Towards Joyce Structures on Moduli Spaces of Meromorphic Quadratic Differentials

Menelaos Zikidis

A thesis submitted in partial fulfilment
of the requirements for the degree of

Doctor of Philosophy

Supervisor: Prof. Tom A. Bridgeland FRS

The University of Sheffield
Faculty of Science
School of Mathematics and Statistics

June 2025

Abstract

In this thesis, we construct a class of geometric structures on moduli spaces of meromorphic quadratic differentials on wild algebraic curves, generalising the work of Bridgeland [15] from the holomorphic to the irregular meromorphic setting. The type of geometric structure we are after is a family of non-linear, flat, meromorphic Ehresmann connections defined on a moduli space X that parametrises irregular, wild quasi-parabolic $SL_2(\mathbb{C})$- Higgs bundles and flat connections on wild curves. The space X fibres over the space of quadratic differentials and resembles a complexification of the Hitchin system for irregular Higgs bundles. The family of Ehresmann connections is the essential object underlying a Joyce structure on spaces of stability conditions in the sense of Bridgeland [14] and arises naturally from the geometry of isomonodromic deformations of irregular connections, prescribing a complex Hyperkähler structure on the underlying integrable system. The main corpus of the thesis is dedicated to building the requisite theory to support the construction of Joyce structures on spaces of meromorphic quadratic differentials with arbitrary poles. Namely, we study decorated wild quasi-parabolic bundles on wild algebraic curves and prove a generalised spectral correspondence between X and the space of anti-invariant, wild branched connections on spectral curves, a result deeply rooted in the ever so popular wild version of the geometric Langlands program. Then we construct an ε-family of flat Ehresmann connections on the resulting integrable system, in the case of wild curves of genus 0. Lastly, we demonstrate the effectiveness of the general theory developed in this thesis by recovering the ad-hoc results of Bridgeland-Masoero [17] as a special case of our hereby established framework.

The research that lead to this thesis was supported by the EPSRC Programme Grant *Enhancing Representation Theory, Noncommutative Algebra And Geometry Through Moduli, Stability And Deformations*, carried out at the University of Sheffield, School of Mathematics and Statistics.

Acknowledgements

First and foremost, I want to thank my advisor, Professor Tom Bridgeland. His supervision was a dream for any student, and I can only wish I was the student he deserved. Sparring with Bridgeland at the blackboard was a highlight of every week and those few occasions when we understood each others thoughts while silently staring at a messy diagram were certainly the intellectual highlights of my life, even if I never stood a chance of telling him anything he didn't already think of. Catching a glimpse of his art of theory-building will be a lesson to accompany me for the rest of my mathematical journey. Most importantly, his attitude and values towards life inspired not just a young mathematician, but a young man seeking to live a life in full. Summarised, he has been to me the epitome of a Mentor. Thank You Sir.

Aside from my mathematical interactions with T.B., my PhD research was an otherwise solitary pursuit. Nonetheless, I benefited immensely from discussions with a number of exceptional mathematicians. I am grateful to T. Pantev, R. Donagi, A. Neitzke, J. Bryan, M. Thaddeus, R. Thomas, E. Shinder, T. Mochizuki, M. Mulase, V. Fock, L. Schaposnick, O. Dumitrescu, A.-P. Nieto, F. Del Monte, P. Descombes, N. Nikolaev, M. Mazzocco, A. Alekseev, C.Y. Mak, E. Franco, A. Brini, R. Hanson, J. Horn and especially my advisors in Heidelberg J. Walcher and A. Wienhard, who have encouraged me to siege war against the forces of my ignorance, since my early undergraduate years and support me to this day. A special thanks goes to A. Neitzke and E. Shinder for taking on the arduous task of evaluating this thesis, a burden no one should have to endure.

Moreover, I want to thank all members of the Algebraic Geometry and Mathematical Physics group in Sheffield. In particular, my office mates and climbing partners who made my time in Sheffield and the gorgeous rugged hills of the Peak District deeply enjoyable: Alberto, Yirui, Yannick, Laura, Keno, Jingxiang, Nadir, Luca, Reinder, Ivan, Alexey, Robert, Kuba, Haoran, Mark, Constantinos and Brad. I am also grateful to my mates with whom I started my mathematical journey in Heidelberg, Milan, Lukas and Max, your friendship will always remain invaluable to me.

At last, I wish to express my deepest gratitude to my family, my mother Marianthi, my father Theodoros, my brother Christodoulos, and Linda. Their devotion and love are the most important things I was ever blessed to enjoy. Particular thanks need to go to Linda for catching a myriad of typos in this thesis, putting up with me and making my life a bliss on a daily basis.

Contents

1. Introduction

The core of the present thesis is dedicated towards building the theory, that generalises the geometric structures and correspondences between them, which enabled Bridgeland to construct a Joyce structure on the space of holomorphic quadratic differentials. This project lies within the realm of Bridgeland's geometrisation programme of Donaldson-Thomas theory, the output of which, is the theory of Joyce structures. Nevertheless, familiarity with Donaldson-Thomas theory, wall-crossing phenomena or stability conditions is not assumed, nor is it necessary to appreciate the value of this thesis.

Although the thesis is guided by the teleological goal of building the correct theory supporting a class of meromorphic Joyce structures, the geometric objects and moduli spaces we construct and study are of great ontological significance themselves. They are deeply connected to the theory of integrable systems, non-abelian Hodge theory, Hyperkähler geometry, and our theory and results are expected to find direct application in the wild version of the geometric Langlands programme, which is currently drawing a lot of attention.

Within the framework of Bridgeland's programme, the value of this thesis is clear, it provides the general theoretic framework for treating a large class of examples, indeed, at present, an essentially exhaustive list of all cases where Joyce structures can be constructed explicitly. For example we treat the cases that arise as isomonodromic flows of Painlevé systems.

Geometry from Donaldson-Thomas Invariants

Recent years have seen the development of a programme initiated by T. Bridgeland aiming to encode Donaldson-Thomas invariants of triangulated Calabi-Yau 3 categories in a geometric structure over their space of stability conditions [13]. These were christened *Joyce Structures* and were introduced by Bridgeland in [14], inspired by prior work of Joyce-Song [45], Kontsevich-Soibelman [52], Gaiotto-Moore-Neitzke [31], as well as the role played by Frobenius manifolds in Gromov-Witten theory [25].

In essence, the variation of Donaldson-Thomas invariants with respect to the parameters on the space of stability conditions gives rise to discontinuous functions, jumping across the walls of marginal stability. Their jumping behaviour is governed by the Wall-crossing formula of Kontsevich-Soibelman, and it is this gadget that allows us to recast the variation of DT invariants into an analytic irregular Riemann-Hilbert type problem, by interpreting the wall-crossing data as Stokes data of some differential equation developing irregular singularities. Solutions to the Riemann-Hilbert problem encode the variation of DT invariants into a complex-analytic geometric structure on the space of stability conditions. Drawing an analogy with the way Frobenius structures on quantum cohomology encode genus 0 GW-invariants, Bridgeland proposed that the relevant geometric structure should consist of a family of flat Ehresmann connections on the tangent bundle of the space of stability conditions, subject to certain naturally occurring symmetries. More precisely, a Joyce structure consists of a pencil of flat, symplectic Ehresmann connections, equipped with a compatible integral affine structure, called a *period structure*. In contrast to the Frobenius case, the connection encoding the variation of the enumerative invariants has a non-linear structure group, namely a group of symplectic automorphisms of an algebraic torus. A Joyce structure naturally induces a complex Hyperkähler structure on the space of stability conditions and admits a description in terms of a twistor space.

What makes Bridgeland's geometrisation programme largely conjectural, is that solutions to the so-occurring non-linear Riemann-Hilbert problems are mostly out of reach, since there is no general theory allowing us to prove existence type results, and no such theory is really expected to exist. It is only in certain examples that we have sufficient existence results and this thesis is a leap towards constructing Joyce structures in a large class of examples, leveraging their relation to complex integrable systems in order to bypass the explicit solution of unwieldy Riemann-Hilbert problems.

Theories of Class-$S[A_1]$

In their seminal work, Gaiotto-Moore-Neitzke [31] realised the relation between wall-crossing phenomena in DT-theory and real Hyperkähler structures arising in the study of BPS-spectra of 4-dimensional $\mathcal{N} = 2$ supersymmetric field theories. Within their framework, the Hitchin moduli space $\mathcal{M}(E, \Phi)$, parametrising Higgs bundles over a Riemann surface, appears as the classical phase space of so-called theories of class-S. This space admits a Lagrangian fibration as a complex integrable system over the space of higher differentials on the curve and serves as the Coulomb branch of the associated field theory. Considering the case with gauge group $G = SL_2(\mathbb{C})$, that is, $SL_2(\mathbb{C})$ Higgs bundles, the space of higher differentials reduces to the space of quadratic differentials $\mathcal{M}(Q)$. We call these theories of class-$S[A_1]$.

Gaiotto, Moore and Neitzke demonstrated that the trajectory structure of quadratic differentials encodes the BPS-spectrum via WKB-analysis. Moreover, they showed that Hitchin's celebrated real Hyperkähler metric on the moduli of Higgs bundles [41], can be reconstructed from Stokes data by means of a Thermodynamic Bethe Ansatz integral equation, whose input consists of the BPS invariants and their wall-crossing behaviour.

In subsequent work, Bridgeland and Smith [18] identified the space $\mathcal{M}(C,Q)$, parametrising complex projective curves of fixed genus g and meromorphic quadratic differentials with simple zeros on those curves, with a space of Bridgeland stability conditions on a class of derived triangulated categories associated to quivers with potential. Crucially, this correspondence relies on results linking the horizontal foliation defined by the quadratic differentials to quivers with potential and interprets finite-length trajectories of the foliation (i.e. BPS-invariants via GMN) as stable objects in the corresponding triangulated category. Note that the space $\mathcal{M}(C,Q)$ has twice the dimension of the Hitchin base $\mathcal{M}(Q)$.

Having a very explicit model of a space of stability conditions in the form of $\mathcal{M}(C,Q)$, where DT invariants and their wall-crossing behaviour are well understood, makes it a natural setting to implement Bridgeland's geometrisation programme. In [15], Bridgeland demonstrated that this is possible in the case of holomorphic quadratic differentials, arguably the simplest, yet rather degenerate case from the point of view of the Bridgeland-Smith correspondence [35].

This thesis aims to generalise the framework that enabled Bridgeland to construct Joyce structures in the case of holomorphic quadratic differentials, to the case of meromorphic quadratic differentials with arbitrary pole orders. As we mentioned above, this pursuit is not merely for the sake of generalisation, but it provides the general theory in order to study of the only non-trivial class of explicit examples, where one can hope to determine the Joyce structure for the time being cf. [16],[17].

Summary of the Holomorphic Case

Here we give a shorthand outline of the framework we are setting off to generalise, as published in [15], which will serve as a scaffolding throughout this thesis. Given a point $(C,Q)\in\mathcal{M}(C,Q)$, where $Q\in H^0(C,\omega_C^{\otimes 2})$, there is an associated smooth curve Σ, cut out by the equation $y^2=\pi^*Q$ inside the surface $\pi: Tot(\omega_C)\to C$. This construction yields a simply ramified double cover $p:=\pi|_\Sigma:\Sigma\to C$. The tangent space $T_{(C,Q)}\mathcal{M}(C,Q)$ can be identified with the anti-invariant part of the first cohomology group of the associated spectral curve $H^1(\Sigma,\mathbb{C})^-$. This cohomology group carries a natural integral structure $T^{\mathbb{Z}}_{(C,Q)}:=(H_1(\Sigma,\mathbb{Z})^-)^*\subset T_{(C,Q)}\mathcal{M}(C,Q)\cong H^1(\Sigma,\mathbb{C})^-$. Thus, the resulting quotient $X^{\#}:=T\mathcal{M}(C,Q)/T^{\mathbb{Z}}$ carries the structure of a $(\mathbb{C}^*)^n$-fibration $X^{\#}\to\mathcal{M}(C,Q)$, with fibre $H^1(\Sigma,\mathbb{C}^*)^-$. These fibres may be interpreted as monodromy groups for anti-invariant abelian flat bundles on the spectral covers via the Riemann-Hilbert correspondence. Inspired by the work of Donagi and Pantev [27], Bridgeland establishes a birational correspondence between the space $\mathcal{M}(\Sigma,L,\partial)$, which parametrises spectral covers equipped with abelian flat bundles, and a space $\mathcal{M}(E,\Phi,\nabla)$, which parametrises $SL_2(\mathbb{C})$-Higgs fields Φ

and $SL_2(\mathbb{C})$-flat connections ∇ on the same holomorphic bundle E. This correspondence extends the well-known spectral construction for holomorphic $SL_2(\mathbb{C})$-Higgs bundles to connections. Through this birational identification of $X^{\#}$ with $\mathcal{M}(C, E, \Phi, \nabla)$, the flows of the ε-family of non-linear Ehresmann connections on $X^{\#} \to M$, are given as the fibres of the map from $\mathcal{M}(C, E, \Phi, \nabla)$ to the character variety $\mathscr{X}_g$, that sends a pair (Φ, ∇) to the monodromy of the connection $\nabla + \varepsilon^{-1}\Phi$. The following diagram summarises these constructions

$$\begin{tikzcd}
 & & \mathcal{M}(C, E, \Phi, \nabla) \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & \\
X^{\#} \arrow[d] & \mathcal{M}(C, Q, \mathcal{L}, \partial) \arrow[l, "\mu_1"'] & & \mathcal{M}(C, E, Q, \nabla) \arrow[r, "\mu_2"] \arrow[d] & \mathscr{X}_g \arrow[d] \\
\mathcal{M}(C, Q) \arrow[rrr, "\sim", "id"'] & & & \mathcal{M}(C, Q) \arrow[r] & \mathcal{M}(C).
\end{tikzcd} \tag{1.1}$$

The nature of the maps α, β allows us to transfer the so constructed family of connections throughout the diagram, ultimately inducing a well defined family of non-linear connections on the bundle $X^{\#} \to \mathcal{M}(C, Q)$.

PDF-Gadget: The table of contents is clickable and one can jump back to the table of contents by clicking on the initials $\boxed{\mathcal{M}.\mathcal{Z}.}$ at the top-right corner of every page.

Notation: Given a stack $\mathcal{M}(A, B, C, \dots)$ by great abuse of notation we write $(A, B, C, \dots) \in \mathcal{M}(A, B, C, \dots)$ to mean $(A, B, C, \dots) \in Obj\,\mathcal{M}(A, B, C, \dots)(\mathbb{C})$.

Summary of Central Constructions and Results in this Thesis

This thesis is a systematic generalisation of diagram 1.1- the spaces and their relations, our starting point being to pass from the space of holomorphic quadratic differentials to meromorphic quadratic differentials with arbitrary irregular poles. This will eventually allow us to construct an ε-family of non-linear, flat connections, the essential ingredient of a Joyce structure. We will do so step by step, and this section is an outline of that pursuit, an *Ariadne's thread* for reading this thesis.

We start by fixing data $(g, \mathscr{m}) := (g, (m_1, \dots, m_d))$ such that $g \in \mathbb{Z}_{\geq 0}$, $(m_i)_i \in \mathbb{Z}^d_{>1}$,

$$6g - 6 + d + \sum_i^d m_i > 0, \tag{1.2}$$

and is **not** one of the following cases: $(0, \{6\}); (0, \{3, 3\}); (1, \{2\})$. We also set $n_i := \lceil \frac{m_i}{2} \rceil$.

$\boxed{\mathcal{M}(\mathscr{C})}$ **Wild Algebraic Curves [Def. 5.1.7]:** Instead of mere algebraic curves we consider *wild algebraic curves*, which we introduce as a tuple $\mathscr{C} = (C, D, \nu, \xi)$ of:

- an algebraic curve C of genus g,
- an effective divisor $D = \sum_i^d n_i \cdot p_i$ in C,
- a section $\nu \in H^0(D, \omega_C(D)^{\otimes 2}|_D)$, which defines a double cover $p_\nu : D_\nu \to D$,
- for the induced covering over the reduced divisors $p_\nu^{red} : D_\nu^{red} \to D^{red}$ by p_ν, a section $\xi : D^{red} \to D_\nu^{red}$ of p_ν, i.e. $p_\nu^{red}(\xi(p)) = p$.

The last ingredient ξ, is essentially fixing a single pre-image of every geometric point of D in the cover D_ν. In particular ξ determines a divisor $D_\xi := \sum_i n_i \cdot \xi(p_i)$. Moreover, we call $\lambda_\xi := \sqrt{p_\nu^* \nu}|_{D_\xi}$ the irregular type of the wild curve and denote the space of wild curves by $\mathcal{M}(\mathscr{C})$.

$\boxed{\mathcal{M}(\mathscr{C}, Q)}$ **Quadratic Differentials on Wild Curves [Def. 5.2.1]:** Here we allow the quadratic differentials on a curve C to develop poles along an effective divisor, i.e. we consider sections $Q \in H^0(C, \omega_C(D)^{\otimes 2})$ and still only allow them to have simple zeros. We say that such a section Q is a *meromorphic quadratic differential on a wild curve* $\mathscr{C} = (C, D, \nu, \xi)$ if $Q|_D = \nu \in H^0(C, \omega_C(D)^{\otimes 2}|_D)$. We prove in Theorem 5.2.22, that the space of quadratic differentials can be realised as a cotangent bundle. A quadratic differential Q determines a unique smooth closed subscheme of $Tot(\omega_C(D))$:

$$\begin{tikzcd}
\Sigma_Q \arrow[r, hook] \arrow[d, dashed, "p"'] & Tot(\omega_C(D)) \arrow[dl, "\pi"] \\
C &
\end{tikzcd}$$

such that $p : \Sigma_Q \to C$ is a double cover of smooth curves ramified over a reduced divisor R, which we call a *spectral cover* of C. The spectral cover comes equipped with a distinguished 1-form $\lambda \in H^0(\Sigma, \omega_\Sigma(p^*D))$, and a Galois involution $\sigma \in Aut(\Sigma)$ switching the sheets of the cover. For a quadratic differential with $Q|_D = \nu$, we have $p^*D = D_\nu$, $p|_D = p_\nu$, $\lambda|_{D_\xi} = \lambda_\xi$. We denote the space parametrising wild curves and quadratic differentials on wild curves by $\mathcal{M}(\mathscr{C}, Q)$.

$\boxed{\mathscr{H}}$ **- Anti-Invariant Cohomology Bundle [Def. 5.2.6]:** For a point $(\mathscr{C}, Q) \in \mathcal{M}(\mathscr{C}, Q)$ with spectral curve Σ, we denote by $H_1(\Sigma, \mathbb{Z})^-$ the subgroup of σ-anti-invariant homology cycles, where the $-$ sign refers to the -1-Eigenspace of the induced action by the involution on cohomology of the spectral cover. We construct a bundle $\mathscr{H} \to \mathcal{M}(\mathscr{C}, Q)$ with fibres $\tilde{H}^1(\Sigma_Q, \mathbb{C})^- := Hom(H_1(\Sigma, \mathbb{Z})^-, \mathbb{C})$. We show that $\mathscr{H}$ admits a unique flat connection preserving the natural integral lattice $\mathscr{H}^{\mathbb{Z}}$ with fibres $Hom_{\mathbb{Z}}(H_1(\Sigma_Q, \mathbb{Z})^-, \mathbb{Z})$, and we build the quotient $\mathscr{H}^\# := \mathscr{H}/\mathscr{H}^{\mathbb{Z}}$, which in turn gives rise to a fibre bundle $\pi : \mathscr{H}^\# \to \mathcal{M}(\mathscr{C}, Q)$ having torus fibres.

$\boxed{\mathcal{M}(\mathscr{C}, \mathscr{E})}$ **Wild Quasi-Parabolic Bundles on Wild Curves [Def. 6.1.4]:** In the holomorphic picture, we considered an $SL_2(\mathbb{C})$-algebraic bundle E on a curve C, i.e. a bundle with trivial determinant $\bigwedge^2 E = \mathcal{O}_C$. Now, given a wild curve $\mathscr{C}$, we equip E with an $\mathcal{O}_{D_\xi}$-submodule $L_\xi < p_\nu^*(E)|_{D_\xi}$. We call the choice of such a submodule a *wild quasi-parabolic structure* on E, and the pair $\mathscr{E} = (E, L_\xi)$ a *wild quasi-parabolic bundle* on $\mathscr{C}$. Unlike a classical quasi-parabolic structure, the filtration of the bundle is not over a closed point, but rather over nilpotent neighbourhoods of points. We denote by $\mathfrak{sl}_2(E)$ the trace-free endomorphisms of E, by $End^{\mathfrak{p}}(\mathscr{E}) \subset \mathfrak{sl}_2(E)$ the subalgebra of endomorphisms of E preserving L_ξ, and by $End^{\mathfrak{n}}(\mathscr{E}) \subset End^{\mathfrak{p}}(\mathscr{E})$ the subalgebra of nilpotent endomorphisms of E preserving L_ξ.

$\boxed{\mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi)}$ **Wild Quasi-Parabolic Higgs Bundles on Wild Curves [Def. 6.2.4]:** Given a wild curve $\mathscr{C} = (C, D, \nu, \xi)$ and a quasi-parabolic bundle $\mathscr{E} = (E, L_\xi)$ on $\mathscr{C}$, we define a *wild quasi-parabolic Higgs bundle on* $\mathscr{E}$ as a section $\Phi \in H^0(C, \mathfrak{sl}_2(E) \otimes \omega_C(D))$ such that $tr(\Phi^2)|_D = \nu$ and $p_\nu^*\Phi|_{D_\xi}(s) = \lambda_\xi \cdot s \; \forall s \in H^0(D_\xi, L_\xi)$. In particular this means that $\Phi \in H^0(C, End^{\mathfrak{p}}(\mathscr{E}) \otimes \omega_C(D))$, and we say that Φ has irregular type λ_ξ. Effectively, this definition allows us to control the irregular part of the eigenvalues and eigensheaf of the Higgs field. Theorem 6.2.8 relates the fibres of the forgetful map $\mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi) \to \mathcal{M}(\mathscr{C}, \mathscr{E})$, consisting of wild quasi-parabolic Higgs bundles on a fixed wild bundle $(\mathscr{C}, \mathscr{E})$, to the cotangent space on the space $\mathcal{M}(\mathscr{C}, \mathscr{E})$. In Theorem 6.2.9, we study the deformation theory of wild quasi-parabolic Higgs bundles on wild curves.

$\boxed{\mathcal{M}(\mathscr{C}, \mathscr{E}, \nabla)}$ **Wild Quasi-Parabolic Flat Bundles on Wild Curves [Def. 6.2.4]:** Analogously, we define flat connections $\nabla : E \to E \otimes \omega_C(D)$, preserving the same quasi-parabolic structure and of the same irregular type, and we denote the space of such by

$\mathcal{M}(\mathscr{C},\mathscr{E},\nabla)$. Moreover, we consider flat connections $\nabla_\infty : E \to E \otimes \omega_C(D)$ that are nilpotent with respect to the quasi-parabolic structure, i.e. $\nabla_\infty|_D \in End^{\mathfrak{n}}(\mathscr{E})$. We denote the space of nilpotent connections by $\mathcal{M}(\mathscr{C},\mathscr{E},\nabla_\infty)$. In Theorem 6.2.9, we study the deformation theory of wild quasi-parabolic flat bundles on wild curves. In proposition 6.2.14, we construct an Atiyah-Bott type skew-symmetric form on the space of such connections.

$\boxed{\mathcal{M}(\mathscr{C},\mathscr{E},\Phi,\nabla_\infty)}$ **Wild Quasi-Parabolic Higgs and Flat Bundles:** We build the fibre product

$$\begin{tikzcd}
\mathcal{M}(\mathscr{C},\mathscr{E},\Phi,\nabla_\infty) \arrow[r] \arrow[d] & \mathcal{M}(\mathscr{C},\mathscr{E},\Phi) \arrow[d] \\
\mathcal{M}(\mathscr{C},\mathscr{E},\nabla_\infty) \arrow[r] & \mathcal{M}(\mathscr{C},\mathscr{E})
\end{tikzcd}$$

consisting of Higgs bundles and nilpotent flat connections on the same quasi-parabolic bundle over the same wild curve $\mathscr{C}$.

$\boxed{\mathcal{M}(\mathscr{C},Q,\mathcal{L},\partial_\infty)}$ **Wild Anti-invariant Abelian Branched Connections [Def. 6.3.1]:** Given a pair $(\mathscr{C},Q)$ and the associated spectral cover $\Sigma \xrightarrow{p} C$, we first construct a canonical connection $\partial_{can} : p^*\omega_C(D) \to p^*\omega_C(D) \otimes \omega_\Sigma(R)$. An *anti-invariant abelian branched connection* is a pair $(\mathcal{L},\partial_\infty)$ consisting of a line bundle $\mathcal{L} \in \mathcal{P}ic(\Sigma_Q)$ and a connection $\partial_\infty : \mathcal{L} \to \mathcal{L} \otimes \omega_{\Sigma_Q}(R)$ such that $(\mathcal{L} \otimes \sigma^*\mathcal{L}, \partial_\infty \otimes \sigma^*\partial_\infty) = (p^*\omega_C(D), \partial_{can})$. We show that this is the Riemann-Hilbert counterpart of the cohomology bundle $\mathscr{H}$, more precisely, Proposition 6.3.11 shows that $\mathscr{H}^{\#}$ arises as an étale quotient of the space of branched connections.

$\boxed{\mathscr{P}}$ **Period Structure on $\mathcal{M}(\mathscr{C},Q)$ [Sec. 5.2.1]:** We recall the construction of a period section of the cohomology bundle $\delta : \mathcal{M}(\mathscr{C},Q) \to \mathscr{H}$. Its Gauß-Manin derivative gives rise to an isomorphism $\mathscr{H} \cong T\mathcal{M}(\mathscr{C},Q)$ that allows us to define a set of local coordinates on $\mathcal{M}(\mathscr{C},Q)$. Then, we describe a period structure on the space of quadratic differentials on wild curves in the sense of [14].

$\boxed{\omega}$ **Poisson and Symplectic Structures [Sec. 5.2.2]:** We describe a natural Poisson structure on $\mathcal{M}(\mathscr{C},Q)$, utilising the identification $\mathscr{H} \cong T\mathcal{M}(\mathscr{C},Q)$ given by the period structure, which arises from the intersection product on homology of spectral curves. We determine the symplectic leaves of this Poisson structure as the submanifolds cut out by fixing the residues of the quadratic differentials at their even order poles.

$\boxed{\alpha}$ **Generalised Spectral Correspondence [Theorem 7.2.12]:** In Theorem 7.2.12, we prove the existence of a map $\alpha : \mathcal{M}(\mathscr{C},\mathscr{E},\Phi,\nabla_\infty) \longrightarrow \mathcal{M}(\mathscr{C},Q,\mathcal{L},\partial_\infty)$, inducing a birational correspondence between the spaces of quasi-parabolic Higgs fields and nilpotent flat connections on one side and the space of anti-invariant branched connections from the other side. This is a central theorem generalising the classical spectral correspondence by abelianising simultaneously irregular quasi-parabolic Higgs and flat bundles on wild curves.

$\boxed{\beta}$ **Generic Finiteness of the Generalised Hitchin Fibre at a fixed Wild Bundle [Prop. 7.3.2]:** We define a map $\beta : \mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla_\infty) \longrightarrow \mathcal{M}(\mathscr{C}, \mathscr{E}, Q, \nabla)$ induced by the Hitchin map on wild Higgs fields and we prove in proposition 7.3.2 that, for genus 0 curves, β is generically étale. We also remark on proof strategies for its generalisation to arbitrary genera.

$\boxed{h_\varepsilon}$ **An ε-family of Ehresmann Connections [Def. 8.3.4]:** First we argue that the natural isomonodromy connection on $\mathcal{M}(\mathscr{C}, \mathscr{E}, \nabla) \to \mathcal{M}(\mathscr{C})$ can be pulled back through the following diagram

$$\begin{tikzcd}
 & & \mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla_\infty) \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & \\
\mathscr{H}^{\#} \arrow[d, "\pi"'] & \mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial_\infty) \arrow[l, "\mu"'] \arrow[rr, dashed, "\gamma"] \arrow[d, "\pi'"] & & \mathcal{M}(\mathscr{C}, \mathscr{E}, Q, \nabla) \arrow[r, "\tau"] \arrow[d, "\varpi'"] & \mathcal{M}(\mathscr{C}, \mathscr{E}, \nabla) \arrow[d, "\varpi"] \\
\mathcal{M}(\mathscr{C}, Q) \arrow[r, "\sim", "id"'] & \mathcal{M}(\mathscr{C}, Q) \arrow[rr, "\sim", "id"'] & & \mathcal{M}(\mathscr{C}, Q) \arrow[r] & \mathcal{M}(\mathscr{C})
\end{tikzcd}$$

to a meromorphic non-linear connection on the fibre bundle $\mathscr{H}^{\#} \to \mathcal{M}(\mathscr{C}, Q)$, developing poles along the exceptional divisors of the maps α and β. Then we deform this connection to an ε-family h_ε of connections with the same property.

$\boxed{A_2}$ **An Example [Chapter 9]:** In chapter 9, we describe the structures introduced in this thesis explicitly in terms of an example associated to the A_2-quiver, i.e. quadratic differentials of type $(g, \mathfrak{m}) = (0, \{7\})$. Importantly we recover the structures that appeared in [17], where Bridgeland studied this example in an ad hoc way without the general theory developed in this thesis. This example serves as a refined testing ground for demonstrating the effectiveness of our theory.

Appendix

$\boxed{\dim(\mathcal{M}(A, \ldots, Z))}$ **Dimensions of Moduli [Appendix Sec. A]:** We include a list of the dimensions of all the moduli spaces involved and the degrees of all relevant divisors to help keep an overview.

$\boxed{\mathcal{M}(A, \ldots, Z)}$ **as Stacks [Appendix Sec. B]:** The spaces $\mathcal{M}(A, \ldots, Z)$ described here are realised as algebraic stacks in section B of the appendix. We opted for omitting the stacky constructions from the core of the thesis, in order to make it more reader friendly.

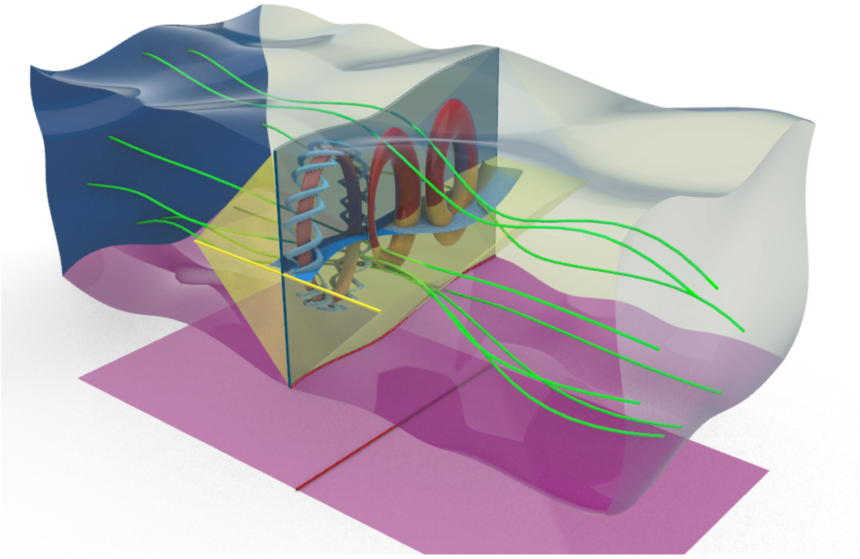
$\boxed{Res(Q) \neq 0}$ **Lifting the Residue Zero Condition [Appendix Sec. C]:** Here we explain how to generalise our setting to include quadratic differentials with arbitrary residues, which are fixed to vanish identically in the main corpus of the thesis for brevity of exposition. In particular we then allow polar types $(m_i)_i \in \mathbb{Z}^d_{>0}$.

Limitations and Remains of the Day

Although this thesis is a big leap towards the construction of Joyce structures on moduli of quadratic differentials with arbitrary irregular singularities, there is still some work left to be done, in order to complete the proof for the existence of a Joyce structure.

Firstly, we make some assumptions on the pole configuration of the quadratic differentials we are considering, cf definition 4.1.7. Namely, we ask for the following conditions to hold for the polar type $(g, \mathscr{m})$: $(m_i)_i \in \mathbb{Z}^d_{>1}$ and $deg(D) \in 2\mathbb{Z}$. Except of only six degenerate cases appearing in genus 0 and genus 1, we expect our results to generalise to any other pole configuration satisfying 1.2. The condition $(m_i)_i \in \mathbb{Z}^d_{>1}$ is only imposed for brevity in exposition, as we discuss in remark 5.2.23, and appendix C explains how to lift that condition to $(m_i)_i \in \mathbb{Z}^d_{>0}$. The condition for $deg(D)$ to be even, appears as a technical condition in our attempt to prove the generic finiteness of the map β. We expect this to be lifted by improving the strategy for this proof. This last condition, brings us to the main limitation of this thesis. Namely, we prove the generic finiteness of the map β, without any further stability assumptions, only for wild curves of genus 0. We conjecture the general statement to hold for any genus and any admissible pole configuration, and give an account of proof strategies in remark 7.3.3. Lastly, one needs to show that the ε-family of connections we constructed is indeed an ε-pencil and that it is compatible with the symplectic form we construct on the $(\mathbb{C}^*)^n$-symplectic fibration $\mathscr{H}^\# \to \mathcal{M}(C, Q)$.

All of these technical remainders are to be studied in the immediate future.



An artistic interpretation of $X^\#$.

2. Background

For the most part, one can value this thesis without really knowing anything about stability conditions nor Donaldson-Thomas theory. In order to appreciate the origin of the ideas that lead to the conception of Joyce Structures and the higher purpose thy aim to serve, it is important to have a fundamental understanding of the concepts that are to be presented in these preliminary remarks. We are by no means to give a thorough account, rather a mere *dessin d'enfant*, that cannot do proper justice to the underlying theory but might equip us with some basic intuition. Readers may well choose to skip chapter 2 altogether.

2.1. Stability Conditions and Quivers

2.1.1. Bridgeland Stability Conditions

In this subsection we recall basic constructions from the paradigm shifting paper of Bridgeland [13]. Being advised by the man himself does not automatically qualify me to understand any of this, as some may presume, and so this presentation is the result of a naive attempt to present the gist of it.

Let $\mathcal{A}$ be an abelian category, e.g. the category of R-modules $Mod(R)$ for a commutative ring, or the category of coherent sheaves $Coh(X)$ on a scheme X. We denote by $C^b(\mathcal{A})$ the category of bounded chain complexes on $\mathcal{A}$, which we localise at quasi-isomorphisms to obtain the bounded derived category $D^b(\mathcal{A})$ of $\mathcal{A}$ by identifying complexes with identical cohomological information.

Remark 2.1.1. $D^b(\mathcal{A})$ carries the structure of a triangulated category, in particular it is additive and so the composition of morphisms:

$$Hom_{D^b(\mathcal{A})}(A^\bullet, B^\bullet) \times Hom_{D^b(\mathcal{A})}(B^\bullet, C^\bullet) \to Hom_{D^b(\mathcal{A})}(A^\bullet, C^\bullet)$$

is a bilinear map. Moreover, there exists a shift endofunctor $[n] : D^b(\mathcal{A}) \to D^b(\mathcal{A})$ moving complexes to the left by n-places. We also use the notation

$$Hom^i_{D^b(\mathcal{A})}(A^\bullet, B^\bullet) := Hom_{D^b(\mathcal{A})}(A^\bullet, B^\bullet[i]).$$

Lastly, note that there exists a tautological embedding $\mathcal{A} \hookrightarrow D^b(\mathcal{A})$ by interpreting short exact sequences as triangles.

Let $\mathcal{D}$ be a triangulated category, not necessarily of the form $D^b(\mathcal{A})$.

Definition 2.1.2. t-Structures and their Hearts

Let $\mathcal{F} \subset \mathcal{D}$ be a full subcategory of the triangulated category that is stable under the shift-endofunctor: $\mathcal{F}[1] \subset \mathcal{F}$, and let

$$\mathcal{F}^{\perp} := \{G \in \mathcal{D} | Hom_{\mathcal{D}}(F, G) = 0;\ \forall F \in \mathcal{F}\}.$$

We call $\mathcal{F}$ a *t-structure* on $\mathcal{D}$ if $\forall E \in \mathcal{D}$, there exists a triangle $F \longrightarrow E \longrightarrow G$ in $\mathcal{D}$ with $F \in \mathcal{F}$ and $G \in \mathcal{F}^{\perp}$. Given a t-structure $\mathcal{F}$, we call the abelian, full subcategory $\mathcal{A} := \mathcal{F} \cap \mathcal{F}^{\perp}[1] \subset \mathcal{D}$, the *heart* of $\mathcal{F}$. Moreover, we call a t-structure $\mathcal{F}$ *bounded* if

$$\mathcal{D} = \bigcup_{i,j \in \mathbb{Z}} \mathcal{F}[i] \cap \mathcal{F}^{\perp}[j].$$

Lemma 2.1.3. ***Characterisation of Hearts***
A full additive subcategory of a triangulated category $\mathcal{A} \subset \mathcal{D}$ is a heart of a bounded t-structure $\mathcal{F} \subset \mathcal{D}$ if and only if the following conditions hold

- *a) if $k_1, k_2 \in \mathbb{Z}$ and $A, B \in \mathcal{A}$, then $k_1 > k_2$ implies $Hom_{\mathcal{D}}(A[k_1], B[k_2]) \overset{!}{=} 0$, i.e. negative Ext-groups vanish,*
- *b) $\forall 0 \neq E \in \mathcal{D}$ there exists:*
  1. *a filtration $0 = E_0 \to E_1 \to \cdots \to E_n = E$,*
  2. *a finite sequence of integers $k_1 > k_2 > \cdots > k_n$ and exact triangles:*

$$E_{j-1} \longrightarrow E_j \longrightarrow F_j \longrightarrow E_{j-1}[1] \longrightarrow \tag{2.1}$$

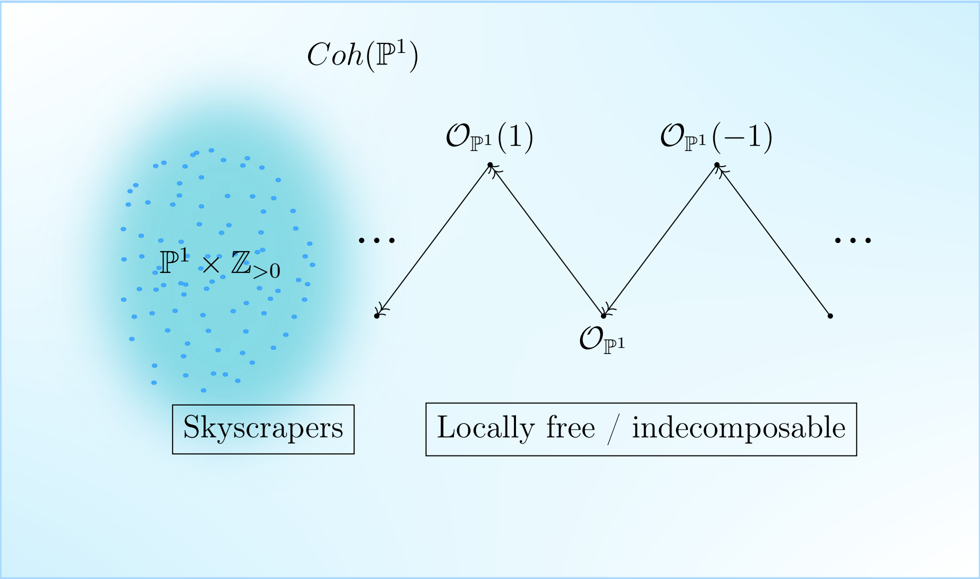
with $F_j \in \mathcal{A}[k_j]$ for all $j \in \{1, \ldots, n\}$.

This last lemma is an abstraction of the properties of the primal model on which a Bridgeland stability condition is based on, namely the notion of stability for vector bundles, represented by the existence and features of Harder-Narasimhan filtrations. Via this characterisation, a heart gives us a way of splitting the triangulated category into pieces $\{\mathcal{A}[k]\}_k$.

Example 2.1.4. **1) $\mathcal{A} \subset D^b(\mathcal{A})$:** The tautological example of a heart is given by embedding an Abelian category in its bounded derived category.
2) $D^b(Coh(\mathbb{P}^1)) \simeq D^b(Rep(\bullet \rightrightarrows \bullet))$: This is a basic example to highlight that a triangulated category can have two inequivalent hearts. By Grothendieck's classification theorem of vector bundles on $\mathbb{P}^1$:

- $\mathbb{Z} \cong \mathcal{P}ic(\mathbb{P}^1); d \mapsto \mathcal{O}_{\mathbb{P}^1}(d)$,
- There exists a sequence $d_1 \geq d_2 \geq \cdots \geq d_n$ in $\mathbb{Z}$ associated to a rank n locally free sheaf $E \in Loc(\mathbb{P}^1)$ such that: $E \cong \bigoplus_i \mathcal{O}_{\mathbb{P}^1}(d_i)$.

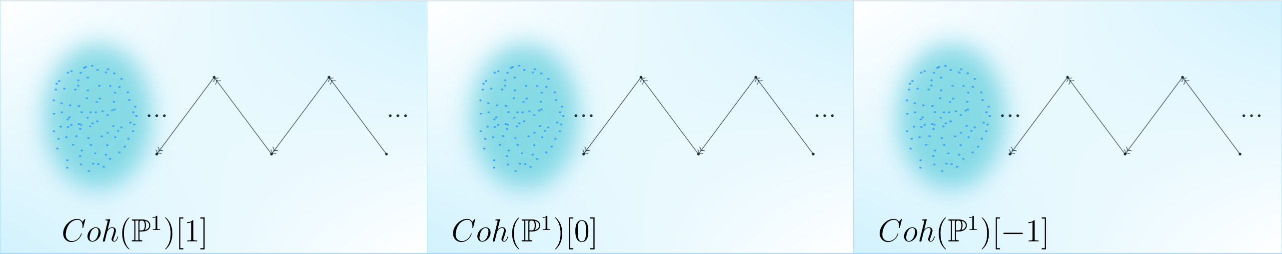
The only stable vector bundles are line bundles and they classify all semistable bundles of a given degree and rank. Adding skyscraper sheaves to locally free sheaves we obtain the abelian category $Coh(\mathbb{P}^1)$. Pictorially:



"Auslander-Reiten representation"

This is a heart for the derived category of coherent sheaves on $\mathbb{P}^1$, which is build by concatenating the image of the heart under the action of the shift endofunctor for all integral values of n:

$D(Coh(\mathbb{P}^1))$



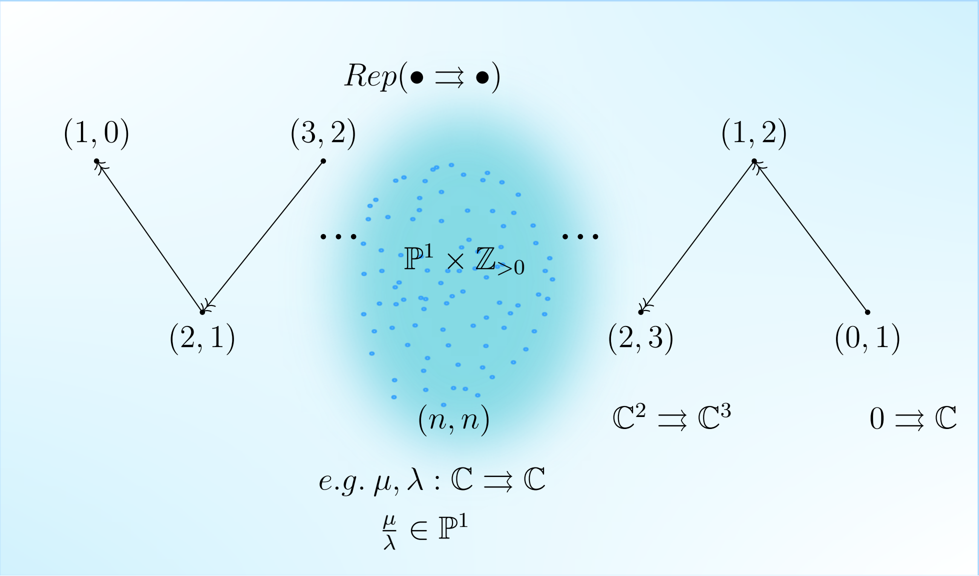
Now consider the category of representations of the Kronecker quiver $Rep(\bullet \rightrightarrows \bullet)$. There is a functor

$$\begin{aligned} \mathcal{F} : Coh(\mathbb{P}^1) &\longrightarrow Rep(\bullet \rightrightarrows \bullet) \\ E &\longmapsto [\Gamma(\mathbb{P}^1, E \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \rightrightarrows \Gamma(\mathbb{P}^1, E \otimes \mathcal{O}_{\mathbb{P}^1})] \end{aligned}$$

where the double map is given by the two distinct global sections of $\mathcal{O}_{\mathbb{P}^1}(1)$. This functor is not a categorical equivalence, e.g. $\mathcal{F}(\mathcal{O}_{\mathbb{P}^1}(-k)) = 0 \; \forall k \geq 1$. Nevertheless, the associated derived functor induces a derived equivalence with

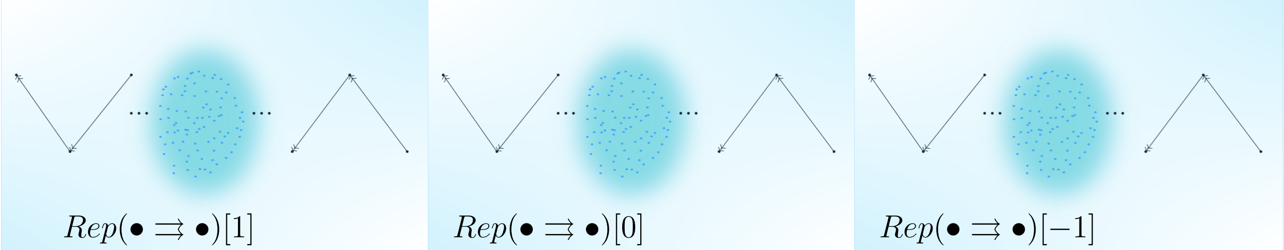
$$\begin{tikzcd}
Coh(\mathbb{P}^1) \arrow[r] \arrow[d, hook] & Rep(\bullet \rightrightarrows \bullet) \arrow[d, hook] \\
D(Coh(\mathbb{P}^1)) \arrow[r, "\simeq"] & D(Rep(\bullet \rightrightarrows \bullet))
\end{tikzcd} \quad . \tag{2.2}$$

In particular, $Rep(\bullet \rightrightarrows \bullet)$ is a heart for $D(Coh(\mathbb{P}^1))$. Pictorially we have



and for the derived category

$$D(Rep(\bullet \rightrightarrows \bullet))$$



It is evident that the two pictorial representations of the derived categories are the same, even if the hearts used to split them are different, indicating the derived equivalence $D^b(Coh(\mathbb{P}^1)) \simeq D^b(Rep(\bullet \rightrightarrows \bullet))$.

3) Given a derived equivalence between two abelian categories $\mathcal{A}, \mathcal{B}$: $\psi : D^b(\mathcal{A}) \simeq D^b(\mathcal{B})$, the preimage $\psi^{-1}(\mathcal{B}) \subset D^b(\mathcal{A})$ is a heart with $\psi^{-1}(\mathcal{B}) \simeq \mathcal{B}$.

The following notion of a slicing, allows for a real-refinement of the splitting of $\mathcal{D}$ given by a heart of a t-structure, and constitutes the basic structure of a stability condition. This is how to break a heart into uncountably many pieces.

Definition 2.1.5. Slices
A *slicing* of a triangulated category is an $\mathbb{R}$-collection of full subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ such that

a) $\mathcal{P}(\phi + 1) \cong \mathcal{P}[1]$ *shift-periodicity,*

b) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$, then $Hom_{\mathcal{D}}(A_1, A_2) = 0$

c) $\forall 0 \neq E \in \mathcal{D}$ there exists:

  1. a filtration $0 = E_0 \to E_1 \to \cdots \to E_n = E$,
  2. a finite sequence of real numbers $\phi_1 > \phi_2 > \cdots > \phi_n$ and exact triangles:

$$E_{j-1} \longrightarrow E_j \longrightarrow F_j \longrightarrow E_{j-1}[1] \longrightarrow \tag{2.3}$$

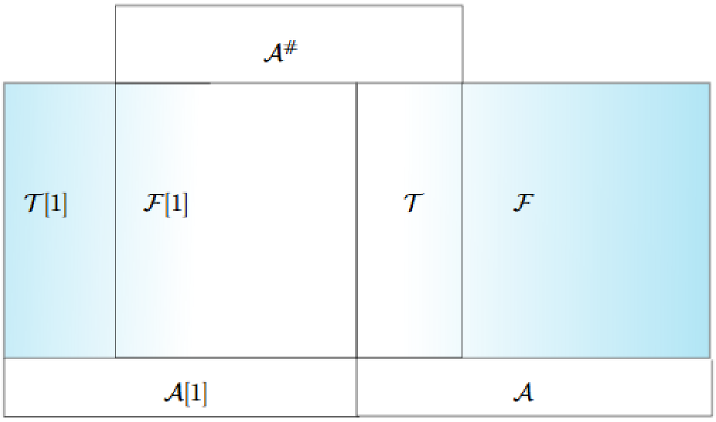
  with $F_j \in \mathcal{P}[\phi_j]$ for all $j \in \{1, \ldots, n\}$.

We write $\mathcal{P} = \bigcup_{\phi \in \mathbb{R}} \mathcal{P}(\phi)$.

Definition 2.1.6. Torsion Pairs
Let $\mathcal{A}$ be an abelian category. A pair of subcategories $(\mathcal{T}, \mathcal{F})$ is called a torsion pair if

- $\forall E \in \mathcal{A}\ \ \exists 0 \to T \to E \to F \to 0$ with $T \in \mathcal{T}, F \in \mathcal{F}$,
- $Hom_{\mathcal{A}}(T, F) = 0$ for $T \in \mathcal{T}, F \in \mathcal{F}$

Lemma 2.1.7. ***Tilting a Heart for a Heart***
Let D be a generic triangulated category, $\mathcal{A} \subset \mathcal{D}$ a heart and $(\mathcal{T}, \mathcal{F})$ a torsion pair, then we can define a new heart by: $\mathcal{A}^{\#} = <\mathcal{T}, \mathcal{F}[1]>$ which we call the tilt of $\mathcal{A}$. (The bracket $< \mathcal{A}, \mathcal{B} >$ denotes the extension-closure full subcategory of $\mathcal{D}$.)



Definition 2.1.8. Stability Conditions on Triangulated Categories
A stability condition on $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{P})$ consisting of:

1) A group homomorphism $Z : K(\mathcal{D}) \longrightarrow \mathbb{C}$, which we call the *central charge* of a stability condition and

2) a slicing $\mathcal{P}$ of $\mathcal{D}$, such that:

$$0 \neq E \in \mathcal{P}(\phi) \implies Z(E) = m(E)\exp(i\pi\phi), \text{ for some } m(E) \in \mathbb{R}_{>0}. \tag{2.4}$$

We call an object E *σ-semistable* of *phase* ϕ, if $E \in \mathcal{P}(\phi)$. If, in addition, E is a simple object in $\mathcal{P}(\phi)$, then we call it *σ-stable* of *phase* ϕ.

Theorem 2.1.9. ***Space of Stability Conditions***
*Consider the subset of stability conditions $Stab(\mathcal{D}) \subset Slices(\mathcal{D}) \times Hom_{\mathbb{Z}}(K_0(\mathcal{D}), \mathbb{C})$. These spaces are equipped with natural topologies and for each connected component $\Sigma \subset Stab(\mathcal{D})$, there is a linear subspace $V(\Sigma) \subset Hom_{\mathbb{Z}}(K_0(\mathcal{D}), \mathbb{C})$ with well defined linear topology and a **local homeomorphism** $\mathcal{Z} : \Sigma \longrightarrow V(\Sigma)$, which maps a stability condition $(Z, \mathcal{P})$ to its central charge Z.*

Proof. Theorem 1.2 in [13] □

This means that, if we forget the semi-stable objects, remembering only the central charge, then a deformation of the central charge lifts to a unique deformation of the subcategory of semi-stable objects. In particular, choosing a basis of the lattice associated to the Grothendieck group $\bigoplus_i \mathbb{Z}\gamma_i$, gives us a set of local coordinates $\{z_i := Z(\gamma_i)\}_i$ on $Stab(\mathcal{D})$.

It is the parameter space of stability conditions that is an invariant of the derived category, and a geometric one at that. The goal of Joyce structures is to equip this already geometric space with additional geometric structure that encodes finer invariants about the underlying CY3-triangulated category, namely, the Donaldson-Thomas invariants.

2.1.2. Natural Structures on Stability Conditions of CY_3-Categories

Now we specialise in the derived categories we are to consider in this thesis, namely Calabi-Yau 3 triangulated categories of finite type.

Definition 2.1.10. **1)** We call a triangulated category $\mathcal{D}$ of *finite type*, if $\forall E, F \in \mathcal{D}$:

$$dim_{\mathbb{C}} \bigoplus_{i \in \mathbb{Z}} Hom^i_{\mathcal{D}}(E, F) < \infty. \tag{2.5}$$

2) If $\mathcal{D}$ is of finite type, then for $E, F \in \mathcal{D}$ we set:

$$\chi(E,F) := \sum_{i\in\mathbb{Z}} (-1)^i Hom^i_{\mathcal{D}}(E,F). \tag{2.6}$$

This can be interpreted as a bilinear form on the Grothendieck group

$$\chi(\cdot,\cdot) : K_0(\mathcal{D}) \times K_0(\mathcal{D}) \longrightarrow \mathbb{Z}, \tag{2.7}$$

which we refer to as the *Euler form.*
3) A triangulated category $\mathcal{D}$ of finite type is said to be *Calabi-Yau 3 (CY_3)*, if $\forall E, F \in \mathcal{D}$:

$$Hom^i_{\mathcal{D}}(E,F) \cong Hom^{3-i}_{\mathcal{D}}(F,E)^*. \tag{2.8}$$

This relation clearly resembles the relation induced by Serre-duality on the coherent sheaves of a Calabi-Yau 3-fold. In the case of a CY_3-category, the Euler form is anti-symmetric.
4) Let $\mathcal{D}$ be triangulated CY_3 category. Then by Serre-duality

$$ker(\chi(\alpha,\cdot)) = ker(\chi(\cdot,\alpha)) \tag{2.9}$$

and we call the free abelian quotient

$$\mathcal{N}(\mathcal{D}) := K_0(\mathcal{D})/ker(\chi(\cdot,\cdot)) \tag{2.10}$$

the *numerical Grothendieck group.* If $\mathcal{N}(\mathcal{D})$ is of finite rank, then we call $\mathcal{D}$ *numerically finite.* If the central charge of a stability condition factors through $\mathcal{N}$, then we call it a *numerical stability condition* and denote the subspace of such by $Stab_{\mathcal{N}}(D) \subset Stab(\mathcal{D})$.

Corollary 2.1.11. ***(to Theorem 2.1.9)***
If $\mathcal{D}$ is numerically finite, then for each connected component $\Sigma \subset Stab_{\mathcal{N}}(\mathcal{D})$, there is a linear subspace $V(\Sigma) \subset Hom_{\mathbb{Z}}(\mathcal{N}(\mathcal{D}), \mathbb{C})$ and a ***local homeomorphism*** *$\mathcal{Z} : \Sigma \longrightarrow V(\Sigma)$ that maps a stability condition $(Z, \mathcal{P})$ to its central charge Z. In particular, Σ is a* ***finite dimensional complex manifold****.*

2.1.3. Derived Categories from Quivers with Potential

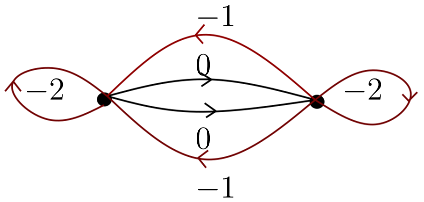
We want to study the spaces of stability conditions of CY_3 categories, but the most natural examples of such categories, the bounded derived category of (non-compact) Calabi-Yau 3-folds, prove to be extremely hard. A more comprehensible, from the point of view of stability conditions, class of triangulated categories, arise from quivers with potential. Such Calabi Yau categories were studied by Ginzburg in [32]. These are the ones that relate to the stability conditions studied in this thesis. We brief review of their construction.

Let $(\mathcal{Q}, \mathcal{W})$ be a quiver with potential, that is, a direct graph $\mathcal{Q}$ and a formal sum $\mathcal{W}$ of cycles in $\mathcal{Q}$. We denote by $\mathbb{C}[\mathcal{Q}]$ the path algebra and by $\mathfrak{p}_Q$ the ideal generated by the arrows. For this we have $Rep(\mathcal{Q}) \cong \mathbb{C}[\mathcal{Q}] - Mod.$

Definition 2.1.12. The Ginzburg Quiver
Given a quiver $\mathcal{Q}$, let $\tilde{\mathcal{Q}}$ denote the graded quiver with the same vertices as $\mathcal{Q}$ and arrows:

- those of $\mathcal{Q}$ [$deg = 0$-arrows],
- for every arrow $\alpha : i \longrightarrow j \in \mathcal{Q}$ there is an additional reverse arrow: $\alpha^* : j \longrightarrow i \in \tilde{\mathcal{Q}}$ [$deg = -1$-arrows],
- a loop $t_i : i \longrightarrow i$ at each vertex [$deg = -2$-arrows].

We call this the *Ginzburg quiver* of $\mathcal{Q}$. E.g. the Ginzburg quiver associated to the Kronecker quiver is depicted below, where the arrows of the Ginzburg enhancement are highlighted in red.



Definition 2.1.13. The Ginzburg Category
Let $(\mathcal{Q}, \mathcal{W})$ be a quiver with potential, the *complete Ginzburg dg-Algebra* consists of the completion of the path algebra of the associated Ginzburg quiver $\hat{\Gamma}(\mathcal{Q}, W) := \varprojlim_{\mathfrak{p}_{\tilde{\mathcal{Q}}}} \tilde{\mathcal{Q}}$, carrying a degree 1 differential: $d(uv) = (du)v + (-1)^p u dv$ which acts by

$$d\alpha = 0, \ \forall \alpha \in \mathcal{Q} \tag{2.11}$$

$$d\alpha^* = \partial_\alpha W, \ \forall \alpha \in \mathcal{Q} \tag{2.12}$$

$$dt_i = \sum_{\alpha \in \mathcal{Q}} [\alpha, \alpha^*]. \tag{2.13}$$

We set $D(\mathcal{Q}, \mathcal{W}) := D^b(\hat{\Gamma}(\mathcal{Q}, \mathcal{W}) - Mod)$.

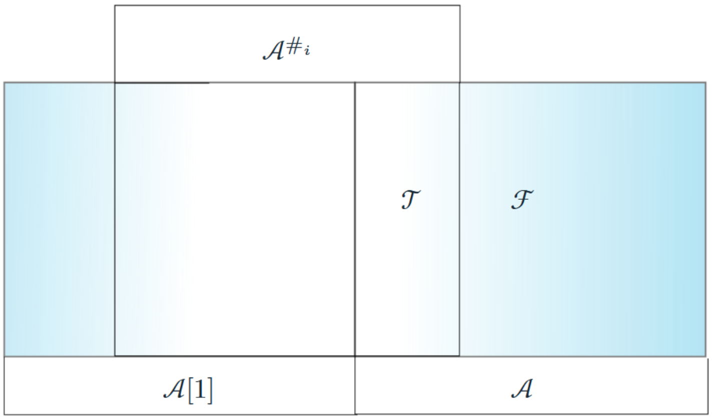
For a detailed study of the Ginzburg category in the context of quivers, we refer to [49]. Keller and Yang give us the following important result, stating that mutations of quivers induce derived equivalences between the Ginzburg categories, and are realised by tilts.

Theorem 2.1.14. ***Keller-Yang [49]***
Let $(\mathcal{Q}, \mathcal{W})$ be a quiver with potential and denote by $\mu_i(\mathcal{Q}, \mathcal{W})$ the mutation of the original quiver with potential at its i-th vertex. Moreover, let $\mathcal{A} \subset D(\mathcal{Q}, \mathcal{W})$ be the canonical heart generated by the finitely many simple objects $\{S_i\}_i$ (vertices of $\mathcal{Q}$). The mutation operation induces an equivalence of derived categories, represented by a tilt

$$D(\mathcal{Q}, \mathcal{W}) \overset{\simeq}{\rightarrow} D(\mu_i(\mathcal{Q}, \mathcal{W}))$$
$$\mathcal{A} \longleftrightarrow \mathcal{A}^{\#_i},$$

where $\mathcal{A}^{\#_i}$ is the following tilt of $\mathcal{A}$ along the simple object S_i(cf. picture)

Proof. [49] □



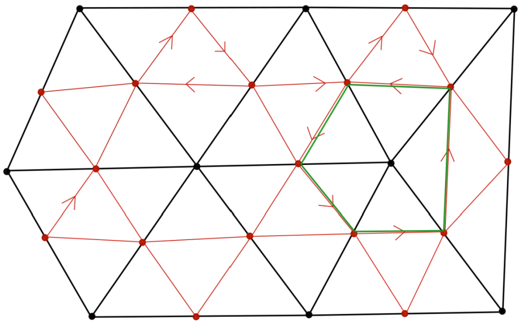
$\mathcal{T} = \{S_i^{\oplus n}\}$ $\mathcal{F} = \{M \in \mathcal{A} | Hom(S_i, M) = 0\}$

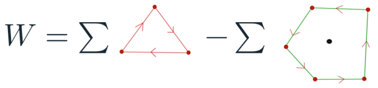
We will be interested in quivers that arise from triangulations of Riemann surfaces.

Definition 2.1.15. Quiver associated to an Ideal Triangulation of a Curve Every ideal triangulation on a Riemann surface $S_{g,d}$ of genus g with d marked points, has a quiver with potential $(\mathcal{Q}, \mathcal{W})$ associated to it in natural way, under the assumption that every puncture is incident to at least three arcs. Then, the quiver $\mathcal{Q}$ is defined by taking the arcs of the triangulation as vertices, while the arrows are drawn in the clockwise direction within every triangle. The potential is defined by the difference of the sum over all cycles of type-1 and the sum over all cycles of the type-2:

- Type 1: 3-cycles arising from triangles,
- Type 2: simple cycles, i.e. without repeated arrows, surrounding the marked points.

The following picture demonstrates what this looks like.





The quiver was constructed in [30] and the potential in [54].

Proposition 2.1.16. *The Ginzburg category $D(\mathcal{Q}_\Delta, \mathcal{W}_\Delta)$ of the quiver with potential associated to a triangulation of a marked surface $S_{g,d}$, is independent of the choice of an ideal triangulation and we write $D_{g,d} = D(\mathcal{Q}_\Delta, \mathcal{W}_\Delta)$.*

Proof. (Sketch) First we note the fact that ideal triangulations on a surface are related by sequences of flips of edges. By work of Ladardini-Fragoso [55] we have that mutations of the quiver $\mathcal{Q}_\Delta$ at a vertex correspond to flipping an edge. The result then follows by applying theorem 2.1.14. □

2.1.4. Bridgeland-Smith Correspondence

For what follows, denote by $S_{g,d}$ a genus g marked bordered Riemann surface, with d marked points, such that all marked points live on the boundary and every boundary component contains at least one marked point. Such a surface is determined up to diffeomorphism by the genus g, the number of marked points d and a collection of integers $k_i \geq 1$ encoding the number of marked points on each boundary component of $S_{g,d}$. Let $\mathcal{M}(C,Q)$ denote the space parametrising equivalence classes of pairs (C,Q), consisting of a projective algebraic curve C of genus g and a quadratic differential Q on C, such that Q has only simple zeros and a collection of d poles with multiplicity $k_i + 2$.

Theorem 2.1.17. ***The Bridgeland-Smith Correspondence***
There is an isomorphism of complex orbifolds

$$\mathcal{M}(C,Q) \cong Stab(D_{g,d})/Aut(g,d). \tag{2.14}$$

Proof. Cf. [18, Theorem 11.2] □

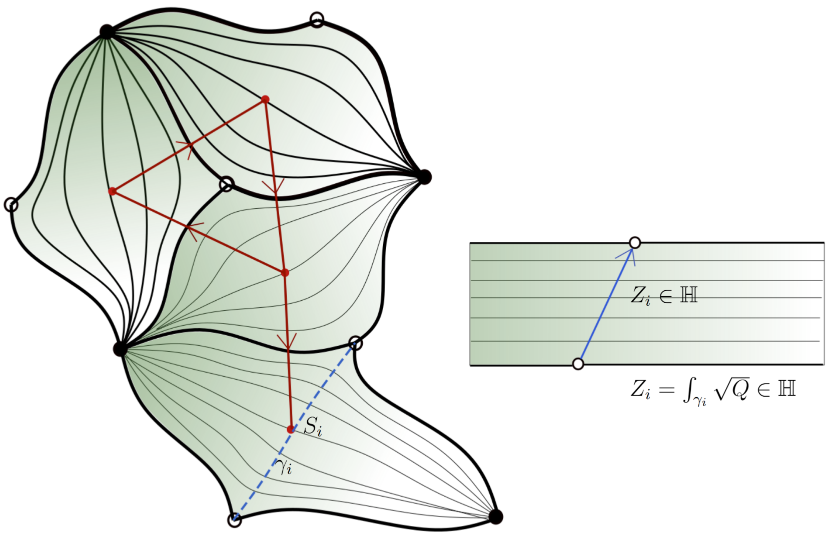
Remark 2.1.18. The Skeleton of the Bridgeland-Smith Correspondence cf. [18]

$$\{\text{(nice) hearts on } D_{g,d}\} \Big/ \mathcal{A}ut^*(D_{g,d}) \longleftrightarrow \{\text{ triangulations of } S_{g,d}\} \, / MCG$$

$$\text{tilt} \longleftrightarrow \text{flip}$$

- $\mathcal{A}ut^*(D_{g,d})$ generated by Keller-Yang cycles.
- MCG Fock-Goncharov show that the group of tilts is the mapping class group.

Given a point $(C,Q) \in \mathcal{M}(C,Q)_{(g,m)}$ that is generic in the sense that Q is saddle-free, i.e. it has no finite-length horizontal trajectories, we obtain a triangulation/ chamber decomposition of the surface by the horizontal trajectories of Q. Locally it looks like the following picture



where the black dots • indicate the positions of the poles of Q, the white dots ∘ indicate the zeros of Q and the lines are the horizontal trajectories emanating from either the zeros or poles and sinking into the poles. Notice that in this generic case, there are no horizontal trajectories between any pair of zeros. This would be a finite length saddle trajectory. An arc S_i between two zeros on the curve C, which corresponds to an edge of the triangulation of definition 2.1.15, lifts to an anti-invariant cycle on a double cover $\gamma_i \in H_1(\Sigma, \mathbb{Z})^-$, something we will explain in detail in the main corpus of the thesis. The oriented edges in red are those of the quiver associated to the triangulation by the $\{S_i\}_i$ as presented in definition 2.1.15. As we vary Q towards a degeneration, the period $\int_{\gamma_i} \sqrt{Q}$ vanishes and a saddle-connection appears in place of the stable arc S_i, for some i. The loci of destabilisation define a wall structure on $\mathcal{M}(C, Q)$.

Quadratic Differentials	**Stability Conditions**
$\mathcal{M}(C, Q)$	$Stab(D_{g,d})/Aut(g, d)$
(C, Q)	$(\mathcal{Z} : K_0(D) \to \mathbb{C}, \mathcal{P})$
$H_1(\Sigma^\circ, \mathbb{Z})^-$	$K_0(D)$
Arcs S_i (edges of triangulation)	Simple objects
Intersection Form on Cohomology	Euler form
Periods of Quadratic Differentials	Central charges
BPS-rays $\mathbb{R}_{>0}e^{i\pi\phi}$	$\mathcal{P}(\phi) \neq \emptyset$
Ideal Triangulations of C	Hearts of $D_{g,d}$
Mapping class group	$Aut(g, d)/Braid(g, d)$

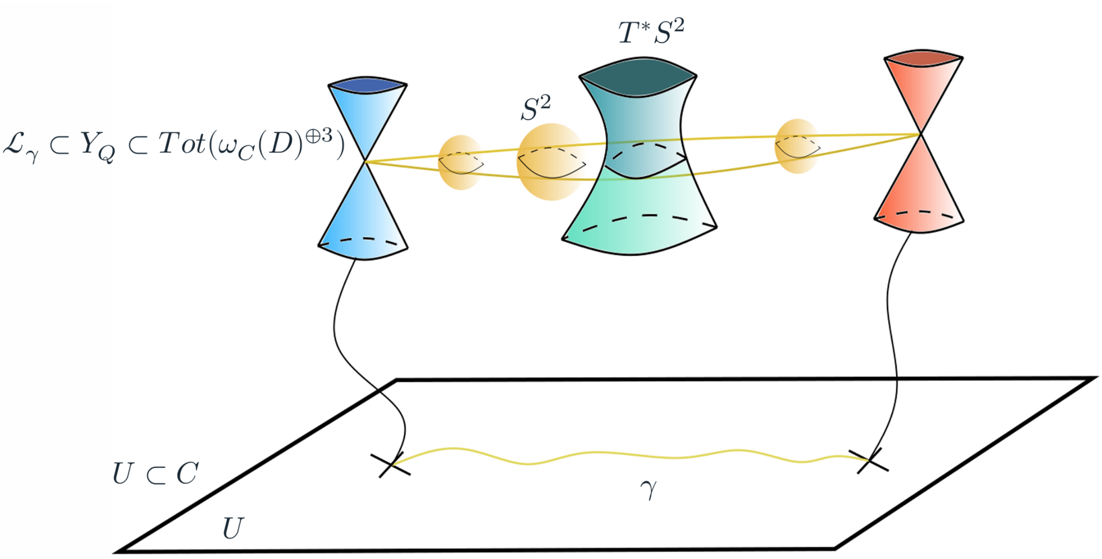
It is worth noting that due to subsequent work of I. Smith [73], we can realise these derived categories of quivers with potential as Fukaya categories of certain quasi-projective

Calabi-Yau 3-folds Y_Q. The 3-folds Y_Q fibering over curve C and are locally described by a symplectic conic fibration as follows

$$Y_Q := \{(a,b,c) \in \omega_C(D)^{\oplus 3} | ab + c^2 = Q\} \subset Tot(\omega_C(D)^{\oplus 3})$$

where $Q \in H^0(C, \omega_C(D)^{\otimes 2})$ is a quadratic differential with distinct zeros. Consider the full CY3 subcategory $\mathcal{F}(Y_Q) \subset D^b Fuk(Y_Q)$ generated by Lagrangian spheres $\mathcal{L}_\gamma$ along paths γ between the zeros of Q, then

$$Stab(\mathcal{F}(Y_Q)) \Big/ Aut(\mathcal{F}) \cong \mathcal{M}(C,Q), \quad K_0(\mathcal{F}(Y_Q)) \cong H_3(Y_Q) \cong H_1(\Sigma_Q, \mathbb{C})^-.$$



2.2. Monodromy

2.2.1. The Regular Riemann-Hilbert Correspondence

Most of this section is the result of a distillation of P.Deligne's lectures [22], but also the notes of B.Conrad [21] and N.Katz's classic [46] on the matter.

Let C be a projective algebraic curve, D^{red} a reduced divisor on C and $C^\circ = C \backslash D^{red}$ the quasi-projective complement. We use GAGA [71] freely and denote by A^{an} the analytification of any algebraic datum A. We start by providing a general working definition for a linear connection on a complex algebraic vector bundle, i.e. a $GL_n(\mathbb{C})$-connection, described by a differential operator acting on the sections of the vector bundle.

Definition 2.2.1. Linear Connection on a $\mathbb{C}$-Vector Bundle
Let M be a $\mathbb{C}$-manifold and $\pi : E \longrightarrow M$ a vector bundle on M. A connection ∇ on π is

a $\mathbb{C}$-linear map:

$$\nabla : E \longrightarrow E \otimes \Omega^1_M \tag{2.15}$$

respecting the Leibniz rule:

$$\nabla(f \cdot s) = \nabla(s) \cdot f + df \otimes s. \tag{2.16}$$

The connection ∇ is called *flat* or *integrable*, if $\nabla \circ \nabla : E \longrightarrow E \otimes \Omega^2_M$ vanishes identically. We call a pair (E, ∇), consisting of a flat connection ∇ on a vector bundle E, a flat bundle.

Theorem 2.2.2. ***Classical Riemann-Hilbert Correspondence***
Let M be a smooth complex algebraic variety and (E, ∇) a pair of a vector bundle with an integrable connection. Then, the sheaf of horizontal sections $ker(\nabla)$ is locally constant with $\underline{\mathbb{C}}_M$-rank equal to the $\mathcal{O}_M$-rank of E, and the natural map $\mathcal{O}_M \otimes_{\underline{\mathbb{C}}_M} ker(\nabla) \longrightarrow E$ is an isomorphism of $\mathcal{O}_M$-modules.
Moreover, the functors $(E, \nabla) \longmapsto \mathrm{ker}(\nabla)$ and $\Lambda \longmapsto (\Lambda \otimes_{\underline{\mathbb{C}}_M}, Id \otimes d)$ induce inverse equivalences of categories between the category of vector bundles with flat connections on M and the category of locally constant sheaves (a.k.a. Local Systems) of finite dimensional complex vector spaces on M.

Proof. (See for example [21].) Sketch: The Frobenius existence theorem shows that the functor that associates to a pair (E, ∇) the sheaf of germs of its flat sections $\mathbb{E} := ker(\nabla)$, gives rise to an equivalence of categories to the category of locally constant sheaves, a.k.a. *local systems*. The inverse of this functor is given by $\mathbb{E} \mapsto (\mathbb{E} \otimes_{\mathbb{C}} \mathcal{O}_M, id \otimes d)$, for the de Rham differential d. ☐

Corollary 2.2.3. *For an algebraic curve C, we have*

$$\{Algebraic\ Flat\ Bundles\ on\ C : (E, \nabla)\} \leftrightarrow Loc(C) \leftrightarrow Hom(\pi_1(C), GL_n(\mathbb{C})) /\!\!/ GL_n(\mathbb{C}).$$

Corollary 2.2.4. *The category of vector bundles with integrable connections is abelian.*

The gist of the result is that the analytically defined problem, i.e. finding solutions to a differential equation on a $\mathbb{C}$-manifold, can be completely encoded into topological data in terms of monodromy, i.e. local systems. These in turn, via GAGA, admit a purely algebraic description in terms of flat algebraic bundles. In what follows, we will sketch how Deligne managed to extend this exact story to differential equations developing certain regular singularities along snc divisors (per definition reduced), thereby answering affirmatively the following question.

Hilbert's 21st Problem- Re-interpreted: Let C be an algebraic curve and C^0 its complement with respect to a reduced divisor. Is any finite dimensional $\mathbb{C}$-representation of $\pi_1((C^\circ)^{an})$ arising as the monodromy representation of a flat bundle with regular singularities?

Remark 2.2.5. Given a quasi-projective algebraic curve C°, there exists a unique completion $C^\circ \hookrightarrow C$ given by the projective non-singular model of the function field of C°.

Deligne, generalising his own algebro-geometric interpretation of Fuchs's regularity criterion for differential equations, provided a regularity criterion of algebraic differential equations on curves of any genus.

Definition 2.2.6. Deligne's Regularity Condition
Given a quasi-projective curve C° with completion C, we say that a flat bundle (E, ∇) on C° has regular singular points along the support of $D^{red} = C \backslash C^\circ$, if and only if there exists a locally free coherent (algebraic) sheaf $\overline{E}$ on C and a connection $\overline{\nabla} : \overline{E} \longrightarrow \overline{E} \otimes \omega_C(D^{red})$ extending (E, ∇).

Lemma 2.2.7. ***Existence of Analytic Extension over Reduced Points***
Let (E^{an}, ∇^{an}) *be an analytic differential equation on* C°*, then there exists an analytic, locally free coherent sheaf* $\overline{E}^{an}$ *on* C *extending* E^{an} *and* $\overline{\nabla}^{an} : \overline{E}^{an} \longrightarrow \overline{E}^{an} \otimes \omega_C(D^{red})$ *such that* $ker(\overline{\nabla}^{an})|_{C^\circ} = ker(\nabla^{an})$.

Proof. In the curve case, the extension can be constructed locally around each puncture explicitly, for this we refer to [46] and the *key lemma* therein.

□

Corollary 2.2.8. *Applying GAGA on lemma 2.2.7, we obtain the existence of an (algebraic) flat bundle* $(\overline{E}, \overline{\nabla})$ *on* C *extending an flat bundle on* (E, ∇) *over* C° *by extending its analytification* $(E^{an}, \nabla^{an}) \rightsquigarrow (\overline{E}^{an}, \overline{\nabla}^{an})$ *and then apply GAGA* $(\overline{E}^{an}, \overline{\nabla}^{an}) \overset{GAGA}{\longleftrightarrow}$: $(\overline{E}, \overline{\nabla})$*. Thereby we have associated a representation of* $\pi_1(C^\circ)$ *to the flat bundle* (E, ∇)*.*

Theorem 2.2.9. ***Riemann-Hilbert for Regular Singular Connections***
There exists an equivalence of categories:

$$\{Flat\ Bundles\ on\ C : (E, \nabla)\ with\ reg.\ singularities\ on\ D^{red}\} \leftrightarrow Hom(\pi_1(C^\circ), GL_n(\mathbb{C}))/\!\!/GL_n(\mathbb{C}) \tag{2.17}$$

Proof. By Lemma 2.2.7 and Corollary 2.2.8, the functor is seen to be essentially surjective. To prove that it is indeed fully faithful, let $(E_1, \nabla_1), (E_2, \nabla_2)$ be two flat bundles on C° with regular singularities and consider the associated internal-Hom flat bundle:

$(\underline{Hom}(E_1, E_2), \nabla_{1\to 2})$, where the connection acts on a local section $s \in \Gamma(U, \underline{Hom}(E_1, E_2))$ and an $e \in \Gamma(U, E)$ by:

$$\nabla_{1\to 2}(s) = \nabla_2(s(e)) - s(\nabla_1(e)). \tag{2.18}$$

The flat sections of the internal Hom flat bundle are the morphisms between (E_1, ∇_1) and (E_2, ∇_2) in the category of flat bundles and they develop regular singular points if an only if both flat bundles do, i.e.

$$ker(\nabla_{1\to 2}) = Hom((E_1, \nabla_1), (E_2, \nabla_2)). \tag{2.19}$$

Then, by applying GAGA on the internal Hom flat bundle we obtain an isomorphism of Hom sets

$$Hom((E_1, \nabla_1), (E_2, \nabla_2)) = Hom((E_1^{an}, \nabla_1^{an}), (E_2^{an}, \nabla_2^{an})) \tag{2.20}$$

and as local systems $ker(\nabla_{1\to 2}) = ker(\nabla_{1\to 2}^{an})$. ☐

To summarise, if the connection is holomorphic or has logarithmic singularities, then local solutions can be analytically continued along paths on C and their multivaluedness is described in terms of classical monodromy. If we analytically continue a fundamental solution ψ along $\gamma \in \pi_1(C^\circ)$, we obtain another solution to the same differential equation and these must be related by an invertible linear transformation. Thereby we obtain a representation $\rho \in Hom(\pi_1(C^\circ), GL_n(\mathbb{C}))$ by considering analytic continuations of ψ along any loop. Starting with a different choice of a fundamental solution ψ', we obtain a representation ρ', that is related to ρ via conjugation. Therefore our ODE really corresponds to a conjugacy class of representations $[\rho] \in Hom(\pi_1(C^\circ), GL_n(\mathbb{C}))/\sim$. Generalising theorem 2.2.9 to irregular singularities, i.e. along non-reduced divisors, is not straightforward. The growth of the solutions to the differential equations around each singularity, even if moderate, will depend on the direction on which one approached the singularity and this leads to sectorial-type solutions. The pairwise relation between the solutions at each sector is the subject of the theory of Stokes data, which we will briefly review in the next section. Crucially, these solutions are cannot be encoded into the topological monodromy.

2.2.2. Generalised Monodromy and Stokes Data

For the sake of simplified notation, we will be discussing the case of $C = \mathbb{P}^1$ and $E = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. On C, consider a rank 2 system of ordinary differential equations $d\psi(z) = A(z)\psi(z)$, where A is an $\mathfrak{sl}_2(\mathbb{C})$-valued 1-form on $\mathbb{P}^1$. We consider the case where A is holomorphic in the complement $C^\circ := \mathbb{P}^1 \backslash \{p_1, \ldots, p_n\}$, for a finite collection of points $p_i \in \mathbb{P}^1 \backslash \{\infty\}$. We can recast this differential equation to a flat connection $\nabla := d + A$ on the trivial bundle

$E = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ and we can associate to ∇ a locally constant sheaf $\mathbb{L}$ of flat sections to ∇, which is the sheaf locally generated by $\psi_U \in \Gamma(U, E)$ s.t. $\nabla|_U \psi_U = 0$. It is useful to think of the flat sections as gauge transformations $\psi_U : U \to SL_2(\mathbb{C})$, intertwining the trivial connection d and $\nabla|_U := d + A$.

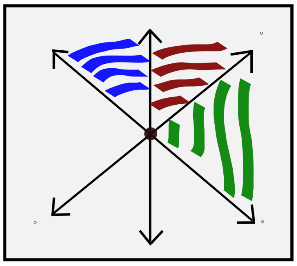
In this generalised case where we consider connections with irregular singularities along the p_i's, we ask for meromorphic solutions to the differential equation, where

$$A(z) = \sum_{i=1}^{d} \sum_{j=1}^{r_i} \frac{A_{i,j}}{(z-p_i)^j} dz. \tag{2.21}$$

Here there are no unique meromorphic solutions, only a formal one of the form:

$$\Psi^i_{\text{formal}}(z) = G_i \left[\mathbb{I} + \sum_{j=0}^{\infty} g_{i,j}(z-p_i)^j \right] \exp\left(\Lambda_{i,0} \cdot \ln(z-p_i) + \sum_{j=1}^{r_i} \frac{\Lambda_{i,j}}{(z-p_i)^j} \right) \tag{2.22}$$

for $\Lambda_{i,j}$ diagonal and G_i consisting of the eigenvectors of A_{i,r_i} cf. [12]. Although this is a formal divergent solution, there exist canonical meromorphic solutions $\Psi^i_j(z)$ once we restrict to certain sectors, cut out by rays centred at a singularity called *(anti-)Stokes rays*



with $\Psi^i_j(z) = \Psi^i_{\text{formal}}(z)$ as we asymptotically approach the singularity $z \to p_i$. Here is a sketch of how these rays are defined. Let's write

$$\exp(Q_i) = \exp\left(\Lambda_{i,0} \cdot \ln(z-p_i) + \sum_{j=1}^{r_i} \frac{\Lambda_{i,j}}{(z-p_i)^j} \right); Q_i = \begin{bmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{bmatrix} \tag{2.23}$$

where $q_\ell(z) \in (z-p_i)^{-r_i}\mathbb{C}[z]$. As we approach p_i, the asymptotic behaviour of an entry e^{q_ℓ} changes depending on the direction we approach the singularity. Stokes rays are the directions where the dominance between the growth rates of e^{q_ℓ} and $e^{q_{\ell'}}$ changes, i.e. where $\mathbb{R}e(q_\ell - q_{\ell'}) = 0$ and anti-Stokes rays are the singular directions along which the difference of the growth rates is the largest. Each pair ℓ, ℓ' gives rise to $\deg(q_\ell - q_{\ell'})$ anti-Stokes directions. The canonical sectorial solutions are now related by linear transformations between the sectors, i.e. there exist matrices S^i_j such that $\Psi^i_{j+1} = \Psi^i_j \cdot S^i_j$. In other words, if we try to analytically continue one sectorial solution, this will jump multiple times before we complete a full circle along a cycle in C°. The collection of these matrices encoding the intermediate jumps, together with the classical monodromy of the differential equation, constitute the generalised monodromy data or *Stokes data* of the connection.

There exists an irregular extension of Deligne's Riemann-Hilbert correspondence between irregular meromorphic connections and Stokes data. The Stokes matrices approach we presented above, can be found in the classical references [72] and [77],[75]. There is a more geometric approach to the subject using filtered local systems initiated by Deligne in a letter to Malgrange [23] and developed further by Malgrange in [62]. Modern approaches to the subject can be found in works of Sabbah [69],[70] and references therein.

3. General Theory of Joyce Structures

In this chapter we aim to give a sufficiently thorough treatment of the ingredients of a Joyce structure.

3.1. Ehresmann Connections

The main ingredient of a Joyce Structure will be a family of Ehresmann connections. Ehresmann connections generalise the notion of a linear connection on a vector bundle to fibre bundles. In [28] Ehresmann identified the key ingredient of linear connections that can be generalised as the notion of a (smooth-)lift of tangent vectors from the base to tangent vectors on the fibres, which in turn defines a notion of parallel transport between the fibres of a fibre bundle.

A linear connection ∇ on a $\mathbb{C}$-vector bundle $E \longrightarrow M$ is a $\mathbb{C}$-linear differential operator that gives a notion of what constitutes a parallel section $s \in \Gamma(E)$ along a vector field $\mathcal{X} \in \Gamma(\mathcal{T}_M)$, namely $\nabla_{\mathcal{X}} s = 0$. This means that s locally solves a system of linear differential equations. Ehresmann generalises the notion of a connection using the concept of a parallel section without making reference to a differential operator. At a glance, given a smooth fibre bundle $\pi : X \longrightarrow M$, an Ehresmann connection defines a sub-bundle of its tangent bundle $\mathcal{H} < \mathcal{T}_X$, i.e. of the linearisation of X. Then, a section of this general fibre bundle $s \in \Gamma(X)$, is said to be parallel to a vector field $\mathcal{X} \in \Gamma(\mathcal{T}_M)$, if $s_*\mathcal{X} \in \mathcal{H}$, where $s_* : \mathcal{T}_M \longrightarrow \mathcal{T}_X$ is the pushforward of vector fields along the (non-linear) section $s : M \longrightarrow X$. This notion will allow us to formalise solutions to non-linear PDE's on fibre bundles and also define parallel transport for points of the fibre bundle, rather than parallel transport of vectors on vector bundles. The parallel transport maps are now inherently non-linear maps.

In what follows, we will be working in the category of complex manifolds with holomorphic morphisms, but all definitions are restrictions of the analogous definitions in the smooth category. Most results in this introductory section are well known and we will not be supplying full proofs, we refer to [50] as the relevant treatise.

Let X, M be $\mathbb{C}$-manifolds and denote by $\mathcal{T}_M$ (resp. $\mathcal{T}_X$) the holomorphic tangent bundle, i.e. $\mathcal{T}_M = (T^{\mathbb{R}}_M \otimes_{\mathbb{R}} \mathbb{C})^{1,0}$. Moreover, let $\pi : X \longrightarrow M$ be a holomorphic submersion, meaning that the map $d\pi$ sitting in the Cartesian diagram

$$\begin{tikzcd}
\mathcal{T}_X \arrow[dr, dashed, "d\pi"] \arrow[drr, bend left, "\pi_*"] \arrow[ddr, bend right] & & \\
& \pi^*\mathcal{T}_M \arrow[r] \arrow[d] & \mathcal{T}_M \arrow[d] \\
& X \arrow[r, "\pi"] & M
\end{tikzcd} \tag{3.1}$$

is surjective. In particular, given a submersion π, we have a short exact sequence (SES)

$$0 \longrightarrow \mathcal{T}_{X/M} \xrightarrow{i} \mathcal{T}_X \xrightarrow{d\pi} \pi^*\mathcal{T}_M \longrightarrow 0, \tag{3.2}$$

where, $\mathcal{T}_{X/M} := ker(d\pi)$ is the relative tangent bundle with respect to π, and i is the natural inclusion.

Definition 3.1.1. Ehresmann Connection
Let X, M and π as above. A *non-linear Ehresmann connection* or *π-connection* on the fibre bundle $\pi : X \longrightarrow M$ is an $\mathcal{O}_X$-morphism $h : \pi^*\mathcal{T}_M \longrightarrow \mathcal{T}_X$ that is a right splitting of the differential SES

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}_{X/M} \arrow[r, "i"] & \mathcal{T}_X \arrow[r, "d\pi"] & \pi^*\mathcal{T}_M \arrow[l, bend right, "h"'] \arrow[r] & 0,
\end{tikzcd} \tag{3.3}$$

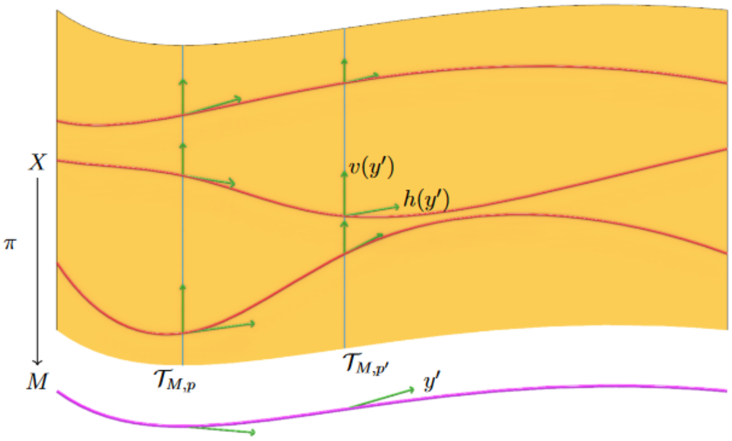
i.e. $d\pi \circ h = id_{\pi^*\mathcal{T}_M}$. We call such a connection *flat or integrable*, if

$$[h(v), h(w)] = h\left([v, w]\right) \quad \forall v, w \in \Gamma(\pi^*\mathcal{T}_M). \tag{3.4}$$

Remark 3.1.2. Conceptually, after identifying fibrewise $\mathcal{T}_M$ with $\pi^*\mathcal{T}_M$, we can think of a π-connection as a rule for lifting tangent vectors on M to tangent vectors on X. More precisely, for every vector $v \in \mathcal{T}_{M,m}$, a π-connection h determines a tangent vector at each point of the fibre X_m. The fact that there is a unique lift $h_x(v) \in \mathcal{T}_{X,x}$ of v at each $x \in X$ follows from h being a section of $d\pi$.

Definition 3.1.3. Horizontal and Vertical Subbundles
Let h be a π-connection, then we call the sub-bundle $\mathcal{H} := Im(h)$, which is transverse to the fibres of π, the *horizontal subbundle* associated to h. We also call the relative tangent bundle $\mathcal{V} := \mathcal{T}_{X/M}$, which is independent of h, the *vertical subbundle.* We call sections of $\mathcal{T}_X$ horizontal (resp. vertical) when they lie in $\mathcal{H}$ (resp. in $\mathcal{V}$).

Remark 3.1.4. Since h is a splitting of the SES 3.3, the tangent bundle on X splits into a direct sum of bundles $\mathcal{T}_X = \mathcal{H} \oplus \mathcal{V}$. A choice of a transversal subbundle to π determines an Ehresmann π-connection.



Lemma 3.1.5. ***Integrability in terms of the Horizontal Distribution***
An Ehresmann π-connection h is flat if and only if the associated horizontal subbundle $\mathcal{H} < \mathcal{T}_X$ is involutive; i.e. $[\mathcal{H},\mathcal{H}] \subset \mathcal{H}$. In other words, the space of sections of the horizontal subbundle carries the structure of a Lie subalgebra of the space of vector fields on X: $(\Gamma(X,\mathcal{H}), [\cdot,\cdot]|_{\mathcal{H}}]) < (\Gamma(X,\mathcal{T}_X), [\cdot,\cdot]])$.

Proof. The integrability (involutivity) of the distribution $\mathcal{H}$ follows tautologically from the integrability of the Ehresmann connection. The inverse direction follows by Frobenius integrability. □

More generally, the sheaf of sections of the subbundle $\mathcal{H}$ is a subsheaf of Lie subalgebras to the sheaf of sections of $\mathcal{T}_X$. The curvature is the obstruction to $\mathcal{H}$ being a Lie subalgebra. In case h is flat, we denote the foliation defined by the integrable distribution $\mathcal{H}$ by $\mathcal{F}_h$ and the space of leaves $X/\mathcal{F}_h$ is then isomorphic to any of the fibres of the bundle π.

Corollary 3.1.6. ***Local Characterisation of Integrability***
If h is flat, then for any $x \in X$ there exist neighbourhoods $x \in U \subset X$, $\pi(x) \in W \subset M$ and local coordinates $(x_1,\dots,x_n) \in U$, $(y_1,\dots,y_d) \in W$ such that $x_i = \pi^*(y_i)$ and

$$h\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial}{\partial x_i} \forall i \in \{1,\dots,d\}.$$

A flat Ehresmann connection provides us with a trivialisation of π over a formal neighbourhood $\mathfrak{U}_m$ of any point $m \in M$; i.e. isomorphisms: $\pi^{-1}(\mathfrak{U}_m) \cong X_m \times \mathfrak{U}_m$.

We can use the Lie derivative on $\mathcal{T}_X$ to differentiate sections of bundles over X in horizontal directions.

Definition 3.1.7. Horizontal Sections of General Bundles
We call a section **horizontal**, if its Lie derivative vanishes along any horizontal direction.

Definition 3.1.8. Horizontal Relative Forms

Let $\Omega_{X/M}$ be the sheaf of relative differential forms on $\pi : X \longrightarrow M$. After identifying $\pi^*\mathcal{T}_M$ with $\mathcal{T}_M$, taking Lie derivatives along h-horizontal vector fields $Y \in \Gamma(X, \mathcal{H})$ defines an action

$$\mathcal{T}_M \curvearrowright \Omega_{X/M}; (Y, \omega) \longmapsto \mathcal{L}_Y(\omega). \tag{3.5}$$

We call a form $\omega \in \Omega_{X/M}$ h-**horizontal**, if $\mathcal{L}_Y(\omega) = 0$ for any horizontal vector field Y.

Proposition 3.1.9. ***Vertical Torsor***

The space of Ehresmann connections on $\pi : X \longrightarrow M$ *is an* $\Omega^1_M \otimes \pi_*\mathcal{T}_{X/M}$*-torsor.*

Proof. Let h be a π-connection and i the natural embedding $\mathcal{T}_{X/M} \hookrightarrow \mathcal{T}_X$. For any section $\mathrm{v} \in \mathcal{H}om(\pi^*\mathcal{T}_M, \mathcal{T}_{X/M}) \cong \mathcal{H}om(\mathcal{T}_M, \pi_*\mathcal{T}_{X/M}) \cong \Omega^1_M \otimes \pi_*\mathcal{T}_{X/M}$, the linear combination $h + i \circ \mathrm{v}$ is again a π-connection. Indeed, by 3.3, we have

$$d\pi \circ (h + i \circ \mathrm{v}) = d\pi \circ h = id_{\pi^*\mathcal{T}_M}. \tag{3.6}$$

□

Definition 3.1.10. Morphisms of Bundles with Ehresmann Connections

Let $\pi_1 : E \longrightarrow M$, $\pi_2 : F \longrightarrow M$ be two submersions and let h_1, h_2 be Ehresmann connections to π_1 and π_2 respectively. Then, a morphism of bundles equipped with Ehresmann connections between (E, h_1) and (F, h_2) is a diagram of bundles

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "\pi_1"'] & & F \arrow[dl, "\pi_2"] \\ & M & \end{tikzcd} \tag{3.7}$$

such that $g_*h_1 = h_2$.

Remark 3.1.11. Descent of π-connections along Étale Quotients

Let G be a discrete group acting freely and properly on X and assume that $\pi : X \longrightarrow M$ is G-invariant, i.e. $\pi \circ g = \pi; \forall g \in G$. In particular, the G-orbits lie in the fibres of π. Let $q : X \longrightarrow Y := X/G$ be the associated étale quotient. Then, the induced map η

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "\pi"'] & Y \arrow[dl, dashed, "\eta"] \\ M & \end{tikzcd} \tag{3.8}$$

is a well defined submersion. This is well defined since π is G-invariant, i.e. $\eta(q(x)) = \pi(x)$. It is also a submersion because $d\pi_x : \mathcal{T}_{X,x} \longrightarrow \mathcal{T}_{M,\pi(x)}$ is surjective, $dq_x : T_{X,x} \longrightarrow T_{Y,q(x)}$ is

an isomorphism and $d\pi_x = d\eta_{q(x)} \circ dq_x$. Moreover, we call a π-connection h G-invariant, if $g_* \circ h = h\ ,\forall g \in G$. In particular, for a G-invariant connection h, we have $g_*(\mathcal{H}_x) = \mathcal{H}_{g\cdot x}$ for the associated horizontal subbundle $\mathcal{H}$.
A G-invariant connection h uniquely defines a connection $j : \eta^*\mathcal{T}_M \longrightarrow \mathcal{T}_Y$ by the identity $q^*(j) = q_* \circ h$. In other words, $dq_x \circ h_x = j_{q(x)} \forall x \in X$. The connection j is well defined due to the G-invariance of h, i.e. $h_x = h_{x'},\ \forall x' \in G \cdot x$, and so the j-lift does not depend on the representative of the G-orbit.

Remark 3.1.12. Composition and Pullback of π-connections
For smooth maps $\pi : X \to Y$ and $\eta : Y \to Z$, a π-connection $h : \pi^*(\mathcal{T}_Y) \to \mathcal{T}_X$ and an η-connection $j : \eta^*(\mathcal{T}_Z) \to \mathcal{T}_Y$, the composite

$$h \circ \pi^*(j) : \pi^*\eta^*(\mathcal{T}_Z) \to \mathcal{T}_X$$

is an $\eta \circ \pi$-connection. Moreover, for a Cartesian diagram

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "\eta"] & X \arrow[d, "\pi"] \\ Z \arrow[r, "f"] & Y \end{tikzcd} \tag{3.9}$$

and a π-connection h, there is a unique η-connection j such that

$$g_* \circ j = g^*(h) \circ \eta^*(f_*) : \eta^*(\mathcal{T}_Z) \to g^*(\mathcal{T}_X). \tag{3.10}$$

The following generalisation of a π-connection allows Ehresmann connections to develop poles, and so they can be thought of as a meromorphic extension of the definition of non-linear connections.

Definition 3.1.13. D-twisted Ehresmann Connections
Let $\pi : X \longrightarrow M$ be a submersion and D be an effective divisor on X. Twisting the divisorial short exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{i_D} \mathcal{O}_X(D) \longrightarrow \mathcal{O}(D)|_D \longrightarrow 0 \tag{3.11}$$

by $d\pi$, we obtain a diagram of sheaves:

$$\begin{tikzcd} \mathcal{T}_X \arrow[r, "id_{\mathcal{T}_X}\otimes i_D"] \arrow[d, "d\pi"'] & \mathcal{T}_X(D) \arrow[r] \arrow[d, "d\pi\otimes id_{\mathcal{O}_X(D)}"] & \mathcal{T}_X(D)|_D \arrow[d, "d\pi\otimes id_{\mathcal{O}_X(D)}|_D"] \\ \pi^*\mathcal{T}_M \arrow[r, "id_{\pi^*\mathcal{T}_M}\otimes i_D"'] & \pi^*\mathcal{T}_M \otimes \mathcal{O}_X(D) \arrow[r] & (\pi^*\mathcal{T}_M \otimes \mathcal{O}_X(D))|_D. \end{tikzcd} \tag{3.12}$$

A *D-twisted Ehresmann π-connection* is a morphism of locally free sheaves

$$h : \pi^*\mathcal{T}_M \longrightarrow \mathcal{T}_X(D),$$

such that the following triangle

$$\begin{tikzcd} & \mathcal{T}_X(D) \arrow[d, "d\pi\otimes id_{\mathcal{O}_X(D)}"] \\ \pi^*\mathcal{T}_M \arrow[ur, "h"] \arrow[r, "id_{\pi^*\mathcal{T}_M}\otimes i_D"'] & \pi^*\mathcal{T}_M \otimes \mathcal{O}_X(D). \end{tikzcd} \tag{3.13}$$

commutes, i.e. $(d\pi \otimes id_{\mathcal{O}_X(D)}) \circ h = id_{\pi^*\mathcal{T}_M} \otimes i_D$.

Strictly speaking, the Ehresmann connection we will construct in this thesis is indeed a D-twisted one, where the pole structure arises as a very interesting exceptional divisor, one worth studying in depth.

3.1.1. Linear Connections as Ehresmann Connections

Now we pass from fibre bundles to the special case of vector bundles. However, this still does not require an Ehresmann connection to be a linear one. For the connection to be linear, one must additionally require that the parallel transport maps we described above respect the vector space structure of the fibres.

Definition 3.1.14. Covariant Derivative of a Linear Connection
Given a linear connection $\nabla : E \to E \otimes \Omega^1_M$ on a vector bundle $\pi : E \longrightarrow M$ and sections $w \in \Gamma(U, \mathcal{T}_M), s \in \Gamma(V, E)$ for open subsets $V \subset U \subset M$, we define the covariant derivative of s in the direction of w at a point $m \in M$ as the contraction of the E-valued 1-form ∇s with the vector field w evaluated at m

$$\nabla_w s(m) := \iota_w \nabla s(m).$$

This forms a fibrewise linear map:

$$\begin{aligned} \nabla_{[\cdot]} s(m) : \mathcal{T}_{M,m} &\longrightarrow E_m \\ w &\longmapsto \iota_w \nabla s(m). \end{aligned}$$

Before addressing the question of how to interpret a linear integrable connection as an Ehresmann connection, we can now formalise the notion of a linear Ehresmann connection utilising definition 3.1.10, as a connection that respects the linear structure of the fibres.

Definition 3.1.15. Linear Ehresmann Connections

Let h be an Ehresmann connection on a vector bundle $E \longrightarrow M$. Moreover, consider the addition map along the fibres:

$$\begin{aligned} \mathrm{A} : E \times_M E &\longrightarrow E \\ (e_1, e_2) &\longmapsto e_1 + e_2 \end{aligned} \tag{3.14}$$

as well as the multiplication map:

$$\begin{aligned} \mathrm{M} : \mathcal{O}_M \times_M E &\longrightarrow E \\ (f, e) &\longmapsto f \cdot e. \end{aligned} \tag{3.15}$$

We call h a linear-π-connection, if the maps A and M induce morphisms of bundles with Ehresmann connections

$$(\pi_{\mathrm{A}} : E \times_M E \longrightarrow M, h_{\mathrm{A}} := (h, h)) \longrightarrow (\pi : E \longrightarrow M, h), \tag{3.16}$$

$$(\pi_{\mathrm{M}} : \mathcal{O}_M \times_M E \longrightarrow E, h_{\mathrm{M}} := (d_{\mathrm{M}}, h)) \to (\pi : E \longrightarrow M, h). \tag{3.17}$$

Explicitly, this condition means:

$$\mathrm{A}_*(h(e_1), h(e_2)) = h(e_1 + e_2) \tag{3.18}$$

$$\mathrm{M}_*(d_{\mathrm{M}}(f), h(e)) = h(f \cdot e). \tag{3.19}$$

As we mentioned before, a differential operator gives rise to the notion of horizontal sections, as those sections which remain constant when derived in a direction parallel to the base. The relation of the linear connections of definition 2.2.1 to Ehresmann connections is to been seen by the following constructive proposition.

Proposition 3.1.16. ***The Horizontal Lift of a Linear Integrable Connection***
A linear integrable connection ∇ on a $\mathbb{C}$-vector bundle $E \longrightarrow M$ defines an integrable linear Ehresmann π-connection.

Proof. The differential operator ∇ determines a horizontal subbundle in a fibrewise manner as follows. Let $x \in E_m$ and $s : M \longrightarrow E$ be the unique horizontal section of E passing through x, i.e. $s \in ker(\nabla)$, such that $s(m) = x$. Then, consider the differential of this section $ds : \mathcal{T}_M \longrightarrow \mathcal{T}_E$, which evaluated at the point $s(m) = x$ gives us a map

$$ds(m) : \mathcal{T}_{M,m} \longrightarrow \mathcal{T}_{E,s(m)}. \tag{3.20}$$

The image of this map is the fibre of the associated horizontal bundle at $x \in E_m$, that is, $\mathcal{H}_x = Im(ds(m))$.

Let $w \in \mathcal{T}_M$, $s \in \Gamma(E)$ and $m \in M$, then $s(m) \in E_m$ and $\nabla_w s(m) = [(ds(m))(w)]^{ver} \in E_m$, where *ver* denotes the projection to the vertical subbundle. Since E is a vector bundle, we have a canonical identification $E_m \cong V_m := \mathcal{T}_{E/M}$. We can then write the horizontal lift morphism h at the point $s(m)$ as

$$h_{s(m)}(w) = (ds(m))(w) - \nabla_w s(m) \in \mathcal{H}_m. \tag{3.21}$$

If s is a horizontal section, then $\nabla_w s(m) = 0 \ \forall w \in \mathcal{T}_M$ and so $ds(m)(w) \in \mathcal{H}_m \ \forall w$. $\square$

3.2. Pre-Joyce Structures

We now further specialise the discussion of Ehresmann connections to the case where the submersion $\pi : X \longrightarrow M$ is the holomorphic cotangent bundle to M, i.e. $X = \mathcal{T}_M$. In that case, there exists a canonical gauge for the torsor of vertical sections of proposition 3.1.9, namely the map $\mathrm{v}_{can} : \pi^*\mathcal{T}_M \longrightarrow \mathcal{T}_{X/M}$ obtained fibrewise from the canonical identifications $\pi^*\mathcal{T}_{M,x} \cong \mathcal{T}_{M,\pi(x)} \cong \mathcal{T}_{(\mathcal{T}_M,\pi(x)),x} \cong \mathcal{T}_{X_{\pi(x)},x} \cong \mathcal{T}_{X/M}$. Fixing this gauge up to a scalar, we arrive at the following structure.

Definition 3.2.1. v-Pencil of π-Connections
Let h be an Ehresmann connection on $\pi : X = \mathcal{T}_M \longrightarrow M$ and denote by $v := i \circ \mathrm{v}_{can}$ the composition of the canonical embedding of the vertical bundle $i : \mathcal{T}_{X/M} \hookrightarrow \mathcal{T}_X$ with the canonical vertical section v_{can}. Then, by rescaling $v \longmapsto \varepsilon^{-1}v, \ \ \varepsilon \in \mathbb{C}^*$ we obtain a $\mathbb{C}^*$-family of π-connections $\{h_\varepsilon := h + \varepsilon^{-1}v\}_{\varepsilon\in\mathbb{C}^*}$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}_{X/M} \arrow[r, "i"] & \mathcal{T}_X \arrow[r] & \pi^*\mathcal{T}_M \arrow[r] \arrow[l, bend right=20, "h"'] \arrow[l, bend right=60, "h_\varepsilon = h + \varepsilon^{-1}v"'] \arrow[ll, bend left=30, "\varepsilon^{-1}\cdot \mathrm{v}_{can}"] & 0.
\end{tikzcd} \tag{3.22}$$

We call this family associated to h a v-pencil of π-Connections based on h.

Definition 3.2.2. Induced Relative Symplectic Form
If M is equipped with a holomorphic symplectic form $\omega \in H^0(M, \Omega^2_M)$, then the canonical vertical section determines a relative symplectic form on the fibre bundle via

$$\Omega_\pi := \mathrm{v}_{can,*} \circ (\pi^*\omega|_{\pi^*\mathcal{T}_M}) \in H^0(X, \Omega^2_{X/M}). \tag{3.23}$$

The following definition is the essence of what a Joyce structure is, first introduced in [14].

Definition 3.2.3. Pre-Joyce Structure
Let M be $\mathbb{C}$-manifold and $\pi : X = \mathcal{T}_M \longrightarrow M$ its holomorphic tangent bundle. A *Pre-Joyce Structure* on M is a pair (ω, h), consisting of

i. a holomorphic symplectic form $\omega \in H^0(M, \Omega^2_M)$ on M,

ii. a flat and symplectic π-connection h, such that $\{h_\varepsilon := h + \varepsilon^{-1} v\}_{\varepsilon \in \mathbb{C}^*}$ is a pencil of flat and symplectic π-connections.

Lemma 3.2.4. ***Locally Hamiltonian Flow***
Let (ω, h) be a pre-Joyce Structure on M. There exists an open covering of X such that for each open subset in the covering $U \subset X$ and local coordinates $(z_i, \theta_i)_i$ on U, there exists a locally defined meromorphic function $W : U \longrightarrow \mathbb{C}$ on X with

$$h\left(\frac{\partial}{\partial z_i}\right) = \frac{\partial}{\partial z_i} + \sum_{p,q} \eta_{pq} \frac{\partial^2 W}{\partial \theta_p \partial z_i} \frac{\partial}{\partial \theta_q} \tag{3.24}$$

satisfying the following PDEs:

$$P1\text{:}\ \frac{\partial^2 W}{\partial \theta_i \partial z_j} - \frac{\partial^2 W}{\partial \theta_j \partial z_i} = \sum_{p,q} \eta_{pq} \frac{\partial^2 W}{\partial \theta_i \partial \theta_p} \cdot \frac{\partial^2 W}{\partial \theta_j \partial \theta_q} \tag{3.25}$$

$$P2\text{:}\ \frac{\partial}{\partial \theta_p}\left(\frac{\partial W_j}{\partial \theta_i} - \frac{\partial W_i}{\partial \theta_j}\right) = 0. \tag{3.26}$$

Proof. Since h is symplectic, it preserves the relative form $\Omega_\pi \in H^0(X, \Omega^2_{X/M})$ associated to $\omega \in H^0(M, \Omega^2_M)$:

$$\mathcal{L}_v \Omega_\pi = 0\ , \forall v \in \Gamma(\mathcal{H}) \implies d(\iota_v \Omega_\pi) = 0 \implies \iota_v \Omega_\pi \text{ closed 1-form.} \tag{3.27}$$

In particular $\iota_v \Omega_\pi$ is *locally exact*, i.e. there exists an open covering of X, such that for each open subset in the covering $U \subset X$, there exists a holomorphic function: $W_v : U \longrightarrow \mathbb{C}$ with $\iota_v \Omega_\pi = dW_v$. In local coordinates, writing W_i for the local Hamiltonians in each horizontal direction $W_{\frac{\partial}{\partial z_i}}$, we have:

$$h\left(\frac{\partial}{\partial z_i}\right) = \frac{\partial}{\partial z_i} + \sum_{p,q} \eta^{pq} \frac{\partial W_i}{\partial \theta_p} \frac{\partial}{\partial \theta_q}. \tag{3.28}$$

Applying the integrability condition for h on the horizontal lifts of the basis $\left\{\frac{\partial}{\partial z_i}\right\}_i$ of $\mathcal{T}_M$, we have

$$\begin{aligned} 0 \overset{!}{=} \left[h\left(\frac{\partial}{\partial z_i}\right), h\left(\frac{\partial}{\partial z_j}\right)\right] = &\sum_{p,q} \eta^{pq} \frac{\partial^2 W_j}{\partial z_i \partial \theta_p} \frac{\partial}{\partial \theta_q} - \sum_{p,q} \eta^{pq} \frac{\partial^2 W_i}{\partial z_j \partial \theta_p} \frac{\partial}{\partial \theta_q} \\ &+ \sum_{p,q} \eta^{pq} \eta^{k\ell} \frac{\partial W_i}{\partial \theta_p} \frac{\partial^2 W_j}{\partial \theta_q \partial \theta_k} \frac{\partial}{\partial \theta_\ell} - \sum_{p,q} \eta^{pq} \eta^{k\ell} \frac{\partial W_i}{\partial \theta_k} \frac{\partial^2 W_i}{\partial \theta_\ell \partial \theta_p} \frac{\partial}{\partial \theta_q}. \end{aligned} \tag{3.29}$$

Identifying the $\frac{\partial}{\partial \theta_q}$-terms, using the anti-symmetry of η and integrating along the θ_p-vertical direction, we arrive at

$$\frac{\partial}{\partial\theta_p}\left(\frac{\partial W_j}{\partial z_i}-\frac{\partial W_i}{\partial z_j}-2\sum_{p,q}\eta^{pq}\frac{\partial W_i}{\partial\theta_p}\frac{\partial W_j}{\partial\theta_q}\right)=0\ \forall p. \tag{3.30}$$

Furthermore, the pre-Joyce structure condition requires that the full pencil of connections h_ε is flat and symplectic for all $\varepsilon\in\mathbb{C}^*$. For the family of connections, we have

$$\begin{aligned} h_\varepsilon\left(\frac{\partial}{\partial z_i}\right) &= h\left(\frac{\partial}{\partial z_i}\right)+\varepsilon^{-1}v\left(\frac{\partial}{\partial z_i}\right)\\ \Longrightarrow \frac{\partial}{\partial z_i}+\sum_{p,q}\eta^{pq}\frac{\partial W_i^\varepsilon}{\partial\theta_p}\frac{\partial}{\partial\theta_q} &= \frac{\partial}{\partial z_i}+\sum_{p,q}\eta^{pq}\frac{\partial W_i}{\partial\theta_p}\frac{\partial}{\partial\theta_q}+\sum_q\varepsilon^{-1}\frac{\partial}{\partial\theta_q}. \end{aligned} \tag{3.31}$$

Running the integrability condition calculation 3.29 for h_ε and using the last equality 3.31, we obtain, in addition to equations (3.30), a collection of equations

$$\varepsilon^{-1}\frac{\partial}{\partial\theta_p}\left(\frac{\partial W_j}{\partial\theta_i}-\frac{\partial W_i}{\partial\theta_j}\right)=0\ \forall p \tag{3.32}$$

from which *P2* follows.

□

3.2.1. Complex Hyperkähler Structures from Pre-Joyce Structures

One of the interesting structures that emerge out of the concept of a (Pre-)Joyce Structure is that of a complex Hyperkähler structure on X.

Definition 3.2.5. $\mathbb{C}$-Hyperkähler structure
Let X be a $\mathbb{C}$-manifold, then a complex Hyperkähler structure on X consists of

- a holomorphic metric $g\in H^0(X,\left(T_X^{\otimes 2}\right)^*)$, and
- endomorphisms of the holomorphic tangent bundle $I,J,K\in End_{\mathcal{O}_X}(\mathcal{T}_X)$, such that
  1. they form a quaternionic triple in the endomorphism algebra, i.e.
  $$I^2=J^2=K^2=IJK=-1, \tag{3.33}$$
  2. they preserve the metric: $g(R(u_1),R(u_2))=g(u_1,u_2),\forall R\in\{I,J,K\}$,
  3. they are horizontal with respect to the holomorphic Levi-Civita connection associated to the metric g: $\nabla^g(R)=0\ ,\forall R\in\{I,J,K\}$.

Proposition 3.2.6. *Let M be a $\mathbb{C}$-manifold and $\pi:X=\mathcal{T}_M\longrightarrow M$. Moreover, let (ω,h) be a pre-Joyce structure on M, then (ω,h) defines a $\mathbb{C}$-Hyperkähler structure on X.*

Proof. (Sketch) This result (even a stronger version thereof) was one of the main outputs of Bridgeland-Strachan [19]. Here we just recall the construction of the complex Hyperkähler metric given a π-connection, without proving the symmetry (3) of definition 3.2.5.

By remark 3.1.4, the π-connection determines a splitting $\mathcal{T}_X = \mathcal{H} \oplus \mathcal{V}$, which, in this special case of the tangent bundle $X \longrightarrow M$, can be identified with $\mathcal{T}_X = \pi^*\mathcal{T}_M \otimes \mathbb{C}^2$. This is because $\mathcal{H} = im(h)$ and $\mathcal{V} = im(v)$ are both copies of $\pi^*\mathcal{T}_M$. We define the metric g as the tensor product of the pulled back symplectic form $\pi^*\omega$ on $\pi^*\mathcal{T}_M$ with the standard symplectic form on $\mathbb{C}^2$:

$$g = \pi^*\omega \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \pi^*\omega \\ \pi^*\omega & 0 \end{bmatrix}. \tag{3.34}$$

Using the Artin-Wedderburn theorem, we can identify the product of the $\mathbb{R}$-division algebras $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ with the product in the endomorphism algebra $End_{\mathbb{C}}(\mathbb{C}^2)$ under which the following I, J give rise to a quaternionic triple $\{I, J, K\} \in End_{\mathbb{C}}(\mathbb{C}^2)$ acting naturally on $\mathbb{C}^2$:

$$I := \begin{bmatrix} i \cdot 1 & 0 \\ 0 & -i \cdot 1 \end{bmatrix}; \; J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; \; K := I \cdot J. \tag{3.35}$$

Using the fact that the splitting induced by the pre-Joyce structure on the tangent sheaf of X respects the symplectic structure of $\mathcal{T}M$, one shows that these automorphisms preserve g in the sense of condition (2) above and are horizontal with respect to ∇^g. □

3.3. Period Structures

Period structures are the ingredient of Joyce structures that resembles a complex integrable system. Let M be a complex manifold.

Definition 3.3.1. (Integral-)Lattice of a Holomorphic Vector Bundle
Let $\mathscr{H}$ be a holomorphic vector bundle on M. A lattice of $\mathscr{H}$ is a locally constant subsheaf $\mathscr{H}^{\mathbb{Z}} < \mathscr{H}$, which is a $\mathbb{Z}$-model for $\mathscr{H}$, i.e. the natural map $\mathscr{H}^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_M \longrightarrow \mathscr{H}$ is an isomorphism of holomorphic vector bundles.

Definition 3.3.2. Period Structure
A period structure $\mathscr{P}$ on a $\mathbb{C}$-manifold M, consists of:

1. An integral lattice $\mathcal{T}_M^{\mathbb{Z}} \subset \mathcal{T}_M$, with associated flat connection ∇,
2. A section $Z \in H^0(M, \mathcal{T}_M)$ satisfying $\nabla(Z) = id$.

Lemma 3.3.3. ***Period Coordinates***
A period structure $\mathscr{P} = (\mathcal{T}_M^{\mathbb{Z}}, \nabla, Z)$ on M induces local-coordinates M.

Proof. Let $p \in M$. By the integrability condition (local constancy of $\mathcal{T}_M^{\mathbb{Z}} \otimes \underline{\mathbb{C}}_M$), a basis of $\mathcal{T}_{M,p}^{\mathbb{Z}}$ gives rise to a local basis of ∇-horizontal sections $\langle \phi_1, \dots, \phi_n \rangle = \Gamma(U, \mathcal{T}_M)$ over some open neighbourhood $p \in U \subset M$. Writing the vector field Z in that basis $Z|_U = \sum_i z_i \cdot \phi_i$, we obtain holomorphic functions $z_i : U \longrightarrow \mathbb{C}$. Then the condition $\nabla(Z) = id$ reads

$$\nabla(Z) = \sum_i dz_i \otimes \phi_i + \nabla(\phi_i) \cdot z_i = \sum_i dz_i \otimes \phi_i \overset{!}{=} id \tag{3.36}$$

which implies that $\left\{\phi_i = \frac{\partial}{\partial z_i}\right\}_i$ is the dual basis to $\{dz_i\}_i$. Now since $\{\phi_i\}_i$ is a local basis for $\Gamma(U, \mathcal{T}_M|_U)$ and thereby linearly independent, the $\{z_i\}_i$ give a local basis for U. □

Corollary 3.3.4. *A flat connection that is a datum in a period structure is necessarily torsion free, since the basis elements of the flat structure now commute:* $[\phi_i, \phi_j] = 0$.

The following notion was defined in [51].

Definition 3.3.5. Integral Affine Structure
An integral affine structure on M is a lattice $\mathcal{T}_M^{\mathbb{Z}} \subset \mathcal{T}_M$ such that the associated flat connection is torsion free. A local coordinate system $(z_1, \dots, z_n)$ is said to be *integral affine*, if $\frac{\partial}{\partial z_i} \in \mathcal{T}_M^{\mathbb{Z}}$ $\forall i$.

Remark 3.3.6. An integral affine coordinate system is uniquely defined up to lattice-preserving affine transformations $z_i \longmapsto \sum_j \alpha_{ij} z_j + v_i$ for $(\alpha_{ij})_{ij} \in GL_n(\mathbb{Z})$ and $v_i \in \mathbb{C}^n$. Fixing a vector field Z amounts to a linearisation of the affine structure, and so a period structure can be interpreted as a *integral linear structure*, or as a fibre bundle $\mathscr{P}$ over $\mathcal{T}_M$ modelled on $GL_n(\mathbb{Z})$, i.e. a bundle of lattice preserving frames. In coordinates, we call an integral affine coordinate system $(z_1, \dots, z_n)$ *integral linear* if Z takes the form $Z = \sum_i z_i \frac{\partial}{\partial z_i}$.

Remark 3.3.7. $\mathbb{C}^*$-action from Euler Vector Field
The fact that the vector field Z in a period structure respects the flat structure globally ($\nabla(Z) = id$), together with the existence of local coordinates such that $Z = \sum_i z_i \frac{\partial}{\partial z_i}$, implies ,by definition, that Z is an Euler vector field. Locally its flow is given by a rescaling of the coordinates:

$$\begin{aligned} \phi_Z|_U : \mathbb{C}^* \times U^* &\longrightarrow U \\ ((z_i)_i, t) &\longmapsto (t \cdot z_i)_i. \end{aligned} \tag{3.37}$$

Globally, an Euler vector field Z is the generator of a $\mathbb{C}^*$-action on M: $\phi_Z : \mathbb{C}^* \times M \longrightarrow M$, which in turn gives rise to a $\mathbb{C}^*$-action on $\mathcal{T}_M$: $d\phi_Z : \mathbb{C}^* \times \mathcal{T}_M \longrightarrow \mathcal{T}_M$. In local coordinates

(z_i, θ_i) on $\mathcal{T}_M$ this action reads

$$\begin{aligned} d\phi_Z|_U : \mathbb{C}^* \times \mathcal{T}_M|_U &\longrightarrow \mathcal{T}_M|_U \\ (t, (z_i, \theta_i)_i) &\longmapsto (t \cdot z_i, t \cdot \theta_i)_i. \end{aligned} \tag{3.38}$$

For a fixed $t \in \mathbb{C}^*$ we also write

$$\begin{aligned} d\phi_{Z,t}|_U : \mathcal{T}_M|_U &\longrightarrow \mathcal{T}_M|_U \\ (z_i, \theta_i)_i &\longmapsto (t \cdot z_i, t \cdot \theta_i)_i. \end{aligned} \tag{3.39}$$

Furthermore, consider the homothetic $\mathbb{C}^*$-action on the fibres of $X = \mathcal{T}_M$

$$\begin{aligned} \rho_{m,t} : \mathcal{T}_{M,m} &\longrightarrow \mathcal{T}_{M,m} \\ x &\longmapsto t \cdot x \end{aligned} \tag{3.40}$$

which, since $\mathcal{T}_M$ is a vector bundle, canonically extends to a map on the total space

$$\begin{aligned} \rho_t : X &\longrightarrow X \\ x &\longmapsto t \cdot x. \end{aligned} \tag{3.41}$$

Now since the connection ∇ is linear, the associated horizontal lift h_∇ is a linear Ehresmann connection. In particular, its parallel transport maps preserve the linear structure of the fibres and $\mathcal{H}^\nabla$ is a linear subbundle. Thereby the $\mathbb{C}^*$-action $d\rho_t$ on $\mathcal{T}_X$, acts on the horizontal bundle by

$$d\rho_t(x)(H_x) = H_{t \cdot x} \ ; \forall x \in X. \tag{3.42}$$

Lemma 3.3.8. *Let $\mathscr{P} = (\mathcal{T}_M^{\mathbb{Z}}, \nabla, Z)$ be a period structure on M and define a vector bundle $E := h_\nabla(Z) \in H^0(X, \mathcal{T}_X)$, i.e. the ∇-horizontal lift of Z. Then E is the generating vector field of the $\mathbb{C}^*$-action*

$$\begin{aligned} m : \mathbb{C}^* \times \mathcal{T}_M &\longrightarrow \mathcal{T}_M \\ (t, s) &\longmapsto m_t(s) := \left(d\phi_{Z,t} \circ \rho_{(t^{-1})}\right)(s) \end{aligned} \tag{3.43}$$

rescaling along the fibres of $X = \mathcal{T}_M$.

Proof. Let $U \subset M$ with $(z_i)_i$ integral linear local coordinates, in particular

$$Z|_U = \sum_i z_i \frac{\partial}{\partial z_i} \in \Gamma(U, \mathcal{T}_M|_U),$$

and let $(z_i, \theta_i)_i$ be the associated coordinates on $\mathcal{T}_M|_U$. Then the leaves of the foliation $\mathcal{F}_\nabla$

associated to ∇ are given locally in these coordinates by

$$\mathscr{L}_U^{\nabla} = \{(z_i, \theta_i) | \theta_i = const.\ \forall i\} \tag{3.44}$$

i.e. the ∇-horizontal lift of $\frac{\partial}{\partial z_i}$ is: $h^{\nabla}\left(\frac{\partial}{\partial z_i}\right) = \frac{\partial}{\partial z_i}$ $\forall i$. In particular

$$E_{\pi^{-1}(U)} = h^{\nabla}(Z|_U) = \sum_i z_i \frac{\partial}{\partial z_i} \in \Gamma(\pi^{-1}(U), \mathcal{T}_X|_{\pi^{-1}(U)})\ \forall i. \tag{3.45}$$

$E_{\pi^{-1}(U)}$ is now the generating vector field of the $\mathbb{C}^*$-action that rescales the coordinates of $\pi^{-1}(U)$ preserving the leaves of the ∇- horizontal foliation $\mathcal{F}_{\nabla}|_{\pi^{-1}(U)}$, that is

$$(t, (z_i, \theta_i)) \longmapsto (t \cdot z_i, \theta_i).$$

Therefore, E is precisely the generating vector field of the action

$$d\phi_{Z,t} \circ \rho_{(t^{-1})}((z_i, \theta_i)) = d\phi_{Z,t}((z_i, t^{-1}\theta_i)) = (t \cdot z_i, \theta_i).$$

This action is the one rescaling the fibres of π, as these are parametrised in local coordinates by $\{(z_i, \theta_i) | z_i = const.\}$. □

Definition 3.3.9. Let $\mathscr{P} = (\mathcal{T}_M^{\mathbb{Z}}, \nabla, Z)$ be a period structure.
1) A skew-symmetric form $\eta : \mathcal{T}_M^* \times \mathcal{T}_M^* \longrightarrow \mathcal{O}_M$ is said to be compatible with $\mathscr{P}$, if $\frac{\eta}{2\pi i}$ takes integral values on the dual lattice $(\mathcal{T}_M^{\mathbb{Z}})^*$.
2) Considering the fibrewise-action of the rescaled lattice $(2\pi i)\mathcal{T}_M^{\mathbb{Z}}$ on $\mathcal{T}_M$ by translations, we write

$$X^{\#} := \mathcal{T}_M / (2\pi i)\mathcal{T}_M^{\mathbb{Z}}, \tag{3.46}$$

and denote by $\pi^{\#} : X^{\#} \longrightarrow M$ the associated $(\mathbb{C}^*)^n$-fibre bundle.

Remark 3.3.10. Note that if η is compatible with $\mathscr{P}$, then it is also parallel with respect to the flat structure of $\mathscr{P}$. The flat structure is defined as the unique one preserving the lattice. A failure of η to be parallel would imply that its integrality property is not stable under parallel transport - *parallel to the lattice*, and fail to be a global property.

3.4. Joyce Structures

All that is left to do, in defining the notion of a Joyce Structure, is to set the proper compatibility conditions, that allow us to glue a Pre-Joyce Structure to a Period structure, i.o.w. impose certain rigidifying symmetries on the data of a Pre-Joyce structure, for it to properly sit on the scaffolding of a period structure. The essential condition we want to impose is that the Pre-Joyce structure on the tangent bundle $\mathcal{T}_M$ descends to the $(\mathbb{C}^*)^n$-fibre bundle $X^{\#}$.

Tom Bridgeland introduced Joyce Structures in [14] as follows (up to notational deviation).

Definition 3.4.1. Joyce Structure
Let M be a complex manifold and let $\pi : \mathcal{T}_M \longrightarrow M$ be the total space of the holomorphic tangent bundle on M. A *Joyce Structure* on M consists of:

1. a period structure $\mathscr{P} = (\mathcal{T}_M^{\mathbb{Z}}, \nabla, Z)$,
2. a $\mathscr{P}$-compatible skew-symmetric form $\eta : \mathcal{T}_M^* \times \mathcal{T}_M^* \longrightarrow \mathcal{O}_M$,
3. a pre-Joyce structure (ω, h),

subject to the following conditions:

[J1] the symplectic form ω is the inverse of η.

[J2] the π-connection h is invariant under the fibrewise-action of the rescaled lattice $(2\pi i)\mathcal{T}_M^{\mathbb{Z}}$ on $\mathcal{T}_M$ by translations, i.e. it descends to a $\pi^{\#}$-connection on $X^{\#}$.

[J3] $[h(Z), h(v)] = [h^{\nabla}(Z), h(v)]$ for any vector field $v \in H^0(M, \mathcal{T}_M)$, where h^{∇} is the linear Ehresmann connection on TM defined by the linear connection ∇.

[J4] the π-connection h is invariant under the global involution $\iota : X \longrightarrow X$ acting by multiplication by -1 on the linear fibres of π.

Furthermore, taking a twisted π-connection, as introduced in definition 3.1.13, instead of a π-connection, we can directly generalise the notion of a (holomorphic) Joyce structure to that of a *Meromorphic Joyce Structure*, developing poles along the twisting divisor.

4. Quadratic Differentials and Spectral Curves

The goal of this chapter is to introduce the notion of *meromorphic quadratic differentials on algebraic curves*, recast their datum into the notion of a *spectral curve* and study the cohomology of those spectral curves in families over the moduli of quadratic differentials.

4.1. Preliminaries on Divisors on Algebraic Curves

Denote by $\mathcal{M}(C) := \mathcal{M}_g$ the moduli space of complex projective curves of genus g. For $C \in \mathcal{M}(C)$ denote by ω_C the canonical sheaf on C.

Throughout this thesis, we will pass seamlessly between the notions of effective Cartier divisors and Weil divisors. We recall here the equivalence of these notions in the case of divisors on smooth curves.

Proposition 4.1.1. *Let C be a smooth algebraic curve, then there is a bijective correspondence between*

1. *Effective Cartier divisors on C, i.e. closed subschemes $D \hookrightarrow C$, whose associated ideal sheaf $\mathcal{I}_D \lhd \mathcal{O}_C$ is an invertible $\mathcal{O}_C$-module, cf.[74, Tag 01WQ].*
2. *Effective Weil divisors on C, i.e. formal sums of closed points $\sum_{p\in C} n_p \cdot p$, with a finite set $\{n_p \neq 0\}_p$ cf.[74, Tag 0BE0].*

Proof. Let D be an effective Cartier divisor. By [74, Tag 01WS], D is effective if and only if for all closed points $p \in D$, there exists an affine neighbourhood $p \in U = Spec(A) \subset C$ is such that $U \cap D = Spec(A/(f))$ for $f \in A$ a non-zero divisor. This means $f \in \Gamma(U, \mathcal{O}_C)$ such that $\mathcal{I}_D|_U = (f) \lhd \mathcal{O}_C(U)$. Since C is smooth, $\mathcal{O}_{C,p}$ is a DVR admitting a uniformiser π_p, and hence $(f) = 0$ or $(f) = \pi_p^{n_p}$ for $n_p = v_p(f) \in \mathbb{Z}_{\geq 0}$. Therefore, we may associate to the subscheme D a Weil-Divisor

$$W(D) = \sum_{p\in C} n_p \cdot p, \text{ with } n_p = v_p(f). \tag{4.1}$$

This formal sum $W(D)$ is well defined, because any two local generators f, g differ by a unit, which in a DVR has vanishing valuation. Clearly, $W(D)$ is effective, since by definition $n_p \in \mathbb{Z}_{\geq 0}$. Conversely, given an effective Weil divisor $W = \sum_{p\in C} n_p \cdot p$, i.e. $n_p \geq 0$ and $n_p \neq 0$ for finitely many $p \in C$, we define $\mathcal{I}_{D,p} := \mathfrak{m}_p^{n_p} \lhd \mathcal{O}_{C,p}$ for $\{\mathfrak{m}_p\} = Spm(\mathcal{O}_{C,p})$, which glues to a global ideal sheaf $\mathcal{I}_D \lhd \mathcal{O}_C$, defining an effective Cartier divisor. □

Remark 4.1.2. From now on, we abuse the notation and denote the Weil divisor $W(D)$ associated to an effective Cartier divisor D also by D. Moreover, we denote the ideal sheaf

$\mathcal{I}_D$ associated to an effective divisor D on a curve C by $\mathcal{O}_C(-D)$ and its structure sheaf by $\mathcal{O}_D$, fitting into the canonical short exact sequence

$$0 \longrightarrow \mathcal{O}_C(-D) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_D \longrightarrow 0. \tag{4.2}$$

Let $(g, \boldsymbol{m}) := (g, (m_1, \ldots, m_d))$ with $g \in \mathbb{Z}_{\geq 0}$ and $(m_i)_i \in \mathbb{Z}^d_{>1}$.

Definition 4.1.3. Divisor of Polar Type $\boldsymbol{m}$

Let $C \in \mathcal{M}(C)$ and let H be an effective divisor on C. We say that H is a *divisor of polar type* $\boldsymbol{m}$ if

$$H = \sum_{i=1}^{d} m_i \cdot p_i \tag{4.3}$$

for a collection of d distinct points $\{p_i\}_i$ on C.

Definition 4.1.4. Effective Square-Root of a Divisor

Let $C \in \mathcal{M}(C)$ and $H = \sum_{i=1}^d m_i \cdot p_i$ a divisor of polar type $\boldsymbol{m}$ on C. We associate to H the effective divisor

$$D = \left\lceil \frac{H}{2} \right\rceil := \sum_{i=1}^{d} \left\lceil \frac{m_i}{2} \right\rceil \cdot p_i \subset C. \tag{4.4}$$

We write $n_i := \lceil \frac{m_i}{2} \rceil$, so that

$$D = \sum_{i=1}^{d} n_i \cdot p_i. \tag{4.5}$$

In particular, H and D have the same support and the sheaf $\mathcal{O}_C(D)$ is an $\mathcal{O}_C$-submodule of $\mathcal{O}_C(H)$.

Definition 4.1.5. Reduction, Even and Odd Components of a Divisor

Let $D = \sum_{i=1}^d n_i \cdot p_i$ be an effective divisor on $C \in \mathcal{M}(C)$. We define the *reduction* of D as the underlying reduced divisor consisting of its support $D^{red} = \sum_{i=1}^d p_i$. Write $D_i := n_i \cdot p_i$ for the component of D supported at the closed point p_i. Moreover, we write D_{ev} (resp. D_{odd}) the divisor consisting of the components of D of even (resp. odd) length, i.e.

$$\begin{aligned} D_i \subset D_{ev} &\iff \deg(D_i) = n_i \equiv 0 \mod 2 \\ \text{(resp. } D_i \subset D_{odd} &\iff \deg(D_i) = n_i \equiv 1 \mod 2). \end{aligned}$$

Hence, we may also write

$$D = D_{ev} + D_{odd} = \sum_i D_i. \tag{4.6}$$

Lastly, we denote the number of even-length components of D by $e := \deg(D_{ev}^{red})$.

Remark 4.1.6. In the notation of definition 4.1.5, we have the relation:

$$2D = H + (H_{odd})^{red}, \quad \text{with } \deg(2D) = \deg(H) + (d - e). \tag{4.7}$$

Definition 4.1.7. Admissible Type
We call a type $(g, m) = (g, \{m_i\}_i)$ an *admissible type*, if the following conditions hold true:

$$6g - 6 + d + \deg(H) > 0, deg(D) \in 2\mathbb{Z}, \tag{4.8}$$

and it is **not** one of the following cases: $(0, \{6\}); (0, \{3, 3\}); (1, \{2\})$. The condition that $deg(D) \in 2\mathbb{Z}$ can be neglected for almost the entirety of the thesis, it only appears in the proof of proposition 7.3.2, which we hope to refine in the near future.

From now on, we fix the data $(g; m)$ of an admissible type, for the entirety of the thesis.

4.2. Meromorphic Quadratic Differentials of fixed Polar Type

Definition 4.2.1. Meromorphic Quadratic Differentials
A meromorphic quadratic differential of type (g, m) is a section

$$Q \in H^0(C, \omega_C^{\otimes 2}(H)) \tag{4.9}$$

where $C \in \mathcal{M}(C)$ and H is a divisor of type m on C.

Although we use the terminology *meromorphic* quadratic differentials, we are really referring to global *regular* sections of the square of the canonical sheaf, twisted by the *polar divisor* $\omega_C^{\otimes 2}(H)$.

Definition 4.2.2. Moduli Space of Quadratic Differentials
We denote by $\mathcal{M}(C, Q)$ the space parametrising pairs (C, Q) of curves $C \in \mathcal{M}(C)$ and quadratic differentials of type $(g; m)$ on C, such that Q has only simple zeros. The defining equivalence relation is given by:

$$(C_1, Q_1) \sim (C_2, Q_2) :\iff \exists f : C_1 \xrightarrow{\sim} C_2 \text{ isomorphism of alg. curves s.t. } f^*Q_2 = Q_1. \tag{4.10}$$

We refer to chapter B in the appendix for the careful construction of this space as a stack. In particular, we prove in lemma B.0.12, that

$$n = dim_{\mathbb{C}}(\mathcal{M}(C, Q)) = 6g - 6 + d + \sum_{i=1}^{d} m_i.$$

Remark 4.2.3. Often times, it will be useful to consider Q as a section of $H^0(C, \omega_C(D)^{\otimes 2})$, where $D = \left\lceil \frac{H}{2} \right\rceil$, instead of $H^0(C, \omega_C^{\otimes 2}(H))$, with extra zeros along H_{odd}^{red}. In what follows, we discuss this in local terms. Since we work locally at an odd order pole of Q, we may assume, without loss of generality, that $H = m \cdot p$ for $m \equiv 1 \mod 2$. Let x be a local coordinate in a neighbourhood $U \subset C$ of p, with $x(p) = 0$. Then locally

$$\begin{aligned} Q(x) &= \left(\frac{q_m}{x^m} + \cdots + \frac{q_1}{x} + q_0 + \mathcal{O}(x)\right) dx^{\otimes 2} \\ &= (q_m + q_{m-1}x + \cdots + q_1 x^{m-1} + q_0 x^m + \mathcal{O}(x^{m+1})) \frac{dx^{\otimes 2}}{x^m} \in H^0(U, \omega_C^{\otimes 2}(H)). \end{aligned}$$

Now viewing Q as a section of $\omega_C(D)^{\otimes 2}$, we write:

$$\begin{aligned} Q(x) &= (q_m x + q_{m-1}x^2 + \cdots + q_1 x^m + q_0 x^{m+1} + \mathcal{O}(x^{m+2})) \frac{dx^{\otimes 2}}{x^{m+1}} \\ &= (q_m x + q_{m-1}x^2 + \cdots + q_1 x^m + q_0 x^{m+1} + \mathcal{O}(x^{m+2})) \left(\frac{dx}{x^n}\right)^{\otimes 2} \in H^0(U, \omega_C(D)^{\otimes 2}), \end{aligned}$$

where $D = n \cdot p = \frac{m+1}{2} \cdot p$. Thereby, the odd order poles become simple zeros once viewed as a section of the appropriate sheaf. In general, we will have additional (simple-)zeros along the reduction of the odd components of H: H_{odd}^{red}.

4.3. Spectral Curves

In this section, we encode the datum of a quadratic differential Q on a curve C into an algebraic subvariety of the total space $Tot(\omega_C(D))$ of the D-twisted canonical sheaf, namely, a ramified covering $\Sigma \to C$, called the *spectral curve*. The nomenclature originates in the theory of integrable systems and will be justified later on, cf. Chapter 7.2.12. This in turn allows us to reinterpret the moduli space of quadratic differentials $\mathcal{M}(C, Q)$ as the base for a family of curves.

Lemma 4.3.1. *Let $f : Y \longrightarrow X$ be a morphism of schemes and let $\mathscr{L}$ be a locally free sheaf on X. There is a canonical isomorphism of algebras of $\mathcal{O}_Y$-modules $Sym(f^*\mathscr{L}) \cong f^*Sym(\mathscr{L})$.*

Proof. First note that given an $\mathcal{O}_Y$-module $\mathcal{M}$, the ($\mathcal{O}_X$-bilinear-) map of $\mathcal{O}_Y$-modules

$$\begin{aligned} \rho : \mathcal{M} \otimes_{\mathcal{O}_X} \mathscr{L} &\longrightarrow \mathcal{M} \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathscr{L}) \\ (m, \ell) &\longmapsto m \otimes_{\mathcal{O}_Y} 1 \otimes_{\mathcal{O}_X} \ell \end{aligned}$$

is an isomorphism with ($\mathcal{O}_X$-bilinear-)inverse given by $(m, \sum_i s_i \otimes \ell_i) \longmapsto \sum_i m \cdot s_i \otimes \ell_i$. This map gives rise to an isomorphism of the tensor algebras

$$T_{\mathcal{O}_Y}(f^*\mathscr{L}) = T_{\mathcal{O}_Y}(\mathcal{O}_Y \otimes_{\otimes \mathcal{O}_X} \mathscr{L}) \cong \mathcal{O}_Y \otimes_{\mathcal{O}_X} T_{\mathcal{O}_X}(\mathscr{L}) = f^*T_{\mathcal{O}_X}(\mathscr{L}). \tag{4.11}$$

Moreover, $\mathcal{O}_Y \otimes_{\mathcal{O}_X} Sym_{\mathcal{O}_X}(\mathscr{L}) = (\mathcal{O}_Y \otimes T_{\mathcal{O}_X}(\mathscr{L}))/\mathcal{I}_{sym}$, for $I = \langle 1 \otimes (x \otimes y - y \otimes x)\rangle$. The image of I under the map ρ is $\rho(I) = \langle (1 \otimes x) \otimes (1 \otimes y) - (1 \otimes y) \otimes 1 \otimes x)\rangle$, which is exactly the ideal of the defining relation of $Sym(f^*\mathscr{L})$ on $T_{\mathcal{O}_Y}(f^*\mathscr{L})$ and the claim follows. □

Definition 4.3.2. The Tautological section
Let $\mathscr{L}$ a locally free sheaf on a scheme X. The tautological section $y \in H^0(Tot(\mathscr{L}), \pi^*\mathscr{L})$ can be defined geometrically as the unique section given by the universal property of fibre products of schemes in the following Cartesian diagram

$$\begin{tikzcd}
Tot(\mathscr{L}) \arrow[rd, dashed, "\exists ! y"] \arrow[rrd, bend left, "id"] \arrow[rdd, bend right, "id"'] & & \\
& Tot(\pi^*\mathscr{L}) \overset{4.3.1}{\cong} \pi^*Tot(\mathscr{L}) \arrow[r] \arrow[d] & Tot(\mathscr{L}) \arrow[d, "\pi"] \\
& Tot(\mathscr{L}) \arrow[r, "\pi"'] & X.
\end{tikzcd} \tag{4.12}$$

Proposition 4.3.3. *Ramified Covers from Quadratic Differentials*
Let $(C, Q) \in \mathcal{M}(C, Q)$ *with* $Q \in H^0(C, \omega_C^{\otimes 2}(H))$ *and* $D = \lceil \frac{H}{2} \rceil$. *Then there exists a smooth subscheme of the total space of the* D*-twisted dualising sheaf of* C

$$\Sigma \hookrightarrow Tot(\omega_C(D))$$

associated to (C, Q), *that induces a double cover* p *on* C

$$\begin{tikzcd}
\Sigma \arrow[r, hook] \arrow[d, dashed, "p"'] & Tot(\omega_C(D)) \arrow[ld, "\pi"] \\
C &
\end{tikzcd} . \tag{4.13}$$

This cover is ramified along the zero-locus of the quadratic differential Q, *viewed as a section of* $H^0(C, \omega_C(D)^{\otimes 2})$. *We call the curve* Σ, *the* ***Spectral Curve*** *associated to* (C, Q) *and* $p : \Sigma \longrightarrow C$ *the associated spectral cover of* C.

Proof. Consider the total space of the D-twisted dualising sheaf $\omega_C(D)$ of C

$$\pi : Tot(\omega_C(D)) := Spec(Sym^\bullet(\omega_C(D)^*)) \longrightarrow C \tag{4.14}$$

and the locally free sheaf $\pi^*\omega_C(D)^{\otimes 2}$ on $Tot(\omega_C(D))$. Note here, that π is flat by definition. Moreover, consider the global sections $y^2, 0, \pi^*Q \in H^0(Tot(\omega_C(D)), \pi^*\omega_C(D)^{\otimes 2})$, where $y \in H^0(Tot(\omega_C(D)), \pi^*\omega_C(D))$ denotes the tautological section of $\pi^*\omega_C(D)$, 0 is the zero section and π^*Q the pullback of Q from C to the total space, after viewing Q as a section of $H^0(C, \omega_C(D)^{\otimes 2})$ instead of $H^0(C, \omega_C^{\otimes 2}(H))$, with additional (simple-)zeros along $(H_{odd})^{red}$, cf. remark 4.2.3.

Now, we define the subvariety Σ of the complex algebraic surface $Tot(\omega_C(D))$ by

$$\Sigma := \{x \in Tot(\omega_C(D)) | y^2(x) - \pi^*Q(x) = 0(x)\} \subset Tot(\omega_C(D)). \tag{4.15}$$

This construction gives rise to a ramified double cover $p : \Sigma \to C$, by restricting the bundle map π to the subscheme Σ, i.e. $p := \pi|_\Sigma$

$$\begin{tikzcd}
\Sigma \arrow[r, hook] \arrow[d, dashed, "p"'] & Tot(\omega_C(D)) \arrow[dl, "\pi"] \\
C &
\end{tikzcd} .$$

The smoothness of Σ follows from the fact that the quadratic differential Q is allowed to develop only simple zeros by definition 4.2.2. □

Lemma 4.3.4. *The resulting morphism of curves $p : \Sigma \longrightarrow C$ is flat.*

Proof. p is flat as a finite map of smooth curves cf.[74, Tag 0BWJ]. □

Definition 4.3.5. Branching and Ramification Divisors
Let $p : \Sigma \longrightarrow C$ denote the spectral cover associated to a point $(C, Q) \in \mathcal{M}(C, Q)$, with $Q \in H^0(C, \omega_C^{\otimes 2}(H))$ and $D = \lceil \frac{H}{2} \rceil$. Then we introduce the *branching divisor* of Σ: $B \subset C$, consisting of the zeros of $Q \in H^0(C, \omega_C(D)^{\otimes 2})$. We denote by $R := p^{-1}(B)^{red} \subset \Sigma$ the *ramification divisor.* A cover with reduced ramification divisor R is also called *simply ramified.* We note here again that, by remark 4.2.3, the ramification locus includes the *odd order poles*, i.e. Σ ramifies over H_{odd}^{red}.

We also denote by $\sigma \in Aut(\Sigma)$ the *covering involution* that exchanges the sheets of the double cover, fixing the ramification divisor. This automorphism is the one induced by the non-trivial representation of the Galois group of the function field of Σ over the function field of C. In particular, it is the restriction of the automorphism of $Tot(\omega_C(D))$ acting on affine coordinates as $(x, y) \longmapsto (x, -y)$.

Remark 4.3.6. On Branching and Ramification

1) By assumption, Q has only simple zeros, therefore B and R are reduced and p is a double-cover ramified exactly over B. In particular, we have

- $\mathcal{O}_\Sigma(p^{-1}B) \cong p^*\mathcal{O}_C(B) \cong \mathcal{O}_C(2R)$,
- $\deg(R) = \deg(B)$ and $\mathcal{Nm}_p(\mathcal{O}_\Sigma(R)) = \mathcal{O}_C(B)$ for the norm map of the cover

$$\begin{aligned} \mathcal{Nm}_p : \mathcal{Pic}(\Sigma) &\longrightarrow \mathcal{Pic}(C) \\ \mathcal{O}_\Sigma(\mathrm{D}) &\longmapsto \mathcal{O}_C(p(\mathrm{D})). \end{aligned} \tag{4.16}$$

2) Equivalently to definition 4.3.5, we could have first defined the ramification divisor R as the support of $coker(dp : T_\Sigma \longrightarrow p^*T_C)$, i.o.w. the locus where the differential of the covering map drops rank, or also as $ker((dp)^*)$ for

$$p^*\omega_C \xrightarrow{(dp)^*} \omega_\Sigma \longrightarrow \omega_{\Sigma/C}, \tag{4.17}$$

for $(dp)^* : p^*\omega_C \to \omega_\Sigma$ the dual map to the differential of p, c.f. [74, Tag 0BWJ]. From this, we also have $p^*\omega_C(R) \cong \omega_\Sigma$. Then we can define the branching divisor as the image of the ramification $B := p(R)$.

3) Note here that if $D \cap B \neq \varnothing$, i.e. the spectral cover is ramified over some point of the divisor D, then the intersection $\Sigma \cap \pi^{-1}D$ is tangential at the ramification locus $p^{-1}(D) \cap R$. Since the spectral curve equation reads $y^2 - \pi^*Q = 0$, the intersection multiplicity is 2, as subschemes of the surface $Tot(\omega_C(D))$.

4) Considering the image of C under the zero-section $0 : C \longrightarrow Tot(\omega_C(D))$ of π as a divisor in the surface $Tot(\omega_C(D))$, we have

$$\mathcal{O}_\Sigma(R) = \mathcal{O}_{Tot(\omega_C(D))}(0(C))|_\Sigma = p^*\omega_C(D) = \omega_\Sigma(p^*D - R), \tag{4.18}$$

as prescribed by Riemann-Hurwitz.

Lemma 4.3.7. *The ramification divisor of the spectral cover has degree*

$$\deg(R) = 4g - 4 + 2\deg(D) = g_\Sigma - 1 + \deg(D). \tag{4.19}$$

Proof. For the direct image of the structure sheaf, we have

$$p_*\mathcal{O}_\Sigma = \mathcal{O}_C \oplus \omega_C(D)^* \tag{4.20}$$

c.f. [4, Lemma 17.2]. Since p is a finite, and therefore affine morphism, its higher direct images vanish, and by [74, Tag 0BEI] we have

$$\begin{aligned} \chi(\Sigma, \mathcal{O}_\Sigma) = \chi(C, p_*\mathcal{O}_\Sigma) &= \chi(C, \mathcal{O}_C) + \chi(C, \omega_C(D)^*) \\ &= 4 - 4g_C - \deg(D). \end{aligned} \tag{4.21}$$

In particular, $g_\Sigma = 4g_C - 3 + \deg(D)$. Applying Riemann-Hurwitz, we obtain

$$\begin{aligned}\deg(R) &= 2g_\Sigma - 2 - \deg(p : \Sigma \to C) \cdot (2g_C - 2) \\ &= 4g_C - 4 + 2 \cdot \deg(D) \\ &= g_\Sigma - 1 + \deg(D).\end{aligned}$$

Since R is reduced Riemann-Hurwitz applies with no modification. $\square$

Definition 4.3.8. The Seiberg-Witten Differential
Considering the pullback p^*Q of the quadratic differential Q on its associated spectral curve and taking the square root gives rise to a section of $p^*\omega_C(D)$

$$\varphi := \sqrt{p^*Q} : \mathcal{O}_\Sigma \longrightarrow p^*\omega_C(D), \tag{4.22}$$

with simple zeros along R. Hence, this section fits into a short exact sequence

$$0 \longrightarrow \mathcal{O}_\Sigma \xrightarrow{\varphi} p^*\omega_C(D) \longrightarrow \mathcal{O}_R \longrightarrow 0. \tag{4.23}$$

Moreover, restricting the tautological 1-form y to the spectral curve gives a section

$$\lambda := i^*y : \mathcal{O}_\Sigma \longrightarrow \omega_\Sigma \otimes p^*\mathcal{O}_C(D), \tag{4.24}$$

with $\lambda^{\otimes 2} = p^*Q$. That is $\lambda = (dp)^* \circ \varphi$ and that it is anti-invariant with respect to the Galois involution σ, i.e. $\sigma^*\lambda = -\lambda$, since σ is the pullback of $(x, y) \longmapsto (x, -y)$ to the spectral curve. In other words, on the spectral curve, the positive square root of the pulled-back quadratic differential is a well defined 1-form on Σ, which we denote by λ.

Remark 4.3.9. General Spectral Construction
The construction of the spectral cover of Proposition 4.3.3 is a general procedure that can be performed on any smooth scheme, given a line bundle and a global section of its square, exactly as we did for quadratic differentials. Let X be a scheme, $\mathscr{L} \in \mathcal{P}ic(X)$, $q \in H^0(X, \mathscr{L}^{\otimes 2})$, $\pi : Tot(\mathscr{L}) \longrightarrow X$ and let $y \in H^0(Tot(\mathscr{L}), \pi^*\mathscr{L})$ be the tautological section. We define the spectral cover of X associated to q as $\Sigma_q := \{y^{\otimes 2} = \pi^*q\} \subset Tot(\mathscr{L}^{\otimes 2})$.

Proposition 4.3.10. ***Functoriality of the Spectral Construction***
Let $f : Y \longrightarrow X$ be a morphism of schemes, $\mathscr{L} \in \mathcal{P}ic(X)$, $q \in H^0(X, \mathscr{L}^{\otimes 2})$. The diagram

$$\begin{tikzcd} \Sigma_{f^*q} \arrow[r] \arrow[d, "p_{f^*q}"'] & \Sigma_q \arrow[d, "p_q"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}, \tag{4.25}$$

is Cartesian, i.e. $Y \times_X \Sigma_q \cong \Sigma_{f^*q}$ as Y-schemes.

Proof. Let $\pi : Tot(\mathscr{L}) \longrightarrow X$ be the canonical projection, $y \in H^0(Tot(\mathscr{L}), \pi^*\mathscr{L})$ the tautological section and Σ_q the spectral cover of X associated to q. First, note that the diagram

$$\begin{tikzcd}
Tot(f^*\mathscr{L}) \arrow[r] \arrow[d, "\pi'"'] & Tot(\mathscr{L}) \arrow[d, "\pi"] \\
Y \arrow[r, "f"'] & X
\end{tikzcd} \tag{4.26}$$

is Cartesian, that is

$$\begin{tikzcd}
Tot(f^*\mathscr{L}) \arrow[rr, "\sim"] \arrow[dr, "\pi'"'] & & Y \times_X Tot(\mathscr{L}) \arrow[dl, "f^*\pi"] \\
& Y &
\end{tikzcd} \tag{4.27}$$

which follows from Lemma 4.3.1. Thereby, we have $f^*(\pi^*q) = (f^*\pi)^*(f^*q)$. Moreover, the tautological section of $f^*\mathscr{L}$, i.e. $y' \in H^0(Tot(f^*\mathscr{L}), \pi'^*f^*\mathscr{L})$, agrees with the pullback of the tautological section of $\mathscr{L}$ to Y, i.e. $f^*y \in H^0(Tot(f^*\mathscr{L}), f^*\pi^*\mathscr{L})$. In total, we have

$$Y \times_X \Sigma_q = \{(f^*y)^{\otimes 2} = f^*(\pi^*q)\} \cong \{y'^{\otimes 2} = (f^*\pi)^*(f^*q)\} = \Sigma_{f^*q} \tag{4.28}$$

and $p_{f^*q} = \pi'|_{\Sigma_{f^*q}} = f^*\pi|_{Y \times_X \Sigma_q} = f^*p_q$, i.e. the following diagram commutes

$$\begin{tikzcd}
& Tot(f^*\mathscr{L}) \arrow[r, "\sim"] \arrow[dd, "\pi'"'] & Y \times_X Tot(\mathscr{L}) \arrow[ddl, "f^*\pi"'] \\
\Sigma_{f^*q} \arrow[ur, hook] \arrow[rr, dashed, "\sim"] \arrow[dr, "p_{f^*q}"'] & & Y \times_X \Sigma_q \arrow[u, hook] \arrow[dl, "f^*p_q"] \\
& Y &
\end{tikzcd} \tag{4.29}$$

.

□

4.4. Cohomology of Spectral Curves

Here we study the cohomology of spectral curves and how it varies in families. The *raison d'être* of section 4.4 is to introduce the cohomology bundles $\mathscr{H}$ over the space of quadratic differentials $\mathcal{M}(C, Q)$, and construct a Gauß-Manin connection on that bundle.

We write $H^*(X, A)$ for the singular cohomology of a smooth $\mathbb{C}$-scheme X instead of $H^*(X(\mathbb{C}), A)$.

Definition 4.4.1. (Anti-)Invariant (Co-)homology Groups
Let $(C, Q) \in \mathcal{M}(C, Q)$ and let $p : \Sigma \longrightarrow C$ be the associated spectral cover with Galois involution σ. We introduce

- the *homology group of σ-invariant cycles*

$$H_1(\Sigma, \mathbb{Z})^+ := \{\gamma \in H_1(\Sigma, \mathbb{Z}) | \sigma^* \gamma = \gamma\}, \tag{4.30}$$

- the *homology group of anti-invariant cycles*

$$H_1(\Sigma, \mathbb{Z})^- := \{\gamma \in H_1(\Sigma, \mathbb{Z}) | \sigma^* \gamma = -\gamma\}, \tag{4.31}$$

- we have analogously for the cohomology groups of (anti-)invariant cocycles $H^1(\Sigma, \mathbb{Z})^{\pm}$, $H^1(\Sigma, \mathbb{C})^{\pm}$.
- Lastly we set $\tilde{H}^1(\Sigma, \mathbb{Z}) := Hom_{\mathbb{Z}}(H_1(\Sigma, \mathbb{Z})^-, \mathbb{Z})$.

Lemma 4.4.2. *The homology group of anti-invariant cycles fits into the following short exact sequence*

$$0 \longrightarrow H_1(\Sigma, \mathbb{Z})^- \longrightarrow H_1(\Sigma, \mathbb{Z}) \xrightarrow{p_*} H_1(C, \mathbb{Z}) \longrightarrow 0. \tag{4.32}$$

Proof. Let $\gamma \in H_1(\Sigma, \mathbb{Z})$, then $p^* p_* \gamma = \gamma + \sigma^* \gamma$, therefore

$$\gamma \in ker(p_*) \implies p^* p_* \gamma \implies \gamma = -\sigma^* \gamma.$$

Moreover, if $\gamma \in H_1(\Sigma, \mathbb{Z})^-$, we have

$$\gamma = -\sigma^* \gamma \implies p_* \gamma = -p_* \sigma^* \gamma = -p_* \gamma \implies 2 p_* \gamma = 0,$$

and since $H_1(C, \mathbb{Z})$ is torsion-free abelian $\gamma \overset{!}{\in} ker(p_*)$. The surjectivity of p_* is precisely the statement of [56, Prop.1.11], for $k = 1, q = 1$ in the notation therein. Note that the fact that our cover p is ramified is absolutely crucial in proving that p_* is surjective. Indeed, this property fails in the case of unramified covers. $\square$

Remark 4.4.3. Dualising the sequence (4.32), we have

$$0 \longrightarrow H^1(C, \mathbb{Z}) \xrightarrow{p^*} H^1(\Sigma, \mathbb{Z}) \longrightarrow \tilde{H}^1(\Sigma, \mathbb{Z})^- \longrightarrow 0, \tag{4.33}$$

where $\tilde{H}^1(\Sigma, \mathbb{Z})^- := Hom(H_1(\Sigma, \mathbb{Z})^-, \mathbb{Z})$. Using the fact that p is simply ramified, Mumford shows in [64, Section 3], that p^* is injective.

Proposition 4.4.4. ***Gauß-Manin Connection on the Family of Spectral Curves*** *Consider $\mathcal{M}(C, Q)$ as the moduli of spectral curves of a fixed type, i.e. as base for a family of curves in the obvious way*

$$f : \underline{\mathbf{\Sigma}} := \coprod_{(C,Q) \in \mathcal{M}(C,Q)} \Sigma_{C,Q} \longrightarrow \mathcal{M}(C, Q) \ ; \ f^{-1}(C, Q) = \Sigma_{(C,Q)}. \tag{4.34}$$

Then, the étalé vector bundle $\mathcal{H}$ *on* $\mathcal{M}(C,Q)$ *with fibres* $H^i(f^{-1}(C,Q),\mathbb{C})$ *admits a unique linear flat connection preserving the natural integral lattice* $\mathcal{H}^{\mathbb{Z}}$ *with fibres* $H^i(f^{-1}(C,Q),\mathbb{Z})$*, for any* i*.*

Proof. The map f is a proper submersion of complex manifolds, so by Ehresmann's theorem cf.[28], f is a locally trivial fibration of complex manifolds. In particular, $H^i(f^{-1}(C,Q),\mathbb{C})$ does not depend on (C,Q), since the fibres of f are locally homeomorphic. Let $\mathcal{F}$ be a sheaf of abelian groups on $\underline{\mathbf{\Sigma}}$. By proper base change, taking higher direct images commutes with base change, i.e. for a Cartesian diagram

$$\begin{tikzcd}
\underline{\mathbf{\Sigma}} \times T \arrow[r, "g'"] \arrow[d, "f'"'] & \underline{\mathbf{\Sigma}} \arrow[d, "f"] \\
T \arrow[r, "g"] & \mathcal{M}(C,Q),
\end{tikzcd}$$

the natural map: $g^* R^i f_* \mathcal{F} \to R^i f'_* g'^* \mathcal{F}$ is an isomorphism, cf. [60, Theorem 6]. In particular for a point on the base $g : (C,Q) \hookrightarrow \mathcal{M}(C,Q)$, we have

$$(R^i f_* \mathcal{F})_{C,Q} \cong H^i(\underline{\mathbf{\Sigma}}_{C,Q}, \mathcal{F}|_{\underline{\mathbf{\Sigma}}_{C,Q}}). \tag{4.35}$$

Furthermore, substituting $\mathcal{F} = \underline{\mathbb{C}}$, we obtain a local system on $\mathcal{M}(C,Q)$ with fibres $H^i(\underline{\mathbf{\Sigma}}_{C,Q},\mathbb{C}) = H^i(f^{-1}(C,Q),\mathbb{C})$. When working in the algebraic category, i.e. f is a proper morphism of algebraic spaces, we use Grothendieck's comparison theorem to identify algebraic and analytic de Rham cohomologies. Using the Riemann-Hilbert correspondence 2.2.2, we associate to the resulting local system $Rf_*\underline{\mathbb{C}}$, a vector bundle $Rf_*\underline{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{M}(C,Q)}$ with a flat connection ∇^{GM}. This is uniquely determined by the sheaf of local sections and by the fact that it preserves the natural integral structure induced by $Rf_*\underline{\mathbb{Z}}$. □

4.5. Prym Varieties of Spectral Covers

Lemma 4.5.1. *Let* $p : \Sigma \longrightarrow C$ *be a simply ramified double cover of smooth algebraic curves and denote by* $Jac(\Sigma), Jac(C)$ *the Jacobian varieties of* Σ *and* C *respectively. The restriction of the pullback of* p *to line bundles of degree zero*

$$p^* : Jac(C) \longrightarrow Jac(\Sigma) \tag{4.36}$$

is an injective map.

Proof. [64, Section 3]. □

Definition 4.5.2. Prym of a Spectral Cover and Anti-Invariant Line Bundles
Let $p : \Sigma \longrightarrow C$ be a simly ramified double cover of smooth algebraic curves. We define by

$$\mathcal{P}rym(\Sigma/C) := Jac(\Sigma)/p^*(Jac(C)) \tag{4.37}$$

the *Prym variety* of Σ as a scheme over C. Moreover, let

$$Jac(\Sigma)^- := \{M \in Jac(\Sigma) | \sigma^* M \cong M^*\}, \tag{4.38}$$

We call $Jac(\Sigma)^-$ the *σ-anti-invariant part* of the Jacobian.

Lemma 4.5.3. ***Characterisation of the Prym Variety***
Consider the 2-torsion subgroup of $Jac(C)$, i.e. $Jac(C)[2] = \{P \in Jac(C) | P^{\otimes 2} = \mathcal{O}_C\}$, then

$$\mathcal{P}rym(\Sigma/C) = Jac(\Sigma)^-/p^*(Jac(C)[2]). \tag{4.39}$$

Proof. This was proven by a short argument in [15], using the results of [64], which we now recall here for completeness. First, note that the natural inclusion $Jac(\Sigma)^- \hookrightarrow Jac(\Sigma)$ gives a map

$$j : Jac(\Sigma)^- \longrightarrow Jac(\Sigma)/p^*(Jac(C)). \tag{4.40}$$

We claim that j is surjective. To see this, let $M \in Jac(\Sigma)$, then, since the endomorphism

$$\begin{aligned} Jac(\Sigma) &\longrightarrow Jac(\Sigma) \\ M &\longmapsto M^{\otimes 2} \end{aligned} \tag{4.41}$$

is an isogeny of abelian varieties, cf. [68], for any $M \in Jac(\Sigma)$, there exists $N \in Jac(\Sigma)$, such that $N^{\otimes 2} = M$. We can split M into a product $M = [N \otimes \sigma^*(N^*)] \otimes [N \otimes \sigma^*(N)]$. The first component $N \otimes \sigma^*(N^*)$ is an element of $Jac(\Sigma)^-$, since $N^* \otimes \sigma^*(N^*)^* = N^* \otimes \sigma^*(N)$. Due to Mumford in [64, Section 3], we also have $N \otimes \sigma^*(N) \in p^*(Jac(C))$, which is a non-trivial result making use of the fact that the ramification locus of $p : \Sigma \longrightarrow C$ is non-trivial but simple. Then the kernel of j is

$$ker(j) = p^*(Jac(C)) \cap Jac(\Sigma)^- = p^*(Jac(C))[2] \subset Jac(\Sigma) \tag{4.42}$$

and the identification 4.39 follows. □

5. Wild Algebraic Curves

In this section, we introduce the notion of a wild algebraic curve, an enhancement of an algebraic curve, that encodes part of the irregular meromorphic data of a quadratic differential.

5.1. Wild Curves and Divisorial Spectral Covers

Let $(C,Q) \in \mathcal{M}(C,Q)$ and consider the *restriction-to-D* exact sequence of of the datum of a quadratic differential

$$0 \longrightarrow \omega_C^{\otimes 2}(D) \longrightarrow \omega_C(D)^{\otimes 2} \xrightarrow{|_D} \omega_C(D)^{\otimes 2}|_D \longrightarrow 0 \tag{5.1}$$

obtained by twisting the sequence

$$0 \longrightarrow \mathcal{O}_C(-D) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_D \longrightarrow 0$$

with $\omega(D)^{\otimes 2}$.

Definition 5.1.1. Enhancement Class
We define the ***enhancement class*** ν of a quadratic differential Q, as the image of Q under the map in cohomology induced by the restriction map $|_D$

$$\nu := Q|_D \in H^0(C, \omega_C(D)^{\otimes 2}|_D).$$

Lemma 5.1.2. ***A Vanishing Lemma***
The group of enhancement classes can be realised as a quotient

$$H^0(C, \omega_C(D)^{\otimes 2}|_D) \cong \frac{H^0(C, \omega_C(D)^{\otimes 2})}{H^0(C, \omega_C^{\otimes 2}(D))}. \tag{5.2}$$

Proof. The long exact cohomology sequence associated to 5.1 reads

$$0 \to H^0(C, \omega_C^{\otimes 2}(D)) \to H^0(C, \omega_C(D)^{\otimes 2}) \to H^0(D, \omega_C(D)^{\otimes 2}|_D) \to H^1(C, \omega_C^{\otimes 2}(D)) \to \cdots \tag{5.3}$$

The cohomology group $H^1(C, \omega_C^{\otimes 2}(D))$ vanishes if $\deg(\omega_C(D)) > 0$. Since D is effective, this condition is fulfilled automatically for $g \geq 2$. For lower genera,we have

$$\deg(\omega_C(D)) \leq 0 \iff \sum_i \left\lceil \frac{m_i}{2} \right\rceil \leq 2 - 2g.$$

Under our admissibility condition 4.1.7, there are no quadratic differentials with these pole configurations allowed and $H^1(C, \omega_C^{\otimes 2}(D))$ vanishes identically. □

Remark 5.1.3. Let $Q \in H^0(C, \omega_C(H))$ be a quadratic differential with $H = m \cdot p$ and $D = n \cdot p = \lceil \frac{m}{2} \rceil \cdot p$. Locally, for a coordinate $x \in U$ at a neighbourhood of a pole $p \in U$ with $x(p) = 0$, we have

$$Q(x) = \Big(\frac{a_m}{x^m} + \cdots \frac{a_1}{x} + a_0\Big)\, dx^{\otimes 2} = (a_m x^{2n-m} + \cdots + a_1 x^{2n-1} + a_0 x^{2n}) \frac{dx^{\otimes 2}}{x^{2n}}.$$

If $m = 2n$, then

$$\nu(x) = Q(x)|_D = [a_m + \cdots + a_{n+1} x^{n-1}] \in \mathbb{C}[x]/x^n$$

and if $m = 2n - 1$, then

$$\nu(x) = Q(x)|_D = [a_m x + \cdots + a_{n+1} x^{n-1}] \in \mathbb{C}[x]/x^n.$$

At an odd order pole of the quadratic differential $m \equiv 1 \mod 2$, the enhancement class has zero constant term.

Since we will be associating enhancement classes to the quadratic differentials of admissible type $(g, \mathscr{m})$ cf.4.1.7, the following technical definition is justified.

Definition 5.1.4. Enhancement Class of type $\mathscr{m}$
Let C, H, D as above. We say that an enhancement class $\nu \in H^0(D, \omega_C(D)^{\otimes 2}|_D)$ is of type $\mathscr{m} = \{m_i\}_i$ if the constant term of $\nu(p_i)$ is zero for all $i : m_i \equiv 1 \mod 2$.

Definition 5.1.5. The Class-ν Cover
Let C, D, ν be as above. We denote by $p_\nu : D_\nu \to D$ the double cover on D defined by the section $\nu \in H^0(C, \omega_C(D)^{\otimes 2}|_D)$ via the spectral curve construction 4.3.9. Moreover, we denote by $\lambda_\nu \in H^0(D_\nu, \omega_\Sigma(D_\nu)|_{D_\nu})$ the associated Seiberg-Witten form, for which $\lambda_\nu^{\otimes 2} = p_\nu^* \nu$.

The last ingredient in defining a wild curve is the choice of a sheet of the $p_\nu : D_\nu \to D$ cover we have just constructed, over each even order pole. Recall that given a divisor H of type $\mathscr{m} = (m_1, \ldots, m_d)$, i.e. $H = \sum_i m_i \cdot p_i$, the points p_i with m_i even have two distinct pre-images in D_ν under p_ν, while the points p_i with m_i odd have only one pre-image, cf. remark 4.2.3. The next definition prescribes the choice of a single preimage over each geometric point of D.

Definition 5.1.6. *"The Signing"*
Let C, D, ν be as above. Consider the associated double cover $p_\nu : D_\nu \longrightarrow D$, as well as the induced cover on the reduction of the divisors $p_\nu^{red} : D_\nu^{red} \longrightarrow D^{red}$. A ***signing*** on D_ν is a choice of a section of the covering p_ν^{red}, that is

$$\xi : D^{red} \hookrightarrow D_\nu^{red}; \text{ s.t. } p_\nu(\xi(p)) = p, \ \forall p \in D^{red}. \tag{5.4}$$

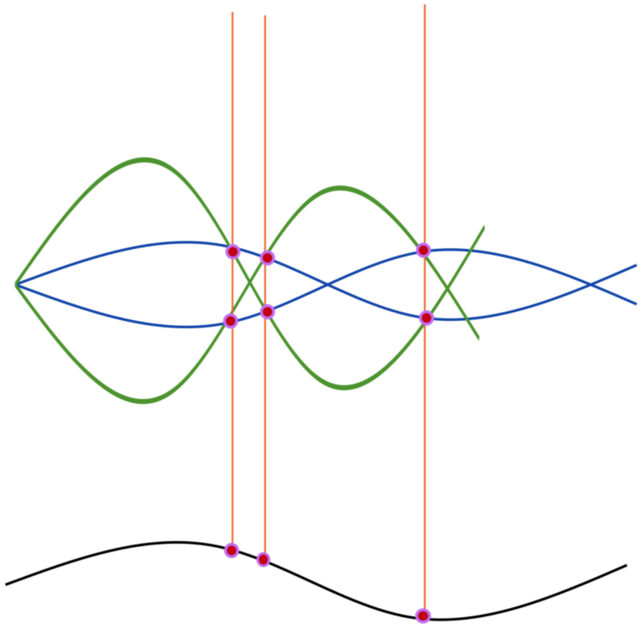


Figure 1: Quadratic Differentials in the same enhancement class give rise to the same 2-cover over the divisor: $D_\nu \to D$.

We collect all of this data into the notion of a wild curve.

Definition 5.1.7. Wild Algebraic Curves

A ***wild curve*** of type $(g, \mathfrak{m})$ is a triple $\mathscr{C} := (C, D, \nu, \xi)$ consisting of:

1) a projective algebraic curve of genus g: $C \in \mathcal{M}(C)$,
2) an effective divisor D on C, such that $D = \lceil \frac{H}{2} \rceil$, for some H of type $\mathfrak{m}$,
3) an *enhancement class* $\nu \in H^0(D, \omega_C(D)^{\otimes 2}|_D)$ of type $\mathfrak{m}$,
4) a *signing* $\xi : D^{red} \hookrightarrow D_\nu^{red}$.

Remark 5.1.8. Note that H does not appear explicitly as part of the data of a wild curve since it can be defined uniquely, given D and the enhancement class ν of type $\mathfrak{m}$. The vanishing order of ν at each point p in the support of D is the length of H at p.

Now, using the enhancement and the signing of a wild curve, we define a last important divisor D_ξ, which is to be thought of as *half*-of-D_ν.

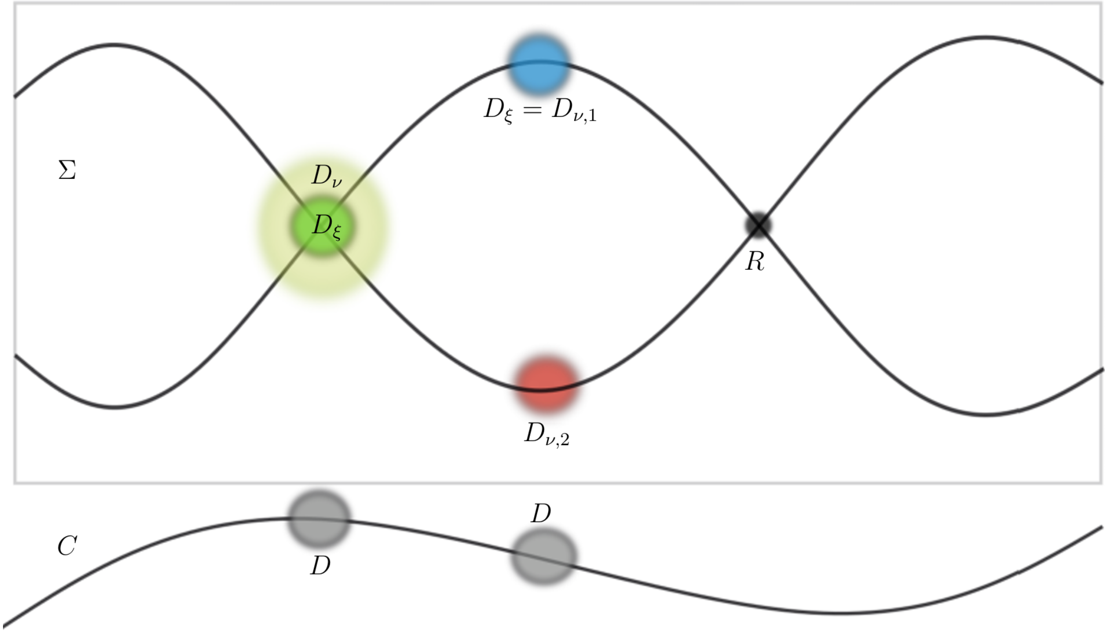
Definition 5.1.9. The Signed Lift of the Polar Divisor-D_ξ

Let $\mathscr{C} = (C, D, \nu, \xi) \in \mathcal{M}(\mathscr{C})$ be a wild curve and $D = \sum_i n_i \cdot p_i$. We define a divisor

$$D_\xi := \sum_i n_i \cdot \xi(p_i). \tag{5.5}$$

Remark 5.1.10. Properties of D_ξ

- $\deg(D_\xi) = \deg(D)$.
- If $D_i = n_i p_i$ a component of D with associated $m_i \equiv 1 \; mod \; 2$, then $\xi(D_i) = \frac{p_\nu^* D_i}{2}$.
- If $D_j = n_j p_j$ a component of D with associated $m_j \equiv 0 \; mod \; 2$, then $\xi(D_j)$ is one of the two connected components of $p_\nu^* D_j$.



Definition 5.1.11. Irregular Type of a Wild Curve
Let $\mathscr{C} = (C, D, \nu, \xi)$ be a wild curve. We call the restriction $\lambda_\xi := \lambda_\nu|_{D_\xi}$ the *irregular type* of the wild curve $\mathscr{C}$.

Definition 5.1.12. Residue of an Enhancement Class
Let $(g, m) = (g, \{m_i\}_i)$ be a polar type and let $\mathscr{C} = (\mathscr{C}, D, \nu, \xi)$ be a wild curve with

$$D = \sum_i n_i \cdot p_i = \sum_i \left\lceil \frac{m_i}{2} \right\rceil \cdot p_i$$

and $\nu \in H^0(C, \omega_C(D)^{\otimes 2}|_D)$. If $m_i \equiv 0 \mod 2$, we have

$$\lambda_\xi(\xi(p))(x) = c_n + c_{n-1}x + \cdots + c_1 x^{n-1}$$

and define the *residue of* ν at p as the leading order term of the polynomial $\lambda_\xi(\xi(p))$, i.e. $Res_p(\nu) = c_1$.

Definition 5.1.13. Moduli of Wild Curves
We denote by $\mathscr{M}(\mathscr{C})$ the space parametrising wild curves $\mathscr{C} = (C, D, \nu, \xi)$, such that $Res_{p_i}(\nu) = 0; \forall p_i \in D$ with $m_i \equiv 0 \mod 2$. An isomorphism between two wild curves

(C, D, ν, ξ) and (C', D', ν', ξ') is given by an isomorphism of algebraic curves $f : C \xrightarrow{\sim} C'$ such that $g^*D' = D$, $g^*\nu' = \nu$ and $g^*\xi' = \xi$.

Remark 5.1.14. Note here that the term *wild* carries multiple meanings throughout the literature, and even more confusingly, some tend to be related. Most notably, the term appears in the theory of ramified covers in finite and mixed characteristic, used to differentiate between covers that are locally described as separable extensions of residue fields (tame case) or inseparable extensions (wild case). In this thesis, all covers are tamely ramified as finite maps between complex curves, and the term *wild* rather refers to the fact that the polar divisor D is non-reduced, a term used mainly in the study of irregular connections, Stokes data and the geometric Langlands programme. For the latter use of the term *wild* cf. [78] and [9].

Remark 5.1.15. The ring parametrising enhancement classes of type $\mathcal{m}$ with vanishing residues is:

$$\mathcal{M}_0^{\mathcal{m}}(\nu) = \bigoplus_{i\in I} \frac{x \cdot \mathbb{C}[x]}{x^{n_i+1}} \oplus \bigoplus_{j\in J} \mathbb{C}[x]/x^{n_j}, \tag{5.6}$$

where $I := \{i \mid m_i \equiv 1 \mod 2\}$ and $J := \{j \mid m_j \equiv 0 \mod 2\}$. As a vector space, it is of dimension

$$dim_{\mathbb{C}}(\mathcal{M}_0^{\mathcal{m}}(\nu)) = deg(D) - d = \sum_i (n_i - 1).$$

Lemma 5.1.16. *The space of enhancement classes $H^0(D, \omega_C(D)^{\otimes 2}|_D)$ on a pair (C, D) of curve C with a divisor D, is dual to the space of sections $H^0(D, T_C(-D)|_D)$ on (C, D).*

Proof. (Sketch) We provide a detailed proof of this result later on as part of the proof of theorem 5.2.22. We remark here, that applying Serre duality [43, Theorem 3.3.4], we have

$$H^0(D, \omega_C(D)^{\otimes 2}|_D) \cong H^0(D, T_C(-D)|_D)^*.$$

The perfect pairing is given by the residue pairing

$$\begin{aligned} H^0(D, \omega_C(D)^{\otimes 2}|_D) \times H^0(D, T_C(-D)|_D) &\longrightarrow \mathbb{C} \\ (\nu, j) &\longmapsto Res(\nu \cdot j). \end{aligned} \tag{5.7}$$

Explicitly, in local terms, choosing a uniformiser x at a point p of D of length n, we have

$$\nu(x) = \sum_{i=0}^{n-1} \alpha_i x^i \left(\frac{dx^{\otimes 2}}{x^{2n}}\right); \quad j(x) = \sum_{i=0}^{n-1} \beta_i x^i (x^n \partial_x)$$

then taking the product and contracting gives

$$\nu(x)\cdot j(x) = \sum_{i=0}^{2n-2} \gamma_i x^i \left(\frac{dx^{\otimes 2}}{x^{2n}}\right)\cdot (x^n\partial_x) = \sum_{i=0}^{2n-2} \gamma_i x^i \frac{dx}{x^n} \implies Res_p(\nu(x)j(x)) = \gamma_{n-1}.$$

□

Definition 5.1.17. Space of Jets
Let (C, D) be a pair of an algebraic curve and an effective divisor, then we denote the space of D-jets on C by $\mathcal{M}_{(C,D)}(j) = H^0(D, T_C(-D)|_D)$ and by $\mathcal{M}(C, D, j) = \mathcal{M}(C, D) \times \mathcal{M}_{(C,D)}(j)$ the trivial bundle of jets over $\mathcal{M}(C, D)$.

5.2. Quadratic Differentials on Wild Curves

Definition 5.2.1. Quadratic Differentials on Wild Curves
Let $\mathscr{C} = (C, D, \nu, \xi) \in \mathcal{M}(\mathscr{C})$. A quadratic differential on $\mathscr{C}$ is a section

$$Q \in H^0(C, \omega_C^{\otimes 2}(H)) \ \text{ with } Q|_D = \nu \in H^0(C, \omega_C(D)^{\otimes 2}|_D). \tag{5.8}$$

We denote the space of quadratic differentials on wild curves by $\mathcal{M}(\mathscr{C}, Q)$.

Lemma 5.2.2. *There is a a forgetful projection*

$$\mathcal{M}(\mathscr{C}, Q) \longrightarrow \mathcal{M}(\mathscr{C}) \tag{5.9}$$

with fibre $H^0(C, \omega_C^{\otimes 2}(D))$. Moreover, $\mathcal{M}(C, Q)$ is a (forgetful) étale quotient of $\mathcal{M}(\mathscr{C}, Q)$ by the moduli of signings $(\mathbb{Z}/2\mathbb{Z})^e$.

Proof. The fact that 5.9 has fibre $H^0(C, \omega_C^{\otimes 2}(D))$ follows from the vanishing lemma 5.1.2. Namely, via the isomorphism $H^0(C, \omega_C(D)^{\otimes 2}|_D) \cong \frac{H^0(C,\omega_C(D)^{\otimes 2})}{H^0(C,\omega_C^{\otimes 2}(D))}$, the enhancement datum ν of a wild curve fixes a quadratic differential up to $H^0(C, \omega_C^{\otimes 2}(D))$. Regarding the second claim quadratic differential Q on a curve C, fixes a divisor D by setting $D = \lceil \frac{H}{2} \rceil$, where H is the polar divisor of Q, as well as an enhancement class by reduction to the divisor D, i.e. $\nu = Q|_D \in H^0(D, \omega_C(D)^{\otimes 2}|_D)$. □

In the last chapter, we saw how a quadratic differential can be interpreted as a spectral cover of the base curve. We highlight the role of the enhancement class in the language of spectral covers, namely, it is the part of the quadratic differential that fixes the geometry of the spectral curve over the polar divisors.

Remark 5.2.3. Applying proposition 4.3.10 to the embedding of D in C, we see that the double cover $p_\nu : D_\nu \to D$ determined by the enhancement class $\nu \in H^0(C, \omega_C(D)^{\otimes 2}|_D)$ of a quadratic differential, is the restriction of the spectral cover $p : \Sigma \to C$ associated to the quadratic differential over the polar divisor D, that is $D_\nu = \iota_D^*\Sigma = \pi^*D \cap_{Tot} \Sigma$. Diagrammatically

$$\begin{tikzcd}
& \pi^*D \arrow[rr, hook] \arrow[dd] & & Tot(\omega_C(D)) \arrow[dd, "\pi"] \\
D_\nu \arrow[ur, hook] \arrow[rr, hook] \arrow[dr, "p_\nu"] & & \Sigma \arrow[ur, hook] \arrow[dr, "p"] & \\
& D \arrow[rr, hook, "\iota_D"] & & C
\end{tikzcd} \quad . \tag{5.10}$$

Remark 5.2.3 shows that the spectral covers of any two quadratic differentials in the same ν-class, restrict to the same D_ν cover of D. Here is an explicit local description of that fact.

Proposition 5.2.4. ***Seiberg-Witten Differential restricts to Irregular type***
Let $\mathscr{C} = (C, D, \nu, \xi)$, let Q a quadratic differential on $\mathscr{C}$ and let λ be the SW-differential on the spectral cover $p : \Sigma \longrightarrow C$ associated to Q, cf. definition 4.3.8. Then, we have

$$\lambda_\nu = \lambda|_{D_\nu} \in H^0(D_\nu, \omega_\Sigma(D_\nu)|_{D_\nu}) = H^0(p^*D, \omega_\Sigma(p^*D)|_{p^*D}), \tag{5.11}$$

$$\lambda_\xi = \lambda|_{D_\xi} \in H^0(D_\xi, \omega_\Sigma(D_\nu)|_{D_\xi}) = H^0(D_\xi, \omega_\Sigma(p^*D)|_{D_\xi}) \tag{5.12}$$

for the λ_ν, λ_ξ of definitions 5.1.5 and 5.1.11. In local terms, this means that the Laurent tail of λ at each pole is fixed by the enhancement datum of the wild curve, that is, it is fixed by half of leading order terms in the Laurent tail of Q.

Proof. We argue locally, so, w.l.o.g., we let the divisor D to have a unique closed point pt, that is, $Q \in H^0(C, \omega_C^{\otimes 2}(m \cdot pt))$, in particular $D = n \cdot pt = \lceil \frac{m}{2} \rceil \cdot pt$. Then, in a local coordinate x around pt with $x(pt) = 0$, the Laurent tail of the quadratic differential at pt is

$$\mathscr{L}_{pt}(Q(x)) = \sum_{j=0}^{2n} \frac{q_j}{x^j} dx^{\otimes 2} = \sum_{j=0}^{2n} q_j x^{2n-j} \frac{dx^{\otimes 2n}}{x^{2n}} \quad \text{and} \quad \nu(x) = \sum_{j=n}^{2n} q_j x^{2n-j} \in \mathbb{C}[x]/(x^n),$$

where crucially $q_{2n} = 0$ if $m \equiv 1 \mod 2$. We prove the statement by giving an explicit formula for the coefficients of $\lambda|_{D_\nu}$ in terms of the coefficients $\{q_n, \dots, q_m\}$ of ν. For notational reasons, we need to differentiate the cases based on whether Σ ramifies over pt or not.

Ramified Case: i.e. $m \equiv 1$ *mod* 2. After passing to a coordinate w on the double cover

$w^2 = x$, for a neighbourhood of $q = p^{-1}(pt)$, with $w(q) = 0$, we write

$$\mathscr{L}_q(p^*Q) = \sum_{j=0}^{2m} \frac{\tilde{q}_j}{w^j} \text{ with } \tilde{q}_j = 0, \forall j : j \mod 2 \equiv 1, \tilde{q}_j = q_{\frac{j}{2}}.$$

The Laurent tail of the square root is

$$\mathscr{L}_q(\lambda) = \sum_{k=0}^{m} \frac{c_k}{w^k}; \text{ where } c_k = \frac{1}{2\sqrt{\tilde{q}_{2m}}} \left[\tilde{q}_{m+k} - \sum_{j=k+1}^{m-1} c_j c_{2m-j} \right].$$

Notice that only the coefficients of the enhancement $\{q_n, \ldots, q_{2n-1}\}$ enter the recursion formula. Analogously, we proceed in the unramified case.
Unramified Case: i.e. $m \equiv 0$ *mod* 2, and let $\{q, \sigma(q)\} = p^{-1}(pt)$. Then, we have at $q \in \Sigma$

$$\mathscr{L}_q(p^*Q) = \sum_{j=0}^{m} \frac{q_j}{w^j} \implies \mathscr{L}_q(\lambda) = \sum_{k=0}^{m/2} \frac{c_k}{w^k}; \text{ where } c_k = \frac{1}{2\sqrt{q_m}} \left[q_{\frac{m}{2}+k} - \sum_{j=k+1}^{m/2-1} c_j c_{m-j} \right]$$

and $\mathscr{L}_{\sigma(q)}(\lambda) = \mathscr{L}_q(\sigma^*\lambda) = \sigma^*\mathscr{L}_q(\lambda)$. □

Remark 5.2.5. If Q is a quadratic differential on a wild curve $\mathscr{C} = (C, D, \nu, \xi)$, with spectral cover $p : \Sigma_Q \longrightarrow C$, carrying an involution σ, then

$$p^*D = D_\nu = D_\xi + \sigma^* D_\xi.$$

In particular if $D_i = n_i \cdot p_i$ and $Q(p_i) = 0$, i.e. p_i is a point on the branching locus $B \subset C$, then $p^*D_i = 2 \cdot (D_\xi|_{p^*D})$.

Definition 5.2.6. The Anti-Invariant Cohomology Bundle $\mathscr{H}$
We denote by $\mathscr{H} \longrightarrow \mathcal{M}(\mathscr{C}, Q)$ the bundle having fibres the anti-invariant cohomology groups $\tilde{H}^1(\Sigma_{(\mathscr{C},Q)}, \mathbb{C})^-$. We denote by $\mathscr{H}^{\mathbb{Z}} \longrightarrow \mathcal{M}(\mathscr{C}, Q)$ the bundle with fibres $\tilde{H}^1(\Sigma_{(\mathscr{C},Q)}, \mathbb{Z})^-$, which is a natural integral lattice of $\mathscr{H}$. Lastly we denote by

$$\pi : \mathscr{H}^{\#} \longrightarrow \mathcal{M}(\mathscr{C}, Q) \tag{5.13}$$

the bundle whose fibres are given by the quotients $\tilde{H}^1(\Sigma_{(\mathscr{C},Q)}, \mathbb{C})^-/\tilde{H}^1(\Sigma_{(\mathscr{C},Q)}, \mathbb{Z})^-$, that is $\mathscr{H}^{\#} = \mathscr{H}/\mathscr{H}^{\mathbb{Z}}$.

Corollary 5.2.7. ***(to proposition 4.4.4)***
The Gauß-Manin local system of proposition 4.4.4, restricts to anti-invariant cohomology and we obtain in exactly the same way a flat Gauß-Manin connection ∇^{GM} on the bundle $\mathscr{H} \longrightarrow \mathcal{M}(\mathscr{C}, Q)$, which is unique with the property that it preserves the integral lattice $\mathscr{H}^{\mathbb{Z}}$.

5.2.1. Period Structure on $\mathcal{M}(\mathscr{C}, Q)$

In this section, we define a period structure on $\mathcal{M}(\mathscr{C}, Q)$ as introduced in definition 3.3.2. We study a period map on $\mathcal{M}(\mathscr{C}, Q)$, which will allow us to relate $\mathscr{H}$ to the tangent bundle $\mathcal{T}\mathcal{M}(C, Q)$, and thereby giving a geometric/moduli theoretic interpretation to the space X of Chapter 3, on which we strive to construct a Joyce Structure.

Definition 5.2.8. Periods of Quadratic Differentials
Let consider a pair of a wild curve and a quadratic differential $(\mathscr{C}, Q) \in \mathcal{M}(\mathscr{C}, Q)$ and recall the 1-form $\lambda \in H^0(\Sigma, \omega_\Sigma(D_\nu))$ on the associated spectral curve from (4.24). We call the associated de Rham cohomology class

$$\mathcal{Z}_Q := [\lambda]_{dR} = \left[\int_* \lambda : H_1(\Sigma, \mathbb{Z})^- \to \mathbb{C}\right] \in \tilde{H}^1(\Sigma, \mathbb{C})^-, \tag{5.14}$$

the *period of the quadratic differential* Q. Note that de Rham class is well-defined since λ is the restriction of the canonically defined tautological 1-form y to the unique spectral curve Σ associated to Q, cf. definition 4.3.8.

Period classes define a section of the cohomology bundle of definition 5.2.6

$$\begin{aligned} \delta : \mathcal{M}(\mathscr{C}, Q) &\longrightarrow \mathscr{H} \\ (\mathscr{C}, Q) &\longmapsto \mathcal{Z}_Q. \end{aligned} \tag{5.15}$$

We call δ the *period section* of $\mathscr{H} \longrightarrow \mathcal{M}(\mathscr{C}, Q)$.

Remark 5.2.9. Note here that, although λ is a meromorphic 1-form on the spectral curve, the residue-zero condition on the enhancement datum tautologically imposes that the residue of λ also vanishes on all anti-invariant homology cycles. Recall that only cycles around even order poles contribute to anti-invariant homology.

Remark 5.2.10. Notation for a Basis of Anti-Invariant Homology
We write $\{\gamma_1, \ldots, \gamma_{2g}\}$ for a basis of the anti-invariant cohomology $H_1(\Sigma, \mathbb{Z})^-$, such that γ_j are A-cycles for $1 \leq j \leq g$, B-cycles for $g+1 \leq j \leq 2g$ and $(\gamma_i, \gamma_j) = \delta_{i,j-g};$ for $1 \leq i \leq g$.

The following result was originally discovered by Veech [76] and was also recovered by Bridgeland-Smith in [18].

Theorem 5.2.11. *The period map δ is a local isomorphism of complex manifolds.*

Proof. This is shown in [76, Theorem 7.22], cf. also [18, Sec. 4]. □

The Gauß-Manin connection on the cohomology bundle $\mathscr{H}$ allows us to recast this result into the following theorem.

Theorem 5.2.12. ***Differential of the Period Section***
The differential of the period section δ with respect to the Gauß-Manin connection yields an isomorphism of $\mathbb{C}$-vector bundles:

$$\begin{aligned}\nabla^{GM}\delta : T\mathcal{M}(\mathscr{C},Q) &\xrightarrow{\sim} \mathscr{H}, \\ w &\longmapsto \nabla^{GM}_{w}\delta(\mathscr{C},Q),\end{aligned} \tag{5.16}$$

that is, $\nabla^{GM}_{w}\delta(\mathscr{C},Q)$ is differentiating the period class in the direction of w.

Proof. (sketch) Let $\{U_i\}$ be a complex analytic cover of $\mathcal{M}(\mathscr{C},Q)$. The flat connection ∇^{GM} on $\mathscr{H} \longrightarrow \mathcal{M}(\mathscr{C},Q)$ prescribes a flat trivialisation of $\mathscr{H}$, i.e. a collection of flat isomorphisms:

$$\mathcal{T}_i : \mathscr{H}|_{U_i} \xrightarrow{\sim} U_i \times \mathscr{H}_x, \text{ for some } x \in U_i \tag{5.17}$$

such that $\nabla^{GM}|_{U_i}$ is identified with the trivial connection d. Let

$$\mathcal{F}_i := \mathcal{T}_i \circ \delta|_{U_i} : U_i \longrightarrow \mathscr{H}_x \tag{5.18}$$

Since $\mathcal{T}$ is flat we have: $d\mathcal{F}_i = \mathcal{T}_i \circ (\nabla^{GM}\delta)|_{U_i}$. Now since δ is known to be a local isomorphism by 5.2.11, $\mathcal{F}_i$ is a (local) biholomorphism and its differential

$$d\mathcal{F}_i : TU_i \xrightarrow{\sim} U_i \times \mathscr{H}_x \tag{5.19}$$

is an isomorphism of complex bundles, therefore composition:

$$\mathcal{T}_i^{-1} \circ d\mathcal{F}_i : T\mathcal{M}(C,Q)|_{U_i} \longrightarrow \mathscr{H}|_{U_i} \tag{5.20}$$

is an isomorphism as well. It remains to see that this construction glues to a global isomorphism of complex vector bundles $T\mathcal{M}(C,Q) \longrightarrow \mathscr{H}$. Let $U_{ij} = U_i \cap U_j$, then $\mathcal{T}_j = (id \times g_{ij}) \circ \mathcal{T}_i$ for some $g_{ij} \in GL(H_x)$ and $x \in U_{ij}$. The differential then reads $d\mathcal{T}_j = (id \times g_{ij}) \circ d\mathcal{T}_i$ and the covariant derivatives agree on the overlaps U_{ij} via:

$$\nabla^{GM}|_{U_j} = \mathcal{T}_j^{-1} \circ d\mathcal{F}_j = \mathcal{T}_i^{-1} \circ (id \times g_{ij}) \circ d\mathcal{F}_i = \mathcal{T}_j^{-1} \circ d\mathcal{F}_j = \nabla^{GM}|_{U_i}$$

□

Lemma 5.2.13. ***Period Coordinates on $\mathcal{M}(\mathscr{C},Q)$***
The isomorphism of Theorem 5.2.12 prescribes a local coordinate system on $\mathcal{M}(\mathscr{C},Q)$.

Proof. Consider a neighbourhood $(\mathscr{C},Q) \in U \subset M$ and a basis $\{\gamma_i\}_i$ of $H_1(\Sigma_{\mathscr{C},Q},\mathbb{Z})^-$. Then, using the Gauß-Manin connection, we extend this basis to $U \times H_1(\Sigma_{\mathscr{C},Q},\mathbb{Z})^-$. Furthermore using the pairing

$$\begin{aligned} H_1(\Sigma_{\mathscr{C},Q},\mathbb{Z})^- \times \tilde{H}^1(\Sigma_{\mathscr{C},Q},\mathbb{C})^- &\longrightarrow \mathbb{C} \\ (\gamma_i, [\lambda]_{dR}) &\longmapsto z_i := \int_{\gamma_i} \mathcal{Z}_Q = \int_{\gamma_i} \delta(\mathscr{C},Q) \end{aligned}$$

we obtain locally defined functions on $\mathcal{M}(\mathscr{C},Q)$, namely the functions $\{z_i\}_i$. The statement of theorem 5.2.12 is that $\left\{\frac{\partial}{\partial z_i}\right\}$ give a local basis for $T\mathcal{M}(\mathscr{C},Q)|_U$ and that the functions $\{z_i\}_i$ can be interpreted as local coordinates on $U \subset \mathcal{M}(\mathscr{C},Q)$. □

Lemma 5.2.14. *Let $v \in T_{(\mathscr{C},Q)}\mathcal{M}_{\mathscr{C}}(\mathscr{C},Q)$, then $\nabla^{GM}_v \mathcal{Z}_Q$ is a de Rham cohomology class of an 1-form that is holomorphic away from ramification.*

Proof. Let $U \subset C$ be a neighbourhood of a geometric point of the polar divisor that is not on the branching locus $p \in D^{red}\backslash B$, and let $x \in U$ be a local coordinate with $x(p) = 0$. Then, a quadratic differential $Q \in \mathcal{M}_{\mathscr{C}}(\mathscr{C},Q)$ is locally represented by a Laurent polynomial

$$Q(x) = \frac{a_{2n}}{x^{2n}} + \cdots + \frac{a_1}{x} + a_0 + O(x). \tag{5.21}$$

Fixing the wild curve $\mathscr{C}$ amounts to fixing the coefficients $\{a_{2n},\ldots,a_{n+1}\}$. The covariant derivative of $\mathcal{Z}_Q$ in the direction of v yields $\nabla^{GM}_v \mathcal{Z}_Q = \frac{1}{\sqrt{p^*Q}}\nabla^{GM}_v p^*Q$, where $(\sqrt{p^*Q})^{-1}$ is a rational function of order $> -n$. This means that $x^n \cdot (\sqrt{p^*Q})^{-1}$ is regular holomorphic. Since v is a tangent vector pointing in a direction vertical to the deformation of the wild curve, $\nabla^{GM}_v p^*Q$ has a pole of order at most n, as $\{a_{2n},\ldots,a_{n+1}\}$ are kept fixed and their moduli is killed by the directional derivative. Therefore, the expression $\nabla^{GM}_v \mathcal{Z}_Q$ represents the de Rham class of an 1-form that is regular holomorphic at the polar point p. □

Definition 5.2.15. Euler Vector field on $\mathcal{M}(\mathscr{C},Q)$

The Euler vector field $Z = \sum_i z_i \frac{\partial}{\partial z_i}$ is the generator of the $\mathbb{C}^*$-action

$$\begin{aligned} \phi : \mathbb{C}^* \times \mathcal{M}(\mathscr{C},Q) &\longrightarrow \mathcal{M}(\mathscr{C},Q) \\ (t,(C,D,\nu,\xi,Q)) &\longmapsto (C,D,t^2\nu,\xi,t^2Q). \end{aligned} \tag{5.22}$$

5.2.2. Symplectic and Cotangent Bundle Structure on $\mathcal{M}(\mathscr{C},Q)$

Let $(C,Q) \in \mathcal{M}(C,Q)$ be a quadratic differential on C and denote by Σ its associated spectral cover and by B its branching divisor on C. Consider the intersection product on homology, restricted on anti-invariant cycles

$$\langle\cdot,\cdot\rangle : H_1(\Sigma,\mathbb{C})^- \times H_1(\Sigma,\mathbb{C})^- \longrightarrow \mathbb{C}. \tag{5.23}$$

We will see now that, upon considering quadratic differentials on wild curves, this pairing gives rise to a symplectic structure on $\mathcal{M}(\mathscr{C}, Q)$.

Definition 5.2.16. Residue of Quadratic differentials on Wild Curves
Let $\mathscr{C} = (C, D, \nu, \xi)$ and Q a quadratic differential on $\mathscr{C}$. Let $q \in H_{ev}^{red} = D^{red} \backslash B$, i.e. a pole of Q that is an even order pole of $Q \in H^0(C, \omega_C(D)^{\otimes 2})$. We define the residue of the quadratic differential Q at the point q

$$\mathcal{R}es_q(Q) := \int_{\gamma_{\xi(q)}} \mathcal{Z}_Q = \int_{\gamma_{\xi(q)}} \delta(\mathscr{C}, Q), \tag{5.24}$$

for $\gamma_{\xi(q)} \in H_1(\Sigma \backslash D_\xi, \mathbb{Z})$, a cycle around $\xi(q)$.

Remark 5.2.17. Let $(\mathscr{C}, Q) \in \mathcal{M}(\mathscr{C}, Q)$. The condition that $Res_{p_i}(\nu) = 0; \forall p_i \in D$, translates to the condition that $\mathcal{R}es_q(Q) = 0$ for $q \in D^{red} \backslash B$.

Definition 5.2.18. A Pairing on $T^*\mathcal{M}(\mathscr{C}, Q)$
Let $(\mathscr{C}, Q) \in \mathcal{M}(\mathscr{C}, Q)$. Since $H^1(\Sigma_{C,Q}, \mathbb{C})^- \cong T_{\mathscr{C},Q}\mathcal{M}(\mathscr{C}, Q)$, due to theorem 5.2.12, using the Gauß-Manin connection, we have a pairing

$$\langle \cdot, \cdot \rangle : T^*_{\mathscr{C},Q}\mathcal{M}(\mathscr{C}, Q) \times T^*_{\mathscr{C},Q}\mathcal{M}(\mathscr{C}, Q) \longrightarrow \mathbb{C}.$$

Explicitly, let $\{\gamma_j\}_j$ be a basis of $H_1(\Sigma, \mathbb{C})^-$ and $\{dz_j\}_j$ its image in $T^*_{C,Q}\mathcal{M}(C, Q)$, then

$$\langle dz_i, dz_j \rangle \longmapsto (\gamma_i \cdot \gamma_j),$$

where $(\cdot, \cdot)$ is the intersection product from which we obtain a bracket

$$\{\cdot, \cdot\} : \mathcal{C}^\infty(\mathcal{M}(\mathscr{C}, Q)) \times \mathcal{C}^\infty(\mathcal{M}(\mathscr{C}, Q)) \longrightarrow \mathcal{C}^\infty(\mathcal{M}(\mathscr{C}, Q)) \tag{5.25}$$

that is defined locally on the period coordinates of $\mathcal{M}(\mathscr{C}, Q)$ by

$$\{z_i, z_j\} := \langle dz_i, dz_j \rangle. \tag{5.26}$$

Lemma 5.2.19. *The bracket $\{\cdot, \cdot\}$ of definition 5.25 is a Poisson structure on $\mathcal{M}(\mathscr{C}, Q)$.*

Proof. This bracket is defined locally on analytic functions in several complex variables, with the following properties:

- Skew-symmetry follows directly from the same property of the intersection pairing.
- Leibniz rule:

$$\begin{aligned}\{f, gh\} = \langle df, d(gh) \rangle = \langle df, dg \cdot h + g \cdot dh \rangle &= h \langle df, dg \rangle + g \langle df, dh \rangle \\ &= h\{f, g\} + g\{f, h\}\end{aligned}$$

- Jacobi-integrability: For $f_i \in \mathcal{C}^\infty(\mathcal{M}(C,Q))$, we have

$$\{f_1,\{f_2,f_3\}\} = \underbrace{\sum_{klij} \epsilon_{kl}\epsilon_{ij} \frac{\partial f_1}{\partial z_k}\frac{\partial^2 f_2}{\partial z_l \partial z_i}\frac{\partial f_3}{\partial z_j}}_{(1)} + \underbrace{\sum_{klij} \epsilon_{kl}\epsilon_{ij} \frac{\partial f_1}{\partial z_k}\frac{\partial f_2}{\partial z_i}\frac{\partial^2 f_3}{\partial z_l \partial z_j}}_{(2)}$$

$$\{f_3,\{f_1,f_2\}\} = \underbrace{\sum_{klij} \epsilon_{kl}\epsilon_{ij} \frac{\partial f_3}{\partial z_k}\frac{\partial^2 f_1}{\partial z_l \partial z_i}\frac{\partial f_2}{\partial z_j}}_{(3)} + \underbrace{\sum_{klij} \epsilon_{kl}\epsilon_{ij} \frac{\partial f_3}{\partial z_k}\frac{\partial f_1}{\partial z_i}\frac{\partial^2 f_2}{\partial z_l \partial z_j}}_{(4)}$$

$$\{f_2,\{f_3,f_1\}\} = \underbrace{\sum_{klij} \epsilon_{kl}\epsilon_{ij} \frac{\partial f_1}{\partial z_k}\frac{\partial^2 f_2}{\partial z_l \partial z_i}\frac{\partial f_3}{\partial z_j}}_{(5)} + \underbrace{\sum_{klij} \epsilon_{kl}\epsilon_{ij} \frac{\partial f_2}{\partial z_k}\frac{\partial f_3}{\partial z_i}\frac{\partial^2 f_1}{\partial z_l \partial z_j}}_{(6)},$$

then, by the antisymmetry of ϵ_{ij}: $(1)+(4) = (2)+(5) = (3)+(6) = 0$ and the Jacobi relation:

$$\{f_1,\{f_2,f_3\}\} + \{f_3,\{f_1,f_2\}\} + \{f_2,\{f_3,f_1\}\} = 0$$

follows.

□

Proposition 5.2.20. *The Poisson structure is non-degenerate, i.e. it is a symplectic structure on $\mathcal{M}(\mathscr{C},Q)$. We denote the symplectic form given by the bracket* (5.25) *by ω.*

Proof. Let $\{z_i\}_i$ be the period coordinates on a neighbourhood $U \subset \mathcal{M}(\mathscr{C},Q)$. Let $\{\gamma_i\}_i$ denote a basis of $H_1(\Sigma,\mathbb{Z})^-$. Given any cycle γ_i, we want to show that

$$\{z_i, z_j\} = \gamma_i \cdot \gamma_j = 0 \ \forall j \iff z_i = 0. \tag{5.27}$$

This is true since all cycles in the anti-invariant cohomology are A, B-cycles, and these all have non-trivial intersections. □

Proposition 5.2.21. *The fibres $\mathcal{M}_{\mathscr{C}}(Q)$ of $\mathcal{M}(\mathscr{C},Q) \longrightarrow \mathcal{M}(\mathscr{C})$ are Lagrangian submanifolds with respect to the symplectic structure ω of* (5.25)*, obtaining a Lagrangian fibration on the space of quadratic differentials.*

Proof. For the period coordinates $(z_i)_i$, let $\omega_{ij} = \omega(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})$ with $\omega_{ij} = -\omega_{ji}$. Moreover, let $(a_\ell)_\ell$ be a local basis on the fibre. For this we have

$$\begin{aligned}\omega\left(\frac{\partial}{\partial a_k}, \frac{\partial}{\partial a_\ell}\right) &= \omega\left(\sum_i \frac{\partial z_i}{\partial a_k}\frac{\partial}{\partial z_j}, \sum_i \frac{\partial z_j}{\partial a_\ell}\frac{\partial}{\partial z_j}\right) = \sum_{ij} \omega_{ij}\frac{\partial z_i}{\partial a_k}\frac{\partial z_j}{\partial a_\ell} \\ &= \sum_{ij} \omega_{ij} \int_{\gamma_i} \nabla^{GM}_{a_k}[\lambda] \int_{\gamma_j} \nabla^{GM}_{a_\ell}[\lambda].\end{aligned}$$

Since $(\gamma_j)_j$ is a basis of anti-invariant cohomology, cycles around poles at ramification are excluded. Since $\nabla^{GM}_{a_k}[\lambda]$ is holomorphic away from ramification for all k by lemma 5.2.14, applying the residue-reciprocity theorem [33, Section 2.3] we conclude

$$\omega\left(\frac{\partial}{\partial a_k}, \frac{\partial}{\partial a_\ell}\right) = 0.$$

□

In the holomorphic setting, the space of quadratic differentials emerges naturally via deformation theory as the cotangent space to the moduli of algebraic curves. To see this, let $C \in \mathcal{M}(C)$ be a curve, denote by TC its tangent sheaf and by ω_C the dualising sheaf. Then we have

$$T_{[C]}\mathcal{M}(C) \cong H^1(C, TC) \implies T^*_{[C]}\mathcal{M}(C) \cong H^0(C, \omega_C^{\otimes 2}). \tag{5.28}$$

The following theorem shows that quadratic differentials can be interpreted as cotangent vectors.

Theorem 5.2.22. ***A Cotangent Bundle structure of*** $\mathcal{M}(\mathscr{C}, Q)$
The space of quadratic differentials on wild curves $\mathcal{M}(\mathscr{C}, Q)$ *is the cotangent bundle to the space of jets* $\mathcal{M}(C, D, j_\nu)$ *of definition 5.1.17.*

Proof. Consider the restriction sequence of the tangent sheaf T_C to the underlying algebraic curve

$$\mathcal{T}^\bullet : \cdots \longrightarrow T_C|_D[-1] \longrightarrow T_C(-D) \longrightarrow T_C \longrightarrow T_C|_D \longrightarrow T_C(-D)[1] \longrightarrow \cdots \tag{5.29}$$

and denote by $\mathbb{D} := \mathcal{R}Hom_{\mathcal{O}_C}(-, \omega_C[1])$ the dualising functor of the Serre-Grothendieck-Verdier duality (SGV). By [43, Theorem 3.3.4], there exists a functorial isomorphism

$$\mathcal{R}\Gamma(C, \mathbb{D}(\mathcal{T}^\bullet)) \xrightarrow{\sim} \mathcal{R}\Gamma(C, \mathcal{T}^\bullet)^*, \tag{5.30}$$

where the dualised complex $\mathbb{D}(\mathcal{T}^\bullet)$ reads

$$\mathbb{D}(\mathcal{T}^\bullet) :\longrightarrow \omega_C(D)^{\otimes 2}|_D[-1] \longrightarrow \omega_C^{\otimes 2}(D) \longrightarrow \omega_C(D)^{\otimes 2} \longrightarrow \omega_C(D)^{\otimes 2}|_D \longrightarrow \omega_C^{\otimes 2}(D)[1] \longrightarrow$$

By the functoriality property of the SGV-duality, we obtain a commutative diagram of LES in cohomology

$$\begin{tikzcd}
H^1(C,T_C)^* \arrow[r] \arrow[d, "\wr"'] & H^1(C,T_C(-D))^* \arrow[r] \arrow[d, "\wr"'] & H^0(D,T_C|_D)^* \arrow[r] \arrow[d, "\wr"'] & H^0(C,T_C)^* \arrow[r] \arrow[d, "\wr"'] & H^0(C,T_C(-D))^* \arrow[d, "\wr"'] \\
H^0(C,\omega_C^{\otimes 2}) \arrow[r] & H^0(C,\omega_C^{\otimes 2}(D)) \arrow[r] & H^0(D,\omega_C^{\otimes 2}(D)|_D) \arrow[r] & H^1(C,\omega_C^{\otimes 2}) \arrow[r] & H^1(C,\omega_C^{\otimes 2}(D))
\end{tikzcd}$$

Note that by our assumption on the degree of D cf. 4.1.7, we have

$$H^1(C,\omega_C^{\otimes 2}(D)) \cong H^0(C,T_C(-D))^* = 0 \tag{5.31}$$

and since $T_C\mathcal{M}(C) = H^1(C,T_C)$ and $T_{(C,D)}\mathcal{M}(C,D) = H^1(C,T_C(-D))$, we obtain a canonical identification: $T^*_{C,D}\mathcal{M}(C,D) \cong H^0(C,\omega_C^{\otimes 2}(D))$. Furthermore, from the LES in cohomology of the short exact sequence: $0 \longrightarrow \omega_C^{\otimes 2}(D) \longrightarrow \omega_C(D)^{\otimes 2} \longrightarrow \omega_C(D)^{\otimes 2}|_D \longrightarrow 0$ we have due to the vanishing lemma 5.1.2

$$\begin{tikzcd}
0 \arrow[r] & H^0(C,\omega_C^{\otimes 2}(D)) \arrow[r] \arrow[d, "\wr"'] & H^0(C,\omega_C(D)^{\otimes 2}) \arrow[r] \arrow[d, "\wr"'] & H^0(C,\omega_C(D)^{\otimes 2}|_D) \arrow[r] \arrow[d, "\wr"'] & 0 \\
0 \arrow[r] & T^*_{(C,D)}\mathcal{M}(C,D) \arrow[r] & T^*_{(C,D,j)}\mathcal{M}(C,D,j) \arrow[r] & T^*_j\mathcal{M}_C(j) \arrow[r] & 0.
\end{tikzcd} \tag{5.32}$$

The left and right vertical maps are the natural isomorphisms between finite dimensional vector spaces and their double-duals $V \xrightarrow{\sim} V^{**}$. $\square$

Remark 5.2.23. We decided to impose the residue-zero condition on quadratic differentials (and wild curves). This decision, however, was only made to simplify an already long and notationally heavy thesis. All aspects of the generalisation to the case of arbitrary residues, have been worked out and are reviewed in the appendix chapter C. Some of highlights towards this generalisation up to this stage are:

- The cohomology bundles have fibres $\tilde{H}^1(\Sigma^\circ,\mathbb{C})^-$ for $\Sigma^\circ := \Sigma\backslash D_\nu$. In particular, we need to restrict the Seiberg-Witten differential to that punctured curve, when defining residues.

- The GM-connection is not God-given, since we want to study the cohomology of families of curves relative to a divisor and this involves the use of advanced machinery of Katz-Kato [47].

- Most importantly, the symplectic form on $\mathcal{M}(\mathscr{C},Q)$ degenerates, and so it only carries

Poisson structure. The symplectic leaves are given by the fixed residue condition and the intersection of the symplectic leaves with the fibres of $\mathcal{M}(\mathscr{C}, Q)$ are Lagrangian. In particular, $\mathcal{M}(\mathscr{C}, Q)$ is not a cotangent bundle.

6. Irregular Higgs Bundles and Flat Connections

The goal of this chapter is to construct certain decorated algebraic $SL_2(\mathbb{C})$-bundles. In particular, we construct a model for the bundle $\pi : X \to M$, on which we aim to construct a family of Ehresmann connections, cf. definition 3.2.3. In Bridgeland's holomorphic theory cf. [15, chapter 5], the correct decorations turned out to be those of holomorphic Higgs fields and flat connections on the same holomorphic bundle. These structures give rise to a fibre bundle $\mathcal{M}(C, E, \Phi, \nabla) \longrightarrow \mathcal{M}(C, Q)$, extending the usual Hitchin fibration. In order to generalise this construction to the meromorphic setting with arbitrary polar divisors, the correct decorations are those of irregular Higgs bundles and irregular flat bundles preserving the same *wild* quasi-parabolic structure over a polar divisor, a notion we introduce in this thesis.

6.1. Irregular Quasi-Parabolic Bundles and Deformation Theory

We start by defining the notion of an algebraic $SL_2(\mathbb{C})$-bundle E on a curve C, as a locally free sheaf, equipped with an equivalence class of $SL_2(\mathbb{C})$-valued local trivialising systems $[(U_\alpha, g_\alpha)]$ and recalling how $SL_2(\mathbb{C})$-Higgs fields on E arise as the dual of the first-order-deformation space of such a bundle. Then, we introduce the notion of a *Wild Quasi-Parabolic Structure*, i.e. an $\mathcal{O}_D$-submodule of $E|_D$: $L_D \overset{\iota}{\hookrightarrow} E|_D$, satisfying a natural compatibility condition with respect to the SL_2-structure on E and study its first-order deformations.

6.1.1. Algebraic G-Bundles

Let C be an algebraic curve, $G < GL_n$ a linear algebraic group and let $\mathcal{F}$ be a locally free $\mathcal{O}_C$-Module of rank n. Moreover, we denote by G_C the sheaf of, in general non-abelian, groups, associated to $[U \subset C \text{ affine open } \longmapsto G(\Gamma(U, \mathcal{O}_C))]$.

Definition 6.1.1. Local Trivialising Systems
A *local trivialising system* for $\mathcal{F}$ is a collection (U_α, g_α), where $\bigcup_\alpha U_\alpha = C$ is a covering of X and $g_\alpha : \mathcal{F}|_{U_\alpha} \xrightarrow{\sim} \mathcal{O}_C^n|_{U_\alpha}$ are isomorphisms for all α, such that the sections

$$g_{\alpha\beta} := g_\alpha \circ g_\beta^{-1} \in \Gamma(U_\alpha \cap U_\beta, GL_{n,C})$$

satisfy the cocycle condition $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma} \in \Gamma(U_{\alpha\beta\gamma}, GL_{n,C})$ on triple intersections $U_\alpha \cap U_\beta \cap U_\gamma =: U_{\alpha\beta\gamma}$.

Definition 6.1.2. G-Structures on Locally Free Sheaves of rank n.
A *G-structure* on $\mathcal{F}$ is a trivialising system (U_α, g_α), such that $g_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, G_C)$. Two G-structures (U_α, g_α),(U'_α, g'_α) are equivalent, if the covering by all U_α, U'_α, together with the collection of all isomorphisms g_α, g'_α is also a G-structure on $\mathcal{F}$.

Definition 6.1.3. Algebraic G-Bundles.
We call a pair $E = (\mathcal{E}, [(U_\alpha, g_\alpha)])$, consisting of a locally free $\mathcal{O}_C$-Module $\mathcal{E}$ of rank n and an equivalence class $[(U_\alpha, g_\alpha)]$ of G-structures on $\mathcal{E}$, an *algebraic G-bundle.* Two algebraic G-bundles E, E' are isomorphic, if there exists:

- a covering $\bigcup_\alpha U_\alpha = C$,
- cocycles $g_{\alpha\beta}, g'_{\alpha\beta}$ on the same covering (U_α) representing the G-structure class of E and E' respectively,
- a collection $h_\alpha \in \Gamma(U_\alpha, G_C)$, such that $h_\alpha \circ g_{\alpha\beta} = g'_{\alpha\beta} \circ h_\beta$ for all $U_{\alpha\beta}$, i.e. the cocycles $g_{\alpha\beta}, g'_{\alpha\beta}$ are cohomologous.

A morphism of locally free sheaves $f : \mathcal{E} \longrightarrow \mathcal{E}'$ is a G-bundle isomorphism, if there exists data as above, such that $f|_{U_\alpha} = g_\alpha \circ h_\alpha \circ (g'_\alpha)^{-1}$. We denote the moduli space of algebraic G-bundles on a fixed curve C by $\mathcal{M}^G_C(E)$ and after allowing the curve to vary, we write $\mathcal{M}^G(C, E)$.

Lemma 6.1.4. *There is an 1:1 correspondence between isomorphism classes of algebraic G-bundles on C and Čech-cohomology classes in $\check{H}^1(C, G_C)$.*

Proof. Follows by the construction of Čech-cohomology. □

From here onwards, we will concern ourselves with the case of $G = SL_2$, and so write economically $\mathcal{M}(C, E)$ instead of $\mathcal{M}^{SL_2}(C, E)$ etc.

Until now, we have been careful in differentiating between the notions of an algebraic bundle and that of a locally free sheaf. From now on we allow ourselves to use the following categorical equivalence between the two notions to simplify notationally-heavy formuli, whenever it is obvious which notion is meant to be used.

Lemma 6.1.5. *There is a categorical equivalence*

$$\{\textit{Vector Bundles on } C\} \longleftrightarrow \{\textit{Locally Free } \mathcal{O}_C\textit{-Modules of finite rank}\}. \tag{6.1}$$

Proof. The left-to-right direction is given by the functor of local sections of a vector bundle $\mathscr{V}(L) : U \mapsto \Gamma(U, L|_U)$. The inverse functor is given by the geometrisation functor

$$\mathscr{L} \longmapsto [\pi : Tot(\mathscr{L}) := Spec(Sym^\bullet(\mathscr{L}^\vee)) \to C]$$

for which we have

$$\Gamma(U, \mathscr{L}|_U) = \Gamma(U, \mathscr{L}^{\vee\vee}|_U) = Hom_C(U, Spec(Sym^\bullet(\mathscr{L}^\vee)|_U)) = Hom_C(U, Tot(\mathscr{L})|_U)$$

that is, $\mathscr{V}(\mathscr{L}(Tot(\mathscr{L}))) = \mathscr{L}$, and the tautological section $y \in H^0(Tot(\mathscr{L}), \pi^*\mathscr{L})$ of definition 4.3.2 is gives rise to an isomorphism $L \xrightarrow{\sim} Tot(\mathscr{V}(L))$. □

Remark 6.1.6. There is an equivalence between the notions of vector bundles with G-structure as presented here and the notion of G-principal bundles. We refer to the appendix, namely B.0.16 for a proof of this at a stacky level.

6.1.2. Deformations of Decorated Algebraic Bundles

We denote the Artin algebra of dual numbers over $\mathbb{C}$ by:

$$\mathbb{C}[\varepsilon] := \mathbb{C}[t]/(t^2); \quad \varepsilon^2 = 0. \tag{6.2}$$

We write $\mathfrak{sl}_2(E)$ for the $\mathfrak{sl}_2$-valued endomorphisms of a bundle E, i.e. the trace free endomorphisms of E.

Lemma 6.1.7. *Consider the space $\mathcal{M}_C(E)$ parametrising isomorphism classes of $SL_2\mathbb{C}$-algebraic bundles on C. The first-order deformation space of $\mathcal{M}_C(E)$ at a point $E \in \mathcal{M}_C(E)$, is given by*

$$T_E\mathcal{M}_C(E) = H^1(C, \mathfrak{sl}_2(E)). \tag{6.3}$$

Proof. For $E \in M_C(E)$, we define the deformation functor

$$\mathscr{F}_E : \mathcal{A}rt_{\mathbb{C}}^{op} \longrightarrow \mathcal{S}et \tag{6.4}$$

$$A \longmapsto [F, \theta], \tag{6.5}$$

where $\mathcal{A}rt_{\mathbb{C}}$ is the category of local Artin $\mathbb{C}$-algebras, and

$$F \in \mathcal{B}un_G(C \times_{\mathbb{C}} Spec(A)) \tag{6.6}$$

is obtained by base change, i.e. there exists isomorphism of G-bundles on C

$$\vartheta : i^*F \xrightarrow{\sim} E. \tag{6.7}$$

Here i denotes the canonical closed embedding

$$i : C \hookrightarrow C \times_{\mathbb{C}} Spec(A) \tag{6.8}$$

associated to the reduction along the *specialisation* morphism

$$A \twoheadrightarrow {}^{A}/_{\mathfrak{m}} \cong \mathbb{C}; \quad \{\mathfrak{m}\} = Spm(A). \tag{6.9}$$

We declare an equivalence between two such pairs by

$$(F, \vartheta) \sim (F', \vartheta') :\Longleftrightarrow \exists\, \eta : F \xrightarrow{\sim} F' \text{ s.t. } \vartheta = \vartheta' \circ i^*(\eta) \tag{6.10}$$

and we denote by $[F, \vartheta]$ the isomorphism classes with respect to this equivalence relation. In order to study first order deformations, we apply the deformation functor on $\mathbb{C}[\varepsilon]$.

Let (F, ϑ) be a representative of $\mathscr{F}_E(\mathbb{C}[\varepsilon])$ and let $(U_\alpha[\varepsilon], f_\alpha)$ be a representative G-structure for F, then we can write the associated cocycles of F as

$$f_{\alpha\beta} = g_{\alpha\beta} + \varepsilon h_{\alpha\beta}, \quad \text{for cocycles } (h_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathfrak{sl}_2(E))), \tag{6.11}$$

where U_α is the reduction of $(U_\alpha[\varepsilon])$ to an affine cover of C:

$$U_\alpha[\varepsilon] = Spec(\mathbb{C}[\varepsilon]) \times U_\alpha \text{ with } U_\alpha = Spec(A_\alpha), \text{ for some } A_\alpha \tag{6.12}$$

and

$$(g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G)_{\alpha\beta} \tag{6.13}$$

is a cocycle associated to a representative G-structure of E. Moreover we show that any Lie algebra-valued 1-cocycle $h_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathfrak{sl}_2(E))$ ($h_{\alpha\beta} + h_{\beta\gamma} = h_{\alpha\gamma}$) on $U_{\alpha\beta\gamma}$, gives rise to a G-bundle automorphism of F that reduces to the identity over the closed subscheme

$$C \hookrightarrow C \times_{\mathbb{C}} Spec(\mathbb{C}[\varepsilon]). \tag{6.14}$$

For this, consider the map

$$\Gamma(U, \mathfrak{sl}_2(E)) \xrightarrow{\psi} \Gamma(U[\varepsilon], F) \tag{6.15}$$

$$s \longmapsto (1 + \varepsilon s) \tag{6.16}$$

for which $\psi(s_1 + s_2) = \psi(s_1) \cdot \psi(s_2)$ holds. Then for a cocycle $h_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathfrak{sl}_2(E))$ and $f_{\alpha\beta} := \psi(h_{\alpha\beta}) = 1 + \varepsilon h_{a\beta}$, we have

$$f_{\alpha\beta} \cdot f_{\beta\gamma} = \psi(h_{\alpha\beta}) \cdot \psi(h_{\beta\gamma}) = \psi(h_{\alpha\beta} + h_{\beta\gamma}) = \psi(h_{\alpha\gamma}) = f_{\alpha\gamma},$$

i.e. $f_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, G_{C[\epsilon]})$ is a group-valued 1-cocycle. Thereby, isomorphism classes of first-order deformations of the G-bundle E are parametrised by $H^1(C, \mathfrak{sl}_2(E))$. □

Corollary 6.1.8. *The cotangent space of $\mathcal{M}_C(E)$ at E is identified via Serre duality with*

$$T^*_E \mathcal{M}_C(E) = H^0(C, \mathfrak{sl}_2(E) \otimes \omega_C), \tag{6.17}$$

after canonically identifying $\mathfrak{sl}_2(E)$ with its dual via the Killing form on $\mathfrak{sl}_2$ (cf.6.1.23).

Definition 6.1.9. **$SL_2\mathbb{C}$-Higgs Bundles**
Let $(C,E) \in \mathcal{M}(C,E)$. We call a section $\Phi \in H^0(C, \mathfrak{sl}_2(E) \otimes \omega_C)$ an *$SL_2(\mathbb{C})$-Higgs field* on E and we refer to the pair (E,Φ) as an *$SL_2(\mathbb{C})$-Higgs bundle* on the curve C.

Definition 6.1.10. Level Structure
If $(C,E) \in \mathcal{M}(C,E)$, then a trivialisation of E over an effective divisor D

$$\eta : E|_D \xrightarrow{\sim} \mathcal{O}_D^{\oplus 2} \tag{6.18}$$

is called a *D-level structure* on E, cf. [63].

Lemma 6.1.11. ***Deformations of D-Level Structures***
The first order deformation space of the space of D-Level Structures $\mathcal{M}_C(E,\eta)$ of $SL_2(\mathbb{C})$-bundles on a fixed curve C is given by

$$T_{(E,\eta)}\mathcal{M}_C(E,\eta) = H^1(C, \mathfrak{sl}_2(E) \otimes \mathcal{O}_C(-D)). \tag{6.19}$$

Proof. Let E be as above, with G-structure (U_α, g_α), such that $g_\alpha|_D = \eta \ \forall \alpha$ with $D \subset U_\alpha$. In analogy to lemma 6.1.7, we consider infinitesimal deformations of the associated cocycles

$$g_{\alpha\beta} \mapsto g_{\alpha\beta} + \varepsilon h_{\alpha\beta}, \ \ h_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathfrak{sl}_2(E)), \tag{6.20}$$

only now we impose the condition that $(g_{\alpha\beta} + \varepsilon h_{\alpha\beta})|_D$ respects η, i.e. $h_{\alpha\beta}|_D = 0$. Then, from the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(-D) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_D \longrightarrow 0$$

we obtain the following LES in cohomology

$$\cdots \longrightarrow H^1(C, \mathfrak{sl}_2(E) \otimes \mathcal{O}_C(-D)) \longrightarrow H^1(C, \mathfrak{sl}_2(E) \otimes \mathcal{O}_C) \longrightarrow H^1(C, \mathfrak{sl}_2(E) \otimes \mathcal{O}_D), \tag{6.21}$$

where the last map is the restriction of 1-cocycles to the 0 dimensional subscheme associated to D, which vanishes identically and therefore the deformations are unobstructed. The deformation space is then identified with cohomology group $H^1(C, \mathfrak{sl}_2(E) \otimes \mathcal{O}_C(-D))$. □

Note that $T^*_E\mathcal{M}_C(E) = H^0(C, \mathfrak{sl}_2(E) \otimes \omega_C) < T^*_{(E,\eta)}\mathcal{M}_C(E,\eta) = H^0(C, \mathfrak{sl}_2(E) \otimes \omega_C(D))$

6.1.3. D-Twisted Higgs Bundles

Definition 6.1.12. D-Twisted $SL_2(\mathbb{C})$-Higgs Bundles
If $(C,E) \in \mathcal{M}(C,E)$, and D an effective divisor on C, then a *D-twisted $SL_2(\mathbb{C})$-Higgs field*

is a section

$$\Phi \in H^0(C, \mathfrak{sl}_2(E) \otimes \omega_C(D)). \tag{6.22}$$

We call the pair (E, Φ) a *D-twisted $SL_2(\mathbb{C})$-Higgs Bundle* on C.

Definition 6.1.13. The Hitchin Map on D-Twisted $SL_2(\mathbb{C})$-Higgs Bundles
For $(C, E) \in \mathcal{M}(C, E)$, we define a map on the space of D-twisted Higgs fields on E

$$\begin{aligned} \chi_{(C,E)} : H^0(C, \mathfrak{sl}_2(E) \otimes \omega_C(D)) &\longrightarrow \mathcal{M}_C(Q) = H^0(C, \omega_C(D)^{\otimes 2}) \\ \Phi &\longmapsto \frac{1}{2} tr(\Phi^2) \end{aligned} \tag{6.23}$$

where Φ^2 denotes the composition

$$\Phi^2 : E \xrightarrow{\Phi} E \otimes \omega_C(D) \xrightarrow{\Phi \otimes id} E \otimes \omega_C(D)^{\otimes 2}, \tag{6.24}$$

rather than the Lie-product on $\mathfrak{sl}_2(E)$, therefore $tr(\Phi^2)$ is generically non-zero. This map gives rise fibre-wise to a morphism

$$\begin{aligned} \chi : \mathcal{M}(C, E, \Phi) &\longrightarrow \mathcal{M}(C, Q) \\ (C, E, \Phi) &\longmapsto (C, \chi_{(C,E)}(\Phi)), \end{aligned} \tag{6.25}$$

acting as the forgetful map on the datum of the $SL_2(\mathbb{C})$-bundle E. We call χ the *Hitchin map* on D-Twisted $SL_2(\mathbb{C})$-Higgs Bundles.

Remark 6.1.14. Consider the $\mathbb{C}^*$-action on $\mathcal{M}_{C,E}(\Phi) = H^0(C, \mathfrak{sl}_2(E) \otimes \omega_C(D))$ induced by the rescaling action on 1-forms, i.e. $\Phi \mapsto \epsilon \cdot \Phi$, as well as the $\mathbb{C}^*$-action on the space of quadratic differentials induced in the same way $Q \longmapsto \epsilon^2 \cdot Q$. Then the Hitchin map is $\mathbb{C}^*$-equivariant with respect to these actions, i.e.

$$\chi_{(C,E)}(\epsilon \Phi) = \frac{1}{2} tr(\epsilon^2 \Phi^2) = \epsilon^2 \chi_{(C,E)}(\Phi) \tag{6.26}$$

Definition 6.1.15. $SL_2(\mathbb{C})$- Higgs Bundles on Wild Curves
Let $\mathscr{C} = (C, D, \nu, \xi)$ be a wild curve. An $SL_2(\mathbb{C})$-Higgs bundle on $\mathscr{C}$ is a pair of an SL_2-bundle E and a D-twisted Higgs field $\Phi \in H^0(C, \mathfrak{sl}_2(E) \otimes \omega_C(D))$ on C such that $\chi(\Phi)|_D = \nu$. Via this definition, we refine the space $\mathcal{M}(C, E, \Phi)$ to $\mathcal{M}(\mathscr{C}, E, \Phi)$ and the Hitchin fibration to $\chi : \mathcal{M}(\mathscr{C}, E, \Phi) \to \mathcal{M}(\mathscr{C}, Q)$.

Recall that, as we have demonstrated before, $\chi(\Phi|_D) = \chi(\Phi)|_D$, cf. proposition 5.2.4.

Definition 6.1.16. Irregular $SL_2(\mathbb{C})$ Flat Connections
For $(C, E) \in \mathcal{M}(C, E)$, an irregular $SL_2(\mathbb{C})$ flat connection on E with poles along an

effective divisor D on C is a $\mathbb{C}$-linear map

$$\nabla : E \longrightarrow E \otimes \omega_C(D), \tag{6.27}$$

such that $\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s$.

Definition 6.1.17. Residue of Irregular Connections
Given an irregular $SL_2(\mathbb{C})$ flat connection $\nabla : E \longrightarrow E \otimes \omega_C(D)$, we call the $\mathcal{O}_D$-linear map $\nabla|_D \in H^0(D, \mathfrak{sl}_2(E) \otimes \omega_C(D)|_D)$ defined by

$$\begin{tikzcd} 0 \arrow[r] & E \otimes \mathcal{O}(-D) \arrow[r] & E \arrow[r] \arrow[d, "\nabla"] & E|_D \arrow[r] \arrow[d, dashed, "\nabla|_D"] & 0 \\ 0 \arrow[r] & E \otimes \omega_C \arrow[r] & E \otimes \omega_C(D) \arrow[r] & E \otimes \omega_C(D)|_D \arrow[r] & 0, \end{tikzcd} \tag{6.28}$$

the residue of ∇ at D. In particular the residue of a flat connection and the residue of a Higgs field take values in the same ring, and so, in exact analogy to the Higgs fields, we define irregular connections on the wild curve.

6.1.4. Irregular Quasi-Parabolic Bundles

Fix a pair (C, D) of an algebraic curve and an effective divisor $D \subset C$.

Definition 6.1.18. Irregular Quasi-Parabolic Bundles on (C, D)
Let E be a G-bundle and $L_D \overset{\iota}{\hookrightarrow} E|_D$ a rank 1 $\mathcal{O}_D$-submodule of $E|_D$. We say that a G-structure $(U_\alpha, g_\alpha)_\alpha$ on E, is compatible with the quasi-parabolic structure $L_D < E|_D$, if the associated cocycles preserve $L_D \subset E|_D$, that is

$$g_{\alpha\beta}(g_\alpha(L_D)) \subset g_\alpha(L_D), \ \ \forall\, \alpha, \beta \ s.t. \ D \subset U_\alpha \cap U_\beta. \tag{6.29}$$

We call a pair (E, L_D) endowed with a class of L_D-compatible G-structures on E an *irregular quasi-parabolic G-bundle* on (C, D).

We formalise the condition on cocycles above to the notion of a global quasi-parabolic endomorphism as follows.

Definition 6.1.19. Endomorphisms of Irregular Quasi-Parabolic Bundles
Let (E, L_D) be an irregular quasi-parabolic bundle. We say that an endomorphism of the bundle $\varphi \in H^0(C, End(E))$ is quasi-parabolic with respect to the filtration $L_D \subset E|_D$, if

$$\varphi|_D(L_D) \subset L_D, \tag{6.30}$$

i.e. $\varphi|_D$ preserves L_D as an $\mathcal{O}_D$-submodule of $E|_D$. We denote the algebra of such endomorphisms by $End^{\mathfrak{p}}_{L_D}(E) \subset End(E)$.

Definition 6.1.20. Nilpotent Endomorphisms of Irregular Bundles
Let (E, L_D) be an irregular quasi-parabolic bundle. We say that an endomorphism of the bundle $\varphi \in H^0(C, End(E))$ is nilpotent with respect to the filtration $L_D \subset E|_D$, if

$$\varphi|_D(E_D) \subset L_D \tag{6.31}$$

$$\varphi|_D(L_D) = 0. \tag{6.32}$$

In particular, it follows that $\varphi|_D \in H^0(D, End(E)|_D)$ is 2-nilpotent, i.e. $\varphi|_D^2 = 0$. We denote the algebra of such nilpotent endomorphisms by $End^{\mathfrak{n}}_{L_D}(E) \subset End(E)$.

Lemma 6.1.21. ***Characterisation of Irregular Nilpotent Endomorphisms***
An endomorphism $\varphi \in H^0(C, End(E))$ is nilpotent with respect to the filtration $L_D \subset E|_D$, if and only if it is parabolic with respect to the filtration $L_D \subset E|_D$ and its restriction to D is 2-nilpotent, that is

$$\varphi \in End^{\mathfrak{n}}_{L_D}(E) \iff [\varphi \in End^{\mathfrak{p}}_{L_D}(E) \text{ and } \varphi|_D^2 = 0]. \tag{6.33}$$

Proof. Let $\varphi \in End^{\mathfrak{n}}_{L_D}(E)$. Tautologically we have $\varphi|_D(L_D) = 0 \subset L_D$ and

$$\varphi|_D^2(E) \subset \varphi(L|_D)L_D = 0.$$

In the other direction, let $\varphi \in End^{\mathfrak{p}}_{L_D}(E)$ with nilpotent polar part $\varphi^2|_D = 0$ and assume that the restriction is not vanishing identically, i.e. $\varphi|_D \neq 0$. Since L_D is a rank 1 $\mathcal{O}_D$-module and the restriction $\varphi|_D$ is 2-nilpotent, either $\varphi|_D(L_D) = 0$ or $\varphi|_D(L_D) \oplus L_D \cong E|_D$, but since $\varphi|_D(L_D) \subset L_D$, we have $\varphi|_D(L_D) \overset{!}{=} 0$. Further, since $\varphi|_D^2 = 0$, the image $\varphi|_D(E|_D) =: L'_D$ is a line with $\varphi|_D(L'_D) = 0$, thereby $L'_D \subset L_D$ and the result follows. ☐

Lemma 6.1.22. ***Deformations of Irregular Quasi-Parabolic Bundles***
The tangent space to the space of quasi-parabolic bundles $\mathcal{M}_C(E, L_D)$, where $E \in \mathcal{M}_C(E)$ and $L_D < E|_D$ an $\mathcal{O}_D$-submodule of $E|_D$, is given by $H^1(C, End^{\mathfrak{p}}_{L_D}(E))$.

Proof. Let E be as above with an L_D-compatible G-structure $(U_\alpha, g_\alpha)_\alpha$. Consider infinitesimal deformations of the G-structure and the associated cocycles

$$g_{\alpha\beta} \mapsto g_{\alpha\beta} + \varepsilon h_{\alpha\beta},\ \ h_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathfrak{sl}_2(E)), \tag{6.34}$$

such that $(g_{\alpha\beta} + \varepsilon h_{\alpha\beta})|_D$ respects the induced deformation of the parabolic structure L_D in the following way

$$(g_{\alpha\beta} + \varepsilon h_{\alpha\beta})((g_\alpha + h_\alpha)L_D) \subset (g_\alpha + h_\alpha)L_D,\ \forall \alpha\beta \text{ with } D \subset U_{\alpha\beta}. \tag{6.35}$$

The cocycle $h_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, End^{\mathfrak{p}}_{L_D}(E))$ is *triangular to order* n in the basis given by L_D and its orthogonal complement over D. The resulting deformation space is $H^1(C, End^{\mathfrak{p}}_{L_D}(E))$. □

Remark 6.1.23. Let R be a commutative algebra over $\mathbb{C}$. Under the duality induced by the Killing form on $\mathfrak{sl}_2(R)$

$$\begin{aligned} B : \mathfrak{sl}_2(R) \times \mathfrak{sl}_2(R) &\longrightarrow R \\ (x, y) &\longmapsto 4 \cdot Tr(x \cdot y), \end{aligned}$$

the direct summands of the Cartan decomposition $\mathfrak{sl}_2 = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ map to

$$\mathfrak{n}_+ \to \mathfrak{n}_-, \quad \mathfrak{h} \to \mathfrak{h}, \quad \mathfrak{n}_- \to \mathfrak{n}_+.$$

Explicitly, given the standard $\mathfrak{sl}_2$-triple e, f, h, i.e. $[h, e] = 2e; [h, f] = -2f; [e, f] = h$, we have

$$e^* = \frac{f}{4}, f^* = \frac{e}{4}, h^* = \frac{h}{8} \tag{6.36}$$

for $x^* := B(x, \cdot)$.

Lemma 6.1.24. *Let (E, L_D) be an irregular quasi-parabolic bundle. Under the duality induced by the Killing form on $\mathfrak{sl}_2(E)$, we have*

$$(End^{\mathfrak{p}}_{L_D}(E))^* \cong End^{\mathfrak{n}}_{L_D}(E) \otimes \mathcal{O}_C(D) \tag{6.37}$$

Proof. Since $\mathfrak{sl}_2(E)$ is (globally) a self-dual sheaf, and we want to show the duality between quasi-parabolic and quasi-nilpotent endomorphisms, we only need to prove the duality locally at a neighbourhood of the divisor D. Let $\varphi \in \Gamma(U, End^{\mathfrak{p}}_{L_D}(E))$. In the local basis prescribed by the quasi-parabolic structure, φ is of the form

$$\varphi = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} \text{ where } \alpha, \beta \in \Gamma(U, \mathcal{O}_C), \gamma \in \Gamma(U, \mathcal{O}_C(-D)).$$

By the Killing duality presented above 6.1.23 and $\mathcal{O}_C(-D) \cong \mathcal{O}_C(D)^*$, we obtain a dual section $\varphi^* \in \Gamma(U, (End^{\mathfrak{p}}_{L_D}(E))^*)$

$$\varphi^* = \begin{bmatrix} \tilde{\alpha} & \tilde{\gamma} \\ \tilde{\beta} & -\tilde{\alpha} \end{bmatrix} \text{ where } \tilde{\alpha}, \tilde{\beta} \in \Gamma(U, \mathcal{O}_C), \tilde{\gamma} \in \Gamma(U, \mathcal{O}_C(D)),$$

which implies $\varphi^* \in \Gamma(U, End^{\mathfrak{n}}_{L_D}(E) \otimes \mathcal{O}_C(D))$. □

Corollary 6.1.25. *Composing the duality on parabolic endomorphisms* (6.37) *with Serre* *duality we have:*

$$H^1(C, End^{\mathfrak{p}}_{L_D}(E)) \cong H^0(C, End^{\mathfrak{n}}_{L_D}(E) \otimes \omega_C(D))^*. \tag{6.38}$$

6.2. Wild Quasi-Parabolic Higgs Bundles and Flat Connections

Let $\mathscr{C} = (C, D, \nu, \xi)$ be a wild curve and $E \in \mathcal{M}_{\mathscr{C}}(E)$ an $SL_2(\mathbb{C})$-bundle on C. Recall from definition 5.1.5, that $\mathscr{C}$ comes equipped with a cover $p_\nu : D_\nu \to D$. When we write $p_\nu^* E$ we mean $p_\nu^*(E|_D)$ and the same shorthand is used for any other structure defined on C, i.e. $p_\nu^*(\cdot) := p_\nu^*(\cdot|_D)$.

Given a wild curve $\mathscr{C}$, we lift the notion of an irregular quasi-parabolic structure on E to that of a *wild* quasi-parabolic structure on E. This is an irregular quasi-parabolic structure on the pullback of E over the divisor D_ξ associated to the wild curve. An $\mathcal{O}_{D_\xi}$-Submodule $L_\xi < p_\nu^*(E)|_{D_\xi}$. We then define *wild-quasi-parabolic Higgs fields* as D-twisted Higgs fields on E, whose spectral data respect the wild structure of the bundle and the curve. We show that these Higgs fields arise naturally as first-order-deformations of wild quasi-parabolic structures. Analogously, we define *wild quasi-parabolic flat bundles.* Lastly, we compute the first-order-deformations of wild quasi-parabolic Higgs bundles and flat bundles.

Definition 6.2.1. Wild Quasi-Parabolic Structure on a Bundle
Let $\mathscr{C} \in \mathcal{M}(\mathscr{C})$ and E an $SL_2(\mathbb{C})$-bundle. A *wild quasi-parabolic structure* on E is an $\mathcal{O}_{D_\xi}$-Submodule $L_\xi < p_\nu^*(E)|_{D_\xi}$. We call the pair $\mathscr{E} := (E, L_\xi)$ a *wild Quasi-Parabolic Bundle* on the wild curve $\mathscr{C}$, and write $\mathcal{M}(\mathscr{C}, \mathscr{E})$ for the space of wild quasi-parabolic bundles on wild curves.

Now we transfer the notions of parabolic and nilpotent endomorphisms of irregular quasi-parabolic bundles of definitions 6.1.19 and 6.1.20, to the wild parabolic setting, asking them to preserve the wild quasi-parabolic structure of a bundle E on C.

Definition 6.2.2. Endomorphisms of Wild Quasi-Parabolic Bundles
Let $\mathscr{E} = (E, L_\xi)$ be a wild quasi-parabolic bundle on the wild curve $\mathscr{C}$. We say that an endomorphism $\varphi \in End(E)$ is *wild parabolic* with respect to $L_\xi < p_\nu^*(E)|_{D_\xi}$, if

$$p_\nu^* \varphi|_{D_\xi}(L_\xi) \subset L_\xi. \tag{6.39}$$

We denote the subalgebra of wild parabolic endomorphisms by $End^{\mathfrak{p}}(\mathscr{E}) < End(E)$. Moreover, we say that an endomorphism φ is *wild nilpotent* with respect to $L_\xi < p_\nu^*(E)|_{D_\xi}$, if

$$p_\nu^* \varphi|_{D_\xi}(p_\nu^*(E)|_{D_\xi}) \subset L_\xi \tag{6.40}$$

$$p_\nu^* \varphi|_{D_\xi}(L_\xi) = 0. \tag{6.41}$$

We denote the subalgebra of wild nilpotent endomorphisms by

$$\mathit{End}^{\mathfrak{n}}(\mathscr{E}) < \mathit{End}^{\mathfrak{p}}(\mathscr{E}) < \mathit{End}(E).$$

Corollary 6.2.3. ***(to Lemma 6.1.21)*** *The analogous characterising lemma to 6.1.21 holds for wild nilpotent endomorphisms, that is*

$$\varphi \in \mathit{End}^{\mathfrak{n}}(\mathscr{E}) \iff [\varphi \in \mathit{End}^{\mathfrak{p}}(\mathscr{E}) \text{ and } (p_\nu^*\varphi \circ p_\nu^*\varphi)|_{D_\xi} = 0]. \tag{6.42}$$

Before passing to Higgs bundles, recall from 5.1.11, that a wild datum (D, ν, ξ) of a wild curve $\mathscr{C}$ uniquely determines a section $\lambda_\xi \in H^0(D_\xi, \omega_\Sigma(D_\nu)|_{D_\xi})$, which we refer to as the irregular type of the wild curve.

Definition 6.2.4. Wild Quasi-Parabolic Higgs Bundles and Flat Connections Let (E, Φ) an irregular Higgs bundle on $\mathscr{C} = (C, D, \nu, \xi)$, cf. definition 6.1.15, and let L_ξ be a wild quasi-parabolic structure on E. We say that Φ is a *wild quasi-parabolic* Higgs field on $\mathscr{E} = (E, L_\xi)$, if

$$p_\nu^*\Phi|_{D_\xi}(s) = \lambda_\xi \cdot s, \ \forall s \in L_\xi. \tag{6.43}$$

In particular, $\Phi \in H^0(C, \mathit{End}^{\mathfrak{p}}(\mathscr{E}) \otimes \omega_C(D))$, i.e. $p_\nu^*\Phi|_{D_\xi}(L_\xi) \subset L_\xi \otimes \omega_\Sigma(D_\xi)|_{D_\xi}$. In that case, we call the residual eigenvalue λ_ξ of Φ the *irregular type* of Φ and we call the tuple $(\mathscr{E}, \Phi)$ a *wild quasi-parabolic Higgs Bundle on* $\mathscr{C}$.

Analogously, a flat bundle (E, ∇) on $\mathscr{C}$ is said to be wild quasi-parabolic on $\mathscr{E}$, if

$$p_\nu^*\nabla|_{D_\xi}(s) = \lambda_\xi \cdot s, \ \forall s \in L_\xi. \tag{6.44}$$

In particular, $p_\nu^*\nabla|_{D_\xi}(L_\xi) \subset L_\xi \otimes \omega_\Sigma(D_\xi)|_{D_\xi}$. We call such a tuple $(\mathscr{E}, \nabla)$ a *wild quasi-parabolic Flat Bundle* on $\mathscr{C}$ and the value λ_ξ the *irregular type of* ∇.

A flat bundle (E, ∇_∞) is said to be *wild nilpotent* on $\mathscr{E}$, if

$$p_\nu^*\nabla_\infty|_{D_\xi}(p_\nu^*(E)|_{D_\xi}) \subset L_\xi \tag{6.45}$$

$$p_\nu^*\nabla_\infty|_{D_\xi}(L_\xi) = 0. \tag{6.46}$$

We denote the space of wild nilpotent flat bundles by $\mathscr{M}(\mathscr{C}, \mathscr{E}, \nabla_\infty)$.

Lastly, we denote by $\mathscr{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla)$ the space of wild quasi-parabolic Higgs fields and flat connections on the same wild bundle $\mathscr{E}$ and wild curve $\mathscr{C}$. Analogously, $\mathscr{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla_\infty)$ denotes the space parametrising wild quasi-parabolic Higgs fields and nilpotent flat connections on the same wild bundle and wild curve.

Remark 6.2.5. Consider a Higgs field Φ with Hitchin-image $\chi(\Phi) = Q$ and associated spectral cover $p : \Sigma \to C$. Then, we have $p^*\Phi|_{D_\nu} = p_\nu^*\Phi$ and in particular, $p^*\Phi|_{D_\xi} = p_\nu^*\Phi|_{D_\xi}$. This follows immediately from the functoriality of the spectral curve construction, cf. proposition 4.3.10.

Lemma 6.2.6. *Let* $(\mathscr{C}, E, Q) \in \mathcal{M}(\mathscr{C}, E, Q)$*, denote by* $p : \Sigma \longrightarrow C$ *the associated spectral cover and consider an endomorphism* $\varphi \in H^0(C, \mathfrak{sl}_2(E))$*. Then,* $p^*\varphi$ *preserves a wild quasi-parabolic structure* $L_\xi < p_\nu^*(E)|_{D_\xi}$ *on* E*, if and only if* $p^*\varphi$ *preserves* $\sigma^*L_\xi < p_\nu^*(E)|_{\sigma(D_\xi)}$*:*

$$p^*\varphi|_{D_\xi}(L_\xi) \subset L_\xi \iff p^*\varphi|_{\sigma(D_\xi)}(\sigma^*L_\xi) \subset \sigma^*L_\xi. \tag{6.47}$$

Proof. Follows directly from the σ-invariance of the pullback section $p^*\varphi \in H^0(\Sigma, \mathfrak{sl}(p^*E))^\sigma$. $\square$

Remark 6.2.7. Here we present explicitly what the residues of the wild connections look like in a neighbourhood of a pole on the spectral cover. Let $(\mathscr{C}, \mathscr{E}, \Phi, \nabla_\infty)$ be a point in $\mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla_\infty)$. For the pullback $p^*\nabla_\infty$ of ∇_∞ to the spectral cover $p : \Sigma \to C$ determined by $(\mathscr{C}, \chi(\Phi)) \in \mathcal{M}(\mathscr{C}, Q)$ expressed in the Eigenbasis of Φ, we have

Unramified Case

$$B^{-1}p^*\nabla_\infty B|_{D_\xi} = B^{-1}dB|_{D_\xi} + \begin{bmatrix} 0 & \beta(w) \\ 0 & 0 \end{bmatrix} \frac{dw}{w^n};\ \beta(w) \in \mathbb{C}[w]/w^n \tag{6.48}$$

and

$$B^{-1}p^*\nabla_\infty B|_{\sigma^*D_\xi} = B^{-1}dB|_{\sigma^*D_\xi} + \begin{bmatrix} 0 & 0 \\ \beta(w) & 0 \end{bmatrix} \frac{dw}{w^n};\ \beta(w) \in \mathbb{C}[w]/w^n \tag{6.49}$$

Ramified Case

$$B^{-1}p^*\nabla_\infty B|_{D_\nu} = B^{-1}dB|_{D_\nu} + \begin{bmatrix} 0 & \beta(w) \\ -\beta(-w) & 0 \end{bmatrix} \frac{dw}{w^{2n}};\ \beta(w) \in w^n\mathbb{C}[w]/w^{2n} \tag{6.50}$$

and

$$B^{-1}p^*\nabla_\infty B|_{D_\xi} = B^{-1}dB|_{D_\xi}. \tag{6.51}$$

Analogously, for $(\mathscr{C}, \mathscr{E}, \Phi, \nabla) \in \mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla)$, the residue of $p^*\nabla$ in the Eigenbasis of Φ reads:

Unramified Case

$$B^{-1}p^*\nabla B|_{D_\xi} = B^{-1}dB|_{D_\xi} + \begin{bmatrix} \lambda_\xi(w) & \frac{\beta(w)}{w^n} \\ 0 & -\lambda_\xi(w) \end{bmatrix} dw;\ \beta(w) \in \mathbb{C}[w]/w^n \tag{6.52}$$

and

$$B^{-1}p^*\nabla B|_{\sigma^* D_\xi} = B^{-1}dB|_{\sigma^* D_\xi} + \begin{bmatrix} \lambda_\xi(w) & 0 \\ \frac{\beta(w)}{w^n} & -\lambda_\xi(w) \end{bmatrix} dw;\ \beta(w) \in \mathbb{C}[w]/w^n \tag{6.53}$$

<u>Ramified Case</u>

$$B^{-1}p^*\nabla B|_{D_\nu} = B^{-1}dB|_{D_\nu} + \begin{bmatrix} \lambda_\nu(w) & \frac{\beta(w)}{w^{2n}} \\ -\frac{\beta(-w)}{w^{2n}} & -\lambda_\nu(w) \end{bmatrix} dw;\ \beta(w) \in w^n\mathbb{C}[w]/w^{2n} \tag{6.54}$$

and

$$B^{-1}p^*\nabla B|_{D_\xi} = B^{-1}dB|_{D_\xi} + \begin{bmatrix} \lambda_\xi(w) & 0 \\ 0 & -\lambda_\xi(w) \end{bmatrix} dw;\ \beta(w) \in w^n\mathbb{C}[w]/w^{2n}. \tag{6.55}$$

Our definition of wild quasi-parabolic Higgs bundles was inspired by [78, Chapters 5 and 6], and was constructed, such that they can be expressed deformation-theoretically as cotangent vectors to wild quasi-parabolic bundles.

Theorem 6.2.8. *The space $\mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi)$ of wild quasi-parabolic Higgs bundles is an affine bundle modelled on $T^*_{(\mathscr{E})}\mathcal{M}_{\mathscr{C}}(\mathscr{E}) = H^0(C, End^{\mathfrak{p}}(E)^*)$. In other words, the fibres $\mathcal{M}_{(\mathscr{C},\mathscr{E})}(\Phi)$ of the forgetful projection*

$$\mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi) \longrightarrow \mathcal{M}(\mathscr{C}, \mathscr{E}) \tag{6.56}$$

are isomorphic to the cotangent space of wild bundles $T^*_{(\mathscr{E})}\mathcal{M}_{\mathscr{C}}(\mathscr{E})$.

Proof. By direct analogy to lemma 6.1.22, we have $T_{(\mathscr{E})}\mathcal{M}_{\mathscr{C}}(\mathscr{E}) = H^1(C, End^{\mathfrak{p}}(\mathscr{E}))$ and via Serre duality $T^*_{(\mathscr{E})}\mathcal{M}_{\mathscr{C}}(\mathscr{E}) = H^0(C, End^{\mathfrak{p}}(\mathscr{E})^* \otimes \omega_C)$. We only need to see that the difference of any two Higgs fields in the fibre $\mathcal{M}_{(\mathscr{C},\mathscr{E})}(\Phi)$ lies in $H^0(C, End^{\mathfrak{n}}(\mathscr{E}) \otimes \omega_C(D))$. This follows by unpacking the definitions. By fixing the irregular type of the curve and the wild quasi-parabolic datum of $\mathscr{E}$, we fix the residual eigenvalues λ_ξ and the positive eigensheaf L_ξ of all Higgs fields in $\mathcal{M}_{(\mathscr{C},\mathscr{E})}(\Phi)$. Hence, their difference yields a nilpotent endomorphism $\Phi_1 - \Phi_2 \in H^0(C, End^{\mathfrak{n}}(\mathscr{E}) \otimes \omega_C(D))$, $\forall \Phi_1, \Phi_2 \in \mathcal{M}_{(\mathscr{C},\mathscr{E})}(\Phi)$, in particular, $\Phi_1|_D - \Phi_2|_D \in H^0(D, End^{\mathfrak{n}}(\mathscr{E}) \otimes \omega_C(D)|_D)$. □

6.2.1. Deformation Theory of Wild Higgs and Flat Bundles

Now we want to study the deformation theory of irregular Higgs bundles and flat connections on wild quasi-parabolic bundles. The two are governed by the same deformation space. For this, we generalise the deformation space calculation from the holomorphic setting of [7, Theorem 2.3] to wild quasi-parabolic Higgs bundles on wild curves.

Theorem 6.2.9. ***Deformations of wild Quasi-Parabolic Higgs and Flat Bundles*** *The tangent spaces of* $\mathcal{M}_{\mathscr{C}}(\mathscr{E},\Phi)$ *and* $\mathcal{M}_{\mathscr{C}}(\mathscr{E},\nabla)$ *are described by the first hypercohomology* $\mathbb{H}^1(\mathcal{C}^\bullet_*)$ *of the complexes*

$$\mathcal{C}^\bullet_\Phi : End^{\mathfrak{p}}(\mathscr{E}) \xrightarrow{ad\Phi} End^{\mathfrak{n}}(\mathscr{E})\otimes\omega_C(D) \tag{6.57}$$

$$\mathcal{C}^\bullet_\nabla : End^{\mathfrak{p}}(\mathscr{E}) \xrightarrow{\nabla\otimes\nabla^*} End^{\mathfrak{n}}(\mathscr{E})\otimes\omega_C(D), \tag{6.58}$$

where $ad_\Phi(f) = [\Phi, f]$ *and* $\nabla\otimes\nabla^* := \nabla\otimes id + id\otimes\nabla^*$*:*

$$T_{(\mathscr{E},\Phi)}\mathcal{M}_{\mathscr{C}}(\mathscr{E},\Phi) \cong \mathbb{H}^1(\mathcal{C}^\bullet_\Phi)$$
$$T_{(\mathscr{E},\nabla)}\mathcal{M}_{\mathscr{C}}(\mathscr{E},\nabla) \cong \mathbb{H}^1(\mathcal{C}^\bullet_\nabla).$$

Proof. Let $(U_\alpha = Spec(A_\alpha))_\alpha$ be an affine cover of C. We proceed using the notation for deformations introduced in the proof of Lemma 6.1.7. First, consider an acyclic resolution complex $\mathcal{K}^{\bullet,\bullet}$ resolving $\mathcal{C}^\bullet_\nabla$:

$$\begin{tikzcd}
\mathcal{K}^{\bullet,\bullet} : & 0 \arrow[r] & End^{\mathfrak{p}}(\mathscr{E}) \arrow[r] \arrow[d] & End^{\mathfrak{n}}(\mathscr{E})\otimes\omega_C(D) \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & \bigoplus_\alpha \tilde{R}^{\mathfrak{p}}_\alpha \arrow[r, "d^{0,0}"] \arrow[d, "\delta^{0,0}"'] & \bigoplus_\alpha \tilde{R}^{\mathfrak{n}}_\alpha \otimes \Gamma(U_\alpha,\omega_C(D)) \arrow[r] \arrow[d, "\delta^{1,0}"'] & 0 \\
& 0 \arrow[r] & \bigoplus_{\alpha\beta} \tilde{R}^{\mathfrak{p}}_{\alpha\beta} \arrow[r, "d^{0,1}"] \arrow[d, "\delta^{0,1}"'] & \bigoplus_{\alpha\beta} \tilde{R}^{\mathfrak{n}}_{\alpha\beta} \otimes \Gamma(U_{\alpha\beta},\omega_C(D)) \arrow[r] \arrow[d, "\delta^{1,1}"'] & 0 \\
& 0 \arrow[r] & \bigoplus_{\alpha\beta\gamma} \tilde{R}^{\mathfrak{p}}_{\alpha\beta\gamma} \arrow[r, "d^{0,2}"] \arrow[d, "\delta^{0,2}"'] & \bigoplus_{\alpha\beta\gamma} \tilde{R}^{\mathfrak{n}}_{\alpha\beta\gamma} \otimes \Gamma(U_{\alpha\beta\gamma},\omega_C(D)) \arrow[r] \arrow[d, "\delta^{1,2}"'] & 0 \\
& & \vdots & \vdots &
\end{tikzcd} \tag{6.59}$$

for resolution modules: $\tilde{R}^{\mathfrak{p}}_\alpha = End^{\mathfrak{p}}(\mathscr{E})|_{U_\alpha}$, $\tilde{R}^{\mathfrak{n}}_\alpha = End^{\mathfrak{n}}(\mathscr{E})|_{U_\alpha}$, where $R^{\mathfrak{p}}_\alpha$,$R^{\mathfrak{n}}_\alpha$ are the associated A_α-modules. The horizontal differential $d^{p,q}$ is induced by $ad_\Phi(\cdot)$ (resp. $\nabla\otimes\nabla^*(\cdot)$) by appropriate restriction to sections on intersections and higher intersections. The vertical differential $\delta^{p,q}$ is the one defined by taking difference of sections, e.g. $\delta^{0,0}(r_\alpha, r_\beta) = r_\alpha - r_\beta$ for pairwise distinct α,β.

We compute the hypercohomology of the double complex as the cohomology of the associated total complex:

$$Tot^n(\mathcal{K}^{\bullet,\bullet}) := \bigoplus_{n=p+q} \mathcal{K}^{p,q};\ D^n := \sum_{n=p+q} [d^{p,q} + (-1)^p \delta^{p,q}]. \tag{6.60}$$

Explicitly we have:

$$\begin{tikzcd}
Tot^0(\mathscr{K}^{\bullet,\bullet}) \arrow[r, equal] \arrow[d, "D^0"] & \mathscr{K}^{0,0} \cong \bigoplus_\alpha \tilde{R}^{\mathfrak{p}}_\alpha \\
Tot^1(\mathscr{K}^{\bullet,\bullet}) \arrow[r, equal] \arrow[d, "D^1"] & \mathscr{K}^{1,0} \oplus \mathscr{K}^{0,1} \cong \bigoplus_{\alpha\beta} \tilde{R}^{\mathfrak{p}}_{\alpha\beta} \oplus \bigoplus_\alpha \tilde{R}^{\mathfrak{n}}_\alpha \otimes \Gamma(U_\alpha, \omega_C(D)) \\
Tot^2(\mathscr{K}^{\bullet,\bullet}) \arrow[r, equal] \arrow[d, "D^2"] & \mathscr{K}^{1,1} \oplus \mathscr{K}^{0,2} \cong \bigoplus_{\alpha\beta\gamma} \tilde{R}^{\mathfrak{p}}_{\alpha\beta\gamma} \oplus \bigoplus_{\alpha\beta} \tilde{R}^{\mathfrak{n}}_{\alpha\beta} \otimes \Gamma(U_{\alpha\beta}, \omega_C(D)) \\
\vdots & \ddots
\end{tikzcd} \tag{6.61}$$

Then, $\mathbb{H}^1(\mathcal{C}^\bullet) = H^1(Tot^\bullet(\mathscr{K}^{\bullet,\bullet})) = ker(D^1)/Im(D^0)$, and:

- $\underline{ker(D^1)}$ consists of pairs of vectors $(r_{\alpha\beta}, \varphi_\alpha)_{\alpha\beta} \in \tilde{R}^{\mathfrak{p}}_{\alpha\beta} \oplus \bigoplus_\alpha \tilde{R}^{\mathfrak{n}}_\alpha \otimes \Gamma(U_\alpha, \omega_C(D))$, such that:

    1. the cocycle conditions $r_{\alpha\gamma} = r_{\alpha\beta} + r_{\beta\gamma}$ hold in the underlying module $R^{\mathfrak{p}}_{\alpha\beta\gamma}$ for all α, β, γ,

    2. the equality $\varphi_\alpha - \varphi_\beta = ad_\Phi(r_{\alpha\beta})$ (resp. $\varphi_\alpha - \varphi_\beta = \nabla \otimes \nabla^*(r_{\alpha\beta})$) holds in $R^{\mathfrak{n}}_{\alpha\beta} \otimes \Gamma(U_{\alpha\beta}, \omega_C(D))$.

- $\underline{Im(D^0) < ker(D^1)}$ consists of pairs of vectors $(r_\alpha - r_\beta, d^{0,1}(r_\alpha) = ad_\Phi(r_\alpha))$ for elements $r_\alpha \in R^{\mathfrak{p}}_\alpha$ (resp. $(r_\alpha - r_\beta, d^{0,1}(r_\alpha) = \nabla \otimes \nabla^*(r_\alpha))$).

Now, we want to construct a map

$$\mathbb{H}^1(\mathcal{C}^\bullet_\Phi) \longrightarrow T_{(\mathscr{E},\Phi)}\mathcal{M}_{\mathscr{C}}(\mathscr{E}, \Phi) \tag{6.62}$$

and respectively for connections

$$\mathbb{H}^1(\mathcal{C}^\bullet_\nabla) \longrightarrow T_{(\mathscr{E},\nabla)}\mathcal{M}_{\mathscr{C}}(\mathscr{E}, \nabla). \tag{6.63}$$

Let $(r_{\alpha\beta}, \varphi_\alpha) \in ker(D^1)$ and let F be a deformation of E as described in Lemma 6.1.7:

- $U_\alpha[\epsilon] := U_\alpha \times Spec(\mathbb{C}[\varepsilon])$,
- $F_\alpha := F|_{U_\alpha[\varepsilon]}$, $\quad F_{\alpha\beta} := F|_{U_{\alpha\beta}[\varepsilon]}$ etc.,
- the cocycle $r_{\alpha\beta}$ gives rise to a group-valued cocycle $1 + \varepsilon r_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, G_{C[\epsilon]})$.

Moreover, φ_α deforms the Higgs field Φ locally by $\Phi_\alpha + \varepsilon\varphi_\alpha \in \Gamma(U_\alpha[\varepsilon], End^{\mathfrak{n}}_{L_\xi}(F) \otimes \omega_C(D))$ for $\Phi_\alpha := \Phi|_{U_\alpha}$. These deformations of Φ glue to a global one, if

$$Ad_{(1+\varepsilon r_{\alpha\beta})}(\Phi_\alpha + \varepsilon\varphi_\alpha) = \Phi_\beta + \varepsilon\varphi_\beta \text{ on } U_{\alpha\beta}[\varepsilon]. \tag{6.64}$$

This identity holds indeed

$$\begin{aligned} Ad_{(1+\varepsilon r_{\alpha\beta})}(\Phi_\alpha + \varepsilon\varphi_\alpha) &= (1+\varepsilon r_{\alpha\beta}) \cdot (\Phi_\alpha + \varepsilon\varphi_\alpha) \cdot (1+\varepsilon r_{\alpha\beta})^{-1} && (6.65)\\ &= (1+\varepsilon r_{\alpha\beta}) \cdot (\Phi_\alpha + \varepsilon\varphi_\alpha) \cdot (1-\varepsilon r_{\alpha\beta}) && (6.66)\\ &= \Phi_\alpha + \varepsilon\varphi_\alpha + \varepsilon[r_{\alpha\beta}, \Phi_\alpha] && (6.67)\\ &= \Phi_\alpha + \varepsilon\varphi_\alpha + \varepsilon\varphi_\beta - \varepsilon\varphi_\alpha && (6.68)\\ &= \Phi_\beta + \varepsilon\varphi_\beta && (6.69) \end{aligned}$$

whereby equality 6.66 can be seen as follows. Let $(1+\varepsilon s) = (1+\varepsilon r)^{-1}$ as elements of a module over the ring of duals $\mathbb{C}[\varepsilon]$, then

$$1 \overset{!}{=} (1+\varepsilon s) \cdot (1+\varepsilon r) = 1 + \varepsilon r + \varepsilon s \implies s \overset{!}{=} -r.$$

The equality 6.68 follows from the cocycle condition 2., and the last equality follows from $\Phi|_{U_{\alpha\beta}} = \Phi_\alpha|_{U_{\alpha\beta}} = \Phi_\beta|_{U_{\alpha\beta}}$.

Thereby, we have associated an infinitesimal deformation of a Higgs bundle (resp. an infinitesimal deformation of a flat connection) to a cocycle. It remains to show that if the cocycle is a coboundary $(r_{\alpha\beta}, \varphi_\alpha) \in Im(D^0)$, then the associated deformation is trivial. Recall that $(r_{\alpha\beta}, \varphi_\alpha) \in Im(D^0)$ is of the form $(r_{\alpha\beta}, \varphi_\alpha) = (r_\alpha - r_\beta, ad_\Phi(r_\alpha))$, in particular, we have for the gluing/transition map $1+\varepsilon r_{\alpha\beta} = 1 + \varepsilon(r_\alpha - r_\beta)$. Therefore, we get a commutative diagram of gluing maps

$$\begin{tikzcd} F_{\alpha\beta} \arrow[r, "1+\varepsilon r_\alpha|_{U_{\alpha\beta}[\varepsilon]}"] \arrow[d, "1+\varepsilon r_{\alpha\beta}"'] & F_{\alpha\beta} \arrow[d, "id"] \\ F_{\alpha\beta} \arrow[r, "1+\varepsilon r_\beta|_{U_{\alpha\beta}[\varepsilon]}"'] & F_{\alpha\beta}. \end{tikzcd} \tag{6.70}$$

The gluing by $1+\varepsilon r_{\alpha\beta}$ is the same as gluing by the identity. This means that, globally, the deformation bundle F is just the pullback of E, $F = \pi_\varepsilon^* E = E[\varepsilon]$, along the nilpotent infinitesimal thickening of the curve

$$\pi_\varepsilon : C \times_{\mathbb{C}} Spec(\mathbb{C}[\varepsilon]) \longrightarrow C. \tag{6.71}$$

Regarding the deformation of the Higgs field, we have

$$\begin{aligned} Ad_{(1+\varepsilon r_{\alpha\beta})}(\Phi_\alpha + \varepsilon\varphi_\alpha) &= \Phi_\alpha + \varepsilon\varphi_\alpha - \varepsilon ad_{\Phi_\alpha}(r_\alpha)\\ &= \Phi_\alpha + \varepsilon\varphi_\alpha - \varepsilon\varphi_\alpha = \Phi_\alpha, \end{aligned}$$

where the second equality follows from the coboundary condition, hence, this deformation is also trivial. Thereby, we have constructed a map

$$\mathbb{H}^1(\mathcal{C}^\bullet_\Phi) \longrightarrow T_{pt}\mathcal{M}_{\mathscr{C}}(\mathscr{E},\Phi), \tag{6.72}$$

and respectively

$$\mathbb{H}^1(\mathcal{C}^\bullet_\nabla) \longrightarrow T_{pt}\mathcal{M}_{\mathscr{C}}(\mathscr{E},\nabla). \tag{6.73}$$

Now we argue that this map is indeed an isomorphism. The inverse map is the obvious one. For a wild Higgs bundle (E,Φ) and a deformation of it (F,Φ') we have locally that

- $F_\alpha = \pi_\varepsilon^* E_\alpha$ with gluing maps determined by those of E and automorphisms $1+\varepsilon r_{\alpha\beta}$ satisfying the cocyle condition,
- $\Phi'_\alpha = \Phi_\alpha + \varepsilon\varphi_\alpha$.

and the condition for the $(\Phi'_\alpha)_\alpha$ to glue to a global section is given, as before, by

$$Ad_{(1+\varepsilon r_{\alpha\beta})}(\Phi'_\alpha) \overset{!}{=} \Phi'_\beta \implies ad_{\Phi_\alpha} \overset{!}{=} \varphi_\alpha - \varphi_\beta \ \ \forall \alpha,\beta, \tag{6.74}$$

which is precisely the second (2.) cocycle condition. Therefore we have that

$$(r_{\alpha\beta},\varphi_\alpha) \overset{!}{\in} ker(D^1). \tag{6.75}$$

This concludes the proof. □

Remark 6.2.10. Tangent Spaces of Fibre Products of Schemes
Let $X,Y,Z \in Sch^{sm}_{/\mathbb{C}}$ be smooth schemes over $\mathbb{C}$, in particular of finite type, and let the following diagram be Cartesian in $Sch^{sm}_{/\mathbb{C}}$

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} . \tag{6.76}$$

For $x \in X(\mathbb{C}), y \in Y(\mathbb{C})$ and $z \in Z(\mathbb{C})$ with $f(x) = g(y) = z$, let $p := (x,y) \in X \times_Z Y(\mathbb{C})$ be the geometric point associated to (x,y). For the tangent space at the point p, we have

$$\begin{aligned} T_p(X \times_Z Y) &= Hom(\mathbb{C}[\varepsilon], X \times_Z Y) \\ &= Hom(\mathbb{C}[\varepsilon], X) \times_{Hom(\mathbb{C}[\varepsilon],X)} Hom(\mathbb{C}[\varepsilon], Y) \\ &= T_xX \times_{T_zZ} T_yY. \end{aligned}$$

Corollary 6.2.11. *Consider the natural submersion*

$$f_\Phi : \mathcal{M}_{\mathscr{C}}(\mathscr{E},\Phi) \longrightarrow \mathcal{M}_{\mathscr{C}}(\mathscr{E}) \tag{6.77}$$

and a point $x \in \mathcal{M}_{\mathscr{C}}(\mathscr{E},\Phi)$*, then the tangent space at that point fits into a short exact sequence*

$$0 \longrightarrow T_x f_\Phi^{-1}(f_\Phi(x)) \longrightarrow T_x\mathcal{M}_{\mathscr{C}}(\mathscr{E},\Phi) \longrightarrow T_{f_\Phi(x)}\mathcal{M}_{\mathscr{C}}(\mathscr{E}) \longrightarrow 0. \tag{6.78}$$

Due to proposition 6.2.8, we have $T^*_{(\mathscr{E})}\mathcal{M}_{\mathscr{C}}(\mathscr{E}) \cong f_\Phi^{-1}(\mathscr{E})$ *and the short exact sequence becomes*

$$0 \longrightarrow T^*_{f_\Phi(x)}\mathcal{M}_{\mathscr{C}}(\mathscr{E}) \longrightarrow T_x\mathcal{M}_{\mathscr{C}}(\mathscr{E},\Phi) \longrightarrow T_{f_\Phi(x)}\mathcal{M}_{\mathscr{C}}(\mathscr{E}) \longrightarrow 0. \tag{6.79}$$

In total, we have an isomorphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H^0(C, End^{\mathfrak{n}}(\mathscr{E}) \otimes \omega_C(D)) \arrow[r] \arrow[d, "\wr"'] & \mathbb{H}^1(\mathcal{C}^\bullet_\Phi) \arrow[r] \arrow[d, "\wr"'] & H^1(C, End^{\mathfrak{p}}(\mathscr{E})) \arrow[r] \arrow[d, "\wr"'] & 0 \\
0 \arrow[r] & T^*_{f_\Phi(x)}\mathcal{M}_{\mathscr{C}}(\mathscr{E}) \arrow[r] & T_x\mathcal{M}_{\mathscr{C}}(\mathscr{E},\Phi) \arrow[r] & T_{f_\Phi(x)}\mathcal{M}_{\mathscr{C}}(\mathscr{E}) \arrow[r] & 0.
\end{tikzcd} \tag{6.80}$$

Remark 6.2.12. We obtain the same picture respectively, when considering the moduli of wild quasi-parabolic connections, by replacing Φ with ∇, ad_Φ with $\nabla \otimes \nabla^*$, and $\mathbb{H}^1(\mathcal{C}^\bullet_\Phi)$ with $\mathbb{H}^1(\mathcal{C}^\bullet_\nabla)$. This last isomorphism of short exact sequences will prove itself useful in section 6.2.2, where we construct a generalised Atiyah-Bott symplectic form on the moduli of Higgs bundles and connections on a fixed wild curve.

6.2.2. Atiyah-Bott Form on Wild Quasi-Parabolic Flat Connections

The goal of this subsection is to construct canonical symplectic forms on the subspaces $\mathcal{M}_{\mathscr{C},Q}(\mathscr{E},\nabla) \subset \mathcal{M}(\mathscr{C},Q,\mathscr{E},\nabla)$ parametrising wild quasi-parabolic flat bundles.

Theorem 6.2.13. ***Duality for Hypercohomology***
Let $\mathscr{K}^\bullet$ *be a finite complex of locally free sheaves on* C

$$\mathscr{K}^\bullet : 0 \longrightarrow \mathscr{K}^0 \longrightarrow \mathscr{K}^1 \longrightarrow \cdots \longrightarrow \mathscr{K}^k \longrightarrow 0$$

and consider its dual complex with respect to the dualising complex $\omega^\bullet_C$ *on* C

$$\check{\mathscr{K}}^\bullet : 0 \longrightarrow (\mathscr{K}^k)^* \otimes \omega_C \longrightarrow (\mathscr{K}^{k-1})^* \otimes \omega_C \longrightarrow \cdots \longrightarrow (\mathscr{K}^0)^* \otimes \omega_C \longrightarrow 0. \tag{6.81}$$

Then we have a duality of hypercohomologies

$$\mathbb{H}^i(\mathscr{K}^\bullet) \cong \mathbb{H}^{k+1-i}(\check{\mathscr{K}}^\bullet)^*. \tag{6.82}$$

In other words, there exists a non-degenerate pairing

$$\mathbb{H}^i(\mathscr{K}^\bullet) \times \mathbb{H}^{k+1-i}(R\mathcal{H}om_{\mathcal{O}_C}(\mathscr{K}^\bullet, \omega_C^\bullet)) \longrightarrow \mathbb{C}. \tag{6.83}$$

Proof. See [36, Section V.2] for the construction of the pairing and the existence of the dualising complex. □

Proposition 6.2.14. ***Atiyah-Bott Form on Wild Quasi Parabolic Connections*** *There exists a canonical symplectic form on the sub-space* $\mathcal{M}_{\mathscr{C},Q}(\mathscr{E},\nabla) \subset \mathcal{M}(\mathscr{C},Q,\mathscr{E},\nabla)$*:*

$$\Omega_{AB,x} : T_x\mathcal{M}_{\mathscr{C},Q}(\mathscr{E},\nabla) \times T_x\mathcal{M}_{\mathscr{C},Q}(\mathscr{E},\nabla) \longrightarrow \mathbb{C}.$$

Proof. Let $(\mathscr{E},\nabla) \in \mathcal{M}_{\mathscr{C},Q}(\mathscr{E},\nabla)$ and consider the associated de Rham complex

$$0 \longrightarrow E \xrightarrow{\nabla} E \otimes \omega_C(D) \xrightarrow{\nabla\circ\nabla} 0, \tag{6.84}$$

as well as the induced 2-term complex on the bundle of wild quasi-parabolic endomorphisms, as we did in Lemma 6.2.9

$$\mathcal{C}^\bullet_\nabla : End^{\mathfrak{p}}(\mathscr{E}) \xrightarrow{\nabla\otimes\nabla^*} End^{\mathfrak{n}}(\mathscr{E}) \otimes \omega_C(D). \tag{6.85}$$

Recall that we have identified the deformation space of wild quasi parabolic connections with the first hypercohomology of this complex $\mathbb{H}^1(\mathcal{C}^\bullet_\nabla)$. An important simplification in our case is that the complex $\mathcal{C}^\bullet_\nabla$ is indeed self dual $\mathcal{C}^\bullet_\nabla = \check{\mathcal{C}}^\bullet_\nabla$. Therefore, we have for the 1st hypercohomology $\mathbb{H}^1(\mathcal{C}^\bullet_\nabla) \cong \mathbb{H}^1(\check{\mathcal{C}}^\bullet_\nabla)$. We define a symplectic form as a composition of the following maps. The first map is the cup product on hypercohomology/ wedge product on tangent spaces

$$\wedge : \mathbb{H}^1(\mathcal{C}^\bullet_\nabla) \otimes \mathbb{H}^1(\check{\mathcal{C}}^\bullet_\nabla) \cong \mathbb{H}^1(\mathcal{C}^\bullet_\nabla) \otimes \mathbb{H}^1(\mathcal{C}^\bullet_\nabla) \longrightarrow \mathbb{H}^2(\mathcal{C}^\bullet_\nabla \otimes \mathcal{C}^\bullet_\nabla). \tag{6.86}$$

The second map is a trace morphism $\mathbb{H}^2(tr)$ acting on $\mathbb{H}^2(\mathcal{C}^\bullet_\nabla \otimes \mathcal{C}^\bullet_\nabla)$ induced on the hypercohomology of the total product complex, by the trace map of complexes

$$\begin{tikzcd}
\mathcal{C}^0_\nabla \otimes \mathcal{C}^0_\nabla \arrow[r] \arrow[d, "\wr"'] & \mathcal{C}^0_\nabla \otimes \mathcal{C}^1_\nabla \oplus \mathcal{C}^1_\nabla \otimes \mathcal{C}^0_\nabla \arrow[r] \arrow[d, "\wr"'] & \mathcal{C}^1_\nabla \otimes \mathcal{C}^1_\nabla \arrow[d, "\wr"'] \\
End^{\mathfrak{p}}(\mathscr{E})^{\otimes 2} \arrow[r] \arrow[d, "tr"] & \begin{bmatrix} End^{\mathfrak{p}}(\mathscr{E}) \\ \otimes \\ End^{\mathfrak{n}}(\mathscr{E}) \otimes \omega_C(D) \end{bmatrix}^{\oplus 2} \arrow[r] \arrow[d, "tr"] & (End^{\mathfrak{n}}(\mathscr{E}) \otimes \omega_C(D))^{\otimes 2} \arrow[d, "tr"] \\
0 \arrow[r] & \omega_C(D) \arrow[r] & 0.
\end{tikzcd} \tag{6.87}$$

Here tr really denotes the map induced on the endomorphism bundles by Killing form on $\mathfrak{sl}_2$. The associated map on hypercohomology gives us

$$\mathbb{H}^2(tr) : \mathbb{H}^2(\mathcal{C}^\bullet_\nabla \otimes \check{\mathcal{C}}^\bullet_\nabla) \longrightarrow \mathbb{H}^2(\omega_C(D)[1]) \cong H^1(\omega_C(D)), \tag{6.88}$$

but since $tr(A \cdot B|_D) = 0$ $\forall A \in End^{\mathfrak{p}}(\mathscr{E})$ and $\forall B \in End^{\mathfrak{n}}(\mathscr{E}) \otimes \omega_C(D)$, the polar part does not contribute to the trace, in other words, it is concentrated away from the *diagonal*, and this map factorises through $H^1(\omega_C)$, i.e.

$$\mathbb{H}^2(tr) : \mathbb{H}^2(\mathcal{C}^\bullet_\nabla \otimes \check{\mathcal{C}}^\bullet_\nabla) \longrightarrow H^1(\omega_C). \tag{6.89}$$

The Atiyah-Bott form reads

$$\Omega_{AB} := \int_C [\mathbb{H}^2(tr) \circ \wedge] : \mathbb{H}^1(\mathcal{C}^\bullet_\nabla) \times \mathbb{H}^1(\mathcal{C}^\bullet_\nabla) \xrightarrow{\wedge} \mathbb{H}^2(\mathcal{C}^\bullet_\nabla \otimes \mathcal{C}^\bullet_\nabla) \xrightarrow{\mathbb{H}^2(tr)} H^1(\omega_C) \xrightarrow[\sim]{\int_C} \mathbb{C}. \tag{6.90}$$

□

6.3. Wild Branched Connections on Spectral Curves

In this subsection, we define what will turn out to be the abelianised counterparts of wild quasi-parabolic flat bundles, namely, anti-invariant branched connections on the spectral curve. These are pairs $(\mathcal{L}, \partial)$ of a line bundle $\mathcal{L}$ on the spectral curve Σ and a irregular connection $\partial : \mathcal{L} \longrightarrow \mathcal{L} \otimes \omega_\Sigma(D_\nu + R)$, satisfying a certain anti-invariance condition with respect to the covering involution.

Definition 6.3.1. Branched Connections on Spectral Covers
Let $(\mathscr{E}, Q) \in \mathcal{M}(\mathscr{E}, Q)$ with associated spectral cover $p : \Sigma \to C$ and let $L \in \mathcal{P}ic(\Sigma)$. A *branched connection* on $(\mathscr{E}, Q)$ is an abelian flat connection, i.e. a $\mathbb{G}_m$- flat connection,

$$\partial_{br} : L \longrightarrow L \otimes \omega_\Sigma(R). \tag{6.91}$$

The term *branched* refers to the fact that ∂_{br} is allowed to develop logarithmic poles along the ramification locus R.

Let λ be the Seiberg-Witten differential associated to $(\mathscr{E}, Q)$. A *wild branched connection* on $(\mathscr{E}, Q)$ is a flat connection

$$\partial^\lambda_{br} : L \longrightarrow L \otimes \omega_\Sigma(D_\nu + R), \tag{6.92}$$

such that $\partial^\lambda_{br} - \lambda$ is a branched connection.

Definition 6.3.2. The Canonical Branched Connection

Let $(\mathscr{C}, Q) \in \mathcal{M}(\mathscr{C}, Q)$ and let $p : \Sigma \longrightarrow C$ be the associated spectral cover with ramification divisor R, and consider the section

$$\varphi := \sqrt{p^* Q} : \mathcal{O}_\Sigma \longrightarrow p^* \omega_C(D). \tag{6.93}$$

We can use this section to perform a meromorphic gauge transformation. For any abelian connection ∂ on a line bundle L on Σ, we have:

$$\partial \longmapsto \varphi \circ \partial \circ \varphi^{-1} \tag{6.94}$$

that is, locally:

$$\partial = d + A(w) \longmapsto d + \varphi A(w) \varphi^{-1} + \frac{d\varphi}{dw} \cdot \varphi^{-1}. \tag{6.95}$$

Applying the gauge transformation on the de Rham differential on Σ, we have

$$[d : \mathcal{O}_\Sigma \longrightarrow \omega_\Sigma] \longmapsto [\partial_{can} := d - \frac{1}{p^* Q} \cdot \frac{d}{dw} p^* Q : p^* \omega_C(D) \longrightarrow p^* \omega_C(D) \otimes \omega_\Sigma(R)] \tag{6.96}$$

and obtain a branched connection on $p^* \omega_C(D)$. This connection ∂_{can} we call the *canonical abelian branched connection.*

Remark 6.3.3. We see from the explicit form of the canonical branched connection that ∂_{can} is σ-invariant, i.e. for $\varphi = \sqrt{p^* q(w) \cdot dw^{\otimes 2}} = \sqrt{p^* q} \cdot dw$ and $\sigma : w \mapsto -w$ we have

$$\sigma^* \partial_{can} = d - \frac{1}{(-\varphi)^2} \cdot \frac{dp^* q}{d(-w)} \cdot d(-w) = \partial_{can}. \tag{6.97}$$

This is as expected, since ∂_{can} is essentially the pullback of a connection from the base curve C to the double cover, possibly developing poles along the ramification locus (recall that $\omega_\Sigma \cong p^* \omega_C \otimes \mathcal{O}_\Sigma(R)$).

Definition 6.3.4. Anti-Invariant Line Bundles
Let $(\mathscr{C}, Q) \in \mathcal{M}(\mathscr{C}, Q)$ and let $p : \Sigma \longrightarrow C$ be the associated spectral cover. An *anti-invariant line bundle* on Σ is a line bundle $\mathcal{L} \in \mathcal{P}ic(\Sigma)$, such that the following identity holds

$$\mathcal{L} \otimes \sigma^* \mathcal{L} = p^* \omega_C(D). \tag{6.98}$$

We denote by $\mathcal{M}(\mathscr{C}, Q, \mathcal{L})$ the space parametrising anti-invariant line bundles.

Remark 6.3.5. We calculate the degree of an anti-invariant line bundle to be

$$\begin{aligned} \deg(\mathcal{L}) + \deg(\sigma^* \mathcal{L}) = 2 \cdot \deg(\mathcal{L}) = \deg(p^* \omega_C(D)) = 2 \cdot \deg(\omega_C(D)) \\ \Longrightarrow \deg(\mathcal{L}) = 2g - 2 + \deg(D). \end{aligned}$$

Definition 6.3.6. Anti-Invariant Branched Flat Bundles

An *anti-invariant branched* flat connection is a branched connection on an anti-invariant line bundle $\mathcal{L}$

$$\partial_\infty : \mathcal{L} \longrightarrow \mathcal{L} \otimes \omega_\Sigma(R), \tag{6.99}$$

such that $\partial_\infty \otimes \sigma^*\partial_\infty = \partial_{can}$. We call a pair $(\mathcal{L}, \partial_\infty)$ with

$$(\mathcal{L} \otimes \sigma^*\mathcal{L}, \partial_\infty \otimes \sigma^*\partial_\infty) = (p^*\omega_C(D), \partial_{can})$$

an anti-invariant branched flat bundle. We denote the space parametrising quadratic differentials and anti-invariant branched flat bundles on wild curves by $\mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial_\infty)$.

An *anti-invariant wild branched* flat connection is a wild branched connection on an anti-invariant line bundle $\mathcal{L}$

$$\partial : \mathcal{L} \longrightarrow \mathcal{L} \otimes \omega_\Sigma(D_\nu + R) \tag{6.100}$$

such that $\partial \otimes \sigma^*\partial = \partial_{can}$. We call a pair $(\mathcal{L}, \partial)$ with $(\mathcal{L} \otimes \sigma^*\mathcal{L}, \partial \otimes \sigma^*\partial) = (p^*\omega_C(D), \partial_{can})$ *an anti-invariant wild branched flat bundle.* We denote the space parametrising quadratic differentials and anti-invariant wild branched flat bundles on wild curves by $\mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial)$.

Lemma 6.3.7. *There exists a canonical isomorphism* $\mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial_\infty) \xrightarrow{\sim} \mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial)$.

Proof. Every point $(\mathscr{C}, Q) \in \mathcal{M}(\mathscr{C}, Q)$ canonically determines a Seiberg-Witten differential $\lambda \in H^0(\Sigma_{(\mathscr{C},Q)}, \omega_\Sigma(D_\nu))$ and so we obtain a canonical isomorphism

$$\begin{aligned} \mathcal{M}_{(\mathscr{C},Q)}(\mathcal{L}, \partial_\infty) &\xrightarrow{\sim} \mathcal{M}_{(\mathscr{C},Q)}(\mathcal{L}, \partial) \\ (\mathcal{L}, \partial_\infty) &\longmapsto (\mathcal{L}, \partial_\infty + \lambda), \end{aligned} \tag{6.101}$$

where the inverse map is given by $[(\mathcal{L}, \partial) \mapsto (\mathcal{L}, \partial - \lambda)]$. □

Definition 6.3.8. The Dual and Square Root of an Abelian Connection
Let C be an algebraic curve, $L \in \mathcal{P}ic(C)$ and $\partial : L \longrightarrow L \otimes \omega_C(D)$ an irregular connection, then we define the dual connection associated to ∂, supported on the dual line bundle

$$\partial^* : L^* \longrightarrow L^* \otimes \omega_C(D) \tag{6.102}$$

via

$$\langle \partial^* t, s \rangle = d \langle t, s \rangle - \langle t, \partial s \rangle \,, \ \forall s \in \Gamma(L), t \in \Gamma(L^*), \tag{6.103}$$

where $\langle \cdot, \cdot \rangle : L^* \otimes L \longrightarrow \mathcal{O}_\Sigma$ denotes the perfect pairing.

Furthermore, let $F \in \mathcal{P}ic(C)$ with $F^{\otimes 2} \cong L$, then ∂ defines via $\partial^{1/2} s \otimes 1 := \partial(s \otimes 1)$ a unique connection

$$\partial^{1/2} : F \longrightarrow F \otimes \omega_C(D), \tag{6.104}$$

such that $(\partial^{1/2})^{\otimes 2} = \partial$.

6.3.1. Abelian Riemann-Hilbert for Branched Connections

Our sole goal here is to relate the space $\mathcal{M}_{(\mathscr{C},Q)}(\mathcal{L}, \partial_\infty)$ of anti-invariant branched connections on a point $(\mathscr{C}, Q) \in \mathcal{M}(\mathscr{C}, Q)$ to the fibres $\tilde{H}^1(\Sigma, \mathbb{C}^*)^-$ of the anti-invariant cohomology bundle $\mathscr{H}^\#$. This is analogous to [15, Section 3.4].

Definition 6.3.9. Let $(\mathscr{C}, Q) \in \mathcal{M}(\mathscr{C}, Q)$ and $p : \Sigma \to C$ the associated spectral cover. We define the following spaces

1. let $Jac^\#(C) = \{(P, \partial^h) | P \in Jac(C), \partial^h : P \to P \otimes \omega_C)\}$
2. let $Jac^\#(\Sigma) = \{(M, \partial^h) | M \in Jac(\Sigma), \partial^h : M \to M \otimes \omega_\Sigma)\}$
3. let $Jac^\#(\Sigma)^- = \{(M, \partial^h) \in Jac^\#(\Sigma) | (M, \partial^h) \otimes \sigma^*(M, \partial^h) = (\mathcal{O}_\Sigma, d)\}$
4. let $Prym^\#(\Sigma) := Jac^\#(\Sigma)^- / Jac(C)[2]$ for the embedding

$$\begin{aligned} Jac(C)[2] &\hookrightarrow Jac^\#(\Sigma)^- \\ P &\mapsto (p^*P, \partial^h_P) \end{aligned}$$

where ∂^h_P is the unique connection on p^*P such that $(p^*P, \partial^h_P)^{\otimes 2} = (\mathcal{O}_\Sigma, d)$, which we constructed in (6.104).

Lemma 6.3.10. *There is an identification* $Prym^\#(\Sigma) = \tilde{H}^1(\Sigma, \mathbb{C}^*)^-$.

Proof. The classical Riemann-Hilbert correspondence [22] gives us identifications

$$Jac^\#(C) = H^1(C, \mathbb{C}^*) \text{ and } Jac^\#(\Sigma) = H^1(\Sigma, \mathbb{C}^*).$$

Due to lemma 4.5.3 we have

$$\begin{aligned} Prym^\#(\Sigma) &= Jac^\#(\Sigma)^- / Jac(C)[2] = Jac^\#(\Sigma) / Jac^\#(C) \\ &= H^1(\Sigma, \mathbb{C}^*) / H^1(C, \mathbb{C}^*) = \tilde{H}^1(\Sigma, \mathbb{C}^*)^- \end{aligned}$$

□

Proposition 6.3.11. *There exists a finite étale map* $\mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial_\infty) \longrightarrow \mathscr{H}^\#$.

Proof. Let $(\mathscr{C}, Q) \in \mathcal{M}(\mathscr{C}, Q)$ and $p : \Sigma \to C$ the associated spectral cover. Choose a square root of $\omega_C(D)$, i.e. a line bundle $P_0 \in Jac(C)$ in the $Jac(C)[2]$-torsor

$$P_0 \in \{P \in Jac(C) | P^{\otimes 2} = \omega_C(D)\} \cong (\mathbb{Z}/2\mathbb{Z})^{2g} .$$

Let $\mathcal{L}_0 := p^* P_0$ and let ∂_0 be the unique connection such that $(\mathcal{L}_0, \partial_0)^{\otimes 2} = (p^*\omega_C(D), \partial_{can})$. Then, up to the finite choice of P_0, the map

$$\begin{aligned} Jac^{\#}(\Sigma) &\longrightarrow \mathcal{M}_{(\mathscr{C},Q)}(\mathcal{L}, \partial_\infty) \\ (\mathcal{M}, \partial^h) &\longmapsto (\mathcal{M} \otimes \mathcal{L}_0, \partial^h \otimes \partial_0) \end{aligned}$$

is a canonical identification. Note that

$$\begin{aligned} (\mathcal{M} \otimes \mathcal{L}_0, \partial^h \otimes \partial_0) \otimes (\sigma^*(\mathcal{M} \otimes \mathcal{L}_0), \sigma^*(\partial^h \otimes \partial_0)) &= (\mathcal{M} \otimes \sigma^*\mathcal{M} \otimes \mathcal{L}^{\otimes 2}, \partial^h \otimes \partial_0 \otimes \sigma^*\partial^h \otimes \sigma^*\partial_0) \\ &= (p^*\omega_C(D), \partial_{can}). \end{aligned}$$

Passing to the quotient by the 2-torsion, we obtain a finite, 2^{2g}-covering map

$$\mathcal{M}_{(\mathscr{C},Q)}(\mathcal{L}, \partial_\infty) \longrightarrow Jac^{\#}(\Sigma)/Jac(C)[2] = \tilde{H}^1(\Sigma, \mathbb{C}^*)^-. \tag{6.105}$$

This gives a finite map to the étalè space

$$\mu : \mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial_\infty) \longrightarrow \mathscr{H}^{\#}. \tag{6.106}$$

□

7. Generalised Spectral Correspondence

In this chapter, we seek to prove a correspondence between, on the one side, the moduli parametrising wild quasi-parabolic Higgs fields and wild quasi-parabolic flat bundles, i.e. $\mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla)$, and on the other side, the moduli of wild anti-invariant flat bundles on spectral covers of wild curves, i.e. $\mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial)$. The holomorphic case of the statement we are after, first appeared in works of Donagi-Pantev in their approach to the Geometric Langlands Program via non-abelian Hodge theory [27, Section 3.2]. We generalise this result to arbitrary irregular poles for the structure group $G = SL_2(\mathbb{C})$ in Theorem 7.2.12.

7.1. Classical Spectral Correspondence for Higgs Bundles

The essence of the original spectral construction that appeared in the fundamental work of Hitchin [40] and was subsequently studied by [5], lies in the correspondences we discuss in this subsection. We also prove a few helpful results for our spacial case of D-twisted $SL_2(\mathbb{C})$-Higgs bundles, to be used later on.

We start with a purely algebraic preamble, explaining how to view a D-twisted Higgs field Φ on a bundle E over a curve C as a $Sym^\bullet(\omega_C(D)^*)$-module structure on E, in other words as a quasi-coherent sheaf on the total space of the twisted dualising sheaf. This structure naturally descends to a generically semisimple $Sym^\bullet(\omega_C(D)^*)/\mathcal{I}$-module structure on E, where $\mathcal{I}$ is the ideal generated by the characteristic polynomial of Φ, and that is the level where one obtains a correspondence between Higgs bundles on the base curve and line bundles on the spectral curve.

Lemma 7.1.1. *Let S be a scheme, then there is an anti-equivalence of categories:*

$$\{\textit{affine morphisms } f : X \longrightarrow S\} \leftrightarrow \{\textit{quasi-coherent sheaves of } \mathcal{O}_S\textit{-Algebras}\}. \tag{7.1}$$

Proof. For $f : X \longrightarrow S$ affine, the sheaf of $\mathcal{O}_S$-algebras $f_*\mathcal{O}_X$ is quasi-coherent and the relative $Spec_S(-)$ functor defines an inverse to the pushforward of the structure sheaf, i.e.

$$X \cong Spec_S(f_*\mathcal{O}_X),$$

cf. [74, Tag 01S5]. □

Lemma 7.1.2. *Let $f : X \longrightarrow S$ be a an affine morphism of schemes and let $\mathcal{A}$ denote the $\mathcal{O}_S$-algebra $f_*\mathcal{O}_X$. Then the pushforward functor f_* on quasi-coherent $\mathcal{O}_X$-Modules induces an equivalence of categories*

$$\{\textit{Quasi-Coherent } \mathcal{O}_X\textit{-Modules}\} \leftrightarrow \{\textit{Quasi-Coherent } \mathcal{A}\textit{-Modules}\}, \tag{7.2}$$

Proof. (Sketch) An $\mathcal{A}$-module M, i.e. an $\mathcal{O}_S$-Module with an $f_*\mathcal{O}_X$-Structure, gives rise to an $\mathcal{O}_X$-Module $\widetilde{M}$ by gluing the $f_*\mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$-Modules $\widetilde{M(U)}$, using gluing properties of the $\mathcal{O}_S(U)$-modules $M(U)$. Quasi-coherence follows analogously from the same property of M. This construction is inverse to the pushforward functor and vice versa, since we have $\widetilde{f_*G}|_U \cong G|_U$ for any affine open $U \subset S$ and any $G \in QCoh_{\mathcal{O}_X}$, as well as $f_*\widetilde{M} \cong M$ by definition. □

Remark 7.1.3. An $f_*\mathcal{O}_X$-structure on a $\mathcal{O}_S$-module E

$$f_*\mathcal{O}_X \otimes_{\mathcal{O}_S} E \longrightarrow E \tag{7.3}$$

can be regarded as an $\mathcal{O}_S$-Algebra homomorphism/representation: $f_*\mathcal{O}_X \longrightarrow End_{\mathcal{O}_S}(E)$.

Example 7.1.4. Let us consider the special case of the spectral curves, as it is the case appearing in our theory. For this, let $\mathscr{C} = (C, D, \nu, \xi)$ be a wild curve, and let $Q \in M_{\mathscr{C}}(Q)$ be a quadratic differential on $\mathscr{C}$ and Σ the associated spectral curve:

$$\begin{tikzcd}
\Sigma \arrow[r, hook] \arrow[d, "p"'] & Tot(\omega_C(D)) \arrow[dl, "\pi"] \\
C &
\end{tikzcd} .$$

We write $\mathscr{J}_Q$ for the defining ideal sheaf of the (embedded) subscheme

$$\Sigma = Spec\,(Sym^\bullet(\omega_C(D)^*)/\mathscr{J}_Q) \hookrightarrow Tot(\omega_C(D)).$$

The morphism p is affine as it is finite of degree 2, and the pushforward of the structure sheaf splits as an $\mathcal{O}_C$-module

$$p_*\mathcal{O}_\Sigma \cong Sym^\bullet(\omega_C(D)^*)/\mathscr{J}_Q \cong_{\mathcal{O}_C\text{-}Mod} \mathcal{O}_C \oplus \omega_C(D)^*. \tag{7.4}$$

By remark 7.1.3, a $p_*\mathcal{O}_\Sigma$-structure on a locally free $\mathcal{O}_C$-Module E, can be interpreted as an algebra homomorphism

$$Sym^\bullet(\omega_C(D)^*)/\mathscr{J}_Q \longrightarrow End_{\mathcal{O}_C}(E). \tag{7.5}$$

In what follows, we explain how Higgs fields are to be interpreted as such $p_*\mathcal{O}_\Sigma$-Module structures on locally free sheaves on the base curve C.

Lemma 7.1.5. *The slice $\mathcal{M}_{\mathscr{C},E,Q}(\Phi)$ of Higgs fields on a bundle E over a wild curve $\mathscr{C}$ with characteristic polynomial described by Q is parametrised by*

$$\mathcal{M}_{\mathscr{C},E,Q}(\Phi) = Hom_{\mathcal{O}_C\text{-}Alg}\left(Sym^\bullet(\omega_C(D)^*)/\mathscr{J}_Q, End(E)\right).$$

Proof. Consider the short exact sequence of sheaves of $\mathscr{O}_C$-Algebras

$$0 \longrightarrow \mathscr{I} \longrightarrow Sym^\bullet(\omega_C(D)^*) \longrightarrow Sym^\bullet(\omega_C(D)^*)/\mathscr{J}_Q \longrightarrow 0 \tag{7.6}$$

and apply the Hom-functor of $\mathscr{O}_C$-Algebras $Hom_{\mathscr{O}_C-Alg}(-, End(E))$, where E is a locally free sheaf on C and $End(E)$ the associated $\mathscr{O}_C$-Algebra of endomorphisms of E. For this, we obtain

$$\begin{tikzcd}
Hom_{\mathscr{O}_C-Alg}(Sym^\bullet(\omega_C(D)^*)/\mathscr{J}_Q, End(E)) \arrow[r] \arrow[dr, dashed, "g"] & Hom_{\mathscr{O}_C-Alg}(Sym^\bullet(\omega_C(D)^*), End(E)) \arrow[r] \arrow[d, "\wr"] & . \\
& Hom_{\mathscr{O}_C-Mod}(\omega_C(D)^*, End(E)) &
\end{tikzcd} \tag{7.7}$$

where the vertical isomorphism is induced by the degree-1 restriction of the graded $\mathscr{O}_C$-algebra $Sym^\bullet(\omega_C(D)^*)$, the inverse of which is provided by the universal property of the symmetric algebra. The Hom-space $Hom_{\mathscr{O}_C-Mod}(\omega_C(D)^*, End(E))$ parametrises, by definition, D-twisted Higgs fields on E, i.e. $\mathscr{O}_C$-linear morphisms: $\Phi : E \longrightarrow E \otimes \omega_C(D)$, and the image of the factorising map $Im(g)$ parametrises exactly the Higgs fields with characteristic polynomial Q, i.e. with associated spectral cover $\Sigma = Spec(Sym^\bullet(\omega_C(D)^*)/\mathscr{J}_Q)$. □

Remark 7.1.6. Let $p : \Sigma \longrightarrow C$ be a smooth covering of curves, ramified over a divisor R. As usual we denote by:

- $p_* : D^bCoh(\Sigma) \longrightarrow D^bCoh(C)$ the direct image (since p is proper $p_! = p_*$),
- $p^* : D^bCoh(C) \longrightarrow D^bCoh(\Sigma)$ inverse image,
- $p^! : D^bCoh(C) \longrightarrow D^bCoh(\Sigma)$ proper inverse image.

For these morphisms, we have the adjunctions $[p^* \dashv p_!]$ and $[p^! \vdash p_*]$. These follow via the Grothendieck duality c.f. [74, Tag 0DWE] or [36].

Lemma 7.1.7. ***Relative Dualising Sheaf of a Spectral Cover.***
Let $p : \Sigma \longrightarrow C$ be a smooth covering of curves, ramified over a divisor R. Then we have

$$p^!F = p^*F \otimes \mathscr{O}_\Sigma(R) \quad \forall F \in Coh(C). \tag{7.8}$$

Proof. Since p is proper finite, the relative dualising sheaf exists and relates the two inverse image functors

$$p^!F = p^*F \otimes \omega_{\Sigma/C} \; \forall F \in Coh(C). \tag{7.9}$$

From the sequence induced by dualising the derivative of the covering map

$$0 \longrightarrow p^*\omega_C(D) \longrightarrow \omega_\Sigma(p^*D) \longrightarrow \mathscr{O}_R \longrightarrow 0,$$

we have $\omega_\Sigma(p^*D) \otimes p^*(\omega_C(D)^*) \cong \mathcal{O}_\Sigma(R)$ and $\omega_{\Sigma/C} \cong \mathcal{O}_\Sigma(R)$, from which we get

$$p^!(F) = p^*(F) \otimes \mathcal{O}_\Sigma(R).$$

$\square$

Proposition 7.1.8. *Let* $p : \Sigma \to C$ *be a ramified double cover of algebraic curves with ramification divisor* R *and a Galois involution* σ *that exchanges the sheets of the cover. Consider* $\mathcal{L} \in \mathcal{P}ic(\Sigma)$*, then there exists a canonical exact sequence:*

$$0 \longrightarrow p^*p_*\mathcal{L} \xrightarrow{i} \mathcal{L} \oplus \sigma^*\mathcal{L} \longrightarrow \mathcal{O}_R \longrightarrow 0. \tag{7.10}$$

Proof. The adjunction $[p^* \dashv p_*]$ gives rise to a map

$$\eta : p^*p_*\mathcal{L} \to \mathcal{L} \tag{7.11}$$

and since $p_*\mathcal{L} = p_*\sigma^*\mathcal{L}$, we also have a map $p^*p_*\mathcal{L} \to \sigma^*\mathcal{L}$, which we combine to obtain a map of locally free sheaves of the same rank $p^*p_*\mathcal{L} \hookrightarrow \mathcal{L} \oplus \sigma^*\mathcal{L}$. We now argue that this map is an isomorphism away from ramification. Denote by $B := p(R) \subset C$ the associated branching divisor of the finite map p and let $U \subset C$, such that $U \cap B = \emptyset$, i.e. U is in the étale locus of p. Then the sheaf $p_*\mathcal{O}_\Sigma|_U$ is an étale algebra over $\mathcal{O}_C|_U$ and $p_*\mathcal{L}|_U$ is a coherent module over this étale algebra, thereby its pullback splits $p^*p_*\mathcal{L}|_{p^{-1}(U)} \cong \mathcal{F}|_{p^{-1}(U)} \oplus \mathcal{G}|_{p^{-1}(U)}$, for some locally free sheaves $\mathcal{F}, \mathcal{G}$ on Σ. Furthermore, by the adjunction $[p^! \vdash p_*]$, we have a canonical monomorphism of locally free sheaves

$$\mathcal{L} \hookrightarrow p^!p_*\mathcal{L} \tag{7.12}$$

and by (7.8) we have $\mathcal{L} \hookrightarrow p^!p_*\mathcal{L} \overset{7.8}{=} p^*p_*\mathcal{L} \otimes \mathcal{O}_\Sigma(R)$. Restricting this morphism gives embeddings $\mathcal{L} \hookrightarrow p^*p_*\mathcal{L}|_{p^{-1}(U)}$ and $\sigma^*\mathcal{L} \hookrightarrow p^*p_*\mathcal{L}|_{p^{-1}(U)}$. In total, we have

$$p^*p_*\mathcal{L}|_{p^{-1}(U)} = (\mathcal{L} \oplus \sigma^*\mathcal{L})|_{p^{-1}(U)}.$$

Let $r \in R$, let U be a neighbourhood of $p(r) \in B$, let z be a local coordinate in U with $z(p(r)) = 0$ and let $w \in p^{-1}(U)$ be a local coordinate with $w(r) = 0$ and $w^2 = z$. Then, we have locally

$$\begin{aligned}
&\Gamma(U, p_*\mathcal{L}|_U) = \mathbb{C}[w],\\
&\Gamma(U, p^*p_*\mathcal{L}|_{p^{-1}(U)}) = \mathbb{C}[w] \otimes_{\mathbb{C}[z]} \mathbb{C}[w] \cong \mathbb{C}[w_1, w_2]/(w_1^2 - w_2^2), \text{ and}\\
&\Gamma(U, \mathcal{L} \oplus \sigma^*\mathcal{L}|_{p^{-1}(U)}) = \mathbb{C}[w] \oplus \mathbb{C}[w]
\end{aligned}$$

For these spaces of sections, there is a short exact sequence

$$0 \longrightarrow \mathbb{C}[w_1, w_2]/(w_1^2 - w_2^2) \xrightarrow{i_r} \mathbb{C}[w] \oplus \mathbb{C}[w] \xrightarrow{\ell_r} \mathbb{C} \longrightarrow 0,$$

where the first map is given by $i_r : h(w_1, w_2) \mapsto (h(w, w), h(w, -w))$ and the second map is the evaluation of the difference at 0, i.e. $\ell_r : (f, g) \mapsto f(0) - g(0)$. First note that i_r is $\mathbb{C}[z]$-linear, as $z = w^2 = (-w)^2$, and σ-equivariant as

$$\begin{aligned}\sigma i_r(h(w_1, w_2)) &= \sigma(h(w, w), h(w, -w)) = (h(w, -w), h(w, w)) \\ &= i_r(h(w_1, -w_2)) = i_r\sigma(h(w_1, w_2)).\end{aligned}$$

Furthermore, we see that i_r is indeed injective. Writing $h(w_1, w_2) = a(w_1) + w_2 b(w_1)$ for $a, b \in \mathbb{C}[w_1]$, we have

$$\begin{aligned}i_r(h) = 0 &\iff a(w) + wb(w) = 0 \text{ and } a(w) + wb(w) = 0 \\ &\iff a(w) = b(w) = 0 \iff h = 0.\end{aligned}$$

Lastly, $Im(i_r) = ker(\ell_r)$. The non-trivial inclusion $ker(\ell_r) \subset Im(i_r)$ follows by associating

$$h_{(f,g)}(w_1, w_2) = \frac{1}{2}(f(w_1) + g(w_2)) + \frac{w_1}{2w_2}(f(w_1) - g(w_2))$$

to any $(f, g) \in ker(\ell_r)$ for which $i_r(h_{(f,g)}) = (f, g)$. Note that $\frac{w_1}{2w_2}(f(w_1) - g(w_2))$ is well defined near $w_2 = 0$, since $f(w_1) - g(w_2)$ has no constant term due to the $(f, g) \in ker(\ell_r)$ condition. □

Remark 7.1.9. This last proposition allows us to think of a section $s \in p^*p_*\mathcal{L}$ as a pair of sections $(s_1, s_2) \in \mathcal{L} \oplus \sigma^*\mathcal{L}$ that agree on the ramification divisor $s_1|_R = s_2|_R$.

Lemma 7.1.10. *Let $p : \Sigma \to C$ be a simply ramified double cover as before and consider an anti-invariant line bundle $\mathcal{L} \in \mathcal{P}ic(\Sigma)$. The determinant bundle of $p_*\mathcal{L}$ is trivial, i.e. $\bigwedge^2 p_*\mathcal{L} \cong \mathcal{O}_C$.*

Proof. Taking the determinant of the short exact sequence 7.10, we have

$$0 \longrightarrow \bigwedge^2 p^*p_*\mathcal{L} \xrightarrow{det(i)} \mathcal{L} \otimes \sigma^*\mathcal{L} \longrightarrow \mathcal{O}_S \longrightarrow 0, \tag{7.13}$$

which, due to the anti-invariance property, becomes

$$0 \longrightarrow \bigwedge^2 p^*p_*\mathcal{L} \longrightarrow \mathcal{O}_\Sigma(R) \longrightarrow \mathcal{O}_S \longrightarrow 0, \tag{7.14}$$

where $\mathcal{O}_S = coker(\det(i))$. The map i fails to be an isomorphism precisely along the ramification locus, and so the determinant vanishes precisely at R, i.e.

$$supp(\mathcal{O}_S) = supp(\mathcal{O}_R).$$

Since $\deg(\mathcal{O}_S) = \deg(\mathcal{O}_R)$ and R is reduced, we get $\mathcal{O}_S \overset{!}{=} \mathcal{O}_R$. Hence, since

$$\deg\left(\bigwedge^2 p^* p_* \mathcal{L}\right) = \deg(\mathcal{O}(R)) - \deg(\mathcal{O}_R) = 0,$$

it follows for degree reasons that $\bigwedge^2 p^* p_* \mathcal{L} = \mathcal{O}_\Sigma$. In Lemma 4.3.1, we proved that the inverse image along a finite map commutes with tensor/wedge/symmetric products of locally free sheaves, i.e. $p^* \bigwedge^2 p_* \mathcal{L} = \mathcal{O}_\Sigma$ and from Lemma 4.5.1, we know the p^* is injective, thereby $\bigwedge^2 p_* \mathcal{L} \overset{!}{=} \mathcal{O}_C$. □

The following Theorem is a refinement of Lemma 7.1.5 and is originally due to Hitchin [40] and formulated in (nearly) the form presented here in [5, Theorem 3.6]. We care about the case of $SL_2(\mathbb{C})$-D-twisted Higgs bundles and we provide a short proof in our special case, since it is instructive for what is to come.

Theorem 7.1.11. ***The Spectral Correspondence***
Let C be an algebraic curve, Q a quadratic differential and $p : \Sigma \to C$ the associated spectral cover. There exists a bijective correspondence between isomorphism classes of anti-invariant line bundles $\mathcal{L}$ on Σ and isomorphism classes of $SL_2(\mathbb{C})$-Higgs bundles (E, Φ) with Hitchin image $\chi_{(C,E)}(\phi) = Q$ cf.6.23. This correspondence induces fibre-wise a map:

$$\vartheta : \mathcal{M}(\mathscr{C}, E, \Phi) \longrightarrow \mathcal{M}(\mathscr{C}, Q, \mathcal{L}). \tag{7.15}$$

Proof. Fix the wild curve $\mathscr{C}$, let (E, Φ) be a D-twisted Higgs bundle on C and consider the lift of the Higgs field to its associated spectral cover $\Sigma = \{y^2 - \pi^*\chi(\Phi) = 0\}$

$$p^*\Phi \in H^0(\Sigma, \mathfrak{sl}_2(p^*E) \otimes \omega_\Sigma(p^*D)). \tag{7.16}$$

Recall our notation where y is the tautological section of $Tot(\omega_C(D))$, $\varphi := \sqrt{p^*\chi(\Phi)}$ and $\lambda := \iota \circ \varphi$ for the canonical sequence

$$0 \longrightarrow p^*\omega_C(D) \xrightarrow{\iota} \omega_\Sigma \otimes p^*\mathcal{O}_C(D) \longrightarrow \mathcal{O}_R \longrightarrow 0. \tag{7.17}$$

We denote by $\mathcal{L}$ the Eigenlinebundle of $p^*\Phi$

$$0 \longrightarrow \mathcal{L}(-R) \longrightarrow p^*E \xrightarrow{p^*\Phi - \lambda} p^*(E \otimes \omega_C(D)) \longrightarrow p^*\omega_C(D) \otimes \mathcal{L} \longrightarrow 0 \tag{7.18}$$

where the second to last map is induced by the counit (7.11) of the adjunction. We remark here that, since $\sigma^*\lambda = -\lambda$, the eigenline bundle associated to this eigenvalue of Φ is precisely $\sigma^*\mathcal{L}$ and $\mathcal{L}$ is distinguished by the choice of a square root φ of p^*Q.

For the inverse direction of the correspondence, let $Q \in H^0(C, \omega_C(D)^{\otimes 2})$ with associated spectral curve $p : \Sigma \longrightarrow C$. The choice of a square root $\varphi := \sqrt{p^*Q} : \mathcal{O}_\Sigma \longrightarrow p^*\omega_C(D)$ induces a morphism of $\mathcal{O}_\Sigma$-Modules $a : \mathcal{L} \longrightarrow \mathcal{L} \otimes p^*\omega_C(D)$. Then, we define the Higgs bundle (E, Φ) to be the pair

$$(E := p_*\mathcal{L}, \Phi := p_*a : E \longrightarrow E \otimes \omega_C(D)). \tag{7.19}$$

This fits into the short exact sequence 7.18 due to the adjunction 7.12. Lastly, by Lemma 7.1.10, we see that the resulting locally free sheaf E is indeed an $SL_2(\mathbb{C})$-bundle. □

7.2. Abelianisation of Irregular Flat Connections on Wild Curves

In this section, we construct a birational correspondence $\alpha : \mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla) \longrightarrow \mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial)$ extending the BNR-correspondence 7.1.11. We do this in 4 steps:

1. Define an irregular abelian connection ∂ on $\mathcal{L}$ from ∇, cf. Definition 7.2.1.
2. Show that the resulting pair $(\mathcal{L}, \partial)$ is anti-invariant, cf. Lemma 7.2.4.
3. Show that it only develops poles along $D_\nu + R$, thereby it is an irregular anti-invariant branched connection, cf. Proposition 7.2.7.
4. Show that generically ∂ arises from a unique ∇ which we prove in Theorem 7.2.12, using Proposition 7.2.9.

Definition 7.2.1. Abelianisation of an Irregular Connection
Let $(\mathscr{C}, \mathscr{E}, \Phi, \nabla) \in \mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla)$ with $\vartheta(\mathscr{C}, E, \Phi) = (\mathscr{C}, Q, \mathcal{L})$ and let $p : \Sigma \to C$ be the spectral cover associated to Q. Consider the Eigenspace-decomposition-SES (7.10)

$$0 \longrightarrow p^*E \xrightarrow{i} \mathcal{L} \oplus \sigma^*\mathcal{L} \longrightarrow \mathcal{O}_R \longrightarrow 0.$$

The connection ∇ pulls back to a connection $p^*\nabla : p^*E \longrightarrow p^*E \otimes p^*\omega_C(D)$ and it extends to a unique connection

$$\tilde{\nabla} : \mathcal{L} \oplus \sigma^*\mathcal{L} \longrightarrow (\mathcal{L} \oplus \sigma^*\mathcal{L}) \otimes \omega_\Sigma(D_\nu + k \cdot R), \tag{7.20}$$

which may develop additional poles along R, $k \in \mathbb{N}$. The connection $\tilde{\nabla}$ is obtained from $p^*\nabla$ by the same meromorphic gauge transformation diagonalising the Higgs field $p^*\Phi$. We define the abelianisation of ∇ as the abelian wild branched connection given by the component

$$\partial : \mathcal{L} \longrightarrow \mathcal{L} \otimes \omega_\Sigma(D_\nu + k \cdot R) \tag{7.21}$$

of the extended connection $\tilde{\nabla}$.

Remark 7.2.2. This extension is unique, such that the embedding i is a flat morphism of bundles with connection, i.e.

$$i(ker(p^*\nabla)) \subset ker(\tilde{\nabla}),$$

sending flat sections to flat sections, cf. 7.2.3. The extra poles of $\tilde{\nabla}$ along R arise from the gauge transformation taking $(p^*E, p^*\nabla)$ to $(\mathcal{L} \oplus \sigma^*\mathcal{L}, \tilde{\nabla})$. We will study this in detail down the line.

Lemma 7.2.3. *The embedding* $i : p^*E \xrightarrow{i} \mathcal{L} \oplus \sigma^*\mathcal{L}$ *is a flat morphism of bundles with connection, if and only if for any neighbourhood* $U \subset \Sigma$ *at ramification and a local coordinate* $w \in U$ *centered at a ramification point, the following equality holds*

$$i(p^*\nabla_{\frac{\partial}{\partial w}}(t)) = \tilde{\nabla}_{\frac{\partial}{\partial w}}(i(t)),\ \forall t \in \Gamma(U, p^*E). \tag{7.22}$$

Proof. Assume that i is flat, let $\{s_i\}_i$ be a local basis of flat sections of $p^*\nabla$ on U and let $t = \sum_j a_j(w)s_j \in \Gamma(U, p^*E)$. Then we have

$$\begin{aligned} i(p^*\nabla_{\frac{\partial}{\partial w}}(t)) &= i\left(\sum_j \frac{\partial a_j(w)}{\partial w} \cdot s_j + \sum_j a_j(w) p^*\nabla_{\frac{\partial}{\partial w}}(s_j)\right) \\ &\overset{s_j \text{ flat}}{=} i\left(\sum_j \frac{\partial a_j(w)}{\partial w} \cdot s_j\right) = \sum_j \frac{\partial a_j(w)}{\partial w} \cdot i(s_j) \end{aligned}$$

$$\tilde{\nabla}_{\frac{\partial}{\partial w}}(i(t)) = \tilde{\nabla}_{\frac{\partial}{\partial w}}\left(\sum_j a_j(w) \cdot i(s_j)\right) \overset{i(s_j) \text{ flat}}{=} \sum_j \frac{\partial a_j(w)}{\partial w} \cdot i(s_j)$$

proving the equality. For the inverse statement, (7.22) implies $i(ker(p^*\nabla)) \subset ker(\tilde{\nabla})$. $\square$

Lemma 7.2.4. ***Anti-Invariance Property***
Let $(\mathscr{E}, \Phi, \nabla) \in \mathcal{M}_{\mathscr{C}}(\mathscr{E}, \Phi, \nabla)$ *and let* $(Q, \mathcal{L}, \partial)$ *be the abelianised data we associated to* $(\mathscr{E}, \Phi, \nabla)$. *The tuple* $(\mathcal{L}, \partial)$ *on* $p : \Sigma \to C$ *is an anti-invariant wild branched connection.*

Proof. We want to show that $(\mathcal{L} \otimes \sigma^*\mathcal{L}, \partial \otimes \sigma^*\partial) = (p^*\omega_C(D), \partial_{can})$. Consider again the determinant of the Eigenspace decomposition sequence 7.10

$$0 \longrightarrow \mathcal{O}_\Sigma \xrightarrow{det(i)} \mathcal{L} \otimes \sigma^*\mathcal{L} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

as well as the SES given by the square root $\varphi = \sqrt{p^*Q}$

$$0 \longrightarrow \mathcal{O}_\Sigma \xrightarrow{\varphi} p^*\omega_C(D) \longrightarrow \mathcal{O}_R \longrightarrow 0.$$

We already know from proposition 7.1.8 and corollary 7.1.10 that $\mathcal{L} \otimes \sigma^*\mathcal{L} \cong p^*\omega_C(D)$. Since line bundle isomorphisms are unique up to invertible global sections of the structure sheaf, there exists a constant section $g \in H^0(\Sigma, \mathcal{O}_\Sigma^\times)$, such that the following diagram of exact sequences commutes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_\Sigma \arrow[r, "det(i)"] \arrow[d, "id"] & \mathcal{L} \otimes \sigma^*\mathcal{L} \arrow[r] \arrow[d, dashed, "\wr"'] & \mathcal{O}_S \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \mathcal{O}_\Sigma \arrow[r, "g\cdot\sqrt{p^*Q}"] & p^*\omega_C(D) \arrow[r] & \mathcal{O}_R \arrow[r] & 0.
\end{tikzcd} \tag{7.23}$$

Recall that the canonical connection ∂_{can} on $p^*\omega_C(D)$ is obtained from d by a meromorphic gauge transform via $\varphi := \sqrt{p^*Q}$, i.e.

$$[d : \mathcal{O}_\Sigma \longrightarrow \omega_\Sigma] \longmapsto [\partial_{can} := d - \frac{1}{p^*Q} \cdot \frac{d}{dw} p^*Q : p^*\omega_C(D) \longrightarrow p^*\omega_C(D) \otimes \omega_\Sigma(R)]. \tag{7.24}$$

The determinant of the extended connection $det(\tilde{\nabla}) = \partial \otimes 1 + 1 \otimes \sigma^*\partial$ is then related to the trivial connection $d = det(p^*\nabla)$ by the meromorphic gauge transform associated to $g \cdot \sqrt{p^*Q}$. The connection ∂_{can} is by definition invariant under the rescaling $\sqrt{p^*Q} \longmapsto g \cdot \sqrt{p^*Q}$, and therefore $\partial \otimes 1 + 1 \otimes \sigma^*\partial$ coincides with ∂_{can}. □

The construction of ∂ above is simple yet rather ad hoc. In what follows, we realise this abelianisation procedure via the Grothendieck-Verdier duality as proposed in [27, Ch. 3].

Lemma 7.2.5. ***Abelianisation via Adjunction***
In the context of definition 7.2.1, given a Higgs pair (E, Φ), we obtain by the spectral correspondence a pair $(\mathcal{L}, Q)$ and in particular a covering $p : \Sigma \longrightarrow C$ ramified over a divisor R. Let $(E, \Phi) \leftrightarrow (\mathcal{L}, Q)$ be related by the spectral correspondence, and let ∇ be an SL_2-flat connection on E. Then ∇ determines a unique meromorphic abelian connection on $\mathcal{L}$

$$\partial : \mathcal{L} \longrightarrow \mathcal{L} \otimes \omega_\Sigma(p^*D + 2R), \tag{7.25}$$

such that $(\mathcal{L}, \partial)$ is an anti-invariant flat line bundle on Σ.

Proof. There exists a canonical embedding

$$\iota : \mathcal{L} \longrightarrow p^! p_* \mathcal{L} \tag{7.26}$$

induced by the universal property for right adjoint functors applied to the adjunction $[p^! \vdash p_*]$. Since $\mathcal{L}$ abelianises E, we have $p_*\mathcal{L} = E$ and therefore

$$p^! p_* \mathcal{L} = p^* p_* \mathcal{L} \otimes \omega_{\Sigma/C} = p^*E \otimes \mathcal{O}_\Sigma(R). \tag{7.27}$$

The flat connection ∇ on E pulls back to a connection on Σ

$$p^*\nabla : p^*E \longrightarrow p^*E \otimes \omega_\Sigma(p^*D). \tag{7.28}$$

Furthermore, we have a canonical abelian connection induced by the action of the de Rham differential on $\mathcal{O}_\Sigma$

$$d : \mathcal{O}_\Sigma(R) \longrightarrow \mathcal{O}_\Sigma(R) \otimes \omega_\Sigma(R). \tag{7.29}$$

Moreover, from the adjunction $[p^* \dashv p_!]$, we have a map $p^*E \longrightarrow \mathcal{L}$. These maps give rise to an abelian connection on $\mathcal{L}$ as follows:

$$\partial : \mathcal{L} \overset{\iota}{\hookrightarrow} p^*E \otimes \mathcal{O}_\Sigma(R) \xrightarrow{p^*\nabla \otimes d} p^*E \otimes \omega_\Sigma(p^*D) \otimes \mathcal{O}_\Sigma(R) \otimes \omega_\Sigma(R) \to \mathcal{L} \otimes \omega_\Sigma(p^*D + 2R). \tag{7.30}$$

□

In order to prove the birational correspondence $\alpha : \mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla) \longrightarrow \mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial)$ we are after, we will need to study the connection $\tilde{\nabla}$ of definition 7.2.1 explicitly at ramification. In the setting of definition 7.2.1, let $r \in R$,let $U \subset \Sigma$ be a σ-invariant neighbourhood of r and w a local coordinate with $\sigma^* w = -w$. In particular, $w(r) = 0$. Moreover, let $s \in \Gamma(U, \mathcal{L})$, such that $s(w) \neq 0$ and fix a trivialisation at U:

$$\begin{aligned} \Gamma(U, \mathcal{O}_U^{\oplus 2}) &\longrightarrow \Gamma(U, \mathcal{L} \oplus \sigma^*\mathcal{L}|_U) \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} &\longmapsto (s, 0) \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} &\longmapsto (0, \sigma^* s). \end{aligned}$$

Writing the extended connection $\tilde{\nabla}|_U$ at U in the local basis induced by this trivialisation, it takes the form:

$$\tilde{\nabla}|_U = d + \begin{bmatrix} a(w) & b(w) \\ c(w) & d(w) \end{bmatrix} dw \tag{7.31}$$

for $a, b, c, d \in \Gamma(U, \mathcal{O}_\Sigma(D_\nu) + k \cdot R)$, for some $k \in \mathbb{Z}_{\geq 0}$.

Lemma 7.2.6. ***The Connection $\tilde{\nabla}$ is σ-Equivariant***

The flat connection $\tilde{\nabla}$ is σ-equivariant. In particular, in a basis of non-vanishing local sections as above, it takes the form

$$\tilde{\nabla}|_U = d + \begin{bmatrix} a(w) & b(w) \\ -b(-w) & -a(-w) \end{bmatrix} dw, \tag{7.32}$$

for $a, b \in \Gamma(U, \mathcal{O}_\Sigma(D_\nu) + k \cdot R)$.

Proof. Away from ramification, $\tilde{\nabla}$ agrees with $p^*\nabla$ and as a pullback connection, it is σ-

invariant on sections over σ-invariant neighbourhoods. At a σ-invariant neighbourhood U of a ramification point r as above, let $[f,g]^T \in \Gamma(U, \mathcal{O}_U^{\oplus^2})$. In this notation, we have

$$\begin{aligned}\sigma^*\left(\tilde{\nabla}_{\frac{\partial}{\partial w}}\left(f(w)\cdot s, g(w)\cdot\sigma^* s\right)\right) &= \sigma^*\left(\left(s\tfrac{\partial}{\partial w}f(w), \sigma^* s\tfrac{\partial}{\partial w}g(w)\right)\right)\\ &\quad +\sigma^*\left(\left(f(w)\cdot s, g(w)\cdot\sigma^* s\right)\begin{bmatrix}a(w) & b(w)\\ c(w) & d(w)\end{bmatrix}\right)\end{aligned}$$

$$= \left(-\tfrac{\partial}{\partial w}(\sigma^* g(w)), -\tfrac{\partial}{\partial w}(\sigma^* f(w))\right) + \begin{bmatrix}\sigma^* d(w)\sigma^* g(w)s(w) + \sigma^* b(w)\sigma^* f(w)\sigma^* s(w)\\ \sigma^* c(w)\sigma^* g(w)s(w) + \sigma^* a(w)\sigma^* f(w)\sigma^* s(w)\end{bmatrix}$$

and

$$\begin{aligned}\tilde{\nabla}_{-\frac{\partial}{\partial w}}\left(\sigma^*\left(f(w)\cdot s, g(w)\cdot\sigma^* s\right)\right) &= \left(s\sigma^*\left(\tfrac{\partial}{\partial w}g(w)\right), \sigma^* s\sigma^*\left(\tfrac{\partial}{\partial w}f(w)\right)\right)\\ &\quad + \left(\sigma^* g(w)\cdot s, \sigma^* f(w)\cdot\sigma^* s\right)\begin{bmatrix}a(w) & b(w)\\ c(w) & d(w)\end{bmatrix}\\ &= -\left(\tfrac{\partial}{\partial w}(\sigma^* g(w)), \tfrac{\partial}{\partial w}(\sigma^* f(w))\right) - \begin{bmatrix}\sigma^* g(w)a(w)s(w) + \sigma^* f(w)c(w)\sigma^* s(w)\\ \sigma^* f(w)b(w)s(w) + \sigma^* f(w)d(w)\sigma^* s(w)\end{bmatrix}.\end{aligned}$$

Since $\tilde{\nabla}$ extends $p^*\nabla$, the covariant derivative of $\tilde{\nabla}$ on σ-invariant sections, i.e. sections $(f,g) \in \Gamma(U, \mathcal{O}_U^{\oplus 2})$ such that $\sigma^* f(w) = g(w)$ must be σ-invariant

$$\sigma^*\tilde{\nabla}_{\frac{\partial}{\partial w}}\left(f(w)\cdot s, g(w)\cdot\sigma^* s\right) \overset{!}{=} \tilde{\nabla}_{-\frac{\partial}{\partial w}}\left(\sigma^*\left(f(w)\cdot s, g(w)\cdot\sigma^* s\right)\right) \implies \begin{cases}a(w) = -d(-w)\\ b(w) = -c(-w)\end{cases}$$

and the form (7.32) follows. $\square$

Although the general constructions 7.2.1 and 7.2.5 allows ∂ to develop poles along $p^*D + 2R$, a local calculation proves that in our case it only develops poles along $p^*D + R$. Using the σ-invariance and the condition that ∇ preserves a given wild quasi-parabolic structure, we can further restrict the form of the connection at the neighbourhood of the ramification. We prove these statements by studying the form of the extended connection $\tilde{\nabla}$ at a neighbourhood of a ramification point explicitly.

Proposition 7.2.7. ***Explicit Description of the Extension $\tilde{\nabla}$ at Ramification*** *Let $r \in R$, $U \subset \Sigma$ be a σ-invariant neighbourhood of r and a local coordinate $w \in U$ with $\sigma^* w = -w$, in particular, $w(r) = 0$. The restriction of the flat connection $\tilde{\nabla}$ to U is of the form*

$$\tilde{\nabla}|_U = d + \begin{bmatrix}a(w) & b(w)\\ -b(-w) & -a(-w)\end{bmatrix}dw, \tag{7.33}$$

for $a \in \Gamma(U, \mathcal{O}_\Sigma(D_\nu + R))$ and $b \in \Gamma(U, \mathcal{O}_\Sigma(D_\xi + R))$.

Proof. Let $\tilde{\nabla}$ be the connection on $\mathcal{L} \oplus \sigma^*\mathcal{L}$ extending $p^*\nabla$ along the exact sequence

$$0 \longrightarrow p^*E \xrightarrow{i} \mathcal{L} \oplus \sigma^*\mathcal{L} \longrightarrow \mathcal{O}_R \longrightarrow 0.$$

Locally, the connection $\tilde{\nabla}$ is related to $p^*\nabla|_U = d + A$ via the gauge transformation

$$\tilde{\nabla}|_U = d + B^{-1}dB + B^{-1}AB, \tag{7.34}$$

where B is the matrix with columns consisting of the eigenvectors of $p^*\Phi|_U$, thereby extending $p^*\nabla$ along i. Since the eigenvectors of $p^*\Phi|_U$ are σ-conjugate to each other, we have

$$B = \begin{bmatrix} v_1^+(w) & v_2^+(w) \\ v_1^-(w) & v_2^-(w) \end{bmatrix}; \text{ with } \sigma^*v_1^+(w) = v_2^+(w), \sigma^*v_1^-(w) = v_2^-(w). \tag{7.35}$$

Note that, due to the σ-conjugacy, the products $v_1^+(w)v_2^-(w)$ and $v_2^+(w)v_1^-(w)$ have the same order of vanishing at 0, and the entries of $B^{-1}dB$ (7.36) have at most a logarithmic pole at $w(r) = 0$

$$\frac{1}{det(B)} \begin{bmatrix} v_2^-(w)\frac{d}{dw}v_1^+(w) + v_2^+(w)\frac{d}{dw}v_1^-(w) & v_2^-(w)\frac{d}{dw}v_2^+(w) - v_2^+(w)\frac{d}{dw}v_2^-(w) \\ v_1^+(w)\frac{d}{dw}v_1^-(w) - v_1^-(w)\frac{d}{dw}v_1^+(w) & -v_1^-(w)\frac{d}{dw}v_2^+(w) + v_1^+(w)\frac{d}{dw}v_2^-(w) \end{bmatrix}, \tag{7.36}$$

$$det(B) = v_1^+(w)v_2^-(w) - v_2^+(w)v_1^-(w). \tag{7.37}$$

Imposing the condition that ∇ preserves the same wild quasi-parabolic structure as the Higgs field cf. (6.54),(6.55), as well as the σ-invariance condition we have

$$B^{-1}AB = \begin{bmatrix} \alpha(w) & \beta(w) \\ -\sigma^*\beta(w) & -\sigma^*\alpha(w) \end{bmatrix} dw; \quad \alpha \in \Gamma(U, \mathcal{O}_\Sigma(D_\nu)), \beta \in \Gamma(U, \mathcal{O}_\Sigma(D_\xi)). \tag{7.38}$$

In total, the connection $\tilde{\nabla}$ is, locally at ramification, of the form

$$\tilde{\nabla}|_U = d + \begin{bmatrix} a(w) & b(w) \\ -b(-w) & -a(-w) \end{bmatrix} dw,$$

where $a \in \Gamma(U, \mathcal{O}_\Sigma(D_\nu + R))$ and $b \in \Gamma(U, \mathcal{O}_\Sigma(D_\xi + R))$. □

Remark 7.2.8. The last proposition shows that the source of the extra poles at ramification are a result of the $B^{-1}dB$ component of the meromorphic gauge transformation. In particular, taking the difference $\tilde{\nabla}_1 - \tilde{\nabla}_2$ for two connections $\nabla_1, \nabla_2 \in \mathcal{M}_{(\mathscr{C},\mathscr{E},\Phi)}(\nabla)$, this term cancels out. We use this fact in the following proposition.

Proposition 7.2.9. *As before, let $r \in R$, $U \subset \Sigma$ be a σ-invariant neighbourhood of r and a local coordinate $w \in U$ with $\sigma^*w = -w$. Let $\nabla_1, \nabla_2 \in \mathcal{M}_{(\mathscr{C},\mathscr{E},\Phi)}(\nabla)$ with extensions $\tilde{\nabla}_1$, $\tilde{\nabla}_2$ giving rise to the same abelian connection $\partial = d + a(w)$ via the abelianisation procedure*

7.2.1. Then, their difference, locally at ramification, is of the form

$$[\tilde{\nabla}_1 - \tilde{\nabla}_2]|_U = \begin{bmatrix} 0 & b_\Delta(w) \\ -b_\Delta(-w) & 0 \end{bmatrix} dw, \tag{7.39}$$

where $b_\Delta(w) \in \Gamma(U, \mathcal{O}_\Sigma(D_\xi - R))$.

Proof. The difference, locally at ramification, reads

$$\begin{aligned} [\tilde{\nabla}_1 - \tilde{\nabla}_2]|_U &= (d + B^{-1}dB + B^{-1}A_1B) - (d + B^{-1}dB + B^{-1}A_2B) && (7.40)\\ &= \begin{bmatrix} \alpha(w) & \beta_1(w) \\ -\sigma^*\beta_1(w) & -\alpha(w) \end{bmatrix} dw - \begin{bmatrix} \alpha(w) & \beta_2(w) \\ -\sigma^*\beta_2(w) & -\alpha(w) \end{bmatrix} dw && (7.41)\\ &= \begin{bmatrix} 0 & \beta_1(w) - \beta_2(w) \\ -\sigma^*\beta_1(w) + \sigma^*\beta_2(w) & 0 \end{bmatrix} dw && (7.42)\\ &=: \begin{bmatrix} 0 & b_\Delta(w) \\ -b_\Delta(-w) & 0 \end{bmatrix} dw, && (7.43) \end{aligned}$$

with $b_\Delta(w) \in \Gamma(U, \mathcal{O}_\Sigma(D_\xi))$. We proceed with a local calculation similar to [15, 4.2]. Since we argue locally, we can assume w.l.o.g. that $D_\xi = n \cdot r$. Then, $N(w) := \frac{w^n}{dw} \cdot [\tilde{\nabla}_1 - \tilde{\nabla}_2]|_U$ is a regular endomorphism of $\mathcal{L}|_U \oplus \sigma^*\mathcal{L}|_U$, in particular, acting with N on a regular section $s \in \Gamma(U, p^*E)$ yields a regular section

$$\left(\left[\tilde{\nabla}_{1\,\frac{\partial}{\partial w}} - \tilde{\nabla}_{2\,\frac{\partial}{\partial w}}\right] \cdot w^n\right)(s) \in \Gamma(U, p^*E). \tag{7.44}$$

If $n \equiv 0 \; mod \; 2$, then, for the regular σ-invariant section $\begin{bmatrix} w^n & w^n \end{bmatrix}^T \in \Gamma(U, p^*E)$, we have

$$\begin{aligned} &\left[\tilde{\nabla}_{1\,\frac{\partial}{\partial w}} - \tilde{\nabla}_{2\,\frac{\partial}{\partial w}}\right] \cdot \begin{bmatrix} w^n \\ w^n \end{bmatrix} = \begin{bmatrix} 0 & b_\Delta(w) \\ -b_\Delta(-w) & 0 \end{bmatrix} \begin{bmatrix} w^n \\ w^n \end{bmatrix} = \begin{bmatrix} b_\Delta(w)w^n \\ -b_\Delta(-w)w^n \end{bmatrix} \overset{!}{\in} \Gamma(U, p^*E) \\ &\implies b_\Delta(w)w^n|_{w(p)=0} \overset{!}{=} -b_\Delta(-w)w^n|_{w(p)=0} \implies b_{\Delta,n} \overset{!}{=} -b_{\Delta,n} \\ &\implies b_{\Delta,n} = 0 \implies \beta_{1,n} = \beta_{2,n}, \end{aligned}$$

and if $n \equiv 1 \; mod \; 2$, then $\begin{bmatrix} w^n & -w^n \end{bmatrix}^T \in \Gamma(U, p^*E)$ is a σ-invariant section, for which we have

$$\begin{aligned} &[\tilde{\nabla}_{1\,\frac{\partial}{\partial w}} - \tilde{\nabla}_{2\,\frac{\partial}{\partial w}}] \cdot \begin{bmatrix} w^n \\ -w^n \end{bmatrix} = \begin{bmatrix} 0 & b_\Delta(w) \\ -b_\Delta(-w) & 0 \end{bmatrix} \begin{bmatrix} w^n \\ -w^n \end{bmatrix} = \begin{bmatrix} -b_\Delta(w)w^n \\ -b_\Delta(-w)w^n \end{bmatrix} \overset{!}{\in} \Gamma(U, p^*E) \\ &\implies b_\Delta(w)w^n|_{w(p)=0} \overset{!}{=} b_\Delta(-w)w^n|_{w(p)=0} \implies b_{\Delta,n} \overset{!}{=} -b_{\Delta,n} \\ &\implies b_{\Delta,n} = 0 \implies \beta_{1,n} = \beta_{2,n}. \end{aligned}$$

In both cases, the difference of the leading coefficients of β_1, β_2 must vanish. Therefore, the difference is a section

$$b_\Delta(w)dw = [\beta_1(w) - \beta_2(w)]dw \in \Gamma(U, \omega_\Sigma(D_\xi - R)).$$

$\square$

Definition 7.2.10. The α-Morphism
We define a morphism

$$\alpha : \mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla) \longrightarrow \mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial) \tag{7.45}$$

by sending the Higgs field Φ to its Hitchin-image $\chi(\Phi) = tr(\Phi^2) =: Q$ and the wild quasi-parabolic connection $(\mathscr{E}, \nabla)$ to the abelian branched flat bundle $(\mathcal{L}, \partial)$ of definition 7.2.1.

Remark 7.2.11. At the level of morphisms, let $f : (\mathscr{E}, \Phi, \nabla) \to (\mathscr{E}', \Phi', \nabla')$ a morphism in $\mathcal{M}_{\mathscr{C}}(\mathscr{E}, \Phi, \nabla)$, i.e. a morphism of SL_2-bundles $f : E \longrightarrow E'$ such that is compatible with the quasi-parabolic structures L_ξ, L'_ξ and intertwines the Higgs fields and flat connections $f\Phi = \Phi' f$ and $f \circ \nabla = \nabla' \circ f$. Then α sends such a morphism to a morphism of the abelianised data $\alpha(f) : (\mathscr{E}, Q, \partial) \to (\mathscr{E}', Q', \partial')$. Explicitly:

- $Q' = tr((f\Phi)^2)$, and $\nu = \nu', p_\nu = p'_\nu$ since $(\mathscr{E}, \Phi)$ $(\mathscr{E}', \Phi')$ are both Higgs fields on the same wild curve.
- $\alpha(f)|_L : L = ker(p^*\Phi - \lambda) \to L' = ker(p'^*(f\Phi) - \lambda')$ and note that $L'_\xi = L'|_{D_\xi}$ due to the compatibility of f with the quasi-parabolic structure.
- for the induced map on abelianised connections we have $p^*f \circ \partial = \partial' \circ p^*f$.

The following theorem is a generalisation of the classical spectral correspondence to include the simultaneous abelianisation of Higgs fields and flat connections. Thereby, flat connections that are wild quasi-parabolic with respect to the residual eigenline of the Higgs field, i.e. have the same wild quasi-parabolic structure as the Higgs field, map to anti-invariant abelian connections on the eigenline $\mathcal{L}$. The interesting fact is that, generically, the fibres of this extended abelianisation map are finite. An abelian branched connection on the spectral curve generically determines an SL_2-connection on the base. This is non-trivial, since, unlike the Higgs field that diagonalises on the spectral cover, the connection does not need be diagonal, but merely wild parabolic. This extends the abelianisation procedure described in the Donagi-Pantev approach to geometric Langlands [27, Chapter 3], and [15, Theorem 5.1.1] from the holomorphic to the wild irregular meromorphic setting. For brevity, we refer to the appendix definition B.0.26, for the definition of a birational morphism between algebraic stacks.

Theorem 7.2.12. ***Generalised Spectral Correspondence***
The morphism α induces a birational correspondence of moduli spaces, extending the spec-

tral BNR-correspondence for D-twisted Higgs bundles to include anti-invariant flat bundles

$$\begin{tikzcd}
\mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla) \arrow[r, "\alpha"] \arrow[d, two heads] & \mathcal{M}(\mathscr{C}, \mathcal{L}, Q, \partial) \arrow[d, two heads] \\
\mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi) \arrow[r, "\vartheta"] & \mathcal{M}(\mathscr{C}, \mathcal{L}, Q).
\end{tikzcd} \tag{7.46}$$

Proof. Given the BNR-correspondence ϑ as described in Theorem 7.1.11, we need to show that the map

$$\mathcal{M}_{(\mathscr{C},\mathscr{E},\Phi)}(\nabla) \longrightarrow \mathcal{M}_{(\mathscr{C},\mathcal{L},Q)}(\partial), \tag{7.47}$$

induced by α, is generically $1:1$. For this, consider two connections $\nabla, \nabla' \in \mathcal{M}_{(\mathscr{C},\mathscr{E},\Phi)}(\nabla)$ on the same wild bundle $\mathscr{E}$ and giving rise to the same abelian connection ∂ on $\mathcal{L}$, i.e. $\nabla, \nabla' \in \alpha^{-1}((\mathscr{C}, \mathcal{L}, Q, \partial))$. We proceed with a local argument, so w.l.o.g., let $D = n \cdot p$. We distinguish between the cases where $p \in B$ and $p \notin B$ when calculating the restriction $(\tilde{\nabla} - \tilde{\nabla}')|_{D_\xi}$, that is, we distinguish by whether the spectral cover ramifies over p or not.

Unramified Case, i.e. $D_\nu = D_\xi + \sigma^*(D_\xi) = p^*D$:

$$(\tilde{\nabla} - \tilde{\nabla}')|_{D_\xi} = \begin{bmatrix} \partial|_{D_\xi} & b \\ 0 & -\partial|_{D_\xi} \end{bmatrix} - \begin{bmatrix} \partial|_{D_\xi} & b' \\ 0 & -\partial|_{D_\xi} \end{bmatrix} = \begin{bmatrix} 0 & b - b' \\ 0 & 0 \end{bmatrix} \tag{7.48}$$

and analogously

$$(\tilde{\nabla} - \tilde{\nabla}')|_{\sigma^*(D_\xi)} = \begin{bmatrix} \partial|_{D_\xi} & 0 \\ b & -\partial|_{D_\xi} \end{bmatrix} - \begin{bmatrix} \partial|_{D_\xi} & 0 \\ b' & -\partial|_{D_\xi} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ b - b' & 0 \end{bmatrix}, \tag{7.49}$$

with $b - b' \in H^0(D_\xi, \omega_\Sigma(D_\xi)|_{D_\xi})$. Recall that $\tilde{\nabla}$ agrees with $p^*\nabla : p^*E \to p^*E \otimes \omega_\Sigma(D_\nu - R)$ away from ramification. At ramification the difference vanishes as we saw in proposition 7.2.9. Therefore, we have globally

$$(\tilde{\nabla} - \tilde{\nabla}') \in H^0(\Sigma, \sigma^*\mathcal{L}^* \otimes \mathcal{L} \otimes \omega_\Sigma(D_\xi - R)).$$

Ramified Case, i.e. $D_\nu/2 = p^*D/2 = D_\xi = \sigma^*(D_\xi)$:

$$(\tilde{\nabla} - \tilde{\nabla}')|_{D_\nu} = \begin{bmatrix} \partial|_{D_\nu} & b \\ \sigma^*b & -\partial|_{D_\nu} \end{bmatrix} - \begin{bmatrix} \partial|_{D_\nu} & b' \\ \sigma^*b' & -\partial|_{D_\nu} \end{bmatrix} = \begin{bmatrix} 0 & b - b' \\ \sigma^*b - \sigma^*b' & 0, \end{bmatrix} \tag{7.50}$$

with $(b - b'), (\sigma^*b - \sigma^*b') \in H^0(D_\xi, \omega_\Sigma(D_\xi - R)|_{D_\xi}) = H^0(\sigma(D_\xi), \omega_\Sigma(\sigma(D_\xi) - R)|_{\sigma(D_\xi)})$ as follows from Proposition 7.2.9. Therefore, we have globally once again

$$(\tilde{\nabla} - \tilde{\nabla}') \in H^0(\Sigma, \sigma^*\mathcal{L}^* \otimes \mathcal{L} \otimes \omega_\Sigma(D_\xi - R)).$$

In both cases, the difference of any two connections in the fibre of α is described by a global section of the sheaf $\sigma^*\mathcal{L}^* \otimes \mathcal{L} \otimes \omega_\Sigma(D_\xi - R)$. Applying Riemann-Roch, we see that the line bundle in question has indeed vanishing Euler characteristic

$$\begin{aligned}\chi(\Sigma, \sigma^*\mathcal{L}^* \otimes \mathcal{L} \otimes \omega_\Sigma(D_\xi - R)) &= \chi(\omega_\Sigma(D_\xi - R)) = \deg(\omega_\Sigma(D_\xi - R)) - g_\Sigma + 1 \\ &= 2g_\Sigma - 2 + \deg(D_\xi) - g_\Sigma + 1 - \deg(D) + 1 - g_\Sigma = 0,\end{aligned}$$

and it follows that, for a generic $\mathcal{L}$, any such section is zero and the difference vanishes. Therefore, the map α is generically 1:1. □

Corollary 7.2.13. *As a direct corollary to Theorem 7.2.12, we get a correspondence*

$$\begin{tikzcd} \mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla_\infty) \arrow[r, "\alpha"] \arrow[d, two heads] & \mathcal{M}(\mathscr{C}, \mathcal{L}, Q, \partial_\infty) \arrow[d, two heads] \\ \mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi) \arrow[r, "\theta"] & \mathcal{M}(\mathscr{C}, \mathcal{L}, Q) \end{tikzcd} \tag{7.51}$$

The proof of this result works in exactly the same way. This is due to the fact that the difference of any two wild quasi-parabolic connections with the same irregular type is equal to a nilpotent quasi-parabolic connection.

7.3. Generic Finiteness of the Hitchin Fibre at a fixed Bundle

Definition 7.3.1. The β-Morphism.
We consider the map of moduli spaces induced by the Hitchin map (6.23) on Higgs fields

$$\begin{aligned}\beta_\infty : \mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla_\infty) &\longrightarrow \mathcal{M}(\mathscr{C}, \mathscr{E}, Q, \nabla_\infty) \\ (\mathscr{C}, \mathscr{E}, \Phi, \nabla_\infty) &\longmapsto (\mathscr{C}, \mathscr{E}, \chi(\Phi), \nabla_\infty).\end{aligned} \tag{7.52}$$

We also write

$$\begin{aligned}\beta : \mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla_\infty) &\longrightarrow \mathcal{M}(\mathscr{C}, \mathscr{E}, Q, \nabla) \\ (\mathscr{C}, \mathscr{E}, \Phi, \nabla_\infty) &\longmapsto (\mathscr{C}, \mathscr{E}, \chi(\Phi), \nabla_\infty + \Phi).\end{aligned} \tag{7.53}$$

Recall that $\mathcal{M}(\mathscr{C}, \mathscr{E}, Q, \nabla_\infty) \cong \mathcal{M}(\mathscr{C}, \mathscr{E}, Q, \nabla)$.

We would like to prove in complete generality that the map β_∞ is generically étale. For the time being, we cannot do so without assuming the existence of a certain schematic point on $\mathcal{M}(\mathscr{C}, \mathscr{E}, Q, \nabla_\infty)$ in the substack cut out by fixing a special wild bundle 7.54. We discuss the origin of this condition and its expected resolution in remark 7.3.3. For now, we can only guarantee via the following proposition 7.3.2, that β_∞ is generically étale for all admissible types of genus 0, i.e types $(0, \mathcal{m})$, in particular, for all examples of Painlevé type.

Proposition 7.3.2. *Let $\omega_C^{1/2}$ be a choice of a θ-characteristic, i.e. a choice of a square root of the canonical bundle on C. Assuming the existence of $\mathscr{C}, Q$ and a connection ∇_∞ on the wild quasi-parabolic bundle $\mathscr{E} = (\omega_C(D)^{1/2} \oplus \omega_C(D)^{-1/2}, p^*\omega_C(D)^{1/2}|_{D_\xi})$, such that the point*

$$(\mathscr{C}, \omega_C(D)^{1/2} \oplus \omega_C(D)^{-1/2}, p^*\omega_C(D)^{1/2}|_{D_\xi}, Q, \nabla_\infty) \in \mathcal{M}(\mathscr{C}, \mathscr{E}, Q, \nabla_\infty) \tag{7.54}$$

is schematic, i.e. has trivial stabilisers, then the map β_∞ is generically étale. The same statement holds tautologically for β.

Proof. We want to argue that the map

$$\psi : \mathcal{M}_{\mathscr{C}}(\mathscr{E}, \Phi) \longrightarrow \mathcal{M}_{\mathscr{C}}(\mathscr{E}, Q)$$

is generically finite. First, note that

$$\dim(\mathcal{M}_{\mathscr{C}}(\mathscr{E}, \Phi)) = 9g - 9 + 3\deg(D) = \dim(\mathcal{M}_{\mathscr{C}}(\mathscr{E}, Q)). \tag{7.55}$$

The source and target of ψ are equidimensional, and its dominance follows from [5, Theorem 1] using the dimension formula [74, Tag 02JT]. We want to show that for every wild curve $\mathscr{C}$, β has a finite fibre, that is, there exists a wild quasi-parabolic bundle $\mathscr{E}$, such that

$$\dim(\psi^{-1}(\mathscr{E}, Q)) = \dim(\mathcal{M}_{(\mathscr{C},\mathscr{E})}(\Phi)) - \dim(\mathcal{M}_{(\mathscr{C},\mathscr{E})}(Q)) = 0.$$

Recall that, by theorem 7.2.12, we have a birational correspondence

$$\alpha : \mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla_\infty) \longrightarrow \mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial_\infty),$$

so we can transfer the discussion from wild quasi-parabolic Higgs bundles $(\mathscr{E}, \Phi)$ to the abelianised data $(\mathcal{L}, Q)$ via α.

Fixing a quadratic differential Q, we assume the existence of a line bundle $\mathcal{L} \in \mathcal{P}ic(\Sigma_Q)$ with the following properties:

1. $\mathcal{L}$ is anti-invariant $\mathcal{L} \otimes \sigma^*\mathcal{L} \cong p^*\omega_C(D)$, i.e. $(Q, \mathcal{L})$ arises as a tuple in $\mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial_\infty)$,
2. $\mathcal{L}$ admits a (non-zero) embedding $\mathcal{L} \hookrightarrow p^*p_*\mathcal{L}$,

and set $\mathscr{E} := (E := p_*\mathcal{L}, L_\xi := \mathcal{L}|_{D_\xi})$ for the de-abelianised wild bundle on C associated to $(Q, \mathcal{L})$. For such a wild quasi-parabolic bundle $\mathscr{E}$ we have a diagram of short exact sequences

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{L} \arrow[r, "id"] \arrow[d, "i"] & \mathcal{L} \arrow[d] & & \\
0 \arrow[r] & p^*E \arrow[r, "j"] \arrow[d] & \mathcal{L}\oplus\sigma^*\mathcal{L} \arrow[r] \arrow[d] & \mathcal{O}_R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{L}^* \arrow[r] \arrow[d] & \sigma^*\mathcal{L} \arrow[r] \arrow[d] & \mathcal{O}_R \arrow[r] & 0 \\
 & 0 & 0 & & .
\end{tikzcd} \tag{7.56}$$

where the left vertical sequence is exact due to the 2nd assumption on $\mathcal{L}$ together with

$$\mathcal{L}\otimes p^*E/\mathcal{L}\cong \det(p^*E)=\mathcal{O}_\Sigma \implies p^*E/\mathcal{L}\cong\mathcal{L}^*.$$

The middle horizontal short exact sequence is the one we studied in 7.10. The bottom horizontal sequence is exact due to the 1st assumption on $\mathcal{L}$, from which we have

$$\mathcal{L}^*=\sigma^*\mathcal{L}\otimes\mathcal{O}_\Sigma(-R),$$

i.e. it arises by twisting $0\to\mathcal{O}_\Sigma(-R)\to\mathcal{O}_\Sigma\to\mathcal{O}_R\to 0$ with $\sigma^*\mathcal{L}$.

Let $\mathcal{M}$ be another line bundle satisfying (1),(2) such that $\mathcal{M}|_{D_\xi}=\mathcal{L}|_{D_\xi}$. Then (7.56) gives a diagram

$$\begin{tikzcd}
 & \mathcal{L} \arrow[d, hook, "i"'] \arrow[ddr, dashed, "\ell\circ j\circ i"] & \\
 & p^*E \arrow[d, hook, "j"'] & \\
\mathcal{M} \arrow[r] & \mathcal{M}\oplus\sigma^*\mathcal{M} \arrow[r, "\ell"] & \sigma^*\mathcal{M}.
\end{tikzcd} \tag{7.57}$$

Since $\deg(\mathcal{L})=\deg(\mathcal{M})=\deg(\sigma^*\mathcal{M})$, the induced map $\ell\circ j\circ i$ is either an isomorphism or zero, as it is a map of line bundles of the same degree. Thereby, either $\mathcal{L}\cong\sigma^*\mathcal{M}$ or $\mathcal{L}\cong\mathcal{M}$. Imposing the condition of the quasi-parabolic structure $\mathcal{M}|_{D_\xi}=L_\xi=\mathcal{L}|_{D_\xi}$ we see that $\mathcal{M}\overset{!}{\cong}\mathcal{L}$.

It remains to show that such a non-empty fibre exists, i.e. that a line bundle $\mathcal{L}$ with those properties exist on the spectral curve Σ_Q. To that end, start by choosing a θ-characteristic $\omega_C^{1/2}$ of the canonical sheaf on the base curve C, which is a finite choice, and let $\mathcal{L}:=p^*(\omega_C(D)^{1/2})$. This line bundle $\mathcal{L}$ satisfies conditions (1),(2), where the 1st property holds by construction

$$p^*(\omega_C(D)^{1/2})^{\otimes 2} = p^*(\omega_C(D))$$

and the 2nd follows by adjunction $\mathcal{L} \hookrightarrow p^*p_*p^*(\omega_C(D)^{1/2}) = p^*p_*\mathcal{L}$. Therefore, the map ψ is generically finite and therefore β_∞ is generically finite morphism. This map is a generically étale map of stacks as defined in the appendix definition B.0.27. For smooth surjective atlases

$$U \longrightarrow \mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla_\infty) \text{ and } V \longrightarrow \mathcal{M}(\mathscr{C}, \mathscr{E}, Q, \nabla_\infty),$$

let $u : U \longrightarrow V$ be the induced morphism on smooth schemes/$\mathbb{C}$, then u is by construction a generically finite morphism between smooth equidimensional irreducible varieties over $\mathbb{C}$ and thereby it is generically étale.

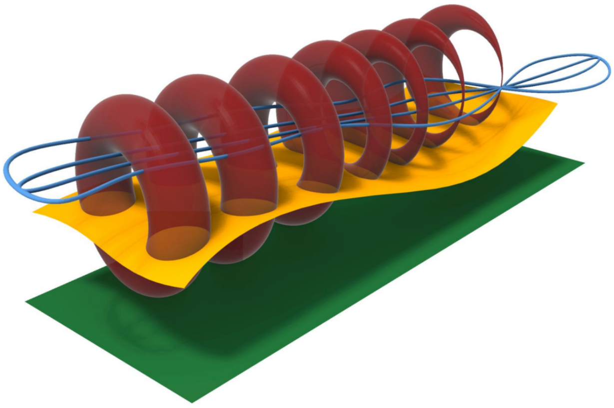
□

Remark 7.3.3. Discussion on Proposition 7.3.2 and beyond.

Without assuming the existence of a schematic point 7.54, the proof above fails to extend the finiteness over the special point $(\mathscr{C}, \omega(D)^{1/2} \oplus \omega(D)^{-1/2}, p^*\omega_C(D)^{1/2}|_{D_\xi}, Q, \nabla_\infty)$ to an open schematic sub-stack using by the dominance of the map β_∞. The underlying holomorphic bundle $E = \omega_C(D)^{1/2} \oplus \omega_C(D)^{-1/2}$, and thereby $p^*E = \mathcal{L} \oplus \mathcal{L}^*$, is unstable as a holomorphic bundle, with non-trivial automorphism group $H^0(C, \omega_C(D))$ of dimension $g - 1 + deg(D)$. The wild quasi-parabolic structure on E, reduces the dimension of the non trivial automorphism group to $\dim(Aut(\mathscr{C}, \mathscr{E})) = g - 1$. These automorphisms rigidify automatically for $g = 0$, and the proof above works without the extra assumption 7.54. In particular all Painlevé-type cases are covered so. For higher genera, the quasi-parabolic structure does not guarantee rigidification, without imposing any stability conditions.

We conjecture that the map β is generically étale for any genus wild curves, and the failure of showing this only lies in the limitations of the proof strategy of proposition 7.3.2. A research direction we are planning to take on in the immediate future is to prove a powerful result generalising [66, Theorem 1.1] and [67, Theorem 3.14] to the wild parabolic setting. In their work, Pauly and Nieto provide the necessary stability conditions characterising very stable and wobbly loci for holomorphic and logarithmic parabolic bundles. Over the dense open very stable loci, the map ψ is shown to be proper finite, and β is indeed generically étale in those cases as well. That is β is also known to be generically étale for types $(g \geq 2, (0))$ and $\{(g = 1, (1, \cdots, 1))\}$. The latter case is included in the extension of our theory described in appendix C.

The theory we developed in this thesis provides the natural setting for the generalisation of this result, after one defines an appropriate (very-)stability/ wobbly condition for wild parabolic endomorphisms. This result would immediately imply the generic finiteness of the map β, without relying on the connection to rigidify any non-trivial automorphisms of the underlying parabolic bundle and it would be an important result in the realm of geometric Langlands and the P=W conjecture, as T.Pantev tried explaining to me.



8. A Family of Ehresmann Connections

8.1. The Generalised Diagram

The theory we developed until now allows us to draw the following diagram

$$\begin{tikzcd}
 & & \mathcal{M}(\mathscr{C},\mathscr{E},\Phi,\nabla_\infty) \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & \\
\mathscr{H}^{\#} \arrow[d, "\pi"'] & \mathcal{M}(\mathscr{C},Q,\mathcal{L},\partial_\infty) \arrow[l, "\mu"'] \arrow[rr, dashed, "\gamma"] \arrow[d, "\pi'"] & & \mathcal{M}(\mathscr{C},\mathscr{E},Q,\nabla) \arrow[r, "\tau"] \arrow[d, "\varpi'"] & \mathcal{M}(\mathscr{C},\mathscr{E},\nabla) \arrow[d, "\varpi"] \\
\mathcal{M}(\mathscr{C},Q) \arrow[r, "\sim", "id"'] & \mathcal{M}(\mathscr{C},Q) \arrow[rr, "\sim", "id"'] & & \mathcal{M}(\mathscr{C},Q) \arrow[r] & \mathcal{M}(\mathscr{C})
\end{tikzcd} \tag{8.1}$$

where

- $\mathcal{M}(\mathscr{C})$ is parametrising wild algebraic curves $\mathscr{C}=(C,D,\nu,\xi)$.
- $\mathcal{M}(\mathscr{C},Q)$ is parametrising wild curves $\mathscr{C}$ and quadratic differentials Q on wild curves.
- $\mathcal{M}(\mathscr{C},\mathscr{E},\nabla)$ is parametrising wild quasi-parabolic flat bundles $(\mathscr{E},\nabla)$ on the wild curve $\mathscr{C}$. This space comes with a forgetful morphism $\varpi:\mathcal{M}(\mathscr{C},\mathscr{E},\nabla)\to\mathcal{M}(\mathscr{C})$.
- $\mathcal{M}(\mathscr{C},\mathscr{E},Q,\nabla)$ is constructed as the fibre product

$$\begin{tikzcd}
\mathcal{M}(\mathscr{C},\mathscr{E},Q,\nabla) \arrow[r, "\tau"] \arrow[d] & \mathcal{M}(\mathscr{C},\mathscr{E},\nabla) \arrow[d, "\varpi"] \\
\mathcal{M}(\mathscr{C},Q) \arrow[r] & \mathcal{M}(\mathscr{C})
\end{tikzcd}$$

- $\mathcal{M}(\mathscr{C},\mathscr{E},\Phi,\nabla_\infty)$ is parametrising pairs of wild quasi-parabolic Higgs fields Φ and wild quasi-parabolic nilpotent connections ∇_∞ on the same wild bundle $\mathscr{E}$ over the same wild curve $\mathscr{C}$. In particular ∇_∞ has vanishing irregular type. On this space we have a map

$$\begin{aligned}\beta:\mathcal{M}(\mathscr{C},\mathscr{E},\Phi,\nabla_\infty) &\longrightarrow \mathcal{M}(\mathscr{C},\mathscr{E},Q,\nabla)\\ (\mathscr{C},\mathscr{E},\Phi,\nabla_\infty) &\longmapsto (\mathscr{C},\mathscr{E},\chi(\Phi),\nabla_\infty+\Phi)\end{aligned}$$

  cf. definition 7.3.1, which we know to be generically étale for types $(0,\boldsymbol{m})$, cf. proposition 7.3.2.

- $\mathcal{M}(\mathscr{C},Q,\mathcal{L},\partial_\infty)$ is parametrising anti-invariant branched flat bundles on spectral curves $\Sigma_{(\mathscr{C},Q)}$ determined by $(\mathscr{C},Q)$. The map $\mu:\mathcal{M}(\mathscr{C},Q,\mathcal{L},\partial_\infty)\to\mathscr{H}^{\#}$ is the one we defined in 6.106.

- $\pi : \mathscr{H} \to \mathcal{M}(\mathscr{C}, Q)$ is the bundle with fibres $\tilde{H}^1(\Sigma_{(\mathscr{C},Q)}, \mathbb{C}^*)^-$.
- α is the abelianisation morphism giving rise to the generalised spectral correspondence 7.2.12.
- $\gamma := \beta \circ \alpha^{-1}$, a map which, due to 7.2.12 and 7.3.2, induces a dominant rational map

  $$\gamma_{(\mathscr{C},Q)} : \mathcal{M}(\mathcal{L}, \partial_\infty) \to \mathcal{M}_{\mathscr{C},Q}(\mathscr{E}, \nabla)$$

  between the space of anti-invariant branched flat bundles on $\Sigma_{(\mathscr{C},Q)}$ and the space of $SL_2(\mathbb{C})$- wild quasi-parabolic flat bundles on $\mathscr{C}$ for types $(0, \mathcal{m})$.

8.2. Remarks on the Isomonodromy Connection

The map $\varpi : \mathcal{M}(\mathscr{C}, \mathscr{E}, \nabla) \to \mathcal{M}(\mathscr{C})$ admits a well-known flat ϖ-connection j, called the isomonodromy connection. By construction, for a point $(\mathscr{C}, \mathscr{E}, \nabla)$, the irregular type of the irregular connection ∇ is controlled by the enhancement datum ν of the underlying wild curve $\mathscr{C}$. Varying the complex structure of the curve C, the positions of the poles (i.e. D of fixed type $\mathcal{m}$) and the irregular type (i.e. ν), results to a system of *isomonodromic deformation equations* originally studied by [61, Malgrange] and [44, Jimbo-Miwa-Ueno]. The flows of this system with respect to the time-parameters (C, D, ν) determined by $\mathscr{C}$, consist of all points $(\mathscr{C}, \mathscr{E}, \nabla) \in \mathcal{M}(\mathscr{C}, \mathscr{E}, \nabla)$ that carry the same generalised monodromy data. We call these *isomonodromic families of flat connections.* Later, these equations were recast into the form of a flat and, crucially, symplectic Ehresmann connection on the bundle $\mathcal{M}(\mathscr{C}, \mathscr{E}, \nabla) \to \mathcal{M}(\mathscr{C})$ and they were studied from an integrable systems perspective in a series of papers by Boalch cf.[8, Theorem 7.1],[10],[11],[9],[6] and simultaneously by Krichever [53], the latter concentrating in the Hamiltonian and Lax formalism of the theory.

Note here that Boalch and Krichever, instead of using our notion of a wild curve that encodes the irregular type into the enhancement, they use the dual notion of jets instead. Of course they studied more general gauge groups and jets are the natural language to use, but in our humble $SL_2(\mathbb{C})$-case, we have seen that the space of wild curve enhancements is isomorphic to the space of jets $\mathcal{M}(C, D, j)$, more accurately we recovered a duality between the space of curve enhancements $\mathcal{M}_{(C,D)}(\nu)$ on a fixed pair (C, D) and the space of jets over D: $\mathcal{M}_{(C,D)}(C, D, j)$.

The resulting Ehresmann connection j is symplectic with respect to the Atiyah-Bott form on the space of irregular connections [8, Theorem 7.3]. Although it is not mentioned, the connections is also algebraic, this is due to the inherent algebro-geometric nature of the isomonodromy condition, which we briefly describe here.

Let $S \hookrightarrow \mathscr{M}(\mathscr{C})$ be a scheme parametrising a a family of wild curves $\mathscr{C} \longrightarrow S$, and denote $\mathscr{C}_s$ the fibre over a point $s \in S$. Further more consider a relative family of flat connections on $\mathscr{C}$, that is for every point $s \in S$ we have a flat bundle $(\mathscr{E}_s, \nabla_s)$ on $\mathscr{C}_s$. Equivalently we can write this family as a relative flat connection

$$\nabla^{rel} : \mathscr{E} \longrightarrow \mathscr{E} \otimes \omega_{\mathscr{C}/S} \tag{8.2}$$

satisfying the Leibniz rule $\nabla^{rel}(f \cdot s) = f \cdot \nabla^{rel}(s) + d^{rel} f \cdot s$, where the relative differential is the image of the de Rham differential under

$$\begin{aligned} 0 \longrightarrow \omega_S \longrightarrow \omega_{\mathscr{C}} &\longrightarrow \omega_{\mathscr{C}/S} \longrightarrow 0 \\ df &\longmapsto d^{rel} f. \end{aligned}$$

Definition 8.2.1. We call an S-family of flat bundles $(\mathscr{C}_s, \mathscr{E}_s, \nabla_s)_{s\in S}$ an *isomonodromic family*, if there exists an absolute flat connection $\nabla : \mathscr{E} \longrightarrow \mathscr{E} \otimes \omega_{\mathscr{C}}$ lifting the relative connection ∇^{rel} associated to the family. We call this absolute connection the *isomonodromy connection.*

It is precisely the flatness condition of the absolute connection that translates to the isomonodromy condition. This is to be seen as follows. Assuming the existence of such an absolute connection ∇, the generalised monodromy of the flat bundle $(\mathscr{E}_s, \nabla_s)$ has to agree with the generalised monodromy of its parallel transport $(\mathscr{C}_{s'}, \mathscr{E}_{s'}, \nabla_{s'}) = \mathcal{P}^{\nabla}_{\gamma}((\mathscr{C}_s, \mathscr{E}_s, \nabla_s))$ with respect to the absolute connection ∇ along any smooth path on the base $\gamma : [0, 1] \to S$ with $s = \gamma(0)$ and $s' = \gamma(1)$. The monodromy of the flat connections ∇_s remains constant along flat sections $\mathcal{H}_{\nabla}$ of the absolute lift determined by ∇. That is

$$Mon(\nabla_s) = Mon(\nabla_{s'}), \forall s, s' \in S : [\nabla_s] = [\nabla_{s'}] \in \mathscr{M}(\mathscr{C}, \mathscr{E}, \nabla)/\mathcal{H}_{\nabla}.$$

8.3. An ε-family of Flat Ehresmann Connections

Here we only consider the cases where β is generically étale.

Definition 8.3.1. A flat π-Connection h

Given the isomonodromy connection j on $\varpi : \mathscr{M}(\mathscr{C}, \mathscr{E}, \nabla) \to \mathscr{M}(\mathscr{C})$, since the right square of (8.1) is Cartesian, we can pullback the ϖ-connection j to a flat ϖ'-connection $\tau^* j$ using remark 3.1.12. Then using remarks 3.1.11 and 3.1.12 we obtain a connection on an open subset of π', which in turn descends along the étale quotient, via remark 3.1.11, to an Ehresmann connection on π. Lastly by remark **??**, we extend the so obtained π-connection on an open subset, to a meromorphic π-connection h on a dense open of π.

Remark 8.3.2. The linear lift E, of definition 3.3.8, of the Euler vector field Z of definition

5.2.15, is the generator of the following $\mathbb{C}^*$-action

$$\begin{aligned} m : \mathbb{C}^* \times \mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial_\infty) &\longrightarrow \mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial_\infty) \\ (t, (C, D, \nu, \xi, Q, \mathcal{L}, \partial_\infty)) &\longmapsto (C, D, t^2\nu, \xi, t^2 Q, \mathcal{L}, \partial_\infty). \end{aligned} \tag{8.3}$$

that is the lift the natural $\mathbb{C}^*$-action on the space of quadratic differentials on wild curves.

Proposition 8.3.3. ***How to build a family***
Let h_ε be an ε-family of flat Ehresmann connections underlying a Joyce structure on a complex manifold M. Then, for the $\mathbb{C}^*$-actions ϕ_t on M cf.3.3.7 and m_t on TM cf.3.3.8, the following identity holds

$$m_{t^{-1}*} h_\varepsilon(\phi_{t*} v) = h_{t\cdot\varepsilon}(v). \tag{8.4}$$

Proof. Let $v = \sum_i \alpha_i \frac{\partial}{\partial z_i} \in TM$. Recall from 3.3.3,3.3.8 the Euler fields Z, E, generating the $\mathbb{C}^*$-actions on TM and $TX = TTM$. By the Joyce structure property [J3] cf. 3.4.1, we have

$$\begin{aligned} &h([Z, v]) = [h_\nabla(Z), h(v)] = [E, h(v)] \implies h(t^{-1}v) = t^{-1}h(v) \\ &\implies t^{-1}\sum_i \alpha_i \frac{\partial}{\partial z_i} + \sum_{p,q} \eta^{pq} \frac{\partial W_i(tz_i, \theta_i)}{\partial \theta_p} \frac{\partial}{\partial \theta_q} = t^{-1}\left(\sum_i \alpha_i \frac{\partial}{\partial z_i} + \sum_{p,q} \eta^{pq} \frac{\partial W_i(z_i, \theta_i)}{\partial \theta_p} \frac{\partial}{\partial \theta_q}\right) \\ &\implies W_i(tz_i, \theta_i) = t^{-1} W_i(z_i, \theta_i) \ \forall i \end{aligned}$$

Applying this result to the ε-family, we obtain

$$\begin{aligned} m_{t*} h_\varepsilon(\phi_{t^{-1}*} \frac{\partial}{\partial z_i}) &= m_{t*} h_\varepsilon(t \cdot \frac{\partial}{\partial z_i}) \\ &= m_{t*}\left(t \cdot \left(\frac{\partial}{\partial z_i} + \sum_{p,q} \eta^{pq} \frac{\partial W_i}{\partial \theta_p} \frac{\partial}{\partial \theta_q} + \sum_q (\varepsilon \cdot t)^{-1} \frac{\partial}{\partial \theta_q}\right)\right) \\ &= \frac{\partial}{\partial z_i} + \sum_{p,q} \eta^{pq} \frac{\partial W_i}{\partial \theta_p} \frac{\partial}{\partial \theta_q} + \sum_q (\varepsilon \cdot t)^{-1} \frac{\partial}{\partial \theta_q} \\ &= h_{t\cdot\varepsilon}\left(\frac{\partial}{\partial z_i}\right) \end{aligned}$$

In other words, we can construct the whole pencil via this action from any element of the ε-family, in particular from $h = h_1$. □

Definition 8.3.4. An ε-family of Flat π-Connections
Let h be the π-connection of definition 8.3.1. We define an ε-family h_ε of flat Ehresmann

connections on the bundle $\pi : \mathscr{H}^{\#} \longrightarrow \mathcal{M}(\mathscr{C}, Q)$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}_{\mathscr{H}/\mathcal{M}} \arrow[r, "i"] & \mathcal{T}_{\mathscr{H}} \arrow[r] & \pi^*\mathcal{T}_{\mathcal{M}} \arrow[r] \arrow[l, bend right, "h"'] \arrow[l, bend right=60, "h_\varepsilon"'] & 0,
\end{tikzcd} \tag{8.5}$$

by setting $h_\varepsilon(v) := m_{\varepsilon^{-1}*}h(\phi_{\varepsilon *}v)$, where m_{t*} and ϕ_{t*} are given by the action of the Euler fields E (8.3.2) and Z (5.2.15) respectively, as prescibed by proposition 8.3.3.

9. An Example

In this chapter, we describe the constructions that appeared in this thesis in a concrete example and in great detail, namely for the data $(g, \mathfrak{m}) = (0, \{7\})$, using this as an opportunity to digest the plethora of definitions and structural results.

9.1. Quadratic Differentials, Spectral and Wild Curves

Consider quadratic differentials of type $(0; 7)$, these are sections

$$Q \in H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(7 \cdot p)); \text{ for some fixed } p \in \mathbb{P}^1.$$

Since $\omega_{\mathbb{P}^1}^{\otimes 2} \cong \mathcal{O}(-4)$, these quadratic differentials are determined by global sections of $\mathcal{O}_{\mathbb{P}^1}(3)$, carrying a single pole. Acting by an element of $Aut(\mathbb{P}^1)$, we can take the pole p to be at $\infty \in \mathbb{P}^1$. These correspond to quadratic differentials associated to cubic polynomial functions on $\mathbb{P}^1$

$$Q(x) = p(x) \cdot dx^{\otimes 2}; \; p(x) \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3)) \tag{9.1}$$

with a pole of order 7 at $\infty \in \mathbb{P}^1$. $\mathcal{M}(C, Q)_{(0;7)}$ is the moduli space parametrising quadratic differentials with simple zeros distinct from ∞ modulo automorphisms, i.e.

$$Q' \sim Q \iff \exists f \in Aut(\mathbb{P}^1) : f^*Q = Q.$$

In particular f is a Möbius transformation of type

$$f : x \longmapsto \alpha \cdot x + \beta; \alpha \in \mathbb{C}^*, \beta \in \mathbb{C},$$

since it must preserve the position of the pole. Explicitly, let $Q(x)$ be quadratic differential of type $(0; 7)$ with three simple roots at $r_1, r_2, r_3 \in \mathbb{C}$

$$Q(x) = \prod_{i=1}^{3} a(x-r_i)dx^{\otimes 2} = (a_3x^3 + a_2x^2 + a_1x + a_0)dx^{\otimes 2},$$

then $f : x \longmapsto \alpha \cdot x$ acts by $f^*Q(x) = \alpha^5 \prod_{i=1}^{3} a_3(x-r_i)dx^{\otimes 2}$ and $f : x \longmapsto x + \beta$ acts by shifting the roots of Q: $r_i \mapsto r_i' = r_i + \beta$. In particular, for $\beta = \frac{a_2}{3a_3}$ we have $\sum r_i' = 0$. Furthermore, observe that

$$\prod_{i=1}^{3}(x-r_i)dx^{\otimes 2} = \left(x^3 - \sum_i r_i x^2 + (r_1r_2 + r_2r_3 + r_1r_3)x - \prod_i r_i\right) dx^{\otimes 2}.$$

Therefore, factoring the set of quadratic differentials with simple poles of type $(0,7)$ by pole preserving automorphisms of $\mathbb{P}^1$, we obtain reduced cubic polynomials with simple roots, up to the action of the group of fifth roots of unity coming from the rescaling action

$$\{x^3 + ax + b \mid (a,b) \in \mathbb{C}^2 \backslash \{4a^3 + 27b^2 = 0\}\} \Big/ \mathbb{Z}/5\mathbb{Z}.$$

Moreover, since monic polynomials with simple roots are determined by their roots we have in total

$$\mathcal{M}(C,Q)_{(0;7)} = \{(r_1, r_2, r_3) \mid \textstyle\sum_i r_i = 0 \text{ and } \nexists i \neq j \text{ s.t. } r_i = r_j\} \Big/ \left(\mathbb{Z}/5\mathbb{Z}\right). \tag{9.2}$$

Note here that since the action of μ_5 is free, the resulting space does not have an orbifold point. Now consider a quadratic differential of this type

$$Q(x) = (x^3 + ax + b)dx^{\otimes 2} \in H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}^{\otimes 2}(7\infty)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$$

Inverting the coordinate x on $\mathbb{P}^1$ by $x \mapsto \frac{1}{z}$, we obtain a coordinate centred at the pole of Q

$$Q(x) = (x^3 + ax + b)dx^{\otimes 2} \mapsto Q(z) = \left(\frac{1}{z^7} + \frac{a}{z^5} + \frac{b}{z^4}\right) dz^{\otimes 2} \tag{9.3}$$

The associated spectral curve Σ is an elliptic curve, which we see as a degree 3 curve in $Tot(\omega_C(4\infty)) \xrightarrow{\pi} C$ cut out by the equation $y^2 = \pi^*Q$. Recall that by remark 4.2.3, since the order of the pole at ∞ is odd, the ramification locus is precisely given by the pre-images of ∞ and the zeros of Q under the covering map. The anti-invariant cohomology of the punctured spectral curve is 2-dimensional $H^1(\Sigma^\circ, \mathbb{C})^- = H^1(\Sigma, \mathbb{C})^-$ and the intersection product agrees with that of the projective elliptic curve. A basis for the de Rham cohomology is given by $\frac{dx}{y}, \frac{xdx}{y}$.

Let us denote by $\tilde{\infty} = supp(p^{-1}(\infty))$ the closed point lying over the pole of Q in the spectral cover by the elliptic curve $p : \Sigma \to \mathbb{P}^1$.

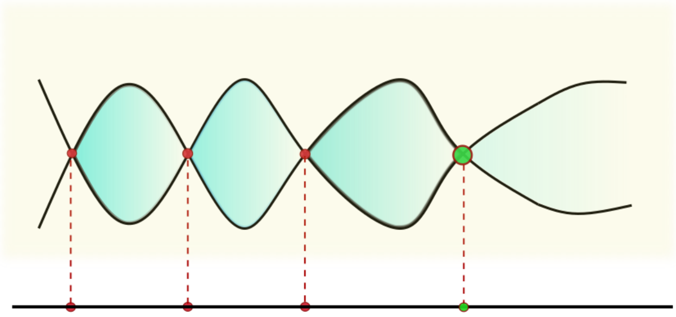


Figure 2: Here the green points are $\infty \in \mathbb{P}^1$ and $\tilde{\infty} \in \Sigma$, and the red points correspond to the rest of the zeros of $Q(x)$.

The data of a wild curve $\mathscr{C} = (C, D, \nu, \xi)$ in this setting are as follows. The underlying algebraic curve is of course $C = \mathbb{P}^1$ and the divisor D is defined in 4.1.4 to be

$$D = \sum_i \left\lceil \frac{m_i}{2} \right\rceil \cdot p_i = \left\lceil \frac{7}{2} \right\rceil \cdot \infty = 4 \cdot \infty.$$

The enhancement ν is defined in 5.1.1 by viewing the quadratic differential as a section of $H^0(C, \omega_C(D)^{\otimes 2}|_D)$ and then restricting it to D, in this case we have

$$\nu = (x^3+ax+b)dx^{\otimes 2}|_{4\cdot\infty} = (z+az^3+bz^4)\frac{dz^{\otimes 2}}{z^8}|_{4\cdot\infty} = z+az^3 \in H^0(D, \omega_{\mathbb{P}^1}^{\otimes 2}(8\cdot\infty)|_{4\cdot\infty}) \quad (9.4)$$

In terms of remark 5.1.5, the enhancement determines a double cover $p_\nu : D_\nu \to D$ with $D_\nu = \iota_D^* \Sigma = \pi^* D \cap_{Tot} \Sigma = 8 \cdot \tilde{\infty} \subset Tot(\omega_{\mathbb{P}^1}(4\infty))$. The enhancement class

$$\nu = [(x + ax^3)dx^{\otimes 2}] \in H^0(C, \omega_C(D)^{\otimes 2}|_D)$$

is the datum fixing the class-ν cover via the defining sequence

$$0 \longrightarrow I_{D_\nu} \longrightarrow \mathcal{O}_\Sigma \longrightarrow \mathcal{O}_{D_\nu} \longrightarrow 0, \quad (9.5)$$

that is $D_\nu = p^* D$ is determined by the curve-enhancement associated to Q: $\nu = Q|_D$.

The last ingredient of a wild curve is the signing. In general this amount to a choice of a pre-image of each closed point of D in the spectral cover $\xi : D^{red} \hookrightarrow D_\nu^{red}$; such that $p_\nu(\xi(p)) = p$, $\forall p \in D^{red}$. In this case, our spectral curve is ramified over the pole of the quadratic differential at ∞, and thereby the choice is unique. What is non-trivial, is the divisor $D_\xi = \sum_i n_i \cdot \xi(p_i) = 4 \cdot \tilde{\infty}$ introduced in definition 5.1.9. In total the wild curve datum reads

$$\mathscr{C} = (\mathbb{P}^1, 4 \cdot \infty, x + ax^3, \infty \mapsto \tilde{\infty}).$$

9.2. Period Structure

For the period coordinates $\{z_i\}_i$ on $\mathcal{M}(\mathscr{C}, Q)$ we have by definition 5.2.8 and lemma 5.2.13

$$\left\{ z_i = \int_{\gamma_i} [\lambda] = \int_{\gamma_i} \left(\sqrt{x^3+ax+b}\right) dx \right\}_{\gamma_i \in \tilde{H}^1(\Sigma,\mathbb{C})^-} \tag{9.6}$$

The anti-invariant cohomology of the spectral curve $\tilde{H}^1(\Sigma,\mathbb{C})^-$ is 2-dimensional and the intersection product agrees with that of the projective elliptic curve. The differential of the period map yields

$$\begin{aligned} \nabla[\delta] : T_{(a,b)}\mathcal{M}(C,Q) &\longrightarrow \tilde{H}^1(\Sigma_{(a,b)},\mathbb{C})^- \\ \frac{\partial}{\partial a} &\longmapsto \nabla_{\partial_a}\lambda = -\frac{x\cdot dx}{2y} \\ \frac{\partial}{\partial b} &\longmapsto \nabla_{\partial_b}\lambda = \frac{dx}{2y}. \end{aligned}$$

In de Rham terms, $\langle \nabla_{\partial_b}\lambda \rangle$ generates the (0,1)-part of the Hodge filtration and $\langle \nabla_{\partial_b}\lambda \rangle$ the (1,0)-part. For the periods of the holomorphic 1-form we have

$$\omega_i = \int_{\gamma_i} \nabla_{\partial_b} = \frac{\partial z_i}{\partial b}. \tag{9.7}$$

Moreover the collection

$$\eta_i := -\int_{\gamma_i} \nabla_{\partial_a} = \frac{\partial z_i}{\partial a} \tag{9.8}$$

are the Weirstraß quasi-periods, which in Betti terms, i.e. after exponentiation, compute the monodromy of the Picard-Fuchs equation/ Gauß-Manin connection. Using the Legendre relation $\omega_2\eta_1 - \omega_1\eta_2 = 2\pi i$, cf. [33, Chapter 2.2], we have

$$\frac{\partial}{\partial a} = -\eta_1\frac{\partial}{\partial z_1} - \eta_2\frac{\partial}{\partial z_2}, \quad \frac{\partial}{\partial b} = \omega_1\frac{\partial}{\partial z_1} + \omega_2\frac{\partial}{\partial z_2} \tag{9.9}$$

which can be inverted to

$$\frac{\partial}{\partial z_1} = -\frac{1}{2\pi i}\left(\omega_2\frac{\partial}{\partial a} + \eta_2\frac{\partial}{\partial b}\right), \quad \frac{\partial}{\partial z_2} = \frac{1}{2\pi i}\left(\omega_1\frac{\partial}{\partial a} + \eta_2\frac{\partial}{\partial b}\right). \tag{9.10}$$

In the period coordinates $\{z_i\}_i$ the symplectic form reads

$$\omega = -\frac{1}{2\pi i} dz_1 \wedge dz_2 = da \wedge db, \tag{9.11}$$

and the skew-symmetric form (Euler product) is $\eta(dz_1, dz_2) = 2\pi i$, which is in accordance with condition **[J1]** for a Joyce structure, cf. 3.4.1.

By definition, the Euler vector field Z of definition 5.2.15, inducing the $\mathbb{C}^*$-action on $\mathcal{M}(\mathscr{C}, Q)$ is

$$Z = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} = \frac{4a}{5} \frac{\partial}{\partial a} + \frac{6b}{5} \frac{\partial}{\partial b} \tag{9.12}$$

9.3. Wild Quasi-Parabolic Higgs Bundles

Now we want to describe the elements of $\mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi)$. These are irregular $Sl_2(\mathbb{C})$ Higgs Bundles that respect a certain wild quasi-parabolic structure L_ξ on $E = \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$, i.e. preserve $\mathscr{E} = (E, L_\xi)$ and such that $Q := tr(\Phi^2)$ is a quadratic differential of type $(0, \{7\})$ with $\nu := tr(\Phi^2)|_{4\cdot\infty} = (x + ax^3)\frac{dx^{\otimes 2}}{x^8}$.

In order to determine what a generic such Higgs field looks like, we first expand it in powers of x: $\Phi = \sum_j \phi_j \cdot x^j dx$ for some $\phi_j \in \mathfrak{sl}_2(\mathbb{C})$. Since $\Phi \in H^0(\mathbb{P}^1, \mathfrak{sl}_2(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}) \otimes \omega_{\mathbb{P}^1}(4\cdot\infty))$ it carries maximally a term of order 2 and so $\Phi = \phi_2 \cdot x^2 + \phi_1 \cdot x + \phi_0$. Recall that the form dx contributes with a pole of order 2 at ∞. Moreover, the Higgs field is an $\mathfrak{sl}_2$-valued 1-form, so after putting ϕ_2 in an upper triangular form, we have

$$\phi_2 = \begin{bmatrix} \alpha_2 & \beta_2 \\ 0 & -\alpha_2 \end{bmatrix}, \text{ with determinant } tr((\phi_2 x^2)^2) = -\alpha_2^2 x^4.$$

The wild structure of the curve $\mathscr{C}$ on which Φ is defined, allows its determinant to develop poles only up to order 3, therefore $\alpha_2 \overset{!}{=} 0$. For ϕ_1, first we rescale the coordinate on $\mathbb{P}^1$ to normalise the left-lower entry $\phi_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ 1 & -\alpha_1 \end{bmatrix}$, then conjugating by the $SL_2(\mathbb{C}[z])$-matrix $T = \begin{bmatrix} 1 & -\alpha_1 \\ 0 & 1 \end{bmatrix}$ allows us to eliminate the diagonal terms $T\phi_1 T^{-1} = \begin{bmatrix} 0 & \beta_1 - \alpha_1^2 \\ 1 & 0 \end{bmatrix}$. Adding the constant term ϕ_0, we have a Higgs field of the form

$$\Phi = \begin{bmatrix} p & x^2 + c \cdot x + d \\ x - q & -p \end{bmatrix}$$

with $tr(\Phi^2) = p^2 + x^3 + (q+\beta)x^2 + (q\beta + \gamma)x + q\gamma$. Imposing the condition of the enhancement type, we see that $c \overset{!}{=} -q$. Lastly setting $a := d - q^2$ and $b := p^2 - q^3 - aq$ we obtain in total

$$\Phi = \begin{bmatrix} p & x^2 + qx + q^2 + a \\ x - q & -p \end{bmatrix} dx, \text{ with } -tr(\Phi^2) = (x^3 + ax + b)dx^{\otimes 2} = Q(x). \tag{9.13}$$

Passing to the associated spectral cover $p : \Sigma \to C$ we can diagonalise this generic Higgs field Φ. Let w be a coordinate on the spectral cover centered at $\tilde{\infty}$ with $w^2 = z$. Pulling back the quadratic differential near the ramification we have

$$p^*Q(w) = 4(w^4 + aw^8 + bw^{10})\frac{dw^{\otimes 2}}{w^{16}}$$

with square root

$$\lambda = \sqrt{p^*Q} = (2w^2 + aw^6 + bw^8 + \mathcal{O}(w^{10}))\frac{dw}{w^8}.$$

Then $D_\nu = 8 \cdot \tilde{\infty} \implies D_\xi = 4 \cdot \tilde{\infty}$ and the irregular types read

$$\lambda_\nu = \lambda|_{D_\nu} = (2w^2 + aw^6)\frac{dw}{w^8}, \tag{9.14}$$

$$\lambda_\xi = \lambda|_{D_\xi} = (2w^2)\frac{dw}{w^8}. \tag{9.15}$$

The pulled-back Higgs field is

$$p^*\Phi(w) = -2\begin{bmatrix} pw^5 & w + qw^3 + (q^2 + a)w^5 \\ w^3 - qw^5 & -pw^5 \end{bmatrix}\frac{dw}{w^8} \tag{9.16}$$

and has Eigenvectors

$$v_\pm = \begin{bmatrix} \frac{pw^4 \mp \lambda}{-1+qw^2} \\ w^2 \end{bmatrix} = \begin{bmatrix} \pm w \pm qw^3 + pw^4 \pm (a + q^2)w^5 + pqw^6 + (p^2 + q^3)w^7 + \mathcal{O}(w^8) \\ w^2 \end{bmatrix} \tag{9.17}$$

Reducing the Eigenvectors to D_ξ we have:

$$\bar{v}_\pm = \begin{bmatrix} \pm w \pm qw^3 \\ w^2 \end{bmatrix} \in H^0(D_\xi, \mathcal{O}_{D_\xi}^{\oplus 2}). \tag{9.18}$$

Therefore the Higgs field Φ lives on the wild quasi-parabolic bundle

$$\mathscr{E} = (E, L_\xi) = \left(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}, \left\langle \bar{v} := \begin{bmatrix} w + qw^3 \\ w^2 \end{bmatrix} \right\rangle\right) \tag{9.19}$$

Let B denote the matrix with columns the Eigenvectors $v_\pm$. The diagonalised Higgs field $p^*\Phi$ reads

$$B^{-1}p^*\Phi B = \begin{bmatrix} 2w^2 + aw^6 + bw^8 + \mathcal{O}(w^9) & 0 \\ 0 & -2w^2 - aw^6 - bw^8 + \mathcal{O}(w^9) \end{bmatrix}\frac{dw}{w^8} \tag{9.20}$$

This restricts over D_ν to

$$B^{-1}p^*\Phi B|_{D_\nu} = \begin{bmatrix} 2w^2 + aw^6 & 0 \\ 0 & -2w^2 - aw^6 \end{bmatrix}\frac{dw}{w^8} = \begin{bmatrix} \lambda_\nu & 0 \\ 0 & -\lambda_\nu \end{bmatrix} \tag{9.21}$$

and over D_ξ to

$$B^{-1}p^*\Phi B|_{D_\xi} = \begin{bmatrix} 2w^2 & 0 \\ 0 & -2w^2 \end{bmatrix} \frac{dw}{w^8} \tag{9.22}$$

as expected from (9.15) of course.

9.4. Wild Quasi-Parabolic Flat Bundles

Now we want to consider a nilpotent quasi-parabolic connection ∇_∞ with respect to L_ξ, that is $\nabla_\infty|_D \in End^{\mathfrak{n}}(\mathscr{E})|_D$. We will first determine the form of a generic such connection. For this let

$$\nabla_\infty(x) = d + \begin{bmatrix} a(x) & b(x) \\ c(x) & d(x) \end{bmatrix} dx, a(x), b(x), c(x), d(x) \in \mathbb{C}[x] \tag{9.23}$$

under the coordinate transformation $x \mapsto \frac{1}{z}$ we get

$$\nabla_\infty(z) = d + \begin{bmatrix} a(z)z^2 & b(z)z^2 \\ c(z)z^2 & d(z)z^2 \end{bmatrix} \frac{dz}{z^4}. \tag{9.24}$$

Since the connection $\nabla_\infty(x)$ is only allowed to have a pole maximally of order 4 at ∞ we see that $a(x), b(x), c(x), d(x) \overset{!}{\in} \mathbb{C}[x]/x^5$, or equivalently $a(z), b(z), c(z), d(z) \overset{!}{\in} z^{-2}\mathbb{C}[z]/z^3$. Pulling the connection back to the spectral cover defined by Φ, $z \mapsto w^2, dz \mapsto 2wdw$,this reads

$$p^*\nabla_\infty(w) = d + \begin{bmatrix} a(w)w^5 & b(w)w^5 \\ c(w)w^5 & d(w)w^5 \end{bmatrix} \frac{dw}{w^8} \tag{9.25}$$

with $a(w), b(w), c(w), d(w) \overset{!}{\in} w^{-4}\mathbb{C}[z]/w^5$. Applying the σ^*-equivariance condition, we obtain

$$p^*\nabla_\infty(w) = d + \begin{bmatrix} a(w)w^5 & b(w)w^5 \\ -b(-w)w^5 & -a(-w)w^5 \end{bmatrix} \frac{dw}{w^8} =: d + A_\infty(w). \tag{9.26}$$

Putting A in the Eigenbasis of Φ results to

$$B^{-1}A_\infty(w)B = \begin{bmatrix} -\frac{b(w)}{2}w^4 + \frac{b(1-q)}{2}w^6 - \frac{a(b-2p)-bq(1+q))}{2}w^8 + \mathcal{O}(w^9) & -/ \\ \frac{b}{2}w^4 - a(w)w^5 + \frac{b(1+q)}{2}w^6 + \frac{(a(b+2p)+b(-1+q)q)}{2}w^8 + \mathcal{O}(w^9) & -/ \end{bmatrix} \frac{dw}{w^8}. \tag{9.27}$$

After imposing the condition that ∇_∞ is nilpotent with respect to the quasi-parabolic structure L_ξ, i.e. the diagonals vanish to order $\mathcal{O}(w^7)$ and the off-diagonals to order $\mathcal{O}(w^3)$, we have that

$$B^{-1}A_\infty(w)B = \begin{bmatrix} -apw^8 + \mathcal{O}(w^9) & aw^5 + apw^8 + \mathcal{O}(w^9) \\ -aw^5 + apw^8 + \mathcal{O}(w^9) & apw^8 + \mathcal{O}(w^9) \end{bmatrix} \frac{dw}{w^8}, \tag{9.28}$$

with $a(w) = a \in \mathbb{C}$. Furthermore, the $\mathcal{O}(w^9)$-terms also need to vanish from every entry, otherwise our connection would develop poles at 0 as well as ∞. Reparametrising $a = r/2p$ where $r \in \mathbb{C}$ and p the constant appearing in the Higgs field Φ we have

$$B^{-1}A_\infty(w)B = \begin{bmatrix} -\frac{r}{2}w^8 & \frac{r}{2p}w^5 + \frac{r}{2}w^8 \\ -\frac{r}{2p}w^5 + \frac{r}{2}w^8 & \frac{r}{2}w^8 \end{bmatrix} \frac{dw}{w^8}. \tag{9.29}$$

That is

$$B^{-1}A_\infty(w)B|_{D_\xi} = 0 \text{ and } B^{-1}A_\infty(w)B|_{D_\nu} = \begin{bmatrix} 0 & \frac{r}{2p}w^5 \\ -\frac{r}{2p}w^5 & 0 \end{bmatrix} \frac{dw}{w^8}. \tag{9.30}$$

The original connection on the base curve takes the form

$$\nabla_\infty(x) = d + \frac{1}{2p} \begin{bmatrix} r & 0 \\ 0 & -r \end{bmatrix} dx, r \in \mathbb{C}, \tag{9.31}$$

i.e. developing at most poles of order 2 at infinity.

9.5. Abelianisation of the flat connection

For such a generic connection ∇_∞, we compute the connection $\tilde{\nabla}_\infty$ obtained by meromorphic gauge transformation from $p^*\nabla_\infty$. We write, as before, $y(x) = \frac{\lambda(x)}{dx}$, i.e.

$$y^2(x) = x^3 + ax + b.$$

We compute the modified connection to be

$$\begin{aligned} \tilde{\nabla}_\infty &= B^{-1}dB + B^{-1}A_\infty(w)B \\ &= d - \frac{1}{2y(w)} \begin{bmatrix} \frac{y'(w)}{2w} - \frac{p+y(w)}{w^2-q} + r & \frac{y'(w)}{2w} + \frac{p-y(w)}{w^2-q} + r + \frac{ry(w)}{p} \\ \frac{y'(w)}{2w} + \frac{p-y(w)}{w^2-q} - r - \frac{ry(w)}{p} & -\frac{y'(w)}{2w} + \frac{p+y(w)}{w^2-q} - r \end{bmatrix} 2wdw \end{aligned}$$

From $\tilde{\nabla}_\infty$ we extract the abelianisation ∂_∞ of ∇_∞, which is given via the 1-form of the left-upper entry of $\tilde{\nabla}_\infty$

$$\begin{aligned} \partial_\infty &= d - \left(\frac{y'(w)}{2y(w)} - \frac{w}{y(w)} \left[\frac{p}{w^2-q} + r - \frac{wy(w)}{w^2-q} \right] \right) dw \\ &= d - \left(\frac{y'(w)}{2y(w)} - \frac{w}{y(w)} \left[\frac{p}{w^2-q} + r \right] \right) dw \end{aligned}$$

Recall that the canonical connection (6.96) in the w coordinate reads $\partial_{can} = d + \frac{y'(w)}{y(w)}dw$, from which we see that, indeed $\partial_\infty \otimes \sigma^*\partial_\infty = \partial_{can}$. Note that ∂_∞ is an abelian connection that develops a logarithmic singularity at $w = 0$, due to the term $\frac{y'(w)}{y(w)}wdw$, that arose in $\tilde{\nabla}_\infty$ from the $B^{-1}dB$-part of the gauge transformation cf. remark 7.2.8.

In order to pass from a point on $\mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial_\infty)$ to the period coordinates $(z_i, \theta_i)_i$ on $T\mathcal{M}(\mathscr{C}, Q)$, we need have to twist $(\mathcal{L}, \partial_\infty)$ by the dual of a square root of the canonical flat bundle $(p^*\omega_C(D), \partial_{can})$, cf. (6.96), as prescribed by proposition 6.3.11 and then integrate along the anti-invariant cycles. Explicitly we obtain the coordinates

$$z_i = \int_{\gamma_i} [\lambda] = \int_{\gamma_i} \left(\sqrt{x^3 + ax + b}\right) dx \tag{9.32}$$

$$\theta_i = -\int_{\gamma_i} \left(\frac{p}{x-q} + r\right) \frac{dx}{2y}. \tag{9.33}$$

In particular $\exp(\theta_i) \in \tilde{H}^1(\Sigma, \mathbb{C}^*)^-$ is the monodromy of $(\mathcal{L} \otimes \mathcal{L}_0^*, \partial_\infty \otimes \partial_0^*)$ along a cycle $\gamma_i \in \tilde{H}^1(\Sigma, \mathbb{C})^-$, obtaining so compatible coordinates on the fibres of $\mathscr{H}^\#$, i.e. on the torus fibres $(\mathbb{C}^*)^{2k} = \tilde{H}^1(\Sigma, \mathbb{C}^*)^-$. This is precisely recovers the flat bundle and monodromy considered in [17][Eq.46,49].

9.6. Relation to the Bridgeland-Masoero Cubic Oscillator

Our fibration $\mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla_\infty) \to \mathcal{M}(\mathscr{C}, Q)$ now fully recovers the fibration $M \to S$ studied in [17, E.(3),(4)], where

$$\begin{aligned} M :&= \{(a, b, q, p, r) \in \mathbb{C}^5 | b = p^2 - q^3 - aq \text{ and } 4a^3 + 27b^2 \neq 0, p \neq 0\} \\ S :&= \{(a, b) \in \mathbb{C}^2 | 4a^3 + 27b^2 \neq 0\} \end{aligned}$$

We summarise the relation between their data and ours in the following table

M	$\mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla_\infty)$	$\mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial_\infty)$
a	$\mathscr{C} = (\mathbb{P}^1, 4\infty, \nu = (x^3 + ax)dx^{\otimes 2})$	$\mathscr{C} = (\mathbb{P}^1, 4\infty, \nu = (x^3 + ax)dx^{\otimes 2})$
b	$b = p^2 - q^3 - aq$	$Q = (x^3 + ax + b)dx^{\otimes 2})$
q	$\mathscr{E} = \left(\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}, \left\langle \bar{v} := \begin{bmatrix} w + qw^3 \\ w^2 \end{bmatrix} \right\rangle\right)$	$\mathcal{L} = \mathcal{O}_\Sigma((q, p) + \infty)$
p	$\Phi = \begin{bmatrix} p & x^2 + qx + q^2 + a \\ x - q & -p \end{bmatrix} dx$	$\mathcal{L} = \mathcal{O}_\Sigma((q, p) + \infty)$
r	$\nabla_\infty = d + \frac{1}{2p} \begin{bmatrix} r & 0 \\ 0 & -r \end{bmatrix} dx$	$\partial_\infty = d - \left(\frac{y'(w)}{2y(w)} - \frac{w}{y(w)} \left[\frac{p}{w^2 - q} + r\right]\right) dw$

Bridgeland and Masoero study the second-order linear differential equation

$$Y''(x) = \tilde{Q}(x,\hbar)Y(x), \tag{9.34}$$

where

$$\tilde{Q}(x,\hbar) := \hbar^{-2}Q_0(x) + \hbar^{-1}Q_1(x) + Q_2(x) \tag{9.35}$$

$$Q_0(x) := x^3 + ax + b = y(x) = \frac{Q(x)}{dx^{\otimes 2}} \tag{9.36}$$

$$Q_1(x) := \frac{p}{x-q} + r \tag{9.37}$$

$$Q_2(x) := \frac{3}{4(x-q)^2} + \frac{r}{2p(x-q)} + \frac{r^2}{4p^2}. \tag{9.38}$$

In the setting of [17], the form of the function $Q_2(x)$ is prescribed by imposing an apparent singularity condition to 9.34, cf. [17, Lemma 2.1]. We can recover this differential equation from our data $(\mathscr{C}, \mathscr{E}, \Phi, \nabla_\infty)$ as follows. Consider the ε-family $\nabla_\infty + \varepsilon^{-1}\Phi$ for generic ∇_∞, Φ on $\mathcal{O}^2_{\mathbb{P}^1}$ and apply the gauge transformation

$$g(x) = \frac{1}{\sqrt{x-q}} \begin{bmatrix} 0 & 1 \\ x-q & -p - \frac{1}{2(x-q)} - \frac{r\varepsilon}{2p} \end{bmatrix} \tag{9.39}$$

this gives

$$\nabla(\varepsilon) = g(x)^{-1}(\nabla_\infty + \varepsilon^{-1}\Phi)g(x) = d + \begin{bmatrix} 0 & 1 \\ \tilde{Q}(x,\varepsilon) & 0 \end{bmatrix} dx. \tag{9.40}$$

Thereby we recover the differential equation 9.34, after replacing $\hbar$ with ε, via

$$\nabla(\varepsilon) \begin{bmatrix} Y(x) \\ Y'(x) \end{bmatrix} = 0. \tag{9.41}$$

Bridgeland and Masoero study isomonodromic deformations of this ε-family of equations, varying, in our language, $(\mathscr{C}, Q)$ and specifying $(\mathscr{E}, \nabla(\varepsilon) = \nabla_\infty + \varepsilon^{-1}\Phi)$ such that the monodromy of $\nabla(\varepsilon)$, i.e. the monodromy of 9.34, remains constant. This isomonodromic problem is closely related to Painlevé I, where one really studies isomonodromic deformations of $d + \Phi$ on the trivial bundle. We are studying a generalised version of this, recovering the Painlevé I case by setting the parameter $r = 0$ (and $\varepsilon = 1$). Explicitly, the isomonodromic flows in our case are

$$-\varepsilon^{-1}\frac{\partial}{\partial r} + \left(\frac{\partial}{\partial b} + \frac{1}{2p}\frac{\partial}{\partial p} + \frac{r}{2p^2}\frac{\partial}{\partial r}\right),$$

$$-\varepsilon^{-1}2p\frac{\partial}{\partial q} - \varepsilon^{-1}(3q^2+a)\frac{\partial}{\partial p} + \left(\frac{\partial}{\partial a} - q\frac{\partial}{\partial b} - \frac{r}{p}\frac{\partial}{\partial q} - \frac{r(3q^2+a)}{2p^2}\frac{\partial}{\partial p} - \frac{r^2}{2p^3}(3q^2 + a\frac{\partial}{\partial r})\right).$$

For $r = 0$ one recovers the Painlevé I flow defined by the Hamiltonian

$$H(a, b, q, p) = q^3 + aq + b - p^2.$$

A. Dimensions of Spaces and Degrees of Divisors

Dimensions of Spaces:

- $\dim(\mathcal{M}_{g,d}) = 3g - 3 + d$
- $\dim(\mathcal{M}(C, D)) = 3g - 3 + deg(D)$
- $\dim(\mathcal{M}(C, D, \nu)) = 3g - 3 + 2\deg(D)$
- $\underline{\dim(\mathcal{M}(\mathscr{C})) = 3g - 3 + \deg(D) =: k}$
- $\dim(\mathcal{M}(C, Q)_{(g,m)}) = 6g - 6 + d + \deg(H)$
- $\dim(\mathcal{M}(C, Q)^{res=0}_{(g,m)}) = 6g - 6 + 2\deg(D)$
- $\underline{\dim(\mathcal{M}(\mathscr{C}, Q)) = 6g - 6 + 2 \cdot \deg(D) = 2k}$
- $\dim(H^1(\Sigma, \mathbb{C})) = 8g - 6 + 2deg(D) = 2g_\Sigma$
- $\dim(H^1(\Sigma^\circ, \mathbb{C})) = 8g - 6 + 2deg(D) + deg(R),\ \Sigma^\circ := \Sigma\backslash R$
- $\dim(\tilde{H}^1(\Sigma, \mathbb{C})^-) = 6g - 6 + 2deg(D) = 2g_\Sigma - 2g = \dim(\tilde{H}^1(\Sigma^\circ, \mathbb{C})^-)$
- $\underline{\dim(\mathscr{H}) = 12g - 12 + 4\deg(D)} = 2 \cdot \dim(\mathcal{M}(\mathscr{C}, Q)) = 4 \cdot \dim(\mathcal{M}(\mathscr{C})) = 4k$
- $\dim(\mathcal{M}(\mathscr{C}, \mathscr{E})) = 6g - 6 + 2\deg(D)$
- $\dim(\mathcal{M}(\mathscr{C}, \mathscr{E}, \nabla)) = 9g - 9 + 3 \cdot \deg(D)$
- $\underline{\dim(\mathcal{M}(\mathscr{C}, Q, \mathscr{E}, \nabla)) = 12g - 12 + 4 \cdot \deg(D) = 4k}$
- $\dim(\mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi)) = 9g - 9 + 3 \cdot \deg(D)$
- $\underline{\dim(\mathcal{M}(\mathscr{C}, \mathscr{E}, \Phi, \nabla)) = 12g - 12 + 4 \cdot \deg(D) = 4k}$
- $\dim(\mathcal{M}(\mathscr{C}, Q, \mathcal{L})) = 7g - 7 + 2\deg(D) = 6g - 6 + 2\deg(D) + g - 1$
- $\underline{\dim(\mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial_\infty)) = 12g - 12 + 4\deg(D)} = 7g - 7 + 2\deg(D) + g + \deg(R) - 1 = 4k$
- $\underline{\dim(\mathcal{M}(\mathscr{C}, Q, \mathcal{L}, \partial)) = 12g - 12 + 4\deg(D)} = 4k$

Degrees of divisors:

- $\deg(H) = \sum_i^d m_i$
- $\deg(H^{red}_{ev}) = e$
- $\deg(H^{red}_{odd}) = d - e$

- $\deg(D)=\sum_i^d n_i=\sum_i\left\lceil\frac{m_i}{2}\right\rceil=\left\lceil\frac{H}{2}\right\rceil=\frac{\deg(H)+(d-e)}{2}$
- $\deg(H^{red})=\deg(D^{red})=d$
- $\deg(D_\nu)=2\deg(D)$
- $\deg(D_\xi)=\deg(D)$
- $\deg(R)=g_\Sigma-1+\deg(D)=4g-4+2\deg(D)$
- $\deg(B)=\deg(R)$

B. The Algebraic Stacks $\mathcal{M}(A,\dots,Z)$

In this chapter, we construct the basic algebraic Artin stacks that appear in this thesis. We start by referencing three lemmata that will help us build the larger stacks out of a few fundamental building blocks, these being the stack of wild algebraic curves and the stack of wild quasi-parabolic bundles.

We write *CFG* for *category fibred in groupoids.*

Lemma B.0.1. ***Fibre Products of Algebraic Stacks***
Let $\mathscr{X}\longrightarrow\mathscr{Y}$ be a morphism of CFG's, fibering over $Sch^{\acute{e}t}/\mathbb{C}$. If $\mathscr{Y}\longrightarrow Sch/S$ is an (algebraic-) stack and $\forall T\in Sch/S$ the fibre product $\mathscr{X}\times_{\mathscr{Y}}T\longrightarrow Sch/S$ is an (algebraic-) stack, then $\mathscr{X}$ is also an (algebraic-) stack.

Proof. cf. [65, Prop.8.1.16] ☐

Lemma B.0.2. ***Relative Algebraicity***
Let $f:\mathscr{X}\longrightarrow\mathscr{Y}$ be a morphism of stacks. If $\mathscr{Y}$ is algebraic and f is a morphism representable by algebraic stacks, then $\mathscr{X}$ is an algebraic stack.

Proof. cf.[74, Tag 05UL] ☐

Lemma B.0.3. *Let $f:\mathscr{X}\longrightarrow\mathscr{Y}$ be a relative Deligne-Mumford stack, i.e. f is a DM-morphism between DM-stacks. For every relative DM-stack $\mathscr{Z}\longrightarrow\mathscr{Y}$ there exists a scheme and a surjective étale morphism $U\longrightarrow\mathscr{X}\times_{\mathscr{Y}}\mathscr{Z}$, giving the structure of a DM-stack on the fibre product.*

Proof. cf. [74, Tag 0CIA]. ☐

Definition B.0.4. The Stack $\mathcal{M}(C)$ of Relative Curves of fixed Genus.
We denote by $\mathcal{M}_g$ the algebraic stack over the étale site:

$$\mathcal{M}_g \longrightarrow Sch^{ét}/_{\mathbb{C}}$$

with groupoid fibers given by:

$$\mathcal{M}_g(S) = \{f : C \to S | f \in Mor_{Sch^{ét}/\mathbb{C}}(C, S) \text{ (smooth), with fibers genus g curves/}\mathbb{C}\}$$

and morphisms $g \in Mor_{Sch^{ét}/\mathbb{C}}(T, S)$ lift to morphism sets:

$$Mor_{\mathcal{M}_g}((S, C), (T, C')) = \left\{ \begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[d] \\ T \arrow[r] & S \end{tikzcd} |\text{Cartesian diagrams}, C \in \mathcal{M}_g(S); C' \in \mathcal{M}_g(T) \right\}.$$

The morphism $\mathcal{M}_g \longrightarrow Sch^{ét}/_{\mathbb{C}}$ is then the canonical forgetful morphism of stacks.

Definition B.0.5. The Stack of pointed curves $\mathcal{M}_{g,d}$
$\mathcal{M}_{g,d}$ is the algebraic stack over the étale site:

$$\mathcal{M}_{g,d} \longrightarrow Sch^{ét}/_{\mathbb{C}}$$

with groupoid fibres:

$$\mathcal{M}_{g,d}(S) = \{(f : C \to S, (p_1, \dots, p_d))\}/Iso$$

where $f \in Mor^{sm}_{Sch^{ét}/\mathbb{C}}(C, S)$ (smooth) with fibres genus g projective curves/$\mathbb{C}$ and $(p_1, \dots, p_d)$ ordered set of smooth sections of f s.t. $p_i(s) \neq p_j(s) \forall i \neq j, \forall s \in S$. The morphisms sets are:

$$Mor_{\mathcal{M}_{g,d}(S)}((C, (p_i)_i), (C', (p'_i)_i)) = \left\{ \begin{tikzcd} C \arrow[rr, "g"] \arrow[dr] & & C' \arrow[dl] \\ & S & \end{tikzcd} |g^*(p'_i) = p_i \right\}.$$

The stack $\mathcal{M}_{g,d}$ is well known to be a smooth connected algebraic stack of finite type/$\mathbb{C}$.

Definition B.0.6. The Stack $\mathcal{M}(C, H)$ of Curves with a Divisor of fixed type m
Let $m = (m_1, \dots, m_d)$. Then we define the stack quotient $\mathcal{M}(C, H) := \mathcal{M}_{g,d}/\mathcal{P}_m$ by enlarging the morphism sets and thereby the isomorphism groups of $\mathcal{M}_{g,d}$:

$$Mor_{\mathcal{M}_{g,d}/\mathcal{P}_m(S)}((C, (p_i)_i), (C', (p'_i)_i)) = \left\{g \in Mor_{\mathcal{M}_{g,d}(S)} | g^*(p_i) \in \{p'_j | \ m_j = m_i\}\right\}.$$

Fixing the type m, the space $\mathcal{M}(C,H)(\mathbb{C})$ parametrizes pairs of complex projective algebraic curves C of genus g with a divisor H of type m.

Remark B.0.7. Analogously to definition of $\mathcal{M}(C,H)$ we can define the stack $\mathcal{M}(C,D)$ for the divisors of definition 4.1.4

Definition B.0.8. Quasi-Coherent Sheaves on a Stack
Let $p : \mathscr{X} \longrightarrow Sch^{ét}/\mathbb{C}$ be stack over the ètale site. Then a (quasi-)coherent sheaf on $\mathscr{X}$ is the following collection of data:

1. (quasi-)coherent sheaves $\mathcal{F}_\xi$ on $p(\xi)$ $\forall \xi \in \mathscr{X}$,
2. for every morphism in $\mathscr{X}$ $H : \xi \longrightarrow \eta$, with associated map of base schemes $h := p(H) : p(\xi) \longrightarrow p(\eta)$, an isomorphism of $\mathcal{O}_{p(\xi)}$-Modules: $\rho_H : h^*\mathcal{F}_\eta \xrightarrow{\sim} \mathcal{F}_\xi$, satisfying a cocycle condition, which for given $H_1 : \xi_1 \longrightarrow \xi_2$, $H_2 : \xi_2 \longrightarrow \xi_3$ reads:

$$h_1^*(\rho_{H_2}) \circ \rho_{H_1} = \rho_{H_2 \circ H_1} : h_1^* h_2^* \mathcal{F}_{\xi_3} \longrightarrow \mathcal{F}_{\xi_1}. \tag{B.1}$$

cf. [ChXIII [2]].

Proposition B.0.9. ***The Relative Stack of Quadratic Differentials of fixed type***
There exists a bundle $\mathscr{F}_{(g,m)} \longrightarrow \mathcal{M}_{g,d}\big/\mathcal{P}_m$ with fibres parametrising meromorphic quadratic differentials on families of marked curves.

Proof. To every S-valued point $(f : C \to S, H/S) \in \mathcal{M}_{g,d}\big/\mathcal{P}_m(S)$ we associate the S-flat sheaf $\mathcal{F} := \omega_{C/S}^{\otimes 2}(H/S)$ on C/S. We claim that $f_*\mathcal{F}$ is a locally free sheaf on S. By the cohomology and base change theorem [38][ChIII-Th.12.11], it is enough to show that: $H^1(C_s, \omega_{C_s}^{\otimes 2}(H_s)) = 0$, $\forall s \in S$, due to Serre duality this translates to $deg(\omega_C(H)) > 0$ for any admissible type (g, m). For this we have $deg(\omega_C(H)) = 2g - 2 + deg(H)$ and since H is effective, we only need to argue that $deg(H) > 2 - 2g$ for $g = 0$ and $g = 1$. In those cases, we have by the admissibility condition that:

$$deg(H) \overset{!}{\geq} 3 > 2 - 2g \text{ for } g = 0,$$
$$\text{and } deg(H) \overset{!}{\geq} 2 > 2 - 2g \text{ for } g = 1.$$

and so $deg(\omega_C(H)) > 0$ follows for all admissible types. Thereby $f_*\mathcal{F}$ is indeed locally free and gives rise to a vector bundle on S with fibres $H^0(C_s, \omega_{C_s}^{\otimes 2}(H_s))$, which in turn gives a bundle:

$$\mathscr{F}_{(g,m)} \longrightarrow \mathcal{M}_{g,d}\big/\mathcal{P}_m \tag{B.2}$$

the $\mathbb{C}$-valued fibres of which, parametrize meromorphic quadratic differentials on a curve C of genus g with polar divisor H of type m. ☐

Definition B.0.10. $\mathcal{M}(C,Q)$- Stack of Quadratic Differentials with Simple Zeros we consider the open substack of the relative stack above:

$$\begin{tikzcd}
\mathcal{M}(C,Q)_{(g,\mathfrak{m})} \arrow[rr, hook] \arrow[dr, "h"'] & & \mathscr{F}_{(g,\mathfrak{m})} \arrow[dl] \\
& {}^{\mathcal{M}_{g,d}}\big/_{\mathcal{P}_{\mathfrak{m}}} &
\end{tikzcd} \tag{B.3}$$

with groupoid fibres:

$$\mathcal{M}(C,Q)_{(g,\mathfrak{m})}(S) = \big\{(C/S, Q/S) \in \mathscr{F}_{(g,\mathfrak{m})}(S)\big\} \tag{B.4}$$

s.t. $Q_s \in H^0(C_s, \omega_{C_s}^{\otimes 2}(H_s = h(Q_s)))$ has only simple zeros and $\{Q_s(X) = 0\} \cap H_s = \emptyset$.

Lemma B.0.11. *The algebraic stack $\mathcal{M}(C,Q)_{(g,\mathfrak{m})}$ is a smooth Deligne-Mumford stack for all admissible types $(g,\mathfrak{m})$ and admits a smooth scheme as open substack, i.e. the generic stabilizer group is finite.*

Proof. We consider the following cases:

- for any type $(g \geq 2, \mathfrak{m})$, the stack $\mathcal{M}_{g,d}$ has finite stabilizers due to Hurwitz automorphisms theorem cf.[42].
- for $g = 1$,the admissibility condition imposes $\sum_i^d m_i + d > 0 \implies \mathfrak{m} \neq 0, d \neq 0$ and the stack $\mathcal{M}_{1,d}$ also has finite stabilizers $\forall d > 0$. Specifically the largest stabilzer subgroups arise for the moduli of elliptic curves $\mathcal{M}_{1,1}$ where $Stab(x) < {}^{\mathbb{Z}}\big/_{6\mathbb{Z}}$.

In those cases we follow that $M(C,H) = {}^{\mathcal{M}_{g,d}}\big/_{\mathcal{P}_{\mathfrak{m}}}$ has finite stabilizers and thereby a vector bundle over this space as well. Resulting to the fact that $\mathcal{M}(C,Q)_{(g,\mathfrak{m})}$ is a DM-stack.

- In the case of $g = 0$,the admissibility condition imposes: $\sum_i^d m_i + d > 6$. Since $PGL_2(\mathbb{C})$ acts sharply 3-transitively on $\mathbb{P}^1$, the generic stabiliser group of $\mathcal{M}(C,Q)_{(g,\mathfrak{m})}$ is infinite if and only if the zeros and poles of a quadratic differential on $\mathbb{P}^1$ do not fix 3 points: #zeros+#poles= $-4 + \sum m_i + d < 3 \implies \sum m_i + d < 7$. Comparing the last inequality with the admissibility condition, we see that the non-trivial automorphisms rigidify.

□

Lemma B.0.12. *Assuming that $\mathcal{M}(C,Q)_{(g;\mathfrak{m})}$ is non-empty, then it is of dimension:*

$$n := dim_{\mathbb{C}}(\mathcal{M}(C,Q)_{(g;\mathfrak{m})}(\mathbb{C})) = 6g - 6 + d + \sum_{i=1}^{d} m_i. \tag{B.5}$$

Proof. This follows from the construction of $\mathcal{M}(C, Q)_{(g;m)}$ and:

- $dim_{\mathbb{C}}(\mathcal{M}_{g,d}) = 3g - 3 + d$
- $h^0(C, \omega_C^{\otimes 2}(H)) = h^0(C, \omega_C^* \otimes \mathcal{O}_C(-H)) + 3g - 3 + \deg(H) = 3g - 3 + \deg(H)$, which holds even for $g = 0$, due to the admissibility condition 4.1.7, imposing $\deg(H) \overset{!}{>} 3$ in the $g = 0$ case.

□

Definition B.0.13. **$\mathbb{C}^*$-action on $\mathcal{M}(C, Q)$**
We define a $\mathbb{C}^*$-action on the groupoid fibres of the stack $\mathcal{M}(C, Q)_{(g,m)} \longrightarrow \mathcal{M}_{g,d}\big/\mathcal{P}_m$ as follows:

$$\begin{aligned} \mathbb{C}^* \times \mathcal{M}(C, Q)_{(g,m)}(S) &\longrightarrow \mathcal{M}(C, Q)_{(g,m)}(S) \\ (\epsilon, (C/S, Q/S)) &\longmapsto (C/S, \epsilon^2 Q/S). \end{aligned} \tag{B.6}$$

This is the $\mathbb{C}^*$-action induced by the natural rescaling action on sections of the sheaf $\omega_{C/S}$ of relative differentials.

Proposition B.0.14. ***The Relative Stack of Wild Curves of fixed type $\mathcal{M}(\mathscr{C})_{g,m}$***
There exists an algebraic stack $\mathcal{M}(\mathscr{C}) \to Sch^{\acute{e}t}$ parametrising families of wild algebraic curves of fixed type, i.e. with groupoid fibres

$$\mathcal{M}(\mathscr{C})(S) = \{(C/S, D/S, \nu/S, \xi/S)|\ \textit{étale families of classes of obj. of Def.5.1.13}\}. \tag{B.7}$$

Proof. As in proposition B.0.9, given an S-valued point $(f : C \to S, H/S) \in \mathcal{M}_{g,d}\big/\mathcal{P}_m$, consider the S-flat sheaf $\mathcal{G} := \omega_{C/S}^{\otimes 2}(D/S)$ with S-fibres $f_*\mathcal{G}_s = \omega_{C_s}^{\otimes 2}(D_s)$. As before we obtain a locally free sheaf $f_*\mathcal{G}$ over S and thereby a vector bundle with fibres $H^0(C_s, \omega_{C_s}^{\otimes 2}(D_s))$. Now consider the short exact sequence of sheaves on stacks

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0. \tag{B.8}$$

Explicitly, let $u : \mathcal{C} \to \mathcal{M}(C, D)$ denote the universal curve and $j : \mathcal{D} \hookrightarrow \mathcal{C}$ the universal divisor, where j is finite, moreover let $\mathcal{F}_{\mathcal{D}} = j^*\omega_{\mathcal{C}/\mathcal{M}(C,D)}(\mathcal{D})^{\otimes 2}$ be the restriction of the the relative sheaf of D-twisted quadratic differentials to $\mathcal{D}$, i.e. for an S-valued point $x = (C/S, D/S) : S \to \mathcal{M}(C, D)$: $\mathcal{F}_{\mathcal{D},x} = \omega_{C/S}(D/S)^{\otimes 2}|_{D/S}$. Due to lemma 5.1.2 $R^1u_*\mathcal{F}_{\mathcal{D}}$ vanishes for all admissible types and the quotient sheaf $\mathcal{Q}$ is the locally free sheaf $\mathcal{Q} = u_*\mathcal{F}_{\mathcal{D}}$. The total space $Tot(\mathcal{Q}) \to \mathcal{M}(C, D)$, is the representable, functor of enhancements. Now we have a universal enhancement $\underline{\nu}$, i.e. the tautological section $\underline{\nu} \in H^0(\mathcal{D}, \mathcal{F}_{\mathcal{D}})$, completing

$(\mathcal{C},\mathcal{D})$ to a universal enhanced curve $(\mathcal{C},\mathcal{D},\underline{\nu}) \to [Tot(\mathcal{Q})]$. By 4.3.10 and 5.1.5, there exists also a universal double cover $\underline{p_\nu} : \underline{\mathcal{D}_\nu} = Spec_{\mathcal{D}}(\mathcal{A}_D) \to \mathcal{D}$ where $\mathcal{A}_{\mathcal{D}} := \mathcal{O}_{\mathcal{D}} \oplus \mathcal{F}_{\mathcal{D}}^{-1}$ carries the $\mathcal{O}_{\mathcal{D}}$-algebra structure given by $\ell \cdot \ell' = \underline{\nu}(\ell \otimes \ell') \forall \ell, \ell' \in \mathcal{F}_{\mathcal{D}}$. The last ingredient is the signing. The space $Sign(\underline{\nu})$ of sections of the universal divisorial spectral cover $\xi : \mathcal{D} \to \underline{\mathcal{D}_\nu}$ give rise to a $(\mathbb{Z}/2\mathbb{Z})^e$-torsor on $Tot(\mathcal{Q})$.

In total we have that the stack of wild curves $\mathcal{M}(\mathscr{C}) \longrightarrow Sch^{ét}/\mathbb{C}$ is a algebraic for all admissible types, as it is by construction a composition of morphisms representable by algebraic spaces

$$\mathcal{M}(\mathscr{C}) \xrightarrow{(\mathbb{Z}/2\mathbb{Z})^e} Tot(\mathcal{Q}) \xrightarrow{V.B.} \mathcal{M}(C,D) \longrightarrow Sch^{ét}/\mathbb{C}. \tag{B.9}$$

Given an étale-atlas $U \longrightarrow \mathcal{M}(C,D)$ by a scheme U, the total space of $\mathbb{V} := Tot(\mathcal{Q}) \times U$ is a smooth U-scheme. The finite étale torsor of signings $\mathbb{V}^\xi \longrightarrow \mathbb{V}$ is again schematic [74, Tag 06RY]. In total we have a schematic étale-surjective morphism $\mathbb{V}^\xi \longrightarrow \mathcal{M}(\mathscr{C})$. It remains to argue that $\mathcal{M}(\mathscr{C})$ has finite stabilisers. Since $\mathcal{M}(\mathscr{C})$ is representable over $\mathcal{M}(C,D)$ there exists, via [74, Tag 0DU6], a monomorphism of the inertia stack

$$\mathcal{I}_{\mathcal{M}(\mathscr{C})} \hookrightarrow \mathcal{I}_{\mathcal{M}(C,D)} \times_{\mathcal{M}(C,D)} \mathcal{M}(\mathscr{C})$$

and since $\mathcal{I}_{\mathcal{M}(C,D)} \to \mathcal{M}(C,D)$ is finite so is the composition

$$\mathcal{I}_{\mathcal{M}(\mathscr{C})} \hookrightarrow \mathcal{I}_{\mathcal{M}(C,D)} \times_{\mathcal{M}(C,D)} \mathcal{M}(\mathscr{C}) \to \mathcal{M}(\mathscr{C}).$$

For any S-valued point $x \in \mathcal{M}(\mathscr{C})(S)$ the automorphisms of x are given by the base change $Aut_{\mathcal{M}(\mathscr{C})}(x) = S \times_{\mathcal{M}(\mathscr{C})} \mathcal{I}_{\mathcal{M}(\mathscr{C})}$. □

Next we want to build the stack of wild quasi-parabolic bundles. We start by recalling the stack of $SL_2(\mathbb{C})$-bundles.

Definition B.0.15. The relative Stack $\mathscr{B}un^G \to \mathcal{M}_g$ of G-Bundles.
We denote the quotient algebraic stack of holomorphic G-torsors on a fixed curve $C \in \mathcal{M}_g(\mathbb{C})$ by:

$$\mathscr{B}un_C^G := \left[{}^{C}/_{G}\right] \longrightarrow {}^{Sch^{ét}}/_{\mathbb{C}}$$

For $S \xrightarrow{f} \mathbb{C} \in {}^{Sch^{ét}}/_{\mathbb{C}}$, a morphism $g \in Mor_{Sch^{ét}/\mathbb{C}}(T,S)$ lifts to:

$$Mor_{\mathscr{B}un_C^G}(\mathcal{E}_S, \mathcal{E}_T') := \left\{\mathcal{E}_T' \xrightarrow{\sim} g^*\mathcal{E}_S\right\}$$

Now, we consider the relative CFG of torsors over $\mathcal{M}_g$, i.e. the forgetful morphism of CFGs:

$$\begin{tikzcd}
\mathscr{B}un^G \arrow[rr] \arrow[dr] & & \mathcal{M}_g \arrow[dl] \\
& Sch^{ét}. &
\end{tikzcd}$$

with $\mathscr{B}un^G(S) := \{(\mathcal{E}, C \to S) | C \in \mathcal{M}_g(S), E \in \mathscr{B}un_C^G(S)\}$. Using lemmata above $\mathscr{B}un^G$ is indeed an algebraic stack, as we have $\forall T \in {}^{Sch^{ét}}/_{\mathbb{C}}$ and $C \in \mathcal{M}_g(T)$:

$$\begin{tikzcd}
\mathscr{B}un_C^G \cong \mathscr{B}un^G \times_{\mathcal{M}_g} T \arrow[r] \arrow[d] & T \arrow[d, "C"] \\
\mathscr{B}un^G \arrow[r] & \mathcal{M}_g.
\end{tikzcd}$$

Proposition B.0.16. ***From G-torsors to Bundles with G-structure***
Let $G < GL_n$ be a linear algebraic group. There is a canonical equivalence of stacks between the stack of G-torsors $\mathscr{B}un_C^G$ and the stack of vector bundles with G-structure $\mathcal{M}_C^G(E)$, cf. definition 6.1.3.

Proof. We construct two mutually quasi-inverse morphisms of stacks

$$\Psi : \mathscr{B}un_C^G \longrightarrow \mathcal{M}_C^G(E), \Psi' : \mathcal{M}_C^G(E) \longrightarrow \mathscr{B}un_C^G \tag{B.10}$$

All constructions are functorial relative to the base S and compatible with pullback so we obtain morphisms of stacks over the étale site ${}^{Sch^{ét}}/_{\mathbb{C}}$.

Let S be a scheme and let $P \in \mathscr{B}un_C^G(S)$ be a G-torsor. Denote by $\rho : G \hookrightarrow GL_n$ the natural representation of G as a linear algebraic group. Then we define the associated rank n vector bundle to P by $E_S := P \times^\rho \mathbb{C}^n$. Étale locally on C_S we trivialise P, i.e. for an étale cover $\{U_\alpha\}_\alpha$ of C_S we have G-equivariant isomorphisms

$$P|_{U_\alpha} \cong U_\alpha \times G. \tag{B.11}$$

This trivialisation provides us with a trivialisation for the associated vector bundle

$$E_S|_{U_\alpha} = P \times^\rho \mathbb{C}^n|_{U_\alpha} = (U_\alpha \times G) \times^\rho \mathbb{C}^n = U_\alpha \times \mathbb{C}^n \tag{B.12}$$

and the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow G$ of P on the intersections $U_\alpha \cap U_\beta =: U_{\alpha\beta}$ induce transition functions on E_S via the composition

$$g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow G \overset{\rho}{\hookrightarrow} GL_n. \tag{B.13}$$

defining a G-structure on the locally free sheaf E_S, and equivalent choices of trivialisations of P give equivalent G-structures. Now given a morphism $f : P \longrightarrow P'$ of G-torsors, we obtain a morphism of between the associated vector bundles

$$f \times^{\rho} id : P \times^{\rho} \mathbb{C}^n \longrightarrow P' \times^{\rho} \mathbb{C}^n \tag{B.14}$$

respecting the induced G-structures. We obtain so a morphism

$$\Psi_S : \mathscr{B}un_C^G(S) \longrightarrow \mathcal{M}_C^G(E)(S). \tag{B.15}$$

This construction is clearly compatible with pullback so we obtain a morphism of stacks over the étale site

$$\begin{tikzcd}
\mathscr{B}un_C^G \arrow[rr, "\Psi"] \arrow[dr] & & \mathcal{M}_C^G(E) \arrow[dl] \\
& Sch^{ét}/\mathbb{C}. &
\end{tikzcd}$$

In the inverse direction, let E_S be a vector bundle with a G-structure on C_S, then by definition there exists an étale cover $\{U_\alpha\}_\alpha$ and trivialisation maps

$$g_\alpha : E_S|_{U_\alpha} \longrightarrow \mathcal{O}_{U_\alpha}^{\oplus n} \tag{B.16}$$

such that the transition functions $g_{\alpha\beta} = g_\alpha \circ g_\beta^{-1} : U_{\alpha\beta} \longrightarrow GL_n$ factorise through G, i.e. $g_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, G_C)$, and respect the Čech cocycle condition on triple intersections. From this we obtain a principal G-bundle P on C_S by gluing the trivial bundles $U_\alpha \times G$ along the same cocycles

$$U_\alpha \times G \ni (x, g) \sim (x, g_{\alpha\beta}(x)g) \in U_\beta \times G \quad \forall x \in U_{\alpha\beta}$$

and since $g_{\alpha\beta}$ respect the cocycle condition the trivialising system glues to a principal G-bundle P on C_S. Restricting a morphism between vector bundles with G-structures on the étale cover we obtain G-equivariant maps between the trivialisations and thereby a morphism between the associated principal G bundles. This gives rise to a map

$$\Psi'_S : \mathcal{M}_C^G(E)(S) \longrightarrow \mathscr{B}un_C^G(S). \tag{B.17}$$

Again this construction is compatible with pullback so we obtain a morphism of stacks over the étale site

$$\begin{tikzcd}
\mathcal{M}_C^G(E) \arrow[rr, "\Psi'"] \arrow[dr] & & \mathscr{B}un_C^G \arrow[dl] \\
& Sch^{ét}/\mathbb{C}. &
\end{tikzcd}$$

It is now easy to see that $\Psi \circ \Psi' \simeq id_{\mathcal{M}^G_C(E)}$. Namely, given a vector bundle with G-structure E_S, the principal G-bundle $\Psi(E_S)$ is constructed by gluing trivial principal G-bundles $U_\alpha \times G$ along $g_{\alpha\beta}$. The associated vector bundle $\Psi'(\Psi(E_S))$ to $\Psi(E_S)$ is obtained by gluing $U_\alpha \times^\rho \mathbb{C}^n$ via the same cocycle $g_{\alpha\beta}$, but the original vector bundle E_S is also a vector bundle with the same G-structure, which results to natural isomorphism $\Psi'_S(\Psi_S(E_S)) \cong E_S$. The identity $\Psi'_S \circ \Psi_S \simeq id_{\mathscr{B}un^G_C(S)}$ follows similarly.

□

Proposition B.0.17. *The stack $\mathscr{B}un^{SL_2\mathbb{C}} \longrightarrow \mathcal{M}_g$ is a smooth algebraic stack.*

Proof. cf. e.g. [39]. □

Proposition B.0.18. ***The Stack of Wild Quasi-Parabolic Bundles on Wild Curves*** *There exists a relative algebraic stack $\mathcal{M}(\mathscr{C},\mathscr{E}) \to \mathcal{M}(\mathscr{C})$ parametrising families of Wild Quasi-Parabolic Bundles, i.e. with groupoid fibres over $Sch^{\acute{e}t}/\mathbb{C}$*

$$\mathcal{M}(\mathscr{C},\mathscr{E})(S) = \{(\mathscr{C}/S, E/S, L_{D/S}/S)|\ \textit{étale families of classes of obj. of Def.6.1.4}\}. \tag{B.18}$$

Proof. Consider the fibre product

$$\mathcal{M}(\mathscr{C},E) := \mathcal{M}(\mathscr{C}) \times_{\mathcal{M}_g} \mathscr{B}un^{SL_2\mathbb{C}} \longrightarrow \mathcal{M}_g.$$

This is a smooth algebraic stack as fibre product of smooth algebraic stacks over a stack with the same property. We define the stack of wild quasi-parabolic bundles as the relative stack

$$\mathcal{M}(\mathscr{C},\mathscr{E}) \longrightarrow \mathcal{M}(\mathscr{C},E), \tag{B.19}$$

with S-valued fibres the rank-1 Grassmanian $\mathbb{P}((p^*_{\nu/S}E/S)|_{D/S})$, parametrising the wild quasi-parabolic structures of def. 6.2.1 on the family E/S with respect to the family of wild structures $\mathscr{C}/S$. For a $\mathbb{C}$-valued point $(\mathscr{C},\mathscr{E}) = (C,D,\nu,\xi,E)$ and $D = n\cdot p$, the fibre is $\mathbb{P}(E|_D) = \mathbb{P}^1_{k[\epsilon]/\epsilon^n}$, parametrising the directions of the nilpotent infinitesimal deformation of a quasi-parabolic structure on $p^*_\nu E$ over the fat point associated to D_ξ. This stack is algebraic, since it is relatively representable by a smooth DM stack, the Grassmanian, over the algebraic stack $\mathcal{M}(\mathscr{C},E)$. □

Definition B.0.19. Stack of Irregular Quasi-Parabolic Higgs Bundles
We define the stack of irregular quasi-parabolic Higgs bundles as a relative stack over the

stack of wild quasi-parabolic bundles

$$\mathcal{M}(\mathscr{C},\mathscr{E},\Phi) \longrightarrow \mathcal{M}(\mathscr{C},\mathscr{E}) \tag{B.20}$$

with fibre over an S-valued point

$$(\mathscr{C}_S,\mathscr{E}_S) = (C/S, D/S, \nu/S, \xi/S, E/S, L_{\xi/S}/S) \in \mathcal{M}(\mathscr{C},\mathscr{E})(S)$$

an S-family of sections of the following evaluation of the algebraic Hom-stack for coherent sheaves of [59]:

$$\Phi/S \in \mathscr{H}om(E/S, E/S \otimes \omega_{C/S}(D/S)) \tag{B.21}$$

such that $p^*_{\nu/S}\Phi/S|_{D_{\xi/S}/S}(\ell) = \lambda_{\xi/S}\cdot\ell \;\forall \ell \in L_{\xi/S}/S$, where $\lambda_{\xi/S}$ is determined fibrewise over S by $\nu/S, s$ via eq.5.12.

Remark B.0.20. The algebraicity of $\mathcal{M}(\mathscr{C},\mathscr{E},\Phi)$ follows from that of $\mathcal{M}(\mathscr{C},\mathscr{E})$ and the fact that the total space over $\mathcal{M}(\mathscr{C},\mathscr{E})$ is a closed sub-stack of a Hom-stack, inheriting an atlas by restriction, see also [20, chapter 7].

Remark B.0.21. The algebraicity of $\mathcal{M}(\mathscr{C},\mathscr{E},\nabla_\infty)$ follows from the algebraicity of the stack of regular flat bundles and lemma B.0.2, utilising the fact that the fibres of the projection $\mathcal{M}(\mathscr{C},\mathscr{E},\nabla_\infty) \to \mathcal{M}(\mathscr{C},\mathscr{E})$ are $\mathbb{W}$-torsors, where $\mathbb{W}$ is the locally free sheaf on $\mathcal{M}(\mathscr{C},\mathscr{E})$ defined fibrewise by

$$\mathbb{W} : (\mathscr{C},\mathscr{E}) \longmapsto H^0(C, End^{\mathrm{n}}(\mathscr{E}) \otimes \omega_C(D)). \tag{B.22}$$

Moreover, the stack $\mathcal{M}(\mathscr{C},\mathscr{E},\Phi,\nabla_\infty)$ is algebraic as a fibre product of algebraic stacks via lemma B.0.1

$$\begin{tikzcd}
\mathcal{M}(\mathscr{C},\mathscr{E},\Phi,\nabla_\infty) \arrow[r] \arrow[d] & \mathcal{M}(\mathscr{C},\mathscr{E},\nabla_\infty) \arrow[d] \\
\mathcal{M}(\mathscr{C},\mathscr{E},\Phi) \arrow[r] & \mathcal{M}(\mathscr{C},\mathscr{E}).
\end{tikzcd}$$

Remark B.0.22. The proof of algebraicity for anti-invariant branched abelian flat bundles $\mathcal{M}(\mathscr{C},Q,\mathcal{L},\partial_\infty)$ follows from the algebraicity of $\mathcal{M}(\mathscr{C},Q)$, the algebraicity of regular flat bundles and the relation between anti-invariant branched abelian connections on $(\mathscr{C},Q)$ and regular connections on the Jacobian of the spectral curve $Jac^{\#}(\Sigma_{\mathscr{C},Q}) \cong \mathcal{M}_{(\mathscr{C},Q)}(\mathcal{L},\partial_\infty)$, as demonstrated in the proof of proposition 6.3.11.

The proof of algebraicity for regular flat bundles is rather cumbersome without assuming any (semi-)stability conditions. It is actually a well-known fact, but it is hard to find in the literature, which led us down the rabbit hole of proving it by hand. We provide an argument for this for the sake of completeness. First, we need a purely algebro-geometric re-characterisation of the notion of a flat connection.

Definition B.0.23. /**Lemma**. **Characterisations of Relative Connections**
Let $(f : C \longrightarrow S) \in \mathscr{M}_g(S)$ and $\mathcal{E} \in \mathscr{B}un_C^G(S)$ then we define a relative regular connection $\mathcal{E}$ to be an $f^*\mathcal{O}_S$-linear morphism:

$$\nabla : \mathcal{E} \longrightarrow \mathcal{E} \otimes \omega_{C/S},$$

such that for any local sections $g \in \Gamma(\mathcal{O}_C)$ and $v \in \Gamma(\mathcal{E})$:

$$\nabla(g \cdot v) = g \otimes \nabla v + dg \otimes v.$$

A morphism between bundles endowed with relative connections is a morphism of bundles that commutes with ∇. We denote the groupoid of relative flat connections over C by $\mathscr{F}\mathscr{B}un_C^G(S)$.

We can construct regular connections in the following way. Given a point $C \in \mathscr{M}_g(S)$, let:

$$\Delta : C \longrightarrow C \times_S C$$

denote the diagonal immersion, and consider the associated short exact sequence:

$$0 \longrightarrow \mathcal{I}_\Delta \longrightarrow \Delta^{-1}(\mathcal{O}_{C\times_S C}) \longrightarrow \mathcal{O}_C \longrightarrow 0$$

i.e. $\mathcal{I}_\Delta$ being the ideal-sheaf defining the diagonal. Then define the sheaf of principal parts, i.e. the $(n+1)$-infinitesimal neighbourhood of the diagonal, by:

$$\mathcal{P}^n_{C/S} := \Delta^{-1}(\mathcal{O}_{C\times_S C}) \Big/ \mathcal{I}_\Delta^{n+1}$$

and further, as usual, we have canonically: $\mathcal{I}_\Delta \Big/ \mathcal{I}_\Delta^2 \cong \omega_{C/S}$. Thereby we have an exact sequence:

$$0 \longrightarrow \omega_{C/S} \longrightarrow \mathcal{P}^1_{C/S} \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

Tensoring this sequence by $\mathcal{E} \in \mathscr{B}un_C^G(S)$ we get:

$$0 \longrightarrow \omega_{C/S} \otimes \mathcal{E} \longrightarrow \mathcal{P}^1_{C/S} \otimes \mathcal{E} \longrightarrow \mathcal{E} \longrightarrow 0,$$

The datum of a connection is equivalent to a right split of this last Atiyah- exact sequence:

$$D : \mathcal{E} \longrightarrow \mathcal{P}^1_{C/S} \otimes \mathcal{E}.$$

thereby the obstruction to the existence of a holomorphic connection on $\mathcal{E}$ is the Atiyah

Class: $[\mathcal{P}^1_{C/S}\otimes\mathcal{E}] \in Ext^1_{\mathcal{O}_X}(\mathcal{E},\mathcal{E}\otimes\omega_{C/S})$. For $S = Spec(\mathbb{C})$ the vanishing of the Atiyah class is ensured by Weil's criterion, i.e. a holomorphic bundle $\mathcal{E}$ on a compact connected Riemann surface admits a holomorphic connection if and only if the degree of every direct summand of $\mathcal{E}$ vanishes. Using the fact that the geometric fibers of $f : C \to S$ are 1-dimensional and applying theorem 12.11 in [38], we obtain a relative version Weil's criterion.

Definition B.0.24. The relative de Rham Stack $\mathscr{F}\mathscr{B}un^G$ of Flat Connections The relative stack of flat bundles on curves of fixed genus is a CFG:

$$\begin{tikzcd} \mathscr{F}\mathscr{B}un^G \arrow[rr] \arrow[dr] & & \mathcal{M}_g \arrow[dl] \\ & Sch^{\acute{e}t}. & \end{tikzcd}$$

with groupoid fibres parametrising triples:

$$\mathscr{F}\mathscr{B}un^G(S) := \left\{(C \xrightarrow{f} S, \mathcal{E}, \nabla) | (C,f) \in \mathcal{M}_g(S), \mathcal{E} \in \mathscr{B}un^G_C(S), \nabla : \mathcal{E} \to \mathcal{E}\otimes\omega_{C/S}\right\}$$

where ∇ is a relative flat connection. As before, we denote by:

$$\mathscr{F}\mathscr{B}un^G_C(S) := \left\{(\mathcal{E}, \nabla) | (C,f) \in \mathcal{M}_g(S), \mathcal{E} \in \mathscr{B}un^G_C(S), \nabla : \mathcal{E} \to \mathcal{E}\otimes\omega_{C/S}\right\}$$

the CFG of relative connections over a fixed curve $C \in \mathcal{M}_g(\mathbb{C})$. Moreover, given scheme $f : S \longrightarrow \mathbb{C}$, a morphism $g \in Mor_{Sch^{\acute{e}t}/\mathbb{C}}(T,S)$ lifts to:

$$Mor_{\mathscr{F}\mathscr{B}un^G_C}((\mathcal{E}_S,\nabla_S),(\mathcal{E}'_T,\nabla'_T)) := \left\{ \begin{tikzcd} \mathcal{E}'_T \arrow[r, "\nabla'_T"] \arrow[d, "\alpha"', "\wr"] & \mathcal{E}'_T \otimes (f\circ g)^*\omega_C \arrow[d] \\ g^*\mathcal{E}_S \arrow[r, "g^*\nabla_S"] & g^*(\mathcal{E}_S \otimes f^*\omega_C) \end{tikzcd} \right\}.$$

Following [57] we construct an atlas on the stack parametrising these splittings.

Proposition B.0.25. *For $C \in \mathcal{M}_g(\mathbb{C})$ the relative stack:*

$$\begin{tikzcd} \mathscr{F}\mathscr{B}un^G_C \arrow[rr] \arrow[dr] & & \mathscr{B}un^G_C \arrow[dl] \\ & Sch^{\acute{e}t}. & \end{tikzcd}$$

is representable over the étale site, i.e. $\forall\ S \in Sch^{\acute{e}t}/\mathbb{C}$ and $\forall\ \mathcal{E} \in \mathscr{B}un^G(S)$, the 2-fibre product:

$$Sch^{\acute{e}t}/S \times_{\mathcal{E}} \mathscr{F}\mathscr{B}un^G_{C_S}$$

is representable by an algebraic space, where $C_S := C \times_{\mathbb{C}} S$ denotes the base change.

Proof. For this consider a morphism in $\mathcal{M}_g$:

$$\begin{tikzcd}
C_{S'} \arrow[r, "g"] \arrow[d, "q"'] & C_S \arrow[d, "p"] \\
S' \arrow[r, "f"] & S.
\end{tikzcd}$$

whereby we will denote by $\omega_{C_S/S}, \omega_{C_{S'}/S'}, q^!\omega_{S'} = \omega_{C_{S'}/C_S}[1]$ the relative dualising sheaves. Now let $\mathcal{E} \in \mathscr{B}un^G_C(S)$, $\mathcal{E}' \in \mathscr{B}un^G_C(S')$, and $\alpha \in Hom_{\mathscr{F}\mathscr{B}un^G_C}((\mathcal{E},\nabla),(\mathcal{E}',\nabla'))$.

By the definition of a relative connection, and passing to the derived category, the datum of α determines a element in:

$$\mathcal{R}^0\mathscr{H}om(g^*\mathcal{E}, \mathcal{P}^1_{C_{S'}/S'} \otimes g^*\mathcal{E}) = \mathcal{R}^0\Gamma(q_*\mathscr{H}om(g^*\mathscr{H}om(\mathcal{E}, \mathcal{P}^1_{C_S/S} \otimes \mathcal{E})^* \otimes \omega_{C_{S'}/C_S}, q^!\omega_{S'}))$$

and namely one such that it maps to the identity in $\mathcal{R}^0\mathscr{H}om(g^*\mathcal{E}, g^*\mathcal{E})$, under the morphism that is induced functorially by the Atiyah sequence. That is because connections are defined as right-splits of that sequence. The functor associated to the 2-fibre product we want to represent is exactly the one sending $S' \in {}^{Sch^{\acute{e}t}}/_{S}$ to the elements of $\mathcal{R}^0\mathscr{H}om(g^*\mathcal{E}, \mathcal{P}^1_{C_{S'}/S'} \otimes g^*\mathcal{E})$ with the property we just described. So we start by showing the representability of:

$$S' \longmapsto \mathcal{R}^0\mathscr{H}om(g^*\mathcal{E}, \mathcal{P}^1_{C_{S'}/S'} \otimes g^*\mathcal{E}).$$

By applying the local Grothendieck-Verdier duality we have:

$$\begin{aligned}
&\mathcal{R}\mathscr{H}om(\mathcal{O}_{S'}, \mathcal{R}q_*\mathscr{H}om(g^*\mathscr{H}om(\mathcal{E}, \mathcal{P}^1_{C_S/S} \otimes \mathcal{E})^* \otimes \omega_{C_{S'}/C_S}), q^!\omega_{S'})\\
\cong&\mathcal{R}\mathscr{H}om(\mathcal{O}_{S'}, \mathscr{H}om(\mathcal{R}q_*g^*\mathscr{H}om(\mathcal{E}, \mathcal{P}^1_{C_S/S} \otimes \mathcal{E})^* \otimes \omega_{C_{S'}/C_S}), \omega_{S'})\\
\cong&\mathcal{R}\mathscr{H}om(\mathcal{R}q_*g^*\mathscr{H}om(\mathcal{E}, \mathcal{P}^1_{C_S/S} \otimes \mathcal{E})^* \otimes \omega_{C_{S'}/C_S}), \mathcal{O}_{S'})
\end{aligned}$$

The spectral sequence: $\mathcal{R}^p\mathscr{H}om(\mathcal{R}^{-r}q_*g^*\mathscr{H}om(\mathcal{E}, \mathcal{P}^1_{C_S/S} \otimes \mathcal{E})^* \otimes \omega_{C_{S'}/C_S}), \mathcal{O}_{S'})$ terminates at $r = -1$, so cutting off at (0,-1)-level we obtain:

$$\begin{aligned}
&\mathcal{R}^0\mathscr{H}om(\mathcal{O}_{S'}, \mathcal{R}^1q_*\mathscr{H}om(g^*\mathscr{H}om(\mathcal{E}, \mathcal{P}^1_{C_S/S} \otimes \mathcal{E})^* \otimes \omega_{C_{S'}/C_S}), q^!\omega_{S'})\\
\cong&\mathcal{R}^0\mathscr{H}om(\mathcal{R}^1q_*g^*\mathscr{H}om(\mathcal{E}, \mathcal{P}^1_{C_S/S} \otimes \mathcal{E})^* \otimes \omega_{C_{S'}/C_S}), \mathcal{O}_{S'})
\end{aligned}$$

Further by derived base change and applying the adjunction $\mathcal{R}f_* \dashv f^!$ on $f : S' \to S$

$$\begin{aligned}
&\mathcal{R}^0\mathscr{H}om(\mathcal{R}^1q_*g^*\mathscr{H}om(\mathcal{E}, \mathcal{P}^1_{C_S/S} \otimes \mathcal{E})^* \otimes \omega_{C_{S'}/C_S}), \mathcal{O}_{S'})\\
\cong&\mathcal{R}^0\mathscr{H}om(\mathcal{R}^1f^*p_*\mathscr{H}om(\mathcal{E}, \mathcal{P}^1_{C_S/S} \otimes \mathcal{E})^* \otimes \omega_{C_{S'}/C_S}), \mathcal{O}_{S'})
\end{aligned}$$

$$\begin{aligned}
&\cong Hom_{S'}(f^*\mathcal{R}^1p_*\mathscr{H}om(\mathcal{E},\mathcal{P}^1_{C_S/S}\otimes\mathcal{E})^*\otimes\omega_{C_{S'}/C_S}),\mathcal{O}_{S'})\\
&\cong Hom_S(\mathcal{R}^1p_*(\mathscr{H}om(\mathcal{E},\mathcal{P}^1_{C_S/S}\otimes\mathcal{E})^*\otimes\omega_{C_S/S})),f_*\mathcal{O}_{S'})\\
&\cong Hom_{\mathcal{O}_S-Alg}(\mathcal{S}ym(\mathcal{R}^1p_*(\mathscr{H}om(\mathcal{E},\mathcal{P}^1_{C_S/S}\otimes\mathcal{E})^*\otimes\omega_{C_S/S})),f_*\mathcal{O}_{S'})\\
&\cong Hom_S(S',Spec(\mathcal{S}ym(\mathcal{R}^1p_*(\mathscr{H}om(\mathcal{E},\mathcal{P}^1_{C_S/S}\otimes\mathcal{E})^*\otimes\omega_{C_S/S}))))
\end{aligned}$$

Thereby the functor on ${Sch^{ét}}/_{S}$:

$$S'\longmapsto\mathcal{R}^0\mathscr{H}om(g^*\mathcal{E},\mathcal{P}^1_{C_{S'}/S'}\otimes g^*\mathcal{E})$$

is schematically representable by

$$X:=Spec(\mathcal{S}ym(\mathcal{R}^1p_*(\mathscr{H}om(\mathcal{E},\mathcal{P}^1_{C_S/S}\otimes\mathcal{E})^*\otimes\omega_{C_S/S}))).$$

Analogously we see that the functor on ${Sch^{ét}}/_{S}$:

$$S'\longmapsto\mathcal{R}^0\mathscr{H}om(g^*\mathcal{E},g^*\mathcal{E})$$

is schematically representable by

$$Y:=Spec(\mathcal{S}ym(\mathcal{R}^1p_*(\mathscr{H}om(\mathcal{E},\mathcal{E})^*\otimes\omega_{C_S/S}))).$$

From the Atiyah Sequence we obtain functorially a morphism of S-schemes:

$$X\longrightarrow Y$$

Furthermore, the identity in $\mathcal{R}^0\mathscr{H}om(g^*\mathcal{E},g^*\mathcal{E})$ determines a section of the S-scheme Y:

$$S\longrightarrow Y$$

Then, the functor associated to the 2-fibre product is represented schematically by the fibre product along the two schematic morphisms we have just described:

$$\begin{tikzcd}
S\times_Y X \arrow[r] \arrow[d] & X \arrow[d]\\
S \arrow[r] & Y,
\end{tikzcd}$$

i.e. the morphism of stacks

$$\begin{tikzcd}
\mathscr{F}\mathscr{B}un^G_C \arrow[rr] \arrow[dr] & & \mathscr{B}un^G_C \arrow[dl]\\
& Sch^{ét}. &
\end{tikzcd}$$

is representable over the étale site, and since $\mathscr{B}un_C^G$ is indeed an algebraic stack, it follows that $\mathscr{F}\mathscr{B}un_C^G$ is an algebraic stack. Furthermore by the algebraicity of $\mathcal{M}_g$ we follow that the relative stack

$$\begin{tikzcd}
\mathscr{F}\mathscr{B}un^G \arrow[rr] \arrow[dr] & & \mathcal{M}_g \arrow[dl] \\
& Sch^{ét}. &
\end{tikzcd}$$

is algebraic as well. □

Definition B.0.26. Birational Morphism of Algebraic Stacks
Let $\alpha : \mathscr{X} \longrightarrow \mathscr{Y}$ be a morphism between integral algebraic stacks of finite type over $\mathbb{C}$. We call α a birational morphism, if there exist a dense open substack $\mathscr{V} \subset \mathscr{Y}$ such that the pre-image $\mathscr{U} := \alpha^{-1}(\mathscr{V}) \subset \mathscr{X}$ is dense in $\mathscr{X}$ and $\alpha|_{\mathscr{U}} : \mathscr{U} \longrightarrow \mathscr{V}$ is an isomorphism of algebraic stacks over $\mathbb{C}$, i.e. in the language of [58] α has an 1-inverse up to a 2-isomorphism.

Definition B.0.27. Generically Étale Morphism of Algebraic Stacks
Let $\beta : \mathscr{X} \longrightarrow \mathscr{Y}$ be a morphism of stacks locally of finite type. Choose a surjective atlas $Y \longrightarrow \mathscr{Y}$ and let $X := \mathscr{X} \times_{\mathscr{Y}} Y$:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathscr{X} \arrow[d, "\beta"] \\
Y \arrow[r] & \mathscr{Y},
\end{tikzcd}$$

Given any smooth atlas $U \longrightarrow X$ we obtain a morphism of smooth schemes (or algebraic spaces) $U \longrightarrow Y$ by composition with the pullback of β. We say that the map of stacks β is generically étale if the map $U \longrightarrow Y$ is generically étale, i.e. if for every irreducible component U_i of U exists a non-empty open subset $V_i \subset U_i$ such that the restriction $V_i \longrightarrow Y$ is étale as a morphism of schemes or algebraic spaces.

C. Lifting the Residue Zero Condition

In this section we discuss the necessary generalisations needed to allow quadratic differentials with arbitrary residues in our theory, cf. remark 5.2.23.

Cohomology of Punctured Spectral Curves

Remark C.0.1. Comparison Theorems
We will proceed by studying the algebraic de Rham cohomology of families of punctured

spectral curves, that being the setting for which the construction of the Gauß-Manin connection is most natural. All results translate to singular cohomology by the comparison results of [34] and [37] which do not make an assumption on the completeness of the smooth varieties considered. Crucially we have the isomorphisms

$$H^*_{dR}(\Sigma^\circ) \cong H^*_{dR}((\Sigma^\circ)^{an}) \cong H^*(\Sigma^\circ(\mathbb{C}), \mathbb{C}). \tag{C.1}$$

All the relevant comparison theorems can be found neatly summarized in [1].

Proposition C.0.2. $dim H^1(\Sigma^\circ, \mathbb{C})^- = 6g - 6 + d + \deg(H)$

Proof. By definition: $H^1(\Sigma^\circ, \mathbb{C}) = H^1(\Sigma^\circ, \mathbb{C})^- \oplus H^1(\Sigma^\circ, \mathbb{C})^+$. Recall that $d := \deg(H^{red}) = \deg(D^{red})$, then:

$$dim H^1(\Sigma^\circ, \mathbb{C}) = 2g_\Sigma + \deg(p^{-1}(D^{red})) - 1 = 8g - 6 + 2 \cdot \deg(D) + d + e - 1,$$

where $e := \#\{p_i \in H^{red} \mid m_i \equiv 0 \ mod \ 2\}$ the number of even order poles. The second equality, follows from Riemann-Hurwitz as well as the fact that even order poles have two distinct preimages in the spectral cover. Furthermore

$$2 \cdot \deg(D) = 2 \cdot \sum_i \left\lceil \frac{m_i}{2} \right\rceil = \sum_i m_i + d - e = \deg(H) + d - e$$

and

$$H^1(\Sigma^\circ, \mathbb{C})^+ \cong H^1(C - D^{red}, \mathbb{C}) \implies dim H^1(\Sigma^\circ, \mathbb{C})^+ = 2g + d - 1.$$

In total we have:

$$dim H^1(\Sigma^\circ, \mathbb{C})^- = dim H^1(\Sigma^\circ, \mathbb{C}) - dim H^1(\Sigma^\circ, \mathbb{C})^+ = 6g - 6 + d + \deg(H).$$

□

Remark C.0.3. Note here that the dimension of $H^1(\Sigma^\circ, \mathbb{C})^-$ agrees with the dimension of $\mathcal{M}(C, Q)$.

Definition C.0.4. Logarithmic de Rham Complex
Let $(\boldsymbol{\Sigma}, Z)$ be a pair of a smooth proper $\mathcal{M}(C, Q)$-Scheme and Z an snc divisor of $\boldsymbol{\Sigma}$. Moreover let $j : \boldsymbol{\Sigma}^0 := \boldsymbol{\Sigma} \backslash Z \longrightarrow \boldsymbol{\Sigma}$ be the canonical embedding of $\mathcal{M}(C, Q)$-schemes. Then the *sheaf of logarithmic differentials* $\Omega_{(\boldsymbol{\Sigma}, Z)/\mathbb{C}}$ is the subsheaf of $j_* \Omega_{\boldsymbol{\Sigma}^0/\mathbb{C}}$, generated by $\Omega_{\boldsymbol{\Sigma}/\mathbb{C}}$ and by sections $\frac{df}{f}$, where f is regular on some open of $\boldsymbol{\Sigma}$ and has zeros restricted along the components of Z. Let $\Omega^\cdot_{(\boldsymbol{\Sigma}, Z)/\mathbb{C}}$ denote the associated *logarithmic de Rham complex* and write $H^i_{dR}(\boldsymbol{\Sigma}, Z) := \mathbb{H}^i(\Omega^\cdot_{(\boldsymbol{\Sigma}, Z)/\mathbb{C}})$ for the associated hypercohomology. Furthermore we

define the sheaf of *relative logarithmic differentials* over the base $\mathcal{M}(C,Q)$ by:

$$\Omega_{(\mathbf{\Sigma},Z)/\mathcal{M}} := \Omega_{(\mathbf{\Sigma},Z)/\mathbb{C}} \Big/ f^*\Omega_{(\mathcal{M}(C,Q)/\mathbb{C})} .$$

Theorem C.0.5. ***Algebraic de Rham Cohomology Relative to an SNC Divisor.*** *The canonical morphism:* $\Omega^{\cdot}_{(\mathbf{\Sigma},Z)/\mathbb{C}} \longrightarrow j_*\Omega_{\mathbf{\Sigma}^0/\mathbb{C}}$ *is a quasi-isomorphism. In particular, in the notation introduced above, we have*

$$H^i_{dR}(\mathbf{\Sigma},Z) \cong H^i_{dR}(\mathbf{\Sigma}^0) \tag{C.2}$$

Proof. The result follows from [22, Prop.6.8]. □

Proposition C.0.6. ***Cohomology Bundles Relative to a Divisor*** *The following functors*

$$H^i_{dR}((\mathbf{\Sigma},Z)/\mathcal{M}(C,Q)) : [V \hookrightarrow \mathcal{M}(C,Q) \text{ affine open}] \longmapsto \mathbb{H}^i(\Omega^{\cdot}_{(f^{-1}(V),f^{-1}(V)\cap Z)/V}) \tag{C.3}$$

are sheaves $\forall i$. *These are, as in the absolute case, locally free and have fibres*

$$H^i_{dR}(\mathbf{\Sigma}_{C,Q} = \Sigma, \Sigma\cap Z = p^{-1}(D)) \cong H^i_{dR}(\Sigma^\circ). \tag{C.4}$$

Proof. cf. [47]. □

The construction of the Gauss-Manin connection on this relative cohomology bundle was accomplished Katz and Oda in [47]. The construction is lengthy and rather technical, here is just the distilled essence of it and a remark on how to compute the GM connection for completeness.

Theorem C.0.7. ***The Relative Gauss-Manin Connection*** *There exists a unique integrable connection on the cohomology bundles of Proposition C.0.6:*

$$\nabla_0^{GM,i} : H^i_{dR}(\mathbf{\Sigma}^0/\mathcal{M}(C,Q)) \longrightarrow H^i_{dR}(\mathbf{\Sigma}^0/\mathcal{M}(C,Q)) \otimes \Omega^1_{\mathcal{M}(C,Q)/\mathbb{C}} \ ; \ \forall i \tag{C.5}$$

Proof. Consider the E_1-page associated to the filtration

$$Im[\Omega^{\cdot-i}_{(\mathbf{\Sigma},Z)/\mathbb{C}} \otimes_{\mathcal{O}_{\mathbf{\Sigma}}} f^*(\Omega^i_{\mathcal{M}(C,Q)/\mathbb{C}}) \longrightarrow \Omega^{\cdot}_{(\mathbf{\Sigma},Z)/\mathbb{C}}] \tag{C.6}$$

i.e. $E_1^{p,q} = \Omega^p_{\mathcal{M}(C,Q)/\mathbb{C}} \otimes_{\mathcal{M}(C,Q)} H^q_{dR}(\mathbf{\Sigma}^0/\mathcal{M}(C,Q))$ with differentials

$$0 \longrightarrow H^q_{dR}(\mathbf{\Sigma}^0/\mathcal{M}(C,Q)) \xrightarrow{d_1^{0,q}} \Omega^1_{\mathcal{M}(C,Q)} \otimes H^q_{dR}(\mathbf{\Sigma}^0/\mathcal{M}(C,Q)) \xrightarrow{d_1^{1,q}} \Omega^2_{\mathcal{M}(C,Q)} \otimes H^q_{dR}(\mathbf{\Sigma}^0/\mathcal{M}(C,Q)) \longrightarrow$$

$d_1^{p,q}(\psi \cdot e) = d\psi \cdot e + (-1)^p \psi \cdot d_1^{0,q} e$ for $\psi \in \Omega^p_{\mathcal{M}(C,Q)}, e \in E_1^{0,q}$. Then the differential $d_1^{0,q}$ is a connection on $H^q_{dR}(\mathbf{\Sigma}^0/\mathcal{M}(C,Q))$ with vanishing curvature

$$d_1^{1,q} \circ d_1^{0,q}(\psi \cdot e) = d^{1,q}(d\psi \cdot e) + d^{1,q}(\psi \cdot d^{0,q} e) = 0.$$

The Gauss-Manin connection on the cohomology bundle of the fibration $\mathbf{\Sigma}^0 \longrightarrow \mathcal{M}(C,Q)$ is defined to be $d_1^{0,q}$, and is the unique one respecting the underlying integral structure. □

Remark C.0.8. Kedlaya explains in [48] how to calculate the relative GM-connection in practice. Let $\psi \in \Omega^p_{(\mathbf{\Sigma},Z)/\mathcal{M}(C,Q)}$ and $\tilde{\psi} \in \Omega^p_{(\mathbf{\Sigma},Z)/\mathbb{C}}$ a lift of ψ to an absolute form, then compute the differential of the lift $d\tilde{\psi} \in \Omega^{p+1}_{(\mathbf{\Sigma},Z)/\mathbb{C}}$ and project to $\Omega^p_{(\mathbf{\Sigma},Z)/\mathcal{M}} \otimes \Omega^1_{\mathcal{M}(C,Q)}$ via

$$0 \longrightarrow f_0^* \Omega^1_{\mathcal{M}(C,Q)} \longrightarrow \Omega^1_{(\mathbf{\Sigma},Z)/\mathbb{C}} \longrightarrow \Omega^1_{(\mathbf{\Sigma},Z)/\mathcal{M}} \longrightarrow 0.$$

This is how we will thinking of, and be performing calculations with this connection down the line, although in a rather more simplified notation.

Remark C.0.9. For our purposes it only remains to restrict the cohomology bundle and connection described in Theorem C.0.7, to the subbundle of anti-invariant cohomology. Before doing so, note that, by the theorems C.0.5, C.0.6 as well as remark C.0.1, we identify the fibres of the locally free sheaf $H^1_{dR}((\mathbf{\Sigma}, Z)/\mathcal{M}(C,Q))$ on the moduli of meromorphic quadratic differentials with $H^1(\Sigma^\circ, \mathbb{C})^-$, where Σ° is the fibre of $\mathbf{\Sigma}^0 \longrightarrow \mathcal{M}(C,Q)$).

Definition C.0.10. Sheaf of Anti-Invariant Cohomology Relative to a Divisor
We denote by $\mathscr{H}$ the *bundle of anti-invariant cohomology groups* of the family of punctured spectral curves:

$$\mathscr{H} \longrightarrow \mathcal{M}(C,Q) \tag{C.7}$$

with fibres $H^1(\Sigma^\circ, \mathbb{C})^-$ and we denote also by ∇^{GM} the restriction of the Gauß-Manin connection of theorem C.0.7, to the anti-invariant cohomology $\mathscr{H}$. By construction, the GM-connection of the last definition is one preserving the integral lattice $\mathscr{H}^{\mathbb{Z}}$ with fibres $Hom_{\mathbb{Z}}(H_1(\Sigma^\circ, \mathbb{Z})^-, \mathbb{Z})$.

Remark C.0.11. Notation for a Basis of Anti-Invartiant Homology
We will be using the following notation if not otherwise specified. We write $\{\gamma_1, \dots, \gamma_{2g+e}\}$ for a basis of anti-invariant cohomology $H_1(\Sigma^\circ, \mathbb{Z})^-$, s.t. γ_j are A-cycles for $1 \leq j \leq g$, B-cycles for $g+1 \leq j \leq 2g$, and cycles around even order poles otherwise, the latter being unordered. For such a basis we have

$$(\gamma_i, \gamma_j) = \begin{cases} \delta_{i,j-g}; & \text{for } 1 \leq i \leq g \\ 0; & \text{otherwise} \end{cases} \tag{C.8}$$

that is, the only non-trivial intersections are between $A-B$-cycle pairs γ_i, γ_{i+g}, and cycles around even order poles do not intersect other cycles.

Period Structure on $\mathcal{M}(\mathscr{C}, Q)$

Definition C.0.1. Periods of Quadratic Differentials
Let $(\mathscr{C}, Q) \in \mathcal{M}(\mathscr{C}, Q)$ and recall from Eq.4.24 the 1-form $\lambda \in H^0(\Sigma, \omega_\Sigma(D_\nu))$ on the associated spectral curve. Restricting $\lambda|_{\Sigma^\circ}$ onto the punctured curve, it becomes an untwisted (i.e. holomorphic) 1-form. We call the associated de Rham cohomology class $\mathcal{Z}_Q := [\lambda|_{\Sigma^\circ}] \in H^1(\Sigma^\circ, \mathbb{C})^-$ the *period of the quadratic differential Q*. Period classes define a section of the cohomology bundle C.0.10:

$$\begin{aligned} \delta : \mathcal{M}(\mathscr{C}, Q) &\longrightarrow \mathscr{H} \\ (\mathscr{C}, Q) &\longmapsto \mathcal{Z}_Q. \end{aligned} \tag{C.9}$$

which we call the *period section* of $\mathscr{H} \longrightarrow \mathcal{M}(\mathscr{C}, Q)$.

The following result was originally discovered by Veech [76] and was also recovered by Bridgeland-Smith in [18].

Theorem C.0.2. *The period map δ is a local isomorphism of complex manifolds.*

Proof. This is result (7.22) in [76], cf. also section 4 in [18]. □

Using the existence of the Gauß-Manin connection on the cohomology bundle $\mathscr{H}$, we can recast this result into:

Theorem C.0.3. *The differential of the period section δ with respect to the Gauß-Manin connection yields an isomorphism of $\mathbb{C}$-vector bundles:*

$$\nabla^{GM}\delta : T\mathcal{M}(\mathscr{C}, Q) \xrightarrow{\sim} \mathscr{H}, \tag{C.10}$$

that is, for $w \in T_{(\mathscr{C},Q)}\mathcal{M}(\mathscr{C}, Q)$, $\nabla^{GM}\delta : w \longmapsto \nabla^{GM}_w \delta(\mathscr{C}, Q)$, differentiating the period class in the direction of w.

Lemma C.0.4. ***Period Coordinates on** $\mathcal{M}(C, Q)$*
The isomorphism of Theorem 5.2.12 prescribes a local coordinate system on $\mathcal{M}(C, Q)$.

Proof. Consider a neighbourhood $(\mathscr{C}, Q) \in U \subset M$ and a basis $\{\gamma_i\}_i$ of $H_1(\Sigma^\circ_{\mathscr{C},Q}, \mathbb{C})^-$, then using the Gauß-Manin connection, we extend this basis to $U \times H_1(\Sigma^\circ_{\mathscr{C},Q}, \mathbb{C})^-$. Furthermore using the pairing:

$$\begin{aligned} H_1(\Sigma^\circ_{\mathscr{C},Q},\mathbb{C})^- \times H^1(\Sigma^\circ_{\mathscr{C},Q},\mathbb{C})^- &\longrightarrow \mathbb{C} \\ (\gamma_i, [\lambda|_{\Sigma^\circ}]_{dR}) &\longmapsto z_i := \int_{\gamma_i} \mathcal{Z}_Q = \int_{\gamma_i} \delta(\mathscr{C},Q) \end{aligned}$$

we obtain locally defined functions on $\mathcal{M}(\mathscr{C},Q)$: $\{z_i\}_i$. The statement of theorem 5.2.12 is then, that $\left\{\frac{\partial}{\partial z_i}\right\}$ give a local basis for $T\mathcal{M}(\mathscr{C},Q)|_U$ and the $\{z_i\}_i$ can be interpreted as local coordinates on $U \subset \mathcal{M}(\mathscr{C},Q)$. □

Poisson Structure on Spaces Quadratic Differentials

Definition C.0.1. Poisson Structure on $\mathcal{M}(C,Q)$
Consider the intersection product on homology, restricted on anti-invariant cycles

$$<\cdot,\cdot> H_1(\Sigma,\mathbb{C})^- \times H_1(\Sigma,\mathbb{C})^- \longrightarrow \mathbb{C}. \tag{C.11}$$

We want to extend this pairing by pullback along the projection $H_1(\Sigma^\circ,\mathbb{C})^- \longrightarrow H_1(\Sigma,\mathbb{C})^-$, but since the cycles in the kernel of this projection, i.e. the cycles around even order poles do not intersect other cycles, the induced pairing

$$<\cdot,\cdot>: H_1(\Sigma^\circ,\mathbb{C})^- \times H_1(\Sigma^\circ,\mathbb{C})^- \longrightarrow \mathbb{C} \tag{C.12}$$

is not perfect, since it degenerates along the kernel. A perfect pairing exists only in the Lefschetz relative sense. Nevertheless, since $H^1(\Sigma^\circ_{C,Q},\mathbb{C})^- \cong T_{C,Q}\mathcal{M}(C,Q)$, using the Gauß-Manin connection we obtain a pairing

$$<\cdot,\cdot>: T^*_{C,Q}\mathcal{M}(C,Q) \times T^*_{C,Q}\mathcal{M}(C,Q) \longrightarrow \mathbb{C}.$$

Explicitly, let $\{\gamma_j\}_j$ be a basis of $H_1(\Sigma^\circ,\mathbb{C})^-$ and $\{dz_j\}_j$ its image in $T^*_{C,Q}\mathcal{M}(C,Q)$, then

$$< dz_i, dz_j > \longmapsto (\gamma_i \cdot \gamma_j)$$

from which we define a degenerate Poisson Structure on $\mathcal{M}(C,Q)$

$$\{\cdot,\cdot\} : \mathcal{C}^\infty(\mathcal{M}(C,Q)) \times \mathcal{C}^\infty(\mathcal{M}(C,Q)) \longrightarrow \mathcal{C}^\infty(\mathcal{M}(C,Q)) \tag{C.13}$$

defined locally on the period coordinates of $\mathcal{M}(C,Q)$ by

$$\{z_i, z_j\} := < dz_i, dz_j > . \tag{C.14}$$

Lemma C.0.2. *The bracket $\{\cdot,\cdot\}$ is a Poisson structure on $\mathcal{M}(C,Q)$.*

Proof. Exactly analogous to 5.2.19. ☐

Definition C.0.3. Residue of Quadratic differentials on Wild Curves
Let $\mathscr{C} = (C, D, \nu, \zeta)$, Q a quadratic differential on $\mathscr{C}$ and $q \in D \backslash B$, i.e. a pole of Q that is not a zero. We define the residue of the quadratic differential Q at the point q

$$\mathcal{R}es_q(Q) := \int_{\gamma_{\zeta(q)}} \mathcal{Z}_Q = \int_{\gamma_{\zeta(q)}} \delta(\mathscr{C}, Q), \tag{C.15}$$

for $\gamma_{\zeta(q)} \in H_1(\Sigma^\circ, \mathbb{Z})$, a cycle around $\zeta(q)$. Setting $\mathcal{R}es_q(Q) = 0$ for $q \in B$, this gives rise to a surjective map:

$$\mathcal{R}es : \mathcal{M}(\mathscr{C}, Q) \longrightarrow \mathbb{C}^e \tag{C.16}$$

$$(C, Q) \longrightarrow (\mathcal{R}es_q(Q))_{q \in D} \tag{C.17}$$

for $e = \deg(D^{red} \backslash B)$, i.e. the number of even order poles.

We want to study the symplectic leaves of the Poisson structure discussed above. The essence of the following characterising lemma is that the symplectic leaves in $\mathcal{M}(C, Q)$ are cut out by fixing the residues of the quadratic differentials.

Lemma C.0.4. ***The Symplectic Fibration over the Space of Residues***
The connected components of the fibres of the surjective map $\mathcal{R}es : \mathcal{M}(\mathscr{C}, Q) \to \mathbb{C}^e$, are symplectic leaves to the Poisson structure on $\mathcal{M}(\mathscr{C}, Q)$, of dimension $6g - 6 + 2 \cdot \deg(D)$.

Proof. Let $\{z_i\}_i$ be the period coordinates on a neighbourhood $U \subset \mathcal{M}(\mathscr{C}, Q)$, then $\mathcal{R}es$ can be expressed in terms of these coordinates by projecting down to the periods around even order poles:

$$\mathcal{R}es : U \longrightarrow \mathbb{C}^e \tag{C.18}$$

$$(z_i)_i \longrightarrow (z_j)_j; \tag{C.19}$$

for $j : \gamma_j \in ker(H_1(\Sigma^\circ)^- \to H_1(\Sigma)^-)$, i.e. in the notation of C.0.11, $2g < j \leq 2g + e$. For a fixed cycle γ_i from the basis of $H_1(\Sigma^\circ, \mathbb{C})^-$; we have:

$$\{z_i, z_j\} = 0 \ \forall j \iff \gamma_i \cdot \gamma_j = 0 \ \forall j \iff \gamma_i \text{ is a cycle around a puncture.}$$

Passing to anti-invariant cycles, we are left precisely with the cycles around even order poles, which by assumption do not intersect other cycles, and non-degeneracy of the restriction of the Poisson form on the fibres of $\mathcal{R}es$ follows. Thereby the dimension of the fibres reads:

$$dim\mathcal{R}es^{-1} = dimH^1(\Sigma^\circ, \mathbb{C})^- - e = 6g - 6 + 2 \cdot \deg(D).$$

□

Now we show that fixing the irregular type ζ, defines Lagrangians of the symplectic leaves of $\mathcal{R}es_q$.

Corollary C.0.5. *We have seen that the enhancement ν determines the residue of quadratic differentials of class ν. In particular, considering the maps:*

$$\begin{tikzcd} \mathcal{M}(\mathscr{C}, Q) \arrow[r, "\mathcal{R}es"] \arrow[d] & \mathbb{C}^e \\ \mathcal{M}(\mathscr{C}), & \end{tikzcd} \tag{C.20}$$

and fixing a wild curve $\mathscr{C}$, there exists a unique vector $v_\zeta \in \mathbb{C}^e$ s.t. $\mathcal{M}_{\mathscr{C}}(Q) \subset (\mathcal{R}es)^{-1}(v_\zeta)$.

Lemma C.0.6. *The fibres $\mathcal{M}_{\mathscr{C}}(Q)$ of $\mathcal{M}(\mathscr{C}, Q) \longrightarrow \mathcal{M}(\mathscr{C})$ are Lagrangian submanifolds in the symplectic leaves with respect to the Poisson structure are given by the fibres of $\mathcal{R}es$ C.16, i.e.*

$$\mathcal{M}_{\mathscr{C}}(Q) \subset (\mathcal{R}es)^{-1}(v_\zeta) \quad \textit{are Lagrangian submanifolds } \forall \mathscr{C} \in \mathcal{M}(\mathscr{C}), \tag{C.21}$$

obtaining a Lagrangian fibration on the space of quadratic differentials.

Proof. Restrict to a fibre of $\mathcal{R}es$ and consider the symplectic form ω on this fibre, and for the period coordinates $(z_i)_i$ let: $\omega_{ij} = \omega(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})$ with $\omega_{ij} = -\omega_{ji}$. Moreover let $(a_\ell)_\ell$ be a local basis on the fibre, then we have:

$$\begin{aligned} \omega\left(\frac{\partial}{\partial a_k}, \frac{\partial}{\partial a_\ell}\right) &= \omega\left(\sum_i \frac{\partial z_i}{\partial a_k}\frac{\partial}{\partial z_j}, \sum_i \frac{\partial z_j}{\partial a_\ell}\frac{\partial}{\partial z_j}\right) = \sum_{ij} \omega_{ij} \frac{\partial z_i}{\partial a_k}\frac{\partial z_j}{\partial a_\ell} \\ &= \sum_{ij} \omega_{ij} \int_{\gamma_i} \nabla^{GM}_{a_k}[\lambda] \int_{\gamma_j} \nabla^{GM}_{a_\ell}[\lambda] \end{aligned}$$

since $(\gamma_j)_j$ is a basis of anti-invariant cohomology, cycles around poles at ramification are excluded. Since $\nabla^{GM}_{a_k}[\lambda]$ is holomorphic away from ramification for all k by lemma 5.2.14, applying the residue-reciprocity theorem [33][Section 2.3] we conclude

$$\omega\left(\frac{\partial}{\partial a_k}, \frac{\partial}{\partial a_\ell}\right) = 0.$$

□

References

[1] Yves André. Comparison theorems between algebraic and analytic de rham cohomology (with emphasis on the p-adic case). *Journal de théorie des nombres de Bordeaux*, 16(2):335–355, 2004.

[2] Enrico Arbarello, Maurizio Cornalba, and Phillip A Griffiths. *Geometry of algebraic curves: volume II with a contribution by Joseph Daniel Harris*. Springer, 2011.

[3] Dmitry Arinkin and Roman Fedorov. An example of the langlands correspondence for irregular rank two connections on p1. *Advances in Mathematics*, 230(3):1078–1123, 2012.

[4] Wolf Barth, Klaus Hulek, Chris Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4. Springer, 2015.

[5] Arnaud Beauville, Sundararaman Ramanan, and Mudumbai S Narasimhan. Spectral curves and the generalised theta divisor. 1989.

[6] Olivier Biquard and Philip Boalch. Wild non-abelian hodge theory on curves. *Compositio Mathematica*, 140(1):179–204, 2004.

[7] Indranil Biswas and Sundararaman Ramanan. An infinitesimal study of the moduli of hitchin pairs. *Journal of the London Mathematical Society*, 49(2):219–231, 1994.

[8] Philip Boalch. Symplectic manifolds and isomonodromic deformations. *Advances in Mathematics*, 163(2):137–205, 2001.

[9] Philip Boalch and Daisuke Yamakawa. Twisted wild character varieties. *arXiv preprint arXiv:1512.08091*, 2015.

[10] Philip P Boalch. G-bundles, isomonodromy, and quantum weyl groups. *International Mathematics Research Notices*, 2002(22):1129–1166, 2002.

[11] Philip Paul Boalch. Geometry and braiding of stokes data; fission and wild character varieties. *Annals of mathematics*, pages 301–365, 2014.

[12] Giulio Bonelli, Oleg Lisovyy, Kazunobu Maruyoshi, Antonio Sciarappa, and Alessandro Tanzini. On painlevé/gauge theory correspondence. *Letters in Mathematical Physics*, 107:2359–2413, 2017.

[13] Tom Bridgeland. Stability conditions on triangulated categories. *Annals of Mathematics*, pages 317–345, 2007.

[14] Tom Bridgeland. Geometry from donaldson-thomas invariants. *arXiv preprint arXiv:1912.06504*, 2019.

[15] Tom Bridgeland. Joyce structures on spaces of quadratic differentials i. *arXiv preprint arXiv:2203.17148*, 2022.

[16] Tom Bridgeland et al. Tau functions from joyce structures. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 20:112, 2024.

[17] Tom Bridgeland and Davide Masoero. On the monodromy of the deformed cubic oscillator. *Mathematische Annalen*, pages 1–66, 2022.

[18] Tom Bridgeland and Ivan Smith. Quadratic differentials as stability conditions. *Publications mathématiques de l'IHÉS*, 121(1):155–278, 2015.

[19] Tom Bridgeland and Ian AB Strachan. Complex hyperkähler structures defined by donaldson–thomas invariants. *Letters in Mathematical Physics*, 111(2):54, 2021.

[20] Sebastian Casalaina-Martin and Jonathan Wise. An introduction to moduli stacks, with a view towards higgs bundles on algebraic curves. In *The geometry, topology and physics of moduli spaces of Higgs bundles*, pages 199–399. World Scientific, 2018.

[21] BRIAN CONRAD. Classical motivation for the riemann–hilbert.

[22] Pierre Deligne. *Équations différentielles à points singuliers réguliers*, volume 163. Springer, 2006.

[23] Pierre Deligne, Brigitte Malgrange, and Jean-Pierre Ramis. Singularités irréguliére: correspondance et documents. *Documents mathématiques*, 5, 2007.

[24] Duiliu-Emanuel Diaconescu, Ron Donagi, and Tony Pantev. Intermediate jacobians and ade hitchin systems. *arXiv preprint hep-th/0607159*, 2006.

[25] Ron Donagi, Boris Dubrovin, Edward Frenkel, Emma Previato, and Boris Dubrovin. *Geometry of 2D topological field theories*. Springer, 1996.

[26] Ron Donagi and Eyal Markman. Spectral curves, algebraically completely integrable hamiltonian systems, and moduli of bundles. *arXiv preprint alg-geom/9507017*, 1995.

[27] Ron Donagi and Tony Pantev. Lectures on the geometric langlands conjecture and non-abelian hodge theory. *Second International School on Geometry and Physics Geometric Langlands and Gauge Theory*, page 129, 2010.

[28] Charles Ehresmann. Les connexions infinitésimales dans un espace fibré différentiable. In *Colloque de topologie, Bruxelles*, volume 29, pages 55–75, 1950.

[29] Roman Fedorov, Alexander Soibelman, and Yan Soibelman. Motivic classes of irregular higgs bundles and irregular connections on a curve. *arXiv preprint arXiv:2404.14549*, 2024.

[30] Vladimir V Fock and Alexander B Goncharov. Dual teichmuller and lamination spaces. *arXiv preprint math/0510312*, 2005.

[31] Davide Gaiotto, Gregory W Moore, and Andrew Neitzke. Wall-crossing, hitchin systems, and the wkb approximation. *Advances in Mathematics*, 234:239–403, 2013.

[32] Victor Ginzburg. Calabi-yau algebras, 2007.

[33] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. John Wiley & Sons, 2014.

[34] Alexander Grothendieck. On the de rham cohomology of algebraic varieties. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 29(1):95–103, 1966.

[35] Fabian Haiden. 3-d calabi–yau categories for teichmüller theory. *Duke Mathematical Journal*, 173(2):277–346, 2024.

[36] Robin Hartshorne. *Residues and duality*, volume 20. Springer, 1966.

[37] Robin Hartshorne. On the de rham cohomology of algebraic varieties. *Publications Mathématiques de l'IHÉS*, 45:5–99, 1975.

[38] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.

[39] Jochen Heinloth. Lectures on the moduli stack of vector bundles on a curve. In *Affine flag manifolds and principal bundles*, pages 123–153. Springer, 2010.

[40] Nigel Hitchin. Stable bundles and integrable systems. 1987.

[41] Nigel J Hitchin, Anders Karlhede, Ulf Lindström, and Martin Roček. Hyperkähler metrics and supersymmetry. *Communications in Mathematical Physics*, 108(4):535–589, 1987.

[42] Adolf Hurwitz and Adolf Hurwitz. Über algebraische gebilde mit eindeutigen transformationen in sich. *Mathematische Werke: Erster Band Funktionentheorie*, pages 391–430, 1932.

[43] Daniel Huybrechts. *Fourier-Mukai transforms in algebraic geometry.* Clarendon Press, 2006.

[44] Michio Jimbo and Tetsuji Miwa. Monodromy perserving deformation of linear ordinary differential equations with rational coefficients. ii. *Physica D: Nonlinear Phenomena*, 2(3):407–448, 1981.

[45] Dominic Joyce and Yinan Song. A theory of generalized donaldson-thomas invariants. *arXiv preprint arXiv:0810.5645*, 2008.

[46] N Katz. An overview of deligne's work on hilbert's twenty-first problem. In *Proceedings of symposia in pure mathematics*, volume 28, pages 537–557, 1976.

[47] Nicholas M Katz and Tadao Oda. On the differentiation of de rham cohomology classes with respect to parameters. *Journal of Mathematics of Kyoto University*, 8(2):199–213, 1968.

[48] Kiran S Kedlaya. p-adic cohomology: from theory to practice. *Arizona Winter School Notes*, 9, 2007.

[49] Bernhard Keller and Dong Yang. Derived equivalences from mutations of quivers with potential. *Advances in Mathematics*, 226(3):2118–2168, 2011.

[50] Ivan Kolár, Peter W Michor, and Jan Slovák. *Natural operations in differential geometry.* Springer Science & Business Media, 2013.

[51] M KONTSEVICH and Y SOIBELMAN. Affine structures and non-archimedean analytic spaces, prépublication sur arxiv, référence. *arXiv preprint math.AG/0406564*, 2004.

[52] Maxim Kontsevich and Yan Soibelman. Stability structures, motivic donaldson-thomas invariants and cluster transformations. *arXiv preprint arXiv:0811.2435*, 2008.

[53] I Krichever. Isomonodromy equations on algebraic curves, canonical transformations and whitham equations. *arXiv preprint hep-th/0112096*, 2001.

[54] Daniel Labardini-Fragoso. Quivers with potentials associated to triangulated surfaces. *Proceedings of the London Mathematical Society*, 98(3):797–839, 2009.

[55] Daniel Labardini-Fragoso. On triangulations, quivers with potentials and mutations. *Mexican Mathematicians Abroad*, 657:103, 2016.

[56] Antonio Lanteri and Daniele C Struppa. Topological properties of cyclic coverings branched along an ample divisor. *Canadian Journal of Mathematics*, 41(3):462–479, 1989.

[57] Yves Laszlo and Christian Pauly. On the hitchin morphism in positive characteristic. *International Mathematics Research Notices*, 2001(3):129–143, 2001.

[58] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39. Springer, 2018.

[59] Max Lieblich. Remarks on the stack of coherent algebras. *International Mathematics Research Notices*, 2006:75273, 2006.

[60] DANIEL LITT. Variations of hodge structure notes for number theory learning seminar on shimura varieties.

[61] Bernard Malgrange. Cours de l'institut fourier. 1971.

[62] Bernard Malgrange. La classification des connexions irrégulières à une variable. *Cours de l'institut Fourier*, 17:1–19, 1982.

[63] Eyal Markman. Spectral curves and integrable systems. *Compositio mathematica*, 93(3):255–290, 1994.

[64] David Mumford. Prym varieties i. In *Contributions to analysis*, pages 325–350. Elsevier, 1974.

[65] Martin Olsson. *Algebraic spaces and stacks*, volume 62. American Mathematical Society, 2023.

[66] Christian Pauly and Ana Peón-Nieto. Very stable bundles and properness of the hitchin map. *Geometriae Dedicata*, 198(1):143–148, 2019.

[67] Ana Peón-Nieto. Equality of the wobbly and shaky loci. *International Mathematics Research Notices*, 2024(8):6710–6730, 2024.

[68] Michael Rosen. Abelian varieties over c. *Arithmetic geometry*, pages 79–101, 1986.

[69] Claude Sabbah. *Isomonodromic deformations and Frobenius manifolds: An introduction.* Springer Science & Business Media, 2007.

[70] Claude Sabbah. *Introduction to Stokes structures*, volume 2060. Springer, 2012.

[71] Jean-Pierre Serre. Géométrie algébrique et géométrie analytique. In *Annales de l'institut Fourier*, volume 6, pages 1–42, 1956.

[72] Yasutaka Sibuya. *Linear differential equations in the complex domain: problems of analytic continuation*, volume 82. American Mathematical Soc., 2008.

[73] Ivan Smith. Quiver algebras as fukaya categories. *Geometry & Topology*, 19(5):2557–2617, 2015.

[74] The Stacks project authors. The stacks project. `https://stacks.math.columbia.edu`, 2022.

[75] Marius Van der Put and Michael F Singer. *Galois theory of linear differential equations*, volume 328. Springer Science & Business Media, 2012.

[76] William A Veech. Moduli spaces of quadratic differentials. *Journal d'analyse mathématique*, 55(1):117–171, 1990.

[77] Wolfgang Wasow. *Asymptotic expansions for ordinary differential equations*. Courier Dover Publications, 2018.

[78] Edward Witten. Gauge theory and wild ramification. *Analysis and Applications*, 6(04):429–501, 2008.