A Study of Stochastic Differential Equations Driven by Cylindrical Lévy Processes

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Abstract

This thesis develops a rigorous framework for stochastic integration in Banach spaces, with a particular focus on integration with respect to cylindrical Lévy processes – an infinite-dimensional generalization of Lévy processes that contains jumps and discontinuities. Motivated by the need to work in non-Hilbertian spaces, the thesis makes original contributions, including a generalization of the Itô integral for Banach space-valued processes, a stochastic Fubini theorem for generalised cylindrical Lévy processes, existence and uniqueness solution for Lévy driven stochastic differential equations in Banach spaces, as well as a key extension of the Brzeźniak and Hausenblas [8] framework to cylindrical Lévy settings using p-summing operators. These results are important in the theory of stochastic partial differential equations and infinite-dimensional stochastic systems. Future work will involve two research articles: one based on Sections 5–7, and another based on the advanced results of Section 8.

Author's Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for a degree or other qualification at this University or elsewhere. All sources are acknowledged as references.

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Contents

1. Introduction

The mathematical modelling of randomness phenomena has become increasingly vital in both theoretical and applied sciences. In particular, stochastic processes – especially those with jumps or discontinuities, such as Lévy processes – provide powerful tools for capturing complex dynamical behavior in various areas, including biology, engineering, finance, and physics. Lévy processes generalize Brownian motion by allowing for discontinuous paths and sudden changes, thereby offering a richer and more flexible analytical and probabilistic framework.

This thesis is devoted to the development of a rigorous theory of stochastic integration in Banach spaces, with a particular focus on cylindrical Lévy processes. It presents original contributions to the theory of stochastic integration, generalizations of Fubini-type results, and an operator-theoretic framework for analyzing infinite-dimensional Lévy driven systems. Such a setting is essential for modelling systems with infinite-dimensional state spaces, which arise naturally in several important applications, such as stochastic partial differential equations (SPDEs), term structure models in mathematical finance, and distributed parameter systems in signal processing.

Extending stochastic calculus to Banach spaces poses substantial technical challenges, as many standard tools from finite-dimensional Itô calculus fail to generalize directly to Banach space settings, especially in the presence of jumps. This thesis systematically addresses these difficulties by developing integration theories suited to Banach spaces of martingale type, identifying and characterizing appropriate classes of operator for integrability, and extending foundational results to the cylindrical Lévy setting.

1.1. **Motivation and Scope.** Stochastic calculus in infinite-dimensional spaces has seen a substantial process, notably in the context of Hilbert spaces. However, there is increasing interest in developing stochastic analysis tools in more general Banach spaces, driven by the need to work with non-Hilbertian function spaces, such as L^p, Sobolev, or Besov spaces, used in SPDEs theory. Unlike Hilbert spaces, Banach spaces lack inner products and the Riesz representation theorem, which necessitates new approaches for defining integrals, establishing measurability, convergence, and regularity of stochastic processes.

A pivotal theme in this thesis is the study of cylindrical Lévy processes, a class of infinite-dimensional Lévy processes that generalize classical finite-dimensional jump processes. By acting on the dual space, these processes allow the construction of well-defined stochastic integrals for a certain class of operator-valued integrands. In this context, the Bochner integral serves as a key tool, extending integration to Banach space-valued functions and facilitating the construction of L^p-spaces. This provides a natural foundation for defining stochastic integrals with respect to Lévy noise. The use of Lévy processes, which encompasses compound Poisson processes, subordinators, and more general infinite activity jump processes, provides a flexible and realistic framework for modelling real-world systems where sudden changes or bursts are observed.

This thesis systematically explores the theory of Banach space-valued stochastic integrals and Lévy processes, starting from the Bochner integral and culminating in more general Lévy stochastic integrals. This framework lays the groundwork for future applications to infinite-dimensional stochastic equations and martingale problems.

1.2. **Thesis Outline and Section Contributions.** The thesis begins by building a robust theoretical foundation and culminates in original results that advance the theory of Lévy driven stochastic calculus in Banach spaces.

Section 2 begins by introducing the Bochner integral, detailing key definitions and results related to measurability, integrability, and L^p spaces for Banach space-valued functions. This sets the stage for the formal development of stochastic integration.

The section then presents a review of Lévy processes, including their construction via characteristic functions, the Lévy–Khintchine formula, and the structure of compound Poisson processes. Key concepts such as infinite divisibility and Lévy measures are explored.

Subsequently, Poisson random measures and compensated Poisson integrals are introduced, which are fundamental to the theory of jump-driven stochastic calculus. Their integration framework is carefully constructed using predictable integrands and square-integrability conditions in Banach spaces.

Finally, the section reviews the Itô formula for Lévy driven stochastic integrals, providing a powerful analytical tool for dealing with nonlinear functionals of stochastic processes.

It closes with a discussion of the Lévy-Itô decomposition and martingale-valued measures associated with Lévy noise.

Section 3 defines cylindrical measures and cylindrical random variables in Banach spaces, extends Lévy processes to infinite dimensions, emphasizing cylindrical Lévy processes and their characteristic functions, and finally discusses weak integrability conditions and cylindrical martingales, linking them to cylindrical Lévy processes.

Section 4 first introduces p-summing operators between Banach spaces, focusing on their norm properties and relationships with L^p spaces. The section then defines p-Radonifying operators, which extends cylindrical measures to Radon measures, and their equivalence to p-summing operators under certain conditions.

Section 5 sets up the framework for stochastic integration in Banach spaces, including assumptions on cylindrical Lévy processes and p-summing operators, then defines stochastic integrals for step process, showing how to approximate general processes in Banach spaces of martingale type $p \in [1, 2]$. A key technical tool here is the Banach space $\Lambda^p(0, T; \Pi_p(\mathrm{K}, \mathrm{E}))$, which accommodates stochastic processes taking values in the space of p-summing operators from a Banach space K into another Banach space E. The section defines the stochastic integral operator $I(\psi)$, of step functions, and extends it to more general integrands. A main contribution, culminating in Theorem 5.36, ensures that under mild assumptions, the stochastic integral exists, has the martingale property, admits càdlàg modification, and it satisfies a stronger inequality, (5.21), i.e.,

$$\mathbb{E} \sup_{t\in[0,T]} \|I_t(\psi)\|^p \le C\mathbb{E} \int_0^T \pi_p^p(\psi(s))\, ds.$$

These results are important for stochastic analysis in infinite-dimensional spaces.

Section 6 extends the classical Fubini theorem to the setting of stochastic integrals with a generalized cylindrical Lévy process in Banach spaces. The main result, Theorem 6.6, allows for interchanging the order of integration with respect to a measure space and a stochastic integral Specially, if the appropriate conditions are met, equality (6.9) holds, i.e.,

$$\int_O \left(\int_0^t \psi(s,y)\, dL(s) \right) d\mu(y) = \int_0^t \left(\int_O \psi(s,y)\, d\mu(y) \right) dL(s), \quad \mathbb{P}-a.s.$$

This extension relies on precise measurability and integrability arguments. The proof involves a careful approximation of integrands and uses tools from Bochner integration and measure theory.

Section 7 applies the theory developed in section 5 to formulate and solve stochastic differential equations (SDEs) of the form:

$$dX(t) = B(t, X(t))\, dt + G(t, X(t))\, dL(t), \quad X(0) = X_0.$$

in Banach spaces, where B and G are strongly measurable functions. The section proves existence and uniqueness of solutions for Lévy driven SDEs in Banach spaces, under Lipschitz and growth conditions on the coefficients B and G, using the Banach Fixed Point Theorem and exploring properties of the stochastic integral.

Section 8 constitutes a significant contribution of this thesis. It extends the work by Brzeźniak and Hausenblas [8], originally developed for Poisson random measures, to the more general cylindrical Lévy processes, thereby extending their results to a broader and more flexible class of stochastic drivers. It introduces a class of Banach spaces $\mathscr{E}_{p,q}$ satisfying specific integrability and continuity properties, and proves that stochastic convolutions and integrals involving p-summing operators are well-defined under these settings. The main result, Theorem 8.8, shows that the stochastic integral mapping satisfies crucial conditions (R1)-(R6) as formulated in [8], such as boundedness, interpolation invariance, and operator liftings from Poisson to Lévy settings. These results substantially broaden the applicability of the theory, particularly for semigroup-based evolution equations and operator-valued integrals in infinite-dimensional settings.

Finally, a few words about future directions. I plan to develop two research papers based on the results of this thesis. The first article will be based on Sections 5, 6, and 7, while the second will be based on Section 8.

2. Foundation

We commence the foundational framework by presenting the Bochner integral, a critical tool for stochastic integration in Banach spaces, which underpins the theoretical development of subsequent stochastic analysis.

2.1. Bochner Integral.

This subsection is based on chapter 1.2 of [12].

Definition 2.1. [40, p. 205] *A set X is a real $\mathbb{R}$ (or complex $\mathbb{C}$) Banach space if X is a vector space over $\mathbb{R}$ (or $\mathbb{C}$) scalar field, equipped with a mapping $\|\cdot\| : X \to \mathbb{R}$, called norm, that satisfies the following properties:*

(i) $\|af\| = |a|\,\|f\|$ *for* $f \in X$, $a \in \mathbb{R}$ *(or* $a \in \mathbb{C}$*),*

(ii) $\|f + g\| \leq \|f\| + \|g\|$ *for* $f, g \in X$,

(iii) $\|f\| = 0$ *if and only if* $f = 0$,

Moreover, if a sequence $\{f_n\} \in X$ *satisfies* $\lim_{n,m\to\infty} \|f_n - f_m\| = 0$, *then there exists an* $f \in X$ *such that* $\lim_{n\to\infty} \|f_n - f\| = 0$.

Definition 2.2. [1, p. 9] *A metric space X is said to be separable if it has a countable dense subset. In particular, a separable Banach space is a Banach space that contains a countable dense subspace with respect to the distance generated by the norm.*

Let us point out that examples of separable Banach spaces include the Lebesgue spaces L^p with $p \in [1, \infty)$ and the Sobolev spaces $H^{s,p}$ with $s \in \mathbb{R}$ and $p \in [1, \infty)$. The space, L^∞, however, is generally not separable (although it is a Banach space).

Building on the notion of separability in Banach spaces, we now turn to the concept of measurability in the context of measurable spaces.

Let X be a Banach space over a scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let $\mathcal{B}(X)$ denote the Borel σ-field of X. Let $(\Omega, \mathcal{F})$ and $(S, \mathcal{S})$ be measurable spaces.

Definition 2.3. [12, Definition 1.2.1, p. 16] *A function* $f : \Omega \to S$ *is* $\mathcal{F}/\mathcal{S}$*-measurable if and only if* $f^{-1}(A) \in \mathcal{F}$ *for every* $A \in \mathcal{S}$.

The Borel measurable, simple measurable, strongly measurable, finitely-valued and separably valued functions are respectively defined as follows:

$$\mathcal{B}(\Omega, \mathcal{F}; X) = \Big\{f : \Omega \to X : f \text{ is } \mathcal{F}/\mathcal{B}(X) - \text{measurable}\Big\}.$$

$$\mathrm{Sim}(\Omega, \mathcal{F}; X) = \Big\{ \sum_{j=1}^{N} x_j \mathbb{1}_{F_j} : N \in \mathbb{N}, x_1, \cdots, x_N \in X, F_1, \cdots, F_N \in \mathcal{F} \Big\}. \tag{2.1}$$

$$\mathrm{Str}(\Omega, \mathcal{F}; X) = \Big\{ f : \Omega \to X : \exists (f_n)_{n=1}^{\infty} \subset \mathrm{Sim}(\Omega, \mathcal{F}; X) : f_n \to f \text{ pointwise} \Big\}.$$

$$\mathrm{Fin}(\Omega, \mathcal{F}; X) = \Big\{ f : \Omega \to X : f \text{ is a finite non-zero constant on each of a finite number of disjoint } \mathcal{F}\text{-measurable sets } F_j; \text{ and } f \text{ is zero on } \Omega \setminus \cup_{j=1}^{n} F_j \Big\}.$$

$$\mathrm{Sep}(\Omega, \mathcal{F}; X) = \Big\{ f : \Omega \to X : \exists \text{ a closed separable subspace } X_0 \subset X : f(\Omega) \subset X_0 \Big\}.$$

Referring to Lemmas 1.2.3-4 and Corollary 1.2.6 in [12], we have the following facts:

(i) $\mathcal{B}(\Omega, \mathcal{F}; X)$, $\mathrm{Str}(\Omega, \mathcal{F}; X)$ and $\mathrm{Sep}(\Omega, \mathcal{F}; X)$ are closed under pointwise limits.

(ii) [31, corollary of Pettis's measurability theorem 1.2.5] The pointwise limit of a sequence of strongly measurable functions is itself strongly measurable.

Theorem 2.4. [31, Theorem 1.1]

$$\mathtt{Str}(\Omega, \mathcal{F}; X) = \mathcal{B}(\Omega, \mathcal{F}; X) \cap \mathit{Sep}(\Omega, \mathcal{F}; X)$$

Corollary 2.5. [12, Corollary 1.2.8] *When X is separable, $Str(\Omega, \mathcal{F}; X) = \mathcal{B}(\Omega, \mathcal{F}; X)$.*

The following definitions and the Alexandroff Theorem mainly follow Definitions 1.1.1-2 and 1.1.6 in [12].

Definition 2.6. [19, Chapter 1, Section 1.1. Def. 1.1] *A set function is a function*

$$\mu : \mathcal{A} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$$

defined on a collection $\mathcal{A}$ of a given set Ω, where $\overline{\mathbb{R}}$ is the extended real number, and $\mu(\mathcal{A})$ assigns a (possibly extended real) number to each set $A \in \mathcal{A}$.

Definition 2.7. [18, Definition III.1.3] *Assume that S is a non-empty set. A family $\mathcal{S}$ of subsets of S is called a field if and only if*

(i) *$\emptyset \in \mathcal{S}$;*

(ii) *if $A \in \mathcal{S}$, then $S \setminus A \in \mathcal{S}$,*

(iii) *if $A, B \in \mathcal{S}$, then $A \cup B \in \mathcal{S}$.*

Lemma 2.8. *(see after* [18, Definition III.1.3]*). If $\mathcal{S}$ is a field of subsets of a non-empty set S, then*

(iv) *if* $A, B \in \mathcal{S}$*, then* $A \cap B \in \mathcal{S}$*,*

(v) *if* $A, B \in \mathcal{S}$*, then* $A \setminus B \in \mathcal{S}$*.*

Definition 2.9. *Assume that* $\mathcal{S}$ *is a field of subsets of a non-empty* S *and* X *is a Banach space. A set function* $\mu : \mathcal{S} \to X$ *is finitely additive if*

(i) $\emptyset \in \mathcal{S}$*;*

(ii) $\mu(\emptyset) = 0$*;*

(iii) $\mu(\sum_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$

for every finite family $\{(A_i)_{i=1}^{n}\} \subset \mathcal{S}$ *of pairwise disjoint sets such that* $\cup_{i=1}^{n} A_i \in \mathcal{S}$*.*

Definition 2.10. *Assume that* $\mathcal{S}$ *is a field of subsets of a non-empty* S *and* X *is a Banach space. Let* $\mu : \mathcal{S} \to X$ *be a finitely additive set function defined on* $\mathcal{S}$*. Then* μ *is countably additive if and only if*

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{n=1}^{\infty} \mu(A_i)$$

for every sequence $\{(A_i)_{i=1}^{\infty}\} \subset \mathcal{S}$ *of pairwise disjoint sets such that* $\cup_{i=1}^{\infty} A_i \in \mathcal{S}$*.*

Theorem 2.11. *(Alexandroff)* [18, Theorem III.5.13, p. 138] *Let* $\mu : \mathcal{S} \to \mathbb{C}$ *be a bounded, see Definition G.3, regular complex valued additive set function defined on a field* $\mathcal{S}$ *of subsets of a compact topological space* S*, see Definition F.7. Then* μ *is countably additive.*

Theorem 2.12. *(Alexandroff)* [12, Theorem 1.1.7] *Let* $\mu : \mathcal{S} \to [0, \infty]$ *be a regular finitely additive set function defined on a field* $\mathcal{S}$ *of subsets of a compact topological space* S*. Then* μ *is countably additive.*

Definition 2.13. [37, Definition 3.6, p. 65] *Let* $(\Omega, \mathcal{F}, \mu)$ *be a measure space and let* $0 < p < \infty$*. For a measurable function* $f : \Omega \to \mathbb{R}$ *(or* $\mathbb{C}$*), define the* L^p*-norm by*

$$\|f\|_{L^p(\mu)} := \Big(\int_{\Omega} |f|^p \, d\mu \Big)^{\frac{1}{p}}. \tag{2.2}$$

Let $L^p(\mu)$ *denote the set of all measurable functions* f *for which*

$$\|f\|_{L^p(\mu)} < \infty.$$

We call $\|f\|_{L^p(\mu)}$*, or simply* $\|f\|_p$*, the* L^p*-norm of* f*.*

Definition 2.14. *Let X be a Banach space, and let $p \in [1, \infty)$. Define*

$$\mathscr{L}^p(\Omega, \mathcal{F}, \mu; X) := \Big\{ f \in Str(\Omega, \mathcal{F}; X) : \|f\|_{\mathcal{L}^p} = \Big(\int_\Omega \|f\|^p \, d\mu(\omega) \Big)^{\frac{1}{p}} < \infty \Big\} \tag{2.3}$$

as a semi-normed vector space, see Definition F.10, and

$$L^p(\Omega, \mathcal{F}, \mu; X) := \Big\{ [f]_\sim : f \in \mathscr{L}^p(\Omega, \mathcal{F}, \mu; X) \Big\} \tag{2.4}$$

is a Banach space, see Definition 2.1, of all equivalence classes of mapping $[f]_\sim : \Omega \to \mathbb{R}^d$ that are equal almost everywhere (a.e.) w.r.t. μ.

Remark 2.15. The symbol $\mathscr{L}$ is used in a very different sense than in Definition 3.10. The present use of this symbol is only temporary and often it is as $\mathscr{L}^p$ and hence the abuse of notation should not cause any problem.

Here the norm on $L^p(\Omega, \mathcal{F}, \mu; X)$ is given by the L^p-norm defined in (2.2)*.*

Lemma 2.16. *Let $\mathscr{L}^p(\Omega, \mathcal{F}, \mu; X)$ be a semi-normed vector space defined in* (2.3)*, Moreover, if $f \sim g$, then $L^p(\Omega, \mathcal{F}, \mu; X)$ defined in* (2.4) *is a Banach space w.r.t. the norm defined by*

$$\|[f]_\sim\|_p : \|f\|_p, \quad f \in \mathscr{L}^p(\Omega, \mathcal{F}, \mu; X).$$

Remark 2.17. $f \sim g$ if and only if $\|f - g\|_p = 0$.

Remark 2.18. Let $f \in \mathscr{L}^p(0, T; X)$ and $\{f_n\}_{n=1}^\infty \subset \mathscr{L}^p(0, T; X)$ be a sequence of functions, such that $f_n \to f$ in $\mathscr{L}^p(0, T; X)$. Then there exists a subsequence $\{f_{n_j}\}_{j=1}^\infty$ such that $f_{n_j} \to f$ almost everywhere.

Lemma 2.19. *$f \in \mathscr{L}^p(\Omega, \mathcal{F}, \mu; X)$ if and only if there exists $\{f_n\}_{n=1}^\infty \subset Fin(\Omega, \mathcal{F}, \mu; X)$ such that $f_n \to f$ pointwise and $\|f - f_n\|_p \to 0$.*

Definition 2.20. [47, Definition, p. 132] *Assume that $(\Omega, \mathcal{F}, \mu)$ is a measure space and X is a Banach space. A function $f : \Omega \to X$ is ($\mu-$) Bochner integrable (or Strongly ($\mu-$) measurable) if and only if there exists a sequence $(f_n)_{n=1}^\infty \subset Fin(\Omega, \mathcal{F}, \mu; X)$ such that $f_n \to f$ pointwise and $\int_\Omega |f - f_n| \, d\mu \to 0$.*

For any set $F \in \mathcal{F}$, the (μ-) Bochner integral of f over F is defined by

$$\int_F f \, d\mu := \lim_{n \to \infty} \int_\Omega f_n \mathbb{1}_F \, d\mu. \tag{2.5}$$

See [47, p. 132-133] for the justification that the limit on the right-hand side of (2.5) exists.

Theorem 2.21. [47, Theorem 1, p. 133] *A strongly $\mathcal{F}$-measurable function f is Bochner μ-integrable if and only if $\|f\|$ is μ-integrable.*

Lemma 2.22. *If $n \in \mathbb{N}$, $(F_i)_{n=1}^n \subset \mathcal{F}$, $\mu(\cup_{i=1}^n F_i) < \infty$ and $f = \sum_{i=1}^n a_i \mathbb{1}_{F_i}$, then $f \in$ $Fin(\Omega, X; \mu)$ and $\int_\Omega f\, d\mu = \sum_{i=1}^n a_i\, \mu(F_i)$.*

2.2. **Lévy processes.**

Having introduced the Bochner integral as a framework for integrating Banach space-valued functions, we are now prepared to study Lévy processes, which will form the foundation for the stochastic models studied in the following sections.

2.2.1. *Basic Definitions and Theorems.*

We begin with a set of classical definitions and theorems about Lévy processes.

Definition 2.23. [6, Chapter 1, Section 4, p. 13] *A random variable is a measurable function from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.*

Definition 2.24. [1, section 1.1.6] *Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.*

Suppose that $X : \Omega \to \mathbb{R}^d$ is a random variable. The characteristic function of X is a function $\phi_X : \mathbb{R} \to \mathbb{C}$ defined by

$$\phi_X(u) = \mathbb{E}(e^{i(u,X)}) = \int_\Omega e^{i(u,X(\omega))}\mathbb{P}(d\omega), \quad u \in \mathbb{R}^d.$$

Let $\mathcal{M}_1(\mathbb{R}^d)$ denote the set of all Borel probability measures on $\mathbb{R}^d$. Note that the subscript "1" in $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ indicates that $\mu(\mathbb{R}^d) = 1$.

Definition 2.25. *If $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ then the characteristic function of μ is a function $\phi_\mu : \mathbb{R} \to \mathbb{C}$ defined by*

$$\phi_\mu(u) = \int_{\mathbb{R}^d} e^{i(u,y)}\, \mu(dy), \quad u \in \mathbb{R}^d.$$

A natural question arises regarding the relationship between the two definitions above. The answer is provided by the following lemma.

Lemma 2.26. *Suppose that $X : \Omega \to \mathbb{R}^d$ is a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mu = \mu_X = \mathrm{L}(X)$ the law of X, i.e., a Borel probability measure on $\mathbb{R}^d$ defined by*

$$\mu_X(A) := \mathbb{P}(X^{-1}(A)), \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Then,

$$\phi_\mu(u) = \phi_X(u), \quad u \in \mathbb{R}^d.$$

Lemma 2.27. [1, Lemma 1.1.11]

(1) *ϕ_X is positive definite for all $u_1, \cdots, u_d \in \mathbb{R}^d$.*

(2) $\phi_X(0) = 1$.

(3) *The map ϕ_X is continuous at the origin.*

Definition 2.28. [1, section 1.1.7, p. 19] *The finite-dimensional distribution of a stochastic process $X = (X(t), t \geq 0)$ is the collection of probability measures $(p_{t_1,t_2,\cdots,t_n}, t_1, t_2, \cdots, t_n \in \mathbb{R}^+, t_1 \neq t_2 \neq \cdots \neq t_n, n \in \mathbb{N})$ defined on $\mathbb{R}^{dn}$ by*

$$p_{t_1,t_2,\cdots,t_n}(A) = \mathbb{P}\left((X(t_1), X(t_1), \cdots, X(t_1)) \in A\right)$$

for each $A \in \mathcal{B}(\mathbb{R}^{dn})$.

Let σ be a permutation of $\{1, 2, \cdots, n\}$; then for each $H_1, H_2, \cdots, H_n \in \mathcal{B}(\mathbb{R}^d)$,

$$p_{t_1,t_2,\cdots,t_n}(H_1 \times H_2 \times \cdots \times H_n) = p_{t_{\sigma(1)},t_{\sigma(2)},\cdots,t_{\sigma(n)}}(H_{\sigma(1)} \times H_{\sigma(2)} \times \cdots \times H_{\sigma(n)}); \tag{2.6}$$

$$p_{t_1,t_2,\cdots,t_n,t_{n+1}}(H_1 \times H_2 \times \cdots \times H_n \times \mathbb{R}^d) = p_{t_1,t_2,\cdots,t_n}(H_1 \times H_2 \times \cdots \times H_n); \tag{2.7}$$

Equations (2.6) *and* (2.7) *are called Kolmogorov's consistency criteria.*

Proof of Equation (2.7). Notations $(\mathbb{R}^d)^n = \mathbb{R}^{dn}$, $A \subset$ Borel, $HA_1 = H \times \mathbb{R}^d$.

$$\begin{aligned}
p_{t_1,t_2,\cdots,t_n,t_{n+1}}(H \times \mathbb{R}^d) =& \mathbb{P}\big((X(t_1), \cdots, X(t_n), X(t_{n+1})) \in H \times \mathbb{R}^d\big) \\
=& \mathbb{P}\big((X(t_1), \cdots, X(t_n)) \in H, X(t_{n+1}) \in \mathbb{R}^d\big) \\
=& \mathbb{P}\big((X(t_1), \cdots, X(t_n)) \in H \cap (X(t_{n+1}) \in \mathbb{R}^d)\big) \\
=& \mathbb{P}\big((X(t_1), \cdots, X(t_n)) \in H \cap \Omega\big) \\
=& \mathbb{P}\big((X(t_1), \cdots, X(t_n)) \in H\big).
\end{aligned}$$

□

Definition 2.29. [1, Section 1.2.2] *Let X be an $\mathbb{R}^d$-valued random variable. We say that X is infinitely divisible if and only if, for each $n \in \mathbb{N}$, there exist i.i.d. random variables $Y_1^{(1)}, \cdots, Y_n^{(n)}$ such that*

$$X = Y_1^{(1)} + \cdots + Y_n^{(n)}.$$

The following result lists two equivalent formulations of infinite divisibility.

Proposition 2.30. [1, Proposition 1.2.6] *Let X be an $\mathbb{R}^d$-valued random variable with the law denoted by μ_X. Then the following three conditions are equivalent.*

(a) *X is infinitely divisible;*

(b) *for every $n \in \mathbb{N}$, μ_X has a convolution n-th root that is itself the law of a random variable, i.e., there exists a measure $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ such that $\nu^{*^n} = \mu$.*

(c) *for every $n \in \mathbb{N}$, $\phi_X(u)$ has an n-th root that is itself the characteristic function of a random variable.*

In general, the convolution n-th root of a probability measure may not be unique. However, it is always unique when the measure is infinitely divisible.

Definition 2.31. [1, Example 1.2.10] *Suppose that $(Z_n, n \in \mathbb{N})$ is a sequence of i.i.d. $\mathbb{R}^d$-valued random variables with common law μ_Z, and that $N \sim Pois(c)$ is a Poisson random variable that is independent of all the $Z_1, Z_2, \cdots, Z_n$. The Compound Poisson random variable X is defined as*

$$X = Z_1 + Z_2 + \cdots + Z_N,$$

so we can think of X as a random walk with a random number of steps, i.e.,

$$X = Z_1(\omega) + Z_2(\omega) + \cdots + Z_N(\omega).$$

Proposition 2.32. [1, Proposition 1.2.11] *In the framework of Definition 2.31, for each $u \in \mathbb{R}^d$,*

$$\phi_X(u) = exp\{ \int_{\mathbb{R}^d} [e^{i(u,y)} - 1] c\, \mu_Z(dy) \}.$$

Proof of Proposition 2.32. Let ϕ_Z be the common characteristic function of $Z_1, Z_2, \cdots, Z_n$. By conditioning on Poisson random variable N and applying the law of total expectation, we obtain

$$\phi_X(u) = \sum_{n=0}^{\infty} \mathbb{E}\Big(exp\{i(u, Z_1 + Z_2 \cdots + Z_N)\} | N = n \Big) \, P(N = n) \cdots$$

Since $N \sim Pois(c)$, we have

$$\begin{aligned}
\cdots &= \sum_{n=0}^{\infty} \mathbb{E}\Big(exp\{i(u, Z_1 + Z_2 \cdots + Z_n)\} \Big) \, e^{-c} \frac{c^n}{n!} \\
&= \sum_{n=0}^{\infty} \mathbb{E}\Big(exp\{i(u, Z_1)\} \, exp\{i(u, Z_2)\} \cdots exp\{i(u, Z_n)\} \Big) \, e^{-c} \frac{c^n}{n!} = \cdots
\end{aligned}$$

Using independence of the $Z_1, Z_2, \cdots, Z_n$, we have

$$\cdots = \sum_{n=0}^{\infty} \mathbb{E}\Big(exp\{i(u, Z_1)\} \Big) \, \mathbb{E}\Big(exp\{i(u, Z_2)\} \Big) \cdots \mathbb{E}\Big(exp\{i(u, Z_n)\} \Big) \, e^{-c} \frac{c^n}{n!} = \cdots$$

Since each Z_k has the same characteristic function, $\phi_Z(u)$, we get

$$\begin{aligned}
\cdots &= \sum_{n=0}^{\infty} \phi_Z(u) \cdot \phi_Z(u) \cdots \phi_Z(u) \, e^{-c} \frac{c^n}{n!} = \sum_{n=0}^{\infty} \left[\phi_Z(u) \right]^n e^{-c} \frac{c^n}{n!} \\
&= e^{-c} \sum_{n=0}^{\infty} \frac{\left[c\phi_Z(u) \right]^n}{n!} = e^{-c} \, exp\{c\phi_Z(u)\} \\
&= exp\{c\phi_Z(u) - c\} = exp\{c[\phi_Z(u) - 1]\} = \cdots
\end{aligned}$$

By the definition of characteristic function of Z and using parts (2) and (3) of Lemma 2.27, we have

$$\begin{aligned}
\cdots &= exp\{c[\int_{\mathbb{R}^d} e^{i(u,y)} \, \mu_Z(dy) - \int_{\mathbb{R}^d} e^{i(0,y)} \, \mu_Z(dy)]\} \\
&= exp\{c[\int_{\mathbb{R}^d} e^{i(u,y)} \, \mu_Z(dy) - \int_{\mathbb{R}^d} 1 \, \mu_Z(dy)]\} \\
&= exp\{c \int_{\mathbb{R}^d} [e^{i(u,y)} - 1] \, \mu_Z(dy)\}.
\end{aligned}$$

□

Definition 2.33. [1, section 1.2.4, p. 29] *A non-negative Borel measure ν on $\mathbb{R}^d \setminus \{0\}$ is called a Lévy measure if and only if*

$$\int_{\mathbb{R}^d \setminus \{0\}} \Big(|y|^2 \wedge 1 \Big) \, \nu(dy) < \infty. \tag{2.8}$$

Exercise 2.34. Suppose that ν is a Borel measure on $\mathbb{R}^d \setminus \{0\}$. Then the following two conditions are satisfied.

(i) ν is a Lévy measure;

(ii)

$$\int_{\mathbb{R}^d\setminus\{0\}} \frac{|y|^2}{1+|y|^2}\,\nu(dy) < \infty. \tag{2.9}$$

Proof of Exercise 2.34. If ν is a Lévy measure, then condition (2.8) holds. We have

$$\frac{|y|^2}{1+|y|^2} \le |y|^2 \wedge 1 \le 2\frac{|y|^2}{1+|y|^2}, \quad y \in \mathbb{R}.$$

The integrand $0 \le |y|^2 \wedge 1 \le 1$ of the sufficient and necessary condition (2.8) is equivalent to the integrand of condition (2.9), $0 < \frac{|y|^2}{1+|y|^2} \le 1$. □

Theorem 2.35. *(Lévy-Khintchine)*[1, p. 29] *A measure $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ is infinitely divisible if there exists a vector $b \in \mathbb{R}^d$, a positive definite symmetric, see Definition F.11, $d \times d$ matrix A and a Lévy measure ν on $\mathbb{R}^d \setminus \{0\}$, such that, for all $u \in \mathbb{R}^d$,*

$$\phi_\mu(u) = exp\{i(b,u) - \frac{1}{2}(u,Au) + \int_{\mathbb{R}^d\setminus\{0\}} [e^{i(u,y)} - 1 - i(u,y)\mathbb{1}_{\hat{B}}(y)]\nu(dy)\},$$

where $\hat{B} := \{x \in \mathbb{R}^d : |x| < 1\} = B_1(0)$.

Conversely, if $\phi : \mathbb{R}^d \to \mathbb{C}$ is a map of the following form

$$\phi(u) = exp\{i(b,u) - \frac{1}{2}(u,Au) + \int_{\mathbb{R}^d\setminus\{0\}} [e^{i(u,y)} - 1 - i(u,y)\mathbb{1}_{\hat{B}}(y)]\nu(dy)\},$$

where $b \in \mathbb{R}^d$, A is a positive definite symmetric $d \times d$ matrix, and ν is a Lévy measure on $\mathbb{R}^d \setminus \{0\}$, then there exists an infinitely divisible probability measure μ on $\mathbb{R}^d$ such that

$$\phi = \phi_\mu.$$

Note that the quadratic term (u, Au) above can be written more explicitly:

(i) In dimension $d = 1$, if $A = [a]$ is a 1×1 matrix, then $(u, Au) = au^2$.

(ii) In general, for $A = [a_{ij}]_{i,j=1}^d$ with $a_{i,j} = a_{j,i}$ (i.e., symmetric), we have $(u, Au) = \sum_{i,j=1}^d a_{ij}u_iu_j$.

Definition 2.36. *A function*

$$\eta : \mathbb{R}^d \ni u \mapsto i(b,u) - \frac{1}{2}(u,Au) + \int_{\mathbb{R}^d\setminus\{0\}} [e^{i(u,y)} - 1 - i(u,y)\mathbb{1}_{\hat{B}}(y)]\nu(dy) \in \mathbb{C}$$

is called a Lévy symbol, where $b \in \mathbb{R}^d$, A is a positive definite symmetric $d \times d$ matrix, ν is a Lévy measure on $\mathbb{R}^d \setminus \{0\}$ as defined in Definition 2.33 and $\hat{B} = \{y \in \mathbb{R}^d : |y| < 1\}$.

If $\mu \in \mathcal{M}_1(\mathbb{R}^d)$ is infinitely divisible, and the parameters b, A, ν are as in Theorem 2.35, then the function η defined above is called the Lévy symbol of μ.
If X is an infinitely divisible random variable and $\mu = \mu_X$ is the law of X, then the Lévy symbol of μ_X is also called the Lévy symbol of X.

Note that for a Lévy process $X = (X(t), t \geq 0)$, when $t = 1$, we have

$$\mathbb{E}\Big[e^{i\mu X(1)}\Big] = \phi_{X(1)}(\mu)$$

where $\phi_{X(1)}$ is the characteristic function of $X(1)$.

Definition 2.37. [1, p. 101] *A set $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ is said to be bounded below if and only if $0 \notin \overline{A}$, where $\overline{A}$ is the closure of A.*

Remark 2.38. One can show that set $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ is said to be bounded below if and only if there exists $r > 0$ such that

$$B(0, r) \cap A = \emptyset.$$

Theorem 2.39. [1, Theorem 2.3.5, p. 101] *Let $N(t, A)$ be the Poisson random measure associated to a Lévy process. Then:*

(a) *If A is bounded below, then $(N(t, A), t \geq 0)$ is a Poisson process with intensity $\mu(A)$.*
(b) *If $A_1, \cdots, A_m \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ are bounded below and pairwise disjoint, then the random variables $N(t, A_1), \cdots, N(t, A_m)$ are independent.*

Theorem 2.40. [1, Theorem 2.3.7, p. 106] *Let A be bounded below, and let N be the Poisson random measure associated to a Lévy process $X = (X(t) : t \geq 0)$. Then:*

(a) *for each $t \geq 0$, $\int_A f(x)N(t, dx)$ has a compound Poisson distribution such that, for each $u \in \mathbb{R}^d$,*

$$\mathbb{E}(exp[i(u, \int_A f(x)\, N(t, dx))]) = exp[t \int_{\mathbb{R}^d} e(^{i(u,x)} - 1)\, \mu_{f,A}(dx)],$$

where $\mu_{f,A}(B) = \mu(A \cap f^{-1}(B))$ for each $B \in \mathcal{B}(\mathbb{R}^d)$;
(b) *if $f \in L^1(A, \mu_A)$, then*

$$\mathbb{E}(\int_A f(x)\, N(t, dx)) = t \int_A f(x)\, \mu(dx);$$

(c) *if* $f \in L^2(A, \mu_A)$, *then*

$$Var\Big(\Big|\int_A f(x)\, N(t,dx)\Big|\Big) = t \int_A |f(x)|^2\, \mu(dx).$$

Proof of Theorem 2.40 (a). For simplicity, we let

$$f = \sum_{j=1}^{n} c_j \mathbb{1}_{A_j} \in L^1(A, \mu_A),$$

where each $c_j \in \mathbb{R}^b, A_j$ is the disjoint Borel subset of A. By [1, Theorem 2.3.5] (see Theorem 2.39 above), the random variables $N(t, A_1), \cdots N(t, A_n)$ are independent Poisson random variables with respective intensities $t\mu(A_j)$, hence we have

$$\begin{aligned}
\mathbb{E}(exp[i(u, \int_A f(x)\, N(t,dx))]) &= \mathbb{E}(exp[i(u, \sum_{j=1}^{n} c_j N(t, A_j))]) \\
&= \mathbb{E}(exp[i(u,\ c_1 N(t, A_1) + \cdots + c_n N(t, A_n))]) \\
&= \mathbb{E}(exp[i(u,\ c_1 N(t, A_1))] \cdots exp[i(u,\ c_n N(t, A_n))]) \\
&= \mathbb{E}(exp[i(u,\ c_1 N(t, A_1))]) \cdots \mathbb{E}(exp[i(u,\ c_n N(t, A_n))]) \\
&= \prod_{j=1}^{n} \mathbb{E}(exp[i(u,\ c_j N(t, A_j))]) \\
&= \prod_{j=1}^{n} \mathbb{E}(exp[i(u, c_j)\, N(t, A_j)]) = \cdots
\end{aligned}$$

Again, by Theorem 2.39, we have $N(t, A) \sim Pois(t\mu_A)$, and by Lemma 2.41 below, $N(t, A_j) \sim Pois(t\mu_{A_j})$, we have

$$\begin{aligned}
\cdots = \prod_{j=1}^{n} exp\{t\mu_{A_j}[e^{i(u,c_j)} - 1]\} &= exp\{t \sum_{j=1}^{n} \mu_{A_j}[e^{i(u,c_j)} - 1]\} \\
= exp\{t \sum_{j=1}^{n} \int_{A_j} [e^{i(u,c_j)} - 1]\, d\mu(x)\} &= exp\{t \int_A [e^{i(u,c_j)} - 1]\, \mu(dx)\}.
\end{aligned}$$

The last equality holds because

$$\begin{aligned}
&\int_A [e^{i(u,c_j)} - 1]\mu(dx) \\
&= \sum_{j=1}^{n} \int_{A_j} [e^{i(u,f(x))} - 1]\, d\mu(x) + \int_{A/(\cup_{j=1}^{n} A_j)} [e^{i(u,f(x))} - 1]\, d\mu(x) \\
&= \sum_{j=1}^{n} \int_{A_j} [e^{i(u,c_j)} - 1]\, d\mu(x) + \int_{A/(\cup_{j=1}^{n} A_j)} [e^{i(u,0)} - 1]\, d\mu(x) \\
&= \sum_{j=1}^{n} \int_{A_j} [e^{i(u,c_j)} - 1]\, d\mu(x) + \int_{A/(\cup_{j=1}^{n} A_j)} [1 - 1]\, d\mu(x) \\
&= \sum_{j=1}^{n} \int_{A_j} [e^{i(u,c_j)} - 1]\, d\mu(x).
\end{aligned}$$

Proofs of Theorem 2.40 parts (b) and (c) are omitted. □

Lemma 2.41. *If $X \sim Pois(\lambda)$, then*

$$\mathbb{E}[e^{i\mu\cdot X}] = e^{\lambda(e^{i\mu}-1)}, \quad \mu \in \mathbb{R}.$$

Proof of Lemma 2.41.

$$\mathbb{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \cdots$$

so

$$\mathbb{E}[e^{i\mu\cdot X}] = \sum_{k=0}^{\infty} e^{i\mu\cdot k}\mathbb{P}(X = k) = \sum_{k=0}^{\infty} e^{i\mu k}e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda e^{i\mu})^k}{k!} = e^{-\lambda}e^{\lambda e^{i\mu}} = e^{\lambda(e^{i\mu}-1)}.$$

□

Theorem 2.42. *Let $\{a_n\}$ be a sequence of real (or complex) numbers.*

(1) *If $\lim_{n\to\infty} a_n$ exists, then*

$$\limsup a_n = \liminf a_n = \lim a_n.$$

(2) *If $\limsup a_n = \liminf a_n$, then $\lim_{n\to\infty} a_n$ exists, and*

$$\limsup a_n = \liminf a_n = \lim a_n.$$

2.2.2. *Subordinators.* We now turn to the concept of subordinators, which are non-decreasing processes that are often used to model time changes in stochastic frameworks, with connections to Lévy processes that will be explored later.

Definition 2.43. *A subordinator is an* $\mathbb{R}$*-valued Lévy process whose trajectories are non-decreasing.*

Remark 2.44. If $T(t)$, $t \geq 0$ is a subordinator on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the following two conditions are satisfied.

(i) $T(t) \geq 0$ almost surely for each $t > 0$;

(ii) $T(t_1) \leq T(t_2)$ almost surely for all $0 \leq t_1 \leq t_2$.

Claim 2.45. If $z = v + iu$ with $u \geq 0$, then

$$\mathbb{E}\Big(|e^{izT(t)}|\Big) \leq 1.$$

Thus $\mathbb{E}\Big(e^{izT(t)}\Big)$ exists. Moreover, the function

$$\{z = v + iu : u > 0\} \ni z \mapsto \mathbb{E}\Big(e^{izT(t)}\Big)$$

is analytic and continuous in the closure of that set.

Finally, we have

$$\mathbb{E}\Big(e^{-uT(t)}\Big) = e^{t\psi(u)}, \quad u > 0,$$

where

$$\psi(u) = bu + \int_0^\infty \Big(1 - e^{-uy}\Big)\lambda(dy), \quad u > 0.$$

Lemma 2.46. *For* $z = v + iu$*, where* $v \in \mathbb{R}$ *and* $u > 0$*,* $\mathbb{E}\Big(e^{izT(t)}\Big)$ *exists. Moreover, we have*

$$zT(t) = vT(t) + iuT(t)$$

Proof of Lemma 2.46. Using basic algebra

$$i(v + iu) = iv - u,$$

we have

$$e^{izT(t)} = e^{(iv-u)T(t)} = e^{-uT(t)} \cdot e^{ivT(t)}.$$

Next using the fact that $|e^{a+ib}| = |e^a| + |e^{ib}|$ and $|e^{ib}| = 1$ for any real b, because it lies on the unit circle, we obtain

$$|e^{izT(t)}| = |e^{-uT(t)} \cdot e^{ivT(t)}| = e^{-uT(t)} \cdot 1 = e^{-uT(t)}.$$

Since $u > 0$ and $T(t) \geq 0$, we have $e^{-uT(t)} \leq 1$. Hence

$$\mathbb{E}|e^{izT(t)}| = \mathbb{E}|e^{-uT(t)}| \leq \mathbb{E}1 = 1.$$

Hence $\mathbb{E}(e^{izT(t)})$ exists. □

Lemma 2.47. *If $z = v + iu$ with $u \geq 0$, then*

$$\int_0^\infty |\left(1 - e^{izy}\right)| \lambda(dy) < \infty.$$

Hence, $\int_0^\infty \left(1 - e^{izy}\right) \lambda(dy)$ exist. Moreover, the function

$$z \mapsto \int_0^\infty \left(1 - e^{izy}\right) \lambda(dy)$$

is analytic in the region $\{z = v + iu : u > 0\}$ and continuous in the region $\{z = v + iu : u \geq 0\}$.

Definition 2.48. [1, p. 56] *A process $T_S = (T_S(t), t \geq 0)$ taking values in $\mathbb{R}^+ \cup \{\infty\}$ is called a killed subordinator if it satisfies the following condition:*

$$T_S(t) = \begin{cases} T(t) & \text{for } 0 \leq t < S, \\ \infty & \text{for } t \geq S. \end{cases}$$

where $T(t)$ is a subordinator and S is a stopping time.

Proposition 2.49. [1, Proposition 1.3.24, p. 56] *There is a one to one correspondence between killed subordinators T_S and Bernstein functions $f \in C^\infty((0,\infty))$ with $f \geq 0$, and $(-1)^n f^{(n)} \leq 0$ for all $n \in \mathbb{N}$, given by*

$$\mathbb{E}(e^{-uT_S(t)}) = e^{-tf(u)}$$

for each $t, u \geq 0$.

Proof of Proposition 2.49. By independence, we have

$$\mathbb{E}(e^{-uT_S(t)}) = \mathbb{E}(e^{-uT_S(t)}\, \mathbb{1}_{[0,S]}(t)) + \mathbb{E}(e^{-uT_S(t)}\, \mathbb{1}_{[S,\infty)}(t)) = \mathbb{E}(e^{-uT(t)})\, P(S \geq t) + e^{-\infty}$$
$$= e^{-t\psi(u)} \int_t^\infty a\, e^{-ax}\, dx + 0 = e^{-t\psi(u)}\, (e^{-ax}|_t^\infty) = e^{-t\psi(u)}\, e^{-at} = e^{-t[\psi(u)+a]} = e^{-tf(u)},$$

where the third equality follows by [1, p. 52],

$$\mathbb{E}(e^{-uT(s)}) = e^{-t\psi(u)},$$

where

$$\psi(u) = -\eta(iu) = bu + \int_0^\infty (1 - e^{-uy})\, \lambda(dy), \quad u > 0.$$

and using the fact that $g_S(x) = a\, e^{-at}$ for each $x \geq 0$. □

2.2.3. *Lévy Processes.*

Definition 2.50. [40, Definition 1.4] *A stochastic process is a family* $\{X_t : t \geq 0\}$ *of* $\mathbb{R}^n$-*valued random variables defined on a common probability space* $(\Omega, \mathcal{F}, \mathbb{P})$.

Let us recall Definition 2.29 of an infinitely divisible random variable. The following definition is taken from [1, Section 1.3, p. 43], where only the case $n = 1$ has been discussed.

Definition 2.51. *Assume that* $n \in \mathbb{N}$, $n \geq 1$. *An* $\mathbb{R}^n$-*valued stochastic process* $X = (X(t) : t \geq 0)$ *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a Lévy process if and only if,*

(L1) $X(0) = 0$ *almost surely;*

(L2) X *has independent stationary increments;*

(L3) X *is stochastically continuous, i.e.,* $\forall a > 0, s \geq 0$,

$$\lim_{t \to s} \mathbb{P}(|X(t) - X(s)| > a) = 0;$$

Definition 2.52. *Assume that* $n \in \mathbb{N}$, $n \geq 1$. *An* $\mathbb{R}^n$-*valued stochastic process* $X = (X(t) : t \geq 0)$ *on a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *with filtration* $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, *is an* $\mathbb{F}$-*Lévy process if and only if it is Lévy process and*

(L4) X *is* $\mathbb{F}$-*adapted, i.e.,* $X(t)$ *is* $\mathcal{F}_t$-*measurable for every* $t \geq 0$;

(L5) *If* $t \geq s \geq 0$, *then the random variable* $X(t) - X(s)$ *is independent of* $\mathcal{F}_s$.

The following result links the previous two definitions.

Proposition 2.53. *Assume that* $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$ *and let* $X = (X(t) : t \geq 0)$ *b an* $\mathbb{R}^n$-*valued Lévy process defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *Let* $\mathbb{F}$ *be the natural filtration generated by* X, *i.e.,*

$$\mathcal{F}_t := \sigma\{X(s) : s \in [0, t]\}, \quad t \geq 0.$$

Then $X = (X(t) : t \geq 0)$ is an $\mathbb{R}^n$-valued $\mathbb{F}$-Lévy process.

Exercise 2.54. Suppose that $X = (X(t) : t \geq 0)$ is a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies conditions (L1) and (L2). Then one can show that condition (L3) is equivalent to the following one:

(L3*) $\forall a > 0$,

$$\lim_{t \to 0} \mathbb{P}(|X(t)| > a) = 0.$$

Proposition 2.55. [1, Proposition 1.3.1, p. 43] *If $X = (X(t) : t \geq 0)$ is a Lévy process, then $X(t)$ is infinitely divisible for each $t \geq 0$.*

By Proposition 2.55, we can write the characteristic function for Lévy process $X(t)$ as $\phi_{X(t)}(u) = e^{\eta(t,u)}$, see Definition 2.24, for each $t \geq 0, u \in \mathbb{R}$, where each $\eta(t, \cdot)$ is a Lévy symbol.

Lemma 2.56. [1, Lemma 1.4.8, p. 67] *If X is a Lévy process and Y is a modification of X, then Y is a Lévy process with the same characteristics as X.*

Definition 2.57. *An* E*-valued process X is said to be progressively measurable with respect to the filtration $\mathbb{F}$ if and only if for every $t \geq 0$, the mapping*

$$[0, t] \times \Omega \ni (s, \omega) \mapsto X_s(\omega) \in \mathrm{E}$$

is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t / \mathcal{B}(\mathrm{E})$-measurable.

Definition 2.58. [40, Definition 1.5] *A stochastic process $\{X_t\}$ on $\mathbb{R}^d$ is stochastically continuous or continuous in probability if and only if, for every $t \geq 0$ and $\varepsilon > 0$,*

$$\lim_{s \to t} \mathbb{P}[|X_s - X_t| > \varepsilon] = 0.$$

Definition 2.59. *A sequence $\{X_n : n = 1, 2, \cdots\}$ of $\mathbb{R}^d$-valued random variables is said to converge stochastically or converge in probability to X if and only if, for each $\varepsilon > 0$,*

$$\lim_{n \to \infty} \mathbb{P}[|X_n - X| > \varepsilon] = 0.$$

This is denoted by

$$X_n \xrightarrow{\mathbb{P}} X.$$

Lemma 2.60. [1, Lemma 1.3.2, p. 43] *If $X = (X(t) : t \geq 0)$ is a stochastically continuous process, then, for each $u \in \mathbb{R}^d$, the map*

$$t \mapsto \phi_{X(t)}(u)$$

is continuous.

Theorem 2.61. [1, Theorem 1.3.3, p. 44] *If X is a Lévy process, then for every $t \geq 0$*

$$\phi_{X(t)}(u) = e^{t\eta(u)} = e^{t\eta(1,u)} = e^{\eta(t,u)}, \quad u \in \mathbb{R}^d,$$

where η is the Lévy symbol of $X(1)$, see Definition 2.36.

Proof of Theorem 2.61. Since X is a Lévy process, hence the properties L(1)-L(3) stated in Definition 2.51 are satisfied. Define the characteristic function

$$\phi_u(t) = \phi_{X(t)}(u), \quad u \in \mathbb{R}^d, t \geq 0.$$

For all $t \geq 0$, from Definition 2.51 (L2), we have

$$\phi_u(t+s) = \mathbb{E}(e^{i(u,X(t+s))}) = \mathbb{E}(e^{i(u,X(t+s)-X(s))}\, e^{i(u,X(s))}) = \cdots \tag{2.10}$$

By independence (a property of Lévy processes),

$$\cdots = \mathbb{E}(e^{i(u,X(t+s)-X(s))})\,\mathbb{E}(e^{i(u,X(s))}) = \mathbb{E}(e^{i(u,X(t))})\,\mathbb{E}(e^{i(u,X(s))}) = \phi_u(t)\,\phi_u(s).$$

Next, by Definition 2.51 (L1), we have

$$\phi_u(0) = \mathbb{E}(e^{i(u,X(0))}) = 1. \tag{2.11}$$

Finally, from Definition 2.51 (L3), we infer that X is stochastic continuous. Hence, by [1, Lemma 1.3.2, p. 43] (see Lemma 2.60 above), the map $t \to \phi_u(t)$ is continuous for each $u \in \mathbb{R}^d$.

Now, the unique continuous solution to (2.10) and (2.11) is given by

$$\phi_u(t) = e^{t\alpha(u)}, \quad \text{where } \alpha : \mathbb{R}^d \to \mathbb{C}.$$

By [1, Proposition 1.3.1, p. 43] (see Proposition 2.55 above), $X(t)$ is infinitely divisible. Therefore, $\phi_{X(t)}(u)$ can be written as

$$\phi_{X(t)}(u) = \phi_u(t) = e^{\alpha(t,u)}, \quad \forall t \geq 0, u \in \mathbb{R}^d,$$

Hence

$$t\,\alpha(u) = t\,\alpha(1,u) = \alpha(t,u)$$

and therefore, $\alpha(u)$ is the Lévy symbol of $X(1)$.

□

2.3. **Poisson Random Measures.** The following definitions are from [1, Section 2.3.1].

Definition 2.62. *Let S be a non-empty set, and let $\mathcal{A}$ be a ring, see Definition F.12 of subsets of S. Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.*
A random measure on $(S, \mathcal{A})$ is a collection $M = (M(B), B \in \mathcal{A})$ such that for every $B \in \mathcal{A}$, $M(B)$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and the following two conditions are satisfied:

(i) *$M(\emptyset) = 0$,*

(ii) *If $A, B \in \mathcal{A}$ are disjoint, then*

$$M(A \cup B) = M(A) + M(B).$$

A σ-additive random measure is a random measure $M = (M(B), B \in \mathcal{A})$ such that

(ii') *if $A_1, A_2, \cdots \in \mathcal{A}$ are disjoint, then*

$$M(\bigcup_{i=1}^{\infty}) = \sum_{i=1}^{\infty} M(A_i)$$

or

$$M(\bigcup_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}} M(A_i).$$

Definition 2.63. *Let S be a non-empty set, and let $\mathcal{A}$ be a ring of subsets of S. A random measure $M = (M(B), B \in \mathcal{A})$ on $(S, \mathcal{A})$ is called independently scattered random measure if and only if for each pairwise disjoint family $\{B_1, \cdots, B_n\}$ in $\mathcal{A}$, the random variables $M(B_1), \cdots, M(B_n)$ are independent.*

Definition 2.64. *Let S be a non-empty set, and let $\mathcal{A}$ be a ring of subsets of S. A random measure $M = (M(B), B \in \mathcal{A})$ on $(S, \mathcal{A})$ is called a Poisson random measure if and only if*

(i) *M is independently scattered;*

(ii) *$M(B) < \infty$ for all $B \in \mathcal{A}$; and for each $B \in \mathcal{A}$, the random variable $M(B) \sim Pois(\lambda(B))$ with $\lambda(B) = \mathbb{E}(M(B))$.*

The next four definitions are from [1, Section 1.1.1, p. 2].

Definition 2.65. *Let S be a non-empty set and $\mathcal{F}$ a collection of subsets of S. We call $\mathcal{F}$ a σ-algebra (or σ-field) if the following conditions are satisfied:*

(i) $S \in \mathcal{F}$.

(ii) *If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.*

(iii) *If $(A_n)_{n\in\mathbb{N}}$ is a sequence of subsets in $\mathcal{F}$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$.*

Definition 2.66. *Let S be a non-empty set, and let $\mathcal{F}$ be a σ-field of subsets of S. A pair $(S, \mathcal{F})$ is called a measurable space if a measure on $(S, \mathcal{F})$ is a mapping $\mu : \mathcal{F} \to [0, \infty)$ that satisfies*

(i) $\mu(\emptyset) = 0$,

(ii) $\mu\big(\bigcup_{n=1}^{\infty} A_n\big) = \sum_{n=1}^{\infty} \mu(A_n)$ *for every sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{F}$.*

The triple $(S, \mathcal{F}, \mu)$ is called a measure space.

Definition 2.67. *Let μ be a measure on a measurable space $(S, \mathcal{F})$.*

(i) *The quantity $\mu(S)$ is called the total mass of μ.*

(ii) *μ is said to be finite if $\mu(S) < \infty$.*

(iii) *μ is said to be σ-finite if there exists a sequence of $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $S = \cup_{n=1}^{\infty} A_n$ and each $\mu(A_n) < \infty$.*

Definition 2.68. [1, Section 1.1.1, p. 2] *The Borel σ-field of $\mathbb{R}^d$, $\mathcal{B}(\mathbb{R}^d)$, is the smallest σ-field of subsets of $\mathbb{R}^d$ that contains all the open sets.*

Definition 2.69. *Elements of $\mathcal{B}(\mathbb{R}^d)$ are called Borel sets, and a measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is called a Borel measure.*

Definition 2.70. *A function $f : \mathbb{R} \to \mathbb{R}$ is called a Borel function if, for every Borel set B in $\mathbb{R}$, the inverse image $f^{-1}(B)$ is also a Borel set in $\mathbb{R}$, i.e., if f is $\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable.*

Definition 2.71. [44, Definition 4.6] *A Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is a measure that assigns to every half-open rectangle $A = [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_d, b_d) \subset \mathbb{R}^d$ the value:*

$$\mu(A) = \prod_{i=1}^{d} (b_i - a_i), \quad -\infty < a_i < b_i < \infty, \; i = 1, 2, \cdots d.$$

Note that Lebesgue measure is σ-finite but not finite.

The following two definitions are from [41, Definition G.1].

Definition 2.72. *The Lebesgue σ-field on $\mathbb{R}^n$, denoted by $\mathcal{B}^*(\mathbb{R}^n)$, is the completion of the Borel σ-field $\mathcal{B}(\mathbb{R}^n)$ with respect to Lebesgue measure μ.*

The Lebesgue σ-field $\mathcal{B}^*(\mathbb{R}^n)$ consists of all sets in $\mathcal{B}(\mathbb{R}^n)$, together with all subsets of sets that have Lebesgue measure zero. In other words, $\mathcal{B}^*(\mathbb{R}^n)$ is the smallest σ-field that contains all Borel sets and all sets of Lebesgue measure zero:

$$\mathcal{B}^*(\mathbb{R}^n) := \{A \subseteq \mathbb{R}^n : \exists B \in \mathcal{B}(\mathbb{R}^n), N \subseteq B, \mu(N) = 0 \text{ and } A = B \cup N\}.$$

Definition 2.73. *All sets in $\mathcal{B}^*(\mathbb{R}^n)$ are called Lebesgue measurable sets.*

Definition 2.74. [44, Definition 10.1] *A function $f : \Omega \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$, on a measure space $(\Omega, \mathcal{F}, \mu)$ is said to be ($\mu-$) integrable, if it is $\mathcal{F}/\mathcal{B}(\overline{\mathbb{R}})-$ measurable and if both*

$$\int f^+ \, d\mu, \int f^- \, d\mu < \infty,$$

where f^+ and f^- denote the positive and negative parts of f, respectively. In this case, we call

$$\int f \, d\mu := \int f^+ \, d\mu - \int f^- \, d\mu \in (-\infty, \infty)$$

the ($\mu-$) integrable of f.

We denote $\mathscr{L}^1(\mu)$ and $\mathscr{L}^1_{\overline{\mathbb{R}}}(\mu)$ as the set of all $\mathbb{R}$-valued and $\overline{\mathbb{R}}$-valued μ-integrable functions, respectively.

Definition 2.75. *If a measure μ, defined as in Definition 2.71, is integrable, then*

$$\int f \, d\mu$$

is called a Lebesgue integral of the function f with respect to the measure μ.

Notation 2.76. *Assume that M is a random measure defined by the property*

$$M((s,t], A) = M(t, A) - M(s, A), \quad \forall\, 0 \le s < t < \infty,\, A \in \mathcal{B}(E).$$

where $M(t, A)$ represents the measure accumulated by time t in the set $A \subseteq E$, and E is a space with a σ-field $\mathcal{B}(E)$. Then the following conditions hold

(1) $M(\{0\}, A) = 0$ *almost surely;*

(2) $M((s,t] \times A)$ *is independent of* $\mathcal{F}_s$*;*

(3) *There exists a* σ*-finite measure* ρ *on* $\mathbb{R}^+ \times E$ *such that*

$$\mathbb{E}(M(t,A)^2) = \rho(t,A), \quad \forall 0 \leq s < t < \infty,\ A \in \mathcal{B}(E).$$

Hence we have

$$\rho((0,t] \times A) = \rho(t,A) = t\,\mu(A),$$

and

$$\mathbb{E}(\langle M(t,A), M(t,A)\rangle) = \rho(t,A),$$

for each $t \geq 0$*,* $A \in \mathcal{B}(E)$*, where* μ *is a* σ*-finite measure on* E*.*

Definition 2.77. [1, Section 2.3.1, p. 105] *Let* U *be a topological space with Borel* σ*-field* $\mathcal{B}(U)$*, and let* $\mathcal{A} \subseteq \mathcal{B}(U)$ *be a ring. Let* $S = \mathbb{R}^+ \times U$ *and let* $\mathcal{I}$ *be the ring comprising finite unions of sets of the form* $I \times A$*, where* $A \in \mathcal{A}$ *and* I *is itself a finite union of intervals. Let* M *be a random measure on* $(S, \mathcal{I})$*. For each* $A \in \mathcal{A}$*, define a process* $M_A = (M_A(t), t \geq 0) = M([0,t), A)$*. We say* M *is a martingale-valued measure if each* M_A *is a martingale.*

The following example illustrates a concrete construction of a martingale-valued measure from a Lévy process.

Example 2.78. [1, Example 4.1.1, p. 215] Let $X = (X(t) : t \geq 0)$ be a Lévy process with the Lévy-Itô decomposition given by

$$X(t) = bt + B_A(t) + \int_{|x|\geq 1} x\,\widetilde{N}(t,dx) + \int_{|x|\geq 1} x\,N(t,dx), \quad t \geq 0, \tag{2.12}$$

where

(1) $b \in \mathbb{R}^d$,

(2) $N(t,dx)$ is an independent Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^d \setminus \{0\})$,

(3) $\widetilde{N}(t,dx)$ is the compensated Poisson random measure,

(4) B_A is a Brownian motion, see Definition F.4, with covariance matrix A, having entries A_{ij}, i.e.,

$$Cov(B_A^i(t), B_A^j(t)) = A_{ij}\,t \quad \forall i,j \in \{1,\cdots,d\}.$$

where $B_A^i(t)$ and $B_A^{eqn-J_st}(t)$ denote the i-th and j-th component of $B_A(t)$, respectively.

Now, let $E = \{x \in \mathbb{R}^d, 0 < |x| < 1\}$. For each $1 \le i \le d$, $A \in \mathcal{B}(E)$, define

$$M_i(t, A) = \alpha\, \widetilde{N}(t,\ A \setminus \{0\}) + \beta\, \sigma_{ij}\, B_j(t)\, \delta_0(A),$$

where

(1) $\alpha, \beta \in \mathbb{R}$ are fixed constants,

(2) $\sigma \in \mathbb{R}^{d \times d}$ is a matrix satisfying $\sigma\sigma^T = A$,

(3) $\sigma\sigma^T = A$ and $\delta_0(A)$ denotes the Dirac measure at 0, see Definition F.5,

(4) σ_{ij} are the entries of σ.

Then each M_i is an $\mathbb{R}$-valued martingale-valued measure, and the associated compensator (or intensity) measure is given by

$$\rho_i(t, A) = t\,[\alpha^2\, \nu(A \setminus \{0\}) + \beta^2\, A_{ii}\, \delta_0(A)], \tag{2.13}$$

where ν is the Lévy measure associated with $N(t, dx)$, and $A_{ii} = Var(B_A^i(t))$ is the variance of the i-th component of $B_A(t)$.

Note that when $A \setminus \{0\}$ is bounded below, $\rho_i(t, A) < \infty$.

Under this construction, we can define a special type of martingale-valued measure associated with Lévy processes.

Definition 2.79. *Under the framework of Definition 2.77, a Lévy martingale-valued measure is a random measure M on $\mathbb{R}^+ \times \mathbb{R}^d$ associated with a Lévy process $X(t)$ with the Lévy-Itô decomposition given by* (2.12)*, such that for each $A \in \mathcal{A}$, the process $M_A := M([0, t), A])$ is a martingale, and the compensator $\rho_i(t, A)$ associated with $M_i(t, A)$ is given explicitly in* (2.13) *of Example 2.78.*

Definition 2.80. [1, Section 4.1] *Fix $E \in \mathcal{B}(\mathbb{R}^d)$ and $0 < T < \infty$, and let $\mathcal{P}$ denote the smallest σ-field with respect to which all mappings $F : [0, T] \times E \times \Omega \to \mathbb{R}$ satisfy the following conditions:*

(1) *For each $0 \le t \le T$, the function $(x, \omega) \to F(t, x, \omega)$ is $\mathcal{B}(E) \otimes \mathcal{F}_t$-measurable;*

(2) *For each $x \in E$, $\omega \in \Omega$, the mapping $t \to F(t, x, \omega)$ is left-continuous.*

We call $\mathcal{P}$ the predictable σ-field. A $\mathcal{P}$-measurable mapping $F : [0,T]\times E\times\Omega \to \mathbb{R}$ is then said to be predictable.

Definition 2.81. [1, Section 4. 1, p. 217, Section 4.2.2, p. 225]

Denote by $\mathcal{P}_2(T,E)$ the space of all equivalence classes of mappings

$$F : [0,T]\times E\times\Omega \to \mathbb{R}$$

which coincide almost everywhere with respect to measure $\mu\times\mathbb{P}$, and satisfy the following two conditions:

(1) *F is predictable;*
(2) $\mathbb{P}\Big(\int_0^T \int_E |F(t,x)|^2\, \mu(dt,\, dx) < \infty\Big) = 1.$

$$\mathbb{E}\left[\int_0^T \int_E |F(t,x)|^2, \mu(dt,dx)\right] < \infty.$$

Denote by $\mathcal{H}_2(T,E)$ the linear space of all equivalence classes of mappings

$$F : [0,T]\times E\times\Omega \to \mathbb{R}$$

which coincide almost everywhere with respect to $\nu\times\mathbb{P}$, and satisfy the following two conditions:

(1) *F is predictable;*
(2) $\int_0^T \int_E |F(t,x)|^2\, \nu(dx)\, dt < \infty.$

$$\int_0^T \int_E \mathbb{E}\left[|F(t,x)|^2\right] \nu(dx), dt < \infty.$$

Definition 2.82. *Poisson stochastic integral Take $E = \{x \in \mathbb{R}^d, 0 < |x| < 1\}$. Let N be a Poisson random measure on $\mathbb{R}^+\times(\mathbb{R}^d\setminus\{0\})$ (time$\times$space) with:*

(i) *Lebesgue measure,* Leb*, on the time component $\mathbb{R}^+$.*
(ii) *intensity measure ν on $\mathbb{R}^d\setminus\{0\}$ describing jump sizes, where ν is assumed to be a Lévy measure (on space), see Definition 2.33*

In this setting, the full intensity measure μ of the time-space Poisson random measure N, is given by

$$\mathrm{Leb}\otimes\nu.$$

That is, the product of the Lebesgue measure in time and the Lévy measure in space. Moreover, μ is a σ-finite measure on the product σ-field $\mathcal{B}(\mathbb{R}^+)\otimes\mathbb{B}(\mathbb{R}^d\setminus\{0\})$. We define the space $\mathcal{P}_2(T,E)$ is equal to the space of all equivalence classes of mappings

$$F : [0,T]\times E\times\Omega \to \mathbb{R}$$

which coincide almost everywhere with respect to measure $\mu\times\mathbb{P}$ and satisfy the following two conditions:

(1) *F is predictable;*
(2) $\mathbb{P}\Big(\int_0^T \int_{\mathrm{E}} |F(t,x)|^2\,\nu(dx)\,dt < \infty\Big) = 1.$

Let $\widetilde{N}$ be the compensated Poisson random measure, defined by

$$\widetilde{N} = N(dt,dx) - \mu(dt,dx).$$

Let H be a vector with components $(H^1, H^2,\cdots,H^d)$, each $H^i \in \mathcal{P}_2(T,E)$.

We can then construct an $\mathbb{R}^d$-valued $Z = (Z_t, t\geq 0)$ with components $Z^1, Z^2,\cdots,Z^d$, where each $1\leq i\leq d$,

$$Z^i(T) = \int_0^T \int_E H^i(t,x)\,\widetilde{N}(dt,dx).$$

Now let $A\subset\mathbb{R}^d\setminus\{0\}$ be a Borel set that is bounded below, and let $P = (P(t), t\geq 0)$ be a compound Poisson process, defined by

$$P(t) = \int_A x\,N(t,dx).$$

Let K be a predictable mapping.

In particular, if H is square-integrable, then for each $1\leq i\leq d$,

$$\int_0^T \int_A H(t,x)\,\widetilde{N}(dt,dx) = \int_0^T \int_A H(t,x)\,N(dt,dx) - \int_0^T \int_A H(t,x)\,\nu(dx)\,dt.$$

Theorem 2.83. [1, Theorem 4.3.4, p. 232]*)*

(1) *If $F\in\mathcal{P}_2(T,E)$, then for every sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(E)$ with $A_n\uparrow E$ as $n\to\infty$, we have*

$$\lim_{n\to\infty}\int_0^T \int_{A_n} F(t,x)\,\widetilde{N}(dt,dx) = \int_0^T \int_E F(t,x)\,\widetilde{N}(dt,dx) \quad \textit{in probability.}$$

(2) *If* $F \in \mathcal{H}_2(T,E)$*, then there exists a sequence* $(A_n)_{n\in\mathbb{N}}$ *with each* $\nu(A_n) < \infty$ *and* $A_n \uparrow E$ *as* $n \to \infty$ *for which*

$$\lim_{n\to\infty}\int_0^T \int_{A_n} F(t,x)\,\widetilde{N}(dt,dx) = \int_0^T \int_E F(t,x)\,\widetilde{N}(dt,dx) \quad a.s..$$

and the convergence is uniform on compact intervals of $[0,T]$.

Definition 2.84. *An* $\mathbb{R}^d$*-valued stochastic process* $Y = (Y(t), t \geq 0)$ *is a Lévy-type stochastic integrals if and only if there exist processes* G_i*,* F_{ij}*,* H_i *and* K_i*, for* $1 \leq i \leq d$*,* $1 \leq j \leq m$*,* $t \geq 0$*, such that* $|G_i|^{\frac{1}{2}}$*,* $F_{ij} \in \mathcal{P}_2(T,E)$*,* $H_i \in \mathcal{P}_2(T,E)$ *and* K *is predictable,* $B = B_j(t))_{1\leq j\leq m}$ *is an m-dimensional standard Brownian motion and* N *is an independent Poisson random measure on* $\mathbb{R}^+ \times (\mathbb{R}^d \setminus \{0\})$ *with compensator* $\widetilde{N}$ *and intensity measure* ν*, which assume to be a Lévy measure. Then, for each* $1 \leq i \leq d$*,* $t \geq 0$*:*

$$\begin{aligned}Y(t) =& Y(0) + \int_0^t G(s)\,ds + \sum_{j=1}^{m}\int_0^t F_j(s)\,dB_j(s)\\ &+ \int_0^t \int_{|x|<1} H(s,x)\,\widetilde{N}(ds,dx)\\ &+ \int_0^t \int_{|x|\geq 1} K(s,x)\,N(ds,dx).\end{aligned}$$

Corollary 2.85. [1, Corollary 4.3.10, p. 235-236]

(1) *If* $H \in \mathcal{P}_2(T,E)$*, then for every sequence* $(A_n)_{n\in\mathbb{N}}$ *in* $\mathcal{B}(E)$ *with* $A_n \uparrow E$ *as* $n \to \infty$*, we have*

$$\lim_{n\to\infty} Y_n(t) = Y(t) \quad \textit{in probability}.$$

(2) *If* $F \in \mathcal{H}_2(T,E)$*, then there exists a sequence* $(A_n)_{n\in\mathbb{N}}$ *with each* $\nu(A_n) < \infty$ *and* $A_n \uparrow E$ *as* $n \to \infty$ *for which*

$$\lim_{n\to\infty} Y_n(t) = Y(t) \quad a.s.,$$

and the convergence is uniform on compact intervals of $[0,T]$.

2.4. **Itô Formula.** After introducing Poisson random measures, which model the jump behavior of stochastic processes, we now proceed to present the Itô formula, a fundamental tool for analyzing the dynamics of processes driven by both continuous and discontinuous trajectories.

Theorem 2.86. *Itô formula 1 for Brownian integrals*[1, Theorem 4.4.3, p. 246] *If* $M = (M(t), t \geq 0)$ *be a Brownian integral with drift of the form*

$$M(t) = \sum_{j=1}^{m} \int_0^t F_{eqn-J_s t}(s)\, dB_j(s) + \int_0^t G(s)\, ds,$$

where each F_{ij}, $(G)^{\frac{1}{2}} \in \mathcal{P}_2(T, E)$ *for all* $t \geq 0$, *then for all* $f \in C^2(\mathbb{R}^d)$, $t \geq 0$ *with probability* 1, *we have*

$$f(M(t)) - f(M(0)) = \int_0^t \partial f(M(s))\, dM(s) + \frac{1}{2} \int_0^t \partial\partial_j f(M(s))\, d[M_j, M_j](s).$$

Lemma 2.87. [1, Lemma 4.4.5, p. 249]*) Assume that* $A \subset \mathbb{R}^d \setminus \{0\}$ *is bounded below and that each* K *is predictable. If* $W = (W(t), t \geq 0)$ *is a stochastic process of the form*

$$W(t) = W(0) + \int_0^t \int_A K(s, x)\, N(ds, dx), \quad t \geq 0,$$

where N *is a Poisson random measure on* $\mathbb{R}^+ \times (\mathbb{R}^d \setminus \{0\})$, *then for all* $f \in C(\mathbb{R}^d)$, $t \geq 0$, *with probability* 1, *we have*

$$f(W(t)) - f(W(0)) = \int_0^t \int_A [f(W(s-) + K(s, x)) - f(W(s-))]\, N(ds, dx).$$

Lemma 2.88. [1, Lemma 4.4.6] *Assume that* $A \subset \mathbb{R}^d \setminus \{0\}$ *is bounded below, each* F_j, $(G)^{\frac{1}{2}} \in \mathcal{P}_2(T, E)$ *and each* K *is predictable. Let* $Y = (W(t), t \geq 0)$ *be a Lévy type stochastic integral of the form*

$$Y(t) = Y(0) + Y_c(t) + \int_0^t \int_A K(s, x)\, N(ds, dx), \quad t \geq 0,\ 1 \leq j \leq m;$$

where N *is a Poisson random measure on* $\mathbb{R}^+ \times (\mathbb{R}^d \setminus \{0\})$, *and* Y_c *is the continuous part of* Y, *given by*

$$Y_c(t) = \int_0^t G(s)\, ds + \sum_{j=1}^{m} \int_0^t F_j(s)\, dB_j(s).$$

Then for each $f \in C^2(\mathbb{R}^d)$*,* $t \geq 0$*, with probability* 1*, we have*

$$f(Y(t)) - f(Y(0)) = \int_0^t \partial f(Y(s-))\, dY_c(s) + \frac{1}{2}\int_0^t \partial\partial_j f(Y(s-))\, d[Y_c, Y_c^{eqn-J_st}](s)$$
$$+ \int_0^t \int_A [f(Y(s-) + K(s,x)) - f(Y(s-))]\, N(ds, dx).$$

Consider a general Lévy-type stochastic integral Y with stochastic differential

$$dY(t) = G(t)\, dt + F(t)\, dB(t) + \int_E H(t,x)\, \widetilde{N}(dt,dx) + \int_{\mathbb{R}^d\setminus E} K(t,x)\, N(dt,dx) \qquad (2.14)$$

where each F_j, $(G)^{\frac{1}{2}} \in \mathcal{P}_2(T,E)$, $H \in \mathcal{P}_2(T,E)$, each K is predictable, $\widetilde{N}(dt,dx) = N(dt,dx) - \nu(dx)dt$ is the compensated Poisson random measure, for all $t \geq 0$ and $E = \{x \in \mathbb{R}^d, 0 < |x| < 1\}$. We can decompose Y as

$$Y(t) = Y(0) + Y_c(t) + Y_d(t).$$

where the continuous part Y_c satisfies

$$dY_c = G(t)\, dt + F(t)\, dB(t);$$

and the discontinuous part Y_d satisfies

$$dY_d = \int_E H(t,x)\, \widetilde{N}(dt,dx) + \int_{\mathbb{R}^d\setminus E} K(t,x)\, N(dt,dx).$$

Theorem 2.89. *Itô formula 2 for Lévy type stochastic integrals* [1, Theorem 4.4.7] *Let* Y *be a Lévy-type stochastic integral of the from* (2.14)*. Then, for each* $f \in C^2(\mathbb{R}^d)$*,* $t \geq 0$*, with probability* 1*, we have*

$$f(Y(t)) - f(Y(0)) = \int_0^t \sum_{i=1}^n \partial_i f(Y(s-))\, dY_c^i(s) + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \partial_i\partial_j f(Y(s-))\, d[Y_c^i, Y_c^j](s)$$
$$+ \int_0^t \int_A [f(Y(s-) + K(s,x)) - f(Y(s-))]\, N(ds,dx)$$
$$+ \int_0^t \int_E [f(Y(s-) + H(s,x)) - f(Y(s-))]\, \widetilde{N}(ds,dx)$$
$$+ \int_0^t \int_E [f(Y(s-) + H(s,x)) - f(Y(s-)) - H(s,x)\,\partial f(Y(s-))]\, \nu(dx)\, ds.$$

Theorem 2.90. *Itô formula 3 for Lévy type stochastic integrals* [1, Theorem 4.4.10] *Let* Y *be a Lévy-type stochastic integral of the from* (2.14)*. Then, for each* $f \in C^2(\mathbb{R}^d)$*,* $t \geq 0$*, with probability* 1*, we have*

$$f(Y(t)) - f(Y(0)) = \int_0^t \sum_{i=1}^n \partial_i f(Y(s-))\, dY_c^i(s) + \frac{1}{2} \sum_{i,j=1}^n \int_0^t \partial_i \partial_j f(Y(s-))\, d[Y_c^i, Y_c^j](s)$$
$$+ \sum_{0 \le s \le t} [f(Y(s)) - f(Y(s-)) - \Delta Y^i(s)\, \partial_i f(Y(s-))].$$

2.5. **Martingales.** To further understand the behavior of stochastic processes, it is crucial to explore the concept of martingales, which formalize the structure of stochastic evolution over time. In what follows, we will always consider finite or infinite sequences of the form $(\xi_n)_{n=0}^N$ (where $N \in \mathbb{N}$) or $(\xi_n)_{n=0}^\infty$.

The following definitions are taken from [11, p. 46, p. 49].

Definition 2.91. *Assume that* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space. A sequence* $(\mathcal{F}_n)_{n \ge 0}$ *of* σ*-fields on* Ω *such that*

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}$$

is called a (discrete time) filtration.

Definition 2.92. *Assume that* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space and* $\mathbb{F} = (\mathcal{F}_n)_{n \ge 0}$ *is a filtration. Assume also that* K *is a separable Banach space. A sequence* $(\xi_n)_{n=0}^N$ *(or* $(\xi_n)_{n=0}^\infty$*) of* K*-valued random variables is called a martingale on* $(\Omega, \mathcal{F}, \mathbb{P})$ *with respect to (w.r.t.)* $\mathbb{F}$ *if and only if*

(i) ξ_n *is Bochner integrable for each* $n = 0, 1, 2, \cdots$*;*

(ii) ξ_n *is adapted to a filtration* $(\mathcal{F}_n)_{n \ge 0}$ *for each* $n = 0, 1, 2, \cdots$*, i.e.,* ξ_n *is* $\mathbb{F}$*-measurable;*

(iii) *For each* $n = 0, 1, 2, \cdots$*,*

$$\mathbb{E}(\xi_{n+1} | \mathcal{F}_n) = \xi_n \quad a.s.$$

In the special case where $\mathrm{K} = \mathbb{R}$, we have the following definitions:

Definition 2.93. *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *be a filtered probability space, i.e.,* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space w.r.t. a filtration* $\mathbb{F}$*. We say that* $(\xi_n)_{n=0}^N$ *(or* $(\xi_n)_{n=0}^\infty$*) is a submartingale (w.r.t.* $\mathbb{F}$*) if and only if*

(i) $\xi_n \in \mathscr{L}^1(\mathcal{F}_n)$ *for each* $n = 0, 1, 2, \cdots$

(ii)

$$\forall n,\ \forall A \in \mathcal{F}_n, \quad \int_A \xi_{n+1}\, d\mu \geq \int_A \xi_n\, d\mu.$$

We say that a discrete time process $(\xi_n)_{n=0}^{N}$, or $(\xi_n)_{n=0}^{\infty}$, is a supermartingale (w.r.t. $\mathbb{F}$) if and only if

(i) *if $\xi_n \in \mathscr{L}^1(\mathcal{F}_n)$ for each $n = 0, 1, 2, \cdots$*

(ii)

$$\forall n,\ \forall A \in \mathcal{F}_n, \quad \int_A \xi_{n+1}\, d\mu \leq \int_A \xi_n\, d\mu.$$

The following definition is based on [7, Definition 2.1][8, p. 629].

Definition 2.94. *Assume that $p \in [1,2]$. A Banach space $(\mathrm{E}, \|\cdot\|)$ is of martingale type p if and only if there exists a constant $C_p > 0$ such that for any discrete and finite* E*-valued martingale $(M_k)_{k=0}^{N}$ one has*

$$\sup_k \mathbb{E}\left[\|M_k\|^p\right] \leq C_p \sum_{k=0}^{N} \mathbb{E}[\|M_k - M_{k-1}\|^p]; \tag{2.15}$$

with the convention that $M_{-1} = 0$.

Remark 2.95. In Definition 2.94, we assume that $p \in [1,2]$ because it is well known that, see e.g., [7, p. 6], if $p > 2$ and K is a Banach space of martingale type p, then $\mathrm{K} = \{0\}$. The basic examples of Banach spaces of martingale type p include the Lebesgue spaces L^q with $q \in [p,\infty)$ and the Sobolev spaces $H^{s,q}$ with $s \in \mathbb{R}$ and $q \in [p,\infty)$. Note that these spaces are separable Banach spaces of martingale type p.

Exercise 2.96. Take $n \in \mathbb{N}$. Take $\mathrm{K} = \mathbb{R}$. Let $(W_t)_{t\geq 0}$ be $\mathbb{R}$-valued standard Brownian Motion. Put $M_k = W_k$. Calculate both sides of (2.15).

See Appendix G for the proof of Exercise 2.96.

Definition 2.97. [11, p. 140] *Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a filtration on it. A stochastic process $(\xi(t))_{t\in[0,T]}$ is called a continuous-time martingale w.r.t. a filtration $\mathbb{F}$ if:*

(i) *$\xi(t)$ is integrable, see Definition F.13, $\forall t \in [0,T]$;*

(ii) *$\xi(t)$ is $\mathbb{F}$-measurable, see Definition 2.3, $\forall t \in [0,T]$;*

(iii) *for all* $0 \leq s \leq t \leq T$,

$$\mathbb{E}(\xi(t)|\mathcal{F}_s) = \xi(s) \quad a.s.$$

3. Cylindrical processes

In this section, we will introduce concepts of cylindrical measures, cylindrical random variables and cylindrical processes, with a particular focus on cylindrical Lévy processes. We begin our exposition with a subsection on conditional expectation.

3.1. **Conditional Expectation.**

Definition 3.1. [14, Proposition 1.10, p. 26] *Assume that* K *is a separable space and that* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space. Let* ξ *be a Bochner integrable* K*-valued random variable on* $(\Omega, \mathcal{F}, \mathbb{P})$*, and let* $\mathcal{A}$ *be a* σ*-field contained in* $\mathcal{F}$*. A Bochner integrable* K*-valued random variable* ζ *on* $(\Omega, \mathcal{A}, \mathbb{P})$ *is said to be the conditional expectation of* ξ *given* $\mathcal{A}$*, denoted by* $\mathbb{E}(\xi|\mathcal{A})$*, if and only if*

$$\int_A \xi \, d\mathbb{P} = \int_A \zeta \, d\mathbb{P}, \quad \forall A \in \mathcal{A}.$$

A special case arises when $\mathrm{K} = \mathbb{R}$, where the conditional expectation coincides with the standard notion of expectation in the real-valued probability theory.

Definition 3.2. *A finite-rank operator* $\psi : U \to V$ *between Banach spaces is a bounded linear operator whose range* $\mathcal{R}(\psi) := \psi(U) = \{\psi(u) : u \in U\}$ *is finite dimensional.*

Exercise 3.3. A bounded linear functional is a particular case of a finite-rank operator, namely of rank one.

See Appendix G for the proof of Exercise 3.3.

Exercise 3.4. Prove the following. If $\psi : U \to V$ is a finite rank operator between Banach spaces U and V, then there exits a natural number $n \in \mathbb{N}$, real numbers $a_1, \cdots, a_n$, vectors $v_1, \cdots, v_n \in V$ and linear functionals $u_1^*, \cdots, u_n^* \in U^*$ such that

$$\psi u = \sum_{i=1}^{n} a_i \langle u_i^*, u \rangle \, v_i, \quad \forall u \in U.$$

Here U^* is the dual space of U. The expression $\langle u_i^*, u \rangle = u_i^*(u) \in \mathbb{R}$ is called the duality between the space U and its dual U^*.

See Appendix G for the proof of Exercise 3.4.

Definition 3.5. [39, Proposition 4.1, p. 72] *A Banach space* K *has the approximation property if and only if for every* $\varepsilon > 0$ *and for every compact set, see Definition F.15,* $\mathrm{K}_0 \subset \mathrm{K}$ *there exists a finite rank operator* $\psi : \mathrm{K} \to \mathrm{K}$ *such that*

$$\|x - \psi x\| \leq \varepsilon, \quad \forall x \in \mathrm{K}_0.$$

Definition 3.6. *Let* $(\Omega, \mathcal{F})$ *be a measurable space, and let* K *be a Banach space. Suppose* μ *is a* K*-valued measure, see Definition F.16, on* $(\Omega, \mathcal{F})$*, and* ν *is a non-negative measure on* $(\Omega, \mathcal{F})$*. We say that measure* μ *is absolutely continuous with respect to measure* ν*, denoted by* $\mu \ll \nu$*, if and only if for any* $A \in \mathcal{F}$*, the following condition holds:*

$$\textit{If } \nu(A) = 0, \textit{ then } \mu(A) = 0.$$

Definition 3.7. [39, p. 102] *A Banach space* K *is said to have the Radon–Nikodym property if and only if for a measurable space* $(\Omega, \mathcal{F})$*, every* K*-valued measure* μ *on* $\mathcal{F}$ *which is absolutely continuous with respect to a finite, positive measure* ν *on* $\mathcal{F}$*, there exists a* ν*-Bochner integrable function* $f : \Omega \to \mathrm{K}$ *such that*

$$\mu(A) = \int_A f \, d\nu, \quad \forall A \in \mathcal{F}.$$

Definition 3.8. [11, Definition 1.7] *Suppose that* ξ *is a* K*-valued random variable on* $(\Omega, \mathcal{F}, \mathbb{P})$*. If there exists a Borel function, see Definition 2.70,* $f_\xi : \mathrm{K} \to \mathbb{R}$*, such that for any Borel set* $B \in \mathcal{B}(\mathbb{R})$

$$\mathbb{P}\{\xi \in B\} = \int_B f_\xi(x) \, dx,$$

then ξ *is said to be a random variable with absolutely continuous distribution and* f_ξ *is called the density function of* ξ*.*

Definition 3.9. [16, p. xiii] *Let* K *and* E *be normed vector spaces. A linear mapping* $f : \mathrm{K} \to \mathrm{E}$ *is isometric if and only if*

$$\|f(x)\|_{\mathrm{E}} = \|x\|_{\mathrm{K}}, \quad \forall x \in \mathrm{K}.$$

Furthermore, f *is called an isometry if it is also surjective, i.e.,* $f(\mathrm{K}) = \mathrm{E}$*.*

3.2. **Cylindrical Processes.**

Definition 3.10. *Assume that X and Y are normed vector spaces. A linear map $A : X \to Y$ is said to be bounded if and only if $\exists\, C > 0$ such that*

$$\|A(x)\| \leq C\|x\|, \quad x \in X.$$

The space of all bounded linear operators from X to Y is denoted by $\mathscr{L}(X, Y)$.
If $A : X \to Y$ is a linear map then we put

$$\|A\|_{\mathscr{L}(X,Y)} := \sup_{\|x\|\leq 1} \|A(x)\|.$$

This definition has to be supplemented by the following result.

Proposition 3.11. *If $A : X \to Y$ is a linear map, then $A \in \mathscr{L}(X, Y)$ if and only if*

$$\|A\|_{\mathscr{L}(X,Y)} < \infty.$$

Moreover, the function

$$\mathscr{L}(X, Y) \ni A \mapsto \|A\|_{\mathscr{L}(X,Y)} \in [0, \infty)$$

is a norm on $\mathscr{L}(X, Y)$. Finally, if Y is a complete vector space, then so is $\mathscr{L}(X, Y)$.

When $Y = \mathbb{R}$ we have the following special case.

Definition 3.12. [47, Definition 1', p. 111][16, p. xiii] *If* K *is a normed vector space over* $\mathbb{R}$*, the dual space of* K*, denoted by* K^**, is the set of all linear bounded, i.e. continuous maps from* K *to* $\mathbb{R}$*, i.e.*

$$\mathrm{K}^* := \mathscr{L}(\mathrm{K}, \mathbb{R}) = \{x^* : \mathrm{K} \to \mathbb{R} \ \textit{ is linear and continuous}\}.$$

It is common to use the notation $\langle x, x^*\rangle$ instead of $x^*(x)$ to represent the action of x^* on x.

We have the following basic result.

Proposition 3.13. *If* K *is a normed vector space over* $\mathbb{R}$*, then the set* K^* *is a vector space. Moreover, the function*

$$\mathrm{K}^* \ni x^* \mapsto \sup\{|\langle x, x^*\rangle| : x \in \mathrm{K},\ |x|_{\mathrm{K}} \leq 1\} \in [0, \infty) \tag{3.1}$$

is a norm and the space K^* *endowed with the norm defined in* (3.1) *is a Banach space.*

Because K^* is a normed vector by Proposition 3.13, the next definition makes sense.

Definition 3.14. *The double dual (also called the bi-dual or second dual) of* K *is the space* K^{**}*, defined as the dual of* K^**, i.e.*

$$\mathrm{K}^{**} = (\mathrm{K}^*)^*.$$

This means that K^{**} *consists of all linear continuous functionals acting on elements of* K^*.

Definition 3.15. [47, Definition 4, p. 113][38, Section 4.5, p. 95] *A normed vector space* K *is said to be reflexive if* K *can be identified with* K^{**}*, i.e., the canonical mapping (or cannoical embedding)*

$$\mathrm{K} \ni x \mapsto \langle x, \cdot \rangle \in \mathrm{K}^{**}$$

is surjective, meaning that every element of K^{**} *corresponds to some* $x \in \mathrm{K}$*. In this case,* K *is bi-dual, implying that* K *and* K^{**} *share the same normed space structure.*

It is important to note that every normed space is double dual, since the canonical mapping from K to K^{**} always exists. However, a normed space is reflexive only if this mapping is surjective, meaning K and K^{**} are essentially the same space.

Proposition 3.16. *The mapping* $x \mapsto \langle x, \cdot \rangle$ *establishes an isometric embedding of* K *into* K^{**}*. This means that the map*

$$\mathrm{K} \to \mathrm{K}^{**}$$

is both linear and norm-preserving, i.e.,

$$\|x\|_{\mathrm{K}} = \|\langle x, \cdot \rangle\|_{\mathrm{K}^{**}}, \quad \forall x \in \mathrm{K}.$$

This ensures that K *is isometrically embedded in* K^{**}*, preserving its normed space structure. If the embedding is surjective, then* K *is reflexive, meaning*

$$\mathrm{K} \cong \mathrm{K}^{**}.$$

The following result is stated in [45, Corollary 2, p. 219].

Corollary 3.17. *Every reflexive Banach space has the Radon–Nikodym property.*

The following definitions are taken from [2] a book by Applebaum.

For each $n \in \mathbb{N}$, let K^{*n} denote the set of all n-tuples of vectors from K^*, i.e.the n-th Cartesian power of K^*:

$$\mathrm{K}^{*n} = \underbrace{\mathrm{K}^* \times \cdots \times \mathrm{K}^*}_{n\text{-times}}.$$

Definition 3.18. [2, p. 698] *Assume that* K *is a Banach space with dual* K^*. *The Borel* σ*-field in* K *is denoted by* $\mathcal{B}(\mathrm{K})$. *For every* $(x_1^*, \cdots, x_n^*) \in \mathrm{K}^{*n}$*, we define a linear map*

$$\psi_{x_1^*,\cdots,x_n^*} : \mathrm{K} \ni x \mapsto (\langle x, x_1^* \rangle, \cdots, \langle x, x_n^* \rangle) \in \mathbb{R}^n,$$

where $\langle x, x_i^* \rangle$ *is the the dual pairing for* $x \in \mathrm{K}$ *and* $x_i^* \in \mathrm{K}^*$.

Now assume that $B \in \mathcal{B}(\mathrm{K})$*,* $n \in \mathbb{N}$ *and* $(x_1^*, \cdots, x_n^*) \in \mathrm{K}^{*n}$. *A cylindrical set (or cylinder) in* K *with basis* B *and generators* $x_1^*, \cdots, x_n^* \in \mathrm{K}^*$ *is a subset of* K *defined by the following formula*

$$\begin{aligned} Z(x_1^*, \cdots, x_n^*; B) : &= \Big\{ x \in \mathrm{K} : (\langle x, x_1^* \rangle, \cdots, \langle x, x_n^* \rangle) \in B \Big\} \\ &= \psi^{-1}_{x_1^*,\cdots,x_n^*}(B). \end{aligned}$$

If Γ *is a subset of* K^**, the family of all cylindrical sets with an arbitrary basis* $B \in \mathcal{B}(\mathrm{K})$*, arbitrary* $x_1^*, \cdots, x_n^* \in \Gamma$*, and arbitrary* $n \in \mathbb{N}$*, is denoted by* $\mathcal{Z}(\mathrm{K}, \Gamma)$. *Any element of this family is called a cylindrical set w.r.t.* (K, Γ).
When $\Gamma = \mathrm{K}^*$*, we write* $\mathcal{Z}(\mathrm{K}) := \mathcal{Z}(\mathrm{K}, \mathrm{K}^*)$.
The σ*-field generated by* $\mathcal{Z}(\mathrm{K}, \Gamma)$ *is called the cylindrical* σ*-field w.r.t.* (K, Γ)*, and its is denoted by* $C(\mathrm{K}, \Gamma)$.
When $\Gamma = \mathrm{K}^*$*, we write* $C(\mathrm{K}) := C(\mathrm{K}, \mathrm{K}^*)$.

Remark 3.19. The family $\mathcal{Z}(\mathrm{K}, \Gamma)$ is generally not a σ-field. It is only an algebra.

Lemma 3.20. [2, Lemma 2.1] *If a set* $\Gamma = \{x_1^*, \cdots, x_n^* \subseteq \mathrm{K}^*\}$ *is finite, and* $\mathcal{F}$ *is an arbitrary generator of* $\mathcal{B}(\mathbb{R}^n)$*, then the cylindrical* σ*-field satisfies:*

$$C(\mathrm{K}, \Gamma) = \mathcal{Z}(\mathrm{K}, \Gamma) = \sigma\Big(Z(x_1^*, \cdots, x_n^*; B) : B \in \mathcal{F}\Big).$$

In particular, if K *is finite dimensional, then* $\mathcal{Z}(\mathrm{K})$ *itself is a* σ*-field.*

Definition 3.21. [2, p. 699] *A function* $\mu : \mathcal{Z}(\mathrm{K}) \to [0, \infty]$ *is called a cylindrical measure on* K *if and only if the restriction of* μ *to the* σ*-field* $C(\mathrm{K}, \Gamma)$ *is a measure for every finite*

$\Gamma \subseteq \mathrm{K}^*$.

If $\mu(\mathrm{K}) < \infty$, *we call* μ *a finite cylindrical measure. If* $\mu(\mathrm{K}) = 1$, *we call* μ *a cylindrical probability measure.*

The following result is usually called the (Lévy-Khintchine for infinitely divisible cylindrical measures). Our presentation follows two sources: [2, Theorem 2.7][34, Theorem 3.4]. We will use the following notation

$$B_1 := \{y \in \mathbb{R} : |y| \leq 1\}$$

Theorem 3.22. *Assume that* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space and* K *is a Banach space. Assume that* μ *is an infinitely divisible cylindrical probability measure on* K.

Then there exist

(i) a linear and continuous map

$$b : \mathrm{K}^* \to \mathbb{R},$$

(ii) a quadratic and continuous form

$$q : \mathrm{K}^* \to \mathbb{R}$$

(iii) and a cylindrical Lévy measure ν *on* K, *see Definition 3.25, such that for all* $x^* \in \mathrm{K}^*$,

$$\begin{aligned}\varphi_\mu(x^*) = exp\Big(& ib(x^*) - \frac{1}{2}q(x^*) \\ &+ \int_{\mathbb{R}\setminus\{0\}} (e^{i\beta} - 1 - i\beta\mathbb{1}_{B_1}(\beta))(\nu \circ (x^*)^{-1})(d\beta)\Big)\end{aligned}$$

where $\nu \circ (x^*)^{-1}$ *is a measure on* $\mathcal{B}(\mathbb{R})$ *defined by*

$$[\nu \circ (x^*)^{-1}](B) = \nu[Z(x^*; B)], \quad B \in \mathcal{B}(\mathbb{R}).$$

Proposition 3.23. *Assume that* K, E *are Banach spaces and* $\psi : \mathrm{K} \to \mathrm{E}$ *is a linear bounded operator. If* μ *is a cylindrical measure on* K, *then its pushforward measure* $\psi(\mu)$ *is a cylindrical measure on* K.

In particular, if E *is finite dimensional, then* $\psi(\mu)$ *is* σ*-additive measure.*

Definition 3.24. *Assume that* K, E *are Banach spaces and* $\psi : \mathrm{K} \to \mathrm{E}$ *is a linear bounded operator. If* μ *is a cylindrical measure on* K*, the pushforward (image) cylindrical measure* $\psi(\mu)$ *(or equivalently* $\mu \circ \psi^{-1}$*) is defined as:*

$$\Big(\psi(\mu)\Big)(B) = \mu\Big(\psi^{-1}(B)\Big), \quad B \in \mathcal{Z}(\mathrm{E}). \tag{3.2}$$

Definition 3.25. [2, Definition 2.5] *Assume that* K *is a Banach space. A cylindrical measure* μ *on* K *is called a cylindrical Lévy measure on* K *if and only if for* $n \in \mathbb{N}$ *and all* $x_1^*, \cdots, x_n^* \in \mathrm{K}^*$*, the image measure* $\mu \circ \psi^{-1}_{x_1^*, \cdots, x_n^*}$ *is a Lévy measure, see Definition 2.33, on* $\mathcal{B}(\mathbb{R}^n)$.

Definition 3.26. *We denote* $L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ *as the space of all* $\mathbb{R}$*-valued random variables on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$.

The following definition is written in the spirit of [2, Definition 3.1].

Definition 3.27. *Assume that* K *is a Banach space. A cylindrical random variable in* K *is a continuous linear operator*

$$X : \mathrm{K}^* \to L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}), \tag{3.3}$$

where K^* *is the dual of* K.

Remark 3.28. The authors of the book [45] do not use the notion "cylindrical random variable". Instead, they use consistently a phrase "a continuous linear operator $X : \mathrm{K}^* \to L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$". In addition, if $p > 0$ and R_p denotes the class of p-Radonifying operators, which will be discussed in next section, see Theorem 4.23, then if $X : \mathrm{K}^* \to R_p$ is a continuous linear operator, then $X : \mathrm{K}^* \to L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ is also a continuous linear operator.

Definition 3.29. *A cylindrical random variable* X *defined in* (3.3) *is said to be induced by a classical/true random variable* $Y : \Omega \to \mathrm{K}$ *if and only if*

$$[X(x^*)](\omega) = \langle Y(\omega), x^* \rangle, \quad \omega \in \Omega,\ x^* \in \mathrm{K}^*.$$

The following result links the two previous definitions by showing that every true random variable induces a cylindrical random variable.

Proposition 3.30. *Assume that* K *is a Banach space. If* X *is true* K*-valued random variable then* $\tilde{X}$ *defined by the following formula*

$$\tilde{X} = \left\{ \mathrm{K}^* \ni x^* \mapsto x^* \circ X = \langle X, x^* \rangle \in L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}) \right\}.$$

is a cylindrical random variable in K *induced by* X*.*

Definition 3.31. [2, Definition 3.1] *Assume that* K *is a Banach space. A cylindrical process in* K *is a family* $(X(t) : t \geq 0)$ *of cylindrical random variables in* K*. In other words, a cylindrical process in a Banach space* K *is a family* $(X(t) : t \geq 0)$ *of continuous linear operators from* K^* *to* $L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$*.*

The following result is similar to Proposition 3.30, while the object is replaced by a process.

Proposition 3.32. *Assume that* K *is a Banach space. If* $L = (L(t) : t \geq 0)$ *is a true* K*-valued process, then* $\tilde{L} = (\tilde{L}(t) : t \geq 0)$ *defined by the following formula*

$$\tilde{L}(t) = \left\{ \mathrm{K}^* \ni x^* \mapsto x^* \circ L(t) = \langle L(t), x^* \rangle \in L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}) \right\}.$$

is a cylindrical process in K *induced by* L*.*

Comments about the proof of Proposition 3.32. If $L(t) : \Omega \to \mathrm{K}$ is a true K-valued random variable, then, by definition, $L(t) : \Omega \to \mathrm{K}$ is $\mathcal{F}/\mathcal{B}(\mathrm{K})$ measurable. Since the map $x^* := \{\mathrm{K} \ni x \mapsto \langle x, x^* \rangle \in \mathbb{R}\}$ is continuous, we infer that for every $x^* \in \mathrm{K}^*$, $\tilde{L}(t)(x^*) = x^* \circ L(t) : \Omega \to \mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$ measurable. Hence $\tilde{L}$ is a well-defined cylindrical process.

□

Definition 3.33. *Assume that* K *is a Banach space. Assume that* $X : \mathrm{K}^* \to L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ *is a cylindrical random variable. This implies that for every* $n \in \mathbb{N}$ *and all* $x_i^* \in \mathrm{K}^*$*,* $i = 1, \cdots, n$*, the function*

$$(X(x_1^*), \cdots, X(x_n^*)) : \Omega \ni \omega \mapsto (X(x_1^*)(\omega), \cdots, X(x_n^*)(\omega)). \in \mathbb{R}^n$$

In this way we can define a map, denoted by $\hat{X}^n$*,*

$$\hat{X}^n : (\mathrm{K}^*)^n \ni (x_1^*, \cdots, x_n^*) \mapsto (X(x_1^*), \cdots, X(x_n^*)) \in L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^n).$$

The construction described in the above definition can be generalized from cylindrical random variables to cylindrical processes.

Definition 3.34. [2, p. 705] *Assume that* K *is a Banach space. Assume that* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space. A cylindrical process* $X = (X(t) : t \geq 0)$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$*, is said to have weakly independent increments if for all* $0 \leq t_0 < t_1 < \cdots < t_n$ *and all* $x_1^*, \cdots, x_n^* \in \mathrm{K}^*$*, the random variables*

$$(X(t_1) - X(t_0))x_1^*, \cdots, (X(t_n) - X(t_{n-1}))x_n^*$$

are independent.

Definition 3.35. [2, p. 705] *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *be a filtered probability space with filtration* $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$*. A cylindrical process* $X = (X(t) : t \geq 0)$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *in a Banach space* K *is said to be weakly* $\mathbb{F}$*-adapted, if for every* $t \geq 0$ *and all* $x^* \in \mathrm{K}^*$*, the random variable* $X(t)(x^*)$ *is* $\mathcal{F}_t$*-measurable.*

Definition 3.36. *Assume that* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space and* K *is a Banach space. A cylindrical Lévy process in* K *is a cylindrical process* $L = (L(t) : t \geq 0)$ *in* K *provided the following condition holds.*
For every $n \in \mathbb{N}$ *and for all* $x_1^*, \cdots, x_n^* \in \mathrm{K}^*$*, the* $\mathbb{R}^n$*-valued process*

$$\Big(L(t)x_1^*, \cdots, L(t)x_n^*\Big), \quad t \geq 0,$$

see Definition 3.33, is an $\mathbb{R}^n$*-valued Lévy process, see Definition 2.51.*

Definition 3.37. *Assume that* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *is a filtered probability space with filtration* $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$ *and* K *is a Banach space. A cylindrical* $\mathbb{F}$*-Lévy process in* K *is a family* $(L(t) : t \geq 0)$ *of cylindrical random variables in* K*, such that for every* $n \in \mathbb{N}$ *and for all* $x_1^*, \cdots, x_n^* \in \mathrm{K}^*$*, the n-dimensional process*

$$\Big(L(t)x_1^*, \cdots, L(t)x_n^*\Big), \quad t \geq 0,$$

see Definition 3.33, is an $\mathbb{R}^n$*-valued* $\mathbb{F}$*-Lévy process, see Definition 2.52.*

Definition 3.38. [2, Definition 3.2], *An* $\mathbb{F}$*-adapted cylindrical process* $(L(t) : t \geq 0)$ *is called a weakly cylindrical* $\mathbb{F}$*-Lévy process if for* $n \in \mathbb{N}$ *and all* $x_1^*, \cdots, x_n^* \in \mathrm{K}^*$*, the stochastic process* $(L(t)x_1^*, \cdots, L(t)x_n^*) : t \geq 0)$ *is an* $\mathbb{F}$*-Lévy process in* $\mathbb{R}^n$*.*

The following lemma explains the relationship between Definitions 3.36 and 3.38.

Lemma 3.39. [2, Lemma 3.8] *For an $\mathbb{F}$-adapted cylindrical process $L = (L(t), t \geq 0)$, the following statements are equivalent:*

(1) *L is a weakly cylindrical $\mathbb{F}$-Lévy process;*

(2) (i) *L has weakly independent increments;*

  (ii) $(L(t)(x^*) : t \geq 0)$ *is an $\mathbb{F}$-Lévy process $\forall x^* \in \mathrm{K}^*$.*

Definition 3.40. *Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, K is a Banach space and*

$$X : \mathrm{K}^* \to L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}),$$

is a cylindrical random variable in K, see Definition 3.26.

A cylindrical distribution of X is a function $\mu : \mathcal{Z}(\mathrm{K}) \to [0, 1]$, i.e., a cylindrical probability measure, defined by the following formula

$$\mu\Big(Z(x_1^*, \cdots, x_n^*; B)\Big) := \mathbb{P}\Big((X(x_1^*), \cdots, X(x_n^*)) \in B\Big) = \mu_{x_1^*, \cdots, x_n^*}(B)$$

for every $n \in \mathbb{N}$, $x_1^*, \cdots, x_n^* \in \mathrm{K}^*$ and $B \in \mathcal{B}(\mathbb{R}^n)$.

Definition 3.41. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let K be a Banach space. Assume that*

$$X : \mathrm{K}^* \to L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}),$$

is a cylindrical random variable in K, and that

$$\mu : \mathcal{Z}(\mathrm{K}) \to [0, 1]$$

is a cylindrical probability measure on $\mathcal{Z}(\mathrm{K})$.

The characteristic function of μ is a function $\varphi_\mu : \mathrm{K}^* \to \mathbb{C}$ defined by

$$\varphi_\mu(x^*) = \int_{\mathrm{K}} e^{i\langle x, x^* \rangle} \mu(dx), \quad x^* \in \mathrm{K}^*. \tag{3.4}$$

The characteristic function of X is a function $\varphi_X : \mathrm{K}^* \to \mathbb{C}$ defined by

$$\varphi_X(x^*) = \mathbb{E}[e^{iX(x^*)}], \quad x^* \in \mathrm{K}^*.$$

Definitions 3.40 and 3.41 tell us the following proposition:

Proposition 3.42. *If X is a cylindrical random variable and μ is the cylindrical distribution of X, then*

$$\varphi_X = \varphi_\mu.$$

Theorem 3.43. [45, Proposition 3.4, p. 231] *A cylindrical random variable $X : \mathrm{K}^* \to L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ is continuous if and only if its characteristic function $\varphi_X : \mathrm{K}^* \to \mathbb{C}$ is continuous.*

Definition 3.44. [2, Definition 2.2] *A cylindrical probability measure μ on $\mathcal{Z}(\mathrm{K})$ is called infinitely divisible if, for all $n \in \mathbb{N}$, there exists a cylindrical probability measure $\mu^{1/n}$ such that*

$$\mu = (\mu^{1/n})^{*n}.$$

Proposition 3.45. [2, p. 700] *A cylindrical probability measure μ with characteristic function φ_μ is infinitely divisible if and only if, for all $n \in \mathbb{N}$, there exists a characteristic function $\varphi_{\mu^{1/n}}$ of a cylindrical probability measure $\mu^{1/n}$ such that*

$$\varphi_\mu(x^*) = \left(\varphi_{\mu^{1/n}}(x^*)\right)^n, \quad x^* \in \mathrm{K}^*.$$

Proposition 3.46. *Assume that X, Y are cylindrical processes, and let* K *be a Banach space. If Y is a* K*-valued Lévy process, then the induced process X is a weakly cylindrical Lévy process with the same characteristic function as Y.*

The following theorem applies the Lévy-Khintchine formula, see Theorem 2.35, to infinitely divisible cylindrical measures.

Theorem 3.47. [2, Theorem 2.7] *Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and* K *is a Banach space. Let μ be an infinitely divisible cylindrical probability measure on* K*. Then there exists a linear and continuous map $b : \mathrm{K}^* \to \mathbb{R}$, a quadratic and continuous form $q : \mathrm{K}^* \to \mathbb{R}$ and a cylindrical Lévy measure ν, see Definition 3.25, on $\mathcal{Z}(\mathrm{K})$ such that for all $x^* \in \mathrm{K}^*$, the characteristic function $\varphi_\mu : \mathrm{K}^* \to \mathbb{C}$ of, as defined in* (3.4) *satisfies the Lévy-Khintchine representation:*

$$\varphi_\mu(x^*) = e^{ib(x^*)} - \frac{1}{2}q(x^*) + \int_{\mathbb{R}\setminus\{0\}} (e^{iy} - 1 - iy\mathbb{1}_{B_1}(y))\,(\nu \circ (x^*)^{-1})\,(dy)),$$

where $B_1 := \{y \in \mathbb{R} : |y| \leq 1\}$, and $\nu \circ (x^*)^{-1}$ is a function on $\mathcal{B}(\mathbb{R})$, as defined in (3.2)*, see Proposition 3.23 and Definition 3.24.*

Proposition 3.48. *Under the assumptions of Theorem 3.47, the characteristic function of the cylindrical Lévy process* $L(t)$ *satisfies*

$$\varphi_{L(t)}(x^*) = e^{t\phi(x^*)}, \quad x^* \in \mathrm{K}^*,$$

where

$$\phi(x^*) = ib(x^*) - \frac{1}{2}q(x^*) + \int_{\mathrm{E}} \left(e^{i\langle x,x^*\rangle} - 1 - \mathbb{1}_{\mathcal{B}_{\mathbb{R}}}(\langle x, x^*\rangle)\, \langle x, x^*\rangle\right) \nu(dx), \quad x^* \in \mathrm{K}^*.$$

Definition 3.49. [45, p. 102] *Let* $p \in (0, \infty)$ *and* K *be a Banach space. A cylindrical measure* μ *on* $\mathcal{Z}(\mathrm{K})$ *is of weak order* p *if*

$$\int_{\mathrm{K}} \left|\langle x, x^*\rangle\right|^p d\mu(x) < \infty, \quad \text{for each } x^* \in \mathrm{K}^*.$$

A Borel measure μ *on* K *is of strong order* p *if*

$$\int_{\mathrm{K}} ||x||^p \, d\mu(x) < \infty.$$

Clearly, the strong order implies the weak order, while the converse is valid only in the finite dimensional case, see the following cited Theorem 3.50.

Theorem 3.50. [45, Theorem 2.1] *For any* $p \in (0, \infty)$*, in every infinite-dimensional normed space* K *there exists a (discrete) probability measure which is of weak order* p *but not of strong order* p*.*

The following definition is based on .....

Definition 3.51. *Let* $p \in (0, \infty)$ *and* K *be a Banach space. A cylindrical process* L *is said to be weakly* p*-integrable if and only if the following condition holds:*

$$\mathbb{E}[\left|L(t)(x^*)\right|^p] < \infty, \quad \forall t > 0,\ x^* \in \mathrm{K}^*.$$

A weakly 1*-integrable cylindrical process* L *is simply called weakly integrable.*
A weakly integrable cylindrical process L *is said to be weakly mean-zero if and only if*

$$\mathbb{E}[|L(t)(x^*)|] = 0, \quad \forall t > 0,\ x^* \in \mathrm{K}^*.$$

Definition 3.52. *Assume that* K *is a Banach space. A cylindrical process* L *in* K *is said to be weakly* $\mathbb{F}$*-martingale if and only if the following three conditions hold:*

(i) L *is* $\mathbb{F}$*-adapted, see Definition 3.35,*

(ii) *L is weakly* 1*-integrable,*

(iii) *For every $x^* \in K^*$, the process $L(t)(x^*)$ is an $\mathbb{R}$-valued martingale, i.e. for all* $0 < s < t$,

$$\mathbb{E}(L(t)(x^*)|\mathcal{F}_s) = L(s)(x^*).$$

Definition 3.53. *Assume that* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space,* $\mathbb{F}$ *is a filtration, and* K *is a Banach space. A cylindrical Lévy process* $L = (L(t) : t \geq 0)$ *in* K *is called a cylindrical* $\mathbb{F}$*-Lévy martingale in* K *if and only if it weakly* $\mathbb{F}$*-martingale.*

Corollary 3.54. *Assume that* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *is a filtered probability space with filtration* $\mathbb{F}$*, and* K *is a Banach space. If* $L = (L(t) : t \geq 0)$ *is a cylindrical* $\mathbb{F}$*-Lévy martingale then L is* $\mathbb{F}$*-progressively measurable.*

Proof of Corollary 3.54. According to Definition of progressively measurability, see Definition 2.57, a process $L = (L(t) : t \geq 0)$ is progressively measurable if, for every $t \geq 0$, the mapping

$$[0, t] \times \Omega \ni (s, \omega) \mapsto L(s, \omega)x^* \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$$

is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t/\mathcal{B}(\mathbb{R})$-measurable for every $x^* \in K^*$.

Since $L = (L(t) : t \geq 0)$ is a cylindrical $\mathbb{F}$-Lévy martingale, we infer that

(a) L is $\mathbb{F}$-adapted, i.e., for every $t \geq 0$, $L(t)$ is $\mathcal{F}_t$-measurable;

(b) L is weakly 1-integrable, i.e., $\mathbb{E}[|L(t)(x^*)|] = 0, \quad \forall t > 0,\ x^* \in K^*$, see Definition 3.53.

To show the progressive measurability, we need to check that for every fixed t, the mapping

$$(s, \omega) \mapsto L(s, \omega)x^*$$

is $\mathcal{B}([0, t]) \otimes \mathcal{F}_t$-measurable.

(i) Since $L(s)$ is $\mathcal{F}_s$-measurable for every s, it follows that $L(s)x^*$ is $\mathcal{F}_s$-measurable for all $x^* \in K^*$.

(ii) The function $s \to L(s)x^*$ is measurable in s, because L is Lévy process, hence has càdlàg paths.

(iii) The composition of a measurable function with an adapted process remains measurable.

Therefore, the mapping $(s,\omega) \mapsto L(s,\omega)x^*$ is measurable w.r.t. $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$.

□

Proposition 3.55. *Assume that* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space,* $\mathbb{F}$ *is a filtration, and* K *is a Banach space. Assume also that* $L = (L(t) : t \geq 0)$ *is a cylindrical* $\mathbb{F}$*-Lévy martingale. Then* L *is weakly mean-zero.*

Proof of Proposition 3.55. Let us choose and fix x^*. According to Definitions 3.52 and 3.53, the $\mathbb{R}$-valued process $L(t)(x^*)$, $t \geq 0$, is a martingale. Hence it is $\mathbb{F}$-adapted and weakly integrable, i.e., $\mathbb{E}\Big[|L(t)(x^*)|\Big] < \infty$, for every $t \geq 0$. Moreover, it satisfies the martingale property, i.e.,

$$\mathbb{E}[L(t)(x^*)|\mathcal{F}_0] = \mathbb{E}[L(0)(x^*)] = 0.$$

Hence the claim of weakly mean zero is proved. □

Definition 3.56. *Suppose* L *is a cylindrical Lévy process. The natural filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ *generated by* L *is defined by*

$$\mathcal{F}_t := \sigma\{L(s)x^* : s \in [0,t],\ x^* \in \mathrm{K}^*\}, \quad t \geq 0. \tag{3.5}$$

The following result is a natural generalization of Proposition 2.53.

Proposition 3.57. *If* L *is a weakly mean-zero cylindrical Lévy process, see Definition 3.52, and* $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ *is the natural filtration generated by* L*, then* L *is a cylindrical* $\mathbb{F}$*-Lévy martingale.*

Remark 3.58. If $t \geq 0$, then the σ-field $\mathcal{F}_t$ defined in formula (3.5) is equal to the smallest σ-field $\mathcal{G}$ satisfying the following property:
$L(s)x^*$ is $\mathcal{G}$ -measurable for all $s \in [0,t]$ and $x^* \in \mathrm{K}^*$.

Proof of Proposition 3.57. We assume that L is a weakly 1-integrable and weakly mean-zero cylindrical Lévy process. We will show that it is a cylindrical $\mathbb{F}$-Lévy martingale. According to Definition 3.52 we need to show that for every $x^* \in \mathrm{K}^*$, the $\mathbb{R}$-valued process $L(t)x^*$, $t \geq 0$ is an $\mathbb{F}$-martingale.

(i) Adaptness: Since L is a cylindrical Lévy process, and $\mathbb{F}$ is the natural filtration generated by L. Therefore, $L(t)x^*$ is adapted to $\mathcal{F}_t$.

(ii) Integrability: the assumption of weakly 1-integrable implies that

$$\mathbb{E}(|L(t)x^*|) < \infty.$$

(iii) For $0 < s < t$, using the fact that Lévy process has independent zero mean increments, see Definition 2.51 (L3), we deduce that $L(t)x^* - L(s)x^*$ and $L(s)x^* - L(0)x^*$ are independent, and that $L(t)x^* - L(s)x^*$ is independent of $\mathcal{F}_s$. In addition, $L(s)x^*$ is $\mathcal{F}_s$-measurable, as proved in part (i), we have

$$\begin{aligned}\mathbb{E}(L(t)x^*|\mathcal{F}_s)) &= \mathbb{E}[(L(t)x^* - L(s)x^*) + L(s)x^*|\mathcal{F}_s] \\ &= \mathbb{E}[(L(t) - L(s))x^*|\mathcal{F}_s] + \mathbb{E}(L(s)x^*|\mathcal{F}_s) \\ &= \mathbb{E}[(L(t) - L(s))x^*] + L(s)x^* \\ &= L(s)x^*\end{aligned}$$

Therefore it is a martingale.

$\square$

Proposition 3.59. *Assume that* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space,* $p \in [1, \infty)$ *and* K *is a Banach space. If a cylindrical Lévy process* L *in* K *is weakly* p*-integrable, then for each* $t \geq 0$*, the mapping*

$$L(t) : \mathrm{K}^* \to L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$$

is continuous.

Proof of Proposition 3.59. Assume that $p \in [1, \infty)$. Let us choose and fix $t \geq 0$. By assumption 3.52 the map

$$L(t) : \mathrm{K}^* \to L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$$

is well-defined. Moreover, it is easy to prove that it is linear. It is also continuous by the closed graph theorem, see e.g., Theorem F.18. Indeed, K^* and $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ are both Banach spaces and it remains to show that the graph G of $L(t)$ is a closed subset of $\mathrm{K}^* \times L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$. For this purpose we take a sequence of element of G, i.e., $(x_n^*, L(t)x_n^*) \in G$ such that

$$(x_n^*, L(t)x_n^*) \to (x^*, y) \in \mathrm{K}^* \times L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}).$$

This means that

$$x_n^* \to x^* \in \mathrm{K}^* \text{ and } L(t)x_n^* \to y \in L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}).$$

We need to show that the limit (x^*, y) belongs to G, i.e.

$$y = L(t)(x^*).$$

Since $L(t)$ is a cylindrical random variable in K, by Definition 3.27, $L(t) : \mathrm{K}^* \to L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ is continuous. Therefore, since $x_n^* \to x^*$ in K^*, $L(t)x_n^* \to L(t)x^*$ in $L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$. On the other hand, $L(t)x_n^* \to y$ in $L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ which implies that $L(t)x_n^* \to y$ in $L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$. Now, we have two limits of the same sequence in $L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$. Hence these limits are equal, i.e., $y = L(t)x^*$.

Hence the assumptions of the closed graph theorem are satisfied and therefore, we infer that the map $L(t)$ is continuous. The proof is complete. □

Proposition 3.60. [23, Proposition 3.1, p. 23] *Suppose that L is a weakly integrable cylindrical Lévy process. Then there exists*

(i) *A linear functional* $B : \mathrm{K}^* \to \mathbb{R}$,

(ii) *A cylindrical Wiener process W in* K,

(iii) *A cylindrical Lévy martingale M,*

such that

$$L(t)x^* = B(x^*)(t) + W(t)x^* + M(t)x^*, \quad t \geq 0, x^* \in \mathrm{K}^*,$$

where M is defined by

$$M(t)x^* := \int_{\mathbb{R}\setminus\{0\}} \beta \widetilde{N}_{x^*}(t,\, d\beta), \quad \beta \in \mathbb{R},$$

and $\widetilde{N}_{x^*}$ *is the compensated Poisson random measure on* $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$ *associated with* $(L(t)x^* : t \geq 0)$*, which is independent of W.*

Remark 3.61. In other words, the characteristics of a cylindrical Lévy process L are captured by a triplet

$$(b, q, \nu),$$

where

(i) $b : \mathrm{K}^* \to \mathbb{R}$ is a continuous linear function,

(ii) $q : \mathrm{K}^* \to \mathbb{R}$ is a quadratic form and

(iii) ν is a cylindrical measure on K.

Moreover, the Lévy-Itô decomposition of L is given by

$$L(t)x^* = b(x^*)t + q(x^*)W_{x^*}(t) + \int_{B_{\mathbb{R}}} y\widetilde{N}_{x^*}(t, dy) + \int_{B_{\mathbb{R}}^c} yN_{x^*}(t, dy), \tag{3.6}$$

where $B_1 := \{y \in \mathbb{R} : |y| < 1\}$.

See [1, Theorem 2.4.16, p. 126, Theorem 2.4.25, p. 129].

Definition 3.62. *A cylindrical Lévy process is called a purely discontinuous cylindrical Lévy process if and only if its Gaussian part, i.e., the cylindrical Wiener process in the decomposition* (3.6)*, is equal to* 0*.*

Remark 3.63. From Remark 3.61, it follows that a cylindrical Lévy process is purely discontinuous if and only if it has characteristics $(b, 0, \nu)$, meaning that the Gaussian part is absent.

4. p-Summing and p-Radonifying Operators

In this section we will use Definition 3.12.

By $\langle x_j, x^*\rangle$ we will denote the duality pairing between K and K^*. We will use notation $\langle x, x^*\rangle$ instead of $x^*(x)$. By B_{K^*} we denote the unit ball in K^*. Note that

$$|\langle x, x^*\rangle| \le |x|_K |x^*|_{K^*}, \quad x \in K, x^* \in K^*.$$

The following definition is from book [16, chapter 2, p. 31] by Diestel, Jarchow and Tonge, see condition (Π_p).

Definition 4.1. *Assume that* K *and* E *are separable Banach spaces and* $p \in [1,\infty)$. *A linear map* $\psi : K \to E$ *is called p-summing (or absolutely p-summing) if and only if there exists a constant* $C \ge 0$ *such that for every* $n \in \mathbb{N}$,

$$\sum_{j=1}^{n} |\psi x_j|_E^p \le C^p \sup\Big\{ \sum_{j=1}^{n} \Big|\langle x_j, x^*\rangle\Big|^p : x^* \in B_{K^*}\Big\}, \qquad \text{for all } x_1, \cdots, x_n \in K. \tag{4.1}$$

The 1*-summing operator is simply called (absolutely) summing operator.*

Here for $x^* \in K^*$, *the norm* $|x^*|_{K^*}$ *is defined by*

$$|x^*|_{K^*} := \sup\Big\{|\langle x, x^*\rangle| : x \in B_K\Big\} = \inf\Big\{C : |\langle x, x^*\rangle| \le C|x|, \quad x \in K\Big\},$$

where B_K *is the closed unit ball in* K.

The p-summing norm of a p-summing operator ψ, *denoted by* $\pi_p(\psi)$ *or* $\|\psi\|_{\Pi_p(K,E)}$, *is the smallest number* $C \ge 0$ *for which the condition* (4.1) *is satisfied. We denote the space of all p-summing linear operators from* K *to* E *by the symbol* $\Pi_p(K,E)$.

Definition 4.2. [45, p. 103] *Assume that* K *is a normed vector space and* $p \in [1,\infty)$. *A sequence* $(x_n)_{n=1}^\infty$ *of elements of* K *is called weakly p-summable if and only if*

$$\sum_{n=1}^{\infty} \Big|\langle x_n, x^*\rangle\Big|^p < \infty, \quad \text{for every } x^* \in K^*.$$

We denote the set of all such sequences in K *by* $\ell_p^{weak}(K)$.

A sequence $(x_n)_{n=1}^\infty$ *of elements of* K *is called strongly p-summable if and only if*

$$\sum_{n=1}^{\infty} |x_n|_K^p < \infty.$$

We denote the set of all such sequences in K *by* $\ell_p^{strong}(\mathrm{K})$.

Corollary 4.3. [45, p. 104] *For any* $p \in [1,\infty)$*, in every infinite-dimensional normed vector space* K*, there exists a sequence which is weakly p-summable but not strongly p-summable.*

Theorem 4.4. $\psi \in \Pi_p(\mathrm{K},\mathrm{E})$ *if and only if the linear operator* $\hat{\psi}$ *defined by*

$$\hat{\psi} : \ell_p^{weak}(\mathrm{K}) \ni (x_n)_{n=1}^{\infty} \mapsto (\psi x_n)_{n=1}^{\infty} \in \ell_p^{strong}(\mathrm{E})$$

is bounded.

Proposition 4.5. [16, p. 31] *Assume that* K *and* E *are separable Banach spaces and* $p \in [1,\infty)$*. If* $\psi \in \Pi_p(\mathrm{K},\mathrm{E})$*, then* $\psi \in \mathscr{L}(\mathrm{K},\mathrm{E})$ *and the following inequality holds*

$$\|\psi\|_{\mathscr{L}(\mathrm{K},\mathrm{E})} \leq \pi_p(\psi).$$

Proof of Proposition 4.5. Let us choose and fix an arbitrary $\psi \in \Pi_p(\mathrm{K},\mathrm{E})$.

Let us recall that the (uniform) norm on $\mathscr{L}(\mathrm{K},\mathrm{E})$ is defined by

$$\|\psi\|_{\mathscr{L}(\mathrm{K},\mathrm{E})} = \inf\Big\{C > 0 : |\psi x|_{\mathrm{E}} \leq C|x|_{\mathrm{K}},\ x \in \mathrm{K}\Big\}$$

Therefore, it is sufficient to prove that

$$|\psi x|_{\mathrm{E}} \leq \pi_p(\psi)|x|_{\mathrm{K}}, \quad x \in \mathrm{K}. \tag{4.2}$$

Let us choose and fix an arbitrary $x \in K$. Applying inequality (4.1) with $n = 1$ and $x_1 = x$ we have

$$\begin{aligned}|\psi x|_{\mathrm{E}}^p &= \sum_{j=1}^{1} |\psi x_j|_{\mathrm{E}}^p \leq \pi_p^p(\psi)\sup\Big\{\sum_{j=1}^{1}\big|\langle x_j, x^*\rangle\big|^p : x^* \in B_{\mathrm{K}^*}\Big\}\\ &= \pi_p^p(\psi)\sup\Big\{\big|\langle x, x^*\rangle\big|^p : x^* \in B_{\mathrm{K}^*}\Big\} \leq \pi_p^p(\psi)\,|x|_{\mathrm{K}}^p.\end{aligned}$$

The proof of (4.2) is thus complete. □

Proposition 4.6. *The space* $\Pi_p(\mathrm{K},\mathrm{E})$ *is a vector space.*

Proof of Proposition 4.6. If $\psi_1, \psi_2 \in \Pi_p(\mathrm{K},\mathrm{E})$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, then we define

$$[\alpha_1\psi_1 + \alpha_2\psi_2](x) = \alpha_1\psi_1(x) + \alpha_2\psi_2(x), \quad x \in \mathrm{K}. \tag{4.3}$$

Our aim is to prove that if linear operators $\psi_1, \psi_2 \in \Pi_p(\mathrm{K},\mathrm{E})$ then

(i) $\alpha_1\psi_1 + \alpha_2\psi_2$ is a linear map and

(ii) $\alpha_1\psi_1 + \alpha_2\psi_2$ belongs to the set $\Pi_p(\mathrm{K},\mathrm{E})$, i.e., it satisfies condition (4.1).

We will show the following.

Step 1. $\alpha_1\psi_1 + \alpha_2\psi_2$ is a linear map.

Firstly show that conditions (V1) and (V2) in Definition F.25 hold for any $x \in \mathrm{K}$.

Next show further that, if we choose and fix $x_1, x_2 \in \mathrm{K}$ and scalars $\beta_1, \beta_2 \in \mathbb{R}$, the following equation holds.

$$(\alpha_1\psi_1 + \alpha_2\psi_2)(\beta_1 x_1 + \beta_2 x_2) = \beta_1(\alpha_1\psi_1 + \alpha_2\psi_2)(x_1) + \beta_2(\alpha_1\psi_1 + \alpha_2\psi_2)(x_2).$$

Step 2. $\alpha_1\psi_1 + \alpha_2\psi_2$ is p-summing.

Proof of Step 1 is standard and is presented in the Appendix G.

Proof of Step 2. We will show that the linear $\alpha_1\psi_1 + \alpha_2\psi_2$ is p-summing.

We need to prove that $\alpha_1\psi_1$ and $\psi_1 + \psi_2$ are p-summing operators when $\alpha_1 \in \mathbb{R}$ and $\psi_1 + \psi_2 \in \Pi_p(\mathrm{K},\mathrm{E})$. For this purpose, let us choose a natural number $n \in \mathbb{N}$ and a finite sequence $x_1, \cdots, x_n \in \mathrm{K}$. By assumptions ψ_i satisfies inequality (4.1) with constant C_i.

First, for $\alpha_1\psi_1$, by the linearity we proved in (i), we have

$$\begin{aligned}\Big(\sum_{j=1}^n |(\alpha_1\psi_1)x_j|_{\mathrm{E}}^p\Big)^{\frac{1}{p}} &= \Big(\sum_{j=1}^n |\alpha_1(\psi_1 x_j)|_{\mathrm{E}}^p\Big)^{\frac{1}{p}} = \Big(\sum_{j=1}^n |\alpha_1|^p\, |\psi_1 x_j|_{\mathrm{E}}^p\Big)^{\frac{1}{p}} = \Big(|\alpha_1|^p \sum_{j=1}^n |\psi_1 x_j|_{\mathrm{E}}^p\Big)^{\frac{1}{p}} \\ &= |\alpha_1|\Big(\sum_{j=1}^n |\psi_1 x_j|_{\mathrm{E}}^p\Big)^{\frac{1}{p}} \le |\alpha_1|\, C_1 \sup\Big\{\sum_{j=1}^n \big|\langle x_j, x^*\rangle\big|^p : x^* \in B_{\mathrm{K}^*}\Big\}^{\frac{1}{p}}\end{aligned}$$

Therefore, $\alpha_1\psi_1$ satisfies inequality (4.1) with constant $|\alpha_1|C_1$.

Secondly, for $\psi_1 + \psi_2$, we consider

$$\Big(\sum_{j=1}^n |(\psi_1 + \psi_2)x_j|_{\mathrm{E}}^p\Big)^{\frac{1}{p}} = \Big(\sum_{j=1}^n |\psi_1 x_j + \psi_2 x_j|_{\mathrm{E}}^p\Big)^{\frac{1}{p}} \le \Big(\sum_{j=1}^n (|\psi_1 x_j|_{\mathrm{E}} + |\psi_2 x_j|_{\mathrm{E}})^p\Big)^{\frac{1}{p}} \cdots$$

by the definition of $\psi_1 + \psi_2$ and by the triangle inequality in E.

Next, applying the Minkowski inequality for $\mathbb{R}$, see [37, Theorem 3.9][41, Corollary 13.4], we have

$$\cdots \le \Big(\sum_{j=1}^n (|\psi_1 x_j|_{\mathrm{E}})^p\Big)^{\frac{1}{p}} + \Big(\sum_{j=1}^n (|\psi_2 x_j|_{\mathrm{E}})^p\Big)^{\frac{1}{p}} \cdots$$

Since ψ_i satisfies inequality (4.1) with constant C_i, we have

$$\cdots \le C_1 \sup\Big\{ \sum_{j=1}^{n} \big|\langle x_j, x^*\rangle\big|^p : x^* \in B_{\mathrm{K}^*}\Big\}^{\frac{1}{p}} + C_2 \sup\Big\{ \sum_{j=1}^{n} \big|\langle x_j, x^*\rangle\big|^p : x^* \in B_{\mathrm{K}^*}\Big\}^{\frac{1}{p}}$$
$$= (C_1 + C_2) \sup\Big\{ \sum_{j=1}^{n} \big|\langle x_j, x^*\rangle\big|^p : x^* \in B_{\mathrm{K}^*}\Big\}^{\frac{1}{p}}.$$

So we proved that that $\psi_1 + \psi_2$ satisfies inequality (4.1) with constant $C_1 + C_2$, i.e., we proved what we wanted in Step 2. □

Hence the proof of Proposition 4.6 is complete. □

Proposition 4.7. *The function π_p is a norm on the vector space $\Pi_p(\mathrm{K}, \mathrm{E})$.*

Proof of Proposition 4.7. We need to prove:

(N1) $\pi_p(\psi) \ge 0$ for every $\psi \in \Pi_p(\mathrm{K}, \mathrm{E})$ and $\pi_p(0) = 0$;

(N2) if $\psi \in \Pi_p(\mathrm{K}, \mathrm{E})$ and $\pi_p(\psi) = 0$ then $\psi = 0$;

(N3) if $\psi \in \Pi_p(\mathrm{K}, \mathrm{E})$ and $\alpha \in \mathbb{R}$ then

$$\pi_p(\alpha\psi) = |\alpha|\, \pi_p(\psi); \tag{4.4}$$

(N4) if $\psi_1 \in \Pi_p(\mathrm{K}, \mathrm{E})$ and $\psi_2 \in \Pi_p(\mathrm{K}, \mathrm{E})$ then

$$\pi_p(\psi_1 + \psi_2) \le \pi_p(\psi_1) + \pi_p(\psi_2). \tag{4.5}$$

(Ad N1) Obviously $\pi_p(\psi) \ge 0$ for every $\psi \in \Pi_p(\mathrm{K}, \mathrm{E})$, because an infimum of a non-empty subset of $[0, \infty)$ belongs to $[0, \infty)$.

If $\psi = 0$, then inequality (4.1) is satisfies with $C = 0$, because the LHS of (4.1) equals $\sum_{j=1}^{n} |0|_{\mathrm{E}}^p = 0$, while its RHS, $0 < |x^*|_{\mathrm{K}} \le 1$. Hence $\pi_p(0) = 0$.

(Ad N2) Let us choose and fix $\psi \in \Pi_p(\mathrm{K}, \mathrm{E})$ such that $\pi_p(\psi) = 0$. This means, by Exercise G.1 in the Appendix, the smallest number C for (4.1) to satisfy is 0.

Hence, by applying inequality (4.1) with $n = 1$ and one-element sequence $x_1 \in \mathrm{K}$, we get $|\psi x_1|_{\mathrm{E}}^p \le 0$. Since the norm is nonnegative, we infer that $|\psi x_j|^p = 0$, which in turn implies that $\psi x_1 = 0$. Since x_1 is an arbitrary element of K we infer that $\psi = 0$.

(Ad N3) Let us choose and fix $\psi \in \Pi_p(\mathrm{K}, \mathrm{E})$. Our aim is to show (4.4). The LHS of inequality (4.4) is the infimum of a certain set of numbers. Thus, see [36], in order to prove the (4.4) it is sufficient that the LHS of inequality (4.4) belong to that set of numbers, i.e., inequality (4.1) for $\alpha\psi$ is satisfied with $C = \pi_p(\alpha\psi)$, i.e., we need to show that for every $n \in \mathbb{N}$ and every finite sequence $x_1, \cdots, x_n \in \mathrm{K}$,

$$\sum_{j=1}^{n} |(\alpha\psi)x_j|_{\mathrm{E}} \leq |\alpha|\pi_p(\psi) \sup\Big\{ \sum_{j=1}^{n} \big|\langle x_j, x^*\rangle\big|^p : x^* \in B_{\mathrm{K}^*}\Big\}^{1/p}. \tag{4.6}$$

Let us prove inequality (4.6).

$$\sum_{j=1}^{n} |(\alpha\psi)x_j|_{\mathrm{E}} = \sum_{j=1}^{n} |\alpha\psi(x_j)|_{\mathrm{E}} = \sum_{j=1}^{n} |\alpha||\psi(x_j)|_{\mathrm{E}} = |\alpha| \sum_{j=1}^{n} |\psi(x_j)|_{\mathrm{E}}$$
$$\leq |\alpha|\pi_p(\psi) \sup\Big\{ \sum_{i=1}^{1} \big|\langle x_j, x^*\rangle\big|^p : x^* \in B_{\mathrm{K}^*}\Big\}^{1/p}.$$

Hence we have

$$\pi_p(\alpha\psi) = |\alpha|\pi_p(\psi).$$

(Ad N4) Let us choose and fix $\psi_1, \psi_2 \in \Pi_p(\mathrm{K}, \mathrm{E})$. Our aim is to show that

$$\pi_p(\psi_1 + \psi_2) \leq \pi_p(\psi_1) + \pi_p(\psi_2).$$

The LHS of inequality (4.5) is the infimum of a certain set of numbers. In order to prove the (4.5) it is sufficient to prove that the RHS of inequality (4.5) belong to that set of numbers. In other words, it is sufficient to show that inequality (4.1) holds for $\psi_1 + \psi_2$ with $C = \pi_p(\psi_1) + \pi_p(\psi_2)$. Thus we need to show that for every $n \in \mathbb{N}$ and every finite sequence $x_1, \cdots, x_n \in \mathrm{K}$,

$$\sum_{j=1}^{n} |(\psi_1 + \psi_2)x_j|_{\mathrm{E}} \leq (\pi_p(\psi_1) + \pi_p(\psi_2)) \sup\Big\{ \sum_{j=1}^{n} \big|\langle x_j, x^*\rangle\big|^p : x^* \in B_{\mathrm{K}^*}\Big\}^{1/p}. \tag{4.8}$$

Let us prove inequality (4.8).

$$\sum_{j=1}^{n} |(\psi_1 + \psi_2)x_j|_{\mathrm{E}} = \sum_{j=1}^{n} |\psi_1 x_j + \psi_2 x_j|_{\mathrm{E}}$$

$$\leq \sum_{j=1}^{n}(|\psi_1 x_j|_{\mathrm{E}} + |\psi_2 x_j|_{\mathrm{E}}) = \sum_{j=1}^{n} |\psi_1 x_j|_{\mathrm{E}} + \sum_{j=1}^{n} +|\psi_2 x_j|_{\mathrm{E}}$$

$$\leq \pi_p(\psi_1) \sup\Big\{ \sum_{j=1}^{n} \big|\langle x_j, x^*\rangle\big|^p : x^* \in B_{\mathrm{K}^*}\Big\}^{1/p} + \pi_p(\psi_2) \sup\Big\{ \sum_{j=1}^{n} \big|\langle x_j, x^*\rangle\big|^p : x^* \in B_{\mathrm{K}^*}\Big\}^{1/p}$$

$$= (\pi_p(\psi_1) + \pi_p(\psi_2)) \sup\Big\{ \sum_{j=1}^{n} \big|\langle x_j, x^*\rangle\big|^p : x^* \in B_{\mathrm{K}^*}\Big\}^{1/p}.$$

Thus, we have proved what we set out to prove.

□

Definition 4.8. *A normed vector space X is a Banach space if and only if every X-valued Cauchy sequence has a limit that also belongs to the space.*

Proposition 4.9. [16, Proposition 2.6, p. 38] *The space* $\Pi_p(\mathrm{K}, \mathrm{E})$ *with norm* π_p *is a Banach space.*

A proof of Proposition 4.9 can be found in the [16, p. 38].

In what follows, we will introduce new concepts related to the spaces of type p and cotype p in preparation for introducing the key concept of radonifying operators.

Let us recall that by $\mathbb{N}$ we denote set of positive integers, i.e., the smallest element of $\mathbb{N}$ is 1. The following definitions are taken from [45]. Let us recall that a ε is Bernoulli random variable if and only if $\mathbb{P}(\varepsilon = 1) = \mathbb{P}(\varepsilon = -1) = \frac{1}{2}$.

Definition 4.10. [45, p. xix-xx, p. 316] *Assume that* $0 \leq p < \infty$ *and* K *is a Banach space. We put*

$$\ell_p(\mathrm{K}) = \{(x_n) \in \mathrm{K}^{\mathbb{N}} : \sum_{n=1}^{\infty} \|x_n\|^p < \infty\}.$$

Let $\varepsilon = (\varepsilon_n)$ *be a sequence of independent Bernoulli random variables and* $0 \leq p < \infty$.

$$s_p[\mathrm{K}] = s_p[\varepsilon; \mathrm{K}] = \{(x_n) \in \mathrm{K}^{\mathbb{N}} : \textit{series} \sum x_n\varepsilon_n \textit{ converges in } \mathrm{K} \textit{ a.s.}$$
$$\textit{and } \sum_{n=1}^{\infty} x_n\varepsilon_n \in L^p(\Omega; \mathrm{K})\}$$

It can be shown, see [45, Theorem V.2, p. 320-321] that the space $s_p[\varepsilon; \mathrm{K}]$ does not depend on index p and is therefore denoted by $s[\mathrm{K}]$.

Definition 4.11. *Assume that* $0 < p < \infty$. *A Banach space* K *is said to be a space of type* p *if and only if* $\ell_p(\mathrm{K}) \subset s[\mathrm{K}]$.

Definition 4.12. *Assume that* $0 < q < \infty$. *A Banach space* K *is said to be a space of cotype* q *if and only if* $s[\mathrm{K}] \subset \ell_q(\mathrm{K})$.

Proposition 4.13. [16, Theorem 11.14, p. 223] *Suppose that* E *has cotype* q *with* $2 \leq q < \infty$, *and that* K *is a compact Hausdorff space, see Definition F.20, Denote by* $C(\mathrm{K})$ *the Banach space of continuous scalar-valued functions on* K *endowed with the* sup *norm.*

If $1 \leq p, q < \infty$, *we denote by* $\Pi_{q,p}(C(\mathrm{K}), \mathrm{E})$ *the space of all* (q, p)*-summing operators from* $C(\mathrm{K})$ *to* E, *that is, bounded linear operators* $T : C(\mathrm{K}) \to \mathrm{E}$ *such that there exists a constant* $C > 0$ *such that*

$$\left(\sum_i \|Tx_i\|^q\right)^{1/q} \leq C \sup_{\varphi \in B_{C(\mathrm{K})^*}} \left(\sum_i |\varphi(x_i)|^p\right)^{1/p}$$

for all finite $C(\mathrm{K})$*-valued sequences* (x_i).

Then following hold:

(a) *if* $q = 2$, *then*

$$\mathscr{L}(C(\mathrm{K}), \mathrm{E}) = \Pi_2(C(\mathrm{K}), \mathrm{E}),$$

(b) *if* $2 < q < r < \infty$ *and* $p < q$, *then*

$$\mathscr{L}(C(\mathrm{K}), \mathrm{E}) = \Pi_{q,p}(C(\mathrm{K}), \mathrm{E}) = \Pi_r(C(\mathrm{K}), \mathrm{E}),$$

Note that, localization allows us to replace $C(\mathrm{K})$ *in this theorem by any* $\mathscr{L}^\infty$*-space.*

Remark 4.14. One can ask a question if the space $\Pi_p(\mathrm{K}, \mathrm{E})$ is separable.

Our understanding of part (a) [16, Theorem 11.14, p. 223] is as follows. If E is any Banach space, then by [16, p. 31], see Proposition 4.5,

$$\Pi_2(C(\mathrm{K}), \mathrm{E}) \subset \mathscr{L}(C(\mathrm{K}), \mathrm{E})$$

and the corresponding embedding

$$i : \Pi_2(C(\mathrm{K}), \mathrm{E}) \hookrightarrow \mathscr{L}(C(\mathrm{K}), \mathrm{E})$$

is bounded. Moreover, according to the proof of part (a) [16, Theorem 11.14, p. 223], if E is a Banach space of cotype $q = 2$, then every $i \in \mathscr{L}(C(\mathrm{K}), \mathrm{E})$ belongs also to $\Pi_2(C(\mathrm{K}), \mathrm{E})$. Hence by the Open Mapping Theorem, it follows that i is an isomorphism of Banach spaces. The same argument applies to part (b).
We also think that the continuity of i^{-1} is proven implicitly while proving that i is a surjection.

The following answer to the above question is based on [46].

On one hand, [16, Theorem 11.14, p. 223]), if $\mathrm{K} = C(\mathrm{K})$ and E has cotype $q = 2$, then the space $\Pi_2(C(\mathrm{K}); Y)$ is isomorphic to the space $\mathscr{L}(C(\mathrm{K}); Y)$. Therefore, the former space is not separable because latter space is not separable neither.

On the other hand, if K is a separable Hilbert space and E is a Banach space of cotype $q = 2$, then for every $p \in [2, \infty)$, $\Pi_p(\mathrm{K}, \mathrm{E}) = \gamma(\ell^2, \mathrm{E})$, see [16, Corollary 12.7 and Theorem 12.12]. Then, by the Hofmann-Jorgenson-Kwapien Theorem which states that $\gamma(\mathrm{K}, \mathrm{E})$ is separable if both K and E are separable, we deduce that the Banach space $\Pi_p(\mathrm{K}, \mathrm{E})$ is separable when both the Hilbert space K and the cotype 2 Banach space E are both separable.

In our understanding the equalities stated in [16, Corollary 12.7] are not only set-theoretical but also in the sense isomorphisms of Banach spaces.

Definition 4.15. [45, p. 27] *A Borel measure $\mu : \mathcal{B}(X) \to [0, \infty)$ on a topological space X is regular if and only if for every $B \in \mathcal{B}(X)$ the following condition holds:*

$$\mu(B) = \sup\{\mu(A) : A \subset B,\ A \textit{ is closed}\}$$

Definition 4.16. [45, p. 28] *A finite Borel measure $\mu : X \to [0, \infty)$ on Hausdorff topological space X is a Radon measure if and only if for every $B \in \mathcal{B}(X)$,*

$$\mu(B) = \sup\{\mu(A) : A \subset B,\ A \textit{ is compact}\} \tag{4.9}$$

We say that a measure μ is tight if and only if condition (4.9) *is satisfied with $B = X$.*

Proposition 4.17. *Each Radon measure is regular and tight and vice versa.*

The following corollary establishes a key relationship between p-Radonifying operators and p-summing operators in the context of Banach spaces.

Definition 4.18. [45, V.5, p. 315-316, p.320-321] *Assume that* K *is a normed vector space,* $0 \leq q \leq p < \infty$. *A cylindrical (probability) measure* μ *on* K *is said to be of type* p *if and only if it is of weak order* p *and the map*

$$\mathrm{K}^* \ni x^* \mapsto \int_{\mathrm{K}} |\langle x, x^* \rangle|^p \, d\mu(x) \in \mathbb{R}$$

is continuous.

Definition 4.19. [45, VI.5.2, p. 416] *Assume that* K *and* E *are two normed vector spaces,* $0 \leq q \leq p < \infty$. *A continuous linear operator*

$$\psi : \mathrm{K} \to \mathrm{E}$$

is (q, p)*-Radonifying if and only if for every cylindrical (probability) measure* μ *on* K *of type* p*, the cylindrical measure* $\psi(\mu) = \mu \circ \psi^{-1}$ *extends to a Radon measure on* E *of strong order* q.

A (p, p)*-Radonifying operator is called* p*-Radonifying. We denote the class of all* (q, p)*-Radonifying operators* $\psi : \mathrm{K} \to \mathrm{E}$ *by the symbol* $R_{p,q}(\mathrm{K}, \mathrm{E})$*, and the class* $R_{p,p}(\mathrm{K}, \mathrm{E})$ *will be denoted by* $R_p(\mathrm{K}, \mathrm{E})$.

Note that if $q > p$, the only (q, p)-Radonifying operator is the null-operator.

The previous definition of a (q, p)-Radonifying operator was given in terms of cylindrical and Radon measures. The result gives an equivalent characterization of such operators in terms of cylindrical and true random variables.

Proposition 4.20. [45, Proposition 5.2, p. 416] *Assume that* K, E *are normed vector spaces and* $p, q \in [0, \infty)$. *Assume also that* $\psi : \mathrm{K} \to \mathrm{E}$ *is a continuous linear operator. Then, the following conditions are equivalent:*

(i) $\psi \in R_{p,q}(\mathrm{K}, \mathrm{E})$;

(ii) *for each probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and for each continuous linear operator*

$$X : \mathrm{K}^* \to L^p(\Omega, \mathcal{F}, \mathbb{P}), \tag{4.10}$$

i.e., for each cylindrical random variable K *such that* (4.10) *holds, the linear operator*

$$X \circ \psi^* : \mathrm{E}^* \to L^q(\Omega, \mathcal{F}, \mathbb{P})$$

i.e., the cylindrical random variable $X \circ \psi^*$ in E is induced by a true E-valued random variable.

See [45, p. 416-417] for proof of Proposition 4.20.

Proposition 4.21. [45, Proposition 5.4, p. 418] *Let K, E be Banach spaces, $p \in [1, \infty)$, and let $\psi : \mathrm{K} \to \mathrm{E}^*$ be a continuous linear operator such that for any cylindrical measure of type p on K the cylindrical measure $\psi(\mu)$ is countably additive on $\mathcal{Z}(\mathrm{E}^*, \mathrm{E})$. Then ψ is p-summing.*

Corollary 4.22. [45, Corollary 2, p. 419] *For Banach spaces K, E and $p \in [1, \infty)$. Each p-Radonifying operator $\psi : \mathrm{K} \to \mathrm{E}$ is p-summing.*

The following result can be found in [45, Theorems 5.4 and 5.5, p. 423, p. 427].

Theorem 4.23. *Assume that K, E are Banach spaces and*

(i) *$p > 1$ or*

(ii) *$p = 1$ and E has the Radon-Nikodym property*

Then $\Pi_p(\mathrm{K}, \mathrm{E}) = R_p(\mathrm{K}, \mathrm{E})$.

Proof of Theorem 4.23. By Corollary 4.22 we have $R_p(\mathrm{K}, \mathrm{E}) \subset \Pi_p(\mathrm{K}, \mathrm{E})$, proof of the reverse inclusion can be found on [45, Theorem 5.4, p. 423 and Theorem 5.5, p. 427]. □

Note that in [45, p. 446] the following is written: Theorem 5.5 was taken from a paper by L. Schwartz [42]. Theorem 5.4 is attributed to S. Kwapien [25] on a theorem of L. Schwartz, without providing a detailed reference.

5. Itô Integral of Cylindrical Lévy Process

5.1. **Preliminaries.** Let us list a family of assumptions that will be used throughout the session. In each result, we will explicitly specify which assumptions are being applied.

Assumption 5.1. *We assume that* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$*, where* $\mathbb{F} = (F_t)_{t\geq 0}$*, is a filtered probability measure space.*

Assumption 5.2. *We assume that* $T > 0$*.*

Assumption 5.3. *We assume that* $p \in [1, 2]$*.*

Assumption 5.4. *We assume that* K, E *are separable Banach spaces.*

Assumption 5.5. *We assume that the Banach space* E *is of martingale type p. If* $p = 1$*, we additionally assume that* E *satisfies the Radon-Nikodym property (RNP), see Definition 3.7.*

Assumption 5.6. *We assume that a cylindrical Lévy measure* ν *in* K*, see Definition 3.25, satisfies the following condition. There exists* $q \in [1, 2]$ *such that*

$$\int_{\mathrm{K}} |x^*(x)|^q \, \nu(dx) < \infty, \quad \forall x^* \in \mathrm{K}^* \tag{5.1}$$

Note that condition (5.1) *can be rewritten as*

$$\int_{\mathbb{R}} |y|^q \Big(\nu \circ (x^*)^{-1}\Big)(dy) < \infty, \quad \forall x^* \in \mathrm{K}^*,$$

where $\nu \circ (x^*)^{-1}$ *denotes the pushforward measure on* $\mathcal{B}(\mathbb{R})$ *given by*

$$\Big(\nu \circ (x^*)^{-1}\Big)(B) = \nu\Big((x^*)^{-1}(B)\Big), \quad B \in \mathcal{B}(\mathbb{R}).$$

Assumption 5.7. *We assume that L is a cylindrical Lévy process, see Definitions 2.51 and 3.36, on* K *with finite q-th weak moments, where* $q \in [1, 2]$*. Furthermore, L is assumed to be* 1*-integrable, weakly mean-zero, purely discontinuous, see Definition 3.62, and associated with the cylindrical Lévy measure* ν *from Assumption 5.6.*

Assumption 5.8. *Assume that* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space,* $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ *is a filtration and that* $(L(t) : t \geq 0)$ *is a cylindrical* $\mathbb{F}$*-Lévy martingale in a Banach space* K*, see Definition 3.53.*

Definition 5.9. *The predictable σ-field on $[0,T]\times\Omega$, denoted by $\mathcal{P}$, is the σ-field generated by all processes*

$$\psi : [0,T]\times\Omega \to \mathbb{R}$$

such that

(1) *ψ is an $\mathbb{F}$-adapted process;*
(2) *for every $\omega\in\Omega$, the path (or trajectory)*

$$[0,T]\ni t\mapsto \psi(t,\omega)\in\mathbb{R}$$

is left-continuous.

In other words, $\mathcal{P}$ is the smallest σ-field such that all left continuous and $\mathbb{F}$-adapted $\mathbb{R}$-valued processes on $[0,T]\times\Omega$ are measurable with respect to it, see Definition 2.80.

Example 5.10. If $t\in[0,T]$ then the function

$$\mathbb{1}_{[0,t]} : [0,T]\times\Omega\ni(s,\omega)\mapsto \mathbb{1}_{[0,t]}(s)\in\mathbb{R},$$

is left-continuous and adapted, and hence it is $\mathcal{P}$-measurable. Therefore, the set

$$[0,t]\times\Omega = (\mathbb{1}_{[0,t]})^{-1}(\{1\}).$$

belongs to the predictable σ-field $\mathcal{P}$.

Since the space $\Pi_p(\mathrm{K},\mathrm{E})$ may not be separable, see Remark 4.14, in the definition below we use the Borel σ-field $\mathcal{B}(\Pi_p(\mathrm{K},\mathrm{E}))$ of subsets of $\Pi_p(\mathrm{K},\mathrm{E})$ and not the so called strong Borel σ-field $\mathcal{B}_s(\Pi_p(\mathrm{K},\mathrm{E}))$.

Definition 5.11. *Assume that* K,E *are separable Banach spaces. We denote by $\mathcal{B}(\Pi_p(\mathrm{K},\mathrm{E}))$ the Borel σ-field on $\Pi_p(\mathrm{K},\mathrm{E})$ with $p\in(1,2]$, i.e., the σ-field generated by the following family of sets*

$$\left\{L\in\Pi_p(\mathrm{K},\mathrm{E}) : L(x)\in B\right\},\quad B\in\mathcal{B}(\mathrm{E}).$$

A process

$$\psi : [0,T]\times\Omega\to\Pi_p(\mathrm{K},\mathrm{E})$$

is called predictable if and only if it is $\mathcal{P}/\mathcal{B}(\Pi_p(\mathrm{K},\mathrm{E}))$-measurable, i.e., for every set $A\in\mathcal{B}(\Pi_p(\mathrm{K},\mathrm{E}))$,

$$\psi^{-1}(A) = \{(s,\omega)\in[0,T]\times\Omega :\ \psi(s,\omega)\in A\}\in\mathcal{P}.$$

Remark 5.12. Let $\mathcal{B}_s(\Pi_p(\mathrm{K},\mathrm{E}))$ be the strong-Borel σ-field on $\Pi_p(\mathrm{K},\mathrm{E})$, i.e., the σ-field generated by the following family of sets

$$\Big\{L \in \Pi_p(\mathrm{K},\mathrm{E}) : L(x) \in B\Big\}, \quad x \in \mathrm{K}, B \in \mathcal{B}(\mathrm{E}).$$

If $\Pi_p(\mathrm{K},\mathrm{E})$ is separable, then, according to Corollary 2.5, the process ψ is $\mathcal{P}/\mathcal{B}_s(\Pi_p(\mathrm{K},\mathrm{E}))$ measurable if and only if the process ψ is $\mathcal{P}/\mathcal{B}(\Pi_p(\mathrm{K},\mathrm{E}))$-measurable.

Remark 5.13. Predictable processes are adapted and measurable w.r.t. the predictable σ-field $\mathcal{P}$ on $[0,\infty)\times\Omega$, where $\mathcal{P}$ is generated by left-continuous, adapted processes with right limits (càglàd). The predictable σ-field $\mathcal{P}$ is the smallest σ-field w.r.t which all such processes measurable. There are examples of predictable processes that are not left-continuous, see [32, Chapter I, Section 2], or [33, Chapter 1], or [35, Chapter II].

Definition 5.14. [23, p. 92] *Let $\lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ be the class of all predictable $\Pi_p(\mathrm{K},\mathrm{E})$-valued processes*

$$\psi : [0,T]\times\Omega \to \Pi_p(\mathrm{K},\mathrm{E})$$

such that

$$\|\psi\|_{\lambda^p} := \Big(\mathbb{E}\int_0^T \|\psi(s)\|_p^p\,ds\Big)^{\frac{1}{p}} < \infty. \tag{5.2}$$

Definition 5.15. *We say that two processes ψ and ϕ belonging to $\lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ are equivalent, and denote by $\psi\sim\phi$, if and only if*

$$\|\psi-\phi\|_{\lambda^p} = 0.$$

Proposition 5.16. *The relation $\psi\sim\phi$ is an equivalence relation in the space $\lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E})$.*

Before we embark on the proof, let us formulate an equivalent version of the Definition 5.15 in the following remark.

Remark 5.17. By Definition 5.14, the norm in $\lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ is given by (5.2). Applying this to $\psi-\phi$, we get

$$\|\psi-\phi\|_{\lambda^p} := \Big(\mathbb{E}\int_0^T \|\psi(s)-\phi(s)\|_p^p\,ds\Big)^{\frac{1}{p}}.$$

By definition, $\psi\sim\phi$, if and only if

$$\|\psi-\phi\|_{\lambda^p} = 0.$$

Since the integrand $\|\psi(s) - \phi(s)\|_p^p$ is nonnegative, the only way for its expectation to be zero is if

$$\|\psi(s) - \phi(s)\|_p^p = 0,$$

for almost all $(s, \omega) \in [0, T] \times \Omega$. Hence

$$\psi(s, \omega) = \phi(s, \omega) \quad \text{for almost all } (s, \omega).$$

Thus, the processes ψ and ϕ are equivalent in $\lambda^p(0, T; \Pi_p(\mathrm{K}, \mathrm{E}))$ if and only if they are equal almost everywhere w.r.t. $[0, T] \times \Omega$.

Proof of Proposition 5.16. To this end, we need to verify the following properties, for all ψ, ϕ and χ.

(1) $\psi \sim \psi$

(2) If $\psi \sim \phi$, then $\phi \sim \psi$

(3) If $\psi \sim \phi$ and $\phi \sim \chi$, then $\psi \sim \chi$.

Ad (1) We need to show $\psi \sim \psi$, i.e., $\|\psi - \psi\|_{\lambda^p} = 0$. But

$$\|\psi - \psi\|_{\lambda^p} = \left(\mathbb{E} \int_0^T \|\psi(s) - \psi(s)\|_p^p \, ds\right)^{1/p} = \left(\mathbb{E} \int_0^T 0 \, ds\right)^{1/p} = 0.$$

Hence $\psi \sim \psi$.

Ad (2) Suppose $\psi \sim \phi$, i.e., $\|\psi - \phi\|_{\lambda^p} = 0$. Since the norm is symmetric, we have

$$\|\psi - \phi\|_{\lambda^p} = \|\phi - \psi\|_{\lambda^p} = 0.$$

Hence $\phi \sim \psi$.

Ad (3) Suppose $\psi \sim \phi$ and $\phi \sim \chi$, i.e.,

$$\|\psi - \phi\|_{\lambda^p} = 0 \quad \text{and} \quad \|\psi - \chi\|_{\lambda^p} = 0.$$

Using the triangle inequality, we have

$$\|\psi - \chi\|_{\lambda^p} \leq \|\psi - \phi\|_{\lambda^p} + \|\phi - \chi\|_{\lambda^p} = 0 + 0 = 0.$$

Hence $\psi \sim \chi$.

□

Definition 5.18. *We denote by* $\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ *the space of equivalence classes of processes in* $\lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$

$$\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E})) := \lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))/\sim,$$

where the space $\lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ *is defined in Definition 5.14 and the equivalence relation* $\sim$ *is defined in Definition 5.15.*

The norm in $\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ *is defined by*

$$\|[\psi]_\sim\|_{\Lambda^p} := \left(\mathbb{E}\int_0^T |\psi(t)|^p_{\Pi_p}\,dt\right)^{1/p} = \|\psi\|_{\lambda^p}$$

The next definition is quickly recalled on pages 49-50 of [17].

Definition 5.19. *By* $L^p([0,T]\times\Omega,\mathcal{P},\mathrm{Leb}\otimes\mathbb{P};\Pi_p(\mathrm{K},\mathrm{E}))$ *we mean the space of all (equivalence classes of) $\mathcal{P}$-measurable functions*

$$\psi : [0,T]\times\Omega \to \Pi_p(\mathrm{K},\mathrm{E})$$

such that

$$\int_{[0,T]\times\Omega} \pi_p^p(\psi(t,\omega))\,(\mathrm{Leb}\otimes\mathbb{P})\,(dt,d\omega) = \mathbb{E}\int_0^T \pi_p^p(\psi(t))\,dt < \infty.$$

This space is equipped with the norm

$$\|\psi\|_{L^p} := \left(\mathbb{E}\int_0^T \pi_p^p(\psi(t))\,dt\right)^{1/p}.$$

As a direct consequence of the above definitions, we obtain the following.

Proposition 5.20. *We have*

$$\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E})) = L^p([0,T]\times\Omega,\mathcal{P},\mathrm{Leb}\otimes\mathbb{P};\Pi_p(\mathrm{K},\mathrm{E})) \tag{5.3}$$

with equal norms.

The following corollary follows from Proposition 4.9 and [17, Chapter 2].

Corollary 5.21. *The space in* (5.3)

$$\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E})) := L^p([0,T]\times\Omega,\mathcal{P},\mathrm{Leb}\otimes\mathbb{P};\Pi_p(\mathrm{K},\mathrm{E}))$$

is a Banach space.

Definition 5.22. [23, p. 92] *A stochastic $\Pi_p(\mathrm{K},\mathrm{E})$-valued step process is a process of the following form*

$$\psi(t) = \psi_0 \mathbb{1}_{\{0\}}(t) + \sum_{k=0}^{N-1} \psi_k \mathbb{1}_{(t_k,t_{k+1}]}(t), \quad t \in [0,T], \tag{5.4}$$

such that

(i) $0 = t_0 < t_1 < \cdots < t_N = T$*;*

(ii) *for each* $k = 0, \cdots, N$*,* $\psi_k : \Omega \to \Pi_p(\mathrm{K},\mathrm{E})$ *is a simple* $\Pi_p(\mathrm{K},\mathrm{E})$*-valued random variable, i.e., finitely valued function, see equation* (2.1)*, of the following form*

$$\psi_k = \sum_{i=1}^{m_k} \mathbb{1}_{A_{k,i}} \psi_{k,i}, \tag{5.5}$$

for some $m_k \in \mathbb{N}$ *and some pairwise disjoint sets* $A_{k,1}, \cdots, A_{k,m_k} \in \mathcal{F}_{t_k}$ *and some* $\psi_{k,1}, \cdots, \psi_{k,m_k} \in \Pi_p(\mathrm{K},\mathrm{E})$.

In particular, each $\psi_k : \Omega \to \Pi_p(\mathrm{K},\mathrm{E})$ *is an* $\mathcal{F}_{t_k}/\mathcal{B}_s(\Pi_p(\mathrm{K},\mathrm{E}))$*-measurable p-integrable simple function.*

We denote the space of all $\Pi_p(\mathrm{K},\mathrm{E})$*-valued stochastic step processes by* $\lambda^p_{\mathrm{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ *and the space of its equivalence classes by* $\Lambda^p_{\mathrm{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$.

Note that we use the following notation

$$\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E})) := \{[\psi] = [\psi]_\sim : \psi \in \lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))\}.$$

$$\Lambda^p_{\mathrm{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E})) := \{[\psi_k] = [\psi_k]_\sim : \psi_k \in \lambda^p_{\mathrm{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))\}.$$

Here, Λ^p and $\Lambda^p_{\mathrm{step}}$ represent the equivalent classes of the spaces λ^p and $\lambda^p_{\mathrm{step}}$, respectively.

From this point onward, we assume that Assumption 5.8 holds.

In the following lemma we will use the following notation, which is a special case of Definition 3.10 and Proposition 3.11, i.e. $\mathscr{L}(\mathrm{K}^*;L^p(\Omega,\mathbb{R}))$ is the Banach space of all bounded linear maps $A : \mathrm{K}^* \to L^p(\Omega,\mathbb{R})$ equipped with the norm defined in two equivalent ways

$$\|A\|_{\mathscr{L}(\mathrm{K}^*;L^p(\Omega,\mathbb{R}))} := \sup\Big\{\|Ax^*\|_{L^p(\Omega,\mathbb{R})} : x^* \in \mathrm{K}^*,\ \|x^*\|_{\mathrm{K}^*} \le 1\Big\},$$

or

$$\|A\|_{\mathscr{L}(\mathrm{K}^*;L^p(\Omega,\mathbb{R}))} := \inf\Big\{C > 0 : \|Ax^*\|_{L^p(\Omega,\mathbb{R})} \le C\|x^*\|_{\mathrm{K}^*} : x^* \in \mathrm{K}^*\Big\}.$$

Lemma 5.23. *Assume that a cylindrical Lévy process L in K satisfies Assumption 5.8. Suppose that $p \in [1,2]$. If L is weakly p-integrable, then for every $T > 0$ there exists a constant C such that for every $t \in [0,T]$, the following inequality holds*

$$\|L(t)\|^p_{\mathscr{L}(\mathrm{K}^*;L^p(\Omega,\mathbb{R}))} \leq Ct,$$

Equivalently, for every element $x^* \in \mathrm{K}^*$, we have

$$\|L(t)x^*\|_{L^p(\Omega,\mathbb{R})} \leq Ct\,\|x^*\|_{\mathrm{K}^*}$$

or

$$(\mathbb{E}\|L(t)x^*\|^p)^{1/p} \leq Ct\,\|x^*\|_{\mathrm{K}^*}.$$

Digression 5.24. *Since $(L(t) : t \geq 0)$ is a cylindrical $\mathbb{F}$-Lévy martingale, see Definition 3.53, for every $t \geq 0$, $L(t)$ is a cylindrical random variable, i.e.*

$$L(t) : \mathrm{K}^* \ni x^* \mapsto L(t)x^* \in L^0(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}) \text{ is a linear map.}$$

In addition, the process is weakly p-integrable, which implies that for every $x^* \in \mathrm{K}^*$, the random variable $L(t)x^*$ is p-integrable. In other words, the $\mathbb{R}$-valued random variable $L(t)x^*$ belongs to $L^p(\Omega,\mathcal{F},\mathbb{P};\mathbb{R})$, abbreviated as $L^p(\Omega;\mathbb{R})$, for $x^* \in \mathrm{K}^*$. Thus, for every $t \geq 0$, the map

$$L(t) : \mathrm{K}^* \to L^p(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}) \tag{5.6}$$

is a linear map.

In the course of the proof of Lemma 5.23 we will show that the norm of the linear map (5.6) is $\leq Ct^{1/p}$.

Proof of Lemma 5.23. Let us choose and fix $p \in [1,2]$. Let $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$ be the filtered probability space as in Assumption 5.1.

Note that assumptions of Proposition 3.59 are satisfied and therefore we infer that for every $t > 0$, $L(t)$ belongs to the space $\mathscr{L}(\mathrm{K}^*;L^p(\Omega,\mathbb{R}))$ and is continuous. We have to find a constant $C > 0$ such that the norm of this linear map is bounded from above by $Ct^{1/p}$ for $t \in [0,T]$.

For this purpose we choose and fix $T > 0$ and define a normed vector space $R_p(T)$, see [30, p.164] for a similar object, by

$$R_p(T) = \{\psi : [0,T]\times\Omega \to \mathbb{R} : \psi \text{ is } \mathbb{F}\text{-progressively measurable and } \|\psi\|_{R_p} < \infty\}, \tag{5.7}$$

where the norm $\|\cdot\|_{R_p}$ is defined by

$$\|\psi\|_{R_p} = \sup_{t\in(0,T]} \frac{1}{t^{1/p}} (\mathbb{E}\Big[\|\psi(t)\|^p\Big])^{1/p}.$$

One can prove, see Lemma G.14, that $R_p(T)$ is a Banach space.

Next, let us observe, that for every $t \in (0, T]$, the following inequality holds

$$\mathbb{E}\Big[\|\psi(t)\|^p\Big] \leq t\,\|\psi\|^p_{R_p}, \quad \text{for every } \psi \in R_p.$$

In particular, since $\mathbb{E}\Big[\|\psi\|^p\Big] = \|\psi\|^p_{L^p(\Omega)}$, for every $t \in (0, T]$, the map

$$R_p \ni \psi \mapsto \psi(t) \in L^p(\Omega)$$

is (obviously) linear and bounded (and hence continuous).

Since, by Assumption 5.8, the process L is a cylindrical $\mathbb{F}$-Lévy martingale, by the properties of $\mathbb{R}$-valued Lévy martingales, see [2, Theorem 2.7], we have

$$\mathbb{E}(L(t)x^*) = Ct.$$

Moreover, by [2, Remark 4.9, eq. (4.2) on p. 713], and by [30, Th. 8.23(i), p. 129], we get

$$\begin{aligned}
\mathbb{E}[\|L(t)x^*\|^p] &= \mathbb{E}\|\int_0^t \int_{\mathbb{R}\setminus\{0\}} x\,\widetilde{N}(ds, dx)\|^p \\
&\leq C\mathbb{E}\int_0^t \int_{\mathbb{R}\setminus\{0\}} |x|^p (\nu\circ(x^*)^{-1})\,dx\,ds \\
&= Ct\int_{\mathbb{R}\setminus\{0\}} |x|^p (\nu\circ(x^*)^{-1})\,dx.
\end{aligned}$$

where $\nu\circ(x^*)^{-1}$ is the pushforward measure of the cylindrical Lévy measure ν, see Definitions 3.21,3.24 and 3.25, which satisfies Assumption 5.6; $\widetilde{N}$ is the compensated Poisson random measure associated with a Lévy process.

In short, there exists a constant $C > 0$ such that

$$\mathbb{E}[\|L(t)x^*\|^p] \leq Ct\int_{\mathbb{R}} |x|^p (\nu\circ(x^*)^{-1})\,dx, \quad \text{for all } x^* \in \mathrm{K}^*,\ t \geq 0. \tag{5.8}$$

Let us temporarily choose and fix $x^* \in \mathrm{K}^*$ and define the process ψ by

$$\psi : [0, T] \times \Omega \ni (t, \omega) \mapsto L(t, \omega)x^* \in \mathbb{R}.$$

We will show that

(i) ψ is progressively measurable.

(ii) ψ belongs to the space R_p.

Ad (i). By Corollary 3.54, we infer that the process ψ is progressively measurable and hence the claim (i). □

Ad (ii). We need to show that $||\psi||_{R_p(T)} < \infty$.

By using inequality (5.8), we have

$$\begin{aligned}
||\psi||_{R_p(T)} &= \sup_{t\in(0,T]} \frac{1}{t^{1/p}} (\mathbb{E}\Big[||\psi(t)||^p\Big])^{1/p} \\
&= \sup_{t\in(0,T]} \frac{1}{t^{1/p}} (\mathbb{E}\Big[||L(t,\omega)x^*||^p])^{1/p} \\
&\le \sup_{t\in(0,T]} \frac{1}{t^{1/p}} \Big(ct \int_{\mathbb{R}} |x|^p (\nu \circ (x^*)^{-1})\, dx\Big)^{1/p} \\
&= c^{1/p} \Big(\int_{\mathbb{R}} |x|^p (\nu \circ (x^*)^{-1})\, dx\Big)^{1/p} < \infty,
\end{aligned}$$

since $\int_{\mathbb{R}} |x|^p (\nu \circ (x^*)^{-1})\, dx < \infty$ under Assumption 5.6. Hence the claim (ii). □

After we have shown (ii) we can define a map $\widehat{\mathbf{L}} : \mathrm{K}^* \to R_p$ by

$$\widehat{\mathbf{L}} x^* = (L(t)x^* : \ t \in [0, T]).$$

We observe that in view of (i) and (ii), (i') $\widehat{\mathbf{L}}$ is well-defined. Moreover, we will show that (ii') $\widehat{\mathbf{L}}$ is a linear map.

To show the linearity, we have to show that for any $x^*, y^* \in \mathrm{K}^*$ and a constant $\alpha \in \mathbb{R}$, the following equality holds

$$\begin{aligned}
\widehat{\mathbf{L}}(\alpha x^* + y^*) &= L(t)(ax^* + y^*) \\
&= L(t)(ax^*) + L(t)y^* \\
&= aL(t)(x^*) + L(t)y^* \\
&= \alpha \widehat{\mathbf{L}} x^* + \widehat{\mathbf{L}} y^*
\end{aligned}$$

Finally, we will show that (iii') $\widehat{\mathbf{L}}$ is a continuous map. For this purpose we will use the closed graph theorem, see Theorem F.18, in the same as the proof to Proposition 3.59.

Indeed, we can easily check the first two assumptions of the closed graph theorem are satisfied, i.e.:

(a) K^* and, by Lemma G.14, R_p are both Banach spaces and hence Fréchet spaces.

(b) the map $\widehat{\mathbf{L}}$ is linear, as proved in part (ii').

It remains to show that the third assumption, i.e.,

(c) the graph G of the map $\widehat{\mathbf{L}}$ is a closed subset of $\mathrm{K}^* \times R_p$. For this purpose we take a sequence of elements of G, i.e., $(x_n^*, L(t)x_n^*) \in G$ such that

$$(x_n^*, L(t)x_n^*) \to (x^*, y) \in \mathrm{K}^* \times R_p$$

This means that

$$x_n^* \to x^* \in \mathrm{K}^*$$

$$L(t)x_n^* \to y \in R_p.$$

We need to show that the limit (x^*, y) belongs to the graph G, i.e.

$$y = L(t)x^*.$$

Since $L(t)x^*$ are cylindrical random variables, by Definition 3.27, it is continuous. Therefore, since $x_n^* \to x^*$ in K^*, $L(t)x_n^* \to L(t)x^*$ in R_p. On the other hand, $L(t)x_n^* \to y$ in R_p which implies that $L(t)x_n^* \to y$ in R_p. Now, we have two limits of the same sequence in R_p. Hence these limits are equal, i.e., $y = L(t)x^*$.

Hence the assumptions of the Closed Graph Theorem, see Theorem F.18, are all satisfied and therefore, we infer that $\widehat{\mathbf{L}} : \mathrm{K}^* \to R_p$ is continuous and hence, since it is linear, bounded. Thus there exists $C > 0$ such that for every $x^* \in \mathrm{K}^*$,

$$\|\widehat{\mathbf{L}}x^*\|_{R_p} \leq C\,|x^*|_{\mathrm{K}^*}.$$

By the definitions of the space $R_p(T)$ and the operator $\widehat{\mathbf{L}}$, we infer that

$$\sup_{t\in(0,T]} \frac{1}{t^{1/p}}\Big(\mathbb{E}\Big[\|L(t)x^*\|^p\Big]\Big)^{1/p} \leq C\,|x^*|_{\mathrm{K}^*}, \quad x^* \in \mathrm{K}^*.$$

Hence,

$$\mathbb{E}\Big[\|L(t)x^*\|^p\Big] \leq Ct\,|x^*|^p_{\mathrm{K}^*}, \quad t \in (0,T],\ x^* \in \mathrm{K}^*.$$

□

Definition 5.25. *Let $0 \leq s < t \leq T$, and assume that Assumptions 5.4 and 5.5 hold with $p \in [1,2]$. Assume that for every $k = 0, \cdots, N-1$, ψ_k is a simple $\Pi_p(\mathrm{K}, \mathrm{E})$-valued* *random variable of the form* (5.5)*, with pairwise disjoint sets* $A_{k,1}, \cdots, A_{k,m_k} \in \mathcal{F}_{t_k}$ *and* $\psi_{k,1}, \cdots, \psi_{k,m_k} \in \Pi_p(\mathrm{K}, \mathrm{E})$.

Assume that L *is a weakly* p*-integrable cylindrical Lévy process in* K*, see Definition 3.52, and that* $\psi : \mathrm{K} \to \mathrm{E}$ *is* p*-summing operator. Then, by Theorem 4.23,* $\psi : \mathrm{K} \to \mathrm{E}$ *is also* p*-radonifying. Hence, according to Definition 3.29, the cylindrical random variable* $(L(t)-L(s))\psi^*$ *in* E *has finite* p*-th weak moments and it is induced by a true random variable*

$$J_{s,t}(\psi) : \Omega \to \mathrm{E}.$$

In other words,

$$(L(t) - L(s))(\psi^* x^*) = x^*(J_{s,t}(\psi)) = \langle J_{s,t}(\psi), x^* \rangle, \quad \forall x^* \in \mathrm{E}^*. \tag{5.9}$$

This construction can be applied to $\psi = \psi_{k,i}$.

We define an E*-valued random variable* $J_{s,t}(\psi_k)$*, when* ψ_k *is of the form* (5.5)*, by the following identity*

$$J_{s,t}(\psi_k) := \sum_{i=1}^{m} \mathbb{1}_{A_{k,i}} J_{s,t}(\psi_{k,i}). \tag{5.10}$$

Proposition 5.26. *Assume that* K, E *are Banach spaces. Let* $p \in [1, 2]$ *and let* L *be a weakly* p*-integrable cylindrical Lévy process in* K*. Assume that* $\psi : \mathrm{K} \to \mathrm{E}$ *is* p*-summing operator. Then the process* $(J_{0,t}(\psi), t \geq 0)$ *is an* E*-valued Lévy process.*

Proof. Let us choose and fix $x^* \in \mathrm{E}^*$. Then by formula (5.9),

$$\langle J_{0,t}(\psi), x^* \rangle = L(t)(\psi^* x^*)$$

By the assumptions on L, we infer that the process $\langle J_{0,t}(\psi), x^* \rangle$ is a Lévy process. Since by assumptions $J_{0,t}(\psi)$ is E-valued, we infer that it is an E-valued Lévy process. □

Remark 5.27. Kosmala in his thesis proved a similar result in the case of Hilbert spaces, see [23, Theorem 3.5, p. 27]. He proved that if L is a weakly square-integrable cylindrical Lévy process and $\psi \in \mathcal{M}^2(0, T; L_{HS}(\mathcal{H}, H)$, where $\mathcal{H} = Q^{1/2}U$ and U, H are separable Hilbert spaces, the process

$$(I(t),\ t \in [0, T])$$

admits a càdlàg modification. Moreover, if L is a cylindrical martingale, then the process I is càdlàg, mean-zero, square-integrable martingale and the following equality holds:

$$\mathbb{E}\Big[\| \int_0^T \psi(s)\, dL(s)\|^2\Big] = \mathbb{E}\Big[\int_0^T \|\psi(s)Q^{1/2}\|^2_{L_{HS}(U,H)}\, ds\Big].$$

Lemma 5.28. *(Radonification of the increments) Under the assumptions listed in Definition 5.25, if* $0 \le s < t \le T$*, the random variable* $J_{s,t}(\psi_k)$ *defined in* (5.10)*, satisfies the following inequality*

$$(\mathbb{E}[\|J_{s,t}(\psi_k)\|^p])^{\frac{1}{p}} \le \|L(t-s)\|_{\mathscr{L}(\mathrm{K}^*,L^p(\Omega;\mathbb{R}))}(\mathbb{E}[\pi_p^p(\psi_k)])^{\frac{1}{p}}. \tag{5.11}$$

Proof of Lemma 5.28. See [23, proof to Lemma 5.6, p. 91-92]. □

5.2. **The Itô Stochastic Integral for Step Processes and Its Properties.** The set of assumptions listed at the beginning of this section will be valid throughout the subsection.

For any $0 \le s < t$, we use the filtration

$$\mathcal{F}_{[0,s]} = \sigma(L_r, 0 \le r \le s).$$

Remark 5.29. Assume that L is a cylindrical Lévy process. The natural filtration $\mathbb{F} = (\mathcal{F}_s : s \ge 0)$, see (3.5), generated by L does not satisfy the usual condition, see Definition F.24. To ensure that standard results from stochastic analysis apply, we work with augmentation $\mathbb{F}^{aug} = (\mathcal{F}_t^{aug} : t \ge 0)$ of this natural filtration, which is defined by

$$\mathcal{F}_t^{aug} := \bigcap_{s>t}(\mathcal{F}_s \vee \mathcal{N}), \quad t \ge 0,$$

where $\mathcal{N}$ denotes the collection of all $\mathbb{P}$-null sets. Let us recall that $A \subset \Omega$ is a $\mathbb{P}$-null set if and only if there exists a set $B \in \mathcal{F}$ such that $A \subset B$ and $\mathbb{P}(B) = 0$.

'Given a filtration $(\mathcal{F}_t)$, its usual augmentation is the smallest right-continuous and complete filtration containing it'[32, Chapter 1, Section 1, p. 4].

The augmented filtration $(\mathcal{F}_t^{aug})_{t\ge 0}$ is the smallest right-continuous and complete filtration containing the natural one. The process L remains adapted to this augmented filtration, and the increments of L continue to be independent of the past, i.e. of $\mathcal{F}_s^{aug}$ for $s < t$.

Therefore, when applying results in Section 7, particularly those concerning stochastic integrals and stochastic differential equations, we use this augmented filtration generated by

L. This ensures that all stochastic integration results, martingale properties and existence-uniqueness theorems apply without restrictions.

To define the stochastic integral, let $\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$, $\Lambda^p_{\mathrm{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ and ψ_k be defined as in Definitions 5.18 and 5.22.

We define the stochastic integral operator as follows.

Definition 5.30. *If $\psi \in \Lambda^p_{\mathrm{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ be a step function of the form* (5.4)*, then the stochastic integral of ψ over the interval $[0,T]$ is a random variable defined by the following formula*

$$I(\psi) := \sum_{k=0}^{N-1} J_{t_k,t_{k+1}}(\psi_k). \tag{5.12}$$

In the next definition we extend Definition 5.30 by defining a family of stochastic integrals over time intervals $[0,t]$, $t \in [0,T]$.

Definition 5.31. *Assume that $\psi \in \Lambda^p_{\mathrm{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ be a step function of the form* (5.4)*. Assume next $t \in [0,T)$. Let the process $\mathbb{1}_{[0,t]}\psi$ be defined by the following formula*

$$[0,T]\times\Omega \ni (s,\omega) \mapsto \mathbb{1}_{[0,t]}(s)\,\psi(s,\omega) \in \Pi_p(\mathrm{K},\mathrm{E}),$$

then $\mathbb{1}_{[0,t]}\psi \in \Lambda^p_{\mathrm{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ of the following form

$$(\mathbb{1}_{[0,t]}\psi)(s) = \psi_0\mathbb{1}_{\{0\}}(s) + \sum_{k=0}^{n-1}\psi_k\mathbb{1}_{(\tilde{s}_k,\tilde{s}_{k+1}]}(s), \quad s\in[0,T],$$

where n is the unique number such that

$$n \in \{1,\cdots,N\} \text{ and } t \in [t_{n-1},t_n),$$

and $(\tilde{s}_k : k = 0,\cdots,n+1)$ is a new auxiliary partition of $[0,T]$ defined by

$$\tilde{s}_k = \begin{cases} t_k & \text{if } k<n, \\ t & \text{if } k=n, \\ T & \text{if } k=n+1. \end{cases}$$

Next we define the the Itô integral of the step process ψ over the time interval $[0,t]$ as:

$$I_t(\psi) := I(\mathbb{1}_{[0,t]}\psi), \quad t\in[0,T).$$

and

$$I_T(\psi) = I(\psi).$$

We also have the following simple observation:

$$I_t(\psi) = I(\mathbb{1}_{[0,t]}\psi) = \sum_{k=0}^{n-1} J_{\tilde{s}_k,\tilde{s}_{k+1}}(\psi_k) = \sum_{k=0}^{n-2} J_{t_k,t_{k+1}}(\psi_k) + J_{t_{n-1},t}(\psi_{n-1}). \tag{5.13}$$

The following results are fundamental in our definition of the stochastic Itô integral for step processes.

Proposition 5.32. *Under Assumptions 5.1-5.7, the following assertions hold. There exists a constant $C > 0$ such that the stochastic integral $I(\psi) = (I_t(\psi) :\ t \in [0,T])$, of every step process $\psi \in \Lambda^p_{\text{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ of the form* (5.4)*, satisfies the following conditions:*

(i) *$I(\psi)$ is an E-valued, $\mathbb{F}$-adapted and càdlàg process.*

(ii) *If $t \in [0,T]$, then*

$$\mathbb{E}\left[I_t(\psi)\right] = 0.$$

(iii) *$I(\psi)$ is a p-integrable $\mathbb{F}$-martingale.*

(iv) *The map I is linear between the space $\Lambda^p_{\text{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ and the space of all E-valued, p-integrable càdlàg $\mathbb{F}$-martingales.*

(v) *If $t \in [0,T]$, then*

$$\sup_{s\in[0,t]} \mathbb{E}\left[\|I_s(\psi)\|^p\right] \leq C\mathbb{E}[\int_0^t \pi_p^p(\psi(s))\, ds]. \tag{5.14}$$

(vi) *If additionally, $p > 1$, then for every $t \in [0,T]$,*

$$\mathbb{E}\left[\sup_{s\in[0,t]} \|I_s(\psi)\|^p\right] \leq C\mathbb{E}[\int_0^t \pi_p^p(\psi(s))\, ds]. \tag{5.15}$$

Proof of Proposition 5.32. Each part of the proposition will be proven separately.

Proof of part (i). Let the time partition be of the form

$$0 = t_0 < t_1 < t_k < \cdots < t_{n-1} < t \leq t_n \leq t_N = T. \tag{5.16}$$

(a) **adaptedness** By Definition 5.30, each ψ_k is $\Pi_p(\mathrm{K},\mathrm{E})$-valued, $\mathcal{F}_{t_k}$-measurable. Since the stochastic integral is constructed as (5.12), i.e.,

$$I(\psi) := \sum_{k=0}^{N-1} J_{t_k,t_{k+1}}(\psi_k),$$

where each $J_{t_k,t_{k+1}}(\psi_k)$ depends only on information up to time t_k, it follows that $I_t(\psi), t \in [0,T]$ is $\mathcal{F}_t$-adapted for all $t \in [0,T]$, and $I_t(\psi) \in \mathrm{E}$, i.e., E-valued, by (5.13).

(b) **càdlàg** Assume that $t \in [t_0, t_1)$. Then by the definition (Definition 5.31), each $J_{s,t}(\psi_k)$ denotes the action of ψ_k on the Lévy process increment $L(t) - L(s)$. By Lemma 5.28, each $J_{s,t}(\psi_k)$ is well-defined, and it has the martingale property . With $n = 1$ and by formula (5.13), $I_t(\psi) = J_{0,t}(\psi_0)$. By Proposition 5.26 this process is càdlàg w.r.t. t. If $n \geq 2$, the proof is similar.

By (5.12) in Definition 5.30, the stochastic integral is given as a sum of increments $J_{t_k,t_{k+1}}(\psi_k)$, where each $J_{s,t}(\psi_k)$ denotes the action of ψ_k on the Lévy process increment $L(t)-L(s)$. By Lemma 5.28, each $J_{s,t}(\psi_k)$ is well-defined and, since L is a weakly p-integrable cylindrical Lévy process, has a martingale property. Moreover, each $J_{s,t}(\psi_k)$ is càdlàg. Hence, $I_t(\psi)$ being a finite sum of càdlàg functions, is itself càdlàg. Finally, since Lévy processes have right-continuous paths with left limits in Banach spaces, and weak p-integrability ensures that jumps are well-defined, it follows that $I(\psi)$ admits left limits. Therefore, $I(\psi)$ is càdlàg.

□

Proof of part (ii). Let us choose and fix a step process $\psi \in \Lambda^p_{\mathrm{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ and $t \in [0,T]$. Without loss of generality, we can assume that the partition corresponding to ψ contains the number t, i.e., is of the form (5.16). Thus the stochastic integral of the process ψ is a stochastic process $I(\psi) = (I_t(\psi) : t \geq 0)$ such that

$$I_t(\psi) = I(\mathbb{1}_{[0,t]}\psi) = \sum_{k=0}^{n-1} \psi_k(L(t_{k+1}) - L(t_k)), \quad t \geq 0.$$

Let us note that ψ_k is $\mathcal{F}_{t_k}$-measurable and $L(t_{k+1}) - L(t_k)$ is independent of the σ-field $\mathcal{F}_{t_k} = \sigma(L_t, 0 \leq t \leq t_k)$, and by using the fact that $\mathbb{E}(L(t_{k+1}) - L(t_k)) = 0$ in the last equality, see Definition 2.51. Thus we deduce the independence of ψ_k and $L(t_{k+1}) - L(t_k)$, see [23, p. 91] [11, Proposition 6.3, p. 154, Corollary 6.2, p. 155]. Therefore, we get

$$\mathbb{E}(I_t(\psi)) = \sum_{k=0}^{n-1} \mathbb{E}(\psi_k(L(t_{k+1}) - L(t_k)) = \sum_{k=0}^{n-1} \mathbb{E}(\psi_k)\mathbb{E}(L(t_{k+1}) - L(t_k)) = 0.$$

□

Proof of part (iii). Let us choose and fix a step process $\psi \in \Lambda^p_{\text{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$. We have, for $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}\|I_t(\psi)\|^p &= \mathbb{E}\Big\|\sum_{k=0}^{n-1}\psi_k\Big(L(t_{k+1})-L(t_k)\Big)\Big\|^p \le \sum_{k=0}^{n-1}\mathbb{E}\Big\|\psi_k\Big(L(t_{k+1})-L(t_k)\Big)\Big\|^p \\
&\le \sum_{k=0}^{n-1}\|L(t_{k+1}-t_k)\|^p_{\mathscr{L}(\mathrm{K}^*,L^p(\Omega;\mathbb{R}))}\mathbb{E}\Big[\pi_p^p(\psi_k)\Big] \le \sum_{k=0}^{n-1}C_L(t_{k+1}-t_k)\mathbb{E}\Big[\pi_p^p(\psi_k)\Big] \\
&= C_L\mathbb{E}\sum_{k=0}^{n-1}(t_{k+1}-t_k)\pi_p^p(\psi_k) = C\mathbb{E}\int_0^T \pi_p^p(\psi(s))\,ds < \infty.
\end{aligned}$$

Therefore, $I_t(\psi)$ is p-integrable, for every $t \in [0,T]$.

Next, we will follow the solutions to [11, Exercise 7.5, p. 211-212], to show that $I(\psi) = (I_t(\psi) : t \in [0,T])$ is a martingale with respect to the filtration $\mathbb{F} = (\mathcal{F}_t : t \in [0,T])$ defined by

$$\mathcal{F}_t = \mathcal{F}_{[0,t]} = \sigma(L_s : 0 \le s \le t).$$

Let the time partition corresponding to ψ be of the following form

$$0 = t_0 < t_1 < \cdots < t_k = s < t_{t+1} < \cdots < t_m = t < t_{m+1} < \cdots < t_n = T.$$

Let us denote the increments $L(t_{j+1}) - L(t_{eqn-J_st})$ by $J_{t_j,t_{j+1}}(\psi_j)$. Then

$$\mathbb{1}_{(0,t]}\psi = \sum_{j=0}^{m-1}\psi_j\mathbb{1}_{[t_j,t_{j+1}]}, \quad 0 < s < t \le T,$$

and

$$I_t(\psi) = I(\mathbb{1}_{(0,t]}\psi) = \sum_{j=0}^{m-1}\psi_j J_{t_j,t_{j+1}}(\psi_j),$$

which is $\mathcal{F}_t$-measurable (and p-integrable).

It remains to compute

$$\mathbb{E}(I_t(\psi)|\mathcal{F}_s) = \mathbb{E}(I(\mathbb{1}_{(0,t]}\psi)|\mathcal{F}_s) = \sum_{j=0}^{m-1}\mathbb{E}(\psi_j J_{t_j,t_{j+1}}(\psi_j)|\mathcal{F}_s).$$

If $j < k$, then ψ_j and $J_{t_j,t_{j+1}}$ are $\mathcal{F}_s$-measurable and therefore

$$\mathbb{E}(\psi_j J_{t_j,t_{j+1}}(\psi_j)|\mathcal{F}_s) = \psi_j J_{t_j,t_{j+1}}(\psi_j).$$

If $j \geq k$, then $\mathcal{F}_s \subset \mathcal{F}_{t_j}$ and

$$\begin{aligned}
\mathbb{E}(\psi_j J_{t_j,t_{j+1}}(\psi_j)|\mathcal{F}_s) &= \mathbb{E}(\mathbb{E}(\psi_j J_{t_j,t_{j+1}}(\psi_j)|\mathcal{F}_{t_j})|\mathcal{F}_s) \\
&= \mathbb{E}(\psi_j \mathbb{E}(J_{t_j,t_{j+1}}(\psi_j)|\mathcal{F}_{t_j})|\mathcal{F}_s) \\
&= \mathbb{E}(\psi_j|\mathcal{F}_s)\mathbb{E}(J_{t_j,t_{j+1}}(\psi_j)) = 0,
\end{aligned}$$

since ψ_j is $\mathcal{F}_{t_j}$-measurable and $J_{t_j,t_{j+1}}(\psi_j)$ is independent of $\mathcal{F}_{t_j}$. It follows that

$$\mathbb{E}(I_t(\psi)|\mathcal{F}_s) = \sum_{j=0}^{k-1} \psi_j J_{t_j,t_{j+1}}(\psi_j) = I(\mathbb{1}_{(0,s]}\psi) = I_s(\psi).$$

Hence we have proved part (iii). □

Proof of part (iv). To prove the linearity of I_t, we need to prove if $\psi, \xi \in \Lambda^p_{\text{step}}(0, T; \Pi_p(\mathrm{K}, \mathrm{E}))$ and $\alpha, \beta \in \mathbb{R}$, then

$$I(\alpha\psi + \beta\xi) = \alpha I(\psi) + \beta I(\xi).$$

Let us choose and fix two step processes $\psi, \xi \in \Lambda^p_{\text{step}}(0, T; \Pi_p(\mathrm{K}, \mathrm{E}))$. Without loss of generality we can assume that the partitions corresponding to ψ and ξ are the same and of the form (5.16). Note that $\alpha\psi + \beta\xi$ is a step process with the same partition. Thus the stochastic integral of the processes ψ, ξ and $\alpha\psi + \beta\xi$ are

$$\begin{aligned}
I(\psi) &= \sum_{k=0}^{n-1} \psi_k\big(L(t_{k+1}) - L(t_k)\big) \\
I(\xi) &= \sum_{k=0}^{n-1} \xi_k\big(L(t_{k+1}) - L(t_k)\big) \\
I(\alpha\psi + \beta\xi) &= \sum_{k=0}^{n-1} (\alpha\psi_k + \beta\xi)\big(L(t_{k+1}) - L(t_k)\big) \\
&= \sum_{k=0}^{n-1} \Big(\alpha\psi_k\big(L(t_{k+1}) - L(t_k)\big) + \beta\xi_k\big(L(t_{k+1}) - L(t_k)\big)\Big) \\
&= \sum_{k=0}^{n-1} \alpha\psi_k\big(L(t_{k+1}) - L(t_k)\big) + \sum_{k=0}^{n-1} \beta\xi_k\big(L(t_{k+1}) - L(t_k)\big) \\
&= \alpha \sum_{k=0}^{n-1} \psi_k\big(L(t_{k+1}) - L(t_k)\big) + \beta \sum_{k=0}^{n-1} \xi_k\big(L(t_{k+1}) - L(t_k)\big) \\
&= \alpha I(\psi) + \beta I(\xi).
\end{aligned}$$

Hence we have proved part (iv). □

Proof of part (v). Using the identity (5.13) of the random variable $I_t(\psi)$ in Definition 5.30, and the fact that $(J_{s_k,s_{k+1}}(\psi_k))_{k=0}^n$ are E-valued martingales with respect to the filtration $(\mathcal{F}_{t_k})_{k=0}^n$, therefore by the martingale type p property of the Banach space E, see Definition 2.94, together with the inequality (5.11) of Lemma 5.28, we have

$$\begin{aligned}
\sup_{t\in[0,T]} \mathbb{E}\|I_t(\psi)\|^p &= \sup_{t\in[0,T]} \mathbb{E}\Big\| \sum_{k=0}^{n-2} J_{s_k,s_{k+1}}(\psi_k) + J_{s_{n-1},t}(\psi_{n-1})\Big\|^p = \sup_{t\in[0,T]} \mathbb{E}\Big\| \sum_{k=0}^{n-1} J_{s_k,s_{k+1}}(\psi_k)\Big\|^p \\
&\le C_p \sum_{k=0}^{n-1} \mathbb{E}\Big\|J_{s_k,s_{k+1}}(\psi_k)\Big\|^p \le C_p \sum_{k=0}^{n-1} \|L(s_{k+1}-s_k)\|^p_{\mathscr{L}(\mathrm{K}^*,L^p(\Omega;\mathbb{R}))} \mathbb{E}\Big[\pi_p^p(\psi_k)\Big] \\
&\le C_p \sum_{k=0}^{n-1} C_L(s_{k+1}-s_k)\mathbb{E}\Big[\pi_p^p(\psi_k)\Big] = C_p C_L \mathbb{E} \sum_{k=0}^{n-1} (s_{k+1}-s_k)\pi_p^p(\psi_k) \\
&= C\mathbb{E}\int_0^T \pi_p^p(\psi(s))\, ds.
\end{aligned}$$

Note that the last inequality is based on the results of Lemma 5.23, see also [23, ineq. (5.17), p. 94][1], while the last equality holds because ψ is a $\Pi_p(\mathrm{K},\mathrm{E})$-valued simple-step process of the form (5.4). Hence the proof of part (v) is complete. □

Proof of part (vi). Since by part (iii), the process $I_t(\psi)$, $t \ge 0$ is an E-valued martingale and E is a Banach space, by [26, Proposition (1) of 1.18, Ch 1, p. 12], we infer that $\|I_t(\psi)\|$, $t \ge 0$ is a nonnegative submartingale. Let us recall that we additionally assume that $p > 1$, then by the Doob martingale, see Theorem C.1 and inequality (5.18), we infer that

$$\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[0,t]} \|I_s(\psi)\|^p\Big] &\le \Big(\frac{p}{p-1}\Big)^p \mathbb{E}\Big\|I_t(\psi)\Big\|^p \\
&\le C_T \mathbb{E}\int_0^t \pi_p^p(\psi(s))\, ds.
\end{aligned}$$

Hence the proof of part (vi) is complete. □

The proof of Proposition 5.32 is thus complete. □

[1] [23, ineq. (5.17), p. 94]

$$\|M(t_{k+1}-t_k)\|^p_{\mathscr{L}(\mathrm{K}^*,L^p(\Omega;\mathbb{R}))} \le \|M\|_{\mathscr{L}(\mathrm{K}^*,R_p)}(t_{k+1}-t_k), \quad \forall t \in [0,T].$$

5.3. **The Itô Stochastic Integral for General Processes and Its Properties.** The aim of this final subsection is to formulate and prove a result about the existence of the Itô stochastic integral of a general process as a càdlàg martingale, see Theorem 5.36.

Proposition 5.33. *If $\psi \in \Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ and $t \in [0,T]$, then*

$$\psi_t : [0,T]\times\Omega \ni (s,\omega) \mapsto \mathbb{1}_{[0,t]}(s)\,\psi(s,\omega) \in \Pi_p(\mathrm{K},\mathrm{E})$$

is also a predictable process and it belongs to the space $\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$.

Proof of Proposition 5.33. Let $t \in [0,T]$. Let us choose and fix a process $\psi \in \Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$. First we will prove that ψ_t is a predictable process.

According to Definition 5.11 we need to show that for every $A \in \mathcal{B}_s(\Pi_p(\mathrm{K},\mathrm{E}))$, $\psi_t^{-1}(A) = \{(s,\omega) \in [0,T]\times\Omega : \psi_t(s,\omega) \in A\} \in \mathcal{P}$. By Example 5.10 the set $[0,t]\times\Omega$ belongs to $\mathcal{P}$. Hence, because

$$\begin{aligned}
\{(s,\omega) \in [0,T]\times\Omega : \psi_t(s,\omega) \in A\} = \;& \{(s,\omega) \in [0,T]\times\Omega : s \in [0,t],\ \psi_t(s,\omega) \in A\} \\
& \cup \{(s,\omega) \in [0,T]\times\Omega : s \in (t,T],\ \psi_t(s,\omega) \in A\} \\
= \;& \{(s,\omega) \in [0,T]\times\Omega : s \in [0,t],\ \psi(s,\omega) \in A\} \\
& \cup \{(s,\omega) \in [0,T]\times\Omega : s \in (t,T],\ 0 \in A\} \\
= \;& ([0,t]\times\Omega \cap \{(s,\omega) \in [0,T]\times\Omega :\ \psi(s,\omega) \in A\}) \\
& \cup (t,T]\times\Omega \cap \{0 \in A\},
\end{aligned}$$

we infer that $\psi_t^{-1}(A)$ belongs to $\mathcal{P}$.

In the above argument we used an observation that the set $\{0 \in A\}$ is either empty or a full set Ω.

Next, we need to show that the condition (5.2) holds, i.e., $\|\psi_t\| < \infty$.

$$\begin{aligned}
\|\psi_t\|_{\Lambda^p} &= (\mathbb{E}[\int_0^T \|\mathbb{1}_{[0,t]}(s)\,\psi(s,\omega)\|_p^p ds])^{\frac{1}{p}} \\
&= (\mathbb{E}[\int_0^t \|\psi(s,\omega)\|_p^p ds + \int_{t^+}^T \|0\|_p^p ds])^{\frac{1}{p}} \\
&= (\mathbb{E}[\int_0^t \|\psi(s,\omega)\|_p^p ds])^{\frac{1}{p}} \\
&\le (\mathbb{E}[\int_0^T \|\psi(s,\omega)\|_p^p ds])^{\frac{1}{p}} < \infty,
\end{aligned}$$

since $\psi \in \Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E})$. Hence the proof is complete. $\square$

In order to be able to define the Itô integrals for a general process ψ we need the following approximation result.

Proposition 5.34. *Under Assumptions 5.4 and 5.5. Then for every process ψ belonging to the Banach space $\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$, there exists a $\Lambda^p_{\mathrm{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$-valued sequence $(\psi_n)_{n=1}^\infty$ such that*

$$\lim_{n\to\infty} \|\psi_n - \psi\|_{\Lambda^p} = 0. \tag{5.17}$$

After we will have prove this the above approximation result we will show how we use it.

Proof of Proposition 5.34. Our aim is to show that any process $\psi \in \Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ can be approximated by a sequence of $(\psi_n)_{n=1}^\infty$ of step processes in $\Lambda^p_{\mathrm{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$, such that (5.17) holds.

For this purpose, let us consider the partition of $[0,T]$,

$$\mathscr{P}_n = \{0 = t_0^n < t_1^n < \cdots < t_{N-1}^n < t_N^n = T\}$$

where $N = N(n)$. Put

$$\mathrm{mesh}(\mathscr{P}_n) := \sup_{0\le k\le N} |t_{k+1}^n - t_k^n|.$$

Let us define the step process ψ_n by

$$\psi_n(t,\omega) = \frac{1}{t_{k+1}^n - t_k^n}\int_{t_k^n}^{t_{k+1}^n} \psi(s)\,ds, \quad \text{for } t \in [t_k^n, t_{k+1}^n),\ \omega \in \Omega.$$

Recall that in (5.3), we have

$$\Lambda^p(0,T;\Pi_p(K,E)) := L^p([0,T]\times\Omega, \mathcal{P}, \mathrm{Leb}\otimes\mathbb{P}; \Pi_p(\mathrm{K},\mathrm{E}))$$

i.e. the space of predictable processes taking values in the Banach space $\Pi_p(\mathrm{K},\mathrm{E})$, such that

$$\mathbb{E}\int_0^T \|\psi(s)\|^p_{\Pi_p(\mathrm{K},\mathrm{E})}\,ds < \infty.$$

Since $\Pi_p(\mathrm{K},\mathrm{E})$ is a Banach space and $\psi \in \Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$, the Bochner integral

$$\int_{t_k^n}^{t_{k+1}^n} \psi(s)\,ds$$

exists in $\Pi_p(\mathrm{K},\mathrm{E})$. Hence, $\psi_n(t,\omega) \in \Pi_p(\mathrm{K},\mathrm{E})$.

As $[t^n_k, t^n_{k+1}) \to 0$, by the Lebesgue Differentiation Theorem (for Banach-valued functions), see Theorem F.34, we have

$$\psi_n(t) \to \psi(t) \text{ in } \Pi_p(\mathrm{K},\mathrm{E}) \text{ for a.e. } t.$$

Moreover, since $\psi \in \Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$, the Bochner-Lebesgue Dominated Convergence Theorem, see Theorem F.35, gives

$$\int_0^T \|\psi_n(s) - \psi\|^p_{\Pi_p(\mathrm{K},\mathrm{E})}\, ds \to 0.$$

Hence,

$$\psi_n \to \psi \text{ in } \Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E})), \quad \text{i.e. } \|\psi_n - \psi\|_{\Lambda^p} \to 0.$$

The proof is complete.

□

Theorem 5.35. *Assume Assumptions 5.4 and 5.5. Then the operator*

$$I : \Lambda^p_{\text{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E})) \to L^p(\Omega,\mathcal{F}_T,\mathbb{P};\mathrm{E}),$$

defined by (5.12)*, is continuous and extends to a unique bounded and linear operator*

$$I : \Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E})) \to L^p(\Omega,\mathcal{F}_T,\mathbb{P};\mathrm{E}).$$

which is also denoted by I.
In particular, if C is the constant from inequality (5.15)*, then for every process $\psi \in \Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$, the Itô integral $I(\psi)$, denoted also by $\int_0^T \psi(s)\, dL(s)$, satisfies the following inequality*

$$\mathbb{E}[\Big\|\int_0^T \psi(s)\, dL(s)\Big\|^p_{\mathrm{E}}] \le C\mathbb{E}[\int_0^T \pi^p_p(\psi(s))\, ds]. \tag{5.18}$$

Theorem 5.35 has been stated and proven in [23, inequality (5.2) and Theorem 5.8].

We now present an extension of Theorem 5.35, which constitutes the main result of this section.

Theorem 5.36. *Under the framework of Proposition 5.34, assume that $p \in (1,2]$, assume also that a process $\psi : [0,T]\times\Omega \to \Pi_p(\mathrm{K},\mathrm{E})$ belongs to $\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$.*

(i) *Assume that* $(\psi_n)_{n=1}^\infty$ *is a* $\Lambda^p_{\text{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$*-valued sequence satisfies condition* (5.17). *Note that for each* $n\in\mathbb{N}$*, we can define a process* $I(\psi_n)=(I_t(\psi_n),\ t\in[0,T])$*, see Definition 5.30, then the sequence of processes* $(I(\psi_n))_{n=1}^\infty$ *is Cauchy in the following sense*

$$\lim_{n,m\to\infty}\mathbb{E}\sup_{t\in[0,T]}\Big[\|I_t(\psi_n)-I_t(\psi_m)\|^p_{\mathrm{E}}\Big]=0. \tag{5.19}$$

(ii) *There exists a unique adapted process* $I(\psi)=(I_t(\psi):t\in[0,T])$ *which has càdlàg* E*-valued trajectories and satisfies*

$$\lim_{n\to\infty}\mathbb{E}\sup_{t\in[0,T]}\Big[\|I_t(\psi_n)-I_t(\psi)\|^p_{\mathrm{E}}\Big]=0. \tag{5.20}$$

for every sequence $(\psi_n)_{n=1}^\infty$ *as in part (i). In particular, the process* $I_t(\psi)$*,* $t\in[0,T]$*, is independent of the approximating sequence* $(\psi_n)_{n=1}^\infty$.

(iii) *The process* $I(\psi)$ *from part (ii) satisfies the following inequality,*

$$\mathbb{E}\sup_{t\in[0,T]}\|I_t(\psi)\|^p\le C\mathbb{E}\int_0^T\pi_p^p(\psi(s))\,ds, \tag{5.21}$$

where the constant $C=(p,\mathrm{K},\mathrm{E})>0$ *is the same constant as in Proposition 5.32.*

(iv) *The process* $I(\psi)$ *from part (ii) is p-integrable martingale.*

Remark 5.37. If $(\psi_n)_{n=1}^\infty$ is a $\Lambda^p_{\text{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$-valued sequence satisfies condition (5.17), then (ψ_n) is Cauchy in Λ^p.

Remark 5.38. In order to have supremum under the expectation, we need $p\in(1,2]$ because of Proposition 5.32 part (vi).

In the following paragraphs, we will prove Theorem 5.36.

Proof of Theorem 5.36. Let us choose and fix $p\in[1,2]$, a process $\psi\in\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ and a $\Lambda^p_{\text{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$-valued sequence $(\psi_n)_{n=1}^\infty$ that satisfies condition (5.17).

For every $n\in\mathbb{N}$, we define a process $(I_t(\psi_n):\ t\in[0,T])$ by Definition 5.30.

Let us first prove that the latter sequence satisfies equality (5.19).

Proof of assertion (5.19). By inequality (5.15) from Proposition 5.32, we infer that

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}\|I_t(\psi_n)-I_t(\psi_m)\|^p_{\mathrm{E}}\Big]&\le C\mathbb{E}\int_0^T\pi_p^p(\psi_n(s)-\psi_m(s))\,ds\\&=C\|\psi_n-\psi_m\|_{\Lambda^p(0,T)}.\end{aligned}$$

By Proposition 5.34, ψ_n is Cauchy in $\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E})$. Therefore,

$$\lim_{n,m\to\infty}\mathbb{E}\Big[\sup_{t\in[0,T]}\|I_t(\psi_n)-I_t(\psi_m)\|^p_{\mathrm{E}}\Big]=0.$$

$\square$

Proof of assertion (5.20). Since $\psi_n\to\psi$ in $\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$, we can find sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ such that

$$\|\psi_{n_k}-\psi\|^p_{\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))}<\frac{1}{10^k},\quad k\in\mathbb{N},$$

where we put

$$\psi_0(s)=0,\quad s\in[0,T].$$

We will write k instead of n_k for brevity.

Note that the series $\sum_{k=1}^{\infty}\frac{1}{10^k}$ is convergent. Since $(a+b)^p\le 2^{p-1}(a^p+b^p)$, for $a,b\ge 0$, $p\ge 1$, and $\pi_p(\cdot)$ is a norm, it follows that

$$\begin{aligned}
&\mathbb{E}\int_0^{\infty}\pi_p^p(\psi_{k+1}(s,\cdot)-\psi_k(s,\cdot))\,ds\\
&\le 2^{p-1}\mathbb{E}\int_0^{\infty}\pi_p^p(\psi_{k+1}(s,\cdot)-\psi(s,\cdot))\,ds+2^{p-1}\mathbb{E}\int_0^{\infty}\pi_p^p(\psi_k(s,\cdot)-\psi(s,\cdot))\,ds\\
&\le 2^p\frac{1}{10^k}\le\frac{4}{10^k},\quad k\in\mathbb{N}.
\end{aligned}$$

By applying the Chebyshev inequality and the Itô inequality (5.15), we infer that for every $k\in\mathbb{N}$,

$$\begin{aligned}
\mathbb{P}\Big\{\sup_{t\in[0,T]}\|I_t(\psi_{k+1}-\psi_k)\|\ge\frac{1}{2^k}\Big\}&\le\frac{1}{(\frac{1}{2^k})^p}\mathbb{E}\Big[\sup_{t\in[0,T]}\|I_t(\psi_{k+1}-\psi_k)\|^p\Big]\\
&\le\frac{C}{(\frac{1}{2^k})^p}\mathbb{E}\int_0^T\pi_p^p(\psi_{k+1}(s)-\psi_k(s))\,ds\\
&\le C(2^p)^k\frac{4}{10^k}\le\frac{C}{2^k}.
\end{aligned}$$

Since the series $\sum_{k=1}^{\infty}\frac{1}{2^k}$ is convergent, we infer that

$$\sum_{k=1}^{\infty}\mathbb{P}\Big\{\sup_{t\in[0,T]}\|I_t(\psi_{k+1}-\psi_k)\|\ge\frac{1}{2^k}\Big\}<\infty. \tag{5.22}$$

Let us put

$$A_k=\Big\{\omega\in\Omega:\sup_{t\in[0,T]}\|(I_t(\psi_{k+1}-\psi_k))(\omega)\|\ge\frac{1}{2^k}\Big\}\Big)$$

with

$$\limsup_k A_k := \bigcap_{n=1}^{\infty}\Big(\bigcup_{k=n}^{\infty} A_k\Big).$$

In the view of the first Borel-Cantelli Lemma, see Proposition D.21, and inequality (5.22), it follows that

$$\mathbb{P}(\limsup_k A_k) = 0.$$

Hence, if

$$\hat{\Omega} = \Omega \setminus \limsup_k A_k,$$

we deduce that $\hat{\Omega} \in \mathcal{F}$ and

$$\mathbb{P}(\hat{\Omega}) = 1.$$

Since

$$\hat{\Omega} = \bigcup_{n=1}^{\infty}\Big(\bigcap_{k=n}^{\infty}(\Omega \setminus A_k)\Big),$$

we infer that, if $\omega \in \hat{\Omega}$, then there exists $n \in \mathbb{N}$, such that for every $k \geq n$,

$$\sup_{t\in[0,T]} \|(I_t(\psi_{k+1} - \psi_k))(\omega)\| < \frac{1}{2^k}.$$

Therefore, we have

$$\|(I_t(\psi_{k+1} - \psi_k))(\omega)\| < \frac{1}{2^k}, \quad t \in [0,T].$$

Hence, if $\omega \in \hat{\Omega}$, then the series

$$\sum_{n=k}^{\infty} (I_t(\psi_{k+1} - \psi_k))(\omega), \quad t \in [0,T],$$

is uniformly absolutely convergent [36, Theorem 7.10, p. 148][2]. Now we define

$$I_t(\psi)(\omega) = \begin{cases} \sum_{k=0}^{\infty} I_t(\psi_{k+1} - \psi_k), & \text{if } \omega \in \hat{\Omega}, \\ 0, & \text{if } \omega \in \Omega \setminus \hat{\Omega}, \end{cases} \qquad t \in [0,T].$$

[2][36, Theorem 7.10, p. 148] Suppose that $\{f_n\}$ is a sequence of functions defined on a set E, $f_n(x) \leq M_n$, where $t \in E$; $n \in \mathbb{N}$; $M_n = \sup_{t\in E} |f_n(x) - \psi(\omega)|$. Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

By the linearity of map I_t for step processes, see Proposition 5.32, we have

$$I_t(\psi)(\omega) = \sum_{k=0}^{\infty} I_t(\psi_{k+1} - \psi_k)(\omega) = \lim_{n\to\infty} \sum_{k=0}^{n-1} I_t(\psi_{k+1} - \psi_k)(\omega) \tag{5.23}$$
$$= \lim_{n\to\infty} I_t\Big[\sum_{k=0}^{n-1}(\psi_{k+1} - \psi_k)\Big](\omega) = \lim_{n\to\infty} I_t(\psi_n)(\omega)$$

Therefore, for every $\omega \in \hat{\Omega}$, $I_t(\psi_n)$ is convergent uniformly with respect to $t \in [0,T]$ to $I_t(\psi)$.

Since for each n, the process $I_t(\psi_n), t \geq 0$ is càdlàg and since the limit of a uniformly convergent sequence of càdlàg-continuous functions preserve the càdlàg-continuity, we deduce $I_t(\psi)$, $t \in [0,T]$ has all càdlàg paths.

After having constructed process $I_t(\psi)$, $t \in [0,T]$, we will show that it satisfies all remaining properties in Theorem 5.36. We begin with the

Proof of assertion (5.20). Again by the Itô inequality (5.15), we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]} \|I_t(\psi_n) - I_t(\psi)\|^p_{\mathrm{E}}\Big] &\leq C\mathbb{E}\int_0^t \pi_p^p(\psi_n(s) - \psi(s))\,ds \\ &\leq C\mathbb{E}\int_0^T \pi_p^p(\psi_n(s) - \psi(s))\,ds \\ &= C\|\psi_n - \psi\|_{\Lambda^p(0,T)}.\end{aligned}$$

By Proposition 5.34, we have

$$\lim_{n\to\infty} \|\psi_n - \psi\|_{\Lambda^p(0,T)} = 0.$$

Therefore,

$$\lim_{n\to\infty} \mathbb{E}\Big[\sup_{t\in[0,T]} \|I_t(\psi_n) - I_t(\psi)\|^p_{\mathrm{E}}\Big] = 0.$$

□

□

We next follow with the

Proof of inequality (5.21). Let us choose and fix $t \in [0,T]$ and $\omega \in \hat{\Omega}$. Then, by (5.24) we have

$$I_t(\psi)(\omega) = \lim_{n\to\infty} I_t(\psi_n)(\omega) \tag{5.24}$$

By taking the expectation, applying the continuity of the norm and using the results that $\lim_{N\to\infty} \|\psi_N\|_{\Lambda^p(0,T)} = \|\psi\|_{\Lambda^p(0,T)}$ from Proposition 5.34, we have

$$\begin{aligned}
\mathbb{E}\|I_t(\omega)\|^p &= \mathbb{E}\|\lim_{N\to\infty} I_t(\psi_N)\|^p = \mathbb{E}\lim_{N\to\infty}\|I_t(\psi_N)\|^p \\
&= \mathbb{E}\liminf_{N\to\infty}\|I_t(\psi_N)\|^p \le \liminf_{N\to\infty}\mathbb{E}\Big\|I_t(\psi_N)\Big\|^p \\
&\le \liminf_{N\to\infty}\mathbb{E}\int_0^t \pi_p^p(\psi_N(s))\,ds = \liminf_{N\to\infty}\|\psi_N\|^p_{\Lambda^p(0,T)} \\
&= \lim_{N\to\infty}\|\psi_N\|^p_{\Lambda^p(0,T)} = \|\psi\|^p_{\Lambda^p(0,T)} = \mathbb{E}\int_0^T \pi_p^p(\psi(s))\,ds.
\end{aligned}$$

Note that the first inequality is due to the inequality (D.3) in the Fatou Lemma, see Corollary D.22, and the second inequality holds by the Itô inequality (5.18). □

Finally, we will present

Proof of part (iv). Let us choose and fix $\psi \in \Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ and $t \in [0,T]$.

Firstly, let us show that the process $(I_t(\psi) : t \ge 0)$ is $\mathcal{F}_t$-adapted.

By Proposition 5.33, the process $\mathbb{1}_{[0,t)}\psi \in \Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$. Hence there exists a sequence $(\psi^n)_{n=1}^\infty$ in $\Lambda^p_{\mathrm{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ approximating $\mathbb{1}_{[0,t)}\psi$, i.e.,

$$\|\psi^n - \mathbb{1}_{[0,t)}\psi\|_{\Lambda^p} \to 0.$$

Then each process $I_t(\psi^n), t \ge 0$ is clearly $\mathcal{F}_t$-adapted.

Moreover, since each $\|I_t(\psi^n) - I_t(\psi)\|_{\Lambda^p} \to 0$, as $n \to \infty$, we can find a subsequence $\psi_{n_k}, n_k \in \mathbb{N}$ such that $\|I_t(\psi^n_k) - I_t(\psi)\|_{\Lambda^p} \to 0$ (almost surely) as $n_k \to \infty$, hence the required results follows.

Finally, let us show the martingale property, i.e.,

$$\mathbb{E}\Big(I_t(\psi)|\mathcal{F}_s\Big) = I_s(\psi) \quad \text{a.s..}$$

For this purpose, let us choose and fix $0 \le s < t \le T$. First, we observe that $I_t(\psi)$ and $I_s(\psi)$ are p-integrable. Moreover we have the following identity

$I_t(\psi) = I(\mathbb{1}_{[0,t]}\psi) = I(\mathbb{1}_{[0,s]}\psi + \mathbb{1}_{(s,t]}\psi) = I(\mathbb{1}_{[0,s]}\psi) + I(\mathbb{1}_{(s,t]}\psi) =: I_s(\psi) + I_{s,t}(\psi)$.

Let us first prove the following two lemmata that we are going to apply.

Lemma 5.39. *If $0 \le s < t \le T$, then the random variable $I(\mathbb{1}_{(s,t]}\psi)$ is independent of the σ-field $\mathcal{F}_s$.*

Proof of Lemma 5.39 . If $0 \leq s < t \leq T$, by Definition 5.41, we have the random variable

$$I(\mathbb{1}_{(0,s]}\psi) = \int_0^T \mathbb{1}_{(0,s]}(r)\psi(r)\, dL(r) \in L^p(\Omega, \mathcal{F}_T, \mathbb{P}; \mathrm{E})$$

generated by the σ-field $\mathcal{F}_{(0,s]} = \sigma(L_r, 0 \leq r \leq s)$, and

$$I(\mathbb{1}_{(s,t]}\psi) = \int_0^T \mathbb{1}_{(s,t]}(r)\psi(r)\, dL(r) \in L^p(\Omega, \mathcal{F}_T, \mathbb{P}; \mathrm{E})$$

generated by the σ-field $\mathcal{F}_{(s,t]} = \sigma(L_{r_2} - L_{r_1}, 0 < s < r_1 < r_2 \leq t)$.

Let us observe that because the increments of our process are independent, the σ-fields $\mathcal{F}_{(0,s]}$ and $\mathcal{F}_{(s,t]}$ are independent as well. Therefore $I(\mathbb{1}_{(0,s]}\psi)$ is independent of $\mathcal{F}_{(s,t]}$, and $I(\mathbb{1}_{(s,t]}\psi)$ is independent of $\mathcal{F}_{(0,s]}$. □

Lemma 5.40. *The process $I(\psi)$ is $\mathbb{F}$-adapted.*

Proof of Lemma 5.40. Let us choose and fix a step process $\psi \in \Lambda^p_{\mathrm{step}}(0, T; \Pi_p(\mathrm{K}, \mathrm{E}))$ and $t \in [0, T]$. Without loss of generality we can assume that the partition corresponding to ψ contains number t, i.e., is of the following form

$$0 = t_0 < t_1 < \cdots < t_k = t < t_{k+1} < \cdots < t_n = T.$$

Thus the stochastic integral of the process ψ is a stochastic process $I(\psi) = (I_t(\psi) : t \geq 0)$ such that

$$I_t(\psi) = I(\mathbb{1}_{[0,t]}\psi) = \sum_{k=0}^{m-1} \psi_k\big(L(t_{k+1}) - L(t_k)\big), \quad t \geq 0.$$

Let us observe that every random variable ψ_k and $L(t_{k+1}) - L(t_k)$ appearing in the formula above is $\mathcal{F}_t$-measurable. In other words, $I_t(\psi)$ is generated by the σ-field $\mathcal{F}_{[0,t]}$. Therefore, by Definition 2.52, the process $I(\psi)$ is $\mathbb{F}$-adapted. □

Let us continue the proof of part (iv) of Theorem 5.36. Using the above two lemmata, we infer that

$$\mathbb{E}\Big(I_t(\psi)|\mathcal{F}_s\Big) = \mathbb{E}\Big(I_s(\psi) + \mathbb{1}_{s,t}(\psi)|\mathcal{F}_s\Big) = \mathbb{E}\Big(I_s(\psi)|\mathcal{F}_s\Big) + \mathbb{E}\Big(I_{s,t}(\psi)|\mathcal{F}_s\Big) = I_s(\psi) + \mathbb{E}(I_{s,t}(\psi)) = I_s(\psi).$$

which shows martingale property for step function. For each n, we have

$$\mathbb{E}\Big(I_{s,t}(\psi)|\mathcal{F}_s\Big) = I_s(\psi^n).$$

Next, by applying the Jensen's inequality, we have

$$\begin{aligned}
\mathbb{E}\Big\|I_s(\psi) - \mathbb{E}\Big(I_t(\psi)|\mathcal{F}_s\Big)\Big\|^p &= \mathbb{E}\Big\|I_s(\psi) - I_s(\psi^n) + \mathbb{E}\Big(I_t(\psi^n)|\mathcal{F}_s\Big) - \mathbb{E}\Big(I_t(\psi)|\mathcal{F}_s\Big)\Big\|^p \\
&\leq 2^p\mathbb{E}\Big\|I_s(\psi) - I_s(\psi^n)\Big\|^p + 2^p\mathbb{E}\Big\|\mathbb{E}\Big(I_t(\psi^n)|\mathcal{F}_s\Big) - \mathbb{E}\Big(I_t(\psi)|\mathcal{F}_s\Big)\Big\|^p \\
&= 2^p\mathbb{E}\|I_s(\psi) - I_s(\psi^n)\|^p + 2^p\mathbb{E}\Big\|\mathbb{E}\Big(I_t(\psi^n) - I_t(\psi)|\mathcal{F}_s\Big)\Big\|^p \\
&\leq 2^p\mathbb{E}\|I_s(\psi) - I_s(\psi^n)\|^p + 2^p\mathbb{E}\Big(\mathbb{E}\Big(\|I_t(\psi^n) - I_t(\psi)\|^p|\mathcal{F}_s\Big)\Big) \\
&= 2^p\mathbb{E}\|I_s(\psi) - I_s(\psi^n)\|^p + 2^p\mathbb{E}\Big(\|I_t(\psi^n) - I_t(\psi)\|^p\Big)
\end{aligned}$$

where the constant 2 comes from the following inequality:

$$(a+b)^p \leq 2^P(a^p + b^p), \quad \text{where } a, b \geq 0,\ p = 1, 2, 3, \cdots .$$

Since for every $t \in [0, \infty]$,

$$\lim_{n\to\infty} \mathbb{E}\|I_t(\psi^n) - I_t(\psi)\|^p = 0.$$

We infer that

$$\lim_{n\to\infty} \mathbb{E}\Big\|I_s(\psi) - \mathbb{E}\Big(I_t(\psi)|\mathcal{F}_s\Big)\Big\|^p = 0.$$

Therefore

$$\mathbb{E}\Big(I_t(\psi)|\mathcal{F}_s\Big) = I_s(\psi) \quad \text{a.s..}$$

Hence we have proved part (iv). □

Hence, the proof of Theorem 5.36 is complete. □

Now we will introduce the useful and very important notation.

Definition 5.41. *Assume that $\psi \in \Lambda^p(0, T; \Pi_p(\mathrm{K}, \mathrm{E}))$. We define*

$$\int_0^t \psi(s)\, dL(s) := I_t(\psi), \quad t \in [0, T].$$

We conclude this section by summarizing the key properties of the stochastic Itô integral, defined in Definition 5.41. In the following theorem we use just introduced symbol $\int_0^t \psi(s)\, dL(s)$ of the stochastic integral.

Theorem 5.42. *Assume that $\psi, \eta \in \Lambda^p(0, T; \Pi_p(\mathrm{K}, \mathrm{E}))$ and $\alpha, \beta \in \mathbb{R}$. Let us denote by $\int_0^t \psi(s)\, dL(s)$, $\int_0^t \eta(s)\, dL(s)$ and $\int_0^t (\alpha\psi(s) + \beta\eta(s))\, dL(s)$ the corresponding processes constructed in Theorem 5.36.*

(i) *For every $t \geq 0$, $\mathbb{P}$-almost surely.*

$$\int_0^t (\alpha\psi(s) + \beta\eta(s))\, dL(s) = \alpha \int_0^t \psi(s)\, dL(s) + \beta \int_0^t \eta(s)\, dL(s),$$

(ii) *For every $t \geq 0$, $\mathbb{E}(\int_0^t \psi(s)\, dL(s)) = 0$.*

(iii) *$\int_0^t \psi(s)\, dL(s) : t \geq 0$ is a p-integrable martingale and*

$$\mathbb{E} \sup_{t\in[0,T]} \| \int_0^t \psi(s)\, dL(s)\|^p \leq C\mathbb{E} \int_0^T \pi_p^p(\psi(s))\, ds,$$

(iv) *$\int_0^t \psi(s)\, dL(s) : t \geq 0$ is càdlàg. More precisely, $\int_0^t \psi(s)\, dL(s)$ has a modification which has càdlàg trajectories.*

Proof of Theorem 5.42. In view of Definition 5.41, this result is an obvious consequence of Theorem 5.36.

□

6. Stochastic Fubini Theorem Applied to a Generalised Cylindrical Lévy Process

In this section, let us fix a number $p \in [1, 2]$. Suppose that K, E are separable Banach spaces, and E is of martingale type p, see Definition 2.94. Let us consider a filtered probability measure space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and a cylindrical Lévy process L on K, see Definition 3.36, which has finite p-th weak moments. Finally let us assume that $(O, \mathscr{O}, \mu)$ is a σ-finite measure space, see Definition 2.67.

Definition 6.1. *Let $(O, \mathscr{O})$ be a measurable space and $\mathcal{P}$ be a predictable σ-field, see Definition 2.80. By $\mathcal{P} \otimes \mathscr{O}$, we denote the smallest σ-field of subsets of $[0, T] \times \Omega \times O$, containing sets of the following form*

$$A \times B : \ A \in \mathcal{P}, \ B \in \mathscr{O}.$$

In other words, $\mathcal{P} \otimes \mathscr{O}$ is the σ-field generated by $\{A \times B : (A, B) \in \mathcal{P} \times \mathscr{O}\}$.

Example 6.2. [39, Section 1] Let $\mathcal{A}$ be the Borel σ-field on $\mathbb{R}$, and let C also be the Borel σ-field on $\mathbb{R}$, i.e.

$$\mathcal{A} = C = \mathcal{B}(\mathbb{R}).$$

We define $\mathcal{A} \otimes C$ the smallest σ-field of subsets of $\mathbb{R} \times \mathbb{R}$ that contains all sets of the form

$$A \times B : \ A \in \mathcal{A}, \ B \in C.$$

It can be shown that $\mathcal{A} \otimes C$ is equal to the Borel σ field on $\mathbb{R}^2$, i.e.,

$$\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2).$$

The following result about product measure is standard, see [3, Section 2.6] [20, Section 1.10, p.23-24].

Theorem 6.3. *There exists a unique measure, denoted by* $\mathrm{Leb} \otimes \mathbb{P}$*, defined on the σ-field $\mathcal{P}$ such that*

$$(\mathrm{Leb} \otimes \mathbb{P})(A \times B) = \mathrm{Leb}(A)\, \mathbb{P}(B),$$

for all measurable sets $A \times B \in \mathcal{P}$.

Definition 6.4. *Assume that* K,E *are separable Banach spaces. The spaces* $\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ *and* $\Lambda^p_{\text{step}}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ *are defined in Definitions 5.18 and 5.22. We call a process* ψ *a simple function if there exists a finite sequence of numbers* $n \in \mathbb{N}\,(0 \le k \le n \le N)$, $0 = t^n_0 < t^n_1 < \cdots < t^n_k = \frac{kT}{n} \le \cdots \le t^n_N$ *such that*

$$\psi(t,\omega) = \sum_{k=0}^{N-1}\sum_{i=1}^{m_k} \psi^n_{k,i}\mathbb{1}_{A^n_{k,i}}(\omega)\mathbb{1}_{(t^n_k,t^n_{k+1}]}(t),$$

for some disjoint sets $A_{k,1},\cdots,A_{k,m_k} \in \mathcal{F}_{t_k}$ *and* $\psi_{k,1},\cdots,\psi_{k,m_k} \in \Pi_p(\mathrm{K},\mathrm{E})$ *for all* $k \in \{0,\cdots,N\}$. *We use* $\Lambda^p_{ss}(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ *to denote the set of all such simple step functions.*

Lemma 6.5. *Let* $(O,\mathscr{O},\mu)$ *be a* σ*-finite measure space. Suppose that a function*

$$\psi : [0,T]\times\Omega\times O \to \Pi_p(\mathrm{K},\mathrm{E})$$

is a $\mathcal{P}\otimes\mathscr{O}/\mathcal{B}_s(\Pi_p(\mathrm{K},\mathrm{E}))$*-measurable such that*

$$\int_O \left(\mathbb{E}\int_0^T \pi_p^p(\psi(t,\cdot,y))\,ds\right)^{\frac{1}{p}} \mu(dy) < \infty. \tag{6.1}$$

Such a function ψ *will be called below a "generalised process".*

Then there exists a sequence $\{\psi^n\}_{n=1}^\infty$ *of simple step functions of the form*

$$\psi^n(t,\omega,y) = \sum_{k=0}^{N-1}\sum_{i=1}^{m_k}\sum_{l=1}^{m_k} \psi^n_{i,l,k}\mathbb{1}_{A^n_{i,l,k}}(\omega)\mathbb{1}_{(t^n_k,t^n_{k+1}]}(t)\mathbb{1}_{B^n_{i,l,k}}(y), \tag{6.2}$$

where for each $n \in \mathbb{N}\,(0 \le k \le n \le N)$, $0 = t^n_0 < t^n_1 < \cdots < t^n_k = \frac{kT}{n} \le \cdots \le t^n_N$*; for some disjoint sets* $A_{1,1,k},\cdots,A_{i,l,m_k} \in \mathcal{F}_{t_k}$, $B_{1,1,k},\cdots,B_{i,l,m_k} \in \mathscr{O}_{t_k}$ *and* $\psi_{k,1},\cdots,\psi_{k,m_k} \in \Pi_p(\mathrm{K},\mathrm{E})$ *for all* $k \in \{0,\cdots,N\}$*, such that*

$$\lim_{n\to\infty}\int_O \left(\mathbb{E}\int_0^T \pi_p^p(\psi^n(t,\cdot,y)-\psi(t,\cdot,y))\,ds\right)^{\frac{1}{p}} \mu(dy) = 0. \tag{6.3}$$

The main result of this section is Theorem 6.6, which provides a generalisation of the stochastic Fubini Theorem from a recent paper [48].

Theorem 6.6. *Suppose that a generalised process*

$$\psi : [0,T]\times\Omega\times\mathscr{O} \to \Pi_p(\mathrm{K},\mathrm{E})$$

is a $\mathcal{P}\otimes\mathscr{O}/\mathcal{B}_s(\Pi_p(\mathrm{K},\mathrm{E}))$*-measurable and satisfies condition* (6.1)*. Then the following holds:*

(1) *There exists a measurable set $O' \subset O$ such that* $\mathrm{Leb}(O \setminus O') = 0$ *and for every* $y \in O'$,

$$\mathbb{E} \int_0^T \pi_p^p(\psi(s, \cdot, y))\, ds < \infty, \tag{6.4}$$

and the process

$$\psi_y := \psi(\cdot, \cdot, y) : [0, T] \times \Omega \ni (t, \omega) \mapsto \psi(t, \omega, y) \in \Pi_p(\mathrm{K}, \mathrm{E}) \tag{6.5}$$

belongs to the space $\Lambda^p(0, T; \Pi_p(\mathrm{K}, \mathrm{E}))$.

In particular, by Theorem 5.36, there exists a process $I(\psi_y) = (I(\psi_y)(t) : t \in [0, T])$

$$[I(\psi_y)](t) := \int_0^t \psi_y(s)\, dL(s), \quad t \in [0, T].$$

(2) *The generalised stochastic process,*

$$[0, T] \times \Omega \times O \ni (t, \omega, y) \mapsto \int_0^t \psi(s, \omega, y)\, dL(s) \in \mathrm{E}$$

is $\mathcal{F}_T \otimes \mathscr{O} / \mathcal{B}_s(\Pi_p(\mathrm{K}, \mathrm{E}))$*-measurable and belongs to* $L^1(\Omega \times O, \mathcal{F}_T, \mathbb{P}; \mathrm{E})$.

Moreover, for $\mathbb{P}$-almost all $\omega \in \Omega$, the function

$$[0, T] \times O \ni (s, y) \mapsto \int_0^t \psi(s, \omega, y)\, dL(s) \in \mathrm{E}$$

is Bochner integrable.

Moreover,

$$\begin{aligned} &\int_O \left(\mathbb{E} \sup_{t \in [0,T]} \| \int_0^t \psi(s, \omega, y)\, dL(s) \|_{\mathrm{E}}^p\, ds \right)^{\frac{1}{p}} \mu(dy) \\ &\qquad \leq C \int_O \left(\mathbb{E} \int_0^T \pi_p^p(\psi(s, \omega, y))\, ds \right)^{\frac{1}{p}} \mu(dy), \end{aligned} \tag{6.6}$$

where $C > 0$ is independent of ψ.

(3) *The process*

$$\psi : [0, T] \times \Omega \ni (s, \omega) \mapsto \int_O \psi(s, \omega, y)\, \mu(dy) \in \Pi_p(\mathrm{K}, \mathrm{E}) \tag{6.7}$$

is $\mathcal{P}$-measurable (i.e., predictable), and

$$\mathbb{E} \int_0^T \left\| \int_O \psi(s, \omega, y)\, \mu(dy) \right\|_{\Pi_p(\mathrm{K},\mathrm{E})}^p ds < \infty. \tag{6.8}$$

Moreover, for (Leb $\otimes \mathbb{P}$)*-almost all* $(t, \omega) \in [0, T] \times \Omega$ [3]*, the function*

$$\psi_{t,\omega} := \psi(t, \omega, \cdot) : O \ni y \mapsto \psi(t, \omega, y) \in \Pi_p(\mathrm{K}, \mathrm{E})$$

is Bochner integrable.

(4) *The following identity is satisfied for every $t \in [0, T]$, $\mathbb{P}$ almost surely:*

$$\int_O \left(\int_0^t \psi(s, y)\, dL(s) \right) d\mu(y) = \int_0^t \left(\int_O \psi(s, y)\, d\mu(y) \right) dL(s). \tag{6.9}$$

Note that the process $\int_0^t \psi(s, y)\, dL(s)$ exists in view of part (1) and the function $O \ni y \mapsto \int_0^t \psi(s, y)\, dL(s) \in$ is Bochner integrable for almost all $\omega \in \Omega$ by part (2). The RHS makes sense because of part (3).

Proof of Theorem 6.6. Let us choose and fix a $\mathcal{P} \otimes \mathscr{O} / \mathcal{B}_s(\Pi_p(\mathrm{K}, \mathrm{E}))$-measurable process $\psi : [0, T] \times \Omega \times \mathscr{O} \to \Pi_p(\mathrm{K}, \mathrm{E})$ satisfying condition (6.1). Note that the spaces $(O, \mathscr{O}, \mu)$ and $([0, T] \times \Omega, \mathcal{P}, \mathrm{Leb} \otimes \mathbb{P})$ are both σ-finite measure spaces, see Definition 5.18.

Proof of part (1). Let us recall, see [20, Theorem 6.1.4, p. 120], that if the integral of a non-negative function is finite, then the function itself is finite almost everywhere. Hence, in view of assumption (6.1), we infer that

$$\left(\mathbb{E} \int_0^T \|\psi(s, \cdot, y)\|_p^p\, ds \right)^{\frac{1}{p}} < \infty \quad \text{for } \mu - a.a.\ y \in O.$$

This obviously implies that

$$\mathbb{E} \int_0^T \pi_p^p(\psi(s, \cdot, y))\, ds < \infty \quad \text{for } \mu - a.a.\ y \in O.$$

and (6.4) follow.

Next, we apply Fubini's theorem, see Theorem D.19, to the function

$$(\omega, y) \mapsto \int_0^T \pi_p^p(\psi(s, \omega, y))\, ds,$$

defined on the product measure space $(\Omega \times O, \mathcal{F}_T \otimes \mathcal{O}, \mathbb{P} \otimes \mu)$.

The measurability of this function follows from the $\mathcal{P} \otimes \mathcal{O}$-measurability of ψ, and its integrability is ensured by (6.4). Therefore, Fubini's theorem yields

$$\int_\Omega \int_O \int_0^T \pi_p^p(\psi(s, \omega, y))\, ds\, \mu(dy)\, d\mathbb{P} = \int_O \int_\Omega \int_0^T \pi_p^p(\psi(s, \omega, y))\, ds\, d\mathbb{P}\, \mu(dy).$$

[3] In this case, it means that there exits a set $A \in \mathcal{P}$ such that $(\mathrm{Leb} \otimes \mathbb{P})([0, T] \times \Omega \setminus A) = 0$ and for every $(t, \omega) \in A$, the function $O \ni y \mapsto \psi(t, \omega, y) \in \Pi_p(\mathrm{K}, \mathrm{E})$ is Bochner integrable.

Hence,

$$\left(\int_\Omega \int_O \int_0^T \pi_p^p(\psi(s,\cdot,y))\,ds\,\mu(dy)\,d\mathbb{P}\right)^{\frac{1}{p}} = \left(\int_O \int_\Omega \int_0^T \pi_p^p(\psi(s,\cdot,y))\,ds\,d\mathbb{P}\,\mu(dy)\right)^{\frac{1}{p}} \leq \cdots$$

By the Minkowski inequality for integrals, see Definition D.18, we have

$$\cdots \leq \int_O \left(\int_\Omega \int_0^T \pi_p^p(\psi(s,\cdot,y))\,ds\,d\mathbb{P}\right)^{\frac{1}{p}} \mu(dy) = \cdots$$

Using the definition of expectation and assumption (6.1), we obtain

$$\cdots = \int_O \left(\mathbb{E}\int_0^T \pi_p^p(\psi(s,\cdot,y))\,ds\right)^{\frac{1}{p}} \mu(dy) < \infty.$$

Hence, we have

$$\left(\int_\Omega \int_O \int_0^T \pi_p^p(\psi(s,\omega,y))\,ds\,\mu(dy)\,d\mathbb{P}\right)^{\frac{1}{p}} = \left(\iint_{\Omega\times O} \int_0^T \pi_p^p(\psi(s,\omega,y))\,ds\,d(\mu\otimes\mathbb{P})\right)^{\frac{1}{p}} < \infty.$$

Therefore, using [20, Theorem 6.1.4, p. 120], we infer that

$$\int_0^T \pi_p^p(\psi(s,\omega,y))\,ds < \infty \quad \text{for } \mu\otimes\mathbb{P} - a.a.\,(\omega,y) \in \Omega\times O.$$

Furthermore, by applying the Minkowski inequality for integrals again, we get

$$\iint_{\Omega\times O}\left(\int_0^T \pi_p(\psi(s,\omega,y))\,ds\right)^p d(\mu\otimes\mathbb{P}) \leq \iint_{\Omega\times O}\int_0^T \pi_p^p(\psi(s,\omega,y))\,ds\,d(\mu\otimes\mathbb{P}) < \infty.$$

Therefore, equation (6.4) is satisfied.

Hence, the process ψ_y belongs to the space $\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ and (6.5) is proved. □

Proof of part (2). By Theorem 5.35, we have

$$\mathbb{E}\left\|\int_0^T \psi(s,y)\,dL(s)\right\|_{\Pi_p(\mathrm{K},\mathrm{E})}^p \leq C\mathbb{E}\int_0^T \|\psi(s,y)\|_{\Pi_p(\mathrm{K},\mathrm{E})}^p\,ds < \infty. \tag{6.10}$$

Hence

$$\int_0^T \psi(s,\omega,y)\,dL(s) < \infty \quad \text{for } \mu\otimes\mathbb{P} - a.a.\,(\omega,y) \in \Omega\times O.$$

Hence $\int_0^T \psi(s,\omega,y)\,dL(s)$ exists.

Hence

$$\psi : \Omega\times O \ni (\omega,y) \mapsto \int_0^T \psi(t,\omega,y)\,dL(s) \in \Pi_p(\mathrm{K},\mathrm{E})$$

is $\mathcal{F}_T\otimes O/\mathcal{B}_s(\Pi_p(\mathrm{K},\mathrm{E}))$-measurable, and this function belongs to $L^1(\Omega\times O,\mathcal{F}_T,\mathbb{P};\mathrm{E})$.

Moreover, from inequality (6.10) and by applying [20, Theorem 6.1.1, p. 117], we have

$$\int_O \left(\mathbb{E} \left\| \int_0^T \psi(s,\omega,y)\, dL(s) \right\|^p_{\Pi_p(\mathrm{K},\mathrm{E})} \right)^{\frac{1}{p}} d\mu(y) \leq C \int_O \left(\mathbb{E} \int_0^T \|\psi(s,\omega,y)\|^p_{\Pi_p(\mathrm{K},\mathrm{E})}\, ds \right)^{\frac{1}{p}} d\mu(y) < \infty.$$

Note that the last inequality holds by assumption (6.1).

Now, applying the Minkowski inequality for integrals, we have

$$\begin{aligned}
\left\| \int_O \mathbb{E} \int_0^T \psi(s,\omega,y)\, dL(s)\, d\mu(y) \right\|_{\mathrm{E}} &= \left(\int_O \left\| \mathbb{E} \int_0^T \psi(s,\omega,y)\, dL(s) \right\|^p_{\mathrm{E}} d\mu(y) \right)^{\frac{1}{p}} \\
&\leq \left(\int_O \mathbb{E} \left\| \int_0^T \psi(s,\omega,y)\, dL(s) \right\|^p_{\mathrm{E}} d\mu(y) \right)^{\frac{1}{p}} \\
&\leq \int_O \left(\mathbb{E} \left\| \int_0^T \psi(s,\omega,y)\, dL(s) \right\|^p_{\mathrm{E}} \right)^{\frac{1}{p}} d\mu(y) < \infty.
\end{aligned}$$

Thus, we have

$$\left\| \int_O \mathbb{E} \int_0^T \psi(s,\omega,y)\, dL(s)\, d\mu(y) \right\|_{\mathrm{E}} < \infty.$$

Hence, by the definition of L^p functions, see Definition 2.14 , we infer that

$$\int_O \mathbb{E} \int_0^T \psi(s,\omega,y)\, dL(s)\, d\mu(y) = \mathbb{E} \int_O \int_0^T \psi(s,\omega,y)\, dL(s)\, d\mu(y) < \infty.$$

Therefore, for $\mathbb{P}$-a.a. $\omega \in \Omega$, the function

$$\psi_y := \psi(\cdot, y) : O \ni y \mapsto \psi(\omega, y) \in \Pi_p(\mathrm{K},\mathrm{E})$$

is Bochner integrable.

So the LHS of (6.9), $\int_O \int_0^T \psi(s,\omega,y)\, dL(s)$ exists.

□

Proof of part (3). By the definition of expectation, we have

$$\mathbb{E} \int_0^T \left(\int_O \pi_p(\psi(s,y))\, \mu(dy) \right)^p ds = \int_\Omega \int_0^T \left(\int_O \pi_p(\psi(s,y))\, \mu(dy) \right)^p ds\, d\mathbb{P} = \cdots$$

Applying the Fubini Theorem, we have

$$\cdots = \iint_{[0,T]\times\Omega} \left(\int_O \pi_p(\psi(s,y))\, \mu(dy) \right)^p d(\mathrm{Leb} \otimes \mathbb{P}).$$

Then, by assumption (6.1), we infer that

$$\iint_{[0,T]\times\Omega} \left(\int_O \pi_p(\psi(s,y))\, \mu(dy) \right)^p d(\mathrm{Leb} \otimes \mathbb{P}) < \infty.$$

Again, by (6.1.9) in [20, Theorem 6.1.3, p. 119], we infer that almost everywhere w.r.t. $\mathrm{Leb}\otimes\mathbb{P}$:

$$\left(\int_O \pi_p(\psi(s,y))\,\mu(dy)\right)^p < \infty.$$

Thus, we have

$$\int_O \pi_p(\psi(s,y))\,\mu(dy) < \infty \quad \text{for} \quad a.a.\,(s,\omega)\in[0,T]\times\Omega \quad \text{w.r.t.} \quad \mathrm{Leb}\otimes\mathbb{P}.$$

Hence, the process

$$\psi : [0,T]\times\Omega \ni (s,\omega) \mapsto \int_O \psi(s,\omega,y)\,\mu(dy)$$

is $\mathcal{P}$-measurable (i.e., predictable). Hence, (6.7) is proved.

Now, applying Jensen's inequality, we have

$$\left\|\int_O \psi(s,\omega,y)\,\mu(dy)\right\|_{\Pi_p(\mathrm{K},\mathrm{E})} \le \int_O \|\psi(s,\omega,y)\|_{\Pi_p(\mathrm{K},\mathrm{E})}\,\mu(dy)$$

Thus

$$\pi_p(\psi(s,\omega)) \le \int_O \pi_p(\psi(s,\omega,y))\,\mu(dy)$$

Moreover,

$$\mathbb{E}\int_0^T \pi_p^p(\psi(s,\omega))\,ds \le \mathbb{E}\int_0^T \left(\int_O \pi_p(\psi(s,\omega,y))\,\mu(dy)\right)^p ds < \infty.$$

The last inequality follows from the results of (6.11) below.

By the definition of expectation, the Minkowski inequality for integrals, and again using the definition of expectation, we obtain, by the assumption (6.1), that

$$\begin{aligned}\left(\mathbb{E}\int_0^T \left(\int_O \pi_p(\psi(s,y))\,\mu(dy)\right)^p ds\right)^{\frac{1}{p}} &= \left(\int_\Omega\int_0^T \left(\int_O \pi_p(\psi(s,\omega,y))\,\mu(dy)\right)^p ds\,\mathbb{P}(d\omega)\right)^{\frac{1}{p}}\\ &\le \int_O \left(\int_\Omega\int_0^T \pi_p^p(\psi(s,\omega,y))ds\,\mathbb{P}(d\omega)\right)^{\frac{1}{p}} \mu(dy)\\ &= \int_O \left(\mathbb{E}\int_0^T \pi_p^p(\psi(s,\cdot,y))\,ds\right)^{\frac{1}{p}} \mu(dy) < \infty.\end{aligned}$$

Hence, we infer that

$$\mathbb{E}\int_0^T \left(\int_O \pi_p(\psi(s,\omega,y))\,\mu(dy)\right)^p ds < \infty. \tag{6.11}$$

Therefore, equation (5.2) is satisfied and (6.8) is proved.

Hence, the process $\psi(s,\omega)$ belongs to $\Lambda^p(0,T;\Pi_p(\mathrm{K},\mathrm{E}))$ and $\psi(s,\omega,y)\in\Pi_p(\mathrm{K},\mathrm{E})$.

By Theorem 5.35, inequality (5.18) holds, i.e.

$$\mathbb{E}\left(\Big\| \int_0^T \psi(s,\omega)\, dL(s)\Big\|^p\right) \le C\left(\mathbb{E}\int_0^T \pi_p^p(\psi(s,\omega))\, ds\right)^{\frac{1}{p}} < \infty.$$

Hence, the Itô integral $\int_0^T \psi(s,\omega)\, dL(s)$ exists.
Hence, the RHS of equation (6.9), $\int_0^T \left(\int_O \psi(s,y)\, d\mu(y)\right) dL(s)$, exists.

□

Proof of part (4). It is sufficient to prove that equality (6.9) holds for $t = T$. To show equality (6.9), let us choose and fix a generalized process ψ satisfying condition (6.1). We will consider a sequence $(\psi^n)_{n=1}^\infty$ of simple step processes as described in Lemma 6.5.

Step 1 We will show here that equality (6.9) holds for each simple step process ψ^n. Indeed, the following equality

$$-\int_O \int_0^T \psi^n(s,y)\, dL(s)\, d\mu(y) + \int_0^T \int_O \psi^n(s,y)\, d\mu(y)\, dL(s) = 0$$

follows from the Fubini Theorem for the simple step processes.

Step 2 We will show here that equality (6.9) holds for each simple step process ψ. Let us observe that

$$\begin{aligned}
\mathbb{E}\Big\| &\int_O \int_0^T \psi(s,y)\, dL(s)\, d\mu(y) - \int_0^T \int_O \psi(s,y)\, d\mu(y)\, dL(s)\Big\| \\
&= \mathbb{E}\Big\| \int_O \int_0^T \psi(s,y)\, dL(s)\, d\mu(y) - \int_O \int_0^T \psi^n(s,y)\, dL(s)\, d\mu(y) \\
&\qquad + \int_0^T \int_O \psi^n(s,y)\, d\mu(y)\, dL(s) - \int_0^T \int_O \psi(s,y)\, d\mu(y)\, dL(s)\Big\| \le \cdots
\end{aligned}$$

Note that here the 3rd, 4th terms are equal by the Fubini Theorem, i.e.

$$-\int_O \int_0^T \psi^n(s,y)\, dL(s)\, d\mu(y) + \int_0^T \int_O \psi^n(s,y)\, d\mu(y)\, dL(s) = 0$$

Next, we apply the Minkowski inequality for integrals,

$$\begin{aligned}
\cdots \le &\mathbb{E}\Big\| \int_O \int_0^T \psi(s,y)\, dL(s)\, d\mu(y) - \int_O \int_0^T \psi^n(s,y)\, dL(s)\, d\mu(y)\Big\| \\
&+ \mathbb{E}\Big\| \int_0^T \int_O \psi^n(s,y)\, d\mu(y)\, dL(s) - \int_0^T \int_O \psi(s,y)\, d\mu(y)\, dL(s)\Big\| = \cdots
\end{aligned}$$

By the linear property of expectation,

$$\begin{aligned}\cdots =&\mathbb{E}\Big\| \int_O \int_0^T (\psi(s,y) - \psi^n(s,y))\ dL(s)\,d\mu(y)\Big\| \\ &+ \mathbb{E}\Big\| \int_0^T \int_O (\psi^n(s,y) - \psi(s,y))\ d\mu(y)\,dL(s)\Big\| = \cdots\end{aligned}$$

By the definition of expectation, this becomes

$$\begin{aligned}\cdots = &\int_\Omega \Big\| \int_O \int_0^T (\psi(s,y) - \psi^n(s,y))\ dL(s)\,d\mu(y)\Big\| d\mathbb{P}(\omega) \\ &+ \mathbb{E}\Big\| \int_0^T \int_O (\psi^n(s,y) - \psi(s,y))\ d\mu(y)\,dL(s)\Big\| = \cdots\end{aligned}$$

By applying the Fubini Theorem again, we get

$$\begin{aligned}\cdots = &\int_O \Big\| \int_\Omega \int_0^T (\psi(s,y) - \psi^n(s,y))\ dL(s) d\mathbb{P}(\omega)\Big\|\, d\mu(y) \\ &+ \mathbb{E}\Big\| \int_0^T \int_O (\psi^n(s,y) - \psi(s,y))\ d\mu(y)\,dL(s)\Big\| = \cdots\end{aligned}$$

Now, by the definition of expectation again, we can write

$$\begin{aligned}\cdots = &\int_O \Big\|\mathbb{E} \int_0^T (\psi(s,y) - \psi^n(s,y))\ dL(s)\Big\|\, d\mu(y) \\ &+ \mathbb{E}\Big\| \int_0^T \int_O (\psi^n(s,y) - \psi(s,y))\ d\mu(y)\,dL(s)\Big\| \le \cdots\end{aligned}$$

Applying Jensen's inequality, we have

$$\begin{aligned}\cdots \le &\int_O \mathbb{E}\Big\| \int_0^T (\psi(s,y) - \psi^n(s,y))\ dL(s)\Big\|\, d\mu(y) \\ &+ \left(\mathbb{E}\Big\| \int_0^T \int_O (\psi^n(s,y) - \psi(s,y))\ d\mu(y)\,dL(s)\Big\|^p\right)^{\frac{1}{p}} \le \cdots\end{aligned}$$

Using (5.21), we obtain

$$\begin{aligned}\cdots \le &\int_O \mathbb{E}\Big\| \int_0^T (\psi(s,y) - \psi^n(s,y))\ dL(s)\Big\|\, d\mu(y) \\ &+ \left(C\mathbb{E} \int_0^T \Big\| \int_O (\psi^n(s,y) - \psi(s,y))\ d\mu(y)\Big\|^p\, dL(s)\right)^{\frac{1}{p}} \le \cdots\end{aligned}$$

Again, applying (5.21), we obtain

$$\cdots \leq \int_O \mathbb{E} \int_0^T \pi_p(\psi(s,y) - \psi^n(s,y))\, dL(s)\, d\mu(y)$$
$$+ \left(C\mathbb{E} \int_O \int_0^T \pi_p^p(\psi^n(s,y) - \psi(s,y))\, dL(s) \right)^{\frac{1}{p}} d\mu(y) \to 0.$$

Finally, the expression goes to zero by Lemma 6.5.

Summing up, we proved

$$\mathbb{E}\Big\| \int_O \int_0^T \psi(s,y)\, dL(s)\, d\mu(y) - \int_0^T \int_O \psi(s,y)\, d\mu(y)\, dL(s) \Big\| = 0.$$

Therefore,

$$\int_O \int_0^T \psi(s,y)\, dL(s)\, d\mu(y) = \int_0^T \int_O \psi(s,y)\, d\mu(y)\, dL(s) \quad \mathbb{P} - a.s.$$

and so equality (6.9) is satisfied. □

Hence, the proof of Theorem 6.6 is complete. □

6.1. **An alternative proof of Theorem 6.6. Step 1** Let us choose and fix a generalised $\mathcal{P} \otimes \mathscr{O} / \mathcal{B}_s(\Pi_p(\mathrm{K}, \mathrm{E}))$-measurable process

$$\psi : [0,T] \times \Omega \times \mathscr{O} \to \Pi_p(\mathrm{K}, \mathrm{E})$$

which satisfies condition (6.1).

Step 2 Let us choose, as in Lemma 6.5, a sequence $\{\psi^n\}_{n=1}^\infty$ of simple step functions of the form (6.2) such that (6.3) holds.

For every $n \in \mathbb{N}$ we define a generalised process

$$I(\psi^n) := \int_0^t \psi^n(s,\omega,y)\, dL(s), \quad (t,\omega,y) \in [0,T] \times \Omega \times O.$$

Since ψ^n is a simple step function, this is well-defined. We also prove that inequality (6.6) is satisfied by all linear combinations of processes ψ^n.

Step 3 We prove that the sequence $I(\psi^n)$ is Cauchy in the following sense

$$\lim_{n,m\to\infty} \int_O \left(\mathbb{E} \sup_{t\in[0,T]} \| \int_0^t \psi^n(s,\omega,y)\, dL(s) - \int_0^t \psi^m(s,\omega,y)\, dL(s) \|_{\mathrm{E}}^p\, ds \right)^{\frac{1}{p}} \mu(dy) = 0.$$

Step 4 We prove that each process $I(\psi^n)$ satisfies all assertions of Theorem 6.6.

Step 5 By considering a subsequence we can assume that

$$\int_O \left(\mathbb{E} \int_0^T \pi_p^p(\psi^n(t,\cdot,y) - \psi(t,\cdot,y))\, ds \right)^{\frac{1}{p}} \mu(dy) \le 10^{-n}, \quad n \in \mathbb{N}.$$

Using the Borel-Cantelli Lemma, as in the proof of Theorem 5.36, we can show that there exist measurable sets $O_1 \subset O$ and $\Omega_1 \subset \Omega$, of full measures, such that for $y \in O_1$ and $\omega \in \Omega_1$, the sequence $\int_0^\cdot \psi^n(s,\omega,y)\, dL(s)$ is uniformly convergent in E. We put, if $y \in O_1$ and $\omega \in \Omega_1$,

$$\int_0^t \psi(s,\omega,y)\, dL(s) := \lim_{n\to\infty} \int_0^\cdot \psi^n(s,\omega,y)\, dL(s)$$

and, otherwise, we put

$$\int_0^t \psi(s,\omega,y)\, dL(s) = 0.$$

Step 6 We show that thus defined generalised process

$$I(\psi) := \int_0^t \psi(s,\omega,y)\, dL(s), \quad (t,\omega,y) \in [0,T] \times \Omega \times O,$$

satisfies all assertions of our Theorem 6.6.

7. Existence and Uniqueness of Solution to Stochastic Differential Equations

7.1. **Preliminaries.** In this section, we apply developed theories about stochastic integrals with respect to cylindrical Lévy process to establish the existence and the uniqueness of the solutions to a stochastic differential equation (SDE) in a Banach space.

We begin by outlining the framework for this section. We assume that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space such that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual condition – right-continuity and completeness, see Definition F.24.

Let us recall several key definitions. Assume that F is a Banach space.

1. An F-valued process $\xi = (\xi(t))_{t\geq 0}$ is $\mathbb{F}$-adapted if and only if $\xi(t)$ is $\mathcal{F}_t$-measurable for each $t \in\geq 0$.
2. An F-valued process $\xi = (\xi(t))_{t\geq 0}$ is càdlàg if and only if $\xi(t)$ is right continuous and has a left limit at every $t \geq 0$.
3. An F-valued process $\xi = (\xi(t))_{t\geq 0}$ is $\mathbb{F}$-predictable if and only if ξ is $\mathbb{F}$-measurable and the mapping
$$\xi : [0, \infty) \times E \times \Omega \to \mathbb{R}$$
satisfies the the following conditions:
   (a) For each $t \geq 0$, the mapping $(x, \omega) \to \xi(t, x, \omega)$ is $\mathcal{B}(E) \otimes \mathcal{F}_t$-measurable;
   (b) For each $x \in \mathrm{E}, \omega \in \Omega$, the mapping $[0, \infty) \ni t \to \xi(t, x, \omega)$ is left-continuous.
   See also Definition 2.80.

7.2. **Formulation.** We assume that F and E are Banach spaces. The norms in these spaces will be denoted by $\|\cdot\|_E$ and $\|\cdot\|_F$, respectively. When there is no danger of ambiguity, these norms will simply be denoted by the notation $\|\cdot\|$.

The space of all p-summing linear operators from E to F is denoted by $\Pi_p(E, F)$ and the norm in this space is denoted by $\pi_p(\cdot)$ or $\|\cdot\|_{\Pi_p(E,F)}$.

Assume that $p \in [1, 2]$. This number will be fixed throughout the whole section.

We assume that L is a cylindrical Lévy process on E such that, for every $t \in \mathbb{R}^+$, the mapping
$$L(t) : E^* \to L^0(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$$

has finite weak p-th moments, i.e.

$$\mathbb{E}[|L(t)x^*|^p] < \infty, \quad \forall t \geq 0, \forall x^* \in E^*.$$

This corresponds to the condition (5.1).

We consider functions

$$B : \mathbb{R}^+ \times F \to F$$

and

$$G : \mathbb{R}^+ \times F \to \Pi_p(E, F)$$

which are assumed to be strongly-measurable. Further assumptions on these functions will be introduced later.

In this section we will study the following SDE:

$$\begin{aligned} dX(t) &= B(t,X(t))\,dt + G(t,X(t))\,dL(t), \quad t \geq 0, \\ X(0) &= X_0, \end{aligned} \tag{7.1}$$

where X_0 is an $\mathcal{F}_0$-measurable, F-valued random variable.

We now introduce the definition of a mild solution.

Definition 7.1. *A solution of the SDE* (7.1) *is a predictable F-valued process $X = (X(t))_{t\geq 0}$ such that for every $T > 0$,*

$$\|X\|_T^p = \sup_{t\in[0,T]} \mathbb{E}[\|X(t)\|^p] = \sup_{t\in[0,T]} \mathbb{E}[\|X(t)\|_F^p] < \infty,$$

and such that, for all $t \geq 0$, we have $\mathbb{P}$-almost surely

$$X(t) = X_0 + \int_0^t B(s,X(s))\,ds + \int_0^t G(s,X(s))\,dL(s).$$

In what follows, we assume the Lipschitz and the linear growth conditions on the coefficients B and G, and an integrability assumption on the initial condition.

In the assumptions below we temporarily replace the time interval $[0, \infty)$ by a bounded interval $[0, T]$.

Assumption 7.2. ***(Initial data)***

We assume that $p \in [1, 2]$ and

(A0) $X_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; F)$.

Assumption 7.3. ***(Lipschitz and Growth Conditions on*** B ***and*** G***)***

We assume that

(B0) *For every* $x \in E$*, the map*

$$[0,T] \ni s \mapsto B(s,x) \in F$$

is Lebesgue measurable;

(B1) *there exists a non-negative function* $b \in L^q([0,T]) = L^q([0,T];\mathbb{R})$*, where* $q = \frac{p}{p-1}$*, such that for all* $x, x_1, x_2 \in E$ *and* $t \in [0,T]$

$$\|B(t,x)\|_F \leq b(t)\,(1+\|x\|),$$

$$\|B(t,x_1) - B(t,x_2)\|_F \leq b(t)\,\|x_1 - x_2\|.$$

(G0) *For every* $x \in E$*, the map*

$$[0,T] \ni s \mapsto G(s,x) \in \Pi_p(E,F)$$

is Lebesgue measurable;

(G1) *there exists a non-negative function* $g \in L^\infty([0,T];\mathbb{R})$ *such that for any* $x, x_1, x_2 \in F$ *and* $t \in [0,T]$

$$\pi_p(G(t,x)) \leq g(t)\,(1+\|x\|),$$

$$\pi_p(G(t,x_1) - G(t,x_2)) \leq g(t)\,\|x_1 - x_2\|.$$

These assumptions ensure well-posedness (existence and uniqueness) of the solution to the SDE.

Theorem 7.4. *Let* $p \in (1,2]$*. Suppose that the Banach spaces E and F satisfy the following conditions:*

(a) *E is reflexive or has a separable dual,*

(b) *F is of martingale type p, and* E^{**} *has the approximation property.*

Assume that L is a cylindrical Lévy process such that Assumptions 7.2-7.3 are satisfied. Then, for every $X_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; F)$*, there exists a unique solution to the SDE* (7.1).

Remark 7.5. Here we need Banach space E^{**} having the approximation property, see Definition 3.5, so as to ensure the stochastic integral with respect to cylindrical Lévy process $L(s)$, i.e., $\int_0^t G(s, X(s)\,dL(s)$, where $G(s, X(s)) \in \Pi_p(E, F)$, is well-defined, i.e., it can be approximated by sums over simple functions, and hence we will be able to apply (5.14) of Proposition 5.32.

Theorem 7.6. *Let $p = 1$. Suppose that the Banach spaces E and F satisfy the following conditions:*

(a) *E is reflexive or has a separable dual,*

(b) *F is of martingale type 1, has the Radon-Nikodym property, and E^{**} has the approximation property.*

Assume that L is a cylindrical Lévy process as before, and that Assumptions 7.2-7.3 are satisfied. Then, for every $X_0 \in L^1(\Omega, \mathcal{F}_0, \mathbb{P}; F)$, there exists a unique solution to the SDE (7.1).

Before we embark on proof of Theorem 7.4, We introduce the following function space:

Definition 7.7. *If $T > 0$ then we define the space*

$$\mathcal{H}_T := \Big\{X : [0, T] \times \Omega \to F : X \text{ is } \mathbb{F}-\text{predictable and}$$
$$\sup_{t\in[0,T]} \mathbb{E}[\|X(t)\|^p] < \infty\Big\}.$$

This space consists of all predictable processes that satisfy a uniform L^p-boundedness condition over $[0, T]$. Obviously, the following lemmas hold.

Lemma 7.8. *The space $\mathcal{H}_T$ is a vector space.*

For the proof of Lemma 7.8 see Appendix G.13.

Let us choose and fix $\beta \geq 0$ and define a norm $\|\cdot\|_{T,\beta}$ on $\mathcal{H}_T$ by

$$\|X\|_{T,\beta} := \left(\sup_{t\in[0,T]} e^{-\beta t}\mathbb{E}[\|X(t)\|^p]\right)^{1/p}, \quad X \in L^p(\Omega, \mathcal{F}, \mathbb{P}; F).$$

Note that

$$\|\cdot\|_T := \|\cdot\|_{T,0}.$$

Lemma 7.9. *The function $\|\cdot\|_{T,\beta}$ is a norm on $\mathcal{H}_T$.*

Proof of Lemma 7.9. Let us recall that a $\mathbb{R}$-valued function $\|\cdot\|$ is a norm on the vector space $\mathcal{H}_T$ if and only if satisfies the following four properties:

(N1) $\|X\|_{T,\beta} \geq 0$ for every $X \in \mathcal{H}_T$ and $\|0\| = 0$;

(N2) If $X \in \mathcal{H}_T$ and $\|X\|_{T,\beta} = 0$ then $X = 0$;

(N3) If $X \in \mathcal{H}_T$ and $\alpha \in \mathbb{R}$ then

$$\|\alpha X\|_{T,\beta} = |\alpha|\,\|X\|_{T,\beta}.$$

(N4) If $X_1, X_2 \in \mathcal{H}_T$, then

$$\|X_1 + X_2\|_{T,\beta} \leq \|X_1\|_{T,\beta} + \|X_2\|_{T,\beta}.$$

(Ad N1) Obviously if $X = 0$, then $\|X\|_{T,\beta} = 0$. Let us choose and fix $X \in \mathcal{H}_T$. Since $\mathbb{E}[\|X(t)\|^p] \geq 0$ for every $t \in [0,T]$, we infer that $\|X\|_{T,\beta} \geq 0$.

(Ad N2) If $\|X\|_{T,\beta} = 0$, then by definition,

$$\sup_{t\in[0,T]} e^{-\beta t}\mathbb{E}[\|X(t)\|^p] = 0.$$

Since $e^{-\beta t}\mathbb{E}[\|X(t)\|^p] \geq 0$ for every $t \in [0,T]$, it follows that

$$\mathbb{E}[\|X(t)\|^p] = 0 \quad \text{for every } t \in [0,T].$$

Therefore,

$$X(t) = 0 \quad \forall t \in [0,T].$$

and hence $X = 0$.

(Ad N3) For $X \in \mathcal{H}_T$ and $\alpha \in \mathbb{R}$, we have

$$\|\alpha X\|_{T,\beta} = \left(\sup_{t\in[0,T]} e^{-\beta t}\mathbb{E}[\|\alpha X(t)\|^p]\right)^{1/p} = |\alpha|\left(\sup_{t\in[0,T]} e^{-\beta t}\mathbb{E}[\|X(t)\|^p]\right)^{1/p} = |\alpha|\,\|X\|_{T,\beta}.$$

(Ad N4) For $X_1, X_2 \in \mathcal{H}_T$, by Minkowski's inequality, we have

$$\begin{aligned}
\|X_1 + X_2\|_{T,\beta} &= \left(\sup_{t\in[0,T]} e^{-\beta t}\mathbb{E}[\|X_1(t) + X_2(t)\|^p]\right)^{1/p} = \sup_{t\in[0,T]}\left(e^{-\beta t}\mathbb{E}[\|X_1(t) + X_2(t)\|^p]\right)^{1/p} \\
&\leq \sup_{t\in[0,T]}\left(e^{-\beta t}\left(\mathbb{E}[\|X_1(t)\|^p]\right)^{1/p} + e^{-\beta t}\left(\mathbb{E}[\|X_2(t)\|^p]\right)^{1/p}\right) \\
&\leq \sup_{t\in[0,T]} e^{-\beta t}\left(\mathbb{E}[\|X_1(t)\|^p]\right)^{1/p} + \sup_{t\in[0,T]} e^{-\beta t}\left(\mathbb{E}[\|X_2(t)\|^p]\right)^{1/p} \\
&= \|X_1\|_{T,\beta} + \|X_2\|_{T,\beta}.
\end{aligned}$$

□

Lemma 7.10. *The norms $\|\cdot\|_T$ and $\|\cdot\|_{T,\beta}$ on $\mathcal{H}_T$ are equivalent.*

Proof Lemma 7.10 . Assume that $\beta > 0$. Then, since $0 < e^{-\beta t} \le 1$ for $t \ge 0$, we infer that

$$\|X\|_{T,\beta} = \left(\sup_{t\in[0,T]} e^{-\beta t}\,\mathbb{E}[\|X(t)\|^p]\right)^{1/p} \le \left(\sup_{t\in[0,T]} \mathbb{E}[\|X(t)\|^p]\right)^{1/p} = \|X\|_{T,0}.$$

Now we will show the inequality in the opposite direction. Since $0 < e^{\beta t} \le e^{\beta T}$ for $t \in [0,T]$, we infer that

$$\begin{aligned}\|X\|_{T,0} &= \left(\sup_{t\in[0,T]} \mathbb{E}[\|X(t)\|^p]\right)^{1/p} = \left(\sup_{t\in[0,T]} e^{\beta t}e^{-\beta t}\,\mathbb{E}[\|X(t)\|^p]\right)^{1/p}\\ &= \left(\sup_{t\in[0,T]} e^{\beta T}e^{-\beta t}\,\mathbb{E}[\|X(t)\|^p]\right)^{1/p} \le e^{\beta T}\left(\sup_{t\in[0,T]} e^{-\beta t}\,\mathbb{E}[\|X(t)\|^p]\right)^{1/p} = e^{\beta T}\,\|X\|_{T,\beta}.\end{aligned}$$

This completes a proof of our equivalence Lemma. Note that it was essential to consider only $T > 0$ and not $T = \infty$. □

Lemma 7.11. *The space $\mathcal{H}_T$ endowed with the norm $\|\cdot\|_T$ is complete.*
Hence, the space $\mathcal{H}_T$ endowed with the norm $\|\cdot\|_{T,\beta}$ is also complete.

Proof of Lemma 7.11. Let (X_n) be a Cauchy sequence in $\mathcal{H}_\mathcal{T}$. Then for every $\varepsilon > 0$, there exists a number $N \in \mathbb{N}$ such that for all $m,n \ge N$,

$$\|X_n - X_m\|_T = \left(\sup_{t\in[0,T]} \mathbb{E}[\|X_n(t) - X_m(t)\|^p]\right)^{\frac{1}{p}} < \varepsilon.$$

Fix any $t \in [0,T]$. Then, for all $m,n \ge N$,

$$(\mathbb{E}\|X_n(t) - X_m(t)\|^p)^{\frac{1}{p}} \le \|X_n - X_m\|_T \le \varepsilon.$$

Observe that the left-hand side is exactly the norm $\|X_n(t) - X_m(t)\|_{L^p(\Omega)}$. Hence for every fixed $t \in [0,T]$, $X_n(t)$ is a Cauchy sequence in $L^p(\Omega,\mathcal{F}_t,\mathbb{P};F)$, i.e.,

$$\|X_n(t) - X_m(t)\|_{L^p(\Omega)} < \varepsilon.$$

Since $L^p(\Omega,\mathcal{F},\mathbb{P};F)$ is complete, hence there exists a process $X(t) \in L^p(\Omega,\mathcal{F},\mathbb{P};F)$ such that

$$\lim_{n\to\infty} \mathbb{E}[\|X_n(t) - X(t)\|^p] \to 0.$$

Define $X : [0,T]\times\Omega \to F$ by setting

$$X(t,\omega) := \lim_{n\to\infty} X_n(t,\omega) \quad \text{in } L^p(\Omega) \text{ for each } t.$$

Since each X_n is $\mathbb{F}$-predictable and $X_n \to X$ in $L^p([0,T]\times\Omega)$, there exists a subsequence that converges to X almost surely. Using the fact that the class of $\mathbb{F}$-predictable processes is closed under almost sure limits, see [32, Theorem IV.35, p. 182], [27, Chapter 1, Section 1.5], it follows that X is also $\mathbb{F}$-predictable.

We now show that $X \in \mathcal{H}_T$, i.e., $\|X\|_T < \infty$.

By the triangle inequality in $L^p(\Omega)$, for each $t \in [0,T]$ and $n \in \mathbb{N}$, we have

$$(\mathbb{E}\|X(t)\|^p)^{1/p} \leq (\mathbb{E}\|X(t) - X_n(t)\|^p)^{1/p} + (\mathbb{E}\|X_n(t)\|^p)^{1/p}$$

Taking the supremum over $t \in [0,T]$, we obtain

$$\sup_{t\in[0,T]} (\mathbb{E}\|X(t)\|^p)^{1/p} \leq \sup_{t\in[0,T]} (\mathbb{E}\|X(t) - X_n(t)\|^p)^{1/p} + \sup_{t\in[0,T]} (\mathbb{E}\|X_n(t)\|^p)^{1/p} .$$

By definition of $\|\cdot\|_T$ norm, this is equivalent to:

$$\|X(t)\|_T \leq \|X(t) - X_n(t)\|_T + \|X_n(t)\|_T.$$

Since (X_n) is Cauchy in $\mathcal{H}_T$, it is bounded, i.e.,

$$\sup_n \|X_n\|_T < \infty.$$

Moreover, for any $\varepsilon > 0$, there exists N such that for all $n \geq N$, we have $\|X - X_n\|_T < \varepsilon$. Hence

$$\|X(t)\|_T \leq \varepsilon + \sup_n \|X_n(t)\|_T < \infty.$$

Hence $X \in \mathcal{H}_T$.

Finally, we show that $X_n \to X$ in $\mathcal{H}_T$. Since X_n is Cauchy in $\mathcal{H}$, for all $n, m \geq N$,

$$\|X_n - X_m\|_T < \varepsilon.$$

Taking the limit as $m \to \infty$, and using the norm continuity of $L^p(\Omega)$, we obtain

$$\|X_n - X\|_T \leq \varepsilon.$$

Hence, $X_n \to X$ in $\mathcal{H}_T$, proving that $\mathcal{H}_T$ is complete under $\|\cdot\|_T$.

Hence, the space $\mathcal{H}_T$, when equipped with the norm $\|\cdot\|_{T,\beta}$, is also complete, as it is equivalent to $\|\cdot\|_T$. $\square$

We are now ready to prove Theorem 7.4, in which we apply the Banach Fixed Point Theorem E.3, see Theorem E.3 in Appendix, to establish the existence and uniqueness of solutions to the SDE.

Proof of Theorem 7.4. Assume that the assumptions of Theorem 7.4 are satisfied. In particular, we choose and fix $X_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; F)$.

We define, for $i = 0, 1, 2$, the maps

$$K_i : \mathcal{H}_T \to \mathcal{H}_T,$$

by, for $X \in \mathcal{H}_T$,

$$K_0(X)(t) := X_0, \quad t \in [0, T],$$

$$K_1(X)(t) := \int_0^t B(s,X(s))\, ds, \quad t \in [0, T], \tag{7.2}$$

$$K_2(X)(t) := \int_0^t G(s,X(s))\, dL(s), \quad t \in [0, T]. \tag{7.3}$$

We first establish the following lemma:

Lemma 7.12. *Each K_i maps $\mathcal{H}_T$ to itself. Hence K maps $\mathcal{H}_T$ to $\mathcal{H}_T$.*

Proof of Lemma 7.12. (1) Case $i = 0$. Let us choose and fix $X \in \mathcal{H}_T$. Define the map $K_0 : \mathcal{H}_T \to \mathcal{H}_T$ by

$$K_0(X)(t) := X_0, \quad t \in [0, T].$$

We want to prove that $K_0(X) \in \mathcal{H}_T$. We need to prove:

(i) $K_0(X)$ is a predictable F-valued process,

(ii) $\sup_{t\in[0,T]} \mathbb{E}[\|K_0(X)(t)\|^p] < \infty$.

Ad (i) Let us observe that the process $K_0(X)$ is constant with respect to time. The random variable X_0 is F-valued and $\mathcal{F}_0$-measurable. Recall that the predictable σ-field on $[0, T] \times \Omega$ is generated by left-continuous adapted processes, hence the process $K_0(X)$, being constant in time and measurable with respect to $\mathcal{F}_0$, is both adapted and left-continuous. Moreover, the Banach space F is equipped with its Borel σ-field $\mathcal{B}(F)$, which is generated by the open sets induced by the norm topology, i.e., the topology defined by the norm $\|\cdot\|$ on F. Hence, the mapping

$t \to K_0(X)(t) = X_0$ is measurable with respect to the predictable σ-field, and takes values in F. Therefore, we infer that the process $K_0(X)$ is predictable.

Ad (ii) For all $t \in [0,T]$, we have

$$\sup_{t\in[0,T]} \mathbb{E}[\|K_0(X)(t)\|^p] = \sup_{t\in[0,T]} \mathbb{E}[\|X_0\|^p] = \mathbb{E}[\|X_0\|^p] < \infty.$$

(2) Case $i = 1$. Let us choose and fix $X \in \mathcal{H}_T$. We want to prove that $K_1X \in \mathcal{H}_T$. We need to prove:

(i) K_1X is a well-defined F-valued process,

(ii) K_1X is a predictable F-valued process,

(iii) K_1X satisfies

$$\sup_{t\in[0,T]} \mathbb{E}|K_1(X)(t)|_F^p < \infty.$$

Ad (i) The Bochner integral in (7.2) is well-defined for each $t \in [0,T]$, because $X \in \mathcal{H}_T$ is $\mathbb{F}$ -predictable and the mappings

$$B(s,\cdot) : E \ni x \mapsto B(s,x) \in F$$

are continuous in x. By Assumption 7.3, and the integrability of X, the mapping $s \mapsto B(s,X(s))$ is Bochner integrable. Hence K_1X is well-defined.

Ad (ii) Since X is predictable, and the map $B(s,\cdot)$ is continuous, the composition $B(s,X(s))$ is again predictable as a function of (s,ω). Thus, K_1X is a predictable.

Ad (iii) By [47, Corollary V.5.1, p. 133], Assumption 7.3 (B1), the Hölder inequality, see Corollary D.17, also we use the fact that, if C is a constant and $g : [0,T] \to [0,\infty)$, then

$$\sup_{t\in[0,T]} (Cg(t)) = C \sup_{t\in[0,T]} g(t).$$

$$\begin{aligned}\sup_{t\in[0,T]} \mathbb{E}\Big[\Big\|K_1(X)(t)\Big\|^p\Big] &= \sup_{t\in[0,T]} \mathbb{E}\Big[\Big\|\int_0^t B(s,X(s))\,ds\Big\|^p\Big]\\ &\le \sup_{t\in[0,T]} \mathbb{E}\Big[\Big(\int_0^t \|B(s,X(s))\|\,ds\Big)^p\Big] \le \cdots\end{aligned}$$

By Assumption 7.3 (B1) and applying Hölder's inequality, we have

$$
\begin{aligned}
\cdots &\leq \sup_{t\in[0,T]} \mathbb{E}\Big[\Big(\int_0^t b(s)\,(1+\|X(s)\|)\,ds\Big)^p\Big] \\
&\leq \sup_{t\in[0,T]} \mathbb{E}\Big[\Big(\int_0^t |b(s)|^q\,ds\Big)^{\frac{p}{q}} \int_0^t (1+\|X(s)\|)^p\,ds\Big] \\
&\leq \Big(\int_0^T |b(s)|^q\,ds\Big)^{\frac{q}{p}} \sup_{t\in[0,T]} \mathbb{E}\Big[\int_0^t (1+\|X(s)\|)^p\,ds\Big] \leq \cdots
\end{aligned}
$$

Using the inequality $(a+b)^p \leq 2^{p-1}(a^p+b^p)$, where $a, b \geq 0$, $p \geq 1$, we have

$$
\begin{aligned}
\cdots &\leq \Big(\int_0^T |b(s)|^q\,ds\Big)^{\frac{q}{p}} \sup_{t\in[0,T]} \mathbb{E}\Big[\int_0^t 2^{p-1}(1^p+\|X(s)\|^p)\,ds\Big] \\
&\leq \Big(\int_0^T |b(s)|^q\,ds\Big)^{\frac{q}{p}} 2^{p-1}\Big[T + \sup_{t\in[0,T]} \mathbb{E}\int_0^t \|X(s)\|^p\,ds\Big] \\
&\leq \|b\|_{L^q}^p 2^{p-1}(T+\|X\|_{T,0}^p) < \infty.
\end{aligned}
$$

(3) Case $i = 2$. Let us choose and fix $X \in \mathcal{H}_T$. We want to prove that $K_2X \in \mathcal{H}_T$. We need to prove

(i) K_2X is a well-defined F-valued process,

(ii) K_2X is a predictable F-valued process,

(iii) K_2X satisfies

$$\sup_{t\in[0,T]} \mathbb{E}|K_2(X)(t)|_F^p < \infty.$$

Ad (i) The stochastic integral in (7.3) above is well-defined because the process X is predictable and the mappings, for every $s \in [0,T]$,

$$G(s,\cdot) : E \ni (s,x) \to G(s,x) \in \Pi_p(E,F)$$

are continuous. By Assumption 7.3, and the integrability of X, the mapping $s \mapsto G(s,X(s))$ is Bochner integrable and takes values in $\Pi_p(E,F)$. Hence K_2X is well-defined.

Ad (ii) Since X is predictable and the map $G(s,\cdot)$ is continuous, the composition $G(s,X(s))$ is again predictable as a function of s, ω. Thus, K_2X is predictable.

Ad (iii) Using the same approach in the case of K_1, we conclude from Assumption 7.3 (G1) and inequality (5.21) from Theorem 5.36 that there exists a constant $C_T > 0$ such that

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\Big\|K_2(X)(t)\Big\|^p\Big] = \sup_{t\in[0,T]} \mathbb{E}\Big[\Big\| \int_0^t G(s,X(s))\, dL(s)\Big\|^p\Big]$$

$$\leq C_T \mathbb{E}\Big[\int_0^T \pi_p^p\Big(G(s, X(s))\Big)\, ds\Big] \leq \cdots$$

By Assumption 7.3 (G1), we have

$$\cdots \leq C_T \mathbb{E}\Big[\int_0^T \Big(g(s)\,(1 + \|X(s)\|)\Big)^p\, ds\Big]$$

$$= C_T \mathbb{E}\Big[\int_0^T g(s)^p\,(1 + \|X(s)\|)^p\, ds\Big]$$

$$\leq C_T \Big[\text{esssup}_{s\in[0,T]} |g(s)|^p\Big] \int_0^T (1 + \mathbb{E}\|X(s)\|)^p\, ds$$

$$\leq C_T \|g\|^p_{L^\infty(0,T)} (T + \|X\|^p_{T,0}) < \infty.$$

□

Let us continue our proof of Theorem 7.4. We will next establish that the process $K_i(X), i = 1,2$ is stochastically continuous. For this purpose, let us choose and fix $t \in [0,T)$ and $\varepsilon > 0$ such that $t + \varepsilon \leq T$. Then we obtain

$$\mathbb{E}[\|K_1(X)(t+\varepsilon) - K_1(X)(t)\|^p] = \mathbb{E}\Big[\Big\| \int_0^{t+\varepsilon} B(s, X(s))\, ds - \int_0^t B(s, X(s))\, ds\Big\|^p\Big]$$

$$= \mathbb{E}\Big[\Big\| \int_t^{t+\varepsilon} B(s, X(s))\, ds\Big\|^p\Big]$$

$$\leq \mathbb{E}\Big[\Big(\int_t^{t+\varepsilon} \|B(s, X(s))\|\, ds\Big)^p\Big]$$

$$\leq \mathbb{E}\Big[\Big(\int_t^{t+\varepsilon} b(s)(1 + \|X(s)\|)\, ds\Big)^p\Big]$$

By applying Hölder's Inequality and the formula $(a+b)^p \leq 2^{p-1}(a^p+b^p)$, where $a, b \geq 0, p \geq 1$, we have

$$\begin{aligned}
\cdots &\leq \Big(\int_t^{t+\varepsilon} |b(s)|^q\, ds\Big)^{\frac{p}{q}} \mathbb{E}\Big[\int_t^{t+\varepsilon} (1+\|X(s)\|)^p\, ds\Big] \\
&\leq \Big(\int_t^{t+\varepsilon} |b(s)|^q\, ds\Big)^{\frac{p}{q}} \mathbb{E}\Big[\int_t^{t+\varepsilon} 2^{p-1}(1^p+\|X(s)\|^p)\, ds\Big] \\
&\leq \Big(\int_t^{t+\varepsilon} |b(s)|^q\, ds\Big)^{\frac{p}{q}} 2^{p-1}\Big[\varepsilon + \mathbb{E}\int_t^{t+\varepsilon} \|X(s)\|^p\, ds\Big] \\
&\leq \|b\|^p_{L^q(t,t+\varepsilon)} 2^{p-1}(\varepsilon + \|X\|^p_{T,0}) \\
&\to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

The last step is based on the following fact:

Digression 7.13. *The assumption that $b \in L^q([0,T];\mathbb{R})$, see Assumption 7.3 (B1), i.e.,*

$$\int_0^T |b(s)|^q\, ds < \infty.$$

implies that, for every $t \in [0,T]$

$$\lim_{\varepsilon\to 0} \int_t^{t+\varepsilon} |b(s)|^q\, ds = 0.$$

Below we will provide the proof to this digression, using the Lebesgue Dominated Convergence Theorem, see Theorem F.32.

Proof to Digression 7.13. Applying the Lebesgue Dominated Convergence to our case, we have $S = [0,T]$, $g(s) = |b(s)|^q$, $\mu :=$ Lebesgue measure, i.e., $\mu(ds) = ds$; $f(s) = 0$, $f_n(s) = \mathbb{1}_{[t,t+\frac{1}{n})}(s)\,|b(s)|^q$. All assumptions of LDCT are satisfied. Hence (F.1) holds, i.e.

$$\int_S f_n(s)\, ds \to \int_S f(s)\, ds,$$

where

$$\int_S f(s)\, ds = \int_S 0\, ds = 0,$$

$$\int_S f_n(s)\, ds = \int_t^{t+\frac{1}{n}} |b(s)|^q\, ds.$$

Hence we proved that

$$\lim_{n\to\infty} \int_t^{t+\frac{1}{n}} |b(s)|^q\, ds = 0.$$

Therefore

$$\lim_{\varepsilon\to 0}\int_t^{t+\varepsilon} |b(s)|^q \, ds = 0.$$

$\square$

Let us continue our proof of Theorem 7.4. For K_2 we obtain by inequality (5.18) from Theorem 5.35 that there exists a constant $C > 0$ such that

$$\begin{aligned}
\mathbb{E}[\|K_2(X)(t+\varepsilon) - K_2(X)(t)\|^p] &= \mathbb{E}\Big[\Big\| \int_0^{t+\varepsilon} G(s, X(s)) \, dL(s) - \int_0^t G(s, X(s)) \, dL(s)\Big\|^p\Big] \\
&= \mathbb{E}\Big[\Big\| \int_t^{t+\varepsilon} G(s, X(s)) \, dL(s)\Big\|^p\Big] \\
&\le C\mathbb{E}\Big[\int_t^{t+\varepsilon} \pi_p^p\Big(G(s, X(s))\Big) \, ds\Big] \\
&\le C\mathbb{E}\Big[\int_t^{t+\varepsilon} \Big(g(s)(1 + \|X(t)\|)\Big)^p \, ds\Big] \le \cdots
\end{aligned}$$

By applying Hölder's Inequality and the formula $(a+b)^p \le 2^{p-1}(a^p+b^p)$, for $a, b \ge 0$, $p \ge 1$, we have

$$\begin{aligned}
\cdots &\le C\Big[\mathrm{esssup}_{s\in[t,t+\varepsilon]}|g(s)|^p\Big]\mathbb{E}\int_t^{t+\varepsilon} (1 + \|X(s)\|)^p \, ds \\
&\le C\Big[\mathrm{esssup}_{s\in[t,t+\varepsilon]}|g(s)|^p\Big]\Big(\varepsilon + \mathbb{E}\int_t^{t+\varepsilon} \|X(s)\|^p \, ds\Big) \\
&\le C\|g\|^p_{L^\infty(t,t+\varepsilon)}(\varepsilon + \|X\|^p_{T,0}) \\
&\to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Similarly, we have for $0 < \varepsilon \leq t$,

$$\begin{aligned}
\mathbb{E}[\|K_1(X)(t-\varepsilon) - K_1(X)(t)\|^p] &= \mathbb{E}\Big[\Big\| \int_0^{t-\varepsilon} B(s, X(s))\, ds - \int_0^t B(s, X(s))\, ds\Big\|^p\Big] \\
&= \mathbb{E}\Big[\Big\| \int_{t-\varepsilon}^t B(s, X(s))\, ds\Big\|^p\Big] \\
&\leq \mathbb{E}\Big[\Big(\int_{t-\varepsilon}^t \|B(s, X(s))\|\, ds\Big)^p\Big] \\
&\leq \mathbb{E}\Big[\Big(\int_{t-\varepsilon}^t b(s)(1 + \|X(s)\|)\, ds\Big)^p\Big] \\
&\leq \Big(\int_0^\varepsilon |b(s)|^q\, ds\Big)^{\frac{p}{q}} \mathbb{E}\Big[\int_0^\varepsilon (1 + \|X(s)\|)^p\, ds\Big] \\
&\leq \Big(\int_0^\varepsilon |b(s)|^q\, ds\Big)^{\frac{p}{q}} \mathbb{E}\Big[\int_0^\varepsilon 2^{p-1}(1^p + \|X(s)\|^p)\, ds\Big] \\
&\leq \Big(\int_0^\varepsilon |b(s)|^q\, ds\Big)^{\frac{p}{q}} 2^{p-1}\Big[\varepsilon + \mathbb{E} \int_0^\varepsilon \|X(s)\|^p\, ds\Big] \\
&\leq \Big(\int_0^\varepsilon |b(s)|^q\, ds\Big)^{\frac{p}{q}} 2^{p-1}\Big[\varepsilon + \sup_{t\in[0,T]} \mathbb{E} \int_0^\varepsilon \|X(s)\|^p\, ds\Big] \\
&\leq \|b\|^p_{L^q(0,\varepsilon)} 2^{p-1}(\varepsilon + \|X\|^p_{T,0}) \\
&\to 0 \quad \text{as}\, \varepsilon \to 0.
\end{aligned}$$

The stochastic continuity of $K_2(X)$ can be shown in a similar way, as follows:

$$\begin{aligned}
\mathbb{E}[\|K_2(X)(t-\varepsilon) - K_2(X)(t)\|^p] &= \mathbb{E}\Big[\Big\| \int_0^{t-\varepsilon} G(s, X(s))\, dL(s) - \int_0^t G(s, X(s))\, dL(s)\Big\|^p\Big] \\
&= \mathbb{E}\Big[\Big\| \int_{t-\varepsilon}^t G(s, X(s))\, dL(s)\Big\|^p\Big] \\
&\leq C\mathbb{E}\Big[\int_{t-\varepsilon}^t \pi_p^p\Big(G(s, X(s))\Big)\, ds\Big] \\
&\leq C\mathbb{E}\Big[\int_{t-\varepsilon}^t \Big(g(s)(1 + \|X(t)\|)\Big)^p\, ds\Big] \\
&\leq C\Big[\operatorname{esssup}_{s\in[0,\varepsilon]} |g(s)|^p\Big] \int_0^\varepsilon (1 + \|X(s)\|)^p\, ds\Big] \\
&\leq C\Big[\operatorname{esssup}_{s\in[0,\varepsilon]} |g(s)|^p\Big]\Big(\varepsilon + \mathbb{E} \int_0^\varepsilon \|X(s)\|^p\, ds\Big) \\
&= C\|g\|^p_{L^\infty(0,\varepsilon)}(\varepsilon + \|X\|^p_{T,0}) \\
&\to 0 \quad \text{as}\, \varepsilon \to 0.
\end{aligned}$$

Thus the process $K(X)$ is stochastically continuous.

In particular, the stochastic continuity implies the existence of a predictable modification of the process $K(X)$ by [74, Proposition 3.12]. In summary, we obtain that K maps from $\mathcal{H}_T$ to $\mathcal{H}_T$.

For applying the Banach Fixed Point Theorem it is enough to show that K is a contraction with respect to $\|\cdot\|_{T,\beta}$ for some β. We have

$$\|K(X_1) - K(X_2)\|_{T,\beta}^p \leq 2^{p-1}\Big(\|K_1(X_1) - K_1(X_2)\|_{T,\beta}^p + \|K_2(X_1) - K_2(X_2)\|_{T,\beta}^p\Big).$$

We estimate each term on the right-hand side separately.

For the part corresponding to the drift, we calculate

$$\begin{aligned}
\|K_1(X_1) - K_1(X_2)\|_{T,\beta}^p &= \sup_{t\in[0,T]} e^{-\beta t}\,\mathbb{E}\Big\|K_1(X_1)(t) - K_1(X_2)(t)\Big\|^p \\
&= \sup_{t\in[0,T]} e^{-\beta t}\,\mathbb{E}\Big\|\int_0^t B(s, X_1(s))\,ds - \int_0^t B(s, X_2(s))\,ds\Big\|^p
\end{aligned}$$

by the definition of K_1. Then by the linearity of integral, we have

$$= \sup_{t\in[0,T]} e^{-\beta t}\,\mathbb{E}\Big\|\int_0^t \Big(B(s, X_1(s)) - B(s, X_2(s))\Big)\,ds\Big\|^p$$

By [47, Corollary V.5.1, p. 133], we have

$$\leq \sup_{t\in[0,T]} e^{-\beta t}\,\mathbb{E}\Big(\int_0^t \Big\|B(s, X_1(s)) - B(s, X_2(s))\Big\|\,ds\Big)^p$$

By Assumption 7.3 (B1) and applying Hölder's inequality, we have

$$
\begin{aligned}
&\leq \sup_{t\in[0,T]} e^{-\beta t}\,\mathbb{E}\Big(\int_0^t b(s)\,\|X_1(s)-X_2(s)\|\,ds\Big)^p\\
&\leq \sup_{t\in[0,T]} e^{-\beta t}\,\mathbb{E}\Big[\Big(\int_0^t |b(s)|^q\,ds\Big)^{\frac{p}{q}}\Big(\int_0^t \|X_1(s)-X_2(s)\|^p\,ds\Big)\Big]\\
&\leq \Big(\int_0^T |b(s)|^q\,ds\Big)^{\frac{p}{q}} \sup_{t\in[0,T]} e^{-\beta t}\,\mathbb{E}\Big[\int_0^t \|X_1(s)-X_2(s)\|^p\,ds\Big]\\
&= \Big(\int_0^T |b(s)|^q\,ds\Big)^{\frac{p}{q}} \sup_{t\in[0,T]} \Big[\int_0^t e^{-\beta(t-s)}\,e^{-\beta s}\mathbb{E}\|X_1(s)-X_2(s)\|^p\,ds\Big]\\
&\leq \Big(\int_0^T |b(s)|^q\,ds\Big)^{\frac{p}{q}} \sup_{t\in[0,T]} \Big[\sup_{s\in[0,t]} e^{-\beta s}\mathbb{E}\|X_1(s)-X_2(s)\|^p \int_0^t e^{-\beta(t-s)}\,ds\Big]\\
&\leq \Big(\int_0^T |b(s)|^q\,ds\Big)^{\frac{p}{q}} \sup_{s\in[0,T]} e^{-\beta s}\mathbb{E}\|X_1(s)-X_2(s)\|^p \sup_{t\in[0,T]} \Big[\int_0^t e^{-\beta(t-s)}\,ds\Big]\\
&= \Big(\int_0^T |b(s)|^q\,ds\Big)^{\frac{p}{q}} \sup_{s\in[0,T]} e^{-\beta s}\mathbb{E}\|X_1(s)-X_2(s)\|^p \sup_{t\in[0,T]} \Big[\frac{1}{\beta}(1-e^{-\beta t})\Big]\\
&= \frac{1}{\beta}\Big(\int_0^T |b(s)|^q\,ds\Big)^{\frac{p}{q}} \sup_{s\in[0,T]} e^{-\beta s}\mathbb{E}\|X_1(s)-X_2(s)\|^p\\
&= \frac{1}{\beta}\Big(\int_0^T |b(s)|^q\,ds\Big)^{\frac{p}{q}} \|X_1-X_2\|_{T,\beta}^p
\end{aligned}
$$

So we proved that

$$
\|K_1(X_1)-K_1(X_2)\|_{T,\beta} \leq \Big(\frac{1}{\beta}\Big)^{\frac{1}{p}} \Big(\int_0^T |b(s)|^q\,ds\Big)^{\frac{1}{q}} \|X_1-X_2\|_{T,\beta}. \tag{7.4}
$$

In the following calculation for the part corresponding to the diffusion, we use the continuity of the stochastic integral formulated in Theorem 5.35 to justify the first inequality.

We begin with the definition of K_2,

$$
\begin{aligned}
\|K_2(X_1)-K_2(X_2)\|_{T,\beta}^p &= \sup_{t\in[0,T]} e^{-\beta t}\,\mathbb{E}\Big\|K_2(X_1)(t)-K_2(X_2)(t)\Big\|^p\\
&= \sup_{t\in[0,T]} e^{-\beta t}\,\mathbb{E}\Big\|\int_0^t G(s,X_1(s))\,ds - \int_0^t G(s,X_2(s))\,ds\Big\|^p
\end{aligned}
$$

Then by the linearity of integral, we have

$$
\begin{aligned}
&= \sup_{t\in[0,T]} e^{-\beta t}\,\mathbb{E}\Big\|\int_0^t \Big(G(s,X_1(s))-G(s,X_2(s))\Big)\,ds\Big\|^p\\
&\leq C \sup_{t\in[0,T]} e^{-\beta t}\,\mathbb{E}\Big[\int_0^t \pi_p^p\Big(G(s,X_1(s))-G(s,X_2(s))\Big)\,ds\Big]
\end{aligned}
$$

By Assumption 7.3 (G1), we have

$$\leq C \sup_{t\in[0,T]} e^{-\beta t}\, \mathbb{E}\Big[\int_0^t \Big(g(s)\,\|X_1(s) - X_2(s)\|\Big)^p \,ds\Big]$$

$$= C \sup_{t\in[0,T]} \Big[\int_0^t g(s)^p\, e^{-\beta(t-s)} e^{-\beta s} \mathbb{E}\|X_1(s) - X_2(s)\|^p \,ds\Big]$$

$$\leq C \sup_{t\in[0,T]} \Big[\Big(\mathrm{esssup}_{s\in[0,t]} g(s)^p \int_0^t e^{-\beta(t-s)}\,ds\Big)\Big(\sup_{0\leq s\leq t} \int_0^t e^{-\beta s}\, \mathbb{E}\|X_1(s) - X_2(s)\|^p \,ds\Big)\Big]$$

$$\leq C \frac{1}{\beta} \mathrm{esssup}_{s\in[0,T]} g(s)^p \|X_1 - X_2\|_{T,\beta}^p$$

$$= \frac{C}{\beta} \|g\|_{L^\infty(0,T)}^p \|X_1 - X_2\|_{T,\beta}^p.$$

So we proved that

$$\|K_2(X_1) - K_2(X_2)\|_{T,\beta} \leq \left(\frac{C}{\beta}\right)^{1/p} \|g\|_{L^\infty(0,T)} \|X_1 - X_2\|_{T,\beta}. \tag{7.5}$$

Combining the above inequalities (7.4) and (7.5), we deduce that we can choose $\beta > 0$ sufficiently large so that K is a strict contraction, and by the Banach fixed point theorem, the SDE (7.1) has a unique fixed point $X \in \mathcal{H}_T$ such that $K(X) = X$ which completes the proof of Theorem 7.4.

□

8. Extension of the BH Work [8] to p-Summing Operators.

In this section we will generalize the results from the paper [8] by Brzeźniak and Hausenblas, where the authors considered Stochastic Evolution Equations (SEEs) driven by Poisson random measures. Their main tool was certain inequality valid for stochastic integrals in martingale type p, $p \in (1,2]$, Banach spaces driven by a class of Poisson random measures. This inequality was further extended to the so called stochastic convolutions. In this section, we show that stochastic integrals driven by Poisson random measures can be replaced with those driven by cylindrical Lévy processes.

This section is organized as follows: in the first subsection, we present preliminary concepts; in the second subsection, we introduce the main results which are based on paper [8, p. 621].

8.1. **Analytic Preliminaries.**

Assumption 8.1. *Assume that* E *is a separable martingale type $p, p \in (1,2]$, Banach space. Let $\mathcal{A}(\mathrm{E})$ denote the class of linear operators A, such that $-A$ is an infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ with the domain $D(A)$ on* E*, see condition* (A.1) *in Theorem A.9. Note that analytic semigroups are not necessarily exponentially stable in general; in this work we consider semigroups of the form $(e^{-tA})_{t\geq 0}$, so any stability depends on the spectral properties of A.*

It is known, that for every $A \in \mathcal{A}(\mathrm{E})$ there exist numbers $C > 0$ and $\omega < 0$ such that

$$\|e^{-tA}\| \leq Ce^{t\omega}, \quad t \geq 0. \tag{8.1}$$

In what follows, we fix $A \in \mathcal{A}(\mathrm{E})$ and numbers $C > 0$ and $\omega < 0$ such that (8.1) is satisfied.

The following inequality is well known, see e.g., [29, Theorem 2.5.6 (d), p. 62; Theorem 2.6.13 (c), p. 74; and p. 196]. There exists $C > 0$ such that

$$|Ae^{-\frac{t}{2}A}| \leq Ct^{-1}, \quad \text{for } t > 0. \tag{8.2}$$

We have decided not to introduce detailed and precise definitions of the real interpolation spaces. Instead, we use the equalities below as such definitions.

Assume that the parameters $\theta \in (0,1)$, $m \in \mathbb{N}$, $\delta \in (0,\infty)$, $q \in [1,\infty)$ are given. Then we define the relevant interpolation spaces accordingly, see Section 1.14.5 in [44], or [13], or [8, p. 620],

$$(\mathrm{E}, D(A^m))_{\theta,q} = (D(A^m), \mathrm{E})_{1-\theta,q} = \{x \in \mathrm{E} : \int_0^\delta \left| t^{m(1-\theta)} A^m e^{-tA} x \right|^q \frac{dt}{t} < \infty\}. \tag{8.3}$$

We denote the associated norm:

$$|x|^q_{(\mathrm{E},D(A^m))_{\theta,q};\delta} = \int_0^\delta \left| t^{m(1-\theta)} A^m e^{-tA} x \right|^q \frac{dt}{t}, \quad x \in \mathrm{E}. \tag{8.4}$$

It follows trivially from the above that

$$(\mathrm{E}, D(A^m))_{\theta,q} = (D(A^m), \mathrm{E})_{1-\theta,q}.$$

Moreover, it is known that the norms defined by the equation (8.3) are equivalent for different values of $\delta \in (0,\infty)$. The spaces $(\mathrm{E}, D(A^m))_{\theta,q} = (D(A^m), \mathrm{E})_{1-\theta,q}$, which we denote by $D_{A^m}(\theta, q)$, depend on m, θ and q. For certain choices of these parameters, however, these spaces are equal, as illustrated by the identity (8.5) below.

The following important results are cited in [8, p. 621] and [44, Theorem 1.15.2(f), p. 101].

Proposition 8.2. *If* $0 < k < m \in \mathbb{N}$, $p \in [1,\infty)$ *and* $\theta \in (0,1)$, *then*

$$(\mathrm{E}, D(A^k))_{\theta,p} = (\mathrm{E}, D(A^m))_{\frac{k}{m}\theta,p}.$$

It follows from Proposition 8.2 that if $p \in [1,\infty)$ and $\theta \in [0, 1 - \frac{1}{p})$, then

$$(E, D(A))_{\theta+\frac{1}{p},q} = D_A(\theta + \frac{1}{p}, q) = D_{A^2}(\frac{\theta}{2} + \frac{1}{2p}, q) = (E, D(A^2))_{\frac{\theta}{2}+\frac{1}{2p},q}. \tag{8.5}$$

with equivalent norms, see Definition F.2.

Equality (8.5) implies the following result:

Corollary 8.3. *Under the above assumptions, there exists a constant* $C > 0$ *such that*

$$\frac{1}{C}|z|_{D_{A^2}(\frac{\theta}{2}+\frac{1}{2p},q)} \le |z|_{D_A(\theta+\frac{1}{p},q)} \le C|z|_{D_{A^2}(\frac{\theta}{2}+\frac{1}{2p},q)}, \quad \forall z \in D_A(\theta + \frac{1}{p}, q). \tag{8.6}$$

Let us now formulate an inequality that will be used in the later part of this section.

Lemma 8.4. [8, Lemma 3.1] *If $q > p \geq 1$, then there exists a constant $C > 0$ such that for all $t > 0$ and $s \in (0,1)$, the following inequality holds:*

$$\Big(\int_0^t \frac{1}{(t-r+s)^{\frac{pq}{q-p}}}\,dr\Big)^{\frac{q-p}{p}} \leq \frac{C}{s^{q(1-\frac{1}{p})+1}}.$$

The main result is stated in the next section.

8.2. **Statements of The Main Results.** In this section, we first define a suitable class of Banach spaces. Next, we present a selection of relevant theorems, propositions, and corollaries based on these conditions to explore additional properties and their connection to p-summing operators. This sets the stage for introducing our main result, Theorem 8.8.

Assume that $p \in (1,2]$ and $q \in [p,\infty)$. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, see Definition F.24.

Definition 8.5. *Consider a "functor" R which associates with a Banach space E of martingale type p a Banach space $R(\mathrm{E})$. We say that this functor is of class $\mathscr{E}_{p,q}$ if and only if the following six conditions are satisfied.*

(R1) *If E is a Banach space of martingale type p, then there exists a family $(I_t)_{t\geq 0}$, of uniformly bounded linear operators from the class $\mathcal{M}^p_{loc}(R(\mathrm{E}))$ of all $\mathbb{F}$-progressively measurable $R(\mathrm{E})$-valued processes, to the spaces $L^p(\Omega, \mathcal{F}_t, \mathbb{P}; \mathrm{E})$, i.e. there exists a constant $C_p > 0$ such that*

$$\mathbb{E}|I_t(\xi)|^p_{\mathrm{E}} \leq C_p \mathbb{E}\int_0^t \|\xi(r)\|^p_{R(\mathrm{E})}\,dr, \quad t \geq 0. \tag{8.7}$$

(R2) *If E_1 and E_2 are isomorphic, then $R(\mathrm{E}_1)$ and $R(\mathrm{E}_2)$ are isomorphic.*

(R3) *If E_1, E_2 are Banach spaces of martingale type p and $\Phi \in \mathscr{L}(\mathrm{E}_1, \mathrm{E}_2)$, then there exists an operator $\hat{\Phi} \in \mathscr{L}(R(\mathrm{E}_1), R(\mathrm{E}_2))$, called a lift of Φ, such that*

$$\|\hat{\Phi}\xi\|_{R(\mathrm{E}_2)} \leq |\Phi|_{\mathscr{L}(\mathrm{E}_1,\mathrm{E}_2)}\,\|\xi\|_{R(\mathrm{E}_1)}, \quad \xi \in R(\mathrm{E}_1). \tag{8.8}$$

(R4) *If E is a Banach space of martingale type p and $\theta \in (0,1)$ then*

$$R(\mathrm{E}_2, \mathrm{E}_1)_{\theta,p} = (R(\mathrm{E}_2), R(\mathrm{E}_1))_{\theta,p}$$

(R5) *There exists a constant $\hat{C}_q > 0$ such that for every E is a Banach space of martingale type p and every $\xi \in \mathcal{M}^p_{loc}(R(\mathrm{E}))$,*

$$\mathbb{E}|I_t(\xi)|^q_{\mathrm{E}} \leq \hat{C}_q \mathbb{E}\Big(\int_0^t \|\xi(r)\|^p_{R(\mathrm{E})}\, dr\Big)^{\frac{q}{p}}, \quad t \geq 0. \tag{8.9}$$

(R6) *If E is a Banach space of martingale type p, and $0 < \theta_1 < \theta_2 < 1$ and $\delta > 0$, there exists a constant $K > 0$ such that*

$$\int_0^\delta \left\| r^{1-\theta_1} A e^{-rA} \xi \right\|^q_{R(\mathrm{E})} \frac{dr}{r} \ \leq\ K^q\ \|\xi\|^q_{R(D_A(\theta_2,q))}, \qquad \xi \in R(\mathrm{E}). \tag{8.10}$$

where by the definition of the norm on the real interpolation space $(\mathrm{E}, D(A))_{\theta,q}$, see equation (8.4),

$$\|x\|^q_{(\mathrm{E},D(A))_{\theta,q};\delta} = \int_0^\delta \|r^{1-\theta} A e^{-rA} x\|^q_{\mathrm{E}} \frac{dr}{r}.$$

We now proceed to formulate the main auxiliary result of this section.

Theorem 8.6. *Assume that* K *is a separable Banach space, $p \in (1, 2]$. The functor R that associates with every separable Banach space* E *of martingale type p, the space $R(\mathrm{E}) := \Pi_p(\mathrm{K}, \mathrm{E})$, equipped with the norm π_p as defined in Definition* (4.1)*, satisfies the conditions (R1), (R3) and (R6) from Definition 8.5.*

Proof of Theorem 8.6. Let us assume that K is a separable Banach space, $p \in (1, 2]$ and E is a separable Banach space of martingale type p. Put $R(\mathrm{E}) := \Pi_p(\mathrm{K}, \mathrm{E})$, equipped with the norm π_p as defined in Definition (4.1).

First we will show that condition (R1) is satisfied. Later we will show that conditions (R3) and (R6) are satisfied.

For simplicity we consider T instead of ∞. Let $\xi : [0, T] \times \Omega \to \Pi_p(\mathrm{K}, \mathrm{E})$ be a progressively measurable process and $t \in [0, T]$. Let the operator I_t be defined by Definition 5.41 and Theorem 5.35, i.e.

$$I_t : \Lambda^p(0, T; \Pi_p(\mathrm{K}, \mathrm{E})) \to L^p(\Omega, \mathcal{F}_T, \mathbb{P}; \mathrm{E})$$

is a bounded linear operator and, by inequality (5.21) , I_t satisfies inequality (8.7) with a constant C_p independent of t.

Now we will prove that (R3) is satisfied. For this aim, let us assume that E_1, E_2 are Banach spaces of martingale type p and $\Phi \in \mathscr{L}(E_1, E_2)$. We define the following map

$$\hat{\Phi} : \Pi_p(K, E_1) \ni L \mapsto \Phi \circ L \in \Pi_p(K, E_2).$$

By [16, Theorem 2.4], the map $\hat{\Phi}$ is well defined. It is obviously linear and again by [16, Theorem 2.4] it is bounded with norm ≤ 1. So we deduce that condition (R3) is satisfied.

The fact that (R6) is satisfied is a consequence of the following result.

Proposition 8.7. *Let* $R(E) = \Pi_p(K, E)$*. For every* $0 < \theta_1 < \theta_2 < 1$ *and* $\delta > 0$*, condition (R6) holds with constant*

$$K^q \;=\; \frac{C_0^{q(1-\theta_2)}\, C_1^{q\theta_2}\, \delta^{q(\theta_2-\theta_1)}}{q(\theta_2-\theta_1)}.$$

Proof. Choose and fix $\xi \in \Pi_p(K, D_A(\theta_2, q))$.

Step 1: Interpolation estimate.

By the real interpolation theorem for operator norms, using the estimate (8.2) and with interpolation parameter θ_2:

$$\left\|Ae^{-rA}\right\|_{\mathcal{L}(D_A(\theta_2,q),\,E)} \;\leq\; C_0^{1-\theta_2}\, C_1^{\theta_2}\, r^{\theta_2-1}, \qquad r \in (0, \delta]. \tag{8.11}$$

Step 2: Pointwise bound via (R3).

Applying inequality (8.8) with $\Phi = Ae^{-rA} \in \mathcal{L}(D_A(\theta_2, q), E)$:

$$\left\|Ae^{-rA}\xi\right\|_{R(E)} \;\leq\; \left\|Ae^{-rA}\right\|_{\mathcal{L}(D_A(\theta_2,q),\,E)}\; \|\xi\|_{R(D_A(\theta_2,q))}\,.$$

Multiplying by $r^{1-\theta_1}$ and substituting (8.11):

$$\left\|r^{1-\theta_1}Ae^{-rA}\xi\right\|_{R(E)} \;\leq\; C_0^{1-\theta_2}\, C_1^{\theta_2}\, r^{\theta_2-\theta_1}\; \|\xi\|_{R(D_A(\theta_2,q))}\,.$$

Since $\theta_2 > \theta_1$, the power $r^{\theta_2-\theta_1} \to 0$ as $r \to 0^+$.

Step 3: Integration. Since $q(\theta_2 - \theta_1) > 0$, the integral converges:

$$\begin{aligned}\int_0^\delta \left\|r^{1-\theta_1}Ae^{-rA}\xi\right\|_{R(E)}^q \frac{dr}{r} \;&\leq\; C_0^{q(1-\theta_2)}\, C_1^{q\theta_2}\; \|\xi\|_{R(D_A(\theta_2,q))}^q \int_0^\delta r^{q(\theta_2-\theta_1)-1}\, dr \\ &=\; \frac{C_0^{q(1-\theta_2)}\, C_1^{q\theta_2}\, \delta^{q(\theta_2-\theta_1)}}{q(\theta_2-\theta_1)}\; \|\xi\|_{R(D_A(\theta_2,q))}^q \;=\; K^q\; \|\xi\|_{R(D_A(\theta_2,q))}^q\,.\end{aligned}$$

□

$\square$

We are now prepared to state our main result of this section, which is inspired by and based on [8, Theorem 2.2, p. 621-622].

Theorem 8.8. *Let* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *be a filtered probability space with filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. *Assume that* $1 < p \leq q \leq 2$ *and that* E*,* A*,* K*,* R(E) *satisfy the previous assumptions with* R(E) *satisfying conditions (R1), (R3), (R5), and (R6). Let* $0 < \theta_1 < \theta_2 < 1$ *with* $\theta_2 < 1 - \frac{1}{p}$. *Then there exists a constant* $\hat{C} > 0$*, depending only on* p*,* q*,* θ_1*,* θ_2*,* T*, and the constants in (R1), (R5), (R6), such that for every progressively measurable* $R(D_A(\theta_2, q))$*-valued process* ξ *with*

$$\mathbb{E} \int_0^T \|\xi(t,\cdot)\|^q_{R(D_A(\theta_2,q))}\, dt \; < \; \infty,$$

the stochastic convolution

$$\mathrm{SC}_t(\xi) \; := \; I_t\Big(e^{-(t-\cdot)A}\xi(\cdot)\Big), \qquad t \geq 0, \tag{8.12}$$

satisfies

$$\mathbb{E} \int_0^T |\mathrm{SC}_t(\xi)|^q_{D_A\left(\theta_1+\frac{1}{p},q\right)}\, dt \; \leq \; \hat{C}\, \mathbb{E} \int_0^T \|\xi(t,\cdot)\|^q_{R(D_A(\theta_2,q))}\, dt.$$

8.3. **Proofs of The main results.** Note that this proof has been modified to include the numbers θ_1 and θ_2 instead of θ.

Proof of Theorem 8.8. Without loss of generality, we consider $\delta = 1$, i.e., the norm $\|\cdot\|_{D_A(\theta_1+\frac{1}{p},p);1}$ defined by formula (8.4) will be denoted simply as $\||\cdot\|_{D_A(\theta_1+\frac{1}{p},p)}$. We also assume, without loss of generality, that A^{-1} exists [4] and is bounded so that the graph norm in $D(A)$ is equivalent to the norm $\|A\cdot\|$. We fix $\theta_2 \in (0, 1-\frac{1}{p})$.

[4]We assume A generates an analytic semigroup $\{S(t)\}_{t\geq 0}$ on E and that 0 lies in the resolvent set $\rho(A)$. This means A is injective and has a bounded inverse, that is,

$$A^{-1} \in \mathcal{L}(\mathrm{E}).$$

Equivalently,

$$0 \in \rho(A) \quad \Rightarrow A^{-1} \text{ exists and is bounded.}$$

In many analytic frameworks, this property is taken as an assumption. Hence, we simply assume A^{-1} exists and is bounded.

Case 1. $q = p \in (1,2]$

Let us choose and fix $\xi \in \mathcal{M}^p_{loc}(R(\mathrm{E}))$ and $T > 0$. By inequality (8.6), there exists a constant $C_1 > 0$ such that the following inequality holds

$$\mathbb{E}\int_0^T \|SC_t(\xi)\|^p_{D_A(\theta_1+\frac{1}{p},p)}\,dt \leq C_1\mathbb{E}\int_0^T \|SC_t(\xi)\|^p_{D_A^2(\frac{\theta_1}{2}+\frac{1}{2p},p)}\,dt = \cdots \tag{8.13}$$

From definition (8.4), the RHS of (8.13) is equal to

$$\cdots = C_1\mathbb{E}\int_0^T\int_0^\delta \|s^{2(1-\frac{\theta_1}{2}-\frac{1}{2p})}A^2e^{-sA}SC_t(\xi)\|^p_{\mathrm{E}}\frac{ds}{s}\,dt = \cdots$$

Since we chose $\delta = 1$, by applying the Fubini Theorem, we infer that

$$\begin{aligned}\cdots =&C_1\int_0^T\int_0^1 \mathbb{E}\|s^{2-\theta_1-\frac{1}{p}}A^2e^{-sA}SC_t(\xi)\|^p_{\mathrm{E}}\frac{ds}{s}\,dt\\ =&C_1\int_0^T\int_0^1 |s^{2-\theta_1-\frac{1}{p}}|^p\mathbb{E}\|A^2e^{-sA}SC_t(\xi)\|^p_{\mathrm{E}}\frac{ds}{s}\,dt = \cdots\end{aligned}$$

Using the definition (8.12) of $SC_t(\xi)$, we have

$$\cdots =C_1\int_0^T\int_0^1 s^{p(2-\theta_1)-1}\mathbb{E}\|A^2e^{-sA}I_t(e^{-(t-\cdot)A}\xi(\cdot))\|^p_{\mathrm{E}}\frac{ds}{s}\,dt = \cdots$$

Using the fact that the stochastic integral commutes with bounded linear operators, we get

$$\cdots =C_1\int_0^T\int_0^1 s^{p(2-\theta_1)-1}\mathbb{E}\|I_t(A^2e^{-sA}e^{-(t-\cdot)A}\xi(\cdot))\|^p_{\mathrm{E}}\frac{ds}{s}\,dt = \cdots$$

Applying the semigroup property of $(e^{-sA})_{s\geq 0}$, we infer that

$$\cdots =C_1\int_0^T\int_0^1 s^{p(2-\theta_1)-1}\mathbb{E}\|I_t(A^2e^{-(t-\cdot+s)A}\xi(\cdot))\|^p_{\mathrm{E}}\frac{ds}{s}\,dt \leq \cdots$$

Using inequality (8.7) in Assumption (R1), we deduce that

$$\cdots \leq C_1C_p\int_0^T\int_0^1 s^{p(2-\theta_1)-1}\mathbb{E}\Big(\int_0^t \|A^2e^{-(t-u+s)A}\xi(u)\|^p_{R(\mathrm{E})}\,du\Big)\frac{ds}{s}\,dt = \cdots$$

By the Fubini Theorem, changing the order of integration, we have

$$\begin{aligned}\cdots =&C_1C_p\int_0^1\int_0^T s^{p(2-\theta_1)-1}\mathbb{E}\Big(\int_0^t \|A^2e^{-(t-u+s)A}\xi(u)\|^p_{R(\mathrm{E})}\,du\Big)\,dt\,\frac{ds}{s}\\ =&C_1C_p\int_0^1 s^{p(2-\theta_1)-1}\int_0^T \mathbb{E}\Big(\int_0^t \|A^2e^{-(t-u+s)A}\xi(u)\|^p_{R(\mathrm{E})}\,du\Big)\,dt\,\frac{ds}{s} \leq \cdots\end{aligned}$$

Using the semigroup property of $(e^{-sA})_{s\geq 0}$ and applying the Fubini Theorem again, we infer that

$$\begin{aligned}\cdots \leq & C_1 C_p \int_0^1 s^{p(2-\theta_1)-1} \int_0^T \mathbb{E}\Big(\int_0^t \|Ae^{-\frac{t-u+s}{2}A}\|^p \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|^p_{R(\mathrm{E})}\, du\Big)\, dt\, \frac{ds}{s} \\ = & C_1 C_p \mathbb{E} \int_0^1 s^{p(2-\theta_1)-1} \int_0^T \Big(\int_0^t \|Ae^{-\frac{t-u+s}{2}A}\|^p \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|^p_{R(\mathrm{E})}\, du\Big)\, dt\, \frac{ds}{s} \leq \cdots\end{aligned}$$

Applying inequality (8.2), we obtain

$$\cdots \leq C_1 C_2 C_p \mathbb{E} \int_0^1 s^{p(2-\theta_1)-1} \int_0^T \int_0^t (t-u+s)^{-p} \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|^p_{R(\mathrm{E})}\, du\, dt\, \frac{ds}{s} \tag{8.14}$$

Since for $0 \leq u \leq t \leq T$, $s > 0$, we have $\frac{1}{(t-u+s)^p} \leq \frac{1}{s^p}$, we infer that

$$\begin{aligned}&\mathbb{E} \int_0^1 s^{p(2-\theta_1)-1} \int_0^T \int_0^t (t-u+s)^{-p} \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|^p_{R(\mathrm{E})}\, du\, dt\, \frac{ds}{s} \\ &\leq \mathbb{E} \int_0^1 s^{p(2-\theta_1)-1} \int_0^T \int_0^t s^{-p} \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|^p_{R(\mathrm{E})}\, du\, dt\, \frac{ds}{s} \\ &= \mathbb{E} \int_0^1 s^{p(1-\theta_1)-2} \int_0^T \int_0^t \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|^p_{R(\mathrm{E})}\, du\, dt\, ds \\ &= \mathbb{E} \int_0^1 s^{p(1-\theta_1)-2} \int_0^T \int_u^T \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|^p_{R(\mathrm{E})}\, dt\, du\, ds \\ &= \mathbb{E} \int_0^1 \int_0^T \int_u^T s^{p(1-\theta_1)-2} \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|^p_{R(\mathrm{E})}\, dt\, du\, ds \\ &= \mathbb{E} \int_0^T \int_0^1 \int_u^T s^{p(1-\theta_1)-2} \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|^p_{R(\mathrm{E})}\, dt\, ds\, du\end{aligned}$$

by applying the Fubini-Tonelli theorem twice.

Hence, following the train of inequalities starting with (8.13) and concluding with (8.14), we deduce that

$$\mathbb{E} \int_0^T \|SC_t(\xi)\|^p_{D_A(\theta_1+\frac{1}{p},p)}\, dt \leq C_1 C_2 C_p \mathbb{E} \int_0^T \int_0^1 \int_u^T s^{p(1-\theta_1)-2} \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|^p_{R(\mathrm{E})}\, dt\, ds\, du. \tag{8.15}$$

Next, in order to estimate the integral on the RHS of (8.15), we introduce the following change of variables

$$\begin{aligned}\rho &= u,\\ \tau &= t\\ \sigma &= t-u+s \end{aligned}\tag{8.16}$$

Then the set

$$\Big\{(t,s,u) : 0<u<t<T, \quad 0<s<1\Big\}$$

over which the last integral is calculated is no larger than the following set expressed in the new variables

$$\Big\{(\rho,\sigma,\tau) : 0<\rho<T,\\ 0<\sigma<T+1-\rho,\\ \rho<\tau<\rho+\sigma\Big\}.$$

Let us show that if (t,s,u) belongs to the first set, then (ρ,σ,τ) belongs to the second set. For this aim, let us choose (t,s,u) belongs to the first set. Since $t>u$ and $s>0$, by definition (8.16) we infer that $\sigma>0$ and $\tau-\rho>0$. Furthermore, since $s>0$, we infer that $\tau<\rho+\sigma$. As $u>0$ it follows that $\rho>0$. Additionally, since $u<T$, we infer that $\rho<T$. Finally, since $s<1$ and $t<T$, we infer that $\sigma<T+1-\rho$. Thus, all the conditions for (ρ,σ,τ) to belong to the second set are satisfied.

Therefore, by the change of measure theorem, see [21, Theorem 5.1, p. 191], and then by the Fubini-Tonelli theorem, we obtain

$$\begin{aligned}&\mathbb{E}\int_0^T\int_0^1\int_u^T s^{p(1-\theta_1)-2}\|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|^p_{R(\mathrm{E})}\,dt\,ds\,du\\ \leq&\mathbb{E}\int_0^T\int_0^{T+1-\rho}\int_\rho^{\rho+\sigma}(\sigma+\rho-\tau)^{p(1-\theta_1)-2}\|Ae^{-\frac{\sigma}{2}A}\xi(\rho)\|^p_{R(\mathrm{E})}\,d\tau\,d\sigma\,d\rho\\ =&\mathbb{E}\int_0^T\int_0^{T+1-\rho}\|Ae^{-\frac{\sigma}{2}A}\xi(\rho)\|^p_{R(\mathrm{E})}\int_\rho^{\rho+\sigma}(\sigma+\rho-\tau)^{p(1-\theta_1)-2}\,d\tau\,d\sigma\,d\rho=\cdots\end{aligned}$$

Define $\tilde{\tau}=\sigma+\rho-\tau$, since $\rho<\tau<\rho+\sigma$, hence $0<\tilde{\tau}<\sigma$, hence

$$\cdots=\mathbb{E}\int_0^T\int_0^{T+1-\rho}\|Ae^{-\frac{\sigma}{2}A}\xi(\rho)\|^p_{R(\mathrm{E})}\int_0^\sigma\tilde{\tau}^{p(1-\theta_1)-2}d\tilde{\tau}\,d\sigma\,d\rho=\cdots$$

By calculating the last integral

$$\begin{aligned}
\cdots =&\mathbb{E}\int_0^T \int_0^{T+1-\rho} \sigma^{p(1-\theta_1)-1} \|Ae^{-\frac{\sigma}{2}A}\xi(\rho)\|^p_{R(\mathrm{E})}\, d\sigma\, d\rho \\
=&\mathbb{E}\int_0^T \int_0^{T+1-\rho} \sigma^{p(1-\theta_1)} \|Ae^{-\frac{\sigma}{2}A}\xi(\rho)\|^p_{R(\mathrm{E})} \frac{d\sigma}{\sigma}\, d\rho \\
=&2^{p(1-\theta_1)}\mathbb{E}\int_0^T \int_0^{\frac{T+1-\rho}{2}} \|\sigma^{1-\theta_1} Ae^{-\tilde{\sigma}A}\xi(\rho)\|^p_{R(\mathrm{E})} \frac{d\tilde{\sigma}}{\tilde{\sigma}}\, d\rho
\end{aligned}$$

Hence, by applying inequality (8.10) in Assumption (R6) we deduce that

$$\cdots \leq 2^{p(1-\theta_1)} K^p \mathbb{E}\int_0^T \|\xi(\rho)\|^p_{R(D_A(\theta_2,p))}\, d\rho$$

Summing up, we have proved

$$\mathbb{E}\int_0^T \int_0^1 \int_u^T s^{p(1-\theta_1)-2} \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|^p_{R(\mathrm{E})}\, dt\, ds\, du \leq 2^{p(1-\theta_1)} K^p \mathbb{E}\int_0^T \|\xi(\rho)\|^p_{R(D_A(\theta_2,p))}\, d\rho. \tag{8.17}$$

Thus, by inequality (8.15), we infer that there exists a constant $\mathbf{C}_p > 0$ such that

$$\mathbb{E}\int_0^T |SC_t(\xi)|^p_{D_A(\theta_1+\frac{1}{p},p)}\, dt \leq \mathbf{C}_p \mathbb{E}\int_0^T \|\xi(\rho)\|^p_{R(D_A(\theta_2,p))}\, d\rho.$$

Hence, the result is proved in the first case. □

Case 2. $q > p$ **and** $p \in (1,2]$

Let us choose and fix $\xi \in \mathcal{M}^p_{loc}(R(\mathrm{E}))$ and $T > 0$. By inequality (8.6), there exists for a constant $C_1 > 0$ such that the following inequality holds:

$$\mathbb{E}\int_0^T |SC_t(\xi)|^q_{D_A(\theta_1+\frac{1}{p},q)}\, dt \leq C_1 \mathbb{E}\int_0^T |SC_t(\xi)|^q_{D_A^2(\frac{\theta_1}{2}+\frac{1}{2p},q)}\, dt = \cdots \tag{8.18}$$

From definition (8.4), and since we chose $\delta = 1$, applying the Fubini Theorem, we infer that the RHS of (8.18) is equal to:

$$\begin{aligned}
\cdots =&C_1 \int_0^T \int_0^\delta \mathbb{E}\|s^{2(1-\frac{\theta_1}{2}-\frac{1}{2p})} A^2 e^{-sA} SC_t(\xi)\|^q_{\mathrm{E}} \frac{ds}{s}\, dt \\
=&C_1 \int_0^T \int_0^1 \mathbb{E}\|s^{2-\theta_1-\frac{1}{p}} A^2 e^{-sA} SC_t(\xi)\|^q_{\mathrm{E}} \frac{ds}{s}\, dt \\
=&C_1 \int_0^T \int_0^1 s^{(2-\theta_1-\frac{1}{p})q}\, \mathbb{E}\|A^2 e^{-sA} SC_t(\xi)\|^q_{\mathrm{E}} \frac{ds}{s}\, dt = \cdots
\end{aligned}$$

Using the definition (8.12) of $SC_t(\xi)$, the fact that the stochastic integral commutes with bounded linear operators, and the semigroup property of $(e^{-sA})_{s\geq 0}$, we infer that

$$
\begin{aligned}
\cdots =& C_1 \int_0^T \int_0^1 s^{q(2-\theta_1)-\frac{q}{p}} \mathbb{E}\|A^2 e^{-sA} I_t(e^{-(t-\cdot)A}\xi(\cdot))\|_{\mathrm{E}}^q \frac{ds}{s}\, dt \\
=& C_1 \int_0^T \int_0^1 s^{q(2-\theta_1)-\frac{q}{p}} \mathbb{E}\|I_t(A^2 e^{-sA} e^{-(t-\cdot)A}\xi(\cdot))\|_{\mathrm{E}}^q \frac{ds}{s}\, dt \\
=& C_1 \int_0^T \int_0^1 s^{q(2-\theta_1)-\frac{q}{p}} \mathbb{E}\|I_t(A^2 e^{-(t-\cdot+s)A}\xi(\cdot))\|_{\mathrm{E}}^q \frac{ds}{s}\, dt \leq \cdots
\end{aligned}
$$

Hence, by first applying inequality (8.9) in Assumption (R5), then using the Fubini Theorem, next invoking the semigroup property of $(e^{-sA})_{s\geq 0}$ and finally applying the Fubini Theorem again, we deduce that

$$
\begin{aligned}
\cdots \leq& C_1 C_q \int_0^T \int_0^1 s^{q(2-\theta_1)-\frac{q}{p}} \mathbb{E}\Big(\int_0^t \|A^2 e^{-(t-u+s)A}\xi(u)\|_{R(\mathrm{E})}^p du\Big)^{\frac{q}{p}} \frac{ds}{s}\, dt \\
\cdots =& C_1 C_q \int_0^1 \int_0^T s^{q(2-\theta_1)-\frac{q}{p}} \mathbb{E}\Big(\int_0^t \|A^2 e^{-(t-u+s)A}\xi(u)\|_{R(\mathrm{E})}^p du\Big)^{\frac{q}{p}} dt\, \frac{ds}{s} \\
=& C_1 C_q \int_0^1 s^{q(2-\theta_1)-\frac{q}{p}} \int_0^T \mathbb{E}\Big(\int_0^t \|A^2 e^{-(t-u+s)A}\xi(u)\|_{R(\mathrm{E})}^p du\Big)^{\frac{q}{p}} dt\, \frac{ds}{s} \\
\leq& C_1 C_q \int_0^1 s^{q(2-\theta_1)-\frac{q}{p}} \int_0^T \mathbb{E}\Big(\int_0^t \|Ae^{-\frac{t-u+s}{2}A}\|^p \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|_{R(\mathrm{E})}^p du\Big)^{\frac{q}{p}} dt\, \frac{ds}{s} \\
=& C_1 C_q \mathbb{E} \int_0^1 s^{q(2-\theta_1)-\frac{q}{p}} \int_0^T \Big(\int_0^t \|Ae^{-\frac{t-u+s}{2}A}\|^p \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|_{R(\mathrm{E})}^p du\Big)^{\frac{q}{p}} dt\, \frac{ds}{s} \leq \cdots
\end{aligned}
$$

Next, by first applying Hölder inequality, see Theorem D.6, and then using Lemma 8.4 in conjuction with inequality (8.2), we obtain

$$
\begin{aligned}
\cdots \leq& C_1 C_q \mathbb{E} \int_0^1 s^{q(2-\theta_1)-\frac{q}{p}} \int_0^T \Big[\Big(\int_0^t \Big\|Ae^{-\frac{t-u+s}{2}A}\Big\|^{\frac{pq}{q-p}} du\Big)^{\frac{q-p}{p}} \times \int_0^t \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|_{R(\mathrm{E})}^q du\Big]\, dt\, \frac{ds}{s} \\
\leq& C_1 C_2 C_q \mathbb{E} \int_0^1 s^{q(2-\theta_1)-\frac{q}{p}} \int_0^T \frac{1}{s^{q(1-\frac{1}{p})+1}} \int_0^t \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|_{R(\mathrm{E})}^q\, du\, dt\, \frac{ds}{s} \\
=& C_1 C_2 C_q \mathbb{E} \int_0^1 s^{q(2-\theta_1)-\frac{q}{p}-q(1-\frac{1}{p})-1} \int_0^T \int_0^t \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|_{R(\mathrm{E})}^q\, du\, dt\, \frac{ds}{s} \\
=& C_1 C_2 C_q \mathbb{E} \int_0^1 s^{q(1-\theta_1)-2} \int_0^T \int_0^t \|Ae^{-\frac{t-u+s}{2}A}\xi(u)\|_{R(\mathrm{E})}^q\, du\, dt\, ds \leq \cdots
\end{aligned}
$$

Note that inequality (8.17) is valid with p replaced by q, because $q \leq 2$. Hence, by applying inequalities (8.17) and (8.18), we infer that there exists a constant K_q such that

$$
\mathbb{E} \int_0^T |SC_t(\xi)|_{D_A(\theta_1+\frac{1}{p},q)}^q\, dt \leq K_q \mathbb{E} \int_0^T \|\xi(\rho)\|_{R(D_A(\theta_2,p))}^q\, d\rho.
$$

Hence, the result is proved in the second case. □

This concludes the proof of the main Theorem in this section. □

Theorem 8.8 represents one of the main achievements of this thesis.

We finish this Section with the following Open Problems.

Is Theorem 8.8 true, in any form, for $q > 2$?

Is Theorem 8.8 true, in any form, for $\theta_1 = \theta_2$?

Appendices

A. Semigroup

Definition A.1. [29, Definition 2.1, p. 4] *A semigroup $T(t), 0 \leq t < \infty$, of bounded linear operators on Banach space X is strongly continuous semigroup of bounded linear operators if*

$$\lim_{t \to 0} T(t)x = x, \quad \textit{for every} \quad x \in X.$$

We denote the strongly continuous semigroup of bounded linear operators on X by C_0 semigroup.

Theorem A.2. [29, Theorem 2.2, p. 4] *Let $T(t)$ be a C_0 semigroup on a Banach space X. Then there exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t}, \quad \textit{for} \quad 0 \leq t < \infty.$$

Definition A.3. [29, p. 8] *If A is a linear, not necessarily bounded, operator in Banach space X, the resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is invertible, i.e., $(\lambda I - A)^{-1}$ is a bounded linear operator in X. The family $R(\lambda : A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ of bounded linear operators is called the resolvent of A.*

Theorem A.4. [29, Theorem 5.2, p. 62] *Let $T(t)$ be a uniformly bounded C_0 semigroup. Let A be the infinitesimal generator of $T(t)$ and assume that $0 \in \rho(A)$. The following statements are equivalent:*

(a) *$T(t)$ can be extended to an analytic semigroup in a sector $\Delta_\delta = (z : |\arg z| < \delta)$ and $\|T(z)\|$ is uniformly bounded in every closed subsector $\Delta'_\delta < \delta$, of Δ_δ.*

(b) *There exists a constant C such that for every $\sigma > 0, \tau \neq 0$*

$$\|R(\sigma + i\tau : A)\| \leq \frac{C}{|\tau|}.$$

(c) *There exist $0 < \delta < \frac{\pi}{2}$ and $M > 0$ such that*

$$\rho(A) \supset \Sigma = \{\lambda : |\arg \lambda| < \frac{\pi}{2} + \delta\} \cup \{0\}$$

and

$$\|R(\lambda : A)\| \leq \frac{M}{|\lambda|} \quad \textit{for} \quad \lambda \in \Sigma, \lambda \neq 0.$$

(d) *$T(t)$ is differentiable for $t > 0$ and there is a constant C such that*

$$\|AT(t)\| \le \frac{c}{t} \quad \textit{for} \quad t > 0.$$

Assumption A.5. [29, Assumption 6.1, p. 69] *Let A be a densely defined closed linear operator for which*

$$\rho(A) \supset \Sigma^+ = \{\lambda : 0 < \omega < |\arg \lambda| < \pi\} \cup V,$$

where V is neighborhood of zero, and

$$\|R(\lambda : A)\| \le \frac{M}{1 + |\lambda|} \quad \textit{for} \quad \lambda \in \Sigma^+.$$

Proposition A.6. [29, p. 70] *By Assumption A.5, $0 \in \rho(A)$, there exists a constant $\delta > 0$ such that $-A + \delta$ is also an infinitesimal generator of an analytic semigroup. This implies the following estimates:*

$$\|T(t)\| \le M^{-\delta t}$$

$$\|AT(t)\| \le M_1 t^{-1} e^{-\delta t}$$

$$\|A^m T(t)\| \le M_m t^{-m} e^{-\delta t}.$$

Theorem A.7. [29, Theorem 6.13, p. 74] *Let $-A$ be the infinitesimal generator of an analytic semigroup $T(t)$. If $0 \in \rho(A)$ then,*

(a) *$T(t) : X \to D(A^\alpha)$ for every $t > 0$ and $\alpha \ge 0$.*

(b) *For every $x \in D(A^\alpha)$ we have $T(t)A^\alpha x = A^\alpha T(t)x$.*

(c) *For every $t > 0$, the operator $A^\alpha T(t)$ is bounded and*

$$\|A^\alpha T(t)\| \le M_\alpha t^{-\alpha} e^{-\delta t}, \quad \delta > 0.$$

(d) *Let $0 < \alpha < 1$ and $x \in D(A^\alpha)$, then*

$$\|T(t)x - x\| \le C_\alpha t^\alpha \|A^\alpha x\|.$$

Assumption A.8. [29, Assumption (F), p. 196] *Let U be an open subset of $\mathbb{R}^+ \times X_\alpha$. the function $f : U \to X$ satisfies the assumption (F) if for every $(t, x) \in U$ there is a neighborhood $V \subset U$ and constants $L \ge 0, 0 < \vartheta \le 1$ such that*

$$\|f(t_1, x_1) - f(t_2, x_2)\| \le L(|t_1 - t_2|^\vartheta + \|x_1 - x_2\|_\alpha)$$

for all $(t_i, x_i) \in V$.

Theorem A.9. [29, Theorem 3.1, p. 196] *Let $-A$ be the infinitesimal generator of an analytic semigroup $T(t)$ satisfying $\|T(t)\| \leq M$, $M \geq 1$, and assume further that $0 \in \rho(-A)$. If f satisfies the Assumption A.8, then every initial data $t_0, x_0 \in U$, the initial value problem:*

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t, u(t)), & t > t_0; \\ u(t_0) = x_0 \end{cases}$$

has a unique local solution $u \in C([t_0, t_1[: X) \cap C^1(]t_0, t_1[: X)$, where $t_1 = t_1(t_0, x_0) > t_0$, and

$$\|A^\alpha T(t)\| \leq C_\alpha t^{-\alpha}, \quad \text{for} \quad t > 0. \tag{A.1}$$

B. Poisson distribution and Poisson process

Definition B.1. [40, p. 7]

(a) *The characteristic function $\phi(z)$ of a probability measure $\mathbb{P}$ on $\mathbb{R}^d$ is*

$$\phi(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle}\, \mathbb{P}(dx), \quad z \in \mathbb{R}^d.$$

(b) *The characteristic function of the law μ_X of a random variable X on $\mathbb{R}^d$ is*

$$\phi_\mu(z) = \int_{\mathbb{R}^d} e^{i\langle z,x\rangle}\, \mu_X(dx) = \mathbb{E}(e^{i\langle z,X\rangle}), \quad z \in \mathbb{R}^d$$

Definition B.2. [40, p. 10] ($d = 1$) *Let $c > 0$. The Poisson distribution with mean c is defined by*

$$\mathbb{P}(X = n) = \frac{e^{-c}c^n}{n!}, \quad \text{for } n = 0, 1, 2, 3, \cdots$$

with $\mathbb{P}(B) = 0$ for any B containing no nonnegative integer. We have

$$\phi(z) = exp(c\,(e^{iz} - 1)), \quad z \in \mathbb{R}.$$

$$L_P(u) = \int_{[0,\infty)} e^{-ux}\, \mathbb{P}(dx) = exp(c\,(e^{-u} - 1)), \quad u \geq 0.$$

$$\psi(u) = -\eta(iu) = bu + \int_0^\infty \left(1 - e^{-uy}\right) \lambda(dy), \quad u > 0.$$

$$\eta(u) = -\psi(-iu) = ibu + \int_0^\infty \left(e^{-iuy} - 1\right) \lambda(dy), \quad u > 0.$$

Definition B.3. *A Poisson distribution with parameter $c > 0$ is a Borel probability measure on $\mathbb{R}$ such that*

$$\mu(\{n\}) = \frac{c^n e^{-c}}{n!}, \quad n = 0, 1, 2, \cdots$$

We say that a random variable X has a Poisson distribution with parameter $c > 0$, and we write $X \sim Pois(c)$, if and only if the the law of X is a Poisson distribution with parameter $c > 0$.

Definition B.4. *A Poisson process with parameter $c > 0$ is stochastic process $\{X_t : t > 0\}$ if it is a Lévy process such that for $t > 0$, the random variable X_t has a Poisson distribution with parameter ct.*

Definition B.5. *A Poisson process with intensity $\lambda > 0$ is a Lévy process N taking values in $\mathbb{N} \cup 0$ such that*

$$\mathbb{P}(N(t) = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n = 0, 1, 2, \cdots$$

Proposition B.6. *Definitions B.4 and B.5 are the same.*

C. Doob and Burkholder-Davis-Gundy inequalities

In this section we assume that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a σ-finite filtered probability space, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, $1 < p < \infty$,

Let us recall that a submartingale has been defined in Section 2.5.

Theorem C.1. [1, Theorem, 2.1.56, p. 86] *Doob martingale inequality: If $X = (X(t) : t \geq 0)$ is a positive submartingale, see Definition 2.93, then for any $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and for all $t > 0$,*

$$\mathbb{E}\Big(\sup_{0\leq s\leq t} X(s)^p\Big) \leq q^p\, \mathbb{E}(X(t)^p).$$

Theorem C.2. [1, Theorem, 2.1.6, p. 87]*Doob tail martingale inequality If $(X(t), t \geq 0)$ is a positive submartingale, then for any $c > 0, p \geq 1$, and $t \geq 0$*

$$\mathbb{P}\Big(\sup_{0\leq s\leq t} X(s) > c\Big) \leq \frac{1}{c^p}\, \mathbb{E}(X(t)^p).$$

Theorem C.3. [41, Theorem 25.12, p. 308] *Doob maximal L^p inequality: Let $(M_t)_{t\in\mathbb{N}}$ be a martingale or a submartingale, see Definition 2.93. Then,*

$$\left\| \max_{1\le t\le T} |M_t| \right\|_p \le \frac{p}{p-1} \|M_T\|_p \le \frac{p}{p-1} \max_{1\le t\le T} \|M_t\|_p.$$

Theorem C.4. *Burkholder-Davis-Gundy inequalities Let $T > 0$ and $(M_t)_{0\le t\le T}$ be a continuous local martingale such that $M_0 = 0$. For every $0 < p < \infty$, there exist constants c_p and C_p such that*

$$c_p \mathbb{E}\left(\langle M \rangle_T^{\frac{p}{2}} \right) \le \mathbb{E}\left(\left(\sup_{0\le t\le T} |M_t| \right)^p \right) \le C_p \mathbb{E}\left(\langle M \rangle_T^{\frac{p}{2}} \right).$$

where $\langle M \rangle$ denotes the quadratic variation of M.

One can find more information about the Burkholder inequality in the lectures by F. Baudoin [4].

Theorem C.5. [41, p. 387-8] *The Burkholder-Davis-Gundy inequalities for a martingale $(M_t)_{t\in T_0}$ on a probability space $(\Omega, \mathcal{F}, \mu)$:*

$$(BDG) \quad c_p \left\| \sup_{0\le t\le T} |M_t| \right\|_p \le \left\| \langle M \rangle_T^{\frac{1}{2}} \right\|_p \le C_p \left\| \sup_{0\le t\le T} |M_t| \right\|_p$$

for all $T \in \mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ (i.e., positive integers), $0 < p < \infty$ and some absolute constants $0 < c_p \le C_p$. $\langle M \rangle_T$ stands for the quadratic variation of the martingale:

$$\langle M \rangle_T := |M_o|^2 + \sum_{t=0}^{T-1} |M_{t+1} - M_t|^2.$$

As cited in [41, p. 388], a proof of (BDG) can be found in [35, Vol. 2, p. 93-5]. If we combine (BDG) with Doob's maximal L^p inequality, we get

$$(BDG') \quad c_p \|M_T\|_p \le \left\| \langle M \rangle_T^{\frac{1}{2}} \right\|_p \le \frac{pC_p}{p-1} \|M_T\|_p$$

for all $T \in \mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ (i.e., positive integers), $1 < p < \infty$-mind the different range for p in (BGM') compared with (BGM).

D. The function Spaces $\mathscr{L}^p$

Throughout this section $(X, \mathcal{A}, \mu)$ will be some measure space.

Theorem D.1. [41, Definition 11.1] *A $(\mu-)$ null set $N \in \mathcal{N}_\mu$ (the family of all μ-null sets) is a measurable set $N \in \mathcal{A}$ satisfying*

$$N \in \mathcal{N}_\mu \Longleftrightarrow N \in \mathcal{A} \quad \textit{and} \quad \mu(N) = 0.$$

Equivalently,

$$\mathcal{N}_\mu := \{N \in \mathcal{A} : \mu(N) = 0\}.$$

If a property $T = T(x)$ is true for all $x \in X$ apart from some x contained in $\mathcal{N}_\mu$, we say that $T(x)$ holds for $(\mu-)$ almost all (a.a.) $x \in X$ or that T holds $(\mu-)$ almost everywhere (a.e.). In other words,

$$T \textit{ holds a.e.} \Longleftrightarrow \{x : T(x) \quad \textit{fails}\} \subset N \in \mathcal{N}_\mu;$$

$$f = g \quad \textit{a.e.} \Longleftrightarrow \{x : f(x) \neq g(x)\} \textit{ is (contained in) a } \mu - \textit{null set}.$$

$$\exists N \in \mathcal{N}_\mu : \{x : f(x) \neq g(x)\} \subset N.$$

Let us recall that by $\mathcal{M}(\mathcal{A})$ we denote the family of measurable functions; and that $\mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ denote the families of (extended) $\mathbb{R}$-valued measurable functions ($\overline{\mathbb{R}} := [-\infty, +\infty]$).

Let us recall that for a function $f : X \to \mathbb{R}$ we put

$$\{f \neq 0\} := \{x \in X : f(x) \neq 0\},$$

$$\{|f| \geq c\} := \{x \in X : |f(x)| \geq c\}$$

Theorem D.2. [41, Theorem 11.2] *Let $f \in \mathcal{M}_{\overline{\mathbb{R}}}(\mathcal{A})$ be a measurable function on a measure space $(X, \mathcal{A}, \mu)$. Then*

(i) $\int |f| \, d\mu = 0 \Longleftrightarrow |f| = 0$ *a.e.* $\Longleftrightarrow \mu\{f \neq 0\} = 0$,

(ii) $\mathbb{1}_N f \in \mathscr{L}^1_{\overline{\mathbb{R}}}(\mu), \quad \forall N \in \mathcal{N}_\mu$ *and* $\int_N f \, d\mu = 0$.

Definition D.3. [41, p. 116] *Define the sets*

$$\mathscr{L}^p(\mu) := \Big\{f : X \to \mathbb{R} : f \in \mathcal{M}(\mathcal{A}), \int |f|^p \, d\mu < \infty\Big\}, \quad p \in [1, \infty),$$

$$\mathscr{L}^\infty(\mu) := \Big\{f : X \to \mathbb{R} : f \in \mathcal{M}(\mathcal{A}), \exists c > 0, \mu\{|f| \geq c\} = 0\Big\}, \quad p = \infty.$$

The $\mathscr{L}^p(\mu)$ space (or simply $\mathscr{L}^p$−space) and the $\mathscr{L}^\infty(\mu)$ space (or simply $\mathscr{L}^\infty$−space) are the spaces consist of all corresponding f.

By definition, a function f belongs to $\mathscr{L}^\infty(\mu)$ if and only if f is $\mu-$ almost everywhere bounded by some constant $c > 0$. This constant may depend on f. When the choice of measure μ is clear, e.g., μ is the Lebesgue measure on $\mathbb{R}^n$, or μ is the counting measure on a set $\mathcal{A}$, and if we want to stress the underlying space or σ-field, then we write $\mathscr{L}^p(\mathbb{R}^n)$ or $\mathscr{L}^p(\mathcal{A})$ (or $\ell^p(\mathcal{A})$) instead of $\mathscr{L}^p(\mu)$.

Definition D.4. *If $0 < p < \infty$ and if f is a measurable function on X, define*

$$\|f\|_p := \Big(\int |f(x)|^p \, \mu(dx)\Big)^{\frac{1}{p}}, \quad p \in [0,\infty),$$

$$\|f\|_\infty := \inf\Big\{c > 0 : \mu\{|f| \geq c\} = 0\Big\}, \quad p = \infty.$$

Clearly, $f \in \mathscr{L}^p$ if and only if $f \in \mathcal{M}(\mathcal{A})$ and $\|f\|_p < \infty$. We call $\|f\|_p$ the $\mathscr{L}^p$−norm of f.

By Theorem D.2 (i) we have the following results.

(i) $\|f\|_p = 0 \Longleftrightarrow f = 0$ a.e.,

(ii) For all $c \in \mathbb{R}$

$$\|cf\|_p = \Big(\int |cf|^p \, d\mu\Big)^{\frac{1}{p}} = \Big(|c|^p \int |f|^p \, d\mu\Big)^{\frac{1}{p}} = |c| \, \|f\|_p,$$

$$\|cf\|_\infty = \inf\Big\{c' : \mu\Big\{|f| \geq \frac{c'}{|c|}\Big\} = 0\Big\} = \inf\Big\{|c|\, c'' : \mu\Big\{|f| \geq c''\Big\} = 0\Big\} = |c| \, \|f\|_\infty.$$

Theorem D.5. *(Young's inequality)*[41, Lemma 13.1] *Let $p, q \in (1,\infty)$ b conjugate numbers, i.e., $\frac{1}{p} + \frac{1}{q} = 1$, hence $q = \frac{p}{p-1}$. Then*

$$AB \leq \frac{A^p}{p} + \frac{B^q}{q}, \quad \forall A, B \geq 0. \tag{D.1}$$

In (D.1) *equality holds if and only if*

$$B = A^{p-1}.$$

Theorem D.6. *(Hölder's inequality)*[41, Theorem 13.2][37, Theorem 3.8]

If $p, q \in [1,\infty]$ are conjugate extended numbers, i.e., $\frac{1}{p} + \frac{1}{q} = 1$, and if $f \in \mathscr{L}^p(\mu)$ and $g \in \mathscr{L}^q(\mu)$, then $fg \in \mathscr{L}^1(\mu)$ and

$$\Big|\int fg \, d\mu\Big| \leq \int |fg| \, d\mu \leq \|f\|_p \cdot \|g\|_q.$$

Hölder's inequality with $p = q = 2$ is usually called the Cauchy-Schwarz inequality. Note that setting $\frac{1}{\infty} = 0$, we see that $p = 1$ and $q = \infty$ are also conjugate numbers.

Corollary D.7. *(Cauchy-Schwartz inequality)*[41, Corollary 13.3] *Let $f, g \in \mathscr{L}^2(\mu)$. Then $fg \in \mathscr{L}^1(\mu)$ and*

$$\int |fg| \, d\mu \leq \|f\|_2 \cdot \|g\|_2;$$

equality holds if and only if

$$\frac{|f(x)|^2}{\|f\|_2^2} = \frac{|g(x)|^2}{\|g\|_2^2} \quad a.e.$$

Another consequence of Hölder's is the Minkowski or triangle inequality for $\| \cdot \|_p$.

Corollary D.8. *(Minkowski's inequality)*[41, Corollary 13.4][37, Theorem 3.9] *If $p \in [1, \infty]$, $f, g \in \mathscr{L}^p(\mu)$, then $f + g \in \mathscr{L}^p(\mu)$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Definition D.9. [41, p. 120] *A sequence $(f_n)_{n\in\mathbb{N}} \subset \mathscr{L}^p(\mu)$ is convergent in $\mathscr{L}^p$-space with $\mathscr{L}^p$-limit f if and only if $\lim_{n\to\infty} \|f_n - f\|_p = 0$, i.e.,*

$$\mathscr{L}^p\text{-}\lim_{n\to\infty} f_n = f \quad \textit{if and only if} \quad \lim_{n\to\infty} \|f_n - f\|_p = 0.$$

Note that $\mathscr{L}^p$-limits are only almost everywhere unique. If f, g are both $\mathscr{L}^p$-limits of the same sequence $(f_n)_{n\in\mathbb{N}}$, we have

$$\|f - g\|_p \leq \lim_{n\to\infty} (\|f - f_n\|_p + \|f_n - g\|_p) = 0,$$

implying only $f = g$ almost everywhere.

Definition D.10. [41, p. 120] *We call $(f_n)_{n\in\mathbb{N}} \subset \mathscr{L}^p(\mu)$ an ($\mathscr{L}^p$-) Cauchy sequence if*

$$\forall \varepsilon > 0, \, \exists N_\varepsilon \in \mathbb{N}, \quad \forall k, n \geq N_\varepsilon : \|f_n - f_k\|_p < \varepsilon.$$

Lemma D.11. [41, Lemma 13.6] *For any sequence $(f_n)_{n\in\mathbb{N}} \subset \mathscr{L}^p(\mu)$, $p \in [1, \infty]$, of positive functions $f_n \geq 0$, we have*

$$\Big\| \sum_{n=1}^{\infty} f_n \Big\|_p \leq \sum_{n=1}^{\infty} \|f_n\|_p.$$

Theorem D.12. *(Riesz-Fischer)*[41, Theorem 13.7] *The spaces $\mathscr{L}^p(\mu)$, $p \in [1, \infty]$, are complete, i.e., every Cauchy sequence $(f_n)_{n\in\mathbb{N}} \subset \mathscr{L}^p(\mu)$ converges to some limit $f \in \mathscr{L}^p(\mu)$.*

Corollary D.13. [41, Corollary 13.8] *Let $(f_n)_{n\in\mathbb{N}} \subset \mathscr{L}^p(\mu)$, $p \in [1,\infty]$ with $\mathscr{L}^p$-$\lim_{n\to\infty} f_n = f$. Then there exists a subsequence $(f_{n(k)})_{k\in\mathbb{N}}$ such that $\lim_{n\to\infty} f_{n(k)}(x) = f(x)$ holds for almost every $x \in X$.*

Theorem D.14. [41, Theorem 13.9] *Let $(f_n)_{n\in\mathbb{N}} \subset \mathscr{L}^p(\mu)$, $p \in [1,\infty]$ be a sequence of functions such that $|f_n| \leq g$ for all $n \in \mathbb{N}$ and some $g \in \mathscr{L}^p(\mu)$. If $f(x) = \lim_{n\to\infty} f_n(x)$ exists for (almost) every $x \in X$, then*

$$f \in \mathscr{L}^p(\mu) \quad \textit{and} \quad \lim_{n\to\infty} \|f - f_n\|_p = 0.$$

Theorem D.15. [41, Theorem 13.10]*(Riesz's convergence theorem)*

Let $(f_n)_{n\in\mathbb{N}} \subset \mathscr{L}^p(\mu)$, $p \in [1,\infty]$ be a sequence such that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in X$ and some $f \in \mathscr{L}^p(\mu)$. Then

$$\lim_{n\to\infty} \|f - f_n\|_p = 0 \quad \Longleftrightarrow \lim_{n\to\infty} \|f_n\|_p = \|f\|_p.$$

Theorem D.16. [41, Theorem 13.13]*(Jensen's inequality) Let $(X, \mathcal{A}, \mu)$ be a measure space and μ a probability measure.*

(i) *Let $V : [0,\infty) \to [0,\infty)$ be a convex function and extend it to $[0,\infty]$:*

$$V\Big(\int f\, d\mu\Big) \leq \int V(f)\, d\mu, \quad f \in \mathcal{M}(\mathcal{A}), f \geq 0.$$

In particular, if $V(f) \in \mathscr{L}^1(\mu)$, $f \geq 0$, then $f \in \mathscr{L}^1(\mu)$.

Note that if $V : [0,\infty) \to \mathbb{R}$ is convex, then the extension $V : [0,\infty] \to (-\infty,\infty]$, where $V(\infty) := +\infty$ is also convex, see [41, p. 126].

(ii) *Let $\Lambda : [0,\infty) \to [0,\infty)$ be a concave function:*

$$\Lambda\Big(\int f\, d\mu\Big) \geq \int \Lambda(f)\, d\mu, \quad f \in \mathscr{L}^1(\mu), f \geq 0.$$

In particular, if $f \in \mathscr{L}^1(\mu)$, $f \geq 0$, then $\Lambda(f) \in \mathscr{L}^1(\mu)$.

Corollary D.17. [41, Corollary 13.18] *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

(i) *(Hölder inequality) Let $p \in (1,\infty)$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in \mathscr{L}^p(\mu)$ and $g \in \mathscr{L}^q(\mu)$, then $fg \in \mathscr{L}^1(\mu)$ and*

$$\int |fg|\, d\mu \leq \|f\|_p \|g\|_q.$$

(ii) *(Reverse Hölder inequality) Let* $p \in (0,1)$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in \mathscr{L}^p(\mu)$ *and* $g \in \mathscr{L}^q(\mu)$, *then*

$$\int |fg|\, d\mu \geq \|f\|_p \|g\|_q.$$

(iii) *(Minkowski inequality) Let* $p \in [1,\infty)$ *and* $f, g \in \mathscr{L}^p(\mu)$, *then* $f + g \in \mathscr{L}^p(\mu)$ *and*

$$\|f+g\|_p \leq \big\||f| + |g|\big\|_p \leq \|f\|_p + \|g\|_p.$$

(iv) *(Reverse Minkowski inequality) Let* $p \in (0,1)$ *and* $f, g \in \mathscr{L}^p(\mu)$, *then*

$$\big\||f| + |g|\big\|_p \geq \|f\|_p + \|g\|_p.$$

(v) *(Hanner's inequality) Let* $p \in [1,2]$ *and* $f, g \in \mathscr{L}^p(\mu)$, *then*

$$\big|\|f\|_p - \|g\|_p\big|^p + (\|f\|_p + \|g\|_p)^p \leq \|f+g\|_p^p + \|f-g\|_p^p.$$

(vi) *(Hanner's inequality) Let* $p \in [2,\infty)$ *and* $f, g \in \mathscr{L}^p(\mu)$, *then and*

$$\big|\|f\|_p - \|g\|_p\big|^p + (\|f\|_p + \|g\|_p)^p \geq \|f+g\|_p^p + \|f-g\|_p^p.$$

Theorem D.18. [41, Theorem 14.16]*(Minkowski's inequality for integrals) Let* $(X, \mathcal{A}, \mu)$ *and* $(Y, \mathcal{B}, \nu)$ *be σ-finite measure spaces and* $f : X \times Y \to \overline{\mathbb{R}}$ *be* $\mathcal{A} \otimes \mathcal{B}$*-measurable and let* $0 < r \leq s < \infty$. *Then*

$$\Big(\int_X \Big(\int_Y |f(x,y)|^r\, \nu(dy)\Big)^{\frac{s}{r}} \mu(dx)\Big)^{\frac{1}{s}} \leq \Big(\int_Y \Big(\int_X |f(x,y)|^r\, \mu(dx)\Big)^{\frac{r}{s}} \nu(dy)\Big)^{\frac{1}{r}}.$$

The following theorem is a general version of Fubini's theorem for product measures involving n factors, taken from the monograph [3] by Ash et al, see Theorem 2.6.7, Comments 2.6.8, pp. 109-111.

Theorem D.19. *(Fubini Theorem) Let* $\mathcal{F}_j$ *be a σ-field of subsets of* Ω_j, $j = 1, \dots, n$. *Let* μ_j *be a σ-finite measure on* $\mathcal{F}_j$, *for* $j = 1, \dots, n$.

Let $\Omega := \prod_{j=1}^n \Omega_j$ *and* $\mathcal{F} := \bigotimes_{j=1}^n \mathcal{F}_j$, *i.e.* $\mathcal{F}$ *is the smallest σ-field of subsets of* $\prod_{j=1}^n \Omega_j$ *containing all sets of the form* $\prod_{j=1}^n A_j$, *with* $A_j \in \mathcal{F}_j$, $j = 1, \dots, n$. *Then*

(a) *there exists a unique measure* μ *on* $\mathcal{F}$ *such that for every set* $\prod_{j=1}^n A_j$, *with* $A_j \in \mathcal{F}_j$, $j = 1, \dots, n$.

$$\mu\Big(\prod_{j=1}^n A_j\Big) = \prod_{j=1}^n \mu(A_j)$$

The measure μ *is σ-finite on* $\mathcal{F}$.

(b) *if a function $f : \Omega \to \mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$ measurable and non-negative, then the function*

$$\prod_{j=1}^{n-1} \Omega_j \ni (\omega_1, \ldots, \omega_{n-1}) \mapsto \int_{\Omega_n} f(\omega_1, \ldots, \omega_{n-1}, \omega_n)\, \mu(d\omega_n)$$

is well defined almost everywhere and $\bigotimes_{j=1}^{n-1} \mathcal{F}_j/\mathcal{B}(\mathbb{R})$ measurable etc, and the following identity holds

$$\int_\Omega f\, d\mu = \int_{\Omega_1} \mu_1(d\omega_1) \int_{\Omega_2} \mu(d\omega_2) \cdots \int_{\Omega_n} f(\omega_1, \ldots, \omega_n)\, \mu(d\omega_n). \tag{D.2}$$

(c) *If a function $f : \Omega \to Y$, where Y is a Banach space, is Bochner integrable then the function*

$$\prod_{j=1}^{n-1} \Omega_j \ni (\omega_1, \ldots, \omega_{n-1}) \mapsto \int_{\Omega_n} f(\omega_1, \ldots, \omega_{n-1}, \omega_n)\, \mu(d\omega_n)$$

is well defined almost everywhere and Bochner integrable etc, and the identity (D.2) *holds in Bochner sense.*

Definition D.20. [40, p. 5] *For a sequence of events $\{A_n\}$, the upper limit event and the lower limit event are defined by*

$$\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \quad \text{and} \quad \liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k,$$

respectively.

Proposition D.21. [45, Proposition 1.3][40, Proposition 1.11]*(Borel-Cantelli lemma)*

(i) *If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(\limsup_{n\to\infty} A_n) = 0$.*

(ii) *If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ and $\{A_n, n \in \mathbb{N}\}$ is independent, then $\mathbb{P}(\limsup_{n\to\infty} A_n) = 1$.*

Therefore, if $\{A_n, n \in \mathbb{N}\}$ is independent, $\mathbb{P}(\limsup_{n\to\infty} A_n)$ is either 0 or 1.

If f and g are non-negative functions, then

$$\int_0^T f(s)\, g(s)\, ds \le \sup_{s\in[0,T]} f(s) \int_0^T g(s)\, ds$$

Corollary D.22. [1, Corollary 1.1.3, p. 8]*(Fatou's lemma) If $(f_n, n \in \mathbb{N})$ is a sequence of non-negative measurable functions on S, then*

$$\int_S \liminf_{n\to\infty} f_n(x)\mu(dx) \le \liminf_{n\to\infty} \int_S f_n(x)\mu(dx).$$

In particular, if $\lim_{n\to\infty} f_n(x)$ *exists* μ*-almost all* $x \in S$*, then*

$$\int_S \lim_{n\to\infty} f_n(x)\mu(dx) \leq \liminf_{n\to\infty} \int_S f_n(x)\mu(dx). \tag{D.3}$$

If f and g are non-negative functions, then

$$\int_0^T f(s)g(s)\, ds \leq \sup_{s\in[0,T]} f(s) \int_0^T g(s)\, ds$$

E. Banach Fix Point Theorem

Definition E.1. [24, p. 299] *Assume that* X *is a non-empty set. A fix point of a mapping* $T : X \to X$ *is any element* $x \in X$ *such that*

$$Tx = x,$$

i.e the image Tx *coincides with* x*.*

Definition E.2. [24, Definition 5.1-1] *Assume that* (X, d) *is a metric space. A mapping* $T : X \to X$ *is called a strict contraction if and only if there is a real number* c *such that* $0 < c < 1$ *and such that*

$$d(Tx, Ty) \leq c\, d(x, y), \quad \forall x, y \in X.$$

A mapping $T : X \to X$ *is called a contraction if and only if*

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in X.$$

Geometrically strict contraction means that any points x and y have images that are closer together than those points x and y, i.e., $\frac{d(Tx,Ty)}{d(x,y)} \leq c$, where $0 < c < 1$.

Theorem E.3. [24, Theorem 5.1-2]*Banach Fixed Point Theorem (Contraction Theorem) Assume that* (X, d) *is a complete metric space. Suppose that* $T : X \to X$ *is a strict contraction on* X*. Then* T *has precisely one fixed point.*

F. Selected Basic and Cited Definitions and Theorems

Definition F.1. [50, p. 12] *Consider two stochastic processes* $X = (X_t)_{t\geq 0}$ *and* $Y = (Y_t)_{t\geq 0}$ *defined on the same filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *and taking values in the same space* E. *Since* X *and* Y *are functions on* $\mathbb{R}^+ \times \Omega$*, the* E*-valued processes* X *and* Y *are:*

- *equal if and only if* $X_t(\omega) = Y_t(\omega)$*, for every* $t \geq 0$ *and every* $\omega \in \Omega$.
- *indistinguishable (or* $\mathbb{P}$*-equivalent) if and only if*

$$\mathbb{P}\{\omega \in \Omega : X_t(\omega) = Y_t(\omega);\ \forall t \in [0, \infty)\} = 1.$$

- *stochastically equivalent if, for each* $t \geq 0$,

$$\mathbb{P}\{\omega : X_t(\omega) = Y_t(\omega)\} = 1 \quad or \quad \mathbb{P}\{\omega : X_t(\omega) \neq Y_t(\omega)\} = 0.$$

  In this case, the process Y *is said to be a modification of* X*. It then follows that* X *and* Y *have the same finite-dimensional distribution* [1, p. 67].

Note that $1 \geq \mathbb{P}\{X_t = Y_t\} \geq \mathbb{P}\{X_t = Y_t;\ \forall t \in \mathbb{R}^+\} = 1$.

Definition F.2. *If* $(X, |\cdot|_X)$ *and* $(Y, |\cdot|_Y)$ *are normed vector spaces, then we say that*

$$X = Y \text{ with equivalent norms}$$

if and only if, $X = Y$ *in the set theoretical sense, and there exists* $C > 0$ *such that*

$$\frac{1}{C}|z|_Y \leq |z|_X \leq C|z|_Y, \quad z \in X.$$

Definition F.3. *If* X *is a vector space, and* $|\cdot|$ *and* $\|\cdot\|$ *are norms on* X*, then we say that these norms are equivalent if and only if and there exists* $C > 0$ *such that*

$$\frac{1}{C}\|z\| \leq |z| \leq C\|z\|, \quad z \in X.$$

Definition F.4. [11, Definition 6.9, p. 151] *The Brownian motion (or Wiener process) is a stochastic process* $B(t), t \geq 0$ *defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *with the following properties:*

(B1) $B(0) = 0$ *almost surely.*

(B2) *Independent increments: For any* $0 \leq t_1 < t_2 < \cdots < t_n$*, the random variables* $B(t_2) - B(t_1), B(t_3) - B(t_2), \cdots, B(t_n) - B(t_{n-1})$ *are independent.*

(B3) *Stationary increments: The distribution of* $B(t+h) - B(t)$ *depends only on h, i.e.,* $B(t+h) - B(t) \sim N(0, h)$*, where* $N(0, h)$ *denotes a Guassian distribution with mean 0 and variance h.*

(B4) *The sample paths of* $B(t)$ *are continuous in t, alsmost surely.*

Definition F.5. [1, p. 19] *The Dirac delta measure* δ_x *on a measurable space* $(S, \mathcal{A})$ *is a measure defined by*

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$

for all $A \in \mathcal{A}$*. It assigns all its mass to the point* $x \in S$*, and assigns* 0 *mass to any set that does not contain x.*

Definition F.6. [36, Definition 4.13] *A mapping* f *of a set E into* $\mathbb{R}^k$ *is said to be bounded if there is a real number C such that* $|f(x)| \leq C$ *for all* $x \in E$*.*

Definition F.7. [38, p. 6-7] *A topological space is a set in which a collection of all open subsets (i.e., topology) has been specified. A topological vector space with topology is a vector space such that:*

(i) *every singleton, i.e., every one-element set, of the vector space is a closed set,*

(ii) *the vector space operations are continuous w.r.t. to the topology.*

Definition F.8. [24, p. 19] *A topological space is a pair* $(X, \mathcal{T})$*, where X is a set and* $\mathcal{T}$ *is a collection of all open subsets of X. We call the collection (or the set)* $\mathcal{T}$ *a topology for X, which has the following properties*

(T1) $\emptyset \in \mathcal{T}, X \in \mathcal{T}$.

(T2) *The union of any member of* $\mathcal{T}$ *is a member of* $\mathcal{T}$*.*

(T3) *The intersection of finitely many members of* $\mathcal{T}$ *is a member of* $\mathcal{T}$*.*

Definition F.9. [47, p. 3] *A topological space* $(X, \mathcal{T})$ *is a set X and a family* $\mathcal{T}$ *of subsets of X satisfying*

- $\emptyset \in \mathcal{T}$.
- $X \in \mathcal{T}$.
- *If* $\tau \in \mathcal{T}$*, then* $\cup\tau \in \mathcal{T}$*.*

- *If $\tau \in \mathcal{T}$ is a finite subfamily, then $\cap\tau \in \mathcal{T}$.*

We shall often omit $\mathcal{T}$ and refer to X as a topological space.

The family $\mathcal{T}$ is called a topology and the sets which belong to the family $\mathcal{T}$ are called open sets.

A subset A of X is called a closed set if and only if the complement $X \setminus A$ is an open set.

One can show, see also [47, p. 3]*, that a subset A of a topological space X is closed if it contains all its accumulation points.*

Let us recall that an element $x \in X$ is an accumulation point or limit point of a subset A of X if and only if every neighbourhood of x contains at least one element (or point) $a \in A$ different from x.

Definition F.10. [47, Definition 1, p. 23] *A real-valued function $f(x)$ defined on a linear space X is called a semi-norm on X, if the following two conditions hold:*

(i) $f(x + y) \leq f(x) + f(y)$ *(subadditivity);*

(ii) $f(\alpha x) = |\alpha| f(x)$.

Definition F.11. *A symmetric $d \times d$ matrix A is positive definite if all the leading principal minor of order r (where $r = 1, \cdots, d$) of A is positive.*

Definition F.12. *A family $\mathcal{A}$ of subsets of S is a ring if and only if*

(i) $\emptyset \in \mathcal{A}$

(ii) $\forall A, B \in \mathcal{A}, A \cup B \in \mathcal{A}, A - B \in \mathcal{A}, A \cap B \in \mathcal{A}$

Definition F.13. *A random variable $\xi : \Omega \to \mathbb{R}$ is said to be integrable if $\int_\Omega |\xi| d\mathbb{P} < \infty$.*

Definition F.14. *If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, a random variable $\xi : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}-$measurable if $\{\xi \in B\} \in \mathcal{F}$, for every Borel set $B \in \mathcal{B}(\mathbb{R})$.*

Definition F.15. *A metric space X is said to be compact if every sequence in X has a convergent subsequence whose limit is also in X. The family of all such compact sequences are then called the compact set.*

Definition F.16. *If $(\Omega, \mathcal{F})$ is a measurable Banach space, a* K*-valued measure on $(\Omega, \mathcal{F})$ is a mapping $\mu : \mathcal{F} \to \mathrm{K}$ such that for every $A \in \mathcal{B}(\mathrm{K})$, the preimage $\mu^{-1}(A) \in \mathcal{F}$, and it satisfies the following conditions:*

(i) *Null set condition:* $\mu(\emptyset) = 0$*;*

(ii) *Countable additivity: for every sequence* $(A_n, n \in \mathbb{N})$ *of mutually disjoint sets in* $\mathcal{F}$*,*

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

Definition F.17. [38, p. 9][18, p. 51] *Let X denotes a topological vector space with topology* τ*. Then X is an Fréchet space (i.e., the union of all linear bounded subsets of X), if its topology* τ *is induced by a complete invariant metric d. That is, Fréchet space is a linear metric space with the following properties:*

(i) *The metric* $d \in X$ *is invariant, i.e.,* $d(x, y) = d(x - y, 0)$*;*

(ii) *The mapping* $(c, x) \to cx$ *of* $\mathbb{K} \times X \to X$*, (where* $\mathbb{K}$ *is a scalar field, and* $c \in \mathbb{K}, x \in X$*), is continuous in c for each x, and continuous in x for each c;*

(iii) *The metric space X is complete.*

Theorem F.18. [38, Theorem 2.15, p. 51] *The Closed Graph Theorem*

Suppose

(a) *X and Y are Fréchet spaces, see Definition F.17;*

(b) $T : X \to Y$ *is linear;*

(c) $G = \{(x, Tx) : x \in X\}$ *is closed in* $X \times Y$*;*

Then T is continuous.

Theorem F.19. [24, Theorem 2.7-9] *Continuity and boundedness*

Let $T : \mathcal{D}(T) \to Y$ *be a linear operator, where* $\mathcal{D}(T) \subset X$ *is a subspace vector and* X, Y *are normed spaces. Then the following assertions hold:*

(a) *T is continuous if and only if T is bounded.*

(b) *If T is continuous at a single point, it is continuous.*

Definition F.20. [18, I.7.2, p. 27-28] *A map* $f : D \to X$ *of a directed set D into a set X is called a generalized sequence in X. That is f converges to the point* $x \in X$*. A topological space X is a Hausdorff space if and only if no generalized sequence in X has more than one limit.*

Definition F.21. [45, p. 27] *A non-empty family* $(A_j)_{j\in J}$ *of sets is called an ascending (resp. descending) directed family* $(A_j)_{j\in J}$ *if for any* $j_1, j_2 \in J \subset \mathbb{R}$*, there exists a* $j_3 \in J$ *such that* $A_{j_1} \cup A_{j_2} \subset A_{j_3}$ *(resp.* $A_{j_1} \cup A_{j_2} \supset A_{j_3}$*).*

Definition F.22. [14, p. 17] *A collection* $\mathcal{K}$ *of subsets of* Ω *is said to be a* π*-system if* $\emptyset \in \mathcal{K}$ *and if* $A, B \in \mathcal{K}$*, then* $A \cap B \in \mathcal{K}$*.*

Definition F.23. *A random field is a random function over an arbitrary domain, e.g., a stochastic process with some restriction on its index set.*

Definition F.24. [1, p. 84] *The usual conditions (or usual hypotheses) are as follows: (see also https://planetmath.org/filteredprobabilityspace)*

- *(completeness) The probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is complete.*
- *The* σ*-fields* $\mathcal{F}_t$ *contain all the sets in* $\mathcal{F}$ *of zero probability.*
- *(right continuity) The filtration* $\mathcal{F}_t$ *is right continuous,* $\mathcal{F}_{t+} = \mathcal{F}_t$*, I.e., for every non-maximal* $t \in T$*, the* σ*-field* $\mathcal{F}_{t+} \equiv \cap_{s>t}\mathcal{F}_s$ *is equal to* $\mathcal{F}_t$*.*

Definition F.25. [24, Definition 2.1-1] *A vector space (or linear space)* X *over a field* $\mathbb{R}$ *of real numbers is a nonempty set whose elements are called vectors endowed with two operations: addition and multiplication by scalars, i.e.,*

$$+ : X \times X \to X$$

$$\cdot : \mathbb{R} \times X \to X$$

which satisfy the following two conditions

(V1) *vector addition is commutative and associative:* $x + y = y + x$ *and* $x + (y + z) = (x + y) + z$*, where* x, y, z *are vectors.*

(V2) *multiplication of vectors by scalars satisfies:* $\alpha(\beta x) = (\alpha\beta)x$*,* $\alpha(x + y) = \alpha x + \alpha y$ *and* $(\alpha + \beta)x = \alpha x + \beta x$*, where* α, β *are scalars and* x, y *are vectors.*

Similarly we define a vector space over $\mathbb{C}$ *of complex numbers.*

Corollary F.26. [16, Corollary 12.7, p. 235]

(a) *If the Banach space* Y *has cotype* 2*, then* $\Pi_{a,s}(X, Y) = \Pi_r(X, Y)$ *for every Banach space* X *and every* $2 \leq r < \infty$*.*

(b) *If the Banach spaces X and Y both have cotype 2, then $\Pi_{a,s}(X,Y) = \Pi_r(X,Y)$ for every $1 \leq r < \infty$.*

In all cases norms are equivalent, due to the Closed Graph Theorem.

Theorem F.27. [16, Theorem 12.12] *Let $u : X \to Y$ be a bounded linear operator between Banach spaces. Then μ is almost summing if and only if there is a constant c such that*

$$\Big(\int_\Omega \| \sum_{k=1}^{n} g_k(\omega) u x_k\|^2 \, d\mathbb{P}(\omega)\Big)^{1/2} \leq c \cdot \|(x_k)_{k=1}^n\|_2^{weak}$$

for every finite sequence $(x_k)_{k=1}^n \subset X$. Here, (g_k) denotes a sequence of independent standard Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\|(x_k)_{k=1}^n\|_2^{weak}$ represents the weak ℓ^2-norm of the sequence (x_k).

Lemma F.28. [14, Lemma 1.3, p. 16] *Let E be a separable metric space with metric ρ and let X be an E-valued random variable. Then there exists a sequence $\{X_m\}$ of simple E-valued random variables such that, for arbitrary $\omega \in \Omega$, the sequence $\{\rho(X(\omega), X_m(\omega))\}$ is monotonically decreasing to 0.*

Proposition F.29. [14, Proposition 1.4, p. 17] *Assume that $\mathcal{K}$ is a π-system and let $\mathcal{G}$ be the smallest family of subsets of Ω such that*

(i) $\mathcal{K} \subset \mathcal{G}$,

(ii) *if $A \in \mathcal{G}$ then $A^c \in \mathcal{G}$,*

(iii) *if $A_i \in \mathcal{G}, \quad \forall i \in \mathbb{N}$ and $A_n \cap A_m = \emptyset$ for $n \neq m$, then $\cup_{n=1}^{\infty} A_n \in \mathcal{G}$.*

Then $\mathcal{G} = \sigma(\mathcal{K})$.

Theorem F.30. [20, Theorem 6.1.3, p. 119] *Let μ be a measure. If f is measurable and non-negative on a measurable set E, then $\int_E f \, d\mu < \infty$ implies that $f(x) < \infty$ a.e. on E.*

Theorem F.31. [20, Theorem 6.1.1, p. 117] *Let μ be a measure. If f, g are measurable and non-negative on a measurable set E, then $\int_E f \, d\mu \leq \int_E g \, d\mu$, whenever $f(x) \leq g(x)$ on E.*

Theorem F.32. *Lebesgue Dominated Convergence Theorem (LCDT) Suppose that $(S, \mathcal{B}, \mu)$ is a measure space where μ is a non-negative measure and supposes also that a sequence of functions $f_n : S \to \mathbb{R}$ is measurable, $f : S \to \mathbb{R}$ and $g : S \to [0, \infty]$ are also measurable, such that*

(i) $f_n(s) \to f(s)$ *is* $\mu - a.e.$;

(ii) $|f_n(s)| \leq g(s)$ *is* $\mu - a.e.$;

(iii) $\int_S g(s)\,\mu(ds) < \infty$;

Then

$$\int_S f_n(s)\,\mu(ds) \to \int_S f(s)\,\mu(ds). \tag{F.1}$$

Theorem F.33. [7, Theorem A.4] *If the Banach spaces $X \subset Y$, with X continuously and densely embedded in Y, are of M-type p, and $p \in (0, 1)$ the the complex interpolation spaces $[Y, X]_\vartheta$ and the real interpolation space $(Y, X)_{\vartheta,p}$ are of M-type p.*

Theorem F.34. [17, Chapter 3]*(Lebesgue Differentiation Theorem) Let $(X, \|\cdot\|)$ be a Banach space, and let $f : \mathbb{R}^n \to X$ be Bochner integrable. Then for almost every $x \in \mathbb{R}^n$,*

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \|f(y) - f(x)\|\,dy = 0.$$

where $B(x, r)$ is the ball centred at x with radius r, and $|B(x, r)|$ is the Lebesgue measure of the ball.

Theorem F.35. *(Bochner-Lebesgue Dominated Convergence Theorem) Let $(X, \mathcal{A}, \mu)$ be a measure space, and E be a Banach space. A function $f : X \to E$ is Bochner integrable if it is strong measurable and $\int_X \|f(x)\|\,d\mu(x)e < \infty$. Suppose (f_n) is a sequence of Bochner integrable functions $f_n : X \to E$ such that the following conditions are satisfied.*

i *(Pointwise convergence) $f_n(x) \to f$ for a.e. $x \in X$;*

ii *(Domination) there exists $g \in L^1(\mu)$ such that $\|f_n(x)\| \leq g(x)$ for all n and a.e. in x.*

Then

a. *f is Bochner integrable, and*

b. $\int_X f_n(x)\,d\mu(x) \to \int_X f(x)\,d\mu(x)$ *in E, and also*

c. $\|f_n - f\|_{L^1(E)} \to 0$.

The following result is from [15, Chapter IV, Section 44, p. 230-233].

Theorem F.36. *Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions (i.e., right-continuous and complete). Consider the product measurable space $(\mathbb{R}_+ \times \Omega, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F})$. Define*

(a) *the predictable σ-field $\mathcal{P}$: the σ-field generated by all adapted, càglàd processes;*

(b) *the optional σ-field $\mathcal{O}$: the σ-field generated by all adapted, càdlàg processes; and*

(c) *the progressively σ-field $\mathcal{P}\mathrm{ro}$: the σ-field generated by all adapted processes that are continuous in time measurement, i.e. generated by adapted processes that are $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$-measurable for every t.*

Moreover, $\mathcal{P} \subset \mathcal{O} \subset \mathcal{P}\mathrm{ro}$. Then,

(a') *A process is predictable if and only if it is measurable w.r.t. $\mathcal{P}$.*

(b') *A process is optional if and only if it is measurable w.r.t. $\mathcal{O}$.*

(c') *A process is progressively measurable if and only if it is measurable w.r.t. $\mathcal{P}\mathrm{ro}$.*

G. Some proofs

Proof of Exercise 2.96. Let us choose and fix $k \in [0, \infty)$, and an $\mathbb{R}$-valued standard Brownian Motion W_k, we have

$$\sup_k \mathbb{E}\left[\|W_k\|^p\right] \leq C_p \sum_{k=0}^{N} \mathbb{E}[\|W_k - W_{k-1}\|^p];$$

For the left-hand side, since $W_k \sim N(0,k)$, the p-th absolute moment is

$$\mathbb{E}|W_k|^p = k^{p/2}\mathbb{E}|Z|^p, \quad \text{where } Z \sim N(0,1)$$

Hence

$$\sup_k \mathbb{E}|W_k|^p = \max_{0\leq k\leq n} k^{p/2}\mathbb{E}|Z|^p. \tag{G.1-LHS}$$

For the right hand side, since $W_k - W_{k-1} \sim N(0,1)$, hence

$$\mathbb{E}|W_k - W_{k-1}|^p = \mathbb{E}|Z|^p, \quad k = 1, \ldots, n$$

Moreover,

$$W_0 - W_{-1} = W_0 - 0 = 0.$$

Hence

$$\sum_{k=0}^{n} \mathbb{E}|W_k - W_{k-1}|^p = n\mathbb{E}|Z|^p. \tag{G.1-RHS}$$

Putting (G.1-LHS) and (G.1-RHS) together, we now need to show

$$\max_{0\leq k\leq n} k^{p/2}\mathbb{E}|Z|^p \leq C_p n\mathbb{E}|Z|^p$$

Dividing both sides by $\mathbb{E}|Z|^p$, which is none zero, gives

$$\max_{0\le k\le n} k^{p/2} \le C_p n$$

Since $\max_{0\le k\le n} k^{p/2} = n^{p/2}$, hence

$$n^{p/2} \le C_p n \Rightarrow C_p \ge n^{p/2-1} \ge 1, \quad \text{for } \ p \in [1,2].$$

Therefore, $C_p \ge 1$ suffices.

$\square$

Proof of Exercise 3.3. Let U be a Banach space and let $u^* \in U^*$, such that $u^* : U \to \mathbb{R}$ (or $\mathbb{C}$) is a bounded linear operator. Define $\psi : U \to \mathbb{R}$ by

$$\psi(u) := \langle u^*, u\rangle.$$

Since u^* is linear and bounded, so is ψ.

Observe that for any $u \in U$, $\psi(u) \in \mathbb{R}$, so the image of ψ lie in a one-dimensional subspace of $\mathbb{R}$, and it has two possibilities for dimension (or rank):

(1) If $u^* = 0$, then the image is just $\{0\}$, and

$$\text{rank}(\psi) = \dim(\{0\}) = 0.$$

(2) If $u^* \neq 0$, then for any scalar $a \in \mathbb{R}$, there exists some $u^* \in U$ such that $a = \langle u^*, u\rangle \neq 0$, so the image $\psi(u) \neq 0$ and has the dimension exactly 1.

$$\text{rank}(\psi) = \dim(\psi(u)) = 1.$$

$\square$

Proof of Exercise 3.4. Since ψ is a finite rank operator, by definition the image $\psi(U)$ is a finite-dimensional. So there exists $n \in \mathbb{N}$ such that

$$\dim \psi(U) = n.$$

Let $\{v_1, \dots, v_n\} \in V$ be a basis of $\psi(U) \subseteq V$. Then for each $u \in U$, the vector $\psi u \in \psi(U)$, so it can be written uniquely as a linear combination of this basis:

$$\psi u := \sum_{i=1}^{n} \alpha_i(u) v_i,$$

where each $\alpha_i : U \to \mathbb{R}$, defined by:

$$\alpha_i(u) = \text{the } i\text{-th coefficient in the expansion of } \psi u,$$

is a linear map, since ψ is linear. Hence, $\alpha_i \in U^*$.

Define $u_i^* := \alpha_i \in U^*$. Then we have:

$$\psi u = \sum_{i=1}^{n} \langle u_i^*, u\rangle v_i.$$

Now for each $i = 1, \dots, n$, we can multiply by scalars $a_i := 1 \in \mathbb{R}$, so we have:

$$\psi u = \sum_{i=1}^{n} a_i \langle u_i^*, u\rangle v_i.$$

□

Exercise G.1. Show that the infimum of the set of all numbers C for which the inequality (4.1) holds is equal to the minimum of the same set.

Proof of Exercise G.1. Let us begin by recalling the definitions of infimum and minimum.

Definition G.2. *$A \subset \mathbb{R}$ is closed if and only if for every A-valued sequence $\{x_n\}_{n=1}^{\infty}$, if $x_n \to x$ in $\mathbb{R}$, then $x \in A$.*
A subset A of a metric space X is called a closed set if and only if every sequence $(x_n) \in A$ which converges to an element $x \in X$, the limit $x \in A$.

Definition G.3. *A set $A \subset \mathbb{R}$ is*

- *bounded above if and only if*

$$\exists a \in \mathbb{R} : \forall x \in A, x \leq a. \tag{G.1}$$

  A number a satisfying (G.1) *is called an upper bound of the set A.*

- *bounded below if and only if*

$$\exists a \in \mathbb{R} : \forall x \in A, x \geq a. \tag{G.2}$$

  A number a satisfying (G.2) *is called a lower bound of the set A.*

- *bounded if and only if it is both lower and upper bound, i.e.,*

$$\exists a_1, a_2 \in \mathbb{R} : \forall x \in A,\ a_1 \leq x \leq a_2.$$

Definition G.4. *The supremum of a non-empty set $A \subset \mathbb{R}$, denoted by* $\sup A$*, is the smallest upper bound of A. In other words,*

$$\sup A = \min B,$$

where B is the set of all upper bounds of A.

The infimum of a non-empty set $A \subset \mathbb{R}$, denoted by $\inf A$*, is the biggest lower bound of A. In other words,*

$$\inf A = \max C,$$

where C is the set of all lower bounds of A.

One can prove the following:

Proposition G.5. *If $A \subset \mathbb{R}$ is a bounded above set, then* $\sup A$ *exists. Similarly, if $A \subset \mathbb{R}$ is a bounded below set, then* $\inf A$ *exists.*

Proposition G.6. *Suppose that $A \subset \mathbb{R}$ is a bounded below set, so that* $\inf A$ *exists. Suppose further that* $\min A$ *exists. Then*

$$\min A \leq \inf A.$$

Definition G.7. *The maximum of a set $A \subset \mathbb{R}$, denoted by* $\max A$ *is, if exists, the biggest number of A.*

The minimum of a set $A \subset \mathbb{R}$, denoted by $\min A$ *is, if exists, the smallest number of A.*

Proposition G.8. *Assume that $A \subset \mathbb{R}$. Then $a = \min A$ if and only if*

(i) *$a \in A$ and*

(ii) *$a \leq x$ for every $x \in A$.*

Similarly, $b = \max A$ if and only if

(i) *$b \in A$ and*

(ii) *$b \geq x$ for every $x \in A$.*

Using the definition of lower and upper bounds, the above proposition can be rewritten as

Proposition G.9. *Assume that $A \subset \mathbb{R}$. Then $a = \min A$ if and only if*

(i) *$a \in A$ and*

(ii') *a is a lower bound of A.*

Hence, we infer that

Proposition G.10. *Assume that $A \subset \mathbb{R}$. If $\min A$ exists, then A is lower bounded and*

$$\min A = \inf A.$$

Proposition G.10 shows that if $\min A$ exists, then it must equal $\inf A$. The following theorem provides a necessary and sufficient condition for the existence of $\min A$ in terms of the infimum.

Theorem G.11. *If A is a bounded below set and $\inf A \in A$, then $\min A$ exists and*

$$\min A = \inf A.$$

Proof of Theorem G.11. Since A is bounded below, by Proposition G.6, $a = \inf A$ exists. By the definition of infimum, a is a lower bound of A and, by assumption $a \in A$. So conditions (ii') and (i) of Proposition G.9 are satisfied and hence $a = \min A$. □

Theorem G.12. *If a set A is (i) bounded below and (ii) closed, then*

$$\min A = \inf A.$$

Proof of Theorem G.12. Since A is lower bounded, $a = \inf A$ exists.

Step 1, we will show that $a \in A$. Since $a + \frac{1}{n} > a$, by the definition of the infimum, $a + \frac{1}{n}$ is not a lower bound of A. Hence there exists $x_n \in A$: $x_n < a + \frac{1}{n}$. Moreover, because a is a lower bound of A, $a \le x_n$. So we constructed a sequence $\{x_n\}_{n=1}^{\infty}$ such that

$$x_n \in A, \quad a \le x_n < a + \frac{1}{n}.$$

Therefore, by the Comparison Theorem (called sometimes the Sandwich Principle)

$$\lim x_n = a.$$

Indeed, both sides of the inequalities satisfy $a \to a$ and $a + \frac{1}{n} \to a$. But, by assumptions, A is closed and by $x_n \in A$, so $a \in A$. We proved what we claimed in Step 1.

Step 2 Thus by Theorem G.11, $\min A$ exists and $\min A = \inf A$. The proof is complete. □

Let us now proceed to the proof of Exercise G.1.

Proof of Exercise G.1. Let us assume that K and E are normed vector spaces and $\psi : \mathrm{K} \to \mathrm{E}$ is a linear bounded map.

Define a set

$$A := \Big\{ C \geq 0 : |\psi x|_{\mathrm{E}} \leq C|x|_{\mathrm{K}},\ x \in \mathrm{K} \Big\}.$$

We will show the equality

$$\inf A = \min A.$$

In view of Theorem G.12, it is sufficient to prove that (i) A is lower bounded. (ii) A is closed.

By definition, every $C \in A$ satisfies $C \geq 0$ so that A is lower bounded by 0 and hence assumption (i) of Theorem G.12 is satisfied.

To prove (ii) A is closed. Let us take a sequence $\{C_n\} : C_n \in A$ and $C_n \to C$. We will show that $C \in A$.

By the definition of set A, because C_n in A,

$$|\psi x|_{\mathrm{E}} \leq C_n|x|_{\mathrm{K}}, \quad \text{for every } x \in \mathrm{K}. \tag{G.3}$$

Let us choose and fix $x \in \mathrm{K}$. By the Comparison Theorem, since $C_n \to C$, and inequality (G.3) is satisfied for every C_n, we infer by the Comparison Theorem, that it is also satisfied by C, i.e.,

$$|\psi x_n|_{\mathrm{E}} \leq C|x_n|_{\mathrm{K}}.$$

Since x was arbitrary we deduce that

$$|\psi x|_{\mathrm{E}} \leq C|x|_{\mathrm{K}}, \qquad \text{for every } x \in \mathrm{K}.$$

Thus we infer that $C \in A$. Hence assumption (ii) of Theorem G.12 is satisfied. By applying Theorem G.12 we infer that $\min A = \inf A$. This completes the proof. □

□

Proof of Proposition 4.6 Step 1. Let us

(Ad i) show that $\alpha_1\psi_1 + \alpha_2\psi_2$ is a linear map.

Ad Condition (V1)

In order to show that $\psi_1 + \psi_2 = \psi_2 + \psi_1$ is sufficient to show that for every $x \in \mathrm{K}$ the value $(\psi_1 + \psi_2)(x)$ is equal to the value $(\psi_2 + \psi_1)(x)$. By the definition of the sum $\psi_1 + \psi_2$ we have the following train of equalities:

$$(\psi_1 + \psi_2)(x) = \psi_1(x) + \psi_2(x) = \psi_2(x) + \psi_1(x) = (\psi_2 + \psi_1)(x).$$

So we prove equality $\psi_1 + \psi_2 = \psi_2 + \psi_1$.

Similarly, if $\psi_3 \in \Pi_p(\mathrm{K}, \mathrm{E})$, we have

$$\begin{aligned}\psi_1(x) + (\psi_2 + \psi_3)(x) =& \psi_1(x) + [\psi_2(x) + \psi_3(x)] \\ &= \psi_1(x) + \psi_2(x) + \psi_3(x) \\ &= [\psi_1(x) + \psi_2(x)] + \psi_3(x) \\ &= (\psi_1 + \psi_2)(x) + \psi_3(x), \quad x \in \mathrm{K}.\end{aligned}$$

Hence, we have proved the equality $\psi_1 + (\psi_2 + \psi_3) = (\psi_1 + \psi_2) + \psi_3$.

Hence, we have proved condition (V1).

Ad Condition (V2)

For $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\psi_1, \psi_2 \in \Pi_p(\mathrm{K}, \mathrm{E})$, to prove $\alpha_1(\alpha_2\psi_1) = (\alpha_1\alpha_2)\psi_1$, $\alpha_1(\psi_1 + \psi_2) = \alpha_1\psi_1 + \alpha_2\psi_2$ and $(\alpha_1 + \alpha_2)\psi = \alpha_1\psi_1 + \alpha_2\psi_2$, we choose and fix $x \in \mathrm{K}$ and then observe that by the properties of real numbers and the definition (4.3),

(a) for identity $\alpha_1(\alpha_2\psi) = (\alpha_1\alpha_2)\psi$, we have

$$LHS = [\alpha_1(\alpha_2\psi)](x) = \alpha_1[\alpha_2\psi(x)] = \alpha_1\alpha_2\psi(x) = (\alpha_1\alpha_2)\psi(x) = RHS.$$

(b) for identity $\alpha_1(\psi_1 + \psi_2) = \alpha_1\psi_1 + \alpha_2\psi_2$, we have

$$LHS = \alpha_1(\psi_1+\psi_2)(x) = \alpha_1[\psi_1(x)+\psi_2(x)] = \alpha_1\psi_1(x)+\alpha_2\psi_2(x) = (\alpha_1\psi_1+\alpha_2\psi_2)(x) = RHS.$$

(c) for identity $(\alpha_1 + \alpha_2)\psi = \alpha_1\psi_1 + \alpha_2\psi_2$, we have

$$\begin{aligned}LHS &= [(\alpha_1 + \alpha_2)\psi](x) = (\alpha_1 + \alpha_2)\psi(x) = \alpha_1\psi(x) + \alpha_2\psi(x) = [\alpha_1\psi](x) + [\alpha_2\psi](x) \\ &= [\alpha_1\psi + \alpha_2\psi](x) = RHS.\end{aligned}$$

This completes the proof to condition (V2).

Finally, for $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and $\psi_1, \psi_2 \in \Pi_p(\mathrm{K}, \mathrm{E})$, we have

$$
\begin{aligned}
(\alpha_1\psi_1 + \alpha_2\psi_2)(\beta_1 x_1 + \beta_2 x_2) &= \alpha_1\psi_1(\beta_1 x_1 + \beta_2 x_2) + \alpha_2\psi_2(\beta_1 x_1 + \beta_2 x_2) \\
&= (\alpha_1\psi_1)(\beta_1 x_1) + (\alpha_1\psi_1)(\beta_2 x_2) + (\alpha_2\psi_2)(\beta_1 x_1) + (\alpha_2\psi_2)(\beta_2 x_2) \\
&= (\alpha_1\psi_1)(\beta_1 x_1) + (\alpha_2\psi_2)(\beta_1 x_1) + (\alpha_1\psi_1)(\beta_2 x_2) + (\alpha_2\psi_2)(\beta_2 x_2) \\
&= \beta_1(\alpha_1\psi_1(x_1)) + \beta_1(\alpha_2\psi_2(x_1)) + \beta_2(\alpha_1\psi_1(x_2)) + \beta_2(\alpha_2\psi_2(x_2)) \\
&= \beta_1(\alpha_1\psi_1(x_1) + \alpha_2\psi_2(x_1)) + \beta_2(\alpha_1\psi_1(x_2) + \alpha_2\psi_2(x_2)) \\
&= \beta_1(\alpha_1\psi_1 + \alpha_2\psi_2)(x_1) + \beta_2(\alpha_1\psi_1 + \alpha_2\psi_2)(x_2).
\end{aligned}
$$

□

Lemma G.13. *The space $\mathcal{H}_T$ is a vector space.*

Proof of Lemma G.13. Let $X_1, X_2 \in \mathcal{H}_T$, and let $\alpha, \beta \in \mathbb{R}$. Hence X_1, X_2 are both predictable F-valued processes. Since F is a Banach space, hence we have

(1) αX_1 is also predictable F-valued. For all $t \in [0, T]$, $\alpha X_1(t) \in F$ and

$$\sup_{t\in[0,T]} \mathbb{E}[\|\alpha X_1(t)\|_F^p] = |\alpha|^p \sup_{t\in[0,T]} \mathbb{E}[\|X_1(t)\|_F^p] < \infty.$$

Hence $\alpha X_1 \in \mathcal{H}_T$.

(2) $\alpha X_1 + \beta X_2$ is also predictable F-valued. For all $t \in [0, T]$, $\alpha X_1(t) + \beta X_2(t) \in F$ and by using the inequality $\|a + b\|^p \leq 2^{p-1}(\|a\| + \|b\|)^p$, where $a, b \in F$, $p \geq 1$, we have

$$\mathbb{E}(\|\alpha X_1(t) + \beta X_2(t)\|_F^p) \leq 2^{p-1}\Big(|\alpha|^p \mathbb{E}(\|X_1(t)\|_F^p) + |\beta|^p \mathbb{E}(\|X_2(t)\|_F^p)\Big)$$

Taking the supremum over $t \in [0, T]$, we get

$$
\begin{aligned}
\sup_{t\in[0,T]} \mathbb{E}(\|\alpha X_1(t) + \beta X_2(t)\|_F^p) &\leq 2^{p-1}\Big(|\alpha|^p \sup_{t\in[0,T]} \mathbb{E}(\|X_1(t)\|_F^p) + |\beta|^p \sup_{t\in[0,T]} \mathbb{E}(\|X_2(t)\|_F^p)\Big) \\
&= 2^{p-1}\Big(|\alpha|^p \|X(t)\|_{\mathcal{H}_T}^p + |\beta|^p \|X_2(t)\|_{\mathcal{H}_T}^p\Big) < \infty.
\end{aligned}
$$

Hence $\alpha X_1 + \beta X_2 \in \mathcal{H}_T$.

□

Lemma G.14. *$R_p(T)$ is a Banach space.*

Proof of Lemma G.14. Recall the definition of $R_p(T)$ as in (5.7):

$$R_p(T) = \{\psi : [0,T] \times \Omega \to \mathbb{R} : \psi \text{ is } \mathbb{F}\text{-progressively measurable and } \|\psi\|_{R_p} < \infty\},$$

where the norm is given by

$$\|\psi\|_{R_p} = \sup_{t\in(0,T]} \frac{1}{t^{1/p}} (\mathbb{E}\Big[\|\psi(t)\|^p\Big])^{1/p}.$$

Let $(\psi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $R_p(T)$. Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$\|\psi_n - \psi_m\|_{R_p(T)} = \sup_{t\in[0,T]} \frac{1}{t^{1/p}} (\mathbb{E}\|\psi_n(t) - \psi_m(t)\|^p)^{\frac{1}{p}} < \varepsilon.$$

Fix any $t \in [0,T]$. Then, for all $m, n \geq N$,

$$(\mathbb{E}\|\psi_n(t) - \psi_m(t)\|^p)^{\frac{1}{p}} \leq t^{1/p} \cdot \|\psi_n - \psi\|_{R_p} \leq t^{1/p}\varepsilon.$$

Observe that the left-hand side is exactly the norm $\|\psi_n(t) - \psi_m(t)\|_{L^p(\Omega)}$. Hence for each fixed t, the sequence $\psi_n(t) \subset L^p(\Omega)$ is Cauchy:

$$\|\psi_n(t) - \psi_m(t)\|_{L^p(\Omega)} < t^{1/p}\varepsilon.$$

Since $L^p(\Omega)$ is complete, there exists $\psi(t) \in L^p(\Omega)$ such that

$$\lim_{n\to\infty} \psi_n(t) = \psi(t) \quad \text{in } L^p(\Omega).$$

Define $\psi : [0,T] \times \Omega \to \mathbb{R}$ by

$$\psi(t,\omega) := \lim_{n\to\infty} \psi_n(t,\omega) \quad \text{in } L^p(\Omega) \text{ for each } t.$$

Since each ψ_n is $\mathbb{F}$-progressively measurable, and convergence in $L^p(\Omega)$ implies convergence in probability, we infer that ψ admits a progressively measurable version. Hence ψ is $\mathbb{F}$-progressively measurable.

We now show that $\psi \in R_p(T)$, i.e., $\|\psi\|_{R_p} < \infty$.

By the triangle inequality in $L^p(\Omega)$, we have

$$(\mathbb{E}\|\psi(t)\|^p)^{1/p} \leq (\mathbb{E}\|\psi(t) - \psi_n(t)\|^p)^{1/p} + (\mathbb{E}\|\psi_n(t)\|^p)^{1/p}$$

Dividing both sides by $t^{1/p}$ gives

$$\frac{1}{t^{1/p}} \left(\mathbb{E}\|\psi(t)\|^p\right)^{1/p} \leq \frac{1}{t^{1/p}} \left(\mathbb{E}\|\psi(t) - \psi_n(t)\|^p\right)^{1/p} + \frac{1}{t^{1/p}} \left(\mathbb{E}\|\psi_n(t)\|^p\right)^{1/p}$$

Taking the supremum over $t \in [0, T]$, we have

$$\sup_{t\in[0,T]} \frac{1}{t^{1/p}} \left(\mathbb{E}\|\psi(t)\|^p\right)^{1/p} \leq \sup_{t\in[0,T]} \frac{1}{t^{1/p}} \left(\mathbb{E}\|\psi(t) - \psi_n(t)\|^p\right)^{1/p} + \sup_{t\in[0,T]} \frac{1}{t^{1/p}} \left(\mathbb{E}\|\psi_n(t)\|^p\right)^{1/p}.$$

By definition of $\|\cdot\|_{R_p}$ norm, this is equivalent to:

$$\|\psi(t)\|_{R_p} \leq \|\psi(t) - \psi_n(t)\|_{R_p} + \|\psi_n(t)\|_{R_p}.$$

Since (ψ_n) is Cauchy in $R_p(T)$, it is bounded, i.e.,

$$\sup_n \|\psi_n\|_{R_p} < \infty.$$

Moreover, for any $\varepsilon > 0$, there exists N such that for all $n \geq N$, we have $\|\psi - \psi_n\|_{R_p} < \varepsilon$.

Therefore,

$$\|\psi(t)\|_{R_p} \leq \varepsilon + \sup_n \|\psi_n(t)\|_{R_p} < \infty.$$

Hence $\psi \in R_p(T)$.

Finally, we show that $\psi_n \to \psi$ in $R_p(T)$.

Since ψ_n is Cauchy in $\mathcal{H}$, for all $n, m \geq N$,

$$\|\psi_n - \psi_m\|_{R_p} < \varepsilon.$$

Taking the limit as $m \to \infty$, and using the norm continuity of $L^p(\Omega)$, we obtain

$$\|\psi_n - \psi\|_{R_p} \leq \varepsilon.$$

Hence, $\psi_n \to \psi$ in $R_p(T)$.

Hence $R_p(T)$ is complete, i.e., a Banach space.

□

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