Monetary Policy and Rational Asset Price Bubbles in a New Keynesian Framework: A Theoretical Analysis

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Abstract

In light of the 2008 financial crisis, this thesis studies the unsettled issue about "should monetary policy lean against rational asset price bubbles" in a New Keynesian (NK) framework, aiming to provide theoretical insights for it.

In the first part of the study (Chapter 2), I adopt a modified NK model with "perpetual youth" overlapping generations as in Galí (2021), where a continuum of equilibria with rational bubbles could exist because of dynamic inefficiency problem. With nominal rigidities and monopolistic competitions, bounded bubble fluctuations affect the model economy mainly on the demand side. It shows that allowing an active policy response to a growing bubble could be more effective in terms of achieving macroeconomic stability, especially when the initial size of the bubble is bigger than a threshold value. However, the leaning-against-the-bubble (LAB) policy may have the risk of being "overreacting" if otherwise.

In the second part of the study (Chapter 3 and 4), I establish (in two steps) an analytically tractable NK model with endogenous capital accumulations, where rational bubbles may exist in equilibrium because of the extra liquidity they generate for financially constrained firms with a lumpy investment opportunity. Under certain conditions, bounded bubble-driven fluctuations (in output) may emerge via both supply- and demand-side mechanisms. A LAB strategy may be indispensable under a strict inflation target, but may risk causing an economic downtown in a more pragmatic setting in the face of a bubble-led boom.

Overall, the analyses of the thesis in different scenarios do not in strong favour of a LAB policy and emphasise in particular a special overreaction risk that it may suffer relative to its conventional counterpart.

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Author's Declaration

I declare that this thesis is a presentation of original work, and I am the sole author. This work has not previously been presented for a degree or other qualification at this, or any other, University. All sources are acknowledged as references.

An earlier version of Chapter 2 has been presented at the York Research Student Workshop in May 2024, and has been submitted and accepted by the 56th Annual Conference of the Money, Macro and Finance Society in 2025. The work of Chapter 3 and 4 has been compressed into an individual paper presented in the Durham-York-Newcastle Centre for Macroeconomic Policy Mini Workshop in January 2025.

Chapter 1

Introduction

1.1 Background

1.1.1 Monetary Policy and Financial Stability

It has now been more than a decade since the 2008 financial crisis. Yet, whether monetary policy should lend a hand by pursuing a financial stability objective in addition to its traditional objectives of inflation and output-gap stabilization and how remains an unsettled issue of recurrent debate.

Prior to the crisis, inflation targeting was the primary mandate of monetary policy, which was mainly inspired by the New Keynesian (NK, for brevity) framework for the conduct of optimal policy, implying that inflation stability was able to keep the output-gap stable simultaneously under a broad range of conditions. Also, it was widely accepted that financial stability should not be taken into account in monetary policy deliberations.[1] This policy framework was supported by three main arguments:

(1) Price instability would be more likely to generate financial crises while financial stability can be achieved as a by-product of price stability, as some empirical work concludes;[2]

(2) Financial crises are unpredictable, and the effects of using monetary policy to address financial risks are hard to justify, so it is more practical for monetary policy to respond only after the realization of crises (the “wait-and-see” approach);

(3) It is a task for prudential policies instead of for monetary policy.

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[1] See, e.g., Bernanke and Gertler 1999, Vickers 1999.

[2] See, e.g., Hardy and Pazarbasioglu 1999, Bordo et al. 2000.

Unfortunately, the crisis serves as a bitter reminder that inflation-targeting alone is not enough simultaneously to guarantee financial stability; crises can still build up unnoticed in a low and stable inflation environment.[3] The costs of "mopping up" after the crisis proved to be very expensive: many economies are still struggling with unsatisfactory economic performance, despite the fact that unconventional policies like quantitative easing and negative nominal interest rates had been employed to provide extra accommodation when the policy rate had run into its zero lower bound, while increasing government debt burdens constrain the further expansion of fiscal policy at the same time, which also raises concerns about the possible after-effects of these unconventional tools.[4] Moreover, even though macro-prudential policies have been added to traditional micro-prudential frameworks to monitor the stability of the financial system as a whole after the crisis,[5] there are still concerns that the prudential framework as a whole is insufficient to maintain financial stability.[6]

Bearing these in mind, the argument that monetary policy should take a role in responding to the build-up of financial instability is strengthened.[7] However, whether it is indeed the case and also the question of how to do so still remain unclear,[8] while divided views have been observed in recent literature as ever. One important reason for this variability is due to the lack of uniformly accepted theoretical frameworks for understanding the build-ups of financial instability and its interaction mechanism with

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[3] See evidence by Borio and Lowe (2002), IMF (2015). Borio and Lowe (2002) and Caballero and Krishnamurthy (2003) provide some possible reasons for this phenomenon. Taylor (2014) also attributes this to excessively low interest rates.

[4] See, Williams (2013), Chen et al. (2014), Chen et al. (2015) for explorations.

[5] As recommended by Viñals (2013), IMF (2014).

[6] See Borio and Lowe (2002), Woodford (2012), Fisher (2016) for some arguments, and Galati and Moessner (2014) for a review of relevant empirical work.

[7] See e.g., Borio and Lowe (2002) and Woodford (2012) for arguments.

[8] Prior to this question, someone would argue that it has to be answered first whether we are able to identify the build-ups of financial risks. The fact is that our ability to deal with this issue is being greatly improved. See Borio and Drehmann (2009) for example for possible monitoring methods. Also see Barlevy (2018) for discussion for policy evaluation.

the real economy and monetary policy as well.[9] Therefore, further research aiming to provide theoretical foundations in this regard for the conduct of monetary policy is worthwhile.

1.2.1 Why Do Rational Bubbles Particularly Matter?

Despite the fact that there are many aspects to and ways of measuring financial instability, rapid increases in asset prices that are not linked to changes in economic fundamentals and which are usually referred to as "bubble" episodes,[10] are particular concerns for policymakers, as they would leave markets vulnerable to shocks and usually indicate an eventual rapid price fall that would disrupt the normal economic order and could cause severe destruction. Indeed, it is widely believed that it is the collapse of the housing bubble that should primarily be to blame for the 2008 crisis and the great recession afterwards.

In practice, managing financial risks posed by bubbles is not supposed to be solely a task for monetary policy, but requires proper cooperative actions of a group of policy tools simultaneously, including fiscal policy and regulations (macroprudential policy). In some situations, monetary policy might even be a poor tool in responding to bubbles, as the adjustment of the systematic policy (rate) might cause unsatisfactory economic fluctuations in other areas (as argued by Bernanke and Gertler (1999), for example), while specific regulations might be able to do a better job. However, if a bubble is suspected to be a systematic phenomenon and thus closely connects different parts of the economy, while it is sensitive to interest rate fluctuations, then monetary policy should be expected to play an important role in managing the risks posed by it.

A rational asset price bubble, a concept which will be elaborated later, is then a relevant concern in this regard, as it has general impact on an economic system while

[9] Another reason is due to the limited relevant empirical work, as crises are rare phenomena.

[10] The work by LeRoy and Porter (1981) and Shiller (1982) identifies that large scale of volatility of actual observed asset prices is hardly explained by their market fundamental, implying the possible existence of bubbles, as also pointed out later by Tirole (1985).

its evolution is affected by policy rates, which will be seen clearly throughout the analyses of this thesis. Therefore, particular attention is worth paying to the study of "how monetary policy should deal with rational bubbles" and which should be among the right directions for addressing the questions I highlighted at the very beginning of this introduction.

1.2 Literature Review

1.2.1 Types of Rational Bubbles[11]

Among the related literature, an asset "bubble" is typically defined as the difference between the observed price of the asset and the expected discounted present value of the dividends it generates, i.e., the "fundamental" value of the asset.[12] By "rational", it means that the bubble is consistent with rational expectations, individual optimisation, and market clearing.[13] Under these definitions and to my current knowledge, there are broadly speaking two dominant types of scenarios that can give rise to a rational bubble among the existing theoretical models, as reviewed below.[14]

1.2.1.1 Dynamic Inefficiency

The most celebrated one among the conditions that cause the equilibrium existence of

[11] Most of the literature on rational bubbles focuses on deterministic ones that feature predictable behaviour and will not have a self-driven burst. Another group of it is stochastic, which was first proposed by Blanchard (1979). Other examples are Blanchard and Watson (1982), Weil (1987), and Miao and Wang (2018). I focus on "deterministic bubbles" throughout the analysis of the thesis.

[12] Recent literature also adopts a "risk premium" approach towards defining a "bubble". See, e.g., Caballero and Simsek 2020.

[13] Its counterpart concept, "irrational bubble", on the contrary, does not require to be so. For example, it can be modelled as an ad hoc deviation of an asset price from its fundamental value (e.g., Bernanke and Gertler (1999, 2001)). See also Brunnermeire and Oehmke (2013), Miao (2014), Martín and Ventura (2018), Barlevy (2018) for surveys on models of rational or irrational bubbles.

[14] There are other types of rational bubbles in the literature, but which are not relevant to my study here, including those characterised an "agency problem" (e.g., Allen and Gorton 1993, Allen and Gale 2000, Barlevy 2014), "information frictions" (e.g., Allen et al. 1993), and "misguided beliefs" (e.g., Hong et al. 2006, 2008). See, Santos and Woodford (1997) for conditions that allow rational bubbles to arise in general equilibrium. Hirano and Toda (2024, 2025) also provide more recent views of the theory of rational bubbles.

a rational bubble is “dynamic inefficiency”, which stems from the work by Samuelson (1958) and Diamond (1965), and which is then developed by Tirole (1985) to address explicitly the issue of rational bubbles. This class of models features a (two-period) overlapping generations (OLG, for brevity) structure with the economy growing over time, where cohorts who born later are richer than their predecessors either because they have higher productivity or because they are larger and can produce more at scale. However, wealth cannot be transferred from young to old agents due to the intertemporal structure of the economy, resulting in the resource allocation being Pareto inefficient.

Bubbles that grow at the rate of interest (but at a rate no greater than the growth rate of the economy) in this case, therefore, serve as a way to improve this inefficient situation, as each generation can transfer wealth to their predecessors by buying intrinsically worthless assets and receive wealth from their successors by selling the bubble assets later on, thus implicitly achieving the intertemporal transfer that makes all agents better off and therefore is desirable for the society as a whole. Recent examples of this class of models include Galí (2014, 2021).

1.2.1.2 Credit (Liquidity) Constraints

Another prevailing approach of modelling rational bubbles features the financial friction of “credit/liquidity constraints”, where economic agents face some borrowing or liquidity restrictions.[15] Bubbles arise in this situation to relax the financing limits either faced by households with volatile income flows or those faced by firms (financial intermediaries) with heterogenous levels of productive efficiency.[16] Notable examples for the former strand include Bewley (1983), Aiyagari (1994) and Huggett (1993),[17] while the latter has been studied extensively in both an OLG framework (e.g., Caballero

[15] Hori and Im (2023) provides an example of where rational bubbles can emerge even when economic agents do *not* face financing constraints but where there are uninsured entrepreneurial risks.

[16] Clain-chamosset-yvrard et al. (2023) pointed out that the liquidity and collateral roles of rational bubbles in credit constraints are equivalent to each other if they are both deterministic, but differ when they are stochastic.

[17] Also, Kocherlakota (1992) and more recently, Bonchi (2023) in a three-period OLG model.

and Krishnamurthy 2006, Farhi and Tirole 2012, Martín and Ventura 2012, 2016, Bengui and Phan 2018, Asriyan et al. 2021) and an infinite-horizon framework (e.g., Kocherlakota 2009, Wang and Wen 2012, Hirano and Yanagawa 2017, Miao, Wang and Xu 2015, Miao and Wang 2015, 2018, Dong et al. 2020, Biswas et al. 2020, Ikeda 2021).

Rational bubbles in this case are generally also beneficial, as they improve the economic agents' financing conditions which yields socially desirable outcomes.[18] Unlike that in the dynamically inefficient case, though, they do not necessarily have to grow at the interest rate to be attractive, as they provide extra liquidity which benefits the bubble holders.

1.2.2 Monetary Policy Implications

1.2.2.1 Models with New Keynesian Features (Sticky Prices)

Despite the fast-growing literature on rational bubbles, the development of relevant monetary policy analysis has lagged far behind.[19] Nonetheless, among the existing studies, Galí (2014) seminally investigates the interaction between monetary policy and rational bubbles arising for dynamic inefficiency reasons in a simple 2-period OLG framework with sticky prices, where both employment and output are constant in equilibrium, with the rational bubble only having redistributive impacts. In the absence of bubble-driven fluctuations (in output), Galí (2014) reaches the conclusion that "leaning-against-the-bubble" (LAB, hereafter) policies may not be desirable, since

[18] If producing the bubble is costly and it is stochastic in the sense that the bubble bursts in the future with a positive probability, e.g., Dong et al. 2022, or if there are other policies distorting economic agents' behaviour, e.g., Miao, Wang, and Zhou 2015, then this claim does not necessarily hold.

[19] A notable phenomenon concerning the research on monetary policy analysis with bubbles is that a considerable amount of effort has been devoted to asset price volatility caused by "irrational bubbles" (e.g., Bernanke and Gertler 1999, 2001, Gilchrist and Leahy 2002, Carlstrom and Fuerst 2007, Nisticò 2012), but little to that caused by "rational" ones. For example, in the influential works by Bernanke and Gertler (1999, 2001) which analyse the desirability of "lean-against-the-wind" monetary policy in a NK framework, ad hoc deviation from the fundamental value of stock prices is assumed and thus this does not belong to the category of "rational bubble" which cannot exist in their models with infinitely lived representative consumers. This implies that volatility in asset markets of which rational bubbles are the major driver should generate distinct feedback mechanism in the face of monetary policy interventions.

raising the policy rate tends to amplify the bubble and may be welfare-reducing in this model economy. However, this stance regarding the LAB policies is challenged by Allen et al. (2017), who reexamine Galí (2014) in a set of mostly non-monetary frameworks. Miao et al. (2019) also extend Galí (2014) by incorporating persistent bubble shocks, suggesting that the LAB policies are favourable in reducing bubble volatility and improving welfare, if E-learnability of the bubbly equilibria approach is adopted instead.

Closest in spirit to Galí (2014), Galí (2021) nests the infinite-horizon NK framework into a "perpetual-youth" type OLG structure as in Yaari (1965) and Blanchard (1985) (referred to as OLG-NK, for brevity), which then admits a continuum of bubbly balanced growth paths (BGP, for short) and allows the possibility of bubble-driven fluctuations in the economy. Galí shows that LAB to a certain extent may be more effective in tackling the bubble-driven fluctuations than a conventional output-gap-focused interest rate rule but may have the risk of overreacting, especially if the bubbles pop up from an originally bubbleless economy. More recently, Bonchi and Nisticò (2024) modify Galí (2021) to incorporate stochastic transitions of economic agents' participations in asset and labour markets as in Nisticò (2016), which, they claim, generates a trade-off in a welfare-based monetary policy analysis, making the conventional inflation targeting regime mostly suboptimal in tackling bubble-driven fluctuations.[20]

There is also work concerning the role of monetary policies in tackling rational bubble fluctuations in a credit-constraint environment in some "optimal" way, although

[20] Even though the model of Bonchi and Nisticò (2024) is not exactly the same as that of Galí (2021), the reason that it gives rise to a rational bubble in equilibrium is essentially equivalent, i.e., the possibility of a declining labour income and a lack of access to proper financing tools for individual households may in some circumstances result in excess savings with a low interest rate in the economy, which is inefficient and could be improved by the bubbles. Most importantly, in those cases where bounded bubble fluctuations could occur (i.e., the system is indeterminate), the corresponding SS should always be *inefficient* (see, particularly Sections 2.3.1-2 of Chapter 2 of the thesis for discussions). Therefore, it is unclear that how it is possible for Bonchi and Nisticò (2024) to always ensure an efficient SS around which bounded bubble fluctuations could emerge.

again this is very scarce. Dong et al. (2020) extends Miao, Wang, and Xu (2015) to provide a model of rational bubbles in a NK framework, showing that a LAB strategy may reduce bubble volatility, but at the cost of making inflation more volatile. Also, somehow peculiarly, their simulation results suggest that the optimal weights of policy feedback on bubbles in alternative simple interest rate rules are always negative in terms of maximising the unconditional mean of household utility, meaning that the policy rate should always go down in the face of a bubble boom, which is against the conventional wisdom regarding confronting a bubble episode.[21]

On the other hand, Ikeda (2021) explores implications of Ramsey-optimal monetary policy and alternative simple interest rate rules in a model a là Christiano et al. (2005) with the presence of both nominal price and wage rigidities and a financial cost channel, where multiple bubbly assets of the type of Miao, Wang, and Xu (2015) serve as a means to mitigate firms' borrowing constraints. In addition to the usual inflationary impact of a bubble induced by its effect on aggregate demand, a bubble also has a deflationary impact in the presence of the financial cost channel, so that inflation remains moderate during a bubble-led boom as a result.[22] Ikeda shows that a LAB strategy is generally either unnecessary or counterproductive: economic inefficiencies caused by a bubble boom should be better addressed by monetary policy focusing on conventional feedback variables (output or inflation). Particularly with the economy featuring the financial cost channel, LAB tends to curb excessively the real economic activities in the face of a bubble boom and reduce the welfare of the households as a consequence.

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[21] This result apparently relies crucially on the fact that rational bubbles in their model are desirable in relaxing agents borrowing constraints and are to a certain extent welfare-enhancing. Based on Dong et al. (2020), Liu and Wang (2024) also studies the interaction between monetary policy and a type of time-varying macroprudential policy.

[22] Intuitively, an increase in the bubble price boost physical capital investment of firms as the bubble boom expands their financing capacity, leading to a higher labour demand and higher production level, resulting in a higher unit labour cost and thus inflation; meanwhile, due to fact that an individual firm must borrow its wage bill at the beginning of each period by assumption, a bubble price rise tends to lower its marginal costs of production by lowering its borrowing cost, thus putting downward pressure on inflation.

1.2.2.2 Others

There are other models which investigate monetary policy implications in the presence of rational bubble but abstract from nominal rigidities. An example is provided by Asriyan et al. (2021), whose model draws heavily on Martín and Ventura (2012, 2016) in an OLG setting. Two types of 'unbacked' assets are assumed in their framework: the "nominal" bubbles created by the private sector and the paper money created by the public sector (central bank). Monetary policy can be non-neutral in their flexible price environment, because money issued by the central bank is considered substitutable for the bubbles in terms of redistributing resources among heterogenous agents confronting financial constraints, thus the central bank is shown to always be capable of affecting the size and dynamics of the bubble by adjusting its money supply. In this sense, money itself could just be viewed as another type of bubbly asset, as in Samuelson (1958) and also Kiyotaki and Moore (2019).[23] In a related spirit, Clain-Chamosset-Yvrard and Seegmuller (2015) show that in a 2-period OLG model with credit market restrictions, additional LAB feedback can restore an inflation-expectation-focused policy to an effective one in ruling out bubble-driven fluctuations of the economy.

Another strand of models also investigate possible interactions between the zero lower bound (ZLB, for short) of monetary policy rates and downward wage rigidity, including Biswas et al. (2020) in an infinite-horizon framework and Bonchi (2023) in a three-period OLG framework.[24] Although monetary policy is not necessarily neutral in their environments, the channel through which the policy effects are transmitted is distinct from that highlighted in models mentioned in the previous section.

1.3 Research Motives and Modelling Approach

As highlighted at the start of this introduction, the general aim of my thesis is to address

[23] Indeed, Kiyotaki and Moore (2019) studies the impact of open-market operations on economic activities in the face of productivity and liquidity shocks under flexible prices.

[24] In both studies bubbles are stochastic.

the unsettled issue regarding "should monetary policy lean against bubbles, and if so, how" by providing rigorous theoretical foundations for it. To begin with, I incline to conduct my study within a NK framework, especially given that it remains the workhorse framework deployed for monetary policy analysis and from which the pre-crisis main stance towards a LAB strategy had been drawn. Hence, discoveries from my research in the thesis are expected to provide insights about restoring the latter as a suitable paradigm for monetary policy in the post-crisis era.

From my point of view, a non-trivial contributory factor to the current disagreement among economists and policymakers about whether monetary policy is a suitable tool to tackle asset market instability is that when financial factors are explicitly modelled in a micro-founded dynamic (stochastic) general equilibrium model (DSGE), the structure soon becomes so complicated that numerical methods then need to be applied to obtain a solution. This is, to my knowledge, mostly true in the existing literature on rational bubbles and monetary policy analysis, particularly for those conducted in a NK framework. Consequently, the underlying mechanisms that drive the apparent dynamics of those models are being left unclear, while simulation results tend to be sensitive to the specific setup, given the intricate interactions between variables in these sophisticated models.

Therefore, in order to understand in depth the economic forces at work, I undertake my analysis throughout the thesis in the context of models that are sparing enough in the sense that they contain only the key elements that I wish to study, so that analytical solutions are mostly possible to be obtained. It is to be hoped that the transmission mechanism and the consequences of various types of policy interventions for the bubbly economies will then be clear, and that policy suggestions based on these results will in turn be fairly transparent as well.

My overall strategy in the following monetary policy discussions also emphasises less the "normative" aspect of a policy design, e.g., whether the policy is "optimal" or not in maximising a proposed social welfare function, but focuses more on the underlying theoretical linkages between monetary interventions and the evolution dynamics of the rational bubbles and the other aggregate variables of interest. In this way I believe that conclusions drawn from my analysis are of broader relevance and would be more robust to different policy objectives within various economic backdrops.

1.4 Organisation of the Thesis (Research Agenda and Findings)

To provide a fuller picture of potential economic mechanisms of monetary policy intervention in a bubbly world, I study two typical situations where two aforementioned types of rational bubbles affect either the demand side (Chapter 2) or the supply and the demand side (Chapter 3 and 4) of an economy. The final chapter (Chapter 5) concludes with some reflections on caveats about the frameworks deployed for the analysis of the thesis and potential directions for future research.

1.4.1 Chapter 2

In Chapter 2, I first review the OLG-NK model of Galí (2021) by elaborating upon its novel feature, the coexistence of a continuum of bubbly BGPs, which underlies in a crucial way its monetary policy analysis but which was not thoroughly explained in the original paper. It highlights that it is the introduction of multiple bubble assets in the model that makes it possible for the multiple bubbly equilibria to exist.

I then extend Galí's (2021) work by investigating how monetary policies might be used to mitigate the effects of bounded rational bubble fluctuations on macroeconomic variables, when these bubble-driven fluctuations cannot be entirely ruled out by the proposed policies in the first place. I have found that in those circumstances, there is generally a trade-off for the policymaker between dampening the bubble impact and reducing the persistence of the resulting output fluctuations under both the conventional and LAB policy regimes.

Also, a LAB policy is shown to be very effective in mitigating the bubble impact on output, but may have the risk of overreacting by causing a potentially severe economic recession if the bubble size in the economy is relatively small in the first place. On the other hand, a conventional output gap-focused policy is also effective in stabilising the economy from a bubble shock, while it does not appear to suffer the type of risk facing the LAB policy.

1.4.2 Chapter 3

In Chapter 3 and 4, I turn to study the implications for formulation of monetary policy with rational bubbles emerging from financial frictions in an infinite-horizon framework with endogenous capital accumulation. As a primary step of this task, in Chapter 3 I develop an analytically tractable flexible-price model whose structure is closely related to Wang and Wen (2012), where individual firms are subject to a type of uninsurable exogenous investment shock a là Kiyotaki and Moore (2019). The assumption that external borrowing channels are completely absent for the firms combined with the lumpy investment opportunity then make it possible for intrinsically useless bubbles to be positively valued, as they serve as a means of transferring resources between heterogenous firms with different investment opportunities.

This neat setup allows me to obtain a whole set of closed-formed solutions of the model, the key one of which clearly demonstrates that rational bubbles in this type of credit-constraint environment do not need to grow at the rate of interest in equilibrium, because they also benefit their holders, the firms, by providing extra liquidity when an investment opportunity comes, thus generating a "liquidity premium" component in the pricing equation. The availability of the closed-formed general equilibrium conditions also enables me to identify three mutually exclusive regions of parameter space, only one of which has a low enough arrival rate of an investment opportunity to permit a bubbly equilibrium.

I then continue to characterise the model dynamics around different types of steady states (SS, for short). In particular, when approximated around a bubbleless SS, the economic system is algebraically shown to be always exposed to bounded bubble fluctuations under flexible prices. The bubble-driven fluctuations are long-lasting even with a transitory bubble shock, since the rational nature of the bubble makes the bubble fluctuation itself intrinsically persistent, but the degree is affected by the arrival rate of an investment opportunity to firms. In addition, investment and output increase with a positive bubble shock, as the larger bubble improves investing firms external financing capacity and then production as a result.

1.4.3 Chapter 4

In Chapter 4, I undertake the task of investigating the desirability of LAB monetary policy by incorporating NK features into the model that I establish in Chapter 3, i.e., monopolistic retailers with Calvo's (1983) staggered price setting, so that meaningful model-based discussions of interactions of the bubble with monetary policy are possible. Instead of searching for some kind of "welfare-optimising" monetary policy rule, I restrict my attention to studying two sets of interest rate rules – the first one is a strict (zero) inflation targeting rule, while the second one is a simple Taylor-type rule with or without direct feedback on variations in the bubble size – to assess the implications of the policy responding explicitly to the bubbles in different scenarios.

It turns out that if a bubble-led economic boom stems from a bubbleless system, then to ensure the successful implementation of the strict inflation target, monetary policy is required to lean against the bubble, as the natural real rate of interest of the economy is varied by the bubble fluctuations along the way. However, in a more practical policy specification, additional feedback on variations in the size of the bubble in the simple Taylor-type rule has a high risk of causing a recession, which relates to the autonomous nature of the evolution of the bubble around a bubbleless SS.

Contrary to the outcome concerning the economy fluctuating around a bubbleless SS, there is no theoretical support for the strict inflation targeting policy to LAB when the model economy fluctuates around a bubbly SS, while adding a direct response to the bubble variations in the simple rule is also not very helpful in terms of ruling out bubble-driven fluctuations of the system. The latter outcome may be due to the fact that the two determinants of the law of motion of the bubble are affected oppositely by interest rate variations, resulting in the LAB policy being ineffective.

Chapter 2

Monetary Policy and Rational Bubbles: A Demand-Side Perspective

2.1 Introduction

Rational bubbles are tricky to incorporate into the standard infinite-horizon New Keynesian (NK, for brevity) paradigm, for the widely-recognised reason that rational bubbles that grow at the rate of interest will simply be ruled out by the transversality condition of the representative consumer.[25] This imposes a dilemma for monetary policy analysis with rational bubbles: even though it is possible to conduct the analysis in a classic OLG model, as in the work done by Galí (2014), Allen et al. (2017), Asriyan et al. (2021), and Bonchi (2023),[26] the short-lifespan structure tends to be too stylised to generate attractive insights or to be reconciled with the data for serious quantitative examination.[27] On the other hand, even when it can be somehow accommodated into an infinite-horizon framework for monetary policy discussions, as in Dong et al. (2020), Ikeda (2021) for rational bubbles affecting the supply side of an economy, or in Miao and Wang (2015) for emergence of bubbles in the financial intermediary sector, the *demand-side effects* of rational bubbles have been ignored, preventing a decent and complete appreciation of the possible impacts of rational bubbles on economic dynamics with monetary policy interventions. This unsatisfactory situation had not been addressed until the work of Galí (2021).

Galí's (2021) analysis is innovative, in the sense that it subtly nests the infinite-

[25] See, Santos and Woodford (1997) for existence conditions of rational bubbles in general equilibrium.

[26] The analysis conducted by Bonchi (2023) is in a three-period OLG model.

[27] For example, in Galí (2014), output is never disturbed by rational bubble fluctuations even in a sticky prices environment, and it is impossible for monetary policy to eliminate the bubble fluctuations themselves in the first place.

horizon NK framework into a "perpetual-youth" type OLG structure as in Yaari (1965) and Blanchard (1985) (referred to as OLG-NK, hereafter), permitting the existence of rational bubbles growing at the rate of interest in equilibrium while maintaining the validity of the transversality condition of households, thus successfully overcoming the key technical problem mentioned above. Also importantly, the model established by Galí (2021) is analytically tractable, enabling precise algebraic investigation of the rich dynamics of the OLG-NK economy.

However, the monetary policy analysis in terms of economic fluctuations driven by rational bubbles in Galí (2021), while inspiring, is incomplete in the sense that in his investigation of alternative monetary policy rules, under certain circumstances, neither the proposed LAB policy nor the conventional policy is capable of completely ruling out bubble-driven fluctuations; but his study is silent about what and how monetary policies could do to mitigate, even when it is impossible or realistic to entirely eliminate, the disruption caused by bubble fluctuations to aggregate demand in these scenarios.

Given the mentioned limitation of Galí (2021), it is my aim to fill this gap in this chapter by extending his analysis, mainly in an analytical way, to investigate how monetary policy might be deployed to mitigate the effects of rational bubble fluctuations on macroeconomic variables, particularly, output and inflation, even when these fluctuations cannot be entirely ruled out by the policy in the first place, so that we could better appreciate the role monetary policy may play in terms of shaping economic dynamics in the face of a rational bubble shock from a demand-side perspective.

I do so by firstly presenting a modified version of the OLG-NK framework as in Galí (2021), where the deterministic growth rate of output is dropped for simplicity. More importantly, I re-specify the individual household's budget constraints, so that the theoretical underpinning of a key feature of the model, namely, the possible coexistence of a continuum of bubbly steady states (SS, for short) is made explicit and clear. After providing the equilibrium conditions and characterising the steady states, I prepare the monetary policy analysis with discussions about potential impacts of the rational bubbles on the equilibrium system, of which some aspects are less or not emphasised in Galí (2021). I then turn to the formal exploration of monetary policy implications particularly in those situations in Galí (2021) where neither the proposed LAB policy nor the output-gap-based rule is workable for insulating the economy from bounded rational bubble fluctuations in combination with sticky prices.

Several discoveries of interest emerge from the analysis, outlined below.

First, it reveals and confirms that it is the introduction of multiple bubble assets and recurrent new bubbles that enables the coexistence of a continuum of bubbly equilibria. If there is only one type of bubble or no emergence of new bubbles, then only a unique bubbly equilibrium could potentially exist. In addition, the endowment of individuals with new bubbles turns out to be a source of *inefficiency* in a bubbly equilibrium, which is not the case in the classic literature on rational bubbles, where a bubbly equilibrium must be dynamically *efficient*.

Second, it generalises the findings of Galí (2021) regarding the local dynamics of the OLG-NK economy under flexible prices to a *global* extent. Under the same threshold value, a bubbly steady state is globally stable as well, while an increase in the natural rate of interest is also a global phenomenon in the face of a bubble boom of *any* size.

Third, when the model economy is exposed to bounded bubble fluctuations, there is generically a monetary policy trade-off between dampening the bubble impact and reducing the persistence of the bubble-led fluctuations in output. On the one hand, an endogenous increase in interest rates induced by a rising bubble always prolongs the resulting output fluctuations, while on the other hand, a stronger policy reaction to the developments on the bubble could effectively mitigate the bubble impact on output. Nonetheless, a special case is that if the size of the aggregate bubble at a steady state around which the economy fluctuates is smaller than a threshold value, then there exists a “surgical” LAB policy under which output could be completely insulated from the bounded bubble fluctuations.

Fourth and most importantly, the policy with a LAB motive, while is very effective in dampening the bubble impact, has an "*overreaction*" risk in causing a recession in the face of a bubble boom, especially when the original size of the bubble in the economy is *smaller* than a threshold value. A key contributory factor to this outcome is that interest rate changes *cannot* affect the size of the bubble *in the impact period*, but the latter *can* have an impact on the former and then aggregate demand via the LAB policy, thus opening a loophole for a too aggressive interest rate adjustment, especially taking into account the imperfect observability of actual bubble episodes. In addition, since the rational bubble fluctuations are *inherently persistent*, *future* interest rates are also expected to be raised higher under the LAB strategy, which tends to crowd out the (total) fundamental wealth in the economy, further depressing current aggregate demand as a result.

Finally, the conventional output gap-focused policy is also effective in stabilising the bounded bubble-driven fluctuations in the economy, while it does not appear to suffer the type of downside risk facing its LAB counterpart, mostly because the endogenous feedback loop between the policy rate and the output gap is now complete.

The remainder of this chapter is organised as follows. Section 2 presents the modified version of the OLG-NK framework of Galí (2021), where equilibrium conditions and steady states are characterised and a log-linearised version of the model is provided. Section 3 discusses preliminarily the equilibrium impacts of the rational bubbles on the economy, when abstracting from price stickiness. Section 4 analyses monetary policy implications in circumstances where the economy cannot be fully stabilised by means of the policy rule. Section 5 concludes with further reflection. Detailed derivations of some of the outcomes are left to the Appendix.

2.2 The OLG-NK Model

The model structure used in this chapter mostly follows Galí (2021). The economy is populated by "perpetual youth" overlapping generations as in Yaari (1965) and Blanchard (1985). An individual is born as an "active" agent, supplying his labour and running his newly set up firm. Each individual has a constant "survival rate", $\gamma \in (0,1)$, and a constant "active rate", $\upsilon \in (0,1)$, governing progression into the next period. Once the individual becomes "inactive" ("retired") with the constant probability $1-\upsilon$, he exits from the labour market permanently; and his firm will be closed either for his death or retirement. The size of the overall labour force at time *t*, and thus the size of the set of firms, are then given by $\alpha \equiv (1-\gamma)/(1-\upsilon\gamma) \in (0,1)$, which is constant but less than the overall size of the population of the economy, which is unity.

2.2.1 Households

A typical household born at time *s* with finite life expectancy seeks to maximise his expected lifetime utility

$$\max E_s \sum_{t=s}^{\infty} \left(\beta\gamma\right)^{t-s} \log C_{t|s} \tag{2.1}$$

by choosing a bundle of consumption goods, $C_{t|s}(i)$, $i \in [0,\alpha]$, at price $P_t(i)$, with $C_{t|s} \equiv \left(\alpha^{-1/\varepsilon} \int_0^{\alpha} C_{t|s}(i)^{1-1/\varepsilon} di\right)^{\varepsilon/\varepsilon-1}$ the consumption index and $\beta \in (0,1)$ the household's discount factor, subject to a sequence of (real-terms) period budget constraints

$$\begin{aligned} &\frac{1}{P_t}\int_0^{\alpha} P_t(i) C_{t|t}(i) di + \left\{ \frac{Z_{t|t}^R}{P_t} + \sum_{k=0}^{\infty} Q_{t|t-k}^B Z_{t|t,t-k}^B + \int_0^{\alpha} \left[Q_t^F(i) - D_t(i) \right] Z_{t|t}^F(i) di \right\} \\ &= \frac{\delta}{1-\gamma} Q_{t|t}^B + Q_{t|t}^F + W_t N_{t|t} \end{aligned} \tag{2.2}$$

for $t = s$, and

$$\begin{aligned} &\frac{1}{P_t}\int_0^{\alpha} P_t(i) C_{t|s}(i) di + \left\{ \frac{Z_{t|s}^R}{P_t} + \sum_{k=0}^{\infty} Q_{t|t-k}^B Z_{t|s,t-k}^B + \int_0^{\alpha} \left[Q_t^F(i) - D_t(i) \right] Z_{t|s}^F(i) di \right\} \\ &= \frac{1}{\gamma} \left\{ \frac{Z_{t-1|s}^R \left(1+i_{t-1}\right)}{P_t} + \left(1-\delta\right) \sum_{k=0}^{\infty} Q_{t|t-k-1}^B Z_{t-1|s,t-k-1}^B + \upsilon\gamma \int_0^{\alpha} Q_t^F(i) Z_{t-1|s}^F(i) di \right\} + W_t N_{t|s} \end{aligned} \tag{2.3}$$

for $t = s+1, s+2, \ldots$, with $P_t \equiv \left(\alpha^{-1} \int_0^{\alpha} P_t(i)^{1-\varepsilon} di\right)^{1/(1-\varepsilon)}$ the aggregate price index. $N_{t|s}$ is the labour supplied by the household, with $N_{t|s}^a > 0$ if he remains active and $N_{t|s}^r = 0$ if otherwise. Furthermore, it is assumed that $N_{t|s}^a = N_t/\alpha$, i.e., aggregate hours, N_t, are assumed to be evenly allocated across active households, with W_t the real wage earned in the competitive and flexible-wage labour market.

Following Blanchard (1985), a complete insurance arrangement is assumed to exist, such that except for the newborns, each household receives a gross annuity of $1/\gamma$ on its security portfolio purchased one period ago, conditional on the household being alive, but otherwise the assets pass to the perfectly competitive insurance company.[28] The budget constraint faced by a newborn household is different from those born in the previous periods as regards the initial financial wealth distribution. Specifically, a household which belongs to cohort $s \le t$ can purchase three types of assets: $Z^R_{t|s}$ denotes the nominal value of the one-period risk-free bonds purchased by the household at the end of period *t*, with i_t the nominally riskless interest rate between period *t* and *t*+1; $Z^F_{t|s}(i)$ is the individual firm *i*'s shares held by the household at the end of period *t*, with its before-dividends price being $Q^F_t(i)$, and $D_t(i)$ the paid-out dividends of firm *i*. Since an individual firm has a constant probability $\upsilon\gamma$ in continuing operating in the next period, the total amount of firms' shares held by the household one period later would thus be reduced by a proportion of $1-\upsilon\gamma$. An individual when born receives a net transfer $Q^F_{t|t} \equiv Q^F_t/\alpha$ with $Q^F_t \equiv \int_0^\alpha Q^F_t(i)di$, which might be thought of as a start-up fund for his newly setup firm. The nominally riskless bonds are in zero net supply in aggregate, while the total outstanding stock of an individual firm's shares is normalised to one.

There is also a series of intrinsically worthless assets, distinguished by the date on which they originated. By "intrinsically worthless", it is meant that these assets do not generate any dividends or rents, etc, yet may be traded at a positive price.[29] For instance,

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[28] See also, Galí (2021) footnote 10 for an alternative interpretation.

[29] Thus the strictly positive price for the intrinsically worthless asset constitutes a pure bubble. In the present framework, the bubble assets could simply be considered as "pieces of papers" (Tirole 1985), or they could also be thought of as attached to the firms' shares during their operations. See also, Galí (2014) footnote 15 and Galí (2021)

$Z^B_{t|s,t-k}$ is the quantity purchased at the end of period t of the bubbly asset introduced on date $t-k$ by a member of cohort $s \le t$ at price $Q^B_{t|t-k} \ge 0$, where non-negativity of the bubble price comes from the assumption of free disposal. In particular, a household born at date t is endowed at his birth with $\delta/(1-\gamma)$ units of a type of the bubbly assets newly introduced at that date, implying that the overall quantity of the new bubble is given by $\delta \in (0,1)$, since the size of the newborn is $1-\gamma$ in each period. Meanwhile, a fraction δ of each type of the pre-existing bubbly assets is assumed to lose its value for whatever reasons in each period.[30] As a result, the total amount of the bubbly assets remains constant and equal to one through the time. For discussion convenience, let's denote the overall value of the new bubbly asset at time t as $U_t \equiv \delta Q^B_{t|t}$, and that of old bubbly assets as $B_t \equiv \sum_{k=1}^{\infty} \delta(1-\delta)^k Q^B_{t|t-k}$. The aggregate value of the bubble in the economy is then given by $Q^B_t \equiv B_t + U_t = \sum_{k=0}^{\infty} \delta(1-\delta)^k Q^B_{t|t-k}$.

As far as I have been concerned, the main technical reason for introducing multiple bubbly assets especially "new" bubbles here (i.e., in Galí's (2021) framework) is to allow for a much wider range of bubbly steady states, which permits potential bounded bubble(-driven) fluctuations that would not otherwise be possible (under sticky prices). More broadly speaking, this setup also captures the re-emergence of a bubbly episode after a hypothetical burst, thus overcoming a criticism of early models of rational bubbles. Consider that if there is only one type of bubbles, then once it bursts with any possibility, it would never come back,[31] or otherwise it would just be rational to hold the bubbly asset forever, so that the bubble would never actually burst in the first place. But if new bubbles which are distinguished from the "old" ones are allowed to emerge, as in the present context, then the recurrence of bubbles becomes possible.[32]

footnote 24-26 for interpretations of (different types of) (pure) bubbles.

[30] For example, in Galí's words, "they are physically destroyed".

[31] E.g., stochastic rational bubbles of the type of Blanchard (1979) or Weil (1987).

[32] See also, Wang and Wen (2012) and Martín and Ventura (2012, 2016), which also feature multiple types of bubbles with "new" bubbles popping up over time.

From the maximisation problem (2.1)-(2.3) of the household, the optimal demand for goods is given by

$$C_{t|s}(i) = \frac{1}{\alpha}\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_{t|s} \tag{2.4}$$

for all $i \in [0, \alpha]$. The remaining optimality conditions which apply to the household's asset holdings are:

$$1 = E_t\left\{\Lambda_{t,t+1}\frac{(1+i_t)P_t}{P_{t+1}}\right\}; \tag{2.5}$$

$$Q_t^F(i) - D_t(i) = \upsilon\gamma E_t\left\{\Lambda_{t,t+1}Q_{t+1}^F(i)\right\}; \tag{2.6}$$

and

$$Q_{t|t-k}^B = (1-\delta)E_t\left\{\Lambda_{t,t+1}Q_{t+1|t-k}^B\right\}, \tag{2.7}$$

for $k = 0, 1, 2, \ldots$, where

$$\Lambda_{t,t+1} \equiv \beta\frac{C_{t|s}}{C_{t+1|s}} \tag{2.8}$$

is the stochastic discount factor (SDF) and it holds for all possible states of natural conditional on the household being alive in period $t+1$, with the real rate of interest r_t satisfying

$$1 + r_t = E_t\left\{\frac{(1+i_t)P_t}{P_{t+1}}\right\}. \tag{2.9}$$

Summing up (2.7) across all types of bubbles, we have

$$\sum_{k=0}^{\infty}\delta(1-\delta)^k Q_{t|t-k}^B = (1-\delta)E_t\left\{\Lambda_{t,t+1}\sum_{k=0}^{\infty}\delta(1-\delta)^k Q_{t+1|t-k}^B\right\}. \tag{2.10}$$

The left-hand-side of (2.10) is the aggregate value of the bubbles, Q_t^B, by definition, while for the right-hand-side of (2.10),

$$(1-\delta)E_t\left\{\Lambda_{t,t+1}\sum_{k=0}^{\infty}\delta(1-\delta)^k Q_{t+1|t-k}^B\right\} = E_t\left\{\Lambda_{t,t+1}\sum_{k=1}^{\infty}\delta(1-\delta)^k Q_{t+1|t-k}^B\right\} = E_t\left\{\Lambda_{t,t+1}B_{t+1}\right\}. \tag{2.11}$$

Therefore, (2.10) is equivalent to

$$
\begin{aligned}
Q_t^B &= E_t\left\{\Lambda_{t,t+1}B_{t+1}\right\} \\
&= E_t\left\{\Lambda_{t,t+1}\left(Q_{t+1}^B - U_{t+1}\right)\right\},
\end{aligned}
\tag{2.12}
$$

which is the key pricing equation for aggregate bubbles.

A comment is worth making about the pricing of the rational bubbles. From (2.7)-(2.12), it can be seen clearly why the assumption of "multiple types of bubble assets" together with the introduction of "new bubbles" is crucial in generating potentially multiple equilibria with rational bubbles. Suppose first there is only one type of bubble; then there should be no actual bubble "bursting" or "depreciation" in any sense, or otherwise aggregate bubble would decay to zero overtime.[33] Thus, in a steady state (SS) equilibrium (without real growth), there can only be a unique value for the bubble with the corresponding real rate of interest being pinned down to zero (i.e., $\Lambda = 1$ according to (2.12) when $\delta = 0$). On the other hand, if there are various bubbly assets but with no replacements occurring between old and new assets, then again there would exist only one type of SS with $\Lambda = 1$ according to (2.7) given $\delta = 0$.[34]

Therefore, it is the legitimate existence of the "new" bubbles in the present framework that allows the potential coexistence of multiple bubbly SSs. In the SS version of (2.7), we have $\Lambda^{-1} = (1-\delta)\bar{Q}_{t+1|t-k}^B / \bar{Q}_{t|t-k}^B$, with a "bar" at the top of a variable denoting its value at a SS. Since it is possible for an *individual* bubble price to vary in a SS, unlike the aggregate one, this provides an extra degree of freedom for the determination of the SS SDF (the real interest rate). As will be further discussed in Section 2.2.4 and as noted by Galí (2021), a bubbly equilibrium featuring $\Lambda > 1$ implies that the size of any old bubble in that case is shrinking over time, with newly emerged bubbles filling up the gap, so that the aggregate size of all these bubbly assets remains constant at the SS.

[33] Again, if the same type of bubble could come back after collapse, then the bubble is never actually burst because it is rational to hold it forever.

[34] For there are no individual bubble "bursts" or "depreciations". Or otherwise, if there are bursts for existing bubbles but no new bubbles emergence to fill the gap left by the bursts of the old bubbles, then the overall size of the bubble would decline to zero overtime.

To facilitate further calculations, note that an individual household's period budget constraints can be re-written as

$$C_{t|s} + \gamma E_t \left\{ \Lambda_{t,t+1} A_{t+1|s} \right\} = A_{t|s} + W_t N_{t|s} \,, \tag{2.13}$$

where

$$A_{t|s} \equiv \frac{Z^R_{t-1|s}\left(1+i_{t-1}\right)}{P_t} + \left(1-\delta\right)\sum_{k=0}^{\infty} Q^B_{t|t-k-1} Z^B_{t-1|s,t-k-1} + \upsilon\gamma \int_0^{\alpha} Q^F_t(i) Z^F_{t-1|s}(i) di \tag{2.14}$$

for $s < t$, and

$$A_{t|t} \equiv \frac{\delta}{1-\gamma} Q^B_{t|t} + Q^F_{t|t} \tag{2.15}$$

for $s = t$ denote the overall (stochastic) payoff at the end of period *t* generated by the asset portfolio purchased by the household at the beginning of the period, by making use of the optimality conditions (2.4)-(2.7). Note that (2.13) is simply the form of the period budget constraints used in Galí (2021).

2.2.2 Firms

Individual firms are finitely-lived, an assumption particularly made in order to ensure a well-defined firm equity price in a bubbly environment, as will be seen later. They are monopolistically competitive and subject to sticky price setting in the style of Calvo (1983). In addition, a newly created firm is assumed to "inherit" the good's index of its predecessor. To streamline the discussions of the present chapter, I assume no economic growth, unlike the case in Galí (2021).

The production function for a firm $i \in [0,\alpha]$ is given by

$$Y_t(i) = N_t(i) \,, \tag{2.16}$$

while the demand schedule facing it is

$$Y_t(i) = \frac{1}{\alpha}\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} Y_t \,, \tag{2.17}$$

which is obtained by summing (2.4) across individual households and imposing the goods' market clearing condition $Y_t(i) = C_t(i)$ as well as $Y_t = C_t$, with Y_t the aggregate output and $C_t \equiv (1-\gamma)\sum_{s=-\infty}^{t} \gamma^{t-s} C_{t|s}$ the aggregate consumption.

The optimal price setting problem facing a firm *i* that can adjust its price in period *t* is to

$$\max_{P_t^*} \sum_{k=0}^{\infty} \left(\upsilon\gamma\theta\right)^k E_t \left\{ \Lambda_{t,t+k} Y_{t+k|t} \left(\frac{P_t^*}{P_{t+k}} - W_{t+k} \right) \right\}, \tag{2.18}$$

subject to

$$Y_{t+k|t} = \frac{1}{\alpha} \left(\frac{P_t^*}{P_{t+k}} \right)^{-\varepsilon} Y_{t+k} \tag{2.19}$$

and $Y_{t+k|t} = N_{t+k|t}$ from the production function (2.16) for $k = 0,1,\ldots$, with $Y_{t+k|t}$ and $N_{t+k|t}$ denoting output and labour input respectively in period $t+k$ for a firm which last adjusted its price in period *t*, and $\Lambda_{t,t+k} \equiv \Lambda_{t,t+1} \times \Lambda_{t+1,t+2} \times \ldots \times \Lambda_{t+k-1,t+k}$. $\theta \in (0,1)$ is the probability that a firm is unable to reset its price of its good in any given period. Specifically, a newly setup firm is assumed to inherit the price of the good it replaces in the previous period with probability θ, and can reset it otherwise. The first order condition associated with the firm's optimal price setting problem takes the form

$$\sum_{k=0}^{\infty} \left(\upsilon\gamma\theta\right)^k E_t \left\{ \Lambda_{t,t+k} Y_{t+k|t} \left(\frac{P_t^*}{P_{t+k}} - \frac{\varepsilon}{\varepsilon-1} W_{t+k} \right) \right\} = 0 \,. \tag{2.20}$$

As in Galí (2021), an ad-hoc assumed real wage schedule is given by

$$W_t = \left(\frac{N_t}{\alpha} \right)^{\varphi}, \tag{2.21}$$

[35]

with $N_t \equiv \int_0^{\alpha} N_t(i)di$ the aggregate work hours. Then solving the firm's optimisation problem and imposing labour market equilibrium yield a version of the New Keynesian Phillips curve (NKPC) (further details can be found in Galí (2021)):

$$\pi_t = \Lambda \upsilon\gamma E_t \pi_{t+1} + \kappa \hat{y}_t \,, \tag{2.22}$$

with $\Lambda = 1/(1+r)$ the SS SDF with r the SS real interest rate (according to (2.5) and

[35] See Galí (2021) footnote 21 for reasons in assuming the ad-hoc wage schedule.

(2.9)), $\pi_t \equiv \log\left(P_t/P_{t-1}\right)$ the inflation rate, $\hat{y}_t \equiv \log\left(Y_t/Y\right)$ the output gap,[36] and $\kappa \equiv \varphi(1-\theta)(1-\Lambda\upsilon\gamma\theta)/\theta$.

2.2.3 General Equilibrium

In equilibrium, goods market clearing requires

$$Y_t = C_t . \tag{2.23}$$

For the labour market, equilibrium implies that $N_t \equiv \int_0^\alpha N_t(i)di = \int_0^\alpha Y_t(i)di \simeq Y_t$, which has been used in deriving the NKPC (2.22).

Asset market clearing requires that for the nominally riskless bonds,

$$\sum_{s=-\infty}^{t}(1-\gamma)\gamma^{t-s}Z_{t|s}^{R} = \sum_{s=-\infty}^{t-1}(1-\gamma)\gamma^{t-s}Z_{t-1|s}^{R} = 0 , \tag{2.24}$$

For individual firms' share holdings,

$$\sum_{s=-\infty}^{t}(1-\gamma)\gamma^{t-s}Z_{t|s}^{F}(i) = \sum_{s=-\infty}^{t-1}(1-\gamma)\gamma^{t-s}Z_{t-1|s}^{F}(i) = 1 \tag{2.25}$$

for $\forall i \in [0,\alpha]$,[37] while for the bubbly assets,

$$\sum_{s=-\infty}^{t}(1-\gamma)\gamma^{t-s}Z_{t|s,t-k}^{B} = \sum_{s=-\infty}^{t-1}(1-\gamma)\gamma^{t-s}Z_{t-1|s,t-1-k}^{B} = \delta(1-\delta)^k \tag{2.26}$$

for $k = 0,1,2,\ldots$. Also, under the assumption that $\lim_{k\to\infty}(\upsilon\gamma)^k E_t\left\{\Lambda_{t,t+k}Q_{t+k}^F(i)\right\} = 0$,

$$Q_t^F(i) = \sum_{k=0}^{\infty}(\upsilon\gamma)^k E_t\left\{\Lambda_{t,t+k}D_{t+k}(i)\right\} \tag{2.27}$$

with $D_{t+k}(i) \equiv Y_{t+k}(i)\left(P_{t+k}(i)/P_{t+k} - W_{t+k}\right)$ the individual firm *i*'s dividends. Thus,

$$Q_t^F \equiv \int_0^\alpha Q_t^F(i)di = \sum_{k=0}^{\infty}(\upsilon\gamma)^k E_t\left\{\Lambda_{t,t+k}D_{t+k}\right\}, \tag{2.28}$$

where $D_{t+k} \equiv \int_0^\alpha D_{t+k}(i)di$ is the aggregate dividends distributed by the firms.

To find the solution for aggregate consumption, first iterating the household's budget

[36] In the present model, output under flexible prices coincides with that in a (zero-inflation) SS.

[37] Note that the first equality simply claims that an individual firm's shares are held by households of all cohorts including the newly born; for the second equality to hold, it implies that although there are "deaths" occurring to old households and old firms, due to the special insurance arrangement, the shares held by those deaths are actually transferred to those alive, while the impact of the potential risk of a shutdown of a firm on the share return is reflected by the discount rate, $\upsilon\gamma$.

constraint (2.13) forward yields

$$\sum_{k=0}^{\infty} \gamma^k E_t \left\{ \Lambda_{t,t+k} C_{t+k|s} \right\} = A_{t|s}^a + \frac{1}{\alpha} \sum_{k=0}^{\infty} (\upsilon\gamma)^k E_t \left\{ \Lambda_{t,t+k} W_{t+k} N_{t+k} \right\} \tag{2.29}$$

for an active household, and

$$\sum_{k=0}^{\infty} \gamma^k E_t \left\{ \Lambda_{t,t+k} C_{t+k|s} \right\} = A_{t|s}^r \tag{2.30}$$

for a retired one, after applying the transversality condition

$$\lim_{T \to \infty} \gamma^T E_t \left\{ \Lambda_{t,t+T} A_{t+T|s} \right\} = 0 , \tag{2.31}$$

with superscripts "*a*" and "*r*" indicating "active" and "retired" respectively.[38] Since $\Lambda_{t,t+k} C_{t+k|s} = \beta^k C_{t|s}$ according to (2.8), the left-hand-side of equations (2.29) and (2.30) is equivalent to $C_{t|s} / (1-\beta\gamma)$. Hence,

$$C_{t|s}^a = (1-\beta\gamma) \left\{ A_{t|s}^a + \frac{1}{\alpha} \sum_{k=0}^{\infty} (\upsilon\gamma)^k E_t \left\{ \Lambda_{t,t+k} W_{t+k} N_{t+k} \right\} \right\} \tag{2.32}$$

and

$$C_{t|s}^r = (1-\beta\gamma) A_{t|s}^r . \tag{2.33}$$

Furthermore, in period *t*, the total value of financial assets in the economy is

$$\begin{aligned} A_t &\equiv (1-\gamma) \sum_{s=-\infty}^{t} \gamma^{t-s} \left[\upsilon^{t-s} A_{t|s}^a + (1-\upsilon^{t-s}) A_{t|s}^r \right] = (1-\gamma) \sum_{s=-\infty}^{t-1} \gamma^{t-s} \left[\upsilon^{t-s} A_{t|s}^a + (1-\upsilon^{t-s}) A_{t|s}^r \right] + (1-\gamma) A_{t|t}^a \\ &= (1-\gamma) \sum_{s=-\infty}^{t-1} \gamma^{t-s} \left\{ \frac{Z_{t-1|s}^R (1+i_{t-1})}{P_t} + (1-\delta) \sum_{k=0}^{\infty} Q_{t|t-k-1}^B Z_{t-1|s,t-k-1}^B + \upsilon\gamma \int_0^\alpha Q_t^F(i) Z_{t-1|s}^F(i) di \right\} + \\ &\quad (1-\gamma) \left\{ \frac{\delta}{1-\gamma} Q_{t|t}^B + Q_{t|t}^F \right\} \\ &= \left\{ \sum_{k=0}^{\infty} \delta (1-\delta)^{k+1} Q_{t|t-k-1}^B + \upsilon\gamma \int_0^\alpha Q_t^F(i) di \right\} + \left\{ \delta Q_{t|t}^B + (1-\upsilon\gamma) Q_t^F \right\} \\ &= Q_t^B + Q_t^F . \end{aligned} \tag{2.34}$$

Therefore, aggregate consumption is given by

[38] These derivations are also provided in Galí (2021) Appendix A.

$$
\begin{aligned}
C_t &\equiv (1-\gamma)\sum_{s=-\infty}^{t}\gamma^{t-s}\left[\upsilon^{t-s}C^{a}_{t|s}+\left(1-\upsilon^{t-s}\right)C^{r}_{t|s}\right] \\
&=(1-\beta\gamma)\left[A_t+\sum_{k=0}^{\infty}\left(\upsilon\gamma\right)^{k}E_t\left\{\Lambda_{t,t+k}W_{t+k}N_{t+k}\right\}\right] \\
&=(1-\beta\gamma)\left[Q^{B}_t+Q^{F}_t+\sum_{k=0}^{\infty}\left(\upsilon\gamma\right)^{k}E_t\left\{\Lambda_{t,t+k}W_{t+k}N_{t+k}\right\}\right] \\
&=(1-\beta\gamma)\left[Q^{B}_t+\sum_{k=0}^{\infty}\left(\upsilon\gamma\right)^{k}E_t\left\{\Lambda_{t,t+k}\left(W_{t+k}N_{t+k}+D_{t+k}\right)\right\}\right] \\
&=(1-\beta\gamma)\left[Q^{B}_t+\sum_{k=0}^{\infty}\left(\upsilon\gamma\right)^{k}E_t\left\{\Lambda_{t,t+k}Y_{t+k}\right\}\right],
\end{aligned}
\tag{2.35}
$$

where obtaining the last equality uses the fact that $Y_t = W_t N_t + D_t$. By defining

$$
\begin{aligned}
X_t &\equiv \sum_{k=0}^{\infty}\left(\upsilon\gamma\right)^{k}E_t\left\{\Lambda_{t,t+k}Y_{t+k}\right\} \\
&=\upsilon\gamma E_t\left\{\Lambda_{t,t+1}X_{t+1}\right\}+Y_t,
\end{aligned}
\tag{2.36}
$$

as the "total fundamental wealth" of the economy, the aggregate consumption function can then be expressed as

$$
C_t=(1-\beta\gamma)\left(Q^{B}_t+X_t\right). \tag{2.37}
$$

The demand-side impact of the bubble on the model economy reflected by the Q^B_t component can then be seen obviously from (2.37). In addition to this direct effect, variations in Q^B_t also affect aggregate consumption through impacting X_t, the present value of current and future fundamental wealth expected to be earned by the currently alive households. Particularly, a rise in the current size of the bubble raises current income which is driven by higher current consumption and thus higher current output, as well as future income affected by the generated movements in the expected future size of the bubble, as in accordance with its law of motion described by equation (2.12). As will be seen in Section 2.4, the latter two layers of the bubble impact on the economy play an important role in explaining the transmission mechanism of monetary policy in the face of a rational bubble shock.

2.2.4 Steady States

In a zero-inflation steady state (ZISS, for short), $P^*_t = P_t = P_{t+k}$ for $k = 0,1,\ldots$. Thus, from the individual firm's optimal price setting problem, aggregate output in a ZISS is $Y = \alpha\left[(\varepsilon - 1)/\varepsilon\right]^{1/\varphi}$ according to (2.20), (2.21) and the production function (2.16).

Define $q_t^B \equiv Q_t^B/Y$ as the aggregate bubble-output ratio, $b_t \equiv B_t/Y$ and $u_t \equiv U_t/Y$ the old and new bubble-output ratio respectively; also, $x_t \equiv X_t/Y$. Then evaluating (2.37) in a SS and imposing the goods market clearing condition yields

$$1 = (1 - \beta\gamma)\left(q^B + x\right), \tag{2.38}$$

while from the bubble pricing equations (2.12), $q^B = \Lambda\left(q^B - u\right)$, i.e.,

$$u = \left(1 - \frac{1}{\Lambda}\right)q^B = -rq^B, \tag{2.39}$$

where $\Lambda = 1/(1+r)$ is the SS SDF and

$$q^B = b + u, \tag{2.40}$$

and

$$x = \frac{1}{1 - \upsilon\gamma\Lambda} \tag{2.41}$$

from (2.36). Combining (2.38) and (2.41) then yields the relationship between the aggregate bubble-output ratio and the SDF:

$$q^B = \frac{1}{1 - \beta\gamma} - x = \frac{\gamma\left(\beta - \upsilon\Lambda\right)}{(1 - \beta\gamma)(1 - \upsilon\gamma\Lambda)}. \tag{2.42}$$

Therefore, for $q^B > 0$, it is required that $\beta > \upsilon\Lambda$. Also, from (2.39), in order for $u \geq 0$, we need $\Lambda \geq 1$ as well. These two conditions imply that a SS with a positive bubble requires that

$$\frac{\beta}{\upsilon} > \Lambda \geq 1, \tag{2.43}$$

i.e., the two exogenous parameters, β and υ, must then satisfy that $\beta > \upsilon$. It is thus evident that, there could exist a continuum of bubbly SSs in the present model where the SDF varies from one to β/υ, or, where $r \in \left(r_0, 0\right]$ with $r_0 \equiv \left(\upsilon/\beta\right) - 1$.

This multiple SS outcome is due to the fact that there are only four linearly independent equations ((2.38)-(2.41)) for five endogenous variables (q^B, x, b, u, Λ), so the system in a SS is indeterminate of degree one, a phenomenon that is caused by the introduction of multiple bubbly assets with re-emergence of new bubbles in the model, as already hinted in Section 2.2.1 before. As a consequence, the variable of particular interest, q^B, could be viewed as a function of Λ and thus r, where $\partial q^B / \partial r > 0$ and $\partial^2 q^B / \partial r^2 < 0$, which in turn implies that there is an upper bound for the SS bubble-output ratio: $\bar{q}^B = \gamma(\beta - \upsilon) / \left[(1-\beta\gamma)(1-\upsilon\gamma) \right]$ when $r = 0$ (and $u = 0$).[39] Thus, we have $q^B \in \left(0, \bar{q}^B\right]$ for $r \in \left(r_0, 0\right]$. A graphical illustration of this is provided by me by Figure 2.1.

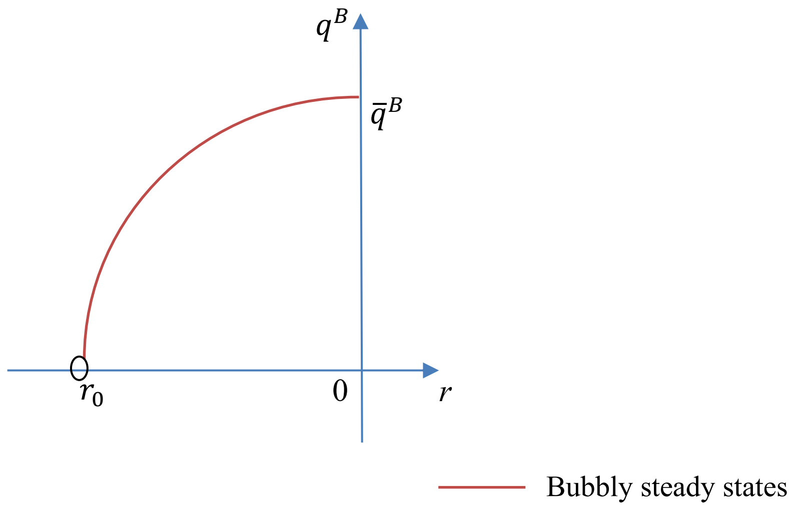

Figure 2.1 A continuum of bubbly SSs

On the other hand, (2.42) implies that when $q^B = u = 0$, $\beta = \upsilon\Lambda$. Collectively, we have two separated regions of parameter space: (1) when $\beta \le \upsilon$, there is a unique SS which is bubbleless, with $q^B = 0$ and $r = r_0 \ge 0$; (2) when $\beta > \upsilon$, there are multiple

[39] It is worth mentioning that although in a SS, the aggregate bubble-output ratio is monotonic in real interest rate, it is not monotonic in the new bubble-output ratio.

SSs, one of which is bubbleless with $q^B = 0$ and $r = r_0 < 0$, while the others are bubbly with $q^B \in \left(0, \bar{q}^B\right]$ and $r \in \left(r_0, 0\right]$. Note that $r < 0$ is relevant to a "dynamic inefficiency" situation, which I will further discuss in Section 2.3.1 below.

2.2.5 Log-linearisation

Define $\hat{q}_t^B \equiv q_t^B - q^B$, $\hat{u}_t \equiv u_t - u$, $\hat{b}_t \equiv b_t - b$ the deviations of the size of the bubble-output ratios from their SS values, and $\hat{c}_t \equiv \log\left(C_t/C\right)$, $\hat{y}_t \equiv \log\left(Y_t/Y\right)$, $\hat{x}_t \equiv x_t - x$. Then approximating (2.37) and (2.36) around a given ZISS and using the fact that $\hat{y}_t = \hat{c}_t$ yield

$$\hat{y}_t = (1-\beta\gamma)\left(\hat{q}_t^B + \hat{x}_t\right), \tag{2.44}$$

$$\begin{aligned}
\hat{x}_t &= \sum_{k=0}^{\infty}\left(\Lambda\upsilon\gamma\right)^k E_t\hat{y}_{t+k} - \frac{\Lambda\upsilon\gamma}{1-\Lambda\upsilon\gamma}\sum_{k=0}^{\infty}\left(\Lambda\upsilon\gamma\right)^k E_t\hat{r}_{t+k} \\
&= \Lambda\upsilon\gamma E_t\hat{x}_{t+1} + \hat{y}_t - \frac{\Lambda\upsilon\gamma}{1-\Lambda\upsilon\gamma}\hat{r}_t \qquad (2.45)\\
&= \frac{\Lambda\upsilon}{\beta}E_t\hat{x}_{t+1} + \frac{1-\beta\gamma}{\beta\gamma}\hat{q}_t^B - \frac{\Lambda\upsilon}{\left(1-\Lambda\upsilon\gamma\right)\beta}\hat{r}_t,
\end{aligned}$$

and

$$\hat{q}_t^B = \Lambda E_t\left\{\hat{q}_{t+1}^B - \hat{u}_{t+1}\right\} - q^B\hat{r}_t, \tag{2.46}$$

where $\hat{r}_t = \hat{i}_t - E_t\pi_{t+1}$ with $\hat{i}_t \equiv \log\left[\left(1+i_t\right)/\left(1+r\right)\right]$. (2.44)-(2.46) and the NKPC (2.22) will be used in the (dynamic) analysis of the model below (especially in Section 2.4).

2.3 Impacts of Bubbles on the Economy (under Flexible Prices): Preliminary Discussions

Before I analyse the monetary policy implications of the presence of bubble-driven fluctuations under sticky prices, in this section I make some preliminary investigations into how the rational bubbles may affect the equilibrium properties of the system. I begin by exploring the phenomenon of "a continuum of bubbly SSs" and the associated "dynamic inefficiency" problem, which is the backbone underlying the monetary policy analysis in the following sections. I then make some analysis of the global dynamics of the system under flexible prices, as part of which I explore more deeply mainly the mechanism underpinning the demand-side impact of the bubble on the economy.

2.3.1 Multiple Bubbly SSs and the Dynamic Inefficiency Problem

We have identified two separate regions of parameter space in Section 2.2.4, under one of which the model economy must be bubbleless, while under the other one it could be bubbly. Galí (2021) does not explore the economic intuition behind this outcome, and their relations with those in the classic literature on rational bubbles by Samuelson (1958) and Tirole (1985). This is hence the task undertaken in this section.

In the Samuelson-Tirole models, a criterion for a possible existence of a rational bubble is that the real rate of interest of the otherwise bubbleless economy must be less than its growth rate,[40] a situation being classified as "*dynamically inefficient*", or otherwise the economy must be bubbleless.[41] A similar narrative applies to the present model: for a rational aggregate bubble to possibly exist in equilibrium, it is necessary for the real interest rate in the bubbleless economy to be less than the growth rate which is zero in the context, i.e., $r = r_0 < 0$ for $q^B > 0$ to be possible, a situation which is only possible when $\beta > \upsilon$. In other words, an equilibrium rational bubble is possible only when the households' discount factor is less than their risk of becoming retired.

Intuitively, an economy being characterised by $\beta > \upsilon$ implies that its households are more patient and prefer to consume more at the later stage of their life, but while the (average) rate of decline of labour income is relatively high – the latter being

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[40] An intrinsically worthless bubble is valuable to someone only when others find it valuable – hence a bubbleless equilibrium always exists. This is always true for *pure* rational bubbles – the ones analysed in the thesis, but may not necessarily be true for bubbles attached to dividend-paying assets, i.e., a bubbleless equilibrium may *not* exist. See, e.g., Hirano and Toda (2025) for a demonstration of the latter.

[41] See also, e.g., Diamond (1965), Gale (1973) for the concept and condition of "dynamic inefficiency".

implied by a non-negligible retirement risk. As a consequence, a household could only achieve desirably higher consumption level when being retired through (high) accumulation of fundamental wealth at the early stage of his life in the absence of the bubble, which pushes down the real interest rate in general equilibrium. If, instead, there is a consistent collective faith in the asset bubbles among individuals, such that $q^B > 0$, then the older generations could always obtain extra resources from the younger ones by selling the bubble assets, a process which is unharmful to the latter but is beneficial to the former, when the economy is assumed to be indefinitely lasting and not shrinking over time. Therefore, the bubbles alleviate the inefficient situation of overaccumulation of wealth in the economy by providing an effective means for intergenerational resource transfers in the present case, resulting in all cohorts being better-off.

This outcome also highlights the crucial assumption of "retirement" of households in the OLG-NK model, i.e., the necessity to have $\upsilon < 1$.[42] Or otherwise, without the declining labour income, households would just smooth their consumption throughout their finite lifespan in accordance with their (expected) labour income stream. On the other hand, if $\beta \leq \upsilon$, i.e., if individual households are impatient in consumption relative to the rate of decline of the labour income over their finite lifetime, then there is less need for the households to accumulate wealth for consumption in their later life to maximise their lifetime utility, when they lack channels to borrow from the younger generations, then, as they would tilt in favour of early-age consumption which can be sufficiently funded (in an average sense) by their labour income. Therefore, in this type of circumstance, there is no room for equilibrium rational bubbles to emerge to function as a channel in transferring resources among generations, because of the lack of a need

[42] Although the finite-lifespan of an individual in the OLG-NK model should result in him accumulating less wealth and thus should have an impact on the equilibrium interest rate as well, this respect is absent here with log sub-utility function of the households, i.e., γ does not play a role in determining the existence of rational bubbles in equilibrium.

for acquiring additional funds for later-life consumption when a household loses his labour income, i.e., the bubbleless economy in equilibrium is already Pareto efficient in a dynamic sense, with $r = r_0 \geq 0$.[43]

Until now, one may incline to conclude that the rational bubbles could thus restore an inefficient economic system an efficient one, indicated by the interest rate being no lower than the growth rate of the economy. This is indeed the case in the Samuelson-Tirole models where an equilibrium with a rational bubble is dynamically efficient with the real interest rate *equalling* the economy's growth rate.[44] However, what is peculiar in the present model is that a bubbly equilibrium could also feature a lower interest rate relative to the growth rate of the economy, i.e., a strictly negative real interest rate for $0 < q^B < \overline{q}^B$, an outcome that turns out to be contributed by the introduction of the "new" bubbles endowed by the newly born.

To better understand how the equilibrium implication of the present model deviates from the classic Samuelson-Tirole one, recall again that a key argument made in the previous paragraphs is that when $r < 0$, there is room for Pareto improvement for the economy, which could be fulfilled by transferring resources from the young to the old by means of the bubbles. In the special case when $u = 0$, i.e., when there are no new bubbles distributed to the newborn, the *youngest* is then a *net* contributor in terms of providing extra wealth to the older ones by buying the bubble assets which are priced up to the point that the desired amount of resource transfers among generations in order for the economy to be efficient are completely carried out by the bubbles, such that in a SS equilibrium, the real rate of interest equals the growth rate of the economy. This is just consistent with the classic models of rational bubbles, with $q^B = \overline{q}^B$ and $r = 0$ in

[43] It is worth noting that the role played here by $(1-\upsilon)$ is similar to that played by "α" (i.e., the rate of smooth decline of labour income over an individual's finite lifespan) in Blanchard (1985). Blanchard shows that $\alpha > 0$ is also necessary for dynamic inefficiency to occur in his model.

[44] The pure bubbles cannot grow faster than the economy, or otherwise they would grow unboundedly, violating the resource constraint.

the present model.

However, if $u > 0$, i.e., if the youngest generation also possesses bubbles upon their birth, this implies, especially in a SS, that the overall funds transferred from the young to the old are less than that required for an efficient system, because the older generations other than the newborn only acquire $q^B < \bar{q}^B$ proportion of the resources. In other words, a proportion of the wealth which should have been redistributed to the old for the system to be dynamically efficient is now left undelivered in the form of the new bubble endowed to the youngest generation. As a result, the equilibrium system is less efficient with $r_0 < r < 0$ for $q^B > 0$ but with $u > 0$; and the lower the equilibrium interest rate (correspondingly, the lower the aggregate size of the bubbles), the lower the implied degree of efficiency of the system.[45] In this type of scenario, old bubbles are shrinking over time (relative to the economy) with the gap being filled up by the newly introduced ones, such that the overall value of all of the bubbles remains unchanged in a SS.

On the other hand, according to (2.8), $\beta C_{t|s} = \Lambda_{t,t+1} C_{t+1|s}$. Hence in a SS,

$$\bar{C}_{t|s} = \left[(1+r) \beta \right]^{t-s} \bar{C}_{s|s} \tag{2.47}$$

for $t > s$, with a "bar" above the variable denoting its SS value. Since $\left[(1+r)\beta\right] \in (0,1)$ for $r \in (r_0, 0]$, (2.47) implies that the higher the SS real interest rate, the "flatter" a typical individual's lifetime consumption profile, with his period consumptions declining less drastically over time – note that again from (2.47), the consumption of the newborn is invariant to their birthdate in a SS and so all households have the same lifetime consumption pattern. Furthermore, aggregate consumption in a SS is given by

[45] The reasoning and conclusion here seem to some extent echo some of the findings claimed by Bonchi and Nisticò (2024), that is a monetary authority with a welfare-optimisation objective should lean against fluctuations in newly created bubbles but be accommodative to those in the old ones. Intuitively, that is because new bubbles are generally inefficient for the system and thus should be "eliminated" from a welfare perspective.

$$C \equiv (1-\gamma)\sum_{s=-\infty}^{t} \gamma^{t-s}\bar{C}_{t|s} = \frac{1-\gamma}{1-\gamma\beta(1+r)}\bar{C}_{t|t}\,, \qquad (2.48)$$

which is derived by making use of (2.47) and the fact that $\bar{C}_{t|t} = \bar{C}_{t-s|t-s}$ for $\forall s < t$.[46] Since $C = Y$ is constant in the present model and is independent of variations in r, (2.48) implies that the higher the real interest rate, the lower the consumption when an individual is born.

Therefore, in the most efficient situation in a bubbly economy where $q^B = \bar{q}^B$ (i.e., $u = 0$) and $r = 0$, individuals are able to smooth their consumption in a desirable way with smaller consumption dispersion throughout their lifetime, regardless of the threat of a quickly declining rate of labour income, which would not be possible in the originally dynamically inefficient environment without the bubbles. Figure 2.2 provides a graphical demonstration for the above discussion, where $0 \geq r_1 > r_2 > r_0$ and $\bar{q}^B \geq q_1^B > q_2^B > 0$.

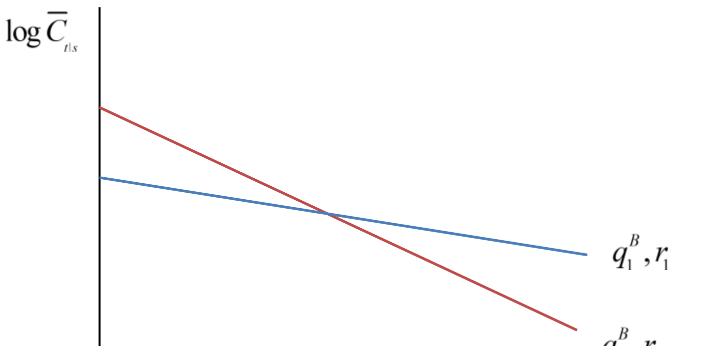

Figure 2.2 Lifetime consumption profile of a typical individual in a SS

It is also worth noting that although the diversion of the resources by a positive new bubble results in the economy being less efficient, it is not the absolute size of the new bubble that matters for the degree of efficiency of a bubbly economy, but the *relative*

[46] This calculation is inspired by Rankin (2014).

size of it to the aggregate bubble. To see this, firstly it is not difficult to check that u is not monotonic in r, but $\partial^2 u/\partial r^2 < 0$, suggesting that for a given value of u, there could be two different efficiency states for the economy. As graphically illustrated by Figure 2.3, except for the top of the $u-q^B$ curve which is associated with the upper bound of the SS new bubble-output ratio, each u value corresponds to two values of q^B, one of which is associated with a higher efficiency of the economy. This is essentially a consequence of the equilibrium requirement for the rational bubbles: since the bubbles are required to grow at the interest rate to be attractive in the present model, a smaller u may correspond to a bigger q^B, in which case the youngest take less of the wealth that should have been transferred to the old, so that the efficiency of the system could be improved to a larger extent; meanwhile, the resulting higher r value indicates that the dying-out fraction of the aggregate bubble that needs to be re-filled by the new bubble is indeed smaller. On the other hand, the smaller u may also be consistent with a smaller q^B, such that the transportation function played by the bubbles are restrained significantly by the implied high shrinkage rate of the pre-existing bubbles, resulting in a more severe degree of inefficiency of the economy and a lower equilibrium interest rate.

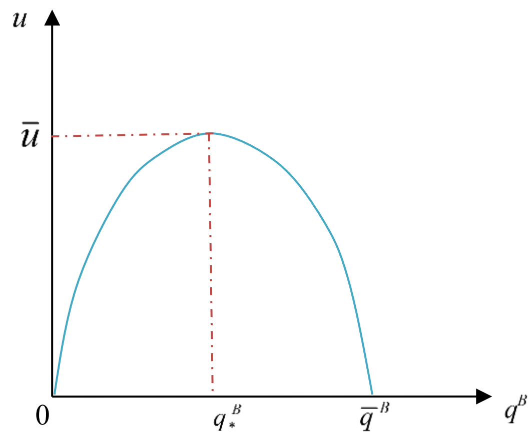

Figure 2.3 The relationships between the new and aggregate bubbles

2.3.2 Equilibrium Properties (under Flexible Prices)

We already know that from (2.8), $\Lambda_{t,t+1}C_{t+1|s} = \beta C_{t|s}$. Also, according to the definition of aggregate consumption, $C_{t+1} = (1-\gamma)\sum_{s=-\infty}^{t}\gamma^{t+1-s}C_{t+1|s} + (1-\gamma)C_{t+1|t+1}$. Thus,

$$\begin{aligned}&\Lambda_{t,t+1}\left\{(1-\gamma)\sum_{s=-\infty}^{t}\gamma^{t-s}C_{t+1|s}\right\}\\ &=\Lambda_{t,t+1}\left\{\frac{1}{\gamma}\left[C_{t+1}-(1-\gamma)C_{t+1|t+1}\right]\right\}\\ &=\beta(1-\gamma)\sum_{s=-\infty}^{t}\gamma^{t-s}C_{t|s}\\ &=\beta C_t,\end{aligned} \tag{2.49}$$

i.e.,

$$\beta\gamma C_t = \Lambda_{t,t+1}C_{t+1} - (1-\gamma)\Lambda_{t,t+1}C_{t+1|t+1} \tag{2.50}$$

by equating the second and the final line of (2.49), which in turn implies that

$$\Lambda_{t,t+1} = \frac{\beta\gamma C_t}{C_{t+1}-(1-\gamma)C_{t+1|t+1}} = \frac{\beta\gamma\left(Q_t^B + X_t\right)}{Q_{t+1}^B - U_{t+1} + \upsilon\gamma X_{t+1}} \tag{2.51}$$

after applying the expression of (2.32) for $C_{t+1|t+1}$ and making use of (2.37) for C_{t+1}. The first equality of (2.51) then reflects the impact of the "perpetual youth" structure on the determination of the SDF: if there were no overlaps of generations, i.e., $\gamma(=\upsilon)=1$, then in equilibrium, $\Lambda_{t,t+1} = \beta C_t / C_{t+1}$ as in a standard NK model; however, since now there is the newly born entering the economy while some of the old exist, variations of individual consumption by age could make a difference in determining the SDF.

Under flexible prices, output remains constant and is equal to $Y = \alpha\left[(\varepsilon-1)/\varepsilon\right]^{1/\varphi}$, which can be detected from the optimal price setting problem of the monopoly firms. Then from (2.51) and (2.37) (with (2.23)),

$$\begin{aligned}\Lambda_{t,t+1} &= \frac{\beta\gamma\, Y/(1-\beta\gamma)}{Q_{t+1}^B - U_{t+1} + \upsilon\gamma\left[Y/(1-\beta\gamma) - Q_{t+1}^B\right]}\\ &= \frac{\beta\gamma}{(1-\upsilon\gamma)(1-\beta\gamma)q_{t+1}^B - (1-\beta\gamma)u_{t+1} + \upsilon\gamma}.\end{aligned} \tag{2.52}$$

Also from the pricing equation for the bubble (equation (2.12)),

$$\Lambda_{t,t+1} = \frac{Q_t^B}{Q_{t+1}^B - U_{t+1}} \tag{2.53}$$

in the absence of uncertainty. Thus, combining (2.52) and (2.53) yields

$$q_{t+1}^B = \frac{\left[(1-\beta\gamma)u_{t+1} - \upsilon\gamma\right]q_t^B - \beta\gamma u_{t+1}}{(1-\upsilon\gamma)(1-\beta\gamma)q_t^B - \beta\gamma}. \tag{2.54}$$

Assume that $u_{t+1} = u > 0$ and through some manipulation,[47] (2.54) becomes

$$q_{t+1}^B = \frac{(1-\beta\gamma)u - \upsilon\gamma}{(1-\upsilon\gamma)(1-\beta\gamma)} + \frac{\dfrac{\beta\upsilon\gamma^2}{(1-\upsilon\gamma)(1-\beta\gamma)} - \dfrac{\beta\upsilon\gamma^2}{(1-\upsilon\gamma)}u}{\beta\gamma - (1-\upsilon\gamma)(1-\beta\gamma)q_t^B} \equiv S\left(q_t^B, u\right), \tag{2.55}$$

A deterministic bubbly equilibrium under flexible prices is then defined by a sequence $\{q_t^B\}$ satisfying (2.55) with $\infty > q_t^B > 0$ for all t and for some given $u \geq 0$, and a bubbly SS is defined by a pair of $\left(q^B, u\right)$ that satisfies (2.55).

Figure 2.4 also provides a depiction of the mapping of (2.55) which is of a rectangular hyperbola with $q_t^B := \beta\gamma / \left[(1-\upsilon\gamma)(1-\beta\gamma)\right] > 0$ its vertical asymptote and $q_{t+1}^B := \left[(1-\gamma\beta)u - \upsilon\gamma\right] / \left[(1-\upsilon\gamma)(1-\gamma\beta)\right]$ the horizontal one.[48] Thus, there is a bunch of trajectories for the aggregate bubble-output ratio consistent with equilibrium for u ranging from 0 to $\bar{u} \equiv \left[1 + 1/(1-\beta\gamma)\right]\sqrt{\beta\upsilon\gamma^2(1-\upsilon\gamma)} - \upsilon\gamma - (1-\upsilon\gamma)\beta\gamma/(1-\beta\gamma)$, with the phase line moving upward as the size of the SS new bubble-output ratio increases. The intersections of each of these phase lines with the 45-degree line are the two bubbly SSs corresponding to a given $u > 0$, one of which is globally stable, denoted by $q^{B(S)}$, while the other one is not, denoted by $q^{B(U)}$, where $q^{B(S)} < q^{B(U)}$. Specifically, given

[47] In the current context, u_{t+1} is treated as a time-invariant parameter, although in principle it is endogenous and does not have to be treated like this.

[48] This is inspired by the graphical approach in Galí (2014). Bonchi and Nisticò (2024) also provides some similar graphical analysis as well as the global stability properties of the bubbly SSs. See Appendix 2.A for more details of the derivations of Figure 2.4.

an initial condition $q_0^B \in \left[0, q^{B(U)}\right)$, the solution to (2.55) converges asymptotically to $q^{B(S)}$, constituting a bounded bubbly equilibrium path; but it becomes unbounded if $q_0^B > q^{B(U)}$.

When u reaches its upper bound $\bar{u}$, the hyperbola becomes tangent to the line with only one bubbly SS denoted as $q_*^B(\bar{u})$. It is then graphically evident that there exists a continuum of stable bubbly SS as well as a continuum of unstable ones, which are partitioned by the point q_*^B, with $\left\{q^{B(S)}(u), u \mid q^{B(S)}(u) = S\left(q^{B(S)}(u), u\right) \text{ for } u \in (0, \bar{u})\right\}$ and $\left\{q^{B(U)}(u), u \mid q^{B(U)}(u) = S\left(q^{B(U)}(u), u\right) \text{ for } u \in [0, \bar{u})\right\}$ respectively.

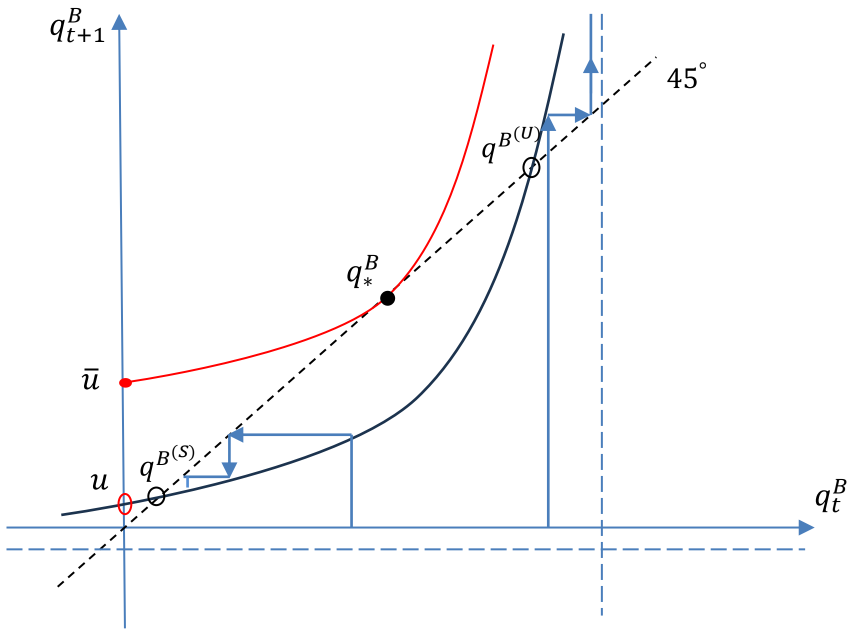

Figure 2.4 Stable and unstable bubbly steady states

Therefore, the above result simply confirms and generalises the finding in Galí (2021) regarding the local stability of a bubbly balanced-growth-path (BGP, for short; under flexible prices): a bubbly BGP (correspondingly, SS, in the present model which has no constant economic growth) with $q^B \in (0, q_*^B)$ is not only locally but *globally* stable, with $q_*^B \equiv \left[(\beta-\upsilon)\gamma+(1-\beta\gamma)\bar{u}\right]/\left[2(1-\upsilon\gamma)(1-\beta\gamma)\right]$,[49] but it is neither locally nor globally stable if with $q^B \in \left(q_*^B, \bar{q}^B\right]$. As will be seen in the later sections where sticky price setting is reintroduced, q_*^B also defines the threshold *below which* bubble-driven fluctuations in output around an exogenously given bubbly SS cannot be ruled out by the proposed conventional policy but can nonetheless be fully stabilised by means of a type of "surgical" LAB policy considered there.

2.3.3 The Mechanism of the Demand-side Impact of the Bubble

It has been learnt from the aggregate consumption function (2.37) in Section 2.2.3 that an increase in the size of the aggregate bubble raises aggregate consumption directly. However, the underlying channel through which the bubbles actually affect aggregate demand in Galí (2021) has not yet been made clear.

To investigate this aspect, it is instructive to combine (2.36) with (2.37) and (2.23) to yield

$$\begin{aligned} E_t\left\{\Lambda_{t,t+1}C_{t+1}\right\} &= (1-\beta\gamma)E_t\left\{\Lambda_{t,t+1}\left(Q_{t+1}^B+X_{t+1}\right)\right\} \\ &= (1-\beta\gamma)\left[Q_t^B+\frac{1}{\upsilon\gamma}\left(X_t-C_t\right)\right], \end{aligned} \tag{2.56}$$

where the second line of (2.56) is valid if I assume the size of the new bubbles is zero at any time in the economy. Making use of (2.37) again to substitute out X_t, (2.56) becomes

$$E_t\left\{\Lambda_{t,t+1}C_{t+1}\right\} = \frac{\beta}{\upsilon}C_t-(1-\beta\gamma)\left(\frac{1}{\upsilon\gamma}-1\right)Q_t^B, \tag{2.57}$$

which is an "aggregate" version of the consumption Euler equation containing a term in financial wealth that would be otherwise *absent* in a standard NK model with infinitely-lived representative household, i.e., if $\upsilon=\gamma=1$ in the present context.

From (2.57), it is then evident to see how the incorporation of the "perpetual youth"

[49] It is not difficult to check that this is exactly the same threshold bubble-output ratio identified in Galí (2021), below which there always exist stationary bubble fluctuations in a local area of the bubbly BGP.

structure in the NK model creates a channel for the rational bubbles to have an impact on aggregate consumption: since there are overlaps and replacements of generations, finitely-lived households obtain a "premium" payoff of their asset portfolio upon their survival under the complete insurance arrangement. This implies that an increase in the asset value – the bubble price here, has a positive impact on the financial wealth of the finitely-lived individuals, because they are *not* the same ones that need to pay the higher price of the portfolio when it is purchased. Therefore, with the "perpetual youth" OLG structure, consumption at the current period will be boosted in the face of a rational bubble boom for a given expectation of future aggregate consumption.[50] The discussion here also highlights the distinctive equilibrium property of rational bubbles compared to their irrational counterparts: while a boom in the latter could have an impact on aggregate consumption in a NK model with an infinitely-lived representative household,[51] it is obviously not the case for the former.

It is also worth noting that, although a growing bubble adds demand pressure to the OLG-NK economy, when prices are fully flexible, the firms that possess market power would immediately adjust their prices to ensure the desirable markup level, instead of expanding production to meet the excessive demand, leaving aggregate output level unchanged. As a consequence, it is necessary for the (real) interest rate to go up to balance the demand and supply of goods through inducing the households to postpone consumption into their later life, where the higher consumption then is feasible because of the rising interest income on the financial portfolio.

Formally speaking, under flexible prices and in the absence of aggregate uncertainty,

$$\frac{1}{1+r_t}C = \frac{\beta}{\upsilon}C - \left(1-\beta\gamma\right)\left(\frac{1}{\upsilon\gamma}-1\right)Q_t^B \tag{2.58}$$

according to (2.57), so that a rise in Q_t^B clearly causes an increase in r_t, provided

[50] See also, Rankin (2023) for a discussion on this type of channel in a NK model with "perpetual youth" generations, albeit there the financial asset consists of government bonds rather than bubbles.

[51] See, for example, Bernanke and Gertler 1999, 2001. Of course, the presumption for the rational bubble to have an impact on aggregate demand is that the conditions for a rational bubble to exist in equilibrium are satisfied.

$\upsilon, \gamma \in (0,1)$. More generally, iterate forward (2.57) T periods and rearrange terms,

$$\frac{\left(\upsilon/\beta\right)^{T}}{\prod_{k=0}^{T-1}\left(1+r_{t+k}\right)}C = C - \frac{\upsilon}{\beta}\left(1-\beta\gamma\right)\left(\frac{1}{\upsilon\gamma}-1\right)Q_t^B \tag{2.59}$$

if assuming for simplicity that $Q_{t+k}^B = 0$ for $k \geq 1$. Therefore, with flexible prices and without aggregate shocks, a rise in Q_t^B may also result in future interest rates adjustments.

Note further that the impact of the movements of the real interest rates is summarised in the changes in the total fundamental wealth. Thus, from (2.37),

$$C = (1-\beta\gamma)\left(Q_t^B + X_t\right) \tag{2.60}$$

under flexible prices, i.e., the dynamic relationship between the bubble and the rates of interest can be summed up as that a rising bubble squeezes out the economy's total fundamental wealth one-for-one. The results above (from (2.58)-(2.60)) echo and generalise the finding in Galí (2021) regarding the equilibrium dynamics in the face of a "small" increase in the bubble size in a local area of a given BGP in a flexible price environment.

2.3.4 Other Impacts of the Bubble

Although the rational bubbles mainly affect aggregate demand in the present model, the supply side of the OLG-NK economy is not completely insulated from their effects. To be specific, according to (2.42) and (2.39),

$$\Lambda = \frac{\gamma\beta - q^B(1-\gamma\beta)}{\left[1-q^B(1-\gamma\beta)\right]\gamma\upsilon} = \frac{q^B}{q^B - u} \tag{2.61}$$

in a SS. Therefore, if we define $\tilde{\beta} \equiv \upsilon\gamma\Lambda$ and $\tilde{\kappa} \equiv \varphi(1-\theta)(1-\tilde{\beta}\theta)/\theta$, then the NKPC (2.22) can be re-expressed as

$$\pi_t = \tilde{\beta}E_t\pi_{t+1} + \tilde{\kappa}\hat{y}_t\,, \tag{2.62}$$

which is isomorphic to that of the textbook NK model, except that there is now the composite effect of both the 'perpetual youth' structure and the rational bubbles on it. Since $\partial\tilde{\beta}/\partial q^B = \left(\partial\tilde{\beta}/\partial\Lambda\right)\left(\partial\Lambda/\partial q^B\right) < 0$ while $\partial\tilde{\kappa}/\partial q^B = \left(\partial\tilde{\kappa}/\partial\tilde{\beta}\right)\left(\partial\tilde{\beta}/\partial q^B\right) > 0$, variations in the size of the aggregate bubble-output ratio have opposite impacts on the coefficients of the NKPC in a ZISS. Thus, the rational bubbles also have an impact on the supply side of the economy albeit in a subtle and indirect way, regardless of the fact that their presence does not generate policy trade-off between the stabilisations of the output gap and inflation.

2.4 Monetary Policy Implications with Bubble-driven Fluctuations

Unlike the case discussed above under flexible prices, when sticky prices are reintroduced, movements in the size of the aggregate rational bubble potentially have an impact on real output and inflation, since then the firms would be willing to adjust their production to meet variations in the goods demand, while monetary policy is no longer neutral and is able to affect the real economy. Therefore, it would be instructive to see how monetary policy may shape the equilibrium dynamics of the OLG-NK economy, especially, in terms of either completely ruling out bubble fluctuations, or mitigating the impact of them on output and inflation if otherwise, to achieve the goal of economic stability.

The analysis throughout this part is restricted to local dynamics around a given ZISS, and under the assumption that the condition for an equilibrium existence of a rational bubble is satisfied, i.e., $\upsilon < \beta$. To streamline the discussion and as done by Galí (2021), I focus on those situations where deviations of newly created bubbles from their SS values are unforecastable, i.e., $E_t\hat{u}_{t+k} = 0$ for $k = 1, 2, ...$ at any time *t*. Under this restriction, equation (2.46) for the law of motion for the aggregate bubble-output ratio becomes

$$\hat{q}_t^B = \Lambda E_t \hat{q}_{t+1}^B - q^B \hat{r}_t . \tag{2.63}$$

The equilibrium of the non-policy block of the OLG-NK model is then represented by (2.44), (2.45), (2.22) and (2.63). To close the model, I follow Galí (2021) to adopt an interest rate rule taking the form of

$$\hat{i}_t = E_t\pi_{t+1} + \phi_y \hat{y}_t + \phi_q \hat{q}_t^B, \tag{2.64}$$

which combines the conventional output gap-stabilisation motive, parameterised by $\phi_y \geq 0$, with a potential leaning-against-the-bubble (LAB) component, parameterised by $\phi_q \geq 0$. The proposed rule (2.64) is very special, in the sense that it can be rewritten as $\hat{r}_t = \phi_y \hat{y}_t + \phi_q \hat{q}_t^B$ and can thus be very helpful in simplifying the structure of the system, as it means that the demand side of the model can now be solved independently of the supply side (i.e., the NKPC equation).

The local equilibrium behaviour of $\hat{q}_t^B, \hat{y}_t$ can then be jointly described by

$$\begin{bmatrix} E_t\hat{q}_{t+1}^B \\ E_t\hat{y}_{t+1} \end{bmatrix} = \begin{bmatrix} \dfrac{1+q^B\phi_q}{\Lambda} & \dfrac{q^B\phi_y}{\Lambda} \\ \dfrac{\beta\Upsilon\phi_q}{\upsilon\Lambda} - \dfrac{(1-\beta\gamma)(1-\upsilon\gamma)}{\upsilon\gamma\Lambda} & \dfrac{(1+\Upsilon\phi_y)\beta}{\upsilon\Lambda} \end{bmatrix} \begin{bmatrix} \hat{q}_t^B \\ \hat{y}_t \end{bmatrix}, \tag{2.65}$$

[52]

where $\Upsilon \equiv \left\{(1-\beta\gamma)\upsilon\left[q^B + \Lambda/(1-\upsilon\gamma\Lambda)\right]\right\}/\beta$.

The system (2.65) being purely forward looking (i.e., neither $\hat{q}_t^B$ nor $\hat{y}_t$ is predetermined) as well as any aggregate fundamental shocks being absent imply that according to the Blanchard and Kahn's (1980) condition, $\left[\hat{q}_t^B, \hat{y}_t\right] = [0,0]$ is the unique bounded rational expectation solution to (2.65) if both eigenvalues of its coefficient matrix lie outside the unit circle. In that case, the economy is insulated from any expectations-driven fluctuations including those in the size of the bubble that are unrelated to its fundamentals.

However, it turns out that those "sunspot" fluctuations cannot always be ruled out under the proposed interest rate rule (2.64), especially from a pragmatic point of view. In the analysis of Galí (2021), the required strength of policy response to variations in

[52] This corresponds to the system (52) in Galí (2021).

the output gap or in the size of the bubble in order to guarantee that the only legitimate solution to the system is the SS itself tends to become unbounded above as the SS size of the aggregate bubble-output ratio approaches zero (or some threshold value).[53] In those circumstances, then, the system is likely to be exposed to bounded bubble-driven fluctuations in output that result from pure mood swings in the market regarding the future value of current aggregate bubbles.[54] But Galí (2021) does not provide monetary policy suggestions should these scenarios occur.

Therefore, in what follows, it is my aim to fill this gap by exploring the potential of monetary policy interventions for stabilising the economy when it confronts bubble fluctuations around a given SS, where $q^B \in \left(0, \overline{q}^B\right)$,[55] concentrating particularly on the output gap, as there is no trade-off between stabilising output and inflation in the present framework. Also, in order to better appreciate the role played by different coefficients in the proposed interest rate rule, I will proceed by considering separately the policy feedback to variations in the output gap and in the size of the aggregate bubble-output ratio.

2.4.1 A LAB Policy

Consider first the role of the bubble coefficient in the policy rule (2.64) in coping with potential bounded bubble-driven fluctuations. When there is no direct policy feedback to deviations in the output gap, i.e., $\phi_y = 0$, the associated characteristic polynomial for the coefficient matrix of (2.65) is

$$\left(\frac{1+q^B\phi_q}{\Lambda}-\lambda\right)\cdot\left(\frac{\beta}{\upsilon\Lambda}-\lambda\right)=0, \tag{2.66}$$

with λ denoting the eigenvalue for the polynomial. The two eigenvalues to (2.66) are

[53] See, Galí (2021) Figure 2 and Figure 3 for a graphical illustration of these results.

[54] As pointed out by Galí (2021), with the proposed interest rate rule, any bounded fluctuations in output must be *bubble-driven* as long as the policy coefficient on the output gap is set non-negative.

[55] As also pointed out by Galí (2021), it is impossible for the economy (under sticky prices and the proposed policy rule) to have bounded bubble fluctuations when the SS bubble-output ratio takes its upper bound value.

thus $\lambda_1 = \left(1+q^B\phi_q\right)/\Lambda$ and $\lambda_2 = \beta/\left(\upsilon\Lambda\right) > 1$ under the conditions for the bubble to possibly exist in equilibrium. If we denote ϕ_q^* as the threshold value for the bubble coefficient above which "sunspot" bubble shocks can be entirely eliminated, it is then obvious that $\phi_q^* \equiv \left(\Lambda - 1\right)/q^B \to +\infty$ as $q^B \to 0$. In practice, however, it might not be feasible for the central bank to react as strongly to the bubble fluctuation as is required to completely eliminate it. Hence, it is worth investigating the impacts of the LAB policy on the evolution of the model economy when it is ineffective at ruling out completely those bubble fluctuations, i.e., when $0 \le \phi_q < \phi_q^*$.

As an illustration, assume that there is a positive but 'small' market sentiment shock to the bubble, such that $\hat{q}_0^B = \varepsilon_0 > 0$ at some time $t = 0$.[56] Given that $0 \le \phi_q < \phi_q^*$, i.e., $0 < \lambda_1 < 1$ for $q^B \in \left(0, \overline{q}^B\right)$, the evolution paths for the bubble and the output gap, as shown in Appendix 2.B to this chapter, can be described respectively by

$$\hat{q}_t^B = \lambda_1^t \hat{q}_0^B, \tag{2.67}$$

$$\hat{y}_t = \Omega_1 \lambda_1^t \hat{q}_0^B, \tag{2.68}$$

where $\Omega_1 \equiv \left[\left(1-\gamma\beta\right)\left(1-\upsilon\gamma\right) - \beta\gamma\Upsilon\phi_q\right]/\left[\gamma\beta - \gamma\upsilon\left(1+q^B\phi_q\right)\right]$.

The impact of the bubble shock on output accompanied by changes in the instrument rate can be separated into two components: the persistence effect, which is represented by λ_1; and the bubble impact, i.e., the "bubble multiplier", measured by Ω_1 in (2.68), reflecting the contemporaneous impact of variations in the size of the bubble on output.

It is evident that, as pointed out by Galí (2021), a moderate LAB policy that fails to rule out bubble fluctuations would end up prolonging the resulting output fluctuations with a higher ϕ_q. This is due to the increasing persistence of the bubble fluctuation, as

56 By "small" I mean that the first-order approximation representation of the system is still valid under the bubble shock. This type of bubble shock may be interpreted as when the investor community becomes more optimistic about the value of the asset, but in a "cautious" way.

the rational bubbles in the model are expected to grow at the interest rate in equilibrium, so that a rise in the latter owing to the LAB policy worsens the volatility of the bubble fluctuation itself at the first place, which then feeds through to the output gap over time through affecting aggregate demand under sticky prices. Despite this fact, for those cases of particular interest here, i.e., when $q^B \to 0$, the impact of the LAB policy on the persistence of the bubble as well as of output fluctuations tends to *vanish* with the persistent root being around υ/β, which may be close to a unit root for any plausible calibration of the model parameters.[57]

On the other hand, it is not so clear how variations in the bubble coefficient in the LAB policy affect the bubble multiplier on output. To investigate this aspect, note first that

$$\frac{\partial \Omega_1}{\partial \phi_q} = -\frac{\beta\upsilon(\Lambda-1)(1-\beta\gamma)}{(1-\upsilon\gamma\Lambda)\left[\beta-\upsilon\left(1+q^B\phi_q\right)\right]^2} < 0 \tag{2.69}$$

for $0 < \phi_q < \phi_q^*$, which implies that a stronger policy reaction directly to the variations in the size of the bubble reduces the bubble multiplier on output. In particular, with the LAB policy intervention and for a given $q^B \in (0, \bar{q}^B)$, the possible value for the bubble multiplier on output ranges from $\underline{\Omega}_1$ to $\bar{\Omega}_1$ as ϕ_q descends from ϕ_q^* to 0, with $\underline{\Omega}_1 \equiv \frac{(1-\gamma\beta)(1-\upsilon\gamma)-\beta\gamma\Upsilon\left[(\Lambda-1)/q^B\right]}{\gamma(\beta-\upsilon\Lambda)}$ and $\bar{\Omega}_1 \equiv \frac{(1-\gamma\beta)(1-\upsilon\gamma)}{\gamma(\beta-\upsilon)}$, i.e., $\Omega_1 \in \left(\underline{\Omega}_1, \bar{\Omega}_1\right)$ for $\phi_q \in \left(0, \phi_q^*\right)$.

Although it is unambiguous that the upper bound of the bubble multiplier is always strictly positive,[58] as shown in Appendix 2.D, the lower bound of it will be less than

[57] When $q^B \to 0$, on the one hand, the impact of ϕ_q on the persistent root λ_1 tends to vanish according to its formula; on the other hand, $\Lambda \to \beta/\upsilon$ as $q^B \to 0$.

[58] Implying a positive impact of an increase in the size of the bubble on actual output under sticky prices.

zero when $q^B \in (0, q_*^B)$.[59] This thus suggests a potential implementation of a "surgical" LAB policy such that output can be insulated from the impact of the bubble fluctuations: if the economy fluctuates in a neighbourhood of a bubbly SS with $q^B \leq q_*^B$, it is then possible for the authority to set $\phi_q = \phi_q^S$ with $\phi_q^S \equiv (1-\beta\gamma)(1-\upsilon\gamma)/(\beta\gamma\Upsilon)$, such that $\Omega_1(\phi_q = \phi_q^S) = 0$.[60] The interest rate is then being controlled in such a way that the actual real interest rate tracks exactly one-for-one its natural counterpart, with $\hat{r}_t = \phi_q^S \hat{q}_t^B = \hat{r}_t^n$ and $\hat{r}_t^n$ the real interest rate under flexible prices.[61] Figure 2.5 provides a sketch in terms of the required feedback strength for the "surgical" LAB policy and its relationships with the size of the bubble around which the economy fluctuates (i.e., ϕ_q^S is increasing in q^B). Thus, the threshold aggregate bubble-output ratio dividing globally stable and unstable branches of the bubbly SSs under flexible prices, q_*^B, turns out to be also the point that defines a region in which a "surgical" LAB policy is possible.

Regardless of the fact that the existence of the "surgical" LAB policy is a theoretically appealing solution for the central bank tacking the bounded bubble-driven fluctuations in the economy, especially when they are unlikely to be ruled out in the present context, the potential downside risk of it may also be considerable in practice. As just hinted, if not precisely calibrated and if the interest rate response to variations in the size of the aggregate bubble turns out to be actually too strong, i.e., $\phi_q^* > \phi_q > \phi_q^S$ for $q^B \in (0, q_*^B)$ in particular, then the economy would be dragged into a recession in

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59 Note that q_*^B is also identified by Galí (2021) as the threshold value below which locally stationary bubble fluctuations cannot be ruled out by means of an output gap-focused interest rate policy

60 For plausible calibrated values for the model parameters, the "surgical" LAB policy only requires the bubble coefficient to be of an order of magnitude of 10^{-5} for all $q^B \in (0, q_*^B)$, which is fairly small.

61 Recall that in Section 2.3.3 above, under flexible prices, when aggregate demand is unaffected by variations in the size of the bubble, a growing bubble crowds out one-for-one the total fundamental wealth of the economy by raising accordingly the real rate of interest.

the face of the bubble boom, with the bubble multiplier on output $\Omega_1 < 0$ being a sign of the central bank's "overreaction".

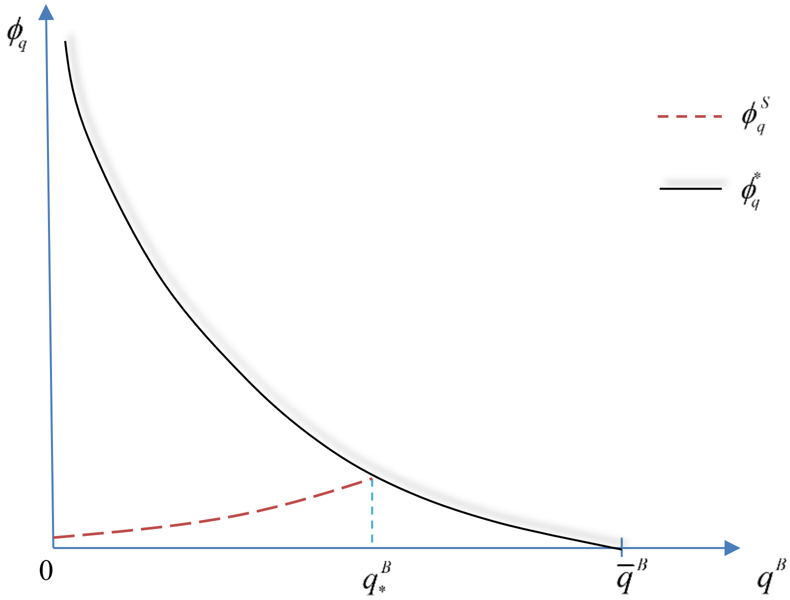

Figure 2.5 The bubble coefficient for a "surgical" LAB policy

To elucidate this phenomenon, it is useful to decompose the bubble impact on output into three channels. As mentioned in Section 2.2.3, there is first of all a *direct effect* of variations in the size of the bubble on output, of size $1-\beta\gamma$ which could be interpreted as the "initial and direct" marginal propensity to consume for the households, as implied by the aggregate consumption function (2.44) together with the goods market clearing condition,

$$\hat{y}_t = (1-\beta\gamma)\left(\hat{q}_t^B + \hat{x}_t\right). \tag{2.70}$$

Clearly, a positive bubble shock immediately increases the financial wealth held by the households, boosting demand for goods and thus current output under sticky prices as a result. This direct channel of the bubble impact is unaffected by the monetary policy.

Secondly, there is also a *static* effect of the bubble shock on aggregate output, which is transmitted by the composite variable, $\hat{x}_t$. As from (2.45),

$$\hat{x}_t = \frac{\Lambda\upsilon}{\beta} E_t\hat{x}_{t+1} + \frac{1-\beta\gamma}{\beta\gamma}\hat{q}_t^B - \frac{\Lambda\upsilon}{(1-\Lambda\upsilon\gamma)\beta}\hat{r}_t, \tag{2.71}$$

where $\hat{x}_t$ reflects the movements of streams of (expected) future output (income) and real interest rates. If the real interest rate were to be kept constant, then an increase in $\hat{q}_t^B$ would also raise $\hat{x}_t$ through the higher contemporaneous output $\hat{y}_t$ as well. In other words, a booming bubble not only boosts output by raising directly the households' financial wealth as the bubble assets' investors, but also by indirectly increasing their overall wealth as workers and firms' owners, from earning higher labour incomes and receiving more dividends, which is driven by the higher output as a result of the direct effect of the bubble shock. In that case, the static multiplier is of size $(1-\beta\gamma)^2/(\beta\gamma)$, which is obtained by looking collectively at (2.71) and (2.44).

However, when we allow endogenous interest rate changes under the LAB policy with $\hat{r}_t = \phi_q \hat{q}_t^B$ for $\phi_q > 0$, (2.71) becomes

$$\hat{x}_t = \frac{\Lambda\upsilon}{\beta} E_t \hat{x}_{t+1} + \left[\frac{1-\beta\gamma}{\beta\gamma} - \frac{\Lambda\upsilon}{(1-\Lambda\upsilon\gamma)\beta}\phi_q \right] \hat{q}_t^B . \tag{2.72}$$

It is then straightforward that the coefficient of $\hat{q}_t^B$ in (2.72) is decreasing in ϕ_q, i.e., a rise in the interest rate called for by the LAB policy in the face of a positive bubble shock reduces the size of the static multiplier by $\Lambda\upsilon\phi_q / [(1-\Lambda\upsilon\gamma)\beta]$. Intuitively, this is because a rise in the real interest rate induces households to postpone immediate consumption due to intertemporal substitution, easing demand pressure at the impact period that arises from the increase in financial wealth. Note also that the lower bound for the coefficient of $\hat{q}_t^B$ in (2.72) is $\frac{(1-\beta\gamma)(\beta-\upsilon\Lambda^2)}{\beta\gamma(\beta-\upsilon\Lambda)}$ for $\phi_q \in (0, \phi_q^*)$, which becomes negative when the economy fluctuates around a SS with $q^B \in (0, \tilde{q}^B)$, where $\tilde{q}^B \equiv \frac{\gamma\left(\beta-\upsilon\sqrt{\beta/\upsilon}\right)}{(1-\beta\gamma)\left(1-\upsilon\gamma\sqrt{\beta/\upsilon}\right)}$.[62] In those cases, if the authority adopts an aggressive LAB

[62] See Appendix 2.D for a proof.

strategy by setting a high ϕ_q value (especially when $q^B \to 0$), then the intertemporal substitution would become so strong that a positive bubble shock even results in a negative static multiplier on output.

Thirdly, there is a *dynamic* channel of the bubble impact on output, which is induced via affecting the expected future output levels, $E_t \hat{y}_{t+k}$, for $k = 1, 2, \ldots$, and is reflected as part of the changes in $E_t \hat{x}_{t+1}$ in (2.71). This dynamic mechanism is closely associated with the *intrinsically persistent* property of the rational bubble fluctuations: even though the initial bubble shock is short-lived, the impact of it on the size of the bubble would persist into the future, which is a consequence of the equilibrium requirement that the bubble must grow at the rate of interest. The persistently higher $E_t \hat{q}^B_{t+k}$ for $k = 1, 2, \ldots$ in turn leads to higher anticipation of future output.

What was discussed above thus implies that the dynamic multiplier of the bubble impact may be stronger the higher the bubble coefficient set in the LAB policy, for both the stronger intertemporal substitution between the current and future consumption and the more persistent bubble fluctuations that would cause higher expected output levels. However, regardless of this positive impact of a stronger policy reaction to the bubble shock on the dynamic effect of the bubble multiplier, the higher expected real interest rates in the transition periods resulting from the stronger monetary policy response to the larger size of the bubble of the initial bubble shock tends to reduce the future fundamental wealth which is a *discounted* sum of future income expected to accrue to currently alive households. Therefore, unlike the impact of the LAB policy on the static effect, the consequence of a higher ϕ_q on the dynamic multiplier appears to be non-monotonic.

Recall now the ultimate bubble impact on output is summarised by adding up the "direct", the "static", and the "dynamic" effects altogether, and taking into account the impact of the interest rate changes on the latter two gives us the net effect of a higher bubble coefficient in the LAB policy on Ω_1. As we have learnt from (2.69), the fact that Ω_1 is monotonically decreasing in ϕ_q implies that the positive aspect of a higher ϕ_q via the dynamic channel of the overall bubble multiplier is never able to be dominant over the negative impacts of it via the other aspects. In particular, the former aspect inclines to be even weaker when the SS bubble size is relatively small. This is because, since from the law of motion for the rational bubble and under the assumption that $E_t\hat{u}_{t+1} = 0$,

$$E_t\hat{q}^B_{t+1} = \frac{1}{\Lambda}\hat{q}^B_t + \frac{q^B}{\Lambda}\hat{r}_t, \tag{2.73}$$

while $\partial\left(q^B/\Lambda\right)/\partial q^B > 0$, so that the bounded bubble fluctuations would be less persistent encountering a same magnitude of an increase in the real interest rate if the original size of the bubble in the economy is smaller.

This thus illuminates to some extent why a negative bubble multiplier on output is more likely (unlikely) to occur when $q^B < q^B_*$ ($q^B \geq q^B_*$), since then the intertemporal substitution effects induced by the policy response to a booming bubble are (not) significantly stronger than the impact of the LAB policy on the persistence of the bubble fluctuations, such that the negative impact of the former aspect of the bubble shock with the monetary intervention (does not) more than offset the positive effect of the bubble fluctuations on current output, generating a negative co-movement between the bubble and output as a result. Other causes that may be responsible for this issue will be further discussed in the next subsection.

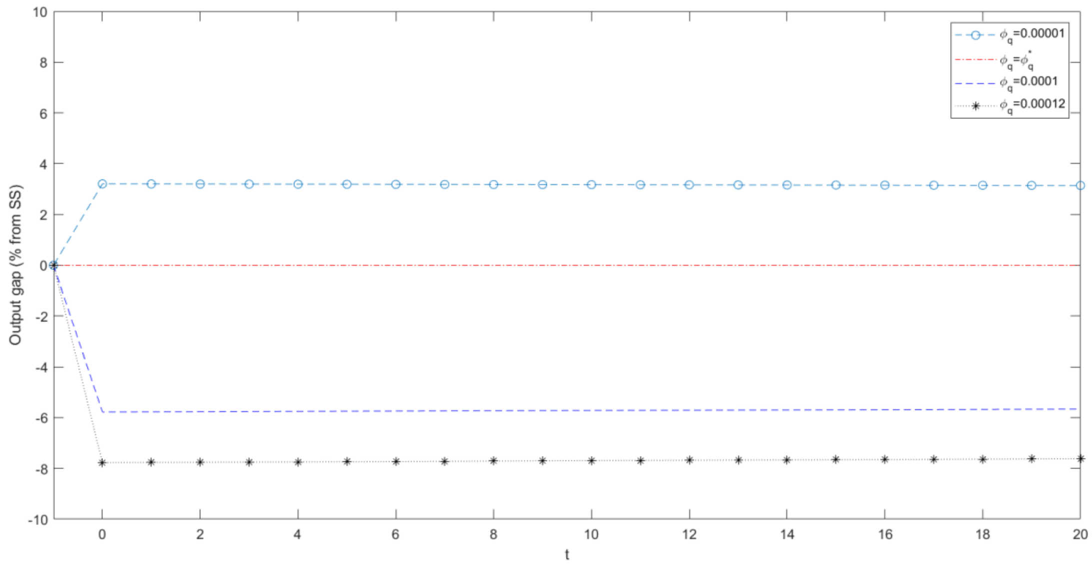

Figure 2.6 The time paths of $\hat{y}_t$ under the LAB policy

As a numerical example, Figure 2.6 depicts the impulse responses of the output gap to a unit of positive sunspot bubble shock (i.e., with $\hat{q}_0^B = 1$) hitting the system at time $t = 0$ with different feedback coefficients in the LAB policy, given that $\beta = 0.998$, $\gamma = 0.996$, $\upsilon = 0.997$ (following Galí (2021), calibrated for quarterly data) and that $q^B \simeq 2.5 \times 10^{-4}$. It can be seen from Figure 2.6 that when the model economy is originally close to being bubbleless, the bubble multiplier on output could become very negative as ϕ_q increases just a tiny bit, while the impact of the transitory bubble shock on output always decays at a very low speed, confirming the theoretical reasoning just made above.

2.4.2 A Conventional Policy

Let's consider now the role played by the output coefficient in the interest rate rule (2.64) in tacking the bubble-driven fluctuations in output. As shown in Galí (2021) (Appendix D), under $\phi_q = 0$, for $q^B \in \left[0, q_*^B\right]$, there is no $\phi_y \geq 0$ which can ensure a unique bonded rational expectation solution to (2.65); for $q^B \in \left(q_*^B, \overline{q}^B\right]$, on the other hand, $\phi_y > \phi_y^* \equiv \left(\Lambda - 1\right)(1 - \Phi)/\left(1 - \Psi\right)$ can guarantee a unique solution to (2.65), with

$\Phi \equiv \Lambda \upsilon \beta^{-1}$ and $\Psi \equiv \Phi\left[1+(\Lambda-1)(1-\beta\gamma)(1-\beta\gamma\Phi)^{-1}\right] > 0$, but the threshold value $\phi_y^* \to +\infty$ as $q_*^B \to q_*^B +$. In other words, for $q^B \in \left[0, q_*^B\right]$ or for q^B larger but close to q_*^B, it may not be practically feasible to rule out completely bounded bubble(-driven) fluctuations of the system by means of a conventional output gap-focused policy. As in the LAB policy case discussed in Section 2.4.1, however, it may still be valuable to investigate to what extent the conventional policy may mitigate the influence of the bounded bubble fluctuations on the economy in those circumstances, especially given that bubbles in reality could not be perfectly observable, i.e., leaning against a bubble might not be robust per se from a policymaker's point of view.

When $\phi_y > 0$, $\phi_q = 0$, the associated characteristic polynomial for the coefficient matrix of (2.65) is given by

$$P(\lambda) = \lambda^2 - \left[\frac{1}{\Lambda} + \frac{(1+\Upsilon\phi_y)\beta}{\upsilon\Lambda}\right]\lambda + \frac{(1+\phi_y)\beta}{\upsilon\Lambda^2} = 0. \tag{2.74}$$

If there is no $\phi_y > 0$ which can guarantee the absence of bounded bubble fluctuations of the system, then as shown in Appendix 2.C, the two eigenvalues for (2.74) in those situations are given by

$$\lambda_1 = \frac{1}{2}\left\{\frac{\upsilon+\beta(1+\Upsilon\phi_y)}{\upsilon\Lambda} - \sqrt{\left[\frac{\upsilon+\beta\left(1+\Upsilon\phi_y\right)}{\upsilon\Lambda}\right]^2 - \frac{4\beta(1+\phi_y)}{\upsilon\Lambda^2}}\right\} \tag{2.75}$$

and

$$\lambda_2 = \frac{1}{2}\left\{\frac{\upsilon+\beta(1+\Upsilon\phi_y)}{\upsilon\Lambda} + \sqrt{\left[\frac{\upsilon+\beta\left(1+\Upsilon\phi_y\right)}{\upsilon\Lambda}\right]^2 - \frac{4\beta(1+\phi_y)}{\upsilon\Lambda^2}}\right\}, \tag{2.76}$$

with $\lambda_1 \in (0,1)$ and $\lambda_2 \in (1,\infty)$.

Similarly to the case in Section 2.4.1, when there is a "small" positive "sunspot" bubble shock $\varepsilon_0 > 0$ hitting the system at some time $t = 0$, the time path for the output gap can be described by

$$\hat{y}_t = \Omega_2 \lambda_1^t \varepsilon_0, \tag{2.77}$$

with $\Omega_2 \equiv \frac{(1-\beta\gamma)(1-\upsilon\gamma)}{\beta\gamma(1+\Upsilon\phi_y)-\upsilon\gamma\Lambda\lambda_1}$ the bubble multiplier under the conventional policy.

Here I consider a special scenario, namely, the impact of a tiny increase in the policy coefficient ϕ_y from zero on $\hat{y}_t$. One aspect is about its impact on the persistence of the output fluctuations. Since, when the economy fluctuates around a given bubbly SS with $q^B \in (0, \bar{q}^B)$,

$$\left.\frac{\partial \lambda_1}{\partial \phi_y}\right|_{\phi_y=0} = \frac{\beta(1-\Upsilon)}{\Lambda(\beta-\upsilon)} > 0, \tag{2.78}$$

an increase in the feedback coefficient on the output gap prolongs the resulting output fluctuations, in a similar spirit to the case discussed in Section 2.4.1. More generally, since the expected return on the rational bubble must in equilibrium equal the rate of interest, an increase in ϕ_y which causes a higher $\hat{r}_t$ given a same size of bubble shock would presumably always raise the persistence of the bubble as well as of output fluctuations.

When it comes to investigating the impact of the conventional policy on the bubble multiplier, note that

$$\frac{\partial \Omega_2}{\partial \phi_y} = -(1-\beta\gamma)(1-\upsilon\gamma)\left[\beta\gamma(1+\Upsilon\phi_y)-\upsilon\gamma\Lambda\lambda_1\right]^{-2}\left(\beta\gamma\Upsilon-\upsilon\gamma\Lambda\frac{\partial \lambda_1}{\partial \phi_y}\right), \tag{2.79}$$

i.e., the first order partial derivative of the bubble multiplier with respect to ϕ_y has an opposite sign to the term $\beta\gamma\Upsilon-\upsilon\gamma\Lambda\frac{\partial \lambda_1}{\partial \phi_y}$. If we evaluate (2.79) at $\phi_y = 0$, then

$$\left.\beta\gamma\Upsilon-\upsilon\gamma\Lambda\frac{\partial \lambda_1}{\partial \phi_y}\right|_{\phi_y=0} = \frac{\beta\gamma}{\beta-\upsilon}(\beta\Upsilon-\upsilon). \tag{2.80}$$

Since $\frac{\partial \Upsilon}{\partial \Lambda} = \frac{(1-\beta\gamma)(1-\upsilon\gamma)\upsilon}{\beta(1-\upsilon\gamma\Lambda)^2} > 0$ and $\frac{\partial q^B}{\partial \Lambda} < 0$ i.e., Υ is decreasing in the SS size of the aggregate bubble-output ratio, $\upsilon/\beta < \Upsilon < 1$ for $q^B \in (0, \bar{q}^B)$, so that the term in (2.80) is always strictly positive. Therefore, in (2.79), we have that $\left.\frac{\partial \Omega_2}{\partial \phi_y}\right|_{\phi_y=0} < 0$ holds for $q^B \in (0, \bar{q}^B)$, i.e., an active conventional stabilisation response is helpful for depressing the impact of the bounded bubble fluctuations on output as well.

Until now, it seems that policy response to the conventional target has similar effects in dealing with bubble-driven fluctuations to the LAB policy: they both face a trade-off between dampening the bubble impact and reducing the persistence of the fluctuations in output. But does the output gap-focused policy also suffer an "overreaction" risk when it attempts to mitigate the bubble multiplier effect on output?

The answer is probably "no". As a preliminary investigation and as an example, Figure 2.7 shows the impulse responses of the output gap to a unit of positive sunspot bubble shock occurring at time $t = 0$ with ϕ_y varying from 0 to 0.125, assuming that $\beta = 0.998, \gamma = 0.996, \upsilon = 0.997$ and that $q^B \simeq 2.56$. Obviously, active policy feedback to variations in the output gap is effective in dampening the bubble impact, but does not seem to risk overreacting by causing an economic downturn in the face of the bubble boom.

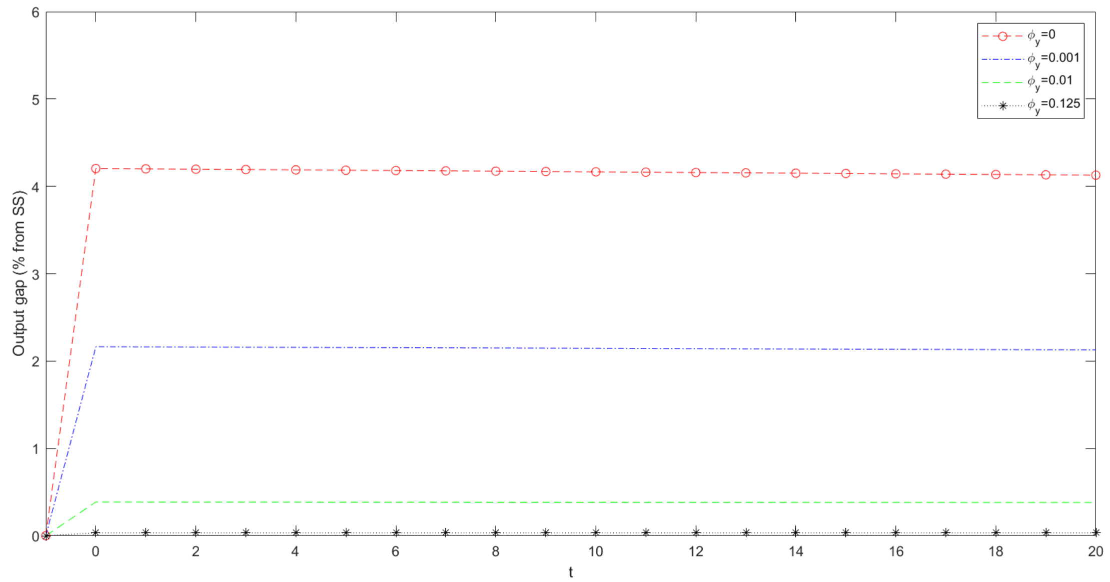

Figure 2.7 The time paths of $\hat{y}_t$ under the conventional policy

To justify this conjecture more formally, recall that it is the static and dynamic channels of the bubble multiplier that interest rate policies can have an impact on. In the present context, $\hat{r}_t = \phi_y \hat{y}_t$, thus from (2.70) and (2.71),

$$\hat{x}_t = \frac{\upsilon\gamma\Lambda}{\beta\gamma + \frac{\upsilon\gamma\Lambda(1-\beta\gamma)}{1-\upsilon\gamma\Lambda}\phi_y} E_t \hat{x}_{t+1} + \left[\frac{1}{\beta\gamma + \frac{\upsilon\gamma\Lambda(1-\beta\gamma)}{1-\upsilon\gamma\Lambda}\phi_y} - 1 \right] \hat{q}_t^B . \tag{2.81}$$

Therefore, it is evident from (2.81) that the coefficient in the square bracket for $\hat{q}_t^B$ is decreasing in ϕ_y, i.e., a stronger feedback on output deviation unambiguously dampens the static effect of the bubble multiplier. However, although it also becomes negative when $\phi_y > 1/(\Lambda\gamma\upsilon) - 1 \equiv \phi_y$, where ϕ_y may just be slightly above zero for any plausible calibration for the parameters and for any bubbly SS, the minimum possible value for the static multiplier is just -1, even when $\phi_y \to +\infty$. In other words, *there is a limit* in terms of the extent the conventional policy can restrain the bubble impact on output.

This outcome is thus in stark contrast to the case with LAB feedback, where the *static* *multiplier* can be as large as negative infinity, as implied by (2.72), especially when the original economy state is nearly bubbleless.[63] An explanation for this difference lies in the fact that with a (sunspot) bubble shock, the size of the *current* bubble is *unaffected* by interest rate changes, but the former can affect the latter and then aggregate demand under sticky prices through the LAB policy – the higher the response coefficient on the bubble, the stronger the negative effect on current output would be. But this is not the case for the conventional policy regime, where the endogenous feedback loop between the output gap and the instrument rate is complete.

Furthermore, if there is explicit feedback on deviations in the size of the bubble, then an interest rate rise called for by the LAB policy exacerbates the volatility of the rational bubbles *over time*, which in turn requires the policy rate to increase endogenously even more *in the subsequent periods*. The latter mechanism is distinctive for the LAB policy, but is the critical force in depressing the *dynamic multiplier* of the bubble impact on output, as discussed before. Although for the policy targeting the output gap, future policy rate may also rise as a result of the amplified bubble fluctuations, the impact of the latter on the former should be much less prominent relative to the policy with an explicit LAB component, as aggregate output adjusts just proportionally to changes in the size of the bubble, especially with the policy interventions (i.e., the effective bubble multiplier on output could be quite small).

Therefore, both the static and dynamic channels of the bubble impact on output are potentially much more negative under the LAB policy regime when confronting a bubble boom, which apparently is the key contributory factor for the "overreaction" risk of the LAB policy. In this sense, a policy with conventional stabilisation target may be a more robust choice regarding stabilising rational bubble-driven fluctuations in the economy.

2.5 Concluding Comments

[63] In those scenarios, the required LAB policy coefficient for ruling out sunspot bubble shock tends to go infinity.

The analysis of Galí (2021) of monetary policy implications with the presence of rational bubbles that have an impact on the economy mainly as aggregate demand shifters is innovative and inspiring, yet it left some issues unaddressed. In the present chapter, I have extended Galí's (2021) work by investigating especially how monetary policy may be used to stabilise the OLG-NK economy when it is exposed to bounded rational bubble fluctuations which are unrelated to variations in fundamentals, so that a fuller picture of this issue could be appreciated.

The study of the chapter generates several findings.

First, the possible coexistence of multiple bubbly equilibria is demonstrated to be a consequence of the introduction of a series of various types of bubble assets together with the emergence of the new bubble. Also, the endowment of the latter to individual households produces a degree of inefficiency in a bubbly equilibrium, which is unlikely to be the case in the classic literature on rational bubbles of this type.

Second, the conclusions drawn by Galí (2021) in terms of the equilibrium properties of the OLG-NK economy in a neighbourhood of a given SS turn out to be also valid for the global dynamics of the model, especially under flexible prices.

Third, when it comes to monetary policy implications regarding tackling potential bounded bubble-driven fluctuations in output and inflation, a policy trade-off is present between reducing the persistence of the fluctuations and reducing the bubble impact, albeit the former aspect may be insignificant in practice relative to the latter one.

Fourth, the interest rate policy with an explicit LAB component is very effective in dampening the bubble impact on output, and could eliminate completely the impact of the bubble fluctuations on output by applying a "surgical" policy, if the original size of the bubble in the economy is smaller than a threshold value. However, if not precisely calibrated, the LAB policy may suffer an "overreaction" problem by unintentionally dragging the economy into a recession in the face of a bubble boom, an outcome that is created by a sort of incomplete feedback loop between the policy rate and the size of the bubble (at the impact period).

Finally, the conventional policy with an output gap-stabilisation motive could also mitigate the bubble impact on the economy when confronting a positive sunspot bubble shock, but does not seem to risk causing an economic downtown compared to its LAB counterpart.

Therefore, from the analysis conducted in this chapter, it can be seen that the rational bubble fluctuations are intrinsically persistent, a distinctive pattern that generates some special implications for monetary policy deliberations. In a world featuring a continuum of bubbly SSs, whether LAB is a desirable strategy or not may come on a case-by-case basis. If the economy is populated by a large size of the bubble at the start, then a LAB policy may be more effective in either completely ruling out the bounded bubble fluctuations or in mitigating their impacts on output or inflation if otherwise. Nonetheless, from a practical point of view, monetary policy with a conventional feedback variable may be a more robust choice in any case.

Appendix

2.A Derivations of Figure 2.4

From (2.55), it is easy to verify that the vertical interception of the mapping is given by $u \geq 0$ when setting $q_t^B = 0$. The vertical asymptote is $\beta\gamma / \left[(1-\upsilon\gamma)(1-\beta\gamma)\right] > 0$, while the horizontal asymptote is $\left[(1-\gamma\beta)u - \upsilon\gamma\right] / \left[(1-\upsilon\gamma)(1-\beta\gamma)\right]$ which could be negative or positive for $u \in (0, \bar{u}]$, where $\bar{u}$ denotes the upper bound of u.

To see whether the right-hand-side of (2.55) is an increasing function of q_t^B or not, we need to see whether

$$\frac{\beta\upsilon\gamma^2}{(1-\upsilon\gamma)(1-\beta\gamma)} - \left(\frac{\beta\upsilon\gamma^2}{1-\upsilon\gamma}\right) u > 0, \tag{2.82}$$

i.e., whether $0 < u < 1/(1-\beta\gamma)$ holds for $\forall u \in (0, \bar{u}]$. As an auxiliary step, consider first the upper bound of u. Since $\frac{\partial u}{\partial q^B} <> 0$, $\frac{\partial^2 u}{\partial \left(q^B\right)^2} < 0$, there must be a unique $\bar{u}$ corresponding to some value $q^{B*} \in \left(0, \bar{q}^B\right]$. From (2.39),

$$\bar{u} = q^{B*} - \frac{q^{B*}}{\Lambda^*} = \frac{\gamma\left(\beta - \Lambda^* \upsilon\right)}{\left(1-\beta\gamma\right)\left(1-\Lambda^*\upsilon\gamma\right)}\left(1 - \frac{1}{\Lambda^*}\right) < \frac{1}{1-\beta\gamma}, \tag{2.83}$$

since $1 \le \Lambda^* < \beta/\upsilon$ for $q^{B*} \in \left(0, \bar{q}^B\right]$, and $\beta \in (0,1), \gamma \in (0,1)$, where Λ^* satisfies

$$\left(\upsilon - \beta\gamma\upsilon - \upsilon^2\gamma\right)\left(\Lambda^*\right)^2 + 2\beta\gamma\gamma\Lambda^* - \beta = 0. \tag{2.84}$$

Therefore, combining (2.82) with (2.83) proves that the right-hand-side of (2.55) is indeed an increasing function of q_t^B.

Furthermore, it is also easy to check that (2.84) is identical to

$$\Psi(\Lambda_*) = \frac{\Lambda_* \upsilon}{\beta}\left[1 + \left(\Lambda_* - 1\right)\frac{1-\beta\gamma}{1-\Lambda_*\upsilon\gamma}\right] = 1, \tag{2.85}$$

implying that $q^{B*} = q_*^B$, with the latter the threshold value identified in Galí (2021).

When $q^{B^S} = q^{B^U} = q_*^B$, we have

$$q_*^B = \frac{\left(\beta - \upsilon\right)\gamma + \left(1-\beta\gamma\right)\bar{u}}{2\left(1-\upsilon\gamma\right)\left(1-\beta\gamma\right)}, \tag{2.86}$$

with $\bar{u}$ satisfying

$$\left[\left(\beta - \upsilon\right)\gamma + \left(1-\beta\gamma\right)\bar{u}\right]^2 = 4\left(1-\upsilon\gamma\right)\left(1-\beta\gamma\right)\beta\gamma\bar{u}.$$

2.B The Time Paths for the Bubble and Output under the LAB Policy

To obtain (2.67) and (2.68), note that when $\phi_q > 0$, $\phi_y = 0$, the two eigenvalues to (2.66) is given by $\lambda_1 = \left(1 + q^B\phi_q\right)/\Lambda$ and $\lambda_2 = \beta/\left(\upsilon\Lambda\right) > 1$. When $\lambda_1 < 1$, the time paths for $\hat{q}_t^B$ and $\hat{y}_t$ are then given by

$$\hat{q}_t^B = a_{11}\lambda_1^t, \tag{2.87}$$

and

$$\hat{y}_t = a_{21}\lambda_1^t , \tag{2.88}$$

respectively, where $[a_{11}, a_{21}]$ is the eigenvector associated with λ_1. If assuming now there is a market sentiment shock occurs at time $t = 0$, and lasts only for one period, i.e., $\hat{q}_0^B = \varepsilon_0$, then we have

$$a_{11} = \varepsilon_0 . \tag{2.89}$$

Hence, plugging (2.89) into (2.87) finds (2.67).

To find the expression for a_{21}, note that we have

$$a_{11}\left[\frac{\beta\Upsilon\phi_q}{\upsilon\Lambda} - \frac{(1-\beta\gamma)(1-\upsilon\gamma)}{\upsilon\Lambda\gamma}\right] + a_{21}\left(\frac{\beta}{\upsilon\Lambda} - \lambda_1\right) = 0 ,$$

so that

$$a_{21} = \frac{\beta\Upsilon\phi_q - (1/\gamma - \beta)(1-\upsilon\gamma)}{(1+q^B\phi_q)\upsilon - \beta}\varepsilon_0 . \tag{2.90}$$

Substituting (2.90) and the expression for λ_1 into (2.88) yields (2.68).

2.C The Eigenvalues for the Solution under the Conventional Policy

The two eigenvalues for (2.74) are given by

$$\lambda_{1,2} = \frac{1}{2}\left\{\frac{\upsilon+\beta(1+\Upsilon\phi_y)}{\upsilon\Lambda} \pm \sqrt{\left[\frac{\upsilon+\beta\left(1+\Upsilon\phi_y\right)}{\upsilon\Lambda}\right]^2 - \frac{4\beta(1+\phi_y)}{\upsilon\Lambda^2}}\right\} . \tag{2.91}$$

If the eigenvalues of (2.91) turn out to be a pair of conjugate of complex numbers, then it must both lie within, in, or outside the unit cycle simultaneously and thus cannot be a solution of saddle-path. Therefore, we can assume that ϕ_y is selected so that

$$\left[\frac{\upsilon+\beta\left(1+\Upsilon\phi_y\right)}{\upsilon\Lambda}\right]^2-\frac{4\beta\left(1+\phi_y\right)}{\upsilon\Lambda^2}\geq 0,$$

i.e., the two eigenvalues are real. Then for a saddle-path solution, $|\lambda_1|<1, |\lambda_2|>1$, and $\lambda\neq 0$ since in (2.74),

$$P(\lambda=0)=\frac{\left(1+\phi_y\right)\beta}{\upsilon\Lambda^2}>0 \tag{2.92}$$

for $\forall\phi_y\geq 0$.

Consider first the possibility that $0<\lambda_1<1$. Then given (2.92), we have

$$P(\lambda=0)=\left(0-\lambda_1\right)\left(0-\lambda_2\right)>0, \tag{2.93}$$

so it must be the case that $\lambda_2>1$ for (2.93) to hold.

Consider now whether it could be the case for $-1<\lambda_1<0$. Then given (2.93), it must have $\lambda_2<-1$, so that in this situation $\lambda_2<\lambda_1$. Therefore,

$$\lambda_1=\frac{1}{2}\left\{\frac{\upsilon+\beta\left(1+\Upsilon\phi_y\right)}{\upsilon\Lambda}+\sqrt{\left[\frac{\upsilon+\beta\left(1+\Upsilon\phi_y\right)}{\upsilon\Lambda}\right]^2-\frac{4\beta\left(1+\phi_y\right)}{\upsilon\Lambda^2}}\right\}>0,$$

but which contradicts the pre-assumption that $-1<\lambda_1<0$.

As a result, for a possible saddle-path solution (with real eigenvalues), it must be the case that $0<\lambda_1<1$ and $\lambda_2>1$.

2.D The Threshold Values under the LAB Policy

(1) The lower bound for Ω_1 is given by

$$\underline{\Omega}_1\equiv\frac{(1-\gamma\beta)(1-\upsilon\gamma)-\beta\gamma\Upsilon\left[(\Lambda-1)/q^B\right]}{\gamma\left(\beta-\upsilon\Lambda\right)}. \tag{2.94}$$

Substituting (2.42) and $\Upsilon\equiv\left\{(1-\beta\gamma)\upsilon\left[q^B+\Lambda/(1-\upsilon\gamma\Lambda)\right]\right\}/\beta\leq 1$ into (2.94) yields

$$\underline{\Omega}_1 \equiv \frac{\left[(1-\gamma\Lambda\upsilon)(\beta-\upsilon\Lambda)-\upsilon\Lambda(\Lambda-1)(1-\beta\gamma)\right](1-\beta\gamma)}{(\beta-\upsilon\Lambda)^2\gamma}. \qquad (2.95)$$

It then turns out that the numerator of $\underline{\Omega}_1$ in (2.95) is increasing in $q^B \in (0,\overline{q}^B]$, because

$$\frac{\partial\left\{\left[(1-\gamma\Lambda\upsilon)(\beta-\upsilon\Lambda)-\upsilon\Lambda(\Lambda-1)(1-\beta\gamma)\right](1-\beta\gamma)\right\}}{\partial\Lambda}$$
$$=(1-\beta\gamma)\left[-\gamma\upsilon(\beta-\upsilon\Lambda)-(1-\gamma\Lambda\upsilon)\upsilon-\upsilon(\Lambda-1)(1-\beta\gamma)-\upsilon\Lambda(1-\beta\gamma)\right]<0$$
$$(2.96)$$

for $q^B \in (0,\overline{q}^B]$, and for the fact that $\frac{\partial q^B}{\partial\Lambda}<0$. Therefore, the smaller the steady state bubble-output ratio, the smaller will be the numerator of the lower bound of Ω_1. When $q^B = q^{B'} = \frac{\gamma(\beta-\upsilon\Lambda^{'})}{(1-\beta\gamma)(1-\upsilon\gamma\Lambda^{'})}$ with $\Lambda^{'}$ satisfying

$$1-\gamma\upsilon\Lambda'-\frac{\upsilon\Lambda'(\Lambda'-1)(1-\beta\gamma)}{\beta-\upsilon\Lambda'}=0, \qquad (2.97)$$

$\underline{\Omega}_1=0$; and when $0<q^B<q^{B'}$, $\underline{\Omega}_1<0$ because now the numerator of $\underline{\Omega}_1$ in (2.95) turns to be negative, while the denominator remains strictly positive as $\beta>\upsilon\Lambda$ for $q^B>0$.

On the other hand, the threshold Λ' corresponding to $q^{B'}$ which satisfies (2.97) is equivalent to

$$(1-\gamma\upsilon\Lambda')(\beta-\upsilon\Lambda')-\upsilon\Lambda'(\Lambda'-1)(1-\beta\gamma)=0, \qquad (2.98)$$

while through some manipulation, the threshold value Λ_* that makes

$$\Psi(\Lambda_*)\equiv\frac{\Lambda_*\upsilon}{\beta}\left[1+(\Lambda_*-1)\frac{1-\beta\gamma}{1-\Lambda_*\upsilon\gamma}\right]=1 \qquad (2.99)$$

can be shown to be equivalent to

$$(1-\gamma\upsilon\Lambda_*)(\beta-\upsilon\Lambda_*)-\upsilon\Lambda_*(\Lambda_*-1)(1-\beta\gamma)=0. \qquad (2.100)$$

Since (2.98) is identical to (2.100), we have $q^{B'}$ identical to q_*^B in the main context.

(2) For the threshold SDF Λ for a possibly negative static channel of the bubble multiplier. Since

$$\frac{\partial\left[\frac{(1-\beta\gamma)(\beta-\upsilon\Lambda^2)}{\beta\gamma(\beta-\upsilon\Lambda)}\right]}{\partial\Lambda}=\left(\frac{1-\beta\gamma}{\beta\gamma}\right)\left[\frac{-\upsilon\Lambda(\beta-\upsilon\Lambda)-(\Lambda-1)\beta\upsilon}{(\beta-\upsilon\Lambda)^2}\right]<0 \quad (2.101)$$

for $q^B>0$, the lower bound of the static multiplier, $\frac{(1-\beta\gamma)(\beta-\upsilon\Lambda^2)}{\beta\gamma(\beta-\upsilon\Lambda)}$, which takes its value when $\phi_q=\phi_q^*$ for a given q^B, is less than zero when $\beta-\upsilon\Lambda^2<0$, i.e., when $(1<)\sqrt{\beta/\upsilon}\equiv\Lambda<\Lambda(<\beta/\upsilon)$.

Chapter 3

Financing Constraints and Rational Bubbles

3.1 Introduction

In Chapter 2, I have investigated the impact of rational bubbles on the formulation of systematic monetary policy from a demand side perspective, where variations in the bubble price mainly affect the economy as an aggregate demand shifter. I now turn to study this issue in a framework where existence of rational bubbles mainly has a supply side impact on an economy instead.

In one important strand of the relevant literature, rational bubbles can exist in equilibrium in a framework with infinitely-lived agents while having a supply-side impact on an economy because of credit constraints, with bubbles serving as a means of enhancing the ability of economic agents (typically, firms) to acquire funds. There are generically three types of circumstances in this type of model: most popularly, firms can raise funds by borrowing but which is constrained (e.g., Dong et al. (2020), Biswas et al. (2020), Ikeda (2021)); or there is a complete absence of any external fund-raising channel, such that bubbles are the only tool for transferring resources between market participants (e.g., Wang and Wen (2012));[64] or entrepreneurs are allowed to finance investment by issuing new equity and/or selling physical capital but are also subject to some sort of liquidity constraint (e.g., Miao et al. (2015), Miao and Wang (2018), Kiyotaki and Moore (2019)).[65]

However, few attempts appear to have been made to discuss monetary policy

[64] Martín and Ventura (2012, 2016) also analyse credit constraints and rational bubbles, but in a 2-period OLG framework with flexible prices. The general equilibrium implications of their models are fundamentally different from the strand of models mentioned in the main text.

[65] In Kiyotaki and Moore's (2019) models, the "rational bubble" is "fiat money" issued by a monetary authority, and they do not treat it as a source of fluctuations.

implications in this type of environment with financial imperfections, especially in a New Keynesian framework.[66] Exceptions are including Dong et al. (2020) and Ikeda (2021),[67] but these models are analytically highly intractable, leading to some ambiguity about the general equilibrium mechanisms at work and about the interactions between their proposed monetary policies and the rational bubbles. It also appears to me that an important feature of actual speculative bubble episodes is absent in the aforementioned research, namely, scenarios where a no-bubble economic state is the resting point.

Therefore, it is my aim to fill these gaps by establishing a model of rational bubbles in which equilibrium mechanisms are in line with the sort of literature mentioned above, but which is analytically tractable for monetary policy discussions, with particular attention paid to bubbly episodes starting from a no-bubble status. I undertake this task in two consecutive steps: in this chapter, I develop a model which permits the potential equilibrium existence of a rational bubble and analyse its long and short run general equilibrium impacts on the economic system with prefer price flexibility; in the next chapter, I introduce nominal rigidities into the model which then gives rise to the non-neutrality of monetary policy and hence enables me to conduct meaningful model-based discussions of monetary policy in the presence of this type of rational bubbles.

Firstly, here in Chapter 3, I set up an infinite-horizon framework with endogenous capital accumulation, whose structure is akin to Wang and Wen (2012) and in which individual firms are exposed to a type of uninsured exogenous investment shock a là Kiyotaki and Moore (2019) at the start of each period. To finance their physical capital investments, firms can only use their own internal funds accumulated from productive activities and/or sell a type of intrinsically worthless asset in which they may have invested in previous periods. The assumptions of the exogenous investment shock and

[66] As illuminated in the Introduction of the thesis, there are other monetary policy analyses with the presence of rational bubbles in the literature, but where New Keynesian features are absent.

[67] There are some other studies, e.g., Liu and Wang (2024), analysing interplays between monetary policy and prudential policies.

the complete absence of external financing channels in the model, although stylised, significantly improve the analytical solvability of the model without a loss of the key mechanism through which rational bubbles of this kind work. Particularly, the present neat framework allows me to obtain a closed-form solution for the key pricing equation for the bubble, which then conveniently enables me to conduct precise qualitative analysis regarding the general equilibrium effect of the bubble.

Given the full set of closed-form general equilibrium conditions, I identify three mutually exclusive regions of parameter space each of which corresponds to different types of steady state equilibria. The bubbly asset can have positive values only when the arrival rate of an investment opportunity is sufficiently low, or otherwise the economy could achieve the first-best resource allocation outcome despite the presence of the financial frictions, i.e., the constraints do not really bind, leaving no room for a bubble to exist. The simple framework also enables me to explore an unaddressed scenario arising particularly in infinite-horizon models of rational bubbles with credit constraints, namely, where the economy remains inefficient but does not permit the existence of a bubble (in the steady state). The analysis and comparison of local dynamics in the neighbourhood of different types of SS of the log-linearised version of the model under flexible prices then provide a fuller picture of the economic mechanisms at work in this class of models of rational bubbles with credit constraints.

Several findings of interest emerging from the study of the model built up in this chapter are summarised as below.

First, the simplification of the model compared to the existing literature does not undermine its capability of obtaining the core equilibrium implications of rational bubbles of this kind, but it significantly improves the analytical tractability of the model, which is a major merit of the present framework. Particularly, it retains the crucial property that the rational bubble is not required to grow at the rate of interest, because its pricing equation includes a "liquidity premium" component arising from the fact that the bubble asset provides extra liquidity to investing firms and facilitates the reallocation of resources among heterogenous agents if the investment opportunity is sufficiently scarce.

Second, unlike models of rational bubbles emerging for dynamic inefficiency reasons, where there are generically only two types of regions of parameter space, one of which is dynamically efficient and permits only a bubbleless steady state equilibrium, while the other one could have either bubbleless or bubbly equilibrium, in the present framework it illustrates explicitly that there is an additional region of parameter space in this type of infinite-horizon models on rational bubbles with credit constraints, in which bubbly equilibrium cannot exist even when the economy is by itself inefficient in allocating resources. The responsibility for this peculiar phenomenon lies critically in the fact that the rational bubbles must generate a certain level of liquidity premium which is a function of Tobin's q as a condition to exist legitimately.

Third, when the arrival rate of an investment opportunity is sufficiently low, the economic system is exposed to bounded bubble fluctuations if it is bubbleless in the steady state. As in the case analysed in Chapter 2, the bubble-driven fluctuations in the present model also display the property of intrinsic persistence, i.e., due to its rational nature, a random innovation in the bubble could have a persistent impact on the economy. Furthermore, the degree of persistence of a bubble-driven fluctuation is algebraically shown to be positively related to the arrival rate of an investment opportunity: the scarcer it is, the less persistent the bubble and the resulting output fluctuations are. This is because in that situation the liquidity premium commanded by a bubble is higher, which in turn entails a lower required growth rate for the bubble price in equilibrium. On the other hand, if the economy sits at a bubbly SS at the start, it would never suffer from bounded economic fluctuations caused by variations in the bubble size.

Fourth, an increase in the bubble price generally enhances the financing capacity of an investing firm, leading to an expansion of capital investment and then output as well. A decline in the implied market value of a unit of capital, Tobin's q, also occurs and reflects an improvement of the efficiency of the economy in allocating resources during the course of a bubble-led boom.

The rest of the chapter is arranged as follows. Section 2 builds up the model framework which underlies the analysis in the remainder of this chapter. Section 3 characterises the steady states of the economy and identifies conditions under which rational bubbles can exist. Section 4 analyses equilibrium dynamics of the model around a given SS in various situations. Section 5 summarises and concludes.

3.2 The Baseline Model without Nominal Rigidities

The model developed in this section is built on Wang and Wen (2012), which features an infinite-horizon production economy populated by two types of economic agents: a unit mass of perfectly competitive firms producing homogeneous consumption goods and a unit mass of identical infinitely-lived households who supply labour inelastically and trade the shares of the firms.

However, crucially for the model's distinctive features relative to Wang and Wen (2012),[68] here firms are assumed to be subject to a type of independent idiosyncratic investment shocks a là Kiyotaki and Moore (2019), which is uninsurable. On the other hand, firms can invest in a type of intrinsically worthless bubbly asset as the households do,[69] which overall supply outstanding is constant and equal to one. As will become clear, these modifications enable me to obtain a full set of closed form solutions of the model without undermining the key mechanisms concerning the general equilibrium implications of rational bubbles of this type,[70] and the significance of this convenience

[68] Specifically, in Wang and Wen (2012), individual firms are subject to an idiosyncratic investment-specific productivity shock which follows a defined distribution; also, the overall supply of bubbly assets varies overall time. There are other differences between my model and theirs, but they are not in the central concerns here.

[69] Again, as noted in Chapter 2, by "intrinsically worthless", it means that it does not deliver any payoff or direct utility. Thus the bubble is also a "pure" one. However, unlike the case in Chapter 2, the bubbles here apparently should not be considered as "stock price bubbles" (as firms do not trade each other's shares but they could buy and sell bubble assets from each other).

[70] Wang and Wen (2012) is able to solve their model analytically, but a complete set of closed-form solutions is

of my model setup will become more obvious when New Keynesian features are introduced back into the framework in Chapter 4 of the thesis.

To streamline the analysis, it is also assumed that there is no aggregate uncertainty about fundamentals. Time is discrete and indexed by $t = 0,1,...$

3.2.1 Firms

At period t, each individual firm $j \in [0,1]$ produces homogenous consumption goods Y_{jt} by hiring labour $N_{jt} \geq 0$ at a competitive real wage rate W_t with physical capital $K_{jt} \geq 0$, according to the Cobb-Douglas production function

$$Y_{jt} = K_{jt}^{\alpha} N_{jt}^{1-\alpha}, \tag{3.1}$$

where $\alpha \in (0,1)$. Solving the static optimal labour demand problem

$$\begin{aligned} &\max_{N_{jt}} Y_{jt} - W_t N_{jt} \\ &s.t.\ Y_{jt} = K_{jt}^{\alpha} N_{jt}^{1-\alpha} \end{aligned} \tag{3.2}$$

yields

$$N_{jt} = \left(\frac{1-\alpha}{W_t} \right)^{\frac{1}{\alpha}} K_{jt}, \tag{3.3}$$

which in turn implies that

$$Y_{jt} = \left(\frac{1-\alpha}{W_t} \right)^{\frac{1-\alpha}{\alpha}} K_{jt}. \tag{3.4}$$

As manifested by (3.3) and (3.4), the constant-returns-to-scale technology implies that both employment and output have a linear relationship with the capital stock, suggesting that aggregate employment and output may depend only on the aggregate capital stock. As a result, the cross-section distribution of individual capital stock, K_{jt}, does not need to be tracked. The gross operating profit for the firm also turns out to be proportional to the capital stock and is given by

$$Y_{jt} - W_t N_{jt} = R_{kt} K_{jt} = \alpha Y_{jt}, \tag{3.5}$$

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not available there.

with

$$\begin{aligned} R_{kt} &\equiv \alpha\left(\frac{1-\alpha}{W_t}\right)^{\frac{1-\alpha}{\alpha}} \\ &= \alpha\frac{Y_{jt}}{K_{jt}} = \frac{\partial Y_{jt}}{\partial K_{jt}} \end{aligned} \tag{3.6}$$

turning out to be the marginal product of capital (MPK, for short).

Firm j is assumed to be subject to an uninsurable idiosyncratic investment shock of the kind proposed by Kiyotaki and Moore (2019), $\tau_{jt} \sim B(1,\eta)$, i.e., the shock follows a Bernoulli distribution

$$\tau_{jt} = \begin{cases} 1, & \text{with Prob. } \eta \\ 0, & \text{with Prob. } 1-\eta \end{cases}, \tag{3.7}$$

which is independently and identically distributed (i.i.d.) over time and across firms and is independent of aggregate shocks, such that the firm can only have an opportunity to install I_{jt} units of new capital from I_{jt} units of the consumption goods with a constant probability η in each period. Therefore, each period only a constant measure $\eta \in (0,1)$ of firms can possibly make capital investment, with η becoming a natural index of scarcity or lumpiness of firm-level investment in this context. Given the capital depreciation rate, $(1-\lambda)$, with $\lambda \in (0,1)$, the law of motion for capital stock accumulation at the beginning of period $t+1$ for the firm is then given by

$$K_{jt+1} = \tau_{jt} I_{jt} + \lambda K_{jt}. \tag{3.8}$$

Firms face extreme financial constraints and have no access to external financing, such that they can neither borrow nor raise new equity to fund their lumpy investments.[71] This also implies that they cannot pay negative dividends, i.e.,

[71] The motive for me assuming the complete absence of external financing channel for firms is mainly for modelling convenience. There are several ways that may be used to justify this assumption in the literature. For example, there may be weak enforcement institutions such that firms cannot commit to making any future repayments of their borrowing, which then effectively prevent firms from issuing any debts (Martín and Ventura 2012); or it may be considered as an extreme case of borrowing restriction due to inalienable entrepreneurial skills

$$D_{jt} \geq 0 \tag{3.9}$$

for $\forall t, j$. Nonetheless, firms can invest in the bubbly asset at price $Q_t^B \geq 0$ at date *t*,[72] although they cannot short sell it. The firm's flow-of-funds (FOF, for short) constraint at period *t* is then given by

$$D_{jt} + \tau_{jt} I_{jt} + Q_t^B \left(Z_{jt+1} - Z_{jt} \right) = R_{kt} K_{jt}, \tag{3.10}$$

where

$$Z_{jt+1} \geq 0 \tag{3.11}$$

is the quantity purchased of the bubbly asset at the end of period *t*. The left-hand side of (3.10) is the firm's expenditure on dividends paid out, investment, and net bubbly asset purchases, while the right-hand side is the operational revenue income which is proportional to its capital stock at the start of that period.

The individual firm *j*'s objective is to maximise the expected present value of the firm:

$$V_{j0} = E_0 \sum_{t=0}^{\infty} \Lambda_{0,t} D_{jt}, \tag{3.12}$$

with $\Lambda_{0,t} = \Lambda_{0,1} \times \Lambda_{1,2} \times ... \times \Lambda_{t-1,t}$ the stochastic discount factor between period 0 and *t*, subject to a sequence of the FOF constraint (3.10), the capital accumulation equation (3.8), and the non-negativity constraints for dividends and bubble holdings, (3.9) and (3.11) respectively, for $t = 0,1,...$, given $K_{j0} > 0$ and $Z_{j0} = 1$. Let q_{jt}, γ_{jt}, and μ_{jt} denote the multipliers associated with the period *t* constraints (3.8), (3.11), and (3.9) respectively. While further details are referred to Appendix 3.A,[73] the optimisation problem of the firm yields the first order conditions for $\{I_{jt}, Z_{jt+1}, K_{jt+1}\}$, *t=0,1,2, …*

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for production in the style of Hart and Moore (1994) (see also, Hirano and Yanagawa 2017). On the other hand, it may be too costly for firms to engage in equity financing so that they choose not to issue any new equities to raise funds for investment (Miao, Wang, and Zhou 2015).

[72] The non-negativity on the bubble price comes from free disposal assumption of the asset.

[73] Inspired by Wang and Wen (2012), here I use a guess-and-verify strategy in deriving the optimality conditions associated with the firm's problem.

which take the form:

$$\left(1+\mu_{jt}\right)\tau_{jt}=\tau_{jt}q_{jt}; \tag{3.13}$$

$$\left(1+\mu_{jt}\right)Q_t^B=\gamma_{jt}+E_t\left\{\Lambda_{t,t+1}\left(1+\overline{\mu}_{t+1}\right)Q_{t+1}^B\right\}; \tag{3.14}$$

$$q_{jt}=E_t\left\{\Lambda_{t,t+1}\left[\left(1+\overline{\mu}_{t+1}\right)R_{kt+1}+\lambda\overline{q}_{t+1}\right]\right\}, \tag{3.15}$$

with

$$\mu_{jt}\geq 0, D_{jt}\geq 0 \quad and \quad \mu_{jt}D_{jt}=0, \tag{3.16}$$

$$\gamma_{jt}\geq 0, Z_{jt+1}\geq 0 \quad and \quad \gamma_{jt}Z_{jt+1}=0, \tag{3.17}$$

and

$$\overline{\mu}_t\equiv\eta\mu_{jt}^i+\left(1-\eta\right)\mu_{jt}^s, \tag{3.18}$$

$$\overline{q}_t\equiv\eta q_{jt}^i+\left(1-\eta\right)q_{jt}^s, \tag{3.19}$$

denote expected values, where the superscripts "*i*" and "*s*" indicate that for firms with and without investment ("investing" and "saving") opportunities at period *t* respectively, while the transversality conditions hold:

$$\lim_{T\to\infty}E_t\left\{\Lambda_{t,t+T}q_{t+T}K_{jt+T}\right\}=\lim_{T\to\infty}E_t\left\{\Lambda_{t,t+T}Q_{t+T}^BZ_{jt+T+1}\right\}=0. \tag{3.20}$$

Note that from (3.15), q_{jt} is independent of the individual firm's investment status, i.e., independent of the idiosyncratic investment shock. Thus,

$$q_{jt}^i=q_{jt}^s=q_{jt}=q_t=\overline{q}_t. \tag{3.21}$$

It is also noticeable that when imposing $\tau_{jt}=1$, (3.13) implies that $1+\mu_{jt}^i=q_t\geq 1$. I next consider the circumstances when q_t is equal to or greater than one sequentially.

<u>Case 1</u> Consider first the case when $q_t=1$ (for all *t*). Then from (3.13), $\mu_{jt}^i=0$, hence $D_{jt}^i\geq 0$ according to the complementary slackness condition (3.16). In this case it is immaterial to a firm whether it has an investment opportunity or not, such that the incomplete market and the financial frictions do not actually matter. Suppose

$D^s_{jt} \geq 0$ and $\mu^s_{jt} = 0$ as well. Then from (3.15) and (3.18) and given (3.21),

$$q_t = E_t\left\{\Lambda_{t,t+1}\left(R_{kt+1} + \lambda q_{t+1}\right)\right\}; \tag{3.22}$$

from (3.14),

$$Q^B_t = \gamma^{i/s}_{jt} + E_t\left\{\Lambda_{t,t+1}Q^B_{t+1}\right\}, \tag{3.23}$$

which implies that $\gamma^i_{jt} = \gamma^s_{jt} = 0$,[74] i.e., $Q^B_t = E_t\left\{\Lambda_{t,t+1}Q^B_{t+1}\right\}$. However, this suggests that in this circumstance, the equilibrium existence of the bubble is simply ruled out by the transversality condition (3.20), since

$$\lim_{T\to\infty} E_t\left\{\Lambda_{t,t+T}Q^B_{t+T}\right\} = Q^B_t = 0, \tag{3.24}$$

where the fact that $\int_0^1 Z_{jt+T+1}dj \equiv 1$ is applied. An alternative way to see this is that if the bubble were to exist, then in a steady state, $Q^B > 0$ implies that $\Lambda = \beta = 1$, as will be seen later, violating the assumption that $\beta < 1$ with β the subjective discount factor for households. Therefore, we have:

Lemma 1

If $q_t = 1$ (for all t), then in equilibrium $Q^B_t = 0$ (for all t) and firms are indifferent to the lumpy investment opportunity.

Case 2 Consider now the case when $q_t > 1$ (for all *t*). Then $\mu^i_{jt} > 0$ according to (3.13), which implies that $D^i_{jt} = 0$. Suppose $D^s_{jt} > 0$, i.e., $\mu^s_{jt} = 0$, while $Z^i_{jt+1} = 0$ and $Z^s_{jt+1} > 0$, i.e., firms with an investment opportunity decide to use all their available resources to invest in physical capital by distributing zero dividends and selling all their bubbly asset, while firms without the investment opportunity choose to do the opposite. Then from (3.10),

[74] Since the overall bubbly asset supply is assumed to be constant, it must be held by some firms for the bubble market to be cleared, so the multipliers cannot be both greater than zero according to the complementary slackness condition.

$$I_{jt} = R_{kt}K^i_{jt} + Q^B_t Z^i_{jt} . \quad (3.25)$$

Meanwhile, for investing firms with $\tau_{jt} = 1$, we have

$$1 + \mu^i_{jt} = q_t ; \quad (3.26)$$

$$\left(1 + \mu^i_{jt}\right) Q^B_t = \gamma^i_{jt} + E_t \left\{ \Lambda_{t,t+1} \left(1 + \eta \mu^i_{jt+1}\right) Q^B_{t+1} \right\} ; \quad (3.27)$$

$$q_t = E_t \left\{ \Lambda_{t,t+1} \left[\left(1 + \eta \mu^i_{jt+1}\right) R_{kt+1} + \lambda q_{t+1} \right] \right\} ; \quad (3.28)$$

for saving firms,

$$Q^B_t = E_t \left\{ \Lambda_{t,t+1} \left(1 + \eta \mu^i_{jt+1}\right) Q^B_{t+1} \right\} , \quad (3.29)$$

with (3.28) still applied. Rewrite (3.29) and (3.28) by making use of (3.26), we can then obtain the key pricing equations for the bubble and Tobin's *q*.

Lemma 2

If $q_t > 1$ (for all t), then in general equilibrium,

(1) the law of motion for the rational bubble is given by

$$Q^B_t = E_t \left\{ \Lambda_{t,t+1} \left[1 + \eta \left(q_{t+1} - 1 \right) \right] Q^B_{t+1} \right\} ; \quad (3.30)$$

(2) for Tobin's q,

$$q_t = E_t \left\{ \Lambda_{t,t+1} \left[R_{kt+1} + \lambda q_{t+1} + \eta (q_{t+1} - 1) R_{kt+1} \right] \right\} . \quad (3.31)$$

(3) The optimal decision rule for investment of a firm with an investment opportunity is given by (3.25).

The pricing equation (3.30) thus indicates that the return on the bubble consists of not only the capital gains, but also the liquidity premium that is represented by the term, $\eta\left(q_{t+1} - 1\right)$, in this situation. In other words, the rational bubble is not required to grow at the rate of interest, unlike the kind of models on rational bubbles studied in Chapter 2, because of the liquidity premium it provides: when an investment opportunity arrives in the next period with probability η, one unit of the bubble can be resold against its value Q^B_{t+1}, financing investment at a marginal cost equal to unity and generating marginal revenues measured by $q_{t+1} > 1$. Thus, contrary to Case 1 just discussed, the rational bubble cannot be ruled out by the transversality condition in the current situation. Ultimately, though, the liquidity premium generated by the bubble relies on a positive market belief about its future value.

A similar narrative applies to the required return for a unit of capital. As indicated by (3.31), i.e., the pricing equation for Tobin's *q*, a unit of capital generates R_{kt+1} units of additional funds which can then be used to finance the lumpy investment, and which in turn produces additional expected return, i.e., the "liquidity premium", measured by $\eta(q_{t+1}-1)R_{kt+1}$. These outcomes concerning the liquidity premium contained in the pricing equations for the bubble and Tobin's *q* are also consistent with those found in the relevant literature (e.g., Wang and Wen 2012, Miao and Wang 2018, Kiyotaki and Moore 2019).

Until now, it may be seen somehow intuitively the condition under which a bubble could exist in equilibrium. When the multiplier for the physical capital accumulation equation (3.8), i.e., the marginal value of possessing one unit of capital – Tobin's *q*, is greater than unity, a bubbly equilibrium can exist because in this case the bubble plays a role in transferring resources between investing and saving firms, alleviating the impact of the financial frictions induced by the lumpy investment opportunity and the absence of an external financing market. On the other hand, when Tobin's *q* is equal to one, it implies that the economy is efficient in allocating resources regardless of the occurrences of the idiosyncratic investment shock and the imperfections of the credit market. As a result, bubbles are not valued by the economic agents and the equilibrium must be bubbleless.

3.2.2 Households

Households are identical and they supply labour inelastically, with labour endowment, N_t, equal to 1. Households can trade three types of assets: riskless bonds which are in zero net supply in aggregate;[75] firms' equities; and the bubbly asset subject to the same no short selling constraint as the firms do. The net supply of each firm's shares is normalised to 1.

The optimisation problem facing the representative household is to maximise his expected lifetime utility

$$E_0 \sum_{t=0}^{\infty} \beta^t \log C_t \,, \tag{3.32}$$

subject to a sequence of budget constraints

$$C_t + B_{t+1} + \int_0^1 Z_{jt+1}^{Sh} \left(V_{jt} - D_{jt}\right) dj + Q_t^B \left(Z_{t+1}^{Bh} - Z_t^{Bh}\right) = W_t N_t + \left(1 + r_{t-1}\right) B_t + \int_0^1 Z_{jt}^{Sh} V_{jt} dj, \tag{3.33}$$

and the short selling constraint for the bubbly asset,

$$Z_{t+1}^{Bh} \geq 0 \,, \tag{3.34}$$

for $t = 0,1,\ldots$, with $\beta \equiv 1/(1+\rho) \in (0,1)$ the subjective discount factor and ρ the time preference rate, C_t the consumption for the household; B_{t+1} the value of the one-period riskless bonds purchased at the end of period *t*, with r_t the (net) risk-free rate between period *t* and *t*+1; Z_{jt+1}^{Sh} the individual firm *j*'s stock shares purchased by the household at the end of period *t*, with V_{jt} the (before dividends) market value of it; Z_{t+1}^{Bh} the quantity purchased of the bubbly asset at the end of period *t*.

The first order conditions for the household's maximisation problem are given by

$$1 = \beta E_t \left\{ \left(1 + r_t\right) \frac{C_t}{C_{t+1}} \right\}, \tag{3.35}$$

$$1 \geq \beta E_t \left\{ \frac{Q_{t+1}^B}{Q_t^B} \frac{C_t}{C_{t+1}} \right\}, \text{ with equality only when } Z_{t+1}^{Bh} > 0 \,; \tag{3.36}$$

and

[75] Since households are identical, there is no actual trading in the riskless bonds among them.in equilibrium. Thus, the introduction of the riskless bond here is purely for pricing purpose, which also facilitates discussions of the model dynamics later.

$$V_{jt} - D_{jt} = \beta E_t \left\{ \frac{C_t}{C_{t+1}} V_{jt+1} \right\}. \tag{3.37}$$

Define also $\Lambda_{t,t+1} \equiv \beta C_t / C_{t+1}$ as the stochastic discount factor between period t and t+1.

3.2.3 General Equilibrium

3.2.3.1 Competitive Equilibrium

A competitive equilibrium consists of sequences of prices $\{W_t, r_t, Q_t^B, V_{jt}\}_{t=0}^{\infty}$, and quantities $\{I_{jt}, N_{jt}, K_{jt+1}, Z_{jt+1}, Y_{jt}, Z_{jt+1}^{Sh}, Z_{t+1}^{Bh}, C_t, B_{t+1}\}_{t=0}^{\infty}$ for $j \in [0,1]$, such that:

(1) Given prices $\{W_t, Q_t^B\}_{t=0}^{\infty}$, the sequence of quantities $\{I_{jt}, N_{jt}, K_{jt+1}, Z_{jt+1}\}_{t=0}^{\infty}$ solves each individual firm *j*'s optimisation problem (3.12) subject to (3.8)-(3.11);

(2) Given prices $\{r_t, Q_t^B, V_{jt}\}_{t=0}^{\infty}$, the sequence of quantities $\{C_t, B_{t+1}, Z_{jt+1}^{Sh}, Z_{t+1}^{Bh}\}_{t=0}^{\infty}$ maximises the representative household's expected lifetime utility (3.32) subject to (3.33)-(3.34);

(3) The markets for labour, riskless bonds, the bubbly asset, individual firms' shares, and consumption goods all clear, i.e., $\int_0^1 N_{jt} dj = N_t = 1$, $B_t = 0$, $\int_0^1 Z_{jt} dj + Z_{t+1}^{Bh} = 1$, $\int_0^1 Z_{jt}^{Sh} dj = 1$ for $\forall j \in [0,1]$, and

$$C_t + \int_0^1 I_{jt} dj = \int_0^1 Y_{jt} dj. \tag{3.38}$$

3.2.3.2 Aggregation and General Equilibrium

Denote Ω^i with measure η the set of individual firms with an investment opportunity and Ω^s with measure $1-\eta$ the set of firms without an investment opportunity in each period. Also define aggregate variables: $I_t \equiv \int_0^1 I_{jt} dj = \int_{j\in\Omega^i} I_{jt} dj$, $K_t \equiv \int_0^1 K_{jt} dj$, $N_t \equiv \int_0^1 N_{jt} dj$, $Y_t \equiv \int_0^1 Y_{jt} dj$. Then the goods market clearing condition (3.38) becomes

$$C_t + I_t = Y_t. \tag{3.39}$$

In aggregation, $N_t = \left[\left(1-\alpha\right)/W_t\right]^{1/\alpha} K_t$ and $Y_t = \left[\left(1-\alpha\right)/W_t\right]^{(1-\alpha)/\alpha} K_t$ from (3.3) and (3.4) respectively, which implies that aggregate output can be expressed as a function of aggregate labour and capital stock,

$$Y_t = K_t^{\alpha} N_t^{1-\alpha} = K_t^{\alpha}, \tag{3.40}$$

where the last equality makes use of the fact that $N_t = 1$ in equilibrium. Also from (3.5),

$$W_t = \left(1-\alpha\right)\frac{Y_t}{N_t} = \left(1-\alpha\right)K_t^{\alpha} \tag{3.41}$$

and

$$R_{kt} = \alpha \frac{Y_t}{K_t} = \alpha K_t^{\alpha-1}, \tag{3.42}$$

which turns out to be the aggregate MPK as well.

Aggregating (3.8) across all firms yields the law of motion for aggregate capital stock:

$$K_{t+1} = I_t + \lambda K_t . \tag{3.43}$$

As implied by (3.30) and (3.36), in general equilibrium households will optimally choose not to hold any of the bubbly asset, i.e., $Z_{t+1}^{Bh} = 0$, even though they are allowed to trade it, because the expected return on the bubble is too low to them due to the presence of the liquidity premium. As a result, all of the bubbles are held by the firms, with $\int_0^1 Z_{jt} dj = 1$.

Given that the arrival of an investment opportunity is i.i.d. across firms and through time, those who turn out to have an investment opportunity must possess η proportion of total capital stock in the economy and of the total amount of the bubbly asset at the start of each period, i.e., $\int_{j\in\Omega^i} K_{jt}^i dj = \eta K_t$ and $\int_{j\in\Omega^i} Z_{jt}^i dj = \eta$. Therefore, for the case when Tobin's q is greater than one, aggregating (3.25) across all investing firms yields the equation for aggregate investment:

$$I_t = \eta R_{kt} K_t + \eta Q_t^B . \tag{3.44}$$

Equation (3.44) thus demonstrates that the bubble influences aggregate investment by increasing directly firms' available funds.

Finally, notice that although the pricing equation for Tobin's *q*, (3.31), is derived under the situation that $q_t > 1$, it still applies when $q_t = 1$.[76] The following proposition summarises the above analysis.

Proposition 1

Given $K_0 > 0$, the general equilibrium paths of the baseline model are characterised by

(1) if $q_t > 1$, $Q_t^B \geq 0$ for all t: nine variables, $\{C_t, I_t, Y_t, K_{t+1}, W_t, R_{kt}, Q_t^B, q_t, r_t\}$, which are fully determined by the system of nine non-linear equations, (3.30), (3.31), (3.35), (3.39), (3.40)-(3.44);

(2) if $q_t = 1$, $Q_t^B = 0$ for all t: eight variables, $\{C_t, I_t, Y_t, K_{t+1}, W_t, R_{kt}, q_t, r_t\}$, which are fully determined by the system of eight non-linear equations, (3.30), (3.31), (3.35), (3.39), (3.40)-(3.43).

3.3 Steady States

3.3.1 Multiple Steady States

From the Euler equation of the household (3.35), $1/(1+r) = \Lambda = \beta$ in a steady state (SS, for brevity). Define $q_t^B \equiv Q_t^B / K$ the bubble-capital ratio with *K* the SS value of the capital stock. Then evaluating (3.31) in a SS yields

$$R_k = \frac{(1-\lambda\beta) q}{\beta\left[(1-\eta) + \eta q\right]}, \tag{3.45}$$

combining which with the SS version of (3.42) we can always recover the SS value of

[76] As a firm without an investment opportunity always pays out dividends.

aggregate capital stock and the other aggregate variables (quantities) of interest.

We have already learnt that (from Lemma 1) when Tobin's q is equal to 1, the economy must be bubbleless, and the pricing equation for the bubble, (3.30), is invalid in that case. By setting $q=1$, we can then use (3.45) to solve for R_k in the SS:

$$R_k = R_k^* \equiv \frac{1}{\beta} - \lambda . \tag{3.46}$$

On the other hand, if the Tobin's q is greater than 1, then the economy could either be bubbleless or bubbly, as hinted by Proposition 1, because the intrinsically worthless bubbly asset can be valuable to someone only if other economic agents find it valuable. Consider first the situation when $q>1$ but $q^B=0$. Combining the SS version of the aggregate investment equation (3.44) with that of (3.43) gives

$$R_k = R_{k_l} \equiv \frac{1-\lambda}{\eta}. \tag{3.47}$$

Then equating (3.47) with (3.45) solves Tobin's q:

$$q = q_l \equiv \frac{\beta(1-\lambda)(1-\eta)}{\eta(1-\beta)}. \tag{3.48}$$

For $q>1$, it implies that the condition

$$\eta_1 > \eta (>0) \tag{3.49}$$

with $\eta_1 \equiv \beta(1-\lambda)/(1-\beta\lambda)$ must be satisfied in this case.

If instead $q^B>0$, then we can solve for Tobin's q from the pricing equation of the bubble, (3.30):

$$q = q_b \equiv 1 + \frac{1}{\eta}\left(\frac{1}{\beta} - 1\right), \tag{3.50}$$

which yields

$$R_k = R_{k_b} \equiv (1-\lambda\beta)\left[1 + \frac{1}{\eta}\left(\frac{1}{\beta} - 1\right)\right] \tag{3.51}$$

by plugging (3.50) into (3.45). We can then make use of (3.51) and the SS version of

(3.43)-(3.44) to obtain

$$q^B = q_b^B \equiv \frac{1}{\eta}\left[(1-\eta)(1-\lambda\beta) - \left(\frac{1}{\beta} - 1\right)\right]. \tag{3.52}$$

Therefore, for $q_b > 1$ and $q^B > 0$, (3.50) and (3.52) implies that for a SS to be possibly bubbly, it must satisfy

$$\eta_2 > \eta (> 0), \tag{3.53}$$

with $\eta_2 \equiv 1 - (1-\beta)/[\beta(1-\lambda\beta)]$.

Proposition 2

<u>Case 1</u> *If* $1-\lambda > \rho^2$*, the parameter space of the model can be divided into three mutually exclusive regions for* η*, in which different types of (deterministic) SS prevail:*

(1a) Region 1, $\eta \in [\eta_1, 1)$*. Only a bubbleless SS exists, with* $q^B = 0$*,* $q = q^* \equiv 1$*,* $R_k = R_k^*$*,* $K = K^*$*, which is the economy's first-best allocation outcome;*

(2a) Region 2, $\eta \in [\eta_2, \eta_1)$*. Only a bubbleless SS can exist, with* $q^B = 0$*,* $q = q_l > 1$*,* $R_k = R_{k_l}$*,* $K = K_l$*;*

(3a) Region 3, $\eta \in (0, \eta_2)$*. Two SSs coexist, one of which is bubbleless with* $q = q_l > 1$*,* $R_k = R_{k_l}$*,* $K = K_l$*, while the other is bubbly with* $q^B = q_b^B > 0$*,* $q = q_b > 1$*,* $R_k = R_{k_b}$*,* $K = K_b$*. In addition,* $q_l > q_b > q^* = 1$*,* $R_{k_l} > R_{k_b} > R_k^*$*, and* $K^* > K_b > K_l$*.*

<u>Case 2</u> *If* $1-\lambda \le \rho^2$*, the parameter space of the model is then divided into two mutually exclusive regions in neither of which a bubbly SS is possible:*

(1b) Region 1, $\eta \in [\eta_1, 1)$*. Only the first-best SS exists;*

(2b) Region 2, $\eta \in (0, \eta_1)$*. Only a bubbleless SS can exist, with* $q = q_l > 1$*,* $R_k = R_{k_l}$*,* $K = K_l$*, where* $R_{k_l} > R_k^*$ *and* $K^* > K_l$*.*

Proof: see Appendix 3.B.

Figure 3.1 also provides a graphical illustration of the regions of parameter space with their associated SS.

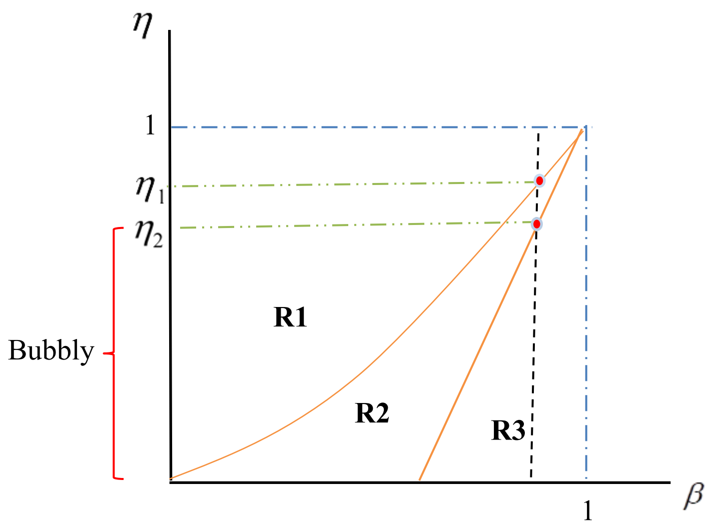

Figure 3.1 Regions of parameter space

Note: R1: efficient and bubbleless; R2: inefficient and bubbleless; R3: inefficient and (possibly) bubbly (2 SSs).

3.3.2 The Arrival Rate of An Investment Opportunity and the Bubbly SS

Proposition 2 thus reveals the crucial role played by the arrival rate of an investment opportunity in admitting the existence of a bubbly equilibrium in the present framework: only when η is sufficiently low can a bubble emerge in equilibrium, i.e., when η lies in the region 3 of the parameter space in case 1. If instead, η is high enough so that it lies within the region 1 of either case 1 or 2, then the economy could reach the first-best allocation outcome by itself, even with the presence of the financial frictions.

To comprehend the economic narrative underneath and to see the general equilibrium effects of the key ingredient of the present model, the idiosyncratic investment shock, consider first the hypothetical situation when $\eta = 1$, i.e., when the exogenously imposed investment barrier for firms is completely lifted. In that case, each firm would invest to the point where the value of a unit of the capital equals the cost of acquiring it at the optimum (around a SS),[77] which is reflected as the marginal q equals one, i.e., $q^* = 1$ in the present context. For reference, also denote the corresponding optimal investment level as I_t^*.

This first-best outcome can still be carried out so long as η is not too low, when firms who are granted with an investment opportunity possess the major proportion of the overall resources of the economy and are able to install new capital by the optimal level I_t^*. To be more specific, recall that overall internal funds for all investing firms are given by $\eta R_{kt} K_t$ in some period t; it is then possible that given a high arrival rate of an investment opportunity, $\eta R_{kt} K_t - I_t^* = D_t^i \geq 0$. In that circumstance, it is just that the amount of investment which should have been carried out by the saving firms is now undertaken entirely by the investing firms as well.[78] Therefore, the rational bubble cannot exist in equilibrium as it provides no function in allocating resources among individual firms.

At the other end of the spectrum, if an investment opportunity to a firm turns out to be terribly rare, such that the desirable amount of investment to the economy as a whole could not be completely carried out by the firms that have an investment opportunity, i.e., $I_t = \eta R_{kt} K_t < I_t^*$, since there is the non-negativity constrain on dividends paid out by firms, then the rational bubble could be valuable, as it serves as a financing vehicle of transferring available funds from saving to investing firms. As implied by (3.44), with the present of the bubble, the actual level of investment would then be boosted by ηQ_t^B, improving the efficiency of the economy as a consequence.

Note, however, there is an additional scenario in the middle, where the economy is

[77] The emphasis of "around a SS" comes from the fact that in the present context firms cannot pay out negative dividends, while in a standard setup, there is equivalently no restriction on distributing "negative dividends". Put this in another way, if the economy starts from a status with an extremely low level of capital stock, then due to the non-negativity constraint on dividends, firms would underinvest, resulting in the marginal q being greater than one.

[78] Consequently, the saving firms are now paying out more dividends than if the investment shock does not present.

inefficient in allocating resources but where the bubble cannot emerge in equilibrium, i.e., when η lies in the region 2 of case 1 or 2 in Proposition 2. In fact, even in the situation just discussed, the bubbly economy remains to be less efficient relative to the first-best outcome, indicated by $q_b > q^*$. This may seem peculiar, as we may expect that the bubble could always play a role in transporting investing funds among firms to the extent that the economy would finally reach the most efficient status.[79] Although this is not an exceptional novel feature for the model, to my best knowledge, it has not yet been properly addressed by relevant studies on rational bubbles with credit constraints.[80] The simple present framework, however, equips me with the potential to make a precise investigation into what may be the contributory factors for this phenomenon, as explored next.

To begin with, recall that from the pricing equation of the bubble, (3.30),

$$1 = E_t \left\{ \Lambda_{t,t+1} \left(1 + LP_{t+1}\right) R_t^B \right\}, \tag{3.54}$$

with $LP_{t+1} \equiv \eta\left(q_{t+1} - 1\right)$ denoting the liquidity premium and $R_t^B \equiv Q_{t+1}^B / Q_t^B$ the capital gain of purchasing the bubble. In a SS and given that the total amount of the bubbly asset outstanding remains constant, it must be that $R^B = 1$ for any $Q^B > 0$.[81] This immediately imposes a restriction on the possible level of the liquidity premium: to ensure that the subjective discount factor of the household satisfies the assumption that it is strictly less than one, LP must be larger than one, i.e., it must be the case that $q > q^*$ in SS. In other words, given the rational nature of the bubble, the economy must be inefficient for it to be possibly bubbly. Indeed, this conclusion just echoes that drawn

[79] As this is the case for models of rational bubbles emerging from dynamic inefficiency: in that strand of models, a bubbly equilibrium could be the most efficient.

[80] For instance, Miao and Wang (2018) and Kiyotaki and Moore (2019) also recognise the existence of this additional region of parameter space, albeit neither of them make analysis or explanation for it.

[81] If the overall supply of the bubbly asset varies overtime, e.g., part of it being destroyed, it is then not necessary for the capital gain on the bubble to be equal to one. A concrete example is provided by Galí (2014) where individual bubbles depreciate at a given rate each period.

in Lemma 1.

However, this alone does not paint the full picture for the existence of the region 2 of case 1 or 2 in Proposition 2. To explain why inefficiency of the economy is a necessary but insufficient condition for the equilibrium existence of a bubble, we may need to trace back to the investment equation (3.44) for an inefficient environment: aggregate investment equals the depreciation of capital in a SS, with $I=\left(1-\lambda\right)K$, which is financed by internal funds accumulated by all investing firms, $\eta R_k K$, and by selling bubbles. If the latter were to be valued positively, then according to (3.54), it is not difficult to show that given a time preference rate of households, the higher the arrival rate of an investment opportunity, the higher fraction of available internal resources would be for keeping the aggregate capital stock at its SS level, i.e., $\frac{\partial\left(\pi R_{k_b}\right)}{\partial\pi}=1-\lambda\beta>0$, so that for some $\eta\geq\eta^b\equiv1-\frac{1-\beta}{\beta(1-\lambda\beta)}$, $\eta R_{k_b}K_b\geq\left(1-\lambda\right)K_b$, which in turn implies a negative bubble but which is illegitimate and contradicts with the presumption made at the start.

Therefore, if it turns out that an investment opportunity in the economy is not that scarce, such that although internal finance of investing firms is not enough to carry out the first-best investment level, it is always sufficient to replenish the capital stock in (the neighbourhood of) a SS, then a bubble must have no value. Put this in another way, it is the special property of rational bubbles which locks down a unique SS level of liquidity premium in the present framework that in turn requires the economy to be inefficient enough for the intrinsically worthless bubbles to be beneficial and valued by infinitely-lived rational agents in generating extra funds for lumpy investments. Similar reasoning applies to explaining the non-existence of bubbles of case 2 in Proposition 2, since when $1-\lambda<\rho^2$, the required investment spending is even easier to fulfil by internal funds in and around a SS while the implied cost of holding the bubble is relatively too high, thus leaving no chance for the rational bubble to emerge around a

SS even for the smallest possible value of η.

Lemma 3

For a deterministic bubbly SS to exist in the present model, it requires that

$$1-\lambda > \rho^2 \quad and \quad \eta \in \left(0, 1-\frac{1-\beta}{\beta\left(1-\lambda\beta\right)}\right).$$

3.3.3 An Extension: Multiple Bubbly Assets

Although the conclusions drawn from the above discussions pertaining to the division of parameter space and the condition for the existence of a bubbly equilibrium are made in the context of a single type of bubbly asset, it still holds even in an environment with multiple bubbly assets, as which is the type of setup in Miao, Wang, and Xu (2015), Dong et al. (2020), and Ikeda (2021) among the relevant literature on rational bubbles emerging from credit constraints. I thus dedicate this subsection to demonstrate that in a formal way.

While other assumptions of the present model remain unchanged, assume instead now there is a series of intrinsically worthless bubbly assets distinguished by the date in which they originate, where $Q^B_{t|t-k} \geq 0$ is used to denote the traded price at time *t* for the bubbly asset introduced in period $t-k$ for $k = 0,1,2,...$ Particularly, each firm is endowed at the start of each period with $\delta \in [0,1)$ units of a new bubbly asset emerging from that period, whose traded price is $Q^B_{t|t}$. The overall value of the new bubble is thus given by $U_t \equiv \delta Q^B_{t|t}$. Meanwhile, a fraction δ of each vintage of those pre-existing bubbles is assumed to lose its value for whatever reason. As a consequence, the total amount of bubbly assets outstanding (the new plus the old ones) remains constant and equal to one through time, and the aggregate value of all these bubbly assets at some date t is denoted as $Q^B_t \equiv U_t + B_t = \sum_{k=0}^{\infty} \delta\left(1-\delta\right)^k Q^B_{t|t-k}$, with $B_t \equiv \sum_{k=1}^{\infty} \delta\left(1-\delta\right)^k Q^B_{t|t-k}$ the overall value of old bubbles.

It can be shown that from an individual firm *j*'s optimisation problem,

$$Q^B_{t|t-k} = (1-\delta) E_t \left\{ \Lambda_{t,t+1} \left[1+\eta\left(q_{t+1}-1\right) \right] Q^B_{t+1|t-k} \right\} \tag{3.55}$$

for $k = 0,1,2,\ldots$. After some manipulations and by making use of the definitions for the aggregate value of all and old bubble assets, we then have

$$Q^B_t = E_t \left\{ \Lambda_{t,t+1} \left[1+\eta\left(q_{t+1}-1\right) \right] B_{t+1} \right\}, \tag{3.56}$$

while the one for Tobin's q, (3.31), remains valid.

In a SS, if $q^B \equiv Q^B/K > 0$ and $b \equiv B/K > 0$ with B the SS value of the old bubbles, then evaluating (3.56) at a SS, (a superscript "s" is now used to indicate the corresponding value in the environment featuring a series of bubbly assets in this subsection)

$$q^s_b = 1 + \frac{1}{\eta}\left(\frac{q^B}{b\beta} - 1 \right) \tag{3.57}$$

which is at least as high as Tobin's q in (3.50) when there is only one type of bubbly asset given other things equal, for $q^B/b \geq 1$. This by itself hints that the economy with various bubbly assets must be more inefficient than if with only one type of them for it to be bubbly, since it is now more costly for the rational agents to hold the bubbles whose values depreciate over time.

From the same aggregate investment equation (3.44), we have

$$q^B = \frac{1-\lambda}{\eta} - R^s_{k_b} = \frac{1-\lambda}{\eta} - (1-\lambda\beta)\left[\left(\frac{1}{\beta} - \frac{b}{q^B} \right)\frac{1}{\eta} + \frac{b}{q^B} \right]. \tag{3.58}$$

Thus, for $q^B > 0$, it is required that

$$1 - \frac{q^B(1-\beta)}{b\left[\beta(1-\lambda\beta)\right]} > \eta (> 0), \tag{3.59}$$

i.e.,

$$\frac{(1-\eta)\beta(1-\lambda\beta)}{1-\beta} > \frac{q^B}{b} (\geq 1). \tag{3.60}$$

Comparing (3.59) with its counterpart of a single bubbly asset environment, (3.53), it is then evident that the set of values of the arrival rate of an investment opportunity within which bubbles could emerge shrinks as the ratio of the aggregate bubble to the old bubble increases. This may be intuitive, as asserted in the last paragraph, since now holding the bubbly assets is less attractive for firms relative to the case with only one type of bubble and thus requires a severer degree of inefficiency of the economy for the bubbles to be beneficial in terms of reallocating resources, which corresponds to a lower η value.

Also, it follows from (3.60) that $1-(1-\beta)/[\beta(1-\lambda\beta)]\equiv\eta_2>\eta(\geq 0)$ for a non-empty set $[1,(1-\eta)\beta(1-\lambda\beta)/(1-\beta))$ of values for the aggregate-old bubble ratio, coinciding exactly the condition (3.53). If this is satisfied, then given an arrival rate of an investment opportunity, there exists a continuum of bubbly SSs indexed by $q^B\in(0,q_b^B]$ with $q_b^B\equiv\frac{1}{\eta}\left[(1-\eta)(1-\lambda\beta)-\left(\frac{1}{\beta}-1\right)\right]$.

On the other hand, since the pricing equation for Tobin's q and the aggregate investment equation are unaffected by the modification of the multiple bubbly assets to the model, the condition for the existence of an inefficient but bubbleless SS remains unchanged, i.e., (3.49) still applies in the present context. Therefore, we are now in the position to claim that:

Proposition 3

Proposition 2 remains valid to the modified model where there is a series of intrinsically worthless bubbly assets, except that for the region 3 of case 1, in addition to the bubbleless SS, there is now a continuum of bubbly SSs with the aggregate bubble-capital ratio ranging from $(0,q_b^B]$, *with* $q_l>q_b^s\geq q_b>q^*$, $R_{k_l}>R_{k_b}^s\geq R_{k_b}>R_k^*$, *and* $K^*>K_b\geq K_b^s>K_l$.

Proof: See Appendix 3.C.

Therefore, what has been discussed in this section provides further support to the arguments made in Section 3.3.2, as well as indicates that the generic conclusions drawn from the present simple framework should be robust to a wider range of relevant models on rational bubbles of this strand with more sophisticated settings.

3.4 Equilibrium Dynamics

In this section, I shift the focus to the analysis of the equilibrium dynamics of the model economy in a neighbourhood of a given SS characterised in Proposition 2. To make progress in this direction, I begin by log-linearising the equilibrium conditions of the model around a SS and then analyse the resulting difference equations system in different circumstances. Throughout the analysis, $\hat{q}_t^B \equiv q_t^B - q^B$ denotes the deviation of the aggregate bubble-capital ratio from its SS value and $\hat{r}_t \equiv \log\left[(1+r_t)/(1+r)\right]$ is the log deviation of the gross real interest rate; for the remining variables of interest, lowercase letters are used to define the log of the original variable while the "^" symbol on top of a variable are used to indicate the deviation from its SS value (except that $\hat{q}_t \equiv \log(q_t/q)$ denotes log deviation of Tobin's q from its SS value). When it is needed, I also continue to use subscript "b" to distinguish a SS value of a variable in a bubbly case, and "l" for that in a bubbleless but inefficient case, and to use the "*" symbol at the top right corner of a SS value of a variable to denote that in the first-best situation.

The resulting log-linearised equilibrium conditions consist of: for the bubble pricing equation (3.30),

$$\hat{q}_t^B = \frac{(1-\eta)(1-\lambda\beta)\beta}{1-\beta} E_t \hat{q}_{t+1}^B \tag{3.61}$$

for $q^B = 0$ with $\hat{q}_t^B \geq 0$ for $\forall t$,[82] and

$$\hat{q}_t^B = E_t \hat{q}_{t+1}^B + \eta\beta q_b q_b^B E_t \hat{q}_{t+1} - q_b^B \hat{r}_t \tag{3.62}$$

for $q^B = q_b^B > 0$; for the pricing equation for Tobin's q (3.31),

[82] Because the rational bubble price is restricted to be nonnegative by free disposal, it is illegitimate to have a negative bubble deviation relative to the bubbleless SS.

$$\hat{q}_t = \beta(\eta R_k + \lambda)E_t\hat{q}_{t+1} + (1-\lambda\beta)E_t\hat{r}_{kt+1} - \hat{r}_t ; \tag{3.63}$$

for the Euler equation of the household (3.35),

$$\hat{c}_t = E_t\hat{c}_{t+1} - \hat{r}_t ; \tag{3.64}$$

for the goods market clearing condition (3.39),

$$\hat{y}_t = \frac{C}{Y}\hat{c}_t + \frac{I}{Y}\hat{i}_t ; \tag{3.65}$$

for the aggregate production function (3.40),

$$\hat{y}_t = \alpha\hat{k}_t ; \tag{3.66}$$

for the marginal product of capital (3.42),

$$\hat{r}_{kt} = \hat{y}_t - \hat{k}_t = \left(1 - \frac{1}{\alpha}\right)\hat{y}_t ; \tag{3.67}$$

for the capital accumulation equation (3.43),

$$(1-\lambda)\hat{i}_t = \hat{k}_{t+1} - \lambda\hat{k}_t ; \tag{3.68}$$

for the aggregate investment equation (3.44),

$$(1-\lambda)\hat{i}_t = \eta R_k(\hat{r}_{kt} + \hat{k}_t) + \eta\hat{q}_t^B . \tag{3.69}$$

3.4.1 Bubbleless Economy

From Proposition 2, we have learnt that if the arrival rate of an investment opportunity falls beyond region 3 of case 1, then the system must be bubbleless around a SS. On the other hand, even when it is possible for a bubble to emerge, the economic agents may happen to have no faith on it throughout the evolution course of the economy. I first show the local dynamics around an efficient SS as a benchmark for comparison, following which inefficient but bubbleless scenarios are analysed.

3.4.1.1 Region 1 of Case 1&2: Efficient and Bubbleless

If an economy is characterised by parameters of region 1 of case 1 or 2 in Proposition 2, then it remains efficient around the SS, i.e., $q_t = 1$ which implies $\hat{q}_t = 0$ (for all *t*). As a result, the log-linearised version of the law of motion of Tobin's *q*, (3.63), becomes

$$\hat{r}_t = (1-\lambda\beta)E_t\hat{r}_{kt+1} = (1-\lambda\beta)(\alpha-1)\hat{k}_{t+1}. \tag{3.70}$$

Thus, (3.70) predicts that, as in line with the standard Ramsey model, a higher level of capital accumulation implies a lower real rate of interest.

By combining (3.70) with (3.64)-(3.66)&(3.68), the local dynamics of the system around the efficient SS could then be represented by

$$\begin{bmatrix} 1 & 0 \\ (1-\lambda\beta)/(1-\alpha) & 1 \end{bmatrix}\begin{bmatrix} \hat{k}_{t+1} \\ E_t\hat{c}_{t+1} \end{bmatrix} = \begin{bmatrix} \lambda+(1-\lambda)\alpha Y^*/I^* & -(1-\lambda)C^*/I^* \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix}, \tag{3.71}$$

which has only one stable eigenvalue corresponding to the state variable $\hat{k}_t$.[83]

Note that the local dynamics of the system in this case is independent of η, because intuitively, the constraint of the lumpy investment opportunity does not actually matter around the SS in this case, as far as achieving the efficient resources allocation of the economy is concerned, as also discussed in Section 3.3.2 above. From its derivation, note that (3.71) actually applies to not only an economy of region 1, but also circumstances with any $\eta \in (0,1)$ so long as the economy starts from having $q_t = 1$ as well.

3.4.1.2 Region 2 of Case 1&2 and Region 3 of Case 1: Inefficient but Bubbleless

If the economy is characterised by a set of parameters of region 2 of case 1 or 2 or of region 3 of case 1 but remaining being bubbleless throughout the course of its evolution in Proposition 2, then with $q_t > 1$, the aggregate investment equation (3.69) applies with $R_k = R_{k_l}$. Combining (3.66)-(3.69) yields

$$\hat{k}_{t+1} = \left[\lambda + \alpha(1-\lambda)\right]\hat{k}_t, \tag{3.72}$$

after imposing $\hat{q}_t^B = 0$.

The law of motion of Tobin's q, (3.63), could now be expressed as

$$-(1-\lambda\beta)(1-\alpha)\hat{k}_{t+1} + \beta E_t\hat{q}_{t+1} = \hat{q}_t + \hat{r}_t, \tag{3.73}$$

83 This is found to be true when studied numerically for plausible calibrated values for the model parameters.

while

$$\alpha \hat{k}_{t+1} = \alpha \hat{k}_t + \hat{r}_t, \tag{3.74}$$

which is obtained by combining (3.64)-(3.67)&(3.69) and imposing $\hat{q}_t^B = 0$. Then from (3.72)-(3.74) and after some manipulation, the equilibrium dynamics of the system in the neighbourhood of an inefficient but bubbleless SS could be jointly described by

$$\begin{bmatrix} \hat{k}_{t+1} \\ E_t \hat{q}_{t+1} \end{bmatrix} = \begin{bmatrix} \lambda + \alpha(1-\lambda) & 0 \\ \dfrac{\alpha + [1-(1-\alpha)\lambda\beta][\lambda + \alpha(1-\lambda)]}{\beta} & \dfrac{1}{\beta} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{q}_t \end{bmatrix}. \tag{3.75}$$

Note that the local dynamics in this case is again independent of the parameter η. This is because up to a first order approximation, the total amount of investment around the inefficient but bubbleless SS is fixed by the depreciation of aggregate capital stock and is funded entirely by internal funds possessed by investing firms.

It turns out that one of the two eigenvalues of the system is $\lambda_1 = \lambda + \alpha(1-\lambda) \in (0,1)$, while the other is $\lambda_2 = 1/\beta > 1$, so that the Blanchard-Kahn condition for determinacy of rational expectation solutions is satisfied, given that $\hat{k}_t$ is the only predetermined variable. If assume further that the economy starts at some point $\hat{k}_0$ away from the SS, the time path for Tobin's q is then given by

$$\hat{\eta}_t = \Omega_1 (\lambda_1)^t \hat{k}_0, \tag{3.76}$$

with $\Omega_1 \equiv \dfrac{[1-(1-\alpha)\lambda\beta][\lambda + \alpha(1-\lambda)] + \alpha}{[\beta(\lambda + \alpha(1-\lambda)) - 1]} < 0$, implying that Tobin's q comoves negatively with the capital stock. Therefore, it appears that if given a "large" enough $\hat{k}_0$, it may be possible that $q_0 = 1$, i.e., the economy may be efficient at that specific point even though the corresponding SS of these regions of parameter space are inefficient.

3.4.2 Bubbly Economy

If the economy is characterised by parameters of region 3 of case 1 in Proposition 2, then it is inefficient but now possible for it to be bubbly near either the bubbleless or the bubbly SS. I analyse the local dynamics of both of these scenarios separately in this section.

3.4.2.1 Around the Bubbleless SS

When the economy features $\eta \in (0,\eta_2)$ and fluctuates around $q^B = 0$, it is noticeable by (3.61) that to a first-order approximation, the expected evolution of the bubble-capital ratio is autonomous and unaffected by changes in the real interest rate or Tobin's *q*. Note also that $\eta \in \left(0,\eta_2\right)$ equivalently implies that $\frac{1-\beta}{\left(1-\eta\right)\left(1-\lambda\beta\right)\beta} \in (0,1)$. This in turn implies that there exist bounded rational expectation solutions to (3.61) other than $\hat{q}_t^B = 0$,[84] which take the form:

$$\hat{q}_{t+1}^B = \frac{1-\beta}{\left(1-\eta\right)\left(1-\lambda\beta\right)\beta}\hat{q}_t^B + \varepsilon_{t+1}, \tag{3.77}$$

where $\hat{q}_t^B = q_t^B = Q_t^B / K_l$, while $\varepsilon_{t+1} \equiv \hat{q}_{t+1}^B - E_t \hat{q}_{t+1}^B$ is thought of as the aggregate bubble innovation (the "bubble shock") in the context and as capturing actual speculative bubble episodes where mood swings in investor communities drive up or down the price for an asset on the sheer basis of expectations of future adjustments in the price of the same asset, regardless of the absence of any news concerning its fundamentals.[85]

Meanwhile, from (3.66)-(3.69), we can obtain the equilibrium condition for output:

$$\hat{y}_{t+1} = \left[\lambda + \alpha(1-\lambda)\right]\hat{y}_t + \alpha\eta\hat{q}_t^B. \tag{3.78}$$

[84] This outcome thus also confirms that if the economy lies within the region 2 in Proposition 2, equilibria in a local area of the necessarily bubbleless SS must always be bubbleless as well, since when $\eta \in [\eta_2, \eta_1)$, $\left(1-\beta\right)/\left[\left(1-\eta\right)\left(1-\lambda\beta\right)\beta\right] > 1$. In other words, there does not exist a continuum of bubbly equilibria converging to the bubbleless SS in the region 2 economy.

[85] Galí (2021) considers similar type of bubble shocks.

The local dynamics of $\left(\hat{y}_t, \hat{q}_t^B\right)$ could then be jointly represented by

$$\begin{bmatrix} \hat{y}_{t+1} \\ E_t \hat{q}_{t+1}^B \end{bmatrix} = \begin{bmatrix} \lambda + \alpha(1-\lambda) & \alpha\eta \\ 0 & (1-\beta)/[(1-\eta)(1-\lambda\beta)\beta] \end{bmatrix} \begin{bmatrix} \hat{y}_t \\ \hat{q}_t^B \end{bmatrix}, \tag{3.79}$$

with two eigenvalues $\lambda_{b1} \equiv \lambda + \alpha(1-\lambda)$ and $\lambda_{b2} \equiv (1-\beta)/[(1-\eta)(1-\lambda\beta)\beta]$, both of which lie within the unit circle. This implies that the Blanchard-Kahn condition for determinacy of rational expectation solutions actually fails in this case, since the aggregate bubble-capital ratio is non-predetermined and should hence correspond to an unstable eigenvalue.[86] In other words, as has already been pointed out, the system in this case is always subject to bounded bubble(-driven) fluctuations. This outcome may be somehow intuitive, given that the bubble is desirable in improving the economy's efficiency within this region but just depends on the "vulnerable" collective faith of the rational agents on it. This type of bubble-driven fluctuations may be considered as a plausible representation of recurrent booms and busts episodes of aggregate bubbles that occur in the real economic world, with the bubbleless status being a resting point.

Let us assume that the system originally sits at its SS, but is hit by a "small" bubble shock, $\hat{q}_0^B = \varepsilon_0 > 0$ at some time $t = 0$, which lasts only for one period. The time paths for output and the aggregate bubble-capital ratio are then given by

$$\hat{y}_t = \alpha\eta \left(\sum_{i=1}^{t} \left(\lambda_{b1}\right)^{t-i} \left(\lambda_{b2}\right)^{i-1} \right) \hat{q}_0^B \tag{3.80}$$

and

$$\hat{q}_t^B = \left(\lambda_{b2}\right)^t \hat{q}_0^B, \tag{3.81}$$

respectively.[87] Noticeably, unlike the bubbleless cases discussed in the last section, now the evolutions of both the bubble and output are affected by η: the higher the arrival rate of an investment opportunity, the more persistent the bubble and the resulting output fluctuations are, as well as the larger is the initial bubble impact on output (which may be interpreted as the "*bubble multiplier*"), which is measured by

[86] Note that output is predetermined by the capital stock as labour supply is inelastic in the present model.

[87] Detailed derivations are referred to Appendix 3.D.

$\alpha\eta$.[88]

The explanation for this phenomenon concerning the persistence respect is twofold. First and foremost, the fact that a transitory bubble shock can generate prolonged fluctuations of the system is a consequence of the equilibrium requirement of the rational bubble that its capital gain matches the rate of interest (after deducting a liquidity premium), so that the initial bubble shock has an impact on the size of the bubble over time and then output as well. Second, a higher value of η implies a lower degree of inefficiency of the economy in equilibrium, given other things equal, which in turn implies a lower liquidity premium contained in the overall return on the bubble.[89] As a result, the bubble price must "grow" at a higher rate in order to be attractive enough in equilibrium alongside a higher arrival rate of an investment opportunity in the context,[90] which contributes further to the persistence of the bubble fluctuation.

On the other hand, the bigger bubble multiplier on output given a higher value of η comes from the fact that the proportion of the bubbly asset that sold for investments is then higher, as indicated clearly by the aggregate investment equation (3.69), hence the same size of a bubble boom would induce a stronger boost on investment if the arrival rate of an investment opportunity is higher, resulting in a higher level of capital stock and thus output (one period later) on impact.

The main result of this section can hence be summarised as below:

Proposition 4

(1) If $1-\lambda > \rho^2$ *and* $\eta \in (0, \eta_2)$*, then the bubbleless SS is locally stable, i.e., bounded rational bubble fluctuations are now possible. This implies that bounded fluctuations in the price of the bubbly asset, driven purely by random jumps in expectations, are possible; and that these cause persistent fluctuations in output by affecting the*

[88] Since output is predetermined, the impact of the bubble shock on output lags one period.

[89] This can be easily shown formally.

[90] In this specific context with asymptotically bubbleless equilibria, the size of the bubble actually declines over time. Therefore, it may be more accurate to say that the bubble price must decline at a lower rate with a higher value of the arrival rate of an investment opportunity.

availability of finance for investment and thus affecting the capital stock;

(2) The degree of persistence of the bubble fluctuations is endogenously determined by the stable eigenvalue λ_{b2} *. Other things being equal, the higher the arrival rate of an investment opportunity, the more persistent the bounded bubble(-driven) fluctuations and the larger the bubble multiplier on output.*

A Numerical Example To provide a numerical illustration of the local dynamics near the bubbleless SS in the face of a transitory bubble shock, I adopt plausible calibrated values at quarterly frequency which are consistent with the relevant literature for the model parameters, with $\beta = 0.99, \alpha = 0.4, \lambda = 0.97$. By setting $\eta = 0.05$, we have $\lambda_{b1} = 0.982, \lambda_{b2} = 0.268$, and the impulse response of the model around the bubbleless SS to a one-period bubble shock with $\hat{q}_0^B = \varepsilon_0 = 1\%$ is depicted in Figure 3.2.

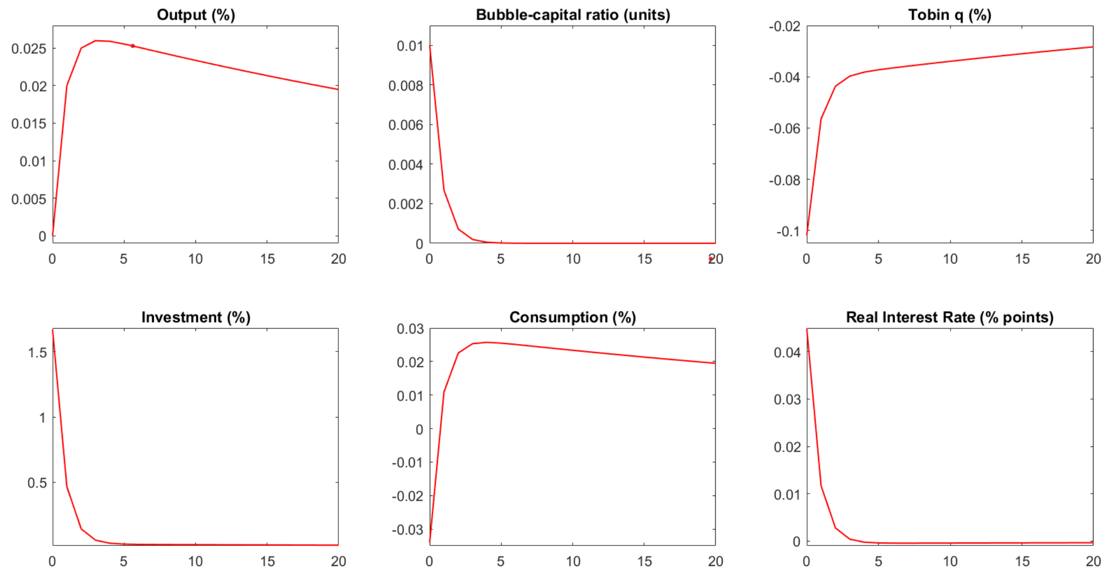

Figure 3.2 Impulse responses of the model to 1% units of positive bubble shock

Under flexible prices, the bubble shock affects the economy through a supply side mechanism. Specifically, investment, capital, and output increase in the face of the bubble shock, as the investing firms are now acquiring more funds from selling the bubble; Tobin's q declines on impact as the marginal product of capital decreases because of a rise in the future capital stock. On the other hand, consumption is pressed down at the early stage of the bubble shock, since the increase in aggregate output does not catch up with the higher goods' demand for investment;[91] as a result, the real interest rate needs to go up to clear the bond market.[92] As the time evolves, investment declines very quickly as the bubble shrinks at a fast speed of λ_{b2} in this example, while output builds up but just declines gradually (reflected by the high λ_{b1}),[93] with consumption picking up and the real interest rate declining accordingly.

3.4.2.2 Around the Bubbly SS

The equilibrium dynamics of the model around a bubbly SS could be jointly described by (3.82)-(3.84):

$$\hat{y}_{t+1} = \left(\lambda + \alpha\eta R_{k_b}\right)\hat{y}_t + \alpha\eta\hat{q}_t^B; \tag{3.82}$$

$$\frac{(1-\lambda\beta)(\alpha-1)}{\alpha}\hat{y}_{t+1} - \frac{1}{q^B}E_t\hat{q}_{t+1}^B + \beta\left(\eta R_{k_b} + \lambda - \eta q_b\right)E_t\hat{q}_{t+1} = -\frac{1}{q^B}\hat{q}_t^B + \hat{q}_t; \tag{3.83}$$

$$\begin{aligned}&\left(1-\alpha\eta\right)\hat{y}_{t+1} - \left(\frac{I_b\eta}{Y_b(1-\lambda)} + \frac{C_b}{Y_b q^B}\right)E_t\hat{q}_{t+1}^B - \left(\frac{C_b}{Y_b}\eta\beta q_b\right)E_t\hat{q}_{t+1}\\ &= \left(1-\alpha\eta\right)\hat{y}_t - \left(\frac{I_b\eta}{Y_b(1-\lambda)} + \frac{C_b}{Y_b q^B}\right)\hat{q}_t^B\end{aligned} \tag{3.84}$$

where (3.82) is derived by combining (3.66)-(3.69), (3.83) is obtained by combining (3.62)-(3.63)&(3.67), and (3.84) is derived by making use of (3.62)&(3.64)-(3.65), with $q_b = 1 + \frac{1}{\eta}\left(\frac{1}{\beta} - 1\right), R_{K_b} = (1-\lambda\beta)q_b, q^B = (1-\lambda)/\eta - R_{K_b}$ and $\hat{q}_t^B = q_t^B - q^B$.

In numerical simulations with a wide range of theoretically valid calibrated values for the model parameters, it turns out that the system now always has one real and stable eigenvalue but two conjugate complex unstable eigenvalues, implying that bounded bubble fluctuations cannot occur around a bubbly SS.[94] Nonetheless, as an example,

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[91] This is also true when labour supply is elastic, i.e., even when output could adjust on impact.

[92] This contributes further to the declining Tobin's q on impact.

[93] In this numerical experiment, output peaks at the fourth period (if the impact period is considered as the first period).

[94] Because in this case the Blanchard-Kahn condition for the determinacy of rational expectation solutions to the

given the same set of calibrated parameters as in Section 3.4.2.1 and assume that $\hat{k}_0 = 1/\alpha\%$ at the initial period, then the local dynamics of the model is shown by Figure 3.3.

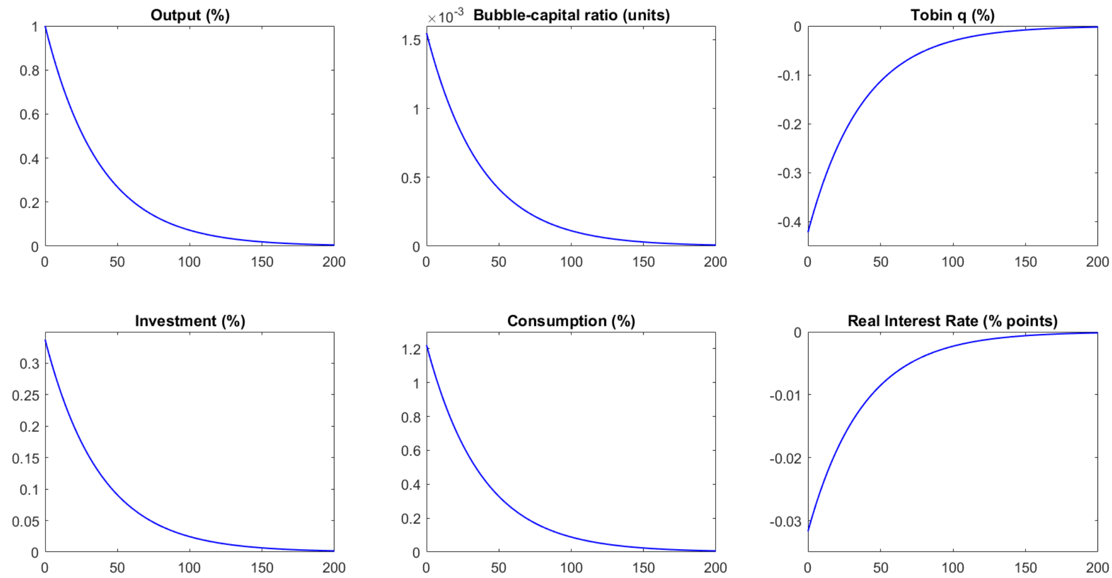

Figure 3.3 Local dynamics of the model given $\hat{k}_0 = 1/\alpha\%$

It is evident from Figure 3.3 that when the capital stock (output) is larger, the degree of inefficiency of the system is lower (indicated by a lower Tobin's q), since in that case the fraction of internal resources available for capital investment, $\eta R_{kt} K_t = \eta \alpha K_t^{\alpha}$, is larger given a higher level of capital stock. Consumption also tends to be higher than the SS level as a consequence of the temporarily stronger production capacity, implying a lower equilibrium rate of interest in order for the goods market to be cleared.

Note that unlike the reasoning in the case (in Section 3.4.2.1) when the economy fluctuates around a bubbleless SS and where the bubble is treated as a "predetermined" variable given a bubble shock, here the economic system is determinate around a bubbly SS, so that the bubble is "determinate" and should now be treated as a non-predetermined variable. The net effect of a lower required liquidity premium (induced

system is satisfied, given that only output is predetermined while the bubble (ratio) and Tobin's q are non-predetermined.

by a lower Tobin's q) and the lower rate of interest (i.e., a higher stochastic discount factor) on the *current* size of the bubble then turns out to be positive (in other words, the positive effect of the lower interest rate dominates the negative effect of the lower Tobin's q on the current size of the bubble, for a given (expectation of) future size of the bubble), i.e., the aggregate size of the bubble would be larger if the model economy starts at a larger than the SS capital stock level.

3.5 Concluding Remarks

The present chapter is conducted as the primary step of my effort to analysing monetary policy implications in an environment where rational bubbles may emerge because of the existence of financing constraints and incomplete financial markets.

To build up the foundations, I have established an analytically tractable infinite-horizon dynamic general equilibrium framework in which individual firms are subject to a type of uninsurable idiosyncratic investment shock a là Kiyotaki and Moore (2019) and are financially constrained. The combination of these two key assumptions then creates room for rational asset price bubbles to possibly exist in equilibrium, because in that environment the bubbles may serve as a financing vehicle transferring resources among heterogenous firms, thus they could be valuable to their rational holders even though they are intrinsically worthless. Furthermore, the rational bubble is not required to grow at the rate of interest to be attractive because of the extra liquidity premium it commands, so that the bubble will not be ruled out by standard transversality conditions in equilibrium.

Several findings have emerged from the analysis of the chapter. First, the neat setup of the present framework makes analytical solutions largely available but without impairing its ability of generating the core equilibrium implications of rational bubbles of this type. Second, unlike rational bubbles in an overlapping generations framework, the present model features three, instead of two, mutually exclusive regions of parameter space in which one of them is inefficient in terms of allocating resources but which does not permit the emergence of rational bubbles. Third, the arrival rate of an investment opportunity in the model economy turns out to be the crucial parameter affecting not only the existence condition of a bubbly SS, but also the degree of persistence and the size of the bubble impact on output when the system is hit by a transitory rational bubble shock. Fourth, capital investments and output are boosted in the face of a bubble boom, because the latter enhances the financing ability of firms, with the efficiency of the economy regarding capital allocations being improved (temporarily).

Albeit the present framework may be too stylised to represent accurately actual economic situations, the analysis of the model provides important clarifications to the mechanism at work of this sort of rational bubbles among the existing literature. The transparency of the model dynamics then lays a solid foundation for the discussions of monetary policy implications in such a bubbly environment in the next chapter.

Appendix

3.A Derivations of the FOCs of Individual Firm

The optimisation problem facing the individual firm j can be equivalently expressed as

$$\max_{\{I_{jt}, Z_{jt+1}, K_{jt+1}\}_{t=0}^{\infty}} \mathcal{L}_0 = E_0 \sum_{t=0}^{\infty} \left\{ \Lambda_{0,t} \left[D_{jt} + \mu_{jt} D_{jt} + \gamma_{jt} Z_{jt+1} + q_{jt} \left(\tau_{jt} I_{jt} + \lambda K_{jt} - K_{jt+1} \right) \right] \right\}, \tag{3.85}$$

with $D_{jt} = R_{kt} K_{jt} - \tau_{jt} I_{jt} - Q_t^B \left(Z_{jt+1} - Z_{jt} \right)$ for $t = 0, 1, 2, \ldots$. The first-order conditions are then given by

$$\frac{\partial \mathcal{L}_t}{\partial I_{jt}} = -\tau_{jt}(1 + \mu_{jt}) + \tau_{jt} q_{jt} = 0 \, , \tag{3.86}$$

$$\frac{\partial \mathcal{L}_t}{\partial Z_{jt+1}} = -(1 + \mu_{jt}) Q_t^B + \gamma_{jt} + E_t \left\{ \Lambda_{t,t+1} \left(1 + \mu_{jt+1} \right) Q_{t+1}^B \right\} = 0 \, , \tag{3.87}$$

$$\frac{\partial \mathcal{L}_t}{\partial K_{jt+1}} = -q_{jt} + E_t\left\{\Lambda_{t,t+1}\left[\left(1+\mu_{jt+1}\right)R_{kt+1} + \lambda q_{jt+1}\right]\right\} = 0\,, \tag{3.88}$$

plus the complementary slackness conditions. Given that the investment shock τ_{jt} is independently and identically distributed across firms and over time and that the production technology is constant return to scale, it is conjectured that the Lagrangian multipliers, $\left\{\mu_{jt}, \gamma_{jt}, q_{jt}\right\}$, rely only on the aggregate state in period *t* and idiosyncratic shocks, so that

$$\mu_{jt}^{i/s} = \mu_t^{i/s}, \gamma_{jt}^{i/s} = \gamma_t^{i/s}, q_{jt}^{i/s} = q_t^{i/s}\,, \tag{3.89}$$

and thus

$$E_t\mu_{jt+1} = \eta\mu_{jt+1}^i + \left(1-\eta\right)\mu_{jt+1}^s \equiv \bar{\mu}_{t+1}\,, \tag{3.90}$$

$$E_t\gamma_{jt+1} = \eta\gamma_{jt+1}^i + \left(1-\eta\right)\gamma_{jt+1}^s \equiv \bar{\gamma}_{t+1}\,, \tag{3.91}$$

$$E_t q_{jt+1} = \eta q_{jt+1}^i + \left(1-\eta\right)q_{jt+1}^s \equiv \bar{q}_{t+1}\,. \tag{3.92}$$

By the independence assumption between the idiosyncratic and aggregate shocks, (3.86)-(3.88) can therefore be rewritten as (3.13)-(3.15) in the main context.

3.B Proof of Proposition 2

For the bubbly SS to exist, it requires that $q > 1$ and $q^B > 0$, i.e.,

$$1 - \frac{1-\beta}{\beta(1-\lambda\beta)} \equiv \eta_2 > \eta\,. \tag{3.93}$$

In order for $\eta > 0$, it then requires that $\beta(2-\lambda\beta) > 1$, i.e.,

$$1-\lambda > \rho^2, \tag{3.94}$$

given that $\beta \equiv 1/\left(1+\rho\right)$. Furthermore, we have

$$\begin{aligned}\eta_1 - \eta_2 &\equiv \frac{\beta(1-\lambda)}{1-\beta\lambda} - \left[1 - \frac{1-\beta}{\beta(1-\lambda\beta)}\right] \\ &= \frac{1}{1-\beta\lambda}\left[\left(1-\beta\right)\left(\frac{1}{\beta}-1\right)\right] > 0,\end{aligned} \tag{3.95}$$

i.e., $\eta_1 > \eta_2$. Therefore, if (3.94) is satisfied, η_1 and η_2 divide the parameter space of the model system into three mutually exclusive regions; or otherwise $\eta_2 \le 0$, there are only two regions of parameter space divided only by η_1, in both of which only a bubbleless SS can exist.

To prove that $q_l > q_b > q^* = 1$, $R_{k_l} > R_{k_b} > R_k^*$, and $K^* > K_b > K_l$ in region 3 of case 1 in Proposition 2, note first that from (3.45)

$$R_k = \frac{(1-\lambda\beta)q}{\beta\left[(1-\eta)+\eta q\right]},$$

$$\left.\frac{\partial R_k}{\partial q}\right|_{given\ \eta} = \frac{\beta(1-\lambda\beta)(1-\eta)}{\beta^2\left[(1-\eta)+\eta q\right]^2} > 0. \tag{3.96}$$

Therefore, since for a given η within the region 3 of case 1,

$$R_{k_l} - R_{k_b} = \frac{1}{\eta}\left[(1-\eta)(1-\lambda\beta) - \left(\frac{1}{\beta}-1\right)\right] > 0, \tag{3.97}$$

i.e., $R_{k_l} > R_{k_b}$, $q_l > q_b$ according to (3.96).[95] Also, given that $0 < \eta < \eta_2 < 1$ in this scenario,

$$R_{k_b} = (1-\lambda\beta)\left[1+\frac{1}{\eta}\left(\frac{1}{\beta}-1\right)\right] > (1-\lambda\beta)\left[1+\left(\frac{1}{\beta}-1\right)\right] = \frac{1}{\beta}-\lambda, \tag{3.98}$$

i.e., $R_{k_b} > R_k^* = \dfrac{1}{\beta} - \lambda$, while it is obvious that $q_b > 1 \equiv q^*$. Providing diminishing marginal product of capital, we have $K^* > K_b > K_l$ for $R_{k_l} > R_{k_b} > R_k^*$.

To show that $R_{k_l} > R_k^*$ and thus $K^* > K_l$ in case 2 of Proposition 2, note that since $\eta \in (0, \eta_1)$ for the bubbleless and inefficient SS to exist,

$$R_{k_l} = \frac{1-\lambda}{\eta} > \frac{1-\lambda}{\eta_1} = \frac{1}{\beta} - \lambda = R_k^*, \tag{3.99}$$

[95] Apply the inverse function rule: since the MPK as a function of Tobin's q is continuous and injective on the defined domain, its inverse function exists and has the same monotonic property as the MPK does.

while $\eta_l > \eta^* = 1$ by assumption. Q.E.D.

3.C Proof of Proposition 3

It is already known from the main context that $\frac{q^B}{b} \in \left[1, \frac{(1-\eta)\beta(1-\lambda\beta)}{1-\beta}\right)$ and that $q^B \in \left(0, q_b^B\right]$. Given that

$$R_{k_b}^s = (1-\lambda\beta)\left[\frac{1}{\eta\beta} - \frac{b}{q^B}\left(\frac{1}{\eta} - 1\right)\right] \tag{3.100}$$

is increasing in $\frac{q^B}{b}$ for a given η level, the lower bound of $R_{k_b}^s$ is obtained when $\frac{q^B}{b} = 1$, i.e., when $R_{k_b}^s = R_{k_b}$. Therefore, $R_{k_b}^s \geq R_{k_b}$ holds. Furthermore, according to (3.58), $R_{k_l} > R_{k_b}^s$ in order for $q^B > 0$.

Given the monotonically increasing relationship between q and R_k for a given level of the arrival rate of an investment opportunity, it is thus straightforward that $q_l > q_b^s \geq q_b$. $K_b \geq K_b^s > K_l$ given the diminishing marginal product of capital.

Finally, from Proposition 2, $q_b > q^*, R_{k_b} > R_k^*, K^* > K_b$. Q.E.D.

3.D Derivations of the Time Paths in the Section 3.4.2.1

The approach adopted here can be considered as a special case of Blanchard and Kahn (1980). Specifically, when there are two stable eigenvalues λ_{b1} and λ_{b2} for the system (3.79), the time paths for $\hat{y}_t$ and $\hat{q}_t^B$ are given by

$$\hat{y}_t = a_{11}\lambda_{b1}^t + a_{12}\lambda_{b2}^t, \tag{3.101}$$

$$\hat{q}_t^B = a_{21}\lambda_{b1}^t + a_{22}\lambda_{b2}^t, \tag{3.102}$$

respectively, where $\left[a_{11}, a_{21}\right]$ is the eigenvector associated with λ_{b1} and $\left[a_{12}, a_{22}\right]$ is the one associated with λ_{b2}.

With a "small" market sentiment shock, $\hat{q}_0^B = \varepsilon_0 > 0$, hitting the system at some time $t = 0$, which lasts only for one period, we have

$$a_{11} + a_{12} = 0 \tag{3.103}$$

and

$$a_{21} + a_{22} = \hat{q}_0^B, \tag{3.104}$$

since $\hat{y}_0 = 0$ as output in the model is predetermined and it is assumed that the economy originally sits at the corresponding SS. For $t = 1$,

$$a_{11}\lambda_{b1} + a_{12}\lambda_{b2} = \hat{y}_1 = \alpha\eta\hat{q}_0^B, \tag{3.105}$$

$$a_{21}\lambda_{b1} + a_{22}\lambda_{b2} = \hat{q}_1^B = \lambda_{b2}\hat{q}_0^B, \tag{3.106}$$

where the second equalities of (3.105) and (3.106) come from (3.78) and (3.77) respectively. Therefore, by combining (3.104) and (3.106), we have $a_{21} = 0$ and $a_{22} = \hat{q}_0^B$; by combining (3.103) and (3.105), we have $a_{11} = \alpha\eta\hat{q}_0^B / (\lambda_{b1} - \lambda_{b2})$ and $a_{12} = \alpha\eta\hat{q}_0^B / (\lambda_{b2} - \lambda_{b1})$. After slight rearrangement we can obtain the time paths for output and the bubble-capital ratio in the main text.

Chapter 4

Monetary Policy, Financing Constraints, and Rational Bubbles

4.1 Introduction

In this chapter, I undertake the task of addressing the question regarding the desirability of adopting a leaning-against-the-bubble (LAB, for brevity) strategy in the formulation of systematic monetary policy with the presence of the type of rational bubbles analysed in the preceding chapter. In order to do so, I introduce monopolistically competitive intermediate goods producers to the Chapter 3 model with Calvo's (1983) style sticky price setting. Instead of searching for some sort of "welfare-optimising" policy rule and hence unlike Dong et al. (2020) and Ikeda (2021), the two papers that are closest to my research in this chapter, I focus on studying two categories of monetary policy rules and their roles in shaping or eliminating potential bubble-driven fluctuations (in output and inflation): a strict (zero) inflation targeting (SIT, for short) rule and a simple Taylor-type interest rate (SIR, for short) rule with or without direct policy feedback on variations in the size of the bubble. While SIT is viewed as a hypothetical objective for the central bank (CB, for brevity) in the analysis of the present chapter,[96] the SIR rules are introduced as an implementable policy regime that aims to stabilise aggregate price (fully) and output in practice.

The rationale for not assessing the design of monetary policies by some defined "welfare-maximisation" criterion but instead concentrating on rules for stabilisation of the model economy is mainly twofold. Firstly, given that stabilising inflation has recently been a practically crucial concern of many central banks around the world,

[96] By "hypothetical", it means that it could not be implemented in practice, as will be discussed later in the chapter.

investigating the potential of a monetary policy and the role of a LAB strategy in ensuring (full) price stability may thus be of special interest and worthwhile.

Secondly, equilibria featuring bubbles in the present framework are all inefficient, as have already been demonstrated in Chapter 3, and to achieve the first-best allocation requires the imperfections of the financial market to be removed or the investment opportunity to be adequately increased,[97] which then also excludes the equilibrium existence of the rational bubble altogether. This somehow illuminates the peculiarity of welfare-relevant monetary policy analysis in the bubbly economy, for which results would need to be obtained through numerical simulations and may also be highly sensitive to specific model assumptions and parameters. Given that the main purpose for me building up the present model is to understand qualitatively, instead of quantitatively, the transmission mechanisms of monetary interventions in a bubbly environment, I leave possible optimal monetary policy analysis to future study.

To preview the main discoveries of interest from the analyses of the chapter, it is first emphasised that in principle inflation can be completely stabilised if a monetary policy is capable of indefinitely anchoring the economy at its "natural" level of allocation – the one that would prevail under flexible prices. In that case, monopolistically competitive intermediate goods producers charge their (constant) optimal markup and have no incentive to adjust their nominal prices either currently or in the future,[98] resulting in the aggregate price level being fully stabilised.

It then turns out that if the inefficient economy is originally bubbleless, LAB is definitely required in the proposed SIT rule to achieve the full stabilisation of inflation, especially in the face of a bubble shock. However, when it comes to the more pragmatic policy deliberation, adopting an explicit LAB motive in the SIR rule may generate an unintentional adverse outcome in terms of coping with bubble-driven fluctuations, the responsibility for which lies critically in the autonomous nature of the evolution of the

[97] Another respect of inefficiency induced by monopolistic competition in the model of the present chapter may be eliminated through an optimal employment subsidy (see, e.g., Chapter 4 of Galí (2015)).

[98] Assume that there are no inherited relative price distortions for simplicity.

rational bubble around a bubbleless SS.

On the other hand, if the economy is bubbly at the very beginning, then adopting a LAB strategy turns out to be neither necessary nor an efficient way for the CB to achieve its policy goal: additional policy responses to variations in the size of the bubble-capital ratio (gap) do not seem to have a significant effect on the policy's success in terms of insulating the system from bubble-driven fluctuations, either in the SIT or in the SIR rule regime, for monetary interventions appear to have opposite impacts on the two determinants of the equilibrium pricing of the bubble. This is a novel feature which is distinguished from those of the rest of the literature on (rational) bubbles with monetary policy designs.

The rest of the chapter is organised as follows. Section 4.2 lays out the full model underlying the monetary policy analyses throughout this chapter, which is based on the framework established in chapter 3 but now with additional New Keynesian ingredients. Sections 4.3 and 4.4 characterise the general equilibrium conditions and (zero-inflation) steady states of the economy, respectively. Section 4.5 turns to study the local dynamics of the model economy around both an (inefficient) bubbleless SS and a bubbly SS and their monetary policy implications for tackling bubble fluctuations which have an impact on output via both supply- and demand-side mechanisms, or for entirely eliminating them altogether. It is done by first deriving the economic system in "gap" terms, i.e., the terms indicating the gaps between the actual and natural values of variables, based on which, candidate interest rate rules for implementing the strict-inflation target are investigated. Since all the considered SIT rules require perfect knowledge of the natural rate of the relevant variables, which is plainly unrealistic, alternative SIR rules with or without a LAB motive are then assessed. Section 4.6 concludes.

4.2 The Full Model with Nominal Rigidities and Monetary Authority

The major ingredient of the full model is inherited that from the baseline one developed in Chapter 3, except that now sectors of intermediate goods and final goods are also introduced into the framework to incorporate New Keynesian (NK, for short) features, i.e., monopolistic competition and nominal rigidity, to allow for analysis of the effects of alternative monetary policies. Specifically, there are four types of agents, namely, households, wholesale firms, intermediate goods producers, and final goods producers. Households are identical and infinitely-lived, supplying labour elastically. Wholesale firms behave competitively, produce homogeneous wholesale goods and sell them to intermediate goods producers, with the latter doing nothing but buying wholesale goods and differentiating them at no cost to re-sell them to perfectly competitive final goods producers. The monopoly power possessed by the intermediate goods producers with differentiated goods thus enables price-setting behaviour, which is necessary in order to enable me to model nominal rigidity in price-setting in the economy.[99] Profits from producing the intermediate goods activity are assumed to be received by the households in a lump-sum manner each period. There is also the central bank setting the nominal interest rate in the market, which will be specified in Section 4.5.

4.2.1 Wholesale Firms

The core spirit of the setup of the wholesale firm sector is essentially the same as of that in Chapter 3. A typical wholesale firm $j \in [0,1]$ produces homogenous wholesale goods Y_{jt}^{W} which are then sold to intermediate goods producers at nominal price, P_t^W, in a competitive market, according to the constant-returns-to-scale production technology

$$Y_{jt}^{W} = K_{jt}^{\alpha} N_{jt}^{1-\alpha}, \quad \alpha \in (0,1), \tag{4.1}$$

[99] The way I introduce price stickiness in the full model is standard in the literature on dynamic NK models (see, e.g., Bernanke et al. 1999, Iacoviello 2005), i.e., the intermediate goods producers are distinguished from the wholesale firms, with the former being monopolistic competitive and thus the source of nominal rigidity. Alternatively, the (wholesale) firms may be assumed to be imperfectly competitive and hence staggered price setting may occur directly within the firm sector (see, e.g., Chari et al. 2000). Given that there is individual heterogeneity (especially in terms of capital accumulation) among firms, the present approach avoids complications to calculations and aggregation relative to the alternative one.

with K_{jt} the physical capital input and N_{jt} the labour input. The optimal labour demand of the firm solves:

$$\begin{aligned} &\max_{N_{jt}} \frac{P_t^W}{P_t} Y_{jt}^W - W_t N_{jt} \\ &s.t.\ Y_{jt}^W = K_{jt}^{\alpha} N_{jt}^{1-\alpha} \end{aligned} \tag{4.2}$$

yielding

$$N_{jt} = \left(\frac{P_t^W (1-\alpha)}{P_t W_t} \right)^{\frac{1}{\alpha}} K_{jt}, \tag{4.3}$$

and then

$$Y_{jt}^W = \left(\frac{P_t^W}{P_t} \frac{1-\alpha}{W_t} \right)^{\frac{1-\alpha}{\alpha}} K_{jt}, \tag{4.4}$$

where P_t is the aggregate nominal price index which will be defined later and W_t is the real wage rate taken as given by the firm. Thus, the gross operating profit for the firm is proportional to its capital stock and is given by

$$R_{kt} K_{jt} = \frac{P_t^W}{P_t} Y_{jt}^W - W_t N_{jt} = \alpha \frac{P_t^W}{P_t} Y_{jt}^W, \tag{4.5}$$

with

$$\begin{aligned} R_{kt} &\equiv \alpha \left(\frac{P_t^W}{P_t} \right)^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{W_t} \right)^{\frac{1-\alpha}{\alpha}} \\ &= \alpha \frac{P_t^W}{P_t} \frac{Y_{jt}^W}{K_{jt}} = \frac{P_t^W}{P_t} \frac{\partial Y_{jt}^W}{\partial K_{jt}} \end{aligned} \tag{4.6}$$

the marginal revenue product of capital.

As in the Chapter 3 model, at the start of each period, the firm j is subject to Kiyotaki-Moore type idiosyncratic investment shocks which are uninsurable and independent of potential aggregate shocks, following the Binomial distribution

$$\tau_{jt} = \begin{cases} 1, & \text{with Prob. } \eta \\ 0, & \text{with Prob. } 1-\eta \end{cases}. \tag{4.7}$$

The firm j seeks to maximise its expected discounted dividends (the stock price of the firm)

$$V_{j0} = E_0 \sum_{t=0}^{\infty} \Lambda_{0,t} D_{jt}^W , \tag{4.8}$$

subject to a sequence of flow-of-funds (FOFs, for short) constraints

$$D_{jt}^W + \tau_{jt} I_{jt} + Q_t^B \left(Z_{jt+1} - Z_{jt} \right) = R_{kt} K_{jt} , \tag{4.9}$$

and

$$K_{jt+1} = \tau_{jt} I_{jt} + \lambda K_{jt} , \tag{4.10}$$

$$D_{jt}^W \geq 0, \quad Z_{jt+1} \geq 0 , \tag{4.11}$$

with $\Lambda_{0,t}$ the stochastic discount factor (SDF, for short) between period 0 and *t*, D_{jt}^W the real dividends paid out by the wholesale firm *j* at period *t*, I_{jt} capital investment, Z_{jt+1} the quantity purchased of the intrinsically worthless bubbly asset at the end of period *t* with $Q_t^B \geq 0$ the market price for the asset in real terms at that period. Since the overall amount of the bubbly asset outstanding in the market is assumed to be constant and equal to one over time, the bubble price, Q_t^B, is thus also a measure of the aggregate size of the rational bubble in the economy. Then the Propositions 1 and 2 derived in Chapter 3 exactly apply here from the firm's optimisation problem, with

$$Q_t^B = E_t \left\{ \Lambda_{t,t+1} \left[1 + \eta \left(q_{t+1} - 1 \right) \right] Q_{t+1}^B \right\} \tag{4.12}$$

and

$$q_t = E_t \left\{ \Lambda_{t,t+1} \left[R_{kt+1} + \lambda q_{t+1} + \eta (q_{t+1} - 1) R_{kt+1} \right] \right\} \tag{4.13}$$

when Tobin's *q* is greater than one, and $Q_t^B = 0$ otherwise.

4.2.2 Households

Households are identical and infinitely-lived, supplying labour elastically through a Walrasian labour market. A representative household then seeks to maximise its expected lifetime utility

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\log C_t - \frac{N_t^{1+\varphi}}{1+\varphi} \right), \quad \varphi > 0 , \tag{4.14}$$

subject to a sequence of period budget constraints,

$$C_t + \frac{B_{t+1}^n}{P_t} + \int_0^1 Z_{jt+1}^{Sh} \left(V_{jt} - D_{jt}^W \right) dj + Q_t^B \left(Z_{t+1}^{Bh} - Z_t^{Bh} \right) = W_t N_t + \frac{\left(1+i_{t-1}\right) B_t^n}{P_t} + \int_0^1 Z_{jt}^{Sh} V_{jt} dj + D_t^R , \tag{4.15}$$

and the short-selling constraint for the bubbly asset,

$$Z_{t+1}^{Bh} \geq 0 \tag{4.16}$$

for $t = 0,1,...$, with $\beta \equiv 1/\left(1+\rho\right) \in \left(0,1\right)$ the household's subjective discount factor, N_t the labour supplied by the household; C_t the household's period *t* consumption expenditure measured in real terms; B_{t+1}^n the nominal value of the one-period riskless bonds purchased at the end of period *t*, with i_t the (net) risk-free nominal interest rate between period *t* and *t*+1; Z_{jt+1}^{Sh} the individual wholesale firm *j*'s shares purchased at the start of period *t*+1, with V_{jt} the (before dividends) real market value of it; Z_{t+1}^{Bh} the holdings of the bubbly asset at the start of period *t*+1; D_t^R the profits received from the intermediate goods producers (specified below).[100]

The optimality conditions for the household's maximisation problem (4.14)-(4.16) are given by

$$1 = E_t \left\{ \Lambda_{t,t+1} \frac{\left(1+i_t\right) P_t}{P_{t+1}} \right\} ; \tag{4.17}$$

$$1 \geq E_t \left\{ \Lambda_{t,t+1} \frac{Q_{t+1}^B}{Q_t^B} \right\}, \text{ with equality when } Z_{t+1}^{Bh} > 0 ; \tag{4.18}$$

$$V_{jt} - D_{jt}^W = E_t \left\{ \Lambda_{t,t+1} V_{jt+1} \right\} , \tag{4.19}$$

[100] Thus I assume that each of the households receives an equal share of intermediate goods producers' overall profits.

with

$$\Lambda_{t,t+1} \equiv \beta \frac{C_t}{C_{t+1}} \tag{4.20}$$

the SDF between period t and t+1, together with its optimal labour supply decision,

$$W_t = C_t N_t^{\varphi} . \tag{4.21}$$

Again, since the return on the bubbly asset is too low for the household given (4.12), it will optimally choose not to invest in it and thus $Z_{t+1}^{Bh} = 0$ in equilibrium.

4.2.3 Intermediate Goods and Final Goods Producers

Assume that there is a continuum of monopolistically competitive intermediate goods producers of measure one. At each period t, an intermediate goods producer $i \in [0,1]$ purchases wholesale goods from the wholesale firms in a competitive market at the price P_t^W and differentiates them (one-for-one) at no cost into specialised intermediate good, $Y_t(i)$, which is then sold at the nominal price $P_t(i)$ to perfectly competitive final goods producers.

The final goods producers choose inputs $Y_t(i)$ for all $i \in [0,1]$, and output Y_t to maximise profits

$$P_t Y_t - \int_0^1 P_t(i) Y_t(i) di \tag{4.22}$$

in each period t, where P_t is the nominal price of the final goods – also considered as the aggregate nominal price index in the economy, subject to the CES aggregate production function

$$Y_t = \left(\int_0^1 Y_t(i)^{\frac{\epsilon-1}{\epsilon}} di \right)^{\frac{\epsilon}{\epsilon-1}}, \quad \epsilon > 1 . \tag{4.23}$$

Households and wholesale firms then purchase these final goods for consumption and investment, with aggregate demand $Y_t = C_t + \int_0^1 I_{jt} dj$. Solving the maximisation problem specified by (4.22)-(4.23) yields the demand function for intermediate good

$i \in [0,1]$:

$$Y_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon} Y_t \,, \tag{4.24}$$

and zero profit condition implies that $P_t \equiv \left(\int_0^1 P_t(i)^{1-\epsilon} di \right)^{1/(1-\epsilon)}$.

As in Calvo (1983), intermediate goods producers are assumed only to be able to change their prices in any period with constant probability $1-\xi$.[101] As a result, under the sticky price setting, the evolution of the aggregate price index satisfies

$$P_t = \left[(1-\xi)\left(P_t^*\right)^{1-\epsilon} + \xi\left(P_{t-1}\right)^{(1-\epsilon)} \right]^{1/(1-\epsilon)}, \tag{4.25}$$

with P_t^* denoting the price chosen by intermediate goods producers setting their price in period t.[102] A first-order Taylor approximation of (4.25) around a zero-inflation steady state (ZISS, for short) is given by

$$p_t = (1-\xi) p_t^* + \xi p_{t-1} \,, \tag{4.26}$$

which can be further expressed as

$$\pi_t = (1-\xi)\left(p_t^* - p_{t-1} \right), \tag{4.27}$$

where $p_t^* \equiv \log P_t^*$, $p_t \equiv \log P_t$, and $\pi_t \equiv p_t - p_{t-1}$ is the inflation (rate).

When an opportunity of resetting price arrives at time t, the intermediate goods producer selling good i chooses the price P_t^* which maximises its expected present value of profits

$$\sum_{k=0}^{\infty} \xi^k E_t \left\{ \Lambda_{t,t+k} Y_{t+k|t} \left(\frac{P_t^* - P_{t+k}^W}{P_{t+k}} \right) \right\} \tag{4.28}$$

subject to the demand schedule (4.24): $Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} Y_{t+k}$, with $Y_{t+k|t}$ indicating the

[101] Implicitly behind the assumption of staggered price setting of the intermediate goods producers may be the "costs of adjusting nominal prices" (Bernanke et al. 1999).

[102] Since all intermediate goods producers choose their price in period t face the same optimisation problem (and the same constraints), the individual index can be omitted here.

output in period $t+k$ of the intermediate goods producer that last resets its price in period t.[103] The first-order condition for the intermediate goods producer's optimisation problem (4.28) is given by

$$\sum_{k=0}^{\infty} \xi^k E_t \left\{ \Lambda_{t,t+k} Y_{t+k|t} \left(\frac{P_t^*}{P_{t+k}} - \frac{\epsilon}{\epsilon-1} \frac{P_{t+k}^W}{P_{t+k}} \right) \right\} = 0, \tag{4.29}$$

i.e.,

$$\sum_{k=0}^{\infty} \xi^k E_t \left\{ \Lambda_{t,t+k} \left(\frac{P_t^*}{P_{t+k}} \right)^{-\epsilon} Y_{t+k} \left(\frac{P_t^*}{P_{t+k}} - \frac{\epsilon}{\epsilon-1} \frac{P_{t+k}^W}{P_{t+k}} \right) \right\} = 0, \tag{4.30}$$

which then yields the optimal price setting rule

$$P_t^* = \frac{\epsilon}{\epsilon-1} \frac{\sum_{k=0}^{\infty} \xi^k E_t \left\{ \Lambda_{t,t+k} P_{t+k}^{\epsilon-1} P_{t+k}^W Y_{t+k} \right\}}{\sum_{k=0}^{\infty} \xi^k E_t \left\{ \Lambda_{t,t+k} P_{t+k}^{\epsilon-1} Y_{t+k} \right\}}. \tag{4.31}$$

In a ZISS, gross inflation $\Pi_{t-1,t+k} \equiv P_{t+k}/P_{t-1} = 1$, $P_t^*/P_{t-1} = 1$, $P_t^*/P_{t+k} = 1$, and $\Lambda_{t,t+k} = \beta^k$, while $P_{t+k}^W/P_{t+k} = (\epsilon-1)/\epsilon$ from (4.29). A first-order approximation of (4.31) around a ZISS yields

$$p_t^* = \mu^p + (1-\xi\beta) \sum_{k=0}^{\infty} (\xi\beta)^k E_t \left\{ p_{t+k}^W \right\}, \tag{4.32}$$

with $p_t^W \equiv \log P_t^W$ and $\mu^p \equiv \log\left[\epsilon/(\epsilon-1)\right]$ the log optimal markup (rate) under flexible prices. Combining (4.32) with (4.27) and after some manipulation then yields a version of the New Keynesian Phillips curve (NKPC, for brevity)

$$\pi_t = \beta E_t \pi_{t+1} - \frac{(1-\xi)(1-\beta\xi)}{\xi} \hat{\mu}_t^p, \tag{4.33}$$

where $\hat{\mu}_t^p \equiv \mu_t^p - \mu^p$ with $\mu_t^p \equiv \log\left(P_t/P_t^W\right)$ the average (log) price markup.

4.3 General Equilibrium

[103] Given that the individual intermediate goods producer is infinitesimal relative to the economy as a whole, it takes as the aggregate price level and aggregate output as given.

Define aggregate variables $K_t \equiv \int_0^1 K_{jt} dj$, $N_t \equiv \int_0^1 N_{jt} dj$. Then in general equilibrium and in aggregate,

$$N_t = \left(\frac{P_t^W}{P_t} \frac{1-\alpha}{W_t} \right)^{\frac{1}{\alpha}} K_t \tag{4.34}$$

and

$$Y_t^W \equiv \int_0^1 Y_{jt}^W dj = \left[\frac{P_t^W (1-\alpha)}{P_t W_t} \right]^{\frac{1-\alpha}{\alpha}} K_t \, , \tag{4.35}$$

which also implies that

$$Y_t^W = K_t^\alpha N_t^{1-\alpha} \, . \tag{4.36}$$

On the other hand, aggregating (4.24) across intermediate goods producers and applying wholesale goods market clearing condition,

$$\int_0^1 Y_t(i) di = \int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon} Y_t di = \Delta_t^P Y_t = Y_t^W \, , \tag{4.37}$$

with $\Delta_t^P \equiv \int_0^1 \left(\frac{P_t(i)}{P_t} \right)^{-\epsilon} di \geq 1$ a measure of price dispersion across individual intermediate goods producers. In a local area of a ZISS, it can be shown that up to a first-order approximation, $\Delta_t^P \simeq 1$.[104] Thus, in an approximate sense,

$$Y_t \simeq \int_0^1 Y_t(i) di = Y_t^W = K_t^\alpha N_t^{1-\alpha} \tag{4.38}$$

from (4.36)-(4.37). Accordingly,

$$\frac{P_t^W}{P_t} Y_t \simeq \frac{P_t^W}{P_t} Y_t^W = R_{kt} K_t + W_t N_t \tag{4.39}$$

from (4.5), with

$$W_t = (1-\alpha) \frac{P_t^W Y_t}{P_t N_t} \tag{4.40}$$

and

[104] See, e.g., Galí (2015) Chapter 3 Appendix 3.4.

$$R_{kt} = \alpha \frac{P_t^W Y_t}{P_t K_t} \tag{4.41}$$

from (4.6). Combining (4.40) (the static labour demand schedule by wholesale firms) with (4.21) (the labour supply decision of households) further yields the labour market equilibrium relationship

$$(1-\alpha)\frac{P_t^W}{P_t} = \frac{C_t}{Y_t} N_t^{1+\varphi}. \tag{4.42}$$

As in Chapter 3, denote Ω^i with measure η the set of individual firms with an investment opportunity, so that aggregate investment $I_t \equiv \int_0^1 I_{jt} dj = \int_{j\in\Omega^i} I_{jt} dj$. Then aggregate capital accumulation evolves according to

$$K_{t+1} = I_t + \lambda K_t \ , \tag{4.43}$$

while

$$I_t = \eta R_{kt} K_t + \eta Q_t^B \tag{4.44}$$

particularly when Tobin's q is greater than 1. Final goods market clearing requires

$$C_t + I_t = Y_t . \tag{4.45}$$

In the asset markets, $B_t^n = 0$ and $\int_0^1 Z_{jt} dj = 1$ since $Z_{t+1}^{Bh} = 0$ in equilibrium, i.e., all the bubbly assets are held by the wholesale firms. For individual wholesale firms' shares market clearing, $\int_0^1 Z_{jt}^{Sh} dj = 1$ for $\forall j \in [0,1]$. For the profits distributed by intermediate goods producers in each period, $D_t^R \equiv \int_0^1 D_t^R(i) di$ with $D_t^R(i) \equiv Y_t(i)\left(P_t(i) - P_t^W\right)/P_t$. Therefore,

$$\begin{aligned} D_t^R &= \int_0^1 \left(\frac{Y_t(i) P_t(i)}{P_t} \right) di - \left(\frac{P_t^W}{P_t} \right) \int_0^1 Y_t(i) di \\ &= Y_t - \left(\frac{P_t^W}{P_t} \right) Y_t^W = Y_t - W_t N_t - R_{kt} K_t . \end{aligned} \tag{4.46}$$

Meanwhile, from individual wholesale firm's FOFs constraint (4.9), we have

$$
\begin{aligned}
D_t^W &\equiv \int_0^1 D_{jt}^W dj \\
&= R_{kt}\int_0^1 K_{jt}dj - \int_0^1\left(\tau_{jt}I_{jt}\right)dj + Q_t^B\left(\int_0^1 Z_{jt+1}dj - \int_0^1 Z_{jt}dj\right) \\
&= R_{kt}K_t - I_t.
\end{aligned}
\tag{4.47}
$$

4.4 Steady States

Recalling the reasoning of Section 3.3.1 of Chapter 3, the derivations of SS values of R_k, q and $q^B \equiv Q^B/K$ are (jointly) determined solely by the pricing equations for the bubbly asset and Tobin's q, together with the investment equation. In other words, the SS values of these three price variables are unaffected by the newly introduced monopolistic competition and nominal rigidities in the present full model. Therefore, the conclusions drawn from Proposition 4 of Chapter 3 still applies here (for the determinations of the SS values for R_k, q and q^B).[105] In a ZISS, prices will be the same as if they were flexible, despite the Calvo-style staggering. Hence, we have $1/(1+r) = \Lambda = \beta$ with r the real rate of interest along the SS, and $P^W/P = (\epsilon - 1)/\epsilon$.

To obtain the values of other aggregate variables of interest along a ZISS, I begin by finding the SS capital-output ratio from (4.41):

$$
\frac{K}{Y} = \alpha\frac{P^W}{PR_k} = \alpha\left(\frac{\epsilon-1}{\epsilon}\right)\frac{1}{R_k}. \tag{4.48}
$$

Since $Y = C + I$ and $I = (1-\lambda)K$, the consumption-output ratio in a ZISS is given by

$$
\frac{C}{Y} = 1-(1-\lambda)\frac{K}{Y} = 1-\frac{\alpha(1-\lambda)(\epsilon-1)}{\epsilon R_k}. \tag{4.49}
$$

Thus, combining (4.49) with the SS version of (4.42) yields the SS employment level:

[105] Note, however, the most efficient situation, i.e., when Tobin's q equals one, in the full model does not lead to the "first-best" allocation outcome in the Chapter 3 economy, because of the presence of the monopoly power of intermediate goods producers. Also, it is now not so straightforward that SS capital stock is still monotonically decreasing in the marginal (revenue) product of capital due to the fact that labour supply is now elastic. See Appendix 4.A for a proof of this point.

$$N=\left[\frac{\epsilon}{(\epsilon-1)(1-\alpha)}-\frac{\alpha(1-\lambda)}{(1-\alpha)R_k}\right]^{-\frac{1}{1+\varphi}}. \tag{4.50}$$

Then from (4.38), SS aggregate capital stock can be solved for, with

$$K=\left(\frac{K}{Y}\right)^{\frac{1}{1-\alpha}}N. \tag{4.51}$$

Finally, the SS output level is given by

$$Y=K^{\alpha}N^{1-\alpha}=\frac{\epsilon}{\alpha(\epsilon-1)}KR_k. \tag{4.52}$$

4.5 Equilibrium Dynamics in a Bubbly Environment

In this section, I turn to undertake the task of studying the equilibrium dynamics in a local area of a given ZISS of the full model, particularly in which emergence of rational bubble(-driven) fluctuations (in output, via both supply- and demand-side mechanisms) is possible, and the role that alternative monetary policy interventions may play in shaping or eliminating those fluctuations. In order to do so, I maintain the assumption that the arrival rate of an investment opportunity of the economy satisfies the condition for a potential existence of a rational bubble in equilibrium, i.e., $\eta\in(0,1-\frac{1-\beta}{\beta(1-\lambda\beta)})$ with $1-\lambda>\rho^2$ throughout the analysis of this section. Fundamental shocks are also excluded from considerations here, so that the discussions can be focused on bubble-driven ones instead.

As a preliminary step, I derive the log-linearised equilibrium conditions around a ZISS for the non-policy block of the system, which are specified by (4.53)-(3.65), (3.66)-(4.61) and (3.68)-(4.64) plus (4.33). In line with the notation used in Chapter 3, here I continue to denote the deviation of the aggregate bubble-capital ratio from its SS value as $\hat{q}_t^B\equiv q_t^B-q^B$, with $q_t^B\equiv Q_t^B/K$ and $\hat{i}_t\equiv\log\left[(1+i_t)/(1+r)\right]$ the deviation of the net nominal interest rate, with $\hat{r}_t\equiv\hat{i}_t-E_t\pi_{t+1}$ that for the net real interest rate; for the remaining variables of interest, lowercase letters are used to define the log of the original variable while the "^" symbol on top of a variable are used to indicate the deviation from its (zero-inflation) SS value, except that $\hat{q}_t \equiv \log\left(q_t/q\right)$ and that the log-deviation of investment is denoted by $\hat{i}_t^R \equiv \log\left(I_t/I\right)$ in order to distinguish it from the nominal interest rate. I also continue to use the subscript "*b*" to distinguish the value of a variable in a bubbly SS, and "*l*" for that in a bubbleless but inefficient one.

For the bubble pricing equation, I obtain

$$\hat{q}_t^B = \frac{(1-\eta)(1-\lambda\beta)\beta}{1-\beta} E_t \hat{q}_{t+1}^B \tag{4.53}$$

when $q^B = 0$ with $\hat{q}_t^B \geq 0$ for $\forall t$, and

$$\hat{q}_t^B = E_t \hat{q}_{t+1}^B + \eta\beta q_b q_b^B E_t \hat{q}_{t+1} - q_b^B \hat{r}_t \tag{4.54}$$

when $q^B = q_b^B > 0$, with $q_b = 1 + \frac{1}{\eta}\left(\frac{1}{\beta} - 1\right)$.

For the pricing equation for Tobin's *q*,

$$\hat{q}_t = \beta(\eta R_k + \lambda) E_t \hat{q}_{t+1} + (1-\lambda\beta) E_t \hat{r}_{kt+1} - \hat{r}_t\,; \tag{4.55}$$

for the Euler equation of the household,

$$\hat{c}_t = E_t \hat{c}_{t+1} - \hat{r}_t\,; \tag{4.56}$$

for the goods market clearing condition,

$$\hat{y}_t = \frac{C}{Y}\hat{c}_t + \frac{I}{Y}\hat{i}_t^R\,. \tag{4.57}$$

Combining (3.64)-(3.65) then yields

$$E_t \hat{y}_{t+1} - \frac{I}{Y} E_t \hat{i}_{t+1}^R = \hat{y}_t - \frac{I}{Y}\hat{i}_t^R + \frac{C}{Y}\hat{r}_t\,. \tag{4.58}$$

For the aggregate production function, I obtain

$$\hat{y}_t = \alpha \hat{k}_t + (1-\alpha)\hat{n}_t\,; \tag{4.59}$$

for the marginal revenue product of capital,

$$\hat{r}_{kt} = -\hat{\mu}_t^p + \hat{y}_t - \hat{k}_t , \tag{4.60}$$

and

$$-\hat{\mu}_t^p + \hat{y}_t = \hat{c}_t + (1+\varphi)\hat{n}_t \tag{4.61}$$

from the labour market equilibrium relationship, (4.42). Therefore, by combining (3.66)-(4.61), we have

$$\left(1 + \frac{Y(1-\alpha)}{C(1+\varphi)}\right)\hat{y}_t = \left(\alpha + \frac{1-\alpha}{1+\varphi}\right)\hat{k}_t + \left(\frac{1-\alpha}{1+\varphi}\right)\hat{r}_{kt} + \left(\frac{I}{C}\frac{1-\alpha}{1+\varphi}\right)\hat{i}_t^R . \tag{4.62}$$

For the capital accumulation equation,

$$(1-\lambda)\hat{i}_t^R = \hat{k}_{t+1} - \lambda\hat{k}_t ; \tag{4.63}$$

for the aggregate investment equation,

$$(1-\lambda)\hat{i}_t^R = \eta R_k(\hat{r}_{kt} + \hat{k}_t) + \eta\hat{q}_t^B . \tag{4.64}$$

Finally, substituting out the log-deviation of the average markup in the inflation equation (4.33) by using the relationship implied by (3.67) yields

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi}\left(\hat{k}_t - \hat{y}_t + \hat{r}_{kt}\right). \tag{4.65}$$

With prices being sticky, the above equilibrium conditions must be complemented with a monetary policy rule in order to close the model. In what follows and for the purpose of this chapter, I analyse two sets of interest rate rules: one of which is a "strict-inflation-targeting" (SIT) rule aiming to fully stabilise aggregate prices despite the possible emergence of bubble fluctuations in the system, while the other one is a "simple Taylor-type interest rate rule" (SIR) with a potential leaning-against-the-bubble (LAB, for short) component.

It is worth noting that the SIT rule which will be studied below is particularly inspired by the approach in Chapter 5 of Woodford (2003) for analysis of natural rate of interest and price stability with endogenous variations in capital stock. Relative to Woodford's analysis, the main complication in the present model is that the equilibrium impact of the bubble on the system needs to be taken into account as well. As in Woodford (2003), to facilitate the analysis of the SIT rule, a corresponding "gap" system, in which endogenous variables are expressed in terms of the gap between the actual and "*natural*" values of them, will need to be derived beforehand. Specifically and in accordance with the definition by Woodford (2003), a "natural" level of an economic variable is the one that would prevail when price is assumed to be flexible *now and in the future*, given all predetermined and exogenous state variables.[106] Note that equations (4.53) through (4.64) describing the equilibrium relationships among the variables of interest also apply to their flexible price counterparts,[107] except that with flexible prices, $\hat{\mu}_t^p = 0$ so that

$$\hat{r}_{kt} = \hat{y}_t - \hat{k}_t \tag{4.66}$$

according to (3.67), since in that case intermediate goods producers would retain the desired markup which is constant in the present model. Throughout I cap a variable with a tilde to denote the "gap" between the actual and natural value of it, where the latter is indicated by a superscript "*n*" of the variable.

I study the circumstances where the model economy fluctuates near an inefficient but bubbleless SS in Section 4.5.1 and that around a bubbly SS in Section 4.5.2.

4.5.1 Fluctuations near a Bubbleless ZISS

As has been pointed out in Section 3.4.2.1 of Chapter 3, when the economy is near an inefficient but bubbleless SS, the fact that $\lambda_{b2} \equiv \dfrac{1-\beta}{(1-\eta)(1-\lambda\beta)\beta} \in (0,1)$ always holds when $\eta \in (0, 1-(1-\beta)/[\beta(1-\lambda\beta)])$, i.e., the condition that it is possible for a rational bubble to emerge in equilibrium is satisfied. This implies that there exist other bounded rational expectation solutions to (4.53) other than the no-bubble-deviation one, these solutions taking the form

---

[106] It may be worth noting again that the defined economy's natural equilibrium level is generally inefficient, both because of the presence of the monopoly power and the rational bubble in the present model.

[107] Obviously, the inflation equation is not valid in a flexible price environment.

$$\hat{q}_{t+1}^{B} = \lambda_{b2}\hat{q}_{t}^{B} + \varepsilon_{t+1}, \tag{4.67}$$

which is a stationary AR(1) process, with $\hat{q}_t^B = q_t^B = Q_t^B / K_l$. $\varepsilon_{t+1} \equiv \hat{q}_{t+1}^B - E_t \hat{q}_{t+1}^B$ with $E_t \varepsilon_{t+1} = 0$ is assumed to be independent of monetary interventions and is viewed as the rational bubble innovation (the "bubble shock"), capturing actual speculative bubble episodes where pure mood swings in the asset market drive up or down the asset price regardless of the absence of any news concerning its fundamentals. More broadly speaking, this scenario with bounded bubble fluctuations near a bubbleless SS may be considered as a plausible representation of a boom and a subsequent bust of bubble episodes similar to those that are apparent in real economies, with the no-bubble state being the resting point.[108]

This outcome together with the fact that monetary policy is incapable of affecting the bubble fluctuations themselves (up to a first-order approximation) in this circumstance suggests that the present system is always subject to intrinsically persistent bounded rational bubble fluctuations, whose degree of persistence is endogenously determined by λ_{b2}. Nevertheless, monetary policy may still be able to mitigate the impacts of the bubble fluctuations on output or inflation through affecting the aggregate demand of the economy, as will be demonstrated in what follows, and based on which I assess the desirability of a LAB strategy.

4.5.1.1 The System in "Gap" Terms

When the system fluctuates near an inefficient but bubbleless SS, the expectation of the aggregate bubble-capital ratio evolves autonomously and is independent of interest rate changes (up to a first-order approximation) according to (4.53), which implies that the actual time path of the rational bubble is identical to its flexible-price counterpart. Consequently,

[108] Therefore, my approach to model "bubble shocks" is distinct from that proposed by Miao, Wang, and Xu (2015), Dong et al. (2020), or Ikeda (2021), in which a different type of "exogenous sentiment shock" is assumed to hit their systems only around bubbly SSs.

$$\tilde{q}_t^B \equiv \hat{q}_t^B - \hat{q}_t^{B^n} = 0 \qquad (4.68)$$

for $\forall t$.

In accordance with the definition by Woodford (2003), $\hat{k}_t = \hat{k}_t^n$ in any *current* period t, i.e., $\tilde{k}_t = 0$, since capital stock is predetermined and is affected by monetary policy among other things when prices are sticky in the *past*. From the aggregate investment equation (4.64) and given that $\tilde{k}_t = 0$ in any *current* period t, it must then be the case that

$$\left(1-\lambda\right)\tilde{i}_t^R = \eta R_{k_l} \tilde{r}_{kt},$$

i.e.,

$$\tilde{i}_t^R = \tilde{r}_{kt}, \qquad (4.69)$$

given that $R_{k_l} = \left(1-\lambda\right)/\eta$. Advancing (4.64) one period and using the fact that

$$\tilde{k}_{t+1} = (1-\lambda)\tilde{i}_t^R, \qquad (4.70)$$

we have $\tilde{r}_{kt+1} = \tilde{i}_{t+1}^R - (1-\lambda)\tilde{i}_t^R$, (4.71)where t is still thought of as the "current" period. Also, from (4.62),

$$\tilde{i}_t^R = \psi_2 \tilde{y}_t \qquad (4.72)$$

by making use of (4.69), with $\psi_2 \equiv \left[\left(1+\varphi\right)C_l\right]/\left[\left(1-\alpha\right)Y_l\right]+1$, and

$$E_t\tilde{i}_{t+1}^R = \psi_2 E_t\tilde{y}_{t+1} - \left[\alpha\psi_2(\psi_2 - 1)\right]\tilde{y}_t \qquad (4.73)$$

after advancing one period (4.62) and making use of (4.70)-(4.72). From the Euler equation and the final goods' market clearing condition,

$$E_t\tilde{y}_{t+1} - \frac{I_l}{Y_l}E_t\tilde{i}_{t+1}^R = \tilde{y}_t - \frac{I_l}{Y_l}\tilde{i}_t^R + \frac{C_l}{Y_l}\tilde{r}_t. \qquad (4.74)$$

With further manipulations,[109] we obtain the following relationships in "gap" terms:

$$\psi_1 E_t\tilde{y}_{t+1} = \left(\psi_1 + \alpha\psi_2\right)\tilde{y}_t + \left(\psi_1 - 1\right)\tilde{r}_t; \qquad (4.75)$$

[109] Further details of derivations are referred to Appendix 4.B.

$$\tilde{q}_t = \beta E_t \tilde{q}_{t+1} + (1-\lambda\beta)\psi_2 E_t \tilde{y}_{t+1} - (1-\lambda\beta)\psi_2 \left[\alpha(\psi_2 - 1) + 1 - \lambda\right]\tilde{y}_t - \tilde{r}_t; \quad (4.76)$$

$$\pi_t = \beta E_t \pi_{t+1} + \psi_3(\psi_2 - 1)\tilde{y}_t, \quad (4.77)$$

with $\psi_1 \equiv 1 - \left[Y_l(1-\alpha)\right] / \left[I_l(1+\varphi)\right]$ and $\psi_3 \equiv (1-\xi)(1-\beta\xi)/\xi$.

At this stage, rewrite (4.75) and iterate it forward, then

$$\begin{aligned} \tilde{y}_t &= \frac{\psi_1}{\psi_1 + \alpha\psi_2} E_t \tilde{y}_{t+1} - \frac{\psi_1 - 1}{\psi_1 + \alpha\psi_2} \tilde{r}_t \\ &= \left(\frac{\psi_1}{\psi_1 + \alpha\psi_2}\right)^T E_t \tilde{y}_{t+T+1} - \frac{\psi_1 - 1}{\psi_1 + \alpha\psi_2} \sum_{T=0}^{\infty} \left(\frac{\psi_1}{\psi_1 + \alpha\psi_2}\right)^T E_t \tilde{r}_{t+T}. \end{aligned} \quad (4.78)$$

By assuming that the impacts of price stickiness on the system vanish asymptotically, i.e., $\lim_{T\to\infty} E_t \tilde{y}_{t+T+1} = 0$, (4.78) becomes

$$\tilde{y}_t = -\frac{\psi_1 - 1}{\psi_1 + \alpha\psi_2} \sum_{T=0}^{\infty} \left(\frac{\psi_1}{\psi_1 + \alpha\psi_2}\right)^T E_t \tilde{r}_{t+T}, \quad (4.79)$$

which implies that the output gap can be explained by current and anticipated deviations between the actual real rate of interest and its natural counterpart. To put this in another way, the real interest rate gaps then summarise the impacts on the actual equilibrium that are generated by nominal rigidities.[110]

4.5.1.2 The Strict-inflation-targeting Rule

The real interest rate value that is consistent with the flexible-price allocation is called the *Wicksellian Natural Rate of Interest* by Woodford (2003) and is denoted by $\hat{r}_t^n$ in the present context. Since the model economy is subject to bounded bubble fluctuations and is abstracted from aggregate fundamental shocks, (the log-linear approximation to) the rational expectation solution to the system (with flexible prices) in any current period *t* depends not only on the (actual existing) capital stock in that period, but also on the size of the bubble-capital ratio as well.[111] Therefore, the equilibrium natural rate

[110] Note that this outcome is the same as in the models without bubble fluctuations and with or without endogenous capital accumulation.

[111] Since again, the capital stock is an endogenous state variable while the bubble is now affected by exogenous shocks (i.e., the bubble innovations) and may thus be considered as an exogenous state variable in the present context.

of interest could be expressed as

$$\hat{r}_t^n = N_r^{k^n} \hat{k}_t^n + N_r^{q^{B^n}} \hat{q}_t^B \tag{4.80}$$

with $N_r^{k^n}$ and $N_r^{q^{B^n}}$ some (endogenously determined) coefficients.[112] Especially at any *current* date *t*,

$$\hat{r}_t^n = N_r^{k^n} \hat{k}_t + N_r^{q^{B^n}} \hat{q}_t^B \tag{4.81}$$

by definition.

Let's now consider an interest-rate feedback rule with a time-varying intercept term of the form

$$\hat{i}_t = \overline{i}_t + \phi_y \tilde{y}_t + \phi_\pi E_t \pi_{t+1}, \quad (\phi_\pi \geq 1)\,,^{113} \tag{4.82}$$

with the short-term nominal interest rate i_t being the monetary policy instrument and

$$\overline{i}_t = \hat{r}_t^n \tag{4.83}$$

Therefore, by proposing a policy rule as (4.82), the CB is assumed to target exactly one-for-one the natural rate of interest which varies due to changes in the capital stock and in the bubble. Equivalently, this implies that in addition to systematic responses to variations in the output gap and in the rate of inflation expected in the next period, the CB also commits to systematic responses to endogenous variations in the capital stock and in the bubble of this special sort.

Examine first a special situation where the policy feedback on the expected inflation is set to be equal to one, i.e., $\phi_\pi = 1$. In that case, the policy rule (4.82) can be written in terms of the (actual) real rate and involves no "direct" response to expected inflation. It is hence possible to solve the local equilibrium dynamics of the system without a reference to the NKPC ((4.77)): one can simply combine (4.75) and (4.76) with (4.82) to obtain (after some algebra)

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[112] The coefficients can be (numerically) calculated out by applying (generalised) Schur decomposition method in solving the rational expectation solutions of the system.

[113] The policy coefficient on expected inflation is assumed always no less than one, so that the real interest rate does not decrease in the face of rising expected inflation.

$$\begin{bmatrix} E_t \tilde{q}_{t+1} \\ E_t \tilde{y}_{t+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \tilde{q}_t \\ \tilde{y}_t \end{bmatrix}, \tag{4.84}$$

where

$$\mathbf{A} \equiv \begin{bmatrix} \frac{1}{\beta} & \frac{1}{\beta}\left\{\Psi_1 - \Psi_2 \frac{(\psi_1 + \alpha\psi_2)}{\psi_1} + \phi_y \left[1 - \Psi_2\left(1 - \frac{1}{\psi_1}\right)\right]\right\} \\ 0 & 1 + \frac{\alpha\psi_2}{\psi_1} + \phi_y\left(1 - \frac{1}{\psi_1}\right) \end{bmatrix}, \tag{4.85}$$

with $\Psi_1 \equiv (1-\lambda\beta)\psi_2\left[\alpha(\psi_2 - 1) + 1 - \lambda\right]$ and $\Psi_2 \equiv (1-\lambda\beta)\psi_2$.

Because the system (4.84) is purely forward looking, $[\tilde{q}_t, \tilde{y}_t] = [0,0]$ for $\forall t$ is the only valid rational expectation solution if and only if both eigenvalues of the coefficient matrix $\mathbf{A}$ lie outside the unit circle, which requires that $A_{22} > 1$, i.e.,

$$\phi_y > \frac{\alpha\psi_2}{1-\psi_1} \equiv \phi_y^*(>0). \tag{4.86}$$

If (4.86) is satisfied, then given the relationship between the inflation and the output gap implied by (4.77), $\pi_t = 0$ for $\forall t$, i.e., full price stability can hence be guaranteed. This in turn implies that

$$i_t = r_t^n \tag{4.87}$$

for $\forall t$ according to the policy rule (4.82), i.e., the instrument rate must be matched one-for-one to the natural rate of interest in equilibrium. The economic system then tracks the natural level of allocation as a consequence of the successful implementation of the SIT rule.

More generally, if the inflation coefficient of the SIT rule takes a value that differs from one, then combining (4.75)-(4.77) with (4.82) yields

$$\begin{bmatrix} E_t \tilde{q}_{t+1} \\ E_t \tilde{y}_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \mathbf{B} \begin{bmatrix} \tilde{q}_t \\ \tilde{y}_t \\ \pi_t \end{bmatrix}, \tag{4.88}$$

where

$$\mathbf{B} \equiv \begin{bmatrix} \beta & (1-\lambda\beta)\psi_2 & 1-\phi_\pi \\ 0 & \psi_1 & (1-\phi_\pi)(\psi_1-1) \\ 0 & 0 & \beta \end{bmatrix}^{-1} \begin{bmatrix} 1 & (1-\lambda\beta)\psi_2\left[\alpha(\psi_2-1)+1-\lambda\right]+\phi_y & 0 \\ 0 & \psi_1+\alpha\psi_2+(\psi_1-1)\phi_y & 0 \\ 0 & \psi_3(1-\psi_2) & 1 \end{bmatrix}. \tag{4.89}$$

Once again, since the system represented by (4.88) is purely forward looking, for a unique bounded rational expectation solution $\left[\tilde{q}_t, \tilde{y}_t, \pi_t\right] = [0,0,0]$ to be guaranteed, we need all three eigenvalues of the coefficient matrix $\mathbf{B}$ lie outside the unit circle. When this is true, $\pi_t = 0$ for $\forall t$, i.e., inflation would be entirely stabilised as well.

As a numerical illustration, I assume plausible calibrated values for the exogenous parameters with one model period being considered to be a quarter, which are broadly consistent with the relevant literature (e.g., Kiyotaki and Moore 2019, Dong et al. 2020). Specifically, I set the subjective discount factor $\beta = 0.99$ (i.e., annual real interest rate of around 4% in SS), capital share $\alpha = 0.4$, one minus capital depreciation rate $\lambda = 0.97$, inverse of the labour supply elasticity $\varphi = 5$; for the intermediate goods sector, I assume that $\xi = 0.75$ (average price duration of one year) and $\epsilon = 11$ (optimal gross markup under flexible prices of 1.1). For the arrival rate of an investment opportunity, I set $\eta = 0.05$, so that the condition for the existence of an equilibrium with a bubble is safely satisfied. Under the chosen parameter values, it turns out that for the determination of the natural interest rate, $N_r^{k^n} = -0.0072$ and $N_r^{q^{B^n}} = 0.0404$, implying that a one unit positive bubble shock (i.e., $\hat{q}_0^B \equiv q_0^B - q^B = q_0^B = 1$) would drive the *Wicksellian* rate and then the nominal rate up by around 4%; also, the threshold value ϕ_y^* is around 0.7868, while e.g., $(\phi_y, \phi_\pi) = (0.8, 1.1)$ is also capable of ensuring determinacy of the system in the more general case.

Therefore, the above outcomes confirm that the strict-inflation policy target could be successfully implemented with a credible threat of the CB adjusting the instrument rate if the actual allocation deviates from the natural one, which merely requires finite policy response coefficients, but conditional on the monetary authority reacting precisely to any variations in the capital stock or in the size of the bubble in order to match one-for-one the natural interest rate.[114] In this sense, LAB is indispensable in attaining full price stability under the proposed SIT rule.

4.5.1.3 The Simple Interest Rate Rule

The SIT rule proposed in the above section, albeit theoretically plausible, is difficult to conduct practically, since it requires accurate knowledge of the structure and state of the economy to precisely pin down the two coefficients for the capital stock and the bubble, and the changes in the economy's aggregate capital stock and the size of the bubble themselves altogether in order to have the CB match one-for-one the natural rate of interest all the way through. Therefore, it is desirable to consider alternative policy rules which do not have these strict requirements for knowledge, but which could be practically implementable for a CB with a central concern of inflation stabilisation in a world exposed to bounded bubble fluctuations.

In this section, I hence adopt a Taylor-type simple interest rate rule (SIR) of the form

$$\hat{i}_t = \phi_y \hat{y}_t + \phi_\pi \pi_t + \phi_q \hat{q}_t^B \qquad (4.90)$$

to investigate if deploying LAB strategy is advisable or not in a more pragmatic setting with regard to attaining the goal of economic, particularly, inflation, stability in a bubbly environment. The SIR rule specified by (4.90) combines the conventional stabilisation motives which are parameterised by ϕ_y and ϕ_π, with a potential LAB desire which is parameterised by $\phi_q \geq 0$. Compared to the SIT rule, it is clear that what makes the SIR rule "simple" is the absence of a (time-varying) intercept term which tracks the natural rate of interest in the formulation.

Equations (4.53), (4.55), (4.58), (4.62)-(4.65) together with the proposed SIR rule (4.90) then jointly describe the equilibrium behaviour of $\hat{k}_t, \hat{q}_t^B, \hat{q}_t, \hat{y}_t, \hat{i}_t^R, \hat{r}_{kt}$ and π_t

[114] This conclusion echoes that drawn by Woodford (2003) with only endogenous capital accumulation, i.e., tracking precisely variations in the natural rate of interest plays a critical role for the monetary authority in pursuing inflation stability.

in a neighbourhood of the bubbleless SS with the representative expectational difference equation system being fifth order.[115] As an illustration, three scenarios with different degrees of policy response to variations in the size of the bubble are examined: $\phi_q = 0, 0.01, 0.06$, while $\phi_y = 0.125, \phi_\pi = 1.5$ for all the three circumstances. Given the same set of calibrated values for the model parameters as in Section 4.5.1.2, the impulse responses of the model for these three cases with 1% units of one-period positive bubble shock hitting the system at some time $t = 0$ is depicted in Figure 4.1.

A prominent feature in Figure 4.1 is that even a minor increase in the strength of the policy response to variations in the size of the bubble could appear to have drastic impact on the dynamics of the system: compared to the conventional policy which coincides with the case when $\phi_q = 0$, the commitment to a tiny response to the bubble with $\phi_q = 0.01$ is able to dampen the bubble impact on inflation and the output gap in an effective way. However, if a more aggressive LAB stance is deployed, e.g., with the bubble coefficient increasing to 0.06, then both *actual output* and the output gap decrease in the face of the positive bubble shock, companied by a deflation on impact, indicating a potential risk of the LAB policy "overreacting". But this type of overreaction risk would hardly happen if the CB sticks to a conventional policy regime and completely ignores the developments in the bubble size.[116]

[115] Note that equations (4.62) and (4.64) are describing static relationships. Note also that in the system, in addition to $\hat{k}_t$, $\hat{q}_t^B$ is considered as "predetermined" as well, for the reason that the economy near a bubbleless SS is exposed to exogenous bubble shocks.

[116] For example, even with raising the inflation coefficient in the SIR to as high as 15, neither output (gap) nor inflation would turn to be decreasing in the face of the bubble shock in the numerical experiments, as long as the bubble coefficient remains zero. At the very least, actual output is unlikely to decline relative to the SS level even with more extreme conventional policy coefficients.

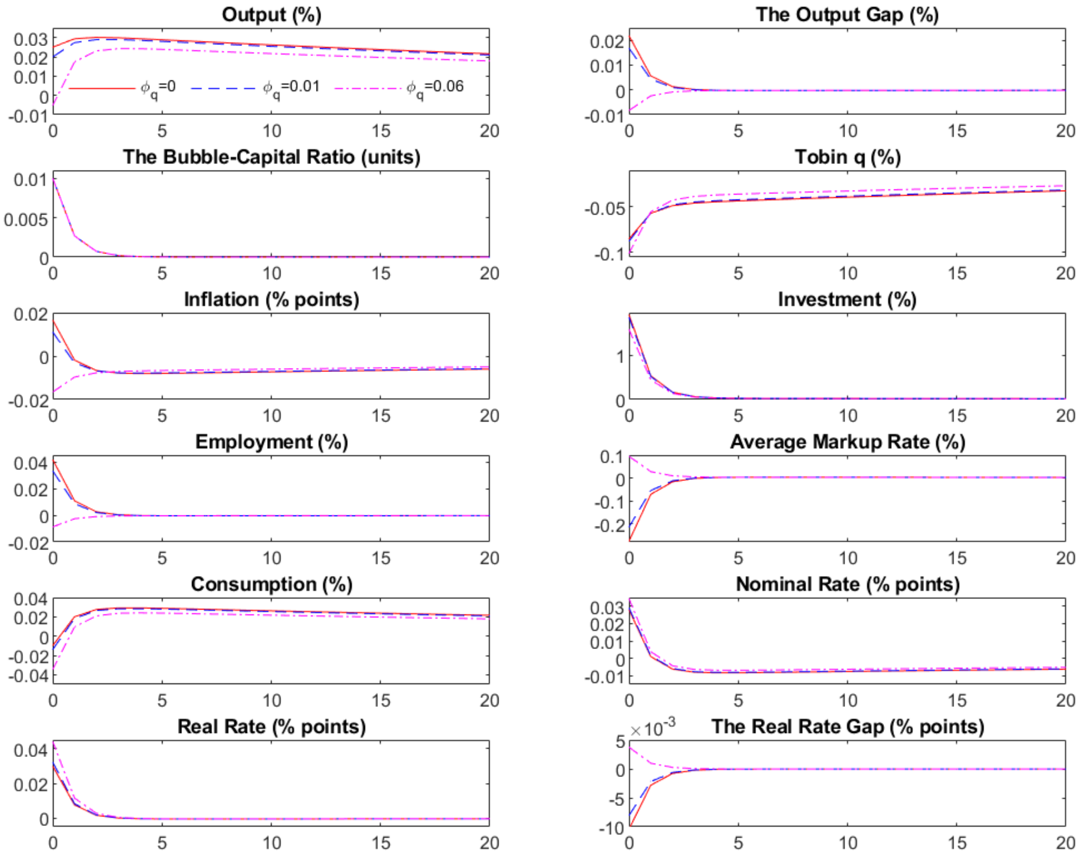

Figure 4.1 Impulse responses of the full model to 1% positive bubble shock

To justify the conjecture just made and to explain the specific patterns emerged under different policy regimes, note first that according to the aggregate investment equation (4.64), an appreciation in the bubble price facilitates aggregate investment immediately, since when the economy is inefficient with Tobin's q remaining greater than one, more capital investment is always desirable. Consequently, consumption tends to be squeezed out, as fewer dividends are now received by the households,[117] which in turn tends to drive up the real interest rate to clear the (nominally) riskless bond market (and the individual wholesale firms' equity market). When consumption declines, labour supply tends to increase due to the income effect, as implied by the optimality condition for labour supply of the household ((4.21)). If prices were fully flexible, the labour demand

[117] Recall that all the wholesale firms' dividends are paid out by the saving firms; thus, when the bubble price goes up, before other things are changed (particularly, the "internal funds", or say, the operational profits of the firms), dividends decrease as a result of the firm spending more on holding the bubbly asset.

curve which is given by (4.34) from (aggregating) the wholesale firm's optimal factor input decision would not be affected by the initial bubble shock, as the price for the wholesale goods in real term then remains constant while capital is predetermined. Thus, equilibrium employment would go up, leading to a higher aggregate output level as well as a higher marginal revenue product of capital (according to (4.41)). Nonetheless, due to the high degree of inelasticity of labour supply,[118] expansions in employment and thus output alongside the bubble boom would be rather mild (on impact) in the flexible price circumstance.

With price stickiness, however, the responsive dynamics of the system to the positive bubble shock could be quite different, as real interest rate is now affected by monetary factors. If the policy feedback to deviations in output or inflation is accommodating and the actual real interest rate is being lower than the natural rate of interest on impact,[119] as in the first case where the bubble coefficient is set to be zero in the SIR rule (4.90), actual consumption would tend to be higher relative to its natural counterpart because of the intertemporal substitution effect. While the higher demand pressure for the final goods tends to be inflationary for the economy, the combination of the increasing goods' demand and the sticky intermediate goods' prices leads to a stronger expansionary need for production, which implies a lower than the (average) desired markup rate level (for the intermediate goods producers) and a higher labour demand from the wholesale firms, which again can be detected from (4.34).[120] As a result, both employment and real wage go up on impact in the labour market equilibrium, with then a higher than the natural rate of aggregate output level, i.e., a positive output gap in general

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[118] As in line with the literature, this aspect is reflected as a high calibrated value of φ in the present context.

[119] If the policy feedback to the conventional targets (i.e., output and inflation) is aggressive, then the responsive pattern could vary somehow. To streamline the analysis here, I skip this part of discussions of the numerical simulations.

[120] Aggregate investment is also higher in equilibrium in this case, because of the higher internal funds possessed by investing firms, which is contributed by a higher marginal revenue product of capital (due to both the higher real wholesale price and the higher labour input) relative to the outcome under flexible prices.

equilibrium.[121]

So far, therefore, given the non-trivial demand side impact of the bubble boom on output with the presence of nominal rigidities, it seems that allowing the policy to respond additionally to developments in the size of the bubble could be helpful in terms of matching the actual real rate of interest to its natural counterpart and then controlling inflation through depressing aggregate demand. This does seem to be proved by the policy experiment with $\phi_q = 0.01$, as it is evident that the real interest rate gap and then the output gap is narrowed by the active policy response to the bubble shock.

However, what is crucial here is that in the present scenario where the model economy fluctuates around the inefficient but bubbleless SS, the evolution of the bubble is independent of the (real) interest rate changes up to a first-order approximation, i.e., monetary interventions have no impact on the movements in the size of the bubble. As a consequence, even when the policy response to variations in the bubble size turns out to be aggressive, i.e., when $\phi_q = 0.06$ in the current experimental context – recall that the bubble coefficient in the natural real interest rate determination equation (4.81) is only around 0.04 – the too high nominal instrument rate would not be automatically "revised" down as it would when it comes to feedback to deviations in output or inflation, because now the bubble remains unaffectedly higher.

Therefore, it is evident that the lack of an "endogenous feedback mechanism" to the variations in the size of the rational bubble of the present sort contributes to the over-reaction risk of the LAB strategy, but this is not very likely to happen in a conventional policy regime. This is because if the interest rate rule involves a direct response to variations in the size of the bubble, although the changes in the policy rate cannot shape the evolution of the bubble itself, the latter can still affect aggregate demand and output via the LAB policy. Thus, additional motive to lean against the bubble in the SIR rule

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[121] The persistent positive output gap in the transition periods in the present scenario is partly explained by the stronger aggregate demand induced by the persistently higher capital investments boosted by the intrinsically prolonged bubble fluctuation with price stickiness.

may *not* be advisable in handling the bubble-driven fluctuations, because it risks causing an unintentional recession in the face of a bubble boom.

Interestingly, the above conclusion echoes to some extent that drawn from Ikeda (2021) from a welfare-maximisation of households' point of view, even though his model setup and the type of "bubble shock" studied there are different from the present one. In Ikeda's (2021) numerical investigations, extra monetary policy feedback on bubble developments is generally counterproductive, as it tends to excessively curb real economic activity and damage the welfare of households severely as a consequence. Similar to my model, bubbles in Ikeda (2021) are also beneficial to the economy, since they relax borrowing constraint of investing firms and hence stimulate capital investment and thus aggregate output. The positive effect of the higher efficiency of the economic system resulting from a bubble boom on the welfare of the households tends to outweigh the induced negative impact of possible higher volatility of inflation or the output gap,[122] so that dampening the bubble-driven boom too much would do more harm than good to the overall welfare of the economy. In fact, if the bubble coefficient in a simple interest rate rule is not restricted to be non-negative, then welfare optimisation may even require the central bank to inflate the size of the bubble from an initial (positive) bubble shock by reducing the interest rate called for by the LAB strategy, as numerically shown to be the case in Dong et al. (2020) in a bubbly economy similar to Ikeda's (2021).[123]

4.5.2 Fluctuations around a Bubbly SS

Next, I turn to study the equilibrium dynamics in a neighbourhood of a bubbly SS, where, unlike the case studied in the previous section, the bubble is no longer evolving autonomously and is affected by changes in the interest rate. Therefore, monetary policy

[122] Since the former positive impact is presumably of first-order, while the latter should be of second-order – recall that in a bubbly economy of the type studied in the present chapter as well as in Ikeda (2021), the corresponding SS must always be *inefficient*.

[123] Apparently, this conclusion challenges the conventional wisdom of "leaning against the wind" which calls for an increase in the instrument rate to restrain a bubble boom.

may now be effective in attaining full price stability by means of completely eliminating potential bubble-driven fluctuations of the system.

4.5.2.1 The System in "Gap" Terms

When the economy fluctuates around a bubbly SS with $q^B > 0$, the bubble evolves according to (4.54). Therefore, the actual bubble size is now in general distinguishable from its natural counterpart, i.e., $\tilde{q}_t^B \equiv \hat{q}_t^B - \hat{q}_t^{B^n}$ is not necessary zero anymore, but is given by

$$\tilde{q}_t^B = E_t \tilde{q}_{t+1}^B + \eta \beta q_b q^B E_t \tilde{q}_{t+1} - q^B \tilde{r}_t \,. \tag{4.91}$$

In what follows, I continue to consider period "*t*" as the "current" period, so that by definition in line with Woodford (2003), $\tilde{k}_t = 0$. Hence, for aggregate investment, we now have

$$(1-\lambda)\tilde{i}_t^R = \eta R_{k_b} \tilde{r}_{kt} + \eta \tilde{q}_t^B \,,$$

i.e.,

$$\tilde{r}_{kt} = \frac{1-\lambda}{\eta R_{k_b}} \tilde{i}_t^R - \frac{1}{R_{k_b}} \tilde{q}_t^B \,, \tag{4.92}$$

while

$$\tilde{r}_{kt+1} = \frac{1-\lambda}{\eta R_{k_b}} \tilde{i}_{t+1}^R - (1-\lambda)\tilde{i}_t^R - \frac{1}{R_{k_b}} \tilde{q}_{t+1}^B \tag{4.93}$$

according to the one-period-ahead aggregate investment equation (4.64) and the fact that $\tilde{k}_{t+1} = (1-\lambda)\tilde{i}_t^R$. Also, from the relationship incorporating the labour market equilibrium ((4.62)),

$$\tilde{i}_t^R = \psi_4 \tilde{y}_t + \psi_5 \tilde{q}_t^B \,, \tag{4.94}$$

with $\psi_4 \equiv \left(\frac{1+\varphi}{1-\alpha} + \frac{Y_b}{C_b}\right) \frac{\eta R_{k_b} C_b}{(1-\lambda) C_b + \eta I_b R_{k_b}}$ and $\psi_5 \equiv \frac{\eta C_b}{(1-\lambda) C_b + \eta I_b R_{k_b}}$; and

$$\tilde{i}_{t+1}^R = \psi_4 \tilde{y}_{t+1} + \psi_5 \tilde{q}_{t+1}^B - \psi_4 \psi_6 \tilde{y}_t - \psi_5 \psi_6 \tilde{q}_t^B \,, \tag{4.95}$$

with $\psi_6 \equiv \alpha(1-\lambda)\frac{1+\varphi}{1-\alpha}\frac{\eta R_{k_b} C_b}{(1-\lambda)C_b + \eta I_b R_{k_b}}$, after substituting out $\tilde{r}_{kt}$ and $\tilde{r}_{kt+1}$ using (4.92), (4.93) and $\tilde{k}_{t+1} = (1-\lambda)\tilde{i}_t^R$. We can now use (4.94) and (4.95) to substitute out $\tilde{i}_{t+1}^R$ and $\tilde{i}_t^R$ in

$$E_t\tilde{y}_{t+1} - \frac{I_b}{Y_b}E_t\tilde{i}_{t+1}^R = \tilde{y}_t - \frac{I_b}{Y_b}\tilde{i}_t^R + \frac{C_b}{Y_b}\tilde{r}_t \tag{4.96}$$

to yield

$$\left(Y_b - I_b\psi_4\right)E_t\tilde{y}_{t+1} - \left(I_b\psi_5\right)E_t\tilde{q}_{t+1}^B = \left[Y_b - I_b\left(\psi_4 + \psi_4\psi_6\right)\right]\tilde{y}_t - \left[I_b\left(\psi_5 + \psi_5\psi_6\right)\right]\tilde{q}_t^B + C_b\tilde{r}_t. \tag{4.97}$$

Also, combining (4.92) with (4.94) to substitute out the investment gap yields

$$\tilde{r}_{kt} = \left(\frac{1-\lambda}{\eta R_{k_b}}\psi_4\right)\tilde{y}_t + \left(\frac{1-\lambda}{\eta R_{k_b}}\psi_5 - \frac{1}{R_{k_b}}\right)\tilde{q}_t^B, \tag{4.98}$$

while

$$\begin{aligned}&\tilde{r}_{kt+1} + \left[\frac{1-\lambda}{\eta R_{k_b}}\psi_4\psi_6 + (1-\lambda)\psi_4\right]\tilde{y}_t + \left[\frac{1-\lambda}{\eta R_{k_b}}\psi_5\psi_6 + (1-\lambda)\psi_5\right]\tilde{q}_t^B \\ &= \left(\frac{1-\lambda}{\eta R_{k_b}}\psi_4\right)\tilde{y}_{t+1} + \left(\frac{1-\lambda}{\eta R_{k_b}}\psi_5 - \frac{1}{R_{k_b}}\right)\tilde{q}_{t+1}^B\end{aligned} \tag{4.99}$$

according to the one-period-ahead aggregate investment equation and by making use of $\tilde{k}_{t+1} = (1-\lambda)\tilde{i}_t^R$ and (4.94)-(4.95).

For the law of motion for Tobin's q and given (4.99), we have

$$\begin{aligned}\tilde{q}_t &= \beta(\eta R_{k_b} + \lambda)E_t\tilde{q}_{t+1} + (1-\lambda\beta)E_t\tilde{r}_{kt+1} - \tilde{r}_t \\ &= \beta(\eta R_{k_b} + \lambda)E_t\tilde{q}_{t+1} + (1-\lambda\beta)\left(\frac{1-\lambda}{\eta R_{k_b}}\psi_4\right)\tilde{y}_{t+1} + (1-\lambda\beta)\left(\frac{1-\lambda}{\eta R_{k_b}}\psi_5 - \frac{1}{R_{k_b}}\right)\tilde{q}_{t+1}^B \\ &\quad - (1-\lambda\beta)\left[\frac{1-\lambda}{\eta R_{k_b}}\psi_4\psi_6 + (1-\lambda)\psi_4\right]\tilde{y}_t - (1-\lambda\beta)\left[\frac{1-\lambda}{\eta R_{k_b}}\psi_5\psi_6 + (1-\lambda)\psi_5\right]\tilde{q}_t^B - \tilde{r}_t.\end{aligned} \tag{4.100}$$

Finally, for the inflation equation, since $\tilde{k}_t = 0$ at current period t,

$$\pi_t = \beta E_t \pi_{t+1} + \psi_3 \left(-\tilde{y}_t + \tilde{r}_{kt} \right)$$
$$= \beta E_t \pi_{t+1} + \psi_3 \left(\frac{1-\lambda}{\eta R_{k_b}} \psi_4 - 1 \right) \tilde{y}_t + \psi_3 \left(\frac{1-\lambda}{\eta R_{k_b}} \psi_5 - \frac{1}{R_{k_b}} \right) \tilde{q}_t^B , \tag{4.101}$$

while the second line of (4.101) is obtained by substituting out $\tilde{r}_{k_t}$ using (4.98).

Before moving into discussing the monetary policy formulations, we may gain some preliminary ideas about the driving forces of the output gap when the system fluctuates around the bubbly SS by rewriting (4.96) and solving it forward as:

$$\tilde{y}_t = E_t \tilde{y}_{t+1} - \frac{I_b}{Y_b} E_t \tilde{i}_{t+1}^R + \frac{I_b}{Y_b} \tilde{i}_t^R - \frac{C_b}{Y_b} \tilde{r}_t$$
$$= \frac{I_b}{Y_b} \tilde{i}_t^R - \frac{C_b}{Y_b} \sum_{T=0}^{\infty} E_t \tilde{r}_{t+T} , \tag{4.102}$$

by assuming $\lim_{T \to \infty} E_t \tilde{y}_{t+T+1} = 0$ and $\lim_{T \to \infty} E_t \tilde{i}_{t+T+1}^R = 0$. Substituting out the investment gap by using (4.94) and rearranging yields

$$\tilde{y}_t = \frac{I_b \psi_5}{Y_b - I_b \psi_4} \tilde{q}_t^B - \frac{C_b}{Y_b - I_b \psi_4} \sum_{T=0}^{\infty} E_t \tilde{r}_{t+T} . \tag{4.103}$$

Comparing (4.103) with (4.79), it can be noticed that, when the economy is bubbly in the steady state, then apart from the impact of current and expected future discrepancies between actual and natural real interest rate, variations in the output gap are now affected also by variations in the size of the bubble (gap), which is in turn affected by the "gaps" in the implied market value of a unit of capital anticipated in the future period as well as the current and future real interest rate gaps, since

$$\tilde{q}_t^B = \left(\pi \beta q_b q^B \right) \sum_{T=1}^{\infty} E_t \tilde{q}_{t+T} - q^B \sum_{T=0}^{\infty} E_t \tilde{r}_{t+T} \tag{4.104}$$

from (4.91).

4.5.2.2 The Strict-inflation-targeting Rule

Similar to the spirit underlying the analysis in Section 4.5.1.2, the discussion of the SIT rule here is in line with the approach in Woodford (2003), Chapter 5. However, unlike the bubbleless case, if prices were fully flexible, then the equilibrium of the economy fluctuating around the bubbly SS would be determinate and should not subject to any "sunspot" type bubble shock.[124] Therefore, the natural real rate of interest depends only on the capital stock in this scenario, which could be expressed as

$$\hat{r}_t^n = N_{r_b}^{k^n} \hat{k}_t^n \tag{4.105}$$

with $N_{r_b}^{k^n}$ again some endogenously determined coefficient. Evaluated at any current date t, (4.105) becomes

$$\hat{r}_t^n = N_{r_b}^{k^n} \hat{k}_t \ . \tag{4.106}$$

Suppose now the CB deploys an interest rate rule of the form

$$\hat{i}_t = \bar{i}_t + \phi_y \tilde{y}_t + \phi_\pi E_t \pi_{t+1} + \phi_q \tilde{q}_t^B, \quad (\phi_\pi \geq 1) \tag{4.107}$$

with

$$\bar{i}_t = \hat{r}_t^n , \tag{4.108}$$

i.e., the CB adjusts its interest-rate operating target to the exact same extent as the natural real rate of interest is changed by variations in the capital stock. An explicit LAB component for the authority is also introduced, parameterised by $\phi_q \geq 0$ in (4.107), in addition to the conventional stabilisation motives based on the output gap and (expected) inflation. A purely forward looking system for the determination of $\{\tilde{q}_t^B, \tilde{q}_t, \tilde{y}_t, \pi_t\}$ (i.e., they are non-predetermined state variables including the bubble-capital gap in the present bubbly case) can then be obtained by combining (4.107) with (4.91), (4.97), (4.100) and (4.101), and can be written in the form

$$\mathbf{B}_b \begin{bmatrix} E_t \tilde{q}_{t+1}^B \\ E_t \tilde{q}_{t+1} \\ E_t \tilde{y}_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \mathbf{A}_b \begin{bmatrix} \tilde{q}_t^B \\ \tilde{q}_t \\ \tilde{y}_t \\ \pi_t \end{bmatrix}, \tag{4.109}$$

with $\mathbf{B}_b$ and $\mathbf{A}_b$ the corresponding coefficient matrices; or simply

[124] Recall the analysis in Section 3.4.2.2 of Chapter 3, when the economy fluctuates around a bubbly SS, it is impossible to have a sunspot bubble shock in the system with flexible prices.

$$\begin{bmatrix} E_t\tilde{q}_{t+1}^B \\ E_t\tilde{q}_{t+1} \\ E_t\tilde{y}_{t+1} \\ E_t\pi_{t+1} \end{bmatrix} = \mathbf{C}_b \begin{bmatrix} \tilde{q}_t^B \\ \tilde{q}_t \\ \tilde{y}_t \\ \pi_t \end{bmatrix}, \tag{4.110}$$

with $\mathbf{C}_b = \mathbf{B}_b^{-1}\mathbf{A}_b$.

To ensure that $[\tilde{q}_t^B, \tilde{q}_t, \tilde{y}_t, \pi_t] = [0,0,0,0]$ is the unique bounded rational expectation solution to the system (4.110) so that the aggregate price is always fully stabilised with potential bubble-driven fluctuations being completely ruled out – because in that case, we would have $\tilde{q}_t^B = \pi_t = 0$ for all t,[125] it requires that all four eigenvalues of $\mathbf{C}_b$ lie outside the unit circle. In the numerical experiments with the same set of calibrated values for the exogenous parameters as in Section 4.5.1.2, $N_{r_b}^{k^n} = -0.0125$, with the bubble-capital ratio along the bubbly SS given by $q^B \simeq 0.552$, i.e., a bubble of size roughly half of the (quarterly) capital stock in the economy. When $(\phi_y, \phi_\pi, \phi_q)$ takes the values of either, e.g., $(0.01, 1.01, 0.5)$, $(0.01, 1.02, 0)$ or $(0.03, 1.01, 0)$, these can guarantee the determinacy of the system.

Although the policy response to the gap in bubble movements is seemingly less effective than those to conventional targets in terms of ensuring full price stability, it does its job anyway in this SIT setting, as far as expanding the determinacy regions of the policy parametric space is concerned.[126] Nonetheless, LAB is no longer indispensable for attaining a strict inflation target in this bubbly situation, unlike the bubbleless case discussed in Section 4.5.1.2 above.

4.5.2.3 The Simple Interest Rate Rule

Let's now turn to investigate the same form of SIR rule specified by (4.90), which again

[125] Recall that under flexible prices, it is impossible to have bounded bubble fluctuations in the system. Therefore, the bubble gap remaining zero all the time implies that there is not any variation in the actual bubble size under sticky prices as well.

[126] Thus, this outcome is contrary to that obtained by Nisticò (2012) where irrational bubble fluctuations are studied, part of which states that LAB policy generally reduces the probability of a system being determinate.

does not require the authority to track precisely the natural rate of interest. Since the evolution of the bubble around a bubbly SS can be shaped by the monetary policy, it should now be possible to eliminate completely potential bubble(-driven) fluctuations in the system to achieve the goal of economic stability by choosing proper policy coefficients in the SIR rule. Specifically, I am interested in the question of whether determinacy of the solution of the system can hold under the SIR rule around a bubbly SS – if it holds, bubble (and other sunspot) shocks as an independent source of macroeconomic disturbances can then be entirely ruled out.

In the present situation, the equilibrium behaviour of the system is jointly represented by equations (4.54), (4.55), (4.58), (4.62)-(4.65) plus (4.90) for $\hat{k}_t, \hat{q}_t^B, \hat{q}_t, \hat{y}_t, \hat{i}_t^R, \hat{r}_{kt}$ and π_t in a neighbourhood of the bubbly SS with the expectational linear difference equation system being again fifth order. Therefore, to insulate the system from bounded bubble-driven fluctuations, it equivalently requires the system to be determinate, i.e., four of the five eigenvalues of it are unstable.[127]

While the fifth order system again cannot be solved for algebraically, it may still be useful to get a sense of the effectiveness of adopting a LAB strategy in the SIR rule with which bounded bubble-driven fluctuations could be completely ruled out. Bearing this purpose in mind, I assess a numerical version of the full model near a bubbly SS with the same set of parameter values as in Section 4.5.2.2. It turns out that, in line with the finding discovered in the SIT case in the above section, although allowing the policy to respond additionally to variations in the size of the aggregate bubble-capital ratio plays a seemingly inefficient role in ensuring a unique bounded solution for the system, it does appear to be helpful with regards to expanding the determinacy region of the parametric space. For instance, while muting the response to movements in the size of the bubble by setting $\phi_q = 0$ and $\left(\phi_y, \phi_\pi\right) = \left(0.01, 0.90\right)$ fails to insulate the system from sunspot (bubble) shocks, re-deploying the LAB strategy by additionally setting

[127] Note that only the capital stock is predetermined among the variables in the present case.

$\phi_q \geq 1.68$ or by increasing slightly the policy coefficient on inflation to 1.01 (with $\phi_y = 0.01$ and $\phi_q = 0$) restores the system to determinacy. In other words, this numerical example demonstrates that it requires a much stronger strength of response in the SIR rule to deviations in the bubble size than those to the conventional policy targets in order to entirely eliminate the bounded bubble-driven fluctuations in the economy, seemingly indicating some sort of ineffectiveness of the LAB strategy.

To better comprehend why this may be the case, particularly, why LAB appears to be only weakly effective as far as ruling out bubble-driven fluctuations is concerned, it may be instructive to consider how the rational bubbles interact with changes in (real) interest rate. Before the sunspot bubble shock is ruled out, an interest rate rise tends to directly drive up the expected future bubble prices according to the law of motion of the bubble (4.54) around a bubbly SS. From this stance, it seems that setting the bubble coefficient $\phi_q > 0$ in the SIR rule should be effective in generating an explosive path given the special rationality property of the bubble.

However, there is another determinant other than the rate of interest in the pricing equation of the bubble (i.e., (4.54)) in the present framework, namely, the "liquidity premium" which is positively correlated to the market value of a unit of capital (in the future) (i.e., $\hat{q}_{t+1}$). The value of the latter is increasing in the rate of interest, meaning that a higher interest rate level would lead to a higher anticipated market value of a unit of capital and hence a higher liquidity premium commanded by the rational bubble. Notably, however, the future bubble price is *decreasing* in the liquidity premium. In other words, a rise in the interest rate would *indirectly* lead to a *lower* bubble price expected in the future through the channel of an increasing future Tobin's q. In fact, if we rewrite (4.54) by using (4.55),

$$\begin{aligned} E_t \hat{q}^B_{t+1} &= \hat{q}^B_t - \eta\beta q_b q^B_b E_t \hat{q}_{t+1} + q^B_b \hat{r}_t \\ &= \hat{q}^B_t - \frac{\eta q_b q^B_b}{\eta R_{k_b} + \lambda}\hat{q}_t + \frac{\eta q_b q^B_b (1-\lambda\beta)}{\eta R_{k_b} + \lambda} E_t \hat{r}_{kt+1} + q^B_b \left(1 - \frac{\eta q_b}{\eta R_{k_b} + \lambda}\right)\hat{r}_t, \end{aligned} \tag{4.111}$$

it is then evident from the second line of (4.111) that the direct impact of real interest rate changes on the expected bubble price in the next period is partially offset by the opposite effect of the interest rate on the expected Tobin's q value. It is also not difficult to show algebraically that the higher the arrival rate of an investment opportunity, the larger the offsetting effect induced by the Tobin's q's response to variations in the real interest rate.

Therefore, the net impact of a higher policy rate on the future bubble price may be insignificant, because the two determinants of the pricing of the rational bubble tend to move in opposite directions in response to the real interest rate changes. Putting this into the context of the LAB policy, this implies that feedback on variations in the bubble size would be less effective in terms of inflating the bubble to the extent that fluctuations of them are then completely ruled out under the Blanchard-Kahn condition for determinacy of the system, since the endogenous rise in the interest rate called for by the LAB policy in the face of a bubble boom has a milder impact on the size of the bubble after initial bubble shock. This mechanism is a consequence of the special equilibrium requirement that the anticipated return on the rational bubble in the present model consists of not only capital gains but also a liquidity premium. This outcome is also different from monetary policy implications in situations where rational bubbles emerge because of dynamic inefficiency.[128]

4.6 Concluding Comments

The present chapter is another part of my effort complementary to the work done in Chapter 2 to enhance our understanding of monetary policy implications in a world with rational asset price bubbles existing because of financing constraints, and in particular, to understand the possible theoretical underpinnings of adopting a "leaning against the bubble" (LAB) strategy in systemic monetary policy deliberations.

[128] In Galí (2021), policy responses to variations in rational bubbles are apparently more effective than those to a conventional one (i.e., the output gap).

The analysis of this chapter has been conducted in a framework with New Keynesian features extended from the baseline one built up in Chapter 3. Specifically, I investigated how alternative interest rate rules may be deployed to tackle or rule out fluctuations in the bubbles which are unrelated to fundamentals but which are a potential threat to economic stability. The neat setup of the model also enables me to make the interactions between monetary interventions and the evolution of the bubble relatively transparent.

Three main insights have emerged from the analysis of the chapter.

First, LAB may be required if full price stability is a central concern of the authority, especially when bubble(-led) booms start from a bubbleless point. A strict-inflation-targeting (SIT) rule, if accurately calibrated, is in principle capable of maintaining zero inflation with the presence of bounded bubble fluctuations by means of the central bank (CB) tracking one-for-one the *Wicksellian* natural rate of interest which varies alongside the capital stock and the size of the bubble.

Second, a simple Taylor-type interest rate (SIR) rule combined with a potential LAB motive may be very effective in dampening bubble impacts on output or inflation, but appears to have a high risk of "overreacting" by causing an economic downturn in the face of a bubble boom. On the other hand, this type of risk is not very likely to occur in a policy regime with only conventional output or inflation stabilisation motives.

Third, if the economy fluctuates in the neighbourhood of a bubbly steady state, then LAB policy appears to be neither necessary nor efficient when it comes to insulating the system from bounded bubble-driven fluctuations. Nonetheless, it is admitted that extra policy feedback to deviations in the size of the bubble under either a SIT rule or a SIR rule may increase the probability of the policy in succeeding in eliminating completely the rational bubble fluctuations in the present model, contrary to some of the findings in the literature.

Overall, the analyses in this chapter do not strongly favour deploying a LAB strategy in monetary policy formulations, especially from a practical standpoint, while conventional policy may be more robust. The conclusions drawn from the analysis in this chapter undoubtedly rely on the specific structure of the present model and the assumptions made; but they are nonetheless in line with the views held by the existing literature to a large extent.

Appendix

4.A Proof of the Validity of Proposition 4 of Chapter 3

The main task here is to show that with labour being elastic in the full model, aggregate capital is still monotonically decreasing in the marginal (revenue) product of capital, so that the conclusion that "$K^* > K_b > K_l$" or "$K^* > K_l$" still holds for "$R_{k_l} > R_{k_b} > R_k^*$" or "$R_{k_l} > R_k^*$". Note that from (4.51) in the main context,

$$\begin{aligned}\frac{\partial K}{\partial R_k} &= N\frac{\partial\left(\frac{K}{Y}\right)^{\frac{1}{1-\alpha}}}{\partial R_k} + \left(\frac{K}{Y}\right)^{\frac{1}{1-\alpha}}\frac{\partial N}{\partial R_k} \\ &= -\left[N\frac{\alpha}{1-\alpha}\left(\frac{K}{Y}\right)^{\frac{\alpha}{1-\alpha}}\left(\frac{\epsilon-1}{\epsilon}\right)\frac{1}{R_k^2}\right] - \left(\frac{K}{Y}\right)^{\frac{1}{1-\alpha}}\left(\frac{1}{1+\varphi}\right)\frac{N}{\Theta_N}\left(\frac{\alpha(1-\lambda)}{(1-\alpha)R_k^2}\right),\end{aligned} \tag{4.112}$$

where $\Theta_N \equiv \frac{\epsilon}{(\epsilon-1)(1-\alpha)} - \frac{\alpha(1-\lambda)}{(1-\alpha)R_k}$ is the term within the square bracket of (4.51) and must be positive for $N > 0$. Therefore, it is evident from (4.112) that $\frac{\partial K}{\partial R_k} < 0$ (within the defined domain). Q.E.D.

4.B Derivations of the "Gap" System in Section 4.5.1.1

To obtain the "gap" relationship (4.75), we substitute out $\tilde{i}_{t+1}^R$ and $\tilde{i}_t^R$ in (4.74) with the right-hand-sides of (4.73) and (4.72).

From the pricing equation for Tobin's q,

$$\tilde{q}_t = \beta E_t \tilde{q}_{t+1} + (1-\lambda\beta) E_t \tilde{r}_{kt+1} - \tilde{r}_t \tag{4.113}$$

when the system fluctuates in the local area of a bubbleless but inefficient SS. Then we use the relationship

$$\tilde{r}_{kt+1} = \tilde{i}_{t+1}^{R} - (1-\lambda)\tilde{i}_t^{R} \tag{4.114}$$

to substitute out $E_t \tilde{r}_{kt+1}$. Further substitutions of $\tilde{i}_{t+1}^{R}$ and $\tilde{i}_t^{R}$ again by making use of (4.73) and (4.72) then reaches (4.76).

From the inflation equation (4.65),

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{\xi}\left(\tilde{y}_t + \tilde{r}_{kt}\right) \tag{4.115}$$

since $\tilde{k}_t = 0$ by definition. Then replace $\tilde{r}_{kt}$ with $\psi_2 \tilde{y}_t$ by making use of (4.69) and (4.72) yields (4.77).

Chapter 5

Conclusions

Whether the objective of financial stability, of which one crucial consideration is large volatilities in asset prices usually viewed as a sign of "bubbles", should be taken into account in the formulation of monetary policy or not is an unsettled issue of recurrent debate. As an effort to enhance the theoretical comprehension of this issue, the present thesis has been devoted to studying the conduct of monetary policies with the presence of *rational* asset price bubbles which are required to be consistent with rational expectations, individual optimisation, and market clearing in equilibrium. Despite its importance for the analysis of the issue at hand, rational bubbles are difficult to be accommodated into a standard New Keynesian (NK) model, the workhorse for monetary policy analysis in modern era, where the former will be ruled out by the transversality condition of the infinitely-lived representative household assumed in the latter in equilibrium. On the other hand, the short life-span overlapping generations (OLG) structure used in the classic literature on rational bubbles is too stylised to be reconciled with data and for serious quantitative analysis.

In the theoretical study in this thesis, I have made use of two types of structures both of which incorporate NK features but allow for the equilibrium existence of rational bubbles. Specifically, in the first part of the study (Chapter 2), I adopt a modified NK model with "perpetual youth" OLG as in Galí (2021), where rational bubbles can emerge in equilibrium because of a dynamic inefficiency problem. With nominal rigidities and monopolistic competition, bounded bubble fluctuations there have an impact on the OLG-NK economy mainly as *aggregate demand shifters*, and the proposed interest rate rules with or without an additional policy response to variations in the size of the bubble can in general be effective in mitigating the bubble's impact on output, albeit at a mild cost of prolonging the bubble-driven fluctuations.

However, in a world featuring a continuum of bubbly equilibria, compared to its counterpart with only a conventional output gap-feedback response, a leaning-against-the-bubble (LAB) strategy may risk causing an economic downturn in the face of a bubble boom, especially if the economy deviates from a steady state with a smaller than a threshold value size of the aggregate bubble. A critical explanation of this outcome lies in the fact that with a sunspot bubble shock, which is much more likely to emerge in a nearly bubbleless economy, changes in the real rate of interest cannot affect the size of the bubble *in the impact period*, but the latter can have an impact on the economy through the LAB policy, thus the feedback routes between these two is asymmetric, in a stark contrast to the case with the conventional policy.

In the second part of the study (Chapter 3 and 4), I establish an analytically tractable NK framework with endogenous capital accumulation, in which firms are subject to a type of uninsurable idiosyncratic investment shocks a là Kiyotaki and Moore (2019) and financing constraints. The combination of these two factors creates an environment for rational bubbles to emerge in equilibrium, because they may serve as a financing vehicle in terms of transferring resources between firms with and without an investment opportunity, and thus may be valuable to the rational agents even though the bubbles are intrinsically worthless. As a consequence, and unlike the case in the OLG-NK model in Chapter 2, rational bubbles in the Chapter 3-4 model are *not* required to grow at the interest rate in equilibrium, because of the "*liquidity premium*" commanded by them resulting from their contribution to the relaxation of firms' financing constraints. In addition, and most importantly, under sticky prices, the developed model enables the potential emergence of rational bubble-driven fluctuations (in output) in equilibrium through both a demand and a supply side mechanism, so that meaningful model-based monetary policy analysis could go ahead.

It turns out that if the model economy rests in a bubbleless state at the beginning, then it is subject to bounded bubble fluctuations while monetary policy is incapable of shaping them (to a first-order approximation). In that case, LAB is necessary in order to match exactly one-for-one the natural rate of interest if the central bank aims to fully stabilise aggregate prices. However, an additional LAB motive may be harmful in a more practical setting where the central bank adopts a simple Taylor-type interest rate rule, in the sense that it may cause an economic recession in the face of a bubble boom. This is because changes in the policy rate cannot affect the trajectory of the bubble when the economy fluctuates around a bubbleless steady state, but the bubble can affect (the demand side of) the economy through the LAB policy. The asymmetric feedback mechanism between the bubble and the interest rate is thus responsible for the downside risk of the extra policy response to variations in the size of the bubble in this case.

On the other hand, if the economy is originally bubbly, then a LAB strategy is neither necessary for achieving a strict inflation target nor effective in completely ruling out bounded bubble(-driven) fluctuations in the economy. A likely reason which is particularly key for the latter phenomenon is that the two determinants of the pricing of the bubble tend to move in opposite directions in the face of policy feedback on variations in the size of the bubble, so that the net effect of an interest rate change induced by the LAB policy on the evolution of the bubble may be insignificant as a result.

Therefore, overall, regardless of the fact that macroeconomic instability caused by rational bubble fluctuations displays distinctive patterns, especially of being intrinsically persistent, the analyses of my thesis do not demonstrate strong theoretical support for adopting a LAB strategy in monetary policy, either from a demand or a supply side perspective. Particularly considering the potential risk of overreacting and unintentionally dragging the economy into a recession in an episode of a bubble boom, a conventional policy regime with only inflation and the output-gap feedback may still be more robust in practice.

Needless to say, although the analytical tractability of the frameworks present in this thesis provides the transparency about the driving factors underlying the conclusions, it is undeniable that the models may not accurately capture the true systematic structure or the nature of asset market volatilities in a real economy. One important caveat is that the type of rational bubbles studied here is “deterministic”, i.e., the bubbles in aggregate will not “burst” with any certain probability once they pop up, and bubbles themselves are beneficial to economic welfare as a whole, which means they may be of less concern to actual policy makers. Instead, scenarios featuring an eventual collapse of speculative bubbles which may themselves be harmful to the efficiency of the economy, for instance, where the bubble exists due to the presence of information frictions in financial markets or of agency problems may also be important considerations for central bankers, but the relevant policy deliberations may be very different to the ones suggested in the thesis in response to a building up bubble episode of those categories. Therefore, further research in these directions is also needed in terms of addressing the ultimate question about “should the central bank lean against asset price bubbles” in practice.

Those limitations notwithstanding, the study of the thesis hopefully provides useful theoretical insights for the policy issue at hand, especially by pointing out a potential downside risk that is special to monetary policies with additional “leaning against the bubble” motive but which is missing in the conventional policy regime.

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