

Some contributions on set-theoretic extensions and consistency strengths

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I confirm that the work submitted is my own, except where work which has formed part of jointly authored publications has been included. My contribution and the other authors to this work has been explicitly indicated below. I confirm that appropriate credit has been given within the thesis where reference has been made to the work of others.

Chapters 2 and 3 are based on work from jointly authored publications.

Chapter 2 is based on *Upwards homogeneity in iterated symmetric extensions*, C. Ryan-Smith, J. Schilhan and Y. Wei [RSSW25], May 2024, accepted by the Journal of Symbolic Logic. I did the first draft of the paper and the first definition of upward homogeneity (though the definition has gone through many versions), Ryan-Smith and Schilhan improved the paper afterwards and added proof for the main theorem.

Chapter 3 is based on *Kreisel–Lévy-type theorems for Kripke–Platek and other set theories*, S. Shu and M. Rathjen [SR22], December 2022, published in Studies in Logic. I wrote out the proof for the main theorem, but Rathjen pointed out to me the application of partial recursive functions in set-theoretic context.

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Abstract

This thesis consists of three chapters, each dealing with a different topic.

Chapter 1 consists of some results in forcings and L -like combinatorics (squares and Suslin trees etc.). We generalise one of Velleman’s results that square sequences can also be defined on limit cardinals not just successor cardinals. We show that a non-trivial ω_2 -c.c. forcing must destroy PFA. We make a tentative definition for the notion of “essential cardinality” of a poset and show that if P is a λ -c.c. forcing then $2^\lambda = |P|^\lambda$ in the generic extension where $|P|$ is the essential size of P .

Chapter 2 concerns a notion called upward homogeneity in iterated symmetric extensions. It is shown that upward homogeneity is the exact condition for a two-step iterated symmetric system to satisfy that if $V \subseteq M \subseteq N$ are models of ZF generated by the symmetric system, then N contains no subsets of V that do not already appear in M . Many known constructions of iterated symmetric extensions satisfy upward homogeneity explicitly.

Chapter 3 proves the result that over Kripke–Platek set theory with infinity (KP), transfinite induction along the ordinal $\varepsilon_{\Omega+1}$ is equivalent to the schema asserting the soundness of KP, where Ω denotes the supremum of all ordinals in the universe; this is analogous to the result that, over Peano arithmetic (PA), transfinite induction along ε_0 is equivalent to the schema asserting the soundness of PA. In the proof we need to code infinitary proofs within KP, and it is done by using partial recursive set functions.

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Chapter 1

Some results regarding forcings and L -like combinatorics

In this chapter we prove some miscellaneous combinatorial results. The first three sections set the background, outlining the relations between trees, square principles, and forcing axioms. More original results start from §1.4.

1.1 Trees

A tree is a partially ordered set with a unique minimal element in which the predecessors of any element are well-ordered; in a sense, they can be regarded as a generalisation of well-ordered sets. While at each level of the tree, there is a path of length of that level, in general there need not be a path whose length is of the height of the tree.

One of the most important applications of trees is that we can translate certain topological problems into combinatorial ones. For instance, a *Suslin line* is a complete dense linear order without end points such that every pairwise disjoint family of open sets is countable, but it is not separable (i.e. not a completion of $\mathbb{Q}$, namely not isomorphic to $\mathbb{R}$). Suslin's problem asks if Suslin lines exist. Kurepa [Kur35] first showed the equivalence between Suslin lines and a certain type of trees in which every pairwise incompatible family is countable. The idea is that the paths of the tree, lexicographically ordered, correspond to the points of the

line, whereas a node of the tree corresponds to the open interval that consists of all paths that pass through it. In general, it is much easier to consider trees than to consider the topological constructions.

Definition 1.1.1. Let κ be a regular cardinal.

- (i) A κ -**tree** is a tree of height κ with levels of size less than κ .
- (ii) A **cofinal branch** through a tree T is a linearly ordered subset of T whose order type is the height of T .
- (iii) A κ -**Aronszajn tree** is a κ -tree with no cofinal branch.
- (iv) A κ -**Suslin tree** is a κ -tree such that every chain and antichain has size $< \kappa$.
- (v) κ has the **tree property** iff there are no κ -Aronszajn trees.
- (vi) On a tree T , a function $f : T \rightarrow T$ is **regressive** if $f(t) <_T t$ for every $t \in T$ that is not a minimal node of T . A tree T of height κ is **special** if there is a regressive map $f : T \rightarrow T$ with the property that for all $t \in T$, $f^{-1}(t)$ can be written as the union of $< \kappa$ antichains of T .

Note that if T is special, then T cannot have cofinal branches: Suppose a special tree T of height κ has a cofinal branch b , so b has order type κ , and the function f which witnesses the specialness of T is regressive on b . By Fodor's lemma, there are κ many nodes that get mapped to a constant value, say t . Then $f^{-1}(t)$ cannot be written as a union of $< \kappa$ many antichains.

So a special κ -tree is Aronszajn. Despite this implication, a special κ -tree is still usually called a special κ -Aronszajn tree.

The better-known version of specialness is defined only when κ is a successor cardinal: If κ is a successor cardinal, then a tree T of height κ is special if T is a union of $< \kappa$ antichains. The generalisation of specialness to all regular κ we have given above is due to Todorćević. Here, we give the proof that the two definitions are equivalent when κ is a successor cardinal.

Lemma 1.1.2. If κ is a successor cardinal then a tree T of height κ is special iff T is the union of $< \kappa$ antichains.

Proof. Let $\mu^+ = \kappa$. One direction is clear: If T is the union of $< \kappa$ antichains, then we simply let f map every node to 0.

For the other direction, given $t \in T$, let $g_t : f^{-1}(t) \rightarrow \mu$ be a decomposition of $f^{-1}(t)$ into μ many antichains. Given $s \in T$ there is an n and a finite sequence $s > f(s) > f^2(s) > \dots > f^n(s) = 0$. We define $h : T \rightarrow \mu^{<\omega}$ by

$$h(s) = \langle g_{f(s)}(s), g_{f^2(s)}(f(s)), \dots, g_{f^n(s)}(f^{n-1}(s)) \rangle.$$

By induction on the level of s we can show that if $t > s$ then $h(t) \neq h(s)$.

Suppose $s = 0$ and $t > s$, then $h(s) = \langle g_0(0) \rangle$ and $h(t) = \langle g_{f(t)}(t), \dots \rangle$. If $f(t) = 0$ then $g_0(t) \neq g_0(s)$ because t and s are comparable, and if $f(t) \neq 0$ then $|h(t)| > 1$, so $h(t) \neq h(s)$.

Suppose $t > s \neq 0$. Then $f(t)$ and $f(s)$ are comparable and $f(s) < s$. If $f(t) = f(s)$, then $g_{f(t)}(t) \neq g_{f(s)}(s)$ since t and s are comparable. If $f(t) \neq f(s)$, then by induction $h(f(t)) \neq h(f(s))$, so again $h(t) \neq h(s)$. $\square$

Remark. If $C \subseteq \kappa$ is club, and $f : T \restriction C \rightarrow T$ is a regressive function such that if $t \in T$ then $f^{-1}(t)$ is a union of $< \kappa$ antichains of T , then we can extend f to the whole T so that f witnesses the specialness of T . Namely, if $\alpha \in \kappa \setminus C$, we send the α -th level to some element at the level $\max(C \cap \alpha)$, and there are only $< \kappa$ more antichains mapped to each point.

We also note that if κ is a successor cardinal, then there is no subtree of a κ -Suslin tree that can be special with height κ . (Here, a subtree of the tree just means a suborder of the tree that is also a tree.)

The tree property at ω is precisely König's lemma. Aronszajn constructed an ω_1 -Aronszajn tree. Generalising Aronszajn's construction, Specker [Spe49] proved that

Theorem 1.1.3. If $\kappa^{<\kappa} = \kappa$, then there is a special κ^+ -Aronszajn tree.

In particular:

Corollary 1.1.4. CH_κ implies that there is a special κ^{++} -Aronszajn tree. $\square$

Mitchell [Mit72] proved that relative to ZFC the tree property at ω_2 is equiconsistent with the existence of a weakly compact cardinal.

There is an old question of Magidor which asks if it is consistent that every regular cardinal greater than ω_1 can have the tree property. It is known that it is

much harder to arrange the tree property at the successors of singular cardinals, or the tree properties at consecutive cardinals. We will see an echo of this fact in the following section.

1.2 Squares

Next we introduce a collection of combinatorial principles which is strongly related to κ -trees that don't have cofinal branches. The following defines what we call "Jensen's squares" (as well as their weakenings).

If X is a set of ordinals, we use $\lim(X)$ to denote the set of limit points of X , i.e. $\lim(X) = \{\alpha \in X \mid \alpha \cap X \text{ is unbounded in } \alpha\}$.

Definition 1.2.1 (Jensen's squares). Let λ be an infinite cardinal and let μ be a cardinal with $2 \leq \mu \leq \lambda^+$. A $\square_{\lambda, < \mu}$ -**sequence** is a sequence $\langle \mathcal{C}_\alpha \mid \alpha \in \lim \lambda^+ \rangle$ such that for each α :

- (i) $\mathcal{C}_\alpha$ is a non-empty family of club subsets of α , each with order type $\leq \lambda$.
- (ii) $|\mathcal{C}_\alpha| < \mu$.
- (iii) For every $C \in \mathcal{C}_\alpha$ and every $\beta \in \lim(C)$, $C \cap \beta \in \mathcal{C}_\beta$.

We say that $\square_{\lambda, < \mu}$ holds if there is a $\square_{\lambda, < \mu}$ -sequence. $\square_{\lambda, \leq \mu}$ -sequences are similarly defined and we write $\square_{\lambda, \mu}$ for $\square_{\lambda, \leq \mu}$.

The square principle $\square_\lambda$ is $\square_{\lambda, 1}$. The principle $\square_{\lambda, \lambda}$ is sometimes also written $\square_\lambda^*$, and called the weak square principle. It is known that in general these principles are strictly decreasing in strength as μ increases ([CFM01] Section 7), i.e., for all infinite λ and $1 \leq \mu_1 < \mu_2 \leq \lambda$, there is a model of ZFC in which $\square_{\lambda, \mu_2} + \neg \square_{\lambda, \mu_1}$ holds.

Observation 1.2.1.1. It is easy to see that $\square_{\kappa, \kappa^+}$ is provable in ZFC, for all κ . (We can define the sequence by choosing a club C_α for each $\alpha < \kappa^+$, then let $\mathcal{C}_\alpha = \{C_\alpha\} \cup \{C_\beta \cap \alpha \mid \beta > \alpha, \alpha \in \lim C_\beta\}$.)

We note the following theorem by Jensen. A proof can be seen in e.g. [Cum05] Section 5.

Theorem 1.2.2. $\square_{\kappa, \kappa}$ holds iff there is a special κ^+ -Aronszajn tree.

Observation 1.2.2.1. Thus, CH_κ implies $\square_{\kappa^+, \kappa^+}$ by Theorem 1.1.3.

Remark. [Kru13] shows that one can define $\square_{\kappa, \kappa}$ -sequences on limit cardinals, and the existence of such a sequence on κ is equivalent to the existence of special κ -Aronszajn trees.

Remark. It is suspected that stronger squares correspond to trees with stronger properties, perhaps trees with ascent paths [LHL18].

$\square_\kappa$ holds everywhere in L as well as some other canonical inner models, but it is independent of ZFC.

Note the following fact, which is easy to see:

Lemma 1.2.3. Let $V \subseteq W$ be two transitive models of ZFC, κ a cardinal of W such that $(\kappa^+)^W = (\kappa^+)^V$. Then if $V \models \square_{\kappa, \lambda}$ for some λ then $W \models \square_{\kappa, \lambda}$. $\square$

This shows that Jensen squares are hard to destroy, and if we want to force the failure of $\square_\kappa$ from a model in which $\square_\kappa$ holds, then we must have collapsed the cardinal. Thus the failure of squares in a model of ZFC can be seen as a detection of how far the model is from the canonical inner models (but it is easy to force the squares, so having squares does not mean the model is close to the canonical inner models).

Squares can be seen as incompactness principles. Here we recall stationary reflection, which is a compactness phenomenon:

Definition 1.2.4. Let κ be an uncountable regular cardinal. Let S be a stationary subset of κ .

- (i) S **reflects at** α if $\alpha < \kappa$, $\text{cf}(\alpha) > \omega$ and $S \cap \alpha$ is stationary in α .
- (ii) $\text{Refl}(S)$ holds if every stationary subset of S reflects at some α .
- (iii) S is **non-reflecting** if S does not reflect at any α .

We cite the following theorems in [CFM01].

Theorem 1.2.5 ([CFM01] Theorem 2.2). Let $\kappa^{<\lambda} = \kappa$ and let $\square_{\kappa, <\lambda}$ hold. If $S \subseteq \kappa^+$ is stationary then there exists a stationary set $T \subseteq S$ such that T does not reflect at any α with $\text{cf}(\alpha) \geq \lambda$. In particular, $\square_{\kappa, <\omega}$ implies the failure of $\text{Refl}(S)$ for every stationary $S \subseteq \kappa^+$.

Theorem 1.2.6 ([CFM01] Theorem 2.3). Let κ be singular. Let $\square_{\kappa, < \text{cf } \kappa}$ hold, and let $S \subseteq \kappa^+$ be stationary. Then there exists a stationary $T \subseteq S$ and $\beta < \kappa$ such that T fails to reflect at points of cofinality greater than β .

Large cardinal axioms with strengths high enough negate weak square principles, the following theorem is an example of such a result:

Theorem 1.2.7 (also from [CFM01]). If κ is supercompact and $\lambda \geq \kappa$, then:

- $\square_{\lambda, < \text{cf } \lambda}$ fails.
- If $\text{cf } \lambda < \kappa$, then $\square_{\lambda, \lambda}$ fails.
- If $\kappa \leq \text{cf } \lambda < \lambda$, then $\square_{\lambda, \text{cf } \lambda}$ can hold.

Observation 1.2.7.1. This theorem is optimal. To see this, assume GCH which is consistent with supercompact. CH_λ implies $\mu^{<\mu} = \mu$ for $\mu = \lambda^+$, so by Theorem 1.2.2 and Theorem 1.1.3, $\square_{\mu, \mu}$ holds in such circumstances. If $\lambda \geq \kappa$ is a regular limit, then it is also a strong limit because of GCH, and then $\lambda^{<\lambda} = \lambda$ and so $\square_{\lambda, \lambda}$ holds.

There are many variants of Jensen's square, and Todorćević's square $\square(\kappa)$ is arguably the most important of them.

Definition 1.2.8 (Todorćević's square). Let $\kappa > \omega_1$ be regular, $2 \leq \mu \leq \kappa$.

$\square(\kappa, < \mu)$ is the following principle: There is a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa \rangle$ such that:

- (i) $\mathcal{C}_{\alpha+1} = \{\{\alpha\}\}$ and $\mathcal{C}_\alpha$ is a family of club subsets of α if α is a limit ordinal.
- (ii) If α is a limit point of $C_\beta \in \mathcal{C}_\beta$, then $C_\beta \cap \alpha \in \mathcal{C}_\alpha$.
- (iii) There is no club $C \subseteq \kappa$ such that for all limit points α in C , $C \cap \alpha \in \mathcal{C}_\alpha$.

Remark. It is easy to see that a $\square_{\kappa, < \mu}$ -sequence is virtually a $\square(\kappa^+, < \mu)$ -sequence, for all κ, μ . But $\square(\kappa^+, < \mu)$ is much weaker than $\square_{\kappa, < \mu}$. In [FH16] it is shown that $\square(\kappa)$ is compatible with stationary reflection, so $\square(\kappa^+)$ does not imply $\square_\kappa$ or even the slightly weaker forms of it. On the other hand, it is shown in [HLH17] that if $\square(\kappa)$ holds then every stationary set of κ can be partitioned into two sets that do not reflect simultaneously, and this is not even implied by $\square(\kappa, 2)$.

Similar to $\square_\kappa$, $\square(\kappa)$ holds almost everywhere in L :

Theorem 1.2.9. In L , for $\kappa > \omega_1$ regular, $\square(\kappa)$ holds iff κ is not weakly compact.

Proof. This can be seen from, for instance, the following two results:

[Tod10] Theorem 8.5 states that the following are equivalent for an inaccessible cardinal θ :

- (i) θ is weakly compact.
- (ii) If $\langle C_\alpha \mid \alpha \in \lim(\theta) \rangle$ is a sequence with $C_\alpha \subseteq \alpha$ club for every $\alpha \in \lim(\theta)$, then there is a club $C \subseteq \theta$ such that for all $\alpha < \theta$ there is a $\beta \geq \alpha$ with $C_\beta \cap \alpha = C \cap \alpha$.

In particular, if θ is weakly compact, then there is no $\square(\theta)$ -sequence.

[SS88] Proposition 6.1 states the following: If $\mu > \omega_1$ is a cardinal which is regular in L , but not weakly compact in L , and $\text{cf } \mu > \omega$, then there is a sequence $\langle C_\alpha \mid \alpha \in \lim(\mu) \rangle$ in L which is a $\square(\mu)$ -sequence in V (and so of course it is also a $\square(\mu)$ -sequence in L). $\square$

And similar to $\square_{\kappa, \kappa}$, there is a characterisation of $\square(\kappa, < \kappa)$ by trees:

Theorem 1.2.10 (Todorćević). $\square(\kappa, < \kappa)$ holds iff there is a κ -Aronszajn tree.

It is natural to ask what are the consistency strengths for failures of squares. For regular κ , it is well-known that the consistency strength for $\neg \square_\kappa$ is a Mahlo cardinal.

Theorem 1.2.11 (Solovay). Let κ be a regular uncountable cardinal and let $\lambda > \kappa$ be a Mahlo cardinal. Then in the Lévy collapse $\text{Col}(\kappa, < \lambda)$, $\lambda = \kappa^+$ and $\square_{\kappa, \mu}$ fails for every $\mu < \kappa$.

Theorem 1.2.12 (Jensen [Jen72]). $\neg \square_\kappa$ implies that κ^+ is Mahlo in L .

Remark. In fact, one can define Jensen square sequences on limit cardinals, and if κ is Mahlo, then there is no Jensen square sequence on κ . This is shown in [Vel83].

The exact consistency strength for the failure of $\square_\kappa$ at singular κ is not known. However, we know that it must be much greater than a Mahlo cardinal. The covering theorem states that if $0^\sharp$ does not exist, then every singular cardinal κ is singular in L , and $(\kappa^+)^L = \kappa^+$. Therefore, if $0^\sharp$ does not exist, then $\square_\kappa$ must hold since it holds in L , by Lemma 1.2.3.

This is similarly true for absence of Woodin cardinals: [Zem10] showed that if $V = K$ where K is the Mitchell–Steel core model, then $\square_\kappa$ holds for every cardinal κ . If there is no Woodin cardinals, then for every singular cardinal κ , $(\kappa^+)^K = \kappa^+$, so $\square_\kappa$ holds for every singular κ .

It is known that failures of consecutive squares have much greater consistency strengths than those of failures of single squares. So far, the best known result is the following:

Theorem 1.2.13 ([Jen+09]). Let $\kappa \geq \omega_3$ be a regular cardinal. Assume that κ is countably closed in the sense that $\eta^\omega < \kappa$ for every $\eta < \kappa$. Suppose that $\square(\kappa)$ and $\square_\kappa$ both fail. Then there is an inner model with a proper class of strong cardinals and a proper class of Woodin cardinals.

1.3 Forcing axioms

In a forcing construction, we start with a ground model V that satisfies ZFC and a forcing (i.e. a partial order with a maximum) $(P, \leq_P, 1_P)$ that lives in V . Elements of P are called conditions and if $p' \leq p$ in P , then we say that p' is stronger than p , or p' extends p . We say that p' and p are compatible, denoted $p' \parallel p$, if they have a common extension.

A subset D of P is said to be dense if for any $p \in P$ there is an extension of p in D . A filter $G \subseteq P$ is a generic filter (with respect to V and P) if for every dense $D \subseteq P$ that lives in V , $D \cap G \neq \emptyset$. In most cases, such generic filters do not live in V , and the smallest ZFC-model that contains V and a generic filter G is called a generic extension of V .

When we consider partial orders as forcings, we are implicitly imposing the left-order topology on the partial orders (for a partial order P , the left-order topology is generated by the base $\{\{x \in P \mid x < p\} \mid p \in P\}$). Then dense sets are exactly those that are dense in this topology.

Definition 1.3.1. Let $\mathcal{K}$ be a class of topological spaces and let κ be a cardinal. The forcing axiom $\text{FA}(\mathcal{K}, \kappa)$ is the following assertion: If $X \in \mathcal{K}$ and $\mathcal{F}$ is a family of $< \kappa$ dense sets of X , then there is a filter that intersects each member in $\mathcal{F}$.

Recall that a forcing P is called c.c.c. if all of its antichains are countable; it is called κ -c.c. if all of its antichains have size $< \kappa$.

Definition 1.3.2. **Martin's axiom** (MA) is $\text{FA}(\mathcal{K}_{\text{c.c.c.}}, 2^\omega)$ where $\mathcal{K}_{\text{c.c.c.}}$ is the class of all c.c.c. forcings.

It is well-known that we can force MA with a c.c.c. forcing without using any large cardinals, therefore $\text{ZFC} + \text{MA}$ is equiconsistent with ZFC, and so naturally one expects that MA is compatible with all Jensen's squares and Todorćević's squares.

Proposition 1.3.3. Suppose P is κ -c.c. If $\square(\kappa)$ holds in V , then $\square(\kappa)$ holds in $V[G]$.

Proof. Recall the following fact: If P is κ -c.c., then every club $C \subseteq \kappa$ in $V[G]$ contains a club from V .

So suppose for a contradiction that in V we have a $\square(\kappa)$ -sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$, but $V[G] \models \neg \square(\kappa)$. So the sequence ceases to be a $\square(\kappa)$ -sequence in $V[G]$, which means that there is a club $C \subseteq \kappa$ such that for all limit points $\alpha \in C$, we have $C_\alpha = C \cap \alpha$. Note that this means $C = \bigcup_{\alpha < \kappa, \alpha \in \text{lim}(C)} C_\alpha$. Since P is κ -c.c., there is a club $D \subseteq \kappa$ in V such that $D \subseteq C$. Define $E = \bigcup_{\alpha < \kappa, \alpha \in \text{lim}(D)} C_\alpha$, then we have $E \subseteq C$. But we actually have $E = C$: indeed, if $\beta \in C_\alpha$ for some $\alpha \in \text{lim}(C)$, let $\alpha' \in \text{lim}(D)$ such that $\alpha' > \alpha$, so $\alpha' \in \text{lim}(C)$ and we have $C_{\alpha'} = C \cap \alpha'$, and then $C_\alpha = C \cap \alpha = C \cap \alpha' \cap \alpha = C_{\alpha'} \cap \alpha$, so $\beta \in C_{\alpha'}$. Thus, E is a club in V such that for all limit points $\alpha \in \text{lim}(E)$, $C_\alpha = E \cap \alpha$. This contradicts $\square(\kappa)$ in V . $\square$

Corollary 1.3.4. MA is compatible with all $\square_\kappa$ for $\kappa > \omega$ and $\square(\kappa)$ for $\kappa > \omega_1$ regular.

Proof. To see this, note that $\square_\kappa$ ($\kappa > \omega$) holds everywhere in L , and the iterated forcing that we use to force MA is c.c.c. and so preserves cardinals. So a $\square_\kappa$ -sequence in L is a $\square_\kappa$ -sequence in the forcing extension by Lemma 1.2.3. Furthermore, $\square(\kappa)$ ($\kappa > \omega_1$ regular) holds in L iff κ is not weakly compact. So if we start with an L that has no weakly compact cardinals, since c.c.c. forcings preserve $\square(\kappa)$, $\square(\kappa)$ holds everywhere in the forcing extension. $\square$

There is an important class of forcings which includes the class of c.c.c. forcings and ω_1 -closed forcings, called proper forcings. Before defining it, we first need to define a generalisation of stationarity.

Definition 1.3.5. Let A be a set and κ be a regular uncountable cardinal such that $|A| \geq \kappa$, and let $[A]^{<\kappa}$ be the set of all subsets of A that has cardinality $< \kappa$.

- (i) A set $X \subseteq [A]^{<\kappa}$ is **unbounded** if for every $x \in [A]^{<\kappa}$ there exists $y \supseteq x$ such that $y \in X$.
- (ii) A set $X \subseteq [A]^{<\kappa}$ is **closed** if for any chain $x_0 \subseteq x_1 \subseteq \dots \subseteq x_\xi \subseteq \dots$ ($\xi < \alpha$, for some $\alpha < \kappa$) of sets in X , the union $\bigcup_{\xi < \alpha} x_\xi$ is in X .
- (iii) $X \subseteq [A]^{<\kappa}$ is called a **club** if it is closed and unbounded.
- (iv) $S \subseteq [A]^{<\kappa}$ is called **stationary** if $S \cap C \neq \emptyset$ for every club $C \subseteq [A]^{<\kappa}$.

For instance, any set C that is club in ω_1 can be seen as a club in $[\omega_1]^\omega$, and any set S that is stationary in ω_1 can be seen as stationary in $[\omega_1]^\omega$ (if C is club in ω_1 and C' is club in $[\omega_1]^\omega$, then $C \cap C'$ is club in $[\omega_1]^\omega$, consisting of ordinals, and thus is club in ω_1 , which implies that $S \cap C \cap C' \neq \emptyset$). Fodor's theorem generalises to this notion of stationarity (as the ordinal ordering is simply the membership relation).

Definition 1.3.6. A forcing is **proper** if it preserves stationary sets of $[A]^\omega$ for any uncountable A .

Remark. C.c.c. forcings are preserved under finite-support iterations, while proper forcings are preserved under countable-support iterations (see [Abr10]).

Definition 1.3.7. Proper forcing axiom (PFA) is $\text{FA}(\mathcal{K}_{\text{proper}}, 2^\omega)$ where $\mathcal{K}_{\text{proper}}$ is the class of all proper forcings.

In contrast to MA being equiconsistent with ZFC, PFA implies failure of $\square_\kappa$ everywhere, and hence has a significant consistency strength:

Theorem 1.3.8 (Todorćević–Magidor). PFA implies $\neg \square_{\kappa, \omega_1}$ for any $\kappa > \omega$.

Remark. Magidor [Mag95] also showed that PFA is consistent with $\square_{\kappa, \omega_2}$ for all $\kappa > \omega_1$.

To force PFA, the only known method carries out an iteration of length a supercompact cardinal κ , and in the forcing extension κ is collapsed to ω_2 which is also the cardinality of the reals in the extension. It is a well-known conjecture that PFA is in fact equiconsistent with a supercompact cardinal. Roughly, the intuitive picture is that PFA implies that there are “supercompactly” many reals, but then also collapses that supercompact cardinal to the least cardinal possible—namely ω_2 —without making PFA trivial. Indeed, there are many results regarding PFA that are guided by the intuition that ω_2 is “close” to being supercompact.

As we have seen in Proposition 1.3.3, if P is a κ -c.c. forcing where κ is a regular cardinal, then P preserves $\square(\lambda)$ for $\lambda \geq \kappa$. Thus, since $\square(\kappa)$ fails everywhere under PFA, if PFA is forced from a universe V , it must be the case that there is some $\kappa \in V$ such that for all regular $\lambda \geq \kappa$, $\square(\lambda)$ fails; and if κ is the least such, then P cannot be λ -c.c. for some $\lambda < \kappa$.

In fact, Viale–Weiß [VW11] proved some significant results which further convince us that a supercompact cardinal is required for the consistency strength of PFA.

Definition 1.3.9. Let κ be a regular uncountable cardinal, $\lambda \geq \kappa$. Let $P_\kappa \lambda$ denote $\{a \subseteq \lambda \mid |a| < \kappa\}$. (So $P_\kappa \lambda$ and $[\lambda]^{<\kappa}$ are just different notations for the same thing.)

- (i) A sequence $\langle d_a \mid a \in P_\kappa \lambda \rangle$ is called a $P_\kappa \lambda$ -**list** if $d_a \subseteq a$ for all $a \in P_\kappa \lambda$.
- (ii) If $D = \langle d_a \mid a \in P_\kappa \lambda \rangle$ is a $P_\kappa \lambda$ -list, D is called **thin** if $|\{d_a \cap x \mid a \in P_\kappa \lambda\}| < \kappa$ for every $x \in P_\kappa \lambda$.
- (iii) Suppose $D = \langle d_a \mid a \in P_\kappa \lambda \rangle$ is a $P_\kappa \lambda$ -list and $d \subseteq \lambda$.
 - d is called a **cofinal branch** of D if for every $a \in P_\kappa \lambda$, the set $\{x \in P_\kappa \lambda \mid d_x \cap a = d \cap a\}$ is unbounded in $P_\kappa \lambda$.
 - d is called an **ineffable branch** of D if the set $\{x \in P_\kappa \lambda \mid d_x = d \cap x\}$ is stationary in $P_\kappa \lambda$.

Lemma 1.3.10. κ has the tree property iff every thin $P_\kappa \kappa$ -list has a cofinal branch.

Proof. ($\Leftarrow$). Suppose every thin $P_\kappa \kappa$ -list has a cofinal branch. Let T be a κ -tree. Then we may assume T is of the following form: each element in T is a subset

of κ of cardinality $< \kappa$, and t is in the α -th level iff t is of order type α ; $s < t$ iff t is an end-extension of s . Let $\langle d_x \subseteq x \mid x \in [\kappa]^{<\kappa} \rangle$ be such that $d_x = x$ if $x \in T$, $d_x = \emptyset$ otherwise. Then, if $a \in [\kappa]^{<\kappa}$, then $|\{d_x \cap a \mid x \in [\kappa]^{<\kappa}\}| < \kappa$, because if $\alpha = \sup a$, the elements $d_x \cap a$ have order type at most α so there are fewer than κ such sets. By our assumption, there is $d \subseteq \kappa$ such that for all $a \in [\kappa]^{<\kappa}$, $\{x \in [\kappa]^{<\kappa} \mid d_x \cap a = d \cap a\}$ is unbounded. This is a branch through T .

($\Rightarrow$). Suppose κ has the tree property. Let $\langle d_x \subseteq x \mid x \in [\kappa]^{<\kappa} \rangle$ be such that $|\{d_x \cap a \mid x \in [\kappa]^{<\kappa}\}| < \kappa$ for all $a \in [\kappa]^{<\kappa}$. Define a κ -tree T by $T \restriction \alpha := \{d_x \cap \alpha \mid x \in [\kappa]^{<\kappa}\}$ ordered by extension; the thinness ensures that every level has cardinality $< \kappa$. Then by the tree property, there is a branch d through T , so $d \subseteq \kappa$. If $\alpha < \kappa$, then $d \cap \alpha$ is an element of T , so there is some $x_\alpha \in [\kappa]^{<\kappa}$ such that $d \cap \alpha = d_{x_\alpha} \cap \alpha$, and $\{x_\alpha \mid \alpha \in \kappa\}$ is unbounded in $[\kappa]^{<\kappa}$, which implies that for any $a \in [\kappa]^{<\kappa}$, $\{x \in [\kappa]^{<\kappa} \mid d_x \cap a = d \cap a\}$ is unbounded in $[\kappa]^{<\kappa}$. $\square$

Recall that an inaccessible κ is weakly compact iff κ has the tree property. Here, we have generalisations to strongly compact and supercompact cardinals.

Theorem 1.3.11 (Jech, Magidor, Weiß [Wei12]). Suppose κ is inaccessible. Then:

- (i) κ is strongly compact iff for all $\lambda \geq \kappa$, every thin $P_\kappa \lambda$ -list has a cofinal branch.
- (ii) κ is supercompact iff for all $\lambda \geq \kappa$, every thin $P_\kappa \lambda$ -list has an ineffable branch.

This gives rise to the following definitions which characterise strongly compact cardinals and supercompact cardinals respectively for inaccessible cardinals.

Definition 1.3.12. Let κ be a cardinal.

- κ has the **strong tree property** if for all $\lambda \geq \kappa$, every thin $P_\kappa \lambda$ -list has a cofinal branch.
- κ has the **super tree property** if for all $\lambda \geq \kappa$, every thin $P_\kappa \lambda$ -list has an ineffable branch.

Theorem 1.3.13 ([VW11] Theorem 4.8). PFA implies that ω_2 has the super tree property.

Theorem 1.3.14 ([VW11] Corollaries 6.6 and 6.11). Suppose κ is inaccessible, and P is a forcing that satisfies the following properties:

- (i) P is the direct limit of an iteration $\langle P_\alpha \mid \alpha < \kappa \rangle$ that takes direct limits stationarily often,
- (ii) P_α has size less than κ for all $\alpha < \kappa$.

Moreover, P collapses κ to ω_2 , and forces PFA in the generic extension.

Then, κ must be strongly compact. If P is also proper, then κ is supercompact.

This is to say that the only known method for forcing PFA requires a strongly compact cardinal, or a supercompact if the forcing is proper. Their argument is based on the covering property and approximation property introduced by [Ham03]. The idea is that if P has the κ -covering and κ -approximation properties, then P does not add new large cardinals $\geq \kappa$, e.g. if $\lambda \geq \kappa$ is measurable in V^P then λ is already measurable in V . (Indeed, if P satisfies the conditions (i) and (ii) stated in the theorem above, then P satisfies both the κ -covering and κ -approximation properties.) Similarly, if W is a forcing extension of V from a forcing that satisfies κ -covering and κ -approximation properties, and κ is inaccessible in V , and if in W , κ has the strong tree property, then κ has the strong tree property in V as well ([VW11] Theorem 6.4).

1.4 Jensen's squares defined on all ordinals

In this section we show that Jensen's squares $\square_{\lambda, < \mu}$ can be expressed as sequences that can also be defined on all ordinals.

Recall the notions of κ -closedness and κ -distributivity in forcing: A forcing P is κ -closed if for any descending sequence of conditions of length $< \kappa$, there is a lower bound to that sequence. P is κ -distributive if for any family of dense open subsets of P of size $< \kappa$, its intersection is also dense.

The following defines a property of forcing that sits between distributivity and closedness.

Definition 1.4.1. Let P be a forcing.

- (i) $G_\alpha(P)$ is a game of length α played by two players as follows. Players INC and COM alternatively write down elements of a descending sequence

$p_0 \geq p_1 \geq \dots$. Player INC writes down p_i for odd i , and Player COM writes down p_i for even and limit i . The first move is always 1_P . Player COM wins if it can always move before the game ends, otherwise Player INC wins.

- (ii) $G'_\alpha(P)$ is similar to $G_\alpha(P)$, except that Player INC plays p_i for all successors i , and Player COM plays p_i only for limits i .

Usually we say that P is α -**strategically closed** if Player COM has a winning strategy for $G_\alpha(P)$.

It is better to interpret (winning) *strategy* intuitively instead of reading set-theoretic definitions: A strategy tells a player how to make the next move at each stage the player should move, given that the player has adhered to the strategy at each previous stage they should move (though of course it cannot dictate how the other player(s) might move). A strategy is winning, if whenever the player has adhered to the strategy throughout the game, they must win.

We will give definitions for (winning) strategies below, in [Definition 1.4.2](#).

Observation 1.4.1.1. It is easy to see that κ -closed posets are κ -strategically closed, and κ -strategically closed posets are κ -distributive.

Remark. In fact, [Jec78] and [For83] showed that if κ is regular, then a poset P is κ -distributive iff for all $\alpha < \kappa$, Player INC does not have a winning strategy in $G_\alpha(P)$.

Definition 1.4.2. In the context of games $G_\alpha(P)$ or $G'_\alpha(P)$, a **strategy** (for either COM or INC) is a function S such that for any ordinal $\beta < \alpha$ and any descending sequence $\vec{p} = \langle p_\gamma \mid \gamma < \beta \rangle$, $S(\vec{p})$ is a (possibly empty) set of lower bounds for $\vec{p}$ —in other words, a set of possible next moves for the player. If $\vec{p} = \langle p_\gamma \mid \gamma < \beta \rangle$ is a partially completed game of $G_\alpha(P)$ (or $G'_\alpha(P)$), we say that the player has **played by** S if for each ordinal $\delta < \beta$ for which they should move, $p_\delta \in S(\vec{p} \restriction \delta)$. S is a **winning strategy** (in $G_\alpha(P)$) for COM if whenever COM has played by S , COM wins, i.e. for each ordinal $\beta < \alpha$ COM should move and each partially completed game $\vec{p} = \langle p_\gamma \mid \gamma < \beta \rangle$ in which COM has played by S , $S(\vec{p}) \neq \emptyset$. Winning strategies for INC, or in other games are similarly defined.

Remark. We note that in this definition, the moves allowed by the strategy at each stage is a set of candidates rather than a single point. Since we are in ZFC, this is the same if the set of candidates consists of only a single point. In the choiceless context, this definition breaks down and we will need to have single candidate at each stage, or else the player might not be able to make a choice and still loses a game when following a winning strategy.

We also note the following facts that given any P , winning strategies for $G'_\alpha(P)$ or $G_\alpha(P)$ propagates upwards along the ordinals until it hits regular ones:

Fact 1.4.3 ([Vel83] Corollary 1.6). If δ is a singular limit ordinal and for all $\gamma < \delta$, COM has winning strategies for $G'_\gamma(P)$, then COM has winning strategies for $G'_{\delta+1}(P)$ as well.

Fact 1.4.4 ([Jak24] Lemma 3.9). If δ is a singular limit ordinal and for all $\gamma < \delta$, COM has winning strategies for $G_\gamma(P)$, then COM has winning strategies for $G_{\delta+1}(P)$ as well.

Therefore, if κ is a cardinal and Player COM has winning strategies for $G'_{\kappa+1}(P)$, then it has winning strategies for $G'_\xi(P)$ and $G_\xi(P)$ for all $\xi < \kappa^+$, by induction on ξ .

Definition 1.4.5. Let $\mathbb{S}_\kappa$ denote the standard forcing of shooting a $\square_\kappa$ -sequence by initial segments. Namely, the elements of $\mathbb{S}_\kappa$ are sequences $\langle C_\alpha \mid \alpha \in \lim(\eta+1) \rangle$ for some limit $\eta < \kappa^+$, satisfying the conditions that each C_α is club in α , if $\alpha < \kappa$ then $\text{otp}(C_\alpha) < \kappa$, and if $\beta \in \lim C_\alpha$ then $\beta \cap C_\alpha = C_\beta$.

$\mathbb{S}_{\kappa, < \mu}$ for shooting a $\square_{\kappa, < \mu}$ -sequence can be defined similarly.

It is clear that if G is a $\mathbb{S}_{\kappa, < \mu}$ -generic then $\bigcup G$ is a $\square_{\kappa, < \mu}$ -sequence in $V[G]$.

Lemma 1.4.6. Given $\kappa > \omega$ a cardinal and any cardinal μ such that $2 \leq \mu \leq \kappa^+$, and $\xi < \kappa^+$, Player COM has winning strategies for $G_\xi(\mathbb{S}_{\kappa, < \mu})$ and $G'_\xi(\mathbb{S}_{\kappa, < \mu})$.

Proof (cf. [CFM01] Lemma 6.1). By Fact 1.4.3 and Fact 1.4.4, it suffices to show that Player COM has winning strategies for $G'_{\kappa+1}(\mathbb{S}_{\kappa, < \mu})$ (and hence for $G_{\kappa+1}(\mathbb{S}_{\kappa, < \mu})$).

Let $\lambda = \text{cf } \kappa$ and let $C \subseteq \kappa$ be a club set of order type λ . We assume that at stage α the players have produced a sequence of conditions $\langle p_\beta \mid \beta < \alpha \rangle$, where $\max \text{dom}(p_\beta) = \gamma_\beta$.

The strategy is defined by induction. Suppose α is a limit ordinal, and $\gamma_\alpha = \sup_{\beta < \alpha} \gamma_\beta$, and suppose that Player COM has adhered to the strategy so far.

If $\sup(C \cap \alpha) = \alpha$, then define p_α by setting $p_\alpha \restriction \gamma_\alpha = \bigcup_{\beta < \alpha} p_\beta$ and $p_\alpha(\gamma_\alpha) = \{\{\gamma_\beta \mid \beta \in C \cap \alpha\}\}$.

If $\sup(C \cap \alpha) < \alpha$, then define p_α by setting $p_\alpha \restriction \gamma_\alpha = \bigcup_{\beta < \alpha} p_\beta$ and $p_\alpha(\gamma_\alpha) = \{\{\gamma_\beta \mid \sup(C \cap \alpha) < \beta < \alpha\}\}$.

Then this is a winning strategy for COM in $G'_{\kappa+1}(\mathbb{S}_{\kappa, < \mu})$. □

Consequently, $\mathbb{S}_{\kappa, < \mu}$ adds no new κ -sequences of ordinals, and so preserves all cardinals and cofinalities up to and including κ^+ .

In [Vel83] Velleman gave an alternative formulation of Jensen's squares and proved the following two theorems, which relate square sequences to strategic closure and forcing axioms on posets that satisfy strategic closure.

Definition 1.4.7. For any infinite ordinals η, τ , $B(\eta, \tau)$ is the statement: There is a sequence $\langle \langle C_\alpha, \tau_\alpha \rangle \mid \alpha < \eta, \alpha \text{ a limit ordinal} \rangle$ such that for all limit ordinals $\alpha < \eta$:

- (i) $C_\alpha \subseteq \alpha$ is club in α ;
- (ii) $\text{otp}(C_\alpha) < \tau_\alpha < \tau$;
- (iii) if β is a limit point of C_α then $C_\beta = C_\alpha \cap \beta$ and $\tau_\beta = \tau_\alpha$.

Theorem 1.4.8. For all infinite cardinal κ , the following are equivalent:

- (i) $B(\kappa, \kappa)$.
- (ii) For every partial order P , if COM has a winning strategy for $G'_\alpha(P)$ for each $\alpha < \kappa$, then COM has a winning strategy for $G'_\kappa(P)$.
- (iii) For every partial order P and family of dense sets $\mathcal{D}$ with $|\mathcal{D}| = \kappa$, if COM has a winning strategy for $G'_\alpha(P)$ for each $\alpha < \kappa$, then there is a filter G of P that intersects every member of $\mathcal{D}$.

Theorem 1.4.9. For all infinite κ , $\square_\kappa \leftrightarrow B(\kappa^+, \kappa^+)$.

We will generalise Theorem 1.4.9. To do so, we introduce a definition of sequences that will correspond to $\square_{\kappa, < \mu}$.

Definition 1.4.10. For any infinite ordinals η, τ and cardinal $\mu > 1$, $B(\eta, \tau, < \mu)$ is the statement: There is a sequence $\langle \langle \mathcal{C}_\alpha, \tau_\alpha \rangle \mid \alpha \in \eta \text{ limit} \rangle$ such that for all limit $\alpha < \eta$:

- (i) $\mathcal{C}_\alpha$ is a non-empty set of clubs of α and $|\mathcal{C}_\alpha| < \mu$.
- (ii) $\text{otp}(C_\alpha) < \tau_\alpha < \tau$ if $C_\alpha \in \mathcal{C}_\alpha$.
- (iii) If $C_\alpha \in \mathcal{C}_\alpha$ and $\beta \in \lim C_\alpha$ then $\beta \cap C_\alpha \in \mathcal{C}_\beta$, and $\tau_\beta = \tau_\alpha$.

Definition 1.4.11. Let $\mu > 1$ be a cardinal and α be an ordinal. We define $G'_{< \mu, \alpha}(P)$, which generalises $G'_\alpha(P)$. Player INC plays at successor stages and Player COM plays at limit stages. Instead of playing a descending sequence, in general the gameplay in $G'_{< \mu, \alpha}(P)$ is going to be a tree.

Suppose α is a successor ordinal, so the gameplay tree T_α at stage α has a maximum level and the elements in that level are leaves. Player INC plays one element of P under each of the leaves which extends the leaf as a condition in P .

Suppose α is a limit ordinal and the gameplay tree T_α at stage α has no maximum level. Player COM plays at least one and fewer than μ elements of P under the tree T_α to extend the tree. Multiple elements may be played to extend the same branch, and some branch may be chosen to be not extended.

Player COM wins if it can always move before the game ends. Otherwise, Player INC wins.

Lemma 1.4.12. Let η, τ be some ordinals and $\mu > 1$ a cardinal. Suppose $B(\eta, \tau, < \mu)$ is true. If P is a partial order and for every $\alpha < \tau$, Player COM has a winning strategy for $G'_\alpha(P)$, then Player COM has a winning strategy for $G'_{< \mu, \eta}(P)$.

We give an intuition of the proof first: At stage α , we need Player COM to make a response according to each of the elements C of $\mathcal{C}_\alpha$. We also assume that by our strategy derived from a $B(\eta, \tau, < \mu)$ -sequence, for all $\beta < \alpha$, at stage β , Player COM has responded “according to” each of the elements C of $\mathcal{C}_\beta$. To continue defining the strategy, for each $C \in \mathcal{C}_\alpha$, we find $\langle C_{i_\beta} \mid \beta \in \lim C \rangle$ such that

$C_{i_\beta} \in \mathcal{C}_\beta$ for each $\beta \in \lim C$ and for all $\beta \in \lim C$, $C \cap \beta = C_{i_\beta}$. By the strategy already defined, there is a path in T_α that Player COM has played according to $\langle C_{i_\beta} \mid \beta \in \lim C \rangle$. We ask Player COM to play under this path “according to” C .

Proof. Let $\langle \langle \mathcal{C}_\alpha, \tau_\alpha \rangle \mid \alpha < \eta \text{ limit} \rangle$ be a $B(\eta, \tau, < \mu)$ -sequence, and for each $\alpha < \tau$, let S_α be Player COM’s winning strategy for $G'_\alpha(P)$. We describe a winning strategy for $G_{<\mu, \eta}(P)$ by induction on $\alpha < \eta$.

Let $\alpha < \eta$ be a limit and suppose T is a play of length α (so T is a tree of height α and each level has cardinality $< \mu$) in which Player COM has played by our strategy derived from $\langle \langle \mathcal{C}_\alpha, \tau_\alpha \rangle \mid \alpha < \eta \text{ limit} \rangle$.

By our strategy, the tree T ’s nodes have been labelled in the following way: If p is a node at level β (a limit) then it is labelled as “played according to C ” for some $C \in \mathcal{C}_\beta$. Conversely, for each $C \in \mathcal{C}_\beta$, there is exactly one p at level β which is labelled by C .

To continue the play, we let Player COM make a move according to each of the elements C of $\mathcal{C}_\alpha$ as follows. Let $\langle \gamma_\zeta \mid \zeta < \text{otp}(C) \rangle$ be the normal enumeration¹ of C . We define a sequence of nodes $\vec{p} = \langle p_\zeta \mid \zeta < \text{otp}(C) \rangle$ as follows.

- If ζ is a limit, then $\gamma_\zeta \in \lim C$ and $C \cap \gamma_\zeta$ is an element of $\mathcal{C}_{\gamma_\zeta}$ so there is exactly one node (at level γ_ζ) that has been played according to $C \cap \gamma_\zeta$. Let p_ζ be that node.
- If ζ is not a limit and there is some $\zeta' < \text{otp}(C)$ greater than ζ that is a limit, let p_ζ be the γ_ζ -th element in the normal enumeration of the branch $\{x \in T \mid x <_P p_{\zeta'}\}$ well-ordered by $<_P$.
- If ζ is not a limit and there is no $\zeta' < \text{otp}(C)$ greater than ζ that is a limit, then choose any one node at level γ_ζ to be p_ζ as long as it is comparable to the previous points.

If we can show that $S_{\tau_\alpha}(\vec{p})$ is non-empty, then we can pick $p \in S_{\tau_\alpha}(\vec{p})$ and play it under the branch $\langle p_\zeta \mid \zeta < \text{otp}(C) \rangle$ and label it as “played according to C ”, and this will complete the induction.

¹A *normal* enumeration means an enumeration that is increasing and continuous, where continuity means, for limit α , the α -th element is the supremum of the β -th element for $\beta < \alpha$.

Therefore it remains to show that $S_{\tau_\alpha}(\vec{p})$ is non-empty; i.e. we need to show that $\langle p_\zeta \mid \zeta < \text{otp}(C) \rangle$ is played by S_{τ_α} , or in other words, if ζ is a limit then $p_\zeta \in S_{\tau_\alpha}(\vec{p} \restriction \zeta)$. So let ζ be a limit in $\text{otp}(C)$ and so $\gamma_\zeta \in \lim C$. Then $C \cap \gamma_\zeta \in \mathcal{C}_{\gamma_\zeta}$ and $\tau_\alpha = \tau_{\gamma_\zeta}$. By our strategy, p_ζ is played according to $C \cap \gamma_\zeta$, and so, looking back to how we define our moves according to the club sets, p_ζ is an element of $S_{\tau_{\gamma_\zeta}}(\vec{q}) = S_{\tau_\alpha}(\vec{q})$, where $\vec{q}$ is the sequence in T we defined by the enumeration of $C \cap \gamma_\zeta$, and p_ζ is the unique node that we have played under the branch $\vec{q}$ at stage γ_ζ according to the club $C \cap \gamma_\zeta$. So $\vec{q} = \vec{p} \restriction \zeta$ and we have $p_\zeta \in S_{\tau_\alpha}(\vec{p} \restriction \zeta)$. $\square$

Theorem 1.4.13. For all infinite cardinal κ and cardinal $\mu > 1$, $\square_{\kappa, < \mu} \leftrightarrow B(\kappa^+, \kappa^+, < \mu)$.

Proof. It suffices to prove $\leftarrow$ because a $\square_{\kappa, < \mu}$ -sequence is virtually a $B(\kappa^+, \kappa^+, < \mu)$ -sequence. By the previous lemma, if $B(\kappa^+, \kappa^+, < \mu)$ is true, then for all partial order P such that for all $\alpha < \kappa^+$, Player COM has winning strategies for $G'_\alpha(P)$, Player COM has a winning strategy for $G'_{< \mu, \kappa^+}(P)$. Set $P = \mathbb{S}_\kappa$, the standard forcing that shoots a $\square_\kappa$ -sequence, and we play a game of $G'_{< \mu, \kappa^+}(P)$ in which Player INC plays elements of the dense set $D_\alpha = \{p \in \mathbb{S}_\kappa \mid \alpha \in \text{dom } p\}$ at stage α , and Player COM plays its winning strategy. Let $\{p_{\alpha, i} \mid \alpha < \kappa^+ \text{ limit}, i < \mu\}$ be the set of Player COM's moves (repetition of the $p_{\alpha, i}$'s allowed). We set $C \in \mathcal{C}_\alpha$ iff C is a club set of α and there is some $p_{\alpha, i}$ played by Player COM such that $C \in p_{\alpha, i}$. Then $\langle \mathcal{C}_\alpha \mid \alpha < \kappa^+ \text{ limit} \rangle$ is a $\square_{\kappa, < \mu}$ -sequence. $\square$

1.5 Forcing axioms and chain conditions

In this section we will show that a non-trivial ω_2 -c.c. forcing must destroy PFA.

We begin with the proposition that a non-trivial κ -c.c. forcing must add a subset of κ (although one can also see this from Bukovsky's theorem [Buk17]).

Lemma 1.5.1. Let κ be a regular cardinal. Let P be a non-trivial κ -c.c. forcing. Then P adds a subset of κ .

Proof. Let λ be the least cardinal such that P adds a new subset $\dot{f} : \lambda \rightarrow 2$ of λ .

Since λ is the least such, for all $\alpha < \lambda$, $\dot{f} \restriction \alpha$ is a set in V . For $\alpha < \lambda$, define

$$T_\alpha = \{t : \alpha \rightarrow 2 \mid (\exists p \in P)p \Vdash t = \dot{f} \restriction \alpha\}.$$

and $T = \bigcup_{\alpha < \lambda} T_\alpha$.

Define a subtree T' with height $\leq \kappa$ as follows (which is not necessarily downward closed). Let $T'_0 = \{\emptyset\}$.

Suppose $\alpha < \kappa$ and T'_α has been defined. If $t \in T'_\alpha$, there is some $p \in P$ such that $p \Vdash t = \dot{f} \restriction \alpha_t$ and there is a least $\alpha'_t > \alpha_t$ such that there are $q_1, q_2 \in P$ such that $q_1 \Vdash t_1 = \dot{f} \restriction \alpha'_t$, $q_2 \Vdash t_2 = \dot{f} \restriction \alpha'_t$, and $t_1 \neq t_2$ both extends t . Let t be immediately succeeded by all the possible values of $\dot{f} \restriction \alpha'_t$ that extend t . This defines $T'_{\alpha+1}$.

Suppose that $\alpha < \kappa$ is a limit and for all $\beta < \alpha$, T'_β has been defined. Then we let

$$T'_\alpha = \{t \in T \mid (\forall \beta < \alpha)(\exists s \in T'_\beta)t \supseteq s, \text{ and if } t' \not\supseteq t, \text{ then } (\exists \beta < \alpha)(\forall s \in T'_\beta)t' \not\supseteq s\}.$$

T'_α may be empty.

This defines $T' = \bigcup_{\alpha < \kappa} T'_\alpha$.

Since $\dot{f}$ is a new subset of λ , $\dot{f}$ defines a cofinal (and downward closed) branch b of T of order type λ such that cofinally many nodes on b split, and b is also new to V .

In V^P (a forcing extension of V by P), we define a branch $b' \subseteq T' \cap b$ by recursion on $\alpha < \kappa$, as follows. Let $t_0 = \emptyset$ be in b' ; so $t_0 \in T'_0 \cap b$.

Suppose $\alpha < \kappa$ and the α -th element t_α of b' has been defined, and $t_\alpha \in T'_\alpha \cap b$. Then there is a least $t \in b$ such that it properly extends t_α and moreover, t is immediately preceded by some $s \in T$ such that s has at least two incompatible immediate successors. By definition of T' , this implies $t \in T'_{\alpha+1}$. Let $t_{\alpha+1} = t$ be the $(\alpha + 1)$ -th element of b' .

Suppose $\alpha < \kappa$ is a limit and for all $\beta < \alpha$, $t_\beta \in T'_\beta$ has been defined. If the union $t = \bigcup_{\beta < \alpha} t_\beta$ is an element of b , then set $t_\alpha = t$ to be the α -th element of b' , and by definition of T' , t_α lies in T'_α . If t is not an element of b , then we terminate the process.

This finishes the construction of b' . Now note that P has κ -c.c., and hence T' has κ -c.c., and therefore $|T'| \leq \kappa$. There are two possibilities:

- (i) If the process defining b' has terminated before κ , then b' is cofinal in b , and so from b' we can construct b , implying that b' is a new subset of T' , hence a new subset of κ .
- (ii) If the process does not terminate before κ , then b' is a branch of order type κ in T' . Since T' is κ -c.c. and ever-splitting at successor levels, T' is a κ -Suslin tree in V . Therefore, b' is a new subset of T' hence of κ . $\square$

Observation 1.5.1.1. Suppose that P is κ -c.c. and does not add a branch of a κ -Suslin tree. Then the second possibility cannot happen. Therefore, κ cannot be the least cardinal which P adds a subset of.

Note that a κ -Suslin tree, when regarded as a forcing, is κ -c.c. and κ -distributive.

Corollary 1.5.2. Let κ be regular uncountable. Suppose that there is no κ -Suslin tree. Then all κ -c.c. κ -distributive forcings are trivial.

Proof. Let P be such a forcing. Suppose $p \in P$ forces that $\dot{f} : \kappa \rightarrow 2$ is a subset of κ . As before, we define

$$T_\alpha = \{t : \alpha \rightarrow 2 \mid (\exists p \in P)p \Vdash t = \dot{f} \restriction \alpha\}$$

and let $T = \bigcup_{\alpha < \kappa} T_\alpha$. By the assumption, T is not κ -Suslin. But every antichain of T has size $< \kappa$, so T has a cofinal branch and moreover the cofinal branch must stop splitting after a certain point (as every antichain of T has size $< \kappa$). We can therefore find a condition p' under p that decides all of $\dot{f}$, so $\dot{f}$ is not new. $\square$

Corollary 1.5.3.

- MA implies that all c.c.c. ω_1 -distributive forcings are trivial.
- PFA implies that all ω_2 -c.c. ω_2 -distributive forcings are trivial.

Proof. MA implies that there is no ω_1 -Suslin tree, and PFA implies that ω_2 has the tree property. $\square$

Remark.

- MA is consistent with existence of ω_2 -Suslin trees, because c.c.c. forcings preserve κ -Suslin trees for $\kappa > \omega_1$ and MA can be forced by a c.c.c. poset.

- PFA is consistent with existence of κ^+ -Suslin trees for $\kappa \geq \omega_2$, so in particular it is consistent with PFA that there are non-trivial κ^+ -c.c. κ^+ -distributive forcings for $\kappa \geq \omega_2$. (In fact this is true for MM in place of PFA, see [BR17] Corollary 1.16.)

Next, we define two properties introduced in [Ham03]. Together they define a tame class of forcings which add no new large cardinals.

Definition 1.5.4. Let $V \subseteq W$ be a pair of transitive inner models of ZFC, μ a cardinal.

- (V, W) satisfies the **μ -covering property** if the class $P_\mu^V V$ is cofinal in $P_\mu^W V$, i.e., if $Y \in W$, $Y \subseteq V$, and $|Y|^W < \mu$, then there is a set $X \in V$ such that $Y \subseteq X$ and $|X|^V < \mu$.
- (V, W) satisfies the **μ -approximation property** if the following holds:
For all $f \in W$ a sequence of sets in V , i.e. $f : \lambda \rightarrow V$ for some λ , f is in V whenever for all $X \subseteq \lambda$, $X \in V$ such that $|X|^V < \mu$, $f \restriction X \in V$.

A forcing P is said to satisfy the **μ -covering property** or the **μ -approximation property** if for every V -generic $G \subseteq P$ the pair $(V, V[G])$ satisfies the μ -covering property or the μ -approximation property respectively.

Observation 1.5.4.1. If P satisfies the μ -covering property, then P is μ -distributive iff it does not add subsets of α for any $\alpha < \mu$.²

Remark. If $V \subseteq W$ has the δ -covering and δ -approximation properties, and $\delta < \gamma$ are both infinite cardinals in V , then it also has the γ -covering and γ -approximation properties [Ham13].

Therefore (since the δ -covering property propagates upward), if P has the κ -covering property, then P preserves cofinalities $\geq \kappa$. The proof is the same as that of κ -c.c. forcings preserve cofinalities $\geq \kappa$.

Lemma 1.5.5. Let κ be regular. Suppose P is κ -c.c. Then P has the κ -covering property.

²We note that, in general, not adding subsets of α for any $\alpha < \mu$ does not mean μ -distributivity (i.e. not adding α -sequences of ground model elements for any $\alpha < \mu$). For instance, a Prikry forcing defined from a normal measure on a measurable cardinal κ does not add subsets of α for any $\alpha < \kappa$, but it does add an ω -sequence that is cofinal in κ .

Proof. If $f \in V^P$ is a function from some $\lambda < \kappa$ to V and $\dot{f}$ is a name for f , for each $\alpha < \lambda$, there are $< \kappa$ possibilities for $\dot{f}(\alpha)$ because they form a maximal antichain. So the union of all such possibilities covers f , and it is a set in V of cardinality $< \kappa$. $\square$

Remark. Note that the converse is not true. For instance, any proper forcing satisfies the ω_1 -covering property (this fact uses another definition of properness, see e.g. [Gol92] Lemma 3.13), but a proper forcing can fail to satisfy c.c.c.

Now we cite a theorem by Caicedo–Veličković.

Theorem 1.5.6 ([CV06]). If W is an outer model of V with $(\omega_2)^W = (\omega_2)^V$, and BPFA holds in both V and W , then $\mathcal{P}(\omega_1)^V = \mathcal{P}(\omega_1)^W$.

Here BPFA is the *bounded* form of PFA: If $\langle A_i \mid i < \omega_1 \rangle$ is a family of maximal antichains of P where P is a proper forcing, and each of A_i has size $\leq \omega_1$, then there is a filter $G \subseteq P$ that meets all A_i . It is also equivalent to that $H(\omega_2) <_{\Sigma_1} V^P$ for any proper forcing P [Bag00]. BPFA also implies $2^\omega = \omega_2$ [CV06].

Corollary 1.5.7.

- (i) Suppose $V \models \text{BPFA}$ and P is a non-trivial c.c.c. forcing. Then $V^P \models \neg \text{BPFA}$.
- (ii) Suppose $V \models \text{PFA}$ and P is a non-trivial ω_2 -c.c. forcing. Then, either ω_1 is collapsed and exactly ω_3^V new reals are added, or $V^P \models \neg \text{BPFA}$.

Proof. Using Theorem 1.5.6, the first assertion follows from the fact that c.c.c. forcings preserve all cardinals and P adds a subset of ω_1 .

If P is a non-trivial ω_2 -c.c. forcing, then under PFA it cannot be ω_2 -distributive by Corollary 1.5.3, which consequently implies that P adds a subset of ω_1 (since P has the ω_2 -covering property). So, if $(\omega_2)^V = (\omega_2)^{V^P}$, then $V^P \models \neg \text{BPFA}$. If $V^P \models \text{BPFA}$, then both V and V^P satisfies $2^\omega = \omega_2$. Since $(\omega_2)^V = (\omega_1)^{V^P}$ and P preserves cardinalities $\geq \omega_2$, P must add exactly ω_3^V many new reals. (Similarly, since $2^{\omega_1} = \omega_2$ holds in both universes, P must add exactly ω_3^V many subsets of ω_1 .) $\square$

In Observation 1.5.1.1, we observed that if a κ -c.c. forcing does not add a branch of a κ -Suslin tree, then κ cannot be the least cardinal the forcing adds a subset of. In [Usu14], such forcings are called *strongly κ -c.c.*

Definition 1.5.8. Let κ be regular. A forcing P is **strongly κ -c.c.** if it is κ -c.c. and does not add a branch of a κ -Suslin tree.

Lemma 1.5.9 ([Usu14]). If P is μ -c.c. for some $\mu < \kappa$, then P is strongly κ -c.c.

Lemma 1.5.10 ([Usu14]). Let κ be regular. Let P be an atomless poset which satisfies the strong κ -c.c. Let $\dot{Q}$ be a P -name for a κ -closed poset ($\dot{Q}$ can be trivial). Then $P * \dot{Q}$ has the κ -approximation property. In particular, P has both the κ -covering and κ -approximation property.

We recall the strong tree property defined in Definition 1.3.12. The following proposition is similar to the result in [VW11], and reciprocates the theme in [Ham03] that if P has the κ -covering and κ -approximation properties then P does not add new large cardinals $\geq \kappa$.

Proposition 1.5.11 ([Usu14]). Let κ be regular. Suppose that there is a poset P which has the κ -covering and κ -approximation properties, and forces that κ has the strong tree property. Then κ has the strong tree property in the ground model.

Corollary 1.5.12. If $V \models \text{PFA}$, and P is a non-trivial ω_2 -c.c. forcing, then $V^P \models \neg \text{PFA}$.

Proof. As deduced before in Corollary 1.5.7, if P is a non-trivial ω_2 -c.c. forcing that does not collapse ω_1 , then $V^P \models \neg \text{PFA}$. If it does, then ω_3^V will become $\omega_2^{V^P}$. But P is ω_2 -c.c., so is strongly ω_3 -c.c. CH_{ω_1} holds in V , so there is a special ω_3 -Aronszajn tree and ω_3 cannot have the tree property, let alone the strong tree property. By the above, $\omega_2^{V^P}$ cannot have the strong tree property, so by Theorem 1.3.13, PFA cannot hold in V^P . $\square$

Remark. This implies that if κ is supercompact and P is a κ -c.c. poset that factors as $Q * \dot{R}$ where Q is a standard forcing that forces PFA and that $\dot{R}$ is a non-trivial forcing in V^Q , then PFA does not hold in V^P .

1.6 Large posets with small chain conditions

It is well-known that, for a forcing P , the number of subsets it can add is limited by its chain condition and its size:

Lemma 1.6.1. Let $\lambda < \kappa$ be regular cardinals. Suppose that P is λ -c.c. Then for each $\mu \leq \lambda$, P adds at most $|P|^\lambda$ many subsets of μ . Thus if P adds κ many subsets of λ , then $|P|^\lambda \geq \kappa$.

Proof. This is done by counting “nice” names of subsets of μ . For each subset x of μ in V^P , there is a name $\dot{x} \in V$ for x . By λ -c.c. of P , for each $\alpha < \mu$, there is a $< \lambda$ -sized maximal antichain $D_{x,\alpha}$ such that $p \in D_{x,\alpha}$ decides the value of $\dot{x}(\alpha)$. Define

$$\dot{x}_* = \{ \langle p, \langle \alpha, \varepsilon \rangle^\bullet \rangle \mid p \in D_{x,\alpha}, p \Vdash \dot{x}(\alpha) = \varepsilon \}$$

where ε is either 0 or 1 according to the value of $\dot{x}(\alpha)$, and $\langle \alpha, \varepsilon \rangle^\bullet$ is the obvious name for $\langle \alpha, \varepsilon \rangle$. Then $1_P \Vdash \dot{x} = \dot{x}_*$. Each $\dot{x}_*$ is a set of size $\leq \lambda$, and is a subset of $|P| \times \mu \times 2$. Therefore there are at most $(|P| \times \mu \times 2)^\lambda = |P|^\lambda$ many distinct names of this form in V . $\square$

In this section, we will prove a kind of reverse to this lemma: If P is λ -c.c., atomless, and determines the number of subsets of λ it adds, then the number of subsets of λ it adds is determined by its “essential” cardinality.

We first show that if a complete boolean algebra has a small chain condition, then the cardinality of complete subalgebra generated by a subset can be bounded.

Lemma 1.6.2. Suppose B is a complete boolean algebra that satisfies κ -c.c. Then, for each $X \subseteq B$, there is $Y \subseteq X$ such that $|Y| < \kappa$ and $\bigvee X = \bigvee Y$.

Proof. Let A be a maximal antichain in B such that for all $a \in A$ there is $x_a \in X$ such that $a \leq x_a$. Let

$$Y = \{x_a \mid a \in A\}.$$

Then $|Y| < \kappa$ since $|A| < \kappa$. To show $\bigvee Y = \bigvee X$, it suffices to show $\bigvee X \leq \bigvee Y$, i.e. for all $x \in X$ we have $x \leq \bigvee Y$. So suppose for contradiction that there is $x \in X$

such that $x \not\leq \bigvee Y$, so $x \wedge \neg \bigvee Y \neq 0$. Let $d = x \wedge \neg \bigvee Y$. Then $d \leq x$, and d is disjoint with all element in A : if there is $e \in A$ such that $d \wedge e \neq 0$, namely $x \wedge \neg \bigvee Y \wedge e \neq 0$, then $e \not\leq x_e$, contradicting the definition of Y . Thus, d can be added to A to make a larger antichain, which is a contradiction to the fact that A is maximal. $\square$

Lemma 1.6.3. If λ is regular, then $(\kappa^{<\lambda})^{<\lambda} = \kappa^{<\lambda}$ for any cardinal κ .

Proof. Let $\theta = \kappa^{<\lambda}$. We show that for any $\mu < \lambda$, $\theta^\mu \leq \theta$. If $f : \mu \rightarrow \kappa^{<\lambda}$, then for each $\alpha < \mu$, $f(\alpha)$ is some $g_\alpha : \delta_\alpha \rightarrow \kappa$ where $|\delta_\alpha| < \lambda$. Let $D_f = \bigcup_{\alpha \in \mu} \{\alpha\} \times \delta_\alpha$. Then $|D_f| = \sup_{\alpha \in \mu} |\delta_\alpha|$. Since λ is regular, we have that $|D_f| < \lambda$. So, the function $h_f : D_f \rightarrow \kappa$ with $h_f((\alpha, \beta)) = g_\alpha(\beta) = f(\alpha)(\beta)$ identifies $f \in \theta^\mu$ as an element in $\kappa^{<\lambda} = \theta$. $\square$

Corollary 1.6.4. If λ is regular, then $\text{cf}(\kappa^{<\lambda}) \geq \lambda$ (by König's theorem). $\square$

Proposition 1.6.5. Let λ be regular. If B is λ -c.c., and $X \subseteq B$, $|X| \leq \kappa$, then $C_B(X)$, the complete subalgebra of B generated by X , has cardinality $\leq \kappa^{<\lambda}$.

Proof. We show that $C_B(X)$ can be written as a union $\bigcup_{\alpha < \kappa^{<\lambda}} C_\alpha$ such that each C_α inductively has cardinality $\leq \kappa^{<\lambda}$. Let $C_0 := X$. If $\alpha < \kappa$ is a limit and C_β , $\beta < \alpha$ has been defined, then $C_\alpha := \bigcup_{\beta < \alpha} C_\beta$. For successor stages, if the ordinal is odd, we take $C_{\alpha+1} := C_\alpha \cup \{-b \mid b \in C_\alpha\}$. If the ordinal is even, we take $C_{\alpha+1} := \{\bigvee Y \mid Y \subseteq C_\alpha\}$. Since B is λ -c.c., $C_{\alpha+1} := \{\bigvee Y \mid Y \subseteq C_\alpha, |Y| < \lambda\}$. So if $|C_\alpha| \leq \kappa^{<\lambda}$, then $|C_{\alpha+1}| \leq (\kappa^{<\lambda})^{<\lambda} = \kappa^{<\lambda}$. Finally, note that $\text{cf}(\kappa^{<\lambda}) \geq \lambda$, so $C_B(X)$ equals to $\bigcup_{\alpha < \kappa^{<\lambda}} C_\alpha$, because if $Y \subseteq \bigcup_{\alpha < \kappa^{<\lambda}} C_\alpha$ and $|Y| < \lambda$ (we can take $|Y| < \lambda$ w.l.o.g. because of Lemma 1.6.2), then there is some $\alpha < \kappa$ such that $Y \subseteq C_\alpha$, and its join and complement exist in the union. $\square$

Remark. We note that a theorem of Gaifman and Hales ([GH67], [Hal64]) says that there are countably generated complete boolean algebras of arbitrarily high cardinality, i.e. for any κ there is a complete boolean algebra B , $|B| \geq \kappa$, such that there is a countable $X \subseteq B$ such that the least complete subalgebra of B that contains X is B itself. Solovay [Sol66] also showed that there are complete κ -distributive boolean algebras on κ generators of arbitrarily high cardinality.

Fact 1.6.6 ([Bel05] Corollary 1.21). Suppose that B, B' are complete boolean algebras and B' is a complete subalgebra of B . Then:

- (i) If G is B -generic over V , then $G \cap B'$ is B' -generic over V .
- (ii) For any Δ_1 -formula $\phi(\dot{x}_1, \dots, \dot{x}_n)$ in the language of the B' -forcing,

$$\llbracket \phi(\dot{x}_1, \dots, \dot{x}_n) \rrbracket_{B'} = \llbracket \phi(\dot{x}_1, \dots, \dot{x}_n) \rrbracket_B.$$

We also need Bukovsky's theorem, from which one can also infer that a non-trivial κ -c.c. poset must add a subset of κ .

Theorem 1.6.7 ([Buk17]). For $M, N \models \text{ZFC}$, κ a cardinal in M , the following are equivalent:

- (i) $N = M(P(\kappa)^N)$ (N is the smallest model of ZFC such that $M \subseteq N$ and $P(\kappa)^N \in N$).
- (ii) There is a κ -c.c. set forcing P in M such that N is P -generic over M .

Definition 1.6.8. P and Q are called **forcing-equivalent** if for any P -generic G there is a Q -generic H such that $V[G] = V[H]$ and vice versa.³

Proposition 1.6.9. Let λ be regular. Suppose that P is an atomless λ -c.c. poset, and P adds exactly κ many subsets of λ (in particular, $\kappa^\lambda = \kappa$ in V). Then there is a forcing equivalent to P which has cardinality at most κ .

Proof. P forces that there is a sequence $\langle \dot{A}_\alpha \mid \alpha < \kappa \rangle$ which enumerates all subsets of λ in V^P . Since $\kappa > \lambda$, we can regard $\langle \dot{A}_\alpha \mid \alpha < \kappa \rangle$ as encoded by a single name $\dot{A}$ which is a name for a subset of κ . For each $\alpha < \kappa$, there is a maximal antichain D_α in P such that each element of D_α determines if $\alpha \in \dot{A}$ or not. Let $X = \{1\} \cup \bigcup_{\alpha < \kappa} D_\alpha$; then $|X| \leq \kappa$. Note that we can regard $\dot{A}$ as an X -name.

In $B := B(P)$, consider $C := C_B(X)$, the complete subalgebra generated by X in B . By Proposition 1.6.5, $|C| \leq \kappa$. Each D_α is a maximal antichain in B , and so each of them is also a maximal antichain in C . Moreover, by (ii) of the fact, if $b \in C$, then $b \Vdash_C \alpha \in \dot{A}$ iff $b \Vdash_B \alpha \in \dot{A}$, similarly for the negation.

Now suppose that G is a generic ultrafilter on B , so $V[G] = V(P(\lambda)^{V[G]})$ by Bukovsky's theorem. Then $G \cap C$ is C -generic over V , and so $G \cap C$ intersects

³Note that this definition of forcing equivalence is weaker than isomorphism between boolean completions. Karagila pointed out to me that this equivalence is not preserved under products: if one takes $P = \{1\}$ and $Q = \{1, a, b\}$ where $a < 1$, $b < 1$, a and b are incomparable, then the finite-support product of ω copies of P is isomorphic to P , but that of Q is isomorphic to adding a Cohen real.

D_α and the intersection agrees with $G \cap D_\alpha$, for every $\alpha < \kappa$. This means that $G \cap C$ determines $P(\lambda)$ in the generic extension and it agrees with $P(\lambda)^{V[G]}$, so $V[G] = V[G \cap C]$.

Conversely, suppose that H is a generic ultrafilter on C , since C is a complete subalgebra of B and so B factors as $C * \dot{D}$ for some $\dot{D}$, taking some $\dot{K}$ a $\dot{D}$ -generic gives us $H * \dot{K}$ a B -generic. H determines all the subsets of λ that $H * \dot{K}$ can possibly add, so $V[H] = V(P(\lambda)^{V[H]}) = V(P(\lambda)^{V[H * \dot{K}]}) = V[H * \dot{K}]$, by Bukovsky. $\square$

This hints that if P is atomless and λ -c.c., then in the generic extension, the cardinality of 2^λ is determined by the “essential size” of P .

Definition 1.6.10. Suppose $\{P_i \mid i \in I\}$ is a family of forcing notions. The **lottery sum** of $\{P_i \mid i \in I\}$ is the partial order $\{\mathbf{1}\} \cup \{\langle i, p \rangle \mid p \in P_i\}$ with the order $\langle i', p' \rangle \leq \langle i, p \rangle$ iff $i = i'$ and $p' \leq_{P_i} p$, and $\mathbf{1}$ is the maximum element. We denote this sum by $\oplus_{i \in I} P_i$.

Lemma 1.6.11. Let P be a forcing. Then P is forcing-equivalent to $\oplus_{p \in P} P \restriction p$.

Proof. If $G \subseteq P$ is generic, then it can be seen as a generic in $\oplus_{p \in P} P \restriction p$. Conversely, suppose $H \subseteq \oplus_{p \in P} P \restriction p$. Then, there is some $\bar{p} \in P$ such that H is generic in $P \restriction \bar{p}$. Define $G = \{p \in P \mid (\forall q \in H)(\exists r \in H)r \leq q \wedge r \leq p\}$. Then G is a filter, and $G \supseteq H$. If $D \subseteq P$ is dense, then $D \cap P \restriction \bar{p}$ is dense in $P \restriction \bar{p}$, so $H \cap D \cap P \restriction \bar{p} \neq \emptyset$, and so $G \cap D \neq \emptyset$ and G is generic. $\square$

Let us make the following definition.

Definition 1.6.12. Let P be a forcing poset. P is said to have **essential cardinality** κ if whenever P is forcing-equivalent to a lottery sum of complete boolean algebras, each of which has cardinality $\leq |B(P)|$ and is not a lottery sum, then each summand has cardinality κ .

The requirement that each of the summands be of cardinality $\leq |B(P)|$ and not a lottery sum is slightly more artificial, but we want to ensure that we are not scanning over equivalents that are uselessly large.

Observation 1.6.12.1. If κ is infinite and P has essential cardinality κ then P is atomless.

Observation 1.6.12.2. In general, P may not have an essential cardinality (e.g. if P is a lottery sum of an atom and an atomless poset). But if P does, then its essential cardinality equals $|B(P)|$.

Corollary 1.6.13. Let λ be regular, P a λ -c.c. forcing. If $|P|^\lambda = \kappa$ essentially (i.e. the essential cardinality ν of P satisfies $\nu^\lambda = \kappa$), then P adds exactly κ many subsets of λ . In particular, in the generic extension, the cardinality of 2^λ is determined by the essential size of P : $2^\lambda = |P|^\lambda$.

Proof. If $|P|^\lambda = \kappa$ essentially, then by Lemma 1.6.1, P adds at most κ many subsets of λ . If there exists a forcing extension in which P could only add exactly $\mu < \kappa$ many subsets of λ , then $\mu^\lambda = \mu$, and let $p \in P$ force that $2^\lambda = \mu$, then by Proposition 1.6.9, $P \restriction p$ can be replaced by a poset which has cardinality at most μ , which violates the condition that $|P|^\lambda = \kappa$ essentially, by Lemma 1.6.11. $\square$

Corollary 1.6.14. Let λ be regular, P a λ -c.c. forcing such that $|P|^\lambda \geq \kappa$ essentially. Then P destroys the inaccessibility of κ . $\square$

Since we have made the definition of essential cardinality, one may ask how we know if a poset has essential cardinality. We show that if B is a complete boolean algebra and $|B| = \kappa$, then B has essential cardinality κ iff for all $b \in B$, $|B \restriction b| = \kappa$.

We need the following theorem.

Theorem 1.6.15 ([Ham14]). If $V[H] \subseteq V[G]$, where $G \subseteq B$ and $H \subseteq C$ are V -generic filters on the complete boolean algebras B and C , respectively, then there are conditions $b \in B$ and $c \in C$ such that $C \restriction c$ is isomorphic to a complete subalgebra of $B \restriction b$.

Observe that for any poset P and $p \in P$, we have $B(P \restriction p) = B(P) \restriction p$, because $B(P) \restriction p$ is the complete boolean algebra such that $P \restriction p$ densely embeds into.

Lemma 1.6.16. P has essential cardinality κ iff for all $p \in P$, $|B(P \restriction p)| = \kappa$.

Proof. For one direction, P is equivalent to $\bigoplus_{p \in P} P \restriction p$ hence to $\bigoplus_{p \in P} B(P \restriction p)$, so P having essential cardinality κ means that for all $p \in P$, $|B(P \restriction p)| = \kappa$. Conversely, suppose that $|B(P)| = \kappa$ but P does not have essential cardinality κ . Then, P is equivalent to a lottery sum of complete boolean algebras $\bigoplus_{i \in I} B_i$ such that one of the summands has cardinality $< \kappa$. Let $i \in I$ be such that $|B_i| < \kappa$. Then, given $G \subseteq B_i$ generic, there is some $H \subseteq B(P)$ generic such that $V[G] = V[H]$. By Theorem 1.6.15, this means that there are $b \in B_i$ and $c \in B(P)$ such that $B_i \restriction b$ is isomorphic to $B(P) \restriction c$. P is dense in $B(P)$, so there is some $p \leq c$. Then $B(P \restriction p) = B(P) \restriction p$ is a complete subalgebra of $B(P) \restriction c$, so $|B(P \restriction p)| < \kappa$. $\square$

Corollary 1.6.17. Let B be a complete boolean algebra, $|B| = \kappa$. B has essential cardinality κ iff for all $b \in B$, $|B \restriction b| = \kappa$. $\square$

The most direct examples would be *homogeneous* complete boolean algebras: a boolean algebra B is **homogeneous** if for every $a \in B \setminus \{0\}$ the relative algebra $B \restriction a = \{b \in B \mid b \leq a\}$ is isomorphic to B .

Question 1.6.18. For a poset P , let $c(P)$ denote the least θ such that P satisfies θ -c.c. Is it true that $|B(P)|$ depends only on $|P|$ and $c(P)$?

If this is true, then suppose that for all $p \in P$ we have $|P \restriction p| = |P|$ and $c(P \restriction p) = c(P)$. Then we have that for all $p \in P$, $|B(P \restriction p)| = |B(P)|$, and so P has essential cardinality $|B(P)|$.

A likely candidate would be $|B(P)| = |P|^{<c(P)}$ if we consider the boolean completion of P as adding suprema for all antichains. The following theorem also slightly supports this speculation:

Theorem 1.6.19 ([CH72]). If B is an infinite boolean algebra such that every countable set has a supremum, then $|B|^\omega = |B|$.

Chapter 2

Upward homogeneity

2.1 Introduction

One generalisation of the axiom of choice is to ask that all sets are in bijection with sets of sets of ordinals. Like AC, this cannot be proved from ZF alone, so one could go one step further and ask that all sets are in bijection with sets of sets of sets of ordinals, and so on. To this end, we say that an α -**set** is an element of $P^\alpha(\text{Ord})$, the α -th iteration of the powerset operator on the class of ordinals taking unions at limit stages. The **Kinna–Wagner Principle** for α , written KWP_α , is the assertion that every set is in bijection with a set of α -sets. In particular, KWP_0 states that every set is in bijection with a set of ordinals and so is equivalent to AC. One may hope that ZF at least asserts that there is some ordinal α such that KWP_α holds, but this is not the case. An early stage of showing this failure was Monroe’s iteration (explored in §2.4 and originally published in [Mon73]), a method by which a strictly increasing chain $\langle M_n \mid n < \omega \rangle$ of models of ZF are constructed such that for all $k > n$, M_k and M_n have the exact same class of n -sets. In particular, since the chain is strictly increasing, $M_k \models \neg\text{KWP}_n$. This stands in stark contrast to Balcar and Vopěnka’s theorem that if V and W are transitive models of ZF, $V \models \text{AC}$ or $W \models \text{AC}$, and $P(\text{Ord})^V = P(\text{Ord})^W$ then $V = W$.¹ Monroe’s construction was later extended by Shani in [Sha18]. Taking

¹This result can be further generalised to “if V and W are transitive models of ZF, $V \models \text{KWP}_\alpha$ or $W \models \text{KWP}_\alpha$, and $P^{\alpha+1}(\text{Ord})^V = P^{\alpha+1}(\text{Ord})^W$ then $V = W$ ”.

the failure of Kinna–Wagner further, the Bristol model, exhibited in [Kar18], is a model of ZF in which KWP_α fails for all ordinals α .

In each case the constructions make use of a certain homogeneity in their forcing: For a notion of forcing $\mathbb{P}$ and an automorphism π of $\mathbb{P}$, one can extend π to act on all $\mathbb{P}$ -names and, as one might expect, if $p \Vdash \phi(\dot{x})$ for some $p \in \mathbb{P}$ and $\mathbb{P}$ -name $\dot{x}$, then $\pi p \Vdash \phi(\pi \dot{x})$. Suppose that for all $q, q' \leq p$ there is π such that $\pi q = q'$ and $\pi \dot{x} = \dot{x}$. Then if $q \Vdash \phi(\dot{x})$ we must have that for all $q' \leq p$, $q' \Vdash \phi(\dot{x})$ as well. Hence we must have that $p \Vdash \phi(\dot{x})$ or $p \Vdash \neg \phi(\dot{x})$. By noting that ground-model sets are not moved by any π , we have that for all ground-model sets y , either $p \Vdash y \in \dot{x}$ or $p \Vdash y \notin \dot{x}$. Therefore, if $p \Vdash \dot{x} \subseteq V$ then we must have $p \Vdash \dot{x} \in V$. A similar process can be carried out in an iteration of symmetric extensions $V \subseteq M \subseteq N$ to ensure that if $\langle p, \dot{q} \rangle \Vdash \dot{x} \subseteq V$ then $p \Vdash \dot{x} \in M$. That is, $P(V)^M = P(V)^N$. We have isolated a homogeneity condition on iterations of symmetric extensions that precisely characterises when the second step does not add subsets of the ground model.

Definition. $\langle \mathbb{P}_0, \mathcal{G}_0, \mathcal{F}_0 \rangle * \langle \dot{\mathbb{P}}_1, \dot{\mathcal{G}}_1, \dot{\mathcal{F}}_1 \rangle^\bullet$ is **upward homogeneous** if for all $\bar{\Gamma} \in \mathcal{F}_0 * \dot{\mathcal{F}}_1$ there is a dense collection of $\langle p^\circ, \dot{q}^\circ \rangle \in \mathbb{P}_0 * \dot{\mathbb{P}}_1$ such that for all $\langle p, \dot{q} \rangle, \langle p, \dot{q}' \rangle \leq \langle p^\circ, \dot{q}^\circ \rangle$ there is $\bar{\pi} \in \bar{\Gamma}$ such that $\bar{\pi} \langle p, \dot{q} \rangle \parallel \langle p, \dot{q}' \rangle$.²

Main theorem. The iteration of symmetric systems $\mathcal{S} = \langle \mathbb{P}_0, \mathcal{G}_0, \mathcal{F}_0 \rangle * \langle \dot{\mathbb{P}}_1, \dot{\mathcal{G}}_1, \dot{\mathcal{F}}_1 \rangle^\bullet$ is upward homogeneous iff for all symmetric extensions $V \subseteq M \subseteq N$ by $\mathcal{S}$, $P(V)^M = P(V)^N$.

This homogeneity is used in several choiceless constructions, some of which we shall exhibit here: Monro’s model of $\neg \text{KWP}_n$ for all $n < \omega$; Karagila–Schlicht’s “forgetting-how-to-count” construction; Morris’s construction of countable unions of countable sets, the power sets of which have arbitrarily large Lindenbaum number; and the Bristol model.

Remark. It is in the first exhibition of the Bristol model ([Kar18] Definition 3.20) that the phrase “upward homogeneity” is used, albeit with a different definition. The definition presented here is a modification of the original that allows it to work in all settings.

²The $\bullet$ -name of the tuple is the obvious name of the tuple; it is explicitly defined in §2.2.

Preliminaries are established in §2.2 to fix notation and to introduce the essential mechanics of symmetric extensions and their iterations. In §2.3 we define upward homogeneity and prove the main theorem. In §2.4 we exhibit some examples that make use of this technique, illustrating the idea behind the definition.

2.2 Preliminaries

Let $\mathbb{P}$ be a forcing. If $\mathbb{1}_{\mathbb{P}}$ is the maximum of $\mathbb{P}$ and $x \in V$ is an element of the ground model, then the canonical $\mathbb{P}$ -name $\check{x}$ of x is defined recursively as

$$\check{x} = \{\langle \mathbb{1}_{\mathbb{P}}, \check{y} \rangle \mid y \in x\}.$$

If $\{\dot{x}_i \mid i \in I\}$ is a family of $\mathbb{P}$ -names, define

$$\{\dot{x}_i \mid i \in I\}^\bullet = \{\langle \mathbb{1}_{\mathbb{P}}, \dot{x}_i \rangle \mid i \in I\};$$

so the $\bullet$ -name of a set is the obvious name for that set. This extends to ordered pairs, sequences, functions, etc. A formula in the language of the forcing $\mathbb{P}$ is a formula in the language of set theory but with every free variable replaced by a $\mathbb{P}$ -name.

In this thesis, the dot notation $\dot{x}$ is always reserved for names (of some forcing). If $\dot{a}$ is a name, then undotted a implicitly denotes the set interpreted from $\dot{a}$ in the corresponding forcing extension, and vice versa. If $\dot{x}, \dot{y}$ are $\mathbb{P}$ -names and there is some $p \in \mathbb{P}$ such that $\langle p, \dot{y} \rangle \in \dot{x}$, then we say that $\dot{y}$ appears in $\dot{x}$.

SYMMETRIC EXTENSION

If V satisfies AC, so does a forcing extension over V . The symmetric-extension construction gives us a way to construct models that violate AC even when the ground model satisfies AC.

We start with our ground model V which may or may not satisfy AC. In V , we may define a symmetric system, i.e. a triple $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, where $\mathbb{P}$ is a forcing, $\mathcal{G}$ a subgroup of the automorphism group $\text{Aut}(\mathbb{P})$ of $\mathbb{P}$, $\mathcal{F}$ a normal filter of subgroups³

³Here, a normal filter of subgroups of a group $\mathcal{A}$ means a filter $\mathcal{B}$ on the set of subgroups of $\mathcal{A}$ such that for all $\pi \in \mathcal{A}$ and $B \in \mathcal{B}$, $\pi B \pi^{-1} \in \mathcal{B}$.

of $\mathcal{G}$. If $\pi \in \text{Aut}(\mathbb{P})$, π acts on the $\mathbb{P}$ -names with the following recursive definition:

$$\pi \dot{x} = \{ \langle \pi p, \pi \dot{y} \rangle \mid \langle p, \dot{y} \rangle \in \dot{x} \}.$$

Easily we have the following (see [Jec08] Lemma 5.13):

Symmetry lemma. Let $\pi \in \text{Aut}(\mathbb{P})$. Then for any $\mathbb{P}$ -name $\dot{x}$ and formula ϕ ,

$$p \Vdash \phi(\dot{x}) \iff \pi p \Vdash \phi(\pi \dot{x}).$$

The symmetry lemma is the cornerstone of arguments in symmetric extensions.

If $\dot{x}$ is a $\mathbb{P}$ -name, we say that $\dot{x}$ is **symmetric** (with respect to $\mathcal{F}$) if $\text{sym}(\dot{x}) := \{ \pi \in \mathcal{G} \mid \pi \dot{x} = \dot{x} \}$ is in $\mathcal{F}$. The class of all hereditarily symmetric names is denoted HS. (We have omitted some dependences in the notation, since in most cases they are clear from context.) If $p \in \mathbb{P}$ and ϕ is a formula in the language of the forcing, $p \Vdash^{\text{HS}} \phi$ means p forces ϕ relativised to the names in HS. If G is a generic filter for $\mathbb{P}$, $\text{HS}^G := \{ \dot{x}^G \mid \dot{x} \in \text{HS} \}$ is the symmetric extension over V defined by $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$; it is a submodel of $V[G]$ satisfying ZF (see [Jec08] Theorem 5.14).

In $\text{HS}^G =: M$ defined as above, we may define a symmetric system $\langle \mathbb{Q}, \mathcal{H}, \mathcal{K} \rangle$ that lives in M , which in turn defines a symmetric extension N over M . The triple $\langle \mathbb{Q}, \mathcal{H}, \mathcal{K} \rangle$ has a $\mathbb{P}$ -name, which will be denoted $\langle \dot{\mathbb{Q}}, \dot{\mathcal{H}}, \dot{\mathcal{K}} \rangle^\bullet$. For a rigorous treatment of iterated symmetric extension, see [Kar16].

In summary, we have extensions of models $V \subseteq M \subseteq N$, defined by the iteration $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle * \langle \dot{\mathbb{Q}}, \dot{\mathcal{H}}, \dot{\mathcal{K}} \rangle^\bullet$, which might be abbreviated to $\mathbb{P} * \dot{\mathbb{Q}}$ as no confusion may occur. Unless otherwise specified, we shall stick to this set of notation throughout.

Example 2.2.1. The canonical example of symmetric extensions is Cohen's first model. The forcing is $\mathbb{P} = \text{Add}(\omega, \omega)$. Define names

$$\begin{aligned} \dot{a}_n &= \{ \langle p, \check{k} \rangle \mid p(n, k) = 1 \}, \\ \dot{A} &= \{ \dot{a}_n \mid n < \omega \}^\bullet. \end{aligned}$$

For $n \neq m \in \omega$, the set $\{ p \in \mathbb{P} \mid (\exists k) p(n, k) \neq p(m, k) \}$ is dense, so $a_n \neq a_m$ in $V[G]$; in particular, A will be an infinite set. The automorphism group $\mathcal{G}$ is

$$\mathcal{G} = S_{<\omega} = \{ \text{finitary permutations of } \omega \},$$

acting on $\mathbb{P}$ via

$$\pi p(\pi n, m) = p(n, m).$$

(When the forcing is of the form $\text{Add}(X, Y)$ defined below, this action is what we will refer to as “the usual action”.) Then we have $\pi \dot{a}_n = \dot{a}_{\pi n}$ and $\pi \dot{A} = \dot{A}$. The filter $\mathcal{F}$ is generated by subgroups of the form

$$\text{fix}(E) = \{\pi \in \mathcal{G} \mid \pi \restriction E = \text{id}\} \quad \text{for } E \in [\omega]^{<\omega}.$$

$\mathcal{F}$ is normal since $\pi \text{fix}(E) \pi^{-1} = \text{fix}(\pi[E])$. So $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a symmetric system. As $\text{fix}(\{n\}) \subseteq \text{sym}(\dot{a}_n)$ and $\mathcal{G} \subseteq \text{sym}(\dot{A})$, we see that $\dot{a}_n, \dot{A}$ are all HS-names and the sets a_n, A are all in the symmetric extension M .

We show that there is no injection from ω into A , so A is Dedekind-finite. If there was an injection $f : \omega \rightarrow A$ in M , let $\dot{f}$ be an HS-name for f , $\text{fix}(E) \subseteq \text{sym}(\dot{f})$ and let $p \Vdash^{\text{HS}} \dot{f} : \check{\omega} \rightarrow \dot{A}$ is an injection”. Extending p if necessary, suppose also that $p \Vdash \dot{f}(\check{n}) = \dot{a}_m$ for some n and $m \notin E$; such an m exists as f is an injection. Then we may find some $m' \neq m$ such that $(\forall k \in \omega) \langle m', k \rangle \notin \text{dom } p$ and $m' \notin E$. So if π is the transposition $(m \ m')$ then $\pi \in \text{fix}(E)$ and $\pi p \parallel p$, but a common extension of πp and p forces that “ $\dot{f}(\check{n}) = \dot{a}_m$ and $\dot{f}(\check{n}) = \dot{a}_{m'}$ ” which is a contradiction.

Definition 2.2.2 (Cohen forcing). We shall make frequent use of the following generalisation of $\text{Add}(\omega, \omega)$: Given sets X and Y we denote by $\text{Add}(X, Y)$ the notion of forcing of partial functions $p : Y \times X \rightarrow 2$ such that $\text{dom}(p)$ is well-orderable and $|\text{dom}(p)| < |X|$. The ordering is given by $q \leq p$ if $q \supseteq p$. A generic filter for $\text{Add}(X, Y)$ will induce a new function $c : Y \times X \rightarrow 2$ and so can be thought of as introducing a Y -indexed family of new subsets of X .

The idea of iterating a symmetric extension, much like iterating forcing extensions, is quite simple at its core. One has a symmetric system $\mathcal{S}_0 = \langle \mathbb{P}_0, \mathcal{G}_0, \mathcal{F}_0 \rangle$ and considers the symmetric extension $M = \text{HS}_{\mathcal{F}_0}^{G_0} \supseteq V$ given by some V -generic filter G_0 for $\mathcal{S}_0$. Then, in M , one can construct a further symmetric system $\mathcal{S}_1 = \langle \mathbb{P}_1, \mathcal{G}_1, \mathcal{F}_1 \rangle$ and can take, over M , a symmetric extension by $\mathcal{S}_1$ to obtain $N = \text{HS}_{\mathcal{F}_1}^{G_1} \supseteq M$ given by some M -generic filter G_1 for $\mathcal{S}_1$. Analogously to iterating notions of forcing we can describe this process as a single symmetric extension in

the ground model. While [Kar19] has a thorough treatment of the topic, we shall only go over what is required for accessing our results.

Definition 2.2.3 (Two-step symmetric extension). Let $\mathcal{S}_0 = \langle \mathbb{P}_0, \mathcal{G}_0, \mathcal{F}_0 \rangle$ be a symmetric system, and let $\dot{\mathcal{S}}_1 = \langle \dot{\mathbb{P}}_1, \dot{\mathcal{G}}_1, \dot{\mathcal{F}}_1 \rangle^\bullet$ be an $\mathcal{S}_0$ -name such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathcal{S}_0}$ “ $\dot{\mathcal{S}}_1$ is a symmetric system” and $\text{sym}(\dot{\mathcal{S}}_1) = \mathcal{G}_0$. We define the iteration $\mathcal{S} = \mathcal{S}_0 * \dot{\mathcal{S}}_1$ to be the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ thusly:

$\mathbb{P}$ is defined similarly to the usual iteration of notions of forcing. Conditions in $\mathbb{P}$ are pairs $\langle p, \dot{q} \rangle$ such that $p \in \mathbb{P}_0$, $\dot{q}$ is an $\mathcal{S}_0$ -name and $\mathbb{1}_{\mathbb{P}_0} \Vdash \dot{q} \in \dot{\mathbb{P}}_1$.⁴ The ordering is given by $\langle p', \dot{q}' \rangle \leq \langle p, \dot{q} \rangle$ if $p' \leq_{\mathbb{P}_0} p$ and $p' \Vdash \dot{q}' \leq_{\dot{\mathbb{P}}_1} \dot{q}$.

For $\langle p, \dot{q} \rangle \in \mathbb{P}$, if $\mathbb{1}_{\mathbb{P}_0} \Vdash \dot{\pi}_1 \in \dot{\mathcal{G}}_1$ then there is a canonical choice for an $\mathcal{S}_0$ -name $\dot{q}'$ such that $\mathbb{1}_{\mathbb{P}_0} \Vdash \dot{\pi}_1(\dot{q}) = \dot{q}'$.⁵ We shall denote this name simply by $\dot{\pi}_1 \dot{q}$ to coincide with the notation $\pi_0 p$ for the action of $\mathcal{G}_0$ on $\mathbb{P}_0$. We set $\mathcal{G} = \mathcal{G}_0 * \dot{\mathcal{G}}_1$ to be the group with elements that are pairs $\bar{\pi} = \langle \pi_0, \dot{\pi}_1 \rangle$ where $\pi_0 \in \mathcal{G}_0$ and $\mathbb{1}_{\mathbb{P}_0} \Vdash \dot{\pi}_1 \in \dot{\mathcal{G}}_1$, with action on $\mathbb{P}$ given by

$$\bar{\pi} \langle p, \dot{q} \rangle = \langle \pi_0 p, \pi_0(\dot{\pi}_1 \dot{q}) \rangle = \langle \pi_0 p, (\pi_0 \dot{\pi}_1)(\pi_0 \dot{q}) \rangle.$$

This action is sometimes denoted $\int_{\bar{\pi}} \langle p, \dot{q} \rangle$ in the literature, especially in longer iterations, and is used in §2.4. Since $\text{sym}(\dot{\mathcal{S}}_1) = \mathcal{G}_0$, we have that $\pi_0 \dot{\mathbb{P}}_1 = \dot{\mathbb{P}}_1$, and by [Kar19] Proposition 3.4, this defines an automorphism.

Whenever $\Gamma_0 \in \mathcal{F}_0$, $\mathbb{1}_{\mathbb{P}_0} \Vdash \dot{\Gamma}_1 \in \dot{\mathcal{F}}_1$ and $\Gamma_0 \leq \text{sym}_{\mathcal{G}_0}(\dot{\Gamma}_1)$, we identify $\bar{\Gamma} = \langle \Gamma_0, \dot{\Gamma}_1 \rangle$ with the group of $\langle \pi_0, \dot{\pi}_1 \rangle$, where $\pi_0 \in \Gamma_0$ and $\mathbb{1}_{\mathbb{P}_0} \Vdash \dot{\pi}_1 \in \dot{\Gamma}_1$. Finally, we set $\mathcal{F} = \mathcal{F}_0 * \dot{\mathcal{F}}_1$ to be the filter generated by subgroups of $\mathcal{G}$ of this form.

WREATH PRODUCTS

In this subsection only, let G and H denote two groups acting on some sets. We define the version of wreath product $G \wr H$ of G by H that we commonly use in symmetric systems.

Definition 2.2.4. Suppose that G acts on a set X and H acts on Y .

⁴As in forcing iterations, since the collection of such names $\dot{q}$ is technically a proper class, we would for instance put a bound on the rank of $\dot{q}$.

⁵For instance, let $\dot{q}' = \{ \langle p, \dot{q} \rangle \mid \dot{q} \text{ is an } \mathcal{S}_0\text{-name and } p \Vdash \dot{q} \in \dot{\pi}_1(\dot{q}) \}$, where we bound the rank of $\dot{q}$ appropriately.

- For each $x \in X$, $h \in H$ is associated with an element in $S_{X \times Y}$:

$$h_x^*(x', y) = \begin{cases} (x', hy) & \text{if } x' = x, \\ (x', y) & \text{if } x' \neq x. \end{cases}$$

Then $H_x^* := \{h_x^* \mid h \in H\}$ is isomorphic to H for each $x \in X$.

- For each $g \in G$, define $g^* \in S_{X \times Y}$ by

$$g^*(x, y) = (gx, y).$$

Then $G^* := \{g^* \mid g \in G\}$ is isomorphic to G .

- The **wreath product** of G by H , denoted $G \wr H$, acting on $X \times Y$ is given by

$$\langle G^*, \langle H_x^* \rangle_{x \in X} \rangle \leq S_{X \times Y}$$

and an element $\pi \in G \wr H$ is of the form

$$\pi = (g^*, (h_x^*)_{x \in X})$$

acting on $X \times Y$ via

$$\pi(x, y) = g^*(h_x^*(x, y)).$$

We denote by π^* the component g^* of π and by π_x^* the component h_x^* of π .

The dependence of the G -set is not shown in the notation, but in our case, there is no need to indicate that as it is always clear in the context. The notation used also differs from the one that is usually given in group-theoretic literatures, but it is more intuitive for our purpose.

It is better to think the group action of $G \wr H$ pictorially: Suppose G acts on a circle X on which we wind many smaller circles, forming a wreath, and H acts on each individual winding Y . Then an element $(g^*, (h_x^*)_{x \in X})$ of $G \wr H$ acts on the wreath by applying h_x^* to the winding on each coordinate $x \in X$, then applying g^* to the whole circle.

2.3 Upwards homogeneity

Recall that we are concerned with iterations of symmetric extensions $\mathcal{S} = \mathcal{S}_0 * \dot{\mathcal{S}}_1$ given by the symmetric systems $\langle \mathbb{P}_0, \mathcal{G}_0, \mathcal{F}_0 \rangle * \langle \dot{\mathbb{P}}_1, \dot{\mathcal{G}}_1, \dot{\mathcal{F}}_1 \rangle^\bullet = \langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$.

Definition 2.3.1. Let $\mathcal{S} = \langle \mathbb{P}_0, \mathcal{G}_0, \mathcal{F}_0 \rangle * \langle \dot{\mathbb{P}}_1, \dot{\mathcal{G}}_1, \dot{\mathcal{F}}_1 \rangle^\bullet = \mathcal{S}_0 * \dot{\mathcal{S}}_1$ be an iteration of symmetric extensions. We say that $\mathcal{S}$ is **upward homogeneous** if, for all $\bar{\Gamma} \in \mathcal{F}$, there is a predense collection of conditions $\langle p^\circ, \dot{q}^\circ \rangle \in \mathbb{P}$ satisfying the following property:

For all $\langle p, \dot{q} \rangle, \langle p', \dot{q}' \rangle \leq \langle p^\circ, \dot{q}^\circ \rangle$ there is $\bar{\pi} \in \bar{\Gamma}$ such that $\bar{\pi}\langle p, \dot{q} \rangle \parallel \langle p', \dot{q}' \rangle$.

That is, there is a predense collection of elements $\langle p^\circ, \dot{q}^\circ \rangle$ such that whenever $p \leq p^\circ$, and $p \Vdash_{\mathcal{S}_0} \dot{q}, \dot{q}' \leq \dot{q}^\circ$ there is $\pi_0 \in \Gamma_0$ and $\mathbb{1}_{\mathbb{P}_0} \Vdash_{\mathcal{S}_0} \dot{\pi}_1 \in \dot{\Gamma}_1$ such that $\pi_0 p \parallel p$ and, for some $p' \leq p, \pi_0(p), p' \Vdash_{\mathcal{S}_0} \pi(\dot{\pi}_1 \dot{q}) \parallel \dot{q}'$.

Note that if $\langle p^\circ, \dot{q}^\circ \rangle$ satisfies this condition for a particular $\bar{\Gamma} \in \mathcal{F}$ then all extensions of $\langle p^\circ, \dot{q}^\circ \rangle$ do as well. Hence, we may equivalently replace “predense” by “dense” or “open dense” in the definition.

Main theorem. An iteration of symmetric extensions $\mathcal{S} = \mathcal{S}_0 * \dot{\mathcal{S}}_1$ is upward homogeneous iff for all symmetric extensions $V \subseteq M \subseteq N$ that $\mathcal{S}$ generates and all $X \in N$ with $X \subseteq V$, we have $X \in M$.

Note that in the case $V \models \text{AC}$ this is equivalent to saying that there are no new sets of ordinals in N that were not already in M .

Proof. ($\Rightarrow$). Let $\dot{X}$ be an $\mathcal{S}$ -name such that $\mathbb{1} \Vdash_{\mathcal{S}} \dot{X} \subseteq \dot{A}$ for some $A \in V$. Letting $\bar{\Gamma} \in \mathcal{F}$ be such that $\bar{\Gamma} \leq \text{sym}(\dot{X})$, we claim that whenever $\langle p^\circ, \dot{q}^\circ \rangle \in \mathbb{P}$ is as in the definition of upward homogeneity, $\langle p, \dot{q} \rangle \leq \langle p^\circ, \dot{q}^\circ \rangle$, and $\langle p, \dot{q} \rangle \Vdash_{\mathcal{S}} \dot{a} \in \dot{X}$, then $\langle p, \dot{q}^\circ \rangle \Vdash_{\mathcal{S}} \dot{a} \in \dot{X}$ as well.

Let $\langle p', \dot{q}' \rangle \leq \langle p, \dot{q}^\circ \rangle$. By the definition of upward homogeneity, there is $\bar{\pi} \in \bar{\Gamma}$ such that $\bar{\pi}\langle p', \dot{q} \rangle \parallel \langle p', \dot{q}' \rangle$. Given that $\langle p', \dot{q} \rangle \leq \langle p, \dot{q} \rangle$, we have that $\langle p', \dot{q} \rangle \Vdash_{\mathcal{S}} \dot{a} \in \dot{X}$ and, indeed, $\bar{\pi}\langle p', \dot{q} \rangle \Vdash_{\mathcal{S}} \dot{a} \in \dot{X}$ as well. Since $\bar{\pi}\langle p', \dot{q} \rangle \parallel \langle p', \dot{q}' \rangle$, we cannot have that $\langle p', \dot{q}' \rangle \Vdash_{\mathcal{S}} \dot{a} \notin \dot{X}$ and so $\langle p, \dot{q}^\circ \rangle \Vdash_{\mathcal{S}} \dot{a} \in \dot{X}$ as required. Similarly, if $\langle p, \dot{q} \rangle \Vdash_{\mathcal{S}} \dot{a} \notin \dot{X}$,

then $\langle p, \dot{q}^\circ \rangle \Vdash_{\mathcal{S}} \check{a} \notin \dot{X}$ as well. Hence, if we let

$$\dot{Y} = \{ \langle \pi p, \check{a} \rangle \mid \pi \in \Gamma_0, a \in A \langle \pi p, \pi \dot{q}^\circ \rangle \Vdash_{\mathcal{S}} \check{a} \in \dot{X} \}$$

then $\Gamma_0 \leq \text{sym}_{\mathcal{S}_0}(\dot{Y})$, so $\dot{Y}$ is an $\mathcal{S}_0$ -name, and $\langle p^\circ, \dot{q}^\circ \rangle \Vdash_{\mathcal{S}} \dot{X} = \dot{Y}$.

($\Leftarrow$). Let $\bar{\Gamma}$ and $\langle p_0, \dot{q}_0 \rangle$ be arbitrary. Consider the $\mathbb{P}$ -name

$$\dot{X} = \{ \langle \bar{\pi} \langle p, \dot{q} \rangle, \langle p, \dot{q} \rangle^\sim \rangle \mid \langle p, \dot{q} \rangle \in \mathbb{P}, \bar{\pi} \in \bar{\Gamma} \},$$

where $\langle p, \dot{q} \rangle^\sim$ refers to the check name of the *ground model set* $\langle p, \dot{q} \rangle$, rather than the condition in $\mathbb{P}$ or the $\mathcal{S}_0$ -name for an element of $\dot{\mathbb{P}}_1$. Note that $\bar{\Gamma} \leq \text{sym}(\dot{X})$ and every $\mathbb{P}$ -name appearing in $\dot{X}$ is a check name, so $\dot{X}$ is a hereditarily $\mathcal{F}$ -symmetric name for a subset of V . Note also that since all $\mathbb{P}$ -names appearing in $\dot{X}$ are check names, for all generic $G \subseteq \mathbb{P}$ we have $\langle p, \dot{q} \rangle \in \dot{X}^G$ iff there is $\bar{\pi} \in \bar{\Gamma}$ such that $\bar{\pi} \langle p, \dot{q} \rangle \in G$. By assumption there is an $\mathcal{S}_0$ -name $\dot{Y}$ and $\langle p^\circ, \dot{q}^\circ \rangle \leq \langle p_0, \dot{q}_0 \rangle$ such that $\langle p^\circ, \dot{q}^\circ \rangle \Vdash_{\mathcal{S}} \dot{X} = \dot{Y}$.

We claim that $\langle p^\circ, \dot{q}^\circ \rangle$ is as in the definition of upward homogeneity. To this end, let $\langle p, \dot{q} \rangle, \langle p, \dot{q}' \rangle \leq \langle p^\circ, \dot{q}^\circ \rangle$ be arbitrary. Since $\langle p, \dot{q} \rangle \Vdash_{\mathcal{S}} \langle p, \dot{q} \rangle^\sim \in \dot{X}$ and $\langle p, \dot{q} \rangle \leq \langle p^\circ, \dot{q}^\circ \rangle$, we have that $\langle p, \dot{q} \rangle \Vdash_{\mathcal{S}} \langle p, \dot{q} \rangle^\sim \in \dot{Y}$ as well. Furthermore, since $\dot{Y}$ and $\langle p, \dot{q} \rangle^\sim$ are $\mathcal{S}_0$ -names, we in fact have that $p \Vdash_{\mathcal{S}_0} \langle p, \dot{q} \rangle^\sim \in \dot{Y}$. If $G \subseteq \mathbb{P}$ is then an arbitrary generic filter with $\langle p, \dot{q}' \rangle \in G$ we have that $\langle p, \dot{q} \rangle \in \dot{Y}^G = \dot{X}^G$ and, by the observation above, there is $\bar{\pi} \in \bar{\Gamma}$ such that $\bar{\pi} \langle p, \dot{q} \rangle \in G$. Since both $\langle p, \dot{q}' \rangle$ and $\bar{\pi} \langle p, \dot{q} \rangle$ lie in the same filter, these conditions must be compatible and we have proved our claim. $\square$

By inspecting the prior proof, we have actually made a somewhat stronger assertion.

Theorem 2.3.2. Let $\bar{\Gamma} \in \mathcal{F}$ and $\langle p^\circ, \dot{q}^\circ \rangle \in \mathbb{P}$. The following are equivalent:

- (i) For all $\langle p, \dot{q} \rangle, \langle p, \dot{q}' \rangle \leq \langle p^\circ, \dot{q}^\circ \rangle$ there is $\bar{\pi} \in \bar{\Gamma}$ such that $\bar{\pi} \langle p, \dot{q} \rangle \parallel \langle p, \dot{q}' \rangle$.
- (ii) $\langle p^\circ, \dot{q}^\circ \rangle$ forces that all $\bar{\Gamma}$ -stable names are equivalent to an $\mathcal{S}_0$ -name.
- (iii) $\langle p^\circ, \dot{q}^\circ \rangle$ forces that the $\bar{\Gamma}$ -symmetrisation of the generic object G ,

$$\bar{\Gamma} \dot{G} = \{ \bar{\pi} \langle \langle p, \dot{q} \rangle, \langle p, \dot{q} \rangle^\sim \rangle \mid \bar{\pi} \in \bar{\Gamma}, \langle p, \dot{q} \rangle \in \mathbb{P} \}$$

is equivalent to an $\mathcal{S}_0$ -name.

2.4 Examples

MONRO'S ITERATION

Monro's article [Mon73] was inspired by a theorem of Balcar and Vopěnka that if M and N are transitive models of ZF such that $P(\text{Ord})^M = P(\text{Ord})^N$ and either $M \models \text{AC}$ or $N \models \text{AC}$, then $M = N$. Monro's construction builds a chain of models $\langle M_n \mid n < \omega \rangle$ of ZF such that for all $n < \omega$, $P^n(\text{Ord})^{M_n} = P^n(\text{Ord})^{M_{n+1}}$, but $M_n \neq M_{n+1}$, where $P^n(\text{Ord})$ refers to the n -th iterate of the power set operation on Ord . This construction is later extended further by [Sha18] to go beyond the ω -th step.

We shall inductively define a sequence of models $\langle M_n \mid n < \omega \rangle$, Dedekind-finite sets $A_n \in M_n$ (except $A_0 = \omega$), and symmetric systems $\mathcal{S}_n = \langle \mathbb{P}_n, \mathcal{G}_n, \mathcal{F}_n \rangle \in M_n$ such that M_{n+1} is a symmetric extension of M_n by $\mathcal{S}_n$. We begin with $M_0 = V$ a model of ZFC and let $A_0 = \omega$. If M_n and A_n have been chosen, the symmetric system $\mathcal{S}_n$ is defined as follows.

The forcing $\mathbb{P}_n$ is $\text{Add}(A_n, \omega)$. Note that since A_n is ω or Dedekind-finite, this is the forcing given by finite partial functions $\omega \times A_n \rightarrow 2$ ordered by reverse inclusion. We define the $\mathbb{P}_n$ -names

$$\dot{a}_k^n = \{ \langle p, \check{x} \rangle \mid p(k, x) = 1, x \in A_n \},$$

$$\dot{A}_{n+1} = \{ \dot{a}_k^n \mid k < \omega \}^\bullet.$$

The group of automorphism is $\mathcal{G}_n = S_{<\omega}$ acting on the forcing by $\pi p(\pi n, x) = p(n, x)$. Then $\pi \dot{a}_k^n = \dot{a}_{\pi k}^n$, and $\pi \dot{A}_{n+1} = \dot{A}_{n+1}$ for all $\pi \in \mathcal{G}_n$.

The filter $\mathcal{F}_n$ is generated by the subgroups $\text{fix}(E) := \{ \pi \in \mathcal{G}_n \mid \pi \restriction E = \text{id} \}$ for $E \in [\omega]^{<\omega}$. Then $\dot{a}_k^n \in \text{HS}_{\mathbb{P}_n}$ for all $k \in \omega$ and hence $\dot{A}_{n+1} \in \text{HS}_{\mathbb{P}_n}$. M_{n+1} is a symmetric extension over M_n given by the symmetric system $\mathcal{S}_n = \langle \mathbb{P}_n, \mathcal{G}_n, \mathcal{F}_n \rangle$.

So M_1 is Cohen's first model defined in Example 2.2.1, and by the same argument the sets A_{n+1} are all infinite Dedekind-finite.

Proposition 2.4.1. For all $n < \omega$, $\mathcal{S}_n * \dot{\mathcal{S}}_{n+1}$ is upward homogeneous. Therefore, M_{n+2} does not have any subsets of M_n that were not already in M_{n+1} and, by induction, $P^{n+1}(\text{Ord})^{M_{n+1}} = P^{n+1}(\text{Ord})^{M_{n+2}}$.

Proof. Here we work in M_n , so $\mathcal{S}_n$ is an element of our ground model M_n and the statement “ $\mathcal{S}_n * \dot{\mathcal{S}}_{n+1}$ is upward homogeneous” makes sense. Note that conditions in $\dot{\mathbb{P}}_{n+1}$ have canonical $\mathbb{P}_n$ -names: if f is a finite function $\omega \times \omega \rightarrow 2$, then there is a $\mathbb{P}_n$ -name $\dot{q}_f$ for a condition in $\dot{\mathbb{P}}_{n+1}$ given by

$$\dot{q}_f = \{ \langle \check{k}, \dot{a}_m^n, (f(k, m)) \rangle^\bullet \mid \langle k, m \rangle \in \text{dom } f \}^\bullet,$$

i.e., $\dot{q}_f(k, \dot{a}_m^n) = f(k, m)$; conversely, whenever $p \Vdash_{\mathcal{S}_n} \dot{q} \in \dot{\mathbb{P}}_{n+1}$ we may extend p to $p' \leq p$ and find some $f : \omega \times \omega \rightarrow 2$ such that $p' \Vdash_{\mathcal{S}_n} \dot{q} = \dot{q}_f$. When $\dot{\mathbb{P}}_{n+1}$ -conditions are cast into the canonical form, elements $\pi \in \mathcal{G}_n$ act on them in a natural way: let $\hat{\pi} \cdot f$ denote the function such that $(\hat{\pi} \cdot f)(k, \pi m) = f(k, m)$ for all $k, m \in \omega$. Then

$$\begin{aligned} \pi \dot{q}_f &= \{ \langle \pi \check{k}, \pi \dot{a}_m^n, \pi(f(k, m)) \rangle^\bullet \mid \langle k, m \rangle \in \text{dom } f \}^\bullet \\ &= \{ \langle \check{k}, \dot{a}_{\pi m}^n, (f(k, m)) \rangle^\bullet \mid \langle k, m \rangle \in \text{dom } f \}^\bullet \\ &= \{ \langle \check{k}, \dot{a}_{\pi m}^n, ((\hat{\pi} \cdot f)(k, \pi m)) \rangle^\bullet \mid \langle k, m \rangle \in \text{dom } f \}^\bullet \\ &= \{ \langle \check{k}, \dot{a}_m^n, ((\hat{\pi} \cdot f)(k, m)) \rangle^\bullet \mid \langle k, m \rangle \in \text{dom}(\hat{\pi} \cdot f) \}^\bullet \\ &= \dot{q}_{\hat{\pi} \cdot f} \end{aligned}$$

We shall still denote by πf the action $\pi f(\pi n, m) = f(n, m)$. In the following, we assume that all conditions in $\dot{\mathbb{P}}_{n+1}$ are in this canonical form.

Let $\text{fix}(E_0) \in \mathcal{F}_n$ and $\text{fix}(E_1) \in \dot{\mathcal{F}}_{n+1}$, where $E_0, E_1 \in [\omega]^{<\omega}$, and let $\langle p_0, \dot{q}_f \rangle$ be a condition in $\mathbb{P}_n * \dot{\mathbb{P}}_{n+1}$. We wish to find a compatible condition $\langle p^\circ, \dot{q}^\circ \rangle$ that witnesses upward homogeneity for $\text{fix}(E_0) * \text{fix}(E_1) \in \mathcal{F}_n * \dot{\mathcal{F}}_{n+1}$. We claim for any f° such that $\text{dom}(f^\circ) = E_1 \times E_0$ and $f^\circ \parallel f$, $\langle p_0, \dot{q}_{f^\circ} \rangle$ is sufficient. Set $\dot{q}^\circ = \dot{q}_{f^\circ}$. Note that in this case we certainly have that $\langle p_0, \dot{q}_f \rangle \parallel \langle p_0, \dot{q}^\circ \rangle$.

Let $\langle p, \dot{q}_g \rangle, \langle p, \dot{q}_h \rangle \leq \langle p_0, \dot{q}^\circ \rangle$. We can find the required $\pi \in \text{fix}(E_0)$ and $\dot{\sigma} \in \text{fix}(E_1)$ as follows:

Firstly, since $\text{dom } g$ and $\text{dom } h$ are finite, there is $\sigma \in \text{fix}(E_1)$ such that $\text{dom}(\sigma g) \cap \text{dom}(h) \subseteq E_1 \times \omega$, akin to Cohen’s first model in [Example 2.2.1](#).

Similarly, given that $\text{dom}(p)$, $\text{dom}(\sigma g)$, and $\text{dom}(h)$ are all finite we can find $\pi \in \text{fix}(E_0)$ such that $\text{dom}(\pi p) \cap \text{dom}(p) \subseteq E_0 \times A_n$ and

$$\text{dom}(\hat{\pi} \cdot (\sigma g)) \cap \text{dom}(h) \subseteq E_1 \times E_0.$$

Given that $g, h \supseteq f^\circ$, and $\text{dom}(f^\circ) = E_1 \times E_0$, we must have that $\hat{\pi} \cdot (\sigma g) \parallel h$, and hence $\langle \pi, \dot{\sigma} \rangle \langle p, \dot{q}_g \rangle \parallel \langle p, \dot{q}_h \rangle$ as required. $\square$

Remark. In the proof we used the fact that there exist canonical $\mathcal{S}_0$ -names for conditions in $\dot{\mathbb{P}}_1$ on which the group $\mathcal{G}_0$ acts nicely. This is the case for all applications of upward homogeneity that we will exhibit and we shall implicitly assume that the names for the conditions of $\dot{\mathbb{P}}_1$ are already cast into such canonical forms. Extending the first co-ordinate to achieve this will not affect if $\mathcal{S}$ is upward homogeneous.

This case serves as an excellent intuitive framework for upward homogeneity. If one only considers $\text{fix}(E_0) \leq \text{sym}_{\mathcal{G}_0}(\dot{X})$ and $\text{fix}(E_1) \leq \text{sym}_{\dot{\mathcal{G}}_1}([\dot{X}])$, where $[\dot{X}]$ is an $\mathcal{S}_0$ -name for an $\dot{\mathcal{S}}_1$ -name $\dot{X}$ of a set X , then any $\dot{q} \in \dot{\mathbb{P}}_1$ that has information filled up on the domain $E_0 \times E_1$ determines the whole content of X . Since E_0 and E_1 are both finite, we can always find some q such that $\text{dom } q \supseteq E_0 \times E_1$, from which we can determine X .

One may view upward homogeneity as a sort of “closure” of $\dot{\mathbb{P}}_1$ with respect to both filters $\mathcal{F}_0$ and $\dot{\mathcal{F}}_1$. In many cases of symmetric extensions, the filter in the symmetric system is taken as generated by $\{\text{fix}(E) \mid E \in \mathcal{I}\}$ where $\mathcal{I}$ is an ideal of subsets of a set related to the forcing. If $\dot{\mathbb{P}}_1$ is not “closed enough” compared to one of the ideals that are used to generate the filters, then upward homogeneity fails.

Example. Let $\mathbb{P}_0 = \text{Add}(\omega, \omega_1)$, $\mathcal{G}_0 = S_{\omega_1}$ acting on $\mathbb{P}_0$ in the usual way ($\pi p(\pi(\alpha), n) = p(\alpha, n)$), $\mathcal{F}_0$ generated by $\{\text{fix}(E) \mid E \in [\omega_1]^{<\omega_1}\}$. Then $\mathbb{P}_0$ adds a set $A = \{a_\alpha \mid \alpha \in \omega_1\}$ of ω_1 -many new reals; if $E \subseteq \omega_1$ is countable in V , then there is an enumeration $\langle a_\alpha \mid \alpha \in E \rangle$ in M , since it is fixed by $\text{fix}(E)$.

Then we let

$$\dot{\mathbb{P}}_1 = \{\text{finite } q : \omega \times A \rightarrow 2\},$$

$\mathcal{G}_1 = S_\omega$ acting on $\dot{\mathbb{P}}_1$ in the usual way, and $\mathcal{F}_1$ generated by $\{\text{fix}(E) \mid E \in [\omega]^{<\omega}\}$. Then $\dot{\mathbb{P}}_1$ adds a set $\{b_n \mid n \in \omega\}$ where each b_n is a subset of A . Let $E \subseteq \omega_1$ be countable, then for each $n \in \omega$, $\dot{\mathbb{P}}_1$ adds $\{\alpha \in E \mid a_\alpha \in b_n\}$ new to M , since for each $r : E \rightarrow 2$ in M , $\{q \in \dot{\mathbb{P}}_1 \mid (\exists \alpha) q(a_\alpha, n) \neq r(n)\}$ is dense.

In this example, $\dot{\mathbb{P}}_1$, consisting of finite conditions, is not closed enough compared to the σ -closedness of the filter $\mathcal{F}_0$.

The reader may also verify that if in Monro's iteration $\mathcal{S}_0 * \dot{\mathcal{S}}_1$ we change the filter of either the first or the second iteration to the improper filter (the filter that contains the trivial subgroup) then we have added new subsets of ω in the second iteration.

Note that when the second iterand is a full forcing extension, this intuition collapses as we may take $\dot{\mathcal{G}}_1$ to be a large permutation group or the trivial group, but the filter is always improper. We have seen that the second iterands may add new sets of ordinals, but the next section shows that there are also cases where they do not.

FORCING OVER SYMMETRIC EXTENSIONS

Here, we consider what happens to [Definition 2.3.1](#) if the second iterand is a full forcing extension. In this case, one may take $\mathcal{G}_1 = \{\text{id}\}$ and upward homogeneity can be simplified: For all $\Gamma \in \mathcal{F}_0$, there is a predense collection of $\langle p^\circ, \dot{q}^\circ \rangle \in \dot{\mathbb{P}}$ such that, for all $\langle p, \dot{q} \rangle, \langle p, \dot{q}' \rangle \leq \langle p^\circ, \dot{q}^\circ \rangle$, there is $\pi \in \Gamma$ such that $\langle \pi p, \pi \dot{q} \rangle \parallel \langle p, \dot{q}' \rangle$. Notationally, we shall identify a (name for a) notion of forcing $\dot{\mathbb{P}}_1$ with the symmetric system $\langle \dot{\mathbb{P}}_1, \{\text{id}\}, \{\{\text{id}\}\}^\bullet \rangle$.

Here is an example from [\[KS19\]](#) Theorem 5.1. Recall Cohen's first model defined in [Example 2.2.1](#) and $A = \{a_n \mid n \in \omega\}$ denotes the set of reals added by $\mathbb{P}_0 = \text{Add}(\omega, \omega)$ in this extension.

Proposition 2.4.2. Let κ be any cardinal. Define

$$\mathbb{P}_1 = \text{Col}(A, \kappa) := \{q \text{ a function } A \rightarrow \kappa \mid \text{dom } q \text{ is well-orderable and } |q| < |A|\}$$

ordered by reverse inclusion. Then $\mathcal{S}_0 * \text{Col}(\dot{A}, \dot{\kappa})^\bullet$ is upward homogeneous. That is, $\mathbb{P}_1$ does not add new sets of ordinals.

Proof. Note that the conditions in $\mathbb{P}_1$ are finite. For $\langle p_0, \dot{q}_0 \rangle \in \dot{\mathbb{P}}_1$ and $E \in [\omega]^{<\omega}$, we find $\langle p^\circ, \dot{q}^\circ \rangle \leq \langle p_0, \dot{q}_0 \rangle$ so that $p^\circ \Vdash_{\mathcal{S}_0} \text{dom}(\dot{q}^\circ) \supseteq \{\dot{a}_n \mid n \in E\}^\bullet$. If $\langle p, \dot{q} \rangle$ and $\langle p, \dot{q}' \rangle$ extend $\langle p^\circ, \dot{q}^\circ \rangle$, let $p' \leq p$ so that $p' \Vdash \text{dom}(\dot{q}) = \{\dot{a}_n \mid n \in F\}^\bullet \wedge \text{dom}(\dot{q}') = \{\dot{a}_n \mid$

$n \in F'\}^\bullet$ for some $F, F' \in [\omega]^{<\omega}$. Then we find $\pi \in \text{fix}(E)$ such that $\text{dom}(\pi p') \cap \text{dom}(p') \subseteq E$ and $\pi[F] \cap F' = E$ and thus $p' \Vdash_{\mathbb{S}_0} \text{dom}(\pi \dot{q}) \cap \text{dom}(\dot{q}') = \{\dot{a}_n \mid n \in E\}^\bullet$. It follows that $\pi \langle p, \dot{q} \rangle \parallel \langle p, \dot{q}' \rangle$. Therefore $\langle p^\circ, \dot{q}^\circ \rangle$ witnesses upward homogeneity as required. $\square$

Remark. Note that if instead the first iterand is a full forcing extension then $\mathbb{P}_0 * \dot{\mathbb{S}}_1$ is upward homogeneous iff $\mathbb{1}_{\mathbb{P}_0}$ forces that $\dot{\mathbb{S}}_1$ is trivial. In the case that the ground model satisfies AC this is an immediate consequence of Balcar and Vopěnka's theorem.

MORRIS'S ITERATION

A remarkable construction that violates choice is showing that countable unions of countable sets need not be countable. For example, Feferman and Lévy constructed a model of ZF in which 2^ω is a countable union of countable sets (and consequently ω_1 is singular). In the same vein, Morris [Mor70] constructed a model of ZF in which, for all ordinals α , there is a set A_α that is a countable union of countable sets admitting a surjection $P(A_\alpha) \rightarrow \aleph_\alpha$. The construction is recast into a modern form in [Kar20b] and corrected in [Kar19], the method of which we follow here.

Let κ be a fixed infinite cardinal. The forcing $\mathbb{P}_0$ is $\text{Add}(\kappa, \omega \times \omega \times \kappa)$. Define

$$\begin{aligned} \dot{x}_{n,m,\alpha} &= \{\langle p, \check{\beta} \rangle \mid p(n, m, \alpha, \beta) = 1\}, \\ \dot{a}_{n,m} &= \{\dot{x}_{n,m,\alpha} \mid \alpha < \kappa\}^\bullet, \\ \dot{A}_n &= \{\dot{a}_{n,m} \mid m < \omega\}^\bullet. \end{aligned}$$

The group $\mathcal{G}_0$ is $(\{\text{id}\} \wr S_\omega) \wr S_\kappa \leq S_{\omega \times \omega \times \kappa}$, so elements are of the form

$$\pi(n, m, \alpha) = \langle n, \pi_n^*(m), \pi_{n,m}(\alpha) \rangle$$

as in Definition 2.2.4, with the action on $\mathbb{P}_0$

$$\pi p(\pi(n, m, \alpha), \beta) = p(n, m, \alpha, \beta).$$

Then:

$$\begin{aligned}\pi \dot{x}_{n,m,\alpha} &= \dot{x}_{\pi(n,m,\alpha)}, \\ \pi \dot{a}_{n,m} &= \dot{a}_{\pi^*(n,m)} = \dot{a}_{n,\pi_n^* m}, \\ \pi \dot{A}_n &= \dot{A}_n.\end{aligned}$$

The filter $\mathcal{F}_0$ is generated by groups of the form

$$\text{fix}(n, E) = \{\pi \in \mathcal{G} \mid \pi \restriction n \times \omega \times E = \text{id}\}$$

for $n < \omega$ and $E \in [\kappa]^{<\kappa}$. Hence all the names defined above are in HS.

Let M be a symmetric extension by the system $\mathcal{S}_0 = \langle \mathbb{P}_0, \mathcal{G}_0, \mathcal{F}_0 \rangle$ and, working in M , let

$$T = \bigcup_{n < \omega} \prod_{k < n} A_k$$

and note that T has a natural structure of a tree. We define $\mathbb{P}_1$ as follows. A condition in $\mathbb{P}_1$ is a finite sequence $t = \langle t_i \mid i \in E \rangle$ such that $E \in [\kappa \times \omega]^{<\omega}$ and $t_i \in T$ for all $i \in E$; $E = \text{dom } t$ is called the *support* of this condition, denoted as $\text{supp } t$. In $\mathbb{P}_1$, $t \leq s$ if the following hold:

1. $\text{supp } t \supseteq \text{supp } s$.
2. For all $i \in \text{supp } s$, $s_i \subseteq t_i$.
3. For all $i \neq j \in \text{supp } s$, if $t_i(k) = t_j(k)$, then $k \in |s_i| (= |s_j|)$.

In other words, if any extensions on the sequences s_i are to be made, then these extensions must be pairwise distinct.

The group $\mathcal{G}_1$ is $\{\text{id}\} \wr S_{<\omega} \leq S_{\kappa \times \omega}$, acting on $\mathbb{P}_1$ via

$$\pi \langle t_i \mid i \in E \rangle = \langle t_{\pi i} \mid i \in E \rangle.$$

The filter $\mathcal{F}_1$ is generated by the subgroups $\text{fix}(E) = \{\pi \in \mathcal{H} \mid \pi \restriction E = \text{id}\}$ with $E \in [\kappa \times \omega]^{<\omega}$.

Let $\mathcal{S}_1 = \langle \mathbb{P}_1, \mathcal{G}_1, \mathcal{F}_1 \rangle$ and let $\dot{\mathcal{S}}_1 = \langle \dot{\mathbb{P}}_1, \dot{\mathcal{G}}_1, \dot{\mathcal{F}}_1 \rangle^\bullet$ be a canonical $\mathcal{S}_0$ -name for $\mathcal{S}_1$.

Proposition 2.4.3. $\mathcal{S}_0 * \dot{\mathcal{S}}_1$ is upward homogeneous.

Proof. Let $\langle \Gamma_0, \dot{\Gamma}_1 \rangle \in \mathcal{F}$ and $\langle p_0, \dot{t}_0 \rangle \in \mathbb{P}$. Then we find $\langle p^\circ, \dot{t}^\circ \rangle \leq \langle p_0, \dot{t}_0 \rangle$, $n < \omega$, $E_0 \in [\kappa]^{<\kappa}$ and $E_1 \in [\kappa \times \omega]^{<\omega}$ such that $\text{fix}(n, E_0) \leq \Gamma_0$ and

$$p^\circ \Vdash_{\mathcal{S}_0} \text{fix}(\check{E}_1) \leq \dot{\Gamma}_1 \wedge \text{supp}(\dot{t}^\circ) = \check{E}_1 \wedge (\forall i \in \check{E}_1)(|\dot{t}_i^\circ| = \check{n}).$$

Suppose that $\langle p, \dot{t} \rangle, \langle p, \dot{t}' \rangle \leq \langle p^\circ, \dot{t}^\circ \rangle$. Then we may find $p' \leq p$, $F, F' \in [\kappa \times \omega]^{<\omega}$ extending E_1 , and $f : F \rightarrow \omega^m$, $f' : F' \rightarrow \omega^{m'}$, for some m, m' such that

$$p' \Vdash_{S_0} \dot{t}_i = \langle \dot{a}_{k, f_i(k)} \mid k < m \rangle^\bullet \wedge \dot{t}'_j = \langle \dot{a}_{k, f'_j(k)} \mid k < m' \rangle^\bullet$$

for all $i \in F$ and $j \in F'$. We first may find $\sigma \in \text{fix}(E_1)$ such that $\sigma[F] \cap F' = E_1$ and thus $p' \Vdash \text{supp}(\sigma \dot{t}) \cap \text{supp}(\dot{t}') = \check{E}_1$. Hence the only reason that $\sigma \dot{t}$ and $\dot{t}'$ might be incompatible is because of their values on E_1 . Note that for all $k \in [n, m)$ and $i \neq j \in E_1$ we have $f_i(k) \neq f_j(k)$, and a similar argument hold for f' . Thus

$$\{\langle f_i(k), f'_i(k) \rangle \mid i \in E_1\}$$

is an injective function and can be extended to $b_k \in S_\omega$. Let $\pi \in \text{fix}(n, E_0)$ be such that for all $k \in [n, m)$, $\pi_k^* = b_k$, taking $m \leq m'$ without loss of generality. By using the last factor of the wreath product which acts on κ we may also ensure that $\pi p' \parallel p'$. Thus, if $r \leq \pi p', p'$,

$$\begin{aligned} r \Vdash (\pi \sigma \dot{t})_i &= \pi \langle \dot{a}_{k, f_i(k)} \mid k < m \rangle^\bullet \\ &= \langle \pi \dot{a}_{k, f_i(k)} \mid k < m \rangle^\bullet \\ &= \langle \dot{a}_{k, f'_i(k)} \mid k < m \rangle^\bullet \\ &= \dot{t}'_i \restriction m, \end{aligned}$$

so $r \Vdash_{S_0} \pi \sigma \dot{t} \parallel \dot{t}'$. Hence also $\langle \pi, \sigma \rangle \langle p, \dot{t} \rangle \parallel \langle p, \dot{t}' \rangle$. Note that while it is not necessarily the case that $\mathbb{1}_{P_0} \Vdash_{S_0} \sigma \in \dot{\Gamma}_1$, we can easily find an S_0 -name $\dot{\tau}$ such that $\mathbb{1}_{P_0} \Vdash_{S_0} \dot{\tau} \in \dot{\Gamma}_1$ and $p^\circ \Vdash_{S_0} \dot{\tau} = \sigma$. Then $\langle \pi, \dot{\tau} \rangle$ is as required. $\square$

THE BRISTOL MODEL

In [VH72] Vopěnka shows that for all set-generic extensions $V \subseteq V[G]$ of models of ZFC and intermediate models $V \subseteq W \subseteq V[G]$, if $W \models \text{ZFC}$ then W is also a set-generic extension of V .⁶ However, if we ask only that W is a model of ZF then this principle no longer holds. The Bristol model is a model $M \models \text{ZF}$ such that $L \subseteq M \subseteq L[\rho_0]$ where ρ_0 is Cohen-generic over L , but M fails to be a symmetric

⁶More precisely they show that $L[A]$ is a generic extension of HOD whenever A is a set of ordinals (see [Jec08], Theorem 15.46), but the theorem generalises easily (see [Jec08], Lemma 15.43).

extension of L . This was first exposited in [Kar18] and later expanded upon in [Kar20a]. While we will not reproduce the entire construction of the Bristol model here, we will describe an upward homogeneity argument made for limit iterands of the construction.

All the definitions here are made in L . If $\bar{a}$ is a sequence labelled by ordinals, a_β denotes its β -th component.

The Bristol model is a class-length iteration of symmetric extensions that lies strictly between L and $L[\rho_0]$, where ρ_0 is a single Cohen real, with the first iterand being a symmetric extension over L . For each $\beta \in \text{Ord}$, there is a name $\dot{\rho}_\beta$: when β is a successor, $\dot{\rho}_\beta$ is a name for a sequence of length ω_β ; we will not define $\dot{\rho}_\beta$ for successors $\beta > 0$, but we will define $\dot{\rho}_\beta$ for limit β when it is due. The β -th iterand is given by the symmetric system denoted $\langle \mathbb{Q}_\beta, \mathcal{G}_\beta, \mathcal{F}_\beta \rangle$; $\mathbb{Q}_\beta$ is a forcing that consists of permutations of fragments of ρ_β .

Fact 2.4.4. Suppose we have constructed $\langle \mathbb{Q}_\gamma, \mathcal{G}_\gamma, \mathcal{F}_\gamma \rangle$ for all $\gamma < \beta$. Then for all $\gamma < \beta$, $q, q' \in \mathbb{Q}_\gamma$, there is some $\pi \in \mathcal{G}_\gamma$ such that $\pi q \parallel q'$. (This is part of the induction hypothesis in [Kar18] §4.2.)

Definition of $\langle \mathbb{P}_\beta, \mathcal{G}_\beta, \mathcal{F}_\beta \rangle$ and relevant facts

Suppose that for all $\gamma < \beta$, $\langle \mathbb{Q}_\gamma, \mathcal{G}_\gamma, \mathcal{F}_\gamma \rangle$ has been constructed. Then these iterations can be regarded as a single symmetric system $\langle \mathbb{P}_\beta, \mathcal{G}_\beta, \mathcal{F}_\beta \rangle$, which we will present.

Definition 2.4.5. The forcing $\mathbb{P}_\beta$ is the finite-support iteration of $\langle \dot{\mathbb{Q}}_\gamma \mid \gamma < \beta \rangle$.

The following fact is given in [Kar18] Proposition 3.8.

Fact 2.4.6. The automorphism group $\mathcal{G}_\beta$ consists of elements of the form $\int_{\bar{\pi}}$ satisfying:

- $\bar{\pi}$ is a sequence $\langle \dot{\pi}_\gamma \mid \gamma < \beta \rangle$ where $\dot{\pi}_\gamma$ appears in $\dot{\mathcal{G}}_\gamma$.
- For all but finitely many $\gamma < \beta$, $\dot{\pi}_\gamma$ is forced to identity function.

Suppose that $\bar{\pi}$ is such sequence and $C(\bar{\pi}) = \{\gamma < \beta \mid \mathbb{1}_{\mathbb{Q}_\gamma} \not\Vdash \dot{\pi}_\gamma = \text{id}^\bullet\}$, then $\int_{\bar{\pi}} \vec{p}$ is given recursively: if $\alpha = \max C(\bar{\pi})$, then $\int_{\bar{\pi}} \vec{p} = \int_{\bar{\pi} \upharpoonright \alpha} \int_{\dot{\pi}_\alpha} \vec{p}$, where

$$\int_{\dot{\pi}_\alpha} \vec{p} = \vec{p} \upharpoonright \alpha \hat{\cdot} \dot{\pi}_\alpha(p_\alpha) \hat{\cdot} \dot{\pi}_\alpha(\vec{p} \upharpoonright (\alpha, \beta)).$$

Fact 2.4.7. Let $\pi_\gamma \in \mathcal{G}_\gamma$ for each γ , and let the length $|\vec{\pi}|$ of $\vec{\pi}$ be greater than β .

- (i) $\int_{\pi_\beta} \int_{\vec{\pi} \upharpoonright \beta} = \int_{\vec{\pi} \upharpoonright \beta \sim \langle \pi_\beta \rangle}$.
- (ii) $\int_{\vec{\pi}} \dot{\rho}_{\beta+1} = \int_{\vec{\pi} \upharpoonright (\beta+1)} \dot{\rho}_{\beta+1}$.

Proof. For (i), see the proof of [Kar18] Proposition 4.5. For (ii), one needs to consider the definition of $\dot{\rho}_{\beta+1}$ and the permutation groups $\mathcal{G}_\alpha$; $\dot{\rho}_{\beta+1}$ is a $\dot{\mathcal{Q}}_{\beta+1}$ -name and if $\pi_\alpha \in \mathcal{G}_\alpha$ for $\alpha > \beta$, then π_α fixes $\dot{\rho}_{\beta+1}$. $\square$

Fact 2.4.8. Suppose that we have constructed $\langle \mathbb{Q}_\gamma, \mathcal{G}_\gamma, \mathcal{F}_\gamma \rangle$ for all $\gamma < \beta$. Then for all $\gamma < \beta$, for all $\vec{p} \in \mathbb{P}_\gamma$, $\dot{q}, \dot{q}' \in \dot{\mathcal{Q}}_\gamma$, there is some $\int_{\vec{\pi}} \in \mathcal{G}_\gamma$ such that $\int_{\vec{\pi}} \vec{p} = \vec{p}$ and $\vec{p} \Vdash \int_{\vec{\pi}} \dot{q} \parallel \dot{q}'$. Moreover, if $\gamma = \gamma' + 1$ is a successor, then the automorphism can be further specified: such $\int_{\vec{\pi}}$ can be taken as one with $\vec{\pi} \upharpoonright \gamma' = \text{id}$ and $\pi_{\gamma'} p_{\gamma'} = p_{\gamma'}$.

Proof. See the proofs of [Kar18] Proposition 4.5 and Proposition 4.14. $\square$

Definition 2.4.9. The filter $\mathfrak{F}_\beta$ consists of the elements $\vec{H} = \langle \dot{H}_\gamma \mid \gamma \in \beta \rangle$ in the following form:

- $\dot{H}_\gamma$ appears in $\dot{\mathcal{F}}_\gamma$ for each $\gamma < \beta$.
- For all but finitely many γ , $\dot{H}_\gamma$ is forced to be $\dot{\mathcal{G}}_\gamma$.

For $\vec{H} \in \mathfrak{F}_\beta$, let $C(\vec{H})$ denote the set $\{\gamma < \beta \mid \dot{H}_\gamma \neq \dot{\mathcal{G}}_\gamma\}$.

Definition of $\langle \mathbb{Q}_\gamma, \mathcal{G}_\gamma, \mathcal{F}_\gamma \rangle$

We now proceed to specify $\langle \mathbb{Q}_\gamma, \mathcal{G}_\gamma, \mathcal{F}_\gamma \rangle$. We will not define $\mathcal{G}_\gamma, \mathcal{F}_\gamma$ for successor γ since they are less relevant for our purpose, but we will define $\mathbb{Q}_\gamma$:

Definition 2.4.10. For all β , $\dot{\mathcal{Q}}_{\beta+1}$ is defined as

$$\dot{\mathcal{Q}}_{\beta+1} = \{ \int_{\vec{\pi}} \dot{\rho}_{\beta+1} \upharpoonright A \mid \int_{\vec{\pi}} \in \mathcal{G}_{\beta+1}, A \subseteq \omega_{\beta+1} \text{ bounded (in } L) \}.$$

Before we define $\langle \mathbb{Q}_\gamma, \mathcal{G}_\gamma, \mathcal{F}_\gamma \rangle$ for limit γ , we need some notations. If β is a limit ordinal, let $\prod \text{SC}(\omega_\beta)$ denote the product of all successor cardinals below ω_β , i.e. an element $f \in \prod \text{SC}(\omega_\beta)$ is such that $f(\gamma) \in \omega_\gamma$ for all successors γ below β . Let $\overline{\prod \text{SC}(\omega_\beta)}$ denote $\prod \text{SC}(\omega_\beta)$ modulo bounded subsets of ω_β (so $f, g \in \prod \text{SC}(\omega_\beta)$ are in the same equivalence class iff they differ up to a bounded subset of ω_β). Fix a scale $F = \{f_\eta \mid \eta < \omega_{\beta+1}\}$ in $\overline{\prod \text{SC}(\omega_\beta)}$, i.e. F is strictly increasing and cofinal in $\overline{\prod \text{SC}(\omega_\beta)}$ with respect to the ordering of domination modulo bounded subsets

of ω_β . Suppose that $\bar{\pi}$ is a sequence $\langle \pi_\theta \mid \theta \in \text{SC}(\omega_\beta) \rangle$ such that $\pi_\theta \in S_\theta$ for all successor cardinals θ below ω_β . We say that $\bar{\pi}$ **implements** $\pi \in S_{\omega_\beta}$ if for every large enough $\theta \in \text{SC}(\omega_\beta)$,

$$f_{\pi(\eta)}(\theta) = \pi_\theta(f_\eta(\theta)).$$

The permutation that $\bar{\pi}$ implements is denoted $\iota(\bar{\pi})$, if it exists. We say that F is a **permutable scale** if every bounded permutation of $\omega_{\beta+1}$ can be implemented by some $\bar{\pi}$, i.e. for every $\eta < \omega_{\beta+1}$, every permutation of η can be implemented. It is a theorem in L that a permutable scale exists for every limit β ([Kar18] Theorem 3.27). In the construction of the Bristol model, we assume that for each limit β we have fixed a corresponding permutable scale.

Now assume that α is a limit ordinal and for each $\beta < \alpha$, $\langle \mathbb{Q}_\beta, \mathcal{G}_\beta, \mathcal{F}_\beta \rangle$ has been defined.

- For $f \in \prod \text{SC}(\omega_\alpha)$, $\dot{\rho}_{\alpha,f}$ denotes $\langle \dot{\rho}_{\beta+1}(f(\beta+1)) \mid \beta < \alpha \rangle^\bullet$.
- $\dot{\rho}_\alpha$ denotes $\langle \dot{\rho}_{\alpha,f} \mid f \in \prod \text{SC}(\omega_\alpha) \rangle^\bullet$.
- For $E \subseteq \prod \text{SC}(\omega_\alpha)$ and $A \subseteq \alpha$, $\dot{\rho}_\alpha \restriction (E, A)$ denotes $\langle \dot{\rho}_{\alpha,f} \restriction A \mid f \in E \rangle^\bullet$.
- For $f \in \prod \text{SC}(\omega_\alpha)$, define

$$\dot{\mathbb{Q}}_{\alpha,f} = \{ \int_{\bar{\pi}} \dot{\rho}_{\alpha,f} \restriction A \mid A \subseteq \alpha \text{ bounded (in } L) \}^\bullet,$$

ordered by reverse inclusion.

We are now ready to define $\langle \mathbb{Q}_\alpha, \mathcal{G}_\alpha, \mathcal{F}_\alpha \rangle$.

Definition 2.4.11.

- $\dot{\mathbb{Q}}_\alpha$ is the following forcing:

$$\{ \int_{\bar{\pi}} \dot{\rho}_\alpha \restriction (E, A) \mid \int_{\bar{\pi}} \in \mathfrak{G}_\alpha, E \subseteq \prod \text{SC}(\omega_\alpha) \text{ bounded}, A \subseteq \alpha \text{ bounded} \}^\bullet.$$

For $\dot{q} \in \dot{\mathbb{Q}}_\alpha$, if $\dot{q} = \langle \int_{\bar{\pi}} \dot{\rho}_{\beta+1}(g(\beta+1)) \mid g \in E, \beta \in A \rangle$, then $\dot{q}(f)$ denotes $\langle \int_{\bar{\pi}} \dot{\rho}_{\beta+1}(f(\beta+1)) \mid \beta \in A \rangle$. Define $\dot{q} \leq_{\dot{\mathbb{Q}}_\alpha} \dot{q}'$ iff for all $f \in \prod \text{SC}(\omega_\alpha)$, $\dot{q}(f) \leq_{\dot{\mathbb{Q}}_{\alpha,f}} \dot{q}'(f)$.

(Note that if $\dot{q} \in \dot{\mathbb{Q}}_\alpha$ and $\dot{q} = \langle \int_{\bar{\pi}} \dot{\rho}_{\beta+1}(g(\beta+1)) \mid \beta \in A, g \in E \rangle^\bullet$, then for each $\beta \in A$, we have $\langle \int_{\bar{\pi}} \dot{\rho}_{\beta+1}(g(\beta+1)) \mid g \in E \rangle^\bullet \in \dot{\mathbb{Q}}_{\beta+1}$, since $\int_{\bar{\pi}} \dot{\rho}_{\beta+1} = \int_{\bar{\pi} \restriction (\beta+1)} \dot{\rho}_{\beta+1}$ and E is bounded.)

- $\mathcal{G}_\alpha$ is defined as $\prod_{\theta \in \text{SC}(\omega_\alpha)} \mathcal{G}_\theta$ the full-support product.

- For $\eta < \omega_{\alpha+1}$ and $f \in \prod \text{SC}(\omega_\alpha)$, let

$$K_{\eta,f} := \{\bar{\pi} \in \mathcal{G}_\alpha \mid \iota(\bar{\pi}) \upharpoonright \eta = \text{id} \text{ and for all } \mu \in \text{SC}(\omega_\alpha), \pi_\mu \upharpoonright f(\mu) = \text{id}\}.$$

Then $\mathcal{F}_\alpha$ is the filter generated by $\{K_{\eta,f} \mid \eta < \omega_{\alpha+1}, f \in \prod \text{SC}(\omega_\alpha)\}$.

It is a fact that $\mathcal{F}_\alpha$ is indeed normal ([Kar18] Proposition 3.28).

Upward homogeneity of $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$

Proposition 2.4.12. $\langle \mathbb{P}_\alpha, \mathfrak{G}_\alpha, \mathfrak{F}_\alpha \rangle * \langle \dot{\mathbb{Q}}_\alpha, \dot{\mathcal{G}}_\alpha, \dot{\mathcal{F}}_\alpha \rangle^\bullet$ is upward homogeneous.

Proof. (See [Kar18] Lemma 4.10.) Let $\vec{H} \in \mathfrak{F}_\alpha$ and $K_{\eta,f} \in \mathcal{F}_\alpha$. Given $\dot{q} \in \dot{\mathbb{Q}}_\alpha$, $\dot{q} = \int_{\bar{\pi}} \dot{\rho}_\alpha \upharpoonright (E, A)$, we let $\dot{q}^\circ$ be an extension of both $\dot{q}$ and $\int_{\bar{\pi}} \dot{\rho}_\alpha \upharpoonright (f \downarrow, \delta + 1)$, where $\delta = \max C(\vec{H})$, and

$$f \downarrow = \{g \in \prod \text{SC}(\lambda) \mid g(\mu) \leq f(\mu) \text{ for all } \mu\}.$$

We show that this is a condition that meets our requirement in Definition 2.3.1.

Let $\vec{p}' \Vdash \dot{q}', \dot{q}'' \leq \dot{q}^\circ$. Suppose $\dot{q}' = \langle \int_{\bar{\pi}'} \dot{\rho}_{\beta+1}(g(\beta+1)) \mid \beta \in A', g \in E' \rangle^\bullet$. By Fact 2.4.4, for each $\beta < \alpha$, for all $r, r' \in \mathbb{Q}_{\beta+1}$, there is some $\tau_\beta \in \mathcal{G}_{\beta+1}$ such that $\tau_\beta r \parallel r'$. Therefore, since $\langle \int_{\bar{\pi}'} \dot{\rho}_{\beta+1}(g(\beta+1)) \mid g \in E' \rangle^\bullet \in \dot{\mathbb{Q}}_{\beta+1}$ for each $\beta \in A' \cup (\delta+1)$, we find a sequence $\vec{\tau}$ such that $\tau_\gamma = \text{id}$ for all $\gamma \notin A' \cap (\delta+1)$ and

$$\vec{\tau} \dot{q}' \upharpoonright (\delta+1) \parallel \dot{q}'' \upharpoonright (\delta+1).$$

Since $\vec{\tau}$ is almost all identity, we get $\iota(\vec{\tau}) = \text{id}$. Moreover, for $\beta \in A' \cap (\delta+1)$, τ_β only need to move coordinates above $f(\beta+1)$, since $\dot{q}', \dot{q}''$ both extend $\int_{\bar{\pi}} \dot{\rho}_\alpha \upharpoonright (f \downarrow, \delta+1)$. This shows that $\vec{\tau} \in K_{\eta,f}$.

CLAIM. For all $\vec{p} \in \mathbb{P}_\alpha$, $\dot{q}, \dot{q}' \in \dot{\mathbb{Q}}_\alpha$ such that $\dot{q} \upharpoonright \delta+1 \parallel \dot{q}' \upharpoonright \delta+1$, there is some $\int_{\bar{\pi}} \in \mathfrak{G}_\alpha$ such that $\bar{\pi} \upharpoonright \delta+1 = \text{id}$, $\int_{\bar{\pi}} \vec{p} = \vec{p}$ and $\int_{\bar{\pi}} \dot{q} \parallel \dot{q}'$.

By the claim, we can find some $\int_{\bar{\pi}} \in \vec{H}$ such that $\int_{\bar{\pi}} \vec{\tau} \dot{q}' \parallel \dot{q}''$, and $\int_{\bar{\pi}}$ also fixes $\vec{p}'$.

Proof of claim. Assume $\dot{q} = \int_{\vec{\sigma}} \langle \dot{\rho}_{\beta+1}(g(\beta+1)) \mid \beta \in A, g \in E \rangle^\bullet$ and $\dot{q}' = \int_{\vec{\sigma}'} \langle \dot{\rho}_{\beta+1}(g(\beta+1)) \mid \beta \in A', g \in E' \rangle^\bullet$ are compatible below $\delta+1$.

We prove the claim by induction on α . There are two cases:

1. α is a limit of limit ordinals, and there is a least limit ordinal θ such that $A, A' \subseteq \theta$.

2. $\alpha = \theta + \omega$ for some limit θ .

In both cases, we may assume that $\delta < \theta$.

Case 1. By the induction assumption on θ , we get that for all $\vec{p} \restriction \theta \in \mathbb{P}_\theta$, $\dot{q} \restriction \theta, \dot{q}' \restriction \theta \in \dot{\mathbb{Q}}_\theta$, there is some $\int_{\vec{\pi} \restriction \theta} \in \mathfrak{G}_\theta$ such that $\vec{\pi} \restriction \delta + 1 = \text{id}$, $\int_{\vec{\pi} \restriction \theta} \vec{p} \restriction \theta = \vec{p} \restriction \theta$, and $\int_{\vec{\pi} \restriction \theta} \dot{q} \restriction \theta \parallel \dot{q}' \restriction \theta$. Since $A, A' \subseteq \theta$, we may take the rest of the sequence $\vec{\pi} \restriction [\theta, \alpha)$ as identity.

Case 2. By the induction assumption on θ , we get part of the sequence $\vec{\pi} \restriction \theta$. Since A, A' are bounded in α , we have only finitely many components left to define. We use [Fact 2.4.8](#) to find π_θ such that $\pi_\theta p_\theta = p_\theta$ and $\int_{\pi_\theta} \int_{\vec{\pi} \restriction \theta} \dot{q} \parallel \dot{q}'$ on the $(\theta + 1)$ -th coordinate, i.e.

$$\int_{\pi_\theta} \int_{\vec{\pi} \restriction \theta} \int_{\vec{\sigma}} \langle \dot{\rho}_{\theta+1}(g(\theta+1)) \mid g \in E \rangle^\bullet \parallel \int_{\vec{\sigma}'} \langle \dot{\rho}_{\theta+1}(g(\theta+1)) \mid g \in E' \rangle^\bullet$$

as conditions in $\dot{\mathbb{Q}}_{\theta+1}$. If $\pi_{\theta+k}$ has been defined, we similarly find $\pi_{\theta+k+1} \in \mathcal{G}_{\theta+k+1}$ such that $\pi_{\theta+k+1} p_{\theta+k+1} = p_{\theta+k+1}$ and $\int_{\pi_{\theta+k+1}} \int_{\vec{\pi} \restriction \theta+k+1} \dot{q} \parallel \dot{q}'$ on the $(\theta + k + 1)$ -th coordinate. This proves the claim and hence the proposition. $\square$

Chapter 3

Kreisel–Lévy-type theorems for set theories

3.1 Introduction

In 1968, [KL68] showed that over Peano arithmetic (PA), the axiom schema which asserts the soundness of PA is equivalent to the schema of transfinite induction along ε_0 , the first ordinal α that satisfies $\alpha = \omega^\alpha$: let

$$\text{RFN}(\text{PA}) = \{ \forall x (\text{Prov}_{\text{PA}}(\ulcorner \phi(\dot{x}) \urcorner) \rightarrow \phi(x)) \mid \phi \text{ a formula in PA} \}$$

be the soundness principle of PA^1 , where $\text{Prov}_{\text{PA}}(\ulcorner \phi(\dot{x}) \urcorner)$ means $\phi(x)$ is PA-provable²; and let

$$\varepsilon_0 = \sup_n \omega_n \quad \text{where } \omega_0 = \omega, \omega_{n+1} = \omega^{\omega_n};$$

the transfinite induction schema along an ordinal α is

$$\text{TI}(\alpha) = \{ \forall \beta ((\forall \gamma < \beta) \phi(\gamma) \rightarrow \phi(\beta)) \rightarrow (\forall \beta < \alpha) \phi(\beta) \mid \phi \text{ a formula in PA} \};$$

then the theorem mentioned above reads as

$$\text{RFN}(\text{PA}) \equiv \text{TI}(\varepsilon_0) \quad \text{over PA,}$$

¹“RFN” represents “reflection” which is the name of the schema used in [KL68]; in set theory, the same phrase can mean something else, so we avoid using this term.

²We assume a standard coding $\ulcorner \cdot \urcorner$ for formulas; $\ulcorner \phi(\dot{x}) \urcorner$ is a primitive-recursive function which on input m outputs $\ulcorner \phi(S^m(0)) \urcorner$ where S is the symbol for successor.

i.e., $\text{PA} + \text{RFN}(\text{PA})$ proves $\text{TI}(\varepsilon_0)$ and vice versa. The direction $\text{PA} + \text{RFN}(\text{PA}) \vdash \text{TI}(\varepsilon_0)$ is essentially from Gentzen’s proof that PA proves $\text{TI}(\alpha)$ for every $\alpha < \varepsilon_0$, and [KL68] used another device to prove the other direction; in [Sch77], the direction $\text{PA} + \text{TI}(\varepsilon_0) \vdash \text{RFN}(\text{PA})$ is proved by transfinite induction along infinitary proof trees as which all PA -proofs can be interpreted with lengths $< \varepsilon_0$.

This paper aims to prove a very similar result in Kripke–Platek set theory $(\text{KP})^3$. KP is a subtheory of ZF consisting of the following axioms:

1. Extensionality. $\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b$.
2. Set induction. $\forall x((\forall y \in x)G(y) \rightarrow G(x)) \rightarrow \forall x G(x)$, for all formulas G .
3. Pair. $\exists x(x = \{a, b\})$.
4. Union. $\exists x(x = \bigcup a)$.
5. Infinity. $\exists x(x \neq \emptyset \wedge (\forall y \in x)(\exists z \in x)(y \in z))$.
6. Δ_0 -separation. $\exists x \forall u(u \in x \leftrightarrow (u \in a \wedge F(u)))$, for all Δ_0 -formulas F .
7. Δ_0 -collection. $(\forall x \in a) \exists y G(x, y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)G(x, y)$, for all Δ_0 -formulas G .

In proof theory, KP is rather similar to PA . For example, it is a well-known result in [Par70] that the primitive-recursive functions on natural numbers are exactly the class of the provably total functions of PA with induction restricted to Σ_1 -formulas. In set theory we may define primitive-recursive functions on sets as well, and by [Rat92] Theorem 1.2, a set function F is primitive-recursive iff it is provably total in KP with (set) induction restricted to Σ_1 -formulas. But more importantly for us, there is a way to transform KP -proofs into infinitary proof trees for which the cuts can at least be partially eliminated. Thus, if we let Ω denote the supremum of all ordinals in the universe of KP , and propose that

$$\text{RFN}(\text{KP}) \equiv \text{TI}(\varepsilon_{\Omega+1}) \quad \text{over } \text{KP}$$

where $\varepsilon_{\Omega+1}$ denotes the $(\Omega + 1)$ -th ordinal α that satisfies $\alpha = \omega^\alpha$, then it is possible to follow the method mentioned in [Sch77] to prove our proposition; the equivalence stated above is what we are trying to prove in this paper.

In Section 3.2 we lay out some preliminary definitions. The direction $\text{KP} + \text{RFN}(\text{KP}) \vdash \text{TI}(\varepsilon_{\Omega+1})$ will be quickly dealt with in 3.3, and all the rest 3.4–3.8 are

³Contrary to convention, we will use KP to denote Kripke–Platek set theory with infinity.

dedicated to proving the converse $KP + TI(\varepsilon_{\Omega+1}) \vdash RFN(KP)$. The length of the latter is partly due to the lengthiness of cut elimination theorem and embedding theorem themselves, but also due to a subtlety in expressing the infinitary proofs within KP , and the need of using partial recursive set functions for which the recursion theorem applies.

3.2 Preliminary definitions

In this paper, our theory T is always assumed to be sufficiently strong, say, it contains primitive-recursive arithmetic.

Definition 3.2.1. Let T be a theory. The **soundness principle** of T , $RFN(T)$, is the schema

$$RFN(T) = \{ \forall x (\text{Prov}_T(\ulcorner \phi(x) \urcorner) \rightarrow \phi(x)) \mid \phi \text{ a formula in } T \}$$

where $\text{Prov}_T(\ulcorner \phi(x) \urcorner)$ means that $\phi(x)$ is T -provable.

Definition 3.2.2. If our background theory is a set theory in which ω is definable, **primitive-recursive set functions**⁴ are the class of functions built up from the initial functions

- $P_{n,i}(x_1, \dots, x_n) = x_i$, $1 \leq i \leq n$,
- $Z(y) = 0$,
- $M(x, y) = x \cup \{y\}$,
- $C(x, y, u, v) = x$ if $u \in v$, y otherwise,
- $\text{Inf}(x) = \omega$

by

- *substitution*: $F(\vec{x}) = K(G_1(\vec{x}), \dots, G_k(\vec{x}))$, and
- *primitive recursion*: $F(z, \vec{x}) = H(\bigcup \{F(u, \vec{x}) \mid u \in z\}, z, \vec{x})$.

Some of the most important examples of primitive-recursive set functions are the characteristic functions of Δ -predicates in KP , and ordinal operations such as addition, multiplication and exponentiation. See [Rat92] 2.2 for details.

⁴The initial function $x \mapsto \omega$ is dispensable in a more general setting.

Definition 3.2.3. Let T be a theory. An **ordinal representation system** $\langle R, \leq \rangle$ in T is a unary relation R and a preorder $\leq$ primitive-recursively coded in T ; members of R are strings of symbols to be interpreted as ordinals and $\leq$ orders the ordinals; the system also comes with basic ordinal operations: addition, multiplication, exponentiation $\alpha \mapsto \omega^\alpha$, all primitive-recursively coded in T .

The strings of symbols in R may be mapped non-injectively into the ordinals and so $\leq$ could be not antisymmetric. For example, ω and ω^1 have the same value, $\omega \leq \omega^1$ and $\omega^1 \leq \omega$, but their notations are different: $\omega \neq \omega^1$.

Definition 3.2.4. Let T be a theory and let $\langle R, \leq \rangle$ be an ordinal representation system in T ; let U be a predicate. The **progressiveness of $<$ for U** , $\text{Prog}_<(U)$, is the formula

$$\text{Prog}_<(U) \equiv \forall x((\forall y < x)y \in U \rightarrow x \in U).$$

If $a \in R$, the **transfinite induction along a for U** , $\text{TI}_<(a, U)$, is the formula

$$\text{TI}_<(a, U) \equiv \text{Prog}_<(U) \rightarrow (\forall x < a)x \in U.$$

If F is a formula, then by $x \in F$ we mean $F(x)$; so, e.g. $\text{Prog}_<(F)$ is the formula $\forall x((\forall y < x)F(y) \rightarrow F(x))$. The **transfinite induction schema along a** , $\text{TI}_<(a)$, is the schema

$$\text{TI}_<(a) = \{\text{TI}_<(a, F) \mid F \text{ a formula in } T\}.$$

If it is clear which ordinal representation system we are using, we often drop the $<$ symbol.

The ordinal representation system in KP we are going to use is rather canonical. In arithmetic, apparently the numbers we have access to are the natural numbers, but by Cantor normal form, every non-zero ordinal α can be written in the form

$$\alpha = \omega^{\alpha_1} + \dots + \omega^{\alpha_n}$$

for some $n \in \omega$ and ordinals $\alpha_1 \geq \dots \geq \alpha_n$; if $\alpha < \varepsilon_0$ then $\alpha_1, \dots, \alpha_n$ must be smaller than α , and we can write $\alpha_1, \dots, \alpha_n$ in Cantor normal forms again, representing them by smaller ordinals. The descending sequences of ordinals must end in

finitely many steps, therefore we may represent every ordinal $\alpha < \varepsilon_0$ by just natural numbers. The idea for representing $\varepsilon_{\Omega+1}$ in KP is exactly the same; we will only sketch the definition.

Definition 3.2.5. Within KP we define an ordinal representation system $\langle R, \leq \rangle$ as follows. Let ON denote the class of ordinal numbers. If a, b are members of R , we use $a \equiv b$ to denote that a, b are exactly the same string of symbols, whereas $a = b$ means the weaker condition $a \leq b$ and $b \leq a$.

- The symbols are $\{\alpha \mid \alpha \in \text{ON}\}, \Omega, +, \omega', \varepsilon..$
- The members a of the class R are in the following forms.
 - $a \equiv \alpha$ for some $\alpha \in \text{ON}$.
 - $a \equiv \Omega$.
 - $a \equiv \omega^{a_0}$ where $a_0 \in R$.
 - $a \equiv a_1 + a_2$ where a_1, a_2 are of the form $\omega^{b_1} + \dots + \omega^{b_n}$ (not necessarily in Cantor normal form) where $b_1, \dots, b_n \in R$.
 - $a \equiv \varepsilon_{a_0}$ where $a_0 \in R$.
- The ordering $<$ is defined as follows. Let $a, b \in R$.
 - $a, b \in \text{ON}$. Then $a < b$ iff $a < b$ as ordinals.
 - $a \in \text{ON}, b \equiv \Omega$. Then $a < b$.
 - $a \in \text{ON}, b \equiv b_1 + b_2$ or $b \equiv \omega^{b_0}$ or $b \equiv \varepsilon_{b_0}$.
 If b contains any Ω symbol, then $a < b$.
 If b does not contain any Ω symbol, then b can be evaluated in ON primitive-recursively, and $a < b$ iff $a < b$ in the evaluation.
 - $a \equiv \Omega, b \equiv b_1 + b_2$.
 If $b_1 < \Omega$, then $b \diamond \Omega$ iff $b_2 \diamond \Omega$, where $\diamond$ is any of $<, =, >$.
 If $b_1 = \Omega$, then $b > \Omega$ (since $b_2 > 0$ always in our convention for $+$).
 If $b_1 > \Omega$, then $b > \Omega$.
 - $a \equiv \Omega, b \equiv \omega^{b_0}$ or ε_{b_0} . Then $b \diamond \Omega$ iff $b_0 \diamond \Omega$ for $\diamond \in \{<, =, >\}$.
 - $a \equiv a_1 + a_2, b \equiv b_1 + b_2$ or $b \equiv \omega^{b_0}$. We may primitive-recursively sort a and b into their Cantor normal form and then compare them in the usual way.
 - $a \equiv a_1 + a_2, b \equiv \varepsilon_{b_0}$. Let ω^{a_0} be the first term of the Cantor normal form of a .

If $a_0 < b$, then $a < b$.

If $a_0 \geq b$, then $a > b$.

– $a \equiv \omega^{a_0}$, $b \equiv \omega^{b_0}$. Then $a \diamond b$ iff $a_0 \diamond b_0$ for $\diamond \in \{<, =, >\}$.

– $a \equiv \omega^{a_0}$, $b \equiv \varepsilon_{b_0}$. Then $a \diamond b$ iff $a_0 \diamond b$ for $\diamond \in \{<, =, >\}$.

The predicates R and $<$ are therefore Δ in KP and are primitive-recursive.

We will be using an auxiliary sum operation for ordinals.

Definition 3.2.6. Define an operation $\#$ as follows: $\alpha \# 0 = 0 \# \alpha = \alpha$. For α with Cantor normal form $\omega^{\alpha_1} + \dots + \omega^{\alpha_m}$ and β with Cantor normal form $\omega^{\alpha_{m+1}} + \dots + \omega^{\alpha_n}$ define

$$\alpha \# \beta := \omega^{\alpha_{\pi(1)}} + \dots + \omega^{\alpha_{\pi(n)}}$$

where π is a permutation of $\{1, \dots, n\}$ such that $\alpha_{\pi(i)} \geq \alpha_{\pi(i+1)}$ for all $1 \leq i < n$. $\alpha \# \beta$ is called the **natural sum** or **Hessenberg sum** of α and β .

3.3 Provable well-orderings in KP

In this section we prove the direction $\text{KP} + \text{RFN}(\text{KP}) \vdash \text{TI}(\varepsilon_{\Omega+1})$; the proof follows almost directly from modification of Gentzen's proof that PA proves $\text{TI}(\alpha)$ for every $\alpha < \varepsilon_0$.

Recall that R denotes the symbol class for our ordinal representation system and $<$ is our ordering on R .

Definition 3.3.1. For a predicate U , let

$$U^J = \{b \in R \mid (\forall a \in R)(R \cap a \subseteq U \rightarrow R \cap a + \omega^b \subseteq U)\},$$

denote the **jump** of U , where $R \cap x$ means $\{b \in R \mid b < x\}$.

Lemma 3.3.2. $\text{KP} \vdash \text{Prog}_{<}(U) \rightarrow \text{Prog}_{<}(U^J)$.

Proof. Assume (1) $\text{Prog}_{<}(U)$ and (2) $(\forall x < b)x \in U^J$, we want to show that $b \in U^J$, i.e.,

$$(\forall a \in R)(R \cap a \subseteq U \rightarrow R \cap a + \omega^b \subseteq U).$$

Assume that (3) $R \cap a \subseteq U$. Let $d \in R \cap a + \omega^b$; we need to show that $d \in U$ under the assumptions (1)–(3).

If $d < a$, then $d \in U$ by (3).

If $d = a$ then using (1) and (3) we have $d \in U$.

If $d > a$, then since $d < a + \omega^b$, we may primitive-recursively find $d_1, \dots, d_k$ such that

$$d = a + \omega^{d_1} + \dots + \omega^{d_k}, \quad d_k \leq \dots \leq d_1 < b.$$

Since $R \cap a \subseteq U$, we get $R \cap a + \omega^{d_1} \subseteq U$ by (2). Using (2) a further $k - 1$ times we obtain

$$R \cap a + \omega^{d_1} + \dots + \omega^{d_k} \subseteq U.$$

Finally, using one application of (1) we have $d \in U$. □

Notation. Let $e_0 \equiv \Omega + 1$, $e_{n+1} \equiv \omega^{e_n}$.

The following lemma shows that $\text{KP} \vdash \text{TI}_<(e_n, U)$ for all $n \in \omega$.

Lemma 3.3.3. For any $n < \omega$ and any definable class U ,

$$\text{KP} \vdash \text{Prog}_<(U) \rightarrow R \cap e_n \subseteq U \wedge e_n \in U.$$

Proof. By induction on n (outside of KP).

For $n = 0$ we need to show that $\forall x((\forall y < x)y \in U \rightarrow x \in U) \rightarrow \Omega + 1 \subseteq U \wedge \Omega + 1 \in U$. Suppose $\text{Prog}_<(U)$ holds but there is some $a < \Omega$ such that $a \notin U$. Then by ϵ -induction of KP, there is a least ordinal α such that there exists some $b \in R$ with value α and $b \notin U$. But this implies $(\forall y < b)y \in U$, and therefore $b \in U$ by $\text{Prog}_<(U)$. Thus, assuming $\text{Prog}_<(U)$, we must have $\Omega \subseteq U$. Then by applying $\text{Prog}_<(U)$ twice we have $\Omega, \Omega + 1 \in U$.

Now suppose the result holds for n . Since the induction hypothesis (i.h.) holds for all definable classes, we have that

$$\text{KP} \vdash \text{Prog}_<(U^J) \rightarrow R \cap e_n \subseteq U^J \wedge e_n \in U^J.$$

Since $\text{KP} \vdash \text{Prog}_<(U) \rightarrow \text{Prog}_<(U^J)$, we have

$$(*) \quad \text{KP} \vdash \text{Prog}_<(U) \rightarrow R \cap e_n \subseteq U^J \wedge e_n \in U^J.$$

Now we argue in KP. Assume $\text{Prog}_<(U)$, then from $(*)$, we obtain

$$R \cap e_n \subseteq U^J \wedge e_n \in U^J.$$

By the definition of U^J , $e_n \in U^J$ implies that $R \cap 0 \subseteq U \rightarrow R \cap 0 + \omega^{e_n} \subseteq U$. Thus $R \cap \omega^{e_n} \subseteq U$, and an application of $\text{Prog}_<(U)$ yields $\omega^{e_n} \in U$ as required. $\square$

Lemma 3.3.4. $\text{KP} \vdash \forall n(\text{Prov}_{\text{KP}}(\ulcorner \text{TI}_<(e_n, F) \urcorner))$ for any formula F .

Proof. Let F be given; we describe the procedure of writing the proof for $\text{TI}_<(e_n, F)$. For any definable class U , let U^{J^n} denote $U^{J \dots J}$ with J applied n times. To find the proof for $\text{TI}_<(e_n, F)$, we start with the proof of $\text{Prog}_<(F^{J^n}) \rightarrow R \cap e_0 \subseteq F^{J^n} \wedge e_0 \in F^{J^n}$. Applying the argument of the previous lemma 3.3.3, we obtain a proof of $\text{Prog}_<(F^{J^{n-1}}) \rightarrow R \cap e_1 \subseteq F^{J^{n-1}} \wedge e_1 \in F^{J^{n-1}}$. Then the argument can be applied again to obtain a proof of $\text{Prog}_<(F^{J^{n-2}}) \rightarrow R \cap e_2 \subseteq F^{J^{n-2}} \wedge e_2 \in F^{J^{n-2}}$, and so on. Eventually, we arrive at $\text{Prog}_<(F) \rightarrow R \cap e_n \subseteq F \wedge e_n \in F$, implying $\text{TI}_<(e_n, F)$. This proof-writing function with argument in n is primitive-recursive, hence provably total in KP, which implies that $\text{KP} \vdash \forall n(\text{Prov}_{\text{KP}}(\ulcorner \text{TI}_<(e_n, F) \urcorner))$. $\square$

Corollary 3.3.5. $\text{KP} + \text{RFN}(\text{KP}) \vdash \forall n \text{TI}_<(e_n, F)$. $\square$

Lemma 3.3.6. $\text{KP} \vdash \forall x(x < \varepsilon_{\Omega+1} \leftrightarrow (\exists n \in \omega)x < e_n)$.

Proof. Clearly $\text{KP} \vdash \forall x((\exists n \in \omega)x < e_n \rightarrow x < \varepsilon_{\Omega+1})$. The other direction is immediate for the cases $a \in \text{ON}$, $a \equiv \Omega$, or $a \equiv \varepsilon_{a_0}$. The cases $a \equiv a_1 + a_2$ or $a \equiv \omega^{a_0}$ are done by induction on the lengths of expressions in R . Let ω^{a_0} be the first term of the Cantor normal form of a . If $a < \varepsilon_{\Omega+1}$, we must have $a_0 < \varepsilon_{\Omega+1}$. Then by the i.h., $a_0 < e_n$ for some n and hence $\omega^{a_0} < e_{n+1}$, so $a < e_{n+1}$. $\square$

From this lemma we obtain

Corollary 3.3.7. $\text{KP} + \text{RFN}(\text{KP}) \vdash \text{TI}_<(\varepsilon_{\Omega+1}, F)$ for any formula F . $\square$

3.4 The infinitary proof system

The rest of the paper is dedicated to prove

$$\text{KP} + \text{TI}(\varepsilon_{\Omega+1}) \vdash \text{RFN}(\text{KP}).$$

This will be done by considering KP-proofs as infinitary proofs with lengths $< \varepsilon_{\Omega+1}$ and with cut complexity at most Π_1/Σ_1 ; an induction along such proof trees will show that the proofs have true conclusions, thus fulfilling $\text{RFN}(\text{KP})$.

We first introduce our infinitary proof system $\text{RS}_\Omega(\mathbf{V})$, which is a Tait-style sequent calculus.

Definitions of $\text{RS}_\Omega(\mathbf{V})$ -terms and -formulas 3.4.1.

- For every set a , the constant c_a is an $\text{RS}_\Omega(\mathbf{V})$ -term.
- If s, t are $\text{RS}_\Omega(\mathbf{V})$ -terms, so are $\{s, t\}$ and $\bigcup s$.
- If $s, t_1, \dots, t_n$ are $\text{RS}_\Omega(\mathbf{V})$ -terms and $A(a, b_1, \dots, b_n)$ is a Δ_0 -formula of KP with all free variables displayed, then $\{x \in s \mid A(x, t_1, \dots, t_n)\}$ is an $\text{RS}_\Omega(\mathbf{V})$ -term.
- If $t_1, \dots, t_n$ are $\text{RS}_\Omega(\mathbf{V})$ -terms and $A(a_1, \dots, a_n)$ is a formula of KP with all free variables displayed, then $A(t_1, \dots, t_n)$ is an $\text{RS}_\Omega(\mathbf{V})$ -formula.

$\text{RS}_\Omega(\mathbf{V})$ -terms are purely symbolic, but we can still evaluate them as sets. Let Comp be a primitive-recursive set function that does the following: for all sets $s, t_1, \dots, t_n$, Δ_0 -formula $A(a, b_1, \dots, b_n)$ with all free variables indicated,

$$\text{Comp}(s, \vec{t}, \ulcorner A \urcorner) = \{x \in s \mid A(x, \vec{t})\};$$

a primitive-recursive computation of such a function follows from, e.g., the proof of [Bar75] I.5.2 (v). Then we can define a primitive-recursive set function Ev which evaluate $\text{RS}_\Omega(\mathbf{V})$ -terms in the set universe:

- $\text{Ev}(u) = a$ if $u \equiv c_a$ is a constant for the set a ;
- if $u \equiv \bigcup s$, then $\text{Ev}(u) = \bigcup \text{Ev}(s)$;
- if $u \equiv \{s, t\}$, then $\text{Ev}(u) = \{\text{Ev}(s), \text{Ev}(t)\}$;
- if $u \equiv \{x \in s \mid A(x, t_1, \dots, t_n)\}$, then $\text{Ev}(u) = \text{Comp}(s, \vec{t}, \ulcorner A \urcorner)$.

Notation.

- If u is an $\text{RS}_\Omega(\mathbf{V})$ -term, $|u|$ denotes the set-theoretic rank of $\text{Ev}(u)$.
- The formula $s = t$ is a shorthand for $(\forall x \in s)x \in t \wedge (\forall x \in t)x \in s$.
- If A is a formula, $\pm A$ denotes formulas both A and $\neg A$.

- If A is a formula, and z is a variable not appearing in A , then A^z denotes A relativised to z : A^z is the result of replacing every unbounded quantifiers $\exists x$ by $(\exists x \in z)$ and $\forall x$ by $(\forall x \in z)$ in A .

Definition of derivability in $\text{RS}_\Omega(\mathbf{V})$ 3.4.2. We give an inductive definition of the relation $\text{RS}_\Omega(\mathbf{V}) \vdash^\alpha \Gamma$ by recursion on α ; this definition is yet outside of KP. The symbol Γ stands for an arbitrary finite set of $\text{RS}_\Omega(\mathbf{V})$ -formulas, and if A is a formula, Γ, A means $\Gamma \cup \{A\}$. $\text{RS}_\Omega(\mathbf{V}) \vdash^\alpha \Gamma$ is meant to express that the system $\text{RS}_\Omega(\mathbf{V})$ proves the disjunction $\bigvee \Gamma$ with a proof of length $\leq \alpha$.

The axioms of $\text{RS}_\Omega(\mathbf{V})$ are of the form $\vdash^\alpha \Gamma, A$ where A is a Δ_0 -formula true in KP (i.e. provable in KP). More precisely, if $A \equiv A(t_1, \dots, t_n)$ is Δ_0 , $t_1, \dots, t_n$ are all the $\text{RS}_\Omega(\mathbf{V})$ -terms appearing in the formula, and $A(\text{Ev}(t_1), \dots, \text{Ev}(t_n))$ is true in KP, then Γ, A is an axiom of $\text{RS}_\Omega(\mathbf{V})$.

The following are the inference rules of $\text{RS}_\Omega(\mathbf{V})$.

$$\begin{array}{ll}
(\wedge) & \frac{\vdash^{\alpha_0} \Gamma, A \quad \vdash^{\alpha_1} \Gamma, B}{\vdash^\alpha \Gamma, A \wedge B} \quad \alpha_0, \alpha_1 < \alpha \\
(\vee) & \frac{\vdash^{\alpha_0} \Gamma, C \quad \text{for some } C \in \{A, B\}}{\vdash^\alpha \Gamma, A \vee B} \quad \alpha_0 < \alpha \\
(b\forall) & \frac{\vdash^{\alpha_s} \Gamma, s \in t \rightarrow A(s) \quad \text{for all terms } s}{\vdash^\alpha \Gamma, (\forall x \in t) A(x)} \quad \alpha_s < \alpha \\
(b\exists) & \frac{\vdash^{\alpha_0} \Gamma, s \in t \wedge A(s)}{\vdash^\alpha \Gamma, (\exists x \in t) A(x)} \quad \alpha_0 < \alpha \\
(\forall) & \frac{\vdash^{\alpha_s} \Gamma, A(s) \quad \text{for all terms } s}{\vdash^\alpha \Gamma, \forall x A(x)} \quad \alpha_s < \alpha \\
(\exists) & \frac{\vdash^{\alpha_0} \Gamma, A(s)}{\vdash^\alpha \Gamma, \exists x A(x)} \quad \alpha_0 < \alpha \\
(\text{Cut}) & \frac{\vdash^{\alpha_0} \Gamma, A \quad \vdash^{\alpha_0} \Gamma, \neg A}{\vdash^\alpha \Gamma} \quad \alpha_0 < \alpha \\
(\Sigma\text{-Ref}) & \frac{\vdash^{\alpha_0} \Gamma, A}{\vdash^\alpha \Gamma, \exists z A^z} \quad \alpha_0, \Omega < \alpha, A \text{ is a } \Sigma\text{-formula}
\end{array}$$

Proof trees that follow the rules of $\text{RS}_\Omega(\mathbf{V})$ are called $\text{RS}_\Omega(\mathbf{V})$ -**proofs** or $\text{RS}_\Omega(\mathbf{V})$ -**derivations**. The **end sequent** of a $\text{RS}_\Omega(\mathbf{V})$ -derivation is the lowest sequent (i.e.

conclusion) of that derivation. The **direct subderivation(s)** of a derivation are the derivation(s) from which the end sequent is inferred, if they exist. The **side formula(s)** of an inference are the formula(s) that are irrelevant to that inference. The **minor formula** of an inference is the formula to which the inference rule is applied, and the **principal formula** of that inference is the formula formed by that inference.

For example, in the inference

$$\frac{\Gamma, \phi \quad \Gamma, \psi}{\Gamma, \phi \wedge \psi} (\wedge)$$

the side formulas are Γ , the minor formulas are ϕ, ψ and the principal formula is $\phi \wedge \psi$. If a derivation D ends with this inference, then the end sequent of D is $\Gamma, \phi \wedge \psi$, and the direct subderivations of D are the derivations which derived Γ, ϕ and Γ, ψ respectively.

The **rank** of a term or formula is defined as follows.

- $\text{rank}(u) = \omega \cdot |u|$.
- $\text{rank}(\pm u \in v) = \max(\text{rank}(u), \text{rank}(v)) + 1$.
- $\text{rank}(A \wedge B) = \text{rank}(A \vee B) = \max(\text{rank}(A), \text{rank}(B)) + 1$.
- $\text{rank}((\exists x \in u)F(x)) = \text{rank}((\forall x \in u)F(x)) = \max(\text{rank}(u)+3, \text{rank}(F(c_\emptyset))+2)$.
- $\text{rank}(\exists x F(x)) = \text{rank}(\forall x F(x)) = \max(\Omega, \text{rank}(F(c_\emptyset)) + 1)$.

$\frac{\alpha}{\rho} \Gamma$ will be used to denote that $\frac{\alpha}{\rho} \Gamma$ and all cut formulas appearing in the derivation have rank $< \rho$.

Observation 3.4.2.1. For each formula A , define

$$k(A) = \{|t| \mid t \text{ occurs in } A\} \cup \{\Omega \mid \text{if } A \text{ contains an unbounded quantifier}\}.$$

- (i) For each formula A , $\text{rank}(A) = \omega \cdot \max(k(A)) + n$ for some $n < \omega$.
- (ii) $\text{rank}(A) < \Omega$ iff A is Δ_0 ; thus A has rank Ω iff A is $\exists x F(x)$ or $\forall x F(x)$ where $F \in \Delta_0$.

Lemma 3.4.3. For each formula $A(s)$, if $|s| < \max(k(A(s)))$, then $\text{rank}(A(s)) = \text{rank}(A(c_\emptyset))$.

Proof. By induction on complexity of A . □

Some formulas can be regarded as generalisations of disjunctions or conjunctions:

- $A_0 \wedge A_1 \simeq \bigwedge_{i \in \{0,1\}} A_i$.
- $A_0 \vee A_1 \simeq \bigvee_{i \in \{0,1\}} A_i$.
- $(\forall x \in t)A(x) \simeq \bigwedge_s (s \in t \rightarrow A(s))$.
- $(\exists x \in t)A(x) \simeq \bigvee_s (s \in t \wedge A(s))$.
- $\forall x A(x) \simeq \bigwedge_s A(s)$.
- $\exists x A(x) \simeq \bigvee_s A(s)$.

Lemma 3.4.4. If A contains an unbounded quantifier and $A \simeq \bigvee_{i \in y} A_i$ or $A \simeq \bigwedge_{i \in y} A_i$ then

$$(\forall i \in y) \text{rank}(A_i) < \text{rank}(A).$$

Proof. Straightforward by using the previous lemma. $\square$

The following lemmas will be formalised in Section 3.7; their proofs outside KP can be carried out similarly to the standard ones (e.g. as in [Sch77]). The parts that need consideration are how to code the infinitary proofs in KP and how to express and prove these lemmas in KP.

Weakening 3.4.5. If $\alpha \leq \alpha'$, $\rho \leq \rho'$, Γ and Γ' are finite sets of formulas, and $\frac{\alpha}{\rho} \Gamma$, then $\frac{\alpha'}{\rho'} \Gamma, \Gamma'$.

Inversion 3.4.6. If A is not Δ_0 , $A \simeq \bigwedge (A_i)_{i \in y}$ and $\frac{\alpha}{\rho} \Gamma, A$ then for all $i \in y$, $\frac{\alpha}{\rho} \Gamma, A_i$.

Reduction 3.4.7. Suppose $\text{rank}(C) = \rho > \Omega$. If $\frac{\alpha}{\rho} \Gamma, \neg C$ and $\frac{\beta}{\rho} \Gamma, C$, then $\frac{\alpha+\beta}{\rho} \Gamma$.

Cut elimination 3.4.8. If $\frac{\beta}{\rho+1} \Gamma$ and $\rho > \Omega$, then $\frac{\omega^\beta}{\rho} \Gamma$.

3.5 Partial recursive set functions

Most naively (and so not quite possibly), one might attempt to define the $\text{RS}_\Omega(\mathbf{V})$ -derivation in such a way: if for all terms s , $D_s \in \text{RS}_\Omega(\mathbf{V})$ -derivation with end sequent $\Gamma, F(s)$, then $\langle \ulcorner \forall \urcorner, \{D_s \mid s \text{ a term}\} \rangle$ is an $\text{RS}_\Omega(\mathbf{V})$ -derivation with end sequent $\Gamma, \forall x F(x)$.

At least two problems are evident here: (1) $\{D_s \mid s \text{ a term}\}$ is a proper class, so we need to enclose this class-many information into a set function expressed as an index; (2) this recursive definition involves an unbounded universal quantifier, which is not allowed in KP in general.

To fix these problems altogether, we use partial recursive set functions, for which the recursion theorem applies. Here we cite [Rat12] Section 2.2 to define partial E -recursive functions; more on partial E -recursive functions can be found in [Nor78] and [Sac17] Chapter X.

Definition 3.5.1. Let $\mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{s}_N, \mathbf{p}_N, \mathbf{d}_N, \bar{\mathbf{0}}, \bar{\omega}, \gamma, \rho, \nu, \pi, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ be distinct natural numbers; they will be indices for initial partial E -recursive functions.

We define a class $\mathbb{E}$ of triples $\langle e, x, y \rangle$ by induction. Instead of $\langle e, x, y \rangle \in \mathbb{E}$ we will write $[e](x) \simeq y$; if $n > 1$, we use $[e](x_1, \dots, x_n) \simeq y$ to convey that

$$[e](x_1) \simeq \langle e, x_1 \rangle \wedge [\langle e, x_1 \rangle](x_2) \simeq \langle e, x_1, x_2 \rangle \wedge \dots \wedge [\langle e, x_1, \dots, x_{n-1} \rangle](x_n) \simeq y.$$

We say that $[e](x)$ is defined, written $[e](x) \downarrow$, if $[e](x) \simeq y$ for some y . Let $\mathbb{N}$ denote ω ; $\mathbb{E}$ is defined by the following clauses:

- $[\mathbf{k}](x, y) \simeq x$.
- $[\mathbf{s}](x, y, z) \simeq [[x](z)]([y](z))$.
 $[\mathbf{s}](x, y, z)$ is not defined unless $[x](z), [y](z)$ and $[[x](z)]([y](z))$ are already defined; the clause for $\mathbf{s}$ should be read as a conjunction of the following clauses: $[\mathbf{s}](x) \simeq \langle \mathbf{s}, x \rangle$, $[\langle \mathbf{s}, x \rangle](y) \simeq \langle \mathbf{s}, x, y \rangle$, and, if there exist a, b, c such that $[x](z) \simeq a$, $[y](z) \simeq b$, $[a](b) \simeq c$, then $[\langle \mathbf{s}, x, y \rangle](z) \simeq c$.
- $[\mathbf{p}](x, y) \simeq \langle x, y \rangle$.
- $[\mathbf{p}_0](x) \simeq (x)_0$.
- $[\mathbf{p}_1](x) \simeq (x)_1$.
- $[\mathbf{s}_N](n) \simeq n + 1$ if $n \in \mathbb{N}$.
- $[\mathbf{p}_N](0) \simeq 0$.
- $[\mathbf{p}_N](n + 1) \simeq n$ if $n \in \mathbb{N}$.
- $[\mathbf{d}_N](n, m, x, y) \simeq x$ if $n, m \in \mathbb{N}$ and $n = m$.
- $[\mathbf{d}_N](n, m, x, y) \simeq y$ if $n, m \in \mathbb{N}$ and $n \neq m$.
- $[\bar{\mathbf{0}}](x) \simeq 0$.
- $[\bar{\omega}](x) \simeq \omega$.

- $[\pi](x, y) \simeq \{x, y\}$.
- $[\nu](x) \simeq \cup x$.
- $[\gamma](x, y) \simeq x \cap (\cap y)$.
- $[\rho](x, y) \simeq \{[x](u) \mid u \in y\}$ if $[x](u)$ is defined for all $u \in y$.

Similarly to the clause for $\mathbf{s}$, this means that $[\rho](x) \simeq \langle \rho, x \rangle$, and if there is a function f with domain y such that $[x](u) \simeq f(u)$ for all $u \in y$, then $[\langle \rho, x \rangle](y) \simeq \{f(u) \mid u \in y\}$.

- $[\mathbf{i}_1](x, y, z) \simeq \{u \in x \mid y \in z\}$.
- $[\mathbf{i}_2](x, y, z) \simeq \{u \in x \mid u \in y \rightarrow u \in z\}$.
- $[\mathbf{i}_3](x, y, z) \simeq \{u \in x \mid u \in y \rightarrow z \in u\}$.

Proposition 3.5.2. The class $\mathbb{E}$ can be defined in KP; $\mathbb{E}$ is a Σ class, and for all e, x, y, y' ,

$$\langle e, x, y \rangle \in \mathbb{E} \wedge \langle e, x, y' \rangle \in \mathbb{E} \Rightarrow y = y'.$$

Proof. See [Rat12]; to prove the definability of $\mathbb{E}$, a theorem ([AR01] Theorem 11.14) involving *inductive definition* is quoted: the definition of $\mathbb{E}$ is an inductive definition by Σ clauses, so the definability of $\mathbb{E}$ in KP follows from that theorem. $\square$

Definition 3.5.3. **Application terms** are defined inductively as follows.

- The constants $\mathbf{k}, \mathbf{s}, \mathbf{p}, \mathbf{p}_0, \mathbf{p}_1, \mathbf{s}_N, \mathbf{p}_N, \mathbf{d}_N, \bar{\mathbf{0}}, \bar{\omega}, \gamma, \rho, \nu, \pi, \mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are application terms.
- Variables are application terms.
- If s and t are application terms, then (st) is an application term.

A **closed** application term is an application term that does not contain variables.

Definition 3.5.4. If r is an application term and u is a variable, we define the formula $[r \simeq u]^\wedge$ inductively as follows.

- If r is a constant or a variable, $[r \simeq u]^\wedge$ is $r = u$.
- If r is (st) , then $[r \simeq u]^\wedge$ is $\exists xy([s \simeq x]^\wedge \wedge [t \simeq y]^\wedge \wedge [x](y) \simeq u)$.

Notation.

- $t \downarrow$ denotes $\exists x[t \simeq x]^\wedge$ (i.e. t is defined).

- $t(a_1, \dots, a_n) \simeq b$ denotes

$$\exists x_1 \dots x_n \exists y (x_1 = a_1 \wedge \dots \wedge x_n = a_n \wedge y = b \wedge [t(x_1, \dots, x_n) \simeq y]^\wedge).$$

- $st_1 \dots t_n$ denotes $((\dots(st_1)\dots)t_n)$ (i.e. it is the functional $s(t_1, \dots, t_n)$).

Definition 3.5.5. A partial n -place (class) function Υ is said to be a **partial E -recursive function** if there exists a closed application term t_Υ such that

$$\text{dom}(\Upsilon) = \{(a_1, \dots, a_n) \mid t_\Upsilon(a_1, \dots, a_n) \downarrow\}$$

and for all sets $(a_1, \dots, a_n) \in \text{dom}(\Upsilon)$,

$$t_\Upsilon(a_1, \dots, a_n) \simeq \Upsilon(a_1, \dots, a_n).$$

In this case, t_Υ is said to be an **index** for Υ .

If Υ_1, Υ_2 are partial E -recursive functions, then write $\Upsilon_1(\vec{a}) \simeq \Upsilon_2(\vec{a})$ if neither $\Upsilon_1(\vec{a})$ nor $\Upsilon_2(\vec{a})$ are defined, or $\Upsilon_1(\vec{a})$ and $\Upsilon_2(\vec{a})$ are both defined and equal.

Remark 3.5.6. Observe that, by using $[s]$ (as well as pairing and projection functions), we have a partial recursive universal computation function U that outputs $[e](x)$ on input e, x .

In the following lemma, the application term $t[x]$ means the application term t with term x indicated; $t[y]$ is the result of replacing every occurrence of x in t by y .

Abstraction lemma 3.5.7. For every application term $t[x]$ there exists an application term $\lambda x.t[x]$ such that the following holds:

$$\forall x_1 \dots x_n (\lambda x.t[x] \downarrow \wedge \forall y (\lambda x.t[x])y \simeq t[y]).$$

Proof. By induction on the term t . If t is the variable x , $\lambda x.x$ is **skk**. If t is a constant or a variable other than x , $\lambda x.t$ is **kt**. If $t \equiv (uv)$, $\lambda x.uv$ is **s**($\lambda x.u$)($\lambda x.v$).

□

Recursion theorem 3.5.8. There exists a closed application term R such that for any application terms f, x ,

$$Rf \downarrow \wedge Rfx \simeq f(Rf)x.$$

Proof. Take R to be $\lambda f.tt$, where t is $\lambda y\lambda x.f(yy)x$. □

In particular, given any partial E -recursive f , there is an index e such that

$$\Phi_e(x) \simeq f(e, x)$$

for all x (Φ_e stands for the same thing as $[e]$).

Remark 3.5.9. The reader may check that the class of partial E -recursive functions include the class of primitive-recursive set functions, defined in 3.2.2.

Remark 3.5.10. We will describe a partial E -recursive function f taking parameters e and x , where e is an index of a yet unknown function. By recursion theorem, we obtain an e such that Φ_e behaves in the same way as $f(e, \cdot)$; thus, if we want to define a partial E -recursive function g , we may assume that g already knows its own index.

For example, if we want to find a partial E -recursive function g such that $g(x) \simeq \langle e, x \rangle$ where e is an index for g , we just define $f(e, x) = \langle e, x \rangle$, and apply R to f to obtain an index e such that $\Phi_e(x) \simeq f(e, x) \simeq \langle e, x \rangle$. Therefore we may define g by simply saying “ $g(x) = \langle e, x \rangle$ where e is an index of g ”.

In the sequel, we will just say “partial recursive” instead of “partial E -recursive”.

3.6 Expressibility of the infinitary derivations

In this section, we show how to code the $RS_\Omega(\mathbf{V})$ -derivations within KP. Definition 3.6.1 gives an idea of how the codes are structured, and the formal definition inside KP is given in 3.6.5.

Definition 3.6.1. Let us fix a natural coding function $\ulcorner \cdot \urcorner$ for formulas and finite sets of formulas, as well as a set of symbols $\ulcorner \text{Axiom} \urcorner$, $\ulcorner \wedge \urcorner$, $\ulcorner \vee \urcorner$ etc.; we give the definition of codes of $RS_\Omega(\mathbf{V})$ -derivations *outside* KP; we will be able to read off primitive-recursively from such a code u

- (i) the name of the last inference of the proof,
- (ii) its principal formula and side formulas,
- (iii) the end sequent,

- (iv) a bound for the length of the proof,
- (v) a bound for the cut rank of the proof.

The corresponding primitive-recursive functions will be denoted by $\text{Rule}(u)$, $\text{PF}(u)$, $\text{SF}(u)$, $\text{End}(u)$, $\text{Length}(u)$, $\text{Rank}(u)$, respectively. The ordinal notation system we use is the one $\langle R, < \rangle$ we defined in 3.2.

- If Γ contains a true Δ_0 -formula, then

$$\langle \ulcorner \text{Axiom} \urcorner, \ulcorner \Gamma \urcorner \rangle \quad (3.6.1)$$

is a member of $\text{RS}_\Omega(\mathbf{V})$ -derivation.

- If $u, v \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(u) = \ulcorner \Gamma, A \urcorner$, $\text{End}(v) = \ulcorner \Gamma, B \urcorner$, and $\text{Length}(u), \text{Length}(v) < a$, $\max(\text{Rank}(u), \text{Rank}(v)) \leq r$, then

$$\langle \ulcorner \wedge \urcorner, \ulcorner A \wedge B \urcorner, \ulcorner \Gamma \urcorner, a, r, u, v \rangle \quad (3.6.2)$$

is a member of $\text{RS}_\Omega(\mathbf{V})$ -derivation.

- If $u \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(u) = \ulcorner \Gamma, A \urcorner$, and $\text{Length}(u) < a$, $\text{Rank}(u) \leq r$, then

$$\langle \ulcorner \vee_0 \urcorner, \ulcorner A \vee B \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle \quad (3.6.3)$$

is a member of $\text{RS}_\Omega(\mathbf{V})$ -derivation.

- If $u \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(u) = \ulcorner \Gamma, B \urcorner$, and $\text{Length}(u) < a$, $\text{Rank}(u) \leq r$, then

$$\langle \ulcorner \vee_1 \urcorner, \ulcorner A \vee B \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle \quad (3.6.4)$$

is a member of $\text{RS}_\Omega(\mathbf{V})$ -derivation.

- Let e be an index for a partial recursive set function. If for all s , $\Phi_e(s) =: u_s \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(u_s) = \ulcorner \Gamma, s \in t \rightarrow F(s) \urcorner$, $\text{Length}(u_s) < a$, $\text{Rank}(u_s) \leq r$, then

$$\langle \ulcorner b\forall \urcorner, \ulcorner (\forall x \in t)F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, e \rangle \quad (3.6.5)$$

is a member of $\text{RS}_\Omega(\mathbf{V})$ -derivation.

- If $u \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(u) = \ulcorner \Gamma, s \in t \wedge F(s) \urcorner$ and $\text{Length}(u) < a$, $\text{Rank}(u) \leq r$, then

$$\langle \ulcorner b\exists \urcorner, \ulcorner (\exists x \in t)F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle \quad (3.6.6)$$

is a member of $\text{RS}_\Omega(\mathbf{V})$ -derivation.

- Let e be an index for a partial recursive set function. If for all t , $\Phi_e(t) =: u_t \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(u_t) = \ulcorner \Gamma, F(t) \urcorner$, $\text{Length}(u_t) < a$, $\text{Rank}(u_t) \leq r$, then

$$\langle \ulcorner \forall \urcorner, \ulcorner \forall x F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, e \rangle \quad (3.6.7)$$

is a member of $\text{RS}_\Omega(\mathbf{V})$ -derivation.

- If $u \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(u) = \ulcorner \Gamma, F(s) \urcorner$, and $\text{Length}(u) < a$, $\text{Rank}(u) \leq r$, then

$$\langle \ulcorner \exists \urcorner, \ulcorner \exists x F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle \quad (3.6.8)$$

is a member of $\text{RS}_\Omega(\mathbf{V})$ -derivation.

- If $u, v \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(u) = \ulcorner \Gamma, A \urcorner$, $\text{End}(v) = \ulcorner \Gamma, \neg A \urcorner$, $\text{Length}(u), \text{Length}(v) < a$, $\max(\text{rank}(A) + 1, \text{Rank}(u), \text{Rank}(v)) \leq r$, then

$$\langle \ulcorner \text{Cut} \urcorner, \ulcorner A \urcorner, \ulcorner \Gamma \urcorner, a, r, u, v \rangle \quad (3.6.9)$$

is a member of $\text{RS}_\Omega(\mathbf{V})$ -derivation. In this case, contrary to others, A is the cut formula, so it does not appear in the end sequent of (3.6.9).

- If $u \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(u) = \ulcorner \Gamma, A \urcorner$, $\max(\text{Length}(u), \Omega) < a$, $\text{Rank}(u) \leq r$, then

$$\langle \ulcorner \Sigma\text{-Ref} \urcorner, \ulcorner \exists z A^z \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle \quad (3.6.10)$$

is a member of $\text{RS}_\Omega(\mathbf{V})$ -derivation.

It is easy to see how the information extraction functions (End , Rank , Length , etc.) can be defined, regardless of such derivation codes being definable or not.

For the moment we haven't seen if KP can express $u \in \text{RS}_\Omega(\mathbf{V})$ -derivation, but KP surely can express if some set looks like a member of $\text{RS}_\Omega(\mathbf{V})$ -derivation (i.e. in one of the forms (3.6.1)–(3.6.10)).

The following definitions are made in KP.

Definition 3.6.2. We say that w is a **quasicode of a $\text{RS}_\Omega(\mathbf{V})$ -derivation**, $w \in \text{RS}_\Omega(\mathbf{V})$ -quasicode, if

$$(w)_0 \in \{ \ulcorner \text{Axiom} \urcorner, \ulcorner \wedge \urcorner, \ulcorner \vee_0 \urcorner, \ulcorner \vee_1 \urcorner, \ulcorner b\forall \urcorner, \ulcorner b\exists \urcorner, \ulcorner \forall \urcorner, \ulcorner \exists \urcorner, \ulcorner \text{Cut} \urcorner, \ulcorner \Sigma\text{-Ref} \urcorner \},$$

and

- if $(w)_0 = \ulcorner \text{Axiom} \urcorner$, then $w = \langle \ulcorner \text{Axiom} \urcorner, \ulcorner \Gamma \urcorner \rangle$ and Γ is a finite set of formulas;
- if $(w)_0 = \ulcorner \wedge \urcorner$, then $w = \langle \ulcorner \wedge \urcorner, \ulcorner A \wedge B \urcorner, \ulcorner \Gamma \urcorner, a, r, u, v \rangle$, where A, B are formulas, Γ is a finite set of formulas, a, r are in our ordinal notation, and u, v are sets;
- if $(w)_0 = \ulcorner \vee_0 \urcorner$, then $w = \langle \ulcorner \vee_0 \urcorner, \ulcorner A \vee B \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$, where A, B are formulas, Γ is a finite set of formulas, a, r are in our ordinal notation, and u is a set;
- if $(w)_0 = \ulcorner \vee_1 \urcorner$, then $w = \langle \ulcorner \vee_1 \urcorner, \ulcorner A \vee B \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$, where A, B are formulas, Γ is a finite set of formulas, a, r are in our ordinal notation, and u is a set;
- if $(w)_0 = \ulcorner b\forall \urcorner$, then $w = \langle \ulcorner b\forall \urcorner, \ulcorner (\forall x \in t)F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, e \rangle$, where t is a set, F is a formula, Γ is a finite set of formulas, a, r are in our ordinal notation, and e is a set;
- if $(w)_0 = \ulcorner b\exists \urcorner$, then $w = \langle \ulcorner b\exists \urcorner, \ulcorner (\exists x \in t)F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$, where t is a set, F is a formula, Γ is a finite set of formulas, a, r are in our ordinal notation, and u is a set;
- if $(w)_0 = \ulcorner \forall \urcorner$, then $w = \langle \ulcorner \forall \urcorner, \ulcorner \forall x F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, e \rangle$, where F is a formula, Γ is a finite set of formulas, a, r are in our ordinal notation, and e is a set;
- if $(w)_0 = \ulcorner \exists \urcorner$, then $w = \langle \ulcorner \exists \urcorner, \ulcorner \exists x F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$, where F is a formula, Γ is a finite set of formulas, a, r are in our ordinal notation, and u is a set;
- if $(w)_0 = \ulcorner \text{Cut} \urcorner$, then $w = \langle \ulcorner \text{Cut} \urcorner, \ulcorner A \urcorner, \ulcorner \Gamma \urcorner, a, r, u, v \rangle$, where A is a formula, Γ is a finite set of formulas, a, r are in our ordinal notation, and u is a set;
- if $(w)_0 = \ulcorner \Sigma\text{-Ref} \urcorner$, then $w = \langle \ulcorner \Sigma\text{-Ref} \urcorner, \ulcorner \exists z A^z \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$, where A is a formula, Γ is a finite set of formulas, a, r are in our ordinal notation, and u is a set.

It is primitive-recursive to determine if $w \in \text{RS}_\Omega(\mathbf{V})$ -quasicode.

If w is a quasicode, we can still read off the information $\text{Rule}(w)$, $\text{PF}(w)$, etc.; this definition exists only to eliminate unnecessarily fussy case distinctions, e.g., if we know that $w \in \text{RS}_\Omega(\mathbf{V})$ -quasicode and $w = \langle \ulcorner \wedge \urcorner, \ulcorner A \wedge B \urcorner, \ulcorner \Gamma \urcorner, a, r, u, v \rangle$, then Γ must be a finite set of formulas.

An $\text{RS}_\Omega(\mathbf{V})$ -derivation figure is a well-founded tree. This means that we can label its nodes by finite sequences of sets.

Notation. Let $\text{Set}^{<\omega}$ denote the class of all finite sequences of sets. If $\sigma \in \text{Set}^{<\omega}$

and x is a set, let σx abbreviate $\sigma^\frown \langle x \rangle$.

It is easy to define a partial recursive set function N such that, if u codes an $\text{RS}_\Omega(\mathbf{V})$ -derivation, $\sigma \in \text{Set}^{<\omega}$, and $N(u, \sigma)$ is a subderivation of u , then for appropriate collections of sets x (depending on the rule by which $N(u, \sigma)$ is derived), $N(u, \sigma x)$ are the direct subderivations of $N(u, \sigma)$.

Definition 3.6.3. We define the partial recursive set function $N(w, \sigma)$, for $w \in \text{RS}_\Omega(\mathbf{V})$ -quasicode and $\sigma \in \text{Set}^{<\omega}$.

- $N(w, \langle \rangle) = w$.
- Suppose $(N(w, \sigma))_0 \in \{ \ulcorner \vee_0 \urcorner, \ulcorner \vee_1 \urcorner, \ulcorner b\exists \urcorner, \ulcorner \exists \urcorner, \ulcorner \Sigma\text{-Ref} \urcorner \}$ and $(N(w, \sigma))_5 = u$. Then $N(w, \sigma 0) = u$.
- Suppose $(N(w, \sigma))_0 \in \{ \ulcorner \wedge \urcorner, \ulcorner \text{Cut} \urcorner \}$ and $(N(w, \sigma))_5 = u$, $(N(w, \sigma))_6 = v$. Then $N(w, \sigma 0) = u$ and $N(w, \sigma 1) = v$.
- Suppose $(N(w, \sigma))_0 \in \{ \ulcorner b\forall \urcorner, \ulcorner \forall \urcorner \}$ and $(N(w, \sigma))_5 = e$. Then $N(w, \sigma x) = \Phi_e(x)$ for all sets x (see Remark 3.5.6).

In any other cases we set $N(w, \sigma) = \emptyset$.

Notation. From now on, we denote $N(w, \sigma)$ by w_σ .

Before we proceed to give the formal definition of the class of $\text{RS}_\Omega(\mathbf{V})$ -derivations, we mention that one can define truth predicates for formulas with bounded complexity.

Definition of the truth predicates 3.6.4. True_Γ , Γ being a syntactical complexity, is defined primitive-recursively.

- $\text{True}_{\Delta_0}(\ulcorner a \in b \urcorner)$ iff $a \in b$ holds; the rest of the cases are $\wedge, \vee, \neg, (\forall x \in t), (\exists x \in t)$ defined similarly to below.
- $\text{True}_{\Sigma_n}(\ulcorner \phi \urcorner)$ iff one of the following holds.
 - For some $\diamond \in \{ \wedge, \vee \}$ and $\psi, \theta \in \Sigma_n$, $\ulcorner \phi \urcorner = \ulcorner \psi \diamond \theta \urcorner$, and $\text{True}_{\Sigma_n}(\ulcorner \psi \urcorner) \diamond \text{True}_{\Sigma_n}(\ulcorner \theta \urcorner)$.
 - For some $\psi \in \Pi_n$, $\ulcorner \phi \urcorner = \ulcorner \neg \psi \urcorner$, and $\neg \text{True}_{\Pi_n}(\ulcorner \psi \urcorner)$.
 - For some $\psi \in \Sigma_n$, $\ulcorner \phi \urcorner = \ulcorner (\exists x \in t) \psi(x) \urcorner$, and $(\exists x \in t) \text{True}_{\Sigma_n}(\ulcorner \psi(x) \urcorner)$.
 - For some $\psi \in \Sigma_n$, $\ulcorner \phi \urcorner = \ulcorner \exists x \psi(x) \urcorner$, and $\exists x \text{True}_{\Sigma_n}(\ulcorner \psi(x) \urcorner)$.
 - $\text{True}_{\Pi_{n-1}}(\ulcorner \phi \urcorner)$.

- $\text{True}_{\Pi_n}(\ulcorner \phi \urcorner)$ is defined in a symmetrical way.

Then in KP we have that for any formula ϕ with complexity Γ ,

$$\text{True}_{\Gamma}(\ulcorner \phi \urcorner) \leftrightarrow \phi.$$

Definition of an $\text{RS}_{\Omega}(\mathbf{V})$ -derivation within KP 3.6.5. A set w is a code of a $\text{RS}_{\Omega}(\mathbf{V})$ -derivation, $w \in \text{RS}_{\Omega}(\mathbf{V})$ -derivation, iff for all $\sigma \in \text{Set}^{<\omega}$, the following holds:

- If $w_{\sigma} \neq \emptyset$ then $w_{\sigma} \in \text{RS}_{\Omega}(\mathbf{V})$ -quasicode.
- If $w_{\sigma} = \langle \ulcorner \text{Axiom} \urcorner, \ulcorner \Gamma \urcorner \rangle$, then Γ contains a true Δ_0 -formula (i.e. there is $A \in \Gamma$ with complexity Δ_0 , and $\text{True}_{\Delta_0}(\ulcorner A \urcorner)$); $w_{\sigma\tau} = \emptyset$ for any $\tau \neq \emptyset$.
- If $w_{\sigma} = \langle \ulcorner \wedge \urcorner, \ulcorner A \wedge B \urcorner, \ulcorner \Gamma \urcorner, a, r, u, v \rangle$, then $\text{End}(w_{\sigma 0}) = \ulcorner \Gamma, A \urcorner$, $\text{End}(w_{\sigma 1}) = \ulcorner \Gamma, B \urcorner$, $\text{Length}(w_{\sigma i}) < a$ and $\text{Rank}(w_{\sigma i}) \leq r$ for $i = 0, 1$; $w_{\sigma x} = \emptyset$ for $x \neq 0, 1$.
- If $w_{\sigma} = \langle \ulcorner \vee_0 \urcorner, \ulcorner A \vee B \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$, then $\text{End}(w_{\sigma 0}) = \ulcorner \Gamma, A \urcorner$, $\text{Length}(w_{\sigma 0}) < a$ and $\text{Rank}(w_{\sigma 0}) \leq r$; $w_{\sigma x} = \emptyset$ for $x \neq 0$.
- If $w_{\sigma} = \langle \ulcorner \vee_1 \urcorner, \ulcorner A \vee B \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$, then $\text{End}(w_{\sigma 0}) = \ulcorner \Gamma, B \urcorner$, $\text{Length}(w_{\sigma 0}) < a$ and $\text{Rank}(w_{\sigma 0}) \leq r$; $w_{\sigma x} = \emptyset$ for $x \neq 0$.
- If $w_{\sigma} = \langle \ulcorner b\forall \urcorner, \ulcorner (\forall x \in t)F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, e \rangle$, then for all s , $\text{End}(w_{\sigma s}) = \ulcorner \Gamma, s \in t \rightarrow A(s) \urcorner$, $\text{Length}(w_{\sigma s}) < a$, $\text{Rank}(w_{\sigma s}) \leq r$.
- If $w_{\sigma} = \langle \ulcorner b\exists \urcorner, \ulcorner (\exists x \in t)F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$, then $\text{End}(w_{\sigma 0}) = \ulcorner \Gamma, s \in t \wedge A(s) \urcorner$ for some s , $\text{Length}(w_{\sigma 0}) < a$ and $\text{Rank}(w_{\sigma 0}) \leq r$; $w_{\sigma x} = \emptyset$ for $x \neq 0$.
- If $w_{\sigma} = \langle \ulcorner \forall \urcorner, \ulcorner \forall x F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, e \rangle$, then for all x , $\text{End}(w_{\sigma x}) = \ulcorner \Gamma, F(x) \urcorner$, $\text{Length}(w_{\sigma x}) < a$, $\text{Rank}(w_{\sigma x}) \leq r$.
- If $w_{\sigma} = \langle \ulcorner \exists \urcorner, \ulcorner \exists x F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$, then $\text{End}(w_{\sigma 0}) = \ulcorner \Gamma, F(s) \urcorner$ for some s , $\text{Length}(w_{\sigma 0}) < a$ and $\text{Rank}(w_{\sigma 0}) \leq r$; $w_{\sigma x} = \emptyset$ for $x \neq 0$.
- If $w_{\sigma} = \langle \ulcorner \text{Cut} \urcorner, \ulcorner A \urcorner, \ulcorner \Gamma \urcorner, a, r, u, v \rangle$, then $\text{End}(w_{\sigma 0}) = \ulcorner \Gamma, A \urcorner$, $\text{End}(w_{\sigma 1}) = \ulcorner \Gamma, \neg A \urcorner$, $\text{Length}(w_{\sigma i}) < a$ and $\text{Rank}(w_{\sigma i})$, $\text{rank}(A)+1 \leq r$ for $i = 0, 1$; $w_{\sigma x} = \emptyset$ for $x \neq 0, 1$.
- If $w_{\sigma} = \langle \ulcorner \Sigma\text{-Ref} \urcorner, \ulcorner \exists z A^z \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$, then $\text{End}(w_{\sigma 0}) = \ulcorner \Gamma, A \urcorner$, $\text{Length}(w_{\sigma 0}) < a$, $\text{Rank}(w_{\sigma 0}) \leq r$; $w_{\sigma x} = \emptyset$ for $x \neq 0$.

The reader may check that this is a Π_2 definition (recall that $\mathbb{E}$ is Σ 3.5.2).

3.7 Formalisation of cut elimination in $KP + TI(\varepsilon_{\Omega+1})$

The lemmas 3.4.5–3.4.8 are now going to be formalised in $KP + TI(\varepsilon_{\Omega+1})$; we will define partial recursive set functions Wkn (for “weakening”), Inv (for “inversion”), Red (for “reduction”), $CutElim$ (for “cut elimination”) that transform members of $RS_{\Omega}(\mathbf{V})$ -derivation into the required forms. We do not care what these functions do to the sets which are not members of $RS_{\Omega}(\mathbf{V})$ -derivation; they may simply diverge. In defining these functions, we may assume that they know their own indices (by 3.5.10).

The reader is advised to familiarise themselves with the standard cut elimination proofs before proceeding; though the reasoning is exactly the same, our proofs may have been obscured in formalisation.

If there is no danger of confusion, we drop the coding notation $\ulcorner \cdot \urcorner$.

Weakening 3.7.1. $KP + TI(\varepsilon_{\Omega+1})$ proves the following. There is a partial recursive set function f taking parameter Γ' such that, whenever Γ' is a finite set of $RS_{\Omega}(\mathbf{V})$ -formulas, $w \in RS_{\Omega}(\mathbf{V})$ -derivation, $\text{End}(w) = \Gamma$, $\text{Length}(w) = a < \varepsilon_{\Omega+1}$, we have $f(\Gamma', w) \in RS_{\Omega}(\mathbf{V})$ -derivation, $\text{End}(f(\Gamma', w)) = \Gamma, \Gamma'$, $\text{Length}(w) \geq \text{Length}(f(\Gamma', w))$ and $\text{Rank}(w) \geq \text{Rank}(f(\Gamma', w))$.

The function f will be referred to as Wkn .

Proof. In the following we suppress the parameter Γ' and write simply $f(w)$.

If $w \notin RS_{\Omega}(\mathbf{V})$ -quasicode, we set $f(w) = \emptyset$.

Now we assume that $w \in RS_{\Omega}(\mathbf{V})$ -quasicode.

If $w = \langle \ulcorner \text{Axiom} \urcorner, \ulcorner \Gamma \urcorner \rangle$, then

$$f(w) = \langle \ulcorner \text{Axiom} \urcorner, \ulcorner \Gamma, \Gamma' \urcorner \rangle.$$

If $w = \langle \ulcorner \wedge \urcorner, \ulcorner A \wedge B \urcorner, \ulcorner \Gamma \urcorner, a, r, u, v \rangle$ and the lengths of u, v are both $< a$, then

$$f(w) = \langle \ulcorner \wedge \urcorner, \ulcorner A \wedge B \urcorner, \ulcorner \Gamma, \Gamma' \urcorner, a, r, f(u), f(v) \rangle \text{ (using the recursion theorem).}$$

If $w = \langle \ulcorner \vee_0 \urcorner, \ulcorner A \vee B \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$ and the length of u is $< a$, then

$$f(w) = \langle \ulcorner \vee_0 \urcorner, \ulcorner A \vee B \urcorner, \ulcorner \Gamma, \Gamma' \urcorner, a, r, f(u) \rangle. \text{ The case of } \vee_1 \text{ is similar.}$$

If $w = \langle \ulcorner b\forall \urcorner, \ulcorner (\forall x \in t)F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, e \rangle$, then

$$f(w) = \langle \ulcorner b\forall \urcorner, \ulcorner (\forall x \in t)F(x) \urcorner, \ulcorner \Gamma, \Gamma' \urcorner, a, r, e' \rangle \text{ where } e' \text{ is the natural index of } f \circ \Phi_e.$$

If $w = \langle \ulcorner b\exists \urcorner, \ulcorner (\exists x \in t)F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$ and the length of u is $< a$, then
 $f(w) = \langle \ulcorner b\exists \urcorner, \ulcorner (\exists x \in t)F(x) \urcorner, \ulcorner \Gamma, \Gamma' \urcorner, a, r, f(u) \rangle$.

If $w = \langle \ulcorner \forall \urcorner, \ulcorner \forall x F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, e \rangle$, then
 $f(w) = \langle \ulcorner \forall \urcorner, \ulcorner \forall x F(x) \urcorner, \ulcorner \Gamma, \Gamma' \urcorner, a, r, e' \rangle$ where e' is the natural index of $f \circ \Phi_e$.

If $w = \langle \ulcorner \exists \urcorner, \ulcorner \exists x F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$ and the length of u is $< a$, then
 $f(w) = \langle \ulcorner \exists \urcorner, \ulcorner \exists x F(x) \urcorner, \ulcorner \Gamma, \Gamma' \urcorner, a, r, f(u) \rangle$.

If $w = \langle \ulcorner \text{Cut} \urcorner, \ulcorner A \urcorner, \ulcorner \Gamma \urcorner, a, r, u, v \rangle$ and the lengths of u, v are $< a$, then
 $f(w) = \langle \ulcorner \text{Cut} \urcorner, \ulcorner A \urcorner, \ulcorner \Gamma, \Gamma' \urcorner, a, r, f(u), f(v) \rangle$ if the cut formula A is not in Γ' , and
 $f(w) = f(u)$ if $A \in \Gamma'$ (in our definition we assume $A \in \text{End}(u)$).

If $w = \langle \ulcorner \Sigma\text{-Ref} \urcorner, \ulcorner \exists z A^z \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$ and the length of u is $< a$, then
 $f(w) = \langle \ulcorner \Sigma\text{-Ref} \urcorner, \ulcorner \exists z A^z \urcorner, \ulcorner \Gamma, \Gamma' \urcorner, a, r, f(u) \rangle$.

One can then prove using $TI(\varepsilon_{\Omega+1})$ that f satisfies the requirements. $\square$

Note that in the cases $b\forall$ and $\forall$, we do not check if say $\Phi_e(s)$ has length $< a$ for every set s . It is unnecessary (since this condition is already checked for $RS_{\Omega}(\mathbf{V})$ -derivation), and we cannot do so recursively either.

Inversion 3.7.2. $KP + TI(\varepsilon_{\Omega+1})$ proves the following. There is a partial recursive set function g taking parameter A such that, whenever A is not Δ_0 , $A \simeq \bigwedge_{i \in y} A_i$, $w \in RS_{\Omega}(\mathbf{V})$ -derivation, $\text{End}(w) = \Gamma, A$, $\text{Length}(w) = a < \varepsilon_{\Omega+1}$, we have $g(A, w, i) \in RS_{\Omega}(\mathbf{V})$ -derivation, $\text{End}(g(A, w, i)) = \Gamma, A_i$, $\text{Length}(w) \geq \text{Length}(g(A, w, i))$ and $\text{Rank}(w) \geq \text{Rank}(g(A, w, i))$, for all $i \in y$.

The function g will be referred to as *Inv*.

Proof. In the following we suppress the parameter $A \simeq \bigwedge_{i \in y} A_i$.

If $w \notin RS_{\Omega}(\mathbf{V})$ -quasicode or $A \notin \text{End}(w)$, then we set $g(w, i) = \emptyset$ for all i . Now we assume that $w \in RS_{\Omega}(\mathbf{V})$ -quasicode and $A \in \text{End}(w)$.

If $w = \langle \ulcorner \text{Axiom} \urcorner, \ulcorner \Gamma, A \urcorner \rangle$, then
 $g(w, i) = \langle \ulcorner \text{Axiom} \urcorner, \ulcorner \Gamma, A_i \urcorner \rangle$ for all $i \in y$.

Now suppose w is not an axiom. We first assume that A is not the principal formula of the last inference of w .

If $w = \langle \ulcorner \wedge \urcorner, \ulcorner B \urcorner, \ulcorner \Gamma, A \urcorner, a, r, u, v \rangle$, and the lengths of u, v are $< a$, then
 $g(w, i) = \langle \ulcorner \wedge \urcorner, \ulcorner B \urcorner, \ulcorner \Gamma, A_i \urcorner, a, r, g(u, i), g(v, i) \rangle$.

If $w = \langle \ulcorner \vee_0 \urcorner, \ulcorner B \urcorner, \ulcorner \Gamma, A \urcorner, a, r, u \rangle$, and the length of u is $< a$, then
 $g(w, i) = \langle \ulcorner \vee_0 \urcorner, \ulcorner B \urcorner, \ulcorner \Gamma, A_i \urcorner, a, r, g(u, i) \rangle$. The case of $\vee_1$ is similar.

If $w = \langle \ulcorner b\forall \urcorner, \ulcorner B \urcorner, \ulcorner \Gamma, A_i \urcorner, a, r, e \rangle$, then
 $g(w, i) = \langle \ulcorner b\forall \urcorner, \ulcorner B \urcorner, \ulcorner \Gamma, A_i \urcorner, a, r, e' \rangle$, where e' is the natural index of $g(\Phi_e(\cdot), i)$.

The rest of the cases are similar to above.

Now assume that A is the principal formula of the last inference of w .

If $A \equiv A_0 \wedge A_1$, $w = \langle \ulcorner \wedge \urcorner, \ulcorner A \urcorner, \ulcorner \Gamma \urcorner, a, r, u, v \rangle$, and the lengths of u, v are $< a$, then
 $g(w, 0) = g(u, 0)$ if $A \in \Gamma$, $g(w, 0) = u$ if $A \notin \Gamma$, and similarly
 $g(w, 1) = g(v, 1)$ if $A \in \Gamma$, $g(w, 1) = v$ if $A \notin \Gamma$.

If $A \equiv (\forall x \in t)F(x)$, $w = \langle \ulcorner b\forall \urcorner, \ulcorner A \urcorner, \ulcorner \Gamma \urcorner, a, r, e \rangle$, then, for each s ,
 $g(w, s) = g(\Phi_e(s), s)$ if $A \in \Gamma$ and $\Phi_e(s)$ has length $< a$, $g(w, s) = \Phi_e(s)$ if $A \notin \Gamma$.

If $A \equiv \forall x F(x)$, $w = \langle \ulcorner \forall \urcorner, \ulcorner \forall x F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, e \rangle$, then, for each s ,
 $g(w, s) = g(\Phi_e(s), s)$ if $A \in \Gamma$ and $\Phi_e(s)$ has length $< a$, $g(w, s) = \Phi_e(s)$ if $A \notin \Gamma$.

Again using $\text{TI}(\varepsilon_{\Omega+1})$ one checks that g satisfies the requirements for $w \in \text{RS}_{\Omega}(\mathbf{V})$ -derivation with length $< \varepsilon_{\Omega+1}$. $\square$

Recall the definition of natural sum $\alpha \# \beta$ for ordinals 3.2.6.

Reduction 3.7.3. $\text{KP} + \text{TI}(\varepsilon_{\Omega+1})$ proves the following. There is a partial recursive set function h taking parameter C such that, whenever $\text{rank}(C) = r > \Omega$, $w_0, w_1 \in \text{RS}_{\Omega}(\mathbf{V})$ -derivation, $\text{End}(w_0) = \Gamma, C$, $\text{End}(w_1) = \Gamma, \neg C$, $\text{Length}(w_0) = a < \varepsilon_{\Omega+1}$, $\text{Length}(w_1) = b < \varepsilon_{\Omega+1}$, $\text{Rank}(w_0)$ and $\text{Rank}(w_1)$ are both $\leq r$, we have $h(C, w_0, w_1) \in \text{RS}_{\Omega}(\mathbf{V})$ -derivation, $\text{End}(h(C, w_0, w_1)) = \Gamma$, $\text{Length}(h(C, w_0, w_1)) \leq a \# b$, $\text{Rank}(h(C, w_0, w_1)) \leq r$.

The function h will be referred to as **Red**.

Proof. We suppress the parameter C with $\text{rank } r > \Omega$. We assume that $w_0, w_1 \in \text{RS}_{\Omega}(\mathbf{V})$ -quasicode satisfy $\text{End}(w_0) = \Gamma, C$, $\text{End}(w_1) = \Gamma, \neg C$, $\text{Length}(w_0) = a < \varepsilon_{\Omega+1}$, $\text{Length}(w_1) = b < \varepsilon_{\Omega+1}$, $\text{Rank}(w_0)$ and $\text{Rank}(w_1)$ are both $\leq r$. The function h is defined by recursion on $a \# b$ along $\varepsilon_{\Omega+1}$.

If $(w_0)_0 = \ulcorner \text{Axiom} \urcorner$ or $(w_1)_0 = \ulcorner \text{Axiom} \urcorner$, then
 $h(w_0, w_1) = \langle \ulcorner \text{Axiom} \urcorner, \ulcorner \Gamma \urcorner \rangle$.

Now we assume that neither of w_0, w_1 is an axiom.

First assume that C is not the principal formula of the last inference of w_0 . We go through the ideas of two representative cases.

Suppose $w_0 = \langle \ulcorner \wedge \urcorner, \ulcorner A_0 \wedge A_1 \urcorner, \ulcorner \Gamma, C \urcorner, a, r, u, v \rangle$; then, assuming this inference is correct, $\text{End}(u) = A_0, \Gamma, C$ and $\text{End}(v) = A_1, \Gamma, C$. By weakening, $\text{Wkn}(A_i, w_1)$ has end sequent $A_i, \Gamma, \neg C$ for $i = 0, 1$, so if both u, v have lengths $< a$, then we may apply h to get $h(u, \text{Wkn}(A_0, w_1))$ and $h(v, \text{Wkn}(A_1, w_1))$ whose end sequents are A_0, Γ and A_1, Γ respectively. Thus we may apply $\wedge$ to $h(u, \text{Wkn}(A_0, w_1))$ and $h(v, \text{Wkn}(A_1, w_1))$ to obtain $h(w_0, w_1)$.

Suppose $w_0 = \langle \ulcorner \forall \urcorner, \ulcorner \forall x F(x) \urcorner, \ulcorner \Gamma, C \urcorner, a, r, e \rangle$. If this inference is correct, then $\Phi_e(s)$ has end sequent $F(s), \Gamma, C$ for every set s ; also $\text{Wkn}(F(s), w_1)$ has end sequent $F(s), \Gamma, \neg C$ for every s . Therefore if $\Phi_e(s)$ has length $< a$ then $h(\Phi_e(s), \text{Wkn}(F(s), w_1))$ has end sequent $F(s), \Gamma$, thus we may apply $\forall$ to the derivations $h(\Phi_e(s), \text{Wkn}(F(s), w_1))$ to obtain $h(w_0, w_1)$.

Now we proceed to the formal definition.

If $w_0 = \langle \ulcorner \wedge \urcorner, \ulcorner A_0 \wedge A_1 \urcorner, \ulcorner \Gamma, C \urcorner, a, r, u, v \rangle$ and u, v have lengths $< a$, then $h(w_0, w_1) = \langle \ulcorner \wedge \urcorner, \ulcorner A_0 \wedge A_1 \urcorner, \ulcorner \Gamma \urcorner, a \# b, r, h(u, \text{Wkn}(A_0, w_1)), h(v, \text{Wkn}(A_1, w_1)) \rangle$.

If $w_0 = \langle \ulcorner \vee_0 \urcorner, \ulcorner A \vee B \urcorner, \ulcorner \Gamma, C \urcorner, a, r, u \rangle$ and u has length $< a$, then $h(w_0, w_1) = \langle \ulcorner \vee_0 \urcorner, \ulcorner A \vee B \urcorner, \ulcorner \Gamma \urcorner, a \# b, r, h(u, \text{Wkn}(A, w_1)) \rangle$. The case of $\vee_1$ is similar.

If $w_0 = \langle \ulcorner b \forall \urcorner, \ulcorner (\forall x \in t) F(x) \urcorner, \ulcorner \Gamma, C \urcorner, a, r, e \rangle$, then, $h(w_0, w_1) = \langle \ulcorner b \forall \urcorner, \ulcorner (\forall x \in t) F(x) \urcorner, \ulcorner \Gamma \urcorner, a \# b, r, e' \rangle$, where e' is the natural index of $\lambda s. h(\Phi_e(s), \text{Wkn}(s \in t \rightarrow F(s), w_1))$.

The other cases, as well as all the cases in which $\neg C$ is not the principal formula of the last inference of w_1 , can be similarly dealt with.

For the remainder we assume that C is the principal formula of the last inference of w_0 and $\neg C$ is the principal formula of the last inference of w_1 . We go through the idea of one representative case.

Suppose $w_0 = \langle \ulcorner \vee_0 \urcorner, \ulcorner A_0 \vee A_1 \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$. Then $\text{Wkn}(C, u)$ has end sequent A_0, Γ, C ; $\text{Wkn}(A_0, w_1)$ has end sequent $A_0, \Gamma, \neg C$. Therefore if u has length $< a$, we may apply h and the derivation $h(\text{Wkn}(C, u), \text{Wkn}(A_0, w_1))$ has end sequent A_0, Γ with length $< a \# b$. Now $\text{Inv}(\neg C, w_1, 0)$ has end sequent $\neg A_0, \Gamma$, so we may apply a cut to $h(\text{Wkn}(C, u), \text{Wkn}(A_0, w_1))$ and $\text{Inv}(\neg C, w_1, 0)$.

The following is the formal definition.

If $w_0 = \langle \ulcorner \vee_0 \urcorner, \ulcorner A_0 \vee A_1 \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$ and u has length $< a$, then
 $h(w_0, w_1) = \langle \ulcorner \text{Cut} \urcorner, \ulcorner A_0 \urcorner, \ulcorner \Gamma \urcorner, a \# b, r, h(\text{Wkn}(C, u), \text{Wkn}(A_0, w_1)), \text{Inv}(\neg C, w_1, 0) \rangle$.

If $w_0 = \langle \ulcorner b\exists \urcorner, \ulcorner (\exists x \in t)F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, u \rangle$, u has length $< a$ and end sequent $s \in t \wedge F(s), \Gamma$ for some set s , then

$h(w_0, w_1) = \langle \ulcorner \text{Cut} \urcorner, \ulcorner s \in t \wedge F(s) \urcorner, \ulcorner \Gamma \urcorner, a \# b, r, h(\text{Wkn}(C, u), \text{Wkn}(s \in t \wedge F(s), w_1)), \text{Inv}(\neg C, w_1, s) \rangle$.

(Note that since $\text{rank}(C) > \Omega$, F contains an unbounded quantifier, so 3.4.4 applies.)

The other cases are similar or symmetrical to the ones above. $\square$

Cut elimination 3.7.4. $\text{KP} + \text{TI}(\varepsilon_{\Omega+1})$ proves the following. There is a partial recursive set function j such that, whenever $w \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(w) = \Gamma$, $\text{Length}(w) \leq a < \varepsilon_{\Omega+1}$, $\text{Rank}(w) \leq r+1$ with $r > \Omega$, we have $j(w) \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(j(w)) = \Gamma$, $\text{Length}(j(w)) \leq \omega^a$, $\text{Rank}(j(w)) \leq r$.

The function j here will be referred to as CutElim .

Proof. We assume $w \in \text{RS}_\Omega(\mathbf{V})$ -quasicode, $\text{End}(w) = \Gamma$, $\text{Length}(w) \leq a < \varepsilon_{\Omega+1}$, $\text{Rank}(w) \leq r+1$ with $r > \Omega$.

We give the definition of j only for the representative cases.

If $w = \langle \ulcorner \text{Axiom} \urcorner, \ulcorner \Gamma \urcorner \rangle$, then $j(w) = w$.

If $w = \langle \ulcorner \wedge \urcorner, \ulcorner A_0 \wedge A_1 \urcorner, \ulcorner \Gamma \urcorner, a, r+1, u, v \rangle$, and u, v have lengths $< a$, then
 $j(w) = \langle \ulcorner \wedge \urcorner, \ulcorner A_0 \wedge A_1 \urcorner, \ulcorner \Gamma \urcorner, \omega^a, r, j(u), j(v) \rangle$. The cases of $\vee_0, \vee_1, b\exists, \exists, \Sigma$ -Ref are similar.

If $w = \langle \ulcorner b\forall \urcorner, \ulcorner (\forall x \in t)F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r+1, e \rangle$, then
 $j(w) = \langle \ulcorner b\forall \urcorner, \ulcorner (\forall x \in t)F(x) \urcorner, \ulcorner \Gamma \urcorner, \omega^a, r, e' \rangle$, where e' is the natural index of $j \circ \Phi_e$.
 The case of $\forall$ is similar.

If $w = \langle \ulcorner \text{Cut} \urcorner, \ulcorner A \urcorner, \ulcorner \Gamma \urcorner, a, r+1, u, v \rangle$ and u, v have lengths $< a$, then
 $j(w) = \langle \ulcorner \text{Cut} \urcorner, \ulcorner A \urcorner, \ulcorner \Gamma \urcorner, a, r, j(u), j(v) \rangle$ if $\text{rank}(A) < r$, $j(w) = \text{Red}(j(u), j(v))$ if $\text{rank}(A) = r$. $\square$

3.8 Embedding KP into $\text{RS}_\Omega(\mathbf{V})$

We are going to formalise, within KP, that KP can be embedded into $\text{RS}_\Omega(\mathbf{V})$; roughly, there is a partial recursive set function P such that, if p is a KP-proof, then $P(p)$ is an $\text{RS}_\Omega(\mathbf{V})$ -derivation with the same conclusion (see 3.8.3). Combining this with our formalisation of cut elimination, every KP-proof can be transformed into an $\text{RS}_\Omega(\mathbf{V})$ -proof with their cuts partially eliminated. Along these proof trees, we are able to carry out a transfinite induction along $\varepsilon_{\Omega+1}$ which gives us the result $\text{KP} + \text{TI}(\varepsilon_{\Omega+1}) \vdash \text{RFN}(\text{KP})$.

Notation.

- If A is any $\text{RS}_\Omega(\mathbf{V})$ -formula, then $\text{no}(A) = \omega^{\text{rank}(A)}$.
- If $\Gamma = \{A_1, \dots, A_n\}$ is a set of $\text{RS}_\Omega(\mathbf{V})$ -formulas, then $\text{no}(\Gamma) := \text{no}(A_1) \# \dots \# \text{no}(A_n)$.
- $\Vdash \Gamma$ abbreviates $\frac{\text{no}(\Gamma)}{0} \Gamma$, and $\Vdash_{\frac{\alpha}{\rho}} \Gamma$ abbreviates $\frac{\text{no}(\Gamma) \# \alpha}{\rho} \Gamma$.

Embedding lemmas 3.8.1.

1. $\Vdash A, \neg A$.
2. (Extensionality). $\Vdash s_1 \neq t_1, \dots, s_n \neq t_n, \neg A(s_1, \dots, s_n), A(t_1, \dots, t_n)$.
3. (Set induction). $\Vdash_{\omega^{\text{rank}(A)}} A \rightarrow \forall x F(x)$, where $A \equiv \forall x ((\forall y \in x) F(y) \rightarrow F(x))$.
4. (Pair). $\Vdash \exists z (s \in z \wedge t \in z)$.
5. (Union). $\Vdash \exists z (\forall y \in s) (\forall x \in y) (x \in z)$.
6. (Infinity). $\Vdash \exists x ((\exists z \in x) z \in x \wedge (\forall y \in x) (\exists z \in x) y \in z)$.
7. (Δ_0 -separation). If A is Δ_0 , $\Vdash \exists y ((\forall x \in y) (x \in s \wedge A(x, \vec{t})) \wedge (\forall x \in s) (A(x, \vec{t}) \rightarrow x \in y))$.
8. (Δ_0 -collection). If F is Δ_0 , $\Vdash (\forall x \in s) \exists y F(x, y, \vec{t}) \rightarrow \exists z (\forall x \in s) (\exists y \in z) F(x, y, \vec{t})$.

Lemma 3.8.1.1. KP proves the following. There is a partial recursive set function f such that, for any $\text{RS}_\Omega(\mathbf{V})$ -formula A , $f(A) \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(f(A)) = A, \neg A$, $\text{Length}(f(A)) \leq \text{no}(A, \neg A)$, $\text{Rank}(f(A)) = 0$.

The f here will be referred to as LEM.

Proof. We define f by recursion on complexity of A .

If A is Δ_0 , we set

$f(A) = \langle \ulcorner \text{Axiom} \urcorner, \ulcorner A, \neg A \urcorner \rangle$. From now on we assume that A is not Δ_0 .

If A is $A_0 \vee A_1$, let

$$w_0 = \langle \ulcorner \vee_0 \urcorner, \ulcorner A_0 \vee A_1 \urcorner, \ulcorner \neg A_0 \urcorner, \text{no}(A_0, \neg A_0) + 1, 0, f(A_0) \rangle,$$

$$w_1 = \langle \ulcorner \vee_1 \urcorner, \ulcorner A_0 \vee A_1 \urcorner, \ulcorner \neg A_1 \urcorner, \text{no}(A_1, \neg A_1) + 1, 0, f(A_1) \rangle, \text{ then set}$$

$$f(A) = \langle \ulcorner \wedge \urcorner, \ulcorner \neg A_0 \wedge \neg A_1 \urcorner, \ulcorner A_0 \vee A_1 \urcorner, \text{no}(A, \neg A), 0, w_0, w_1 \rangle.$$

If A is $(\exists x \in t)F(x)$, write $B(s)$ for $s \in t \wedge F(s)$, and let e be the natural index of the function

$$\Phi_e(s) = \langle \ulcorner b\exists \urcorner, \ulcorner (\exists x \in t)F(x) \urcorner, \ulcorner s \in t \rightarrow \neg F(s) \urcorner, \text{no}(B(s), \neg B(s)) + 1, 0, f(B(s)) \rangle;$$

then we set

$$f(A) = \langle \ulcorner b\forall \urcorner, \ulcorner (\forall x \in t)\neg F(x) \urcorner, \ulcorner (\exists x \in t)F(x) \urcorner, \text{no}(A, \neg A), 0, e \rangle.$$

If A is $\exists x F(x)$, let e be the natural index of the function

$$\Phi_e(s) = \langle \ulcorner \exists \urcorner, \ulcorner \exists x F(x) \urcorner, \ulcorner \neg F(s) \urcorner, \text{no}(F(s), \neg F(s)) + 1, 0, f(F(s)) \rangle, \text{ and set}$$

$$f(A) = \langle \ulcorner \forall \urcorner, \ulcorner \forall x \neg F(x) \urcorner, \ulcorner \exists x F(x) \urcorner, \text{no}(A, \neg A), 0, e \rangle.$$

The other cases are symmetrical. □

Extensionality 3.8.1.2. KP proves the following. There is a partial recursive set function Ext such that, whenever $A(a_1, \dots, a_n)$ is a KP-formula with free variables among $a_1, \dots, a_n$, $\vec{s} := \langle s_1, \dots, s_n \rangle$, $\vec{t} := \langle t_1, \dots, t_n \rangle$ are sequences of terms with length n , we have $\text{Ext}(A, \vec{s}, \vec{t}) \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(\text{Ext}(A, \vec{s}, \vec{t})) = s_1 \neq t_1, \dots, s_n \neq t_n, \neg A(\vec{s}), A(\vec{t})$, $\text{Length}(\text{Ext}(A, \vec{s}, \vec{t})) \leq \text{no}(s_1 \neq t_1, \dots, s_n \neq t_n, \neg A(\vec{s}), A(\vec{t}))$, $\text{Rank}(\text{Ext}(A, \vec{s}, \vec{t})) = 0$.

Proof. Define Ext by recursion on complexity of A ; we will write $\vec{s} \neq \vec{t}$ for $s_1 \neq t_1, \dots, s_n \neq t_n$.

If A is Δ_0 , set $\text{Ext}(A, \vec{s}, \vec{t}) = \langle \ulcorner \text{Axiom} \urcorner, \ulcorner \vec{s} \neq \vec{t}, \neg A(\vec{s}), A(\vec{t}) \urcorner \rangle$. In this case, if $\text{Ext}(A, \vec{s}, \vec{t}) \notin \text{RS}_\Omega(\mathbf{V})$ -derivation, then by definition of $\text{RS}_\Omega(\mathbf{V})$ -derivation, we have

$$\neg \text{True}_{\Delta_0}(\ulcorner \vec{s} \neq \vec{t} \urcorner) \wedge \neg \text{True}_{\Delta_0}(\ulcorner \neg A(\vec{s}) \urcorner) \wedge \neg \text{True}_{\Delta_0}(\ulcorner A(\vec{t}) \urcorner),$$

which implies $\vec{s} = \vec{t} \wedge A(\vec{s}) \wedge \neg A(\vec{t})$ and yields a contradiction in KP.

From now on we assume that A is not Δ_0 .

If A is $A_0 \wedge A_1$, $\text{Ext}(A, \vec{s}, \vec{t})$ is defined in the following way:

$$\begin{array}{c} (\vee) \frac{\text{Ext}(A_0, \vec{s}, \vec{t}) \parallel \neg \vec{s} \neq \vec{t}, \neg A_0(\vec{s}), A_0(\vec{t})}{\vec{s} \neq \vec{t}, \neg A_0(\vec{s}) \vee \neg A_1(\vec{s}), A_0(\vec{t})} \quad (\vee) \frac{\text{Ext}(A_1, \vec{s}, \vec{t}) \parallel \neg \vec{s} \neq \vec{t}, \neg A_1(\vec{s}), A_1(\vec{t})}{\vec{s} \neq \vec{t}, \neg A_0(\vec{s}) \vee \neg A_1(\vec{s}), A_1(\vec{t})} \\ (\wedge) \frac{}{\vec{s} \neq \vec{t}, \neg A_0(\vec{s}) \vee \neg A_1(\vec{s}), A_0(\vec{t}) \wedge A_1(\vec{t})} \end{array}$$

If A is $(\forall x \in t)F(x)$, we derive $\vec{s} \neq \vec{t}, (\exists x \in t(\vec{s})) \neg F(x, \vec{s}), s \notin t(\vec{t}) \vee F(s, \vec{t})$ uniformly for all s :

$$\begin{array}{c} (\vee) \frac{\text{Ext}(s \in t(\vec{a}), \vec{s}, \vec{t}) \parallel \neg \vec{s} \neq \vec{t}, s \in t(\vec{s}), s \notin t(\vec{t})}{\vec{s} \neq \vec{t}, s \in t(\vec{s}), s \notin t(\vec{t}) \vee F(s, \vec{t})} \quad (\vee) \frac{\text{Ext}(F(s, \vec{a}), \vec{s}, \vec{t}) \parallel \neg \vec{s} \neq \vec{t}, \neg F(s, \vec{s}), F(s, \vec{t})}{\vec{s} \neq \vec{t}, \neg F(s, \vec{s}), s \notin t(\vec{t}) \vee F(s, \vec{t})} \\ (\wedge) \frac{}{\vec{s} \neq \vec{t}, s \in t(\vec{s}) \wedge \neg F(s, \vec{s}), s \notin t(\vec{t}) \vee F(s, \vec{t})} \\ (b\exists) \frac{}{\vec{s} \neq \vec{t}, (\exists x \in t(\vec{s})) \neg F(x, \vec{s}), s \notin t(\vec{t}) \vee F(s, \vec{t})} \end{array}$$

If this proof tree is given by $\Phi_e(s)$, we set $\text{Ext}(A, \vec{s}, \vec{t}) = \langle \ulcorner b\forall \urcorner, \ulcorner (\forall x \in t(\vec{t}))F(x, \vec{t}) \urcorner, \ulcorner \vec{s} \neq \vec{t}, (\exists x \in t(\vec{s})) \neg F(x, \vec{s}) \urcorner, \text{no}(\vec{s} \neq \vec{t}, \neg A(\vec{s}), A(\vec{t})), 0, e \rangle$.

If A is $\forall x F(x)$, $\text{Ext}(A, \vec{s}, \vec{t})$ is defined as follows. For every s , $\text{Ext}(F(s, \vec{a}), \vec{s}, \vec{t})$ derives $\vec{s} \neq \vec{t}, \neg F(s, \vec{s}), F(s, \vec{t})$. Applying $\exists$ gives us $\vec{s} \neq \vec{t}, \exists x \neg F(x, \vec{s}), F(s, \vec{t})$ for all s , and by $\forall$ we obtain $\vec{s} \neq \vec{t}, \exists x \neg F(x, \vec{s}), \forall x F(x, \vec{t})$.

The other cases are symmetrical. $\square$

Set induction 3.8.1.3. KP proves the following. There is a partial recursive set function Ind such that, for any $\text{RS}_\Omega(\mathbf{V})$ -formula F and $A \equiv \forall x((\forall y \in x)F(y) \rightarrow F(x))$, $\text{Ind}(F) \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(\text{Ind}(F)) = A \rightarrow \forall x F(x)$, $\text{Length}(\text{Ind}(F)) \leq \text{no}(A \rightarrow \forall x F(x)) \# \omega^{\text{rank}(A)}$, $\text{Rank}(\text{Ind}(F)) = 0$.

Proof. We define a partial recursive set function f in KP such that for any term s , $f(F, s) \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(f(F, s)) = \neg A, F(s)$, $\text{Length}(f(F, s)) \leq \omega^{\text{rank}(A)} \# \omega^{|s|+1}$, $\text{Rank}(f(F, s)) = 0$; f is defined by recursion on $|s|$. Given the index of $f(F, \cdot)$, it is then clear how to define Ind .

If $\frac{\omega^{\text{rank}(A)} \# \omega^{|t|+1}}{0} \neg A, F(t)$ has been proved by $f(F, t)$ for all $|t| < |s|$, then there is a partial recursive set function g such that $g(F, t)$ proves $\frac{\omega^{\text{rank}(A)} \# \omega^{|s|+1}}{0} \neg A, t \in s \rightarrow F(t)$ for all terms t (if $|t| < |s|$, this follows from $f(F, t)$; if $|t| \geq |s|$, then $t \notin s$ is an axiom). Now the derivation continues as follows:

$$\begin{array}{c} \frac{\neg A, t \in s \rightarrow F(t) \text{ for all } t}{\neg A, (\forall y \in s)F(y)} \quad \neg F(s), F(s) \\ \frac{}{\neg A, (\forall y \in s)F(y) \wedge \neg F(s), F(s)} \\ \frac{}{\neg A, \exists x((\forall y \in x)F(y) \wedge \neg F(x)), F(s)} \end{array}$$

where weakening is implied where appropriate; note that $\text{no}(\neg F(s), F(s)) < \omega^{\text{rank}(A)}$, so the end sequent has length $\omega^{\text{rank}(A)} \# \omega^{|s|} + 4 < \omega^{\text{rank}(A)} \# \omega^{|s|+1}$; the end sequent is $\neg A, F(s)$ by contraction. $\square$

Pair 3.8.1.4. KP proves the following. There is a partial recursive set function Pair such that, if s, t are terms, then $\text{Pair}(s, t) \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(\text{Pair}(s, t)) = \exists z(s \in z \wedge t \in z)$, $\text{Length}(\text{Pair}(s, t)) = 1$, $\text{Rank}(\text{Pair}(s, t)) = 0$.

Proof. $s \in \{s, t\} \wedge t \in \{s, t\}$ is a Δ_0 -formula true in KP. $\square$

Union 3.8.1.5. KP proves the following. There is a partial recursive set function Union such that, if s is a term, then $\text{Union}(s) \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(\text{Union}(s)) = \exists z(\forall y \in s)(\forall x \in y)(x \in z)$, $\text{Length}(\text{Union}(s)) = 1$, $\text{Rank}(\text{Union}(s)) = 0$.

Proof. $(\forall y \in s)(\forall x \in y)x \in \cup s$ is a Δ_0 -formula true in KP. $\square$

Infinity 3.8.1.6. KP proves the following. There is $w \in \text{RS}_\Omega(\mathbf{V})$ -derivation such that $\text{End}(w) = \exists x((\exists z \in x)z \in x \wedge (\forall y \in x)(\exists z \in x)y \in z)$ and $\text{Length}(w) = 1$, $\text{Rank}(w) = 0$; we also write w as Inf.

Proof. $(\exists z \in \omega)z \in \omega \wedge (\forall y \in \omega)(\exists z \in \omega)y \in z$ is a Δ_0 -formula true in KP. $\square$

Δ_0 -separation 3.8.1.7. KP proves the following. There is a partial recursive set function Sep such that, given $A(a, b_1, \dots, b_n)$ a Δ_0 -formula of KP with all free variables indicated, $s, t_1, \dots, t_n$ terms, $\text{Sep}(A, \langle s, \vec{t} \rangle) \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(\text{Sep}(A, \langle s, \vec{t} \rangle)) = \exists y((\forall x \in y)(x \in s \wedge A(x, \vec{t})) \wedge (\forall x \in s)(A(x, \vec{t}) \rightarrow x \in y))$, $\text{Length}(\text{Sep}(A, \langle s, \vec{t} \rangle)) = 1$, and $\text{Rank}(\text{Sep}(A, \langle s, \vec{t} \rangle)) = 0$.

Proof. $(\forall x \in \{x \in s \mid A(x, \vec{t})\})(x \in s \wedge A(x, \vec{t})) \wedge (\forall x \in s)(A(x, \vec{t}) \rightarrow x \in \{x \in s \mid A(x, \vec{t})\})$ is a Δ_0 -formula true in KP. $\square$

Δ_0 -collection 3.8.1.8. KP proves the following. There is a partial recursive set function Col such that, given $F(a, b, c_1, \dots, c_n)$ a Δ_0 -formula of KP with all free variables indicated, $s, t_1, \dots, t_n$ terms, $\text{Col}(F, \langle s, \vec{t} \rangle) \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(\text{Col}(F, \langle s, \vec{t} \rangle)) = (\forall x \in s)\exists y F(x, y, \vec{t}) \rightarrow \exists z(\forall x \in s)(\exists y \in z)F(x, y, \vec{t})$,

$$\begin{aligned} \text{Length}(\text{Col}(F, \langle s, \vec{t} \rangle)) &= \text{no}((\forall x \in s) \exists y F(x, y, \vec{t}) \rightarrow \exists z (\forall x \in s) (\exists y \in z) F(x, y, \vec{t})), \\ \text{Rank}(\text{Col}(F, \langle s, \vec{t} \rangle)) &= 0. \end{aligned}$$

Proof. By LEM we have

$$\Vdash \neg(\forall x \in s) \exists y F(x, y, \vec{t}), (\forall x \in s) \exists y F(x, y, \vec{t}).$$

Applying Σ -Ref gives

$$\frac{\alpha+1}{0} \neg(\forall x \in s) \exists y F(x, y, \vec{t}), \exists z (\forall x \in s) (\exists y \in z) F(x, y, \vec{t})$$

where $\alpha = \text{no}(\neg(\forall x \in s) \exists y F(x, y, \vec{t}), (\forall x \in s) \exists y F(x, y, \vec{t}))$. Two applications of $\vee$ gives

$$\frac{\alpha+3}{0} (\forall x \in s) \exists y F(x, y, \vec{t}) \rightarrow \exists z (\forall x \in s) (\exists y \in z) F(x, y, \vec{t});$$

note also that $\alpha + 3 < \text{no}((\forall x \in s) \exists y F(x, y, \vec{t}) \rightarrow \exists z (\forall x \in s) (\exists y \in z) F(x, y, \vec{t}))$. $\square$

Thus we have all the proofs for [Embedding lemmas 3.8.1](#).

Next we define a finitary proof system for KP.

Definition of the finitary sequent calculus of KP 3.8.2. Let a, b etc. denote free variables, s, t etc. set terms, Γ denote a finite set of formulas. KP has the following axioms:

- (Logical axioms). $\Gamma, A, \neg A$ for any formula A .
- (Extensionality). $\Gamma, a = b \wedge B(a) \rightarrow B(b)$ for any formula $B(a)$.
- (Set induction). $\Gamma, \forall x ((\forall y \in x) F(y) \rightarrow F(x)) \rightarrow \forall x F(x)$ for any formula $F(a)$.
- (Pair). $\Gamma, \exists z (a \in z \wedge b \in z)$.
- (Union). $\Gamma, \exists z (\forall y \in z) (\forall x \in y) x \in z$.
- (Infinity). $\Gamma, \exists x ((\exists z \in x) z \in x \wedge (\forall y \in x) (\exists z \in x) y \in z)$.
- (Δ_0 -separation). $\Gamma, \exists y ((\forall x \in y) (x \in a \wedge B(x)) \wedge (\forall x \in a) (B(x) \rightarrow x \in y))$ for any Δ_0 -formula B .
- (Δ_0 -collection). $\Gamma, (\forall x \in a) \exists y G(x, y) \rightarrow \exists z (\forall x \in a) (\exists y \in z) G(x, y)$ for any Δ_0 -formula G .

The rules of inference are

$$\begin{array}{ll}
(\wedge) & \frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \\
(\vee) & \frac{\Gamma, A}{\Gamma, A \vee B} \quad \frac{\Gamma, B}{\Gamma, A \vee B} \\
(b\forall) & \frac{\Gamma, a \in t \rightarrow F(a)}{\Gamma, (\forall x \in t)F(x)} \\
(b\exists) & \frac{\Gamma, s \in t \wedge F(s)}{\Gamma, (\exists x \in t)F(x)} \\
(\forall) & \frac{\Gamma, F(a)}{\Gamma, \forall x F(x)} \\
(\exists) & \frac{\Gamma, F(s)}{\Gamma, \exists x F(x)} \\
(\text{Cut}) & \frac{\Gamma, A \quad \Gamma, \neg A}{\Gamma}
\end{array}$$

In inferring a sequent by $b\forall$ or $\forall$, the variable a cannot appear in the lower sequents.

We say that KP proves Γ if Γ can be derived from these axioms and inference rules.

Notation. Let ϕ_0 denote the exponentiation $\alpha \mapsto \omega^\alpha$.

Embedding theorem 3.8.3. KP proves the following. There is a partial recursive set function P such that:

- (i) If p is a code of an axiom of KP with end sequent $\Gamma(a_1, \dots, a_n)$ where $a_1, \dots, a_n$ are all the free variables that Γ has, then for all terms $s_1, \dots, s_n$,
 $P(p, \langle s_1, \dots, s_n \rangle) \in \text{RS}_\Omega(\mathbf{V})$ -derivation,
 $\text{End}(P(p, \langle s_1, \dots, s_n \rangle)) = \Gamma(s_1, \dots, s_n)$,
 $\text{Length}(P(p, \langle s_1, \dots, s_n \rangle)) < \Omega \cdot \omega^\omega$.
- (ii) If p is a code of a KP-proof that is not an axiom and uses $k \geq 0$ instances of $b\forall/\forall$ -inferences, with end sequent $\Gamma(a_1, \dots, a_n)$ where $a_1, \dots, a_n$ are all the free variables that Γ has, then, there is some $m < \omega$ such that for all terms $s_1, \dots, s_n$,
 $P(p, \langle s_1, \dots, s_n \rangle) \in \text{RS}_\Omega(\mathbf{V})$ -derivation,
 $\text{End}(P(p, \langle s_1, \dots, s_n \rangle)) = \Gamma(s_1, \dots, s_n)$,

$$\begin{aligned}\text{Length}(P(p, \langle s_1, \dots, s_n \rangle)) &< \Omega \cdot \phi_0^{k+1}(\omega), \\ \text{Rank}(P(p, \langle s_1, \dots, s_n \rangle)) &= \Omega + m.\end{aligned}$$

Proof. P is defined by recursion on the length of p .

If p is an axiom, the value of P is given by the previously defined functions LEM, Ext, Ind, Pair, Union, Inf, Sep, Col, along with appropriate weakening. Note that, if we write $\Gamma = A_1, \dots, A_n$, by 3.4.2.1 (i), for some $m_1, \dots, m_n$, we have $\text{rank}(A_i) \leq \omega \cdot \Omega + m_i$ for $i = 1, \dots, n$. Then $\text{no}(\Gamma) = \omega^{\text{rank}(A_1)} \# \dots \# \omega^{\text{rank}(A_n)} = (\Omega \cdot \omega^{m_1}) \# \dots \# (\Omega \cdot \omega^{m_n}) = \Omega \cdot (\omega^{m_1} \# \dots \# \omega^{m_n}) < \Omega \cdot \omega^m$ for $m = \max(m_1, \dots, m_n) + 1$.

Now assume p is not an axiom.

If the last inference of p is $\wedge$, and we are in the situation

$$\frac{p_0 \vdash \Gamma, A \quad p_1 \vdash \Gamma, B}{p \vdash \Gamma, A \wedge B}$$

then

$$P(p, \langle s_1, \dots, s_n \rangle) = \langle \ulcorner \wedge \urcorner, \ulcorner A \wedge B \urcorner, \ulcorner \Gamma \urcorner, a, r, P(p_0, \langle s_1, \dots, s_n \rangle), P(p_1, \langle s_1, \dots, s_n \rangle) \rangle,$$

where

$$\begin{aligned}a &= \max(\text{Length}(P(p_0, \langle s_1, \dots, s_n \rangle)), \text{Length}(P(p_1, \langle s_1, \dots, s_n \rangle))) + 1, \\ r &= \max(\Omega, \text{Rank}(P(p_0, \langle s_1, \dots, s_n \rangle)), \text{Rank}(P(p_1, \langle s_1, \dots, s_n \rangle))).\end{aligned}$$

The cases $\vee$ and Cut are similarly dealt with.

If the last inference of p is $\exists$, and we are in the situation

$$\frac{p_0 \vdash \Gamma, F(s(a_1, \dots, a_n, b_1, \dots, b_m))}{p \vdash \Gamma, \exists x F(x)}$$

and $b_1, \dots, b_m$ are free variables that do not appear in $\Gamma, \exists x F(x)$, then we can define

$$P(p, \langle s_1, \dots, s_n \rangle) = \langle \ulcorner \exists \urcorner, \ulcorner \exists x F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, P(p_0, \langle s_1, \dots, s_n, c_\emptyset, \dots, c_\emptyset \rangle) \rangle, \text{ where}$$

$$\begin{aligned}a &= \text{Length}(P(p_0, \langle s_1, \dots, s_n, c_\emptyset, \dots, c_\emptyset \rangle)) + 1, \\ r &= \max(\Omega, \text{Rank}(P(p_0, \langle s_1, \dots, s_n, c_\emptyset, \dots, c_\emptyset \rangle))).\end{aligned}$$

The case $b\exists$ is similarly dealt with.

If the last inference of p is $\forall$, and we are in the situation

$$\frac{p_0 \vdash \Gamma, F(a)}{p \vdash \Gamma, \forall x F(x)}$$

and so p_0 uses $k-1$ instances of $b\forall/\forall$ -inferences, if p uses k of them. By induction, we have some $m < \omega$ such that for all terms $s_1, \dots, s_n, t$,

$$\text{End}(P(p_0, \langle s_1, \dots, s_n, t \rangle)) = \Gamma(s_1, \dots, s_n), F(s_1, \dots, s_n, t),$$

$$\text{Length}(P(p_0, \langle s_1, \dots, s_n, t \rangle)) < \Omega \cdot \phi_0^k(\omega),$$

$$\text{Rank}(P(p_0, \langle s_1, \dots, s_n, t \rangle)) = \Omega + m.$$

Therefore, if e is an index for $\lambda t.P(p_0, \langle s_1, \dots, s_n, t \rangle)$, we may define

$$P(p, \langle s_1, \dots, s_n \rangle) = \langle \ulcorner \forall \urcorner, \ulcorner \forall x F(x) \urcorner, \ulcorner \Gamma \urcorner, \Omega \cdot \phi_0^k(\omega), \text{Rank}(P(p_0, \langle s_1, \dots, s_n, c_\emptyset \rangle)), e \rangle. \quad {}^5$$

If the last inference of p is $b\forall$, and we are in the situation

$$\frac{p_0 \vdash \Gamma(\vec{s}), a \in t(\vec{s}) \rightarrow F(\vec{s}, a)}{p \vdash \Gamma(\vec{s}), (\forall x \in t(\vec{s})) F(\vec{s}, x)}$$

and p_0 uses $k-1$ instances of $b\forall/\forall$ -inferences if p uses k of them. By induction, there is some $m < \omega$ such that for all terms $\vec{s}, r$,

$$\text{End}(P(p_0, \langle \vec{s}, r \rangle)) = \Gamma(\vec{s}), r \in t(\vec{s}) \rightarrow F(\vec{s}, r),$$

$$\text{Length}(P(p_0, \langle \vec{s}, r \rangle)) < \Omega \cdot \phi_0^k(\omega),$$

$$\text{Rank}(P(p_0, \langle \vec{s}, r \rangle)) = \Omega + m.$$

Therefore, if $e(\vec{s})$ is an index (function) for $\lambda r.P(p_0, \langle \vec{s}, r \rangle)$, we may define

$$P(p, \vec{s}) = \langle \ulcorner b\forall \urcorner, \ulcorner (\forall x \in t) F(x) \urcorner, \ulcorner \Gamma \urcorner, \Omega \cdot \phi_0^k(\omega), \text{Rank}(P(p_0, \langle \vec{s}, c_\emptyset \rangle)), e(\vec{s}) \rangle. \quad \square$$

Corollary 3.8.4. For any finite set Γ of formulas, $\text{KP} + \text{TI}(\varepsilon_{\Omega+1})$ proves that, for all sets $x_1, \dots, x_n$, if $\Gamma(c_{x_1}, \dots, c_{x_n})$ is KP -provable, then there is $w \in \text{RS}_\Omega(\mathbf{V})$ -derivation with end sequent $\Gamma(c_{x_1}, \dots, c_{x_n})$, $\text{Length}(w) < \varepsilon_{\Omega+1}$, $\text{Rank}(w) \leq \Omega + 1$.

Proof. Given a KP -proof p of $\Gamma(c_{x_1}, \dots, c_{x_n})$, we have $P(p, \langle \rangle) \in \text{RS}_\Omega(\mathbf{V})$ -derivation, $\text{End}(P(p, \langle \rangle)) = \Gamma(c_{x_1}, \dots, c_{x_n})$, $\text{Length}(P(p, \langle \rangle)) < \Omega \cdot \varepsilon_0$, $\text{Rank}(P(p, \langle \rangle)) = \Omega + m$ for some m which we can read off from $P(p, \langle \rangle)$. Thus $w := \text{CutElim}^{m-1}(P(p, \langle \rangle)) \in \text{RS}_\Omega(\mathbf{V})$ -derivation has the same end sequent as $P(p, \langle \rangle)$, the length of w is $\leq \phi_0^{m-1}(\Omega \cdot \varepsilon_0) < \varepsilon_{\Omega+1}$, and $\text{Rank}(w) = \Omega + 1$. $\square$

Lemma 3.8.5. $\text{KP} + \text{TI}(\varepsilon_{\Omega+1})$ proves that, for any finite set Γ of Π_n -formulas ($n \geq 2$), for all sets $x_1, \dots, x_n$, if $w \in \text{RS}_\Omega(\mathbf{V})$ -derivation has end sequent $\Gamma(c_{x_1}, \dots, c_{x_n})$, $\text{Length}(w) < \varepsilon_{\Omega+1}$, $\text{Rank}(w) \leq \Omega + 1$, then $\text{True}_{\Pi_n}(\ulcorner \forall \Gamma(x_1, \dots, x_n) \urcorner)$.

⁵The uniformity of the ranks of the premises allows us to use $\text{Rank}(P(p_0, \langle s_1, \dots, s_n, c_\emptyset \rangle))$ as the rank of the derivation $P(p, \langle s_1, \dots, s_n \rangle)$.

Proof. By induction on a along $\varepsilon_{\Omega+1}$.

If $w = \langle \ulcorner \wedge \urcorner, \ulcorner A \wedge B \urcorner, \ulcorner \Gamma \urcorner, a, r, u, v \rangle$, by induction, the disjunctions of $\text{End}(u) = \Gamma, A$ and $\text{End}(v) = \Gamma, B$ are true, so the disjunction of $\Gamma, A \wedge B$ is true.

If $w = \langle \ulcorner \forall \urcorner, \ulcorner \forall x F(x) \urcorner, \ulcorner \Gamma \urcorner, a, r, e \rangle$, by induction the disjunctions of $\text{End}(\Phi_e(s)) = \Gamma, F(s)$ are true for all s , so the disjunction of $\Gamma, \forall x F(x)$ is true.

The other cases of logical rules are similar.

If the last inference is a cut, then the cut formula must be Σ_1/Π_1 and so the equivalence $\text{True}_{\Pi_n}(\ulcorner \forall \Gamma \urcorner) \leftrightarrow \forall \Gamma$ is not violated. If the last inference is Σ -Ref, the induction is completed by the fact that KP proves $A \leftrightarrow \exists z A^z$ for any Σ -formula A . $\square$

This concludes the proof of our theorem:

Theorem 3.8.6. $\text{KP} + \text{TI}(\varepsilon_{\Omega+1}) \vdash \text{RFN}(\text{KP})$. $\square$

Since the proof of Lemma 3.8.5 simply hinges on the fact that we may eliminate cuts above certain complexity that is only determined by the complexity of axioms in $\text{RS}_\Omega(\mathbf{V})$, we may deduce that, if T stands for $\text{KP} + \Gamma$ -separation + Γ -collection where Γ is any given syntactic complexity,

$$\text{RFN}(T) \equiv \text{TI}(\varepsilon_{\Omega+1}) \quad \text{over } T$$

as well; we may add the axiom schemata Γ -separation and Γ -collection into the system $\text{RS}_\Omega(\mathbf{V})$, forgo the comprehension terms $\{x \in s \mid A(x, \vec{t})\}$, and the cut elimination (now above the complexity of Γ -separation and Γ -collection) and the embedding theorem go through as usual.

This argument does not apply to ZF; indeed, we can show that ZF proves $\text{TI}(\varepsilon_{\Omega+1})$. Suppose it doesn't, then there is a formula F and a model M of ZF such that $\text{TI}(\varepsilon_{\Omega+1}, F)$ is false in M . By the reflection theorem of ZF, there is a set model N in M such that $\text{TI}(\varepsilon_{\Omega+1}, F)$ is absolute for N , and so $\neg \text{TI}(\varepsilon_{\Omega+1}, F)^N$. But as $(\varepsilon_{\Omega+1})^N$ is an ordinal in M , this contradicts foundation in M .

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