm{L}_{a,b} \cap \mathrm{J}$.

   Let ρ be a diagram in J. We must show that $\rho\nu_{a,b} \in \mathrm{L}_{a,b} \cap \mathrm{J}$. Since $\rho \in \mathrm{J}$, for each $(i,j) \in T$, the vertices $\overline{i}$ and $\overline{j}$ are connected in ρ. Since the vertices i and $\overline{i}$ form a connected component of $\nu_{a,b}$, as do j and $\overline{j}$, we see that $\overline{i}$ and $\overline{j}$ are connected in the composite $\rho\nu_{a,b}$. Therefore $\rho\nu_{a,b} \in \mathrm{J}$. Since $\mathrm{L}_{a,b}$ is a left ideal and $\nu_{a,b} \in \mathrm{L}_{a,b}$, the composite $\rho\nu_{a,b} \in \mathrm{L}_{a,b}$.
   (b) We observe that right multiplication by $\nu_{a,b}$ gives a retraction of the inclusion map $\mathrm{L}_{a,b} \cap \mathrm{J} \to \mathrm{J}$. Right multiplication by $\nu_{a,b}$ acts on an n-diagram, d, by merging the component of d containing $\overline{a}$ with the component containing $\overline{b}$, whilst preserving all other connected components of d. For any $\rho \in \mathrm{L}_{a,b}$, $\overline{a}$ and $\overline{b}$ already lie in the same connected component and so $\rho\nu_{a,b} = \rho$.
   (c) By repeating the argument of the two previous points for each $(i,j) \in T$, we see that the right multiplication of the product of all $\nu_{i,j}$ is a retraction $\mathcal{T}_n(\delta, r) \to \mathrm{J}$ since the retraction is given by right multiplication of an element of J. Since J is a left A-module retract of A itself, it then follows from [Boy24, Lemma 2.5] that J is the principal ideal generated by the product of all $\nu_{i,j}$ for $(i,j) \in T$, and this generator is idempotent.

Therefore, the family of left ideals $\mathrm{L}_{i,j}$ forms an idempotent left cover of I_{n-1} whose height is equal to its width. □

We now show that the (co)homology of the Tanabe algebras, for $r \geqslant 2$, coincides with the (co)homology of the symmetric groups, independently of the parameter δ and the parity of the index n. We also include the proof for the totally propagating partition algebras and the uniform block permutation algebras at the same time.

Theorem 6.2.0.3. *Let $\delta \in R$. Let $r \geqslant 2$. There exist isomorphisms of graded R-modules*

1. $\mathrm{Tor}_\star^{\mathcal{T}_n(\delta,r)}(\mathbb{1},\mathbb{1}) \cong H_\star(\Sigma_n,\mathbb{1})$ *and* $\mathrm{Ext}^\star_{\mathcal{T}_n(\delta,r)}(\mathbb{1},\mathbb{1}) \cong H^\star(\Sigma_n,\mathbb{1})$*;*
2. $\mathrm{Tor}_\star^{TPP_n}(\mathbb{1},\mathbb{1}) \cong H_\star(\Sigma_n,\mathbb{1})$ *and* $\mathrm{Ext}^\star_{TPP_n}(\mathbb{1},\mathbb{1}) \cong H^\star(\Sigma_n,\mathbb{1})$*;*
3. $\mathrm{Tor}_\star^{U_n}(\mathbb{1},\mathbb{1}) \cong H_\star(\Sigma_n,\mathbb{1})$ *and* $\mathrm{Ext}^\star_{U_n}(\mathbb{1},\mathbb{1}) \cong H^\star(\Sigma_n,\mathbb{1})$*.*

Proof. We observe that we have isomorphisms of R-algebras

$$\begin{aligned}\mathcal{T}_n(\delta)/\mathrm{I}_{n-1} &\cong R[\Sigma_n]\\ TPP_n/\mathrm{I}_{n-1} &\cong R[\Sigma_n]\\ U_n/\mathrm{I}_{n-1} &\cong R[\Sigma_n].\end{aligned}$$

By Lemma 6.2.0.2, the two-sided ideal I_{n-1} has an R-free idempotent left cover with height equal to the width in all three cases. The homological statements now follow from [Boy24, Theorem 1.7] (recalled as Proposition 5.2.2.1) and the cohomological statements follow from Proposition 5.2.2.4. □

6.3 Cohomology of Rook-Brauer Algebras and Their Subalgebras

In this section, we prove the cohomological analogues of results on the homology of Brauer algebras [BHP21; Boy25], Temperley–Lieb algebras [BH24; Sro24; Boy25] and rook algebras [Boy25]. We also prove results on the (co)homology of the rook-Brauer algebras and on the (co)homology of the Motzkin algebras.

We recall from Definition 5.1.0.6 that for $0 \leqslant i \leqslant n-1$, I_i is the two-sided ideal of $\mathcal{RB}_n(\delta,\varepsilon)$ spanned by rook-Brauer n-diagrams having at most i propagating edges. We also recall from Definitions 4.6.1.9 and 4.6.1.10 that for $0 \leqslant i \leqslant n$ and $p \in P_i$ a right link state of a rook-Brauer n-diagram with precisely i defects, J_p is the left ideal of $\mathcal{RB}_n(\delta,\varepsilon)$ spanned by rook-Brauer n-diagrams having right link state obtained from p by (possibly empty) sequences of splices and deletions.

6.3.1 Cohomological Analogues of Known Results

In this subsection, we collect some cohomological analogues of known results on the homology of Brauer algebras, Temperley–Lieb algebras and rook algebras. These proofs apply Theorem 5.2.3.4 and Corollary 5.2.3.2 using arguments in [Boy25]. The details of the proofs in the invertible parameter case for Proposition 6.3.1.1 and Proposition 6.3.1.2 are similar to what is covered in Subsection 6.4.1 and the proofs in the odd n case are similar to Subsection 6.4.2.

Proposition 6.3.1.1 is the cohomological analogue of [BHP21, Theorem A] and [Boy25, Theorem 1.3], Proposition 6.3.1.2 is the cohomological analogue of [Sro24, Theorem A] and the rook algebra part of Proposition 6.3.1.3 is the cohomological analogue of [Boy25, Theorem 5.4].

Proposition 6.3.1.1. *If $\delta \in R$ is invertible or if n is odd, then there is an isomorphism of graded R-modules* $\operatorname{Ext}^{\star}_{\mathcal{B}_n(\delta)}(\mathbb{1}, \mathbb{1}) \cong H^{\star}(\Sigma_n, \mathbb{1})$.

Proof. For the δ invertible case, [Boy25, Lemma 7.1 and Proof of Theorem 7.5] construct an idempotent element that generates $\mathcal{B}_n(\delta) \cap J_p$ where $p \in P_i$ has no isolated vertices and is the right link state of a Brauer n-diagram. This idempotent element is the Brauer n-diagram whose right link state is p and whose left link state is the mirror image of p, premultiplied by a particular inverse power of δ. The result then follows from Theorem 5.2.3.4 with $l = 0$ and $m = n - 1$, noting that $\mathcal{B}_n(\delta)/\left(\mathcal{B}_n(\delta) \cap I_{n-1}\right) \cong R[\Sigma_n]$.

For the n odd case, [Boy25, Lemmas 8.6 and 8.7, Proof of Theorem 1.4] construct an idempotent element that generates $\mathcal{B}_n(\delta) \cap J_p$ where $p \in P_i$ has at least one defect. The result then follows from Theorem 5.2.3.4 with $l = 1$ and $m = n - 1$, noting that $\mathcal{B}_n(\delta) \cap I_0 = 0$ when n is odd. □

Proposition 6.3.1.2. *If n is odd, then* $\operatorname{Ext}^{0}_{\mathcal{TL}_n(\delta)}(\mathbb{1}, \mathbb{1}) \cong R$ *and* $\operatorname{Ext}^{q}_{\mathcal{TL}_n(\delta)}(\mathbb{1}, \mathbb{1}) = 0$ *for* $q \geqslant 1$.

Proof. [Boy25, Lemmas 8.6 and 8.12, Proof of Theorem 1.2] construct an idempotent element that generates $\mathcal{TL}_n(\delta) \cap J_p$ where $p \in P_i$ is the right link state of a Temperley–Lieb n-diagram that has at least one defect. The result then follows from Theorem 5.2.3.4 with $l = 1$ and $m = n - 1$, noting that $\mathcal{TL}_n(\delta) \cap I_0 = 0$ when n is odd and $\mathcal{TL}_n(\delta)/\left(\mathcal{TL}_n(\delta) \cap I_{n-1}\right) \cong R$. □

Proposition 6.3.1.3. *Let $\varepsilon \in R$ be invertible. There exists an isomorphism of graded R-modules* $\operatorname{Ext}^{\star}_{\mathcal{R}_n(\varepsilon)}(\mathbb{1}, \mathbb{1}) \cong H^{\star}(\Sigma_n, \mathbb{1})$. *Furthermore, the graded R-modules* $\operatorname{Tor}^{\mathcal{PR}_n(\varepsilon)}_{\star}(\mathbb{1}, \mathbb{1})$ *and* $\operatorname{Ext}^{\star}_{\mathcal{PR}_n(\varepsilon)}(\mathbb{1}, \mathbb{1})$ *are both isomorphic to a copy of R concentrated in degree zero.*

Proof. The statement for rook algebras follows directly from [Boy25, Lemmas 5.2 and 5.3, Proof of Theorem 5.4] by applying Corollary 5.2.3.2. [Boy25, Lemmas 5.2 and 5.3, Proof of Theorem 5.4] construct a finite number of idempotent elements that commute with each other and that generate $\mathcal{R}_n(\varepsilon) \cap I_{n-1}$. These idempotent elements are the rook n-diagrams obtained from the identity diagram by deleting exactly one propagating edge. The result for rook algebras then follows from Corollary 5.2.3.2 with $A = \mathcal{R}_n(\varepsilon)$ and $I = \mathcal{R}_n(\varepsilon) \cap I_{n-1}$, noting that $\mathcal{R}_n(\varepsilon)/I \cong R\Sigma_n$.

The only difference to note for planar rook algebras is that $\mathcal{PR}_n(\varepsilon)/\left(\mathcal{PR}_n(\varepsilon) \cap I_{n-1}\right) \cong R$, since the only permutation diagram in $\mathcal{PR}_n(\varepsilon)$ is the identity diagram. □

6.3.2 Rook-Brauer Algebras and Motzkin Algebras

In this subsection, we prove that if ε is invertible then the (co)homology of the rook-Brauer algebra $\mathcal{RB}_n(\delta, \varepsilon)$ is isomorphic to the (co)homology of the symmetric group Σ_n and that the (co)homology of the Motzkin algebras vanishes in positive degrees.

Remark 6.3.2.1. Immediately prior to the submission of this thesis, the author was made aware of another paper [Ta25] that proves the homological parts of the results described above (Theorem 6.3.2.5 and Theorem 6.3.2.7) using an alternative method. [Ta25] uses inductive resolutions as in [BHP23] to prove these results, and it was uploaded to the arXiv after the submission of [FG25a] in March 2025.

Definition 6.3.2.2. Let $p \in P_i$ be a right link state of a rook-Brauer n-diagram. Let d_p be the rook-Brauer n-diagram defined as follows:

- if $\bar{i}$ is an isolated vertex in p, then i and $\bar{i}$ are isolated in d_p;
- if $\bar{i}$ and $\bar{j}$ are connected by a non-propagating edge in p, then they are also connected by a non-propagating edge in d_p;
- if there is a defect at $\bar{i}$ in p then there is a propagating edge between i and $\bar{i}$ in d_p;
- there are no non-propagating edges in the left-hand column of d_p.

Lemma 6.3.2.3. *Let $p \in P_i$ be a right link state of a rook-Brauer n-diagram. Let α be the number of isolated vertices in p and let β be the number of non-propagating edges in p. Then, for any $y \in J_p$, we have $yd_p = \varepsilon^{\alpha+\beta} y$.*

Proof. We will check the following equivalent statement: the double diagram $y * d_p$ has all the vertex pairings and isolated vertices of y with no loops and precisely $\alpha+\beta$ contractible components in the middle column.

Any non-propagating edge or isolated vertex in the left-hand column of y appears in the left-hand column of $y * d_p$.

Suppose y has a propagating edge from i to $\bar{j}$. Therefore the double diagram $y * d_p$ has an edge from i to j'. A propagating edge in y leads to a defect in its right link state at $\bar{j}$. Since $y \in J_p$, its right link state is obtained from p by a sequence of splices and deletions. Therefore p must have a defect at $\bar{j}$. By definition of d_p, there is an edge from j' to $\bar{j}$ in the double diagram $y * d_p$. In other words i and $\bar{j}$ are connected in the double diagram as required.

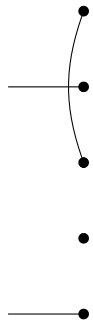
Suppose y has isolated vertices in the right-hand column. There are precisely α of these that are isolated in p, giving a factor of ε^{α} in the product. Any other isolated vertex in the right-hand column of y is the result of a deletion of a defect in its right link state. By definition of d_p, there is an edge joining such an isolated vertex to the right-hand column in the double diagram. This leads to the vertex being isolated in the product without generating any extra factors of ε.

Suppose y has a non-propagating edge in its right-hand column. There are precisely β of these that appear in p, giving a factor of ε^{β} in the product. Any other non-propagating edge in the right-hand column of y is the result of a splice in its right link state. By definition of d_p, there are edges joining the two vertices to the right-hand column in the double diagram. This leads to the non-propagating edge existing in the right-hand column of the product.

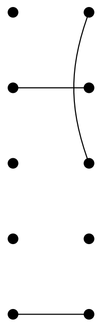
Finally, in order to form a loop, we would need non-propagating edges in the left-hand column of d_p but, by definition, there are none.

Therefore, the double diagram $y * d_p$ has all the vertex pairings and isolated vertices of y with no loops and precisely $\alpha + \beta$ contractible components in the middle column, as required. ☐

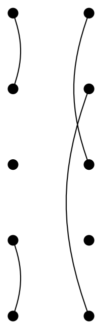
Example 6.3.2.4. Suppose we start with the following right link state, p, of a rook-Brauer 5-diagram:



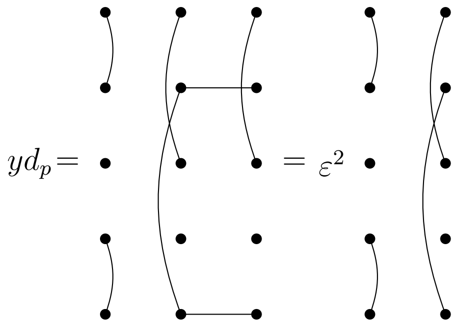
We see that p has a single isolated vertex and a single non-propagating edge. Then by Definition 6.3.2.2, we have the following rook-Brauer 5-diagram d_p:



Now, we take y to be any diagram in the ideal J_p, for example we can take y to be the following diagram (where we have spliced the defects at $\overline{2}$ and $\overline{5}$):



We then see that $yd_p = \varepsilon^2 y$ as claimed in Lemma 6.3.2.3, where we have one contractible component in the middle column coming from the single isolated vertex in p, and the other contractible component in the middle column coming from the single non-propagating edge in p.



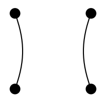
Theorem 6.3.2.5. *Let ε be invertible. For any δ, there exist isomorphisms of graded R-modules*

$$\mathrm{Tor}_\star^{\mathcal{RB}_n(\delta,\varepsilon)}(\mathbb{1},\mathbb{1}) \cong H_\star(\Sigma_n,\mathbb{1}) \quad \textit{and} \quad \mathrm{Ext}^\star_{\mathcal{RB}_n(\delta,\varepsilon)}(\mathbb{1},\mathbb{1}) \cong H^\star(\Sigma_n,\mathbb{1}).$$

Proof. We apply Theorem 5.2.3.4 with $l = 0$, $m = n-1$ and $A = \mathcal{RB}_n(\delta,\varepsilon)$. As an R-module, the rook-Brauer algebra is free on a subset of rook-Brauer diagrams (all of them, in fact).

Let $e_p = \varepsilon^{-(\alpha+\beta)} d_p$. Lemma 6.3.2.3 implies that e_p is idempotent. It also implies that for any $y \in J_p$, $ye_p = \varepsilon^{-(\alpha+\beta)} yd_p = y$ and so $J_p \subseteq A \cdot e_p$. The reverse inclusion, $A \cdot e_p \subseteq J_p$, holds since e_p has right link state p by construction. □

Remark 6.3.2.6. [Boy25, Theorem 7.3 and Corollary 7.4] claim that if ε is invertible in R and $\delta \in R$ is any element, then the homology of the rook-Brauer algebra $\mathcal{RB}_n(\delta,\varepsilon)$ is isomorphic to the homology of the Brauer algebra $\mathcal{B}_n(\delta)$. The justification for this claim occurs before the statement of [Boy25, Theorem 7.3]. It is implicitly claimed that the quotient of the rook-Brauer algebra $\mathcal{RB}_n(\delta,\varepsilon)$ by a certain two-sided ideal (generated by the diagrams ρ_i defined in Section 5 of that paper) is isomorphic (as an algebra) to the Brauer algebra $\mathcal{B}_n(\delta)$. However, this is not the case. For $n = 2$, one can show that the Brauer 2-diagram



also lies in the given two-sided ideal so the isomorphism of algebras does not hold.

Finally, we show that the (co)homology of the Motzkin algebras vanishes in positive degrees.

Theorem 6.3.2.7. *Let ε be invertible. For any δ, the graded R-modules $\mathrm{Tor}_\star^{\mathcal{M}_n(\delta,\varepsilon)}(\mathbb{1},\mathbb{1})$ and $\mathrm{Ext}^\star_{\mathcal{M}_n(\delta,\varepsilon)}(\mathbb{1},\mathbb{1})$ are isomorphic to R concentrated in degree zero.*

Proof. The method for the Motzkin algebras is exactly the same as for the rook-Brauer algebras. The key point to note is that if we start with a right link state of a Motzkin diagram, then the diagram d_p will be a Motzkin diagram. □

6.4 Cohomology of Walled Brauer Algebras

In Subsection 6.4.1, we prove an analogue of [Boy25, Lemma 7.1] for the walled Brauer algebra $\mathcal{B}_{r,s}(\delta)$ with δ invertible and use this to show that the (co)homology of $\mathcal{B}_{r,s}(\delta)$ is isomorphic to the (co)homology of $\Sigma_r \times \Sigma_s$.

In Subsection 6.4.2, we prove analogues of [Boy25, Lemmas 8.6 and 8.7] for the walled Brauer algebra $\mathcal{B}_{r,s}(\delta)$ and we use these in Subsection 6.4.3 to construct an R-free idempotent left cover of the two-sided ideal I_{r+s-1}. We then show that the (co)homology of the walled Brauer algebra $\mathcal{B}_{r,s}(\delta)$ is isomorphic to the (co)homology of $\Sigma_r \times \Sigma_s$ when $r \neq s$ for any $\delta \in R$ and that such an isomorphism holds in a range when $r = s$. We show that this range is sharp in the case $r = s = 1$.

6.4.1 Walled Brauer Algebras with Invertible Parameter

Definition 6.4.1.1. Fix $p \in P_i$, a right link state in $\mathcal{B}_{r,s}(\delta)$ with precisely i defects. Define a diagram $d_p \in \mathcal{B}_{r,s}(\delta)$ as follows:

- if there is a non-propagating edge between $\overline{a}$ and $\overline{b}$ in p then there is a non-propagating edge between a and b and a non-propagating edge between $\overline{a}$ and $\overline{b}$ in d_p;
- if there is a defect at vertex $\overline{a}$ in p, then there is a propagating edge between a and $\overline{a}$ in d_p.

One can think of the diagram d_p as being the rook-Brauer $(r+s)$-diagram whose right link state is p and whose left link state is the mirror image of p.

Lemma 6.4.1.2. *Fix $p \in P_i$, a right link state in $\mathcal{B}_{r,s}(\delta)$ with precisely i defects. If y is a diagram in $\mathcal{B}_{r,s}(\delta) \cap J_p$, then $yd_p = \delta^{\alpha} y$, where α is the number of non-propagating edges in p.*

Proof. In order for y to lie in $\mathcal{B}_{r,s}(\delta) \cap J_p$, the right link state of y must be obtained from p by a (possibly empty) sequence of splices. We note the following facts:

- if $\overline{a}$ and $\overline{b}$ are connected by a non-propagating edge in p then they are also connected by a non-propagating edge in y;
- if there is a defect at vertex $\overline{a}$ in p then there are two possibilities:
  - the vertex $\overline{a}$ is part of a propagating edge in y (which results in a defect in its right link state);
  - the vertex $\overline{a}$ is part of a non-propagating edge in the right-hand column of y (which is the result of a splice).

Now consider the double diagram $y * d_p$. We will show that if two vertices are connected in y then they are connected in the double diagram $y * d_p$ and that there are precisely α loops in the middle column of $y * d_p$.

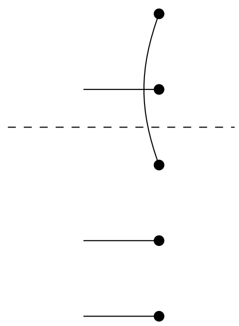
Any non-propagating edges in the left-hand column of y are also present in the left-hand column of the double diagram $y * d_p$.

There are precisely α non-propagating edges that appear in both the right-hand column of y and the right link state p. For each of these there is a corresponding non-propagating edge in the left-hand column of d_p by construction. These form α loops in the middle column of $y * d_p$. Any other non-propagating edge in the right-hand column of y, say between the vertices $\overline{a}$ and $\overline{b}$, is the result of splicing two defects together. Since there were defects at $\overline{a}$ and $\overline{b}$ in the right link state p, there are edges from a' to $\overline{a}$ and from b' to $\overline{b}$ in the double diagram $y * d_p$. Combining these with the edge from a' to b' in the left-hand part of the double diagram $y * d_p$ formed by the splice, we see that $\overline{a}$ and $\overline{b}$ are connected in $y * d_p$.

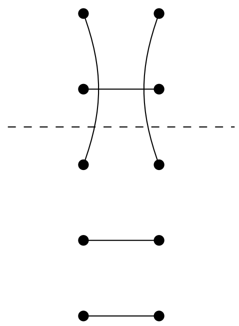
Finally, if there is a propagating edge from a to $\overline{b}$ in y then there is an edge from a to b' in the double diagram $y * d_p$. Since this propagating edge would lead to a defect in the right link state of y, there is an edge from b' to $\overline{b}$ in $y * d_p$ by the construction of d_p.

Combining all of this, we see that all the vertex pairings in y are present in $y * d_p$ with precisely α loops in the middle column. Therefore, $yd_p = \delta^{\alpha} y$. □

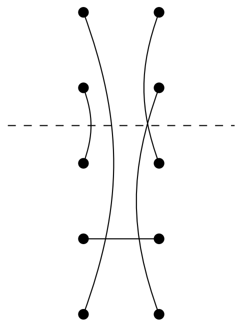
Example 6.4.1.3. Suppose we start with the following right link state, p, of a walled Brauer $(2+3)$-diagram:



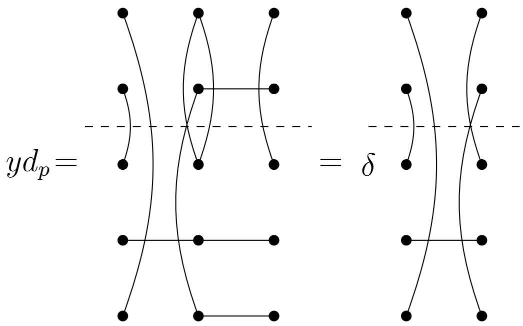
We see that p has a single non-propagating edge. Then by Definition 6.4.1.1, we have the following walled Brauer $(2+3)$-diagram d_p:



Now, we take y to be any diagram in the ideal $\mathcal{B}_{2,3}(\delta) \cap J_p$, for example we can take y to be the following diagram (where we have spliced the defects at $\overline{2}$ and $\overline{5}$):



We then see that $yd_p = \delta y$ as claimed in Lemma 6.4.1.2, where we have one loop in the middle column formed from the single non-propagating edge in p and the corresponding non-propagating edge in the left hand column of d_p.



Theorem 6.4.1.4. *Let r and s be non-negative integers. Let $\delta \in R$ be invertible. There exist isomorphisms of graded R-modules*

$$\mathrm{Tor}_\star^{\mathcal{B}_{r,s}(\delta)}(\mathbb{1}, \mathbb{1}) \cong H_\star(\Sigma_r \times \Sigma_s, \mathbb{1}) \quad \textit{and} \quad \mathrm{Ext}^\star_{\mathcal{B}_{r,s}(\delta)}(\mathbb{1}, \mathbb{1}) \cong H^\star(\Sigma_r \times \Sigma_s, \mathbb{1}).$$

Proof. The walled Brauer algebra $\mathcal{B}_{r,s}(\delta)$ is a subalgebra of the rook-Brauer algebra $\mathcal{RB}_{r+s}(\delta, \varepsilon)$.

We will apply [Boy25, Theorem 1.11] for the homological statement and Theorem 5.2.3.4 for the cohomological statement with $l = 0$, $m = r + s - 1$ and $A = \mathcal{B}_{r,s}(\delta)$.

We note that $\mathcal{B}_{r,s}(\delta)/\left(\mathcal{B}_{r,s}(\delta) \cap I_{-1}\right) = \mathcal{B}_{r,s}(\delta)$ and $\mathcal{B}_{r,s}(\delta)/\left(\mathcal{B}_{r,s}(\delta) \cap I_{r+s-1}\right) \cong R[\Sigma_r \times \Sigma_s]$. The latter holds since the quotient is spanned R-linearly by walled Brauer $(r+s)$-diagrams having $r + s$ propagating edges. The conditions of walled Brauer diagrams tells us that such a diagram consists of a permutation of the first r vertices and a permutation of the last s vertices.

The first condition of the theorem is satisfied since the walled Brauer algebra $\mathcal{B}_{r,s}(\delta)$ is free on a subset of rook-Brauer $(r+s)$-diagrams, namely the walled Brauer $(r+s)$-diagrams.

Let $e_p = \delta^{-\alpha} d_p$. Lemma 6.4.1.2, with $y = d_p$, implies that e_p is idempotent. There is an inclusion $\mathcal{B}_{r,s}(\delta) \cdot e_p \subset \mathcal{B}_{r,s}(\delta) \cap J_p$ since e_p lies in J_p. For the reverse inclusion, take $y \in \mathcal{B}_{r,s}(\delta) \cap J_p$. Lemma 6.4.1.2 tells us that $ye_p = \delta^{-\alpha} y d_p = y$, so $y \in \mathcal{B}_{r,s}(\delta) \cdot e_p$ as required. $\square$

6.4.2 Idempotents and Ideals for Walled Brauer Algebras

We prove versions of [Boy25, Lemmas 8.6 and 8.7] for the walled Brauer algebra $\mathcal{B}_{r,s}(\delta)$. The proofs proceed as in [Boy25] but for our version of [Boy25, Lemma 8.7] (Lemma 6.4.2.2), we have to carefully ensure that we do not break the conditions of walled Brauer diagrams.

Lemma 6.4.2.1. *Let p be the right link state of a walled Brauer $(r+s)$-diagram. Suppose that there exists a walled Brauer $(r+s)$-diagram e such that*

1. *e has right link state p;*
2. *if p has a defect at vertex $\bar{b}$ then there is a sequence of edges in the sesqui-diagram (p, e) that connects b' and $\bar{b}$;*
3. *each vertex b' in (p, e) is connected to the right-hand column of vertices.*

Then $ye = y$ for all $y \in \mathcal{B}_{r,s}(\delta) \cap J_p$.

Proof. We will show that if two vertices are connected in the diagram y then they are connected in the double diagram $y * e$ and that the double diagram $y * e$ contains no loops. If there is a non-propagating edge between a and b in the left-hand column of y then there must also be a non-propagating edge between a and b in the left-hand column of the double diagram $y * e$.

If y has a propagating edge from a to $\bar{b}$, then there is an edge from a to b' in the double diagram $y * e$. Condition 2 tells us that b' is connected to $\bar{b}$ in the double diagram $y * e$ as required.

Suppose y contains a non-propagating edge between $\bar{a}$ and $\bar{b}$ in its right-hand column. If this non-propagating edge is present in the right link state p then $\bar{a}$ and $\bar{b}$ are connected in the double diagram $y * e$ by Condition 1. If the non-propagating edge is not present in the right link state p, it is the result of splicing two defects together. In this case a' and b' are connected in the double diagram $y * e$. Condition 2 tells us that a' is connected to $\bar{a}$ and that b' is connected to $\bar{b}$ and so $\bar{a}$ and $\bar{b}$ are connected in the double diagram $y * e$ as required.

Finally, Condition 3 tells us that every vertex in the middle column of the double diagram $y*e$ is connected to the right-hand column and therefore $y*e$ cannot contain any loops. □

Lemma 6.4.2.2. *Let p be the right link state of a walled Brauer $(r + s)$-diagram with at least one defect. There exists a diagram e_p in $\mathcal{B}_{r,s}(\delta)$ satisfying the conditions of Lemma 6.4.2.1.*

Proof. We start with two copies of the right link state p placed side-by-side. We will explain how to extend the right-hand copy of p to the necessary diagram e_p. We note that Condition 1 of Lemma 6.4.2.1 is automatically satisfied since we are starting with p as the right link state. There is an illustrative example demonstrating the steps of this lemma following this proof.

Label the vertices of the left-hand copy by $1, 2, \dots, r + s$ and label the vertices of the right-hand copy by $\overline{1}, \overline{2}, \dots, \overline{r + s}$.

Extend all but one of the defects in the right-hand copy of p to be connected to the corresponding defect in the left-hand copy of p by a horizontal edge (since these propagating edges are horizontal, they satisfy the conditions on edges in a walled Brauer diagram). We note that the choice of defect that we leave unconnected is immaterial, the following method applies equally well for any choice.

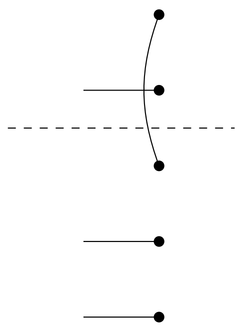
We will join the final defect in the right-hand column of p to the corresponding defect in the left-hand column of p, but via a sequence of edges that passes through every non-propagating edge in the left-hand copy of p. We note that when we have done this, each defect in the right-hand copy of p will be connected to the corresponding defect in the left-hand copy (so Condition 2 of Lemma 6.4.2.1 will be satisfied) and every vertex in the left-hand copy of p will be connected to a vertex in the right-hand copy of p (so Condition 3 of Lemma 6.4.2.1 will be satisfied). We proceed as in [Boy25, Lemma 8.7], but we must take extra care to make sure that the non-propagating edges that we add to form our path between the two remaining defects satisfy the conditions of a walled Brauer diagram.

Suppose the right link state p has x non-propagating edges. By the conditions on walled Brauer diagrams, there are x vertices amongst the first r vertices and x vertices amongst the last s vertices that are part of non-propagating edges. Choose a total ordering $1 < \cdots < x$ on the non-propagating edges in the left-hand copy of p (any total ordering will do). Label the vertices corresponding to the total ordering $r_1, \dots, r_x$ for the vertices in the top part of the diagram and $s_1, \dots, s_x$ for the vertices in the bottom part of the diagram.

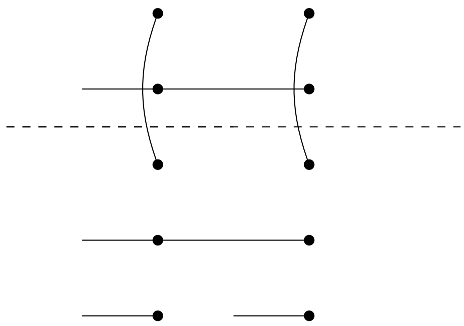
Suppose that our final unconnected defect in the right-hand copy of p is amongst the last s vertices, at vertex $\overline{a}$. Connect our last remaining defect in the right-hand copy of p to the vertex s_1 in the left-hand copy of p. Now join vertex r_i to s_{i+1} (for $1 \leqslant i \leqslant x - 1$) by non-propagating edges to the right of the vertices in the left-hand copy of p. Finally, join r_x to the vertex a.

We note that had our remaining defect been amongst the first r vertices (still at some vertex $\overline{a}$) we would join the defect to vertex r_1, join s_i to r_{i+1} for $1 \leqslant i \leqslant x - 1$ and then join s_x to vertex a. □

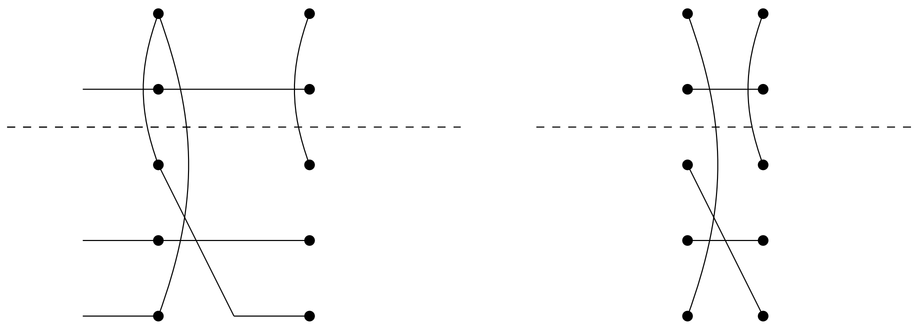
Example 6.4.2.3. Suppose we start with the following right link state, p, of a walled Brauer $(2 + 3)$-diagram:



We place two copies side by side and extend all but one of the defects in the right-hand copy to horizontal edges. In this picture, we have extended the two top-most defects.



Finally, we join the remaining defect in the right-hand copy to the corresponding defect in the left-hand copy via a path that passes through the non-propagating edge in the left-hand copy of the link state. We now have a sesqui-diagram (p, e_p) where the diagram e_p satisfies the conditions of Lemma 6.4.2.1. The diagram on the right is e_p.



6.4.3 A Free Idempotent Left Cover for Walled Brauer Algebras

Definition 6.4.3.1. Let U denote the set of elements (i, j) where i and j are positive integers satisfying $1 \leqslant i \leqslant r$ and $r + 1 \leqslant j \leqslant r + s$.

Consider a walled Brauer $(r + s)$-diagram d. Multiplying on the left by any other walled Brauer $(r + s)$-diagram preserves non-propagating edges in the right-hand column of d. We can therefore make the following definition.

Definition 6.4.3.2. For $(i, j) \in U$, let $L_{i,j}$ be the left ideal of $\mathcal{B}_{r,s}(\delta)$ spanned R-linearly by walled Brauer $(r + s)$-diagrams such that the vertices $\bar{i}$ and $\bar{j}$ are joined by a non-propagating edge.

Lemma 6.4.3.3. *The collection of left ideals $L_{i,j}$ (for $(i, j) \in U$) cover the two-sided ideal I_{r+s-1} in $\mathcal{B}_{r,s}(\delta)$.*

Proof. A basis diagram in $L_{i,j}$ contains at least one non-propagating edge, and so cannot have $r + s$ propagating edges. Hence each $L_{i,j}$ is contained in I_{r+s-1}.

Conversely, a basis diagram in I_{r+s-1} has fewer than $r + s$ propagating edges. Therefore, we must have at least one non-propagating edge in each column of vertices and so the basis diagram must lie in some $L_{i,j}$. □

Lemma 6.4.3.4. *Let $T \subseteq U$. Then*

$$\bigcap_{(i,j)\in T} L_{i,j} = 0$$

if and only if there exist distinct elements $a = (i_1, j_1)$ and $b = (i_2, j_2)$ in T such that one of the co-ordinates of a is equal to one of the co-ordinates of b.

Proof. If there exist distinct elements $a = (i_1, j_1)$ and $b = (i_2, j_2)$ in T such that one of the co-ordinates of a is equal to one of the co-ordinates of b, then a diagram in the intersection would have a vertex in the right-hand column with two distinct edges incident to it, which contradicts the definition of a walled Brauer n-diagram.

Conversely, suppose there are not distinct elements $a = (i_1, j_1)$ and $b = (i_2, j_2)$ in T such that one of the co-ordinates of a is equal to one of the co-ordinates of b. We construct a diagram in the intersection as follows. For each $(i, j) \in T$, we have a non-propagating edge joining i and j and a non-propagating edge joining $\overline{i}$ and $\overline{j}$. For every co-ordinate v of an element in U that is not a co-ordinate of an element in T, we have a propagating edge joining v and $\overline{v}$. □

Lemma 6.4.3.5. *Suppose $r \neq s$ (with $r, s \geqslant 1$). Suppose there are not distinct elements $a = (i_1, j_1)$ and $b = (i_2, j_2)$ in T such that one of the co-ordinates of a is equal to one of the co-ordinates of b. Then the intersection*

$$\bigcap_{(i,j)\in T} L_{i,j}$$

is principal and generated by an idempotent.

Proof. Let q be the right link state such that for each $(i, j) \in T$, we have a non-propagating edge joining $\overline{i}$ and $\overline{j}$ and for every co-ordinate v of an element in U that is not a co-ordinate of an element in T we have a defect. We have

$$\bigcap_{(i,j)\in T} L_{i,j} = J_q.$$

Since $r \neq s$, q must have at least one defect by the condition on non-propagating edges. Lemmas 6.4.2.1 and 6.4.2.2 tell us that there exists a diagram e_q such that right multiplication by e_q gives a retraction $\mathcal{B}_{r,s}(\delta) \to \mathcal{B}_{r,s}(\delta) \cap J_q$. It follows from [Boy24, Lemma 2.6] that J_q is principal and generated by an idempotent. □

Lemma 6.4.3.6. *Suppose $r = s$ (with $r, s \geqslant 1$). Suppose there are not distinct elements $a = (i_1, j_1)$ and $b = (i_2, j_2)$ in T such that one of the co-ordinates of a is equal to one of the co-ordinates in b. Suppose that*

$$\bigcup_{(i,j)\in T} \{i, j\} \neq \underline{r + s}.$$

Then the intersection

$$\bigcap_{(i,j)\in T} L_{i,j}$$

is principal and generated by an idempotent.

Proof. Since the union of all indices in T does not include every vertex in the right-hand column, the same proof as Lemma 6.4.3.5 applies here, since we must have at least one defect. $\square$

Proposition 6.4.3.7. *Suppose $r \neq s$ (with $r, s \geqslant 1$). The left ideals $L_{i,j}$ (for $(i,j) \in U$) form an R-free idempotent left cover of I_{r+s-1} such that the height is equal to the width.*

Proof. By Lemma 6.4.3.3, the collection of left ideals $L_{i,j}$ (for $(i,j) \in U$) cover I_{r+s-1}. By Lemmas 6.4.3.4 and 6.4.3.5, then for any $T \subseteq U$ the intersection

$$\bigcap_{(i,j)\in T} L_{i,j}$$

is either zero or principal and generated by an idempotent. Therefore the collection of left ideals $L_{i,j}$ (for $(i,j) \in U$) is an R-free idempotent left cover of I_{r+s-1} in $\mathcal{B}_{r,s}(\delta)$ with height equal to the width. $\square$

Proposition 6.4.3.8. *Suppose $r = s$ ($r, s \geqslant 1$). The left ideals $L_{i,j}$ (for $(i,j) \in U$) form an R-free idempotent left cover of I_{r+s-1} of height $\frac{r+s}{2} - 1$.*

Proof. By Lemma 6.4.3.3, the collection of left ideals $L_{i,j}$ (for $(i,j) \in U$) cover I_{r+s-1}. Let $T \subseteq U$. If there are distinct elements $a = (i_1, j_1)$ and $b = (i_2, j_2)$ in T such that one of the co-ordinates of a is equal to one of the co-ordinates in b then by Lemma 6.4.3.4 the intersection

$$\bigcap_{(i,j)\in T} L_{i,j}$$

is zero. Assume then that $T \subseteq U$ such that there are not distinct elements $a = (i_1, j_1)$ and $b = (i_2, j_2)$ in T such that one of the co-ordinates of a is equal to one of the co-ordinates in b, and such that

$$\bigcup_{(i,j)\in T} \{i, j\} \neq \underline{r+s}.$$

Then by Lemmas 6.4.3.4 and 6.4.3.6, the intersection

$$\bigcap_{(i,j)\in T} L_{i,j}$$

is either zero or principal and generated by an idempotent. Hence an intersection of at most $\frac{r+s}{2} - 1$ ideals from $L_{i,j}$ (for $(i,j) \in U$) is either zero or principal and generated by an idempotent. This is because if we instead have $T \subseteq U$ such that there are not distinct elements $a = (i_1, j_1)$ and $b = (i_2, j_2)$ in T such that one of the co-ordinates of a is equal to one of the co-ordinates in b, and such that

$$\bigcup_{(i,j)\in T} \{i, j\} = \underline{r+s},$$

then $|T| = \frac{r+s}{2}$, but we cannot say whether or not the intersection

$$\bigcap_{(i,j)\in T} L_{i,j}$$

is principal and generated by an idempotent, and we know that the intersection is not zero by Lemma 6.4.3.4.

Therefore the collection of left ideals $L_{i,j}$ (for $(i,j) \in U$) is an R-free idempotent left cover of height $\frac{r+s}{2} - 1$. □

6.4.4 Walled Brauer Algebras with Any Parameter

Theorem 6.4.4.1. *Let $r, s \geqslant 1$. For any $\delta \in R$ and for $r \neq s$, there exist isomorphisms of graded R-modules*

$$\mathrm{Tor}_{\star}^{\mathcal{B}_{r,s}(\delta)}(\mathbb{1}, \mathbb{1}) \cong H_{\star}(\Sigma_r \times \Sigma_s, \mathbb{1}) \quad \text{and} \quad \mathrm{Ext}^{\star}_{\mathcal{B}_{r,s}(\delta)}(\mathbb{1}, \mathbb{1}) \cong H^{\star}(\Sigma_r \times \Sigma_s, \mathbb{1}).$$

Furthermore, when $r = s$, these isomorphisms hold in the range $0 \leqslant \star \leqslant \frac{r+s}{2} - 1$.

Proof. This follows from Propositions 5.2.2.1 and 5.2.2.4 with the idempotent left covers of Proposition 6.4.3.7 (when $r \neq s$) and Proposition 6.4.3.8 (when $r = s$), together with the isomorphism of R-algebras $\mathcal{B}_{r,s}(\delta)/I_{r+s-1} \cong R[\Sigma_r \times \Sigma_s]$ from Proof of Theorem 6.4.1.4. □

We will now show that the range found in Theorem 6.4.4.1 for the case $r = s$ cannot be extended in general by showing that it is sharp for the case of $\mathcal{B}_{1,1}(\delta)$.

Proposition 6.4.4.2. *There is an isomorphism of R-algebras $\mathcal{B}_{1,1}(\delta) \cong \mathcal{TL}_2(\delta)$. In particular, there exist isomorphisms of R-modules*

$$\mathrm{Tor}_{q}^{\mathcal{B}_{1,1}(\delta)}(\mathbb{1}, \mathbb{1}) \cong \begin{cases} R & q = 0 \\ R/\delta R & q > 0,\ q \text{ odd} \\ R_\delta & q > 0,\ q \text{ even} \end{cases}$$

and

$$\mathrm{Ext}^{q}_{\mathcal{B}_{1,1}(\delta)}(\mathbb{1}, \mathbb{1}) \cong \begin{cases} R & q = 0 \\ R/\delta R & q > 0,\ q \text{ even} \\ R_\delta & q > 0,\ q \text{ odd} \end{cases}$$

where R_δ is the kernel of the map $R \to R$ given by multiplication by δ.

Proof. We note that both algebras have a basis consisting of two diagrams. The isomorphism of R-algebras is determined by sending the identity diagram to the identity diagram, the non-identity diagram to the non-identity diagram and noting in both cases that the non-identity diagram squares to δ lots of itself.

The isomorphisms on (co)homology follow from the isomorphism of R-algebras together with the calculations of Boyd and Hepworth [BH24, Proposition 7.1]. □

Corollary 6.4.4.3. *The range in Theorem 6.4.4.1 is sharp when $r = s = 1$.*

Proof. Theorem 6.4.4.1 tells us that $\mathrm{Tor}_q^{\mathcal{B}_{1,1}(\delta)}(\mathbb{1}, \mathbb{1})$ and $\mathrm{Ext}^q_{\mathcal{B}_{1,1}(\delta)}(\mathbb{1}, \mathbb{1})$ agree with the (co)homology of $\Sigma_1 \times \Sigma_1$ when $q = 0$. However, this isomorphism does not extend beyond $q = 0$. The group $\Sigma_1 \times \Sigma_1$ is isomorphic to the trivial group and therefore its (co)homology is trivial in positive degrees. On the other hand, Proposition 6.4.4.2 tells us that the (co)homology of $\mathcal{B}_{1,1}(\delta)$ is non-trivial in positive degrees. □

Chapter 7

Cohomology of Diagram Algebras II

In this chapter, we continue on from Chapter 6 to apply results from Chapter 5 to the remainder of the diagram algebras defined in Chapter 4. In particular, we will apply results to the dilute Temperley–Lieb algebras and to the blob algebras. Similarly to the preceding chapter, if we work over a diagram algebra with invertible parameter(s), then we prove global isomorphisms on (co)homology by adapting arguments first used in [Boy25] to show that certain left ideals defined in Section 4.6 are generated by idempotents and then applying Proposition 5.2.2.4 for dilute Temperley–Lieb algebras or Theorem 7.2.1.1 for blob algebras.

If we instead work over a diagram algebra with arbitrary parameter(s) (that is, not necessarily invertible in the base ring), then we prove stable isomorphisms on (co)homology by adapting arguments in [Boy24; Sro24] to construct free idempotent left covers (see Subsection 5.2.1) of certain two-sided ideals defined in Section 5.1 and then applying Proposition 5.2.2.4. If the height of the idempotent left cover is equal to the width, then in fact we get a global isomorphism on (co)homology.

Finally, we define a notion of Tate cohomology for the diagram algebras considered in this thesis.

As noted in Section 1.3, the contents of this chapter are adapted from joint work of the author with Daniel Graves, and parts of Section 7.2 are part of an ongoing collaboration between Guy Boyde, the author and Daniel Graves [BFG25]. Original results of this chapter first appeared in [FG24a, Sections 9–11] and [FG26, Sections 4 and 5], and other original results on the blob algebras will appear in [BFG25].

7.1 Cohomology of Dilute Temperley–Lieb Algebras

Our results on the (co)homology of dilute Temperley–Lieb algebras mirror a number of recent results on the homology of Temperley–Lieb algebras. Vanishing results on the (co)homology of Temperley–Lieb algebras were first proved by Boyd and Hepworth [BH24]. The vanishing of the homology of Temperley–Lieb algebras when the parameter δ is invertible has been proved in different ways by Boyd and Hepworth [BH24, Theorem A], Sroka [Sro24, Theorem C] and Boyde [Boy25, Theorem 7.6]. The global vanishing of homology for n odd, independently of the parameter, was first demonstrated by Sroka [Sro24, Theorem A] with an alternative proof provided by Boyde [Boy25, Theorem 1.1]. Finally, a result which showed vanishing in a range for n even whilst identifying higher homology groups in terms of a cup module was proved by Sroka [Sro24, Theorem B]. We note that the vanishing range we find in Theorem 7.1.4.3 for the (co)homology of a dilute Temperley–Lieb algebra with n even is one greater than the range in Sroka's paper for the Temperley–Lieb algebras with n even. This is due to the fact that the product in the dilute Temperley–Lieb algebra allows for a straightforward check that the first (co)homology group expressed in terms of the cup module is zero (see Lemma 7.1.4.2).

In this section, we start by defining the cup module for the dilute Temperley–Lieb algebras which we will use in Subsection 7.1.4 in our description of the higher (co)homology groups of $d\mathcal{TL}_n(\delta)$ for n even. We then prove some technical results that are required to show that certain intersections of ideals in our cover are principal and generated by an idempotent.

Finally, we define the ideals we use to cover I_{n-1} (see Definition 5.1.0.8) and prove that they form an idempotent left cover.

7.1.1 The Cup Module

The following left $d\mathcal{TL}_n(\delta)$-module will arise when calculating the (co)homology of the dilute Temperley–Lieb algebras. It is the analogue of Sroka's cup module for the Temperley–Lieb algebras [Sro24, Definition 7].

Definition 7.1.1.1. Let n be even. Let $\mathrm{Cup}(n)$ denote the left $d\mathcal{TL}_n(\delta)$-module spanned by all dilute Temperley–Lieb n-diagrams with non-propagating edges between $\overline{i}$ and $\overline{i+1}$ for $i \in \{1, 3, \ldots, n-1\}$.

7.1.2 Idempotents and Ideals for Dilute Temperley–Lieb Algebras

Boyde proves two lemmas ([Boy25, Lemmas 8.6 and 8.12]) which allow us to identify the necessary idempotents to prove parameter-independent results for the Temperley–Lieb algebras. In this subsection, we prove the analogues of these lemmas for the dilute Temperley–Lieb algebras. Suppose we have a right link state, p. Recall from Definition 4.6.1.11 that J_p denotes the left ideal of $d\mathcal{TL}_n(\delta)$ with basis given by the diagrams having right link state obtained from p by a (possibly empty) sequence of splices. Lemma 7.1.2.1, below, provides conditions under which a diagram e will satisfy $ye = y$ for all $y \in \mathsf{J}_p$. In other words, if there exists such a diagram e, it is a principal idempotent generator of J_p. Following that, Lemma 7.1.2.2 shows that such a diagram always exists. We note that three of our conditions in Lemma 7.1.2.1 (Conditions 1, 3 and 4) correspond precisely to the conditions given by [Boy25, Lemma 8.6]. Our extra condition ensures compatibility with the isolated vertices which are allowed in dilute Temperley–Lieb diagrams.

Lemma 7.1.2.1. *Let p be a right link state of a dilute Temperley–Lieb n-diagram. Suppose that e is a diagram in $d\mathcal{TL}_n(\delta)$ such that*

1. *e has right link state p,*
2. *if the vertex $\overline{j}$ is isolated in p then the vertices j and $\overline{j}$ are isolated in e,*
3. *if p has a defect at vertex $\overline{j}$ then there is a sequence of edges in the sesqui-diagram (p, e) that connects the vertices j' and $\overline{j}$,*
4. *in the sesqui-diagram (p, e), every non-propagating edge in the left-hand column appears in precisely one of the sequences of edges arising in Condition 3.*

Then for any $y \in \mathsf{J}_p$, we have $ye = y$.

Proof. We will prove the following equivalent statement. We will show that if two vertices in the set $\{1, \overline{1}, \ldots, n, \overline{n}\}$ are connected in y, then they are also connected in the double diagram $y * e$ and that the double diagram $y * e$ contains no loops, no floating edges and no edges lying entirely within the middle column. We break the proof up into parts.

1. In the first part, we show that if two vertices in the set $\{1, \overline{1}, \ldots, n, \overline{n}\}$ are connected in y, then they are also connected in the double diagram $y * e$.

   Any non-propagating edge in the left hand column of y is present in the double diagram $y * e$.

   Suppose y has a non-propagating edge in the right-hand column between vertices $\overline{i}$ and $\overline{j}$. There are two cases. Since $y \in \mathsf{J}_p$, its right link state is formed from p by a (possibly empty) number of splices. Therefore, a non-propagating edge in the right hand column of y is either also a non-propagating edge in p or it is the result of a splice. If the edge exists in p, then the non-propagating edge exists in the right-hand column of the double diagram $y * e$ since e has right link state p by Condition 1. If the non-propagating edge in y is the result of splicing two defects in p, then Condition 3 tells us that there is a sequence of edges joining the vertex $\overline{i}$ in the right column of the double diagram to the vertex i'. This is joined to vertex j' in the double diagram by the non-propagating edge in y and Condition 3 tells us that vertex j' is connected to the vertex $\overline{j}$ in the right-hand column of $y * e$. Therefore the vertices $\overline{i}$ and $\overline{j}$ are connected in the right-hand column of the double diagram $y * e$.

   Suppose y has a propagating edge from i to $\overline{k}$, so that in the double diagram $y * e$ there is an edge from the vertex i to the vertex k'. Therefore, in the right link state of y there is a defect at $\overline{k}$. Since $y \in \mathsf{J}_p$, this means that there is a defect at $\overline{k}$ in p as well. Condition 3 tells us that there is a sequence of edges in the sesqui-diagram (p, e) joining the vertices k' and $\overline{k}$ and therefore the vertex i in the left-hand column of the double diagram $y * e$ is connected to $\overline{k}$ in the right-hand column.

   We have shown that all vertices that were connected in y are connected in the double diagram $y * e$.

2. Condition 2 tells us that a vertex is isolated in p if and only if it is isolated in the right-hand column of e if and only if it is isolated in the middle column of the double diagram $y * e$. Therefore the isolated vertices in the left-hand column of y are precisely the isolated vertices in the left-hand column of $y * e$ and the isolated vertices in the right-hand column of y are precisely the isolated vertices in the right-hand column of $y * e$.

3. We now show that there can be no floating edges. We have already shown that any propagating edge in y is connected to the right-hand column of the double diagram $y * e$ and therefore cannot be a floating edge in the double diagram. The only other possibility is a floating edge arising from the diagram e. Condition 3 tells us that if e has a propagating edge terminating at $\overline{j}$, then there is a sequence of edges connecting the vertices j' and $\overline{j}$ in the sesqui-diagram (p, e). Since p is the right link state of e it also has a defect at $\overline{j}$ and since $y \in \mathsf{J}_p$, it follows that every propagating edge in e is part of a sequence of edges in $y * e$ that connect the left-hand column to the right-hand column (if the defect at $\overline{j}$ is present in the right link state of y) or connect the right-hand column to the right-hand column (if there is a non-propagating edge connected to $\overline{j}$ in the right link state of y coming from the splice of the defect at $\overline{j}$ with another defect in p). Therefore we can have no

floating edges.

4. We now show that there can be no loops. A loop in the double diagram $y * e$ must be formed by a non-zero number of non-propagating edges in the right-hand column of y and a non-zero number of non-propagating edges in the left-hand column of e. However, Condition 4 tells us that every non-propagating edge in the left-hand column of e must be part of a sequence of edges connecting a defect in (p, e) at some vertex j' to the vertex $\overline{j}$. Since $y \in \mathsf{J}_p$, in the double diagram $y * e$ this sequence of edges must either connect to the left-hand column (if the defect at $\overline{j}$ in p is present in the right link state of y) or the right-hand column (if there is a non-propagating edge connected to $\overline{j}$ in the right link state of y coming from the splice of the defect at $\overline{j}$ with another defect in p) and so no loops can be formed.

5. Finally, we show that there are no edges which are connected to neither the left-hand nor the right-hand column. We have already dealt with non-propagating edges in the left-hand column of e in the previous point. We have also shown that any non-propagating edge in the right-hand column of y arising from a splice of two defects in its right link state must have both vertices connected to the right-hand column of the double diagram $y * e$, so cannot lie entirely within the middle column. Finally, if we have a non-propagating edge in the right-hand column of y which also exists in the link state p, Conditions 3 and 4, together with the fact $y \in \mathsf{J}_p$ tells us that this edge must be connected to the right-hand column of the double diagram $y * e$ and therefore cannot lie entirely within the middle column.

Therefore, for any $y \in \mathsf{J}_p$, $ye = y$ as required. □

Lemma 7.1.2.2. *Let p be a right link state of a dilute Temperley–Lieb n-diagram with at least one defect. There exists a diagram $e_p \in d\mathcal{TL}_n(\delta)$ satisfying the conditions of Lemma 7.1.2.1.*

Proof. We start with our link state p. Since p has at least one defect, it can have at most $n-1$ isolated vertices. Suppose we have x isolated vertices with $0 \leqslant x \leqslant n-1$. We temporarily forget the isolated vertices. The remaining vertices and edges form the right link state of Temperley–Lieb $(n-x)$-diagram with at least one defect. Applying [Boy25, Lemma 8.12] yields a Temperley–Lieb $(n-x)$-diagram satisfying Conditions 1, 3 and 4 of Lemma 7.1.2.1. Replacing the x isolated vertices in the right-hand column and inserting x isolated vertices in the left-hand column symmetrically yields a dilute Temperley–Lieb n-diagram which also satisfies Condition 2 of Lemma 7.1.2.1. □

7.1.3 An Idempotent Left Cover for Dilute Temperley–Lieb Algebras

In this section we provide an idempotent left cover of the two-sided ideal I_{n-1}. Recall from Definition 5.1.0.8 that this is the two-sided ideal spanned R-linearly by all dilute Temperley–Lieb n-diagrams having fewer than n propagating edges.

Definition 7.1.3.1. For each non-empty subset $S \subseteq \{\overline{1}, \dots, \overline{n}\}$, let K_S be the left ideal of $d\mathcal{TL}_n(\delta)$ spanned R-linearly by dilute Temperley–Lieb n-diagrams such that the isolated vertices in the right-hand column correspond precisely to the elements of S.

Definition 7.1.3.2. For $1 \leqslant i \leqslant n-1$, let L_i denote the left ideal of $d\mathcal{TL}_n(\delta)$ spanned R-linearly by dilute Temperley–Lieb n-diagrams such that there are no isolated vertices in the right-hand column and the vertices $\overline{i}$ and $\overline{i+1}$ are connected by a non-propagating edge.

We note that each K_S for any non-empty subset $S \subseteq \{\overline{1}, \ldots, \overline{n}\}$ is seen to be a left ideal because isolated vertices and non-propagating edges in the right-hand column of a basis diagram in any K_S are preserved when multiplying on the left by a dilute Temperley–Lieb diagram, and if there is a propagating edge from a to $\overline{b}$ in a basis diagram in any K_S, then multiplying on the left by a dilute Temperley–Lieb diagram with an isolated vertex at $\overline{a}$ would result in a floating edge and therefore the product is 0, i.e. we cannot increase the number of isolated vertices in the right-hand column of a basis diagram in any K_S. The argument is similar for any L_i.

Lemma 7.1.3.3. *The collection of left ideals consisting of K_S for all non-empty subsets S of $\{\overline{1}, \ldots, \overline{n}\}$ and L_i for $1 \leqslant i \leqslant n-1$ cover the two-sided ideal I_{n-1}.*

Proof. A basis diagram in any K_S must have at least one isolated vertex. Therefore this diagram cannot have n propagating edges and so $\mathsf{K}_S \subset \mathsf{I}_{n-1}$. A basis diagram of L_i must have at least one non-propagating edge and therefore cannot have n propagating edges. Therefore $\mathsf{L}_i \subset \mathsf{I}_{n-1}$.

Conversely, a basis element of I_{n-1} must either have isolated vertices in the right-hand column or, if it has no isolated vertices in the right-hand column, it must have at least one non-propagating edge in the right-hand column. If the diagram contains isolated vertices in the right-hand column then it lies in some K_S. Now suppose that the basis diagram has no isolated vertices in the right-hand column but has at least one non-propagating edge. The planarity condition on diagrams ensures that the diagram has a non-propagating edge between two consecutive vertices in the right-hand column. □

Lemma 7.1.3.4. *The following intersections of ideals are zero.*

1. *An intersection involving an ideal of the form K_S and an ideal of the form L_i is zero.*
2. *Let S and T be two distinct, non-empty subsets of $\{\overline{1}, \ldots, \overline{n}\}$. The intersection $\mathsf{K}_S \cap \mathsf{K}_T$ is zero.*
3. *Let $U \subseteq \underline{n-1}$. We have*
$$\bigcap_{i \in U} \mathsf{L}_i = 0$$
if and only if U contains consecutive elements of $\underline{n-1}$.

Proof. For Part 1, a diagram in the intersection would need to have a non-zero number of isolated vertices in the right-hand column (since it lies in some K_S) but would also need to have no isolated vertices in the right-hand column (since it lies in some L_i). This is clearly a contradiction, so such an intersection is empty.

For Part 2, the definitions of K_S and K_T dictate precisely which vertices in the right-hand column are isolated. If the two sets S and T are distinct, there can be no intersection.

For Part 3, on the one hand, if U contains two consecutive elements of $\underline{n-1}$, say j and $j+1$, then a diagram in the intersection must have a non-propagating edge between $\overline{j}$ and $\overline{j+1}$ and also a non-propagating edge between $\overline{j+1}$ and $\overline{j+2}$. In other words, such a diagram would have two edges incident to vertex $\overline{j+1}$ which contradicts the definition of a dilute Temperley–Lieb diagram.

On the other hand, suppose U contains no consecutive elements. Let d be the diagram such that

- for each $i \in U$, d has a non-propagating edge from i to $i+1$ and a non-propagating edge from $\overline{i}$ to $\overline{i+1}$ and
- for each vertex j in the left-hand column which is not part of such a non-propagating edge, there is a propagating edge between j and $\overline{j}$.

This lies in the intersection and so the intersection is non-zero. $\square$

Lemma 7.1.3.5. *For each non-empty subset $S \subseteq \{\overline{1}, \dots, \overline{n}\}$, the ideal K_S is principal and generated by an idempotent.*

Proof. Suppose $S = \{\overline{1}, \dots, \overline{n}\}$. Let q be the right link state consisting of n isolated vertices. Then $\mathsf{K}_S = \mathsf{J}_q$, the left ideal spanned R-linearly by all dilute Temperley–Lieb n-diagrams whose right link state is q. This ideal is generated by the dilute Temperley–Lieb n-diagram with no edges whatsoever and one observes that this diagram is idempotent.

Now suppose that S is a non-empty strict subset of $\{\overline{1}, \dots, \overline{n}\}$. Let q_1 denote the right link state such that the vertices labelled by elements of S are isolated and vertices labelled by elements in the complement of S have defects.

We have $\mathsf{K}_S = \mathsf{J}_{q_1}$, the left ideal spanned R-linearly by all dilute Temperley–Lieb n-diagrams whose right link state can be obtained from q_1 by a valid sequence of splices. Since S is non-empty and $S \neq \{\overline{1}, \dots, \overline{n}\}$, the right link state q_1 contains at least one defect. By combining Lemmas 7.1.2.1 and 7.1.2.2, there exists an element e_{q_1} such that right multiplication by e_{q_1} gives a retraction $d\mathcal{TL}_n(\delta) \to \mathsf{J}_{q_1}$. Therefore, by [Boy24, Lemma 2.6], $\mathsf{K}_S = \mathsf{J}_{q_1}$ is principal and generated by an idempotent. $\square$

Lemma 7.1.3.6. *Let n be odd. Let $U \subset \underline{n-1}$ be a subset containing no consecutive elements. Then*

$$\bigcap_{i \in U} \mathsf{L}_i$$

is principal and generated by an idempotent.

Proof. Let q_2 denote the right link state such that there is a non-propagating edge between $\overline{i}$ and $\overline{i+1}$ for each $i \in U$ and defects at all other vertices.

We have

$$\bigcap_{i \in U} \mathsf{L}_i = \mathsf{J}_{q_2},$$

the left ideal spanned R-linearly by all dilute Temperley–Lieb n-diagrams whose right link state can be obtained from q_2 by a valid sequence of splices. Since n is odd we must have at least one defect.

By combining Lemmas 7.1.2.1 and 7.1.2.2, there exists an element e_{q_2} such that right multiplication by e_{q_2} gives a retraction $d\mathcal{TL}_n(\delta) \to \mathsf{J}_{q_2}$. Therefore, by [Boy24, Lemma 2.6], $\bigcap_{i \in U} \mathsf{L}_i = \mathsf{J}_{q_2}$ is principal and generated by an idempotent. □

Lemma 7.1.3.7. *Let n be even. Let $U \subset \underline{n-1}$ be a subset containing no consecutive elements such that $U \neq \{1, 3, \ldots, n-1\}$. Then*

$$\bigcap_{i \in U} \mathsf{L}_i$$

is principal and generated by an idempotent.

Proof. The same proof as Lemma 7.1.3.6 applies here since we must have at least one defect. □

Lemma 7.1.3.8. *Let n be even and let δ be invertible. Then*

$$\bigcap_{i \in \{1,3,\ldots,n-1\}} \mathsf{L}_i = \mathrm{Cup}(n)$$

is principal and generated by an idempotent.

Proof. Let d_p be the dilute Temperley–Lieb n-diagram with non-propagating edges between i and $i+1$ and between $\overline{i}$ and $\overline{i+1}$ for $i \in \{1, 3, \ldots, n-1\}$. Then right multiplication by $e_p = \delta^{-n/2} d_p$ gives a retraction $d\mathcal{TL}_n(\delta) \to \mathrm{Cup}(n)$ and so $\mathrm{Cup}(n)$ is principal and generated by an idempotent by [Boy24, Lemma 2.6]. □

Proposition 7.1.3.9. *Consider the collection of left ideals containing K_S for all non-empty subsets S of $\{\overline{1}, \ldots, \overline{n}\}$ and L_i for all $1 \leqslant i \leqslant n-1$.*

1. *For n odd, these ideals form an R-free idempotent cover of I_{n-1} with height equal to the width.*
2. *For n even and any $\delta \in R$, these ideals form an R-free idempotent cover of I_{n-1} with height $\frac{n}{2} - 1$.*
3. *For n even with $\delta \in R$ invertible, these ideals form an R-free idempotent cover of I_{n-1} with height equal to the width.*

Proof. We first make two observations, equally valid for Parts (1), (2) and (3). Firstly, by Lemma 7.1.3.3, this collection of left ideals cover I_{n-1}. Secondly, as Lemma 7.1.3.4 shows that we have a number of intersections of ideals being equal to zero and Lemma 7.1.3.5 shows that each ideal K_S is principal and generated by an idempotent for a non-empty subset $S \subseteq \{\overline{1}, \ldots, \overline{n}\}$, it only remains to consider the intersection

$$\bigcap_{i \in U} \mathsf{L}_i$$

for $U \subset \underline{n-1}$ a subset containing no consecutive elements.

We now prove Part (1). Let n be odd and let $U \subset \underline{n-1}$ be a subset containing no consecutive elements. Then by Lemma 7.1.3.6,

$$\bigcap_{i \in U} \mathsf{L}_i$$

is principal and generated by an idempotent. Together with the two observations above, this means that in this case each possible intersection of ideals in the cover is either zero or principal and generated by an idempotent. Hence the collection of left ideals containing K_S for all non-empty subsets S of $\{\overline{1}, \ldots, \overline{n}\}$ and L_i for all $1 \leqslant i \leqslant n-1$ form an R-free idempotent cover of I_{n-1} with height equal to the width.

For Part (2), we now let n be even and take any $\delta \in R$, and let $U \subset \underline{n-1}$ be a subset containing no consecutive elements such that $U \neq \{1, 3, \ldots, n-1\}$. Then by Lemma 7.1.3.7,

$$\bigcap_{i \in U} \mathsf{L}_i$$

is principal and generated by an idempotent. Together with the two observations above, this means that in this case any $(\frac{n}{2}-1)$-fold intersection of ideals in the cover is either zero or principal and generated by an idempotent. In addition, we cannot say whether or not the intersection

$$\bigcap_{i \in \{1,3,\ldots,n-1\}} \mathsf{L}_i$$

is zero or principal and generated by an idempotent. Hence the collection of left ideals containing K_S for all non-empty subsets S of $\{\overline{1}, \ldots, \overline{n}\}$ and L_i for all $1 \leqslant i \leqslant n-1$ form an R-free idempotent cover of I_{n-1} with height $\frac{n}{2}-1$.

Finally, for Part (3), we again let n be even and now take $\delta \in R$ invertible. By Lemma 7.1.3.7, the two observations above, and the reasoning for Part (2), it only remains to consider the intersection

$$\bigcap_{i \in \{1,3,\ldots,n-1\}} \mathsf{L}_i.$$

By Lemma 7.1.3.8, since $\delta \in R$ is invertible this is principal and generated by an idempotent. Therefore each possible intersection of ideals in the cover is either zero or principal and generated by an idempotent. Hence the collection of left ideals containing K_S for all non-empty subsets S of $\{\overline{1}, \ldots, \overline{n}\}$ and L_i for all $1 \leqslant i \leqslant n-1$ form an R-free idempotent cover of I_{n-1} with height equal to the width. □

7.1.4 (Co)homological Stability of Dilute Temperley–Lieb Algebras

We now prove our main results.

Theorem 7.1.4.1. *Suppose that either*

1. *n is odd and δ is any element of R or*

2. n is even and δ is invertible.

Then both the graded R-modules $\mathrm{Tor}_\star^{d\mathcal{TL}_n(\delta)}(\mathbb{1},\mathbb{1})$ and $\mathrm{Ext}^\star_{d\mathcal{TL}_n(\delta)}(\mathbb{1},\mathbb{1})$ are isomorphic to R concentrated in degree zero.

Proof. This follows from Propositions 5.2.2.1 and 5.2.2.4 with the idempotent left covers of Proposition 7.1.3.9 Parts 1 and 3, together with the isomorphism of R-algebras $d\mathcal{TL}_n(\delta)/\mathsf{I}_{n-1} \cong R$. □

In order to prove our (co)homological result for the dilute Temperley–Lieb algebras for n even and any parameter δ, we need the following lemma.

Lemma 7.1.4.2. *Let n be even. Then*

$$\mathbb{1} \otimes_{d\mathcal{TL}_n(\delta)} \mathrm{Cup}(n) = 0 \quad \textit{and} \quad \mathrm{Hom}_{d\mathcal{TL}_n(\delta)}(\mathrm{Cup}(n), \mathbb{1}) = 0.$$

Proof. A basis diagram, d, in $\mathrm{Cup}(n)$ has no propagating edges and so can be written as a product $d = d_l d_r$, where d_l is the dilute Temperley–Lieb n-diagram whose left link state is the left link state of d and whose right link state consists solely of isolated vertices, and d_r is the dilute Temperley–Lieb n-diagram whose right link state is the right link state of d and whose left link state consists solely of isolated vertices. In $\mathbb{1} \otimes_{d\mathcal{TL}_n(\delta)} \mathrm{Cup}(n)$ we have

$$1 \otimes d = 1 \otimes (d_l d_r) = 1 \cdot d_l \otimes d_r = 0.$$

Similarly, in $\mathrm{Hom}_{d\mathcal{TL}_n(\delta)}(\mathrm{Cup}(n), \mathbb{1})$ we have

$$f(d) = f(d_l d_r) = d_l f(d_r) = 0$$

as required. □

Theorem 7.1.4.3. *Let n be even. For any $\delta \in R$ we have isomorphisms of R-modules*

$$\mathrm{Tor}_q^{d\mathcal{TL}_n(\delta)}(\mathbb{1},\mathbb{1}) \cong \begin{cases} R & q = 0 \\ 0 & 1 \leqslant q \leqslant \frac{n}{2} \\ \mathrm{Tor}_{q-\frac{n}{2}}^{d\mathcal{TL}_n(\delta)}(\mathbb{1}, \mathrm{Cup}(n)) & q > \frac{n}{2} \end{cases}$$

and

$$\mathrm{Ext}^q_{d\mathcal{TL}_n(\delta)}(\mathbb{1},\mathbb{1}) \cong \begin{cases} R & q = 0 \\ 0 & 1 \leqslant q \leqslant \frac{n}{2} \\ \mathrm{Ext}^{q-\frac{n}{2}}_{d\mathcal{TL}_n(\delta)}(\mathrm{Cup}(n), \mathbb{1}) & q > \frac{n}{2}. \end{cases}$$

Proof. We start with the homological statement. Let $P_\star$ be a projective resolution of $\mathbb{1}$ by right $d\mathcal{TL}_n(\delta)$-modules. Consider the bicomplex $P_\star \otimes_{d\mathcal{TL}_n(\delta)} C_\star$ where $C_\star$ is our Mayer–Vietoris complex.

Consider the vertical-homology-first spectral sequence associated to our double complex. The vE^1-page takes the form:

$$vE^1_{\alpha,\beta} = H_\beta(P_\alpha \otimes_{d\mathcal{TL}_n(\delta)} C_\star) \cong P_\alpha \otimes_{d\mathcal{TL}_n(\delta)} H_\beta(C_\star)$$

since each term in $P_\star$ is projective.

By Lemma 5.2.1.3 ([Boy24, Lemma 2.2]), the Mayer–Vietoris complex is acyclic and so the vE^1-page is concentrated in the row $\beta = 0$, where it is given by the complex $P_\star \otimes_{d\mathcal{TL}_n(\delta)} \mathbb{1}$. Since $P_\star$ is a projective resolution we see that

$$vE^2_{\alpha,0} \cong \mathrm{Tor}_\alpha^{d\mathcal{TL}_n(\delta)}(\mathbb{1}, \mathbb{1})$$

and the spectral sequence collapses on the vE^2-page.

Now consider the horizontal-homology-first spectral sequence associated to the double complex $P_\star \otimes_{d\mathcal{TL}_n(\delta)} C_\star$. Since $P_\star$ is a projective resolution of $\mathbb{1}$ by right $d\mathcal{TL}_n(\delta)$-modules, we have

$$hE^1_{\alpha,\beta} \cong \mathrm{Tor}_\alpha^{d\mathcal{TL}_n(\delta)}(\mathbb{1}, C_\beta).$$

We note that when n is even we have an R-free idempotent left cover of I_{n-1} of height $\frac{n}{2} - 1$ by Proposition 7.1.3.9. By Proposition 5.2.1.4, the Mayer–Vietoris complex, $C_\star$, is a length $\frac{n}{2} - 1$ partial projective resolution of $\mathbb{1}$ by left $d\mathcal{TL}_n(\delta)$-modules which satisfies $\mathbb{1} \otimes_{d\mathcal{TL}_n(\delta)} C_\beta = 0$ for $1 \leqslant \beta \leqslant \frac{n}{2} - 1$.

It follows from this that $\mathrm{Tor}_0^{d\mathcal{TL}_n(\delta)}(\mathbb{1}, C_0) \cong R$, $\mathrm{Tor}_\alpha^{d\mathcal{TL}_n(\delta)}(\mathbb{1}, C_0) = 0$ for $\alpha > 1$ and $\mathrm{Tor}_\alpha^{d\mathcal{TL}_n(\delta)}(\mathbb{1}, C_\beta) = 0$ for $1 \leqslant \beta \leqslant \frac{n}{2} - 1$.

The only non-zero $\frac{n}{2}$-fold intersection of ideals of the form K_S and L_i is

$$\bigcap_{i \in \{1,3,\ldots,n-1\}} \mathsf{L}_i = \mathrm{Cup}(n).$$

Therefore, the hE^1-page takes the form

$$hE^1_{\alpha,\beta} \cong \begin{cases} R & (\alpha,\beta) = (0,0) \\ \mathrm{Tor}_\alpha^{d\mathcal{TL}_n(\delta)}(\mathbb{1}, \mathrm{Cup}(n)) & (\alpha,\beta) = (\alpha, \frac{n}{2}) \\ 0 & \text{otherwise.} \end{cases}$$

However, $\mathrm{Tor}_0^{d\mathcal{TL}_n(\delta)}(\mathbb{1}, \mathrm{Cup}(n)) = \mathbb{1} \otimes_{d\mathcal{TL}_n(\delta)} \mathrm{Cup}(n) = 0$ by Lemma 7.1.4.2 and so the spectral sequence collapses on the hE^1-page and we can read off the homology.

The cohomological argument is similar. Let $I^\star$ be an injective resolution of $\mathbb{1}$ by left $d\mathcal{TL}_n(\delta)$-modules. Consider the double complex $\mathrm{Hom}_{d\mathcal{TL}_n(\delta)}(C_\star, I^\star)$. The horizontal-homology-first spectral sequence has the form

$$hE_1^{\alpha,\beta} \cong H^\alpha(\mathrm{Hom}_{d\mathcal{TL}_n(\delta)}(C_\star, I^\beta)) \cong \mathrm{Hom}_{d\mathcal{TL}_n(\delta)}(H_\alpha(C_\star), I^\beta)$$

since each term of $I^\star$ is injective. As before, the spectral sequence collapses on the hE_2-page and

$$hE_2^{0,\beta} \cong \mathrm{Ext}^\beta_{d\mathcal{TL}_n(\delta)}(\mathbb{1}, \mathbb{1}).$$

On the other hand, using Lemma 5.2.1.5 in place of the vanishing on tensor products, the vertical-homology-first spectral sequence has

$$vE^1_{\alpha,\beta} \cong \begin{cases} R & (\alpha,\beta) = (0,0) \\ \operatorname{Ext}^{\beta}_{d\mathcal{TL}_n(\delta)}(\operatorname{Cup}(n), \mathbb{1}) & (\alpha,\beta) = (\frac{n}{2},\beta) \\ 0 & \text{otherwise.} \end{cases}$$

However, $\operatorname{Ext}^0_{d\mathcal{TL}_n(\delta)}(\operatorname{Cup}(n), \mathbb{1}) = \operatorname{Hom}_{d\mathcal{TL}_n(\delta)}(\operatorname{Cup}(n), \mathbb{1}) = 0$ by Lemma 7.1.4.2 and so the spectral sequence collapses on the vE^1-page and we can read off the cohomology. □

Hence the maps

$$\operatorname{Tor}_q^{d\mathcal{TL}_{n-1}(\delta)}(\mathbb{1},\mathbb{1}) \to \operatorname{Tor}_q^{d\mathcal{TL}_n(\delta)}(\mathbb{1},\mathbb{1}) \text{ and } \operatorname{Ext}^q_{d\mathcal{TL}_n(\delta)}(\mathbb{1},\mathbb{1}) \to \operatorname{Ext}^q_{d\mathcal{TL}_{n-1}(\delta)}(\mathbb{1},\mathbb{1})$$

induced by the inclusions $d\mathcal{TL}_{n-1}(\delta) \hookrightarrow d\mathcal{TL}_n(\delta)$ are isomorphisms for $n \geqslant 2q+1$ and hence the dilute Temperley–Lieb algebras satisfy (co)homological stability and the stable (co)homology is trivial.

7.2 Cohomology of Blob Algebras

In this section, we prove an analogue of [Boy25, Lemma 7.1] for blob algebras and use it to show that the (co)homology of blob algebras with both parameters invertible vanishes in positive degrees. We also prove analogues of [Boy25, Lemmas 8.6 and 8.12] and use these to construct an idempotent left cover of the two-sided ideal $\mathcal{I}_n$. We then prove that the (co)homology of $\mathbf{Bl}_n(\delta,\gamma)$ for any parameters δ and γ vanishes in positive degrees if n is odd and that the (co)homology vanishes in a range if n is even. We subsequently deduce (co)homological stability.

7.2.1 Blob Algebras with Invertible Parameters

In this subsection, we begin by showing that the (co)homology of the blob algebra $\mathbf{Bl}_n(\delta,\gamma)$ vanishes in positive degrees if both the parameters, δ and γ, are invertible.

We recall from Definition 5.1.0.9 that for $0 \leqslant i \leqslant n-1$, $\mathcal{I}_i$ is the two-sided ideal of $\mathbf{Bl}_n(\delta,\gamma)$ spanned by blob n-diagrams that either have at most $i-1$ propagating edges, or have precisely i propagating edges, at least one blob and have a blob on the top-most propagating edge. We also recall from Definition 4.6.1.12 that for $2 \leqslant i \leqslant n$, R_i is the set of right link states of blob n-diagrams with either precisely i defects with a blob on the top-most, or precisely $i-1$ defects, none of them blobbed. In addition, we recall that R_1 is the set of right link states of blob n-diagrams with precisely 1 defect, which has a blob on it, and that R_0 is the set of right link states of blob n-diagrams with precisely 0 defects. We recall again from Definition 4.6.1.12 that for $0 \leqslant i \leqslant n$ and $q \in R_i$, $\mathcal{J}_q$ is the left ideal of $\mathbf{Bl}_n(\delta,\gamma)$ spanned by blob n-diagrams having right link state obtained from q by (possibly empty) sequences of splices and blobbings.

We begin by showing that we have a chain of isomorphisms on the (co)homology of blob algebras under certain conditions. This result is similar to Theorem 5.2.3.4 for subalgebras of the rook-Brauer algebras.

Theorem 7.2.1.1. *For ease of notation, let $A = \mathbf{Bl}_n(\delta, \gamma)$. Let $0 \leqslant l \leqslant m \leqslant n$. Suppose, for each i in the range $l \leqslant i \leqslant m$ and for each right link state $q \in R_i$, that there exists an idempotent e_q in A such that we have an equality of left ideals $A \cdot e_q = \mathcal{J}_q$. Then there exist chains of isomorphisms of graded R-modules*

$$\mathrm{Tor}_\star^{A/\mathcal{I}_{l-1}}(\mathbb{1}, \mathbb{1}) \cong \mathrm{Tor}_\star^{A/\mathcal{I}_l}(\mathbb{1}, \mathbb{1}) \cong \cdots \cong \mathrm{Tor}_\star^{A/\mathcal{I}_m}(\mathbb{1}, \mathbb{1})$$

and

$$\mathrm{Ext}^\star_{A/\mathcal{I}_{l-1}}(\mathbb{1}, \mathbb{1}) \cong \mathrm{Ext}^\star_{A/\mathcal{I}_l}(\mathbb{1}, \mathbb{1}) \cong \cdots \cong \mathrm{Ext}^\star_{A/\mathcal{I}_m}(\mathbb{1}, \mathbb{1}).$$

Proof. We wish to apply [Boy25, Corollary 4.4] (in the homological case) and Corollary 5.2.3.3 (in the cohomological case) to the chain of two-sided ideals

$$0 = \mathcal{I}_{-1} \leqslant \mathcal{I}_0 \leqslant \cdots \leqslant \mathcal{I}_{m-1} \leqslant \mathcal{I}_m \leqslant A.$$

As R-modules, A and each of the ideals $\mathcal{I}_i$ is free on a basis of diagrams. In order to complete the proof, we must check the two conditions on [Boy25, Corollary 4.4] and Corollary 5.2.3.3. We note that each $\mathcal{I}_i$ for $-1 \leqslant i \leqslant n$ acts as multiplication by $0 \in R$ on $\mathbb{1}$ since none of them contain the identity diagram. Therefore the first condition is satisfied.

For the second condition, in the case $l = 0$ or $l = 1$ we note that there are isomorphisms of left A-modules

$$\mathcal{I}_0/\mathcal{I}_{-1} \cong \mathcal{I}_0 \cong \bigoplus_{q \in R_0} \mathcal{J}_q \quad \text{and} \quad \mathcal{I}_1/\mathcal{I}_0 \cong \bigoplus_{q \in R_1} \mathcal{J}_q.$$

The left-hand isomorphism holds since both sides are spanned R-linearly by the blob n-diagrams with no propagating edges. The right-hand equality holds since both sides are spanned R-linearly by the blob n-diagrams having precisely one blobbed propagating edge and no other propagating edges. Since we assume that $\mathcal{J}_q$ is generated by an idempotent, the second condition of [Boy25, Corollary 4.4] and Corollary 5.2.3.3 holds for these quotients.

For $2 \leqslant i \leqslant m$, the quotient $\mathcal{I}_i/\mathcal{I}_{i-1}$ is spanned R-linearly by the blob n-diagrams, d, such that

- d has precisely $i - 1$ propagating edges, none of which are blobbed or
- d has precisely i propagating edges and the top-most is blobbed.

Since each of these diagrams has some right link state we have an equality

$$\frac{\mathcal{I}_i}{\mathcal{I}_{i-1}} = \sum_{q \in R_i} \frac{\mathcal{J}_q}{\mathcal{I}_{i-1}}.$$

We have assumed that each $\mathcal{J}_q$ is generated by an idempotent so it suffices to show that this sum is direct. Suppose that q and q' are distinct elements of R_i. It suffices to show that $\mathcal{J}_q \cap \mathcal{J}_{q'} \subset \mathcal{I}_{i-1}$. An element in the intersection can be obtained from both q and q' via splicing and blobbing. Since q and q' are distinct we must perform at least one these operations, the result of which for both would correspond to a diagram in $\mathcal{I}_{i-1}$. In the case of a splice this is because performing a splice reduces the number of propagating edges. In the case of adding a blob to the top-most defect, this yields a diagram in $\mathcal{I}_{i-1}$ (see Definition 5.1.0.9). ☐

Definition 7.2.1.2. Fix $q \in R_i$. Define a diagram $d_q \in \mathbf{Bl}_n(\delta, \gamma)$ as follows:

- if there is a non-propagating edge between $\overline{a}$ and $\overline{b}$ in q then there is a non-propagating edge between a and b and a non-propagating edge between $\overline{a}$ and $\overline{b}$ in d_q. Furthermore, if there is a blob on the non-propagating edge in q, then the two non-propagating edges in d_q are also decorated with blobs;
- if there is a defect at vertex $\overline{a}$ in q, then there is a propagating edge between a and $\overline{a}$ in d_q. If the defect in q was decorated with a blob then so is the propagating edge in d_q.

One can think of the diagram d_q as being the blob n-diagram whose right link state is q and whose left link state is the mirror image of q.

Lemma 7.2.1.3. *Fix $q \in R_i$. If y is a diagram in the left ideal $\mathcal{J}_q$ of $\mathbf{Bl}_n(\delta, \gamma)$, then $yd_q = \delta^\alpha \gamma^\beta y$, where α is the number of non-propagating edges without blobs in q and β is the number of non-propagating edges with a blob in q.*

Proof. In order for y to lie in $\mathcal{J}_q$, the right link state of y must be obtained from q by a (possibly empty) sequence of splices and blobbings. We note the following facts:

- if $\overline{a}$ and $\overline{b}$ are connected by a non-propagating edge in q then they are also connected by a non-propagating edge in y (with blobs preserved);
- if there is a defect at vertex $\overline{a}$ in q then there are two possibilities:
  - the vertex $\overline{a}$ is part of a propagating edge in y (which results in a defect in its right link state);
  - the vertex $\overline{a}$ is part of a non-propagating edge in the right-hand column of y (which is the result of a splice).

Now consider the double diagram $y * d_q$. We will show that if two vertices are connected in y then they are connected in the double diagram $y * d_q$ and that there are precisely α unblobbed loops and precisely β blobbed loops in the middle column of $y * d_q$.

There are precisely α unblobbed non-propagating edges that appear in both the right-hand column of y and the right link state q. For each of these there is a corresponding non-propagating edge in the left-hand column of d_q by construction. These form α unblobbed loops in the middle column of $y * d_q$. Similarly, there are precisely β blobbed non-propagating edges that appear in both the right-hand column of y and the right link state q. For each of these there is a corresponding non-propagating edge in the left-hand column of d_q by construction. These form β unblobbed loops in the middle column of $y * d_q$.

Any other non-propagating edge in the right-hand column of y, say between the vertices $\overline{a}$ and $\overline{b}$, is the result of splicing two defects together. Since there were defects at $\overline{a}$ and $\overline{b}$ in the right link state q, there are propagating edges from a' to $\overline{a}$ and from b' to $\overline{b}$ in the double diagram $y * d_q$. Combining these with the non-propagating edge in y formed by the splice, we see that $\overline{a}$ and $\overline{b}$ are connected in $y * d_q$. We note that if we spliced two unblobbed defects, the edges in this path will all be unblobbed by construction of d_q. Similarly, if the splice included a blobbed defect, then the path will be blobbed.

All the non-propagating edges in the left-hand column of y are also present in the left-hand column of the double diagram of $y * d_q$.

Finally, if there is a propagating edge from a to $\overline{b}$ in y then there is a propagating edge from a to b' in the double diagram $y * d_q$. Since this propagating edge would lead to a defect in the right link state of y, there is a propagating edge from b' to $\overline{b}$ in $y * d_q$ by the construction of d_q. If the propagating edge in y was unblobbed, then the edge from b' to $\overline{b}$ in $y * d_q$ is also unblobbed by the construction of d_q.

Combining all of this, we see that all the vertex pairings in y are present in $y * d_q$ with precisely α unblobbed loops and β blobbed loops in the middle column. Therefore, $yd_q = \delta^\alpha \gamma^\beta y$. □

Theorem 7.2.1.4. *Let $\delta, \gamma \in R$ be invertible. There exist isomorphisms of graded R-modules*

$$\mathrm{Tor}_\star^{\mathbf{Bl}_n(\delta,\gamma)}(\mathbb{1},\mathbb{1}) = \begin{cases} R & \star = 0 \\ 0 & \star > 0 \end{cases} \quad \textit{and} \quad \mathrm{Ext}^\star_{\mathbf{Bl}_n(\delta,\gamma)}(\mathbb{1},\mathbb{1}) = \begin{cases} R & \star = 0 \\ 0 & \star > 0. \end{cases}$$

Proof. We will apply Theorem 7.2.1.1 with $l = 0$ and $m = n$ noting that $\mathbf{Bl}_n(\delta,\gamma)/\mathcal{I}_{-1} \cong \mathbf{Bl}_n(\delta,\gamma)$ and $\mathbf{Bl}_n(\delta,\gamma)/\mathcal{I}_n \cong R$.

Take $0 \leqslant i \leqslant n$. Fix $q \in R_i$. Let α be the number of unblobbed non-propagating edges in q and β be the number of blobbed non-propagating edges in q.

Let $e_q = \delta^{-\alpha}\gamma^{-\beta} d_q$. Lemma 7.2.1.3, with $y = d_q$, implies that e_q is idempotent. There is an inclusion $\mathbf{Bl}_n(\delta,\gamma) \cdot e_q \subset \mathcal{J}_q$ since e_q lies in $\mathcal{J}_q$. For the reverse inclusion, take $y \in \mathbf{Bl}_n(\delta,\gamma)$. Lemma 7.2.1.3 tells us that $ye_q = \delta^{-\alpha}\gamma^\beta y d_q = y$, so $y \in \mathbf{Bl}_n(\delta,\gamma) \cdot e_q$ as required. □

7.2.2 Idempotents and Ideals for Blob Algebras

We prove some technical lemmas that are required to prove that we have an R-free idempotent left cover. The lemmas in this subsection are key to showing that certain intersections of ideals in our idempotent cover (to be constructed in the following subsection) are principal and generated by an idempotent.

Lemma 7.2.2.1. *Let q be a right link state of a blob n-diagram with at least one defect and no blobbed non-propagating edges. The vertices of q can be divided into mutually disjoint sets of consecutive vertices such that each set contains precisely one defect and possibly some non-propagating edges which do not intersect any other set.*

Proof. We split the vertices after each defect, starting from the top, with the last set containing all the vertices below the penultimate defect. We note that other choices will usually be available. In particular, if q contains only one defect, we take the whole set of vertices. □

Lemma 7.2.2.2. *Let q be a right link state of a blob n-diagram with at least one defect and no blobbed non-propagating edges. Suppose that there exists a diagram e in $\mathbf{Bl}_n(\delta,\gamma)$ such that*

1. *e has right link state q,*
2. *if q has a defect at $\overline{b}$ there is a sequence of edges in the sesqui-diagram (q,e) joining b' and $\overline{b}$,*
3. *for each c there is a sequence of edges joining c' to the right-hand column of the sesqui-diagram (q,e).*

If $y \in \mathcal{J}_q$, then $ye = y$.

Proof. We will argue as in [Boy25, Lemma 8.6]. We will show the equivalent statement that the double diagram $y * e$ has all the pairings of vertices present in the diagram y with no unblobbed loops and no blobbed loops.

Any non-propagating edge in the left-hand column of y appears in the left-hand column of the double diagram $y * e$.

Suppose y has a propagating edge from a to $\overline{b}$. Therefore, in the double diagram $y * e$, there is an edge from a to b'. Since $y \in \mathcal{J}_q$, its right link state is obtained from q by a valid sequence of splices and blobbings. Since the right link state of y has a defect at $\overline{b}$, so does the right link state q and Condition 2 tells us that there is sequence of edges from b' to $\overline{b}$ in the double diagram $y * e$. Therefore a and $\overline{b}$ are connected in the double diagram $y * e$. Condition 2 also tells us that if the propagating edge in y was decorated with a blob, then the path connecting the vertices in the double diagram will also be decorated with a blob.

If there is a non-propagating edge in the right-hand column of y which is also in the right link state q, then this edge exists in the right-hand column of the double diagram $y * e$ by Condition 1.

Suppose there is a non-propagating edge between $\overline{a}$ and $\overline{b}$ in the right-hand column of y that appears as the result of either a splice or a blobbing followed by a splice in the right link state, q. In this case there is a non-propagating edge from a' to b' in the double diagram $y * e$ and there are defects at $\overline{a}$ and $\overline{b}$ in the right link state q. Condition 2 tells us that a' is connected to $\overline{a}$ and b' is connected to $\overline{b}$ in the double diagram $y * e$, so $\overline{a}$ and $\overline{b}$ are connected in the double diagram and because we assume that q is the right link state of a blob diagram with no blobbed non-propagating edges, the path is decorated with a blob if and only if the original non-propagating edge in y is decorated with a blob.

Finally, Condition 3 tells us that all vertices in the middle column are connected to the right-hand column and so we cannot form any loops in the double diagram $y * e$.

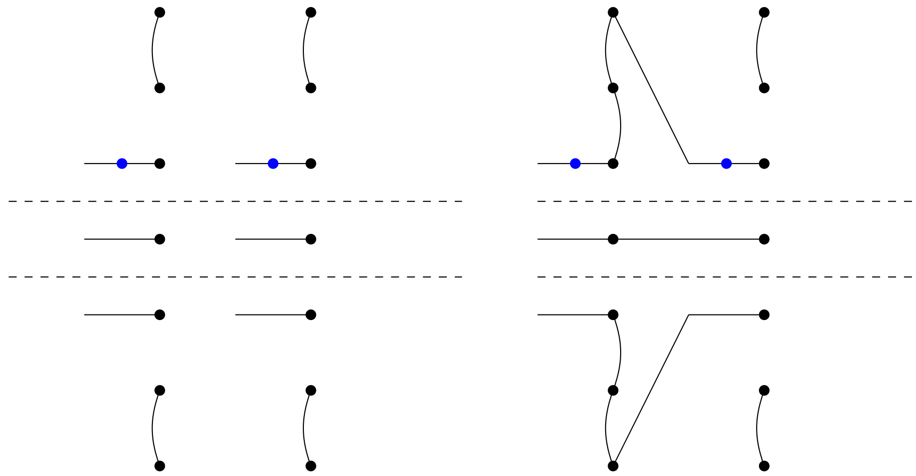
Therefore, the composite ye will have all the vertex pairings of y, with blobs only occurring on a path corresponding to a blobbed edge in y and we do not form any loops (blobbed or unblobbed). □

Lemma 7.2.2.3. *Let q be a right link state of a blob n-diagram with at least one defect. If q does not contain a blobbed non-propagating edge then there is a diagram e_q in $\mathbf{Bl}_n(\delta, \gamma)$ satisfying the conditions of Lemma 7.2.2.2.*

Proof. There is an illustrative example of the steps in this lemma following the proof. We will argue as in [Boy25, Lemma 8.12]. We start with two copies of the right link state q next to one another and aim to extend the right hand copy to the necessary diagram, e_q. Boyde's algorithm works as follows. We split the vertices of the right link state up into sets following Lemma 7.2.2.1. Each set of vertices will contain precisely one defect in the left-hand copy and the right-hand copy. [Boy25, Lemma 8.12] tells us that for each set of vertices we can connect the defect in the right-hand copy to the defect in the left-hand copy via a sequence of edges that passes through every non-propagating edge in the corresponding set of vertices in the left-hand copy. Recall our choice of vertex splitting in the proof of Lemma 7.2.2.1. The top-most defect will occur below any non-propagating edges in its set. Boyde's algorithm tells us that the top-most defect is therefore connected to the top-most vertex in the left-hand copy. Therefore, if the top-most defect is decorated with a blob, the resulting diagram e_q is a valid blob n-diagram because the resulting blobbed propagating edge will be the top-most propagating edge.

Following this method we get the necessary diagram e_q. We note that e_q has right link state q by construction, so Condition 1 of Lemma 7.2.2.2 is satisfied. Furthermore, every defect in the right-hand copy is connected to the corresponding defect in the left-hand copy, so Condition 2 of Lemma 7.2.2.2 is satisfied. Finally, since every non-propagating edge in the left-hand copy is part of a path joining two defects, every vertex in the left-hand copy is connected to a vertex in the right-hand copy so Condition 3 of Lemma 7.2.2.2 is satisfied. □

Example 7.2.2.4. We provide an illustrative example for a right link state with a blob on the top-most defect. By omitting the blob, we would obtain an example for a Temperley–Lieb right link state. In this first diagram (on the left) we have two copies of a right link state of a blob 7-diagram together with a separation of its vertices into sets following Lemma 7.2.2.1. In the second diagram (on the right) we follow Boyde's algorithm for each set of vertices, connecting the defect in the right-hand copy to the defect in the left-hand copy via a sequence of non-crossing edges that passes through every non-propagating edge in the corresponding set of vertices in the left-hand column.



7.2.3 A Free Idempotent Left Cover for Blob Algebras

In this subsection we construct an R-free idempotent left cover of the two-sided ideal $\mathcal{I}_n$, the two-sided ideal of $\mathbf{Bl}_n(\delta, \gamma)$ spanned by all non-identity blob n-diagrams (see Definition 5.1.0.9).

Definition 7.2.3.1. For $1 \leqslant i \leqslant n-1$, let $\mathcal{K}_i$ be the left ideal of $\mathbf{Bl}_n(\delta, \gamma)$ spanned R-linearly by blob n-diagrams such that the vertices $\overline{i}$ and $\overline{i+1}$ are joined by an unblobbed, non-propagating edge.

Definition 7.2.3.2. Let $\mathcal{L}$ be the left ideal of $\mathbf{Bl}_n(\delta, \gamma)$ spanned R-linearly by blob n-diagrams such that the edge incident to the vertex $\overline{1}$ is decorated by a blob.

Lemma 7.2.3.3. *The left ideals $\mathcal{K}_i$ $(1 \leqslant i \leqslant n-1)$ and $\mathcal{L}$ cover the two-sided ideal $\mathcal{I}_n$ of $\mathbf{Bl}_n(\delta, \gamma)$.*

Proof. Each $\mathcal{K}_i$ and $\mathcal{L}$ are contained inside $\mathcal{I}_n$ since a basis element in either ideal must either have a non-propagating edge or a blob. Therefore these basis elements cannot be the identity element, and so they lie in $\mathcal{I}_n$.

The only diagram in $\mathcal{I}_n$ with no non-propagating edges is the diagram with n propagating edges with a blob on the top-most. This diagram lies in $\mathcal{L}$.

Any other diagram in $\mathcal{I}_n$ must have at least one non-propagating edge in both the left-hand column and the right-hand column. We have three cases:

- the diagram has no blobbed non-propagating edges in the right-hand column;
- the diagram has a blobbed non-propagating edge incident to vertex $\overline{1}$ and
- the diagram has at least one blobbed non-propagating edge in the right-hand column, but no blobbed non-propagating edge incident to $\overline{1}$.

Any basis diagram with no blobbed non-propagating edges in its right-hand column and at least one non-propagating edge in its right link state must have an unblobbed non-propagating edge linking $\overline{i}$ and $\overline{i+1}$ for some i by planarity. Hence, such a basis diagram lies in some $\mathcal{K}_i$.

Any blobbed non-propagating edge must appear above the topmost propagating edge. If the diagram has a blobbed non-propagating edge incident to $\overline{1}$ then it lies in $\mathcal{L}$.

If the diagram has at least one blobbed non-propagating edge, but no blobbed non-propagating edge incident to $\overline{1}$, then there must be an unblobbed non-propagating edge incident to $\overline{1}$. If this is connected to $\overline{2}$ then the diagram lies in $\mathcal{K}_1$ and we are done. If not, it is the outermost unblobbed non-propagating edge with other unblobbed non-propagating edges nested within it. By planarity, at least one of these must join vertices $\overline{i}$ and $\overline{i+1}$ for some i, and so the diagram lies in some $\mathcal{K}_i$.

This gives the required coverage. □

Lemma 7.2.3.4. *Let* $S \subseteq \underline{n-1}$*. We have*

$$\bigcap_{i \in S} \mathcal{K}_i = 0$$

if and only if S *contains consecutive elements of* $\underline{n-1}$*.*

Proof. If S contains consecutive elements, say i and $i+1$, an element in the intersection must have an edge from $\overline{i}$ to $\overline{i+1}$ and from $\overline{i+1}$ to $\overline{i+2}$. This is forbidden and so the intersection is zero.

Conversely, suppose S contains no consecutive elements. Let d be the diagram such that

- for each $i \in S$, d has a non-propagating edge from i to $i+1$ and a non-propagating edge from $\overline{i}$ to $\overline{i+1}$ and
- for $j \notin S$, d has a propagating edge from j to $\overline{j}$.

This lies in the intersection and so the intersection is non-zero. □

Similarly, we have the following.

Lemma 7.2.3.5. *Let* $S \subseteq \underline{n-1}$*. We have*

$$\mathcal{L} \cap \bigcap_{i \in S} \mathcal{K}_i = 0$$

if and only if one of the following conditions holds:

- S *contains consecutive elements of* $\underline{n-1}$*,*
- $1 \in S$*.*

Proof. If S contains consecutive elements of $\underline{n-1}$ then

$$\bigcap_{i \in S} \mathcal{K}_i = 0$$

by Lemma 7.2.3.4 and so

$$\mathcal{L} \cap \bigcap_{i \in S} \mathcal{K}_i = 0.$$

Furthermore, if $1 \in S$, then an element in the intersection must have an unblobbed non-propagating edge joining $\overline{1}$ and $\overline{2}$ (since it lies in $\mathcal{K}_1$), but the edge incident to $\overline{1}$ must be decorated with a blob (since it lies in $\mathcal{L}$). This is a contradiction and so the intersection is zero in this case as well.

Conversely, suppose neither condition holds. Let S be a subset of $\underline{n-1}$ which contains no consecutive elements and does not contain 1. Let d be the diagram such that

- d has a blobbed propagating edge from 1 to $\overline{1}$;
- for each $i \in S$, d has an unblobbed non-propagating edge from i to $i+1$ and an unblobbed non-propagating edge from $\overline{i}$ to $\overline{i+1}$;
- an unblobbed propagating edge from j to $\overline{j}$ for all $j \notin S$ ($j \neq 1$).

The diagram d lies in the intersection

$$\mathcal{L} \cap \bigcap_{i \in S} \mathcal{K}_i$$

as required. □

Lemma 7.2.3.6. *The left ideal $\mathcal{L}$ is principal and generated by an idempotent.*

Proof. One observes that this ideal is generated by the diagram with n propagating edges with a blob on the top-most. This diagram is idempotent since blobs are idempotent. □

Lemma 7.2.3.7. *Let n be odd. Let $S \subset \underline{n-1}$ be a subset containing no consecutive elements of $\underline{n-1}$. The ideal*

$$\bigcap_{i \in S} \mathcal{K}_i$$

is principal and generated by an idempotent.

Proof. Let q_1 be the right link state with:

- an unblobbed non-propagating edge from $\overline{i}$ to $\overline{i+1}$ for each $i \in S$
- an unblobbed defect at vertex $\overline{j}$ for all $j \notin S$.

The intersection

$$\bigcap_{i \in S} \mathcal{K}_i$$

is spanned R-linearly by all blob n-diagrams whose right link states are obtained from q_1 by a valid sequence of splices and blobbings. In other words, in the notation of Definition 4.6.1.12, we have

$$\bigcap_{i \in S} \mathcal{K}_i = \mathcal{J}_{q_1}.$$

Lemmas 7.2.2.2 and 7.2.2.3 tell us that there exists an element e_{q_1} in $\mathcal{J}_{q_1}$ such that right multiplication by e_{q_1} gives a retraction $\mathbf{Bl}_n(\delta, \gamma) \to \mathcal{J}_{q_1}$. Therefore, $\mathcal{J}_{q_1}$ is principal and generated by an idempotent by [Boy24, Lemma 2.5]. □

Definition 7.2.3.8 ([Sro24, Definition 11]). Let n be even. Let $M = \{1, 3, \ldots, n-1\}$. We call M the *innermost maximal subset.*

Lemma 7.2.3.9. *Let n be even. Let $S \subset \underline{n-1}$ be a subset containing no consecutive elements of $\underline{n-1}$ and such that $S \neq M$. The ideal*

$$\bigcap_{i \in S} \mathcal{K}_i$$

is principal and generated by an idempotent.

Proof. The same proof as Lemma 7.2.3.7 applies here, since we must have at least one defect. □

Lemma 7.2.3.10. *Let n be odd. Let $S \subset \underline{n-1}$ be a subset containing no consecutive elements and such that $1 \notin S$. The intersection*

$$\mathcal{L} \cap \bigcap_{i \in S} \mathcal{K}_i$$

is principal and generated by an idempotent.

Proof. Let q_2 be the right link state with:

- a blobbed defect at vertex $\overline{1}$;
- an unblobbed non-propagating edge from $\overline{i}$ to $\overline{i+1}$ for each $i \in S$
- an unblobbed defect at vertex $\overline{j}$ for all $j \notin S$ $(j \neq 1)$.

The intersection

$$\mathcal{L} \cap \bigcap_{i \in S} \mathcal{K}_i$$

is spanned R-linearly by all blob n-diagrams whose right link state can be obtained from q_2 by a valid sequence of blobbings and splicings. In other words, in the notation of Definition 4.6.1.12, we have

$$\mathcal{L} \cap \bigcap_{i \in S} \mathcal{K}_i = \mathcal{J}_{q_2}.$$

Lemmas 7.2.2.2 and 7.2.2.3 tell us that there exists an element e_{q_2} in $\mathcal{J}_{q_2}$ such that right multiplication by e_{q_2} gives a retraction $\mathbf{Bl}_n(\delta, \gamma) \to \mathcal{J}_{q_2}$. Therefore, $\mathcal{J}_{q_2}$ is principal and generated by an idempotent by [Boy24, Lemma 2.5]. □

Lemma 7.2.3.11. *Let n be even. Let $S \subset \underline{n-1}$ be a subset containing no consecutive elements of $\underline{n-1}$ such that $1 \notin S$ (in, particular, we note that $S \neq M$). The ideal*

$$\mathcal{L} \cap \bigcap_{i \in S} \mathcal{K}_i$$

is principal and generated by an idempotent.

Proof. The same proof as Lemma 7.2.3.10 applies here, since we must have at least one defect. □

Proposition 7.2.3.12. *Let n be odd. The left ideals $\mathcal{K}_i$ $(1 \leqslant i \leqslant n-1)$ and $\mathcal{L}$ form an R-free idempotent left cover of $\mathcal{I}_n$ of height n and the height equal to its width.*

Proof. By Lemma 7.2.3.3, the ideals $\mathcal{K}_i$ $(1 \leqslant i \leqslant n-1)$ and $\mathcal{L}$ cover $\mathcal{I}_n$. Then Lemmas 7.2.3.4, 7.2.3.5, 7.2.3.6, 7.2.3.7 and 7.2.3.10 show that each possible intersection of ideals in the cover is either zero or principal and generated by an idempotent. Therefore the ideals $\mathcal{K}_i$ $(1 \leqslant i \leqslant n-1)$ and $\mathcal{L}$ form an R-free idempotent left cover of $\mathcal{I}_n$ of height n and the height equal to its width. □

Proposition 7.2.3.13. *Let n be even. The left ideals $\mathcal{K}_i$ $(1 \leqslant i \leqslant n-1)$ and $\mathcal{L}$ form an R-free idempotent left cover of $\mathcal{I}_n$ of height $\frac{n}{2}-1$.*

Proof. By Lemma 7.2.3.3, the ideals $\mathcal{K}_i$ $(1 \leqslant i \leqslant n-1)$ and $\mathcal{L}$ cover $\mathcal{I}_n$. Then Lemmas 7.2.3.4, 7.2.3.5, 7.2.3.6, 7.2.3.9 and 7.2.3.11 show that each possible $(\frac{n}{2}-1)$-fold intersection of ideals in the cover is either zero or principal and generated by an idempotent. In addition, we cannot say whether or not the intersection

$$\bigcap_{i \in M} \mathcal{K}_i$$

is zero or principal and generated by an idempotent where M is the innermost maximal subset. Therefore the ideals $\mathcal{K}_i$ $(1 \leqslant i \leqslant n-1)$ and $\mathcal{L}$ form an R-free idempotent left cover of $\mathcal{I}_n$ of height $\frac{n}{2}-1$. □

7.2.4 Blob Algebras with Any Parameters

In this subsection we show that the blob algebras exhibit (co)homological stability. We do this by proving vanishing results on (co)homology that are independent of both parameters.

Theorem 7.2.4.1. *Let n be odd. For any $\delta, \gamma \in R$ we have isomorphisms of R-modules*

$$\mathrm{Tor}_{\star}^{\mathbf{Bl}_n(\delta,\gamma)}(\mathbb{1},\mathbb{1}) = \begin{cases} R & \star = 0 \\ 0 & \star > 0 \end{cases} \quad \textit{and} \quad \mathrm{Ext}^{\star}_{\mathbf{Bl}_n(\delta,\gamma)}(\mathbb{1},\mathbb{1}) = \begin{cases} R & \star = 0 \\ 0 & \star > 0. \end{cases}$$

Proof. This follows from Propositions 5.2.2.1 and 5.2.2.4 with the idempotent left cover of Proposition 7.2.3.12. □

Theorem 7.2.4.2. *Let n be even. For any $\delta, \gamma \in R$ we have isomorphisms of R-modules*

$$\mathrm{Tor}_{\star}^{\mathbf{Bl}_n(\delta,\gamma)}(\mathbb{1},\mathbb{1}) = \begin{cases} R & \star = 0 \\ 0 & 1 \leqslant \star < \frac{n}{2} \end{cases} \quad \textit{and} \quad \mathrm{Ext}^{\star}_{\mathbf{Bl}_n(\delta,\gamma)}(\mathbb{1},\mathbb{1}) = \begin{cases} R & \star = 0 \\ 0 & 1 \leqslant \star < \frac{n}{2}. \end{cases}$$

Proof. This follows from Propositions 5.2.2.1 and 5.2.2.4 with the idempotent left cover of Proposition 7.2.3.13. □

Corollary 7.2.4.3. *The blob algebras exhibit (co)homological stability and*

$$\operatorname*{colim}_{n\to\infty} \operatorname{Tor}_q^{\mathbf{Bl}_n(\delta,\gamma)}(\mathbb{1},\mathbb{1}) = \begin{cases} R & q=0 \\ 0 & q>0 \end{cases}$$

and

$$\lim_{n\to\infty} \operatorname{Ext}^q_{\mathbf{Bl}_n(\delta,\gamma)}(\mathbb{1},\mathbb{1}) = \begin{cases} R & q=0 \\ 0 & q>0. \end{cases}$$

Proof. This follows directly from Theorems 7.2.4.1 and 7.2.4.2. □

7.3 Towards Tate Cohomology of Diagram Algebras

For many of the diagram algebras that we have considered in this thesis, we have shown that the (co)homology of the algebra is isomorphic to the (co)homology of a finite group. In a setting where we have the homology and the cohomology of a finite group, it is natural to consider Tate cohomology.

We begin by defining a norm map for augmented algebras whose zeroth homology and cohomology coincide with the zeroth homology and cohomology of a group.

Definition 7.3.0.1. Let A be an augmented R-algebra. Suppose that there exist isomorphisms of R-modules $f\colon \operatorname{Tor}_0^A(\mathbb{1},\mathbb{1}) \xrightarrow{\cong} H_0(G,\mathbb{1})$ and $g\colon H^0(G,\mathbb{1}) \xrightarrow{\cong} \operatorname{Ext}^0_A(\mathbb{1},\mathbb{1})$ for some finite group G. We define the *A-norm map for a group G*

$$\mathbf{N}\colon \operatorname{Tor}_0^A(\mathbb{1},\mathbb{1}) \to \operatorname{Ext}^0_A(\mathbb{1},\mathbb{1})$$

to be the composite

$$\operatorname{Tor}_0^A(\mathbb{1},\mathbb{1}) \xrightarrow{f} H_0(G,\mathbb{1}) \xrightarrow{N} H^0(G,\mathbb{1}) \xrightarrow{g} \operatorname{Ext}^0_A(\mathbb{1},\mathbb{1}),$$

where N is the usual norm map $R_G \to R^G$ from coinvariants to invariants induced by $\lambda \mapsto \sum_{g\in G} g\lambda$ (see [Bro82, Section III.1] for instance).

We can now splice together the homology and cohomology of such augmented algebras.

Definition 7.3.0.2. Let A be an augmented R-algebra. Suppose that there exist isomorphisms of R-modules $f\colon \operatorname{Tor}_0^A(\mathbb{1},\mathbb{1}) \xrightarrow{\cong} H_0(G,\mathbb{1})$ and $g\colon H^0(G,\mathbb{1}) \xrightarrow{\cong} \operatorname{Ext}^0_A(\mathbb{1},\mathbb{1})$ for some finite group G. Let $\mathbf{N}$ denote the A-norm map. We define

$$\widehat{\operatorname{Ext}}^p_A(\mathbb{1},\mathbb{1}) = \begin{cases} \operatorname{coker}(\mathbf{N}) & p=0 \\ \ker(\mathbf{N}) & p=-1 \\ \operatorname{Ext}^p_A(\mathbb{1},\mathbb{1}) & p\geqslant 1 \\ \operatorname{Tor}^A_{-p-1}(\mathbb{1},\mathbb{1}) & p\leqslant -2. \end{cases}$$

We call $\widehat{\operatorname{Ext}}^p_A(\mathbb{1},\mathbb{1})$ the *Tate cohomology of A*.

The following definition and examples demonstrate how we identify the known homology and cohomology of families of diagram algebras with the Tate cohomology of certain groups.

Definition 7.3.0.3. We say that an augmented R-algebra, A, is *G-centred* if there is a finite group G and an inclusion of augmented R-algebras $R[G] \hookrightarrow A$ which induces isomorphisms $\mathrm{Tor}^A_\star(\mathbb{1},\mathbb{1}) \cong H_\star(G,\mathbb{1})$ and $\mathrm{Ext}^\star_A(\mathbb{1},\mathbb{1}) \cong H^\star(G,\mathbb{1})$.

Examples 7.3.0.4. The following algebras are Σ_n-centred:

- The partition algebra $P_n(\delta)$ with δ invertible ([BHP23, Theorem A], Theorem 6.1.1.6);
- The Tanabe algebra $\mathcal{T}_n(\delta, r)$ for any δ and for $r \geqslant 2$ (Theorem 6.2.0.3);
- The totally propagating partition algebra TPP_n (Theorem 6.2.0.3);
- The uniform block permutation algebra U_n (Theorem 6.2.0.3);
- The Brauer algebra $\mathcal{B}_n(\delta)$ for n odd or for δ invertible ([BHP21, Theorem A], [Boy25, Theorem 1.3], Theorem 6.3.1.1);
- The rook-Brauer algebra $\mathcal{RB}_n(\delta,\varepsilon)$ for any δ and for ε invertible (Theorem 6.3.2.5);
- The rook algebra $\mathcal{R}_n(\varepsilon)$ with ε invertible ([Boy25, Theorem 5.4], Theorem 6.3.1.3).

The walled Brauer algebra $\mathcal{B}_{r,s}(\delta)$ with δ invertible or $r \neq s$ odd is $R[\Sigma_r \times \Sigma_s]$-centred (Theorem 6.4.1.4, Theorem 6.4.4.1).

The following algebras are 1-centred, that is G-centred for the trivial group:

- The Temperley–Lieb algebra $\mathcal{T}_n(\delta)$ with either δ invertible or n odd ([BH24, Theorem A], [Sro24, Theorem A], Theorem 6.3.1.2);
- The Motzkin algebra $\mathcal{M}_n(\delta,\varepsilon)$ with δ, ε invertible (Theorem 6.3.2.7);
- The planar rook algebra $\mathcal{PR}_n(\varepsilon)$ with ε invertible (Theorem 6.3.1.3);
- The blob algebra $\mathcal{B}_n(\delta,\gamma)$ if both δ and γ are invertible (Theorem 7.2.1.4);
- The blob algebra $\mathcal{B}_n(\delta,\gamma)$ for any δ,γ with n odd (Theorem 7.2.4.1);
- The dilute Temperley–Lieb algebra $d\mathcal{TL}_n(\delta)$ with δ invertible (Theorem 7.1.4.1).

Proposition 7.3.0.5. *Let A be a G-centred augmented R-algebra. There is an isomorphism of $\mathbb{Z}$-graded R-modules*

$$\widehat{\mathrm{Ext}}^\star_A(\mathbb{1},\mathbb{1}) \cong \widehat{H}^\star(G,\mathbb{1}),$$

where $\widehat{H}^\star(G,\mathbb{1})$ is the Tate cohomology of G with coefficients in $\mathbb{1}$.

Proof. This is true more or less by construction. The isomorphisms in degrees 0 and -1 follow directly from the definition of the A-norm map $\mathbf{N}$ in Definition 7.3.0.1. The isomorphisms in all other degrees follow by combining the definition of our $\mathbb{Z}$-graded theory in Definition 7.3.0.2 and the definition of a G-centred algebra in Definition 7.3.0.3. □

Proposition 7.3.0.5 gives a way of identifying the homology and cohomology of many diagram algebras with the Tate cohomology of a group as $\mathbb{Z}$-graded R-modules. However, the Tate cohomology of a group is known to carry more structure, such as a graded-commutative cup product. We therefore conjecture that there should exist a more structured version of this result. We also conjecture that we should be able to obtain the Tate cohomology of these diagram algebras as the cohomology of an appropriate complete resolution in the sense of [AM02, Section 3].

Appendix A

Future Questions

We discuss questions and ideas for future areas of research for the topics considered in Part I and Part II.

A.1 Generalised Tate Cohomology of Hopf Algebras

Question A.1.1. In [Hed20, Section I.2.5], Hedenlund defines a notion of cup product for the definition of generalised Tate cohomology of Hopf algebras used therein. Hedenlund also shows that that definition of generalised Tate cohomology is additively isomorphic to the definition used in [CK97] (and therefore also to those used in [BC92; CK97; Hed20; Mis94; Goi92] by arguments in [Ghe24a] — see Subsection 2.2.2).

With the cup product on the version of generalised Tate cohomology of Hopf algebras used in this thesis (Definition 3.2.0.9), can we prove that the additive isomorphism due to Hedenlund also preserves the multiplicative structure?

Question A.1.2. Can we remove the dependence on the initial choice of contracting homotopy h for $(\hat{P}_*)_k$ in the construction of a complete diagonal approximation $\hat{\Delta}$ (Remark 3.2.0.8)? That is, if we have contracting homotopies g, h, then they are homotopic (via the map $H = h(h-g)$ for example). Can we show that if we instead start the construction from $g = h - (dH - Hd)$ then the resulting complete diagonal approximation $\hat{\Delta}'$ is chain homotopic to $\hat{\Delta}$?

Question A.1.3. To define the cup product for generalised Tate cohomology, we establish the existence of a complete diagonal approximation, for which it is necessary for us to work over a Hopf algebra which is finite-dimensional over a field (see Section 3.2). However, for the cup product defined in [Hed20, Section I.2.5], it is sufficient instead to work over a finitely-generated Hopf algebra which is projective over the base ring. Is it possible for us to prove the existence of a complete diagonal approximation over these Hopf algebras?

Question A.1.4. One equivalent way of viewing usual $\mathrm{Ext}_A^n(M,N)$ not studied in this thesis is via equivalence classes of extensions of M by N [Ben91, Section 2.6]. This is equipped with a product which splices extensions together called the *Yoneda product*, and when calculating Ext with trivial coefficients, this agrees with the cup product [Ben91, Proposition 3.2.1]. In [Lan09, Section 3.1], it is claimed that one can view the negative complete Ext groups $\widehat{\mathrm{Ext}}_A^{-n}(M,N)$ as equivalence classes of "negative extensions" of N by M and that this has a Yoneda product. Can it be proven that this agrees with the cup product on complete Ext?

Question A.1.5. Can we find any examples of Hopf algebras where the A_∞-structure on *usual* Ext is trivial but is non-trivial on *complete* Ext?

Question A.1.6. Can we show that Steenrod operations act on complete Ext in an analogous manner to usual Ext, using arguments similar to [May70], for example? Can we compute examples, and moreover can we find an example of Hopf algebras A, B where $A \cong B$ as algebras so that additively and multiplicatively, complete Ext on A and B agree but Steenrod operations on complete Ext differ?

In this thesis, we only applied the Steenrod operations P^0 and βP^0 to examples. For usual Ext or complete Ext, can we find an example where P^i or βP^i for $i > 1$ detects the coalgebra structure in the base Hopf algebra?

Question A.1.7. For cohomology of spaces with coefficients in $\mathbb{F}_2$, Steenrod squares (that is, mod 2 Steenrod operations) can be calculated using an explicit construction called *cup-i products* [Ste47]. This has been generalised to mod p Steenrod operations for any p on certain operads [KM21, Section 5]. The explicit constructions and examples in [KM21] are for simplicial sets and cubical sets, but a priori it could be possible to do this in other contexts. Can we use these in examples of Steenrod operations on the cohomology of cocommutative Hopf algebras?

A.2 Cohomology of Diagram Algebras

Question A.2.1. This question follows descriptions used in [BS18] related to a categorification of Temperley–Lieb algebras. In [GL98], Graham–Lehrer show that there is a category whose objects are the natural numbers $\mathbb{N}$ and whose morphisms in $\mathrm{Hom}(n, m)$ are planar 'link diagrams' from n vertices to m vertices. The Temperley–Lieb algebra $\mathcal{TL}_n$ is then identified with $\mathrm{Hom}(n, n)$, and this category is called the *Temperley–Lieb category*, and [BS18] shows it to be a braided monoidal category. [GP24] construct Frobenius algebra objects in this category. Does this provide a route to using results in Part I to study Tate cohomology of these objects which are closely related to Temperley–Lieb algebras?

Question A.2.2. All known calculations of cohomology of diagram algebras in this thesis and in the literature are purely additive. In Section 7.3, we define a notion of Tate cohomology for diagram algebras, and show that we can define it for all the diagram algebras considered in this thesis. As in Part I, one advantage that (Tate) cohomology has over homology is that we can also consider multiplicative structures. Can we calculate, for example, the composition product or the cup product on the (Tate) cohomology of some diagram algebra?

Question A.2.3. This question concerns work in an ongoing collaboration between Guy Boyde, the author and Daniel Graves.

Recall from Definition 5.1.0.1 that we can equip the blob algebra $\mathbf{Bl}_n(\delta, \gamma)$ with the augmentation that sends every non-identity blob n-diagram to $0 \in R$ and the identity diagram to $1 \in R$, and the trivial module $\mathbb{1}$ consists of a copy of the ground ring where $\mathbf{Bl}_n(\delta, \gamma)$ acts via the augmentation τ.

There is another augmentation which we can equip the blob algebra $\mathbf{Bl}_n(\delta, \gamma)$ with, namely the map that sends the "blobbed identity" diagram with n propagating edges with a blob on the top-most to $1 \in R$ and all other blob n-diagrams to $0 \in R$. There is also a trivial module corresponding to $\mathbf{Bl}_n(\delta, \gamma)$ acting on R via this augmentation which we denote by $\mathbb{2}$.

Let I be the two-sided ideal of $\mathbf{Bl}_n(\delta, \gamma)$ spanned R-linearly by all blob n-diagrams having fewer than n propagating edges. Can we construct an R-free idempotent left cover of I using techniques from Chapter 7 to get a (co)homological stability result where the stable (co)homology is trivial and the (co)homology in degree 0 is a copy of $R \oplus R$?

Question A.2.4. The original proof of homological stability for the Temperley–Lieb algebras in [BH24] uses the complex of planar injective words, which is a Temperley–Lieb analogue of a technique used for proving homological stability of the symmetric groups [Ker05; Ran11]. Our proofs of (co)homological stability for the two variants on these algebras in this thesis, namely the dilute Temperley–Lieb algebras and the blob algebras, rely on very different techniques. Is it possible to adapt Boyd and Hepworth's argument to these algebras?

This is related to two questions in [BH24, Section 1.7] asking about how homological stability for diagram algebras algebras fits into the frameworks of [RW17] and [GKR25]. As noted therein, it seems likely that this would have to involve some variant of the complex of planar injective words, and so investigating the adaptation of the techniques of Boyd and Hepworth to variants of the Temperley–Lieb algebras could prove fruitful.

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Table of Notation

Notation	Description	Page List
R-**mod**	category of (left) R-modules	11
C_n	cyclic group of order n	11
Σ_n	symmetric group of degree n	11
(A, ∇, η)	algebra with multiplication ∇ and unit η	14
τ	twist map of R-modules $M \otimes N \to N \otimes M$	14
(C, Δ, ε)	coalgebra with comultiplication Δ and counit ε	14
$\mathbf{Alg}_R$	category of R-algebras	15
$\mathbf{Coalg}_R$	category of R-coalgebras	15
$(A, \nabla, \eta, \Delta, \varepsilon, S)$	Hopf algebra with multiplication ∇, unit η, comultiplication Δ, counit ε and antipode S	16
$M^\vee$	R-linear dual module of the A-module M, for A a Hopf algebra over R	17
$\mathbf{Ch}(R)$	category of chain complexes of R-modules	18
$C_{\geqslant 0}$	non-negative truncation of a chain complex C_*	18
$(C \otimes D)_*$	tensor product of chain complexes C_*, D_*	18
$(C \hat{\otimes} D)_*$	complete tensor product of chain complexes C_*, D_*	19
$\underline{\mathrm{Hom}}^\bullet_R(C_*, D_*)$	hom cochain complex on C_*, D_*	19
P_*	projective resolution of R-modules	22
I^*	injective resolution of R-modules	22
$\mathrm{Ext}^i_R(M, N)$	i^{th} cohomology of M with coefficients in N	23
$\widehat{\mathrm{Ext}}^i_A(R, M)$	i^{th} generalised Tate cohomology of R with coefficients in N	24
$\smile$	cup product on (complete) Ext	25, 41
m_n	structure maps $A^{\otimes n} \to A$ of degree $2 - n$ on an A_∞-algebra A	26
P^i	mod p Steenrod operations on the cohomology of a cocommutative Hopf algebra over $\mathbb{F}_p$	27
βP^i	mod p Steenrod operations on the cohomology of a cocommutative Hopf algebra over $\mathbb{F}_p$	27
$I(A^\vee)$	augmentation ideal of the linear dual $A^\vee$ for A a Hopf algebra	28

Notation	Description	Page List
$\tilde{\Delta}$	reduced comultiplication on $I(A^\vee)$	28
$B^\bullet$	cobar construction of $I(A^\vee)$	28
$[x_1\|x_2\|\dots\|x_m]$	basic tensor elements in B^m	28
$\hat{\Delta}$	complete diagonal approximation	37
$P_n(\delta)$	partition algebra with parameter δ in ground ring	60
$\mathcal{T}_n(\delta, r)$	Tanabe algebra with parameter δ in ground ring, $r \geqslant 1$	62
TPP_n	totally propagating partition algebra	62
U_n	uniform block permutation algebra	63
$\mathcal{RB}_n(\delta, \varepsilon)$	rook-Brauer algebra with parameters δ, ε in ground ring	64
$\mathcal{M}_n(\delta, \varepsilon)$	Motzkin algebra with parameters δ, ε in ground ring	65
$\mathcal{R}_n(\varepsilon)$	rook algebra with parameter ε in ground ring	65
$\mathcal{PR}_n(\varepsilon)$	planar rook algebra with parameter ε in ground ring	65
$\mathcal{B}_n(\delta)$	Brauer algebra with parameter δ in ground ring	65
$\mathcal{TL}_n(\delta)$	Temperley–Lieb algebra with parameter δ in ground ring	65
$\mathcal{B}_{r,s}(\delta)$	walled Brauer algebra with parameter δ in ground ring	66
$d\mathcal{TL}_n(\delta)$	dilute Temperley–Lieb algebra with parameter δ in ground ring	67
$\mathbf{Bl}_n(\delta, \gamma)$	blob algebra with parameters δ, γ in ground ring	69
P_i	for $0 \leqslant i \leqslant n$, set of right link states of rook-Brauer n-diagrams with precisely i defects	73
J_p	for a right link state p of a rook-Brauer n-diagram, left ideal of $\mathcal{RB}_n(\delta, \varepsilon)$ with basis given by the diagrams having right link state obtained from p by (possibly empty) sequences of splices and deletions	73
J_p	for a right link state p of a dilute Temperley–Lieb n-diagram, left ideal of $d\mathcal{TL}_n(\delta)$ with basis given by the diagrams having right link state obtained from p by a (possibly empty) sequences of splices	73
R_0	set of right link states of blob n-diagrams with precisely 0 defects	73

Notation	Description	Page List
R_1	set of right link states of blob n-diagrams with precisely 1 blobbed defect and no unblobbed defects	73
R_i	for $2 \leqslant i \leqslant n$, set of right link states of blob n-diagrams with either precisely $i-1$ unblobbed defects, or with precisely i defects with a blob on the top-most	73
$\mathcal{J}_q$	for a right link state q of a blob n-diagram, left ideal of $\mathbf{Bl}_n(\delta,\gamma)$ with basis given by the diagrams having right link state obtained from q by a (possibly empty) sequence of blobbing and splice operations	73
$d_1 * d_2$	double diagram	74
(p, d)	sesqui-diagram	74
$\mathbb{1}$	trivial A-module for augmented algebra A	77
$\mathrm{Tor}^A_\star(\mathbb{1},\mathbb{1})$	homology of A with trivial coefficients for A an augmented (diagram) algebra	78
$\mathrm{Ext}^\star_A(\mathbb{1},\mathbb{1})$	cohomology of A with trivial coefficients for A an augmented (diagram) algebra	78
I_{n-1}	two-sided ideal of $P_n(\delta)$ spanned k-linearly by all n-diagrams having having at most $n-1$ propagating components	78
I_i	for $0 \leqslant i \leqslant n-1$, two-sided ideal of $\mathcal{RB}_n(\delta,\varepsilon)$ spanned k-linearly by all n-diagrams having having at most i propagating edges	78
I_{n-1}	two-sided ideal of $d\mathcal{TL}_n(\delta)$ spanned k-linearly by dilute Temperley–Lieb n-diagrams having at most $n-1$ propagating edges	78
$\mathcal{I}_0$	two-sided ideal of $\mathbf{Bl}_n(\delta,\gamma)$ spanned k-linearly by all blob n-diagrams having no propagating edges	78
$\mathcal{I}_i$	for $1 \leqslant i \leqslant n$, two-sided ideal of $\mathbf{Bl}_n(\delta,\gamma)$ spanned k-linearly by all blob n-diagrams having either at most $i-1$ propagating edges, or having precisely i propagating edges, at least one blob and a blob on the top-most propagating edge	78
$\mathrm{L}_{i,j}$	for $1 \leqslant i < j \leqslant n$ and $r \geqslant 2$, left ideal in $\mathcal{T}_n(\delta,r)$ (resp. TPP_n, U_n) spanned k-linearly by the diagrams where $\bar{i}$ and $\bar{j}$ are in the same connected component	91

Notation	Description	Page List
$L_{i,j}$	for $1 \leqslant i \leqslant r$ and $r+1 \leqslant j \leqslant r+s$, left ideal of $\mathcal{B}_{r,s}(\delta)$ spanned k-linearly by walled Brauer $(r+s)$-diagrams such that the vertices $\bar{i}$ and $\bar{j}$ are joined by a non-propagating edge	103
$\mathrm{Cup}(n)$	for n even, left $d\mathcal{TL}_n(\delta)$-module spanned by all dilute Temperley–Lieb n-diagrams with non-propagating edges between $\bar{i}$ and $\overline{i+1}$ for $i \in \{1, 3, \ldots, n-1\}$	110
K_S	for non-empty subset $S \subseteq \{\bar{1}, \ldots, \bar{n}\}$, left ideal of $d\mathcal{TL}_n(\delta)$ spanned k-linearly by dilute Temperley–Lieb n-diagrams such that the isolated vertices in the right-hand column correspond precisely to the elements of S	112
L_i	for $1 \leqslant i \leqslant n-1$, left ideal of $d\mathcal{TL}_n(\delta)$ spanned k-linearly by dilute Temperley–Lieb n-diagrams such that there are no isolated vertices in the right-hand column and the vertices $\bar{i}$ and $\overline{i+1}$ are connected by a non-propagating edge	113
$\mathcal{K}_i$	for $1 \leqslant i \leqslant n-1$, left ideal of $\mathbf{Bl}_n(\delta, \gamma)$ spanned k-linearly by blob n-diagrams such that the vertices $\bar{i}$ and $\overline{i+1}$ are joined by an unblobbed, non-propagating edge	125
$\mathcal{L}$	left ideal of $\mathbf{Bl}_n(\delta, \gamma)$ spanned k-linearly by blob n-diagrams such that the edge incident to the vertex $\bar{1}$ is decorated by a blob	125
$\mathbf{N}$	for A an augmented algebra, G a finite group, A-norm map for a group G $\mathrm{Tor}_0^A(\mathbb{1}, \mathbb{1}) \to \mathrm{Ext}_A^0(\mathbb{1}, \mathbb{1})$	130