

A Transversality Method for Homotopy Groups of Stable Loci in Affine GIT

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Abstract

This thesis studies the homotopy groups of stable loci arising in affine Geometric Invariant Theory (GIT). We begin by establishing the foundational theory of reductive linear algebraic groups and their actions on affine varieties. In particular, we review Nagata's theorem on the finite generation of invariant rings.

The main contribution of this work is to extend the transversality framework of Daskalopoulos–Uhlenbeck and Wilkin to the affine setting. We construct a G -equivariant holomorphic vector bundle $\pi : W \rightarrow G \cdot \lambda$ over the conjugation orbit of a 1-PS, where the fibres are negative weight spaces $V(\lambda)_-$. We prove that the evaluation map D is transverse to the zero section of this bundle. Consequently, we establish that for generic homotopies, the preimage of the zero section is empty under specific dimensional inequalities, i.e. these homotopies avoid non-stable strata.

Our main theorem asserts that the stable locus V^{st} is $(d_{min} - 2)$ -connected, where d_{min} is the minimal real codimension of the unstable strata. The applications of this result are demonstrated by several examples including the 2-Kronecker quiver, the space of controllable linear systems, and the uniqueness of Maximum Likelihood Estimates for Gaussian graphical models.

Contents

Abstract

Declarations

1	Introduction	1
2	Reductive Linear Algebraic Groups	2
3	Geometric Invariant Theory	13
3.1	Affine GIT Quotients	15
3.2	Hesselink Stratifications of Affine GIT Quotients	24
4	Transversality	26
5	Homotopy Groups	32
6	Homotopy Groups of Stable Loci through Transversality	36

Declarations

I declare that this thesis is a presentation of my original research work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature, and acknowledgement of collaborative research and discussions.

Material contained in Section 6 of this thesis is based on, and in part reproduces, the preprint authored by the candidate

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The version presented in this thesis may include additional remarks and more explicit constructions of examples.

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1 Introduction

The construction of quotient spaces for varieties under the action of an algebraic group is a fundamental problem in algebraic geometry. When the group G is not finite, the topological quotient X/G , namely the space of orbits equipped with the quotient topology, might fail to have the structure of an algebraic variety. An obstruction, discussed in Section 3, is the existence of non-closed orbits of the G -action.

Geometric Invariant Theory constructs a good quotient $X // G$ by considering the algebraic structure of the ring of invariant polynomials $\mathbb{K}[X]^G$. Since this thesis studies homotopy groups of stable loci of an affine GIT quotient, a natural question is how its fundamental group relates to that of the original space. In the projective setting, Biswas, Hogadi, and Parameswaran [4] prove that if a connected reductive group G acts on a smooth connected projective variety M and the GIT quotient $M // G$ is nonempty, then the induced homomorphism $\pi_1(M) \rightarrow \pi_1(M // G)$ on fundamental groups is an isomorphism. In particular, over $\mathbb{C}$, the corresponding map of topological fundamental groups is also an isomorphism [4, Theorem 4.4]. This phenomenon has a symplectic analogue. For Hamiltonian actions of connected compact Lie groups, Li [24] shows that the fundamental group of a compact symplectic manifold identifies with that of its symplectic reduction. Moreover, in the projective setting, the Kempf-Ness theorem [27, Theorem 8.3] identifies the projective GIT quotient with the symplectic reduction by a maximal compact subgroup, which allows one to consider the π_1 -invariance results for projective GIT in the symplectic context.

By contrast, the present thesis studies the homotopy groups of the stable locus V^{st} inside an affine representation space. While Biswas et al. [4] show that, in the projective setting, the fundamental group of a GIT quotient is the same as that of the original space, our main result Theorem 6.6 of Section 6 gives a connectivity result for the stable locus itself, namely if

$$d_{\min} = \min_j \{2 \dim_{\mathbb{C}} V(\lambda_j)_- - 2 \dim_{\mathbb{C}} (G \cdot \lambda_j)\},$$

then the stable locus V^{st} is $(d_{\min} - 2)$ -connected, where $V(\lambda_j)_-$ denotes the negative weight space, and $G \cdot \lambda_j$ denotes its conjugation orbit. Thus the present thesis computes homotopy groups of the stable locus inside the semistable locus up to an explicit dimensional bound. A further point of context is provided by Hoskins [16], where she proves an affine version of the Kempf-Ness theorem with respect to a character ρ . In the present thesis, however, the ambient space is $V \cong \mathbb{C}^n$, which is contractible, and the nontriviality of homotopy groups arises from removing unstable and strictly semistable strata.

This work is structured as follows. We first establish the necessary algebraic and geometric foundations in Section 2 and 3, reviewing the theory of linear algebraic groups and the construction of affine GIT quotients. We then introduce the tools required for the main results in Sections 4 and 5, covering transversality and homotopy groups. Finally, Section 6 presents a general framework for computing the connectivity of stable loci, based on an infinite-dimensional transversality argument, followed by applications to several significant problems in quiver representations and statistical theory.

Section 2 provides a self-contained exposition of the algebraic objects related to the theory. The section starts with the foundational definitions of affine algebraic sets, the Zariski topology, and affine varieties. It then defines linear algebraic groups and establishes their properties. Proposition 2.19 shows that any affine algebraic group is isomorphic to a closed subgroup of some general linear group $\mathrm{GL}_n(\mathbb{K})$. Then we discuss the structure of these groups, which includes the properties of the identity component G° and the definitions of solvable and nilpotent groups. This leads to the concept of a reductive group, and Nagata's theorem (Theorem 2.34) states that the action of a reductive group on a finitely generated $\mathbb{K}$ -algebra gives a finitely generated invariant algebra. This makes the quotient $X // G$ an affine variety.

Section 3 begins with a detailed motivating example from quiver representations, namely the McKay quiver of type $\tilde{A}_n$, to demonstrate the construction of the GIT quotient $M(\Gamma) // G(\Gamma)$ from the invariant ring and how this solves the problem of non-closed orbits by identifying them with the unique closed orbit in their closure. The concept of affine GIT quotients is subsequently formally defined as the spectrum of the invariant ring, and the notion of a good categorical quotient is also introduced.

The theory is extended by discussing a linearisation of the action, which is defined by a character $\chi : G \rightarrow \mathbb{G}_m$. This allows us to define the notions of χ -unstable, χ -semistable, and χ -stable points. We then state the Hilbert-Mumford criterion, which reduces the topological definitions of (semi)stability to computations involving the weights of one-parameter subgroups (1-PSs) $\lambda : \mathbb{G}_m \rightarrow G$. This section ends with the construction of Hesselink stratification, which provides a finite and disjoint decomposition of the null cone into G -invariant, locally closed subsets $S_{[\lambda]}$, called strata. The index is the conjugacy class of the most destabilising 1-PSs.

Section 4 defines smooth manifolds, vector bundles, and transversality. A smooth map $f : M \rightarrow N$ is

transverse to a submanifold $S \subset N$ if their tangent spaces span the ambient tangent space at every point of intersection. Theorem 4.12 states that if f is transverse to S , its preimage $f^{-1}(S)$ is a well-defined submanifold. Section 5 defines the concepts of fundamental groups and higher homotopy groups, where an n -connected space is defined as one with $\pi_i(X) = 0$ for all $i \leq n$.

Section 6 contains the main original research contribution of [32]. It develops a transversality method for studying the homotopy groups of stable loci in affine GIT, adapting the infinite-dimensional transversality framework of Daskalopoulos-Uhlenbeck [7] and Wilkin [33] to the affine setting. This yields an explicit connectivity bound for the stable locus V^{st} , together with corresponding information on the homotopy groups of the quotient $\mathcal{M}^{st} = V^{st}/G$.

We finish this thesis by demonstrating the applicability of the framework with three concrete examples. It is applied to the 2-Kronecker quiver, where we recover the known topology of its stable locus $V^{st} \simeq S^3$. It is also applied in the context of controllable linear systems, where GIT stability is equivalent to controllability, obtaining a connectedness bound $d_{min} = 2m$. Finally, we apply the framework to a problem in statistics, which connects GIT stability to the existence of a unique Maximum Likelihood Estimate (MLE) for Directed Acyclic Graph (DAG) models. The connectedness of the stable locus is computed in terms of the number of samples n and model parameters k , $d_{min} = 2n - 4k + 4$. This establishes the connectedness of the space of good data samples.

In summary, the thesis provides a detailed discussion from the foundations of reductive groups and GIT to an original framework of transversality methods. It gives a computable formula for the connectedness of stable loci of affine GIT quotients, and the applicability of this method is demonstrated through different concrete examples.

2 Reductive Linear Algebraic Groups

We provide the necessary definitions and state Nagata's theorem of finite generation of invariant rings so that the affine GIT quotients, which will be discussed in the next section, are well-defined algebraic varieties. The main references of this section are [17], [21], and [30].

Definition 2.1. An *affine algebraic set* is a subset $X \subseteq \mathbb{K}^n$ that can be written as the set of common zeros of a collection of polynomials. For any subset $S \subseteq \mathbb{K}[x_1, \dots, x_n]$ of polynomials, the associated affine algebraic set $X = V(S) := \{(a_1, \dots, a_n) \in \mathbb{A}^n(\mathbb{K}) \mid f(a_1, \dots, a_n) = 0 \text{ for all } f \in S\}$.

Proposition 2.2. *The union of two algebraic sets is an algebraic set, the intersection of two algebraic sets is an algebraic set, and the empty set and the entire space are algebraic sets.*

Proof. We start by proving that if $X_1 = V(S_1)$ and $X_2 = V(S_2)$ are two algebraic sets, where $S_1, S_2 \subseteq \mathbb{K}[x_1, \dots, x_n]$, then the union $X_1 \cup X_2$ is also an algebraic set. Consider the set of polynomials $S := \{f \cdot g \mid f \in S_1, g \in S_2\}$. Let $p \in X_1 \cup X_2$, then either $p \in X_1$ or $p \in X_2$. If $p \in X_1 = V(S_1)$, then $f(p) = 0$ for all $f \in S_1$. Hence for any $h = f \cdot g \in S$, we have $h(p) = f(p)g(p) = 0$. Similarly, if $p \in X_2 = V(S_2)$, then for any $h = f \cdot g \in S$, we have $h(p) = f(p) \cdot 0 = 0$. In either case, $h(p) = 0$ for all $h \in S$, and thus $p \in V(S)$. Hence $X_1 \cup X_2 \subseteq V(S)$. For the reverse inclusion, let $p \in V(S)$. This means that we have $(f \cdot g)(p) = f(p)g(p) = 0$ for all $f \in S_1, g \in S_2$. Assume, for contradiction, that $p \notin X_1 \cup X_2$, which implies $p \notin X_1$ and $p \notin X_2$. Since $p \notin X_1$, there exists some polynomial $f_0 \in S_1$ such that $f_0(p) \neq 0$. But $p \in V(S)$ implies that we have $f_0(p) \cdot g(p) = 0$ for any $g \in S_2$, consequently $g(p) = 0$ for all $g \in S_2$. This means that $p \in V(S_2) = X_2$, a contradiction. Therefore, the initial assumption must be false, and we have $p \in X_1 \cup X_2$. Combining both inclusions, we conclude that $X_1 \cup X_2 = V(S)$, which is an algebraic set by definition.

Next, we proceed with the proof that $X_1 \cap X_2$ is an algebraic set. Consider $S = S_1 \cup S_2$. Note that a point $p \in \mathbb{A}^n(\mathbb{K})$ lies in the intersection $X_1 \cap X_2$ if and only if $p \in X_1$ and $p \in X_2$. Then this is equivalent to the condition that $f(p) = 0$ for all $f \in S_1$, and $g(p) = 0$ for all $g \in S_2$, which precisely means that $h(p) = 0$ for all $h \in S = S_1 \cup S_2$. Therefore, $p \in X_1 \cap X_2$ if and only if $p \in V(S_1 \cup S_2)$. Hence $X_1 \cap X_2 = V(S)$, which is an algebraic set.

Lastly, observe that the entire space $\mathbb{A}^n(\mathbb{K}) = V(\{0\})$, so it is clearly an algebraic set. Since a non-zero constant polynomial never vanishes, $\emptyset = V(\{1\})$. Thus $\emptyset$ is an algebraic set. $\square$

To define a topology formed by closed sets requires that any finite union and any arbitrary intersection of closed sets are closed. We have demonstrated, in the preceding proposition, the cases for the union and intersection of two sets. The results can be generalised as follows.

The result that the union of two algebraic sets is algebraic can be extended by induction to show that any finite union of algebraic sets is also an algebraic set. Furthermore, the argument for the intersection of two algebraic sets generalises directly to arbitrary intersections. If $\{X_i = V(S_i)\}_{i \in I}$ is any collection of algebraic sets, then their intersection is given by $\bigcap_{i \in I} X_i = V(\bigcup_{i \in I} S_i)$, which is a consequence of that fact that a point p lies in the intersection $\bigcap_{i \in I} X_i$ if and only if it is a zero of every polynomial in every set S_i .

Definition 2.3 (The Zariski Topology). The *Zariski topology* on an affine space $\mathbb{A}^n$ is the topology defined by letting its closed sets to be the affine algebraic sets.

Definition 2.4. A non-empty topological space X is *irreducible* if it cannot be expressed as the union of two proper closed subsets. A non-empty subset $Y \subseteq X$ is irreducible if it is irreducible as a topological space with the subspace topology.

Definition 2.5. An *affine variety* is a Zariski-closed subset $X \subseteq \mathbb{A}^n(\mathbb{K})$ endowed with the induced Zariski topology. Its vanishing ideal $I(X) := \{f \in \mathbb{K}[x_1, \dots, x_n] \mid f|_X = 0\}$ is a radical ideal.

In particular, if X is irreducible, then $I(X)$ is prime, in which case we call X an *irreducible affine variety*.

Proposition 2.6. Let $X \subseteq \mathbb{A}^n$ be a closed subset. Then X is irreducible if and only if the ideal $I(X)$ is a prime ideal in $R = \mathbb{K}[x_1, \dots, x_n]$.

Proof. Assume that X is irreducible, then X is non-empty. Note that the constant polynomial 1 does not vanish on any non-empty set, so $1 \notin I(X)$. Thus $I(X)$ is a proper ideal of R . Let $f, g \in R$ such that $f \cdot g \in I(X)$. Then we have $f(p)g(p) = 0$ for all $p \in X$. Define the closed sets $Y_1 = X \cap V(f)$, and $Y_2 = X \cap V(g)$. It follows that $X = Y_1 \cup Y_2$. The irreducibility of X implies that $X = Y_1$ or $X = Y_2$. If $X = Y_1$, then $X \subseteq V(f)$, which means $f \in I(X)$. Similarly, if $X = Y_2$, then $X \subseteq V(g)$, thus $g \in I(X)$. Therefore, $I(X)$ is a prime ideal.

Conversely, suppose that $I(X)$ is a prime ideal, which means that it is a proper ideal. If X were empty, then we would have $I(X) = R$, contradicting the properness. Thus X is non-empty. Let $X = X_1 \cup X_2$, where X_1 and X_2 are closed subsets of X . Assume for contradiction that $X_1 \subsetneq X$ and $X_2 \subsetneq X$. Note that the inclusions imply the reverse inclusions of ideals, namely $I(X) \subseteq I(X_i)$ where $i = 1, 2$. Since X_i is a proper subset of X , there exists a point $p \in X \setminus X_i$. X_i being closed implies that there exists some polynomial $h \in I(X_i)$ such that $h(p) \neq 0$. Note that p is a point in X , so $h \notin I(X)$. Thus we have an element $h \in I(X_i) \setminus I(X)$, proving $I(X)$ is a proper subset of $I(X_i)$.

We can therefore choose polynomials $f \in I(X_1) \setminus I(X)$ and $g \in I(X_2) \setminus I(X)$. For any point $p \in X = X_1 \cup X_2$, we have $f(p) = 0$ when $p \in X_1$ or $g(p) = 0$ when $p \in X_2$. In either case, $(fg)(p) = f(p)g(p) = 0$, thus the product fg vanishes on all of X , contradicting the primeness of $I(X)$. Hence $X = X_1$ or $X = X_2$, which shows the irreducibility of X . $\square$

To equip an affine variety with the structure of a group in a compatible way, we need to define the structure-preserving maps between affine varieties.

Definition 2.7. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties. A function $\phi : X \rightarrow Y$ is a *morphism* or a *regular map* if there exist m polynomials, $p_1, \dots, p_m \in \mathbb{K}[x_1, \dots, x_n]$, such that for every point $a = (a_1, \dots, a_n) \in X$, its image is given by $\phi(a) = (p_1(a), \dots, p_m(a)) \in Y$. Furthermore, ϕ becomes an isomorphism if it has a two-sided inverse $\psi : Y \rightarrow X$ that is also a morphism.

In addition, we say that a function $f : X \rightarrow \mathbb{A}^1$ is a *regular function* if it is a regular map from X to the affine line $\mathbb{A}^1$. In other words, there exists a polynomial P in the ambient coordinates such that $f(x) = P(x)$ for all $x \in X$.

The product of two affine varieties is defined as

$$X \times Y := \{(x, y) \in \mathbb{A}^{n+m} \mid x \in X, y \in Y\}.$$

Suppose $X = V(S)$ and $Y = V(T)$ where $S \subseteq \mathbb{K}[x_1, \dots, x_n]$ and $T \subseteq \mathbb{K}[y_1, \dots, y_m]$. Note that any point $(a, b) \in \mathbb{A}^{n+m}$ lies in $X \times Y$ if and only if $a \in X$, $b \in Y$, which occurs when $f(a) = 0$ for all $f \in S$ and $g(b) = 0$ for all $g \in T$. Regarding polynomials in S as functions of $(x_1, \dots, x_n, y_1, \dots, y_m)$ that are independent of the y_i variables, we have $f(a, b) = f(a) = 0$. Similarly, $g(a, b) = g(b) = 0$. Therefore, $(a, b) \in V(S \cup T)$, which implies that $X \times Y = V(S \cup T)$. Moreover, since X, Y are irreducible, it follows that $V(S \cup T)$ is irreducible. Hence the product of two affine varieties is also an affine variety.

Definition 2.8. An *algebraic group* over $\mathbb{K}$ is an algebraic variety G with the structure of a group such that the multiplication $(g, h) \mapsto g \cdot h$, and the inversion $g \mapsto g^{-1}$ are morphisms (or regular maps). If furthermore the underlying variety is affine, then G is called a *linear algebraic group*.

For an algebraic group G , we claim that only one irreducible component of G contains the identity element e of G . Note that as an algebraic variety, G admits a unique and finite decomposition into its irreducible components. Suppose that we have distinct irreducible components $X_1, \dots, X_m$ containing e . Then the product $X_1 \times \dots \times X_m$ is also irreducible, whose image under the multiplication morphism is the irreducible set

$$X_1 X_2 \cdots X_m = \{x_1 x_2 \cdots x_m \mid x_i \in X_i \text{ for } i = 1, \dots, m\},$$

which still contains e .

Thus the image $X_1 \cdots X_m \subseteq X_i$ for some i . However, we observe that each of the irreducible components $X_1, \dots, X_m$ lies in $X_1 \cdots X_m$. This implies that $m = 1$, proving the claim. We denote by G^0 the *identity component* of G , which is the unique irreducible component containing e .

Proposition 2.9. *Let G be an algebraic group.*

- (i) *The identity component G^0 is a closed normal subgroup of finite index, whose cosets are the connected and irreducible components of G containing e ;*
- (ii) *any closed subgroup of finite index in G contains G^0 .*

Proof. (i) The closedness of G^0 follows from the fact that it is an irreducible component. We have established that G^0 contains the identity e . Note that the inversion map $i : g \mapsto g^{-1}$ is an isomorphism of varieties, so the image $i(G^0) = (G^0)^{-1}$ is an irreducible component, which also contains e . By the above uniqueness argument, we have $(G^0)^{-1} = G^0$, which shows that G^0 is closed under taking inverses.

Now let $x \in G^0$ be an arbitrary element, and consider the set xG^0 . Since $x \mapsto xg$ is a homeomorphism, it follows that xG^0 is irreducible. Note that $x^{-1} \in G^0$, so the set $x^{-1}G^0$ is also irreducible, which contains $x^{-1}x = e$. The uniqueness of G^0 then implies that $x^{-1}G^0 = G^0$, and we can therefore conclude that G^0 is a subgroup of G .

Consider the conjugation map $G^0 \mapsto gG^0g^{-1}$, which is an automorphism of G . So gG^0g^{-1} inherits the irreducibility, and it contains e . By the uniqueness of the identity component, we have $gG^0g^{-1} = G^0$ for all $g \in G$. Thus G^0 is a closed normal subgroup.

Observe that the cosets of G^0 are sets of the form gG^0 . Since left translation is a homeomorphism, each coset gG^0 is an irreducible component of G . An algebraic variety is a Noetherian topological space, which means it has a finite number of irreducible components. Hence there can only be finitely many of them. Since they are disjoint, these are also the connected components of G . This proves that the index of G^0 is finite.

- (ii) Let H be a closed subgroup of G with finite index, i.e. $[G : H] = m$. This means we have a finite disjoint union of cosets

$$G = H \cup g_2 H \cup \dots \cup g_m H.$$

Note that each coset $g_i H$ is also a closed set. Consider $U = g_2 H \cup \dots \cup g_m H$, which is closed by construction. Then $H = G \setminus U$, implying that H is open. Hence H is both closed and open.

We now obtain a partition of G into finite number of disjoint open sets, as each $g_i H$ is a translate of H and is thus open. Therefore, we can express the identity component G^0 as

$$\begin{aligned} G^0 &= G^0 \cap (H \cup g_2 H \cup \dots \cup g_m H) \\ &= (G^0 \cap H) \cup (G^0 \cap g_2 H) \cup \dots \cup (G^0 \cap g_m H). \end{aligned}$$

Since each set $(G^0 \cap g_i H)$ is an open set in the subspace topology of G^0 , we have written G^0 as a finite disjoint union of open subsets. Since G^0 is connected, it cannot be written as a disjoint union of two or more non-empty open subsets. Note also that $e \in G^0$ and $e \in H$, so $G^0 \cap H$ is non-empty. Hence we must have $G^0 \cap g_i H = \emptyset$ for all $i = 2, \dots, m$, which means that no element of G^0 lies in any coset other than H . Therefore, we conclude that $G^0 \subseteq H$. $\square$

Example 2.10. Here are some examples of algebraic groups.

- The affine line $\mathbb{A}^1 = \text{Spec}(\mathbb{K}[x])$ with the structure of the additive group $\mathbb{G}_a$.
- The affine line without the origin $\mathbb{A}^1 \setminus \{0\}$ with the structure of the multiplicative group $\mathbb{G}_m$.
- The general linear group $\text{GL}_n(\mathbb{C})$ with the operation defined as matrix multiplication, which is isomorphic to a Zariski-closed set in $\mathbb{A}^{n^2+1}$, and hence an affine variety.

We define a *G-action* on a variety X to be a morphism $a : G \times X \rightarrow X$. A *rational representation* of a linear algebraic group G is a morphism of algebraic varieties $\rho : G \rightarrow \text{GL}(V)$, where V is a finite dimensional vector space over $\mathbb{K}$. Note that $\text{GL}(V)$ is a subset of the affine space $\text{End}(V) \cong \mathbb{A}^{n^2}$, where $n = \dim V$.

Definition 2.11. A $\mathbb{K}$ -vector space V is a *rational G-module* if it is equipped with a group action $G \times V \rightarrow V$, denoted $(g, v) \mapsto g \cdot v$, that is also a morphism of varieties.

Proposition 2.12. A finite dimensional vector space V has the structure of a rational G -module if and only if there exists a rational representation $\rho : G \rightarrow \text{GL}(V)$.

Proof. Suppose that we are given a rational representation $\rho : G \rightarrow \text{GL}(V)$, which is a morphism of varieties. Then we can define a group action $a : G \times V \rightarrow V$ by $(g, v) \mapsto \rho(g)(v)$. Consider the evaluation map $\text{ev} : \text{End}(V) \times V \rightarrow V$, given by $(A, v) \mapsto A(v)$, which is, by construction, a morphism. By fixing a $\mathbb{K}$ -basis $\{e_1, \dots, e_n\}$ of V , we have $V \cong \mathbb{A}^n$ and $\text{End}(V) \cong \mathbb{A}^{n^2}$. Thus in terms of basis, the evaluation map is defined by $((a_{ij}), (v_j)) \mapsto (\sum_j a_{ij}v_j)_i$, whose components are polynomials in the basis. So its restriction to the open subset $\text{GL}(V) \times V$ is also a morphism.

Now consider the map $\phi : G \times V \rightarrow \text{GL}(V) \times V$ defined by $\phi(g, v) = (\rho(g), v)$. This is a morphism as its components are the morphism ρ and the identity morphism id_V . Thus the action a is the same as the composition $\text{ev} \circ \phi$, which implies that $a : G \times V \rightarrow V$ is a morphism. Therefore, V is a rational G -module.

Conversely, suppose V is a rational G -module. So the action $a : G \times V \rightarrow V$ given by $a(g, v) = g \cdot v$ is a morphism. Note that for each $g \in G$, the map $v \mapsto g \cdot v$ is linear, which thus defines $\rho : G \rightarrow \text{GL}(V)$ by setting $\rho(g)(v) = g \cdot v$. It is clear that $\rho(g_1g_2) = \rho(g_1)\rho(g_2)$ for all $g_1, g_2 \in G$, so ρ is a group homomorphism.

To show that ρ is indeed a morphism of varieties, we choose the basis $\{e_1, \dots, e_n\}$ for V and let $\{f_1, \dots, f_n\}$ be the corresponding dual basis for V^* . Note that the entries of the matrix for $\rho(g)$ in the chosen basis are given by $c_{ij} : G \rightarrow \mathbb{K}$ where $c_{ij}(g) = f_i(\rho(g)(e_j)) = f_i(g \cdot e_j)$. Now for each e_j , consider the map $\psi_j : G \rightarrow V$ defined by $\psi_j(g) = g \cdot e_j$. Then ψ_j is the composition of the morphism $g \mapsto (g, e_j)$ from $G \rightarrow G \times V$ and the action a , which implies that ψ_j is a morphism. Thus for each j , the coordinate functions of ψ_j are regular functions on G .

Moreover, since the dual basis f_i is a linear functional, it is a polynomial of degree 1, and therefore a regular function. Hence the composition $c_{ij} = f_i \circ \psi_j$ is regular, which means all the matrix entries of $\rho(g)$ are regular functions on G . Therefore, we can conclude that ρ is a morphism of varieties and hence a rational representation. $\square$

Definition 2.13. Let $X \subseteq \mathbb{A}^n$ be an affine variety. The set of all regular functions $f : X \rightarrow \mathbb{K}$ forms a ring, called the *coordinate ring* of X , and is denoted by $\mathbb{K}[X]$.

Consider the natural restriction map $\Phi : \mathbb{K}[x_1, \dots, x_n] \rightarrow \mathbb{K}[X]$ defined by $P \mapsto P|_X$, where P is any polynomial in $\mathbb{K}[x_1, \dots, x_n]$. This map is clearly a ring homomorphism, and the surjectivity of Φ follows from the fact that $\mathbb{K}[X]$ consists of regular functions. Note that the kernel $\ker(\Phi) = \{P \in \mathbb{K}[x_1, \dots, x_n] \mid P|_X = 0\}$ is an ideal of the variety of X , i.e. $\ker(\Phi) = I(X)$. Thus by the First Isomorphism Theorem, we obtain

$$\mathbb{K}[X] \cong \mathbb{K}[x_1, \dots, x_n]/I(X).$$

The Hilbert Basis Theorem [2, Theorem 7.5] implies that $\mathbb{K}[X]$ is finitely generated.

Definition 2.14. Let R be a commutative ring. A *chain of prime ideals* of length d in the ring is a sequence of distinct prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_d.$$

The *Krull dimension* of R , denoted $\dim R$, is the supremum of the lengths of all such chains of prime ideals in R .

The *dimension* of a non-empty affine variety X , $\dim X$, is the Krull dimension of its coordinate ring.

Let a linear algebraic group G acts on an affine variety X . The induced action on the coordinate ring $\mathbb{K}[X]$ is a group homomorphism from G to $\mathbb{K}[X]$ given by

$$(g \cdot f)(x) := f(g^{-1} \cdot x)$$

for any $g \in G, f \in \mathbb{K}[X]$, and $x \in X$. In particular, this defines a left action in the sense that for any $g_1, g_2 \in G$, we have

$$((g_1 g_2) \cdot f)(x) = f((g_1 g_2)^{-1} x) = f(g_2^{-1} g_1^{-1} x) = (g_1 \cdot (g_2 \cdot f))(x).$$

Proposition 2.15. *Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties whose coordinate rings are $\mathbb{K}[X]$ and $\mathbb{K}[Y]$, respectively. Then the coordinate ring, $\mathbb{K}[X \times Y]$, of the product $X \times Y \subseteq \mathbb{A}^{n+m}$ is isomorphic to the tensor product $\mathbb{K}[X] \otimes_{\mathbb{K}} \mathbb{K}[Y]$.*

Proof. We note that

$$\mathbb{K}[X] \otimes_{\mathbb{K}} \mathbb{K}[Y] \cong \frac{\mathbb{K}[x_1, \dots, x_n]}{I(X)} \otimes_{\mathbb{K}} \frac{\mathbb{K}[y_1, \dots, y_m]}{I(Y)} \cong \frac{\mathbb{K}[x_1, \dots, x_n] \otimes_{\mathbb{K}} \mathbb{K}[y_1, \dots, y_m]}{\langle I(X) \otimes 1, 1 \otimes I(Y) \rangle}.$$

Since the definition of tensor products gives $\mathbb{K}[x_1, \dots, x_n] \otimes_{\mathbb{K}} \mathbb{K}[y_1, \dots, y_m] \cong \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_m]$, it follows that the ideal in the denominator is the ideal generated by $I(X)$ and $I(Y)$, which is precisely the ideal $I(X \times Y)$. Therefore, we obtain $\mathbb{K}[X] \otimes_{\mathbb{K}} \mathbb{K}[Y] \cong \mathbb{K}[X \times Y]$. $\square$

Definition 2.16. Let $X \subseteq \mathbb{A}^n$ be an affine variety and let $p = (p_1, \dots, p_n)$ be a point on X . Let $I(X) = \langle f_1, \dots, f_m \rangle$ be the ideal of X . The *tangent space* to X at p , denoted $T_p(X)$, is the affine linear subspace of $\mathbb{A}^n$ defined by the vanishing of the linear parts of the polynomials in $I(X)$ at p , i.e.

$$T_p(X) := V \left(\left\{ \sum_{i=1}^n \frac{\partial f_j}{\partial x_i}(p)(x_i - p_i) \mid j = 1, \dots, m \right\} \right).$$

The *dimension* of the tangent space, $\dim T_p(X)$, is its dimension as a vector space.

Definition 2.17. Let X be an affine variety. A point $p \in X$ is *nonsingular* (or *smooth*) if $\dim T_p(X) = \dim(X)$. A point that is not nonsingular is called *singular*. The variety X is said to be *nonsingular* if all of its points are nonsingular.

Proposition 2.18. *Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties with coordinate rings $\mathbb{K}[X]$ and $\mathbb{K}[Y]$, respectively. Then there is a natural bijective correspondence between sets*

$$\text{Hom}(X, Y) \cong \text{Hom}(\mathbb{K}[Y], \mathbb{K}[X]),$$

where Hom on the left denotes the set of morphisms of varieties, and the right Hom denotes the set of morphisms of $\mathbb{K}$ -algebras.

Proof. A proof of a more general version of this proposition can be found in Hartshorne [14, Chapter I, Proposition 3.5].

For our proof, we want to construct mutually inverse maps $\Phi : \text{Hom}(X, Y) \rightarrow \text{Hom}(\mathbb{K}[Y], \mathbb{K}[X])$ and $\Psi : \text{Hom}(\mathbb{K}[Y], \mathbb{K}[X]) \rightarrow \text{Hom}(X, Y)$. Let $f : X \rightarrow Y$ be a morphism of varieties. Define

$$\Phi(f) = f^* : \mathbb{K}[Y] \rightarrow \mathbb{K}[X], \quad h \mapsto h \circ f.$$

This is well-defined since the composition of a regular function on Y with a morphism f is a regular function on X . More specifically, write $f = (f_1, \dots, f_m)$ with $f_i = y_i \circ f \in \mathbb{K}[X]$. Note that any $h \in \mathbb{K}[Y]$ can be expressed by a polynomial $H \in \mathbb{K}[t_1, \dots, t_m]/I(Y)$, and hence

$$h \circ f = H(f_1, \dots, f_m) \in \mathbb{K}[X].$$

The map f^* is a $\mathbb{K}$ -algebra homomorphism because pointwise addition and multiplication of functions commute with precomposition. Thus Φ is well-defined.

Now let $\varphi : \mathbb{K}[Y] \rightarrow \mathbb{K}[X]$ be a $\mathbb{K}$ -algebra homomorphism, our goal is to construct a corresponding morphism of varieties from X to Y . Define the map $f_\varphi : X \rightarrow \mathbb{A}^m$ by

$$f_\varphi(x) := (\varphi(y_1)(x), \dots, \varphi(y_m)(x)) \quad \text{for } x \in X.$$

Since each component function $\varphi(y_i)$ is regular on X , the map f_φ is a morphism of varieties from X to $\mathbb{A}^m$. Note that a point lies in Y if and only if it is a common zero of all polynomials in the ideal $I(Y) \subseteq \mathbb{K}[t_1, \dots, t_m]$. Let $F \in I(Y)$ be such a polynomial. By definition of $\mathbb{K}[Y]$, the function $F(y_1, \dots, y_m)$ is the zero element in $\mathbb{K}[Y]$.

Since φ is a $\mathbb{K}$ -algebra homomorphism, we have

$$0 = \varphi(0) = \varphi(F(y_1, \dots, y_m)) = F(\varphi(y_1), \dots, \varphi(y_m)) \in \mathbb{K}[X],$$

which implies that the regular function $F(\varphi(y_1), \dots, \varphi(y_m))$ is the zero map on X , i.e. $F(f_\varphi(x)) = 0$ for any $x \in X$. As F were arbitrary, it follows that $f_\varphi(x) \in Y$. Therefore, $\text{Im}(f_\varphi) \subseteq Y$, and f_φ is a well-defined morphism of varieties. Then we define $\Psi(\varphi) := f_\varphi$.

Let $\varphi = \Phi(f) = f^*$, then $\Psi(\Phi(f))(x) = f_\varphi(x) = ((y_1 \circ f)(x), \dots, (y_m \circ f)(x))$. But y_i is the i -th coordinate function on $Y \subseteq \mathbb{A}^m$, hence $f_\varphi = f$. This means $\Psi \circ \Phi = \text{id}_{\text{Hom}(X, Y)}$. Conversely, let $f = \Psi(\varphi) = f_\varphi$. Since $\mathbb{K}[Y]$ is generated as a $\mathbb{K}$ -algebra by $y_1, \dots, y_m$, we have $f^*(y_i) = y_i \circ f = y_i \circ f_\varphi = \varphi(y_i)$. Thus $f^* = \varphi$, meaning $\Phi \circ \Psi = \text{id}_{\text{Hom}(\mathbb{K}[Y], \mathbb{K}[X])}$.

Therefore, Φ and Ψ are mutually inverses. $\square$

For an algebraic group G , the multiplication map $m : G \times G \rightarrow G$ given by $(g_1, g_2) \mapsto g_1 g_2$ induces a pullback homomorphism of coordinate rings

$$m^* : \mathbb{K}[G] \rightarrow \mathbb{K}[G \times G] \cong \mathbb{K}[G] \otimes_{\mathbb{K}} \mathbb{K}[G],$$

which is defined by $(m^*(f))(g_1, g_2) = f(m(g_1, g_2)) = f(g_1 g_2)$ for any regular function $f \in \mathbb{K}[G]$ and any pair of points $(g_1, g_2) \in G \times G$. The existence of such a pullback homomorphism is a consequence of the contravariant equivalence of categories between the category of affine varieties over $\mathbb{K}$ and the category of finitely generated, integral $\mathbb{K}$ -algebras [14, Corollary 3.8].

Furthermore, the pullback m^* is unique. This is because the multiplication map m is a uniquely defined morphism of varieties, and that the contravariant equivalence establishes a bijective mapping between the set variety morphisms and the set of $\mathbb{K}$ -algebra homomorphisms.

Proposition 2.19. *Any affine algebraic group G admits a morphism of algebraic groups $\rho : G \rightarrow \text{GL}_n(\mathbb{K})$ for some $n \in \mathbb{N}$ such that it is a closed embedding. In other words, every affine algebraic group is isomorphic to a linear algebraic group.*

Proof. We follow the method of [17, Section 8.6]. Let G be an affine algebraic group, then it is an affine variety. This means that its coordinate ring $\mathbb{K}[G]$ is finitely generated as a $\mathbb{K}$ -algebra. For any $g, x \in G$ and any $f \in \mathbb{K}[G]$, define the left action of G by $(g \cdot f)(x) := f(g^{-1}x)$. This is a homomorphism of groups from G to the group of automorphism of $\mathbb{K}[G]$ since $(gh \cdot f) = g \cdot (h \cdot f)$. Note that for any $f \in \mathbb{K}[G]$, the set of its left translates $\{g \cdot f \mid g \in G\}$ spans a finite-dimensional, G -stable subspace $W_f \subseteq \mathbb{K}[G]$. The finite-dimensionality of W_f is a consequence of the map $\lambda^* : \mathbb{K}[G] \rightarrow \mathbb{K}[G] \otimes_{\mathbb{K}} \mathbb{K}[G]$, which is the pullback of the multiplication map $\lambda : G \times G \rightarrow G$ defined by $\lambda(x, y) = xy$. Writing $\lambda^*(f) = \sum_{k=1}^r a_k \otimes b_k$, we obtain

$$(g \cdot f)(x) = f(g^{-1}x) = (\lambda^* f)(g^{-1}, x) = \sum_{k=1}^r a_k(g^{-1})b_k(x).$$

This shows that every translate $g \cdot f$ is a linear combination of the fixed finite set of functions $\{b_1, \dots, b_r\}$. Thus, $\dim W_f \leq r < \infty$. We then set $W = \sum_{i=1}^m W_{f_i}$, which is a finite-dimensional, G -stable subspace of $\mathbb{K}[G]$ that contains the generators $\{f_1, \dots, f_m\}$.

Let $n = \dim W$ and fix a basis $\{u_1, \dots, u_n\}$ for W . Since W is G -stable, for any $g \in G$, the function $g \cdot u_j$ is in W and can be expressed as a linear combination of the basis vectors, namely

$$g \cdot u_j = \sum_{i=1}^n \rho_{ij}(g)u_i,$$

where $\rho_{ij} \in \mathbb{K}[G]$. Define a map $\rho : G \rightarrow \text{GL}_n(\mathbb{K})$ by $\rho(g) = (\rho_{ij}(g))$. This map is a group homomorphism

since

$$\begin{aligned}
\sum_i \rho_{ij}(gh)u_i &= (gh) \cdot u_j = g \cdot (h \cdot u_j) = g \cdot \left(\sum_k \rho_{kj}(h)u_k \right) \\
&= \sum_k \rho_{kj}(h)(g \cdot u_k) = \sum_k \rho_{kj}(h) \left(\sum_i \rho_{ik}(g)u_i \right) \\
&= \sum_i \left(\sum_k \rho_{ik}(g)\rho_{kj}(h) \right) u_i.
\end{aligned}$$

Comparing coefficients of u_i gives $\rho(gh) = \rho(g)\rho(h)$. Since the functions $g \mapsto \rho_{ij}(g)$ are the coordinates of the vector $g \cdot u_j$, which are regular, it follows that each ρ_{ij} is a regular function, ρ is a morphism of affine varieties.

We now show that ρ is a closed embedding by proving it is injective and $\rho^* : \mathbb{K}[\mathrm{GL}_n(\mathbb{K})] \rightarrow \mathbb{K}[G]$ is surjective. Suppose $\rho(g) = I_n$. Then $g \cdot u_j = u_j$ for all j , which implies $u_j(g^{-1}x) = u_j(x)$ for all $x \in G$. Thus we obtain $u_j(e) = u_j(g)$. By construction of W , it follows that $f_i(e) = f_i(g)$ for all $i = 1, \dots, m$. Hence for any $F \in \mathbb{K}[G]$, we have $F(g) = P(f_1(g), \dots, f_m(g)) = P(f_1(e), \dots, f_m(e)) = F(e)$ for some polynomial P . This means g and e cannot be separated by regular functions, implying $g = e$. Hence ρ is injective.

Note that as an affine variety in $\mathbb{A}^{n^2+1}$, $\mathrm{GL}_n(\mathbb{K})$ has coordinates $(A, \frac{1}{\det(A)})$, where $A = (a_{ij}) \in \mathrm{GL}_n(\mathbb{K})$. This gives that $\mathbb{K}[\mathrm{GL}_n(\mathbb{K})] = \mathbb{K}[(a_{ij}), \frac{1}{\det(A)}]$. Then ρ^* acts by sending $a_{ij} \mapsto \rho_{ij}$. Choose the basis $\{u_1, \dots, u_n\}$ for W to be compatible with the evaluation at the identity $e \in G$. Assuming W is not the zero space, there exists a function in W that does not vanish at e . Now evaluating at the identity gives

$$u_j(g^{-1}e) = \sum_{i=1}^n \rho_{ij}(g)u_i(e).$$

This means that the composition $u_j \circ \mathrm{inv}$ can be expressed as a $\mathbb{K}$ -linear combination of the regular functions ρ_{ij} , where $\mathrm{inv} : g \mapsto g^{-1}$ is the inversion map.

Note that $u_j(g^{-1})$ can be written as $\mathrm{inv}^*(u_j) \in \mathbb{K}[\{\rho_{ij}\}]$ for each $j = 1, \dots, n$, where $\mathrm{inv}^* : \mathbb{K}[G] \rightarrow \mathbb{K}[G]$ is the pullback of the inversion map. Because the set $\{u_1, \dots, u_n\}$ is a basis for W , the set $\{\mathrm{inv}^*(u_1), \dots, \mathrm{inv}^*(u_n)\}$ is a basis for the vector space $\mathrm{inv}^*(W) \subseteq \mathbb{K}[\{\rho_{ij}\}]$. Moreover, since inv^* is an isomorphism, $\mathrm{inv}^*(W)$ contains the set $\{\mathrm{inv}^*(f_1), \dots, \mathrm{inv}^*(f_m)\}$, which is also a generating set for the coordinate ring $\mathbb{K}[G]$. Since $\mathbb{K}[\{\rho_{ij}\}]$ is a subalgebra of $\mathbb{K}[G]$ that contains a generating set for $\mathbb{K}[G]$, we obtain $\mathbb{K}[G] = \mathbb{K}[\mathrm{inv}^*(f_1), \dots, \mathrm{inv}^*(f_m)] \subseteq \mathbb{K}[\{\rho_{ij}\}]$. As $\mathbb{K}[\{\rho_{ij}\}]$ is also a subalgebra of $\mathbb{K}[G]$, equality follows. This proves the surjectivity of ρ^* .

Hence we conclude that ρ is a closed embedding. Thus G is isomorphic, as an algebraic group, to its image $\rho(G)$, which is a closed subgroup of $\mathrm{GL}_n(\mathbb{K})$. $\square$

Corollary 2.20. *Any affine algebraic group is a nonsingular algebraic variety.*

Proof. Let G be an affine algebraic group. By the proposition above, there exists a closed embedding $\rho : G \hookrightarrow \mathrm{GL}_n(\mathbb{K})$ for some $n \in \mathbb{N}$. This means that ρ is an isomorphism of algebraic groups from G onto its image, $\rho(G)$, which is a closed subgroup of $\mathrm{GL}_n(\mathbb{K})$. To prove that G is nonsingular, it suffices to show that any closed subgroup $\rho(G)$ is nonsingular.

Now we show that $\rho(G)$ contains at least one nonsingular point. An algebraic group is an irreducible variety, so is $\rho(G)$. This implies the set of its nonsingular points, $\rho(G)_{ns}$, is a non-empty, Zariski-open subset [14, Theorem I 5.3 & Corollary II 8.16], which guarantees the existence of at least one nonsingular point. We denote such a nonsingular point by y_0 .

For any element $g \in \rho(G)$, consider the action of $\rho(G) \cong G$ on itself by left multiplication $L_g : \rho(G) \rightarrow \rho(G)$ defined by $L_g(z) = gz$. This map is an automorphism of the variety $\rho(G)$, and thus an isomorphism that preserves nonsingularity. Hence for any other arbitrary point $y \in \rho(G)$, we choose $g = yy_0^{-1}$, which gives

$$L_{yy_0^{-1}}(y_0) = (yy_0^{-1})y_0 = y.$$

Since y_0 is nonsingular and $L_{yy_0^{-1}}$ is an isomorphism, its image y is also a nonsingular point of $\rho(G)$. As y was arbitrary, we conclude that $\rho(G)$ is nonsingular. $\square$

Definition 2.21. Let A, B be arbitrary closed subgroups of an algebraic group G , the *commutator group* $[A, B]$ is the group generated by commutators $aba^{-1}b^{-1}$, where $a \in A, b \in B$.

Note that, in general, the commutator group does not necessarily inherit the closedness. For example, consider the cyclic subgroups of $\mathrm{GL}_2(\mathbb{C})$ generated by $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$. Let $A = \langle M_1 \rangle$ and $B = \langle M_2 \rangle$. Observe that

$$M_1^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \text{ and } M_2^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I,$$

thus $A = \{I, M_1\}$, $B = \{I, M_2\}$. This means both A and B are finite groups of order 2, and therefore they are closed subgroups of G .

By definition, the commutator subgroup $[A, B]$ is generated by the single commutator $[M_1, M_2]$, given that A, B are generated by single elements. Since M_1 and M_2 are their own inverses, we have $[M_1, M_2] = M_1 M_2 M_1 M_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Then $[A, B]$ is the cyclic group generated by $[M_1, M_2]$, that is,

$$[A, B] = \left\{ \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}.$$

This implies that $[A, B]$ is contained in the group U of unipotent upper-triangular matrices (i.e. the upper-triangular matrices where all the entries on the main diagonal are equal to 1), which is isomorphic to the additive group and is a closed subgroup of $\mathrm{GL}_2(\mathbb{C})$.

In the Zariski topology on $\mathbb{A}^1 \cong U$, the closed sets are the entire space, the empty set, and finite sets of points. But $[A, B]$ corresponds to the infinite set $\{\dots, -4, -2, 0, 2, 4, \dots\} \subset \mathbb{C}$, so its closure is the entire affine line, which means $[A, B]$ is the entire group U . Furthermore, $[A, B]$ is clearly not equal to U , for example, there is no integer for which $2n = 1$. Therefore, $[A, B]$ is a proper subset of its closure, which implies that it is not a closed subgroup.

Theorem 2.22. Let G be an algebraic group and let A, B be closed subgroups of G . If A is also connected, then the commutator subgroup $[A, B]$ is a closed and connected subgroup of G .

We note that Proposition 17.2 in [17] provides an algebro-geometric approach for the proof, which concerns subgroups generated by families of irreducible varieties. Moreover, the algebraic approach relies on the result [17, Proposition 7.5] that if $\{W_i\}_{i \in I}$ is a family of irreducible subvarieties of G , and each of which contains the identity element $e \in G$, then the subgroup generated by this family is a closed and connected subgroup of G .

Definition 2.23. For an algebraic group G , the Lie algebra $\mathfrak{g}$ is the tangent space to the variety G at the identity element e , which is endowed with a bilinear, anti-symmetric map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the Jacobi identity.

The *derived series* of a group G is defined inductively by

- $\mathfrak{D}^0(G) = G$,
- $\mathfrak{D}^{i+1}(G) = [\mathfrak{D}^i(G), \mathfrak{D}^i(G)]$,

where $[-, -]$ denotes the commutator group. If G is an algebraic group, $\mathfrak{D}^1(G) = [G, G]$ is a closed subgroup of G , which is connected if G is [17, Proposition 17.2]. We can show the result holds for all $\mathfrak{D}^i(G)$ by induction.

The *lower central series* of a group G is defined inductively by $\mathfrak{C}^0(G) = G$, $\mathfrak{C}^{i+1}(G) = [G, \mathfrak{C}^i(G)]$. Similarly each term in the lower central series is a closed subgroup when G is an algebraic group [17, Proposition 17.2]. Moreover, note that $\mathfrak{C}^{k+1}(G) = [G, \mathfrak{C}^k(G)]$, which is generated by commutators of the form $[x, y]$, where $x \in G$ and $y \in \mathfrak{C}^k(G)$. Then by induction we can show that each $\mathfrak{C}^k(G)$ is normal in G .

Definition 2.24. A group G is *solvable* if and only if the derived series terminates at the trivial group, i.e. there exists some integer k such that $\mathfrak{D}^k(G) = \{e\}$. A group G is *nilpotent* if and only if $\mathfrak{C}^n(G) = \{e\}$.

Proposition 2.25. An algebraic group G is solvable if and only if there exists a chain $G = G_0 \supset G_1 \supset \dots \supset G_n = \{e\}$ of closed subgroups, where $[G_i, G_i] \subset G_{i+1}$ for $0 \leq i < n$.

Proof. ($\Leftarrow$) Suppose such a chain of closed subgroups of G exists. Let $\mathfrak{D}^k(G)$ denote the k -th term in the derived series of G . We will show by induction that $\mathfrak{D}^k(G) \subseteq G_k$ for all $k \geq 0$. For $k = 0$, we have $\mathfrak{D}^0(G) = G = G_0$, so the inclusion holds. Assume $\mathfrak{D}^k(G) \subseteq G_k$ for some $k \geq 0$. Then for $k + 1$, we obtain

$$\mathfrak{D}^{k+1}(G) = [\mathfrak{D}^k(G), \mathfrak{D}^k(G)] \subseteq [G_k, G_k] \subseteq G_{k+1}.$$

Thus the inclusion $\mathfrak{D}^{k+1}(G) \subseteq G_{k+1}$ holds. By induction, we conclude that $\mathfrak{D}^k(G) \subseteq G_k$ for all $k = 0, 1, \dots, n$, and for $k = n$, we have $\mathfrak{D}^n(G) \subseteq G_n = \{e\}$. Hence G is solvable

($\Rightarrow$) Conversely, we assume that G is solvable. Define

$$G_i := \mathfrak{D}^i(G) \text{ for } 0 \leq i \leq n.$$

By definition $G_0 = G$ is a closed subgroup. Assume $G_i = \mathfrak{D}^i(G)$ is a closed subgroup, which is automatically an algebraic group. Then the commutator subgroup $G_{i+1} = [G_i, G_i]$ is also a closed subgroup [17]. Therefore, we conclude that every term in the derived series is a closed subgroup of G . Furthermore, since G is solvable, $G_n = \mathfrak{D}^n(G) = \{e\}$ for some n . Hence there indeed exists such a chain of closed subgroups satisfying $[G_i, G_i] \subseteq G_{i+1}$. $\square$

Remark 2.26. The above proposition can be rephrased to give an analogous criterion for G to be nilpotent, that is, an algebraic group G is *nilpotent* if and only if there exists a finite chain of closed subgroups

$$G = G_0 \supset G_1 \supset \dots \supset G_n = \{e\}$$

such that $[G, G_i] \subseteq G_{i+1}$ for all $0 \leq i < n$.

Moreover, since $[G_i, G_i] \subseteq [G, G_i]$, every nilpotent group is solvable. $\triangle$

Proposition 2.27. *Every algebraic group G contains a unique maximal connected solvable normal subgroup, which is closed.*

Proof. Let $\mathcal{S}$ be the set of all closed, connected, solvable, normal subgroups of G , namely,

$$\mathcal{S} := \{H \leq G \mid H \text{ is closed, connected, solvable, and normal}\}.$$

Note that the trivial subgroup $\{e\} \in \mathcal{S}$, so this set is non-empty. Since the dimension of any subgroup in $\mathcal{S}$ is an integer between 0 and $\dim(G)$, Zorn's lemma implies that there exists an element $R \in \mathcal{S}$ whose dimension is maximal among all elements of $\mathcal{S}$. We now prove that $R \in \mathcal{S}$ is the unique maximal element with respect to inclusion.

Let $S \subseteq \mathcal{S}$ be any other subgroup. Consider $RS := \{rs \mid r \in R, s \in S\}$, we claim that $\overline{RS} \in \mathcal{S}$. We note that RS is normal because R and S are normal. According to Lemma 17.3(c) of [17], it is also solvable. The connectedness of RS follows from the fact that it is the image of the connected variety $R \times S$ under the multiplication morphism, and the closure $\overline{RS}$ inherits these properties [17, Corollary 7.4]. Thus $\overline{RS} \in \mathcal{S}$.

By construction, we have the inclusions $R \subseteq RS \subseteq \overline{RS}$, which implies that $\dim(R) \leq \dim(\overline{RS})$. But R is an element of $\mathcal{S}$ with maximal dimension, so we must have $\dim(R) = \dim(\overline{RS})$. Note that both R and $\overline{RS}$ are connected, hence they are irreducible. Since $R \subseteq \overline{RS}$ is an inclusion of irreducible closed varieties of the same dimension, it follows that $R = \overline{RS}$. This then implies that $S \subseteq RS \subseteq \overline{RS} = R$. Since S was an arbitrary element of $\mathcal{S}$, we have shown that R contains every element of $\mathcal{S}$. Moreover, the uniqueness is an immediate consequence of the maximality. $\square$

Definition 2.28. The identity component of such the maximal connected solvable normal subgroup of G is called the *radical* of G , and is denoted $\text{Rad}(G)$. A linear algebraic group T is called a *torus* if it is isomorphic to $(\mathbb{C}^\times)^n$. A connected linear algebraic group G is *semisimple* if its radical is trivial.

Definition 2.29. A linear algebraic group G is called

- (i) *reductive* if the radical of G is a torus,
- (ii) *linearly reductive* if every rational representation of G is completely reducible,
- (iii) *geometrically reductive* if for every finite-dimensional rational representation $\rho : G \rightarrow \text{GL}(V)$ and every non-zero vector $v \in V^G$, there exists a homogeneous G -invariant polynomial $f \in \mathbb{K}[V]^G$ of positive degree such that $f(v) \neq 0$.

In a field $\mathbb{K}$ of characteristic $p = 0$, (i) and (ii) of the above definition have been known to be equivalent [27, Appendix to Chapter 1]. Moreover, given that we are working on an arbitrary field $\mathbb{K}$ with $p = 0$ throughout the thesis, all three properties are in fact equivalent [27, Theorem A.1.0].

Theorem 2.30 (Haboush's Theorem [13]). *Let G be a linear algebraic group. Then G is reductive if and only if it is geometrically reductive.*

Proposition 2.31. *A finite group G over a field of characteristic $p > 0$ is linearly reductive if and only if its order is prime to p . Moreover, any finite group is always geometrically reductive.*

Proof. We note that Nagata [28] proved that in characteristic $p > 0$, the only linearly reductive groups are extensions of a torus by a finite group of order that is prime to p . So we now prove that any finite group G is geometrically reductive, regardless of the characteristic of the field.

Let $\rho : G \hookrightarrow \mathrm{GL}(V)$ be a finite-dimensional rational representation of G . Let $v \in V^G$ be a non-zero vector that is fixed by every element of G , i.e. $\rho(g)(v) = v$ for all $g \in G$. By definition, we need to find a homogeneous, G -invariant polynomial $f \in \mathbb{K}[V]^G$ of positive degree such that $f(v) \neq 0$.

Since $v \neq 0$, the set $\{v\}$ is linearly independent, which can be extended to form a basis for the entire vector space V . Without loss of generality, let $\{v, u_2, u_3, \dots, u_n\}$ be a basis for V . Define a linear functional $\lambda \in V^*$ on this basis such that $\lambda(v) = 1$, and $\lambda(u_i) = 0$ for all other basis vectors u_i . Note that an arbitrary vector $x \in V$ can be written as $x = c_1 v + c_2 u_2 + \dots + c_n u_n$ for some scalars $c_1, c_2, \dots, c_n \in \mathbb{K}$, hence we have $\lambda(x) = c_1 \cdot 1 = c_1$. In particular, the linear functional λ is a homogeneous polynomial of degree 1.

Consider the action of G on the dual space V^* defined as

$$(g \cdot \lambda)(v) = \lambda(\rho(g^{-1})(v)) = \lambda(g^{-1} \cdot v),$$

which is a well-defined left G -action, and subsequently, it extends to the coordinate ring $\mathbb{K}[V]$. We can now define a polynomial $f \in \mathbb{K}[V]$ by taking the product over G , namely,

$$f(x) = \prod_{g \in G} (g \cdot \lambda)(x) = \prod_{g \in G} \lambda(g^{-1} \cdot x).$$

The product f is a homogeneous polynomial of degree $|G|$ since each $(g \cdot \lambda)$ is a linear functional that is a homogeneous polynomial of degree 1. Because G is a non-trivial finite group, its order $|G|$ is finite and is a positive integer, and so f has a positive degree.

Moreover, for any element $h \in G$, we have

$$(h \cdot f)(x) = \prod_{g \in G} \lambda(g^{-1} \cdot (h^{-1} \cdot x)) = \prod_{g \in G} \lambda((hg)^{-1} \cdot x).$$

Let $k = hg$. Since left multiplication by h is a bijection from G to itself, as g was arbitrary, k denotes any element in G . Reindexing the product yields

$$\prod_{k \in G} \lambda(k^{-1} \cdot x) = f(x).$$

This means that f is G -invariant, i.e. $f \in \mathbb{K}[V]^G$. Finally, we evaluate f at the G -invariant vector v , which gives

$$f(v) = \prod_{g \in G} \lambda(g^{-1} \cdot v) = \prod_{g \in G} \lambda(v) = 1^{|G|} = 1.$$

Thus we have constructed a polynomial f which is homogeneous, G -invariant, of positive degree, and does not vanish at v . Therefore, we conclude any finite group G is geometrically reductive. $\square$

Remark 2.32. Note that in the above proof, we assumed that the finite group G is non-trivial. This is because the proposition is clearly true when $G = \{e\}$, which means the order $|G| = 1$, so the order of G is prime to any p . Thus G is linearly reductive. Furthermore, any non-zero invariant vector v can be separated by a linear functional λ , which itself is G -invariant (as $e \cdot \lambda = \lambda$, homogeneous of positive degree 1, and non-vanishing at v). Thus the trivial group is also geometrically reductive. $\triangle$

Note that a complex linear algebraic group is an affine variety G over $\mathbb{C}$ equipped with group operations that are morphisms of varieties. We have also shown, in Proposition 2.19, that any such group is isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathbb{C})$ for some n .

Since the determinant function $\det : M_n(\mathbb{C}) \rightarrow \mathbb{C}$ is a polynomial in the matrix entries, it is continuous with respect to the standard topology on $\mathbb{C}^{n^2}$. The preimage of the closed set $\{0\}$ is the set of non-invertible matrices, which is thus closed. Consequently, $\mathrm{GL}_n(\mathbb{C})$ is an open subset of the space $M_n(\mathbb{C})$. Hence $\mathrm{GL}_n(\mathbb{C})$ is a complex manifold, which means it is a complex Lie group.

Assume G is the zero locus of a set of polynomials $\{P_1, \dots, P_k\}$, and define a map

$$\Phi : \mathrm{GL}_n(\mathbb{C}) \rightarrow \mathbb{C}^k, \quad \Phi(X) = (P_1(X), \dots, P_k(X)).$$

Since each P_i is a polynomial, Φ is a holomorphic map, and by definition, we have $G = \Phi^{-1}(0)$. Since any algebraic group (over a field of characteristic 0) is a smooth variety [26, Theorem 3.23], the rank of the Jacobian matrix is constant over the entire group. Hence the closed subset $G \subset \mathrm{GL}_n(\mathbb{C})$ is smooth, which implies that the Jacobian $d\Phi_X$ has constant rank for all $X \in G$. By the implicit function theorem, which states that if the Jacobian matrix $d\Phi_X$ of Φ has constant rank in a neighbourhood of a point $X \in G$, then G is a complex submanifold near X , we can conclude that G is a complex submanifold of the complex Lie group $\mathrm{GL}_n(\mathbb{C})$.

Moreover, it is clear to see that the operations of multiplication and inversion of G are holomorphic. Therefore, we have established that any complex linear algebraic group is naturally a complex Lie group.

Proposition 2.33. *Let G be a connected complex linear algebraic group. Then the following conditions are equivalent:*

- (i) G is reductive (i.e. its radical $\mathrm{Rad}(G)$ is a torus).
- (ii) G is isomorphic to the complexification of a maximal compact subgroup $K \subset G$.

Proof. Suppose that G is reductive, then its radical $\mathrm{Rad}(G)$ is a torus $T \cong (\mathbb{C}^\times)^n$, which is the largest connected, solvable, normal subgroup of G . As a complex Lie group, G also has a maximal compact subgroup K . Let $\mathfrak{g}$ and $\mathfrak{k}$ be the respective Lie algebras of G and K , where $\mathfrak{k}$ is a real Lie subalgebra of $\mathfrak{g}$. Note that we can also view $\mathfrak{g}$ as a real Lie algebra of twice the dimension.

Since G is a reductive complex Lie group, its Lie algebra $\mathfrak{g}$ admits a Cartan involution $\theta : \mathfrak{g} \rightarrow \mathfrak{g}$, which is a $\mathbb{R}$ -linear Lie algebra automorphism of the underlying real Lie algebra structure such that $\theta^2 = \mathrm{id}$ [21, Section 2 Chapter VII]. The subalgebra of fixed points of θ is the Lie algebra of a maximal compact subgroup (see [21, Prop. 6.16, Cor. 6.22 & Chap. VII Section 2] for more details). Thus we can choose θ such that $\mathfrak{k} = \{X \in \mathfrak{g} \mid \theta(X) = X\}$.

The Cartan decomposition of $\mathfrak{g}$ with respect to θ is

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the (+1)-eigenspace and $\mathfrak{p} = \{X \in \mathfrak{g} \mid \theta(X) = -X\}$ is the (-1)-eigenspace. So we have $\theta(iX) = -i\theta(X)$ for any $X \in \mathfrak{g}$. Furthermore, because we have $\theta(iX) = -iX$ for $X \in \mathfrak{k}$, and $\theta(iX) = -i(-X) = iX$ for $X \in \mathfrak{p}$, we have an $\mathbb{R}$ -linear isomorphism $X \mapsto iX$ from $\mathfrak{k}$ to $\mathfrak{p}$. This implies that $\mathfrak{p}$ can be written as $i\mathfrak{k}$, consequently, the decomposition of $\mathfrak{g}$ becomes

$$\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}.$$

Hence we have precisely obtained the structure of the complexification of the real Lie algebra $\mathfrak{k}$.

Consider the inclusion map $K \hookrightarrow G$ induces a homomorphism $\phi : K_{\mathbb{C}} \rightarrow G$, where $K_{\mathbb{C}}$ is the complexification of K [21, Proposition 7.5]. Then its differential $d\phi : \mathrm{Lie}(K_{\mathbb{C}}) \rightarrow \mathrm{Lie}(G)$ is the isomorphism established above, thus $G \cong K_{\mathbb{C}}$.

Conversely, suppose that G is the complexification of a compact Lie group K . Then the complex Lie algebra $\mathfrak{g} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{k} \oplus i\mathfrak{k}$. Since K is compact, its real Lie algebra $\mathfrak{k}$ is reductive, and in particular, we have $\mathfrak{k} = Z\mathfrak{k} \oplus [\mathfrak{k}, \mathfrak{k}]$, where $Z\mathfrak{k}$ denotes the centre of $\mathfrak{k}$ and $[\mathfrak{k}, \mathfrak{k}]$ is semisimple [21, Theorem 1.54 & Corollary 4.25].

Note that $\mathfrak{g} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$, we can complexify $\mathfrak{k}$ as

$$\mathfrak{g} \cong (Z\mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}) \oplus ([\mathfrak{k}, \mathfrak{k}] \otimes_{\mathbb{R}} \mathbb{C}),$$

which is the direct sum of a complex abelian ideal $\mathfrak{a} := Z\mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ and a complex semisimple ideal $\mathfrak{s} := [\mathfrak{k}, \mathfrak{k}] \otimes_{\mathbb{R}} \mathbb{C}$. Since $\mathfrak{s}$ has no non-zero solvable ideals, the radical $\mathrm{rad}(\mathfrak{g}) \subset \mathfrak{a}$. Since the ideal $\mathfrak{a}$ is abelian, it is solvable, and thus we have $\mathrm{rad}(\mathfrak{g}) = \mathfrak{a}$. Moreover, $[\mathfrak{a}, \mathfrak{s}] = 0$ implies that $\mathrm{rad}(\mathfrak{g}) = Z(\mathfrak{g})$. Therefore, we have showed that $\mathfrak{g}$ is a reductive complex Lie algebra.

Corollary 7.11 in [21] implies that G can be written as the commuting product $G = (Z(G))^\circ G_{ss}$, where $(Z(G))^\circ$ denotes the identity component of the centre of G , and G_{ss} is the closed analytic subgroup with the Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Since G_{ss} is semisimple, any connected normal solvable subgroup of G lies in $(Z(G))^\circ$, which is solvable. Hence we obtain $\text{Rad}(G) = (Z(G))^\circ$.

Now consider the Cartan involution θ such that $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$, it follows from Proposition 7.21 in [21] that

$$Z\mathfrak{g} = (Z\mathfrak{g} \cap \mathfrak{k}) \oplus (Z\mathfrak{g} \cap i\mathfrak{k}) \quad \text{and} \quad (Z(G))^\circ = \exp(Z\mathfrak{g} \cap \mathfrak{k}) \exp(Z\mathfrak{g} \cap i\mathfrak{k}).$$

Let $T := \exp(Z\mathfrak{g} \cap \mathfrak{k}) = (Z(K))^\circ$, a compact torus in K . Since the identity component $(Z(G))^\circ$ is abelian, the map

$$T \times (Z\mathfrak{g} \cap i\mathfrak{k}) \rightarrow (Z(G))^\circ, \quad (t, X) \mapsto t \exp X,$$

is a Lie group isomorphism. In particular, we obtain

$$(Z(G))^\circ = \exp(Z\mathfrak{k}) \exp(iZ\mathfrak{k}).$$

This means that $(Z(G))^\circ$ is the complexification of the compact torus $T = (Z(K))^\circ$. Hence we have established that $\text{Rad}(G)$ is a complex torus. $\square$

We now have all the ingredients for the statement concerning the finite generation of invariant rings.

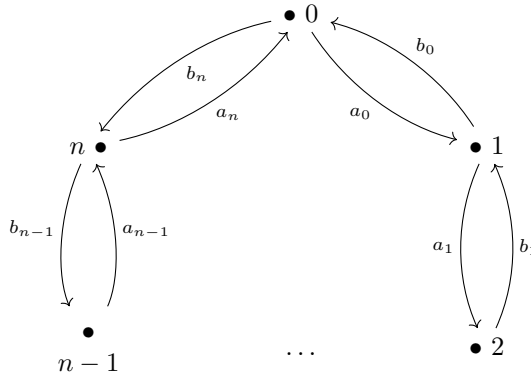
Theorem 2.34 (Nagata [28]). *Let G be a geometrically reductive group over an algebraically closed field $\mathbb{K}$, and let A be a finitely generated commutative $\mathbb{K}$ -algebra upon which G acts rationally by $\mathbb{K}$ -algebra automorphisms. Then the ring of invariants, A^G , is a finitely generated $\mathbb{K}$ -algebra.*

3 Geometric Invariant Theory

Geometric invariant theory emerges from the problem of constructing quotient spaces for an algebraic variety X under the action of a reductive linear algebraic group G , where the group action induces an equivalence relation on the points of X . A primary obstruction to this construction is that the topological quotient X/G may not exist as an algebraic variety. This is because a necessary condition for the canonical projection $\pi : X \rightarrow X/G$ to be a morphism of varieties is that its fibres, namely the orbits of the G -action, must be closed in the Zariski topology. However, when G is noncompact, there exist nonclosed orbits, which implies that a non-trivial G -action is not proper and the quotient is not Hausdorff.

To surmount this difficulty, GIT proposes an approach by considering an appropriate G -invariant Zariski-open subset $U \subseteq X$ so that U/G exists as a quasi-projective variety. This section concerns the construction of GIT quotients, denoted $X // G$, which resolves the problem of non-closed orbits when forming such quotients. We assume throughout the remaining sections that $\mathbb{K}$ is an arbitrary algebraically closed field of characteristic zero. The main references of this section are [9], [11], [16], and [27].

To illustrate how GIT deals with a space with non-closed orbits, we consider a non-trivial quiver variety arising from the McKay correspondence [5]. Fix the finite cyclic subgroup $\Gamma = C_{n+1} \subset \text{SL}_2(\mathbb{C})$ with $n \geq 1$, that is generated by $\zeta = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$ with $\omega^{n+1} = 1$. In other words, ω is a primitive $(n+1)$ -th root of unity. Then the extended Dynkin diagram $\tilde{\Delta}(\Gamma)$ is of type $\tilde{A}_n$ [5], and its McKay quiver (I, A) has vertices $I = \{0, 1, \dots, n\}$ and arrows $a_i : i \rightarrow i+1$, $b_i : i+1 \rightarrow i$.



Choose the dimension vector $\mathbf{d} = (1, 1, \dots, 1)$. The space of representations of (I, A) of dimension $\mathbf{d}$ is

$$M(\Gamma) = \text{Rep}((I, A), \mathbf{d}) = \bigoplus_{i=0}^n (\text{Hom}(\mathbb{C}, \mathbb{C})_{a_i} \oplus \text{Hom}(\mathbb{C}, \mathbb{C})_{b_i}) \cong \mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1},$$

which is the same as the set $\{(a_0, \dots, a_n, b_0, \dots, b_n) \mid a_i, b_i \in \mathbb{C}\}$. The change of basis action of $G(\Gamma) = (\mathbb{C}^\times)^{n+1} / \mathbb{C}^\times \cong (\mathbb{C}^\times)^n$ on $M(\Gamma)$ is given by

$$t \cdot (a_0, \dots, a_n, b_0, \dots, b_n) = (t_1 t_0^{-1} a_0, \dots, t_0 t_n^{-1} a_n, t_0 t_1^{-1} b_0, \dots, t_n t_0^{-1} b_n),$$

where $t = (t_0, \dots, t_n)$. We note that the group $G(\Gamma)$ is reductive since it is by construction a torus, this implies the existence of the affine quotient $M(\Gamma) // G(\Gamma) = \text{Spec } \mathbb{C}[M(\Gamma)]^{G(\Gamma)}$.

We now compute the ring $\mathbb{C}[M(\Gamma)]^{G(\Gamma)}$ of invariants under the $G(\Gamma)$ -action. Observe from the structure of the quiver action, the obvious invariant polynomials correspond to cycles, i.e. for each vertex $i = 0, \dots, n$, we consider the loop $i \rightarrow i+1 \rightarrow i$. We claim that the ring of invariants is generated by

$$z_i := a_i b_i, \quad x := \prod_{i=0}^n a_i, \quad \text{and } y := \prod_{i=0}^n b_i, \quad (3.1)$$

satisfying $z_0 z_1 \dots z_n = xy$. Note that $z_i \mapsto (t_{i+1} t_i^{-1} a_i)(t_i t_{i+1}^{-1} b_i)$, and the coordinates are scalars in $\mathbb{C}$, we yield $(t_{i+1} t_i^{-1})(t_i t_{i+1}^{-1})(a_i b_i) = a_i b_i$. Similar arguments hold for x and y , hence the polynomials given in (3.1) are indeed invariant. Therefore,

$$\mathbb{C}[M(\Gamma)]^{G(\Gamma)} \cong \frac{\mathbb{C}[z_0, \dots, z_n, x, y]}{\langle z_0 z_1 \dots z_n - xy \rangle},$$

and the spectrum of which is precisely the affine variety defined by the vanishing of the ideal $\langle z_0 z_1 \dots z_n - xy \rangle$. Thus the GIT quotient $M(\Gamma) // G(\Gamma)$ is given by the hypersurface $\{(z_0, \dots, z_n, x, y) \in \mathbb{C}^{n+3} \mid z_0 z_1 \dots z_n = xy\}$, namely

$$M(\Gamma) // G(\Gamma) \cong \{(z_0, \dots, z_n, x, y) \in \mathbb{C}^{n+3} \mid z_0 z_1 \dots z_n = xy\}.$$

Now we consider the quotient of a subvariety of $M(\Gamma)$, namely the zero level of the complex moment map $\mu_{\mathbb{C}} : M(\Gamma) \rightarrow \text{Lie}(G(\Gamma))^* \cong \mathbb{C}^{n+1}$. This approach is motivated by the Kempf-Ness theorem, which asserts that the GIT quotient of the semistable locus is homeomorphic to the symplectic quotient of the zero level set of the moment map. See also Kirwan [20] for the symplectic and Morse-theoretic perspective on such quotients.

Note that for the $\tilde{A}_n$ quiver, each vertex $i \in I$ has the relation $a_i b_i - b_{i-1} a_{i-1} = 0$. The zero level set $\mu^{-1}(0)$ is then defined as

$$\mu_{\mathbb{C}}^{-1}(0) = \{(a, b) \in \mathbb{C}^{2(n+1)} \mid a_i b_i = b_{i-1} a_{i-1}\}.$$

The moment map condition imposed to the relations in (3.1) yields $z_i = z_{i-1}$ for all $i \in \{0, \dots, n\}$, which implies that $z_0 = z_1 = \dots = z_n$. Let $z := z_i$ be a new coordinate. It follows that

$$xy = \prod_{i=0}^n (a_i b_i) = \prod_{i=0}^n z_i = z^{n+1}.$$

Hence the coordinate ring of the quotient is

$$\mathbb{C}[\mu_{\mathbb{C}}^{-1}(0)]^{G(\Gamma)} \cong \frac{\mathbb{C}[x, y, z]}{\langle xy - z^{n+1} \rangle}.$$

Therefore, we conclude that

$$\mu_{\mathbb{C}}^{-1}(0) // G \cong \text{Spec } \frac{\mathbb{C}[x, y, z]}{\langle xy - z^{n+1} \rangle} \cong \mathbb{C}^2 / \mathbb{Z}_{n+1}.$$

An issue that may arise in forming quotient spaces is that not all orbits of the group action are closed sets in the Zariski topology. In the above quiver example, if we take a point $p \in M(\Gamma)$ where $a_0 = 1$

with all other $a_i = 0$ and all $b_i = 0$. Then consider the action of an element $t(\epsilon) = (t_0, t_1, \dots, t_n) = (1, \epsilon, \dots, \epsilon) \in (\mathbb{C}^\times)^{n+1}$ for some $\epsilon \in \mathbb{C}^\times$. The action of a_0 is

$$t(\epsilon) \cdot a_0 = t_1 t_0^{-1} a_0 = \epsilon \cdot 1^{-1} \cdot 1 = \epsilon,$$

which implies that $\lim_{\epsilon \rightarrow 0} (t(\epsilon) \cdot a_0) = 0$. Since all other components are 0 by construction, it follows that the entire representation approaches the zero representation. This means it is a closed orbit that lies in the closure of the orbit of p , and so the orbit of p is not closed.

The GIT technique involved in the example shifts focus from the points and orbits themselves to the ring of invariant polynomials. The values of the invariant polynomials take on the non-closed orbit of our chosen point p are $z_i = a_i b_i = 0$, $x = \prod a_i = 0$, and $y = \prod b_i = 0$. Now consider the closed orbit at the origin $q = (0, \dots, 0)$, the invariant polynomials are $z_i = 0$, $x = 0$, $y = 0$. Since a point in the GIT quotient corresponds to a set of values that the invariant generators can take, it follows that the points p and q are mapped to the point $(z_0, \dots, z_n, x, y) = (0, \dots, 0)$ in the GIT quotient $M(\Gamma) // G(\Gamma)$.

Therefore, in this quiver example, affine GIT resolves non-closed orbits by identifying each with the closed orbit in its closure. The quotient map $\pi : M(\Gamma) \rightarrow M(\Gamma) // G(\Gamma)$ sends every point in the space to the unique closed orbit contained within its orbit closure, i.e. π has the property that every fibre contains a unique closed $G(\Gamma)$ -orbit. In particular, for any $p, q \in M(\Gamma)$, we have

$$\pi(p) = \pi(q) \iff \overline{G(\Gamma) \cdot p} \cap \overline{G(\Gamma) \cdot q} \neq \emptyset.$$

Equivalently, all invariant polynomials take the same values at p and q .

3.1 Affine GIT Quotients

We start by discussing the construction of the prime spectrum, which is the procedure of associating a geometric object to a commutative ring.

Definition 3.1. Let A be a commutative ring. The *spectrum* of A , denoted $\text{Spec}(A)$, is the set of all prime ideals of A .

We may endow $\text{Spec}(A)$ with the Zariski topology. Specifically, for any ideal $I \subseteq A$, we define the set $V(I) \subseteq \text{Spec}(A)$ by

$$V(I) := \{\mathfrak{p} \in \text{Spec}(A) \mid I \subseteq \mathfrak{p}\}.$$

The sets $V(I)$ for all ideals I of A form the closed sets of the Zariski topology on $\text{Spec}(A)$.

Let X be an affine variety with coordinate ring $\mathbb{K}[X]$. By Hilbert's Nullstellensatz (see [2] for more details), the maximal ideals of $\mathbb{K}[X]$ are in one-to-one correspondence with the points of X . Since every maximal ideal is prime, we have a natural inclusion of sets $X \hookrightarrow \text{Spec}(\mathbb{K}[X])$. In this sense, $\text{Spec}(\mathbb{K}[X])$ can be seen as a geometric object which contains the classical variety X as the subset of its closed points. The non-maximal prime ideals correspond to the irreducible subvarieties of X (see Chapter 1, Section 1 of Hartshorne's book [14] for more details).

Let G be a complex reductive algebraic group acting rationally on an affine space V over $\mathbb{K}$. This means that the action is a morphism $a : G \times V \rightarrow V$. The induced action on the coordinate ring $\mathbb{K}[V]$ by $\mathbb{K}$ -algebra automorphisms, which is given by

$$g \cdot f(v) = f(a(g^{-1} \cdot v)) = f(g^{-1} \cdot v)$$

for every $f \in \mathbb{K}[V]$, $g \in G$ and $v \in V$.

Definition 3.2. The *ring of invariants* is the subalgebra of $\mathbb{K}[X]$ consisting of all functions fixed by the action

$$\mathbb{K}[X]^G := \{f \in \mathbb{K}[X] \mid g \cdot f = f \text{ for all } g \in G\}.$$

As established in Theorem 2.34, since the group G is reductive, the ring of invariants $\mathbb{K}[V]^G$ is finitely generated as a $\mathbb{K}$ -algebra. This guarantees that it is the coordinate ring of some affine variety.

Definition 3.3. The *affine GIT quotient* of V by G is the affine variety whose coordinate ring is the ring of invariants. We denote it by $V // G$.

$$V // G := \text{Spec}(\mathbb{K}[V]^G).$$

Furthermore, the natural inclusion of rings $\pi^* : \mathbb{K}[V]^G \hookrightarrow \mathbb{K}[V]$ is a $\mathbb{K}$ -algebra homomorphism. Then it induces a morphism $\pi : V \rightarrow V // G$ between the affine varieties, which is called the *quotient morphism*.

Definition 3.4. Let a group G act on an algebraic variety X . A *categorical quotient* of X by G is a pair (Y, ϕ) , where Y is a variety and $\phi : X \rightarrow Y$ is a G -invariant morphism, such that it satisfies the universal property, i.e. for any other G -invariant morphism $\psi : X \rightarrow Z$, there exists a unique morphism $\eta : Y \rightarrow Z$ satisfying $\psi = \eta \circ \phi$.

Theorem 3.5. *The quotient morphism $\pi : V \rightarrow V // G$ has the following properties:*

- (i) π is constant on G -orbits, i.e. π is a G -invariant morphism.
- (ii) π is surjective.
- (iii) If $\phi : X \rightarrow Y$ is any morphism of affine varieties that is constant on G -orbits, then there exists a unique morphism $\psi : X // G \rightarrow Y$ such that $\phi = \psi \circ \pi$.
- (iv) π separates disjoint closed G -invariant subsets of V . That is, if $Z_1, Z_2 \subset X$ are closed, G -invariant, and the intersection $Z_1 \cap Z_2 = \emptyset$, then $\overline{\pi(Z_1)} \cap \overline{\pi(Z_2)} = \emptyset$.

For any two points $v, w \in V$, we have $\pi(v) = \pi(w)$ if and only if $\overline{G \cdot v} \cap \overline{G \cdot w} \neq \emptyset$.

Proof. (i) Note that a point in $X // G$ is a maximal ideal of $\mathbb{K}[X]^G$. The point $\pi(x)$ corresponds to the ideal $\mathfrak{m}_x = \{f \in \mathbb{K}[X]^G \mid f(x) = 0\}$. Since we have $f(gx) = (g^{-1} \cdot f)(x) = f(x)$, it follows that $f(x) = 0$ if and only if $f(gx) = 0$, implying $\mathfrak{m}_x = \mathfrak{m}_{gx}$. Hence $\pi(x) = \pi(gx)$.

- (ii) We expand the concise proof of Proposition 6.2 in [9]. To show π is surjective, it suffices to show that π is dominant (i.e. the image of π is dense in $V // G$) and the image of π is closed.

We first prove that π is dominant. Because V is an irreducible affine variety, its coordinate ring $\mathbb{K}[V]$ is an integral domain. So the ring of invariants $\mathbb{K}[V]^G$ is also an integral domain. Consequently, the quotient $V // G$ is also irreducible. Note that π is induced by the natural inclusion $\pi^* : \mathbb{K}[V]^G \hookrightarrow \mathbb{K}[V]$, and π^* is clearly injective, hence π is a dominant morphism (An injective homomorphism between integral domains corresponds to a dominant morphism between their affine varieties [14, Chapter II, Section 2, Exercise 2.18(b)]). Therefore, $\pi(V)$ is a dense subset of $V // G$.

We then prove that $\pi(V)$ is closed in $V // G$. Let Z be any closed, G -invariant subset of V . Assume, for contradiction, that the image $\pi(Z)$ is not closed in $V // G$. Then there exists a point $y \in \overline{\pi(Z)} \setminus \pi(Z)$. Consider the set $Z_1 = \pi^{-1}(y)$, which is closed in V and is also G -invariant as for any $x \in Z_1$, we have $\pi(g \cdot x) = \pi(x) = y$. Note that $Z \cap Z_1 = \emptyset$, since if the intersection were not empty, let x' be an element in the intersection, then $x' \in Z$ with its image $\pi(x') = y$, but this would imply $y \in \pi(Z)$, contradicting the choice of y .

By Lemma 6.1 of [9], there is a G -invariant function $f \in \mathbb{K}[V]^G$ such that $f(Z) = 0$ and $f(Z_1) = 1$. Note that $f = \pi^*(g) = g \circ \pi$, where $g \in \mathbb{K}[V]^G$ is a unique regular function. Then by construction, we obtain $g(\pi(z)) = f(z) = 0$ for all $z \in Z$. At y , we have $g(y) = g(\pi(z')) = f(z') = 1$ for all $z' \in Z_1$. Thus $g(\pi(Z)) = 0$ and $g(y) = 1$. However, g is a regular function so it is also zero on $\pi(Z)$, since we set $y \in \pi(Z)$, it follows that $g(y) = 1$, giving a contradiction.

Therefore, the initial assumption that $\pi(Z)$ is not closed is false, and we conclude that $\pi(Z)$ is closed for any closed G -invariant subset Z .

- (iii) Recall that Proposition 2.18 implies that ϕ induces a pullback homomorphism $\phi^* : \mathbb{K}[Y] \rightarrow \mathbb{K}[X]$. Since ϕ is constant on G -orbits, for any $h \in \mathbb{K}[Y]$, the function $\phi^*(h) = h \circ \phi$ is also constant on orbits, hence it is G -invariant. This means the image of ϕ^* is contained in $\mathbb{K}[X]^G$, and thus ϕ^* can be viewed as a homomorphism $\phi^* : \mathbb{K}[Y] \rightarrow \mathbb{K}[X]^G$. Again by Proposition 2.18, this algebra homomorphism corresponds to the morphism of varieties $\psi : \text{Spec}(\mathbb{K}[X]^G) \rightarrow \text{Spec}(\mathbb{K}[Y])$.

- (iv) Dolgachev [9, Proposition 6.2] provides a detailed proof for this condition. □

Definition 3.6. Let a reductive group G act on an affine variety X . A categorical quotient $\phi : X \rightarrow Y$ is a *good quotient* if it is surjective, G -invariant, and the induced ring homomorphism $\phi^* : \mathbb{K}[Y] \rightarrow \mathbb{K}[X]$ identifies $\mathbb{K}[Y]$ with the invariant ring $\mathbb{K}[X]^G$. Furthermore, it separates disjoint closed G -invariant subsets. That is, if Z_1, Z_2 are closed G -invariant subsets with $Z_1 \cap Z_2 = \emptyset$, then $\phi(Z_1) \cap \phi(Z_2) = \emptyset$.

Remark 3.7. For an affine space V , property (ii) of Theorem 3.5 is equivalent to the statement that the coordinate ring of $V // G$ is the ring of invariants $\mathbb{K}[V]^G$. Moreover, property (iii) of the theorem implies that the morphism $\pi : V \rightarrow V // G$ is indeed the categorical quotient.

Therefore, the affine GIT quotient $V // G$ we have defined is a categorical good quotient. For more general and categorical treatment on the construction of categorical quotient and the property of goodness, we refer Chapter 6 of Dolgachev's book [9]. $\triangle$

Corollary 3.8. *For any two points $v, w \in V$, we have $\pi(v) = \pi(w)$ if and only if $\overline{G \cdot v} \cap \overline{G \cdot w} \neq \emptyset$. Moreover, every fibre of π contains exactly one closed orbit, and thus $V // G$ serves as the space of closed orbits.*

Proof. Suppose $\pi(v) = \pi(w)$. Let $Z_1 = \overline{G \cdot v}$ and $Z_2 = \overline{G \cdot w}$. Both Z_1 and Z_2 are closed, G -invariant subsets of V . Assume, for contradiction, that $Z_1 \cap Z_2 = \emptyset$. By property (iv) of the above theorem, this implies that $\pi(Z_1) \cap \pi(Z_2) = \emptyset$.

However, since π is constant on G -orbits, $\pi(Z_1) = \{\pi(v)\}$ and $\pi(Z_2) = \{\pi(w)\}$. As these are singletons, they are closed sets. Our assumption $\pi(v) = \pi(w)$ means that $\pi(Z_1) = \{\pi(v)\}$ and $\pi(Z_2) = \{\pi(w)\}$, and their intersection is clearly non-empty. Therefore, our initial assumption is false, and we have $Z_1 \cap Z_2 \neq \emptyset$.

Conversely, suppose that $\overline{G \cdot v} \cap \overline{G \cdot w} \neq \emptyset$. Let z be an element in the intersection. Since $z \in \overline{G \cdot v}$, any continuous function on V that is also constant on the orbit has the same value at z . Let $f \in \mathbb{K}[V]^G$. Then f is continuous and constant on G -orbits. Thus $f(z) = f(v)$. Similarly, since $z \in \overline{G \cdot w}$, we have $f(z) = f(w)$. It follows that $f(v) = f(w)$ for all $f \in \mathbb{K}[V]^G$.

Note that the points $\pi(v)$ and $\pi(w)$ in $V // G$ correspond to the maximal ideals $\mathfrak{m}_v = \{f \in \mathbb{K}[V]^G \mid f(v) = 0\}$ and $\mathfrak{m}_w = \{f \in \mathbb{K}[V]^G \mid f(w) = 0\}$. Since $f(v) = f(w)$ for all invariant functions, these two sets of functions are identical. Therefore, $\mathfrak{m}_v = \mathfrak{m}_w$ and hence $\pi(v) = \pi(w)$.

Now we prove the second statement. Let $y \in V // G$ and consider the fibre $\pi^{-1}(y)$, which is non-empty since π is surjective. Let $x \in \pi^{-1}(y)$ be an arbitrary point such that $\pi(x) = y$, then $\overline{G \cdot x}$ contains a unique closed orbit O . Since π is constant on $\overline{G \cdot x}$ with $\pi(x) = y$, it follows that $O \subset \pi^{-1}(y)$. Hence the fibre contains at least one closed orbit.

Suppose O_1 and O_2 are two distinct closed orbits in the same fibre $\pi^{-1}(y)$. Then for any $v_1 \in O_1$ and $v_2 \in O_2$, we have $\pi(v_1) = \pi(v_2) = y$, implying the intersection $\overline{O_1} \cap \overline{O_2} \neq \emptyset$. This means that $O_1 \cap O_2 \neq \emptyset$ since they are closed. Because orbits form a partition of V , we get $O_1 = O_2$, which contradicts the assumption. Therefore, each fibre has only one closed orbit. $\square$

More generally, we have the following theorem.

Theorem 3.9 ([27, Theorem 1.1]). *Let G be a reductive group that acts on an affine scheme X , the affine GIT quotient $X \rightarrow X // G$ is a good quotient and hence a categorical quotient.*

Definition 3.10. Let G act linearly on a vector space V . The *null cone*, $\mathcal{N}$, is the set of points that cannot be distinguished from the origin $0 \in V$ by invariant polynomials. It is the fibre

$$\mathcal{N} := \pi^{-1}(\pi(0)) = \{v \in V \mid f(v) = f(0) \text{ for all } f \in \mathbb{K}[V]^G\}.$$

Equivalently, by the previous corollary, $\mathcal{N} = \{v \in V \mid 0 \in \overline{G \cdot v}\}$. We call the points in the null cone *unstable* points.

An example of the case when the closure of one G -orbit is contained within the closure of another is the action of the special linear group SL_2 on $\mathbb{A}^2$ by matrix multiplication, whose ring of invariant polynomials consists of only the constants as no non-constant polynomial is SL_2 -invariant. Consequently, the affine GIT quotient $\mathbb{A}^2 // \mathrm{SL}_2$ is a single point.

Definition 3.11. Let G be an algebraic group. A *character* $\chi : G \rightarrow \mathbb{G}_m$ is an algebraic group morphism from G to the multiplicative group $\mathbb{G}_m$.

To deal with such collapse of orbits, and hence obtain a well-defined quotient space, we need a non-trivial character $\chi : G \rightarrow \mathbb{G}_m$ to linearise the G -action so that we can restrict our attention to an open subset of well-behaved points. (For a more complete discussion on the linearisation of actions, a reference is Chapter 7 of Dolgachev's book [9], where the Lemma 3.5 shows that a rational G -action on an affine variety X can always be linearised).

Definition 3.12. Let G be a linear algebraic group, and X be a projective variety. Let L be a line bundle on X . A *linearisation* of the G -action on X with respect to L is a G -action $\bar{\varphi} : G \times L \rightarrow L$ such that

(i) the diagram

$$\begin{array}{ccc} G \times L & \xrightarrow{\bar{\varphi}} & L \\ \text{id} \times p \downarrow & & \downarrow p \\ G \times X & \xrightarrow{\varphi} & X \end{array}$$

is commutative, where $p : L \rightarrow X$ is the projection map;

(ii) the zero section $s_0 : X \rightarrow L$ is G -invariant.

For any element $(g, l) \in G \times L$, the commutativity means that $(p \circ \bar{\varphi})(g, l) = (\varphi \circ (\text{id} \times p))(g, l)$, which gives $p(\bar{\varphi}(g, l)) = \varphi(g, p(l))$. Since φ and $\bar{\varphi}$ denote group actions, the equation can be rewritten as $p(g \cdot l) = g \cdot p(l)$. This precisely implies that the projection map p is G -equivariant.

This G -equivariance implies that for any $g \in G$ and $x \in X$, the fibre $L_x : p^{-1}(x)$ over x is mapped to the fibre $L_{g \cdot x}$ over $g \cdot x$ under the G -action. It follows that the action $\bar{\varphi}$ induces a map $\phi : L_x \rightarrow L_{g \cdot x}$ between fibres, which is defined by $l \mapsto g \cdot l$.

Note that $s_0(x) = 0_x$ for all $x \in X$, where 0_x denotes the zero vector in the fibre L_x . Since the zero section s_0 is G -invariant, it follows that $g \cdot s_0(x) = s_0(g \cdot x)$. In terms of the fibre map ϕ , this means $\phi(0_x) = 0_{g \cdot x}$. Since L is a line bundle, every fibre L_x is isomorphic to the one-dimensional space $\mathbb{A}^1$. Because the action $\bar{\varphi}$ is a morphism of varieties, the induced map ϕ is also a morphism between $\mathbb{A}^1$ and itself. Moreover, since $\bar{\varphi}$ is invertible, the map ϕ is given by invertible polynomials, and thus the map ϕ is linear. Hence the G -invariance of the zero section s_0 implies the action $\bar{\varphi}$ is linear on the fibre.

Since we work with an affine space V , the line bundle we are considering is the trivial line bundle $L = V \times \mathbb{K}$. A *linearisation* of the G -action on this trivial bundle is given by a character $\chi : G \rightarrow \mathbb{G}_m$, namely

$$g \cdot (v, x) := (g \cdot v, \chi(g)^{-1}x) \text{ for all } g \in G, v \in V, x \in \mathbb{K}.$$

This means that the character χ lifts the G -action on V to an action on the trivial line bundle $L = V \times \mathbb{K}$. A section of L is then a regular function $f : V \rightarrow \mathbb{K}$ given by $v \mapsto (v, f(v))$. Consequently, the condition for f to be G -invariant with respect to the linearised action is

$$g \cdot (v, f(v)) = (g \cdot v, \chi(g)f(v)) = (g \cdot v, f(g \cdot v)),$$

which yields $f(g \cdot v) = \chi(g)f(v)$. This leads to the following notion of semi-invariant functions.

Definition 3.13. Let $\chi : G \rightarrow \mathbb{G}_m$ be a character. A regular function $f \in \mathbb{K}[V]$ is a *semi-invariant* of weight χ if for all $g \in G$ and $v \in V$, one has

$$f(g \cdot v) = \chi(g)f(v).$$

The space of such functions is denoted by $\mathbb{K}[V]_\chi^G$.

Definition 3.14. Let χ be a character of G . A point $v \in V$ is called

- χ -*semistable* if there exists a semi-invariant function $f \in \mathbb{K}[V]_\chi^{G_n}$ for some integer $n > 0$ such that $f(v) \neq 0$. The set of all χ -semistable points is denoted $V^{ss}(\chi)$.
- χ -*unstable* if it is not χ -semistable, which is equivalent to $f(v) = 0$ for all semi-invariants $f \in \mathbb{K}[V]_\chi^{G_n}$ for some integer $n > 0$.
- χ -*stable* if it is χ -semistable, its stabiliser G_v is finite (i.e. $\dim G_v = 0$), and its orbit $G \cdot v$ is closed in the semistable locus $V^{ss}(\chi)$.

Proposition 3.15 (Topological Criterion of Semistability [16, Lemma 2.4]). *Let L^{-1} denote the dual linearisation, which is the trivial line bundle $V \times \mathbb{K}$ with the G -action $g \cdot (v, x) = (g \cdot v, \chi(g)^{-1}x)$. Let $\tilde{v} = (v, x) \in L^{-1}$ be a lifted point of v with $x \neq 0$. Then*

- *v is χ -semistable if and only if the orbit closure $\overline{G \cdot \tilde{v}}$ of $\tilde{v}$ has an empty intersection with the zero section $V \times \{0\} \subset L^{-1}$.*
- *v is χ -stable if and only if the orbit $G \cdot \tilde{v}$ is closed in L^{-1} and the dimension of the stabiliser $\dim G_v = 0$.*

Proof. The proof follows the arguments of the original projective version (see [19, Lemma 2.2] and [27, Proposition 2.2]) in a similar way. $\square$

Example 3.16. Recall the affine GIT quotient $M(\Gamma) // G(\Gamma)$ that we constructed at the beginning, where we have the space of representations $V = M(\Gamma) \cong \mathbb{C}^{2n+2}$, whose points are tuples of scalars $(a_0, b_0, \dots, a_n, b_n)$. The reductive algebraic group that acts on V is $G = G(\Gamma) \cong (\mathbb{G}_m)^n$. Note also that the notion of (semi)stability depends on a character $\chi : G \rightarrow \mathbb{G}_m$, and the character in the example is trivial, i.e. $\chi = 1$. So we need to choose another non-trivial character of $G(\Gamma)$.

For our torus G , any character is determined by a vector $m = (m_0, \dots, m_n)$ with $\sum m_i = 0$. Let's fix the character χ defined by $m = (-1, 1, 0, \dots, 0)$. For any element $t = (t_0, \dots, t_n) \in G$, the character is given by

$$\chi(t) = t_0^{-1} t_1.$$

From the above topological criterion, we can deduce that a point $v \in V$ is χ -unstable if and only if the closure $\overline{G \cdot v}$ intersects the zero section. So to show that a point v is unstable, we need to find a sequence of group elements $g_k \in G$ such that the limit of the lifted point $g_k \cdot (v, 1)$ exists and is a point of the form $(v', 0)$. This requires that the sequence $g_k \cdot v$ converges to some point $v' \in V$, and the sequence $\chi(g_k)^{-1}$ converges to 0, which is equivalent to that $|\chi(g_k)| \rightarrow \infty$.

Let $\lambda(t) = (t^{r_0}, \dots, t^{r_n})$ be a 1-PS, where r_i are integers with $\sum r_i = 0$. Then $\chi(\lambda(t)) = (t^{r_0})^{-1} t^{r_1} = t^{r_1 - r_0}$. For $|\chi(\lambda(t))| \rightarrow \infty$ as $t \rightarrow 0$, the weight $r_1 - r_0$ needs to be negative. This implies that we need to find a vector r such that $r_1 - r_0 < 0$. A possible choice is $r = (n, -1, \dots, -1)$, which gives $r_1 - r_0 = -1 - n = -(n+1) < 0$.

Recall that the action of $\lambda(t)$ on the coordinates of $V = M(\Gamma)$ involves multiplying them by powers of t . Specifically, the action on a_i has weight $r_{i+1} - r_i$ and the action on b_i has weight $r_i - r_{i+1}$. So for $r = (n, -1, \dots, -1)$, we have negative weights for both a_0 and b_n , and non-negative weights for all other a_i and b_i . Let the point $v = (0, 0, 1, 0, \dots, 0)$, i.e. it has $a_1 = 1$ and all other coordinates are zero.

By definition, for the lifted point $\tilde{v} = (v, 1)$,

$$\lim_{t \rightarrow 0} \lambda(t) \cdot \tilde{v} = \lim_{t \rightarrow 0} (\lambda(t) \cdot v, \chi(\lambda(t))^{-1}).$$

Since the action on a_1 is $r_2 - r_1 = (-1) - (-1) = 0$, the action is trivial and $\lim_{t \rightarrow 0} \lambda(t) \cdot v = \lim_{t \rightarrow 0} (t^0 \cdot v) = v$. Note that $\chi(\lambda(t)) = t^{r_1 - r_0} = t^{-(n+1)}$, hence we yield $\lim_{t \rightarrow 0} \chi(\lambda(t))^{-1} = \lim_{t \rightarrow 0} (t^{-(n+1)})^{-1} = \lim_{t \rightarrow 0} t^{n+1} = 0$. Combining the two parts, we have $\lim_{t \rightarrow 0} \lambda(t) \cdot \tilde{v} = (v, 0) \in V \times \{0\}$. This implies that $\overline{G \cdot \tilde{v}}$ has a non-empty intersection with the zero section. Therefore, the chosen v is not χ -semistable.

Definition 3.17. A *one-parameter subgroup* (1-PS) of G is a non-trivial morphism of algebraic groups $\lambda : \mathbb{G}_m \rightarrow G$.

Proposition 3.18. Let G be a complex reductive linear algebraic group, and let $\lambda : \mathbb{G}_m \rightarrow G$ be a one-parameter subgroup. Then $\lambda(\mathbb{G}_m)$ is contained in a maximal torus of G . In particular, λ is diagonalisable in every rational representation of G .

Proof. Note that we work over $\mathbb{C}$. So by a torus we mean a connected diagonalisable algebraic group that is isomorphic to $(\mathbb{G}_m)^r$ for some $r \geq 0$.

By Proposition 2.19, there exists a faithful rational representation of G , which is a closed immersion $\iota : G \hookrightarrow \mathrm{GL}(V)$ for some finite dimensional complex vector space V . Consider the homomorphism $\sigma : \iota \circ \lambda : \mathbb{G}_m \rightarrow \mathrm{GL}(V)$. Take an arbitrary vector $v \in V$ and define $\psi_v : \mathbb{G}_m \rightarrow V$ by $\psi_v(t) = \sigma(t)v$. Since σ is a homomorphism of algebraic groups, its matrix entries with respect to any basis of V are regular functions on $\mathbb{G}_m$.

Consequently, there exists a finite set of integers $N \subset \mathbb{Z}$ and vectors $\{v_n\}_{n \in N}$ in V such that for all $t \in \mathbb{G}_m$, we have the decomposition

$$\psi_v(t) = \sum_{n \in N} t^n v_n.$$

As σ is a group homomorphism, we have $\psi_v(st) = \sigma(s)\psi_v(t)$, which gives

$$\sum_{n \in N} s^n t^n v_n = \sum_{n \in N} t^n (\sigma(s)v_n).$$

Note that this equality holds for all $t \in \mathbb{G}_m$, comparing the coefficients yields $\sigma(s)v_n = s^n v_n$ for every $n \in N$ and every $s \in \mathbb{G}_m$. It follows that each non-zero vector $v_n \in V$ is a weight vector for the action of $\mathbb{G}_m$.

The map $\psi_v(t) = \sum t^n v_n$ shows that any vector in the cyclic submodule generated by v is a linear combination of such weight vectors. Since V is spanned by a finite basis, it follows that V is the direct

sum of its weight spaces $V_n = \{v \in V \mid \sigma(t)v = t^n v \text{ for all } t \in \mathbb{G}_m\}$. So we have a weight decomposition

$$V = \bigoplus_{n \in \mathbb{Z}} V_n.$$

Consequently, there exists a basis for V consisting of weight vectors, and in this basis, every linear map $\sigma(t)$ is represented by a diagonal matrix of the form $\text{diag}(t^{n_1}, \dots, t^{n_m})$ for integers $n_i \in \mathbb{Z}$.

Let $D \subset \text{GL}(V)$ be the maximal torus consisting of all diagonal matrices in this basis. We have shown that $\sigma(\mathbb{G}_m) = (\iota \circ \lambda)(\mathbb{G}_m)$ is a subgroup of D . Hence $\sigma(\mathbb{G}_m)$ is a torus. Now define $T := \iota^{-1}(\sigma(\mathbb{G}_m))$. Since ι is a closed immersion, T is a closed subgroup of G , and note also that ι induces an isomorphism from T to the torus $\sigma(\mathbb{G}_m)$, hence T is a torus in G . By construction, $\lambda(\mathbb{G}_m) \subseteq T$.

Consider the collection $\mathcal{T}$ of all tori in G that T , which is clearly non-empty. If $S_1 \subset S_2$ is a proper inclusion of tori, then it is a proper inclusion of connected closed subvarieties, which implies that $\dim S_1 < \dim S_2$. Since the dimension of any torus in G is bounded by $\dim G$, any strictly ascending chain of tori in $\mathcal{T}$ is finite. Therefore, there exists a maximal element in $\mathcal{T}$. Thus we have shown that $\lambda(\mathbb{G}_m)$ is contained in a maximal torus of G .

Denote by $T_{\max}$ the maximal torus in G . Then for any rational representation $\rho : G \rightarrow \text{GL}(V)$, the restriction $\rho|_{T_{\max}}$ is also a rational representation. This means that the space V decomposes into a direct sum of weight spaces for the action of $T_{\max}$, so there exists a basis of V in which every element $\rho(t)$ for $t \in T_{\max}$ can be written as a diagonal matrix. Since $\lambda(\mathbb{G}_m) \subseteq T_{\max}$, this holds for all elements of the form $\rho(\lambda(z))$ for $z \in \mathbb{G}_m$. Therefore, the homomorphism $\rho \circ \lambda$ is diagonalisable. $\square$

Let $T \cong (\mathbb{G}_m)^r$ be an algebraic torus of dimension r . The set of all algebraic group homomorphisms from the torus T to the multiplicative group $\mathbb{G}_m$ forms the *character lattice* $X^*(T) := \text{Hom}(T, \mathbb{G}_m)$. We note that this set forms a group under pointwise multiplication of functions $(\chi_1 \cdot \chi_2)(t) := \chi_1(t)\chi_2(t)$. An element $\chi \in X^*(T)$ is a *character*.

Similarly, the set of all algebraic group homomorphisms from $\mathbb{G}_m$ into the torus T consists of the one-parameter subgroups of T . That is,

$$X_*(T) := \text{Hom}(\mathbb{G}_m, T).$$

This set also forms a group, and an element $\lambda \in X_*(T)$ is a 1-PS (or a *cocharacter*). We refer the reader Humphrey's book [17] for more detailed background constructions of both the notions.

Proposition 3.19. *The character lattice $X^*(T)$ is a free abelian group of rank r , i.e. $X^*(T) \cong \mathbb{Z}^r$.*

Proof. We note that in Humphrey's book [17], Section 16.1 (page 101-103) discusses the character lattice $X^*(\mathbb{G}_m) \cong \mathbb{Z}$, and subsequently, the relation $X^*(T) \cong \mathbb{Z}^r$ for any torus $T \cong \mathbb{G}_m^r$ is made clear in Section 16.2 (page 103-105). We follow the discussions in the corresponding chapter of the book, and give an more integrated proof.

Let $\chi : \mathbb{G}_m \rightarrow \mathbb{G}_m$ be a character. Note that $\mathbb{G}_m$ can be written as the set $\{(x, y) \in \mathbb{A}^2 \mid xy - 1 = 0\}$, so it is an affine variety. Thus the coordinate ring is

$$\mathbb{K}[\mathbb{G}_m] \cong \frac{\mathbb{K}[x, y]}{(xy - 1)} \cong \mathbb{K}[x, x^{-1}],$$

where x, y are coordinates of $\mathbb{A}^2$. By Proposition 2.18, there exists a map $\chi^* : \mathbb{K}[x, x^{-1}] \rightarrow \mathbb{K}[x, x^{-1}]$ given by $\chi^*(f) = f \circ \chi$ for all regular functions $f \in \mathbb{K}[x, x^{-1}]$. Note that $x \in \mathbb{K}[x, x^{-1}]$ is a unit, so its image $\chi^*(x)$ is also a unit. It follows that $\chi^*(x) = cx^n$ for some non-zero constant c and $n \in \mathbb{Z}$. Moreover, since χ is a group homomorphism, $\chi(1) = 1$, which implies that $c = 1$. Therefore, for any $t \in \mathbb{G}_m$, a character $\chi_n : \mathbb{G}_m \rightarrow \mathbb{G}_m$ is given by $\chi_n(t) = t^n$ for some $n \in \mathbb{Z}$. Hence $X^*(\mathbb{G}_m) \cong \mathbb{Z}$.

Now consider an r -dimensional torus T , which is connected and diagonalisable, so we have $T \cong (\mathbb{G}_m)^r$. Note that the isomorphism $\phi : (\mathbb{G}_m)^r \rightarrow T$ of algebraic groups induces an isomorphism $\phi^* : X^*(T) \rightarrow X^*((\mathbb{G}_m)^r)$ [17, Page 102].

Let $\chi \in X^*((\mathbb{G}_m)^r)$. For each $i \in \{1, \dots, r\}$, let $\iota_i : \mathbb{G}_m \rightarrow (\mathbb{G}_m)^r$ be the inclusion map into the i -th coordinate of $(\mathbb{G}_m)^r$. Note that $\chi \circ \iota_i : \mathbb{G}_m \rightarrow \mathbb{G}_m$ is a character, then there exists an $n_i \in \mathbb{Z}$ such that $(\chi \circ \iota_i)(t) = t^{n_i}$ for all $t \in \mathbb{G}_m$. Note also that since ι_i is an inclusion, any element $(t_1, \dots, t_r) \in (\mathbb{G}_m)^r$ can be written as the product $\prod_{i=1}^r \iota_i(t_i)$. It follows that

$$\chi(t_1, \dots, t_r) = \chi\left(\prod_{i=1}^r \iota_i(t_i)\right) = \prod_{i=1}^r \chi(\iota_i(t_i)) = \prod_{i=1}^r t_i^{n_i}.$$

Because $\chi \in X^*((\mathbb{G}_m)^r)$ were arbitrary, we conclude that $X^*(T) \cong X^*((\mathbb{G}_m)^r) \cong \mathbb{Z}^r$. $\square$

For a 1-PS $\lambda : \mathbb{G}_m \rightarrow G$ and a vector $v \in V$, write the λ -weight decomposition

$$V = \bigoplus_{n \in \mathbb{Z}} V_n, \quad \lambda(t) \cdot v_n = t^n v_n \text{ where } v_n \in V_n.$$

For $v = \sum_n v_n$ with $v_n \in V_n$, define

$$m(v, \lambda) := \min\{n \in \mathbb{Z} \mid v_n \neq 0\}.$$

So for a $v \in V \setminus \{0\}$ the limit $\lim_{t \rightarrow 0} \lambda(t) \cdot v$ exists and equals 0 if and only if all weights n are greater than 0, which is equivalent to $m(v, \lambda) > 0$. A tool that can be used to determine the semistability of points is the Hilbert-Mumford Criterion, which reduces the problem of checking orbit closures to an analysis of the action of one-parameter subgroups (1-PSs).

Theorem 3.20 (Projective Hilbert-Mumford Criterion [31, Theorem 5.17]). *Let G be a reductive group acting linearly on a vector space V . For a point $[v] \in \mathbb{P}(V)$ and a 1-PS λ , let $\mu([v], \lambda) := -m(v, \lambda)$. Then a point $[v] \in \mathbb{P}(V)$ is*

- *unstable if and only if there exists a one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow G$ with $\mu([v], \lambda) < 0$.*
- *semistable if and only if $\mu([v], \lambda) \geq 0$ for all 1-PS λ .*
- *stable if and only if $\mu([v], \lambda) > 0$ for all non-trivial 1-PS λ , and the stabiliser $G_{[v]}$ is finite. This is equivalent to that the G -orbit of $[v]$ is closed in the semistable locus and $G_{[v]}$ is finite.*

Before proving the theorem, we consider the case of torus actions. A detailed exposition on this construction can be found in Dolgachev's book [9]. Let the torus $T = (\mathbb{G}_m)^r$ act linearly on V , which induces a weight space decomposition

$$V = \bigoplus_{\xi \in X^*(T)} V(\xi),$$

where $X^*(T) \cong \mathbb{Z}^r$ is the character lattice of T , and $V(\xi) = \{w \in V \mid t \cdot w = \xi(t)w \text{ for all } t \in T\}$ is the weight space corresponding to the character ξ . For a point $[v] \in \mathbb{P}(V)$, let $\hat{v} = \sum_{\xi} \hat{v}_{\xi} \in V(\xi)$ be its lift in V . Define the *weight set* of $[v]$ as $W_T([v]) := \{\xi \in X^*(T) \mid \hat{v}_{\xi} \neq 0\}$. Then the weight polytope $\Delta([v])$ is the convex hull of the weight set in $\mathfrak{t}^*$, where $\mathfrak{t}$ is the Lie algebra of the torus T . That is,

$$\Delta([v]) := \text{conv}_{\mathbb{R}}(W_T([v])) \subset \mathfrak{t}^*.$$

For a 1-PS $\lambda \in X_*(T) \cong \mathbb{Z}^r$, we have $\lambda(t) \cdot w = t^{\langle \xi, \lambda \rangle} w$ for any $w \in V(\xi)$, where $\langle \cdot, \cdot \rangle$ is the natural pairing. Then we define the numerical weight as

$$\mu([v], \lambda) = - \min_{\xi \in W_T([v])} \langle \xi, \lambda \rangle.$$

The Hilbert-Mumford criterion for torus actions can be stated as follows.

Proposition 3.21 ([9, Theorem 9.2]). (i) $[v]$ is semistable if and only if $0 \in \Delta([v])$.

(ii) $[v]$ is stable if and only if $\Delta([v])$ contains the origin in its interior, i.e. $0 \in \text{int } \Delta([v])$.

Proof. (i) We consider the contrapositive, which is the point $[v]$ is unstable if and only if $0 \in \overline{T \cdot \hat{v}}$. This happens if there is a 1-PS $\lambda \in X_*(T)$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot \hat{v} = 0$, which then implies that every weight $\xi \in W_T([v])$, the corresponding pairing $\langle \xi, \lambda \rangle > 0$.

By the separating hyperplane theorem (detailed constructions can be found in Section 5.3 of [18] and Chapter IV Section 3 of [6]), the existence of such a λ , which defines a hyperplane between the origin and the convex hull $\Delta([v])$, implies that $0 \notin \Delta([v])$. Hence the semistability of $[v]$ means $\min_{\xi} \langle \xi, \lambda \rangle \leq 0$ for all $\lambda \in X_*(T)$, which implies that $\mu([v], \lambda) \geq 0$ for all λ . Therefore, we can conclude that $0 \in \Delta([v])$.

Conversely, if $0 \notin \Delta([v])$, then by the separating hyperplane theorem, there exists a linear functional $\lambda \in \mathfrak{t}$ with $\langle \xi, \lambda \rangle > 0$ for all $\xi \in W_T([v])$. Since $\Delta([v])$ is a weight polytope whose vertices are in $X^*(T)$, we may choose a λ such that $\min_{\xi} \langle \xi, \lambda \rangle > 0$. Hence $\mu([v], \lambda) < 0$ and $[v]$ is unstable.

- (ii) Assume $[v]$ is stable. Then $0 \in \Delta([v])$ as $[v]$ is also semistable, and the orbit $T \cdot [v]$ is closed in the semistable locus with the stabiliser $T_{[v]}$ finite. Suppose, for contradiction, that $0 \notin \text{int}(\Delta([v]))$. Then the origin is on the boundary of $\Delta([v])$, and the separating hyperplane theorem implies that there is a supporting hyperplane to the convex hull $\Delta([v])$ at the origin. This implies that there is a non-zero linear functional on $\mathfrak{t}$, which is also a non-trivial 1-PS $\lambda \in X_*(T)$ (since $\Delta([v])$ is a weight polytope whose vertices are in $X^*(T)$), such that $\Delta([v])$ is contained within one of the closed half-space defined by this functional. This means $\langle \xi, \lambda \rangle \leq 0$ for all $\xi \in \Delta([v])$, and there is at least one $\xi_0 \in W_T([v])$ with $\langle \xi_0, \lambda \rangle = 0$, which gives $\mu([v], \lambda) = -\min \langle \xi, \lambda \rangle = 0$.

Since the weight is zero, $t^{-\min \langle \xi, \lambda \rangle} = t^0 = 1$, and so the limit is

$$\lim_{t \rightarrow 0} \lambda(t) \cdot \hat{v} = \lim_{t \rightarrow 0} \sum_{\xi \text{ with } \langle \xi, \lambda \rangle = 0} t^{\langle \xi, \lambda \rangle} \hat{v}_\xi = \sum_{\xi \text{ with } \langle \xi, \lambda \rangle = 0} \hat{v}_\xi.$$

Let $[v_0] \in \mathbb{P}(V)$ denote such a limit point, which is also in the closure $\overline{T \cdot [v]}$. Note that the decomposition of weight spaces induces a decomposition for every $\hat{v}$ based on their pairing with λ , namely,

$$\hat{v} = \sum_{\xi \text{ with } \langle \xi, \lambda \rangle = 0} \hat{v}_\xi + \sum_{\eta \text{ with } \langle \eta, \lambda \rangle > 0} \hat{v}_\eta.$$

It follows that the limit point $[v_0] = [\sum_{\xi \text{ with } \langle \xi, \lambda \rangle = 0} \hat{v}_\xi]$. Since λ is a non-trivial 1-PS, the point $[v_0]$ is fixed by $\lambda(\mathbb{G}_m)$, and hence the stabiliser $T_{[v_0]}$ is of positive dimension. If $[v_0] \in T \cdot [v]$, then $T_{v_0} = T_v$ as T is abelian, contradicting the finiteness of $T_{[v]}$. But stability requires the stabiliser $T_{[v]}$ to be finite, hence the point $[v_0] \notin T \cdot [v]$. Since $[v_0] \in \overline{T \cdot [v]}$, the orbit is not closed in the semistable locus, contradicting the definition of stability. Therefore, our initial assumption is false, and $0 \in \text{int}(\Delta([v]))$.

Conversely, assume that $0 \in \text{int}(\Delta([v]))$, which clearly implies that $0 \in \Delta([v])$, so $[v]$ is semistable. Moreover, the origin is not in any supporting hyperplane. For any non-trivial λ , let $H_\lambda = \{x \in \mathfrak{t}^* \mid \langle x, \lambda \rangle = 0\}$ be a hyperplane through the origin. If $\langle \xi, \lambda \rangle \geq 0$ for all $\xi \in W_T([v])$, then H_λ would be a supporting hyperplane of $\Delta([v])$, which contradicts that 0 is in the interior. Hence for any non-trivial λ , the pairing $\langle \cdot, \lambda \rangle$ takes both positive and negative values on $W_T([v])$.

In particular, there exists some weight ξ such that $\langle \xi, \lambda \rangle < 0$, which then implies that

$$\min_{\xi} \langle \xi, \lambda \rangle < 0 \text{ for all non-trivial } \lambda.$$

Consequently, we obtain that $\mu([v], \lambda) > 0$ for all non-trivial λ , which implies the orbit $T \cdot [v]$ is closed in the semistable locus.

The stabiliser of $[v]$ under the T -action is

$$T_{[v]} = \{t \in T \mid t \cdot \hat{v} = c\hat{v} \text{ for some } c \in \mathbb{G}_m\}.$$

So we have $\sum_{\xi} \xi(t) \hat{v}_\xi = c \sum_{\xi} \hat{v}_\xi$, which implies $\xi(t) = c$ as the weight spaces $V(\xi)$ are linearly independent. Hence for any two weights $\xi, \eta \in W_T([v])$, we have $\xi(t) = \eta(t) = c$, i.e. $t \in \ker(\xi\eta^{-1})$. Thus the stabiliser can be written as

$$T_{[v]} = \bigcap_{\xi, \eta} \ker(\xi\eta^{-1}).$$

Since the convex hull $\Delta([v])$ has a non-empty interior, the weight space $W_T([v])$ is not contained in any proper affine subspace of $\mathfrak{t}^*$. Hence the set $\{\xi - \eta \mid \xi, \eta \in W_T([v])\}$ generates $X^*(T)$, and thus $T_{[v]}$ is finite. $\square$

Proof of Theorem 3.20. If there exists a 1-PS λ of G with $\mu([v], \lambda) < 0$, then $m(v, \lambda) > 0$. This means that for the decomposition $v = \sum_n v_n$ with respect to λ , all non-zero v_n have $n > 0$. Thus

$$\lim_{t \rightarrow 0} \lambda(t) \cdot v = \lim_{t \rightarrow 0} \sum_n t^n v_n = 0,$$

which shows that $0 \in \overline{G \cdot v}$. Hence $[v]$ is unstable.

Since a point $[v]$ is unstable for G if and only if there exists $g \in G$ such that $g \cdot [v]$ is unstable for a fixed maximal torus $T \subset G$ [9, Theorem 9.3], the converse direction can be approached based on the torus case. $\square$

For a character $\chi \in X^*(T)$, and a 1-PS $\lambda \in X_*(T)$, the composition $\chi \circ \lambda : \mathbb{G}_m \rightarrow \mathbb{G}_m$ is a homomorphism of tori. Since an algebraic group homomorphism from $\mathbb{G}_m$ to itself is a power map, there is a unique $q \in \mathbb{Z}$ such that $(\chi \circ \lambda)(t) = t^q$. Now we define the *natural pairing*

$$\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \rightarrow \mathbb{Z}$$

by $\langle \chi, \lambda \rangle = q$. This means we can write $\chi(\lambda(t)) = t^{\langle \chi, \lambda \rangle}$ for all $t \in \mathbb{G}_m$. This gives an analogue of the criterion for the case of affine GIT quotients.

Theorem 3.22 (Affine Hilbert-Mumford Criterion [16, Proposition 2.5]). *Let G be a reductive group that acts linearly on an affine space $V \subseteq \mathbb{A}^n$, and let χ be a character of G . On the trivial line bundle $V \times \mathbb{A}^1$, we have a linearisation of the G -action given by*

$$g \cdot (v, a) = (g \cdot v, \chi(g)^{-1}a).$$

For $v \in V$, define $\mu_\chi(v, \lambda) := \langle \chi, \lambda \rangle$. Then

- v is χ -semistable if and only if for every 1-PS λ of G for which the limit $\lim_{t \rightarrow 0} \lambda(t) \cdot v$ exists, we have $\mu_\chi(v, \lambda) \geq 0$.
- v is χ -stable if and only if for every non-trivial 1-PS λ of G for which the limit $\lim_{t \rightarrow 0} \lambda(t) \cdot v$ exists, we have $\mu_\chi(v, \lambda) > 0$.

Example 3.23. This example demonstrates the scenario described in Exercise 3.14 of Hoskins's survey in [11]. Consider the GL_2 -action on the space $\mathrm{Mat}_{2 \times n}$ of $2 \times n$ matrices, which is by left multiplication. We want to show that the semistable locus for the character $\chi := \det$ is the matrices of maximal rank i.e. of rank 2.

Note that we proved earlier in Proposition 3.18 that for a reductive group G , any 1-PS λ of G is contained in a maximal torus, and hence acts diagonalisably in every rational representation. Then we can find an invertible matrix $g \in \mathrm{GL}_2$ such that

$$\lambda(t) = g \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix} g^{-1}$$

for some $a, b \in \mathbb{Z}$. Then applying the character gives $\det(\lambda(t)) = t^{a+b}$, which means the pairing $\langle \det, \lambda \rangle = a + b$. The Hilbert-Mumford criterion requires the existence of limits of 1-PS actions. For a general 1-PS $\lambda(t)$ acting on a matrix $A \in \mathrm{Mat}_{2 \times n}$, we have

$$\lambda(t) \cdot A = \left(g \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix} g^{-1} \right) \cdot A.$$

Set $B := g^{-1} \cdot A$ and let $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Since left multiplication by the fixed g is an isomorphism, the limit $\lim_{t \rightarrow 0} \lambda(t) \cdot A$ exists if and only if $\lim_{t \rightarrow 0} \mathrm{diag}(t^a, t^b) \cdot B$ exists.

Suppose that the matrix A has maximal rank, i.e. $\mathrm{rank}(A) = 2$. Then $\mathrm{rank}(B) = \mathrm{rank}(g^{-1} \cdot A) = 2$, which means that neither row b_1, b_2 of B vanishes. Moreover, the existence of the limit $\lim_{t \rightarrow 0} \mathrm{diag}(t^a, t^b) \cdot B$ implies that $a \geq 0$ and $b \geq 0$. Hence $\mu_\chi(A, \lambda) = a + b \geq 0$, proving the semistability. In particular, if λ is nontrivial and the limit exists then at least one of a, b is strictly positive, which gives $a + b > 0$, hence A is stable.

Conversely, assume that $\mathrm{rank} A \leq 1$. Choose any $g \in \mathrm{GL}_2$ such that $B = g^{-1} \cdot A$ and that it takes the form $\begin{pmatrix} 0 \\ b_2 \end{pmatrix}$. Consider the 1-PS $\lambda(t) = \mathrm{diag}(t^{-1}, 1)$, it is clear that the limit exists and $\mu_\chi(A, \lambda) = -1 + 0 = -1 < 0$. Thus A is unstable. It follows that we obtain the following equalities

$$V^{ss}(\chi) = V^s(\chi) = \{A \in \mathrm{Mat}_{2 \times n} \mid \mathrm{rank} A = 2\},$$

where V denotes $\mathrm{Mat}_{2 \times n}$. Note that this result also implies that there are no strictly semistable points in this example.

Example 3.24. Consider the action of $\mathrm{SL}_n(\mathbb{C})$ on the space $\mathrm{Mat}_{n \times n}(\mathbb{C})$ by conjugation, i.e. a linear representation $\mathrm{SL}_n(\mathbb{C}) \times \mathrm{Mat}_{n \times n}(\mathbb{C}) \rightarrow \mathrm{Mat}_{n \times n}(\mathbb{C})$ given by $(g, A) \mapsto gAg^{-1}$.

Note that any character $\chi : \mathrm{SL}_n \rightarrow \mathbb{G}_m$ admits a Lie algebra character $d\chi : \mathfrak{sl}_n \rightarrow \mathfrak{g}_m \cong \mathbb{C}$. Since $\mathfrak{sl}_n$ is a semisimple Lie algebra, every Lie algebra homomorphism into $\mathfrak{g}_m$ is the zero map, which means $d\chi = 0$. Hence χ is locally constant, and consequently, the character χ is trivial that is defined by $\chi(g) = 1$ for all $g \in \mathrm{SL}_n$. It follows that the linearisation on the trivial line bundle $\mathrm{Mat}_{n \times n}(\mathbb{C}) \times \mathbb{A}^1$ is given by

$$g \cdot (A, a) = (gAg^{-1}, \chi(g)^{-1}a) = (gAg^{-1}, a).$$

For any $t \in \mathbb{G}_m$, we have $(\chi \circ \lambda)(t) = 1 = t^0$. This implies that for any matrix $A \in \mathrm{Mat}_{n \times n}$ and any 1-PS λ of SL_n , we have $\mu_\chi(A, \lambda) = \langle \chi, \lambda \rangle = 0$.

Recall that by the criterion, a point $A \in \mathrm{Mat}_{n \times n}$ is semistable if and only if for every 1-PS $\lambda : \mathbb{G}_m \rightarrow \mathrm{SL}_n$ for which the limit $\lim_{t \rightarrow 0} \lambda(t) \cdot A$ exists, we have $\mu_\chi(A, \lambda) \geq 0$. Since we have established that $\mu_\chi(A, \lambda) = 0$, the condition for semistability is satisfied for any $n \times n$ complex matrix A . Therefore, we conclude that the semistable locus is the entire space, i.e. $V^{ss} = \mathrm{Mat}_{n \times n}(\mathbb{C})$. Moreover, since there does not exist a matrix $A \in \mathrm{Mat}_{n \times n}$ such that $\mu(A, \lambda) > 0$ with the existence of the limit $\lim_{t \rightarrow 0} \lambda(t) \cdot A$, the stable locus is the empty set, that is, $V^{st} = \emptyset$.

3.2 Hesselink Stratifications of Affine GIT Quotients

In this section, we define and discuss the necessary notions of Hesselink stratifications of the unstable locus of affine GIT quotients, which follows the framework presented in Hoskins's paper [16].

Definition 3.25. A one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow G$ is called *destabilising* for a point $v \in V$ if the limit $\lim_{t \rightarrow 0} \lambda(t) \cdot v$ exists in V and $\mu_\chi(v, \lambda) = \langle \chi, \lambda \rangle < 0$.

Given an unstable point v , there might exist many destabilising 1-PSs associated to this point. The following process is used to choose the one that is the most destabilising. This uses a G -invariant norm $\|\cdot\|$ on the set $\chi_*(G)/G$ of conjugacy classes of 1-PSs of G . For a complex reductive group G , this can be achieved by fixing a maximal compact subgroup $K \subset G$ and a K -invariant inner product on its Lie algebra $\mathfrak{k}$ [16, Remark 2.11].

Definition 3.26. Let $v \in V$ be χ -unstable point. The *normalised Hilbert-Mumford function* for v is given by

$$M_G^\chi(v) := \inf \frac{\mu_\chi(v, \lambda)}{\|\lambda\|},$$

where the infimum is taken over all 1-PSs λ of G with the limit $\lim_{t \rightarrow 0} \lambda(t) \cdot v$ exists. Note that $M_G^\chi < 0$ since v is unstable.

A one-parameter subgroup λ is called χ -*adapted* to v if it achieves the infimum, which means

$$M_G^\chi(v) = \frac{\mu_\chi(v, \lambda)}{\|\lambda\|}.$$

Definition 3.27. A one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow G$ is *indivisible* (or *primitive*) if it is not a non-trivial positive multiple of another 1-PS. In other words, there does not exist a $\lambda_1 \in X_*(G)$ such that $\lambda(t) = \lambda_1(t^n)$ for all $t \in \mathbb{G}_m$ and $n \geq 2$. We write Λ^χ for the set of indivisible 1-PSs adapted to v

Hesselink stratification partitions the null cone $\mathcal{N}$ into strata, where each stratum consists of points that have the same conjugacy class of adapted 1-PSs.

Definition 3.28. Let $[\lambda]$ denote the conjugacy class of a non-trivial, 1-PS of G . Then the associated *stratum* is the set

$$S_{[\lambda]} := \{v \in \mathcal{N} \mid \Lambda^\chi(v) \cap [\lambda] \neq \emptyset\}.$$

For a specific λ of the conjugacy class $[\lambda]$, we define the *blade* to be the subset

$$S_\lambda := \{v \in \mathcal{N} \mid \lambda \in \Lambda^\chi(v)\} \subseteq S_{[\lambda]},$$

and the *limit set* for the blade S_λ is the subset

$$Z_\lambda := \{v \in \mathcal{N} \mid \lambda \in \Lambda^\chi(v) \cap G_v\} \subseteq S_\lambda,$$

i.e. Z_λ consists of points in the blade S_λ that are fixed by λ . It follows from a theorem of Kempf [16, Theorem 2.15] that $S_{[\lambda]} = GS_\lambda$, and that the map $p_\lambda : S_\lambda \rightarrow Z_\lambda$ defines a retraction by $p_\lambda(v) = \lim_{t \rightarrow 0} \lambda(t) \cdot v$. We then have the following theorem concerning the structure of the null cone $\mathcal{N}$.

Theorem 3.29 ([16, Theorem 2.16]). *Let G be a reductive group acting on an affine space V with respect to a character χ . Then the null cone $\mathcal{N} = V \setminus V^{\chi-ss}$ admits a decomposition*

$$\mathcal{N} = \bigsqcup_{[\lambda]} S_{[\lambda]}$$

of a finite disjoint union of G -invariant locally closed subvarieties $S_{[\lambda]}$ of V .

In addition, there is a strict partial ordering on the index of conjugacy classes, which is given by

$$[\lambda] < [\lambda'] \text{ if and only if } \frac{\langle \chi, \lambda \rangle}{\|\lambda\|} > \frac{\langle \chi, \lambda' \rangle}{\|\lambda'\|}.$$

This ordering implies that the boundary of a stratum can only intersect another stratum $S_{[\lambda']}$ if $[\lambda] < [\lambda']$. Then we obtain the stratification of V that is given by

$$V = \bigsqcup S_{[\lambda]}.$$

Since the minimal stratum $S_{[0]} := V^{\chi-ss}$ is contained in V , which is indexed by the trivial 1-PS, this stratification of V is well-defined, and is called *Hesselink stratification*.

Example 3.30. We consider the stratification of $V = \mathbb{C}^2$ with coordinates (x, y) under the action of a two-dimensional torus $G = \mathbb{G}_m \times \mathbb{G}_m$. Let $g \in G$ be $g = (g_1, g_2)$, and define the G -action on V by $(g_1, g_2) \cdot (x, y) = (g_1 x, g_2 y)$. A 1-PS $\lambda \in X_*(G) \cong \mathbb{Z}^2$ is given by $k = (k_1, k_2) \in \mathbb{Z}^2$, where $\lambda(t) = (t^{k_1}, t^{k_2}) \in G$ for any $t \in \mathbb{G}_m$. Note that λ is indivisible if $\gcd(k_1, k_2) = 1$. Choose a non-trivial character $\chi : G \rightarrow \mathbb{G}_m$ that is given by $\chi(g_1, g_2) = g_1 g_2$, and consequently, for $v \in V$, we have

$$\mu_\chi(v, \lambda) = \langle \chi, \lambda \rangle = (1, 1) \cdot (k_1, k_2) = k_1 + k_2.$$

The Hilbert-Mumford function for this case is

$$M_G^\chi(v) = \inf \frac{k_1 + k_2}{\sqrt{k_1^2 + k_2^2}},$$

where the infimum is taken over all $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that the limit $\lim_{t \rightarrow 0} \lambda(t) \cdot v$ exists. Since $\lambda(t) \cdot v = (t^{k_1}, t^{k_2}) \cdot (x, y) = (t^{k_1} x, t^{k_2} y)$, the limit exists if and only if the limit exists for each component, that is, $x = 0$ (or $k_1 \geq 0$) and $y = 0$ (or $k_2 \geq 0$). So we have the following cases

- $v = (x, y)$ with $x \neq 0$ and $y \neq 0$. Then $\lim_{t \rightarrow 0} \lambda(t) \cdot v$ exists only if $k_1 \geq 0$ and $k_2 \geq 0$. To find the infimum in $M_G^\chi(v)$, consider the function $f(\theta) = \cos \theta + \sin \theta$ with $\theta \in [0, \frac{\pi}{2}]$. Then f achieves its minimum at the endpoints $f(0) = f(\frac{\pi}{2}) = 1$. Hence

$$M_G^\chi(v) = 1,$$

and in particular, these points are χ -semistable and we have the stratum $S_{[0]} = V^{\chi-ss} = \{(x, y) \in \mathbb{C}^2 \mid x \neq 0, y \neq 0\}$.

- $v = (x, 0)$ with $x \neq 0$. The limit exists if $k_1 \geq 0$, and we consider $f(\theta)$ with $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, whose minimum occurs at $\theta = -\frac{\pi}{2}$, which is $f(-\frac{\pi}{2}) = -1$. Note that the infimum is achieved by 1-PSs with $k = (0, -m)$ for some $m > 0$, so the unique indivisible 1-PS is $\lambda_{(0, -1)}$. Moreover, since $M_G^\chi(v)$ is negative, the points are unstable.
- $v = (0, y)$ with $y \neq 0$. The argument for this case is symmetric to the previous one, so we have $M_G^\chi(v) = -1 < 0$, which is obtained by 1-PSs with $k = (-m, 0)$ for some $m > 0$, and the points are also unstable. The unique indivisible 1-PS is then $\lambda_{(-1, 0)}$.
- $v = (0, 0)$. The limit exists for all $k \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. The minimum of $f(\theta)$ is obtained at $\theta = -\frac{3\pi}{4}$, which is $-\sqrt{2}$. This is achieved by 1-PSs with $k = (-m, -m)$ for some $m > 0$, and it follows that the indivisible 1-PS is $\lambda_{(-1, -1)}$.

Therefore, we have obtained four distinct strata, namely the semistable stratum $S_{[0]} = \{M_G^X(v) \geq 0\} = \{(x, y) \mid x \neq 0, y \neq 0\}$, the $\lambda_{(0,-1)}$ -stratum $S_{\lambda_{(0,-1)}} = \{v \mid \Lambda^X(v) = \{\lambda_{(0,-1)}\}\} = \{(x, 0) \mid x \neq 0\}$, the $\lambda_{(-1,0)}$ -stratum $S_{\lambda_{(-1,0)}} = \{v \mid \Lambda^X(v) = \{\lambda_{(-1,0)}\}\} = \{(0, y) \mid y \neq 0\}$, and the $\lambda_{(-1,-1)}$ -stratum $S_{\lambda_{(-1,-1)}} = \{v \mid \Lambda^X(v) = \{\lambda_{(-1,-1)}\}\} = \{(0, 0)\}$. The null cone $\mathcal{N} = V \setminus S_{[0]}$ is the union of the two axes, moreover, the Hesselink stratification partitions the entire space V as

$$V = S_{[0]} \sqcup S_{\lambda_{(0,-1)}} \sqcup S_{\lambda_{(-1,0)}} \sqcup S_{\lambda_{(-1,-1)}}.$$

To see the partial ordering on the strata, observe that $1 > -1 > -\sqrt{2}$, which are the corresponding values of $M_G^X(v)$ for v in the respective cases, implies the following partial ordering

$$[0] < [\lambda_{(0,-1)}], \quad [0] < [\lambda_{(-1,0)}], \quad [\lambda_{(0,-1)}] < [\lambda_{(-1,-1)}], \quad [\lambda_{(-1,0)}] < [\lambda_{(-1,-1)}].$$

Furthermore, the blade $S_{\lambda_{(-1,-1)}}$ coincides with the limit set $Z_{\lambda_{(-1,-1)}}$ as they both are the origin $(0, 0)$, thus the retraction $p_{\lambda_{(-1,-1)}}(0, 0) = \lim_{t \rightarrow 0} (t^{-1} \cdot 0, t^{-1} \cdot 0) = (0, 0)$ is the identity map on the origin.

4 Transversality

In this chapter, we introduce the notion of transversality, which is used to understand the topology of affine GIT quotients in later chapters. Our primary reference on this topic that is mainly referred to is the book by Guillemin & Pollack [12].

Definition 4.1. A *smooth manifold* of dimension n is a second-countable, Hausdorff topological space M equipped with a maximal smooth atlas. A *smooth atlas* $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ is a collection of *charts* such that

- each chart $(U_\alpha, \varphi_\alpha)$ consists of an open set $U_\alpha \subseteq M$ and a homeomorphism φ_α from U_α onto an open subset $\varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ of Euclidean space;
- the collection of open sets $\{U_\alpha\}_{\alpha \in I}$ covers M , namely, $\bigcup_{\alpha \in I} U_\alpha = M$;
- for any two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) , if $U_\alpha \cap U_\beta \neq \emptyset$, the *transition map*

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

is a diffeomorphism, i.e. a smooth map with a smooth inverse, between open subsets of $\mathbb{R}^n$, and we say the two charts are *compatible*.

An atlas is *maximal* if it is not properly contained in any other smooth atlas. A maximal smooth atlas is also called a *smooth structure*.

Proposition 4.2. Let M be a manifold of dimension n , and let $\mathcal{A}$ be a smooth atlas on M . Then the atlas $\mathcal{A}$ is contained in a unique maximal atlas and so determines a unique smooth structure on M .

Proof. Denote $\mathcal{A}_{max}$ the maximal atlas, and define

$$\mathcal{A}_{max} := \{(U, \varphi) \mid (U, \varphi) \text{ is compatible with every chart in } \mathcal{A}\}.$$

By construction, we have $\mathcal{A} \subset \mathcal{A}_{max}$, and consequently $\bigcup_{(U, \varphi) \in \mathcal{A}} U = M$. Now let $(U, \varphi), (V, \psi) \in \mathcal{A}_{max}$ be any two charts. If $U \cap V = \emptyset$, then they are trivially compatible.

Suppose $U \cap V \neq \emptyset$, and let p be an arbitrary point in the intersection. Since $\mathcal{A}$ covers M , there is a chart $(W, g) \in \mathcal{A}$ with $p \in W$. Because (U, φ) is compatible with $\mathcal{A}$, the map $\varphi \circ g^{-1}$ is smooth on $g(W \cap U)$, similarly, the map $g \circ \psi^{-1}$ is smooth on $\psi(W \cap V)$. Then on the open set $\psi(W \cap U \cap V)$, we obtain the identity

$$\varphi \circ \psi^{-1} = (\varphi \circ g^{-1}) \circ (g \circ \psi^{-1}),$$

hence $\varphi \circ \psi^{-1}$ is smooth on a neighbourhood of $\psi(p)$. Since $p \in U \cap V$ was arbitrarily chosen, it follows that the composition is smooth on $\psi(U \cap V)$. Similarly, we also have $\psi \circ \varphi^{-1}$ is smooth on $\varphi(U \cap V)$, which implies that the two charts (U, φ) and (V, ψ) are compatible. Thus $\mathcal{A}_{max}$ is an atlas.

The maximality of $\mathcal{A}_{max}$ follows from the observation that if (U, φ) is a chart that is compatible with any chart in $\mathcal{A}_{max}$, then it is in particular compatible with every chart in an atlas $\mathcal{A} \subset \mathcal{A}_{max}$. Thus there is no atlas that is strictly larger than $\mathcal{A}_{max}$, which means it is indeed maximal by construction.

If $\mathcal{B}$ is another maximal atlas with $\mathcal{A} \subset \mathcal{B}$, then every chart in $\mathcal{B}$ is compatible with $\mathcal{A}$, hence $\mathcal{B} \subset \mathcal{A}_{max}$. Conversely, $\mathcal{A}_{max}$ is an atlas that contains $\mathcal{A}$, the maximality of $\mathcal{B}$ implies that $\mathcal{A}_{max} \subset \mathcal{B}$. Therefore, $\mathcal{B} = \mathcal{A}_{max}$, and so $\mathcal{A}_{max}$ is unique. $\square$

Definition 4.3. Let M and N be smooth manifolds. A continuous map $f : M \rightarrow N$ is *smooth* if for every point $p \in M$, there exist charts (U, φ) containing p and (V, ψ) containing $f(p)$ such that the composition

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \rightarrow \psi(V)$$

is a smooth map between open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$

Definition 4.4. Let B be a smooth manifold. A smooth manifold E with a smooth map $\pi : E \rightarrow B$ is called a *smooth vector bundle* of rank k over B if

- there exists a k -dimensional vector space F such that for all $p \in B$, the fibre $\pi^{-1}(p)$ is isomorphic to F as a vector space,
- any point $p \in B$ has an open neighbourhood U and a diffeomorphism $\varphi_U : \pi^{-1}(U) \cong U \times F$ such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi_U} & U \times F \\ \pi \downarrow & & \downarrow \text{pr}_U \\ U & \xrightarrow{\text{id}_U} & U \end{array}$$

is commutative, and $\varphi_U|_{\pi^{-1}(p)} : \pi^{-1}(p) \rightarrow \{p\} \times F$ is a vector space isomorphism for all points $p \in U$.

We call B the *base space*, and E the *total space*. φ_U is a *local trivialisation* of E . A *line bundle* is a vector bundle of rank 1.

Example 4.5. The *trivial line bundle* of rank k over a base space B is the triple $(B \times \mathbb{R}^k, B, \pi : B \times \mathbb{R}^k \rightarrow B)$. Note that $B \times \mathbb{R}^k$ is equipped with the product topology, so its smooth structure inherits from the given smooth structure on B and the usual smooth structure (i.e. defined by the identity chart) on $\mathbb{R}^k$.

Definition 4.6. Let (E, B, π) be a smooth vector bundle. A smooth map $s : B \rightarrow E$ such that $\pi \circ s = \text{id}_B$ is called a *section* of the vector bundle.

The set of sections $\Gamma(E)$ can be given the structure of a real vector space by defining pointwise addition and pointwise scalar multiplication in each fibre. The *zero section* is the section that maps to the origin in each fibre.

Definition 4.7. Let (E, B, π) and (E', B', π') be two smooth vector bundles. A pair of smooth maps (f, g) where $f : E \rightarrow E'$ and $g : B \rightarrow B'$, is called a *(smooth) vector bundle morphism* if $\pi' \circ f = g \circ \pi$ and for every point $b \in B$, $f_b := f|_{\pi^{-1}(b)} : \pi^{-1}(b) \rightarrow \pi'^{-1}(g(b))$ is a linear map of vector spaces.

If in addition the maps $f : E \rightarrow E'$ and $g : B \rightarrow B'$ in the pair (f, g) are diffeomorphisms, then such a pair is called a *(smooth) vector bundle isomorphism*.

Note that an equivalent way of defining a bundle isomorphism is that (f, g) is an isomorphism if and only if $f : E \rightarrow E'$ is a diffeomorphism, and for all $b \in B$, f_b is a linear isomorphism. The second condition follows from the fact that if a map is an isomorphism locally in trivialising charts, and the transition functions between charts are smooth, then the map is a global diffeomorphism.

Proposition 4.8. A rank k vector bundle (E, B, π) is isomorphic to a trivial vector bundle if and only if it admits k linearly independent sections.

Proof. Assume that the vector bundle (E, B, π) is trivial. By definition, this means there exists a vector bundle isomorphism $\Phi : E \rightarrow B \times \mathbb{R}^k$. Let $\pi_1 : B \times \mathbb{R}^k \rightarrow B$ be the projection onto the first factor. It follows that Φ is a homeomorphism such that $\pi_1 \circ \Phi = \pi$, and for each $b \in B$, the restriction $\Phi_b : \pi^{-1}(b) \rightarrow \{b\} \times \mathbb{R}^k$ to the fibre $\pi^{-1}(b)$ is a linear isomorphism.

Let $\{e_1, e_2, \dots, e_k\}$ be a basis for $\mathbb{R}^k$. For each $i \in \{1, 2, \dots, k\}$, we define a global section σ_i of the trivial bundle $B \times \mathbb{R}^k$ by

$$\sigma_i : B \rightarrow B \times \mathbb{R}^k, \quad b \mapsto (b, e_i).$$

Note that each σ_i is continuous because its components, the identity map on B and a constant map to e_i are continuous. Moreover, we have $\pi_1 \circ \sigma_i = \text{id}_B$, so each σ_i is indeed a section. For any $b \in B$, the set $\{\sigma_1(b), \dots, \sigma_k(b)\}$ corresponds to the set $\{(b, e_1), \dots, (b, e_k)\}$, which forms a basis for the fibre $\{b\} \times \mathbb{R}^k$.

Consider the inverse $\Phi^{-1} : B \times \mathbb{R}^k \rightarrow E$, and define $s_i : B \rightarrow E$ by composing Φ^{-1} with the sections σ_i , i.e. $s_i := \Phi^{-1} \circ \sigma_i$ for $i = 1, \dots, k$. We want to show these maps are linearly independent sections of E . Since both Φ^{-1} and σ_i are continuous maps, their composition s_i is continuous. Furthermore, for any $b \in B$ we have

$$(\pi \circ s_i)(b) = \pi(\Phi^{-1}(\sigma_i(b))) = \pi(\Phi^{-1}(b, e_i)).$$

Because Φ^{-1} is also a bundle map, the property $\pi_1 \circ \Phi = \pi$ implies that $\pi \circ \Phi^{-1} = \pi_1$, it follows that $(\pi \circ s_i)(b) = \pi_1(b, e_i) = b$. This means that $\pi \circ s_i = \text{id}_B$, thus each s_i is a section of E .

To show the linear independence of the sections s_i , let $b \in B$ be an arbitrary point. Consider a linear combination $\{s_1(b), \dots, s_k(b)\}$ in the fibre $\pi^{-1}(b)$ such that

$$\sum_{i=1}^k c_i s_i(b) = 0_{\pi^{-1}(b)}$$

for some scalars $c_i \in \mathbb{R}$. Applying the linear isomorphism Φ_b yields $\Phi_b(0_{\pi^{-1}(b)}) = 0_{\mathbb{R}^k}$, and by linearity, we obtain

$$\sum_{i=1}^k c_i \Phi_b(s_i(b)) = 0_{\mathbb{R}^k}.$$

By definition, $s_i(b) = \Phi^{-1}(b, e_i)$, hence $\Phi_b(s_i(b)) = e_i$, which implies that $\sum_{i=1}^k c_i e_i = 0_{\mathbb{R}^k}$. Since $\{e_1, \dots, e_k\}$ is a basis for $\mathbb{R}^k$, it follows that $c_1 = c_2 = \dots = c_k = 0$. As this holds for an arbitrary $b \in B$, the sections $s_1, \dots, s_k$ are linearly independent.

Conversely, suppose that the vector bundle (E, B, π) admits k linearly independent sections $s_j : B \rightarrow E$ for $j = 1, 2, \dots, k$. By definition, each s_j is a continuous map satisfying $\pi \circ s_j = \text{id}_B$. Since the rank of the bundle is k , each fibre $\pi^{-1}(b)$ is a k -dimensional real vector space. It follows that for every $b \in B$, the set $\{s_1(b), s_2(b), \dots, s_k(b)\}$ is a basis for the fibre $\pi^{-1}(b)$.

Now for any point $(b, v) \in B \times \mathbb{R}^k$, where $v = (v_1, \dots, v_k) \in \mathbb{R}^k$, we define $\Psi : B \times \mathbb{R}^k \rightarrow E$ by $\Psi(b, v) = \sum_{j=1}^k v_j s_j(b)$. It is clear that $\Psi(b, v)$ is a linear combination of vectors in $\pi^{-1}(b)$, so $\pi(\Psi(b, v)) = b$. This means that $\pi \circ \Psi = \pi_1$. Moreover, for a fixed $b \in B$, consider the restriction $\Psi_b : \mathbb{R}^k \rightarrow \pi_b^{-1}$ given by $\Psi_b(v) = \sum v_j s_j(b)$, which is clearly linear. Note also that it sends the basis $e_1, \dots, e_k$ of $\mathbb{R}^k$ to the basis $\{s_1(b), \dots, s_k(b)\}$ of the fibre. Thus, Ψ_b is an isomorphism for all $b \in B$.

Fix a local trivialisation $\varphi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$. For each section s_i and $b \in U$, let $\varphi(s_i(b)) = (b, u_i(b))$. Define the column matrix $S(b) := (u_1(b) \cdots u_k(b)) \in \text{GL}_k(\mathbb{R})$. It follows that $(\varphi \circ \Psi)(b, v) = (b, S(b)v)$ with the inverse defined by $(b, v) \mapsto (b, S(b)^{-1}v)$. Since $b \mapsto u_i(b)$ is continuous, and inversion on GL_k is continuous, both coordinate maps are continuous.

Therefore, Ψ is a bundle isomorphism, and hence the bundle (E, B, π) is trivial. $\square$

Note that for every point in the smooth loci of an algebraic variety, the tangent space defined in 2.16 is equivalent to which in the differential-geometric setting. This then leads to the following definition:

Definition 4.9. Let $f : M \rightarrow N$ be smooth and $p \in M$ be a point. The *differential* of f at p is the linear map

$$df_p : T_p(M) \rightarrow T_{f(p)}(N), \quad v \mapsto Jf(p)v,$$

where $Jf(p) = \frac{\partial f_i}{\partial x_i}(p)$ is the $m \times n$ Jacobian matrix.

Definition 4.10. Let $f : M \rightarrow N$ be a smooth map of manifolds. M is said to be a *embedded submanifold* of N if

- the differential $df_p : T_p M \rightarrow T_{f(p)} N$ is injective for all $p \in M$,
- the map f is an injection, and it is a homeomorphism onto its image $f(M)$ with the subspace topology.

The *codimension* of N in M is given by $\text{codim}_N(M) := \dim M - \dim N$.

Definition 4.11 (Transversality). Let $f : M \rightarrow N$ be a smooth map, and let $S \subseteq N$ be an embedded submanifold. The map f is said to be *transverse* to S if for every point $p \in M$ with $f(p) \in S$, the image

of the differential df_p and the tangent space of the submanifold together spans the tangent space $T_{f(p)}N$. That is,

$$\text{Im}(df_p) + T_{f(p)}S = T_{f(p)}N.$$

Furthermore, two embedded submanifold $S_1, S_2 \subseteq N$ are said to *intersect transversely* if for every $q \in S_1 \cap S_2$, their tangent spaces at q span the tangent space of N , namely,

$$T_q S_1 + T_q S_2 = T_q N.$$

Theorem 4.12 ([12, Theorem 1 & 2 Chapter 1 Section 5]). *If the smooth map $f : M \rightarrow N$ is transversal to a submanifold $S \subset N$, then the preimage $f^{-1}(S) \subset M$ is also a submanifold. Consequently, we have*

$$\text{codim}_M(f^{-1}(S)) = \text{codim}_N S.$$

Similarly, if the embedded submanifolds S_1 and S_2 are transversal, then their intersection $S_1 \cap S_2$ is also an embedded submanifold of N . Moreover, we have the following codimension property

$$\text{codim}_N(S_1 \cap S_2) = \text{codim}_N S_1 + \text{codim}_N S_2.$$

Example 4.13. Let

$$S_1 := \{(x, y, z) \in \mathbb{R}^3 \mid F_1(x, y, z) = x^2 + y^2 + z^2 - 1 = 0\}$$

be a unit 2-sphere centred at the origin, and

$$S_2 := \{(x, y, z) \in \mathbb{R}^3 \mid F_2(x, y, z) = (x - 1)^2 + y^2 + z^2 - 1 = 0\}$$

be a unit 2-sphere centred at $(1, 0, 0)$. Then $S_1 \cap S_2 := \{(\frac{1}{2}, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 = \frac{3}{4}\}$, which represents a circle of radius $\frac{\sqrt{3}}{2}$ in the plane $x = \frac{1}{2}$.

At a point $q = (\frac{1}{2}, y, z) \in S_1 \cap S_2$, we have

$$\nabla F_1(q) = (1, 2y, 2z), \quad \nabla F_2(q) = (-1, 2y, 2z),$$

which are linearly independent since y, z do not vanish simultaneously. So $\text{span}\{\nabla F_1(q), \nabla F_2(q)\}$ is 2-dimensional, which implies that the intersection is indeed transversal. Moreover, transversality gives

$$\text{codim}_{\mathbb{R}^3}(S_1 \cap S_2) = \text{codim}_{\mathbb{R}^3} S_1 + \text{codim}_{\mathbb{R}^3} S_2 = 1 + 1 = 2,$$

hence $\dim(S_1 \cap S_2) = 3 - 2 = 1$, which matches the fact that $S_1 \cap S_2$ is a circle.

Now let $S_3 := \{(x, y, z) \in \mathbb{R}^3 \mid (x - 2)^2 + y^2 + z^2 - 1 = 0\}$ be a unit 2-sphere centred at $(2, 0, 0)$. Then the intersection $S_1 \cap S_3 = \{(1, 0, 0)\}$, and so $T_q S_1 = T_q S_3 = \{x = 1\}$. Thus $T_q S_1 + T_q S_3 \neq T_q \mathbb{R}^3$, and the intersection is not transversal.

Example 4.14 (S^1 -action on $\mathbb{C}$). Consider the manifold $\mathbb{C} \cong \mathbb{R}^2$ with standard coordinates (x, y) , where a point is represented as $z = x + iy$. We endow $\mathbb{C}$ with the symplectic form $\omega = dx \wedge dy$. Consider the Lie group $S^1 = U(1) = \{e^{i\theta} \in \mathbb{C} \mid \theta \in \mathbb{R}\}$, whose Lie algebra is $\mathfrak{u}(1) \cong i\mathbb{R}$, which is isomorphic to $\mathbb{R}$ via the map $i\alpha \mapsto \alpha$. Consequently, the dual of the Lie algebra, $\mathfrak{u}(1)^*$ is also isomorphic to $\mathbb{R}$.

Define the action of S^1 on $\mathbb{C}$ by complex multiplication

$$\Phi : S^1 \times \mathbb{C} \rightarrow \mathbb{C}, \quad (e^{i\theta}, z) \mapsto e^{i\theta} z.$$

In real coordinates, if $z = x + iy$, then $e^{i\theta} z = (\cos \theta + i \sin \theta)(x + iy) = (x \cos \theta - y \sin \theta) + i(x \sin \theta + y \cos \theta)$. It is clear that this action preserves the area form $dx \wedge dy$, and so it is a symplectic transformation.

For each element $X \in \mathfrak{u}(1)$, we have a vector field $X^\#$ on $\mathbb{C}$ defined by $X^\#(p) = \frac{d}{dt}|_{t=0}(\exp(tX) \cdot p)$. Take the basis element $X = i \in \mathfrak{u}(1)$, the corresponding 1-PS is $\exp(ti)$. Then the vector field at a point (x, y) is

$$X^\#(x, y) = (-y, x) = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y},$$

which is Hamiltonian. The moment map $\mu : \mathbb{C} \rightarrow \mathfrak{u}(1)^* \cong \mathbb{R}$ is defined by the relation $d\langle \mu, X \rangle = \iota_{X^\#} \omega$ for all $X \in \mathfrak{u}(1)$, where $\langle \cdot, \cdot \rangle$ is the pairing between $\mathfrak{u}(1)^*$ and $\mathfrak{u}(1)$. Let $\mu(x, y)$ be the function on $\mathbb{C}$

corresponding to the basis element $X = i$. We have

$$d\mu = \iota_{X\#}\omega = (-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}) \lrcorner (dx \wedge dy) = -y(-dy) + x(dx) = xdx + ydy.$$

Hence $d\mu = d(\frac{1}{2}(x^2 + y^2))$, and we can choose the moment map to be $\mu(x, y) = \frac{1}{2}(x^2 + y^2) = \frac{1}{2}|z|^2$.

Let $\{\xi\} \subset \mathbb{R}$ be a point for some $\xi \in \mathbb{R}$, where $\{\xi\}$ is clearly a 0-dimensional embedded submanifold of $\mathbb{R}$. The level set of the moment map μ is

$$\mu^{-1}(\xi) = \{z \in \mathbb{C} \mid \frac{1}{2}|z|^2 = \xi\}.$$

If $\xi < 0$, then $\mu^{-1}(\xi)$ is the empty set, which is trivially a submanifold. If $\xi = 0$, then $\mu^{-1}(0) = \{0\}$, a single point. If $\xi > 0$, then $\mu^{-1}(\xi) = \{z \in \mathbb{C} \mid |z|^2 = 2\xi\}$, which is a circle of radius $\sqrt{2\xi}$ centred at the origin.

At a point $p = (x, y) \in \mathbb{C}$, the differential $d\mu$ is the linear map $d\mu_p : T_p\mathbb{C} \rightarrow T_{\mu(p)}\mathbb{R}$. By definition, the moment map μ is transverse to $\{\xi\}$ if for every point $p \in \mu^{-1}(\xi)$, we have $\text{Im}(d\mu_p) + T_\xi\{\xi\} = T_\xi\mathbb{R}$, which is equivalent to $d\mu_p$ being surjective. For $\xi > 0$, the origin $(0, 0)$ is not in the level set $\mu^{-1}(0)$, which implies that the differential $d\mu_p = xdx + ydy$ is a surjective linear map. Therefore, μ is transverse to the submanifold $\{\xi\}$ for any $\xi > 0$. It follows that $\mu^{-1}(\xi)$ is a submanifold of $\mathbb{C}$, i.e. a circle of radius $\sqrt{2\xi}$, which is indeed a 1-dimensional submanifold.

For $\xi = 0$, the level set is $\mu^{-1}(0) = \{(0, 0)\}$. The differential $d\mu_p$ is then the zero map at this point, and its image $\text{Im}(d\mu_{(0,0)}) = \{0\}$, which means that μ is not transverse to $\{0\}$. Moreover, the point $0 \in \mathbb{R}$ is a critical value of μ and the point $(0, 0)$ is a critical point. Note that although in this example the level set $\mu^{-1}(0) = \{0\}$ is a 0-dimensional submanifold, this is not guaranteed in general when transversality fails.

Proposition 4.15. *Let $f : M \rightarrow N$ be a map that is transverse to a submanifold $S \subseteq N$, and let $W := f^{-1}(S)$. Then the tangent space $T_p(W)$ is the preimage of $T_{f(p)}(S)$ under the differential $df_p : T_p(M) \rightarrow T_p(N)$.*

Proof. Since f is transverse to the submanifold S , by definition, we have $df_p(T_pM) + T_{f(p)}S = T_{f(p)}N$ for every $p \in W = f^{-1}(S)$, which is also a submanifold. Let $v \in T_p(W)$ be a tangent vector. Since $f(W) \subset S$, $df_p(v)$ lies in $T_{f(p)}S$. Hence $T_p(W) \subseteq df_p^{-1}(T_{f(p)}S)$. The codimension condition implies that

$$\dim(df_p(T_pM) + T_{f(p)}S) = \dim T_{f(p)}N = \dim N,$$

which means that

$$\dim(df_p(T_pM)) + \dim(T_{f(p)}S) - \dim(df_p(T_pM) \cap T_{f(p)}S) = \dim N.$$

Because df_p is also a linear operator, it follows that $\dim(df_p^{-1}(T_{f(p)}S)) = \dim M + \dim S - \dim N = \dim W$. Therefore, we conclude that $df_p(T_p(W)) = T_p(W)$. $\square$

Definition 4.16. Let $f : M \rightarrow N$ be a smooth map. A point $p \in M$ is *regular* if the differential $df_p : T_pM \rightarrow T_pN$ is surjective. A point $p \in M$ is called *critical* if it is not a regular point. A point $q \in N$ is a *regular value* of f if $q \in f(M)$ and p is a regular point for all $p \in f^{-1}(q)$. A point $q \in N$ is called a *critical value* otherwise.

Theorem 4.17 (Sard's Theorem [12]). *If $f : M \rightarrow N$ is a smooth map of manifolds, then almost every point in N is a regular value of f . Equivalently, the set of critical values of f has measure zero.*

Theorem 4.18 (Parametric Transversality Theorem [22, Theorem 6.35]). *Suppose M and N are smooth manifolds, $S \subseteq N$ is an embedded submanifold, and $\{F_j : j \in J\}$ is a family of smooth maps from $M \rightarrow N$. If the map $F : M \times J \rightarrow N$ is transverse to S , then for almost every $j \in J$, the map $F_j : M \rightarrow N$ is transverse to S .*

Proposition 4.19 (General Position Lemma [12]). *Let M and N be submanifolds of $\mathbb{R}^n$. Then for almost every $a \in \mathbb{R}^n$ the translate $M + a$ intersects N transversely.*

Proof. Let $F : M \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be given by $(m, a) \mapsto m + a$. This map is clearly smooth as it is the addition in $\mathbb{R}^n$. In particular, we denote $F_a : M \rightarrow \mathbb{R}^n$ with $m \mapsto m + a$ for each fixed $a \in \mathbb{R}^n$.

We first show that F is a submersion, i.e. the differential of F is surjective. Observe that $\dim(M \times \mathbb{R}^n) = \dim M + \dim \mathbb{R}^n = \dim M + n$. Let $((m, a), (u, v)) \in T(M \times \mathbb{R}^n)$, where $u \in T_m M$ and $v \in T_a \mathbb{R}^n \cong \mathbb{R}^n$. Thus $D_{(m,a)}F(u, v) = u + v$, where $u + v \in T_{(m+a)}\mathbb{R}^n \cong \mathbb{R}^n$. Note that for any point $v \in \mathbb{R}^n$, setting $u = 0$ yields $D_{(m,a)}F(0, v) = 0 + v = v$, which implies that for every $v \in \mathbb{R}^n$, there exists a pair $(0, v)$ in the domain of the differential $D_{(m,a)}F$. Hence it is surjective and therefore a submersion., which implies that F is transverse to N .

Now by the parametric transversality theorem (Theorem 4.18), the map $F_a : M \rightarrow \mathbb{R}^n$ with $F_a(m) = m + a$ is transverse to N for almost every $a \in \mathbb{R}^n$, which means

$$\text{Im}(D_m F_a) + T_{F_a(m)} N = T_{F_a(m)} \mathbb{R}^n \cong \mathbb{R}^n$$

Since F_a is a translation, we have $D_m F_a(u) = u$ for all $u \in T_m M$, hence the image $\text{Im}(D_m F_a) \cong T_m M$. Moreover, the tangent space $T_x(M + a)$ to the translated manifold $M + a$ at $x = m + a$ is the translation of $T_m M$ to the point x , it follows that $T_x(M + a) = T_m M$. Therefore, we can conclude that $M + a$ intersects N transversely. $\square$

Proposition 4.20. *Let $X \subseteq Y$ be a compact submanifold that intersects another submanifold Z , with $\dim X + \dim Z < \dim Y$. Then X can be pulled away from Z by an arbitrarily small deformation, i.e. given $\epsilon > 0$ there exists a deformation $X_t = i_t(X)$ such that X_1 does not intersect Z and $|x - i_1(x)| < \epsilon$ for all $x \in X$.*

Proof. Note that two submanifolds X, Z of Y are transverse at $y \in Y$ if $T_y X + T_y Z = T_y Y$, which requires that $\dim X + \dim Z - \dim(T_y X \cap T_y Z) = \dim Y$. Since $\dim X + \dim Z < \dim Y$, it follows that the intersection of X and Z cannot be transverse. Given that X is a compact submanifold of Y , we may apply the ϵ -neighbourhood theorem [12, Chapter 2 Section 3].

Define the normal bundle of X in Y as

$$N_{X/Y} := \{(x, v) \mid x \in X, v \in T_x Y / T_x X\},$$

where $T_x Y / T_x X$ is the normal space to the tangent space $T_x X$ of X at x in $T_x Y$. Then there is an open neighbourhood $U \subset Y$ of X and a diffeomorphism $\varphi : N_{X/Y} \rightarrow U$, which satisfies $\varphi(x, 0) = x$ for all $x \in X$, i.e. φ maps the zero section of $N_{X/Y}$ onto X [12, Chapter 2]. (Note that the compactness of X implies that U can be chosen to be sufficiently small.)

To deform X , we consider a smooth section $\sigma : X \rightarrow N_{X/Y}$, and a family of maps

$$i_t : X \rightarrow Y, \quad i_t(x) = \varphi(t\sigma(x)), \quad \text{for all } t \in [0, 1],$$

such that for $t = 0$, $i_0(x) = \varphi(0 \cdot \sigma(x)) = \varphi(x, 0) = x$, which is the inclusion. For $t = 1$, $i_1(x) = \varphi(\sigma(x))$, which perturbs X along the direction of $\sigma(x)$, representing the deformed submanifold. Furthermore, if the section σ is chosen with a sufficiently small norm, then each i_t is an embedding, and thus $X_t = i_t(X)$ is a submanifold of Y that is diffeomorphic to X .

Now we need to choose a σ such that $X_1 \cap Z = \emptyset$ and the deformation is arbitrarily small, where $X_1 := i_1(X)$. Consider $W := \varphi^{-1}(Z \cap U) \subset N_{X/Y}$. Since φ is a diffeomorphism and $Z \cap U$ is a submanifold, it follows that W is a submanifold of $N_{X/Y}$ with $\dim W = \dim Z$. So X_1 being disjoint from Z is equivalent to $\sigma(X) \cap W = \emptyset$.

By the parametric transversality theorem, the set of smooth sections of $N_{X/Y}$ that are transverse to W is residual, and thus a dense subset of the space of all smooth sections of the normal bundle. We can therefore choose a section σ_0 from this dense subset whose image $\sigma_0(X)$ is transverse to W . Then we obtain

$$\dim(\sigma_0(X) \cap W) = \dim(\sigma_0(X)) + \dim W - \dim N_{X/Y}.$$

Observe that $\dim \sigma_0(X) = \dim X$ (as φ is a diffeomorphism), and that $\dim N_{X/Y} = \dim Y$, we have

$$\dim(\sigma_0(X) \cap W) = \dim X + \dim Z - \dim Y.$$

Note that the hypothesis $\dim X + \dim Z < \dim Y$ implies that the dimension $\dim(\sigma(X) \cap W) < 0$, hence $\sigma_0(X) \cap W = \emptyset$. Thus we have shown that $X_1 = \varphi(\sigma_0(X))$ does not intersect $Z \cap U$.

Moreover, note that given any $\epsilon > 0$, we can always find a transverse section σ_0 that is arbitrarily close to the zero section in the uniform norm. Without loss of generality, let σ_0 be such that $\sup_{x \in X} \|\sigma_0(x)\| < \delta$ for some $\delta > 0$, where δ is small enough such that the $|x - i_1(x)| = |x - \varphi(\sigma_0(x))| < \epsilon$. Note also that the compactness of $X \subset Y$ implies that the distance $\text{dist}(X, Y \setminus U)$ from X to the closed set $Y \setminus U$ is

positive, since if the distance were zero, then the compactness of X would imply the existence of a point in X that is also a limit point of $Y \setminus U$, contradicting the closedness of the complement. Now consider $\epsilon' = \min(\epsilon, \text{dist}(X, Y \setminus U))$, then we can similarly find a section such that $|x - i_1(x)| < \epsilon'$ so that the entire deformed manifold $i_1(X) = X_1$ remains in U . Since $\sigma_0(X) \cap W = \emptyset$, it follows that $X_1 \cap Z = \emptyset$. $\square$

Proposition 4.21. *Let X be a submanifold of $\mathbb{R}^n$. Then almost every vector space V of any fixed dimension ℓ in $\mathbb{R}^n$ intersects X transversely.*

Proof. The transversality of X and V means that $T_p V + T_p X = \mathbb{R}^n$ for all $p \in V \cap X$, which implies that for all such p , we have $V + T_p X = \mathbb{R}^n$. Let $\text{Gr}(\ell, n)$ be the set of all ℓ -dimensional subspaces of $\mathbb{R}^n$ with $\dim(\text{Gr}(\ell, n)) = \ell(n - \ell)$. We want to show that the set of V that is not transverse to X is a set of measure zero in the Grassmannian $\text{Gr}(\ell, n)$.

Let $S \subset (\mathbb{R}^n)^\ell$ be an open subset of ℓ -tuples $(v_1, \dots, v_\ell)$. Then every ordered ℓ -tuple $s = (v_1, \dots, v_\ell) \in S$ spans an ℓ -dimensional subspace $V_s = \text{span}(v_1, \dots, v_\ell)$ of $\mathbb{R}^n$. Furthermore, note that $\text{GL}_\ell(\mathbb{R})$ acts on S by $(v_1, \dots, v_\ell) \cdot A = (v_1, \dots, v_\ell)A$ for all $A \in \text{GL}_\ell(\mathbb{R})$, so the right action induces new linear combinations of $v_1, \dots, v_\ell$. Thus we have $\text{Gr}(\ell, n) \cong S / \text{GL}_\ell(\mathbb{R})$. It follows that the projection map $\pi : S \rightarrow \text{Gr}(\ell, n)$, that is given by $\pi(s) = V_s$, is a smooth submersion.

Suppose that $0 \in X$. Note that a subspace V_s is transverse to X at the origin if $V_s + T_0 X = \mathbb{R}^n$ for $s \in S$. Let $\dim T_0 X = \dim X = k$, and consider the projection $\text{pr} : \mathbb{R}^n \rightarrow \mathbb{R}^n / T_0 X$, where $\dim(\mathbb{R}^n / T_0 X) = n - k$. Then the condition

$$V_s + T_0 X = \mathbb{R}^n$$

is equivalent to the surjectivity of the restriction map $\text{pr}|_{V_s} : V_s \rightarrow \mathbb{R}^n / T_0 X$. This requires that

$$\dim \text{Im}(\text{pr}|_{V_s}) = \text{rank}(\text{pr}(v_1), \dots, \text{pr}(v_\ell)) = \dim \mathbb{R}^n / T_0 X = n - k.$$

Note that this equality holds for $\ell \geq n - k$, which also aligns the transversality condition $\dim V + \dim X \geq \dim \mathbb{R}^n$.

So it can be deduced that V_s is not transverse if and only if $\text{rank}\{\text{pr}(v_1), \dots, \text{pr}(v_\ell)\} < n - k$, which holds if and only if all $(n - k) \times (n - k)$ minors of the matrix representation of these vectors vanish. The vanishing of these minors defines a proper real variety of S that is closed. Hence the set

$$S_0 := \{s \in S \mid V_s + T_0 X = \mathbb{R}^n\}$$

is the complement of the proper subset of S . Thus S_0 is open and dense, and its complement $S_0^\perp$ is of measure zero.

For all non-zero intersection points, consider the map $\Phi : \mathbb{R}^\ell \times S \rightarrow \mathbb{R}^n$ defined by $\Phi(t, s) = \sum_{i=1}^\ell t_i v_i$. Let (t, s) be any point such that $p = \Phi(t, s) \in X \setminus \{0\}$. Clearly $p \neq 0$, which implies $t \neq 0$. It follows that the differential $d\Phi_{(t,s)}(\dot{t}, \dot{s}) = \sum \dot{t}_i v_i + \sum t_i \dot{v}_i$ is surjective, and hence Φ is a submersion at (t, s) . Therefore, the preimage

$$\Phi^{-1}(X \setminus \{0\}) = \left\{ ((t_1, \dots, t_\ell), (v_1, \dots, v_\ell)) \mid \sum_{i=1}^\ell t_i v_i \in X \setminus \{0\} \right\}$$

is a submanifold of $\mathbb{R}^\ell \times S$, and Φ is transverse to $X \setminus \{0\}$.

By the parametric transversality theorem, for almost every $s \in S$ (i.e. the set of $s \in S$ for which the map Φ_s is not transverse to $X \setminus \{0\}$ is a set of measure zero), the map $\Phi_s : \mathbb{R}^\ell \setminus \{0\} \rightarrow \mathbb{R}^n$ with $\Phi_s(t) = \Phi(t, s)$ is transverse to $X \setminus \{0\}$. Hence we obtain

$$\text{Im}(d\Phi_s(t)) + T_p(X \setminus \{0\}) = T_p \mathbb{R}^n \cong \mathbb{R}^n$$

for every $t \in \mathbb{R}^\ell \setminus \{0\}$ such that $p = \Phi_s(t) \in X \setminus \{0\}$. Since $\text{Im}(\Phi_s) = V_s$, we thus yield $V_s + T_p X = \mathbb{R}^n$ for all $p \in V_s \cap (X \setminus \{0\})$, and we can conclude that V intersects $X \setminus \{0\}$ transversely. $\square$

5 Homotopy Groups

In this chapter, we recall some basic notions in the homotopy theory and define n -th homotopy groups. The main reference is Hatcher's book [15].

Let X be a topological space. A continuous map $f : I \rightarrow X$ is called a *path*, where $I := [0, 1]$. If a path f has the same starting and ending point, i.e. $f(0) = f(1) = x_0 \in X$, then it is called a *loop*.

Definition 5.1. A topological space X is *path-connected* if for any two points $x, y \in X$, there exists a path $f : I \rightarrow X$ such that $f(0) = x$ and $f(1) = y$.

Definition 5.2. Let $f, g : X \rightarrow Y$ be continuous maps between topological spaces. A *homotopy* from f to g is a continuous map $H : X \times I \rightarrow Y$ such that $H|_{X \times \{0\}} = f : X \rightarrow Y$ and $H|_{X \times \{1\}} = g : X \rightarrow Y$. We say the maps f, g are *homotopic*, and denote $f \sim g$.

Example 5.3 (Straight Line Homotopy). Let X be a topological vector space and let $C \subset X$ be a convex subset. For any continuous maps $f, g : X \rightarrow C$, define

$$H : X \times I \rightarrow C, \quad H(x, t) = (1 - t)f(x) + tg(x).$$

Then we have $H(x, 0) = f(x)$, and $H(x, 1) = g(x)$, so H is a homotopy from f to g . Such a homotopy is called the *straight line homotopy*.

Note that the convexity of C guarantees the definition of the homotopy $H(x, t)$ is valid. If C is non-convex, for example $S^1 \subset \mathbb{R}^n$, then $H(x, t)$ need not take values in C .

Proposition 5.4. *Homotopy is an equivalence relation on continuous maps $X \rightarrow Y$.*

Proof. For any continuous $f : X \rightarrow Y$, the map $H(x, t) := f(x)$ is continuous and satisfies $H(x, 0) = H(x, 1) = f(x)$. Hence $f \sim f$, proving the reflexivity. If H is a homotopy from $f \rightarrow g$, define $H'(x, t) = H(x, 1 - t)$. The map $t \mapsto 1 - t$ is continuous, so H' is continuous, with $H'(x, 0) = H(x, 1) = g(x)$ and $H'(x, 1) = H(x, 0) = f(x)$. Thus $g \sim f$, showing the symmetry.

Let H be a homotopy from f to g and K a homotopy from g to h . Define

$$L(x, t) = \begin{cases} H(x, 2t) & 0 \leq t \leq \frac{1}{2}, \\ K(x, 2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

At $t = \frac{1}{2}$, we have $H(x, 1) = g(x) = K(x, 0)$, so the two maps agrees. Furthermore, $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are closed in I , hence L is continuous on $X \times I$. It is clear that $L(x, 0) = H(x, 0) = f(x)$ and $L(x, 1) = K(x, 1) = h(x)$, so we can conclude that $f \sim h$, which proves the transitivity. $\square$

Definition 5.5 (Path Homotopy). Let $f, g : I \rightarrow X$ be two paths in a topological space X with $f(0) = g(0) = x_0$ and $f(1) = g(1) = x_1$. A *path homotopy* between f and g is a homotopy $H : I \times I \rightarrow X$ such that

- $H(s, 0) = f(s)$ and $H(s, 1) = g(s)$ for all $s \in I$;
- $H(0, t) = x_0$ and $H(1, t) = x_1$ for all $t \in I$ i.e. the endpoints of the path remain fixed throughout the homotopy.

Similarly, path homotopy is an equivalence relation on the set of paths from x_0 to x_1 , and we denote such an equivalence class of f by $[f]$.

Definition 5.6. Let $x_0 \in X$ be a chosen basepoint. The *fundamental group* $\pi_1(X, x_0)$ is the set of path homotopy classes of loops in X based at x_0 . That is,

$$\pi_1(X, x_0) := \{[\gamma] \mid \gamma : I \rightarrow X \text{ is a loop with } \gamma(0) = \gamma(1) = x_0\}.$$

We can indeed equip the set $\pi_1(X, x_0)$ a group structure. Given two loops γ_1, γ_2 based at x_0 , their *concatenation* $\gamma_1 \cdot \gamma_2$ is the loop defined by

$$(\gamma_1 \cdot \gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \gamma_2(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

The concatenation is well-defined in the sense that if $\gamma_1 \sim \gamma'_1$ and $\gamma_2 \sim \gamma'_2$, then $\gamma_1 \cdot \gamma_2 \sim \gamma'_1 \cdot \gamma'_2$. Moreover, the group operation is defined by $[\gamma_1] \cdot [\gamma_2] := [\gamma_1 \cdot \gamma_2]$.

Proposition 5.7 ([23, Theorem 7.13]). *Let X be a path-connected space, and let $x_0, x_1 \in X$ be distinct points. Then for any integer $n \geq 1$, there is a group isomorphism*

$$\pi_1(X, x_0) \cong \pi_1(X, x_1).$$

In other words, the fundamental group of a path-connected space does not depend on basepoints.

Proof. Since X is path-connected, there exists a path $\gamma : I \rightarrow X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$, which also induces a map from the loops based at x_0 to the loops based at x_1 .

Let γ be a path that induces a map $\Phi_\gamma : \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$ defined by conjugation. Then for any $[f] \in \pi_1(X, x_1)$, we have

$$\Phi_\gamma([f]) := [\gamma \cdot f \cdot \gamma^{-1}],$$

where $\gamma^{-1}(t) = \gamma(1 - t)$ and $\cdot$ denotes path concatenation. It is clear that $\gamma \cdot f \cdot \gamma^{-1}$ is a loop based at x_0 . Note that for any $[f], [g] \in \pi_1(X, x_1)$,

$$\Phi_\gamma([f] \cdot [g]) = [\gamma \cdot (f \cdot g) \cdot \gamma^{-1}] = [(\gamma \cdot f \cdot \gamma^{-1}) \cdot (\gamma \cdot g \cdot \gamma^{-1})] = \Phi_\gamma([f]) \cdot \Phi_\gamma([g]).$$

Hence the map Φ_γ is indeed a group homomorphism. Moreover, the reverse path γ^{-1} induces a homomorphism $\Phi_{\gamma^{-1}} : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$. It is straightforward to show that $\Phi_\gamma \circ \Phi_{\gamma^{-1}}$ and $\Phi_{\gamma^{-1}} \circ \Phi_\gamma$ are the respective identity maps. For instance, for $[f] \in \pi_1(X, x_1)$,

$$\Phi_{\gamma^{-1}}(\Phi_\gamma([f])) = [\gamma^{-1} \cdot (\gamma \cdot f \cdot \gamma^{-1}) \cdot (\gamma^{-1})^{-1}] = [(\gamma^{-1} \cdot \gamma) \cdot f \cdot (\gamma^{-1} \cdot \gamma)] = [f].$$

So we can conclude that Φ_γ is an isomorphism. $\square$

Definition 5.8. A topological space X is *simply-connected* if it is path-connected and its fundamental group is trivial for any basepoint $x_0 \in X$.

Proposition 5.9. Let X be a path-connected space. Then X is simply connected if and only if any two paths in X with the same starting and end points are path-homotopic.

Proof. Assume that X is simply connected. Let $\alpha, \beta : I \rightarrow X$ be two paths that have the same starting and end points, i.e. $\alpha(0) = \beta(0) = x_0$ and $\alpha(1) = \beta(1) = x_1$. Denote the inverse path of β by β^{-1} , where $\beta^{-1}(s) = \beta(1 - s)$, then it is a path from x_1 to x_0 . Consider a path $\gamma : \alpha \cdot \beta^{-1}$ formed by the concatenation of paths. Since both the starting and end points of γ are x_0 , it is a loop based at x_0 . Note that a consequence of simply-connectedness of X is that every loop based at x_0 is null-homotopic. Therefore, γ is path-homotopic to a constant loop c_{x_0} at x_0 . Since

$$(\alpha \cdot \beta^{-1}) \cdot \beta \sim c_{x_0} \cdot \beta \sim \beta,$$

and

$$\alpha \cdot (\beta^{-1} \cdot \beta) \sim \alpha \cdot c_{x_1} \sim \alpha,$$

it follows that $\alpha \sim \beta$.

Conversely, assume that for any $x_0, x_1 \in X$, two arbitrary paths from x_0 to x_1 are path-homotopic. Let $\gamma : I \rightarrow X$ be any loop based at x_0 , then we have $\gamma(0) = \gamma(1) = x_0$. Consider the constant path $c_{x_0} : I \rightarrow X$, which is also a path from x_0 to x_0 . So we have two paths, γ and c_{x_0} , that are both based at x_0 . Consequently, we conclude $\gamma \sim c_{x_0}$, which means γ is null-homotopic. Moreover, since the choice of γ were arbitrary, every loop based at x_0 is null-homotopic. This implies that the fundamental group $\pi_1(X, x_0)$ is trivial. Therefore, X is simply connected. $\square$

Example 5.10 ($\mathbb{R}^n$ is simply connected). We show that $\mathbb{R}^n$ is simply connected by first establishing that every convex subset of $\mathbb{R}^n$ is simply connected. An alternative approach of proving this is via the strong deformation retraction $F(x, t) = (1 - t)x + tx_0$, where x_0 is a basepoint.

A subset $C \subseteq \mathbb{R}^n$ is *convex* if for any two points $x, y \in C$, the linear combination $(1 - t)x + ty \in C$ for all $t \in I$. Consider the map $f : I \rightarrow C$ defined by $f(t) = (1 - t)x + ty$, which is the straight line segment between x and y . Furthermore, observe that $f(0) = x$ and $f(1) = y$, so the set C is path-connected.

Let $\gamma : I \rightarrow C$ be a loop based at x_0 . Consider the straight line homotopy $H : I \times I \rightarrow C$ defined by $H(s, t) = (1 - t)\gamma(s) + tx_0$. Then this gives a homotopy between the loop γ and the constant loop c_{x_0} , which implies that the fundamental group $\pi_1(C, x_0)$ is trivial. Hence C is simply connected.

Now that the line segment joining two arbitrary points $x, y \in \mathbb{R}^n$ is the set $\{(1 - t)x + ty \mid t \in [0, 1]\}$, which is a subset of $\mathbb{R}^n$ because $\mathbb{R}^n$ is a vector space. Hence $\mathbb{R}^n$ is a convex set, and thus is simply connected.

Definition 5.11. A topological space X is *contractible* if there exists a point $x_0 \in X$ and a homotopy (a *contraction*) $H : X \times I \rightarrow X$ such that $H(x, 0) = x$ and $H(x, 1) = x_0$ for all $x \in X$. This is equivalent of saying that the identity map id_X is homotopic to the constant map $f : X \rightarrow X$, $f(x) = x_0$.

Example 5.12 ($\mathbb{R}^n$ is contractible). Define $H : \mathbb{R}^n \times I \rightarrow \mathbb{R}^n$ by $H(x, t) = (1 - t)x$, which is the straight line homotopy by setting $f = \text{id}_{\mathbb{R}^n}$ and $g \equiv 0$. Then $H(x, 0) = x = \text{id}_{\mathbb{R}^n}(x)$ and $H(x, 1) = 0 = f(x)$, where f is a constant function. So H is a homotopy such that $\text{id}_{\mathbb{R}^n} \sim f$, i.e. it is a contraction of $\mathbb{R}^n$ to 0. Hence $\mathbb{R}^n$ is contractible.

Proposition 5.13. *Let X be a topological space, then the following conditions are equivalent:*

- (i) Every map $S^1 \rightarrow X$ is homotopic to a constant map whose image is a point.
- (ii) Every map $S^1 \rightarrow X$ extends to a map $D^2 \rightarrow X$, where D^2 is the closed 2-ball.
- (iii) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Proof. (i) $\Rightarrow$ (ii): Suppose that $f : S^1 \rightarrow X$ is a continuous map that is homotopic to a constant $x_0 \in X$, via the homotopy $H : S^1 \times I \rightarrow X$ with $H(s, 0) = f(s)$ for all $s \in S^1$, and $H(s, 1) = x_0$ for some constant $x_0 \in X$. Consider the quotient map $q : S^1 \times I \rightarrow (S^1 \times I)/(S^1 \times \{1\})$, then H is constant on the fibre of q . Thus H can be obtained by the composition

$$S^1 \times I \rightarrow (S^1 \times I)/(S^1 \times \{1\}) \rightarrow X.$$

Note that $(S^1 \times I)/(S^1 \times \{1\})$ is homeomorphic to the closed 2-disk D^2 via the map $\varphi : (S^1 \times I)/(S^1 \times \{1\}) \rightarrow D^2$ given by $\varphi(\theta, t) = (1 - t)e^{i\theta}$. Therefore, every map f extends to a map $D^2 \rightarrow X$.

(ii) $\Rightarrow$ (iii): Let $\pi_1(X, x_0)$ be the set of basepoint preserving homotopy class of maps $(S^1, s_0) \rightarrow (X, x_0)$, and let $\psi : (S^1, s_0) \rightarrow (X, x_0)$ be such a map, i.e. $\psi(s_0) = x_0$. Then ψ extends to a map $F : (D^2, s_0) \rightarrow (X, x_0)$ such that $F|_{S^1} = \psi$.

Define a map $H_1 : D^2 \times I \rightarrow D^2$ by $(x, t) \mapsto (1 - t)x + ts_0$, where s_0 is a boundary point on $S^1 \cong \partial D^2$. Then $H_1(x, 0) = x = \text{id}_{D^2}$ and $H_1(x, 1) = s_0$. Furthermore, for every $(x, t) \in D^2 \times I$, since

$$\|H_1(x, t)\| = \|(1 - t)x + ts_0\| \leq (1 - t) \cdot \|x\| + t\|s_0\| \leq (1 - t) + t = 1,$$

it follows that $H_1(x, t) \in D^2$, hence H_1 is a deformation retraction of D^2 onto $\{s_0\}$. Consider a homotopy $H_2 : S^1 \times I \rightarrow X$ with $H_2(s, t) = F(H_1(s, t)) = F((1 - t)s + ts_0)$. Because $H_2(s, 0) = F(s) = \psi(s)$ and $H_2(s, 1) = F(s_0) = \psi(s_0) = x_0$, we can conclude that H_2 is a basepoint-preserving homotopy of ψ to the constant map x_0 . Hence all loops in X are nullhomotopic, which proves that $\pi_1(X, x_0) = 0$.

(iii) $\Rightarrow$ (i): Because the space X is path-connected, any loop in X is nullhomotopic regardless of its basepoint. Hence the triviality of $\pi_1(X, x_0)$ implies that all maps $S^1 \rightarrow X$ are nullhomotopic, i.e. homotopic to a constant map, which proves (i). $\square$

We can deduce from the above proposition that a space X is simply-connected if and only if all maps $S^1 \rightarrow X$ are homotopic. Simply-connectedness of X implies that it is path-connected and its fundamental group is trivial, i.e. $\pi_1(X) = 0$. We have shown that trivial fundamental group implies every map $S^1 \rightarrow X$ is homotopic to a constant map ((iii) $\Rightarrow$ (i)). Path-connectedness of X implies that all constant maps to X are homotopic.

Conversely, assume that all maps $S^1 \rightarrow X$ are homotopic, then so are the constant maps. Hence X is path-connected, and by the above equivalent conditions we have $\pi_1(X) = 0$. Thus X is simply connected.

Definition 5.14. Let (X, x_0) be a pointed topological space. Let $I^n = [0, 1]^n$ be the n -cube and ∂I^n its boundary. For an integer $n \geq 1$, let $\text{Map}(X, x_0)$ be the set of all maps $f : I^n \rightarrow X$ with $f(\partial I^n) = x_0$. Two maps $f, g \in \text{Map}(X, x_0)$ are homotopic if there exists a map $H : I^n \times I \rightarrow X$ such that $H(s, 0) = f(s)$ for all $s \in I^n$, $H(s, 1) = g(s)$ for all $s \in I^n$, and $H(s, t) = x_0$ for all $s \in \partial I^n$ and all $t \in I$.

The n -th homotopy group of X based at x_0 , denoted $\pi_n(X, x_0)$, is the set of all equivalence classes

$$\pi_n(X, x_0) = \text{Map}(X, x_0) / \sim,$$

where $f \sim g$ if they are homotopic.

Definition 5.15. A space X with a basepoint x_0 is n -connected if its i -th homotopy group is trivial for $i \leq n$. In particular, 0-connected is called *path-connected* and 1-connected is called *simply-connected*.

Furthermore, there is a generalised version of Proposition 5.13, which is stated as follows.

- (i) Every map $S^i \rightarrow X$ is homotopic to a constant map.
- (ii) Every map $S^i \rightarrow X$ extends to a map $D^{i+1} \rightarrow X$.

(iii) $\pi_i(X, x_0) = 0$ for all $x_0 \in X$.

See Section 4.1 of Hatcher's book [15] for more details.

Example 5.16 (Homotopy groups of $\mathbb{R}P^n$). Note that $\mathbb{R}P^0$ is a singleton set and all its homotopy groups are trivial, so $\pi_0(\mathbb{R}P^0)$ is a singleton set. Note that the real projective space $\mathbb{R}P^n$ is obtained by identifying antipodal points on the n -sphere S^n via the projection $p : S^n \rightarrow \mathbb{R}P^n$, which also induces a homomorphism p_* between homotopy groups. Thus it is also path-connected and $\pi_0(\mathbb{R}P^n)$ is a singleton set.

$\mathbb{R}P^1$ is homeomorphic to S^1 via the map given by $z \mapsto z^2$, where S^1 is viewed as the set of unit complex numbers. This implies that their fundamental groups are isomorphic, hence $\pi_1(\mathbb{R}P^1) \cong \pi_1(S^1) \cong \mathbb{Z}$. For $n \geq 2$, the simply-connectedness of S^n implies that $\pi_1(S^n) = \{0\}$. To determine $\pi_1(\mathbb{R}P^n)$, we consider the following long exact sequence

$$\cdots \rightarrow \pi_1(S^n) \xrightarrow{p_*} \pi_1(\mathbb{R}P^n) \xrightarrow{\partial} \pi_0(S^0) \xrightarrow{i_*} \pi_0(S^n) \rightarrow \cdots,$$

where i_* is the map induced by the inclusion $i : S^0 \rightarrow S^n$, and we have $\pi_1(S^n) = \{0\}$ and $\pi_0(S^n) = \{*\}$. The space $S^0 = \{-1, 1\}$ has two path components, so the set $\pi_0(S^0)$ has two elements, which means that $\pi_0(S^0) \cong \mathbb{Z}_2$. Consequently, the map ∂ is injective since $\ker(\partial) = \text{im}(p_*) = \{0\}$, and it is also surjective since $\text{im}(\partial) = \ker(i_*) = \mathbb{Z}_2$. Hence ∂ is an isomorphism and we can conclude that $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$ for $n \geq 2$.

For higher homotopy groups, we claim that $\pi_q(S^n) \cong \pi_q(\mathbb{R}P^n)$ for all $q \geq 2$. To prove the claim, it suffices to show that the map $p : S^n \rightarrow \mathbb{R}P^n$ is a covering space projection, which then induces isomorphisms $p_* : \pi_q(S^n) \rightarrow \pi_q(\mathbb{R}P^n)$ for all $q \geq 2$ [15, Proposition 4.1].

Observe that $\mathbb{R}P^n := S^n / \sim$, where the equivalence relation $\sim$ identifies antipodal points, i.e. $x \sim y$ if and only if $y = x$ or $y = -x$. We endow $\mathbb{R}P^n$ with the quotient topology induced by the map $p : S^n \rightarrow \mathbb{R}P^n$, where $p(x) = [x] = \{x, -x\}$. By definition, the map p is surjective. To establish that p is a covering map, we need to show that for each $[x] \in \mathbb{R}P^n$, there exists an open neighbourhood U of $[x]$ such that the preimage $p^{-1}(U) \subset S^n$ is a disjoint union of open sets, i.e. $p^{-1}(U) = \bigsqcup_{i \in I} V_i$, and that the restriction $p|_{V_i}$ is a homeomorphism.

Let $[x_0] \in \mathbb{R}P^n$ be an arbitrary point. The fibre over this point is $p^{-1}([x_0]) = \{x_0, -x_0\} \subset S^n$. Consider the point $x_0 \in S^n$, and define an open neighbourhood of x_0 as

$$V_{x_0} = \{y \in S^n \mid y \cdot x_0 > 0\},$$

which is the open hemisphere centred at x_0 . Similarly, define an open neighbourhood of $-x_0$

$$V_{-x_0} = \{y \in S^n \mid y \cdot (-x_0) > 0\} = \{y \in S^n \mid y \cdot x_0 < 0\}.$$

It is clear to see that $V_{x_0} \cap V_{-x_0} = \emptyset$, and let $U := p(V_{x_0})$. Since its preimage $p^{-1}(U) = V_{x_0} \cup (-V_{x_0}) = V_{x_0} \cup V_{-x_0}$, which is open in S^n , it follows that U is an open neighbourhood of $[x_0]$ in $\mathbb{R}P^n$. Moreover, this implies that the restriction $p|_{V_{x_0}}$ is an open map.

We now show that $p|_{V_{x_0}} : V_{x_0} \rightarrow U$ and $p|_{V_{-x_0}} : V_{-x_0} \rightarrow U$ are homeomorphisms. We shall prove this for V_{x_0} as the argument for V_{-x_0} is identical. Note that the continuity and the surjectivity follow from p is a quotient map. For injectivity, suppose $y_1, y_2 \in V_{x_0}$ such that $p(y_1) = p(y_2)$. This implies that $y_1 = \pm y_2$, but we have $y_1 \cdot x_0 > 0$ and $y_2 \cdot x_0 > 0$, hence $y_1 = y_2$. So $p|_{V_{x_0}}$ is a homeomorphism.

Therefore, we conclude that the projection $p : S^n \rightarrow \mathbb{R}P^n$ is a covering map.

Theorem 5.17 (Extension Theorem [12]). *Let M be a smooth manifold with boundary. Let N be a smooth manifold and $Z \subset N$ a closed embedded submanifold. Let $C \subset M$ be a closed subset. Suppose $f : M \rightarrow N$ is smooth and is transverse to Z along C , i.e. we have $df_x(T_x M) + T_{f(x)}Z = T_{f(x)}N$ for all $x \in C \cap f^{-1}(Z)$, and when $\partial M \neq \emptyset$, the restriction $f|_{\partial M}$ is transverse to Z . Then there exists a smooth map $g : M \rightarrow N$ such that g and $g|_{\partial M}$ are transverse to Z , g is homotopic to f , and $g = f$ on a neighbourhood of C .*

6 Homotopy Groups of Stable Loci through Transversality

This section presents the main original research contribution of the thesis, developed in [32], which concerns the homotopy groups of stable loci in affine GIT quotients. The aim is to compute the homotopy groups by showing that maps of spheres into the ambient space V can be generically deformed to avoid

the non-stable locus $V \setminus V^{st}$ under a dimensional inequality. The strategy of this section is developed by extending the transversality approach of Daskalopoulos-Uhlenbeck [7] and its finite-dimensional analogue due to Wilkin [33], but the key constructions are adapted to the affine GIT setting.

In particular, the approach applies to cases where semistability does not coincide with stability. The Morse-theoretic arguments using the norm square of the moment map $\|\mu\|^2$, as described in [16], cannot be used to compute the homotopy groups of the stable locus. This is because the level set $\mu^{-1}(0)$ contains orbits for all polystable points (namely points with closed orbits in semistable loci), which by definition include both the stable orbits and the strictly semistable orbits. The negative gradient flow of $\|\mu\|^2$ retracts the stable loci and the strictly semistable loci together onto the level set. Hence the Morse-theoretic method cannot be used to deduce the homotopy groups of the stable loci alone. This approach thus differs from the Morse theoretic approach in the sense that it focuses on the stable locus itself rather than the deformation retract of the semistable locus onto $\mu^{-1}(0)$.

When removing a submanifold $N \subset M$ of codimension k from a manifold M , the theory of transversality (see for example [12, Chapter 2]) ensures that the homotopy groups $\pi_\ell(M \setminus N)$ are the same as $\pi_\ell(M)$ for all $\ell \leq k - 2$. In our case, we deal with the problem of computing homotopy groups when removing a subvariety from a manifold. More specifically, we extend the infinite-dimensional transversality framework of Daskalopoulos-Uhlenbeck [7] and Wilkin [33] through the construction of a holomorphic vector bundle that is equivariant under the group action over the conjugation orbit $G \cdot \lambda$ of a 1-PS. The fibre of this bundle over any $\lambda' \in G \cdot \lambda$ is the negative weight space $V(\lambda')_-$. A key observation is that the zero section $\mathcal{O}_W \subset W$ corresponds to points that are not stable (i.e. unstable or strictly semistable) with respect to λ' .

The main constructions are a space of homotopies $\mathcal{F}$ and an associated evaluation map $D : \mathcal{F} \times S^n \times I \times (G \cdot \lambda) \rightarrow W$. Proposition 6.4 shows that the map D is transverse to the zero section $\mathcal{O}_W$, and by the parametric transversality theorem, a generic homotopy $f \in \mathcal{F}$ induces a restriction D_f that is also transverse to $\mathcal{O}_W$. Following this, Corollary 6.5 provides an inequality $n+1 < 2(\dim_{\mathbb{C}} V(\lambda)_- - \dim_{\mathbb{C}}(G \cdot \lambda))$, and if this holds for all Hesselink strata, then a generic map $f_0 : S^n \rightarrow V^{st}$ can be deformed via a homotopy to a map whose image is in V^{st} . Our main result is Theorem 6.6, where we define $d_{min} = \min_j \{2 \dim_{\mathbb{C}} V(\lambda_j)_- - 2 \dim_{\mathbb{C}}(G \cdot \lambda_j)\}$ and conclude that V^{st} is $(d_{min} - 2)$ -connected, i.e. $\pi_n(V^{st}) = 0$ for all $n \leq d_{min} - 2$. Consequently, we obtain $\pi_n(\mathcal{M}^{st}) \cong \pi_{n-1}(G)$ for $1 \leq n < d_{min} - 1$, where $\mathcal{M}^{st} = V^{st}/G$ (Corollary 6.7).

Moreover, this inequality relates the dimension of the space of homotopies, the dimension of the 1-PS orbit, and the rank of the vector bundle (which coincides with the dimension of the negative weight spaces). This provides a lower bound for which the stable loci are connected, and we will illustrate the applicability of the framework with concrete examples.

Let G be a complex reductive linear algebraic group that acts on a complex affine space $V \cong \mathbb{C}^n$ via a linear representation $\sigma : G \rightarrow \mathrm{GL}(V)$. Then the G -action on V induces an action on its ring of regular functions $\mathbb{C}[V]$, which is given by $(g \cdot f)(v) = f(g^{-1} \cdot v)$ for $g \in G, v \in V, f \in \mathbb{C}[V]$. The subring consisting of G -invariant functions is denoted by $\mathbb{C}[V]^G$, consequently, the affine GIT quotient is $V // G := \mathrm{Spec} \mathbb{C}[V]^G$. This quotient is well-defined as $\mathbb{C}[V]^G$ is finitely generated (see Theorem 2.34).

Let $K \subset G$ be the maximal compact subgroup of G , and endow V with a K -invariant Hermitian inner product. Fix a non-trivial one-parameter subgroup $\lambda : \mathbb{G}_m \rightarrow G$. The conjugation orbit of λ is

$$G \cdot \lambda := \{g\lambda g^{-1} \mid g \in G\} \cong G/C_G(\lambda),$$

where $C_G(\lambda)$ is the centraliser of $\lambda(S^1)$ in G . In particular, $G \cdot \lambda$ is a smooth projective variety. For each $\lambda' \in G \cdot \lambda$, the representation V decomposes into

$$V = \bigoplus_{i \in \mathbb{Z}} V_i(\lambda'), \quad \text{where } V_i(\lambda') = \{v \in V \mid \lambda'(t) \cdot v = t^i v\},$$

and so we can denote

$$V(\lambda')_- = \bigoplus_{i < 0} V_i(\lambda'), \quad V(\lambda')_0 = V_0(\lambda'), \quad V(\lambda')_+ = \bigoplus_{i > 0} V_i(\lambda').$$

Let

$$W := \{(\lambda', v) \in (G \cdot \lambda) \times V \mid v \in V(\lambda')_-\}. \quad (1)$$

The projection $\pi : W \rightarrow G \cdot \lambda$ defined by $(\lambda', v) \mapsto \lambda'$ then defines a vector bundle, and the *zero section* of W is $\mathcal{O}_W := \{(\lambda', 0) \mid \lambda' \in G \cdot \lambda\}$.

Remark 6.1. By the Hilbert-Mumford criterion, a point $v \in V$ is unstable if and only if there is a 1-PS $\lambda' : \mathbb{G}_m \rightarrow G$ such that $\lim_{t \rightarrow 0} \lambda'(t) \cdot v = 0$. In the weight space decomposition $V = \bigoplus_{i \in \mathbb{Z}} V_i(\lambda')$, the condition $\lim_{t \rightarrow 0} \lambda'(t) \cdot v = 0$ is equivalent to that v has components only in the positive weight space $V(\lambda')_+ = \bigoplus_{i > 0} V_i(\lambda')$.

More generally, v is not stable (i.e. it is unstable or strictly semistable) if and only if there exists a λ' such that $\lim_{t \rightarrow 0} \lambda'(t) \cdot v$ exists, which is equivalent to v having no components in negative weight spaces, i.e. $v \in V(\lambda')_{\geq 0} = \bigoplus_{i \geq 0} V_i(\lambda')$.

Observe the bundle $W \rightarrow G \cdot \lambda$ is constructed using the negative weight spaces as its fibres, and the zero section $\mathcal{O}_W$ consists of zero vectors in each fibre. As we will see, we will construct a map D_f in Proposition 6.4 which, for a given homotopy $f(s, t)$ and a 1-PS λ' , evaluates the projection of $f(s, t)$ onto $V(\lambda')_-$. Thus an intersection with the zero section occurs if and only if the point $v = f(s, t)$ is mapped to the zero vector, which means that v is not stable with respect to the corresponding 1-PS λ' .

Therefore, $D_f^{-1}(\mathcal{O}_W)$ consists of the points (s, t) in the homotopy and 1-PSs λ' in the orbit $G \cdot \lambda$ such that $f(s, t)$ is not stable with respect to λ' . The subsequent argument will show that for a generic f , this intersection is empty under certain dimensional conditions, which implies that the image of f avoids the non-stable condition for all $\lambda' \in G \cdot \lambda$. $\triangle$

Proposition 6.2. $\pi : W \rightarrow G \cdot \lambda$ is a holomorphic G -equivariant vector bundle of rank $m := \dim_{\mathbb{C}} V(\lambda)_-$, which is isomorphic to the associated bundle $G \times_{C_G(\lambda)} V(\lambda)_- \rightarrow G/C_G(\lambda)$.

Proof. Consider the map $\Phi : G \times V(\lambda)_- \rightarrow W$ defined by

$$\Phi(g, v) = (g\lambda g^{-1}, g \cdot v).$$

To check that the codomain is W , let $\lambda' = g\lambda g^{-1}$, and consider the action of $\lambda'(t)$ on $g \cdot v_i$, which is given by $\lambda'(t)(g \cdot v_i) = (g\lambda(t)g^{-1})(g \cdot v_i) = g\lambda(t)(g^{-1}g)v_i = g\lambda(t)v_i = g(t^i v_i) = t^i(g \cdot v_i)$. Hence for any $(g, v) \in G \times V(\lambda)_-$, the vector $g \cdot v$ lies in $V(g\lambda g^{-1})_-$, so $\Phi(g, v) \in W$. Moreover, let $n \in C_G(\lambda)$, then

$$\Phi(gn^{-1}, n \cdot v) = (gn^{-1}\lambda ng^{-1}, gn^{-1}n \cdot v) = (g\lambda g^{-1}, g \cdot v) = \Phi(g, v),$$

which implies that Φ is invariant under the $C_G(\lambda)$ -action.

Note that $G \times_{C_G(\lambda)} V(\lambda)_- := (G \times V(\lambda)_-)/\sim$, where $(g_1, v_1) \sim (g_2, v_2)$ if and only if there exists an $n \in C_G(\lambda)$ such that $(g_2, v_2) = (g_1 n^{-1}, n \cdot v_1)$, therefore Φ descends to a well-defined map

$$\tilde{\Phi} : G \times_{C_G(\lambda)} V(\lambda)_- \rightarrow W, \quad \tilde{\Phi}([g, v]) = \Phi(g, v) = (g\lambda g^{-1}, g \cdot v),$$

and we shall prove that $\tilde{\Phi}$ is bijective.

For the surjectivity, let $(\lambda', w) \in W$ be an arbitrarily chosen element, then there exists some $g \in G$ such that $\lambda' = g\lambda g^{-1}$. Define $v := g^{-1} \cdot w$, then for $v_j = g^{-1}w_j$ we have $\lambda(t)v_j = \lambda(t)(g^{-1}w_j) = g^{-1}(g\lambda(t)g^{-1})w_j = g^{-1}(t^j w_j) = t^j(g^{-1}w_j) = t^j v_j$. Thus $v_j \in V(\lambda)_-$, which implies that $v = g^{-1}w \in V(\lambda)_-$. Now consider $[g, v] \in G \times_{C_G(\lambda)} V(\lambda)_-$, observe that $\tilde{\Phi}([g, v]) = (g\lambda g^{-1}, g \cdot v) = (\lambda', g \cdot (g^{-1}w)) = (\lambda', w)$. Hence $\tilde{\Phi}$ is surjective.

To prove the injectivity, assume $\tilde{\Phi}([g_1, v_1]) = \tilde{\Phi}([g_2, v_2])$. This means

$$(g_1\lambda g_1^{-1}, g_1 \cdot v_1) = (g_2\lambda g_2^{-1}, g_2 \cdot v_2).$$

From the first component $g_1\lambda g_1^{-1} = g_2\lambda g_2^{-1}$ implies that $(g_2^{-1}g_1)\lambda(g_2^{-1}g_1)^{-1} = \lambda$. Let $h = g_2^{-1}g_1 \in C_G(\lambda)$, which gives $g_1 = g_2 h$. Furthermore, from the second component $g_1 \cdot v_1 = g_2 \cdot v_2$, we have $(g_2 h) \cdot v_1 = g_2 \cdot v_2$, implying $h \cdot v_1 = v_2$. Thus we have $[g_1, v_1] = [g_2 h, v_1]$, which is equivalent to $((g_2 h)h^{-1}, h \cdot v_1) = (g_2, h \cdot v_1) = (g_2, v_2)$. Therefore, we have shown that $[g_1, v_1] = [g_2, v_2]$, establishing the injectivity, and we can thus conclude that $\tilde{\Phi}$ is bijective.

Now we show the holomorphicity of $\tilde{\Phi}$. Note that G is complex Lie group, of which the group multiplication, inversion, and the conjugation are holomorphic maps, and the G -action on V is a holomorphic representation, thus Φ is a holomorphic map. Since $C_G(\lambda)$ is a closed complex subgroup of G , the $C_G(\lambda)$ -action is also holomorphic, which is in addition free and proper, hence the quotient $G \times_{C_G(\lambda)} V(\lambda)_-$ is a complex manifold and the quotient map $q : G \times V(\lambda)_- \rightarrow G \times_{C_G(\lambda)} V(\lambda)_-$ is a holomorphic submersion. Because $\Phi = \tilde{\Phi} \circ q$, it follows that $\tilde{\Phi}$ is holomorphic. It is furthermore a biholomorphism by the Inverse Function theorem, where it suffices to show the complex differential $d\tilde{\Phi}_{[g, v]}$ is a $\mathbb{C}$ -linear isomorphism at every point $[g, v] \in G \times_{C_G(\lambda)} V(\lambda)_-$, and thus W can be endowed with a complex manifold structure making $\pi : W \rightarrow G \cdot \lambda$ a holomorphic map.

Since $\tilde{\Phi}$ is a biholomorphism that is also linear on the fibres and commutes with projections, it is an isomorphism of holomorphic vector bundles. Thus W inherits the structure of a holomorphic vector bundle from $G \times_{C_G(\lambda)} V(\lambda)_-$. To see the G -equivariance of $\tilde{\Phi}$, let $g_0 \in G$ be arbitrary. Then we have $\tilde{\Phi}(g_0 \cdot [g, v]) = ((g_0 g) \lambda (g_0 g)^{-1}, (g_0 g) \cdot v) = (g_0 (g \lambda g^{-1}) g_0^{-1}, g_0 (g \cdot v)) = g_0 \cdot (g \lambda g^{-1}, g \cdot v) = g_0 \cdot \tilde{\Phi}([g, v])$.

Note that the fibre $W_{\lambda'} = V_{i < 0}(\lambda')$. If $\lambda' = g \lambda g^{-1}$, the map $v \mapsto g \cdot v$ restricts to an isomorphism from $V(\lambda)_-$ to $V(\lambda')_-$. Hence $\dim_{\mathbb{C}} V(\lambda')_- = \dim_{\mathbb{C}} V(\lambda)_-$ for all $\lambda' \in G \cdot \lambda$. Therefore, the rank of the vector bundle W is $m = \dim_{\mathbb{C}} V(\lambda)_-$.

Denote $E = G \times_{C_G(\lambda)} V(\lambda)_-$, $p_E : E \rightarrow G/C_G(\lambda)$, then we have the following commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\Phi}} & W \\ p_E \downarrow & & \downarrow \pi \\ G/C_G(\lambda) & \xrightarrow{\phi} & G \cdot \lambda \end{array},$$

where ϕ is the canonical isomorphism. So $\pi = \phi \circ p_E \circ \tilde{\Phi}^{-1}$, and thus we can conclude that π is a holomorphic G -equivariant vector bundle that is isomorphic to the associated bundle, as stated. $\square$

Remark 6.3. To verify that the above diagram is indeed commutative, it suffices to show that $\phi \circ p_E = \pi \circ \tilde{\Phi}$. Note that $\tilde{\Phi}([g, v]) = (g \lambda g^{-1}, g \cdot v) \in W$, so $(\pi \circ \tilde{\Phi})([g, v]) = \pi((g \lambda g^{-1}, g \cdot v)) = g \lambda g^{-1}$. Moreover, we have $p_E([g, v]) = g C_G(\lambda) \in G/C_G(\lambda)$, and thus $(\phi \circ p_E)([g, v]) = \phi(g C_G(\lambda)) = g \lambda g^{-1}$ given that ϕ acts on the coset representative by conjugation.

Therefore, we have shown that $\pi \circ \tilde{\Phi} = \phi \circ p_E$, which confirms the commutativity. $\triangle$

Fix $n \geq 0$, consider a based map $f_0 : S^n \rightarrow V^{st}$ with $f_0(s_0) = x_0$, and let

$$\mathcal{F} := \{f \in C^{r+1}(S^n \times I, V) \mid f(s, 0) = f_0(s), f(s_0, t) = x_0, f(s, 1) = x_0\},$$

where $s \in S^n$, and $t \in I$, then its tangent space is

$$T_f \mathcal{F} = \{\dot{f} \in C^r(S^n \times I, V) \mid \dot{f}(s, 0) = 0, \dot{f}(s_0, t) = 0, \dot{f}(s, 1) = 0\}.$$

For $\lambda' \in G \cdot \lambda$ let $\pi_{\lambda'}^\perp : V \rightarrow V(\lambda')_-$ denote the orthogonal projection. Define

$$D : \mathcal{F} \times S^n \times I \times (G \cdot \lambda) \rightarrow W, \quad D(f, s, t, \lambda') = (\lambda', \pi_{\lambda'}^\perp(f(s, t))).$$

Let $p = (f, s, t, \lambda') \in \mathcal{F} \times S^n \times I \times (G \cdot \lambda)$, and let $x := f(s, t) \in V$. Then a tangent vector at p in $T_p(\mathcal{F} \times S^n \times I \times G \cdot \lambda)$ can be written as $(\dot{f}, v_s, v_t, \xi)$. Then

$$dD_p(\dot{f}, v_s, v_t, \xi) = (\xi, \pi_{\lambda'}^\perp(\dot{f}(s, t) + v_s \partial_s f(s, t) + v_t \partial_t f(s, t)) + (d\pi_\bullet^\perp)_{(\lambda', x)}(\xi)),$$

where $(d\pi_\bullet^\perp)_{(\lambda', x)}(\xi) \in V(\lambda')_-$ denotes the derivative, with respect to λ' , of $\pi_{\lambda'}^\perp$, evaluated at $x \in V$ and at direction $\xi \in T_{\lambda'}(G \cdot \lambda)$. To deduce the above formula, we start by writing the evaluation map

$$\text{ev} : \mathcal{F} \times S^n \times I \rightarrow V, \quad (f, s, t) \mapsto f(s, t).$$

Its differential at (f, s, t) is

$$d(\text{ev}_{(f, s, t)})(\dot{f}, v_s, v_t) = \dot{f}(s, t) + v_s \partial_s f(s, t) + v_t \partial_t f(s, t) \in V.$$

Next define $\Phi : (G \cdot \lambda) \times V \rightarrow W$ by $\Phi(\lambda', v) := (\lambda', \pi_{\lambda'}^\perp(v))$, and thus by definition $D = \Phi \circ (\text{id}_{G \cdot \lambda}, \text{ev})$.

Pick an arbitrary (λ', x) , we have

$$d\Phi_{(\lambda', x)}(\xi, u) = (\xi, \pi_{\lambda'}^\perp(u) + (d\pi_\bullet^\perp)_{(\lambda', x)}(\xi)),$$

where u is a tangent vector of V . Then at the point $p = (f, s, t, \lambda')$ we have $dD_p = d\Phi_{(\lambda', x)} \circ (\text{id}, d(\text{ev})_{(f, s, t)})$. Substituting $(\dot{f}, v_s, v_t, \xi)$ and use the above equations yields

$$\begin{aligned} dD_p(\dot{f}, v_s, v_t, \xi) &= d\Phi_{(\lambda', x)}(\xi, \dot{f}(s, t) + v_s \partial_s f(s, t) + v_t \partial_t f(s, t)) \\ &= (\xi, \pi_{\lambda'}^\perp(\dot{f}(s, t) + v_s \partial_s f(s, t) + v_t \partial_t f(s, t)) + (d\pi_\bullet^\perp)_{(\lambda', x)}(\xi)), \end{aligned} \quad (2)$$

as desired.

Proposition 6.4. *The map D is transverse to the zero section $\mathcal{O}_W \subset W$; i.e. for every $p = (f, s, t, \lambda')$ satisfying $D(p) \in \mathcal{O}_W$ we have*

$$dD_p(T_p M) + T_{D(p)} \mathcal{O}_W = T_{D(p)} W,$$

where $M = \mathcal{F} \times S^n \times I \times (G \cdot \lambda)$. Moreover, D_f is transverse to the zero section $\mathcal{O}_W$ for f in a residual subset of $\mathcal{F}$.

Proof. Choose $p = (f, s, t, \lambda')$ with $D(p) = (\lambda', 0) \in \mathcal{O}_W$. Since $W \rightarrow G \cdot \lambda$ is a vector bundle with fibre $V(\lambda')_-$, we have splittings

$$T_{(\lambda', 0)} W = T_{\lambda'}(G \cdot \lambda) \oplus V(\lambda')_-, \quad T_{(\lambda', 0)} \mathcal{O}_W = T_{\lambda'}(G \cdot \lambda) \oplus \{0\}.$$

Recall that the previous computations (2) imply that the differential at p , where $x = f(s, t)$ and $\pi_{\lambda'}^\perp(x) = 0$, is given by

$$dD_p(\dot{f}, v_s, v_t, \xi) = (\xi, \pi_{\lambda'}^\perp(\dot{f}(s, t) + v_s \partial_s f(s, t) + v_t \partial_t f(s, t)) + (d\pi_{\bullet}^\perp)_{(\lambda', x)}(\xi)).$$

Now fix an arbitrary vector $v \in V(\lambda')_-$, and choose a bump function $\beta : S^n \times I \rightarrow \mathbb{R}$ such that

- $\beta(s, t) = 1$,
- the support of β , $\text{supp } \beta$, is contained in a small ball that does not intersect the boundary $(S^n \times \{0, 1\}) \cup (\{s_0\} \times I)$.

Consequently, to show surjectivity onto $V(\lambda')_-$, we consider variations with $\xi = 0$. In this case, $(d\pi_{\bullet}^\perp)_{(\lambda', x)}(0) = 0$ because this term is linear in ξ . Thus, the differential simplifies to

$$dD_p(\dot{f}, v_s, v_t, 0) = (0, \pi_{\lambda'}^\perp(\dot{f}(s, t) + v_s \partial_s f(s, t) + v_t \partial_t f(s, t))).$$

Now setting $\dot{f} = \beta v$, $v_s = v_t = 0$ gives $\pi_{\lambda'}^\perp(\dot{f}(s, t)) = \pi_{\lambda'}^\perp(v) = v$. Therefore the differential of D at p becomes

$$dD_p(\dot{f}, 0, 0, 0) = (0, v).$$

Since v is arbitrary, we have $V(\lambda')_- \subset \text{Im}(dD_p)$. Hence we compute

$$dD_p(T_p M) + T_{(\lambda', 0)} \mathcal{O}_W = \text{Im}(dD_p) + (T_{\lambda'}(G \cdot \lambda) \oplus \{0\}) = T_{\lambda'}(G \cdot \lambda) \oplus V(\lambda')_- = T_{(\lambda', 0)} W,$$

which then implies that D is transverse to $\mathcal{O}_W$ at p . And since p was arbitrarily chosen, we conclude that D is transverse to $\mathcal{O}_W$ globally.

Consider $X = S^n \times I \times G \cdot \lambda$, and define

$$D_f := D|_{\{f\} \times X} : X \rightarrow W, \quad (s, t, \lambda') \mapsto (\lambda', \pi_{\lambda'}^\perp(f(s, t))).$$

Then the subset $\mathcal{F}^\natural = \{f \in \mathcal{F} \mid D_f \pitchfork \mathcal{O}_W\}$ is residual in $\mathcal{F}$. Therefore D_f is transverse to the zero section $\mathcal{O}_W$ for f in a residual subset by the parametric transversality theorem [22, Theorem 6.35]. $\square$

Corollary 6.5. *Let $f \in \mathcal{F}$ be a generic homotopy. If the rank $m = \dim_{\mathbb{C}} V(\lambda)_-$ of the vector bundle $W \rightarrow G \cdot \lambda$ satisfies the condition*

$$n + 1 + 2 \dim_{\mathbb{C}}(G \cdot \lambda) < 2m, \tag{3}$$

then the preimage $D_f^{-1}(\mathcal{O}_W)$ is empty.

Proof. Assume the preimage $D_f^{-1}(\mathcal{O}_W)$ is non-empty. Take any point $x \in D_f^{-1}(\mathcal{O}_W) \subset X = S^n \times I \times (G \cdot \lambda)$, and set $y = D_f(x) \in \mathcal{O}_W$. Because D_f is transverse to $\mathcal{O}_W$, we have the equality of tangent spaces

$$T_y W = dD_f(T_x X) + T_y \mathcal{O}_W.$$

Hence

$$\dim_{\mathbb{R}} T_y W = \dim_{\mathbb{R}}(dD_f(T_x X)) + \dim_{\mathbb{R}} T_y \mathcal{O}_W - \dim_{\mathbb{R}}(dD_f(T_x X) \cap T_y \mathcal{O}_W).$$

Note that $\dim_{\mathbb{R}} T_y W = 2m + 2 \dim_{\mathbb{C}}(G \cdot \lambda)$, $\dim_{\mathbb{R}} T_y \mathcal{O}_W = 2 \dim_{\mathbb{C}}(G \cdot \lambda)$, and since the dimension of the intersection $dD_f(T_x X) \cap T_y \mathcal{O}_W$ is non-negative, we obtain the inequality

$$2m \leq \dim_{\mathbb{R}}(dD_f(T_x X)) \leq \dim_{\mathbb{R}} T_x X = n + 1 + 2 \dim_{\mathbb{C}}(G \cdot \lambda),$$

where the second inequality follows from the rank-nullity theorem as the differential dD_f is a linear map.

The original hypothesis of the corollary is the negation of the inequality obtained above, therefore we have established that $D_f^{-1}(\mathcal{O}_W)$ must be empty. $\square$

$D_f^{-1}(\mathcal{O}_W)$ being empty for the chosen $G \cdot \lambda$ means that for any point (s, t) in the homotopy and any $\lambda' \in G \cdot \lambda$, the point $f(s, t)$ is non-zero. Consequently, the Hilbert-Mumford weight $\mu(f(s, t), \lambda') > 0$, which implies that no λ' from this particular class can be used to show that $f(s, t)$ is unstable or strictly semistable.

To ensure a homotopy lies entirely within the stable locus, this must hold for every class of 1-PS that can characterise non-stable points. Note that the Hesselink stratification of the non-stable locus $V^{us}(\rho) \cup (V^{ss}(\rho) \setminus V^{st}(\rho))$ is a finite union of strata, each associated with a particular conjugacy class of 1-PSs that can cause instability [16].

Let $\{[\lambda_1], [\lambda_2], \dots, [\lambda_k]\}$ be the finite set of conjugacy classes of 1-PSs that characterise all possible ways a point can be ρ -unstable or strictly ρ -semistable. For each class $[\lambda_j]$, let W_j be the corresponding vector bundle defined in (3.1) whose fibre rank $m_j = \dim_{\mathbb{C}} V(\lambda_j)_-$ satisfies inequality (3). If the condition

$$n + 1 + 2 \dim_{\mathbb{C}}(G \cdot \lambda_j) < 2m_j$$

holds for each class $[\lambda_j]$ with $j = 1, \dots, k$, then Corollary 6.5 implies that there is a residual set $\mathcal{F}_j^{\text{th}} \subset \mathcal{F}$ of homotopies f such that $D_f^{-1}(\mathcal{O}_{W_j})$ is empty. The intersection $\bigcap_{j=1}^k \mathcal{F}_j^{\text{th}}$ is also a residual subset of $\mathcal{F}$ because a countable intersection of residual sets is residual. Any homotopy f in the intersection avoids every Hesselink stratum, which implies that its image lies entirely in the stable locus $V^{st}(\rho)$.

Theorem 6.6. *Define $d_{\min} = \min_{j \in \{1, \dots, k\}} \{2m_j - 2 \dim_{\mathbb{C}}(G \cdot \lambda_j)\}$. Let n be a non-negative integer such that $n + 1 < d_{\min}$, then the n -th homotopy group of the stable locus is trivial, in other words,*

$$\pi_n(V^{st}(\rho)) = 0.$$

Equivalently, $V^{st}(\rho)$ is $(d_{\min} - 2)$ -connected.

This theorem allows us to deduce the homotopy groups of the moduli space of stable orbits, $\mathcal{M}^{st}(\rho) := V^{st}(\rho)/G$. Note that the principal G -bundle $G \rightarrow V^{st}(\rho) \rightarrow \mathcal{M}^{st}(\rho)$ induces a long exact sequence in homotopy groups

$$\cdots \rightarrow \pi_n(G) \rightarrow \pi_n(V^{st}(\rho)) \rightarrow \pi_n(\mathcal{M}^{st}(\rho)) \rightarrow \pi_{n-1}(G) \rightarrow \pi_{n-1}(V^{st}(\rho)) \rightarrow \cdots,$$

we have also assumed that $\pi_k(V^{st}(\rho)) = 0$ for $k \leq d_{\min} - 2$. Then if $k = 0$, which means $0 \leq d_{\min} - 2$, then $\pi_0(V^{st}(\rho)) = 0$. Thus we have $0 \rightarrow \pi_0(\mathcal{M}^{st}(\rho)) \rightarrow 0$, which then implies that $\pi_0(\mathcal{M}^{st}(\rho)) = 0$. Hence $\mathcal{M}^{st}(\rho)$ is path-connected.

For higher homotopy groups, consider $1 \leq k < d_{\min} - 2$. We observe from the long exact sequence that

- if $k \leq d_{\min} - 2$, then $\pi_k(V^{st}(\rho)) = 0$;
- if $i - 1 \leq d_{\min} - 3 < d_{\min} - 2$, then $\pi_{k-1}(V^{st}(\rho)) = 0$. The corresponding segment of the long exact sequence $\pi_k(V^{st}(\rho)) \rightarrow \pi_k(\mathcal{M}^{st}(\rho)) \rightarrow \pi_{k-1}(G) \rightarrow \pi_{k-1}(V^{st}(\rho))$ becomes $0 \rightarrow \pi_k(\mathcal{M}^{st}(\rho)) \rightarrow \pi_{k-1}(G) \rightarrow 0$. The exactness implies that the connecting homomorphism $\partial_k : \pi_k(\mathcal{M}^{st}(\rho)) \rightarrow \pi_{k-1}(G)$ is an isomorphism.

This leads to the following corollary:

Corollary 6.7. *Assume G acts freely on $V^{st}(\rho)$ with quotient $\mathcal{M}^{st}(\rho) := V^{st}(\rho)/G$, and let $d_{\min}$ be defined as in the preceding theorem. If n is a non-negative integer such that $n + 1 < d_{\min}$, then the homotopy groups of the quotient $\mathcal{M}^{st}(\rho)$ are related to the homotopy groups of G by an isomorphism*

$$\pi_n(\mathcal{M}^{st}(\rho)) \cong \pi_{n-1}(G)$$

for $1 \leq n < d_{\min} - 1$. Furthermore, if $d_{\min} > 1$, then $\mathcal{M}^{st}(\rho)$ is path-connected.

Example 6.8 (Stable Locus of the 2-Kronecker Quiver). Consider the *2-Kronecker quiver* with vertices 1, 2 and two arrows $a, b : 1 \rightarrow 2$. Let the dimension vector $\mathbf{v} := (\dim V_1, \dim V_2) = (1, 1)$, fix a Hermitian structure on V_1, V_2 , and choose the stability parameter $\alpha = (1, -1)$. Then this gives

$$\alpha_1 \dim V_1 + \alpha_2 \dim V_2 = 1 + (-1) = 0,$$

which means α is admissible.

A representation in this case is a pair $(A, B) \cong \text{Hom}(\mathbb{C}, \mathbb{C})^2 \cong \mathbb{C}^2$. Each subrepresentation is determined by the orthogonal projection associated to the Hermitian structure on the vector spaces V_1, V_2 (see [33, Section 2] for more details). In this case, the non-trivial subrepresentations are $\pi : V_1 \oplus V_2 \rightarrow V_1$ supported at vertex 1, and $\pi^\perp : V_1 \oplus V_2 \rightarrow V_2$ supported at vertex 2, whose slopes are $\mu_\alpha(\pi) = \alpha_1 \cdot 1 = 1 > 0$ and $\mu_\alpha(\pi^\perp) = \alpha_2 \cdot 1 = -1 < 0$, respectively. Therefore, the subrepresentation π of type $(1, 0)$ is destabilising [33]. Moreover, note that given a representation (A, B) , such a representation exists if and only if

$$A = 0 \text{ and } B = 0,$$

which implies that the unstable locus is $\{(0, 0)\}$, and the stable locus is thus $\mathbb{C}^2 \setminus \{(0, 0)\} \simeq S^3$.

We now apply the transversality framework. The non-stable locus $V^{us} = \{(0, 0)\}$ consists of a single Hesselink stratum, which is associated to the destabilising subrepresentation π of type $(1, 0)$. This corresponds to a specific 1-PS $\lambda : \mathbb{G}_m \rightarrow G$ that is given by $\lambda(t) = (t, 1) \in G = \text{GL}_1(\mathbb{C}) \times \text{GL}_1(\mathbb{C})$. The action of this 1-PS on $v = (A, B)$ is

$$\lambda(t) \cdot (A, B) = (1 \cdot A \cdot t^{-1}, 1 \cdot B \cdot t^{-1}) = (t^{-1}A, t^{-1}B) = t^{-1}(A, B).$$

So the weight of this action is -1 , which means that the entire space V is the negative weight space, so $V(\lambda)_- = V \cong \mathbb{C}^2$, and $V(\lambda)_{\geq 0} = \{0\}$. Hence $\dim_{\mathbb{C}} V(\lambda)_- = 2$.

To determine the orbit of this 1-PS, note that $G = \text{GL}_1(\mathbb{C}) \times \text{GL}_1(\mathbb{C})$ is abelian, thus the centraliser $C_G(\lambda)$ is the entire group G . It follows that the conjugation orbit is

$$G \cdot \lambda = G/C_G(\lambda) = G/G,$$

which is a single point. Hence $\dim_{\mathbb{C}}(G \cdot \lambda) = 0$. Moreover, this calculation is confirmed by the following quiver-specific formulae.

Define the spaces Hom^0 and Hom^1 for the destabilising subrepresentation π with dimension vector $(1, 0)$ with complement $\pi^\perp$ with dimension vector $(0, 1)$ as follows:

$$\text{Hom}^0(\pi, \pi^\perp) := \text{Hom}(\pi(V_1), \pi^\perp(V_1)) \oplus \text{Hom}(\pi(V_2), \pi^\perp(V_2))$$

and

$$\text{Hom}^1(\pi, \pi^\perp) := \text{Hom}(\pi(V_1), \pi^\perp(V_2)) \oplus \text{Hom}(\pi(V_2), \pi^\perp(V_1)).$$

We can now compute the dimension of these spaces by substituting the vector spaces at each vertex, where $\pi(V_1) = \mathbb{C}$, $\pi(V_2) = 0$, $\pi^\perp(V_1) = 0$, and $\pi^\perp(V_2) = \mathbb{C}$. Therefore,

$$\dim_{\mathbb{C}}(G \cdot \lambda) = \dim_{\mathbb{C}} \text{Hom}^0(\pi, \pi^\perp) = 0 \text{ and } \dim_{\mathbb{C}} V(\lambda)_- = \dim_{\mathbb{C}} \text{Hom}^1(\pi, \pi^\perp) = 2.$$

Therefore, the vector bundle $\pi : W \rightarrow G \cdot \lambda$ is a bundle over a single point, and W is the fibre $W \cong V(\lambda)_- \cong \mathbb{C}^2$. The zero section $\mathcal{O}_W$ is the origin $\{0\} \in \mathbb{C}^2$. For a homotopy $f \in \mathcal{F}$ (i.e. a map from $S^n \times I$ to V), the map D_f from Corollary 6.5 is defined on $X = S^n \times I \times (G \cdot \lambda)$. Since $G \cdot \lambda = \{\lambda\}$, we have $X \cong S^n \times I$. Thus the map D_f becomes

$$D_f : (S^n \times I) \times \{\lambda\} \rightarrow W \cong \mathbb{C}^2, \quad D_f(s, t, \lambda) = (\lambda, \pi_\lambda^\perp(f(s, t))).$$

As $V = V(\lambda_-)$, the projection $\pi_\lambda^\perp$ onto $V(\lambda)_-$ is the identity map on V , and the map D_f is the homotopy f itself. Since the intersection with the zero section, $D_f^{-1}(\mathcal{O}_W)$, is the set of (s, t, λ) such that $D_f(s, t, \lambda) = (\lambda, 0) \in \mathcal{O}_W$, which requires $\pi_\lambda^\perp(f(s, t)) = 0$, i.e. $f(s, t) = 0$, it follows that $D_f^{-1}(\mathcal{O}_W)$ consists of points $(s, t) \in S^n \times I$ that are mapped to $(0, 0)$ under the homotopy f .

By Corollary 6.5, for a generic f , $D_f^{-1}(\mathcal{O}_W)$ is empty if the dimensional inequality holds, namely

$$\dim_{\mathbb{R}}(S^n \times I) + 2 \dim_{\mathbb{C}}(G \cdot \lambda) < 2 \dim_{\mathbb{C}} V(\lambda)_-.$$

This implies that $n + 1 < 4 \Rightarrow n < 3$. Therefore, we conclude that for $n \leq 2$, any generic homotopy $f : S^n \times I \rightarrow V$ can be deformed to an f' such that $D_{f'}^{-1}(\mathcal{O}_W) = \emptyset$, which means that the image of f' never intersects $V^{us} = \{(0, 0)\}$. This shows that for $n \leq 2$, any map $S^n \rightarrow V^{st}$ is null-homotopic, and in particular, $\pi_n(V^{st}) = 0$ for $n \leq 2$.

Alternatively, we can obtain the connectivity bound via the Euler characteristic formula developed in [33]. Let $d = (1, 0)$ denote the dimension vector of π , and $e = (0, 1)$ denote the dimension vector of $\pi^\perp$. It follows that the Euler form χ is given by

$$\begin{aligned}\chi(\pi, \pi^\perp) &= \dim \operatorname{Hom}^0(\pi, \pi^\perp) - \dim \operatorname{Hom}^1(\pi, \pi^\perp) \\ &= d(I - A)e^T \\ &= -2,\end{aligned}$$

where I is the identity matrix and $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ is the corresponding adjacency matrix. So the minimal dimension of the quiver representation is

$$d_{\min} = (-2)\chi(\pi, \pi^\perp) = 4,$$

which matches the result obtained via our approach.

Example 6.9 (Stability and Controllability of Linear Control Systems). This example arises from the action of the complex reductive group $G = \operatorname{GL}_n(\mathbb{C})$ on the affine space $V = M_{n \times n}(\mathbb{C}) \times M_{n \times m}(\mathbb{C})$ by conjugation, where GIT stability condition coincides with the classical controllability condition for the linear control system $\dot{x} = Ax + Bu$.

Let $(A, B) \in V$, then the action is $g \cdot (A, B) = (gAg^{-1}, gB)$ for all $g \in G$. The pair (A, B) is said to be *controllable* if and only if the *controllability matrix*

$$C(A, B) = [B, AB, A^2B, \dots, A^{n-1}B] \in M_{n \times nm}(\mathbb{C})$$

has rank n [10]. Controllability is the fundamental property of a control system that ensures it can be driven from any initial state to any desired final state within a finite time.

Choose the character $\chi : \operatorname{GL}_n(\mathbb{C}) \rightarrow \mathbb{C}^\times$ defined by $\chi(g) = \det(g)$, and V can be endowed with the χ -linearisation, in other words, the G -action can be lifted from V to $V \times \mathbb{C}$ by defining $g \cdot (v, \zeta) := (g \cdot v, \chi(g)^{-1}\zeta)$ for any $(v, \zeta) \in V \times \mathbb{C}$. Then a point $v \in V$ is χ -semistable if

$$\overline{G \cdot (v, 1)} \cap V \times \{0\} = \emptyset.$$

Further, a point $v \in V$ is χ -stable if it is χ -semistable, its orbit $G \cdot v$ is closed within $V^{ss} \subset V$, and its stabiliser subgroup $G_v = \{g \in G \mid g \cdot v = v\}$ is finite. Equivalently, if considering the lifted space, χ -stability then requires that $G \cdot (v, 1)$ is closed in $V^{ss} \times \mathbb{C}^\times$, and G_v is finite.

Proposition 4.1 in [10] gives the criterion that links the GIT stability with the controllability of linear control systems. A pair (A, B) is stable in the sense of GIT if and only if it is controllable, which is equivalent to the condition that no proper A -invariant subspace contains $\operatorname{Im} B \subset \mathbb{C}^n$. This can be seen by considering the contrapositive. If (A, B) is not controllable, there exists a proper A -invariant subspace $0 \neq W \subset \mathbb{C}^n$ containing $\operatorname{Im} B$. Write $r = \dim W$, where $1 \leq r \leq n - 1$. Then we can choose a basis of $\mathbb{C}^n$ so that $W = \operatorname{Span}\{e_1, \dots, e_r\}$, $W^\perp = \operatorname{Span}\{e_{r+1}, \dots, e_n\}$. In this adapted basis, the matrices A and B take the block form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

where $A_{11} \in M_{r \times r}(\mathbb{C})$, $A_{12} \in M_{r \times (n-r)}(\mathbb{C})$, $B_1 \in M_{r \times m}(\mathbb{C})$, and the zero block in B comes from $\operatorname{Im} B \subseteq W$.

Consider the one-parameter subgroup $\lambda : \mathbb{C}^\times \rightarrow \operatorname{GL}_n(\mathbb{C})$ with $\lambda(t) = \operatorname{diag}(1, \dots, 1, \zeta^{-1}, \dots, \zeta^{-1})$, where 1 appears r times and ζ^{-1} appears $n - r$ times. Clearly λ preserves the block decomposition. Then under conjugation, we have $\lambda(\zeta) \cdot (A, B) = (\lambda(\zeta)A\lambda(\zeta)^{-1}, \lambda(\zeta)B)$, where in blocks we yield

$$\lambda(\zeta)A\lambda(\zeta)^{-1} = \begin{pmatrix} A_{11} & \zeta A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad \lambda(\zeta)B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}.$$

Then the limit $\lim_{\zeta \rightarrow 0} \lambda(\zeta) \cdot (A, B)$ exists and the numerical invariant $\mu^x((A, B), \lambda) = \langle \chi, \lambda \rangle$ is $-(n - r)$,

which is negative since $r < n$. Thus, by the Hilbert-Mumford criterion, (A, B) is not χ -semistable and thus not χ -stable. Therefore, we can define the unstable stratum associated to the subspace W as

$$S_W := \{(A, B) \in V \mid A(W) \subset W, \text{Im } B \subset W\}.$$

Conversely, assume there is no proper A -invariant subspace W containing $\text{Im } B$. Proposition 2.5 in [19] gives two conditions for stability. Firstly, note that the kernel of the G -action is the subgroup of $\text{GL}_n(\mathbb{C})$ that acts trivially on V . For $n \geq 1$, if $g \cdot B = B$ for all $B \in M_{n \times m}(\mathbb{C})$, then g must be I_n . So the kernel of the action is trivial, which is $\{I_n\}$. And so $\chi(I_n) = \det(I_n) = 1$, the first condition holds.

Secondly, consider an arbitrary 1-parameter subgroup λ for which the limit $\lim_{\zeta \rightarrow 0} \lambda(\zeta) \cdot (A, B)$ exists in V . Similarly, we can choose a basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$ such that $\lambda(\zeta)$ is diagonal with respect to this basis, i.e., $\lambda(\zeta)e_j = \zeta^{w_j}e_j$ for integer weights w_j , which can be ordered as $w_1 \geq w_2 \geq \dots \geq w_n$.

Then the existence of limit implies that $V_{\geq 0} = \text{Span}\{e_j \mid w_j \geq 0\}$ is an A -invariant subspace that contains $\text{Im } B$, which contradicts our initial assumption. Thus $V_{\geq 0}$ cannot be a proper subspace of $\mathbb{C}^n$, so $V_{\geq 0} = \mathbb{C}^n$, which means all basis vectors e_j are in $V_{\geq 0}$. Consequently, all weights $w_j \geq 0$. Since for the character $\chi(g) = \det(g)$, the pair $\langle \chi, \lambda \rangle = \sum_{j=1}^n w_j$, which is clearly always non-negative. Moreover, if $\langle \chi, \lambda \rangle = 0$, then every w_j must be 0, which means $\lambda(\zeta)e_j = \zeta^0 e_j = e_j$ for all j . Hence the 1-PS $\lambda(\zeta) = I_n$ for all $\zeta \in \mathbb{C}^\times$, thus its image $\lambda(\mathbb{C}^\times) = \{I_n\}$ coincides with the kernel of the G -action on V . Therefore, we can conclude that (A, B) is χ -stable.

We define the stable locus as

$$V^{st}(\chi) = \{(A, B) \in V \mid (A, B) \text{ controllable}\}.$$

Consequently, the complement $V^{us}(\chi) = V \setminus V^{st}(\chi)$ is the locus of uncontrollable systems.

Fix an $(n-1)$ -dimensional subspace $W \subset \mathbb{C}^n$, and pick the 1-PS $\mu_W(\zeta) = \text{diag}(I_{n-1}, \zeta^{-1})$ in a basis $(w_1, \dots, w_{n-1}, v)$, where $W = \text{Span}\{w_1, \dots, w_{n-1}\}$. Then conjugating μ_W by G yields

$$[\lambda] = \{\mu_W \mid W \in \text{Gr}(n-1, n)\} \cong \mathbb{CP}^{n-1}, \quad \text{with } \dim_{\mathbb{C}}[\lambda] = n-1.$$

For any $\mu = \mu_W \in [\lambda]$, the vector space $V = M_{n \times n}(\mathbb{C}) \times M_{n \times m}(\mathbb{C})$ decomposes under the action of $\mu(\zeta)$ into weight spaces $V_i(\mu)$. We define the negative weight space as

$$V(\mu)_- = \bigoplus_{i < 0} V_i(\mu).$$

To determine its dimension, we start by writing an arbitrary element $(A, B) \in V$ in block form corresponding to the decomposition $\mathbb{C}^n = W \oplus W^\perp$, namely

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

We observe that for $\mu_W(\zeta) = \text{diag}(I_{n-1}, \zeta^{-1})$, the components of (A, B) that transform with negative powers of ζ are those entries in A that map from W to $W^\perp$, and entries in B that map from the input space to $W^\perp$, which are $1 \times (n-1)$ block and $1 \times m$ block, respectively. The complex dimension of the negative weight space is therefore

$$\dim_{\mathbb{C}} V(\mu)_- = \dim_{\mathbb{C}} A_{21} + \dim_{\mathbb{C}} B_2 = (n-1) + m.$$

Let $\mathcal{W} \rightarrow G \cdot \lambda$ be the G -equivariant holomorphic vector bundle whose fibre over $\mu \in G \cdot \lambda$ is $V(\mu)_-$ with rank $\dim_{\mathbb{C}} V(\mu)_-$. Fix $k \geq 0$ as the dimension of the sphere S^k . We consider the space of homotopies $\mathcal{F}$ and map $D : \mathcal{F} \times S^k \times I \times (G \cdot \lambda) \rightarrow \mathcal{W}$ as defined previously. The conditions for Proposition 6.4 are satisfied by this setup, and hence the map D is transverse to the zero section $\mathcal{O}_{\mathcal{W}}$ of $\mathcal{W}$.

Recall that $D(F, s, t, \mu) \in \mathcal{O}_{\mathcal{W}}$ if and only if $\pi_\mu^\perp(F(s, t)) = 0$, which means $F(s, t) \in \bigoplus_{i \geq 0} V_i(\mu)$. This is the necessary condition for the limit $\lim_{\zeta \rightarrow 0} \mu(\zeta) \cdot F(s, t)$ to exist in V .

We can now apply Corollary 6.5. For a generic homotopy $F \in \mathcal{F}$, the preimage $D_F^{-1}(\mathcal{O}_{\mathcal{W}})$ is empty if $\dim_{\mathbb{R}} S^k + 1 + 2(\dim_{\mathbb{C}}(G \cdot \lambda)) < 2 \dim_{\mathbb{C}} V(\mu)_-$, which means

$$(k+1) + 2(n-1) < 2((n-1) + m),$$

so we obtain

$$k + 1 < 2m.$$

This implies that $F(s, t)$ has a non-zero component in $V(\mu)_- = \bigoplus_{i < 0} V_i(\mu)$ for any (s, t) in the homotopy and any $\mu \in G \cdot \lambda$. Consequently, the limit $\lim_{\zeta \rightarrow 0} \mu(\zeta) \cdot F(s, t)$ does not exist for any $\mu \in G \cdot \lambda$.

More generally, note that the unstable locus $V^{us}(\chi)$ is a finite union of strata, each associated with a conjugacy class of 1-PSs $[\lambda_j]$ that can achieve χ -instability. For each such class $[\lambda_j]$, we have a corresponding rank $\dim_{\mathbb{C}} V(\lambda_j)_-$ and dimension $\dim_{\mathbb{C}}(G \cdot \lambda_j)$. By applying the transversality argument to each of these finitely many classes, and since d_{min} is the minimum of $2 \dim_{\mathbb{C}} V(\lambda_j)_- - 2 \dim_{\mathbb{C}}(G \cdot \lambda_j)$ over all such destabilising conjugacy classes, if $k + 1 < d_{min}$, then for a generic F , we have $F(S^k \times I) \subset V^{st}(\chi)$. This indicates that the k -th homotopy group $\pi_k(V^{st}(\chi))$ would then be trivial.

For controllable linear systems, the unstable systems are precisely those for which a proper A -invariant subspace W contains $\text{Im } B$. The stratification of the unstable locus $V^{us}(\chi)$ is by the dimension r of the smallest such subspaces W . The most generic unstable systems correspond to $r = n - 1$. For this stratum, we have shown that the corresponding component of the connectivity bound is $d_{min} = 2m$. Let this be the minimal bound over all strata, the space of controllable systems has trivial homotopy groups $\pi_k(V^{st}(\chi)) = 0$ for $k + 1 < 2m$, that is, for $k \leq 2m - 2$.

Example 6.10 (Directed Gaussian Graphical Models and MLE). The work of Améndola et al. [1] provides a novel connection between Maximum Likelihood Estimation (MLE) in statistics and the principles of Geometric Invariant Theory (GIT). Their idea is to translate statistical properties, such as the existence and uniqueness of the MLE, into the language of GIT stability by using the Kempf-Ness functional. They apply this to Gaussian group models, where maximising the log-likelihood is shown to be equivalent to a norm minimisation problem over a group orbit [1, Proposition 3.4, Proposition 3.13]. This relates the boundedness of the likelihood to semistability, the existence of an MLE to polystability, and the uniqueness of the MLE to stability [1, Theorem 3.10, Theorem 3.15].

Building upon this framework, Derksen and Makam [8] employed the machinery of quiver representations to solve the MLE threshold problem for matrix normal models, which is a key example of a Gaussian group model. They provided exact formulae for the number of samples required to ensure an MLE exists and is unique [8, Theorem 1.2, Theorem 1.3], and thus proving a conjecture of Drton, Kuriki, and Hoff [8, Corollary 1.4]. Related work of Makam, Reichenbach, and Seigal [25] studies Gaussian graphical models on directed acyclic graphs with symmetry constraints, namely RDAG (restricted directed acyclic graph) models, and characterises existence and uniqueness of the MLE via linear independence conditions. The following example applies this GIT perspective to a specific Directed Acyclic Graph (DAG) model, demonstrating the topological properties of the GIT-stable space of data samples for which a unique MLE exists.

We consider a directed Gaussian graphical model on a star-shaped Directed Acyclic Graph (DAG), which is a connected DAG that has a unique child vertex, which is a vertex that has a parent, meaning there is an edge pointing to it from another vertex. This graph has k parent vertices $\{1, \dots, k\}$ and a single child vertex $k + 1$, representing a linear regression problem [3].

Suppose that sample data are collected into a matrix $Y = [Y^{(1)} \dots Y^{(k)} \mid Y^{(k+1)}]$ of size $n \times (k + 1)$, where n denotes the number of observations. The Maximum Likelihood Estimation (MLE) for the regression coefficients exists and is unique if and only if the matrix formed by the parent columns, $[Y^{(1)} \dots Y^{(k)}]$, has full column rank [3, Theorem 3.5 (c)]. We consider a degenerate sample $f = [Y^{(1)} \dots Y^{(k)} \mid Y^{(k+1)}]$, where the parent block $X := [Y^{(1)} \dots Y^{(k)}]$ is rank-deficient. Thus there are infinitely many regression coefficients β satisfying $X^\top(Y^{(k+1)} - X\beta) = 0$ [3, Section 3.2 & Section 7]. We refer to Section 5 of [3] for the complete construction of the f -stabilisation, where Lemma 5.3 shows $\hat{f}(\epsilon)$ has full column rank, hence lies in V^{st} and has a unique MLE.

Let $V = M_{n \times (k+1)}(\mathbb{C})$ be the sample space, in which a point is a block $[Y^{(1)} \dots Y^{(k)} \mid Y^{(k+1)}]$ with each column $Y^{(i)} \in \mathbb{C}^n$. Let the reductive group be $G := \text{GL}_k(\mathbb{C}) \times \mathbb{C}^\times$, which is embedded in $\text{GL}_{k+1}(\mathbb{C})$ via

$$g = (A, t) \mapsto \begin{pmatrix} A & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

We define the right action of G on the sample space V by matrix multiplication as $Y \mapsto Yg$. The character $\rho : G \rightarrow \mathbb{C}^\times$ is then defined by $\rho(g) = t$. Note that the 1-PS λ in G takes the general form

$$\lambda(t) = \text{diag}(t^{w_1}, t^{w_2}, \dots, t^{w_k}, t^{-\sum_{i=1}^k w_i}),$$

where the integers $\{w_1, \dots, w_k, -\sum_{i=1}^k w_i\}$ are the weights of the action.

There exist three equivalences between the notions of GIT stability and the existence and uniqueness of the MLE in the setting of DAG models [3, Remark 3.6], which can be stated as follows

- A sample Y is unstable if no MLE exists;
- a sample Y is polystable if an MLE exists, but possibly not unique;
- a sample Y is stable if the MLE is unique.

A 1-PS is destabilising if the limit $\lim_{t \rightarrow 0} Y \cdot \lambda(t)$ exists for a non-zero Y . The minimal destabilisation corresponds to making a single parent column redundant, namely the corresponding data of this parent column is zero for the limit to exist. We consider the specific 1-PS λ_1 defined as $\lambda_1(t) := \text{diag}(t^{-1}, 1, \dots, t)$, where the only negative weight is -1 , this means that $\lambda_1(t)$ acts on the first parent column $Y^{(1)}$. The corresponding negative weight space is

$$V(\lambda_1)_- = \{[Y^{(1)}0 \dots 0] \mid Y^{(1)} \in \mathbb{C}^n\} \cong \mathbb{C}^n,$$

as each sample Y has n observations. Therefore, the dimension of the negative weight space is $\dim_{\mathbb{C}} V(\lambda_1)_- = n$.

Observe that the 1-PS λ_1 gives the weight decomposition as

$$V_- = \text{span}\{e_1\}, \quad V_0 = \text{span}\{e_2, \dots, e_k\}, \quad V_+ = \text{span}\{e_{k+1}\},$$

and any matrix commuting with λ_1 preserves each weight space. Let $g = (g_{ij})_{1 \leq i, j \leq k+1}$ be an arbitrary element in G , then the commutator relation $t^{w_i} g_{ij} = t^{w_j} g_{ij}$ for all $t \in \mathbb{C}^\times$ implies that the matrix A is of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix},$$

where $a \in \mathbb{C}^\times$ and $B \in \text{GL}_{k-1}(\mathbb{C})$. Hence the centraliser $C_G(\lambda_1)$ is isomorphic to the product $\mathbb{C}^\times \times \text{GL}_{k-1}(\mathbb{C}) \times \mathbb{C}^\times$. Thus we have $\dim_{\mathbb{C}} C_G(\lambda_1) = 1 + (k-1)^2 + 1 = (k-1)^2 + 2$. Furthermore, since $G = \text{GL}_k(\mathbb{C}) \times \mathbb{C}^\times$, its dimension is the sum of the dimensions of the two Lie groups, namely $\dim_{\mathbb{C}} G = k^2 + 1$. Note that the quotient $G/C_G(\lambda_1) \cong G \cdot \lambda_1$ is irreducible, it follows that $\dim_{\mathbb{C}}(G \cdot \lambda_1) = \dim_{\mathbb{C}} G - \dim_{\mathbb{C}} C_G(\lambda_1) = 2k - 2$.

Recall that Proposition 2.3 establishes that $D_f : S^\ell \times I \times G \cdot \lambda_1$ is transverse to the zero section $\mathcal{O}_W$ for generic homotopies f , where $W \cong (G \cdot \lambda_1) \times \mathbb{C}^n$. Moreover, if $\ell + 1 + 2 \dim_{\mathbb{C}}(G \cdot \lambda_1) = \ell + 4k - 3 < 2n$, then Corollary 2.4 implies that $D_f^{-1}(\mathcal{O}_W)$ is empty, and so f can be deformed such that its image remains in the stable locus V^{st} .

By Theorem 2.5, we have $d_{min} = 2 \dim_{\mathbb{C}} V(\lambda_1)_- - 2 \dim_{\mathbb{C}}(G \cdot \lambda_1) = 2n - 4k + 4$. Thus to ensure $d_{min} > 0$, one requires that $n > 2k - 2$. Consequently, we have $\pi_q(V^{st}) = 0$ for every $0 \leq q < d_{min} - 2$, which implies that V^{st} is $(2n - 4k + 2)$ -connected. In particular, the stable locus V^{st} is path-connected if $n \geq 2k - 1$, and it is simply connected if $n \geq 2k$. These results imply that the space of samples with a unique MLE is connected, which means that any sample with a unique MLE can be continuously deformed into any other such sample through a path consisting entirely of samples with unique MLEs. Furthermore, the algebraic parameter space X_f of all f -stabilisations [3, Definition 5.8] lies in a connected ambient space and inherits the connectedness properties.

Consequently, observe from Corollary 6.7, we obtain $\pi_q(V^{st}/G) \cong \pi_{q-1}(G)$ for all $1 \leq q < d_{min} - 1$. Note also that $G = \text{GL}_k(\mathbb{C}) \times \mathbb{C}^\times$ is homotopy equivalent to $U(k) \times S^1$, we can therefore conclude that for $1 \leq q < d_{min} - 1$,

$$\pi_q(V^{st}/G) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & q = 2, \\ \mathbb{Z} & q \text{ is even, } 4 \leq q < 2k, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we have obtained the homotopy groups of the moduli space of stable samples.

In particular, $\pi_2(V^{st}/G) \cong \mathbb{Z} \oplus \mathbb{Z}$ implies that there can be non-trivial families of f -stabilisations parametrised by a 2-sphere, S^2 , that cannot be continuously contracted to a single model within the moduli space. Moreover, the non-trivial homotopy group $\pi_q(V^{st}/G) \cong \mathbb{Z}$ implies that for each even q with $4 \leq q < 2k$, one can construct a family of f -stabilisations parametrised by a q -sphere, S^q , in a non-trivial way, i.e. they are not null-homotopic. Therefore, while the f -stabilisation method of Bérczi et al. [3] guarantees a solution with a unique MLE, the topological properties of the solution space are intricate. The existence of these non-trivial homotopy groups proves that there are entire families of

such stabilisations, parametrised by spheres, which are topologically distinct and cannot be continuously deformed into one another.

From a statistical point of view, the stable locus V^{st} is the space of samples for which the model has a unique MLE. The connectedness result thus shows that if the sample size is sufficiently large compared to the number of parameters, the locus of samples admitting a unique MLE is highly connected. Moreover, this example is also related to the stabilisation procedure of Derksen and Makam’s quiver-based statistics work [8] and to the f -stabilisations of Bérczi et al. [3], which produce perturbations of non-identifiable samples into the stable locus, where the MLE is unique.

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