Robust Stability Analysis of Cyber-Physical Systems Controlled by Model Predictive Control

by

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ABSTRACT

This thesis presents novel methods for analysing linear and nonlinear of constrained Networked Control Systems (NCS) controlled by Model Predictive Control (MPC), subject to a cyber-attack in the form of random packet losses and (bounded) additive disturbances. Conditions for (robust) stability and criteria for the Separation Principle design conditions of controllers and estimators are investigated. Stability conditions are presented through lemmas and theorems, with numerical examples provided to verify the proposed results.

Using a counterexample, this thesis shows that the Separation Principle does not hold when a TCP-like protocol is used. Further analysis uncovers a trade-off between estimation and controller prediction errors, suggesting that improving estimation performance can affect controller performance. Conditions are established under which the explicit control law is Piecewise Affine (PWA) and the cost function is Piecewise Quadratic (PWQ).

For discrete-time linear and nonlinear MPC-controlled systems with a buffer mechanism for packet losses mitigation, subject to input constraints and random packet losses, it is shown that, at best, the use of a buffer facilitates the transfer of the initial state to the terminal region but does not guarantee stability under consecutive packet losses. The number of consecutive packet losses the system can tolerate while maintaining stability is upper bounded by expressions dependent on system and controller parameters.

The last part extends the results by analysing the robustness of an MPC-controlled system under bounded additive disturbances. Conditions are derived under which the state remain in the region of attraction under consecutive packet losses. The results show that the use of the buffer does not provide the transfer of an initial state to the terminal region. However, we derive upper bounds on the number of consecutive packet losses that can be tolerated while maintaining robust stability if the disturbance is sufficiently small.

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To my family: Gerardo, Silvia and Pricila . . .

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Acronyms

ACK Acknowledgment

AWGN Additive White Gaussian Noise

C-A Controller-Actuator

CLF Control Lyapunov Function

CPS Cyber-Physical System

DARE Discrete Algebraic Ricatti Equation

DCS Distributed Control System

DDD Disruption Deception Disclousure

DMPC Distributed Model Predictive Control

DoS Denial of Service

FDI False Data Injection

FH-LQ Finite-Horizon Linear Quadratic

GM Gaussian-mixture

gPCE generalized Polynomial Chaos Expansion

i.i.d. independent and identically distributed

ICT Information Communication Technology

IDoS Informational Denial-of-Service

IPM Interior Point Method

ISpS Input-to-State practical Stability

KF Kalman Filter

LMI Linear Matrix Inequality

LMV Linear-minimum-variance

LQG Linear Quadratic Gaussian

LTI Linear Time-Invariant

MHE Moving Horizon Estimation

mp-QP multi-parametric Quadratic Programming

MPC Model Predictive Control

NCS Networked Control System

PID Proportional Integral Derivative

PWA Piecewise Affine

PWM Pulse Width Modulation

PWQ Piecewise Quadratic

QP Quadratic Programming

RH Receding Horizon

RR Round-Robin

S-C Sensor-Controller

SCADA Supervisory Control and Data Acquisition

SINR Signal to Interference plus Noise Ratio

TCP Transmission Control Protocol

TOD Try-Once-Discard

UAV Unmanned Aerial Vehicle

UDP User Datagram Protocol

WSN Wireless Sensor Network

Notation and Variables

Notation

Sets

$\mathbb{R}$	the set of real numbers
$\mathbb{N}$	the set of integers
$\mathbb{R}_{\geq 0}$	the set of non-negative real numbers
$\mathbb{N}_{\geq 0}$	the set of non-negative integers
$\mathbb{R}^n$	the set of real-valued n-dimensional vectors
$\mathbb{R}^{n \times m}$	the set of $n \times m$ matrices with real numbers
$\emptyset$	the empty set

Algebraic operators and matrices

A^{-1}	inverse of matrix A
$A^\top$	transpose of matrix A
$A \succeq 0$	for a matrix A denotes positive semidefiniteness
$A \succ 0$	for a matrix A denotes positive definiteness
$\mathrm{tr}(A)$	the trace of matrix A
$A \otimes B$	the Kronecker operator of matrix A and B
$\bar{\lambda}(A)$	max eigenvalue of matrix A

$\underline{\lambda}(A)$	min eigenvalue of matrix A
A^i	i-th row of matrix A
$\text{diag}(r)$	diagonal matrix whose diagonal entries are given by the vector r
$\mathbf{I}_{n\times n}$	$n \times n$ identity matrix
$\mathbf{1}_{n\times n}$	$n \times n$ one matrix
$\mathbf{0}_{n\times n}$	$n \times n$ zero matrix
x	vector $x \in \mathbb{R}^n$
$\|x\|$	absolute value of x
$\\|x\\|$	denotes a generic vector norm
$\\|x\\|_p$	the L_p-norm of x
$\\|x\\|_2^2$	the L_2-norm of x (Euclidean norm)
$\\|x\\|_Q^2$	the quadratic form $x^\top Qx$

Probabilities and expectations

$\Pr(x)$	the probability of an event x
$\mathbb{E}\{x \mid y\}$	the conditional expectation of x given y
$\mathbb{E}\{x\}$	the unconditional expectation of x
$\mathcal{N}(\mu, \Sigma)$	the (multivariate) normal distribution with mean μ and covariance matrix Σ
$\mathcal{B}(\mu)$	the (univariate) Bernoulli distribution with mean μ
Σ_{xy}	the covariance matrix between x and y

Other

∇_x	Nabla operator with respect to x
$f^i(x) = x_i$	$f^i(x) = x$ for $i = 0$, and $f^i(x) = f(f^{i-1}(x))$ for $i \in \mathbb{N}_{\geq 1}$
$\mathcal{L}(\cdot)$	Lagrange multiplier function
λ	Lagrange multiplier

λ_i i-th row of the Lagrange multiplier

$\hat{\lambda}, \check{\lambda}$ Lagrange multiplier for active and inactive constraints

$\hat{A}, \check{A}$ active and inactive constraints

Variables

System

k discrete-time index, $k \in \mathbb{N}_{\geq 0}$

N prediction horizon, $N \in \mathbb{N}_{\geq 0}$

y_k system output at time k, $y_k \in \mathbb{R}^p$

A, B state-space system matrices, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$

Q, R, Q_f state, input, and terminal weighting matrices

States

x_k system state at time k, $x_k \in \mathbb{R}^n$

$\hat{x}_k$ system state estimate at time k

$\bar{x}_k$ nominal system state at time k

$x_{k+j|k}$ the j-step ahead prediction of x made at time $k \in \mathbb{N}_{\geq 0}$

x^+ shorthand for x_{k+1} when $x = x_k$

x_j shorthand for the system state j-step ahead in prediction of x at time k, *i.e.* $x_{k+j|k}$

$\mathbf{x}$ system state sequence, $\mathbf{x} = \{x_0, \ldots, x_N\}$

x_i system state reached after i subsequent steps, $i \in \mathbb{N}_{\geq 0}$

$x_{i|N}$ system state reached after i subsequent steps given step N

x_{-1} shorthand for the system state at time $k-1$, *i.e.* x_{k-1}

z_k open-loop system state prediction at time k

$\mathbf{z}_{\cdot|k}$ sequence of predicted states $\mathbf{z}_{\cdot|k} = \left\{z_{k|k}, z_{k+1|k}, \ldots\right\}$

Input

u_k	control input applied at time k, $u_k \in \mathbb{R}^m$
$\bar{u}_k$	nominal control input applied at time k
u_k^0	optimal control input applied at time k
$\mathbf{u}_k$	control input sequence computed at time k, $\mathbf{u}_k \in \mathbb{R}^{mN}$
$\mathbf{u}_k^0$	optimal control input sequence computed at time k
$\mathbf{u}$	control input sequence, $\mathbf{u} = \{u_0, \ldots, u_{N-1}\}$
$\mathbf{u}(x)$	control input sequence that depends on x, $\mathbf{u}(x) = \{u_0(x), \ldots, u_{N-1}(x)\}$
u_j	shorthand for the input j-step ahead in prediction of u at time k, *i.e.* $u_{k+j\mid k}$
$u_{j\mid N}$	input at step j given step N
$\mathcal{U}$	input constraint set

Stochastic

w_k	bounded additive process disturbance at time k, $w_k \in \mathcal{W} \subset \mathbb{R}^n$
w_k	unbounded additive process disturbance at time k, $w_k \in \mathbb{R}^n$ with $w_k \sim \mathcal{N}(0, Q_w)$
Q_w	process disturbance covariance
$\{w\}_i^j$	disturbance sequence $\{w\}_i^j = \{w_i, w_{i+1}, \ldots, w_j\}$ if $j > i$
$\mathcal{W}$	process disturbance constraint set
s_k	unbounded additive measurement disturbance at time k, $s_k \in \mathbb{R}^p$ with $s_k \sim \mathcal{N}(0, R_s)$
R_s	measurement disturbance covariance
ν_k	input packet loss variable, $\nu_k \in \{0, 1\}$ with $\nu_k \sim \mathcal{B}(\bar{\nu})$
γ_k	output packet loss variable, $\gamma_k \in \{0, 1\}$ with $\gamma_k \sim \mathcal{B}(\bar{\gamma})$
$\bar{\nu}, \bar{\gamma}$	probabilities of input and output successful packet delivery
$\mathcal{I}_k$	information set
ε_k	prediction error at time k

e_k	estimation error at time k
P_k	state covariance at time k
$\hat{x}_{k\|k-1}$	state estimate at time k given time $k-1$
$e_{k\|k-1}$	estimation error at time k given time $k-1$
$P_{k\|k-1}$	state covariance at time k given time $k-1$
$\mathcal{P}_k$	open-loop state covariance prediction at time k

Other

Γ_N^{β}	region of attraction defined by N and β
$J_N(\cdot, \cdot)$	cost function
$V_N^0(\cdot)$	value function
$\mathbb{P}_N(\cdot)$	optimal control problem

Chapter 1

Introduction

Contents

1.1 Motivation

A Cyber-Physical System (CPS) is a type of large-scale feedback system in which the physical process is tightly integrated with control, communication and computation technologies . Typical examples are industrial process plants, power transmission and distribution networks, water treatment and distribution facilities, autonomous vehicles in transportation systems, industrial robotics systems, building automation, and smart grids.

Within the CPS framework, the architecture has a physical plant layer composed by interconnected plants with their corresponding actuators and transducers, and a controller layer composed by wired or wireless interconnected remote controllers which are permanently sharing information through a communication network layer. The CPS can be designed and implemented with flexibility, robustness, and scalability, together with fault-tolerant, disturbance rejection, and noise suppression techniques under the main assumption that the data over the communication network layer is reliable [1, 2].

A cyber-threat in CPS refers to any potential danger or risk to one or several cyber assets in the CPS architecture, including data, software, hardware or communication channels. They can be intentional (*e.g.* malicious actions) or unintentional (*e.g.* software bugs, human errors, or natural disasters) disruptions. On the other hand, a cyber-attack can be considered a specialization of cyber-threats that specifically involves strategic, coordinated and stealthy malicious actions by one or many adversaries with the objective to compromise the confidentiality, integrity, or availability of the CPS. Thus, not all cyber-threats can be cyber-attacks but all cyber-attacks can be cyber-threats [2].

Cybersecurity of CPS from the viewpoint of control theory studies the vulnerabilities against cyber-attacks, as well as the stealthiness of the attacker's techniques which reduce the performance and become potential dangers for the Networked Control Systems (NCS). The Stuxnet cyber-attack on an Iranian nuclear plant in 2011 [3, 4], the Maroochy water breach [5] in 2000, RQ-170 attack on an Unmanned Aerial Vehicle (UAV) in 2011 [6], and the Ukraine attack on the power distribution system in 2015 [7], are well-known events which indicated that successful cyber-attacks may have serious economic, industrial or social security impact. Therefore, the cybersecurity threats of CPS has been studied over the past two decades with topics such as resilient control strategies, secure estimation, stochastic stability analysis, and the confidentiality, integrity and availability of the data over the communication channels [8, 9, 10].

Control theory has been developing successful fault-tolerant and disturbance rejection techniques. Although they can handle cyber-threats in some sense, a fault and a cyber-attack are different. A fault is a physical event that is dependent of the system dynamics, and does not have a malicious intention like a cyber-attack does, and a cyber-attack can remain undetected, be coordinated and is not constrained by the system dynamics [11].

All the techniques in control theory which are formulated, analysed or solved based on subsets in the state-space can be referred as set-theoretic methods for control systems. The constraints, uncertainties and disturbances of the dynamic systems in the CPS can be described in terms of sets. Therefore, topics such as Lyapunov functions, invariance, constraints satisfaction, convexity, contractiveness, convergence, stability and receding horizon-based design are powerful tools to analyse cyber-attacks and design resilient controllers for CPS [12].

Model Predictive Control (MPC), a subset of set-theoretic control, is a natural candidate for addressing these challenges. It explicitly incorporates state and input constraints, predicts the future evolution over a finite horizon, applies receding horizon strategy, and integrates the concepts of terminal cost, invariant sets, regions of attractions, Lyapunov-like functions which guarantee recursive feasibility and stability [13, 14]. Complementary, robust control theory aims to guarantee stability and performance under uncertainty in the form of the following types: deterministic or stochastic, additive or multiplicative, parametric, model mismatch, external disturbances, and measurement noise [15]. Thus, the combination of both makes a powerful tool to analyse robustness guarantees (stability) with constraint satisfaction under uncertainty, and cyber-threats or cyber-attacks.

Among the various types of cyber-attacks discussed in the literature review (Chapter 2, *e.g.* [16, 17]) –including False Data Injection (FDI), bias injection, replay or rerouting– Denial of Service (DoS) is specially disruptive in NCS. Such cyber-attack, targeting the availability layer in the NCS, is often difficult to distinguish from packet losses caused by congestion, wired or wireless interference, or hardware faults, particularly when they occur intermittently. From the controller's viewpoint, the system may be forced to operate in open loop for multiple consecutive periods of time that can last from seconds to minutes, with a small fraction persisting significantly longer. During this time, uncertainties, as mentioned before, can accumulate dangerously and drive the system towards constraint violations or instability regions. Moreover, the detection mechanisms typically act on the order of seconds to a few minutes, but the may lag behind of the attack and the mitigation tools may be not sufficient (*e.g.* persistently stealthy attacks). Thus, offer the controller designer analysis of the worst-case scenario under consecutive packet losses, finite attack durations, and non-instantaneous detection and mitigation, improves the resilient design [18].

In addition, existing results on packet dropouts and intermittent communications–formulated in the framework of switching systems or stochastic NCS–typically focus on controllability, observability, or mean-square stability under probabilistic packet dropouts models (Chapter 2, *e.g.* [19, 20]). While these works provide important insights, some of them neglect input and state constraints, relying on strong assumptions such as global stability or the Separation Principle. On the other hand, in constrained MPC settings, the interaction between packet losses, disturbances, prediction horizons, terminal ingredients, and buffering strategies for mitigation can alter closed-loop stability properties. In particular, it is difficult to determine under which conditions robust stability can be guaranteed, how many consecutive packet losses can be tolerated, or whether the controller and estimator can be designed independently.

The motivation of this project is that, despite the significant amount of research that was developed around cybersecurity on CPS there is little research about stability analysis in cybersecurity from the viewpoint of constrained MPC. The challenge likely stems from the absence of a generalized set-theoretic framework and corresponding techniques for addressing key issues such as robust stability guarantees, and constraint satisfaction in resilient cyber-secure controllers. This includes controllers designed to withstand simultaneous integrity and availability attacks, as well as resilient cooperative and non-cooperative distributed controllers operating under such cyber-threats.

The focus is on DoS attacks modelled as random packet losses over the Controller-Actuator (C-A) and Sensor-Controller (S-C) channels, in the presence of bounded and unbounded uncertainty. While other types of attacks such as FDI are important, availability cyber-attacks are fundamentally relevant threats that directly challenges the stability and constraint satisfaction guarantees in MPC through worst-case scenarios such as consecutive packet losses. Furthermore, understanding (robust) stability and Separation Principle under random packet dropouts is a necessary step towards the design of resilient controllers capable of handling more complex, combined and distributed attack scenarios.

The aim of this thesis is to provide novel methods for the analysis of linear and nonlinear NCS controlled by nominal and stochastic input-constrained MPC subject to availability DoS attacks in the form of random packet losses, and uncertainty as additive bounded disturbance and unbounded noise as Additive White Gaussian Noise (AWGN). Specifically, we seek to identify: (i) conditions under which stability is guaranteed in the presence of unbounded disturbances, and random packet losses over the controller and sensor communication channels, and determine when the Separation Principle between the controller and estimator design holds or fails; and (ii) establish nominal and robust stability guarantees in the presence of bounded additive disturbances, and worst-case scenario of consecutive packet losses, highlighting the MPC design ingredients together with the buffer-based mitigation mechanism over the control communication channel.

1.2 Outline and summary of contributions

The thesis concerns the stability analysis of nonlinear and linear systems controlled by MPC technique, subject to input constraints, random packet losses over the communication channels and additive uncertainty. Stability and robustness conditions are formally presented through lemmas and theorems, with numerical examples provided to verify the proposed results.

Before the summary of contributions is presented, we outline the types of controllers, estimators, mitigation mechanisms, uncertainties, protocols and channels, and where they are considered on each chapter, see Table 1.1.

<table>
<tr><th>Chapter</th><th>Controller</th><th>Estimator</th><th>Mitigation Mechanism</th><th>Protocol</th><th>Channel</th></tr>
<tr><td>4</td><td>Stochastic MPC</td><td>Kalman Filter</td><td>Stochastic MPC plus Kalman Filter</td><td>TCP/UDP</td><td>C-A / S-C</td></tr>
<tr><td>5, 6</td><td>Nominal MPC</td><td>–</td><td>Buffer</td><td>UDP</td><td>C-A</td></tr>
<tr><th colspan="6">Uncertainty</th></tr>
<tr><th>Chapter</th><th>Source</th><th>Location (Protocol)</th><th>Stochastic/ Nondeterministic</th><th>Signal Type</th><th>Channel</th></tr>
<tr><td rowspan="2">4</td><td>Measurement noise</td><td>Sensor</td><td rowspan="2">Unbounded stochastic AWGN</td><td rowspan="3">Continuous</td><td rowspan="2">C-A / S-A</td></tr>
<tr><td>Process noise</td><td rowspan="2">Plant</td></tr>
<tr><td>5</td><td>Process disturbance</td><td>Bounded stochastic additive</td><td>C-A</td></tr>
<tr><td>4</td><td rowspan="2">Packet dropouts</td><td>Network (TCP/UDP)</td><td rowspan="2">Nondeterministic</td><td rowspan="2">Discrete (binary)</td><td>C-A / S-C</td></tr>
<tr><td>5, 6</td><td>Network (UDP)</td><td>S-C</td></tr>
</table>

Table 1.1: Controllers, estimators, mitigation mechanisms, uncertainties, protocols and channels considered throughout the thesis

In what follows we summarize each chapter.

Chapter 2 summarizes the literature of relevant research about cybersecurity on CPS from the viewpoint of control theory. Formal definitions of CPS, NCS, together with a taxonomy of cyber-attacks. A block diagram describes the location of the three generalized groups of attacks; disclosure, deception and disruption (availability). The availability DoS and FDI cyber-attacks models are presented; since the DoS attack is the primary cyber-threat used in this thesis, a block diagram indicates the location of this attack within the NCS.Other attack types, such as bias injection, replay, and rerouting attacks, are also reviewed, along with the associated detection and mitigation techniques reported in the literature. In particular, works on LQG-based control systems under DoS attacks, MPC-based control systems subject to DoS and FDI, and the combination of both, are discussed, owing to their importance in Chapters 4–6. Finally, set-theoretic control, controllability, observability and stability analysis subject to cyber-attacks are reviewed to motivate and contextualize the stability analysis of MPC schemes subject to DoS attacks.

Chapter 3 introduces fundamental definitions, concepts and results required throughout the thesis. It presents the system settings for linear, nonlinear and stochastic formulations, reviews MPC theory, Quadratic Programming (QP), Lyapunov-based stability analysis, stability conditions in MPC, and the Separation Principle theory. Most of the material in this chapter is summarized from existing literature and serves to establish notation and technical preliminaries used in the subsequent chapters.

Chapter 4 investigates an input-constrained Linear Quadratic Gaussian (LQG) problem under random packet losses in both the sensing and controller channels. In Section 4.4, a counterexample illustrates that, unlike in the unconstrained case, the Separation Principle does not hold when a Transmission Control Protocol (TCP) protocol is used in the channels. Further analysis uncovers a relationship between estimation errors and controller prediction errors, indicating that improving estimation performance can potentially worsen controller performance. In Section 4.5, we outline the conditions under which the explicit control law is Piecewise Affine (PWA) and the cost function is Piecewise Quadratic (PWQ). We identified that the effect of the probability of packet losses on the explicit PWA solution is due to the simultaneous influence on both the unconstrained and constrained solutions. By numerical examples, it is shown that the associated polyhedral partitions are affected by the probability of packet losses and penalization of the terminal cost.

Chapter 5 analyses an MPC-controlled nominal discrete-time nonlinear system subject to input constraints and random packet losses in the controller loop over a User Datagram Protocol (UDP) communication channel. The MPC formulation incorporates a terminal cost function that satisfies a local Control Lyapunov Function (CLF), but does not include explicit terminal constraints. It is shown, at best, the use of a buffer facilitates the transfer of the initial state to the terminal region associated with the CLF-based terminal cost. The number of consecutive packet losses the system can tolerate while maintaining stability is upper bounded by expressions dependent on system and controller parameters. The specialized analysis in the linear section evaluates these bounds and assesses their conservatism by approximating them using the open-loop stage-cost and terminal-cost parameters. The numerical examples illustrate that, despite the conservatism of these bounds, they can be used to fine-tune the controller's design parameters and a minimum buffer size, acknowledging that the buffer alone is insufficient to guarantee stability under consecutive packet losses.

Chapter 6 extends the analysis to MPC-controlled discrete-time nonlinear system subject to input constraints, random packet losses over a UDP-based controller channel, and bounded additive disturbances. The controller formulation is a nominal MPC which includes a terminal cost function that satisfies a local CLF without explicit terminal constraints, however, its operation in closed-loop differs from the one used in Chapter 5 because the nominal MPC aims to control an uncertain system. The buffer stores the nominal MPC optimal sequence, which is delivered to the plant during random packet losses and in the presence of disturbances. We show the conditions under which the system state remains in the region of attraction despite the combined effects of uncertainty and consecutive packet losses. When there are additive disturbances, the use of the buffer does not provide the transfer of an initial state to the terminal region, however, we analyse the case under the assumption that the buffer transferred an initial state to the terminal region if and only if the disturbance is sufficiently small. Furthermore, we derived an upper bound on the number of consecutive packet losses that can be tolerated while maintaining robust stability. For the linear case, the bounds are approximated using the open-loop stage-cost and terminal-cost parameters. Numerical examples verify the bounds for the nonlinear and linear cases. As in Chapter 5, the results suggest a minimum buffer size for tuning guideline, however, in the presence of additive disturbances, this guideline is more conservative and restrictive, and should be used strictly as a design heuristic rule.

Chapter 7 presentes a summary of the main contributions and discussions on some directions for future work.

Finally, we provide a list of publications in which the results of Chapter 4 and Chapter 5 have been published.

- **Chapter 4:** Paul Trodden, Paulo Loma-Marconi, and Iñaki Esnaola. "Stability and the Separation Principle in Output-Feedback Stochastic MPC with Random Packet Losses". In: *IFAC-PapersOnLine* 56.2 (2023), pp. 3818–3823. ISSN: 2405-8963. DOI: 10.1016/j.ifacol.2023.10.1312, [21]

- **Chapter 5:** Paulo Loma-Marconi, Paul Trodden, and Iñaki Esnaola. “Stability of Nonlinear Model Predictive Control Under Consecutive Packet Losses”. In: *2024 UKACC 14th International Conference on Control (CONTROL)*. IEEE, Apr. 2024, pp. 295–300. DOI: 10.1109/control60310.2024.10531842, [22]

Chapter 2

Literature review

Contents

The literature review presents a summary of the state of the art and relevant research in the last two decades of cybersecurity from the viewpoint of control theory and set-theoretic methods on CPS. The sections are focused on the resilient strategies and analysis techniques for LQG, MPC, Distributed MPC (DMPC) and set-theoretic methods for control. Also, there is a section covering controllability/stabilizability and observability analysis under DoS cyber-attack.

2.1 Cybersecurity on CPS

A good amount of surveys, books and papers describe the architecture of the CPS as the complex integration of different layers of computation and communication technologies with physical systems which interact with humans. The state-of-the-art paper [8] classifies around 138 selected studies of CPS from the viewpoint of automatic control. It presents a comprehensive comparison framework of actual and future research directions for both industry and academia, and classifies the literature according to the type of attack and defence (or both) strategies, security properties such as availability, integrity and confidentiality, and CPS components types such as plant, sensors, controllers, actuators, network, and detection and estimation processes. Although, there is a variety of proposed controllers such as event-triggered, Linear Quadratic Regulators (LQR), Proportional Integral Derivative (PID) and sliding mode control. MPC and set-theoretic methods are not categorized among the controllers for distributed systems.

Cybersecurity on CPS is presented as an architecture of three layers: physical layer, communication network layer and control layer. For example, the security of Wireless Sensor Network (WSN), Information Communication Technology (ICT), Supervisory Control and Data Acquisition (SCADA), and Embedded Systems can be categorized in the context of risk assessments, that is, the probability of the occurrence of events that could compromise the architecture and lead it to an unwanted outcome or performance.

Resilient strategies refer to techniques or methods to ensure a CPS continues to operate safely and maintain acceptable performance despite faults, disturbances, uncertainties, cyber-attacks or unexpected changes on the system dynamics through detection, adaptation, mitigation and recovery mechanisms [23, 24, 25]. Two types of resilient techniques are very popular among cybersecurity for CPS: (i) the reactive approach acts on the identification and mitigation of the risks after they occur, and (ii) the proactive approach predetermines the risk before they occur and mitigate them [1, 9, 26, 27]. In this thesis the reactive approach is analysed since there are some assumptions on the type of risks and the use of some techniques to mitigate them.

In [1], the cybersecurity risks of Distributed Control Systems (DCS) for CPS are listed as follows: integrity, confidentiality, availability, and authentication and validation. And in [28], the authors mentioned five stages a resilient control design for distributed CPS should have in order recognize and respond to known and unknown threats which are recon, resist, respond, restore and recover. However, from control theory point of view they cover only the surface.

Other relevant literature [29, 11, 30] which give an overview of modern and recent advances and proposes frameworks for network threats such as signal sampling, data quantization, communication delay, packet dropouts, access constraints, channel fading, power constraints, and methodologies for attack and defence.

A comprehensive survey of recent literature in NCS is presented in [31]. The authors develop an attack space model to illustrate how adversarial resources are allocated in common cyber-attacks. The paper reviews three types of attacks: FDI, DoS, and replay, and their corresponding detection and mitigation strategies, including residual-based anomaly detection, robust and resilient controller design.

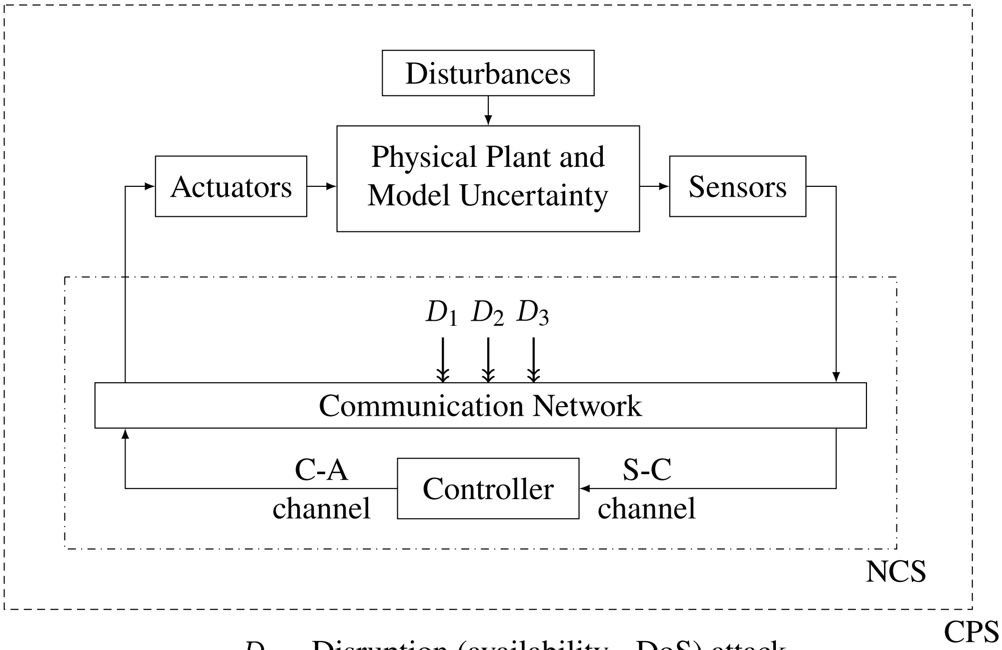
A framework to estimate the impact of cyber-attacks on stochastic linear NCS is studied in [17]. By using a detector stealthiness constraint based on the Kullback–Leibler-divergence, the authors define two impact metrics that evaluate the probability that the critical states leave a safety region and the expected value of the infinity norm of the critical states. Moreover, they derive convex optimization bounds to compute worst-case impact under some types of attacks such as FDI, DoS, replay, and bias injection.

Another related work is [16], the work presents a framework for evaluating the impact of various cyber-attack strategies on NCS protected by an anomaly detector. It considers more attacks such as DoS, FDI, sign alternation, rerouting, replay, and bias injection. In order to quantify the attack impact, the infinity norm of critical states after a fixed number of time steps is used.

From the viewpoint of stochastic MPC for CPS, [32] is a review paper where resilient stochastic MPC and resilient DMPC strategies are revised for both non-linear and linear CPS. The review highlights the importance of adequate probability distribution formulation of the cyber-attacks, and the application of probabilistic constraints which allows constraint violations caused by uncertainties or attacks are beneficial for the resilience of the formulation. Technologies that could handle cyber-attacks such as tube-based MPC and scenario-based approach for linear systems, and generalized polynomial chaos expansions (gPCE) and the Gaussian-mixture (GM) approximation approaches for non-linear systems are described, see the references therein [33, 34, 35, 36].

2.2 Availability and integrity cyber-attacks

In CPS, three distinctive groups are classified: disclosure (confidentiality), deception (integrity), and disruption (availability) attacks, also called DDD attacks [11]. These can be located over the C-A and S-C channels, and they can be independent of each other, mixed, and triggered at the same time Fig. 2.1.



D_1 = Disruption (availability - DoS) attack
D_2 = Deception (integrity - FDI) attack
D_3 = Disclousure (confidentiality) attack

Figure 2.1: Location of DDD attacks.

A confidentiality attack access the communication channels and obtains information about the NCS [37, 11], the other two types of cyber-attacks are revised in the next sections where we focus on integrity and availability attacks.

2.2.1 DoS attack models

A DoS attack randomly prevents the successful data delivery over the communication channels that potentially derive the CPS to an unstable region [38, 39]. They can be dangerous due to the packet loss in the communication channels, and they do not need knowledge about the system dynamics.

Fig. 2.2 shows the DoS attack location over the communication channel, where the packet dropouts can be represented by the switch symbols. The figure also shows that the controller and estimator may or may not have knowledge about the packet dropouts on both channels, this is represented by dashed lines.

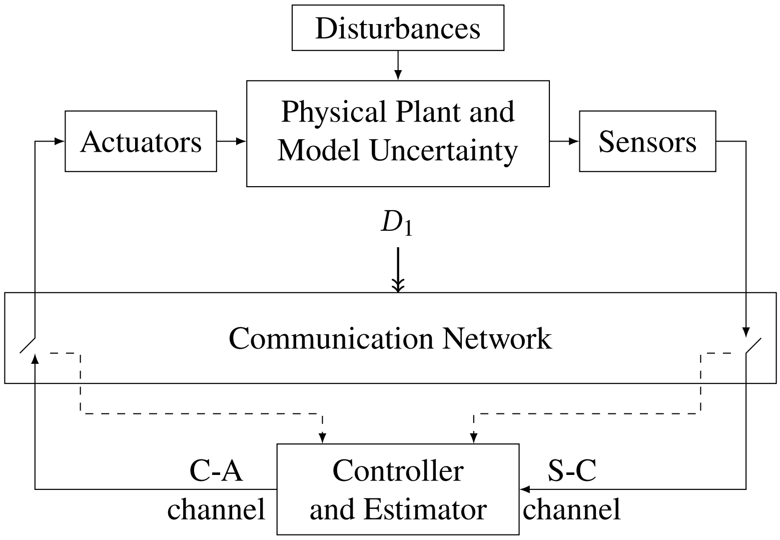


Figure 2.2: Location of the DoS attack and packet dropouts.

It is important to establish the difference between a DoS attack and a packet dropout. The DoS attack is a general threat that describes strategies, methods or mechanisms intentionally designed to induce unsuccessful data delivery over the communication channels. The packet dropouts are the consequences of these unsuccessful data deliveries. Therefore, a DoS attack can be analysed from the perspectives of the attacker or the system under attack. In this thesis, we focus on the analysis of the attacked system under to packet dropouts.

The following references consider the DoS attack model described previously. In [40], the authors provide an overview of known and recent research approaches of the DoS attacks models on NCS such as deterministic constraint model, game-theoretic optimization model, and probabilistic model for multi-hop networks. A simple probabilistic model that captures the uncertainty of the DoS attack is a Bernoulli process [41], and a complex model is a Markov process [42, 43]. There are other models such as Pulse Width Modulation (PWM) signal model [44], time delay models [45], and Signal to Interference plus Noise Ratio (SINR) model [46, 47].

Another game-theoretic work related to DoS models is [48], the authors introduce Informational Denial-of-Service (IDoS) attacks, which overwhelm human operators with deceptive alerts to exploit cognitive limitations and obscure genuine attacks. It formalizes IDoS using semi-Markov models and proposes attention-management strategies that selectively highlight alerts to mitigate risk. Data-driven evaluations and a case study show that controlled, intentional inattention can minimize overall security risk.

2.2.2 LQG subject to DoS attacks

Given the relevance of the following works to Chapter 4, some of their technical aspects are discussed in what follows.

Kalman filtering, optimal control law and Separation Principle

The authors in [41] propose a stochastic optimal LQG controller applying a modified time-variant Kalman Filter (KF) based on the results in [49, 50]. The resultant framework is able to handle Bernoulli packet losses of a LQG output-feedback problem for a stochastic LTI system subject to AWGN in both S-C and C-A channels under TCP and UDP protocols. No input or states constraints are considered.

The main results are highlighted as follows:

- The derivation of optimal estimation schemes for both TCP and UDP protocols based on time-variant KF theory.
- The proposition of optimal control laws for both protocols using dynamic programming technique.
- The conditions under which the Separation Principle holds for the TCP case and does not hold for the UDP case.
- The derivation of stability conditions for both TCP and UDP protocols based on the largest eigenvalue of the dynamic matrix and the probabilities of successful packet delivery.

Complementary, a two-variable optimization stochastic LQG compensator which encloses the zero-input and hold-input attack under TCP protocol is presented in [51]. The solution can be viewed as a general optimization problem of the LQG formulation presented in the previous papers but only for the TCP case.

Stability analysis and constrained LQG

In [52], stability is analysed for the same stochastic LQG framework, *i.e.* subject to AWGN and without constraints. The objective is to mean-square stabilize the system while minimizing a quadratic performance criteria under Bernoulli packet dropouts. The authors perform the analysis for both TCP and UDP protocols.

The work on [53] formulates a stochastic LQG problem with power constraints and safety constraints imposed on the attacker actions. Power constraints are defined as state and input constraints in an expected sense that limit the energy of state and input at each time step, and safety constraints are defined as both state and input constraints in a probabilistic sense that the state and input remain within specified hyperplanes.

Optimal estimation

From the viewpoint of optimal linear estimation only, Kalman filtering with a probabilistic perspective is considered in [54]. Instead of using the asymptotic behaviour of the expectation of the error state covariance matrix as a prior performance metric, as in [49], the bounded probability of the error state covariance matrix can be viewed as a posteriori performance metric.

In [55], a Linear-Minimum-Variance (LMV) filter using the orthogonality principle is proposed to give a solution in terms on Lyapunov equation. The problem formulation is a stochastic LTI systems subject to Bernoulli packet dropouts, and AWGN. No constraints are considered. Sufficient condition for the convergence of the steady-state LMV filter is also given.

2.2.3 MPC subject to DoS attacks

In [56], the authors review some resilient MPC dividing them by category: lossy communication, transmission delays, and resource awareness. The authors review the advantages of implementing MPC to tackle cyber-attacks by showing examples of time-triggered, event-triggered and self-triggered control laws in deterministic and stochastic fashion with efficient implementations on embedded systems. Time-triggered control also known as sampled-data control is where at fixed sampling time the sensors measurements are sent to the controller, the controller calculates a new input value and is sent to the actuators. However, the event-triggered MPC and self-triggered MPC methods are proposed as robust control designs that requires the solution of three optimization problems at each time instant which can be computationally inefficient. Therefore, for some cases, time-triggered control is still a desirable approach at the expense of transmission delays due to the synchronized clocks for the controller, sensor and actuator.

Buffer mechanism integration

It is in the interest of this thesis to discuss the following related works. Some time-triggered control propositions are presented as a packetized MPC for non-linear systems with integrated logic selector buffer and *limited consecutive* Bernoulli packet dropouts over the C-A channel [57]. The buffer is set before the actuator and acts as a firewall protection: if there is a missing input signal the buffer sends the previous valid input signal to the actuator, thereby preventing the plant going on unstable open-loop dynamics Fig. 2.3.

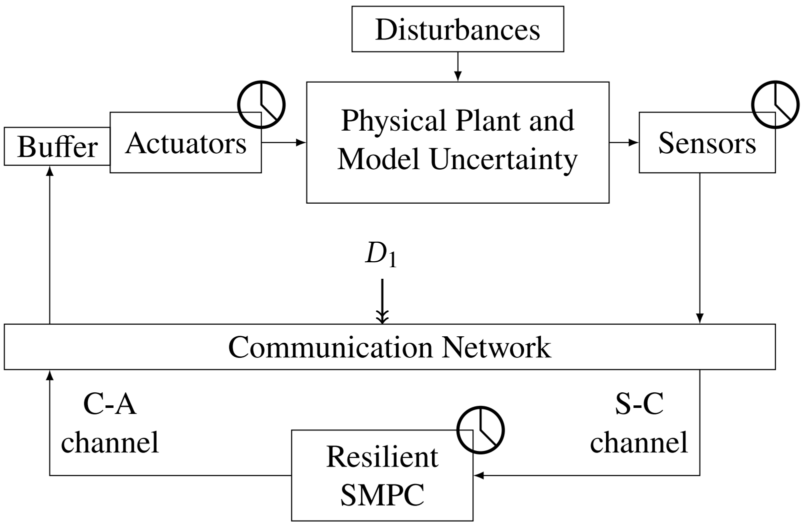


Figure 2.3: Time-triggered MPC with smart buffering. It is assumed the synchronized clocks for the controller, sensors and actuators.

Extensions of this approach that incorporates stochastic stability notions and allow for *unlimited consecutive* Bernoulli packet dropouts are studied in [58, 59]. The analysis of robust stochastic stability under Markovian packet dropouts model–a more general formulation than the Bernoulli model–with unlimited dropout length is addressed in [42]. Similar to earlier works, the authors present bounds under which stochastic stability can be guaranteed.

Finally, [60] uses a stochastic MPC formulation based on an input parametrized model. The main difference with respect to the aforementioned problem settings is that the authors consider ACK of the packet dropouts available to the controller, along with uncertainties. This results in a more robust analysis and improves or extends some results previously reported by the same authors.

Other techniques are listed as follows: controller design based on an augmented closed-loop dynamics subject to logarithmic data quantization (packetized data) errors and packet dropouts [61], an extension of the latter with Markov dropout model [62], and extending the previous to non-linear systems, [63] approach uses a Tagaki-Sugeno fuzzy MPC controller and closed-loop stability guarantee by solving Linear Matrix Inequalities (LMI).

A packet-based MPC with C-A and S-C packet losses for non-linear systems is shown in [64]. The formulation is an extension of [59], a buffer in the actuator and another buffer in the estimator. The stochastic stability conditions are generalization forms presented in [60]. Other formulation in [65] is an output feedback stochastic MPC with additive disturbances and noisy measurements, the minimization is subject to a discounted sum of second moments of an auxiliary output.

2.2.4 FDI attack models

An integrity attack, called FDI attack, corrupts the S-C and C-A communication channels by introducing false data which drive the NCS to an undesired state with little knowledge of the plant [66, 67, 68]. The traditional approach relies on data encryption to protect the S-C channel, while the latter technique depends on identification methods, see [69, 26] for extensive review.

An objective of the FDI attack is to maximize the damage while remains undetectable. The authors in [67] analyse an FDI attack for linear systems with LQG controller by assuming there is a packet dropout detector in the S-C channel. Sufficient conditions are proved under which the attack destabilizes the system while avoiding the detection. Attacks on the state estimation process is studied in [68], the authors prove that the residual detection technique is not enough when the attacker has knowledge about the measurement data. Similar to [68], the time-invariant state estimator proposition in [70] finds conditions under which FDI attacks are undetectable.

2.2.5 MPC subject to FDI attacks

In [10] is studied the FDI attacks with delay transportation and disturbances. The authors describe secure networked predictive control for UDP channel based on a secure UDP receiver that verifies the integrity of the information. However, this method uses a secure encryption UDP algorithm.

In [71], an FDI attack detection and rejection method based on MPC is proposed. The false data is injected into the control variable, if the attack is detected, a traditional state-feedback backup controller takes over the control of the CPS. The optimal solution of the proposition is obtained by the integration of the main controller and the backup controller into a single MPC controller. If there is no attack, the optimal solution is the main controller, if there is an attack, the optimal solution is obtained by the constraints of both dynamics.

In [72], the authors propose a moving-window attack-detection based on MPC. The formulation uses past samples data of the corrupted states and it compares them with the nominal past samples in a temporal sliding window. The recorded data of the moving-window is used to obtain a probabilistic performance index which guarantees the information integrity and make a decision about the existence of the attack. The decision is a hypothesis test over the window of observation.

2.2.6 MPC subject to simultaneous DoS and FDI attacks

There not too much work related with the design and analysis of resilient MPC for CPS, but the following are worth mentioning. In [73], it is proposed a secure observer-based controller for linear systems subject to DoS and FDI attacks. Sufficient conditions under which the observer-based controller is guaranteed using stochastic analysis. The controller gain is obtained applying LMI technology. The extension of the previous work is presented in [74], it includes noisy measurements and actuation time-delays. The solution is also obtained by LMI.

In [75], the authors develop an observer-based fuzzy MPC scheme for uncertain discrete-time nonlinear NCS under FDI and DoS, using interval type-2 Takagi–Sugeno fuzzy theory. A secure observer resilient to FDI attacks and a fuzzy MPC algorithm ensuring recursive feasibility are proposed, with simulations demonstrating effectiveness.

In DMPC systems it is common to assume the state measurements are available all the time for all the subsystems, but in attacked DMPC systems the loss of communication is no more guaranteed. For DMPC systems, various algorithms have been proposed to solve problems of maintenance and tolerance to faults, such as the type of local or global cost function, non-iterative or single-iterative solution procedure, and the degree of information exchange between the subsystems. However, DMPC under cyber-attacks can be very dangerous, as an example, in [76], a cooperative-based DMPC scheme composed by nonconforming local controllers are submitted under two types of FDI attacks: fake weights attacks and fake constraints attacks. The following references propose formulations to countermeasure DoS and FDI attacks in DMPC.

In [77], a robust cooperative DMPC algorithm using an observer within each subsystem is presented. The observer is continuously and recursively estimating the bounds of the states for all the plant states which are used as extra state constraints within a LMI formulation to calculate the control gains. An iterative DMPC formulation wide area measurement power systems based in cooperative control strategy subject to FDI attacks is proposed in [78]. The FDI attacks are described as delayed input states, and sufficient condition under which the closed-loop system stability is guaranteed is obtained by using Lyapunov LMI approach.

In these works [79, 80], the DMPC formulation is subject to DoS attacks, it solves a min-max worst-case optimization problem considering the packet dropouts probabilities and input constraints. Also, recursive feasibility and closed-loop mean-square stability are proved. A resilient stochastic DMPC subject to DoS attacks and input saturation constraints is presented in [81], the authors characterized the DoS attack as a Bernoulli-distributed white sequence which provides a trade-off between input saturation constraint satisfaction and control performance. Recursive feasibility and closed-loop stability are proved.

A secure distributed output-feedback MPC framework for leader-following consensus in disturbed linear multi-agent systems facing simultaneous FDI and DoS attacks is presented in [82]. The scheme integrates a robust multivariate observer, a DMPC controller, and actuator buffering to maintain recursive feasibility and achieve consensus.

2.2.7 Set-theoretic control for CPS under cyber-attacks

Applied set-theoretic methods is presented in [83], the authors present a stochastic MPC with information sets divided in several subsets such that the information set with all possible packet losses, the infinite-horizon and the finite-horizon information set help to establish a relationship between the infinite-horizon quadratic performance index and the prediction horizon performance index. A dual-mode linear MPC formulation subject to input and state constraints, an attack-resilient terminal set and stability analysis is shown in [84], the authors propose conditions under which the constraints are satisfied and exponential stability is guaranteed by the duration of the DoS attacks and MPC prediction horizon.

In [85] is presented a set-theoretic control framework subject to DoS and FDI attacks, an anomaly detector module and a resilient pre-check/post-check module, and the scheme ensures uniformly ultimate boundedness and constraints satisfaction. The authors in [86] analyse a constrained linear system subject to FDI attack from the viewpoint of set-theoretic analysis where the control input plus the attacked control input satisfies the hard input constraints. It is proposed condition under which it is not possible to satisfy state constraints based on the spectral radius of the system, the relative sizes of the input and state constraints, and the portion of the input constraint set that is allowed to the cyber-attack.

The work in [87] addresses a class of perturbed Linear Time-Invariant (LTI) systems and introduce two security metrics that quantify the potential impact of stealthy attacks on the sensor measurement loop. Analytical tools (based on reachable sets theory) are provided to evaluate these metrics based on given system dynamics, control strategies, and monitoring mechanisms. Synthesis methods, formulated as semidefinite programs, to redesign controllers and monitors that minimize the effects of stealthy attacks while ensuring the desired performance of the system in the absence of attacks.

A framework for modelling attack scenarios in cyber-physical control systems by representing them as constrained linear switching systems is presented in [88]. The authors embed in a single model the dynamics of the physical process, attack patterns, and attack detection mechanisms, more specifically FDI attacks on both control and sensor loops. The framework accommodates a broad class of non-deterministic attack strategies and enables system safety to be characterized as an asymptotic property. By computing the maximal safe set, the proposed impact metrics provide an intuitive quantification of safety degradation and the influence of cyber-attacks on system safety.

In the domain of hybrid CPS, a system under dwell-time constraints can be modelled as a constrained switching system. A dwell-time constraint enforces or imposes a restriction on how fast the switching process is allowed from one mode to another, in other words, the minimum time duration a system must remain in a given mode before switching to the other [89]. Under this perspective, packet losses in NCS can be modelled as a constrained switching system with minimum and maximum dwell time properties. In [90], the authors present combinatorial methods for hybrid systems based on automata theory, in which this general methodology is used for invariant sets computation for systems with dwell-time constraints.

2.3 Controllability, observability and stability analysis subject to cyber-attacks

Since the study of controllability, observability and stability analysis in NCS subject to cyber-attacks are related with the title of this thesis, some relevant works are presented in the following subsection. Moreover, a subsequent subsection is dedicated to stability of MPC.

2.3.1 Controllability and observability subject to DoS attack

A series of works develop a unified framework for analysing fundamental system-theoretic properties of discrete-time linear NCS subject to packet losses. In [91], the work focuses on controllability under packet dropouts, modelling the loss patterns with an automaton that constrains the number of allowable consecutive losses, and reformulating the problem as a constrained switching system that is algebraically characterized and algorithmically feasible through a connection to Skolem's theorem.

Extending this methodology, [92] addresses observability under similar packet loss constraints, demonstrating that the controllability framework directly generalizes to observability and remains valid even for non-invertible system matrices, thereby improving upon the earlier results. Finally, the authors in [20], generalize these ideas by providing algorithmically verifiable necessary and sufficient conditions not only for observability and controllability, but also for properties such as reachability, detectability, and stabilizability, unifying them through algebraic relations and Skolem's theorem while also establishing connections to models with time-varying delays in wireless control networks.

The work in [93] investigates the problem of quantized output-feedback stabilization in NCS subject to DoS attacks. The authors first consider the case where the average duration and frequency of DoS attacks are bounded and an initial estimate of the plant state is available, proposing an output encoding strategy that ensures exponential convergence with finite data rates. They further demonstrate that an appropriate state transformation can eliminate the assumption on attack frequency. Finally, it derives bounds on the system state under DoS conditions and establishes sufficient conditions on attack duration and frequency that guarantee Lyapunov stability of the closed-loop system.

2.3.2 Stability of MPC subject to DoS attack

The conditions for stability guarantee based on a known upper bound of the attack and the spectral radius of the dynamic matrix of a linear system by means of mean-square stability analysis is studied in [94]. As an extension of [59], in [95] the authors present Input-to-State Stability (ISS) conditions such as terminal control law, Lipschitz continuity, robust positive invariance for closed-loop stability with bounded consecutive packet dropouts. The authors in [61] present a notion of closed-loop stability of a linear system with input and state constraints applying LMI stability analysis for packet loss and data quantization error.

A Lyapunov function based on conditions similar to [95] is presented in [96] to ensure regional Input-to-State practical Stability (ISpS) analysis for non-linear NCS. In [97, 38], the authors established conditions to guaranteed feasibility and ISpS based in the attack duration, disturbance bound and prediction horizon.

Similar to [94], the authors in [98] studied the conditions of closed-loop stability guarantee using an augmented model to bound the maximum dropout rate before a linear system goes to unstable mode. An output quantized feedback with encoded scheme is presented in [93], the authors encodes the data flow of the S-A and C-A channels to remove the DoS the maximum packet losses frequency to guarantee Lyapunov stability for the closed-loop system.

Another stability analysis is studied in [19], a system operating under time-varying transmission intervals, time-varying delays, and strict communication constraints. The proposition employs node analysis where only one node may transmit at a time according to scheduling protocols such as Round-Robin (RR) or Try-Once-Discard (TOD). The authors extend these classical protocols to new "periodic" and "quadratic" variants, and propose a unified modelling framework for linear plants and controllers using discrete-time switched linear uncertain systems, accommodating both continuous- and discrete-time controllers. To address the stability analysis over a range of transmission intervals and delays (including those with a nonzero lower bound), they introduce a procedure for constructing a convex polytopic overapproximation with tight norm-bounded uncertainty.

2.4 Summary and research opportunities

Most of the resilient controller and resilient estimator propositions presented in the literature review relies on the stochastic modelling of the cyber-attacks, *i.e.* Bernoulli and Markov processes. Especially for the availability and integrity attacks, the filtering theory is extensively used to detect or reject attacks, and the controller design theory is mostly used to propose resilient controllers. In most of the cases, for both S-C and C-A channels, the resilient formulations are different. The controller and estimator need to be design separately, but the downside is that the CPS scheme has to be studied under one type of cyber-attack, that is why there are not so many works mixing different types of attacks, *e.g.* both FDI and DoS. The following were identified in the literature review.

Control design: For the CPS in Fig. 2.1, subject to disturbances, noisy measurements and stochastic uncertainties, the resilient (stochastic) controller is designed to satisfy the following requirements:

- Handle threats or attacks in both S-C and C-A channels.
- Closed-loop stability guarantee under different types of threats or attacks.
- Ensure closed-loop constraints satisfaction.

Stability analysis: The stability analysis can be studied from the viewpoint of Lyapunov analysis and stochastic (mean-square) analysis. The conditions for linear and non-linear CPS are continuity and positive invariance satisfaction. Most of the propositions do not account closed-loop stability with input and state constraint satisfaction, most of them only account the input constraints, and use the duration of the DoS attack as upper bound to ensure convergence. From the viewpoint of set-theoretic design and analysis two techniques are used:

- Resilient terminal set with resilient terminal cost.
- Stochastic tube MPC to deal with DoS attacks by the addition of extra constraints to guarantee recursive feasibility and quadratic stability.

In summary, it is still theoretically challenging to propose a unified resilient stochastic set-theoretic (MPC-based) control framework with recursive feasibility and closed-loop stability guarantees due to the complex integration of the attack detection, resilient control and resilient estimation. Some identified challenges are still open according to the literature review:

- Robust stability analysis of nonlinear and linear system under MPC-controlled systems with constraints satisfaction.
- Integrate zero-input DoS attack and hold-input DoS attack models in a CPS with asynchronous clocks in the controllers, sensors and actuators.
- Propose secure sensor fusion framework for multiple sensor channels.
- Formulate set-theoretic information approach to model invariant sets for DMPC.
- Resilient controllers with attack detection of simultaneous availability and integrity attacks.
- Recursive feasibility guarantees under different types of threats or attacks.
- Ensure closed-loop probabilistic or chance constraints satisfaction, *i.e.* handling uncertainty by requiring constraints to be satisfied with *high probability* rather than deterministically [99].
- Use of hybrid systems and automata theory frameworks to model DoS attacks as dwell-time constraints for constrained switching systems.

Given the summary and research opportunities identified above, this thesis addresses the problem of robust closed-loop stability analysis for linear and nonlinear systems under the DoS cyber-attack, in the form of random packet losses affecting both actuation and sensor channels. State constraints are not explicitly considered, since guaranteeing recursive feasibility jointly with closed-loop stability remains technically challenging and is still an open problem, as discussed in the literature review. Instead, we focus the effort on the non-trivial integration of input constraints into the analysis of robust closed-loop stability and the conditions under which the Separation Principle holds or fails, in the presence of DoS attacks. The resulting analysis has the objective to provide methods that may facilitate the systematic inclusion of both state and input constraints in future works on resilient MPC frameworks.

Chapter 3

Background Theory

Contents

This chapter presents the basic definitions and results on MPC theory, QP, Lyapunov stability for MPC, and Separation Principle. It serves as an introduction to material that is related in the subsequent chapters. Most of the definitions and results can be found in the literature.

In what follows, the notation $x_{k+j|k}$ is the j-step ahead prediction of x, and $x_{k|k}$ is the value of x at time $k \in \mathbb{N}_{\geq 0}$. x^+ is shorthand for x_{k+1} when $x = x_k$.

3.1 System setup

The thesis considers the nonlinear, linear and linear stochastic MPC formulation. The linear stochastic system is used in Chapter 4, the linear and nonlinear systems are used in Chapter 5, and the nonlinear stochastic system is used Chapter 6.

3.1.1 Nonlinear system

Consider the following discrete-time nonlinear system

$$x_{k+1} = f(x_k, u_k) \tag{3.1a}$$

$$y_k = g(x_k). \tag{3.1b}$$

3.1.2 Nonlinear stochastic system

Consider the following discrete-time nonlinear system with additive uncertainty

$$x_{k+1} = f(x_k, u_k) + w_k \tag{3.2a}$$

$$y_k = g(x_k) + s_k. \tag{3.2b}$$

3.1.3 Linear system

The following discrete-time linear system

$$x_{k+1} = Ax_k + Bu_k \tag{3.3a}$$

$$y_k = Cx_k. \tag{3.3b}$$

3.1.4 Linear stochastic system

And the following discrete-time linear system with additive uncertainty

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{3.4a}$$

$$y_k = Cx_k + s_k. \tag{3.4b}$$

For all the cases, $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, and $y_k \in \mathbb{R}^p$ are, respectively, the state, input, and output of the system at sample time $k \in \mathbb{N}_{\geq 0}$. The system input, u_k, is constrained to take values in a set $\mathcal{U} \subset \mathbb{R}^m$ but the states and outputs are unconstrained for all the settings. The nonlinear and linear stochastic system are subject to uncertainty in the form of process noise w_k and measurement noise s_k.

We remark that for all the cases, the output y_k is defined for completeness but is not used in Chapters 5 and 6.

Also, some assumptions are introduced.

Assumption 3.1 (Continuity of the system)**.** *The functions* $f(\cdot,\cdot)$ *and* $g(\cdot)$ *are continuous and satisfies* $0 = f(0,0)$ *and* $0 = g(0)$.

Assumption 3.2. *The matrices* A, B *and* C *are known, the pair* (A,B) *is stabilizable, and the pair* (C,A) *is observable.*

Assumption 3.3. *The set* $\mathcal{U}$ *is known and compact, containing the origin in its interior.*

Assumption 3.4 (Noise for the linear stochastic system)**.** *The process noise* $w_k \in \mathbb{R}^n$ *and measurement noise* $s_k \in \mathbb{R}^p$ *are independent and identically distributed (i.i.d.) random variables, with* $w_k \sim \mathcal{N}(0, Q_w)$ *and* $s_k \sim \mathcal{N}(0, R_s)$.

Assumption 3.5 (Noise for the nonlinear stochastic system)**.** *The process noise* w_k *and measurement noise* s_k *are bounded additive disturbances and can take any values in the sets* $\mathcal{W} \subset \mathbb{R}^n$ *and* $\mathcal{S} \subset \mathbb{R}^p$. *The sets* $\mathcal{W}$ *and* $\mathcal{S}$ *are compact, and contain the origin in their interiors.*

Assumption 3.6. *Let* $\mathcal{I}_k := \{\mathbf{y}_k\}$ *denote the information set available to the controller at time*

$k \in \mathbb{N}_{\geq 0}$, where $\mathbf{y}_k = \{y_k, y_{k-1}, \cdots, y_1\}$. Additional vectors may be included in $\mathcal{I}_k$ as required by the problem formulation.

3.2 Model Predictive Control (MPC)

MPC is an optimization-based control strategy that solves a Finite-Horizon Linear Quadratic (FH-LQ) problem subject to constraints. A model of predicted states based on the system dynamics is used to minimize an objective function that satisfies constraints given the actual state. The obtained solution is a sequence of open-loop controls from which only the first one is implemented on the system, inducing feedback. The rest of the sequence is discarded and the process is repeated for the next sampling time, this method is often called Receding Horizon (RH) principle. Advantages of MPC against other control strategies are the handling of constraints, and the system's nonlinearities [100, 101].

3.2.1 Optimal control problem

The general optimal control problem for MPC includes the state and terminal state constraints. In this work we omit the use of both since our objective of stability analysis requires input constraints only. Under this setting, recursive feasibility of the optimization problem is trivially guaranteed, as no state constraints are imposed and the admissible control law is defined over the entire state space. Therefore, if the problem is feasible at the initial time, it remains feasible at all subsequent time steps. However, we still need terminal ingredients to guarantee stability, this is exposed in a later section.

The following definitions and assumptions are required before properly presenting the problem.

Definition 3.1 (Cost function)**.** *Let $J_N(\cdot,\cdot) : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}_{\geq 0}$ denote the cost function defined as*

$$J_N(x_k, \mathbf{u}_k) \coloneqq \sum_{j=0}^{N-1} \ell(x_{k+j|k}, u_{k+j|k}) + V_f(x_{k+N|k}), \tag{3.5}$$

where $N \in \mathbb{N}_{\geq 0}$ is the horizon length, $\ell(\cdot,\cdot)$ is the stage cost and $V_f(\cdot)$ is terminal cost functions.

Definition 3.2. *The decision variable of the optimal problem is the finite sequence of future control input,* i.e.

$$\mathbf{u}_k \coloneqq \left\{u_{k|k}, u_{k+1|k}, \ldots, u_{k+N-1|k}\right\}. \tag{3.6}$$

Definition 3.3. *The stage cost function* $\ell(\cdot,\cdot)\colon \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}_{\geq 0}$ *and terminal cost function* $V_f(\cdot)\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *are defined as*

$$\ell(x,u) = \|x\|_Q^2 + \|u\|_R^2 \tag{3.7a}$$

$$V_f(x) = \|x\|_{Q_f}^2, \tag{3.7b}$$

where $Q \in \mathbb{R}^{n\times n}$ *and* $R \in \mathbb{R}^{m\times m}$ *are the state penalty, and input penalty matrices respectively, and* $Q_f \in \mathbb{R}^{n\times n}$ *is the terminal penalty matrix, and* $Q \succ 0$, $R \succ 0$, *and* $Q_f \succ 0$.

Definition 3.4. *The value function* $V_N^0(\cdot)\colon \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}_{\geq 0}$ *of the optimal problem is*

$$V_N^0(x_k) = \min_{\mathbf{u}_k \in \mathbb{U}} J_N(x_k, \mathbf{u}_k). \tag{3.8}$$

Assumption 3.7. *The constraint set is*

$$\mathbb{U} := \mathcal{U} \times \cdots \times \mathcal{U}. \tag{3.9}$$

Now, we are in the position to present the optimal control problems for the different settings that are used in the Chapters 4, 5 and 6.

In what follows we say $\mathbb{P}_N(x_k)$ is the optimal problem to be solved, and $x_{k+j|k}$ is the j-step ahead prediction of x made at time $k \in \mathbb{N}_{\geq 0}$

3.2.1.1 Nominal nonlinear MPC

Before we proceed, we must remark that the following control problem is used for both the nonlinear (Section 3.1.1) and nonlinear stochastic (Section 3.1.2) systems, which are presented in the Chapters 5 and 6, respectively.

Given the system (3.1) (or (3.2)) at a state x_k, the problem is to determine the optimal control law such that the state x_k is transferred to (nearby) the origin, subject to the input constraint set (3.9) (and uncertainty given by Assumption 3.5), while minimizing the cost function (3.5) through the following optimal control problem.

$$\mathbb{P}_N(x_k): \quad V_N^0(x_k) = \min_{\mathbf{u}_k \in \mathbb{U}} J_N(x_k, \mathbf{u}_k), \tag{3.10a}$$

subject to, for $j \in \mathbb{N}_{[0,N-1]}$,

$$x_{k|k} = x_k \tag{3.10b}$$

$$x_{k+j+1|k} = f(x_{k+j|k}, u_{k+j|k}) \tag{3.10c}$$

$$u_{k+j|k} \in \mathcal{U}. \tag{3.10d}$$

3.2.1.2 Nominal linear MPC

Similarly, given the system (3.3) at a state x_k, the problem is to determine the optimal control law such that the state x_k is transferred to the origin, subject to the input constraint set (3.9), while minimizing the cost function (3.5) through the following optimal control problem.

$$\mathbb{P}_N(x_k): \quad V_N^0(x_k) = \min_{\mathbf{u}_k \in \mathbb{U}} J_N(x_k, \mathbf{u}_k), \tag{3.11a}$$

subject to, for $j \in \mathbb{N}_{[0,N-1]}$,

$$x_{k|k} = x_k \tag{3.11b}$$

$$x_{k+j+1|k} = Ax_{k+j|k} + Bu_{k+j|k} \tag{3.11c}$$

$$u_{k+j|k} \in \mathcal{U}. \tag{3.11d}$$

3.2.1.3 Stochastic linear MPC

Finally, given the system (3.4) at a state x_k, the problem is to determine the optimal control law such that the state x_k is transferred to a neighbourhood of the origin, subject to the input constraint set (3.9) and uncertainty (Assumption 3.4), while minimizing the cost function (3.5) through the following optimal control problem.

$$\mathbb{P}_N(x_k): \quad V_N^0(\mathcal{I}_k) = \min_{\mathbf{u}_k \in \mathbb{U}} \mathbb{E}\{J_N(x_k, \mathbf{u}_k) \,|\, \mathcal{I}_k\}, \tag{3.12a}$$

subject to, for $j \in \mathbb{N}_{[0,N-1]}$,

$$x_{k|k} = x_k \tag{3.12b}$$

$$x_{k+j+1|k} = Ax_{k+j|k} + Bu_{k+j|k} \tag{3.12c}$$

$$u_{k+j|k} \in \mathcal{U}. \tag{3.12d}$$

We need to remark that all the optimal control problems do not have state, output and terminal constraints.

Definition 3.5 (Optimal solution)**.** *Solving* $\mathbb{P}_N(x_k)$ *at* x_k *yields the solution*

$$\mathbf{u}_k^0(x_k) := \{u_{k|k}^0, u_{k+1|k}^0, \ldots, u_{k+N-1|k}^0\}. \tag{3.13}$$

The application of the first control in the optimal sequence to the plant, followed by a repetition of the whole process at the next sampling instant, defines the implicit control law.

Definition 3.6 (Optimal (implicit) control law)**.** *The optimal (implicit) control law for the optimal problem* $\mathbb{P}_N(x_k)$ *is defined as*

$$u_k = \kappa_N(x_k) := u_{k|k}^0. \tag{3.14}$$

3.2.2 Linear Quadratic Programming (QP) problem

The linear QP problem formulation is a widely used technique for solving MPC problems. Its primary advantage lies in its direct compatibility with QP solvers for online implementation, as well as its ability to provide an explicit solution for offline implementation [102]. This compact, dense formulation reduces the number of decision variables and constraints, making it well-suited for active-set methods. However, it tends to be numerically ill-conditioned, particularly for unstable systems. Therefore, while the problem size is smaller, performance may degrade for large N, leading to significant computational costs.

An alternative approach is the sparse QP formulation, which retains state predictions within the decision variables. Although both formulations yield the same theoretical solution, their practical performance differs considerably. The sparse formulation results in a larger problem but is better conditioned, making it more suitable for interior-point methods. However, for small values of N, this advantage may be negligible [103].

First, we need to restate the nominal linear MPC as defined in Section 3.2.1.2. By using the quadratic norms for the stage and terminal costs defined in Definition 3.3, and using the cost function (3.1), the optimal control problem $\mathbb{P}_N(x_k)$ for the nominal linear MPC can be formulated as

$$\mathbb{P}_N(x_k): \quad V_N^0(x_k) = \min_{\mathbf{u}_k \in \mathbb{U}} J_N(x_k, \mathbf{u}_k) \tag{3.15a}$$

$$\begin{aligned} J_N(x_k, \mathbf{u}_k) &= \sum_{j=0}^{N-1} \left(\left\| x_{k+j|k} \right\|_Q^2 + \left\| u_{k+j|k} \right\|_R^2 \right) + \left\| x_{k+N|k} \right\|_{Q_f}^2 \\ &= \sum_{j=0}^{N-1} \left(x_{k+j|k}^\top Q x_{k+j|k} + u_{k+j|k}^\top R u_{k+j|k} \right) + x_{k+N|k}^\top Q_f x_{k+N|k}, \end{aligned} \tag{3.15b}$$

subject to, for $j \in \mathbb{N}_{[0,N-1]}$,

$$x_{k|k} = x_k \tag{3.15c}$$

$$x_{k+j+1|k} = A x_{k+j|k} + B u_{k+j|k} \tag{3.15d}$$

$$u_{k+j|k} \in \mathcal{U}. \tag{3.15e}$$

Second, we need the following definition, and then, we are in conditions to formulate the QP problem by vectorizing the prediction and cost function equations.

Definition 3.7 (Quadratic Programming (QP) [102])**.** *A QP problem with constraints is defined by*

$$\min_z \frac{1}{2} z^\top H z + c^\top z + d, \tag{3.16}$$

subject to,

$$Pz \leq q, \tag{3.17}$$

where $z \in \mathbb{R}^n$ is the decision variable, $H \in \mathbb{R}^{n \times n}$ is the Hessian matrix, $c \in \mathbb{R}^{n \times 1}$, and $d \in \mathbb{R}$. The constraints $P \in \mathbb{R}^{c_z \times n}$, $q \in \mathbb{R}^{c_z}$, and c_z is the number of constraints for z.

3.2.2.1 Vectorizing the prediction equation

Setting $x_k = x_{k|k}$, and $u_k = u_{k|k}$ and applying the model (3.3) recursively, we have that

$$\begin{aligned} x_{k+1|k} &= A x_k + B u_k \\ x_{k+2|k} &= A x_{k+1|k} + B u_{k+1|k} = A^2 x_k + AB u_{k|k} + B u_{k+1|k} \\ \vdots \;\; &= \quad \vdots \quad = \quad \vdots \\ x_{k+N|k} &= A x_{k+N-1|k} + B u_{k+N-1|k} = A^N x_k + A^{N-1} B u_{k|k} + \cdots + B u_{k+N-1|k}, \end{aligned}$$

stacking

$$\underbrace{\begin{bmatrix} x_{k+1|k} \\ x_{k+2|k} \\ \vdots \\ x_{k+N|k} \end{bmatrix}}_{\mathbf{x}_k} = \underbrace{\begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}}_{F} x_k + \underbrace{\begin{bmatrix} B & 0 & \dots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & B \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} u_{k|k} \\ u_{k+1|k} \\ \vdots \\ u_{k+N-1|k} \end{bmatrix}}_{\mathbf{u}_k},$$

the predicted equality constraint is

$$\mathbf{x}_k = Fx_k + G\mathbf{u}_k, \tag{3.18}$$

where $\mathbf{x}_k$, and $\mathbf{u}_k$ are the state, and input sequence predictions over all steps $j \in \mathbb{N}_{[0,N]}$.

3.2.2.2 Vectorizing the cost function in prediction

Rewriting the cost function (3.15) without the input constraint

$$x_{k|k}^\top Q x_{k|k} + \begin{bmatrix} x_{k+1|k} \\ x_{k+2|k} \\ \vdots \\ x_{k+N|k} \end{bmatrix}^\top \underbrace{\begin{bmatrix} Q & 0 & \dots & 0 \\ 0 & Q & \ddots & \vdots \\ \vdots & \ddots & Q & 0 \\ 0 & \dots & 0 & Q_f \end{bmatrix}}_{\tilde{Q}} \begin{bmatrix} x_{k+1|k} \\ x_{k+2|k} \\ \vdots \\ x_{k+N|k} \end{bmatrix} + \begin{bmatrix} u_{k|k} \\ u_{k+1|k} \\ \vdots \\ u_{k+N-1|k} \end{bmatrix}^\top \underbrace{\begin{bmatrix} R & 0 & \dots & 0 \\ 0 & R & \ddots & \vdots \\ \vdots & \ddots & R & 0 \\ 0 & \dots & 0 & R \end{bmatrix}}_{\tilde{R}} \begin{bmatrix} u_{k|k} \\ u_{k+1|k} \\ \vdots \\ u_{k+N-1|k} \end{bmatrix}$$

with $x_{k|k} = x_k$, now the problem is to minimize

$$J_N(x_k, \mathbf{u}_k) = x_k^\top Q x_k + \mathbf{x}_k^\top \tilde{Q} \mathbf{x}_k + \mathbf{u}_k^\top \tilde{R} \mathbf{u}_k, \tag{3.19a}$$

subject to,

$$\mathbf{x}_k = Fx_k + G\mathbf{u}_k. \tag{3.19b}$$

Substituting the equality constraint (3.18) in (3.19a),

$$J_N(x_k, \mathbf{u}_k) = x_k^\top Q x_k + (Fx_k + G\mathbf{u}_k)^\top \tilde{Q}(Fx_k + G\mathbf{u}_k) + \mathbf{u}^\top(k)\tilde{R}\mathbf{u}_k,$$

omitting k only for the algebraic operations

$$
\begin{aligned}
&= x^\top Q x + ((Fx)^\top + (G\mathbf{u})^\top)\tilde{Q}(Fx + G\mathbf{u}) + \mathbf{u}^\top \tilde{R}\mathbf{u} \\
&= x^\top Q x + (Fx)^\top \tilde{Q} F x + (Fx)^\top \tilde{Q} G\mathbf{u} + (G\mathbf{u})^\top \tilde{Q} F x + (G\mathbf{u})^\top \tilde{Q} G\mathbf{u} + \mathbf{u}^\top \tilde{R}\mathbf{u},
\end{aligned}
$$

grouping terms

$$
\begin{aligned}
&= \underbrace{x^\top Q x + x^\top F^\top \tilde{Q} F x}_{} + \underbrace{x^\top F^\top \tilde{Q} G\mathbf{u} + \mathbf{u}^\top G^\top \tilde{Q} F x}_{} + \underbrace{\mathbf{u}^\top G^\top \tilde{Q} G\mathbf{u} + \mathbf{u}^\top \tilde{R}\mathbf{u}}_{} \\
&= x^\top (Q + F^\top \tilde{Q} F) x \quad + \quad 2\mathbf{u}^\top G^\top \tilde{Q} F x \quad + \mathbf{u}^\top (G^\top \tilde{Q} G + \tilde{R})\mathbf{u} \\
&= x^\top (Q + F^\top \tilde{Q} F) x + (2G^\top \tilde{Q} F x)^\top \mathbf{u} + \mathbf{u}^\top (G^\top \tilde{Q} G + \tilde{R})\mathbf{u} \\
&= x^\top \underbrace{(Q + F^\top \tilde{Q} F)}_{M} x + (\underbrace{2G^\top \tilde{Q} F}_{L} x)^\top \mathbf{u} + \frac{1}{2}\left(\mathbf{u}^\top \underbrace{2(G^\top \tilde{Q} G + \tilde{R})}_{H} \mathbf{u} \right) \\
&= \frac{1}{2}\mathbf{u}_k^\top H \mathbf{u}_k + \underbrace{(L x_k)^\top}_{c^\top} \mathbf{u}_k + \underbrace{x_k^\top M x_k}_{\alpha}.
\end{aligned}
$$

Hence, the cost function as QP problem (without constraints) in compact form is

$$
J_N(x_k, \mathbf{u}_k) = \frac{1}{2}\mathbf{u}_k^\top H \mathbf{u}_k + c^\top \mathbf{u}_k + \alpha, \tag{3.20a}
$$

where

$$
\begin{aligned}
H &= 2(G^\top \tilde{Q} G + \tilde{R}) && (3.20b) \\
c &= L x_k && (3.20c) \\
L &= 2G^\top \tilde{Q} F && (3.20d) \\
\alpha &= x_k^\top M x_k && (3.20e) \\
M &= Q + F^\top \tilde{Q} F. && (3.20f)
\end{aligned}
$$

We still need to incorporate the constraints in the previous formulation. The following section describes the constraint modelling.

3.2.2.3 Constraint modelling

There are different types of defining the input constraint $\mathcal{U}$ which can be found on the literature (*e.g.* [102, 14, 104]). For the analysis in Chapter 4, we consider linear inequalities constraints in the form of *box constraints*. The reason to use this formulation is the simplicity since it can be integrated easily in the QP problem in compact form, facilitating both analysis and implementation.

Definition 3.8. *Let $\mathcal{U}$ be the input constraint set defined as*

$$\mathcal{U} = \{u_k \colon P_u u_k \leq q_u\} \subseteq \mathbb{R}^m, \tag{3.21}$$

where the matrices $P_u \in \mathbb{R}^{c_u \times m}$ and $q_u \in \mathbb{R}^{c_u \times 1}$, and c_u is the number of constraints for the input.

Let the box constraint for the input be

$$u_{\min} \leq u \leq u_{\max}, \tag{3.22}$$

where $u_{\min}, u_{\max} \in \mathbb{R}^m$.

The matrices P_u and q_u are defined as

$$\underbrace{\begin{bmatrix} +\mathbf{I}_{m\times m} \\ -\mathbf{I}_{m\times m} \end{bmatrix}}_{P_u} u_{k+j|k} \leq \underbrace{\begin{bmatrix} +u_{\max} \\ -u_{\min} \end{bmatrix}}_{q_u}. \tag{3.23}$$

From Definition 3.8 we have that

$$P_u u_{k+j|k} \leq q_u, \tag{3.24}$$

stacking

$$\underbrace{\begin{bmatrix} P_u & 0 & \dots & 0 \\ 0 & P_u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_u \end{bmatrix}}_{\tilde{P}_u} \underbrace{\begin{bmatrix} u_{k|k} \\ u_{k+1|k} \\ \vdots \\ u_{k+N-1|k} \end{bmatrix}}_{\mathbf{u}_k} \leq \underbrace{\begin{bmatrix} q_u \\ q_u \\ \vdots \\ q_u \end{bmatrix}}_{\tilde{q}_u}$$

results in

$$\tilde{P}_u \mathbf{u}_k \leq \tilde{q}_u. \tag{3.25}$$

3.2.2.4 Constrained QP problem in compact form

Finally, with the previous results and by Definition 3.7, the constrained QP problem in compact form is

$$\mathbb{P}_N(x_k): \quad V_N^0(x_k) = \min_{\mathbf{u}_k \in \mathbb{U}} J_N(x_k, \mathbf{u}_k) \tag{3.26a}$$

$$= \min_{\mathbf{u}_k \in \mathbb{U}} \frac{1}{2}\mathbf{u}_k^\top H \mathbf{u}_k + c^\top \mathbf{u}_k + \alpha, \tag{3.26b}$$

subject to, for $j \in \mathbb{N}_{[0,N-1]}$,

$$\tilde{P}_u \mathbf{u}_k \leq \tilde{q}_u. \tag{3.26c}$$

The optimal solution obtained by numerical methods or techniques such as multi-parametric Quadratic Programming (mp-QP) [102] and Interior Point Methods (IPM) [103], is defined as

$$\begin{aligned} \mathbf{u}_k^0 &= \arg\min_{\mathbf{u}_k \in \mathbb{U}} \left\{ \frac{1}{2}\mathbf{u}_k^\top H \mathbf{u}_k + c^\top \mathbf{u}_k + \alpha \,:\, \tilde{P}_u \mathbf{u}_k \leq \tilde{q}_u \right\} \\ &= \left\{ u_{k|k}^0, u_{k+1|k}^0, \ldots, u_{k+N-1|k}^0 \right\}. \end{aligned} \tag{3.27}$$

The application of the first control (RH principle) in the optimal sequence to the real plant, followed by a repetition of the whole process at the next sampling instant, defines the implicit control law

$$u_k = \kappa_N(x_k) := u_{k|k}^0. \tag{3.28}$$

Unconstrained solution

Although, it is not practical to implement the unconstrained MPC solution in real (industrial) systems–since most of them require constraints satisfaction–we make use of its derivation in Chapter 4 to analytically link the implicit constrained solution and explicit unconstrained solution.

Minimizing (3.20a) with the gradient and solving for $\mathbf{u}_k$

$$V_N^0(x_k, \mathbf{u}_k) = \min_{\mathbf{u}_k} \frac{1}{2}\mathbf{u}_k^\top H \mathbf{u}_k + c^\top \mathbf{u} + \alpha \tag{3.29a}$$

$$\nabla_{\mathbf{u}} V_N^0(x_k, \mathbf{u}_k) = 0 \tag{3.29b}$$

$$H\mathbf{u}_k^0 + c = 0 \tag{3.29c}$$

$$\mathbf{u}_k^0 = -H^{-1}c \tag{3.29d}$$

the optimal solution results in

$$\mathbf{u}_k^0 = -H^{-1}Lx_k, \tag{3.29e}$$

The optimal $\mathbf{u}_k^0$ for any x_k is unique if $H \succ 0$ is invertible.

Applying the RH principle means that only the first control input of the vector (3.29e) is applied to the real system and defines the explicit control law

$$u_k^0 = K_N x_k \tag{3.30}$$

where

$$K_N = [I \quad 0 \quad 0 \quad \dots \quad 0](-H^{-1}L). \tag{3.31}$$

Notice that u_k^0 is an explicit linear time-invariant control law because H and L do not depend on x_k.

3.3 Stability in MPC

MPC stability analysis requires the application of Lyapunov theory, as the presence of constraints makes the closed-loop system nonlinear. As discussed in [13], the key idea is to adapt the basic MPC framework so that the cost function can serve as a Lyapunov function to ensure stability in closed-loop operation. These adaptations typically involve introducing stability conditions in the stage cost, terminal cost and the value function, and some terminal ingredients with or without terminal sets.

In this work, we avoid the use of terminal sets, then, it is sufficient to provide terminal conditions for nonlinear systems and the Lyapunov equation for linear systems. The following are standard definitions commonly used in MPC and Lyapunov stability theory [100, 102, 12].

Definition 3.9 ($\mathcal{K}, \mathcal{K}_\infty$ and $\mathcal{KL}$ functions)**.** *A function $\alpha(\cdot)\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is class of $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. A function $\alpha(\cdot)\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is class of $\mathcal{K}$ and unbounded,* i.e. *$\alpha(s) \to \infty$ as $s \to \infty$. A function $\beta(\cdot,\cdot)\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is class of $\mathcal{KL}$ if it is continuous, and if, for each $k \geq 0$, $\beta(\cdot, k)$ is a class $\mathcal{K}$ function, and for each $s \geq 0$, $\beta(s, \cdot)$ is* *non-increasing and* $\beta(s,k) \to 0$ *as* $k \to \infty$.

Definition 3.10 (Positively Invariant set (PI))**.** *A compact set* $\mathcal{X} \subseteq \mathbb{R}^n$ *is a positively invariant set for the system* $x^+ = f(x)$ *if* $x \in \mathcal{X}$ *implies* $f(x) \in \mathcal{X}$.

Definition 3.11 (Control positive invariant set)**.** *The set* $\mathcal{X}$ *is a control positive invariant set for* $x^+ = f(x,u)$ *if for all* $x \in \mathcal{X}$ *there exists an admissible control input* $u \in \mathcal{U}$ *such that* $x^+ \in \mathcal{X}$.

Definition 3.12 (Robust Positively Invariant set (RPI))**.** *The set* $\mathcal{X}$ *is robust positively invariant for* $x^+ = f(x,u) + w$ *if* $x^+ \in \mathcal{X}$ *for all* $x \in \mathcal{X}$ *and for all* $w \in \mathcal{W}$.

Definition 3.13 (Robust control positive invariant set)**.** *The set* $\mathcal{X}$ *is a control positive invariant set for* $x^+ = f(x,u) + w$ *if for all* $x \in \mathcal{X}$ *there exists an admissible control input* $u \in \mathcal{U}$ *such that* $x^+ \in \mathcal{X}$ *and for all* $w \in \mathcal{W}$.

3.3.1 Lyapunov stability

Definition 3.14 (Lyapunov stability)**.** *Let* x^* *be an equilibrium point of the system* $x_{k+1} = f(x_k)$ *if* $f(x^*) = x^*$.

a. *The* x^* *is Lyapunov stable if for any* $\epsilon > 0$ *there exists a* $\delta = \delta(\epsilon) > 0$ *such that*

$$\|x_0 - x^*\| \leq \delta \implies \|x_k - x^*\| \leq \epsilon, \quad \textit{for all } k \in \mathbb{N}_{\geq 0}. \tag{3.32}$$

b. *The* x^* *is asymptotically stable in the Lyapunov sense in* $\mathcal{X}$ *if it is Lyapunov stable and*

$$\lim_{k\to\infty} \|x_k - x^*\| = 0, \quad \textit{for all } x_0 \in \mathcal{X}. \tag{3.33}$$

c. *The* x^* *is globally asymptotically stable if it is asymptotically stable in the Lyapunov sense and* $\mathcal{X} = \mathbb{R}^n$.

d. *The* x^* *is exponentially stable if it is Lyapunov stable and there exist a* $\tau > 0$ *and* $\varrho \in (0,1)$ *such that*

$$\|x_0 - x^*\| \leq \delta \implies \|x_k - x^*\| \leq \tau\|x_0 - x^*\|\varrho^k, \quad \textit{for all } k \in \mathbb{N}_{\geq 0}. \tag{3.34}$$

Definition 3.15 (Lyapunov function)**.** *Suppose* $\mathcal{X}$ *is a positively invariant set for* $x^+ = f(x)$.

A function $V(\cdot)\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be Lyapunov function in $\mathcal{X}$ if there exist functions $\alpha_1(\cdot), \alpha_2(\cdot), \alpha_3(\cdot) \in \mathcal{K}_\infty$ such that

$$V(x) \geq \alpha_1(\|x\|) \tag{3.35a}$$

$$V(x) \leq \alpha_2(\|x\|) \tag{3.35b}$$

$$V(f(x)) - V(x) \leq -\alpha_3(\|x\|). \tag{3.35c}$$

3.3.2 Stability conditions in MPC

Since in this thesis we consider input constraints but not state and output constraints, the following assumption and properties are presented as follows.

Assumption 3.8 (Bounds on the stage and terminal costs, [100])**.** *Suppose the following.*

a. The stage cost function $\ell(\cdot,\cdot)\colon \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}_{\geq 0}$ and terminal cost function $V_f(\cdot)\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are continuous, with $\ell(0,0) = 0$ and $V_f(0) = 0$, and satisfy

$$\ell(x,u) \geq \alpha_1(\|x\|), \tag{3.36a}$$

for all $x \in \mathbb{R}^n$, for all $u \in \mathcal{U}$,

$$V_f(x) \leq \alpha_2(\|x\|), \tag{3.36b}$$

for all $x \in \mathbb{R}^n$, where $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are $\mathcal{K}_\infty$ functions.

b. The stage cost function $\ell(\cdot,\cdot)$ and terminal cost function $V_f(\cdot)$ satisfy

$$\ell(x,u) \geq c_1\|x\|_2^2, \tag{3.37a}$$

for all $x \in \mathbb{R}^n$, for all $u \in \mathcal{U}$, and some $c_1 > 0$,

$$V_f(x) \leq c_2\|x\|_2^2, \tag{3.37b}$$

for all $x \in \mathbb{R}^n$, and some $c_2 > 0$.

c. There exists a control law $\kappa_f\colon \mathbb{R}^n \to \mathbb{R}^m$ such that, for all $x \in \mathcal{X}_f$,

$$V_f\big(f(x,\kappa_f(x))\big) \leq V_f(x) - \ell(x,\kappa_f(x)) \tag{3.38a}$$

$$\kappa_f(x) \in \mathcal{U}, \tag{3.38b}$$

where $\mathcal{X}_f$ is the terminal region.

Lemma 3.1 (Value function properties)**.** *If Assumption 3.8 holds, using Definition 3.15 and knowing that $u = \kappa_N(x)$, the following properties are defined as follows.*

a. Suppose $\mathcal{X}$ is a control positive invariant set for $x^+ = f(x, \kappa_N(x))$. The function $V_N^0(\cdot)$ is said to be Lyapunov function in $\mathcal{X}$ if there exist functions $\alpha_1(\cdot), \alpha_2(\cdot) \in \mathcal{K}_\infty$ such that

$$V_N^0(x) \geq \alpha_1(\|x\|) \tag{3.39a}$$

$$V_N^0(x) \leq \alpha_2(\|x\|) \tag{3.39b}$$

$$V_N^0(f(x, \kappa_N(x))) - V_N^0(x) \leq -\alpha_1(\|x\|), \tag{3.39c}$$

for all $x \in \mathbb{R}^n$ and for all $\kappa_N(x) \in \mathcal{U}$.

b. Suppose $\mathcal{X}$ is a control positive invariant set for $x^+ = f(x, \kappa_N(x))$. The function $V_N^0(\cdot)$ is said to be Lyapunov function in $\mathcal{X}$ if there exist constants $c_1, c_2 > 0$ such that

$$V_N^0(x) \geq c_1\|x\|_2^2 \tag{3.40a}$$

$$V_N^0(x) \leq c_2\|x\|_2^2 \tag{3.40b}$$

$$V_N^0(f(x, \kappa_N(x))) - V_N^0(x) \leq -c_1\|x\|_2^2, \tag{3.40c}$$

for all $x \in \mathbb{R}^n$ and for all $\kappa_N(x) \in \mathcal{U}$.

c. Suppose $\mathcal{X}$ is a control positive invariant set for $x^+ = f(x, \kappa_N(x))$. The function $V_N^0(\cdot)$ is said to be Lyapunov function in $\mathcal{X}$ and $x^+ = f(x, \kappa_N(x))$ is exponentially stable in $\mathcal{X}$, such that

$$V_N^0(f(x, \kappa_N(x))) - V_N^0(x) \leq -\ell(x, \kappa_N(x)). \tag{3.41}$$

Proof. See [100]. □

For linear systems, we can guarantee Assumption 3.8c with the terminal ingredient defined by the following lemma.

Lemma 3.2 (Lyapunov equation)**.** *If Assumption 3.2 holds, then there exists a unique Q_f such* *that*

$$(A + BK)^\top Q_f (A + BK) - Q_f \leq -(Q + K^\top RK), \tag{3.42}$$

where K is a control law such that $(A + BK)$ is stable, and $Q \succ 0$, $R \succ 0$, and $Q_f \succ 0$.

Proof. See [14]. □

3.4 The Separation Principle

In this section we present the fundamentals of the Separation Principle, which is an important result in modern control theory. This principle states that, under certain conditions, the design of an optimal controller and the design of an optimal state estimator can be performed independently. Originally developed for the LQG regulator, it shows that the optimal output-feedback controller can be designed by combining an optimal state-feedback LQR with an optimal state estimator, generally in the form of a KF [105, 106, 107].

We begin by deriving the LQR and Kalman gains separately, and subsequently formulate the LQG control problem.

The LQR problem

Consider the system (3.3). The objective is to minimize the following cost function

$$J(x_k, u_k) = \sum_{k=0}^{\infty} \left(x_k^\top Q x_k + u_k^\top R u_k \right), \tag{3.43a}$$

subject to, for $k \in \mathbb{N}_{\geq 0}$,

$$x_{k+1} = Ax_k + Bu_k, \quad x(0) = x_0, \tag{3.43b}$$

with $Q \succeq 0$ and $R \succ 0$.

If Assumption 3.2 holds, then

$$u_k = -Kx_k \tag{3.44a}$$

minimizes (3.43), where

$$K = (R + B^\top PB)^{-1} B^\top PA, \tag{3.44b}$$

is the optimal LQR, and

$$P = A^\top PA - A^\top PB(R + B^\top PB)^{-1}B^\top PA + Q, \tag{3.44c}$$

solves the Discrete Algebraic Ricatti Equation (DARE), where $P \succeq 0$.

The Kalman filter

Let

$$\hat{x}_k \coloneqq \mathbb{E}\{x_k\} \tag{3.45a}$$

$$e_k \coloneqq x_k - \hat{x}_k \tag{3.45b}$$

$$P_k \coloneqq \mathbb{E}\left\{e_k e_k^\top\right\}, \tag{3.45c}$$

where $\hat{x}_k$ is the state estimate, e_k is the estimation error and P_k is the state covariance at time k.

Consider the system (3.4). The aim of the KF is to find an estimate state $\hat{x}$ that minimizes the error covariance P_k given y_k. If Assumption 3.2 and Assumption 3.4 hold, then the following is defined as follows.

Innovation step:

$$\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + Bu_k \tag{3.46a}$$

$$P_{k|k-1} = AP_{k-1|k-1}A^\top + Q_w \tag{3.46b}$$

$$\hat{y}_k = C\hat{x}_{k|k-1}. \tag{3.46c}$$

Correction step:

$$L_k = P_{k|k-1}C^\top(R_s + CP_{k|k-1}C^\top)^{-1} \tag{3.47a}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k(y_k - \hat{y}_k) \tag{3.47b}$$

$$P_{k|k} = P_{k|k-1} - L_kCP_{k|k-1}. \tag{3.47c}$$

If $P_k = P$ then

$$L = APC^\top (R_s + CPC^\top)^{-1} \tag{3.48a}$$

$$P = APA^\top - APC^\top (R_s + CPC^\top)^{-1} CPA^\top + Q_w, \tag{3.48b}$$

where L is the optimal Kalman gain, and P is the unique solution for the DARE problem.

The LQG problem

Consider the system (3.4) and Assumption 3.4. The objective is to minimize the following stochastic cost function

$$J(x_k, u_k) = \mathbb{E}\left\{ \sum_{k=0}^{\infty} \left(x_k^\top Q x_k + u_k^\top R u_k \right) \right\}, \tag{3.49}$$

with $Q \succeq 0$ and $R \succ 0$.

If Assumption 3.2 holds, then

$$u_k = -K\hat{x}_k, \tag{3.50}$$

is the minimizer of (3.49), where K is given by the LQR solution and the state estimate $\hat{x}_k$ is given by the KF.

Let

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - \hat{y}_k) \tag{3.51a}$$

$$\tilde{x}_{k+1} = x_{k+1} - \hat{x}_{k+1}. \tag{3.51b}$$

Then, for the closed-loop dynamics we have that

$$\begin{bmatrix} x_{k+1} \\ \tilde{x}_{k+1} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x_k \\ \tilde{x}_k \end{bmatrix}, \tag{3.52}$$

is block upper-triangular, where the characteristic polynomial $p(\cdot)$ in the z-plane is

$$p(z) = \det(zI - A + BK)\det(zI - A + LC), \tag{3.53}$$

showing that the eigenvalue assignment for the output-feedback can be separated into the eigenvalue assignment for the state-feedback LQR and the eigenvalue assignment for the observer (KF).

Chapter 4

Input-constrained output-feedback stochastic MPC

Contents

This chapter considers a LQG control problem with constraints on system inputs and random packet losses occurring on the communication channel between plant and controller. It is well known that [41], in the absence of constraints, the Separation Principle between estimator and controller holds when the channel employs a TCP-like protocol but not so under a UDP-like protocol. This chapter gives a counterexample that shows that, under an MPC scheme that handles the constraints, not only does the Separation Principle not hold in the TCP-like case, there exist instances where stability is lost in the TCP case but maintained in the UDP case; thus the stability region for TCP does not contain that of UDP (c.f. [41]). Theoretical analysis characterizes and reveals a trade-off between estimation errors in the estimator and prediction errors in the controller. Counterintuitively, the poorer on-average performance of the estimator in the UDP case may be compensated by smaller prediction errors in the controller. Also, the impact of random packet losses over the control law can be identified by analysing the partitions and the piecewise solution in explicit form.

4.1 Introduction

The Separation Principle is a cornerstone result in modern control theory [108]. In its simplest form, it says that the design of a state-feedback controller and an output-measurement state estimator may be executed independently while guaranteeing stability of the overall loop. The focus of control research over the last few decades has arguably been allowed to focus more on the simpler case of state feedback because the Separation Principle is either known, or assumed, to hold in the considered setting; once a state feedback control law has been designed, a state estimator can always be designed and deployed on the real, output-measured system.

Another important consideration for modern control system design is the communication channel between controller and plant; in the advent of new and emerging control technologies such as smart grids, robotics, and advanced autonomous systems, it is a realistic proposition that controller and plant are not co-located and/or physically connected. In such cases, sensor measurements and control inputs are sent and received over communication channels and may be subject to noise, delays, and packet losses. It may be necessary to consider such effects when designing the controller and estimator.

An important line of research [49, 50], collected in [41], discovered that whether the Separation Principle holds depends on the communication channel protocol employed. In particular, the authors considered a discrete-time LTI system with Gaussian process noise and sensing noise on the output measurement—a classical LQG-type problem—and modelled the channel between controller and plant as being subject to random packet losses. A key result was to show not just that the Separation Principle holds when the channel employs a TCP-like communication protocol—*i.e.* where an acknowledgement of a received packet is transmitted—but also that it *does not* hold when the channel is UDP-like, *i.e.* absent of any acknowledgement. Moreover, the *stability region* of the TCP-based controller strictly contains that of the UDP-based controller, in the sense that the former stabilizes an LTI system when the latter is unable to.

It is interesting to enquire whether the same result holds in the presence of constraints. In this chapter, therefore, we consider the same LQG-type setting albeit with the addition of (general) constraints on the system inputs. To handle these constraints, we replace the classical LQG controller based on dynamic programming with a (conventional) stochastic model predictive controller, wherein the constrained optimal control problem is solved at each new state estimate computed by the KF. In other words, we consider a standard output-feedback MPC design that may be found in many industrial applications: MPC in the state estimates, with a Kalman filter in the loop.

The contribution of this chapter is to show the existence of a counterexample where the stability of the predictive controller under the TCP-like protocol depends on the gain of the estimator. Thus, the Separation Principle *does not* hold for the TCP-like case when constraints are present and the controller employs a receding horizon, in direct contrast to the fact that the Separation Principle does hold for a (static) finite-horizon implementation of LQG [109]. While this is not surprising in itself—for it is well known that the Separation Principle does not hold in general in constrained MPC—an interesting observation is that the UDP-based controller stabilizes the same example. This establishes that UDP-based estimation and constrained control may outperform a TCP-based scheme and, moreover, confirms that the TCP-like stability region no longer contains the UDP-like stability region when constraints are present. Finally, we provide a theoretical analysis that shows an interesting relation and trade-off between the estimation errors and prediction errors in both schemes; owing to an information asymmetry between estimator and controller in the TCP-like case, the on-average poorer performance of the UDP-like estimator may be compensated for by smaller prediction errors in the controller. The last section is about the conditions under which the explicit control law of the problem is PWA and the corresponding cost function is PWQ. The resulting polyhedral partitions offer geometric insight into the closed-loop behaviour, demonstrating that the trajectories originated from the same regions for TCP and UDP, drives the state towards the origin rapidly in the TCP-like estimation than the UDP-like estimation.

4.2 Problem formulation

We consider the following discrete-time linear system

$$x_{k+1} = Ax_k + \nu_k Bu_k + w_k \tag{4.1a}$$

$$y_k = \gamma_k (Cx_k + s_k), \tag{4.1b}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, and $y_k \in \mathbb{R}^p$ are, respectively, the state, input, and output of the system at sample time $k \in \mathbb{N}_{\geq 0}$. The system is subject to uncertainty in the form of (i) process noise w_k and measurement noise s_k, and (ii) random packet losses affecting the input and output channels, via the variables ν_k and γ_k. The system input, u_k, is constrained to take values in a set $\mathcal{U} \subset \mathbb{R}^m$ but the states and outputs are unconstrained.

We make the following standing assumptions.

Assumption 4.1. *The matrices A, B and C are known, the pair (A, B) is stabilizable, and the pair (C, A) is observable.*

Assumption 4.2. *The set $\mathcal{U}$ is known and compact, containing the origin in its interior.*

Assumption 4.3. *The process noise $w_k \in \mathbb{R}^n$ and measurement noise $s_k \in \mathbb{R}^p$ are i.i.d. random variables, with $w_k \sim \mathcal{N}(0, Q_w)$ and $s_k \sim \mathcal{N}(0, R_s)$.*

Assumption 4.4. *The input packet loss variable $\nu_k \in \{0, 1\}$ and output packet loss variable $\gamma_k \in \{0, 1\}$ are i.i.d random variables with $\nu_k \sim \mathcal{B}(\bar{\nu})$ and $\gamma_k \sim \mathcal{B}(\bar{\gamma})$, where $\bar{\nu}$ and $\bar{\gamma}$ are the respective probabilities of successful packet delivery.*

Assumption 4.5. *The information set available to the controller at time $k \in \mathbb{N}_{\geq 0}$ is*

$$\mathcal{I}_k = \begin{cases} \mathcal{F}_k := \{\mathbf{y}_k, \boldsymbol{\gamma}_k, \boldsymbol{\nu}_{k-1}\} & \textit{TCP-like protocol} \\ \mathcal{G}_k := \{\mathbf{y}_k, \boldsymbol{\gamma}_k\} & \textit{UDP-like protocol} \end{cases} \tag{4.2}$$

where $\mathbf{y}_k = \{y_k, y_{k-1}, \cdots, y_1\}$, $\boldsymbol{\gamma}_k = \{\gamma_k, \gamma_{k-1}, \cdots, \gamma_1\}$, and $\boldsymbol{\nu}_k = \{\nu_k, \nu_{k-1}, \cdots, \nu_1\}$.

Assumptions 4.1 and 4.2 are mild and standard. Assumptions 4.3–4.5 imply the same setting studied in the literature (e.g. [49, 41]), wherein the actuation and sensing channels are either TCP-like—in which an Acknowledgement (ACK) of successful or unsuccessful packet delivery is sent—or UDP-like where no such acknowledgement is sent. Fig. 4.1 illustrates the setup in the TCP-like case, showing also the controller (MPC-TCP) and estimator (KF-TCP); the lack of state measurements (Assumption 4.5) motivates the need for the latter, and Fig. 4.2 illustrates the setup in the UDP-like case. The difference between this setup and that of [41] is the presence of input constraints.

The formulation of problem is as follows. Given the system (4.1) at a state x_k, the problem is to determine the optimal control law such that the state estimate $\hat{x}_k$ (obtained in Section 4.3.1) is transferred to a neighbourhood of the origin, subject to the input constraint set (Assumption 4.2), packet losses (Assumption 4.4), and uncertainty (Assumption 4.3). The control law is obtained by minimizing the expectation of a cost function associated with the stochastic optimal control problem in Section 4.3.2.

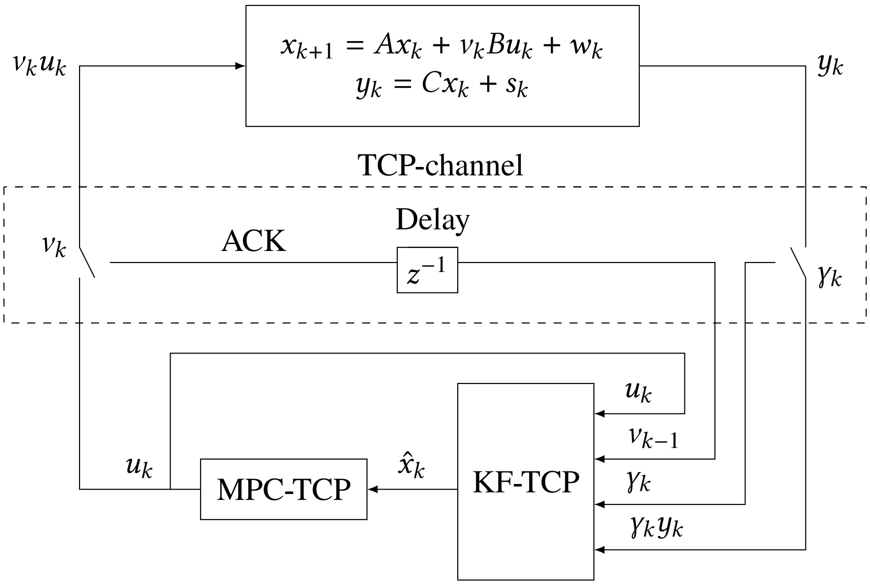


Figure 4.1: Problem setting, including the uncertain system, TCP-like channel, and control and estimation modules.

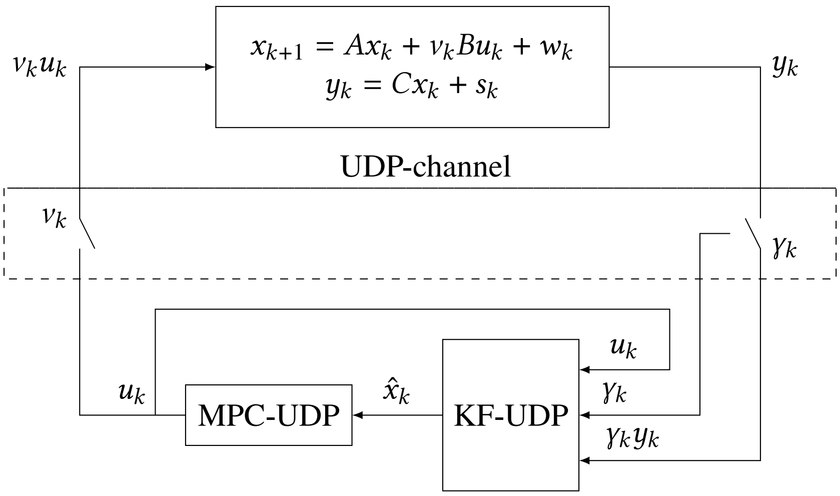


Figure 4.2: Problem setting, including the uncertain system, UDP-like channel, and control and estimation modules.

We remark that even though only input constraints are considered and the system is linear, this is not a trivial problem; a common approach to establishing stability even with *state* measurements available is to assume the existence of a *global* CLF [100], which is restrictive in the presence of constraints. The aim of this chapter is to analyse the closed-loop stability of the system in the described setting, and establish if, and under which conditions, the Separation Principle—which allows independent design of estimator and controller—holds.

4.3 Output-feedback stochastic MPC formulation

The control scheme we study is composed of two steps: first, a KF performs state estimation conditioned on the information set $\mathcal{I}_k$ [41]. Subsequently, a stochastic MPC computes and sends an optimal control input to the plant, based on minimizing the expectation of a cost function conditioned on the state estimate and covariance.

4.3.1 Estimator formulation

We briefly recall the KF conditioned on either TCP-like or UDP-like information sets, as given in [41]. Let

$$\hat{x}_k := \mathbb{E}\{x_k \,|\, \mathcal{I}_k\} \tag{4.3a}$$

$$e_k := x_k - \hat{x}_k \tag{4.3b}$$

$$P_k := \mathbb{E}\big\{e_k e_k^\top \,|\, \mathcal{I}_k\big\}, \tag{4.3c}$$

where $\hat{x}_k$ is the state estimate, e_k is the estimation error and P_k is the state covariance at time k.

Before we proceed, it should be emphasized that the ACK signal (ν_{k-1}) received by the KF-TCP estimator–shown in Fig. 4.1 and defined within the set $\mathcal{F}_k$ in (4.5) for the TCP-like protocol–is the realization of successful or unsuccessful packet delivery that has already happened at time $k-1$. Since ν_{k-1}, ν_k and $\mathcal{F}_k$ are i.i.d, and following the remark on the notation in [41], we use ν_k instead of ν_{k-1} to maintain notation consistency, *i.e.* at time k we refer to ν_k, and at time $k-1$ we refer to ν_{k-1}. This clarification does not change the following analysis.

Considering the problem of estimating the state x_k at time k, knowing that $\mathbb{E}\{w_{k-1}\} = 0$ from Assumption 4.3, $\mathbb{E}\{\nu_{k-1}^2\} = \mathbb{E}\{\nu_{k-1}\} = \bar{\nu}$ from Assumption 4.4, and $\mathbb{E}\{e_{k-1}\} = 0$, the two cases differ on whether the value of ν_{k-1} is available for the following **innovation step**.

TCP-like protocol:

$$\begin{aligned}\hat{x}_{k|k-1} &= \mathbb{E}\{x_k \,|\, \mathcal{F}_{k-1}\}\\ &= \mathbb{E}\{Ax_{k-1} + \nu_{k-1}Bu_{k-1} + w_{k-1} \,|\, \mathcal{F}_{k-1}\}\\ &= A\hat{x}_{k-1|k-1} + \nu_{k-1}Bu_{k-1}, \end{aligned} \tag{4.4a}$$

$$e_{k|k-1} = x_k - \hat{x}_{k|k-1}$$

$$
\begin{aligned}
&= Ax_{k-1} + \nu_{k-1}Bu_{k-1} + w_{k-1} - \big(A\hat{x}_{k-1|k-1} + \nu_{k-1}Bu_{k-1}\big) \\
&= Ae_{k-1} + w_{k-1}, && \text{(4.4b)}\\
P_{k|k-1} &= \mathbb{E}\big\{e_{k|k-1}e_{k|k-1}^\top \mid \mathcal{F}_{k-1}\big\} \\
&= \mathbb{E}\big\{(Ae_{k-1} + w_{k-1})(Ae_{k-1} + w_{k-1})^\top \mid \mathcal{F}_{k-1}\big\} \\
&= \mathbb{E}\big\{Ae_{k-1}e_{k-1}^\top A^\top + w_{k-1}w_{k-1}^\top \mid \mathcal{F}_{k-1}\big\} \\
&= AP_{k-1}A^\top + Q_w. && \text{(4.4c)}
\end{aligned}
$$

UDP-like protocol:

$$
\begin{aligned}
\hat{x}_{k|k-1} &= \mathbb{E}\{x_k \mid \mathcal{G}_{k-1}\} \\
&= \mathbb{E}\{Ax_{k-1} + \nu_{k-1}Bu_{k-1} + w_{k-1} \mid \mathcal{G}_{k-1}\}, \\
&= A\hat{x}_{k-1|k-1} + \bar{\nu}Bu_{k-1}, && \text{(4.5a)}\\
e_{k|k-1} &= x_k - \hat{x}_{k|k-1} \\
&= Ax_{k-1} + \nu_{k-1}Bu_{k-1} + w_{k-1} - \big(A\hat{x}_{k-1|k-1} + \bar{\nu}Bu_{k-1}\big) \\
&= Ae_{k-1} + (\nu_{k-1} - \bar{\nu})Bu_{k-1} + w_{k-1}, && \text{(4.5b)}\\
P_{k|k-1} &= \mathbb{E}\big\{e_{k|k-1}e_{k|k-1}^\top \mid \mathcal{G}_{k-1}\big\} \\
&= \mathbb{E}\big\{Ae_{k-1}e_{k-1}^\top A^\top + (\nu_{k-1} - \bar{\nu})^2 Bu_{k-1}u_{k-1}^\top B^\top + w_{k-1}w_{k-1}^\top \mid \mathcal{G}_{k-1}\big\} \\
&= \mathbb{E}\big\{Ae_{k-1}e_{k-1}^\top A^\top + (\nu_{k-1}^2 - 2\nu_{k-1}\bar{\nu} + \bar{\nu}^2) Bu_{k-1}u_{k-1}^\top B^\top + w_{k-1}w_{k-1}^\top \mid \mathcal{G}_{k-1}\big\} \\
&= AP_{k-1}A^\top + \bar{\nu}(1 - \bar{\nu})Bu_{k-1}u_{k-1}^\top B^\top + Q_w. && \text{(4.5c)}
\end{aligned}
$$

In both cases, and because y_k, γ_k, w_k and $\mathcal{I}_k$ are independent, the **correction step** gives

$$
\begin{aligned}
\hat{x}_k &= \hat{x}_{k|k-1} + \gamma_k K_k\big(y_k - C\hat{x}_{k|k-1}\big), && \text{(4.6a)}\\
e_k &= x_k - \hat{x}_k \\
&= x_k - \hat{x}_{k|k-1} - \gamma_k K_k\big(Cx_k + s_k - C\hat{x}_{k|k-1}\big) \\
&= (\mathbf{I} - \gamma_k K_k C)e_{k|k-1} - \gamma_k K_k s_k, && \text{(4.6b)}\\
P_k &= (\mathbf{I} - \gamma_k K_k C)P_{k|k-1}, && \text{(4.6c)}\\
K_k &= P_{k|k-1}C^\top (CP_{k|k-1}C^\top + R_s)^{-1}. && \text{(4.6d)}
\end{aligned}
$$

In both cases the Kalman gain K_k is time-varying and stochastic, given its dependency on γ_k; it is well known that K_k, even for a stable process, does not converge to a steady value [41].

It is also well known and easy to see from the innovation equations that the state error covariance $P_{k|k-1}$ is independent of the control input in the TCP-like case but not so in the UDP-like case. Indeed, a cornerstone result of [41] and its underlying work was to establish that the Separation Principle holds in the TCP-like case but does not in the UDP-like case. In particular, [41] considered a classical LQG setup and showed that in the TCP-like case the optimal controller is a linear function of the state estimate and the optimal estimator is independent of this; on the other hand, the optimal controller in the UDP-like case is a *nonlinear* function of the state estimate and the optimal estimator depends in a non-straightforward way on this control law.

We aim to study the same issue, albeit in the context of an *input-constrained* LQG setting. To deal with the input constraints systematically, we employ a conventional model predictive controller in the loop. The next subsection describes the formulation of the controller.

4.3.2 Controller formulation

The stochastic optimal control problem we consider, for the system at a state x_k and the information $\mathcal{I}_k$ available to the controller, is

$$V_N^0(\mathcal{I}_k) = \min_{\mathbf{u}_k \in \mathbb{U}} \mathbb{E}\big\{J_N(x_k, \mathbf{u}_k, \boldsymbol{\nu}_{\cdot|k}) \,|\, \mathcal{I}_k\big\}, \tag{4.7}$$

where the decision variable

$$\mathbf{u}_k := \big\{u_{k|k}, u_{k+1|k}, \ldots, u_{k+N-1|k}\big\}, \tag{4.8}$$

is the finite sequence of future control inputs, selected such that it lies in the constraint set

$$\mathbb{U} := \mathcal{U} \times \cdots \times \mathcal{U} \tag{4.9}$$

and minimizes the expectation of a cost function

$$J_N(x_k, \mathbf{u}_k, \boldsymbol{\nu}_{\cdot|k}) := \beta V_f(x_{k+N|k}) + \sum_{j=0}^{N-1} \ell(x_{k+j|k}, \nu_{k+j|k} u_{k+j|k}) \tag{4.10a}$$

with

$$\ell(x,u) = \|x\|_Q^2 + \|u\|_R^2 \tag{4.10b}$$

$$V_f(x) = \|x\|_{Q_f}^2. \tag{4.10c}$$

Assumption 4.6. *$Q \succ 0$, $R \succ 0$, $Q_f \succ 0$ and $\beta \geq 1$.*

The purpose of β is to enlarge the Region of Attraction (RoA) by weighting the terminal cost such that $\beta V_f(\cdot)$ and N can be used as stabilizing ingredients, [110]. In the subsequent sections, by simulations and analysis, we see the effect of incrementing β and N for the problem formulation.

The expectation in (4.7) is to be taken over predicted states and the actuation channel packet loss variable ν:

$$\boldsymbol{\nu}_{\cdot|k} := \left\{\nu_{k|k}, \nu_{k+1|k}, \dots\right\}. \tag{4.11}$$

This motivates the consideration of the two different information sets, TCP-like and UDP-like, and how they affect the formulation of the optimal control problem.

- In the UDP-like case, $\mathcal{I}_k = \mathcal{G}_k = \left\{\mathbf{y}_k, \boldsymbol{\gamma}_k\right\}$ contains no additional information on which to condition the expectation in (4.7) beyond the state estimate and covariance—provided by the estimator—and, as in the estimator for the UDP-like protocol (4.5), the expected value $\mathbb{E}\{\nu\} = \bar{\nu}$.
- In the TCP-like case, $\mathcal{I}_k = \mathcal{F}_k = \left\{\mathbf{y}_k, \boldsymbol{\gamma}_k, \boldsymbol{\nu}_{k-1}\right\}$ contains information of the past realizations of ν. Since the predictive control formulation would require information on *future* realizations of ν_k, which is not possible, the expected value $\mathbb{E}\{\nu\} = \bar{\nu}$ is used.

In both cases, therefore, and since ν is i.i.d. with $\mathbb{E}\{\nu u\} = \bar{\nu}u$, the expectations over $\nu_{k+j|k}$ are replaced by $\bar{\nu}$:

$$\begin{aligned}\mathbb{E}\left\{x_{k+j+1} \mid \mathcal{I}_k\right\} &= z_{k+j+1|k} \\ &= Az_{k+j|k} + \bar{\nu}Bu_{k+j|k}\end{aligned} \tag{4.12a}$$

$$\mathcal{P}_{k+j+1|k} = A\mathcal{P}_{k+j|k}A^\top + Q_w, \tag{4.12b}$$

and from Lemma 1 in [50],

$$\mathbb{E}\left\{\left\|x_{k+j}\right\|_Q^2 \mid \mathcal{I}_k\right\} = \left\|z_{k+j|k}\right\|_Q^2 + \operatorname{tr}\left(Q\mathcal{P}_{k+j|k}\right), \tag{4.12c}$$

with $z_{k|k} = \hat{x}_k$ and $\mathcal{P}_{k|k} = P_k$. Note that we use $z_{\cdot|k}$ and $\mathcal{P}_{\cdot|k}$ to denote open-loop predictions by the controller, and reserve $\hat{x}_{\cdot|k}$ and $P_{\cdot|k}$ for the estimator; as we will show, the prediction $z_{k+1|k}$ is not necessarily equal to innovation $\hat{x}_{k+1|k}$.

The optimal control problem (4.7) may be rewritten in the deterministic form

$$V_N^0(\mathcal{I}_k) = V_N^0(\hat{x}_k, P_k) = \min_{\mathbf{u}_k \in \mathbb{U}} J_N(\hat{x}_k, \mathbf{u}_k, \bar{\nu}) + c(P_k) \tag{4.13}$$

subject to, for $j \in \mathbb{N}_{[0,N-1]}$,

$$z_{k|k} = \hat{x}_k \tag{4.14a}$$

$$z_{k+j+1|k} = Az_{k+j|k} + \bar{\nu}Bu_{k+j|k} \tag{4.14b}$$

$$u_{k+j|k} \in \mathcal{U}, \tag{4.14c}$$

where

$$c(P_k) = \sum_{j=0}^{N} \operatorname{tr}(Q_j \mathcal{P}_{k+j|k}), \quad Q_j = \begin{cases} Q & j \in \mathbb{N}_{[0,N-1]}, \\ \beta Q_f & j = N, \end{cases} \tag{4.15}$$

is a constant term that may be omitted from the optimization, but is required to determine the value function.

Remark 4.1. *The additional information contained in the TCP-like case benefits the estimator but provides no additional information for use by the predictive controller. Problem* (4.13) *subject to* (4.14) *is therefore of a form close to a conventional input-constrained MPC problem; the only difference is the inclusion of the mean of the input packet loss variable in the dynamic model.*

Remark 4.2. *It is well known [111] that the value of* $c(P_k)$ *may be reduced by parametrizing the control input as* $u_{k+j|k} = Kz_{k+j|k} + v_{k+j|k}$*,* K *stabilizing for* (A, B)*. This, however, replaces pure input constraints with state constraints, resulting in the recursive feasibility of the controller being non-trivial to establish.*

Solving $\mathbb{P}_N(\hat{x}_k, P_k)$ yields the (unique) optimal solution

$$\mathbf{u}_k^0(\hat{x}_k) = \left\{u_{k|k}^0(\hat{x}_k), \ldots, u_{k+N-1|k}^0(\hat{x}_k)\right\}, \tag{4.16}$$

with associated optimal cost value $V_N^0(\hat{x}_k, P_k)$; $\mathbf{u}_k^0(\hat{x}_k)$ does not depend on P_k but $V_N^0(\hat{x}_k, P_k)$ does. The application of the first control in the optimal sequence to the plant, followed by a repetition of the process at the next sampling instant, defines the implicit control law

$$u_k = \kappa_N(\hat{x}_k) \coloneqq u_{k|k}^0(\hat{x}_k). \tag{4.17}$$

In view of the lack of state constraints, the domain of the value function $V_N^0(\cdot, P)$ and control law $\kappa_N(\cdot)$ is the whole state space, meaning that recursive feasibility of the optimal control problem is trivially established. Stability of the closed-loop system, including the KF in the loop, is much harder to establish, exacerbated by the lack of terminal state constraints [100].

4.4 Stability and the Separation Principle

We open this section with an interesting example. Under a particular choice of parameters, we find an instance where the TCP-like scheme loses stability while the UDP-like one retains it. We find the stability of the TCP-like controller depends on the gain of the estimator, showing that the controller and estimator cannot necessarily be designed separately. This serves as a counterexample to show that the Separation Principle does not necessarily hold in the presence of constraints.

4.4.1 A counterexample

Consider a system with

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

the noise covariances $Q_w = 0.0001\mathbf{I}_{2\times 2}$, $R_s = 0.0001\mathbf{I}_{1\times 1}$, and the input constraint set $\mathcal{U} = \{u \colon |u| \leq 1\}$. The expected values of ν and γ are $\bar{\nu} = 0.95$ and $\bar{\gamma} = 0.7$.

We design the controller with $Q = \mathbf{I}_{2\times 2}$, $R = 1$, $N = 3$, $\beta = 1$, and

$$Q_f = \begin{bmatrix} 12.70 & 4.86 \\ 4.86 & 3.71 \end{bmatrix}.$$

It is easily verified that this choice satisfies Assumption 4.7, given in the next section.

Let $x_0 = \begin{bmatrix} 0.731 & 0.7 \end{bmatrix}^\top$, the initial state estimate $\hat{x}_0 = \begin{bmatrix} 3.66 & 0.7 \end{bmatrix}^\top$, and the covariance

$$P_0 = \begin{bmatrix} 2.6 & 0 \\ 0 & 2 \end{bmatrix}.$$

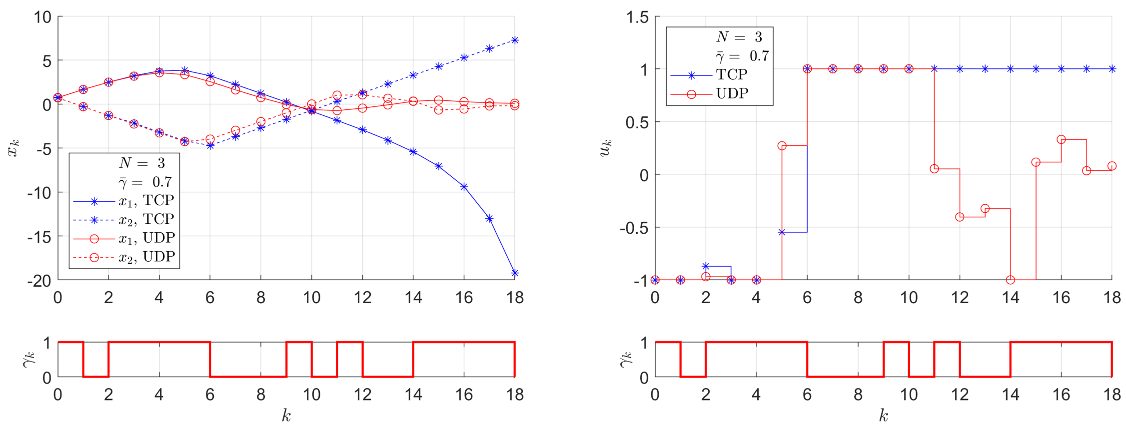
Let $\nu_k = 1, k = 0, 1, \ldots$ (*i.e.* no packet losses on the actuation channel) and

$$\{\gamma_k\} = \{0, 1, 1, 1, 1, 0, 0, 0, 1, 0, 1, 0, 0, 1, 1, 1, 1, 0, 0\}.$$

In what follows, we show a series of five simulations: the first simulation is subject to the previous conditions and the rest have slightly different conditions such as no input constraints, no packet losses, different P_0, and increased β, but there is no simulation of increased N since that scenario is analysed in the next section.

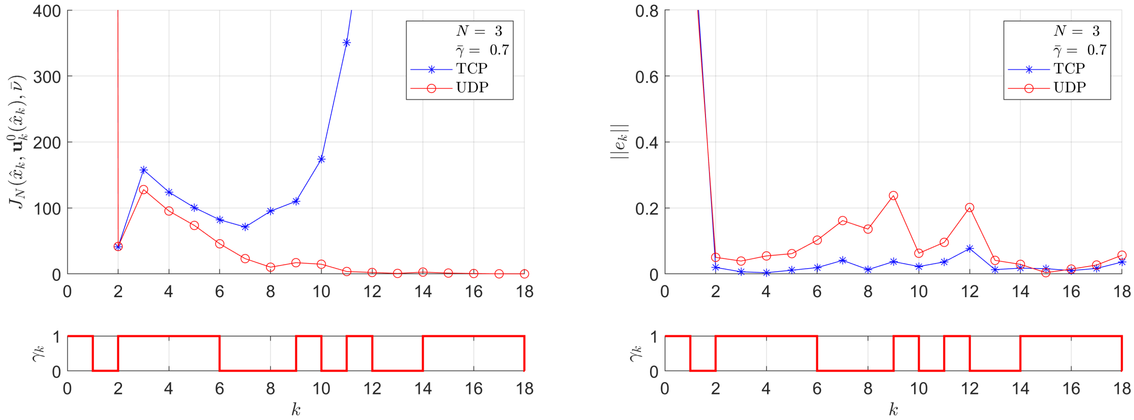
4.4.1.1 First simulation: TCP-like scheme loses stability

Fig. 4.3 shows the true state and the input control, and Fig. 4.4 shows the optimal cost and estimation error trajectories under TCP-like and UDP-like state estimation schemes. Note the TCP-based trajectory diverges but the UDP-based scheme converges to the origin, and the same behaviour is shown in the optimal cost trajectories. Nevertheless, the estimation error in the TCP-like scheme is generally smaller compared to the UDP-like scheme.



(a) States (b) Inputs

Figure 4.3: True state trajectories and applied controls under packet losses with input constraints.

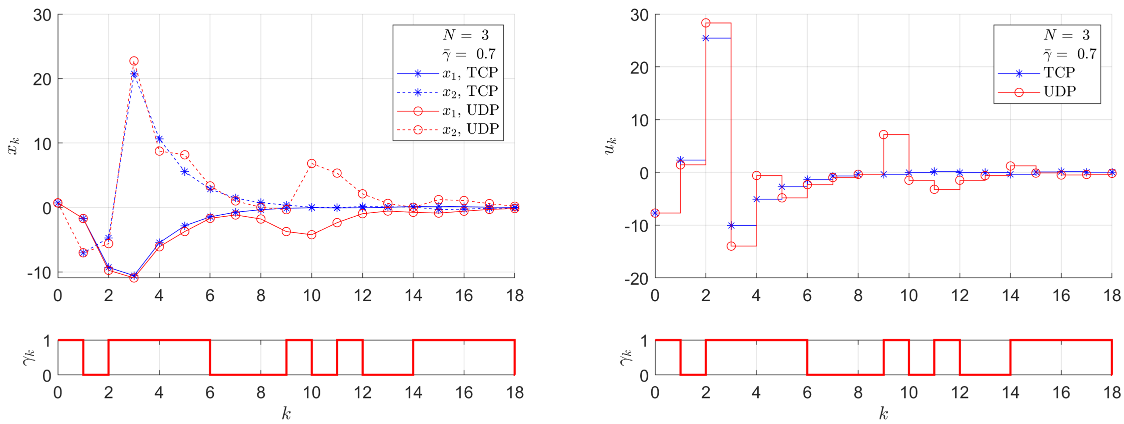


(a) Optimal cost (b) Estimation errors

Figure 4.4: Optimal cost and estimation errors under packet losses with input constraints.

4.4.1.2 Second simulation: no input constraints

When there are no input constraints both controllers maintain stability. Fig. 4.5 shows the true state trajectories and input control, and Fig. 4.6 shows the optimal cost and estimation errors. We can verify that despite the packet losses over the sensor channel, the stability is guaranteed for both schemes and the Separation Principle holds.

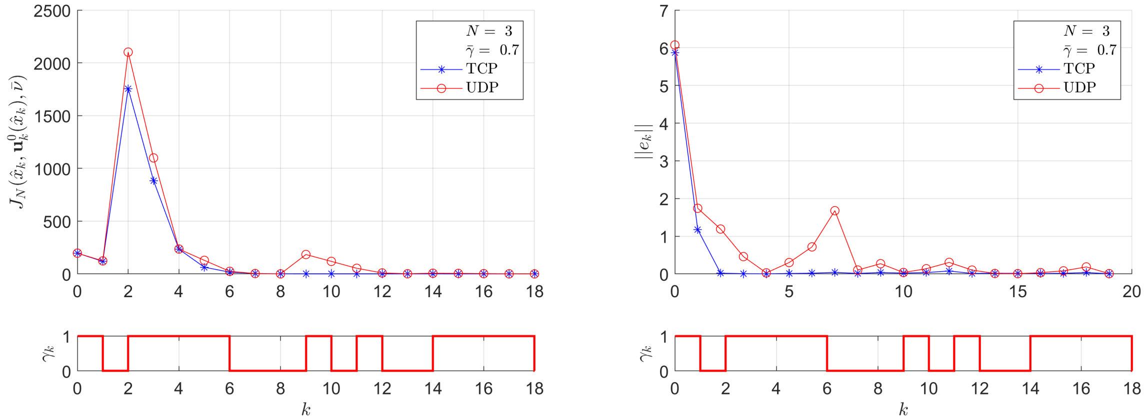


(a) States (b) Inputs

Figure 4.5: True state trajectories and applied controls under packet losses without input constraints.

4.4.1.3 Third simulation: no packet losses

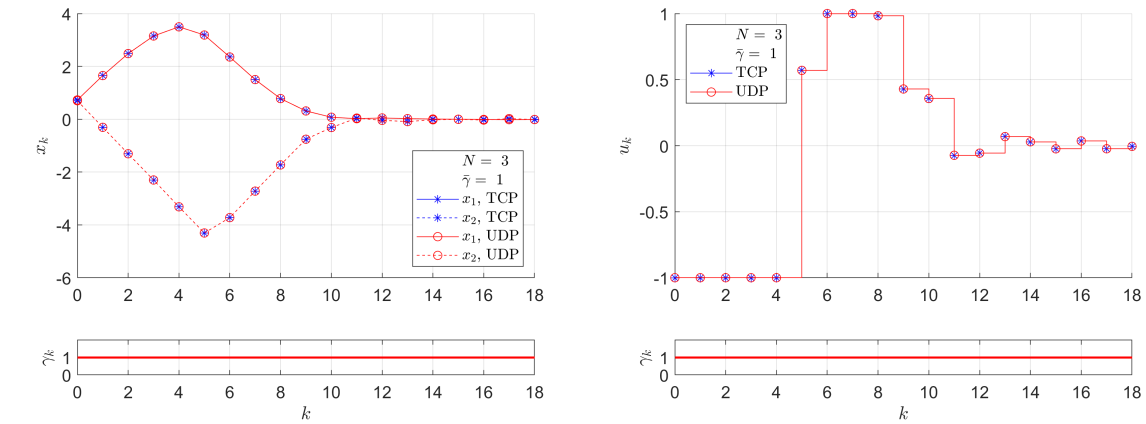
When there are no packet losses over the sensor channel, *i.e.* $\gamma_k = 1$ and $\bar{\gamma} = 1$, the system behaves like a traditional input-constrained stochastic MPC where the TCP-like and UDP-like



(a) Optimal cost (b) Estimation errors

Figure 4.6: Optimal cost and estimation errors under packet losses without input constraints.

schemes behave the same, Fig. 4.7 and Fig. 4.8. Moreover, we must recall that there are no packet losses over the actuation channel and $\bar{\nu} = 1$.

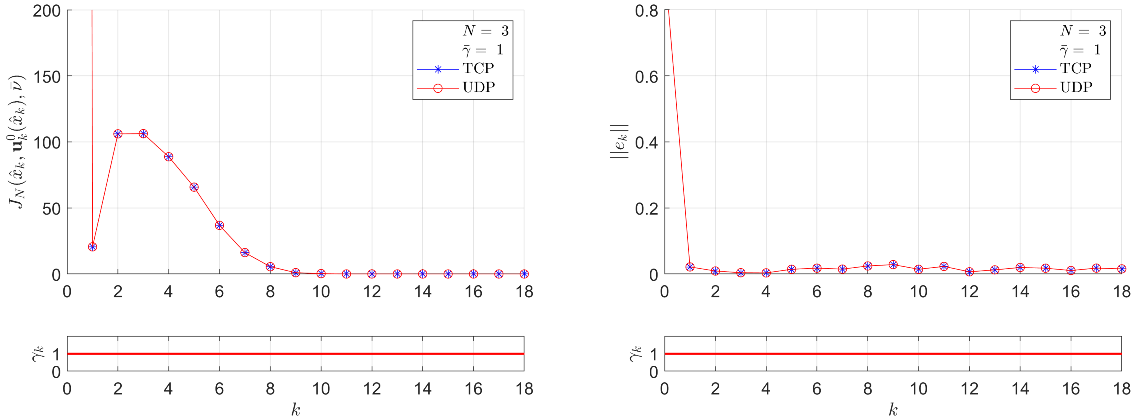


(a) States (b) Inputs

Figure 4.7: True state trajectories and applied controls without packet losses under input constraints.

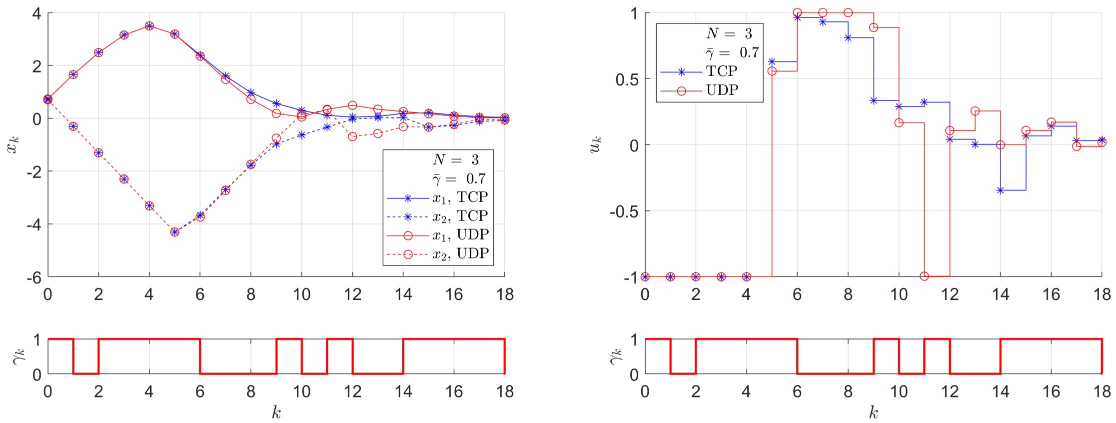
4.4.1.4 Fourth simulation: different initial covariance P_0

Changing the initial covariance to $P_0 = \text{diag}(8.579, 0)$, and repeating the same simulation under the same realizations of random variables, finds that both controllers maintain stability. Since K_k depends on P_0, this shows that the stability of the TCP-based controller can depend on the estimator gain, Fig. 4.9 and Fig. 4.10.



(a) Optimal cost (b) Estimation errors

Figure 4.8: Optimal cost and estimation errors without packet losses under input constraints.



(a) States (b) Inputs

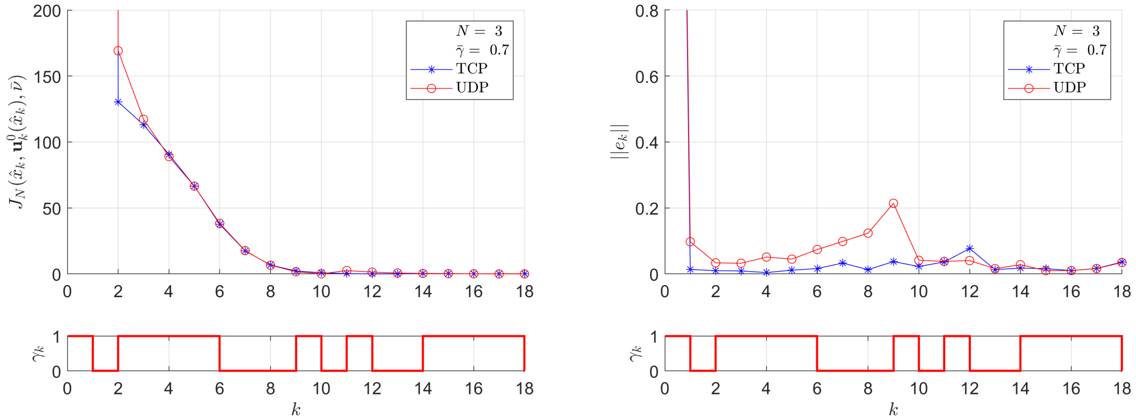
Figure 4.9: True state trajectories and applied controls under $P_0 = \text{diag}(8.579, 0)$ with packet losses.

4.4.1.5 Fifth simulation: increased β but the TCP-like scheme is unstable

Changing the design parameter β to 5 shows that the TCP-like controller losses stability, Fig. 4.11, meaning that increasing the terminal cost $\beta V_f(\cdot)$ is not sufficient to guarantee stability. In the next section we explore what happens if N is also increased.

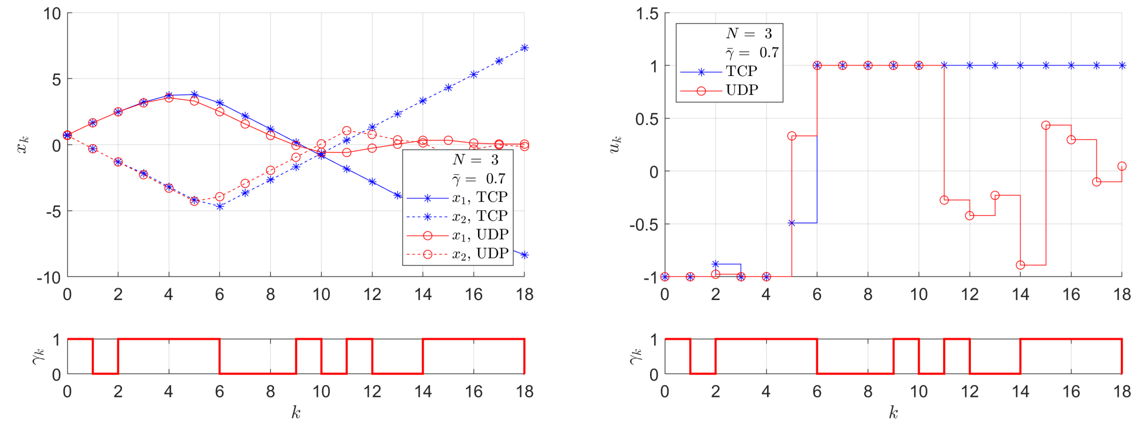
4.4.2 Analysis

Adhering to the aim of analysing stability of a formulation that omits state constraints, we consider the use of just the terminal cost $\beta V_f(\cdot)$ and horizon length N as stabilizing ingredients. These and the cost function are supposed to satisfy certain assumptions and definitions [110],



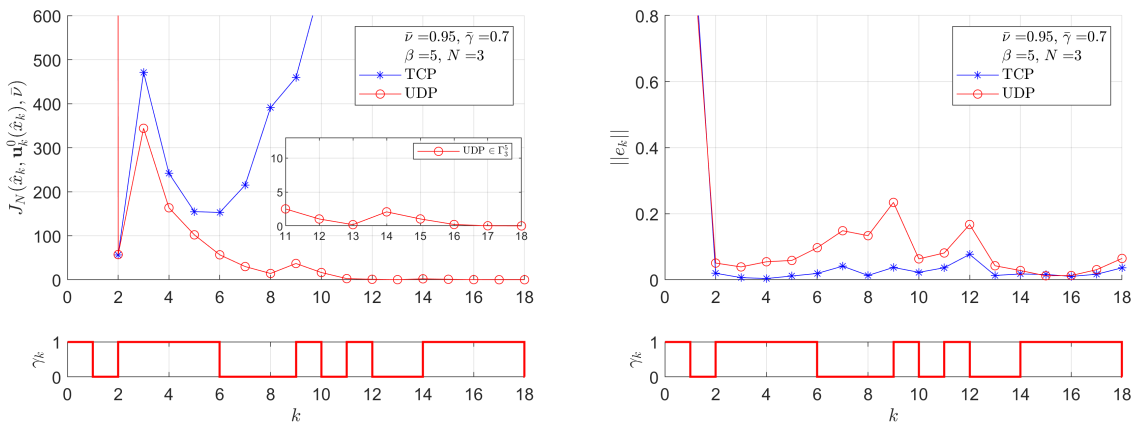
(a) Optimal cost (b) Estimation errors

Figure 4.10: Optimal cost and estimation errors under $P_0 = \text{diag}(8.579, 0)$ with packet losses.



(a) States (b) Inputs

Figure 4.11: True state trajectories and applied controls under $\beta = 5$ with packet losses.



(a) Optimal cost (b) Estimation errors

Figure 4.12: Optimal cost and estimation errors under $P_0 = \text{diag}(8.579, 0)$ with packet losses.

outlined in the next subsection.

4.4.2.1 Preliminaries: stability without a terminal set

Assumption 4.7. *The matrix $Q_f \succ 0$ is such that*

$$\big(A + \bar{v}BK_f\big)^\top Q_f\big(A + \bar{v}BK_f\big) - Q_f \leq -\Big(Q + \bar{v}^2 K_f^\top R K_f\Big) \tag{4.18}$$

for some K_f that stabilizes the pair (A, B).

The assumption says that $V_f(x) = x^\top Q_f x$ is a CLF for the expected terminal dynamics; it follows that $\beta V_f(\cdot), \beta \geq 1$, is also a CLF.

Definition 4.1. *Let $d_1 > 0$ be such that $K_f x \in \mathcal{U}$ for all*

$$x \in \mathcal{X}_f(d_1) := \big\{x \colon V_f(x) \leq d_1\big\}. \tag{4.19}$$

Such a d_1 is guaranteed to exist in view of the positive definiteness of Q_f (Assumption 4.6) and the fact that $\mathcal{U}$ contains the origin in its interior (Assumption 4.2).

The following result is an immediate consequence of the latter.

Lemma 4.1. *For all $x \in \mathcal{X}_f(d_1)$,*

$$V_f\big(Ax + \bar{v}BK_f x\big) - V_f(x) \leq -\ell\big(x, \bar{v}K_f x\big). \tag{4.20}$$

We require one more definition:

Definition 4.2. *Let $d_2 > 0$ be such that*

$$d_2 \leq \ell(x, \bar{v}u) \tag{4.21}$$

for all $x \notin \mathcal{X}_f(d_1)$ and $u \in \mathcal{U}$, and the given $\bar{v} \in (0, 1)$.

Such a d_2 is guaranteed to exist in view of positive definiteness of Q and R (Assumption 4.6) and compactness of $\mathcal{U}$ (Assumption 4.2).

Finally, given Definition 4.1 and Definition 4.2, we define the following set of states:

$$\Gamma_N^\beta := \{x \in \mathbb{R}^n : J_N^0(x, \mathbf{u}^0(z), \bar{v}) \leq \beta d_1 + N d_2\}. \tag{4.22}$$

By construction,

$$\hat{x}_k = \mathbb{E}\{x_k \mid \mathcal{I}_k\} \in \Gamma_N^\beta \iff J_N(\hat{x}_k, \mathbf{u}_k^0, \bar{v}) \leq \beta d_1 + N d_2.$$

The following result is adapted from [110], and concerns the evolution of the system $x^+ = Ax + \bar{v}Bu$ and optimal cost function $J_N(x, \mathbf{u}^0(x), \bar{v})$ when the loop is closed with $u = \kappa_N(x)$.

Lemma 4.2. *If $x \in \Gamma_N^\beta$, then the successor state $x^+ = Ax + \bar{v}B\kappa_N(x) \in \Gamma_N^\beta$ and*

$$J_N(x^+, \mathbf{u}^0(x^+), \bar{v}) - J_N(x, \mathbf{u}^0(x), \bar{v}) \leq -\ell(x, \bar{v}\kappa_N(x)). \tag{4.23}$$

Proof. See Appendix 4.A. □

4.4.2.2 Closed-loop analysis: prediction and estimation errors

In our setting, according to Lemma 4.2,

$$\hat{x}_k = \mathbb{E}\{x_k \mid \mathcal{I}_k\} \in \Gamma_N^\beta \implies z_{k+1|k} = \mathbb{E}\{x_{k+1} \mid \mathcal{I}_k\} = A\hat{x}_k + \bar{v}B\kappa_N(\hat{x}_k) \in \Gamma_N^\beta. \tag{4.24}$$

This successor state $z_{k+1|k} = \mathbb{E}\{x_{k+1} \mid \mathcal{I}_k\}$ is, however, the state estimate predicted by the controller at time k, using information $\mathcal{I}_k$, while the actual estimated state determined by the estimator at time $k+1$ is $\hat{x}_{k+1} = \mathbb{E}\{x_{k+1} \mid \mathcal{I}_{k+1}\}$:

$$\hat{x}_{k+1} = \hat{x}_{k+1|k} + \gamma_{k+1}K_{k+1}(y_{k+1} - C\hat{x}_{k+1|k}) \tag{4.25}$$

We note that the innovation $\hat{x}_{k+1|k}$ in this equation is not necessarily equal to the $z_{k+1|k}$ predicted by the controller:

$$z_{k+1|k} = A\hat{x}_k + \bar{v}Bu_{k|k}^0 \qquad \text{MPC prediction} \tag{4.26a}$$

$$\hat{x}_{k+1|k} = A\hat{x}_k + \nu_k Bu_{k|k}^0 \neq z_{k+1|k} \qquad \text{TCP-like innovation} \tag{4.26b}$$

$$\hat{x}_{k+1|k} = A\hat{x}_k + \bar{v}Bu_{k|k}^0 = z_{k+1|k} \qquad \text{UDP-like innovation} \tag{4.26c}$$

Note that the MPC prediction and TCP-like innovation are equal only if $\bar{\nu} = 1$ or $\bar{\nu} = 0$ but neither of both scenarios are interesting cases.

Therefore, we define the following errors terms.

Definition 4.3. *The **prediction error at** $k+1$ is the error between the new state estimate given by the estimator at $k+1$ and the one-step ahead state prediction at $k+1$ given by the controller at k, defined as*

$$\varepsilon_{k+1} \coloneqq \hat{x}_{k+1} - z_{k+1|k}, \tag{4.27}$$

which differs according to the protocol employed.

Definition 4.4. *The **estimation error at** $k+1$ is the error between the true new state at $k+1$ and the new state estimate given by the estimator at $k+1$, that is*

$$e_{k+1} \coloneqq x_{k+1} - \hat{x}_{k+1}, \tag{4.28}$$

again, which depends on the protocol employed.

In what follows we use the superscript `tcp` and `udp` to distinguish the protocol employed in order to obtain the relations between the prediction errors $\varepsilon_{k+1}^{\text{tcp}}$, $\varepsilon_{k+1}^{\text{udp}}$, and estimation errors e_{k+1}^{tcp}, e_{k+1}^{udp} respectively. The estimation error $e_k = x_k - \hat{x}_k$ at time k given by the KF in (4.3), is employed when required.

TCP-like protocol:

Since $\hat{x}_{k+1|k}$ is computed (at time $k+1$) using the available ν_k but the prediction $z_{k+1|k}$ used only $\bar{\nu}$, we have

$$\begin{aligned}
\varepsilon_{k+1}^{\text{tcp}} &= \hat{x}_{k+1} - z_{k+1|k} \\
&= \hat{x}_{k+1|k} + \gamma_{k+1} K_{k+1}\left(y_{k+1} - C\hat{x}_{k+1|k}\right) - \left(A\hat{x}_k + \bar{\nu} B u_{k|k}^0\right) \\
&= A\hat{x}_k + \nu_k B u_{k|k}^0 + \gamma_{k+1} K_{k+1}\left(Cx_{k+1} + s_{k+1} - C\hat{x}_{k+1|k}\right) - \left(A\hat{x}_k + \bar{\nu} B u_{k|k}^0\right) \\
&= \nu_k B u_{k|k}^0 + \gamma_{k+1} K_{k+1} C(A(x_k - \hat{x}_k) + w_k) + \gamma_{k+1} K_{k+1} s_{k+1} - \bar{\nu} B u_{k|k}^0 \\
&= (\nu_k - \bar{\nu}) B u_{k|k}^0 + \gamma_{k+1} K_{k+1} C(Ae_k + w_k) + \gamma_{k+1} K_{k+1} s_{k+1},
\end{aligned} \tag{4.29}$$

while the estimation error is

$$\begin{aligned}
e_{k+1}^{\text{tcp}} &= x_{k+1} - \hat{x}_{k+1} \\
&= Ax_k + \nu_k B u_{k|k}^0 + w_k - \big(\hat{x}_{k+1|k} + \gamma_{k+1} K_{k+1}\big(y_{k+1} - C\hat{x}_{k+1|k}\big)\big) \\
&= Ax_k + \nu_k B u_{k|k}^0 + w_k - \Big(A\hat{x}_k + \nu_k B u_{k|k}^0 + \gamma_{k+1} K_{k+1}\big(Cx_{k+1} + s_{k+1} - C\hat{x}_{k+1|k}\big)\Big) \\
&= Ax_k + \nu_k B u_{k|k}^0 + w_k - A\hat{x}_k - \nu_k B u_{k|k}^0 \\
&\qquad\qquad - \gamma_{k+1} K_{k+1}\Big(C\Big(Ax_k + \nu_k B u_{k|k}^0 + w_k\Big) + s_{k+1} - C\Big(A\hat{x}_k + \nu_k B u_{k|k}^0\Big)\Big) \\
&= (\mathbf{I} - \gamma_{k+1} K_{k+1} C)(Ae_k + w_k) - \gamma_{k+1} K_{k+1} s_{k+1}.
\end{aligned} \tag{4.30}$$

Thus note that

$$e_{k+1}^{\text{tcp}} + \varepsilon_{k+1}^{\text{tcp}} = Ae_k + (\nu_k - \bar{\nu}) B u_{k|k}^0 + w_k. \tag{4.31}$$

UDP-like protocol:

In this case both $\hat{x}_{k+1|k}$ and the prediction $z_{k+1|k}$ are computed using knowledge of only $\bar{\nu}$:

$$\begin{aligned}
\varepsilon_{k+1}^{\text{udp}} &= \hat{x}_{k+1} - z_{k+1} \\
&= \hat{x}_{k+1|k} + \gamma_{k+1} K_{k+1}\big(y_{k+1} - C\hat{x}_{k+1|k}\big) - \Big(A\hat{x}_k + \bar{\nu} B u_{k|k}^0\Big) \\
&= A\hat{x}_k + \bar{\nu} B u_{k|k}^0 + \gamma_{k+1} K_{k+1}\big(Cx_{k+1} + s_{k+1} - C\hat{x}_{k+1|k}\big) - \Big(A\hat{x}_k + \bar{\nu} B u_{k|k}^0\Big) \\
&= \gamma_{k+1} K_{k+1} C(\nu_k - \bar{\nu}) B u_{k|k}^0 + \gamma_{k+1} K_{k+1} C(Ae_k + w_k) + \gamma_{k+1} K_{k+1} s_{k+1},
\end{aligned} \tag{4.32}$$

while the estimation error is

$$\begin{aligned}
e_{k+1}^{\text{udp}} &= x_{k+1} - \hat{x}_{k+1} \\
&= Ax_k + \nu_k B u_{k|k}^0 + w_k - \big(\hat{x}_{k+1|k} + \gamma_{k+1} K_{k+1}\big(y_{k+1} - C\hat{x}_{k+1|k}\big)\big) \\
&= Ax_k + \nu_k B u_{k|k}^0 + w_k - \Big(A\hat{x}_k + \bar{\nu} B u_{k|k}^0 + \gamma_{k+1} K_{k+1}\big(Cx_{k+1} + s_{k+1} - C\hat{x}_{k+1|k}\big)\Big) \\
&= Ax_k + \nu_k B u_{k|k}^0 + w_k - A\hat{x}_k - \bar{\nu} B u_{k|k}^0 \\
&\qquad\qquad - \gamma_{k+1} K_{k+1}\Big(C\Big(Ax_k + \nu_k B u_{k|k}^0 + w_k\Big) + s_{k+1} - C\Big(A\hat{x}_k + \bar{\nu} B u_{k|k}^0\Big)\Big) \\
&= A(x_k - \hat{x}_k) + (\nu_k - \bar{\nu}) B u_{k|k}^0 + w_k \\
&\qquad\qquad - \gamma_{k+1} K_{k+1}\Big(C\Big(A(x_k - \hat{x}_k) + (\nu_k - \bar{\nu}) B u_{k|k}^0 + w_k\Big) + s_{k+1}\Big) \\
&= (\mathbf{I} - \gamma_{k+1} K_{k+1} C)(\nu_k - \bar{\nu}) B u_{k|k}^0 + (\mathbf{I} - \gamma_{k+1} K_{k+1} C)(Ae_k + w_k) - \gamma_{k+1} K_{k+1} s_{k+1}.
\end{aligned} \tag{4.33}$$

Thus note that, again,

$$e_{k+1}^{\text{udp}} + \varepsilon_{k+1}^{\text{udp}} = Ae_k + (\nu_k - \bar{\nu})Bu_{k|k}^0 + w_k. \tag{4.34}$$

This, together with the fact that if x_k and e_k are given, then $u_{k|k}^0 = \kappa_N(\hat{x}_k = x_k - e_k)$ is the same control in both UDP and TCP cases, proves the following.

Proposition 4.1. *For a given x_k and e_k, the following statements are true:*

$$e_{k+1}^{\text{udp}} + \varepsilon_{k+1}^{\text{udp}} = e_{k+1}^{\text{tcp}} + \varepsilon_{k+1}^{\text{tcp}}, \tag{4.35}$$

and

$$e_{k+1}^{\text{udp}} = e_{k+1}^{\text{tcp}} + (\mathbf{I} - \gamma_{k+1}K_{k+1}C)(\nu_k - \bar{\nu})Bu_{k|k}^0 \tag{4.36}$$

$$\varepsilon_{k+1}^{\text{udp}} = \varepsilon_{k+1}^{\text{tcp}} - (\mathbf{I} - \gamma_{k+1}K_{k+1}C)(\nu_k - \bar{\nu})Bu_{k|k}^0. \tag{4.37}$$

Proof. Comparing (4.31) and (4.34) leads to (4.35). From (4.33), we have that

$$e_{k+1}^{\text{udp}} = (\mathbf{I} - \gamma_{k+1}K_{k+1}C)(Ae_k + w_k) - \gamma_{k+1}K_{k+1}s_{k+1} + (\mathbf{I} - \gamma_{k+1}K_{k+1}C)(\nu_k - \bar{\nu})Bu_{k|k}^0,$$

and using (4.30), it follows that

$$= e_{k+1}^{\text{tcp}} + (\mathbf{I} - \gamma_{k+1}K_{k+1}C)(\nu_k - \bar{\nu})Bu_{k|k}^0,$$

proving (4.36). Applying the same procedure for (4.29) and (4.32) completes the proof of (4.37). □

The result characterizes a trade-off between the estimation error and prediction error depending on the channel protocol employed; if the effect of the input $u_{k|k}^0$ is to increase the estimation error in the UDP case compared with the TCP case, then a counter effect is to reduce the prediction error by the same margin. It is also worth pointing out that

$$\begin{aligned} e_{k+1} + \varepsilon_{k+1} &= (x_{k+1} - \hat{x}_{k+1}) + (\hat{x}_{k+1} - z_{k+1|k}) \\ &= x_{k+1} - z_{k+1|k}, \end{aligned} \tag{4.38}$$

so this quantity represents the (unknown) total error between true state and MPC prediction.

It is now clear that monotonicity of the value function cannot be assured since, in general, $\hat{x}_{k+1} \neq z_{k+1|k}$. Indeed, we may write for the cost function [100]

$$J_N\big(\tilde{x}_{k+1}, \mathbf{u}^0_{k+1}(\tilde{x}_{k+1}), \bar{v}\big) \leq J_N\big(z_{k+1|k}, \mathbf{u}^0_{k+1}(z_{k+1|k}), \bar{v}\big) + \sigma(\|\varepsilon_{k+1}\|), \tag{4.39}$$

where $\sigma(\cdot)$ is a function of class $\mathcal{K}$, and so, for all $\hat{x}_k \in \Gamma_N^\beta$,

$$J_N\big(\tilde{x}_{k+1}, \mathbf{u}^0_{k+1}(\tilde{x}_{k+1}), \bar{v}\big) - J_N\big(\hat{x}_k, \mathbf{u}^0_k(\hat{x}_k), \bar{v}\big) \leq -\ell(\hat{x}_k, \bar{v}\kappa_N(\hat{x}_k)) + \sigma(\|\varepsilon_{k+1}\|). \tag{4.40}$$

It is then of interest to determine when $\|\varepsilon_{k+1}\|$ is zero (or small), in order that

$$\tilde{x}_k \in \Gamma_N^\beta \implies \tilde{x}_{k+1} \in \Gamma_N^\beta, \tag{4.41}$$

as a key step towards ensuring stability of the controller.

The next result is an immediate result of the developed expressions (4.29) and (4.32).

Proposition 4.2. *If* $\gamma_{k+1} = 0$ *then*

1. ***UDP-like protocol:*** $\varepsilon^{\text{udp}}_{k+1} = 0$ *necessarily, so for all* $\hat{x}_k \in \Gamma_N^\beta$,

$$J_N\big(\tilde{x}_{k+1}, \mathbf{u}^0_{k+1}(\tilde{x}_{k+1}), \bar{v}\big) - J_N\big(\hat{x}_k, \mathbf{u}^0_k(\hat{x}_k), \bar{v}\big) \leq -\ell(\hat{x}_k, \bar{v}\kappa_N(\hat{x}_k)), \tag{4.42}$$

   and (4.41) *holds. However,*

$$e^{\text{udp}}_{k+1} = Ae_k + (v_k - \bar{v})Bu^0_{k|k} + w_k. \tag{4.43}$$

2. ***TCP-like protocol:*** $\varepsilon^{\text{tcp}}_{k+1} = (v_k - \bar{v})Bu^0_{k|k} \neq 0$ *whenever* $Bu^0_{k|k} \neq 0$. *Therefore, for all* $\hat{x}_k \in \Gamma_N^\beta$,

$$J_N\big(\tilde{x}_{k+1}, \mathbf{u}^0_{k+1}(\tilde{x}_{k+1}), \bar{v}\big) - J_N\big(\hat{x}_k, \mathbf{u}^0_k(\hat{x}_k), \bar{v}\big) \leq -\ell(\hat{x}_k, \bar{v}\kappa_N(\hat{x}_k)) + \sigma\Big(\Big\|(v_k - \bar{v})Bu^0_{k|k}\Big\|\Big), \tag{4.44}$$

   and (4.41) *does not necessarily hold. Moreover,*

$$e^{\text{tcp}}_{k+1} = Ae_k + w_k. \tag{4.45}$$

Proof. For the UDP-like protocol, substituting $\gamma_{k+1} = 0$ in (4.32) leads to $\sigma(\|\varepsilon_{k+1}\|) = 0$ in (4.40), and applying the same $\gamma_{k+1} = 0$ in (4.33) results in (4.43). Following the same procedure for the TCP-like protocol by substituting $\gamma_{k+1} = 0$ in (4.29) leads to $\sigma(\|\varepsilon_{k+1}\|) \neq 0$ in (4.40), and $\gamma_{k+1} = 0$ in (4.30) results in (4.45). This completes the proof. □

This result depicts a kind of reverse separation principle wherein, in the case of sensor dropouts, the UDP-MPC cost function enjoys a monotonic decrease, independent of the estimator, if $\hat{x}_k \in \Gamma_N^\beta$. The estimator performance is, however, dependent on the control input. In the TCP case, on the other hand, the estimator is independent of the controller (*c.f.* the separation observed by [41]), but the monotonicity of the controller cost function is now assured only for suitably small inputs.

4.4.3 Revisiting the counterexample

The designed controller in Section 4.4.1 satisfies Assumptions 4.6–4.2, the latter with $d_1 = 1.85$ and $d_2 = 1.2$; therefore, with $N = 3$ and $\beta = 1$,

$$\Gamma_3^1 = \{x \colon J_N(x, \mathbf{u}^0(x), \bar{\nu}) \leq 5.45\}.$$

The initial state estimate $\hat{x}_0 \notin \Gamma_3^1$; however, in the UDP case the state estimate enters Γ_3^1 at $k = 11$ and remains therein. In the TCP case, the state estimate never enters Γ_3^1; the cost reaches a minimum of 41.16 at $k = 2$ before diverging, Fig. 4.13 and Fig. 4.14.

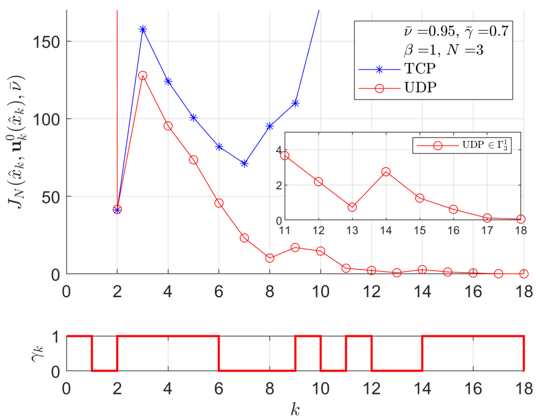
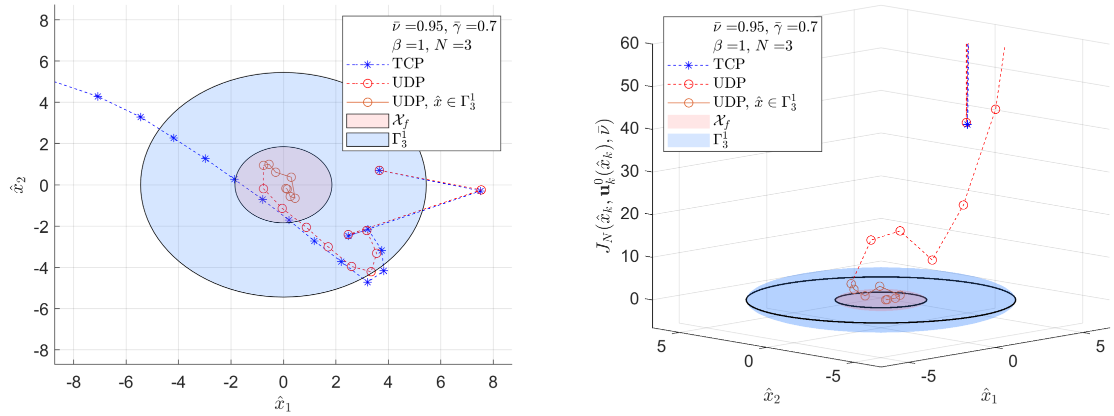


Figure 4.13: Optimal cost value in the UDP case.



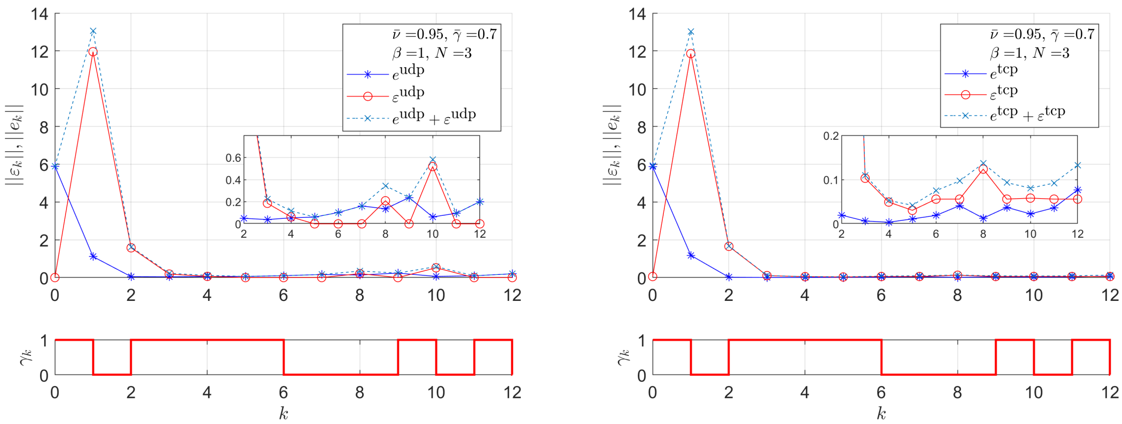
(a) Phase-portrait (b) Phase-portrait vs optimal cost value

Figure 4.14: Phase-portrait and optimal cost value.

4.4.3.1 Verifying Proposition 4.1 and Proposition 4.2

In Fig. 4.13, it can be seen that whenever $\gamma_{k+1} = 0$, $J_N\left(\hat{x}_{k+1}, \mathbf{u}_{k+1}^0(\hat{x}_{k+1}), \bar{\nu}\right) < J_N\left(\hat{x}_k, \mathbf{u}_k^0(\hat{x}_k), \bar{\nu}\right)$ (Proposition 4.2). The increase at $k = 14$ (when γ_k rises from 0 to 1) is explained by Proposition 4.1: the state estimate $\hat{x}_{14}$ is improved over the prediction $\bar{x}_{14|13} = z_{14|13}$ at the expense of higher prediction error.

We can also verify (4.36) and (4.37) (Proposition 4.1) by using the input values shown in Fig. 4.3(b) and the estimation and prediction errors for the TCP and UDP cases in Fig. 4.15 whenever $\gamma_{k+1} = 0$. For the positive input value $u_{6|6}^0 = 1$ at $k = 6$, and $\gamma_{6+1} = 0$, the estimation and prediction errors satisfy $e_{6+1}^{\text{udp}} > \varepsilon_{6+1}^{\text{udp}}$, and $e_{6+1}^{\text{tcp}} < \varepsilon_{6+1}^{\text{tcp}}$.



(a) UDP case (b) TCP case

Figure 4.15: Estimation and prediction errors for the TCP and UDP cases.

4.4.3.2 Increasing N

We have seen before (Fig. 4.13 and Fig. 4.14), in the TCP case, the state estimate never enters Γ_3^1. However, increasing N to 4 results in the state estimate entering

$$\Gamma_4^1 = \left\{x \colon J_N\left(x, \mathbf{u}^0(x), \bar{\nu}\right) \leq 6.65\right\}$$

at $k = 9$ on both protocols, and subsequently maintaining stability, see Fig. 4.16 and Fig. 4.17.

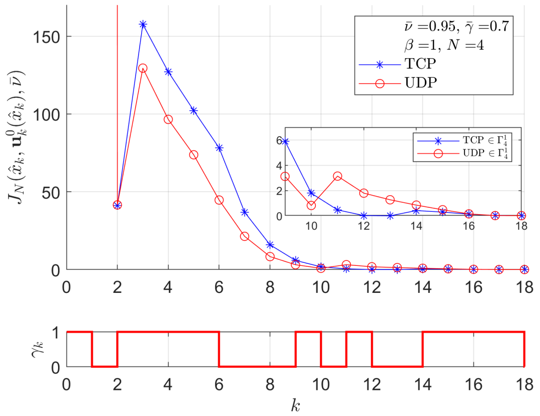
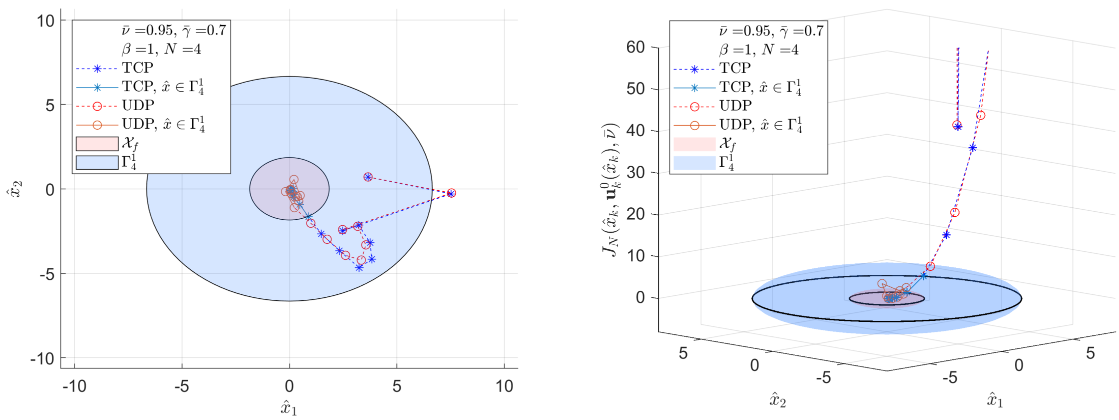


Figure 4.16: Optimal cost value in the UDP case.



(a) Phase-portrait

(b) Phase-portrait vs optimal cost value

Figure 4.17: Phase-portrait and optimal cost value.

4.4.3.3 Increasing N and β

Increasing β to 3 and keeping N equals to 4 results in the state estimate entering

$$\Gamma_4^3 = \left\{x \colon J_N\left(x, \mathbf{u}^0(x), \bar{\nu}\right) \leq 10.35\right\}$$

at $k = 8$ over the UDP protocol, and at $k = 9$ over the TCP protocol, see Fig. 4.18 and Fig. 4.19.

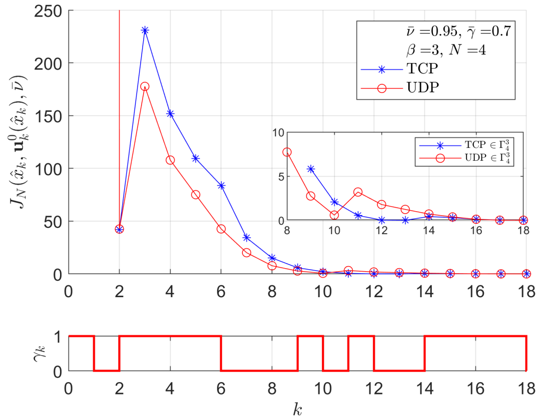
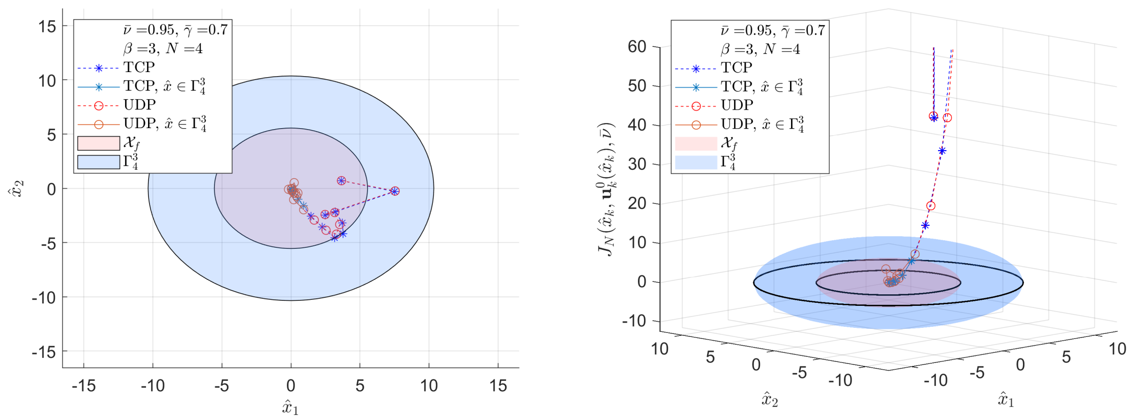


Figure 4.18: Optimal cost value in the UDP case.



(a) Phase-portrait

(b) Phase-portrait vs optimal cost value

Figure 4.19: Phase-portrait and optimal cost value.

4.5 Explicit input-constrained solution

In the previous section we performed an initial analysis of the problem using the established properties of the MPC control law and value function; that control law, however, is merely implicit. It is well known that when the objective function is quadratic [102], the system dynamics are linear, and the constraints are polyhedral or polytopic, the value function becomes PWQ, and the control law is PWA. The aim in this section is to show the effect of $\bar{\nu}$ on the PWA control law and to investigate whether any additional insight emerges by using these facts.

We start by substituting (4.14b) and (4.14a) into the optimal (4.13)

$$V_N^0(\mathcal{I}_k) = \min_{\mathbf{u}_k \in \mathbb{U}} z_{k|k}^\top Q z_{k|k} + \mathbf{z}_k^\top \tilde{Q} \mathbf{z}_k + \bar{v}^2 \mathbf{u}_k^\top \tilde{R} \mathbf{u}_k + c(P_k), \tag{4.46a}$$

subject to, for $j \in \mathbb{N}_{[0,N-1]}$,

$$\mathbf{z}_k = F z_{k|k} + G \bar{v} \mathbf{u}_k \tag{4.46b}$$

$$\mathbf{u}_k \in \mathcal{U}, \tag{4.46c}$$

where $\tilde{Q} = \operatorname{diag}(\mathbf{I}_{(N-1)\times(N-1)} \otimes Q, \beta Q_f)$, $\tilde{R} = \mathbf{I}_{N\times N} \otimes R$, $F = \left[A, A^2, \ldots, A^N\right]^\top$,

$$G = \begin{bmatrix} B & 0 & \ldots & 0 \\ AB & B & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B & A^{N-2}B & \ldots & B \end{bmatrix},$$

$\mathbf{z}_k = \mathbb{E}\{\mathbf{x}_k \,|\, \mathcal{I}_k\} = \mathbb{E}\left\{F x_{k|k} + G \boldsymbol{\nu}_k \mathbf{u}_k \,|\, \mathcal{I}_k\right\} = \left[z_{k|k}, \ldots, z_{k+N|k}\right]^\top$, $\mathbf{x}_k = \left[x_{k|k}, \ldots, x_{k+N|k}\right]^\top$, $\boldsymbol{\nu}_k = \left[\nu_{k|k}, \ldots, \nu_{k+N-1|k}\right]^\top$, and $\mathbf{u}_k = \left[u_{k|k}, \ldots, u_{k+N-1|k}\right]^\top$, allow us to formulate the mp-QP.

Before we proceed, we need the following.

Definition 4.5. *For $\mathbf{u}_k \in \mathcal{U}$ and $\mathbf{z}_k \in \mathbb{R}^n$, let $H = 2(G^\top \tilde{Q} G + \tilde{R})$, $H = H^\top \succ 0$, $M = Q + F^\top \tilde{Q} F$, and $L = 2G^\top \tilde{Q} F$. P_u and Q_u are inequality constraints of appropriate dimensions.*

By Definition 4.5, if $\bar{v} \in (0, 1]$ and $\beta \geq 1$, then for all $z_{k|k} \in \mathbb{R}^n$ and $u_k \in \mathcal{U} \subseteq \mathbb{R}^m$, the quadratic problem

$$V_N^0(z_{k|k}, P_k) = \min_{\mathbf{u}_k \in \mathbb{U}} \frac{1}{2} \mathbf{u}_k^\top \bar{v}^2 H \mathbf{u}_k + \bar{v} c^\top \mathbf{u}_k + \alpha, \tag{4.47a}$$

subject to,

$$P_u \mathbf{u}_k \leq Q_u, \tag{4.47b}$$

where $\alpha = z_{k|k}^\top M z_{k|k} + c(P_k)$, and $c = L z_{k|k}$, has an optimal (unique) solution

$$\mathbf{u}_k^0(z_{k|k}) = \arg\min_{\mathbf{u}_k \in \mathbb{U}} \left\{ \frac{1}{2} \mathbf{u}_k^\top \bar{v}^2 H \mathbf{u}_k + \bar{v} c^\top \mathbf{u}_k + \alpha \colon P_u \mathbf{u}_k \leq Q_u \right\} \tag{4.48a}$$

$$= \{u^0_{k|k}(z_{k|k}), \ldots, u^0_{k+N-1|k}(z_{k|k})\}, \tag{4.48b}$$

over the feasible set $\mathcal{U}_u = \{u_k \in \mathbb{R}^m : P_u \mathbf{u}_k \leq Q_u\}$.

Solving (4.47) using numerical methods leads to an implicit on-line solution. Our objective is to obtain an explicit PWA off-line solution for all possible initial states. For this matter, the mp-QP formulation [102] is presented as follows.

4.5.1 mp-QP formulation

Applying the Lagrange multiplier function to the problem (4.47)

$$\mathcal{L}(z_{k|k}, \mathbf{u}_k, \lambda) = \frac{1}{2}\mathbf{u}_k^\top \bar{\nu}^2 H \mathbf{u}_k + \bar{\nu} c^\top \mathbf{u}_k + \alpha + \lambda^\top (P_u \mathbf{u}_k - Q_u), \tag{4.49a}$$

such that

$$\frac{\partial \mathcal{L}}{\partial \mathbf{u}_k} = 0, \quad \bar{\nu}^2 H \mathbf{u}_k + \bar{\nu} c + P_u^\top \lambda = 0, \tag{4.49b}$$

and if there exists H^{-1},

$$\mathbf{u}_k + \left(\bar{\nu}^2 H\right)^{-1} \bar{\nu} c = -\left(\bar{\nu}^2 H\right)^{-1} P_u^\top \lambda, \tag{4.49c}$$

allow us to define

$$\boldsymbol{\mu}_k = \mathbf{u}_k + (\bar{\nu} H)^{-1} L z_{k|k}. \tag{4.49d}$$

Therefore, by Definition 4.5, if $\bar{\nu} \in (0, 1]$ and $\beta \geq 1$, then for all $z_{k|k} \in \mathbb{R}^n$ and $u_k \in \mathcal{U} \subseteq \mathbb{R}^m$, the equivalent mp-QP problem of (4.47) is

$$\tilde{V}_N^0(z_{k|k}) = \min_{\boldsymbol{\mu}_k \in \mathbb{U}} \frac{1}{2}\boldsymbol{\mu}_k^\top \bar{\nu}^2 H \boldsymbol{\mu}_k, \tag{4.50a}$$

subject to,

$$P_u \boldsymbol{\mu}_k \leq Q_u + S_u z_{k|k}, \tag{4.50b}$$

where $S_u = P_u(\bar{\nu}H)^{-1}L$, $\tilde{V}_N^0(z_{k|k}) = V_N^0(z_{k|k}, P_k) - \rho$ and $\rho = \alpha - \frac{1}{2}c^\top H^{-1}c$, has an optimal (unique) solution

$$\boldsymbol{\mu}_k^0(z_{k|k}) = \arg\min_{\boldsymbol{\mu}_k \in \tilde{\mathcal{U}}_\mu} \left\{ \frac{1}{2}\boldsymbol{\mu}_k^\top \bar{\nu}^2 H \boldsymbol{\mu}_k : P_u \boldsymbol{\mu}_k \leq Q_u + S_u z_{k|k} \right\} \tag{4.51a}$$

$$= \left\{ \mu_{k|k}^0(z_{k|k}), \ldots, \mu_{k+N-1|k}^0(z_{k|k}) \right\}, \tag{4.51b}$$

over the feasible set $\tilde{\mathcal{U}}_\mu = \left\{ \mu_k \in \mathbb{R}^m : P_u \boldsymbol{\mu}_k \leq Q_u + S_u z_{k|k} \right\}$.

The explicit solution $\boldsymbol{\mu}_k^0$ can be obtained using the first-order Karush-Kuhn-Tucker (KKT) optimally conditions where we exploit the convexity of the quadratic function of the problem and the linearity of the constraints.

4.5.2 Explicit solution

Before we proceed with the solution, we define the following.

Definition 4.6 (Active and inactive constraints, adapted from [102]). *Let $\bar{\boldsymbol{\mu}}$ denote a set of feasible points in $\tilde{\mathcal{U}}_\mu$. The i-th inequality constraint $g_i(\boldsymbol{\mu}) \leq 0$ is active at $\bar{\boldsymbol{\mu}}$ if $g_i(\bar{\boldsymbol{\mu}}) = 0$, and is inactive at $\bar{\boldsymbol{\mu}}$ if $g_i(\bar{\boldsymbol{\mu}}) < 0$.*

Definition 4.7 (Sets of active and inactive constraints, [102]). *If P_{u_j}, S_{u_j} and Q_{u_j} denote the j-th row of P_u, Q_u and S_u respectively, where $j \in \mathbb{N}_{[1,m]}$, then $\hat{A}$ be the set of active constraints for which the sub-matrices of P_u, Q_u and S_u are active at the optimum $\boldsymbol{\mu}_k^0(z_{k|k})$,* i.e.

$$\hat{A} = \left\{ z_{k|k} \in \mathbb{R}^n, \boldsymbol{\mu}_k^0(z_{k|k}) \in \tilde{\mathcal{U}}_\mu : P_{u_j}\boldsymbol{\mu}_k^0(z_{k|k}) - S_{u_j} z_{k|k} = Q_{u_j} \right\}, \tag{4.52}$$

and let $\check{A}$ be the set of inactive constraints for which the sub-matrices of P_u, Q_u and S_u are inactive at the optimum $\boldsymbol{\mu}_k^0(z_{k|k})$, i.e.

$$\check{A} = \left\{ z_{k|k} \in \mathbb{R}^n, \boldsymbol{\mu}_k^0(z_{k|k}) \in \tilde{\mathcal{U}}_\mu : P_{u_j}\boldsymbol{\mu}_k^0(z_{k|k}) - S_{u_j} z_{k|k} < Q_{u_j} \right\}. \tag{4.53}$$

The union of the active constraint set $\hat{A}$ and the inactive constraint set $\check{A}$ is the set of all constraints, and their intersection is empty.

Using the previous definitions, we are able to formulate the following.

Definition 4.8 (Critical Region, [102])**.** *The critical region CR_0 associated with the set of active constraints $\hat{A}$ is the set of all $z_{k|k} \in \mathbb{R}^n$ such that the constraints indexed by $\hat{A}$ are active at the optimum $\boldsymbol{\mu}_k^0$.*

Considering the previous, in what follows we show the KKT conditions under which the PWA optimizer $\boldsymbol{\mu}_k^0$ exists in the $\mathcal{H}$-polyhedral representation of the CR_0.

Theorem 4.1. *Let $\hat{P}_u$, $\hat{Q}_u$ and $\hat{S}_u$ denote the sub-matrices of P_u, Q_u and S_u in $\hat{A}$, then the optimal $\boldsymbol{\mu}_k^0(z_{k|k})$ and the corresponding Lagrange multiplier $\hat{\lambda}$ for the active constraints in the problem* (4.50) *are continuous PWA functions of the state $z_{k|k}$ over the CR_0, such that*

$$\boldsymbol{\mu}_k^0 = H^{-1}\hat{P}_u^\top\left(\hat{P}_u H^{-1}\hat{P}_u^\top\right)^{-1}\left(\hat{Q}_u + \hat{S}_u z_{k|k}\right) \tag{4.54a}$$

$$\hat{\lambda} = -\left(\hat{P}_u^\top\left(\bar{v}^2 H\right)^{-1}\hat{P}_u^\top\right)^{-1}\left(\hat{Q}_u + \hat{S}_u z_{k|k}\right). \tag{4.54b}$$

Proof. Defining the Lagrange multiplier function for (4.50)

$$\mathcal{L}(z_{k|k}, \boldsymbol{\mu}_k, \lambda) = \frac{1}{2}\boldsymbol{\mu}_k^\top \bar{v}^2 H \boldsymbol{\mu}_k + \lambda^\top\left(P_u\boldsymbol{\mu}_k - Q_u - S_u z_{k|k}\right),$$

then, the optimal $\boldsymbol{\mu}_k^0$ satisfies the following KKT conditions:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{\mu}_k} = 0, \quad \bar{v}^2 H \boldsymbol{\mu}_k^0 + P_u^\top \lambda = 0, \quad \lambda \in \mathbb{R}^q, q \in \mathbb{N}_{>0} \tag{4.55a}$$

$$\lambda_i\left(P_u^i \boldsymbol{\mu}_k^0 - Q_u^i - S_u^i z_{k|k}\right) = 0, \quad i \in \mathbb{N}_{[1,q]} \tag{4.55b}$$

$$\lambda \geq 0, \quad \text{dual feasibility condition,} \tag{4.55c}$$

$$P_u \boldsymbol{\mu}_k^0 - Q_u - S_u z_{k|k} \leq 0, \quad \text{primal feasibility condition,} \tag{4.55d}$$

where the superscript denotes the i-th row, [112].

Solving (4.55a) leads to

$$\boldsymbol{\mu}_k^0 = -\left(\bar{v}^2 H\right)^{-1} P_u^\top \lambda, \tag{4.56}$$

and replacing the result in (4.55b) we have the complementary slackness conditions

$$\lambda\left(-P_u\left(\bar{v}^2 H\right)^{-1} P_u^\top \lambda - Q_u - S_u z_{k|k}\right) = 0.$$

Let us define the primal feasibility condition (4.55d) for the corresponding sets of active $\hat{A}$ and inactive $\check{A}$ constraints, in which $\hat{P}_u, \hat{Q}_u$ and $\hat{S}_u$ correspond to the sub-matrices of P_u, Q_u and $S_u \in \hat{A}$, and $\check{P}_u, \check{Q}_u$ and $\check{S}_u$ correspond to the sub-matrices of P_u, Q_u and $S_u \in \check{A}$, *i.e.*

$$\begin{aligned}\hat{P}_u \boldsymbol{\mu}_k^0 - \hat{S}_u z_{k|k} &= \hat{Q}_u \\ \check{P}_u \boldsymbol{\mu}_k^0 - \check{S}_u z_{k|k} &< \check{Q}_u,\end{aligned}$$

and let $\hat{\lambda}$ and $\check{\lambda}$ denote the Lagrange multipliers for the active and inactive constraints respectively, we have that from (4.55b) $\check{\lambda} = 0$ and

$$\hat{\lambda} = -\left(\hat{P}_u \left(\bar{\nu}^2 H\right)^{-1} \hat{P}_u^\top\right)^{-1} \left(\hat{Q}_u + \hat{S}_u z_{k|k}\right)$$

is a PWA function of $z_{k|k}$.

Therefore, substituting the latter in (4.56) leads to

$$\boldsymbol{\mu}_k^0 = H^{-1} \hat{P}_u^\top \left(\hat{P}_u H^{-1} \hat{P}_u^\top\right)^{-1} \left(\hat{Q}_u + \hat{S}_u z_{k|k}\right) \tag{4.58}$$

is a PWA function of $z_{k|k}$. □

We must remark that both $\hat{\lambda}$ and $\boldsymbol{\mu}_k^0$ depend on $\bar{\nu}$ due to $S_u = P_u(\bar{\nu}H)^{-1}L$ from (4.50) implies $\hat{S}_u = \hat{P}_u(\bar{\nu}H)^{-1}L$.

Theorem 4.2 (Critical Region CR_0)**.** *Let $\mathcal{P}_d(\hat{A}, \hat{\lambda})$ denote the set of full-rank linear independent combinations of $\hat{A}$ and the Lagrange multiplier $\hat{\lambda}$ corresponding to the active constraints for the dual feasibility conditions, and let $\mathcal{P}_p(\hat{A}, \check{A}, \hat{\lambda})$ denote the set of full-rank linear independent combination of $\hat{A}$, $\check{A}$ and $\hat{\lambda}$ for the primal feasibility conditions, then there exists a critical region for all $z_{k|k} \in \mathbb{R}^n$ and $\bar{\nu} \in (0, 1]$ such that the combination of $\mathcal{P}_p$ and $\mathcal{P}_d$ is active at the optimum* $\boldsymbol{\mu}_k^0$, i.e.

$$CR_0 = \left\{z_{k|k} \in \mathbb{R}^n \colon \mathcal{P}_p \bigcap \mathcal{P}_d\right\}. \tag{4.59}$$

Proof. The optimizer $\boldsymbol{\mu}_k^0$ from (4.54a) must satisfy the primal feasibility conditions (4.55d)

$$\mathcal{P}_p = \left\{z_{k|k} \in \mathbb{R}^n \colon P_u H^{-1} \hat{P}_u^\top \left(\hat{P}_u H^{-1} \hat{P}_u^\top\right)^{-1} \left(\hat{Q}_u + \hat{S}_u z_{k|k}\right) \le Q_u + S_u z_{k|k}\right\} \tag{4.60}$$

and the Lagrange multipliers $\hat{\lambda}$ from (4.54b) must satisfy the dual feasibility conditions (4.55c)

$$\mathcal{P}_d = \left\{ z_{k|k} \in \mathbb{R}^n \colon\ -\left(\hat{P}_u^\top \left(\bar{\nu}^2 H\right)^{-1} \hat{P}_u^\top\right)^{-1} \left(\hat{Q}_u + \hat{S}_u z_{k|k}\right) \geq 0 \right\}, \tag{4.61}$$

which completes the Proof. □

As shown in [113], after obtaining CR_0 the rest of the critical regions $CR_j = \tilde{\mathcal{U}}_\mu \setminus CR_0$ must be explored. In the following lemma we show the conditions for partitioning the rest of the space.

Lemma 4.3 (Critical Regions). *Let us define* $CR_0 \coloneqq \left\{ z_{k|k} \in \mathbb{R}^n \colon Az_{k|k} \leq b \right\}$ *and* $CR_j \coloneqq \left\{ z_{k|k} \in \mathbb{R}^n \colon A_j z_{k|k} \leq b_j \right\}$*, then* CR_0 *and* CR_j *are polytopes mutually disjoint partitions of* $\tilde{\mathcal{U}}_\mu$ *such that*

$$\tilde{\mathcal{U}}_\mu \coloneqq CR_0 \bigcup \left(\bigcup_{j=1}^{n_r} CR_j \right) \subset \mathcal{U}_u, \quad j \in \mathbb{N}_{[1,n_r]}, \tag{4.62}$$

where $n_r \coloneqq \dim(b)$ *is the number of critical regions generated.*

CR_0 is obtained by following Theorem 4.2 and the rest CR_j follows the same procedure for each new region until the whole state space is covered.

Proof. The proof is presented in [112]. □

4.5.3 Control law

Now we are in conditions to present the corresponding PWA control law $\mathbf{u}_k^0$ and the PWQ value function $V_N^0(z_{k|k})$ in the $\mathcal{H}$-polyhedral representation of the critical regions for the problem (4.50).

Theorem 4.3. *If there exists H^{-1} and $\bar{\nu} \in (0, 1]$, then the explicit optimal sequence $\mathbf{u}_k^0(z_{k|k})$ of the problem* (4.13) *is continuous PWA function on polyhedra for $z_{k|k}$ such that*

$$\mathbf{u}_k^0(z_{k|k}) = K_{N_j} z_{k|k} + g_j, \quad \textit{if} \quad z_{k|k} \in CR_j,\ j \in \mathbb{N}_{[0,n_r]}, \tag{4.63a}$$

where

$$K_{N_j} = F_j - (\bar{\nu} H)^{-1} L \tag{4.63b}$$

$$F_j = H^{-1} \hat{P}_u^\top (\hat{P}_u H^{-1} \hat{P}_u^\top)^{-1} \hat{P}_u (\bar{\nu} H)^{-1} L \tag{4.63c}$$

$$g_j = H^{-1}\hat{P}_u^\top(\hat{P}_u H^{-1}\hat{P}_u^\top)^{-1}\hat{Q}_u, \tag{4.63d}$$

and the control law

$$u_{k|k}^0(z_{k|k}) = K_{N_0} z_{k|k} + g_0, \tag{4.64a}$$

where

$$K_{N_0} = [\mathbf{I}, 0, \ldots, 0] K_{N_j} \tag{4.64b}$$

$$g_0 = [\mathbf{I}, 0, \ldots, 0] g_j, \tag{4.64c}$$

is an explicit continuous PWA function on polyhedra for all $z_{k|k}$.

Proof. It follows directly using (4.58) in (4.49d) such that

$$\begin{aligned} \mathbf{u}_k^0 &= \boldsymbol{\mu}_k^0 - (\bar{\nu} H)^{-1} L z_{k|k} \\ &= H^{-1}\hat{P}_u^\top \left(\hat{P}_u H^{-1}\hat{P}_u^\top\right)^{-1}\left(\hat{Q}_u + \hat{S}_u z_{k|k}\right) - (\bar{\nu} H)^{-1} L z_{k|k}, \end{aligned}$$

and since $S_u = P_u(\bar{\nu} H)^{-1} L$ from (4.50) implies $\hat{S}_u = \hat{P}_u(\bar{\nu} H)^{-1} L$, then

$$= H^{-1}\hat{P}_u^\top \left(\hat{P}_u H^{-1}\hat{P}_u^\top\right)^{-1}\left(\hat{Q}_u + \hat{P}_u(\bar{\nu} H)^{-1} L z_{k|k}\right) - (\bar{\nu} H)^{-1} L z_{k|k},$$

reordering completes the proof. □

Theorem 4.4. *If* $\bar{\nu} \in (0, 1]$ *and* $\beta \geq 1$, *then the value function* V_N^0 *and the cost function* J_N *for the problem* (4.50) *related as*

$$V_N^0(z_{k|k}, P_k) = J_N\left(z_{k|k}, \mathbf{u}_k^0(z_{k|k}), \bar{\nu}\right) + c(P_k), \tag{4.65}$$

has a PWQ function on polyhedra for $z_{k|k}$ *such that*

$$J_N(z_{k|k}, \mathbf{u}_k^0(z_{k|k}), \bar{\nu}) = \frac{1}{2} z_{k|k}^\top E_j z_{k|k} + d_j^\top z_{k|k} + e_j, \quad \textit{if} \quad z_{k|k} \in CR_j, j \in \mathbb{N}_{[0,n_r]}, \tag{4.66a}$$

where

$$E_j = K_{N_j}^\top \bar{v}^2 H K_{N_j} + 2\bar{v} L^\top K_{N_j} + 2M \tag{4.66b}$$

$$d_j = \left(K_{N_j}^\top \bar{v}^2 H + \bar{v} L^\top\right) g_j \tag{4.66c}$$

$$e_j = g_j^\top \bar{v}^2 H g_j. \tag{4.66d}$$

Proof. It follows directly by using (4.63a) in (4.47). □

4.5.4 Effects of the design parameters over the explicit control law and the CR_0

The results from Theorem 4.2 and Theorem 4.3 reveal the actual effects of the design parameters β, N and $\bar{v}$ over the critical region CR_0 and the explicit control law $\mathbf{u}^0$.

First, since $S_u = P_u(\bar{v}H)^{-1}L$ from (4.50) implies $\hat{S}_u = \hat{P}_u(\bar{v}H)^{-1}L$, and according to (4.60), the set $\mathcal{P}_p$ is penalized by $\bar{v}$, *i.e.*

$$\begin{aligned}
\mathcal{P}_p &= \left\{z_{k|k} \in \mathbb{R}^n : P_u H^{-1} \hat{P}_u^\top \left(\hat{P}_u H^{-1} \hat{P}_u^\top\right)^{-1} \left(\hat{Q}_u + \hat{S}_u z_{k|k}\right) \leq Q_u + S_u z_{k|k}\right\} \\
&= \left\{z_{k|k} \in \mathbb{R}^n : P_u H^{-1} \hat{P}_u^\top \left(\hat{P}_u H^{-1} \hat{P}_u^\top\right)^{-1} \left(\hat{Q}_u + \hat{P}_u(\bar{v}H)^{-1} L z_{k|k}\right) \leq Q_u + P_u(\bar{v}H)^{-1} L z_{k|k}\right\} \\
&= \left\{z_{k|k} \in \mathbb{R}^n : P_u H^{-1} \hat{P}_u^\top \left(\hat{P}_u H^{-1} \hat{P}_u^\top\right)^{-1} \left(\bar{v}\hat{Q}_u + \hat{P}_u H^{-1} L z_{k|k}\right) \leq \bar{v} Q_u + P_u H^{-1} L z_{k|k}\right\}.
\end{aligned} \tag{4.67}$$

Second, according to (4.61), the set $\mathcal{P}_d$ is also penalized by $\bar{v}$, *i.e.*

$$\begin{aligned}
\mathcal{P}_d &= \left\{z_{k|k} \in \mathbb{R}^n : -\bar{v}^2 \left(\hat{P}_u^\top H^{-1} \hat{P}_u^\top\right)^{-1} \left(\hat{Q}_u + \hat{S}_u z_{k|k}\right) \geq 0\right\} \\
&= \left\{z_{k|k} \in \mathbb{R}^n : -\bar{v}^2 \left(\hat{P}_u^\top H^{-1} \hat{P}_u^\top\right)^{-1} \left(\hat{Q}_u + \hat{P}_u(\bar{v}H)^{-1} L z_{k|k}\right) \geq 0\right\} \\
&= \left\{z_{k|k} \in \mathbb{R}^n : -\bar{v} \left(\hat{P}_u^\top H^{-1} \hat{P}_u^\top\right)^{-1} \left(\bar{v}\hat{Q}_u + \hat{P}_u H^{-1} L z_{k|k}\right) \geq 0\right\},
\end{aligned} \tag{4.68}$$

and the matrix H–dependent on β and N–is affecting both sets $\mathcal{P}_p$ and $\mathcal{P}_d$, then we can say that

$$CR_0 = \left\{z_{k|k} \in \mathbb{R}^n : \mathcal{P}_{p_{(\beta,N,\bar{v})}} \bigcap \mathcal{P}_{d_{(\beta,N,\bar{v})}}\right\}, \tag{4.69}$$

which leads us to notice that the CR_0 evaluated at $\bar{v} = 1$, depicts the *maximal* critical region when there are no packet losses over the actuator implying

$$CR_0\Big|_{\bar{v}\in(0,1)} \subset CR_0\Big|_{\bar{v}=1}, \tag{4.70}$$

when β and N are fixed. We can also observe that if we vary β, the sets $\mathcal{P}_p$ and $\mathcal{P}_d$ change which means the partitions change too. On the next section we show an example of the effect of $\bar{v}$ and β over the partitions.

Before presenting the effects on the explicit control law, we need an additional lemma.

Lemma 4.4 (Unconstrained solution)**.** *If* $\bar{v} \in (0, 1]$ *and* $\beta \geq 1$*, then the QP problem from* (4.47) *without input constraints,* i.e. $u_k \in \mathbb{R}^m$,

$$V_N^0(z_{k|k}, P_k) = \min_{\mathbf{u}_k \in \mathbb{R}^m} \frac{1}{2}\mathbf{u}_k^\top \bar{v}^2 H \mathbf{u}_k + \bar{v} c^\top \mathbf{u}_k + \alpha, \tag{4.71a}$$

has the unconstrained optimal

$$\mathbf{u}_k^*(z_{k|k}) = -(\bar{v}H)^{-1} L z_{k|k}, \tag{4.71b}$$

for all $z_{k|k} \in \mathbb{R}^n$.

Proof. Applying the gradient to (4.71a), leads to

$$\begin{aligned} \nabla_{\mathbf{u}} V_N^0(z_{k|k}, P_k) &= 0 \\ \bar{v}^2 H \mathbf{u}_k^* + \bar{v} c &= 0, \end{aligned}$$

a simple rearrangement completes the proof. □

Therefore, according to Theorem 4.3, the explicit control law is a linear combination of the constrained solution $\boldsymbol{\mu}_k^0$ depicted in (4.58) and the unconstrained solution $\mathbf{u}_k^*$ of the problem (4.47), *i.e.*

$$\begin{aligned} \mathbf{u}_k^0(z_{k|k}) &= \boldsymbol{\mu}_k^0(z_{k|k}) + \mathbf{u}_k^*(z_{k|k}) && (4.72a) \\ &= F_j z_{k|k} + g_j - (\bar{v}H)^{-1} L z_{k|k} \\ &= H^{-1}\hat{P}_u^\top(\hat{P}_u H^{-1}\hat{P}_u^\top)^{-1}\hat{P}_u(\bar{v}H)^{-1} L z_{k|k} + g_j - (\bar{v}H)^{-1} L z_{k|k}. && (4.72b) \end{aligned}$$

This indicates that the impact of $\bar{\nu}$ over the explicit PWA solution (4.3) is due to the simultaneous effect–by the same magnitude–on both constrained $\mathbf{u}_k^0$ and unconstrained $\mathbf{u}_k^*$ solutions. Then, this implies that for a given $z_{k|k}$, and knowing that $\left\|(\bar{\nu}H)^{-1}\right\| \geq \left\|H^{-1}\right\|$ and $H \succ 0$,

$$\left\|\mathbf{u}_k^*(z_{k|k})\right\|\Big|_{\bar{\nu}=1} \leq \left\|\mathbf{u}_k^*(z_{k|k})\right\|\Big|_{\bar{\nu}\in(0,1)}, \tag{4.73}$$

but we can not say the same about $\mathbf{u}_k^0(z_{k|k})$ since its evaluation depends on the critical regions CR_j which depend on $\bar{\nu}$.

In addition, if we vary β in (4.63a), we cannot guarantee that incrementing β will compensate the effect of $\bar{\nu}$, certainly it helps due to the fact that H is penalized by $\bar{\nu}$,

$$\bar{\nu}H = \bar{\nu}\big(2(G^\top \tilde{Q} G + \tilde{R})\big), \tag{4.74}$$

where β is defined in $\tilde{Q}$ (4.46a).

The results in this section show the conditions under which the explicit optimal control law is a PWA and the cost function is a PWQ for the given problem, and some notions of the effects of β, N and $\bar{\nu}$ over the CR_0 and the explicit control law. In what follows, we present some examples of the impacts of β, N and specially $\bar{\nu}$, over the PWA control and the partitions.

4.6 Numerical examples

The first example illustrates the closed-loop behaviour under the explicit control law action subject to random packet losses. It visualizes the partitions of the PWA control, the corresponding cost and estimate error.

The second example compares the effects of varying β for a fixed $\bar{\nu}$, and varying $\bar{\nu}$ for a fixed β, highlighting the impact of $\bar{\nu}$ and β in closed-loop system under random packet losses.

4.6.1 Example 1

Consider a system with

$$A = \begin{bmatrix} 1 & -1.2 \\ 1.2 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix},$$

noise covariances $Q_w = 0.0001\mathbf{I}_{2\times 2}$, $R_s = 0.0001\mathbf{I}_{1\times 1}$, and input constraint set $\mathcal{U} = \{u\colon |u| \le 1\}$. The expected values of ν and γ are $\bar{\nu} = 0.8$ and $\bar{\gamma} = 0.8$.

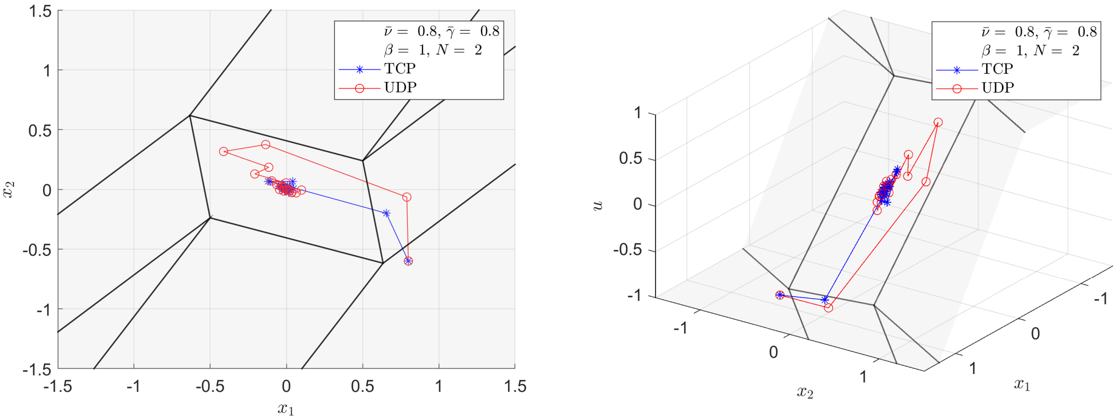
We design the controller with $Q = \mathbf{I}_{2\times 2}$, $R = 1$, $N = 2$, $\beta = 1$, and

$$Q_f = \begin{bmatrix} 6.78 & 5.06 \\ 5.06 & 12.94 \end{bmatrix}.$$

The explicit MPC law associated according to Theorem 4.3 is

$$u_k^0 = \begin{cases}
1.0000 & \text{if} \begin{bmatrix} 0.9881 & 0.1540 \\ 0.6907 & -0.7231 \\ -0.6907 & 0.7231 \\ -1.0000 & 0 \end{bmatrix} \hat{x}_k \le \begin{bmatrix} -0.5317 \\ -0.1732 \\ 0.8853 \\ 10000 \end{bmatrix} \quad \text{Region 1} \\
\begin{bmatrix} -1.8583 & -0.2897 \end{bmatrix} \hat{x}_k & \text{if} \begin{bmatrix} 0.9881 & 0.1540 \\ -0.3169 & -0.9485 \\ -0.9881 & -0.1540 \\ 0.3169 & 0.9485 \end{bmatrix} \hat{x}_k \le \begin{bmatrix} 0.5317 \\ 0.3853 \\ 0.5317 \\ 0.3853 \end{bmatrix} \quad \text{Region 2} \\
1.0000 & \text{if} \begin{bmatrix} 0.8559 & -0.5172 \\ -0.6907 & 0.7231 \\ -1.0000 & 0 \\ 0 & -1.0000 \end{bmatrix} \hat{x}_k \le \begin{bmatrix} -0.3051 \\ 0.1732 \\ 10000 \\ 10000 \end{bmatrix} \quad \text{Region 3} \\
1.0000 & \text{if} \begin{bmatrix} 0.8559 & -0.5172 \\ 0.6907 & -0.7231 \\ -1.0000 & 0 \\ 0 & 1.0000 \end{bmatrix} \hat{x}_k \le \begin{bmatrix} -0.8628 \\ -0.8853 \\ 10000 \\ 10000 \end{bmatrix} \quad \text{Region 4} \\
-1.0000 & \text{if} \begin{bmatrix} 0.6907 & -0.7231 \\ -0.6907 & 0.7231 \\ -0.9881 & -0.1540 \\ 1 & 0 \end{bmatrix} \hat{x}_k \le \begin{bmatrix} 0.8853 \\ -0.1732 \\ -0.5317 \\ 10000 \end{bmatrix} \quad \text{Region 5} \\
\begin{bmatrix} -1.4656 & 0.8857 \end{bmatrix} \hat{x}_k + 0.4775 & \text{if} \begin{bmatrix} 0.8559 & -0.5172 \\ -0.8559 & 0.5172 \\ 0.3169 & 0.9485 \\ 0 & -1 \end{bmatrix} \hat{x}_k \le \begin{bmatrix} 0.8628 \\ 0.3051 \\ 0.3853 \\ 10000 \end{bmatrix} \quad \text{Region 6} \\
\begin{bmatrix} -1.4656 & 0.8857 \end{bmatrix} \hat{x}_k - 0.4775 & \text{if} \begin{bmatrix} -0.3169 & -0.9485 \\ 0.8559 & -0.5172 \\ -0.8559 & 0.5172 \\ 0 & 1 \end{bmatrix} \hat{x}_k \le \begin{bmatrix} -0.3853 \\ 0.3051 \\ 0.8628 \\ 10000 \end{bmatrix} \quad \text{Region 7} \\
-1.0000 & \text{if} \begin{bmatrix} -0.8559 & 0.5172 \\ -0.6907 & 0.7231 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \hat{x}_k \le \begin{bmatrix} -0.8628 \\ 0.8853 \\ 10000 \\ 10000 \end{bmatrix} \quad \text{Region 8} \\
-1.0000 & \text{if} \begin{bmatrix} 0.6907 & -0.7231 \\ -0.8559 & 0.5172 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \hat{x}_k \le \begin{bmatrix} 0.1732 \\ -0.3051 \\ 10000 \\ 10000 \end{bmatrix} \quad \text{Region 9}
\end{cases}$$

The corresponding polyhedral partitions of the state space depicted in Fig. 4.20 shows the trajectories of the system's states under TCP-like and UDP-like channels. Starting in Region 8, the states under UDP-like estimation struggles to convergence around the origin in Region 2, while the states under TCP-like estimation reaches near the origin faster. This was expected according to (4.44) where the error in estimation for TCP-like is smaller compared to the UDP-channel.



(a) Phase plot (b) PWA

Figure 4.20: Phase-plot state and PWA control with TCP-like and UDP-like estimation.

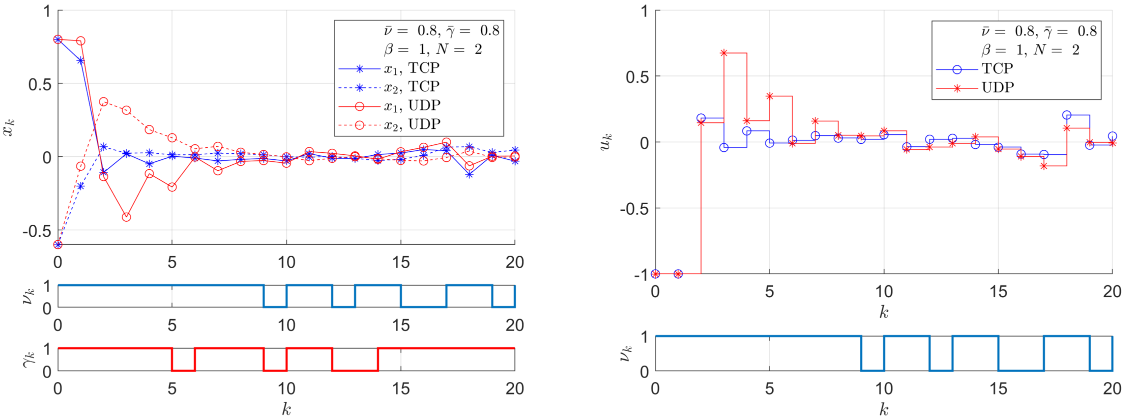
The applied control for TCP-like channel depicted in Fig. 4.21 shows less effort to stabilize the states trajectories compared to the UDP-like channel. However, between time step 15 and 20, when there are packet losses over the actuator but there are not over the sensor, the control effort for the TCP-like channel is slightly higher than the UDP-like channel. In Fig. 4.22, this is more evident with the cost function behaviour where the cost in TCP-like channel is a little higher than the cost in UDP-like channel, and the error in estimation verifies (4.42) where the error in UDP-like estimation is higher than the error in TCP-like estimation.

4.6.2 Example 2

Consider a system with

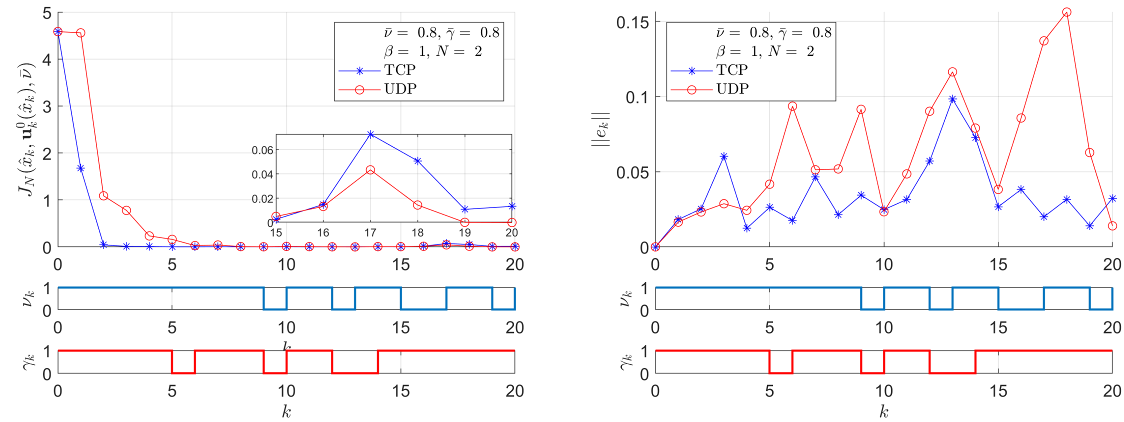
$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1.5 \\ -0.1 \end{bmatrix},$$

noise covariances $Q_w = 0.0001\mathbf{I}_{2\times 2}$, $R_s = 0.0001\mathbf{I}_{1\times 1}$, and input constraint set $\mathcal{U} = \{u \colon |u| \leq 1\}$. The expected values of ν and γ are $\bar{\nu} = 0.8$ and $\bar{\gamma} = 0.8$.



(a) States (b) Input

Figure 4.21: True state trajectories and applied controls with TCP-like and UDP-like estimation.



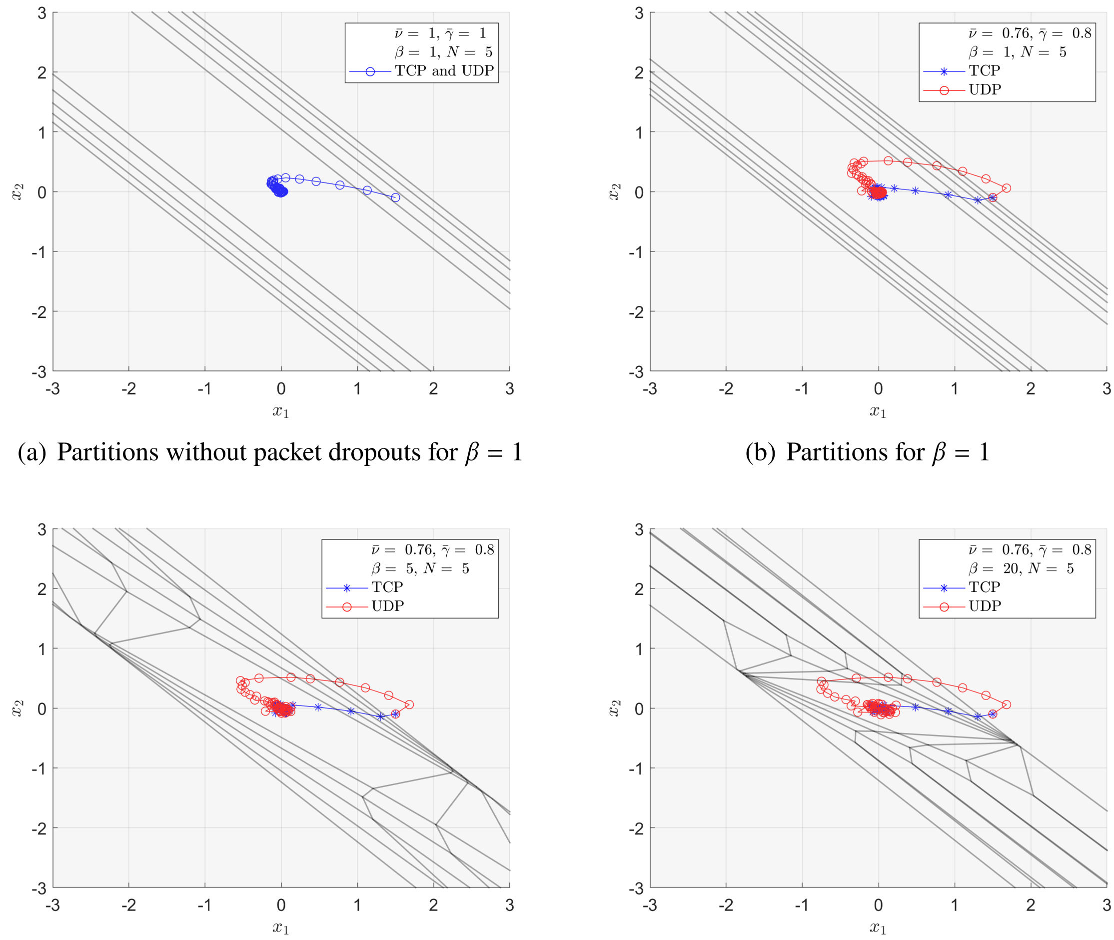
(a) Cost function (b) Estimate error

Figure 4.22: Cost function and estimated error with TCP-like and UDP-like estimation.

We design the controller with $Q = \mathbf{I}_{2\times2}$, $R = 0.1$, $N = 5$,

$$Q_f = \begin{bmatrix} 6.78 & 6.26 \\ 6.26 & 16.78 \end{bmatrix}.$$

Fig. 4.23 illustrates the effect of varying β while keeping $\bar{\nu}$ fixed. As β increases, CR_0 decreases, leading to changes in the partition configurations. Nevertheless, the control law successfully drives the system states to the origin. In Fig. 4.24, where β is fixed and $\bar{\nu}$ is varied, the impact of $\bar{\nu}$ is evident, particularly as it decreases, this highlights the significant role of $\bar{\nu}$ in the system's behaviour. Finally, Fig. 4.25 demonstrates that a sufficiently large β can, in principle, counteract the effects of small $\bar{\nu}$. Specifically, comparing Fig. 4.25 and Fig. 4.24 reveals a clear improvement when increasing β from 1 to 10.



(a) Partitions without packet dropouts for $\beta = 1$

(b) Partitions for $\beta = 1$

(c) Partitions for $\beta = 5$

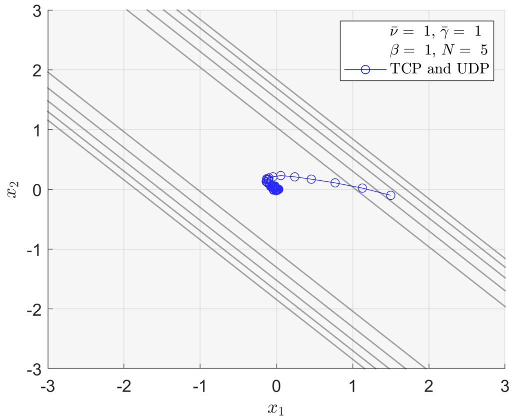
(d) Partitions for $\beta = 20$

Figure 4.23: Polyhedral partitions under different β values with fixed $\bar{\nu} = 0.76$.

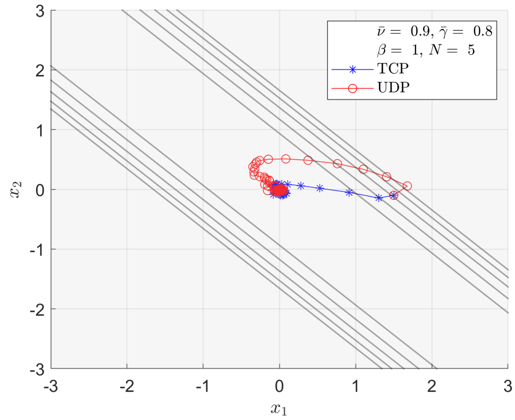
4.7 Conclusions

This chapter has considered an input-constrained LQG-type problem under random packet losses on the sensing and actuation channels. In Section 4.4, a counterexample established that, unlike in the unconstrained case, the Separation Principle does not hold when a TCP-like protocol is employed on the channels. In other words, for specific parameter values, the TCP-like scheme loses stability while the UDP-like scheme remains stable. Further analysis identified a relationship between the estimation and prediction errors, suggesting that controller performance may be worsened by improving estimation performance.

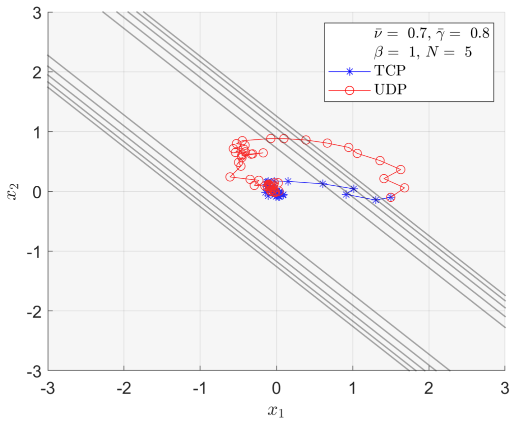
A closed-loop analysis of prediction and estimation errors between TCP-like and UDP-like



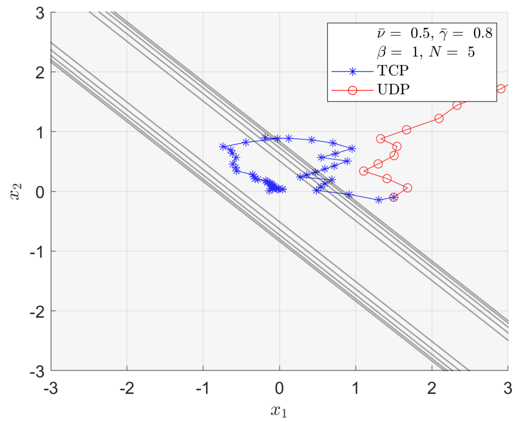
(a) Partitions without packet dropouts for $\beta = 1$



(b) Partitions for $\bar{\nu} = 0.9$



(c) Partitions for $\bar{\nu} = 0.7$



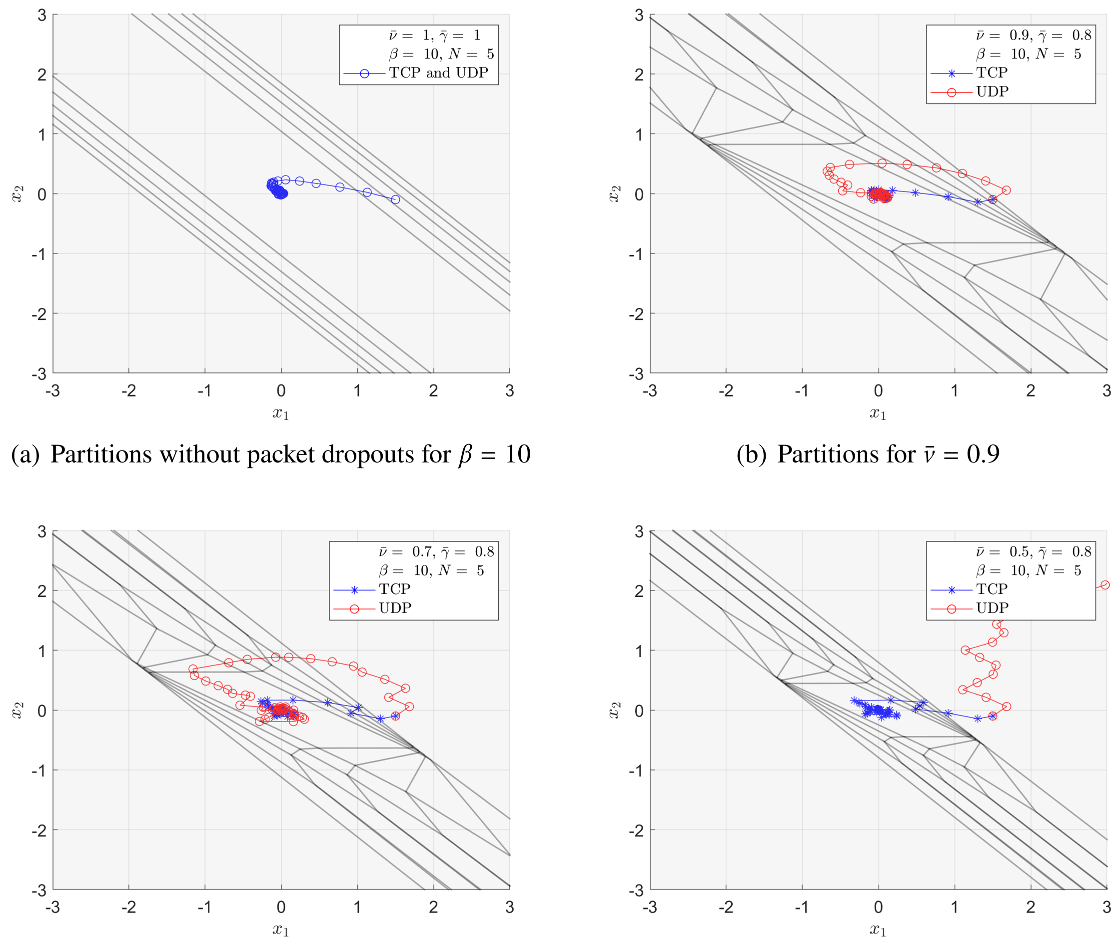
(d) Partitions for $\bar{\nu} = 0.5$

Figure 4.24: Polyhedral partitions under different $\bar{\nu}$ values with fixed $\beta = 1$.

protocols, reveals a form of reverse Separation Principle: if the state estimate is in the RoA, the UDP-MPC cost function decreases monotonically, independently of the estimator, while the estimator performance remains dependent on the applied control input. However, for the TCP case, the estimator evolves independently of the controller–consistent with unconstrained separation results–but the monotonic descent property of the cost is guaranteed only for sufficiently small control inputs.

Moreover, these small control inputs needed to guarantee monotonicity in the TCP case are difficult to satisfy in practice, since the input constraints may prevent the controller from providing enough energy to effectively reduce the error in prediction. Unlike the unconstrained control that could complete this objective. This is evident by comparing the counterexample simulations with and without input constraints.

At the same time, the effect of the implicit constrained control is to increase the estimation



(a) Partitions without packet dropouts for $\beta = 10$

(b) Partitions for $\bar{\nu} = 0.9$

(c) Partitions for $\bar{\nu} = 0.7$

(d) Partitions for $\bar{\nu} = 0.5$

Figure 4.25: Polyhedral partitions under different $\bar{\nu}$ values with fixed $\beta = 10$.

error in the UDP, causing to reduce the prediction error in the TCP case by the same margin, and conversely. These observations reveal an asymmetry between estimation and control roles under different communication protocols in the input constrained setting.

In Section 4.5, we presented the conditions under which the constrained explicit control law is PWA and the cost is PWQ. The associated polyhedral partitions provide geometric insight into the closed-loop behaviour. Numerical trajectories show that, starting from identical regions, TCP-like estimation drives the state closer to the origin rapidly than UDP-like estimation, in agreement with the smaller estimation errors predicted for TCP-like channels.

The numerical examples clarify the roles of the design parameters β and $\bar{\nu}$. Increasing β alters the partition structure by shrinking the critical regions while still enabling convergence to the origin. Variations in $\bar{\nu}$ have a pronounced impact on performance, highlighting the sensitivity of the closed-loop operation subject to packet dropouts. Moreover, sufficiently large β can partially counteract the effects of $\bar{v}$. In addition, we identified the impact of $\bar{v}$ over the explicit PWA solution is due to the simultaneous effect–by the same magnitude–on both constrained and unconstrained solutions.

4.A Proof of Lemma 4.2

Following the procedure in [13]. Let

$$\mathbf{u}^0(x) = \left\{u_0^0(x), u_1^0(x), \ldots, u_{N-1}^0(x)\right\},$$

in which $u_0^0(x) = \kappa_N(x)$, is an optimal control sequence for $J_N\left(x, \mathbf{u}^0(x), \bar{v}\right)$, and

$$\mathbf{x}^0 = \left\{x_0^0, x_1^0, \ldots, x_N^0\right\}$$

is the resultant optimal state sequence, in which $x_0^0 = x$ and $x_1^0 = x^+$.

Let

$$\mathbf{u}^0(x^+) = \left\{u_0^0(x^+), u_1^0(x^+), \ldots, u_{N-1}^0(x^+)\right\}$$

is an optimal control sequence for the cost $J_N\left(x^+, \mathbf{u}^0(x^+), \bar{v}\right)$, and let

$$\tilde{\mathbf{u}}(x) = \left\{u_1^0(x), u_2^0(x), \ldots, u_{N-1}^0(x), u_N^0(x)\right\},$$

in which $u_N^0(x) = K_f x_N^0$, is a feasible (not necessarily) optimal control sequence for $J_N(x^+, \tilde{\mathbf{u}}(x), \bar{v})$, that results in the state sequence

$$\tilde{\mathbf{x}} = \left\{x_1^0, x_2^0, \ldots, x_N^0, x_{N+1}^0\right\},$$

where $x_{N+1}^0 = (A + \bar{v}BK_f)x_N^0$.

Since it is difficult to compare $J_N\left(x, \mathbf{u}^0(x), \bar{v}\right)$ and $J_N\left(x^+, \mathbf{u}^0(x^+), \bar{v}\right)$ directly, but

$$J_N\left(x^+, \mathbf{u}^0(x^+), \bar{v}\right) \leq J_N(x^+, \tilde{\mathbf{u}}(x), \bar{v}).$$

By simply rearrangement

$$J_N(x^+, \tilde{\mathbf{u}}(x), \bar{v}) = \sum_{j=1}^{N-1} \ell\big(x_j^0, \bar{v}u_j^0\big) + \ell\big(x_N^0, \bar{v}u_N^0\big) + V_f\big(x_{N+1}^0\big), \tag{4.75a}$$

and

$$J_N\big(x, \mathbf{u}^0(x), \bar{v}\big) = \ell(x, \bar{v}\kappa_N(x)) + \sum_{j=1}^{N-1} \ell\big(x_j^0, \bar{v}u_j^0\big) + V_f\big(x_N^0\big),$$

such that

$$\sum_{j=1}^{N-1} \ell\big(x_j^0, \bar{v}u_j^0\big) = J_N\big(x, \mathbf{u}^0(x), \bar{v}\big) - \ell(x, \bar{v}\kappa_N(x)) - V_f\big(x_N^0\big). \tag{4.75b}$$

Hence, (4.75b) in (4.75a) results in

$$J_N(x^+, \tilde{\mathbf{u}}(x), \bar{v}) = J_N\big(x, \mathbf{u}^0(x), \bar{v}\big) - \ell(x, \bar{v}\kappa_N(x)) + V_f\big(x_{N+1}^0\big) - V_f\big(x_N^0\big) + \ell\big(x_N^0, \bar{v}u_N^0\big),$$

and by applying Lemma 4.1, it follows that

$$J_N\big(x^+, \mathbf{u}^0(x^+), \bar{v}\big) \leq J_N(x^+, \tilde{\mathbf{u}}(x), \bar{v}) = J_N\big(x, \mathbf{u}^0(x), \bar{v}\big) - \ell(x, \bar{v}\kappa_N(x)),$$

which completes the proof.

Chapter 5

Nominal stability of State-Feedback MPC under consecutive packet losses

Contents

The aim of this chapter is to simplify the problem formulation of Chapter 4 such that the Separation Principle is no longer a problem, and the focus is now the stability analysis from the controller's viewpoint, *i.e.* in the presence of random packet dropouts over the C-A channel under the UDP-like protocol. This simplification eliminates the ACK of the packet losses at the controller, input constraints are still present, and there are no uncertainties. However, to mitigate the packet dropouts, a buffer is introduced. These relaxations enable us to determine conservative bounds on the number of consecutive packet losses that linear and nonlinear systems can tolerate before losing stability. And the results help to extend the analysis to the system under disturbances, which are studied in the subsequent chapter.

In this chapter, we study the stability of a discrete-time nonlinear and linear networked system controlled by MPC without explicit terminal constraints. The system is subject to input constraints and random packet losses on the actuation communication channel between the controller and the plant. The scenario where a buffer–intended to store transmitted control sequences and provide some robustness to packet losses–is present, and also the scenario where a buffer is *not* present. The stability is analysed of the closed-loop system in these stochastic scenarios, employing the assumption that the terminal cost is merely a local, rather than global CLF. We develop conditions that characterize an upper bound on the number of consecutive packet losses in order that stability is maintained.

5.1 Introduction

NCS pose a significant challenge because of the effects that imperfect communication between actuators, sensors, and controllers has on system stability and performance [56]. When communication imperfections have a malicious cause, *cybersecurity* of the system becomes the main concern and an even more pressing challenge. For instance, a DoS attack can interrupt the actuator and sensor communication channels by random packet flooding, consuming resources such as network bandwidth and CPU cycles. As a result, the elevated level of packet loss reduces control and estimation performance, and could even cause instability or–if constraints are present–infeasibility. Although many fault-tolerant and disturbance rejection algorithms have been developed to reduce such large-scale disturbances, these typically do not consider the fault or disturbance to be of malicious intent. Moreover, malicious cyberattacks can be coordinated, remain undetected, and are not constrained by the system dynamics [11]. Consequently, robust control and estimation algorithms may need refinement to handle this class of disturbances in a satisfactory manner.

Following the idea of the reduction on control and estimation performance due to the packet losses, Chapter 2 discusses the consequences of random packet losses on controllability and observability in linear NCS. The authors in [91, 92, 20] establish that controllability and observability must be revisited under packet losses rather than being purely structural properties, *e.g.* fixed pairs (A, B) or (A, C) for linear systems. Meaning that packet losses do not only degrade the performance but can alter the system's ability to be controlled or observed. In this sense, this chapter can be viewed as a complementary analysis to the cited works.

Various methods have been proposed and studied in the context of MPC under packet losses. Quevedo and Nešić [58] proposed a packetized MPC approach for nonlinear systems, including a buffer to provide some protection against dropouts on the C-A channel; conditions for stochastic stability were established, and later improved [42, 60]. In [64], a buffered actuator and a buffered estimator are used to extend the approach of [42] to consider packet losses on both the C-A and S-C channels. A Lyapunov function, based on conditions similar to those in [95], is developed in [96] to ensure regional ISpS. In [38, 97], conditions are developed to guarantee feasibility and ISpS based on attack duration, disturbance bounds and controller prediction horizon.

A common feature of the stability analyses in the preceding works is the assumption of a *global* CLF for the terminal cost of the system. Yet it is known and accepted that this is a strong, and perhaps impossible to meet, assumption for systems subject to constraints. We seek to relax this assumption in this chapter. A discrete-time nonlinear NCS subject to random packet losses and controlled by an MPC formulation without explicit terminal constraints is considered. Similar to [58] and related works, the controller transmits the optimal control sequence over a lossy UDP-like communication channel, *i.e.* without any ACK of packets received. The stability analysis of [58] is improved by relaxing the assumption of the existence of a global CLF. This is achieved by exploiting the stability analysis for constrained MPC without terminal constraints [110].

We consider the situation where the initial state is in the RoA yet relatively close to the origin and there are subsequent packet losses, and the situation where the initial state is anywhere in the RoA and there are subsequent packet losses. These two cases can represent the two different scenarios where there is, or is not, a buffer on the plant side in order to improve robustness. For each scenario, an upper bound on the number of consecutive losses in order that system state remains within the RoA is derived. The developed conditions characterize a relationship between stability and the properties of the system and controller; namely, the prediction horizon, and the design parameters of the controller.

5.2 Problem formulation

We consider the following discrete-time nonlinear system

$$x_{k+1} = f(x_k, v_k u_k), \tag{5.1}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$ are, respectively, the state and the input of the system at sample time $k \in \mathbb{N}_{\geq 0}$. The system input, u_k, is constrained to take values in a set $\mathcal{U} \subset \mathbb{R}^m$ but the states are unconstrained. Moreover, the input is subject to random packet losses via v_k.

Assumption 5.1 (Continuity of the system)**.** *The function $f(\cdot,\cdot)$ is continuous and satisfies* $0 = f(0,0)$.

Assumption 5.2. *The set $\mathcal{U}$ is compact and contains the origin in its interior.*

The system is connected to a controller via a partially lossy UDP-like communication channel. While the state measurement is communicated perfectly to the controller, communication between the controller and system actuator is subject to random packet losses as depicted in Fig. 5.1.

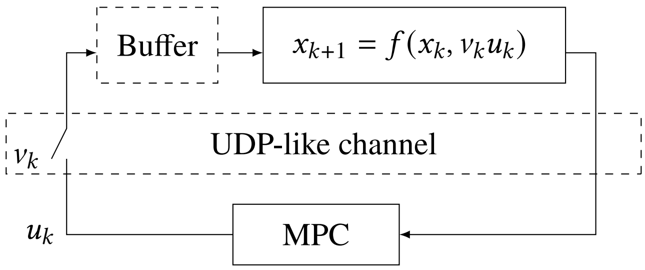


Figure 5.1: A system in connection with a controller via a UDP-like channel. The C-A channel is subject to random packet losses.

Assumption 5.3. *The input packet loss variable $v_k \in [0,1]$ forms a sequence of i.i.d random variables with $v_k \sim \mathcal{B}(\bar{v})$, where $\bar{v} = \Pr[v_k = 1]$ is the probability of successful packet delivery.*

Assumption 5.3 depicts a common NCS setting studied in the literature (*e.g.* [58, 42]). We also consider that a buffer may (or may not) be present at the input to the system; the buffer stores control sequences transmitted by the controller and its precise operation is described in Section 5.3.

The formulation of problem is as follows. Given the system (5.1) at a state x_k, the problem is to determine the optimal control law such that the state x_k is transferred to the origin, subject to the input constraint set (Assumption 5.2) and packet losses (Assumption 5.3), while minimizing a cost function through the following optimal control problem.

5.3 Controller formulation

5.3.1 Optimal control problem

With the system at a state x_k, the optimal control problem to be solved is

$$\mathbb{P}_N(x_k): \quad V_N^0(x_k) = \min_{\mathbf{u}_k \in \mathbb{U}} J_N(x_k, \mathbf{u}_k), \tag{5.2}$$

where

$$\mathbf{u}_k := \{u_{k|k}, u_{k+1|k}, \dots, u_{k+N-1|k}\}, \tag{5.3}$$

is the finite sequence of future control inputs. The constraint set is

$$\mathbb{U} := \mathcal{U} \times \cdots \times \mathcal{U}, \tag{5.4}$$

while the cost function is

$$J_N(x_k, \mathbf{u}_k) := \beta V_f(x_{k+N|k}) + \sum_{j=0}^{N-1} \ell(x_{k+j|k}, u_{k+j|k}), \tag{5.5}$$

where $\beta \geq 1$ and the stage $\ell(\cdot,\cdot)$ and terminal cost $V_f(\cdot)$ functions satisfy the following assumption.

Assumption 5.4. *The stage cost function $\ell(\cdot,\cdot)\colon \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}_{\geq 0}$ and terminal cost function $V_f(\cdot)\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are continuous, with $\ell(0,0) = 0$ and $V_f(0) = 0$, and satisfy*

$$\ell(x,u) \geq c_1 \|x\|_2^2, \tag{5.6a}$$

for all $x \in \mathbb{R}^n$, for all $u \in \mathcal{U}$, and some $c_1 > 0$,

$$V_f(x) \leq c_2 \|x\|_2^2, \tag{5.6b}$$

for all $x \in \mathbb{R}^n$, and some $c_2 > 0$.

We remark that this represents a conventional input-constrained nonlinear MPC formulation [100]; however, we also emphasize that no terminal constraints are present but a terminal cost is.

Solving $\mathbb{P}_N(x_k)$ at x_k yields the solution

$$\mathbf{u}_k^0(x_k) := \left\{u_{k|k}^0, u_{k+1|k}^0, \ldots, u_{k+N-1|k}^0\right\}. \tag{5.7}$$

The application of the first control in the optimal sequence to the plant, followed by a repetition of the whole process at the next sampling instant, defines the implicit control law

$$u_k = \kappa_N(x_k) := u_{k|k}^0. \tag{5.8}$$

In view of the lack of state constraints, the domain of the value function $V_N^0(\cdot)$ and control law $\kappa_N(\cdot)$ is the whole state space $\mathbb{R}^n$, meaning that the optimal control problem is (trivially) recursively feasible.

5.3.2 Buffering mechanism

Similar to the buffering mechanism described in [42], at each time instant k, the controller transmits the whole sequence $\mathbf{u}_k^0(x_k)$ and not just the input $u_{k|k}^0$. If $\nu_k = 1$ then this sequence is received by a buffer at the input to the plant, and this buffer acts as a parallel-in–serial-out shift register. If $\nu_k = 0$, the buffer outputs the first element of a *previously received* and shifted sequence to the system; this is repeated until a new sequence is successfully received, overwriting any previous sequence.

This process is formally modelled as follows. Let b_k denote the contents of the buffer at time k. Then,

$$b_k = (1 - \nu_k)Sb_{k-1} + \nu_k \mathbf{u}_k^0(x_k) \tag{5.9a}$$

$$u_k = e^\top b_k, \tag{5.9b}$$

with $b_0 = \{0, 0, \dots, 0\}$ and S and e defined as

$$S := \begin{bmatrix} \mathbf{0}_{m\times m} & \mathbf{I}_{m\times m} & \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0}_{m\times m} & \cdots & \mathbf{0}_{m\times m} & \mathbf{I}_{m\times m} & \mathbf{0}_{m\times m} \\ \mathbf{0}_{m\times m} & \cdots & \cdots & \mathbf{0}_{m\times m} & \mathbf{I}_{m\times m} \\ \mathbf{0}_{m\times m} & \cdots & \cdots & \cdots & \mathbf{0}_{m\times m} \end{bmatrix} \tag{5.9c}$$

$$e^\top := [\mathbf{I}_{m\times m} \quad \mathbf{0}_{m\times m} \quad \cdots \quad \mathbf{0}_{m\times m}]. \tag{5.9d}$$

Thus, S removes the first element from a control sequence, shifting it one step forward in time, while e extracts the first element from the buffered sequence.

5.4 Stability analysis of nonlinear systems

To establish stability of the closed-loop system under random packet losses and in the absence of a terminal constraint, we combine the stability analysis of the value function with the stability conditions for input-constrained MPC without terminal constraints [110]. This combination permits a significant relaxation of a technical assumption in [58], which requires the existence of a *global* CLF for use as the MPC terminal cost function; this assumption is impossible to meet for many input-constrained systems [100].

We first present some relevant results and necessary assumptions from the literature that aid our developments. In what follows, a variable without a subscript denotes its current value—for example, x means x_k—while the subscript j denotes the j-step ahead prediction: *e.g.* x_j means $x_{k+j|k}$.

5.4.1 Preliminaries: stability without terminal constraint set

The following assumptions and definitions are recalled from [110]. First we define a sublevel set of the terminal cost function as follows.

Definition 5.1. *Let $d_1 > 0$ be such that for all*

$$x \in \mathcal{X}_f(d_1) := \{x \colon V_f(x) \le d_1\}. \tag{5.10}$$

Such d_1 is guaranteed since $V_f(x)$ is positive definite (Assumption 5.4).

Assumption 5.5. *There exists a control law* $\kappa_f\colon \mathbb{R}^n \to \mathbb{R}^m$ *and a constant* $d_1 > 0$ *such that, for all* $x \in \mathcal{X}_f(d_1)$*,*

$$V_f\big(f(x,\kappa_f(x))\big) \le V_f(x) - \ell(x,\kappa_f(x)) \tag{5.11a}$$

$$\kappa_f(x) \in \mathcal{U}. \tag{5.11b}$$

This assumption indicates that $V_f(\cdot)$ is a local CLF for the terminal dynamics, a relaxation of the global CLF assumption in [58, 42]. Since $\beta \ge 1$, it follows that $\beta V_f(\cdot)$ is also a local CLF. Recall that the set $\mathcal{X}_f(d_1)$ acts as an implicit terminal region for the controller (5.10); no terminal constraint is enforced, but $x_{k+N|k} \in \mathcal{X}_f(d_1)$ is implied if the closed-loop system is stable [110].

The $\mathcal{X}_f(d_1)$ satisfies the property of boundedness, since it is contained within some finite-radius ball in the state-space. In [110], the authors demonstrate that increasing β increases the RoA, *i.e.* $\beta_1 \le \beta_2$ implies the RoA for β_1 is a subset of the RoA for β_2. The connectedness property is satisfied by establishing recursive feasibility of $\mathcal{X}_f(d_1)$. Specifically, they show that for any state x that can be steered to the interior of $\mathcal{X}_f(d_1)$ by a feasible control sequence, there exists a β such that x is in the RoA.

Definition 5.2. *There exists a* $d_2 > 0$ *such that, for all* $x \notin \mathcal{X}_f(d_1)$ *and* $u \in \mathcal{U}$*,*

$$d_2 \le \ell(x,u). \tag{5.12}$$

Such d_2 is guaranteed because $\ell(x,u)$ is positive definite (Assumption 5.4) and the fact $\mathcal{U}$ contains the origin in its interior (Assumption 5.2).

The previous definitions 5.1–5.2 and Assumption 5.5 allow us to define the set

$$\Gamma_N^{\beta} := \big\{x \in \mathbb{R}^n \colon V_N^0(x) \le \beta d_1 + N d_2\big\}. \tag{5.13}$$

In [110], it is established that the controlled system $x^+ = f(x,\kappa_N(x))$ is exponentially stable with region of attraction Γ_N^β, and, for all $x \in \Gamma_N^\beta$,

$$V_N^0(f(x, \kappa_N(x))) - V_N^0(x) \leq -\ell(x, \kappa_N(x)). \tag{5.14}$$

Moreover, some further salient and relevant facts follow directly, concerning the value, stage and terminal cost functions.

Lemma 5.1. *There exist constants* $\gamma \in (0, 1]$, *and* $\varrho \in (0, 1]$ *such that*

$$V_f\big(f(x, \kappa_f(x))\big) \leq \gamma V_f(x) \tag{5.15a}$$

$$V_f\big(f(x, \kappa_f(x))\big) - V_f(x) \leq (\gamma - 1)V_f(x), \tag{5.15b}$$

for all $x \in \mathcal{X}_f(d_1)$, *and*

$$V_N^0(f(x, \kappa_N(x))) \leq \varrho V_N^0(x), \tag{5.16a}$$

for all $x \in \Gamma_N^\beta$, *where*

$$\varrho = 1 - \frac{c_1}{c_2}, \tag{5.16b}$$

for some $c_1, c_2 > 0$.

Proof. (5.15b) is obtained by subtracting $V_f(x)$ from (5.15a). We obtain (5.16a) by applying the procedure in [100] as follows. From Assumption 5.4, the value function satisfies

$$c_1\|x\|_2^2 \leq \ell(x, \kappa_N(x)) \leq V_N^0(x) \leq V_f(x) \leq c_2\|x\|_2^2.$$

From the latter,

$$\begin{aligned} V_N^0(x) &\leq c_2\|x\|_2^2 \\ c_1 V_N^0(x) &\leq c_1 c_2\|x\|_2^2 \\ -\frac{c_1}{c_2}V_N^0(x) &\geq -c_1\|x\|_2^2, \end{aligned}$$

adding $V_N^0(x)$ on both sides,

$$V_N^0(x) - c_1\|x\|_2^2 \leq \left(1 - \frac{c_1}{c_2}\right)V_N^0(x),$$

and recalling (5.14), we have that,

$$V_N^0(f(x, \kappa_N(x))) \leq V_N^0(x) - c_1\|x\|_2^2 \leq \left(1 - \frac{c_1}{c_2}\right) V_N^0(x),$$

which completes the proof. □

5.4.2 Stability analysis of the nonlinear NCS

The previous assumptions establish stability of conventional MPC for an input-constrained nonlinear system in the absence of packet losses and the buffering mechanism. In what follows, we consider the analysis where the buffer is present and there are consecutive packet losses. We examine two scenarios: first, where the initial state is in the region of attraction Γ_N^β but is not necessarily in the terminal region $\mathcal{X}_f(d_1)$; secondly, where the initial state is in the terminal region. For both cases, we show that there exists a finite upper bound on the number of consecutive packet losses in order that the state remains in Γ_N^β.

In the sequel we suppose that Assumptions 5.1–5.5 hold even if not explicitly stated. Furthermore, we define some additional assumptions similar to those in [58]. These concern the behaviour of the value, stage and terminal cost functions when the system is operating in open loop.

Assumption 5.6 ($V_N^0(\cdot)$ in open-loop)**.** *There exists a* $\zeta \in [1, +\infty)$ *such that*

$$V_N^0(f(x, 0)) \leq \zeta V_N^0(x), \tag{5.17}$$

for all $x \in \Gamma_N^\beta$.

Assumption 5.7 ($\ell(\cdot, \cdot)$ and $V_f(\cdot)$ in open-loop)**.** *There exist* $\sigma \in [1, +\infty)$ *and* $\rho \in [1, +\infty)$ *such that*

$$\ell(f(x, 0), 0) \leq \sigma \ell(x, u), \tag{5.18}$$

for all $x \in \Gamma_N^\beta$ *and* $u \in \mathcal{U}$,

$$V_f(f(x, 0)) \leq \rho V_f(x), \tag{5.19}$$

for all $x \in \mathcal{X}_f(d_1)$.

Both Assumptions 5.6 and 5.7 hold, as long as the $V_N^0(\cdot)$ in open-loop, and the $\ell(\cdot,\cdot)$ and $V_f(\cdot)$ in open-loops exhibit exponential dynamics along their trajectories, otherwise, these assumptions do not hold.

We will use these two assumptions as *alternatives* in the analyses that follow, rather than assume that both hold simultaneously.

5.4.2.1 Scenario 1

Now, as previously described, we define two scenarios. First consider that the system with initial state $x_0 \in \Gamma_N^\beta$ experiences no packet losses for i consecutive steps. That is, the closed-loop control law $u_j = \kappa_N(x_j)$ is applied to the plant for steps $j = 0, 1, \ldots, i-1$. The initial state x_0 is, according to (5.16a), transferred to

$$x_i \in \varrho^i \Gamma_N^\beta. \tag{5.20}$$

We cannot say whether or not $x_i \in \mathcal{X}_f$; this is our first scenario.

5.4.2.2 Scenario 2

For the second scenario, consider that the system with an initial state $x_0 \in \Gamma_N^\beta$ experiences packet losses for i consecutive steps. That is, system is controlled according to the buffering mechanism described in Section 5.3.2, using the optimal control sequence computed and stored at time 0,

$$\mathbf{u} = \{u_0, \ldots, u_{N-1}\}. \tag{5.21a}$$

During consecutive packet losses, no re-optimization is performed and the buffered control inputs are applied sequentially. After i steps, the remaining control sequence

$$\mathbf{u}^* = \{u_i, \ldots, u_{N-1}\} \tag{5.21b}$$

has length $N - i$ of admissible inputs.

Therefore, the initial state x_0 is transferred to

$$x_i \in \begin{cases} \Gamma^{\beta}_{N-i} & \text{if } i < N, \\ \Gamma^{\beta}_{0} = \mathcal{X}_f(d_1) & \text{if } i = N, \end{cases} \tag{5.22}$$

where the subscript $N - i$ means the sublevel set of Γ^{β}_{N} for $N - i$ remaining prediction steps, while the subscript 0 is the sublevel set of Γ^{β}_{N} for 0 remaining prediction steps, which coincides with $\mathcal{X}_f(d_1)$. By construction, $\Gamma^{\beta}_{N} \supseteq \Gamma^{\beta}_{N-1} \supseteq \dots \Gamma^{\beta}_{N-i} \supseteq \cdots \supseteq \Gamma^{\beta}_{0}$.

5.4.3 Generalization of both scenarios: Case 1 and Case 2

Note that both scenarios generalize to

$$x_i \in \mu\Gamma^{\beta}_{N}, \mu \in (0, 1], \quad \text{or} \quad x_i \in \mathcal{X}_f(d_1), \tag{5.23}$$

where these two possibilities are not mutually exclusive.

This allows us to generalize further, considering

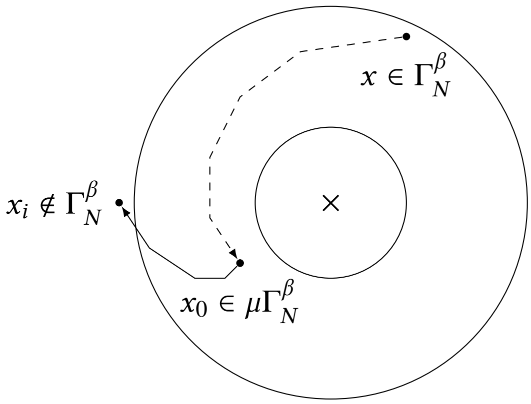
$$x_0 \in \mu\Gamma^{\beta}_{N}, \mu \in (0, 1], \quad \text{or} \quad x_0 \in \mathcal{X}_f(d_1), \tag{5.24}$$

without explicitly considering how the state arrived there, and analysing how the system behaves subsequently in these *two different cases* when it experiences i consecutive packet losses (with the buffer already exhausted in the latter case; its benefit has already been realized by transferring an initial state to $\mathcal{X}_f(d_1)$), see Fig. 5.2.

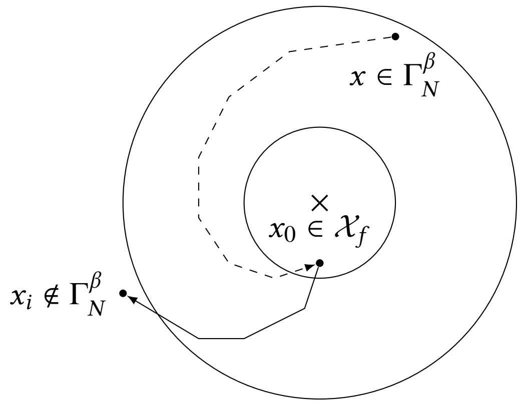
5.4.3.1 Case 1: bound on i given $x_0 \in \mu\Gamma^{\beta}_{N}$ and the open-loop cost value parameter

Let $x_0 \in \mu\Gamma^{\beta}_{N}$, $\mu \in (0, 1]$, denote an initial state reached via closed-loop control or buffered control, and let $x_i = f^i(x_0, 0)$ denote a state reached after i subsequent steps of open-loop operation. In what follows, we present the conditions under which the state x_i is contained in Γ^{β}_{N}.

Theorem 5.1. *Suppose Assumption 5.6 holds. If* $\mu \in (0, 1]$ *and the number of i consecutive*



(a) Case 1



(b) Case 2

Figure 5.2: Two cases for the i consecutive packet losses.

open-loop steps satisfies

$$i \leq \left\lfloor \frac{\ln(1/\mu)}{\ln \zeta} \right\rfloor \tag{5.25}$$

then $x_i \in \Gamma_N^\beta$.

Proof. By Assumption 5.6,

$$V_N^0(x_i) \leq \zeta^i V_N^0(x_0) \leq \zeta^i \mu(\beta d_1 + N d_2).$$

Therefore, if $\zeta^i \mu \leq 1$ then

$$V_N^0(x_i) \leq \beta d_1 + N d_2 \iff x_i \in \Gamma_N^\beta.$$

A simple rearrangement for i completes the proof. □

5.4.3.2 Case 2: bound on i given $x_0 \in \mathcal{X}_f(d_1)$ and the open-loop cost value parameter

Now let $x_0 \in \mathcal{X}_f(d_1)$ denote an initial state reached via closed-loop control or buffered control, and let $x_i = f^i(x_0, 0)$ denote a state reached after i subsequent steps of open-loop operation. The main result here depicts a sufficient condition under which $x_i \in \Gamma_N^\beta$. It uses the following lemma.

Lemma 5.2. *For all* $x_0 \in \mathcal{X}_f(d_1)$,

$$V_N^0(x_0) \leq \xi V_f(x_0), \tag{5.26}$$

where

$$1 \leq \xi := \big(\beta - (\beta - 1)\big(1 - \gamma^N\big)\big) \leq \beta. \tag{5.27}$$

The latter inequality is strict if $\beta > 1$.

Proof. See Appendix 5.A. □

Theorem 5.2. *Suppose Assumption 5.6 holds. If $\xi \geq 1$ and the number of i consecutive open-loop steps satisfies*

$$i \leq \left\lfloor \ln\!\left(\frac{\beta + Nd_2/d_1}{\xi}\right) \Big/ \ln \zeta \right\rfloor, \tag{5.28}$$

then $x_i \in \Gamma_N^\beta$.

Proof. Since $x_0 \in \mathcal{X}_f(d_1)$,

$$V_N^0(x_0) \leq \xi V_f(x_0) \leq \xi d_1,$$

and by Assumption 5.6,

$$V_N^0(x_i) \leq \zeta^i V_N^0(x_0).$$

By comparison, it follows that

$$V_N^0(x_i) \leq \zeta^i \xi V_f(x_0) \leq \zeta^i \xi d_1.$$

Therefore, $x_i \in \Gamma_N^\beta$ is ensured if

$$\zeta^i \xi d_1 \leq \beta d_1 + N d_2,$$

which, with simple rearrangement, completes the proof. □

In both cases we have presented bounds on the number of consecutive packet losses in terms of the constants depicted in Lemma 5.1 (which is known to exist) and Assumption 5.6 (which is assumed to exist). However, both need to be estimated by analysing the behaviour of the *value function* along (closed- and open-loop) system trajectories. A more practical option is to estimate similar constants for the *stage* and *terminal* costs; hence, Assumption 5.7 becomes relevant. In what follows we present the analysis for Case 1 and 2.

5.4.3.3 Case 1: bound on i given $x_0 \in \mu\Gamma_N^\beta$, and the open-loop stage-cost and terminal-cost parameters

Let $x_0 \in \mu\Gamma_N^\beta$, $\mu \in (0, 1]$, be an initial state reached via closed-loop control or buffered control, and let $x_i = f^i(x_0, 0)$ be a state reached after i subsequent steps of open-loop operation. In what follows, we show the conditions under which x_i is contained in Γ_N^β.

First, the lemma that links the value function at x_i for i consecutive packet losses, and the stage cost and terminal cost functions at x_0 under the assumption of open-loop operation ($u = 0$) in the interim.

Lemma 5.3. *Suppose Assumption 5.7 holds. For all $x_0 \in \mu\Gamma_N^\beta$ and all $u_0 \in \mathcal{U}$,*

$$V_N^0(x_i) \leq \Psi\ell(x_0, u_0) + \Phi V_f(x_0), \tag{5.29}$$

where

$$\Psi := \sigma^i \frac{\sigma^N - 1}{\sigma - 1} > 1, \quad \Phi := \beta\rho^{N+i} > 1. \tag{5.30}$$

Proof. The proof is in Appendix 5.B. □

Theorem 5.3. *Suppose Assumption 5.7 holds with $\sigma > 1$. If*

$$\beta\rho^{N+i} d_1 + \sigma^i \frac{\sigma^N - 1}{\sigma - 1}\ell(x_0, u_0) \leq \beta d_1 + N d_2 \tag{5.31}$$

is satisfied, then $x_i \in \Gamma_N^\beta$.

Proof. By (5.29), for any $x_0 \in \mu\Gamma_N^\beta$,

$$V_N^0(x_i) \leq \Psi\ell(x_0, u_0) + \Phi V_f(x_0) \leq \Psi\ell(x_0, u_0) + \Phi d_1,$$

since $x_0 \in \mu\Gamma_N^\beta$, then $x_i \in \Gamma_N^\beta$ if

$$\Phi d_1 + \Psi\ell(x_0, u_0) \leq \beta d_1 + N d_2.$$

Rearrangement establishes the result. □

The problem with (5.31) is that it does not provide an explicit bound on i. However, the expression here can be analysed to provide a lower bound on the i that satisfies the condition.

Theorem 5.4. *There exists an i that satisfies* (5.31) *bounded as*

$$i \geq \left\lfloor \ln\left(\frac{\beta d_1 + N d_2}{\beta \rho^N d_1}\right) \Big/ \ln \rho \right\rfloor =: i^*. \tag{5.32}$$

Proof. See Appendix 5.C. □

This gives a conservative result that says the system is stable if the number of consecutive packet losses $i \leq i^*$.

5.4.3.4 Case 2: bound on i given $x_0 \in \mathcal{X}_f(d_1)$, and the open-loop stage-cost and terminal-cost parameters

Now let $x_0 \in \mathcal{X}_f(d_1)$ denote an initial state reached via closed-loop control or buffered control, and let $x_i = f^i(x_0, 0)$ denote a state reached after i subsequent steps of open-loop operation. The result below shows a sufficient condition under which $x_i \in \Gamma_N^\beta$.

The lemma corresponding to a link between the value function at x_i after i consecutive open-loop steps and terminal cost function at x_0 under the assumption of open-loop operation ($u = 0$) in the interim.

Lemma 5.4. *Suppose Assumption 5.7 holds. For all* $x_0 \in \mathcal{X}_f(d_1)$*,*

$$V_N^0(x_i) \leq \varphi V_f(x_0), \tag{5.33}$$

where

$$\varphi := \sigma^N \frac{\sigma^i - 1}{\sigma - 1}(1 - \gamma) + \beta \rho^{N+i} > 1. \tag{5.34}$$

Proof. The proof is in Appendix 5.D. □

Theorem 5.5. *Suppose Assumption 5.7 holds with $\sigma > 1$. If*

$$\sigma^N \frac{\sigma^i - 1}{\sigma - 1}(1-\gamma) + \beta\rho^{N+i} \leq \beta + N\frac{d_2}{d_1} \tag{5.35}$$

is satisfied, then $x_i \in \Gamma_N^\beta$.

Proof. By (5.33), for any $x_0 \in \mathcal{X}_f(d_1)$,

$$V_N^0(x_i) \leq \varphi V_f(x_0) \leq \varphi d_1,$$

since $x_0 \in \mathcal{X}_f(d_1)$ if and only if $V_f(x_0) \leq d_1$. Therefore, $x_i \in \Gamma_N^\beta$ if

$$\varphi d_1 \leq \beta d_1 + N d_2.$$

Rearrangement establishes the result. □

Analysing (5.35) we provide a lower bound on the i that satisfies the condition.

Theorem 5.6. *The exists an i that satisfies* (5.35) *is bounded as*

$$i \geq \left\lfloor \ln\left(\frac{\sigma^N(1-\gamma)}{\beta\rho^N(\sigma-1)} + \frac{\beta + N d_2/d_1}{\beta\rho^N}\right) \Big/ \ln\rho \right\rfloor =: i^*. \tag{5.36}$$

Proof. See Appendix 5.E. □

This gives a conservative result that says the system is stable if the number of consecutive packet losses $i \leq i^*$.

Remark 5.1. *As discussed in Chapter 2, dwell-time constraints for switching systems can be interpreted as a general way to model packet losses, specifically consecutive packet losses. A bound on the number of consecutive packet dropouts could correspond to a maximum dwell-time in the loss mode, while a minimum frequency of successful transmissions can be interpreted as a minimum dwell-time in the communication-available mode. The tools developed in [90] could be used to analyse not only minimum and maximum dwell-time constraints arising from consecutive packet losses, but also stability, invariance and boundedness analysis from the viewpoint of switching systems and automata theory.*

In particular, when the packet loss process is bounded by a maximum number of consecutive losses ("at worst"), the admissible switching sequences form a finite automaton, and a minimum dwell-time in the successful transmission mode ("at best") corresponds to restricting the automaton so that recovery transitions occur sufficiently often. The existence of an invariant multi-set for this constrained system ([90]) implies a characterization of where the trajectories lie at all times for both worst-case and best-case scenarios.

Therefore, the proposed theorems for Case 1 and Case 2, could be compared directly with dwell-time results by interpreting bounded consecutive packet loss sequences as automaton constraints, under which boundedness follows from the existence of invariant (or uniformly bounded) sets across all admissible switching sequences.

5.5 Stability analysis of linear systems

The stability analysis of linear systems can be done as a specialization of the analysis for nonlinear system. However, the results are more restrictive, some insights and practical aspects are improved, for example, it is relatively easy to compute or estimate the various constants and parameters involved in the different bounds.

5.5.1 Problem formulation

Now we consider that $f(x, \nu u) = Ax + \nu Bu$, that is the system

$$x_{k+1} = Ax_k + \nu_k Bu_k \tag{5.37}$$

is LTI.

Assumption 5.8. *The matrices A and B are known and the pair (A, B) is stabilizable.*

We also consider the same setting in Fig. 5.1, *i.e.* the linear system is connected to a controller via a partially lossy UDP-like communication channel. While the state measurement is communicated perfectly to the controller, communication between the controller and system actuator is subject to random packet losses as depicted in Fig. 5.3.

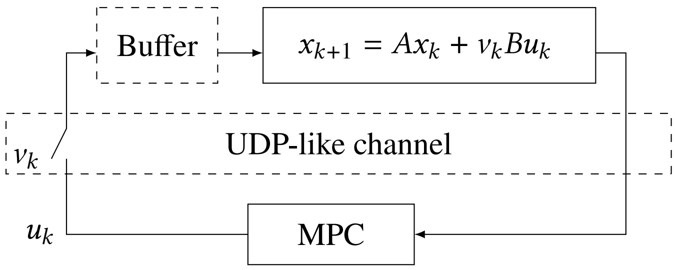


Figure 5.3: A system in connection with a controller via a UDP-like channel. The C-A channel is subject to random packet losses.

5.5.2 Controller formulation

With the system at a state x_k, the optimal control problem $\mathbb{P}_N(x_k)$ to be solved for the linear case is

$$V_N^0(x_k) = \min_{\mathbf{u}_k \in \mathbb{U}} J_N(x_k, \mathbf{u}_k) := \beta V_f(x_{k+N|k}) + \sum_{j=0}^{N-1} \ell(x_{k+j|k}, u_{k+j|k}), \tag{5.38a}$$

subject to, for $j \in \mathbb{N}_{[0,N-1]}$

$$x_{k|k} = x_k \tag{5.38b}$$

$$x_{k+j+1|k} = Ax_{k+j|k} + Bu_{k+j|k} \tag{5.38c}$$

$$u_{k+j|k} \in \mathcal{U}, \tag{5.38d}$$

with

$$\ell(x,u) = \|x\|_Q^2 + \|u\|_R^2 \tag{5.38e}$$

$$V_f(x) = \|x\|_{Q_f}^2, \tag{5.38f}$$

and $\beta \geq 1$.

Where

$$\mathbf{u}_k := \{u_{k|k}, u_{k+1|k}, \dots, u_{k+N-1|k}\}, \tag{5.39}$$

is the finite sequence of future control inputs and constraint set is

$$\mathbb{U} := \mathcal{U} \times \cdots \times \mathcal{U}. \tag{5.40}$$

Assumption 5.9. *$Q \succ 0$, $R \succ 0$, $Q_f \succ 0$ and $\beta \geq 1$.*

Solving $\mathbb{P}_N(x_k)$ at x_k yields the solution

$$\mathbf{u}_k^0(x_k) := \{u_{k|k}^0, u_{k+1|k}^0, \ldots, u_{k+N-1|k}^0\}. \tag{5.41}$$

Also, the buffered mechanism depicted in Section 5.3.2 is used in the same way for the linear system case.

5.5.3 Preliminaries: stability without terminal constraint set

The following assumption it is a specialization of Assumption 5.5 in which the control law was a function $\kappa_f(x)$, but now we can define the control K_f explicitly.

Assumption 5.10. *The matrix $Q_f \succ 0$ is such that*

$$(A + BK_f)^\top Q_f (A + BK_f) - Q_f \leq -(Q + K_f^\top R K_f)$$

for some $K_f : \mathbb{R}^n \to \mathbb{R}^m$ that stabilizes the pair (A, B).

The assumption says that the choice of Q_f means $V_f(x)$ is a global CLF for the dynamics $x^+ = (A + BK_f)$, but it does consider the input constraint. Therefore, we need to limit x to be in a neighbourhood of the origin in order to ensure $V_f(x)$ is a local CLF in this neighbourhood, and such that $u = K_f x$ is constraint admissible. Since $\beta \geq 1$ it follows that $\beta V_f(\cdot)$, is also a CLF.

Definition 5.3. *Let $d_1 > 0$ be such that $K_f x \in \mathcal{U}$ for all*

$$x \in \mathcal{X}_f(d_1) := \left\{x \colon \|x\|_{Q_f}^2 \leq d_1\right\}. \tag{5.42}$$

Since we are considering a linear quadratic setting, $\mathcal{X}_f(d_1)$ is defined as an ellipsoid. Also, such a d_1 is guaranteed to exist in view of the positive definiteness of Q_f (Assumption 5.9) and the fact that $\mathcal{U}$ contains the origin in its interior (Assumption 5.2).

The following result is an immediate consequence of Assumption 5.10 and Definition 5.3.

Lemma 5.5. *For all $x \in \mathcal{X}_f(d_1)$ and for all $K_f \in \mathcal{U}$,*

$$V_f(Ax + BK_f x) - V_f(x) \leq -\ell(x, K_f x). \tag{5.43}$$

Such $V_f(\cdot)$ is a local CLF in $\mathcal{X}_f(d_1)$, and $\mathcal{X}_f(d_1)$ is an invariant and constraint admissible set for the terminal dynamics $x^+ = Ax + BK_f x$.

We require one more ingredient [110]:

Definition 5.4. *Let $d_2 > 0$ be such that for all $x \notin \mathcal{X}_f(d_1)$ and $u \in \mathcal{U}$,*

$$d_2 \leq \ell(x, u). \tag{5.44}$$

Such a d_2 is guaranteed to exist in view of positive definiteness of Q and R (Assumption 5.9) and compactness of $\mathcal{U}$ (Assumption 5.2).

The previous defines the set

$$\Gamma_N^{\beta} := \left\{x \in \mathbb{R}^n \colon V_N^0(x) \leq \beta d_1 + N d_2\right\}, \tag{5.45}$$

such that the controlled system $x^+ = Ax + B\kappa_N(x)$ is exponentially stable with region of attraction Γ_N^{β}, and, for all $x \in \Gamma_N^{\beta}$,

$$V_N^0(Ax + B\kappa_N(x)) - V_N^0(x) \leq -\ell(x, \kappa_N(x)). \tag{5.46}$$

In addition, the following lemmas concerning the value, stage and terminal cost functions are defined as follows.

Lemma 5.6. *There exists a constant $\gamma \in (0, 1]$, such that for all $x \in \mathcal{X}_f(d_1)$*

$$V_f\big(Ax + BK_f x\big) \leq \gamma V_f(x). \tag{5.47}$$

The latter allow us to define the following lemma, the proof can be obtained by applying the Rayleigh-Ritz theorem, *i.e.* $\underline{\lambda}(A)\|x\|_2^2 \leq \|x\|_A^2 \leq \bar{\lambda}(A)\|x\|_2^2$, if A is symmetric.

Lemma 5.7. *There exists a $\bar{\gamma} > \gamma$ such that for all $x \in \mathcal{X}_f(d_1)$*

$$\bar{\gamma} := \frac{\bar{\lambda}\big((A + BK_f)^\top Q_f (A + BK_f)\big)}{\bar{\lambda}(Q_f)}, \tag{5.48}$$

where

$$\gamma \geq \frac{\big\|(A + BK_f)x\big\|^2_{Q_f}}{\|x\|^2_{Q_f}}, \quad \|x\|^2_{Q_f} \neq 0. \tag{5.49}$$

5.5.4 Stability analysis of the linear NCS

Adhering to the same procedure we did in Section 5.4.2, we examine directly the two cases using the behaviour concerning the open-loop operations for the value, stage and terminal costs.

Definition 5.5 ($V_N^0(\cdot)$ in open-loop)**.** *There exists a $\zeta \in [1, +\infty)$ such that for all $x \in \Gamma_N^\beta$*

$$V_N^0(Ax) \leq \zeta V_N^0(x). \tag{5.50}$$

Definition 5.6 ($\ell(\cdot,\cdot)$ and $V_f(\cdot)$ in open-loop)**.** *There exist $\sigma \in [1, +\infty)$ and $\rho \in [1, +\infty)$ such that*

$$\ell(Ax, 0) \leq \sigma \ell(x, u), \tag{5.51}$$

for all $x \in \Gamma_N^\beta$ and $u \in \mathcal{U}$,

$$V_f(Ax) \leq \rho V_f(x), \tag{5.52}$$

for all $x \in \mathcal{X}_f(d_1)$.

Such σ and ρ are guaranteed to exist in view of the positive definiteness of Q and R (Assumption 5.9) and compactness of $\mathcal{U}$ (Assumption 5.2).

The previous Definitions 5.5–5.6 allow us to define the following lemmas. Again, the proof of both can be obtained by applying the Rayleigh-Ritz theorem.

Lemma 5.8. *There exists a $\bar{\rho} > \rho$ such that for all $x \in \mathcal{X}_f(d_1)$*

$$\bar{\rho} := \frac{\bar{\lambda}(A^\top Q_f A)}{\bar{\lambda}(Q_f)}, \tag{5.53}$$

where

$$\rho \geq \frac{\|Ax\|^2_{Q_f}}{\|x\|^2_{Q_f}}, \quad \|x\|^2_{Q_f} \neq 0. \tag{5.54}$$

Lemma 5.9. *There exists a $\bar{\sigma} > \sigma$ such that for all $x \in \Gamma^{\beta}_{N}$*

$$\bar{\sigma} := \frac{\bar{\lambda}(A^{\top}QA)}{\bar{\lambda}(Q)}, \tag{5.55}$$

where

$$\sigma \geq \frac{\|Ax\|^2_{Q}}{\|x\|^2_{Q}} \geq \frac{\|Ax\|^2_{Q}}{\|x\|^2_{Q} + \|u\|^2_{R}}, \quad \|x\|^2_{Q} \neq 0, \|u\|^2_{R} \neq 0, \tag{5.56}$$

for all $u \in \mathcal{U}$.

5.5.4.1 Case 1: bound on i given $x_0 \in \mu\Gamma^{\beta}_{N}$, and the linear open-loop stage-cost and terminal-cost parameters

Let $x_0 \in \mu\Gamma^{\beta}_{N}$, $\mu \in (0, 1]$, be an initial state reached via closed-loop control or buffered control, and let $x_i = A^i x_0$ be a state reached after i subsequent steps of open-loop operation. In what follows, we show the conditions under which x_i is contained in Γ^{β}_{N}.

First, the lemma that links the value function at x_i, the stage cost and terminal cost function at x_0 under the assumption of open-loop operation in the interim.

Lemma 5.10. *For all $x_0 \in \Gamma^{\beta}_{N}$ and all $u_0 \in \mathcal{U}$*

$$V^0_N(x_i) \leq \Psi\ell(x_0, u_0) + \Phi V_f(x_0), \tag{5.57}$$

where

$$\Psi := \bar{\sigma}^i \frac{\bar{\sigma}^N - 1}{\bar{\sigma} - 1} > 1, \quad \Phi := \beta\bar{\rho}^{N+i} > 1. \tag{5.58}$$

Proof. By applying Lemma 5.3, and Lemmas 5.8–5.9. □

Theorem 5.7. *If*

$$\beta\bar{\rho}^{N+i} d_1 + \bar{\sigma}^i \frac{\bar{\sigma}^N - 1}{\bar{\sigma} - 1}\left(\|x_0\|^2_Q + \|u_0\|^2_R\right) \leq \beta d_1 + N d_2 \tag{5.59}$$

is satisfied with $\bar{\sigma} > 1$, then $x_i \in \Gamma^{\beta}_{N}$.

Theorem 5.8. *The i that satisfies* (5.59) *is bounded as*

$$i \geq \left\lfloor \ln\left(\frac{\beta d_1 + N d_2}{\beta \bar{\rho}^N d_1}\right) \Big/ \ln \bar{\rho} \right\rfloor =: i^*. \tag{5.60}$$

This gives a conservative result that says the system is stable if the number of consecutive packet losses $i \leq i^*$.

5.5.4.2 Case 2: bound on i given $x_0 \in \mathcal{X}_f(d_1)$, and the linear open-loop stage-cost and terminal-cost parameters

The lemma corresponding to a link between the value function at x_i and terminal cost function at x_0 under the assumption of open-loop operation in the interim.

Lemma 5.11. *For all $x_0 \in \mathcal{X}_f(d_1)$,*

$$V_N^0(x_i) \leq \varphi V_f(x_0), \tag{5.61}$$

where

$$\varphi := \bar{\sigma}^N \frac{\bar{\sigma}^i - 1}{\bar{\sigma} - 1}(1 - \bar{\gamma}) + \beta \bar{\rho}^{N+i} > 1. \tag{5.62}$$

Proof. By applying Lemma 5.4, and Lemmas 5.7–5.9. □

Theorem 5.9. *If*

$$\bar{\sigma}^N \frac{\bar{\sigma}^i - 1}{\bar{\sigma} - 1}(1 - \bar{\gamma}) + \beta \bar{\rho}^{N+i} \leq \beta + N \frac{d_2}{d_1} \tag{5.63}$$

is satisfied with $\bar{\sigma} > 1$, then $x_i \in \Gamma_N^\beta$.

Then, we provide a lower bound on the i that satisfies the condition.

Theorem 5.10. *The i that satisfies* (5.63) *is bounded as*

$$i \geq \left\lfloor \ln\left(\frac{\bar{\sigma}^N (1 - \bar{\gamma})}{\beta \bar{\rho}^N (\bar{\sigma} - 1)} + \frac{\beta + N d_2 / d_1}{\beta \bar{\rho}^N}\right) \Big/ \ln \bar{\rho} \right\rfloor =: i^*. \tag{5.64}$$

This gives a conservative result that says the system is stable if the number of consecutive packet losses $i \leq i^*$.

5.6 Numerical example 1: nonlinear system

Consider the nonlinear system $x_{k+1} = f(x_k, v_k u_k)$

$$x_{k+1} = r\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}x_k + \epsilon\|x_k\|_2^2 + v_k\begin{bmatrix}1.5\\ 0\end{bmatrix}u_k,$$

where $r = 1.05$, $\theta = 0.12$, $\epsilon = 0.0001$, subject to the input constraint set $\mathcal{U} = \{u\colon |u| \leq 1\}$.

The controller is designed as $\ell(x,u) = x^\top Qx + u^\top Ru$, $V_f(x) = x^\top Q_f x$ and $\kappa_f(x)$, with $R = 1$, $\beta = 1$, horizon $N = 3$, and

$$Q = \begin{bmatrix}0.4 & 0\\ 0 & 1\end{bmatrix}, \quad Q_f = \begin{bmatrix}1.06 & 1.59\\ 1.58 & 9.61\end{bmatrix}.$$

Q_f is a local approximation solution that satisfies Assumption 5.10 given a local approximation of $\kappa_f(x) \approx K_f x$ around the origin.

The controller satisfies Definition 5.1 and Definition 5.2 with $d_1 = 2.89$ and $d_2 = 1.19$; therefore, for $N = 3$ and $\beta = 1$,

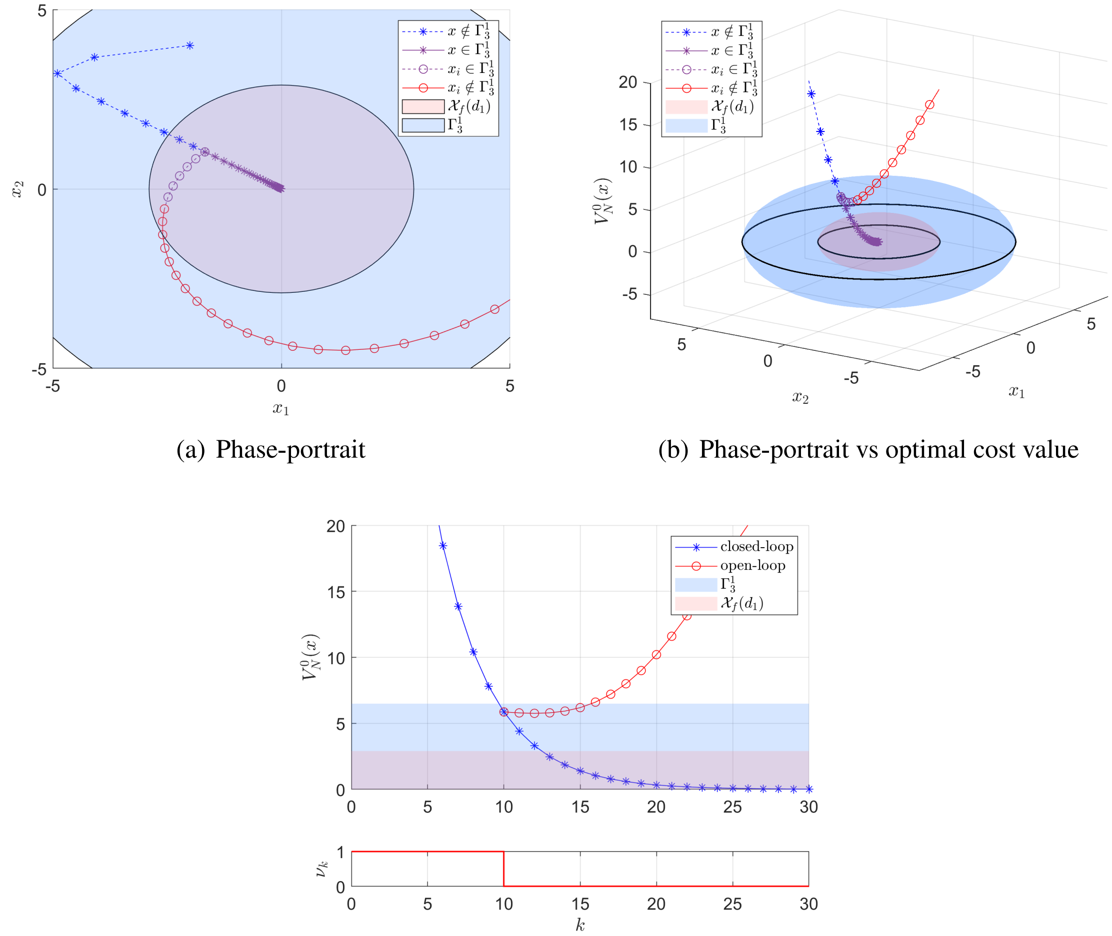
$$\Gamma_3^1 = \left\{x\colon V_N^0(x) \leq 6.49\right\}.$$

5.6.1 Case 1: $x_0 \in \mu\Gamma_N^\beta$

Let $x = \begin{bmatrix}-2 & 4\end{bmatrix}^\top$ denote the initial state not in Γ_3^1. After closed-loop operation without packet losses, x reaches the state $x_0 = \begin{bmatrix}-1.67 & 1.03\end{bmatrix}^\top \in \Gamma_3^1$, and it successfully converges towards the origin. However, supposing that all i subsequent packets are instead dropped ($v_k = 0$), for $1 \leq i \leq 5$, the states x_1 to x_5 in open-loop operation remain in Γ_3^1 but x_6 leaves the set, see Fig. 5.4.

By reviewing the closed-loop trajectory we obtain $\mu = 0.72$, and reviewing the open-loop trajectory give us $\zeta = 1.06$ (Assumption 5.6). Theorem 5.1 predicts $i^* = \lfloor 5.64 \rfloor = 5$, meaning that x_i, for $i = 5$ steps, is guaranteed to remain within Γ_3^1. Thus, the main result for the nonlinear Case 1 is verified, see Fig. 5.4.

Now, we present the behaviour of the state while there are packet dropouts before reaching the Γ_3^1. In Fig. 5.5, from time step $k = 4$ to $k = 6$, the system suffers from $i = 3$ consecutive packet



(a) Phase-portrait

(b) Phase-portrait vs optimal cost value

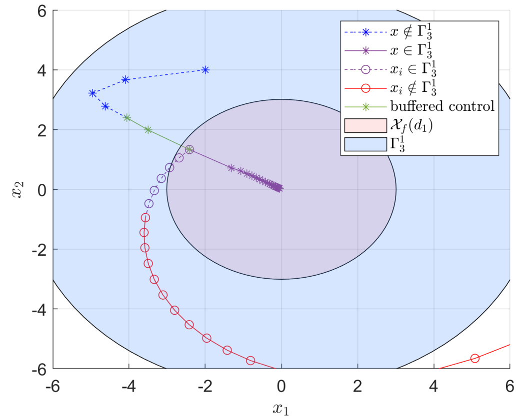
(c) Optimal cost value

Figure 5.4: Case 1 - nonlinear system: phase-portrait and optimal cost value under closed-loop control.

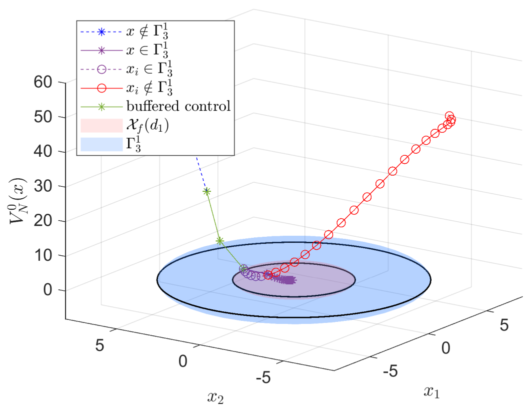
dropouts and the buffer enters in action. However, the size of the buffer is $N = 3$, and it only transfers x to $x_0 = \begin{bmatrix} -2.41 & 1.33 \end{bmatrix}^\top \in \Gamma_3^1$. Supposing that all i subsequent packets are dropped, for $1 \leq i \leq 5$, the states x_1 to x_5 in open-loop operation remain in Γ_3^1 but x_6 leaves the set. Again, Theorem 5.1 predicts $i^* = \lfloor 5.64 \rfloor = 5$, meaning that x_i, for $i = 5$ steps, is guaranteed to remain within Γ_3^1.

Design choice of the buffer size

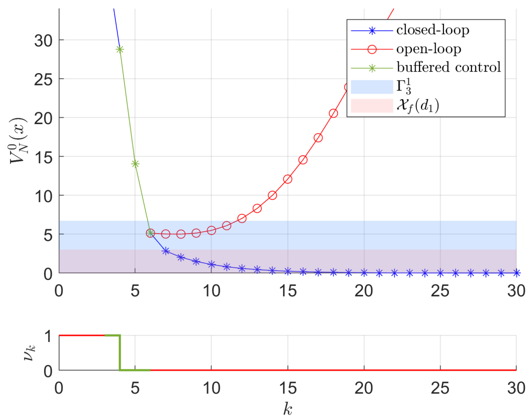
With these results, we can suggest that the approximate minimum size of the buffer such that x reaches Γ_N^β is $N_b \geq N + i^* = 8$. In Fig. 5.6, x under buffered control operation is transferred to $x_0 = \begin{bmatrix} -2.09 & 1.14 \end{bmatrix}^\top \in \Gamma_8^1$ despite consecutive packet dropouts, verifying the design choice of the buffer size.



(a) Phase-portrait



(b) Phase-portrait vs optimal cost value



(c) Optimal cost value

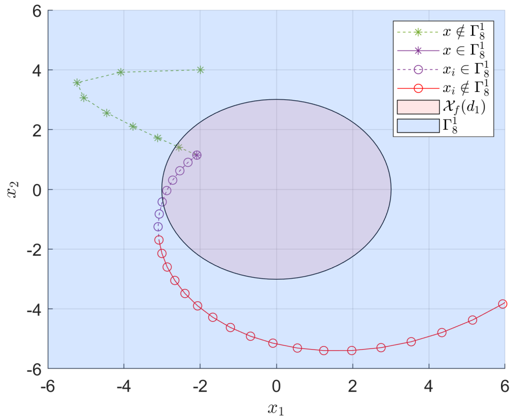
Figure 5.5: Case 1 - nonlinear system: phase-portrait and optimal cost value under closed-loop and buffered control.

5.6.2 Case 2: $x_0 \in \mathcal{X}_f(d_1)$

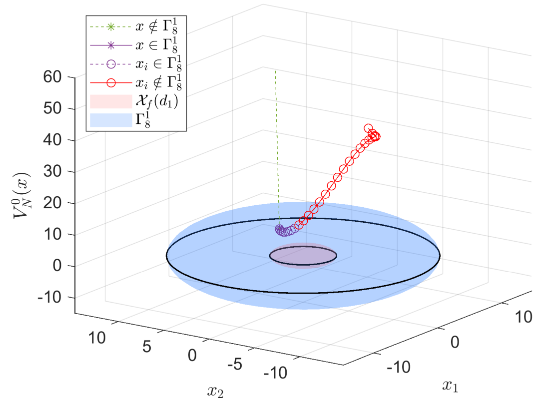
For Case 2, under closed-loop operation without packet losses, x reaches the state $x_0 = \begin{bmatrix} -1.09 & 0.67 \end{bmatrix}^\top \in \mathcal{X}_f(d_1)$ and converges successfully towards the origin. However, supposing that all i subsequent packets are instead dropped ($\nu_k = 0$), for $1 \leq i \leq 13$, the states x_1 to x_{13} in open-loop operation remain in Γ_3^1 but x_{14} leaves the set, see Fig. 5.7.

Theorem 5.2 predicts $i^* = \lfloor 13.85 \rfloor = 13$, meaning that x_i, for $i = 13$ steps, is guaranteed to remain within Γ_3^1. Thus, the main result for the nonlinear Case 2 is verified, see Fig. 5.7.

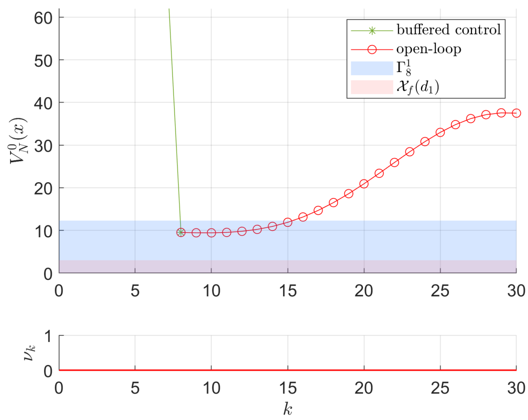
In Fig. 5.8, from time step $k = 5$ to $k = 7$, while in closed-loop, the system suffers from $i = 3$ consecutive packet dropouts and the buffer enters in action. However, the size of the buffer is $N = 3$, and it is only able to transfer x to $x_0 = \begin{bmatrix} -1.78 & 0.98 \end{bmatrix}^\top \in \mathcal{X}_f(d_1)$.



(a) Phase-portrait



(b) Phase-portrait vs optimal cost value



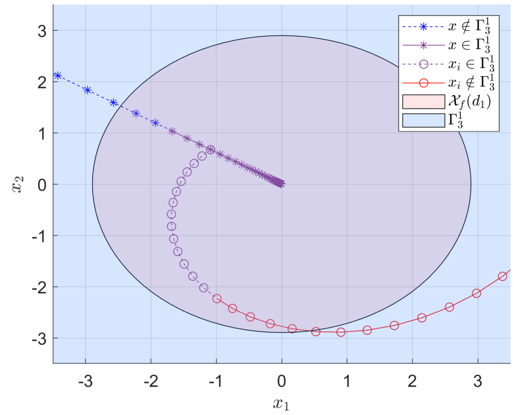
(c) Optimal cost value

Figure 5.6: Case 1 - nonlinear system: phase-portrait and optimal cost value under buffered control only.

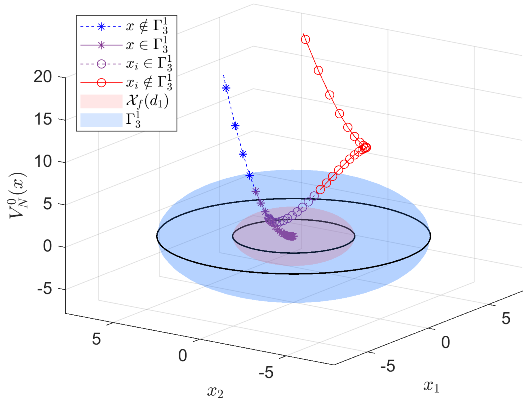
Supposing that all i subsequent packets are dropped, for $1 \leq i \leq 9$, the states x_1 to x_9 in open-loop operation remain in Γ_3^1 but x_{10} leaves the set. Theorem 5.2 predicts $i^* = \lfloor 10.5 \rfloor = 10$, meaning that x_i, for $i = 10$ steps, is guaranteed to remain within Γ_3^1.

Design choice of the buffer size

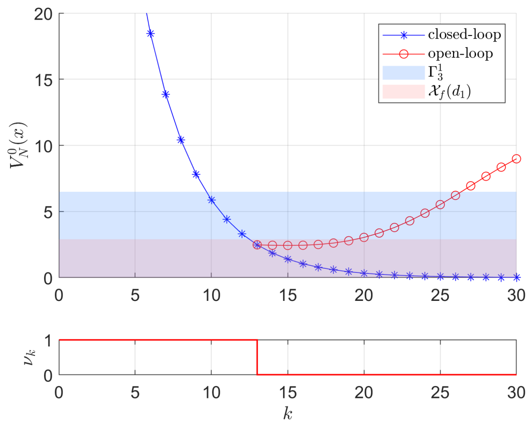
As we did in Case 1, the approximate minimum size of the buffer necessary for x to reach $\mathcal{X}_f(d_1)$ is $N_b \geq N + i^* = 11$. In Fig. 5.9, x under buffered control operation is transferred to $x_0 = \begin{bmatrix} -1.12 & 0.61 \end{bmatrix}^\top \in \mathcal{X}_f(d_1)$ despite consecutive packet dropouts; thus, verifying the design choice.



(a) Phase-portrait



(b) Phase-portrait vs optimal cost value



(c) Optimal cost value

Figure 5.7: Case 2 - nonlinear system: phase-portrait and optimal cost value under closed-loop control.

5.7 Numerical example 2: linear system

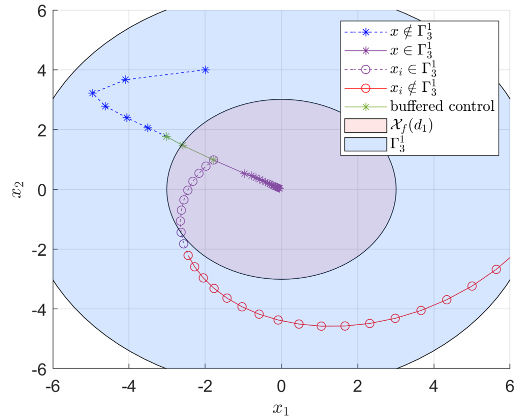
Consider the linear system $x_{k+1} = Ax_k + \nu_k Bu_k$ with

$$A = \begin{bmatrix} 1.04 & -0.13 \\ 0.13 & 1.04 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

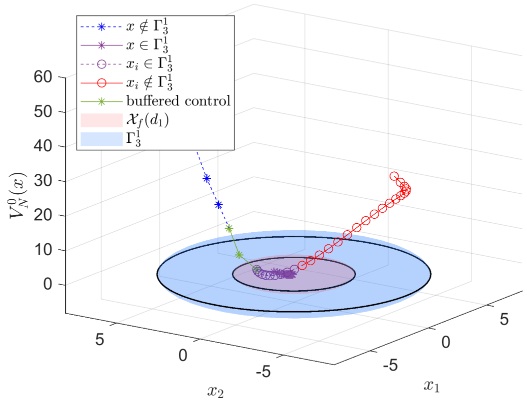
and input constraint set $\mathcal{U} = \{u \colon |u| \leq 1\}$. The controller has $\ell(x,u) = x^\top Qx + u^\top Ru$, $V_f(x) = x^\top Q_f x$ and $\kappa_f(x) = K_f x$, with $R = 1$, $K_f = \begin{bmatrix} -0.71 & -0.77 \end{bmatrix}$, $\beta = 1$, prediction horizon $N = 3$, and

$$Q = \begin{bmatrix} 0.41 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_f = \begin{bmatrix} 1.39 & 1.93 \\ 1.93 & 10.25 \end{bmatrix}.$$

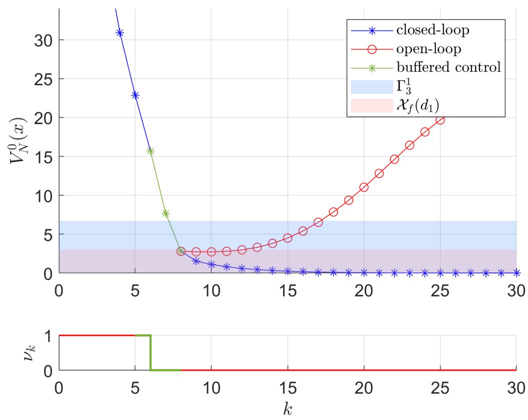
The controller satisfies Assumptions 5.10 and 5.4 with $d_1 = 2.73$ and $d_2 = 1.25$; therefore, with



(a) Phase-portrait



(b) Phase-portrait vs optimal cost value



(c) Optimal cost value

Figure 5.8: Case 2 - nonlinear system: phase-portrait and optimal cost value under closed-loop and buffered control.

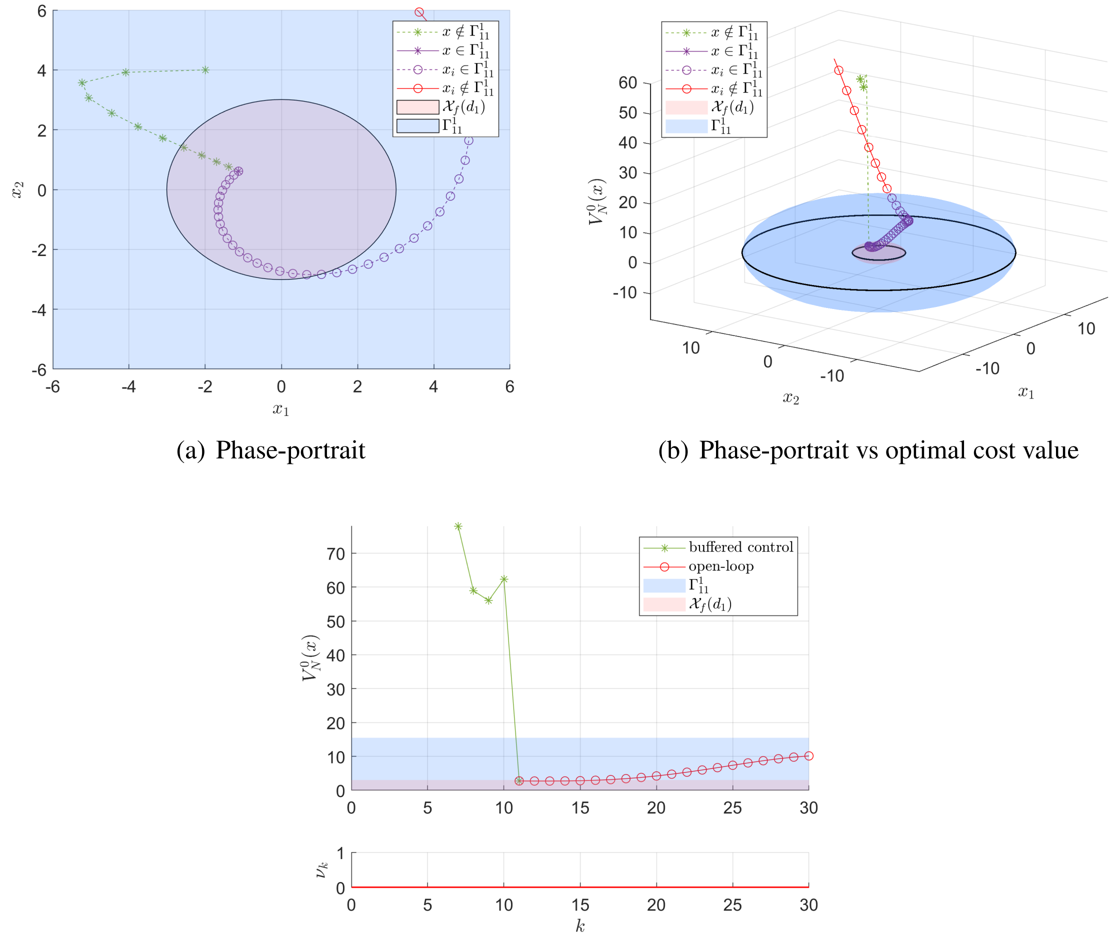
$N = 3$ and $\beta = 1$,

$$\Gamma_3^1 = \left\{x \colon V_N^0(x) \leq 6.49\right\}.$$

5.7.1 Case 1: $x_0 \in \mu\Gamma_N^\beta$

Let $x = \begin{bmatrix} 2 & -4 \end{bmatrix}^\top$ denote the initial state not in Γ_3^1. After closed-loop operation without packet losses, x has reached the state $x_0 = \begin{bmatrix} 1.90 & -1.02 \end{bmatrix}^\top \in \Gamma_3^1$, and it successfully converges towards the origin. However, supposing that all i subsequent packets are instead dropped ($\nu_k = 0$), for $1 \leq i \leq 5$, the states x_1 to x_5 in open-loop operation remain in Γ_3^1 but x_6 leaves the set, see Fig. 5.10.

The constants in Lemmas 5.7–5.9 are evaluated such that $\bar{\gamma} = 0.83$, $\bar{\rho} = 1.10$ and $\bar{\sigma} = 1.10$, and since we know x_0, we have that $\gamma = 0.6$, $\rho = 1.03$ and $\sigma = 1.04$, confirming $\bar{\gamma} > \gamma$, $\bar{\rho} > \rho$ and



(a) Phase-portrait

(b) Phase-portrait vs optimal cost value

(c) Optimal cost value

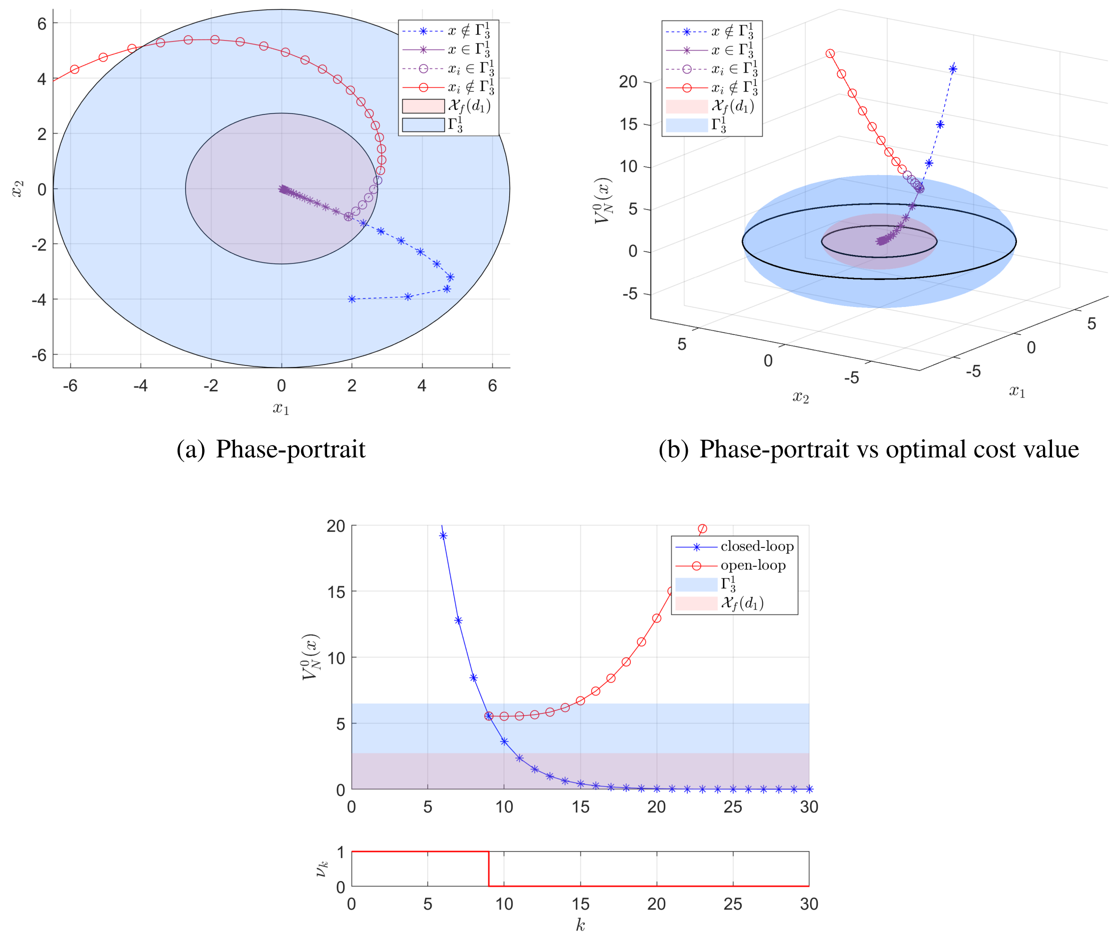
Figure 5.9: Case 2 - nonlinear system: phase-portrait and optimal cost value under buffered control only.

$\bar{\sigma} > \sigma$.

Theorem 5.8 predicts $i^* = \lfloor 5.87 \rfloor = 5$ given $\bar{\rho}$, meaning that x_i, for $i = 5$ steps, is guaranteed to remain within Γ^1_3. Thus, the main result for Case 1 is verified, see Fig. 5.10.

Design choice of the buffer size

The approximately minimum size of the buffer in order to reach Γ^1_3 is $N_b \geq N + i^* = 8$. In Fig. 5.11, x under buffered control operation is transferred to $x_0 = \begin{bmatrix} 1.90 & -1.02 \end{bmatrix}^\top \in \Gamma^1_9$ despite consecutive packet dropouts, verifying the design choice of the buffer size.



(a) Phase-portrait

(b) Phase-portrait vs optimal cost value

(c) Optimal cost value

Figure 5.10: Case 1 - linear system: phase-portrait and optimal cost value under closed-loop control.

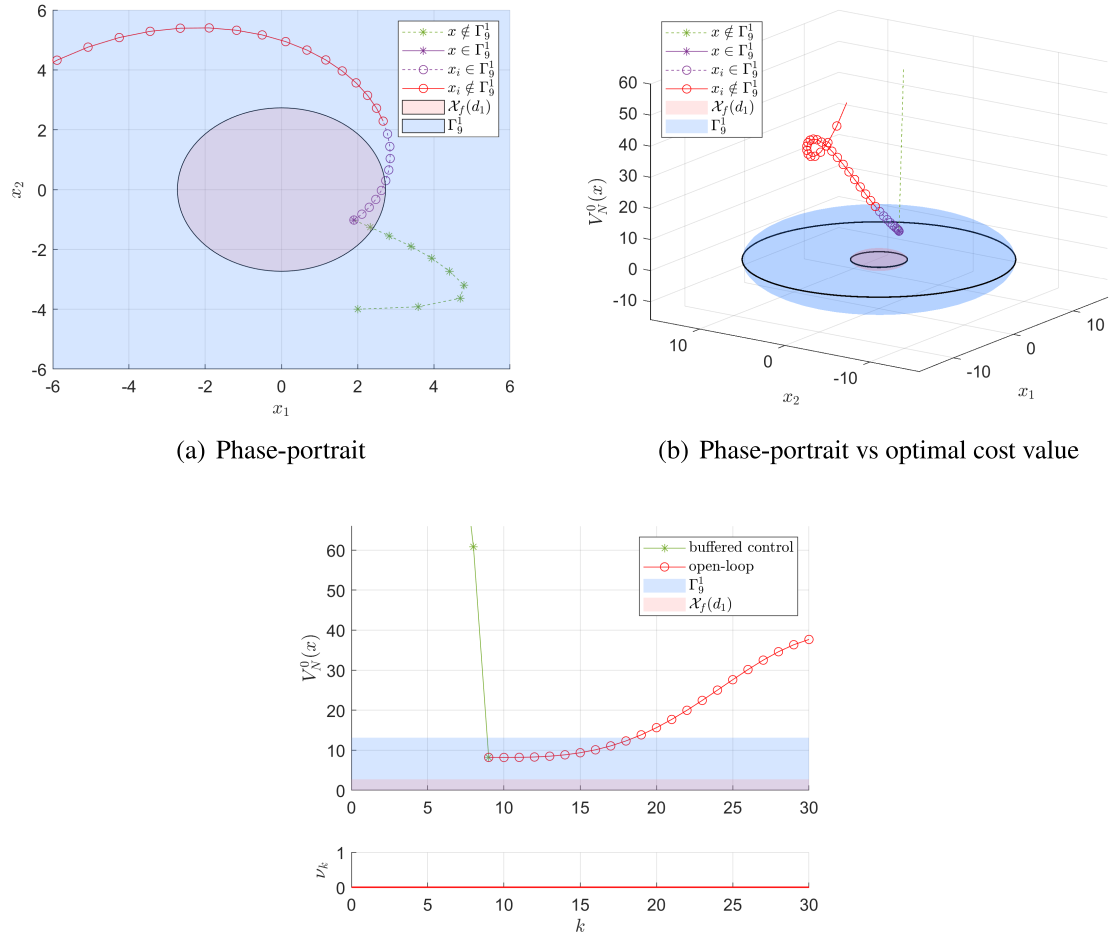
5.7.2 Case 2: $x_0 \in \mathcal{X}_f(d_1)$

For Case 2, under closed-loop operation without packet losses, x reaches the state $x_0 = \begin{bmatrix} 0.81 & -0.87 \end{bmatrix}^\top \in \mathcal{X}_f(d_1)$ and converges successfully towards the origin. However, supposing that all i subsequent packets are instead dropped ($\nu_k = 0$), for $1 \leq i \leq 12$, the states x_1 to x_{12} in open-loop operation remain in Γ_3^1 but x_{13} leaves the set, see Fig. 5.12.

Theorem 5.10 predicts $i^* = \lfloor 12.50 \rfloor = 12$ given $\bar{\gamma}, \bar{\rho}$ and $\bar{\sigma}$, meaning that x_i, for $i = 12$ steps, is guaranteed to remain within Γ_3^1. Thus, the main result for Case 2 is verified, see Fig. 5.12.

Design choice of the buffer size

As we did in Case 1, the approximately minimum size of the buffer in order to reach $\mathcal{X}_f(d_1)$ is $N_b \geq N + i^* = 11$. In Fig. 5.13, x under buffered control operation is transferred to



(a) Phase-portrait

(b) Phase-portrait vs optimal cost value

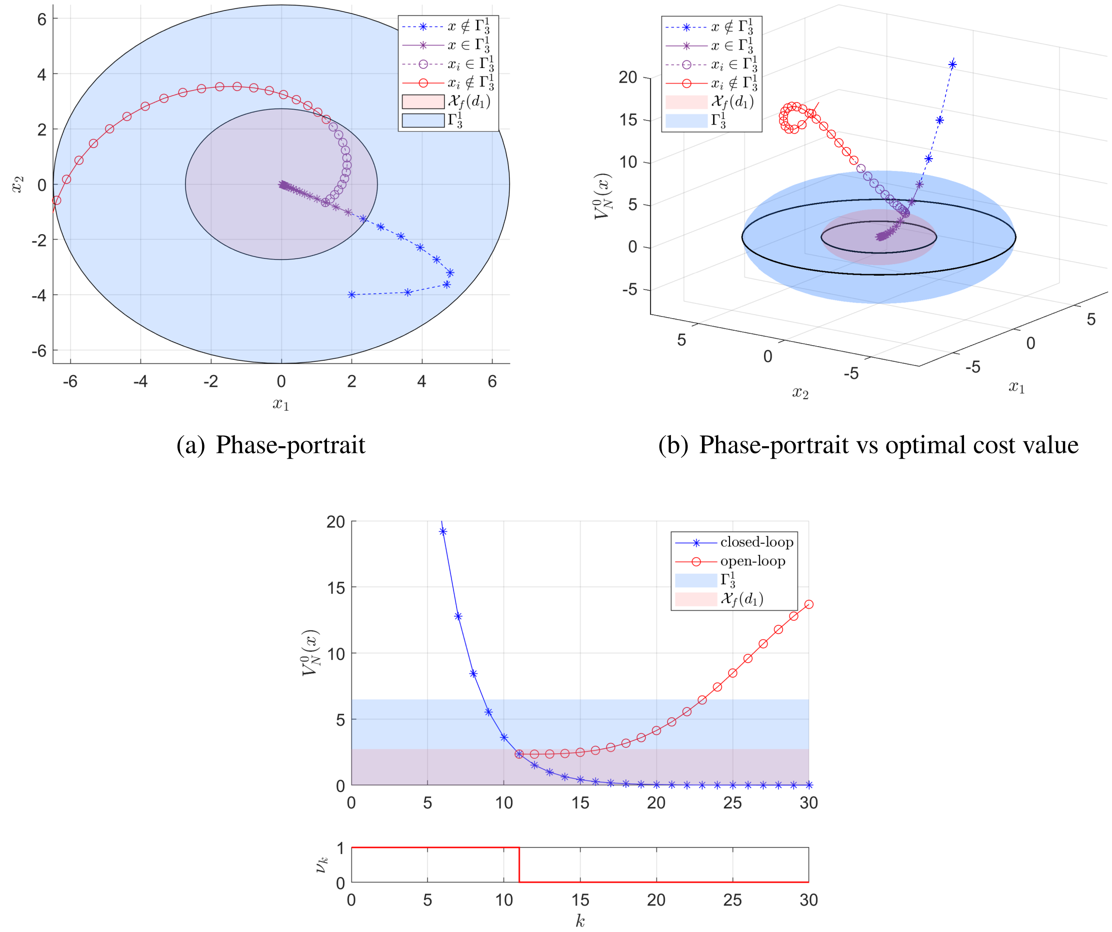
(c) Optimal cost value

Figure 5.11: Case 1 - linear system: phase-portrait and optimal cost value under buffered control only.

$x_0 = \begin{bmatrix} 1.01 & -0.53 \end{bmatrix}^\top \in \mathcal{X}_f(d_1)$ despite consecutive packet dropouts; thus, verifying the design choice.

5.8 Conclusions

This chapter has studied MPC-controlled discrete time-invariant nonlinear and linear systems subject to input constraints and random packet losses over the C-A channel. The MPC formulation includes a terminal cost function that satisfies a local control Lyapunov condition, but no explicit terminal constraints. It is shown that the use of a buffer provides, at best, the transfer of an initial state to the terminal region associated with the control Lyapunov terminal cost. The number of subsequent consecutive packet losses that the system may experience in order that stability is maintained is upper bounded by expressions that depend on system and controller



(a) Phase-portrait

(b) Phase-portrait vs optimal cost value

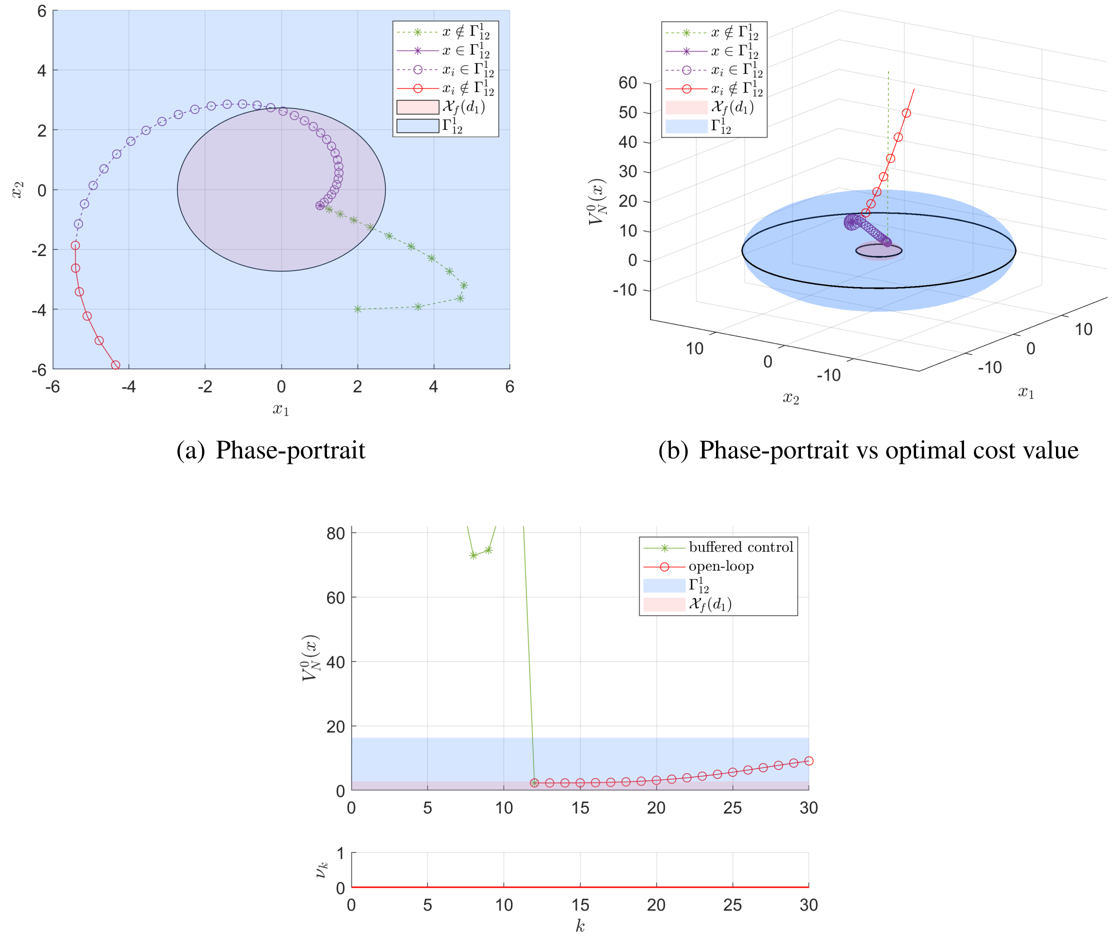
(c) Optimal cost value

Figure 5.12: Case 2 - linear system: phase-portrait and optimal cost value under closed-loop control.

parameters. Numerical example 1 verifies the existence of these bounds for a nonlinear system by analysing the value function along the closed-loop and open-loop system trajectories.

The specialized analysis in the linear section evaluates these bounds and verifies their conservativeness by approximating them using the open-loop stage-cost and terminal-cost parameters. It was shown by the numerical example 2 that the existence of these bounds–despite their conservatism–can be used to tune the design parameters of the controller and the selection of the buffer size; knowing that the buffer with the size of the horizon length is not sufficient to guarantee stability when it is exhausted.

For both scenarios, Case 1 and Case 2, the numerical examples suggest that the minimum buffer size is approximately equal to the sum of the prediction horizon and the bound on the number of consecutive packet losses in open-loop operation before the state leaves the RoA. However,



(a) Phase-portrait

(b) Phase-portrait vs optimal cost value

(c) Optimal cost value

Figure 5.13: Case 2 - linear system: phase-portrait and optimal cost value under buffered control only.

this design choice should be interpreted as a tuning guideline rather than a definitive rule, since further theoretical analysis is required when the buffer size is decoupled from the prediction horizon.

In order to emphasize the central findings, Table 5.1 summarizes the main technical results, highlighting the main assumptions, setting, and associated estimated and computable parameters.

5.A Proof of Lemma 5.2

First, let us define the following lemma.

Lemma 5.12 ($V_f(\cdot)$ in closed-loop)**.** *Suppose* $x_j \in \mathcal{X}_f(d_1)$, *then* $x_{j+1} = f(x_j, \kappa_f(x_j)) \in \mathcal{X}_f(d_1)$,

Aspect	Main Assumptions / Setting	Key Technical Results
Control design	State feedback MPC with input constraints and buffering	Buffer mitigates packet losses but cannot guarantee stability when exhausted.
Terminal ingredients	Local CLF terminal cost $V_f(x)$, no terminal constraint	Local stability ensured via terminal cost without imposing terminal constraints.
RoA Γ_N^{β} and $\mathcal{X}_f(d_1)$	Implicitly defined by MPC design (N, Q, R, Q_f) and $\beta V_f(x)$	The buffer guarantees reachability of the state to the terminal region $\mathcal{X}_f(d_1)$.
Stability mechanism	Value-function-based analysis	Stability is maintained only if the number of consecutive packet losses is bounded.
Nonlinear closed-loop behaviour	Nonlinear dynamics, MPC feedback applied	Value function decrease is characterized by **estimated parameters** μ, γ, ξ
Nonlinear open-loop behaviour	Buffer exhausted, open-loop evolution	Value function growth is characterized by **estimated parameters** ζ, σ, ρ, verified numerically (Example 1).
Linear closed-loop behaviour	Linear dynamics, MPC feedback applied	Value function decrease is characterized by **computable parameters** $\bar{\gamma} > \gamma$, conservative but explicit bounds.
Linear open-loop behaviour	Buffer exhausted, open-loop evolution	Value function growth is characterized by **computable parameters** $\bar{\rho} > \rho, \bar{\sigma} > \sigma$, derived from stage and terminal costs.
Conservatism	Analytical upper bounds	Linear bounds are conservative but informative for tuning horizon length and buffer size (Example 2).
Design guideline	Empirical observation from simulations	Minimum buffer size ≈ prediction horizon + open-loop packet-loss bound.

Table 5.1: Summary of technical results.

for some $j \in \mathbb{N}_{\geq 0}$. *Then, for* $i \geq j$,

$$V_f(x_i) \leq \gamma^{i-j} V_f(x_j) \implies x_i \in \mathcal{X}_f(d_1), \tag{5.65a}$$

and

$$\sum_{j=i}^{N+i} V_f(x_j) \leq \sum_{j=0}^{N} \gamma^j V_f(x_i). \tag{5.65b}$$

Proof. It follows by applying (5.15a). □

Let us consider a state $x_{i|N} = x_i \in \mathcal{X}_f(d_1)$ and controls $u_{j|N} = u_j = \kappa_f(x_j) \in \mathcal{U}$, $j \in \mathbb{N}_{[i,N+i-1]}$. After $N+i$ steps without consecutive packet losses, the value function is

$$V_N^0(x_i) \leq \sum_{j=i}^{N+i-1} \Big(\ell(x_j, u_j) + \beta V_f(x_{j+1}) - \beta V_f(x_j)\Big) + \beta V_f(x_i), \tag{5.66a}$$

applying Assumption 5.5, we have that

$$\leq (1-\beta) \sum_{j=i}^{N+i-1} \Big(V_f(x_j) - V_f(x_{j+1})\Big) + \beta V_f(x_i), \tag{5.66b}$$

owing to (5.15b), we can write

$$\leq (1-\beta)(1-\gamma) \sum_{j=i}^{N+i-1} V_f(x_j) + \beta V_f(x_i), \tag{5.66c}$$

using (5.65b), we can say that

$$\leq \left(\beta + (1-\beta) \sum_{j=0}^{N-1} \gamma^j\right) V_f(x_i), \tag{5.66d}$$

because of $\sum_{j=m}^{n} z^j = \frac{z^m - z^{n+1}}{1-z}$, for $m \in \mathbb{N}_{[0,n]}$ and $|z| < 1$, then

$$= \left(\beta - (1-\gamma)(\beta-1)\frac{1-\gamma^N}{1-\gamma}\right) V_f(x_i). \tag{5.66e}$$

Since $V_N^0(x_i)$ represents where $x_i \in \mathcal{X}_f(d_1)$ has reached after $N+i$ steps without consecutive packet losses, this will be our starting state $x_0 \in \mathcal{X}_f(d_1)$ from where the value function experiences consecutive packet losses. Therefore, $x_i \in \mathcal{X}_f(d_1) \implies x_0 \in \mathcal{X}_f(d_1)$, such that

$$V_N^0(x_0) \leq \big(\beta - (\beta-1)(1-\gamma^N)\big) V_f(x_0), \tag{5.66f}$$

which completes the proof. □

5.B Proof of Lemma 5.3

First, let us define the following lemmas.

Lemma 5.13 ($\ell(\cdot,\cdot)$ in open-loop)**.** *Let denote $x_j \in \Gamma_N^\beta$, $u_j \in \mathcal{U}$ and $x_{j+1} = f(x_j, 0)$, $j \in \mathbb{N}_{\geq 0}$, if* (5.18) *holds, then*

$$\ell(x_i, 0) \leq \sigma^{i-j}\ell(x_j, u_j), \tag{5.67a}$$

for $i \geq j$. Also, it allows us to define

$$\sum_{j=i}^{N+i} \ell(x_j, 0) \leq \sum_{j=0}^{N} \sigma^j \ell(x_i, 0). \tag{5.67b}$$

Proof. It follows by applying (5.18). □

Lemma 5.14 ($V_f(\cdot)$ in open-loop)**.** *Let denote the states $x_j \in \mathcal{X}_f(d_1)$ and $x_{j+1} = f(x_j, 0)$, $j \in \mathbb{N}_{\geq 0}$, if* (5.19) *holds, then*

$$V_f(x_i) \leq \rho^{i-j} V_f(x_j) \tag{5.68a}$$

for $i \geq j$. Moreover, it allows us to define

$$\sum_{j=i}^{N+i-1} V_f(x_j) \leq \sum_{j=0}^{N-1} \rho^j V_f(x_i). \tag{5.68b}$$

for $x_i \in \mathcal{X}_f(d_1)$.

Proof. It follows by applying (5.19). □

Now, let us consider $x_i \in \Gamma_N^\beta$, after $N + i$ steps for i consecutive packet losses the value function is

$$V_N^0(x_i) \leq \sum_{j=i}^{N+i-1} \ell(x_j, 0) + \beta V_f(x_{N+i}), \tag{5.69a}$$

owing to (5.67b) and (5.68a), we can say that

$$\leq \sum_{j=0}^{N-1} \sigma^j \ell(x_i, 0) + \beta\rho^N V_f(x_i), \tag{5.69b}$$

knowing that $\sum_{j=0}^{n} z^j = \frac{z^{n+1}-1}{z-1}$ for $z \in \mathbb{R}_{>1}$ and $n \in \mathbb{N}_{>0}$

$$\leq \frac{\sigma^N - 1}{\sigma - 1}\ell(x_i, 0) + \beta\rho^N V_f(x_i). \tag{5.69c}$$

The latter represents the value function at $x_i = f^i(x_0, 0)$. Since our task is to approximate the exponential growth of ζ starting from $x_0 \in \mu\Gamma_N^\beta$ up to x_i (Assumption 5.6) by applying Assumption 5.7 for the stage cost and terminal cost, then from (5.67a) and (5.68a) we have that

$$\leq \frac{\sigma^N - 1}{\sigma - 1}\sigma^i \ell(x_0, u_0) + \beta\rho^{N+i} V_f(x_0), \tag{5.69d}$$

for $x_0 \in \mu\Gamma_N^\beta$. □

5.C Proof of Theorem 5.4

Rearranging (5.31) for i leads to

$$\sigma^i\left(1 + \frac{b\rho^i}{a\sigma^i}\right) \leq \frac{c}{a} \tag{5.70a}$$

$$\ln(\sigma^i) + \ln\left(1 + \frac{b\rho^i}{a\sigma^i}\right) \leq \ln\left(\frac{c}{a}\right), \tag{5.70b}$$

where

$$a = \frac{\sigma^N - 1}{\sigma - 1}\ell(x_0, u_0), \quad b = \beta\rho^N d_1, \quad c = Nd_2 + \beta d_1, \tag{5.70c}$$

knowing that $\ln\left(\frac{x}{y}\right) \leq \ln\left(1 + \frac{x}{y}\right)$, if $x > y$, let i^* denote the lower bound of i, for which (5.31) holds *i.e.* $i^* \leq i$, then

$$\ln(\sigma^{i^*}) + \ln\left(\frac{b\rho^{i^*}}{a\sigma^{i^*}}\right) - \ln\left(\frac{c}{a}\right) \leq \ln(\sigma^i) + \ln\left(1 + \frac{b\rho^i}{a\sigma^i}\right) - \ln\left(\frac{c}{a}\right) \leq 0 \tag{5.70d}$$

$$\ln(\sigma^{i^*}) + \ln\left(\frac{b\rho^{i^*}}{a\sigma^{i^*}}\right) - \ln\left(\frac{c}{a}\right) \leq 0, \tag{5.70e}$$

reordering we have that

$$i^* \ln(\rho) \leq \ln\left(\frac{c}{a}\right) - \ln\left(\frac{b}{a}\right) \tag{5.70f}$$

leads to the solution

$$i^* \leq \frac{\ln\left(\frac{c}{b}\right)}{\ln \rho}, \tag{5.70g}$$

which completes the proof. □

5.D Proof of Lemma 5.4

Since the closed-loop control or the benefit of the buffered control was to transfer an initial state to the state $x_0 \in \mathcal{X}_f(d_1)$, it also means that after N steps that initial state was transfer to $x_N = x_{N|N} \in \mathcal{X}_f(d_1)$ by the closed-loop control or the buffered control. Hence, starting from x_N we determine the value function after $N + i$ steps for i consecutive packet losses and then we relate with $x_0 \in \mathcal{X}_f(d_1)$ according to Assumption 5.7. The value function for x_i is

$$V_N^0(x_i) \leq \sum_{j=N}^{N+i-1} \ell(x_j, 0) + \beta V_f(x_{N+i}), \tag{5.71a}$$

owing to (5.67b) and (5.68a), we can say that

$$\leq \sum_{j=0}^{i-1} \sigma^j \ell(x_N, 0) + \beta\rho^i V_f(x_N), \tag{5.71b}$$

knowing that $\sum_{j=0}^{n} z^j = \frac{z^{n+1}-1}{z-1}$ for $z \in \mathbb{R}_{>1}$ and $n \in \mathbb{N}_{>0}$

$$\leq \frac{\sigma^i - 1}{\sigma - 1} \ell(x_N, 0) + \beta\rho^i V_f(x_N), \tag{5.71c}$$

since we can relate x_N with x_0 by applying (5.67a) and (5.68a), it leads to

$$\leq \frac{\sigma^i - 1}{\sigma - 1} \sigma^N \ell(x_0, u_0) + \beta\rho^i \rho^N V_f(x_0), \tag{5.71d}$$

where $x_0 \in \mathcal{X}_f(d_1)$ and $u_0 = \kappa_f(x_0) \in \mathcal{U}$. Now, applying Assumption 5.5, it follows that

$$\leq \sigma^N \frac{\sigma^i - 1}{\sigma - 1}\Big(V_f(x_0) - V_f(x_1)\Big) + \beta\rho^{N+i} V_f(x_0), \tag{5.71e}$$

applying (5.15b), we have that

$$\leq \sigma^N \frac{\sigma^i - 1}{\sigma - 1}(1-\gamma) V_f(x_0) + \beta\rho^{N+i} V_f(x_0), \tag{5.71f}$$

then

$$= \left(\sigma^N \frac{\sigma^i - 1}{\sigma - 1}(1-\gamma) + \beta\rho^{N+i}\right) V_f(x_0), \tag{5.71g}$$

completes the proof. □

5.E Proof of Theorem 5.6

Rearranging (5.35) for i leads to

$$\sigma^i \left(1 + \frac{b\rho^i}{a\sigma^i}\right) \leq 1 + \frac{c}{a} \tag{5.72a}$$

$$\ln(\sigma^i) + \ln\left(1 + \frac{b\rho^i}{a\sigma^i}\right) \leq \ln\left(1 + \frac{c}{a}\right), \tag{5.72b}$$

where

$$a = \frac{\sigma^N(1-\gamma)}{\sigma - 1}, \quad b = \beta\rho^N, \quad c = \beta + N\frac{d_2}{d_1}, \tag{5.72c}$$

knowing that $\ln\left(\frac{x}{y}\right) \leq \ln\left(1 + \frac{x}{y}\right)$, if $x > y$, let i^* denote the lower bound of i, for which (5.35) holds, *i.e.* $i^* \leq i$, then

$$\ln(\sigma^{i^*}) + \ln\left(\frac{b\rho^{i^*}}{a\sigma^{i^*}}\right) - \ln\left(1 + \frac{c}{a}\right) \leq \ln(\sigma^i) + \ln\left(1 + \frac{b\rho^i}{a\sigma^i}\right) - \ln\left(1 + \frac{c}{a}\right) \leq 0 \tag{5.72d}$$

$$\ln(\sigma^{i^*}) + \ln\left(\frac{b\rho^{i^*}}{a\sigma^{i^*}}\right) - \ln\left(1 + \frac{c}{a}\right) \leq 0, \tag{5.72e}$$

reordering we have that

$$i^* \ln(\rho) \leq \ln\left(\frac{a+c}{a}\right) - \ln\left(\frac{b}{a}\right) \tag{5.72f}$$

leads to the solution

$$i^* \leq \frac{\ln\left(\frac{a+c}{b}\right)}{\ln \rho}, \tag{5.72g}$$

which completes the proof. □

Chapter 6

Robust Stability of State-Feedback MPC under consecutive packet losses

Contents

In this chapter, we analyse the stability of a discrete-time nonlinear NCS with additive uncertainty, building on the problem setting from Chapter 5. The system is controlled using MPC without terminal constraints, subject to input constraints and random packet losses in the actuation communication channel. As in Chapter 5, we consider two scenarios: one where a buffer stores transmitted control sequences to enhance robustness against packet losses, and one without a buffer, which are affected by additive uncertainty. These two scenarios are generalized and presented in four cases, depending on whether the initial state lies in the RoA or in the terminal region, and whether control is executed in closed-loop mode or via buffered control during packet dropouts.

To assess closed-loop stability under these stochastic conditions, we assume that the terminal cost functions as a local, rather than global, CLF. Also, we establish conditions under which the terminal region and RoA remain robust positively invariant in the presence of bounded additive disturbances. Additionally, for each of the four cases, we derive an upper bound on the number of consecutive packet losses that can be tolerated while maintaining stability in the presence of uncertainty. These results highlight the relations between the controller design parameters, and the local stability properties of the nominal system.

6.1 Introduction

As reviewed in Chapter 5, stochastic NCS with unreliable communication between actuators, sensors, and controllers are challenging to analyse. Introducing uncertainty further complicates the problem, as conditions derived for a nominal plant are no longer valid [56].

Methods for MPC under packet losses and uncertainty are particularly relevant from a practical perspective. [42, 60] investigate a packetized MPC approach for nonlinear systems with process noise, incorporating a buffer to mitigate dropouts in the C–A channel. Extending on this, [64] extends the approach of [42] by introducing buffers for both the actuator and estimator, addressing packet losses in both the C–A and S–C channels. In [65], an MPC control law that minimizes a discounted cost subject to a discounted expectation constraint is examined. This approach does not use a buffer but accounts for additive disturbances. Additionally, [84] examines the exponential stability of the closed-loop system, considering the impact of random packet loss duration and MPC design parameters.

A common feature of the stability analyses in the preceding works is the assumption of a *global* CLF for the terminal dynamics of the system. It is known and accepted that this is a strong, and perhaps impossible to meet, assumption for systems subject to constraints. We relax this assumption in this chapter as we did in Chapter 5. We consider a discrete-time nonlinear NCS subject to random packet losses and controlled by an MPC formulation without explicit terminal constraints. Similar to [58] and related works, we consider that the controller transmits the optimal control sequence over a lossy UDP-like communication channel, *i.e.* without any ACK of packets received. We improve the stability analysis of [58] by relaxing the assumption of the existence of a global CLF. This is achieved by exploiting the stability analysis for constrained MPC without terminal constraints [110].

We examine a discrete nonlinear time-invariant NCS subject to random packet losses and uncertainty, controlled using an MPC formulation without explicit terminal constraints. Following the approach of Chapter 5, we assume the controller transmits the optimal control sequence over a lossy UDP-like communication channel with no ACK of received packets and extends the study by incorporating robust stability analysis.

The analysis is structured in four distinct cases, defined by whether the initial state lies in the RoA or in the terminal region, and by whether the plant operates purely in closed-loop mode or uses buffered control before packet losses occur. Specifically, the four cases are as follows: (i) closed-loop control with the initial state in the RoA, (ii) closed-loop control with the initial state in the terminal region, (iii) buffered control with the initial state in the RoA, and (iv) buffered control with the initial state in the terminal region. The presence or absence of a buffer on the plant side directly influences robustness to packet losses. For each case, we establish an upper bound on the number of consecutive packet losses that can occur while keeping the system state within the region of attraction subject to uncertainty in the process. The resulting conditions reveal the relationship between stability of nominal nonlinear systems and the stability of nonlinear systems subject to additive disturbances.

The analysis for a discrete LTI system is obtained as a specialization of the nonlinear case corresponding to closed-loop operation with the initial state in the RoA. The resulting bound depends explicitly on the open-loop stage-cost and terminal-cost parameters, meaning that the result is more conservative but easy to compute.

6.2 Problem formulation

We consider the following discrete-time uncertain nonlinear system

$$x_{k+1} = f(x_k, v_k u_k) + w_k, \tag{6.1}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $w_k \in \mathbb{R}^n$ are, respectively, the state, the input and the process noise of the system at sample time $k \in \mathbb{N}_{\geq 0}$. The system input, u_k, is subject to random packet losses via v_k, and is constrained to take values in a set $\mathcal{U} \subset \mathbb{R}^m$ but the states are unconstrained.

Assumption 6.1 (Continuity of the system)**.** *The function $f(\cdot,\cdot)$ is continuous and satisfies* $0 = f(0, 0)$.

Assumption 6.2 (Constraints)**.** *The sets $\mathcal{U}$ and $\mathcal{W}$ are compact, and contain the origin in their interiors.*

Assumption 6.3. *The process noise, w_k, is a bounded additive disturbance and can take any value in a set $\mathcal{W} \subset \mathbb{R}^n$.*

The system is connected to a controller via a partially lossy UDP-like communication channel. While the state measurement is communicated with additive disturbance to the controller, communication between the controller and system actuator is subject to random packet losses as depicted in Fig. 6.1.

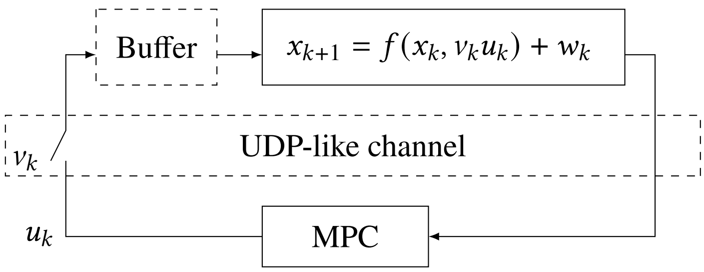


Figure 6.1: A system in connection with a controller via a UDP-like channel. The C-A channel is subject to random packet losses.

Assumption 6.4. *The input packet loss variable $v_k \in [0, 1]$ forms a sequence of i.i.d random variables with $v_k \sim \mathcal{B}(\bar{v})$, where $\bar{v} = \Pr[v_k = 1]$ is the probability of successful packet delivery.*

As we did in Chapter 5, Assumption 6.4 describes a common NCS setting studied in the literature (*e.g.* [58, 42]), and we also consider that a buffer may (or may not) be present at the input to the system; the buffer stores *nominal* control sequences transmitted by the controller and its precise operation is described in Section 5.3.2.

Therefore, the problem formulation is as follows. Given the system (6.1) at a state x_k, the problem is to determine the optimal control law such that the state x_k is transferred to the neighbourhood of the origin, subject to the constraints (Assumption 6.2), uncertainty (Assumption 6.3), and packet losses (Assumption 6.4), while minimizing a cost function through the following optimal control problem.

6.3 Controller formulation

Before we present of the optimal control problem, we need to consider that the nominal system, defined as

$$\bar{x}_{k+1} = f(x_k, u_k), \tag{6.2}$$

denotes the behaviour if the initial state is x_k and the nominal input u_k is the first input of the optimal sequence obtained as the solution of the optimal control problem (6.3a).

This represents a significant change in the setting compared to the deterministic case in (5.1), since now there is a *nominal MPC* in the loop aiming to control the uncertain system (6.1).

In what follows we formulate the optimal control problem, but for the sake of consistency, we preserve the notation of the predicted states and inputs.

6.3.1 Optimal control problem

With the system at a state x_k, the optimal *nominal* control problem to be solved is defined by

$$\mathbb{P}_N(x_k): \quad V_N^0(x_k) = \min_{\mathbf{u}_k \in \mathbb{U}} J_N(x_k, \mathbf{u}_k), \tag{6.3a}$$

subject to, for $j \in \mathbb{N}_{[0,N-1]}$,

$$x_{k|k} = x_k \tag{6.3b}$$

$$x_{k+j+1|k} = f(x_{k+j|k}, u_{k+j|k}) \tag{6.3c}$$

$$u_{k+j|k} \in \mathcal{U}, \tag{6.3d}$$

where the finite sequence of future control inputs is

$$\mathbf{u}_k := \{u_{k|k}, u_{k+1|k}, \ldots, u_{k+N-1|k}\}, \tag{6.4}$$

and the constraint set is

$$\mathbb{U} := \mathcal{U} \times \cdots \times \mathcal{U}, \tag{6.5}$$

The *nominal* cost function is

$$J_N(x_k, \mathbf{u}_k) := \beta V_f(x_{k+N|k}) + \sum_{j=0}^{N-1} \ell(x_{k+j|k}, u_{k+j|k}), \tag{6.6}$$

where $\beta \geq 1$ and the stage $\ell(\cdot,\cdot)$ and terminal cost $V_f(\cdot)$ functions satisfy the following assumption for quadratic cost norms.

Assumption 6.5. *The stage cost function $\ell(\cdot,\cdot)\colon \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}_{\geq 0}$ and terminal cost function $V_f(\cdot)\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ are continuous, with $\ell(0,0) = 0$ and $V_f(0) = 0$, and satisfy*

$$\ell(x,u) \geq c_1 \|x\|_2^2, \tag{6.7a}$$

for all $x \in \mathbb{R}^n$, for all $u \in \mathcal{U}$, and some $c_1 > 0$,

$$V_f(x) \leq c_2 \|x\|_2^2, \tag{6.7b}$$

for all $x \in \mathbb{R}^n$, and some $c_2 > 0$.

We remark that this represents a conventional *nominal* input-constrained nonlinear MPC formulation [100]; however, as we did in Chapter 5, we emphasize that there is a terminal cost but there is no terminal constraint.

Solving $\mathbb{P}_N(x_k)$ at x_k yields the *nominal* solution

$$\mathbf{u}_k^0(x_k) := \{u_{k|k}^0, u_{k+1|k}^0, \ldots, u_{k+N-1|k}^0\}. \tag{6.8}$$

The first control in the optimal sequence is applied to the uncertain plant (6.1), followed by a repetition of the whole process at the next sampling time, defines the implicit control law

$$u_k = \kappa_N(x_k) := u^0_{k|k}. \tag{6.9}$$

Due to the lack of state constraints, the domain of the value function $V_N^0(\cdot)$ and control law $\kappa_N(\cdot)$ is the whole state space $\mathbb{R}^n$, meaning that the optimal control problem is (trivially) recursively feasible.

6.3.2 Buffer

Again, we need to emphasize that the buffer, described in Section 5.3.2, stores the *nominal* control sequence (6.4) transmitted by the controller at each time step k. If $\nu_k = 1$ then this sequence is received by the buffer and acts as a parallel-in–serial-out shift register. If $\nu_k = 0$, the buffer outputs the *previously optimal sequence received* and applied to the uncertain system until the buffer runs out of data or a new sequence is successfully received.

6.4 Stability analysis of nonlinear systems

To establish stability of the closed-loop system under random packet losses, uncertainty and in the absence of a terminal constraint, we combine the stability analysis of the value function with the stability conditions for nominal input-constrained MPC without terminal constraints [110], and the analysis of robustness of nominal MPC for the Region of Attraction (RoA) and the implicit terminal region [13].

This combination permits (i) the relaxation of the technical assumption in [58, 42] for both deterministic and uncertain nonlinear systems, which requires the existence of a *global* CLF terminal cost function, an assumption that is impossible to meet for many input-constrained systems [100], and (ii) the analysis of conditions under which the RoA and the terminal region exhibit positive invariance for the controlled uncertain system.

We first present some relevant results and necessary assumptions. In what follows, a variable without a subscript denotes its current value, *e.g.* x means x_k, and a variable with a subscript -1 denotes its past value, *e.g.* x_{-1} means x_{k-1}, while the subscript j denotes the j-step ahead prediction: *e.g.* x_j means $x_{k+j|k}$.

6.4.1 Preliminaries: stability without terminal constraint set

First, we briefly recall the conditions of nominal stability presented in Chapter 5. These results then serve as key ingredients in demonstrating the robustness of nominal MPC with additive disturbances in the absence of random packet losses.

6.4.1.1 Nominal stability conditions

We recall the assumptions and definitions presented in Chapter 5, [110].

Definition 6.1. *There exist a $d_1 > 0$ and $d_2 > 0$ such that, for all*

$$x \in \mathcal{X}_f(d_1) := \{x \colon V_f(x) \leq d_1\}, \tag{6.10a}$$

and for all $x \notin \mathcal{X}_f(d_1)$ and $u \in \mathcal{U}$,

$$d_2 \leq \ell(x, u). \tag{6.10b}$$

Such d_1 and d_2 are guaranteed since $V_f(x)$ and $\ell(x, u)$ are positive definite (Assumption 6.2 and Assumption 6.5).

Assumption 6.6. *There exists a control law $\kappa_f \colon \mathbb{R}^n \to \mathbb{R}^m$ and a constant $d_1 > 0$ such that, for all $x \in \mathcal{X}_f(d_1)$,*

$$V_f\big(f(x, \kappa_f(x))\big) \leq V_f(x) - \ell(x, \kappa_f(x)) \tag{6.11a}$$

$$\kappa_f(x) \in \mathcal{U}. \tag{6.11b}$$

As we did in Chapter 5, let us recall that $V_f(\cdot)$ is a local CLF for the *nominal* terminal dynamics, which relaxes the global CLF assumption in [58, 42], and $\beta V_f(\cdot)$ is also a local CLF since $\beta \geq 1$, and the set $\mathcal{X}_f(d_1)$ serves as an implicit terminal region for the controller.

The previous Definition 6.1 allow us to define the (nominal) set

$$\Gamma_N^{\beta} := \{x \in \mathbb{R}^n \colon V_N^0(x) \leq \beta d_1 + N d_2\}. \tag{6.12}$$

It is established in [110] that the controlled system $x^+ = f(x, \kappa_N(x))$ is asymptotically stable with region of attraction Γ_N^β, and, for all $x \in \Gamma_N^\beta$,

$$V_N^0(f(x,\kappa_N(x))) - V_N^0(x) \le -\ell(x,\kappa_N(x)). \tag{6.13}$$

We also recall the following lemma concerning the cost value, the stage and terminal cost functions.

Lemma 6.1. *Suppose Assumptions 6.1–6.6 hold. There exist constants* $\gamma \in (0,1]$*, and* $\varrho \in (0,1]$ *such that*

$$V_f\big(f(x,\kappa_f(x))\big) \le \gamma V_f(x), \tag{6.14a}$$

for all $x \in \mathcal{X}_f(d_1)$*, and*

$$V_N^0(f(x,\kappa_N(x))) \le \varrho V_N^0(x), \tag{6.14b}$$

for all $x \in \Gamma_N^\beta$*, where*

$$\varrho = 1 - \frac{c_1}{c_2}, \tag{6.14c}$$

for some $c_1, c_2 > 0$*.*

Proof. Refer to the proof in Lemma 5.1, Chapter 5. □

Armed with the previous, we are now in the position to extend the analysis for uncertain system.

6.4.1.2 Robustness of nominal stability conditions

We want to show the effect on applying the nominal control law to the uncertain system (6.1) if $w \in \mathcal{W}$ is sufficiently small. We begin with the following assumption.

Assumption 6.7 (Lipschitz continuity [42, 64])**.** *There exist constants* $\lambda_\ell, \lambda_f, \lambda_V > 0$ *such that for all* $(x,x') \in \mathbb{R}^n$*, and for all* $(u,u') \in \mathcal{U}$

$$|\ell(x,u) - \ell(x',u')| \le \lambda_\ell \|x - x'\|_2^2 \tag{6.15a}$$

$$\left|V_f(x) - V_f(x')\right| \le \lambda_f \|x - x'\|_2^2 \tag{6.15b}$$

$$\left|V_N^0(x) - V_N^0(x')\right| \le \lambda_V \|x - x'\|_2^2. \tag{6.15c}$$

Then, we can define the following lemmas.

Lemma 6.2. *If Assumptions 6.6–6.7 hold, from* (6.14a) *we say that there exist* $\gamma \in (0,1]$ *and* $\bar{w}_{\lambda_f} > 0$ *such that for all* $x \in \mathcal{X}_f(d_1)$ *and for all* $w \in \mathcal{W}$

$$V_f\big(f(x,\kappa_f(x)) + w\big) \leq \gamma V_f(x) + \bar{w}_{\lambda_f}, \tag{6.16a}$$

and

$$V_f\big(f(x,\kappa_f(x)) + w\big) \leq V_f(x) - \ell(x,\kappa_f(x)) + \bar{w}_{\lambda_f} \tag{6.16b}$$

where

$$\bar{w}_{\lambda_f} \coloneqq \max_{w\in\mathcal{W}} \lambda_f \|w\|_2^2. \tag{6.16c}$$

Proof. Let $x^+ = f(x,\kappa_f(x)) + w$ and $\bar{x}^+ = f(x,\kappa_f(x))$ be the uncertain state and nominal state, respectively. From (6.15b)

$$\begin{aligned}\big|V_f(x^+) - V_f(\bar{x}^+)\big| &\leq \lambda_f \|x^+ - \bar{x}^+\|_2^2 \\ &= \lambda_f \big\|f(x,\kappa_f(x)) + w - f(x,\kappa_f(x))\big\|_2^2,\end{aligned}$$

we have that

$$V_f(x^+) \leq V_f(\bar{x}^+) + \lambda_f \|w\|_2^2, \tag{6.17a}$$

and using (6.14a) such that $V_f(\bar{x}^+) \leq \gamma V_f(\bar{x})$ proves (6.16a). Applying (6.11a) in (6.17a) such that

$$\begin{aligned}V_f(x^+) &\leq V_f(\bar{x}^+) + \lambda_f \|w\|_2^2, \\ &\leq V_f(x) - \ell(x,\kappa_f(x)) + \lambda_f \|w\|_2^2,\end{aligned}$$

completes the proof of (6.16b). □

Lemma 6.3. *From* (6.14b) *we say that there exists a* $\bar{w}_{\lambda_V} > 0$ *such that for all* $x \in \Gamma_N^\beta$*, and* $w \in \mathcal{W}$*,*

$$V_N^0(f(x,\kappa_N(x)) + w) \leq \varrho V_N^0(x) + \bar{w}_{\lambda_V}, \tag{6.18a}$$

where

$$\bar{w}_{\lambda_V} \coloneqq \max_{w\in\mathcal{W}} \lambda_V \|w\|_2^2. \tag{6.18b}$$

Proof. Let $x^+ = f(x,\kappa_N(x)) + w$ and $\bar{x}^+ = f(x,\kappa_N(x))$ be the uncertain state and nominal state, respectively. From (6.15c)

$$\begin{aligned} \left|V_N^0(x^+) - V_N^0(\bar{x}^+)\right| &\leq \lambda_V \|x^+ - \bar{x}^+\|_2^2, \\ &= \lambda_V \|f(x, \kappa_N(x)) + w - f(x, \kappa_N(x))\|_2^2, \end{aligned}$$

we have that

$$V_N^0(x^+) \leq V_N^0(\bar{x}^+) + \lambda_V \|w\|_2^2,$$

and applying $V_N^0(\bar{x}^+) \leq \varrho V_N^0(\bar{x})$ from (6.14b), completes the proof. □

Given these facts, in what follows we show that Γ_N^β and $\mathcal{X}_f(d_1)$ are robust positively invariant for the uncertain system (6.1) if $w \in \mathcal{W}$ is sufficiently small.

Lemma 6.4 (Robust positive invariance of $\mathcal{X}_f(d_1)$ [13])**.** *The terminal region $\mathcal{X}_f(d_1)$ is robust positively invariant for $x^+ = f(x, \kappa_f(x)) + w \in \mathcal{X}_f(d_1)$ if*

$$\lambda_f \|w\|_2^2 \leq (1 - \gamma) d_1, \tag{6.20}$$

for all $w \in \mathcal{W}$ if w is sufficiently small.

Proof. From Lemma 6.2, we wish to show $x^+ \in \mathcal{X}_f(d_1)$. Then, from (6.17a) we have that

$$V_f(x^+) \leq \gamma V_f(x) + \lambda_f \|w\|_2^2 \leq \gamma d_1 + \lambda_f \|w\|_2^2.$$

Since we need $x^+ \in \mathcal{X}_f(d_1)$, it means $V_f(x^+) \leq d_1$

$$\gamma d_1 + \lambda_f \|w\|_2^2 \leq d_1.$$

Hence, $x \in \mathcal{X}_f(d_1)$ implies $x^+ \in \mathcal{X}_f(d_1)$ if

$$\lambda_f \|w\|_2^2 \leq (1 - \gamma) d_1,$$

completes the proof. □

Lemma 6.5 (Robust positive invariance of Γ_N^β [13])**.** *The Γ_N^β is robust positively invariant for*

$x^+ = f(x, \kappa_N(x)) + w \in \Gamma_N^\beta$ *if*

$$\lambda_V \|w\|_2^2 \leq (1 - \varrho)(\beta d_1 + N d_2), \tag{6.22a}$$

and

$$\lambda_f \|w\|_2^2 \leq (1 - \gamma) d_1, \tag{6.22b}$$

for all $w \in \mathcal{W}$ *if* w *is sufficiently small, where*

$$\varrho = 1 - \frac{c_1}{c_2}, \tag{6.22c}$$

for some $c_1, c_2 > 0$. *The* ϱ *is defined as in Lemma 5.1.*

Proof. From Lemma 6.3, we also wish to show $x^+ \in \Gamma_N^\beta$, that is

$$V_N^0(x^+) \leq \varrho V_N^0(x) + \lambda_V \|w\|_2^2 \leq \varrho(\beta d_1 + N d_2) + \lambda_V \|w\|_2^2.$$

Since we wish to show $x^+ = f(x, \kappa_N(x)) + w \in \Gamma_N^\beta$, it means $V_N^0(x^+) \leq \beta d_1 + N d_2$

$$\varrho V_N^0(x) + \lambda_V \|w\|_2^2 \leq \beta d_1 + N d_2.$$

Hence, $x \in \Gamma_N^\beta$ implies $x^+ \in \Gamma_N^\beta$ if

$$\lambda_V \|w\|_2^2 \leq (1 - \varrho)(\beta d_1 + N d_2),$$

completes the proof. □

6.4.2 Stability analysis of the nonlinear NCS

The previous assumptions and lemmas establish stability of conventional nominal MPC for an input-constrained nonlinear system in the absence of packet losses and buffering, and provide conditions for the robustness of nominal MPC. In what follows, we consider the analysis where the buffer is present, there are consecutive packet losses and uncertainty.

We examine two scenarios: (i) the initial state is in the region of attraction Γ_N^β, but it is not necessarily in the terminal region $\mathcal{X}_f(d_1)$; (ii) the initial state is in the terminal region. For both scenarios, we show through four cases that there exists a finite upper bound on the number of consecutive packet losses in order that the state remains in Γ_N^β with uncertainty.

In the sequel we suppose that Assumptions 6.1–6.7 hold even if not explicitly stated. Furthermore, we define some additional assumptions and lemmas similar to those in [58, 42, 64, 13]. These concern the behaviour of the value, stage and terminal cost functions when the system is operating in open loop with and without uncertainties.

6.4.2.1 Nominal open-loop

We start by recalling the assumptions of the deterministic case presented in Chapter 5.

Assumption 6.8 ($V_N^0(\cdot)$ in open-loop)**.** *There exists a* $\zeta \in [1, +\infty)$ *such that*

$$V_N^0(f(x,0)) \leq \zeta V_N^0(x), \tag{6.24}$$

for all $x \in \Gamma_N^\beta$*.*

Assumption 6.9 ($\ell(\cdot,\cdot)$ and $V_f(\cdot)$ in open-loop)**.** *There exist* $\sigma \in [1, +\infty)$ *and* $\rho \in [1, +\infty)$ *such that*

$$\ell(f(x,0),0) \leq \sigma \ell(x,u), \tag{6.25a}$$

for all $x \in \Gamma_N^\beta$ *and* $u \in \mathcal{U}$*,*

$$V_f(f(x,0)) \leq \rho V_f(x), \tag{6.25b}$$

for all $x \in \mathcal{X}_f(d_1)$*.*

6.4.2.2 Open-loop with uncertainty

Now we formulate the following for uncertain open-loop behaviour.

Assumption 6.10 (Lipschitz continuity (open-loop))**.** *If Assumption 6.1 holds, there exists a constant* $\eta_\ell > 0$ *such that for all* $(x, x') \in \mathbb{R}^n$

$$|\ell(x,0) - \ell(x',0)| \leq \eta_\ell \|x - x'\|_2^2, \tag{6.26}$$

Lemma 6.6 ($V_N^0(\cdot)$ in open-loop with uncertainty)**.** *If Assumption 6.10 and Assumption 6.8* *hold, there exists a constant* $\bar{w}_{\lambda_V} > 0$ *such that for all* $x \in \Gamma_N^\beta$ *and all* $w \in \mathcal{W}$

$$V_N^0(f(x,0)+w) \leq \zeta V_N^0(x) + \bar{w}_{\lambda_V}, \tag{6.27a}$$

where

$$\bar{w}_{\lambda_V} := \max_{w \in \mathcal{W}} \lambda_V \|w\|_2^2. \tag{6.27b}$$

Lemma 6.7 ($\ell(\cdot,\cdot)$ and $V_f(\cdot)$ in open-loop with uncertainty)**.** *If Assumption 6.10 and Assumption 6.9 hold, there exist constants* $\bar{w}_{\eta_\ell}, \bar{w}_{\lambda_f} > 0$ *such that*

$$\ell(f(x,0)+w,0) \leq \sigma\ell(x,u) + \bar{w}_{\eta_\ell}, \tag{6.28a}$$

for all $x \in \Gamma_N^\beta$ *and* $u \in \mathcal{U}$*,*

$$V_f(f(x,0)+w) \leq \rho V_f(x) + \bar{w}_{\lambda_f}, \tag{6.28b}$$

for all $x \in \mathcal{X}_f(d_1)$*, where*

$$\bar{w}_{\eta_\ell} := \max_{w \in \mathcal{W}} \eta_\ell \|w\|_2^2, \quad \bar{w}_{\lambda_f} := \max_{w \in \mathcal{W}} \lambda_f \|w\|_2^2, \tag{6.28c}$$

for all $w \in \mathcal{W}$*.*

We will use these assumptions and the lemmas as *alternatives* in the analyses that follow. Now, as previously described in the introduction of this section, we define two scenarios.

6.4.2.3 First scenario

Consider that the system with initial state $x_0 \in \Gamma_N^\beta$ experiences no packet losses for i consecutive steps subject to disturbances. That is, the closed-loop control law $u_j = \kappa_N(x_j)$ is applied to the plant for steps $j = 0, 1, \ldots, i-1$.

Lemma 6.8 ($V_N^0(\cdot)$ in closed-loop)**.** *If* $w \in \mathcal{W}$ *is sufficiently small, then*

$$V_N^0(x_i) \leq \varrho^i V_N^0(x_0) + \bar{w}_{\lambda_V}\frac{1-\varrho^i}{1-\varrho} \tag{6.29}$$

for all $x_0 \in \Gamma_N^\beta$*.*

Proof. From Lemma 6.3, we iterate x_0 for i steps, such that

$$\begin{aligned} V_N^0(x_i) &\leq \varrho^i V_N^0(x_0) + \varrho^{i-1}\lambda_V \|w_0\|_2^2 + \varrho^{i-2}\lambda_V \|w_1\|_2^2 + \dots \\ &= \varrho^i V_N^0(x_0) + \lambda_V \sum_{j=0}^{i-1} \varrho^{i-1-j} \|w_j\|_2^2, \end{aligned}$$

since $\lambda_V \|w\|_2^2 \leq \bar{w}_{\lambda_V}$, then

$$\leq \varrho^i V_N^0(x_0) + \bar{w}_{\lambda_V} \frac{1-\varrho^i}{1-\varrho},$$

completes the proof. □

It allows us to define the following sublevel set.

$$\Omega := \left\{ x \in \mathbb{R}^n \colon V_N^0(x_i) \leq \varrho^i V_N^0(x_0) + \bar{w}_{\lambda_V} \frac{1-\varrho^i}{1-\varrho} \right\}. \tag{6.31}$$

Because of the disturbances, we do not know if $x_i \in \Omega \subset \Gamma_N^\beta$, but since $\lambda_V \|w\|_2^2$ is bounded by (6.22a) we can determine the conditions under which $x_i \in \Omega \subset \Gamma_N^\beta$. We first define the following lemma to guarantee the descent property of $V_N^0(x^+) \in \Gamma_N^\beta$, the procedure is similar to [13].

Lemma 6.9 (Descent property of $V_N^0(x^+) \in \Gamma_N^\beta$)**.** *If $x \in \Gamma_N^\beta$, and by the application of Lemma 6.5, let*

$$\lambda_V \|w\|_2^2 \leq (\delta - \varrho)(\beta d_1 + N d_2), \tag{6.32a}$$

for some $\delta \in (\varrho, 1]$. Then,

$$V_N^0(x^+) \leq \delta V_N^0(x). \tag{6.32b}$$

Proof. From Lemma 6.3 and knowing that $V_N^0(x) \leq \beta d_1 + N d_2$,

$$\begin{aligned} V_N^0(x^+) &\leq \varrho V_N^0(x) + \lambda_V \|w\|_2^2 \\ &\leq \varrho(\beta d_1 + N d_2) + \lambda_V \|w\|_2^2, \end{aligned}$$

and by (6.32a), we have that

$$\leq \varrho(\beta d_1 + N d_2) + (\delta - \varrho)(\beta d_1 + N d_2),$$

which completes the proof. □

Then, we can define the following for i steps.

Lemma 6.10. *There exists a $\mu \in (0,1]$ such that the initial state $x_0 \in \Gamma_N^\beta$ transferred to $x_i \in \mu\Gamma_N^\beta \implies x_i \in \Omega \subset \Gamma_N^\beta$ such that*

$$V_N^0(x_i) \leq \mu V_N^0(x_0), \tag{6.34a}$$

where

$$\mu = \frac{\varrho^i(1-\delta)+\delta-\varrho}{1-\varrho}, \tag{6.34b}$$

for some $\delta \in (\varrho, 1]$.

Proof. From Lemma 6.8, Lemma 6.9, and knowing that $V_N^0(x_0) \leq \beta d_1 + N d_2$

$$\begin{aligned} V_N^0(x_i) &\leq \varrho^i V_N^0(x_0) + \lambda_V \|w\|_2^2 \frac{1-\varrho^i}{1-\varrho} \\ &\leq \varrho^i(\beta d_1 + N d_2) + (\delta-\varrho)(\beta d_1 + N d_2)\frac{1-\varrho^i}{1-\varrho}, \end{aligned}$$

reordering, we have that

$$= \frac{\varrho^i(1-\delta)+\delta-\varrho}{1-\varrho}(\beta d_1 + N d_2),$$

therefore, $x_0 \in \Gamma_N^\beta \implies x_i \in \mu\Gamma_N^\beta$. □

Hence, the initial state x_0 is, according to the latter, transferred to

$$x_i \in \Omega \subset \Gamma_N^\beta. \tag{6.36}$$

We cannot say whether or not $x_i \in \mathcal{X}_f(d_1)$ but from Lemma 6.4 we know that $\mathcal{X}_f(d_1)$ is robust positively invariant for $x_1 = f(x_0, u_0) + w_0$, then we can imply that $\mathcal{X}_f(d_1)$ is also positive invariance for x_i, *i.e.* $x_i \in \mathcal{X}_f(d_1)$.

6.4.2.4 Second scenario

Consider that the system with an initial state $x_0 \in \Gamma_N^\beta$ experiences packet losses for i consecutive steps, then the system is controlled according to the buffering mechanism Section 5.3.2, using the optimal control sequence computed and stored at time 0.

Initially, we cannot say the initial state x_0 is transferred to $x_i \in \mu\Gamma_N^\beta$ because of the nominal optimal sequence stored in the buffer and the disturbances, but we can assume that if $w \in \mathcal{W}$ is small enough, and if there exists a $\delta^* \in (\varrho, 1]$, where $\delta^* \neq \delta$, then $x_i \in \mu^*\Gamma_N^\beta$ for some $\mu^* \in (0, 1]$ after $i < N$ steps of consecutive packet losses.

First, we use analysis from the first scenario to define the following for the second scenario.

Lemma 6.11. *If $x \in \Gamma_N^\beta$, and by applying Lemma 6.5, let*

$$\lambda_V \|w\|_2^2 \leq (\delta^* - \varrho)(\beta d_1 + N d_2), \tag{6.37a}$$

for some $\delta^* \in (\varrho, 1]$, where $\delta^* \neq \delta$. Then,

$$V_N^0(x^+) \leq \delta^* V_N^0(x). \tag{6.37b}$$

Proof. Similar to Lemma 6.9. □

Then, we define the following.

Lemma 6.12. *There exists a $\mu^* \in (0, 1]$ such that the initial state $x_0 \in \Gamma_N^\beta$ transferred to $x_i \in \mu^*\Gamma_N^\beta \implies x_i \in \Omega \subset \Gamma_N^\beta$ such that*

$$V_N^0(x_i) \leq \mu^* V_N^0(x_0), \tag{6.38a}$$

where

$$\mu^* = \frac{\varrho^i(1 - \delta^*) + \delta^* - \varrho}{1 - \varrho}, \tag{6.38b}$$

for some $\delta^* \in (\varrho, 1]$, and $\delta^* \neq \delta$.

Proof. The proof is similar to Lemma 6.10. □

We cannot say if $x_i \in \mathcal{X}_f(d_1)$, but for the purpose of the analysis we will *assume* that, under the buffer, the initial condition $x_0 \in \Gamma_N^\beta$ has been transferred to $x_i \in \mathcal{X}_f(d_1)$ *if and only if* $w \in \mathcal{W}$ is small enough for $i = N$ consecutive packet losses.

Then, for the buffered mechanism, we can say that

$$x_i \in \begin{cases} \mu^* \Gamma_N^\beta & \text{if } \ i < N, \ \text{for } \ \delta^* \in (\varrho, 1], \\ \mathcal{X}_f(d_1) & \text{if } \ i = N, \end{cases} \tag{6.39}$$

if and only if $w \in \mathcal{W}$ is small enough.

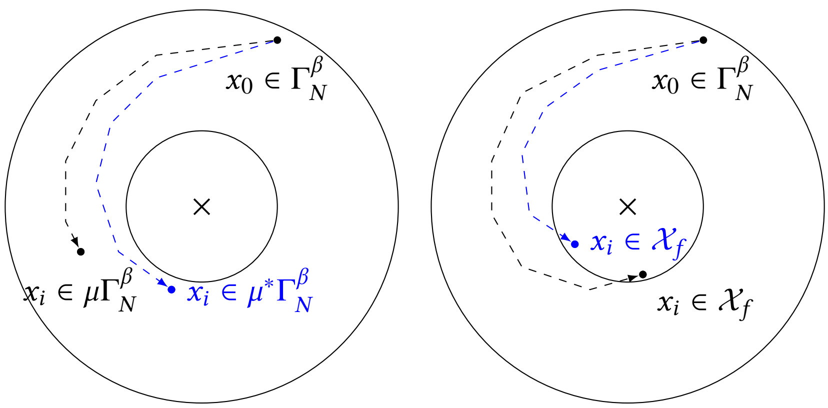
6.4.3 Generalized scenarios: Case 1a, Case 1b, Case 2a and Case 2b

We are now in the position to generalize both scenarios, where these four possibilities are not mutually exclusive.

Given $x_0 \in \Gamma_N^\beta$, x_i reaches either $\mu\Gamma_N^\beta$ or $\mathcal{X}_f(d_1)$ under closed-loop control *without* packet losses, or x_i reaches $\mu^*\Gamma_N^\beta$ or $\mathcal{X}_f(d_1)$ under buffered control *with* consecutive packet losses; see Table 6.1 and Fig. 6.2.

Closed-loop control		**Buffered control**	
Case 1a:	$x_i \in \mu\Gamma_N^\beta$, $\mu \in (0, 1]$	Case 2a:	$x_i \in \mu^*\Gamma_N^\beta$, $\mu^* \in (0, 1]$
Case 1b:	$x_i \in \mathcal{X}_f(d_1)$	Case 2b:	$x_i \in \mathcal{X}_f(d_1)$

Table 6.1: Four cases of x_i with and without packet losses.



(a) Case 1a (black) and Case 2a (blue) (b) Case 1b (black) and Case 2b (blue)

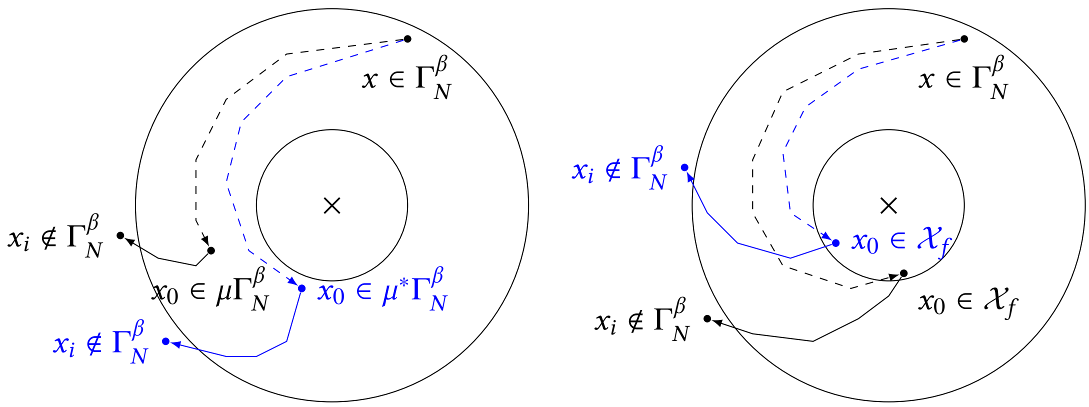
Figure 6.2: Behaviour of x_i for the four cases with and without packet losses.

Our objective is to analyse the behaviour of x_i after it has reached $\mu\Gamma_N^\beta$ or $\mathcal{X}_f(d_1)$ in closed-loop control and subsequently experiences i consecutive packet losses. Also, we want to analyse the behaviour of x_i after it has reached $\mu^*\Gamma_N^\beta$ or $\mathcal{X}_f(d_1)$ in buffered control (with consecutive packet losses), and there are still packet losses and the buffered has already been exhausted.

Therefore, x_i is set to x_0 and x_i is now the state reached in open-loop operation. This allows us to generalize further; see Table 6.2 and Fig. 6.3.

Closed-loop control	Buffered control
Case 1a: $x_0 \in \mu\Gamma_N^\beta,\ \mu \in (0,1]$	Case 2a: $x_0 \in \mu^*\Gamma_N^\beta,\ \mu^* \in (0,1]$
Case 1b: $x_0 \in \mathcal{X}_f(d_1)$	Case 2b: $x_0 \in \mathcal{X}_f(d_1)$

Table 6.2: Generalization of the four cases of x_i with consecutive packet losses.



(a) Case 1a (black) and Case 2a (blue)

(b) Case 1b (black) and Case 2b (blue)

Figure 6.3: Behaviour of x_i for the four cases when there are still consecutive packet losses.

Without explicitly saying how the state arrived there, we analyse how the system behaves subsequently in these *four different cases* when it experiences i consecutive packet losses. In the latter case, under the assumption that the buffer has already been exhausted since its benefit was realized by transferring the initial state to $\mathcal{X}_f(d_1)$) if w is small enough.

Before we start, let us define the following sequence in open-loop.

If

$$x_1 = f(x_0, 0) + w_0,$$

the sequence

$$\begin{aligned} x_2 &= f(x_1, 0) + w_1 = f(f(x_0, 0) + w_0, 0) + w_1, \\ x_3 &= f(x_2, 0) + w_2 = f(f(f(x_0, 0) + w_0, 0) + w_1, 0) + w_2, \\ \vdots &= \qquad \vdots \qquad = \qquad\qquad \vdots \end{aligned}$$

is generalized as

$$\begin{aligned} x_2 &:= f^2\big(x_0, 0, \{w\}_0^1\big), \\ x_3 &:= f^3\big(x_0, 0, \{w\}_0^2\big), \\ \vdots &:= \qquad \vdots \\ x_i &= f^i\big(x_0, 0, \{w\}_0^{i-1}\big). \end{aligned} \tag{6.40}$$

6.4.3.1 Case 1a - closed-loop control: bound on i given $x_0 \in \mu\Gamma_N^\beta$ and the open-loop cost value parameter

Let $x_0 \in \mu\Gamma_N^\beta$, $\mu \in (0, 1]$, denote an initial state reached via closed-loop control only, and let $x_i = f^i\big(x_0, 0, \{w\}_0^{i-1}\big)$ denote a state reached after i subsequent steps of open-loop operation, In what follows, we present the conditions under which the state x_i is contained in Γ_N^β.

Theorem 6.1. *If*

$$i \leq \left\lfloor \ln\!\left(\frac{(\zeta - 1)(\beta d_1 + N d_2) + \bar{w}_{\lambda_V}}{\mu(\zeta - 1)(\beta d_1 + N d_2) + \bar{w}_{\lambda_V}} \right) \Big/ \ln \zeta \right\rfloor \tag{6.41}$$

then $x_i \in \Gamma_N^\beta$.

Proof. By Lemma 6.6,

$$V_N^0(x_i) \leq \zeta^i V_N^0(x_0) + \lambda_V \sum_{j=0}^{i-1} \zeta^{i-1-j} \|w_j\|_2^2 \leq \zeta^i V_N^0(x_0) + \bar{w}_{\lambda_V} \frac{\zeta^i - 1}{\zeta - 1}.$$

By comparison, it follows that

$$V_N^0(x_i) \leq \zeta^i V_N^0(x_0) + \bar{w}_{\lambda_V} \frac{\zeta^i - 1}{\zeta - 1} \leq \zeta^i \mu(\beta d_1 + N d_2) + \bar{w}_{\lambda_V} \frac{\zeta^i - 1}{\zeta - 1}.$$

Therefore, $x_i \in \Gamma_N^\beta$ is ensured if

$$\zeta^i \mu(\beta d_1 + N d_2) + \bar{w}_{\lambda_V} \frac{\zeta^i - 1}{\zeta - 1} \leq \beta d_1 + N d_2. \tag{6.42}$$

A simple rearrangement for i completes the proof. □

Similar to the latter, we present the following, where the proof is also similar.

6.4.3.2 Case 2a - buffered control: bound on i given $x_0 \in \mu^* \Gamma_N^\beta$ and the open-loop cost value parameter

Let $x_0 \in \mu^* \Gamma_N^\beta$, $\mu^* \in (0, 1]$, denote an initial state reached via buffered control, and let $x_i = f^i\big(x_0, 0, \{w\}_0^{i-1}\big)$ denote a state reached after i subsequent steps of open-loop operation, We present the conditions under which the state x_i is contained in Γ_N^β.

Theorem 6.2. *If*

$$i \leq \left\lfloor \ln\left(\frac{(\zeta - 1)(\beta d_1 + N d_2) + \bar{w}_{\lambda_V}}{\mu^*(\zeta - 1)(\beta d_1 + N d_2) + \bar{w}_{\lambda_V}} \right) \Big/ \ln \zeta \right\rfloor \tag{6.43}$$

then $x_i \in \Gamma_N^\beta$.

Proof. Similar to Theorem 6.1. □

6.4.3.3 Case 1b - closed-loop control: bound on i given $x_0 \in \mathcal{X}_f(d_1)$ and the open-loop cost value parameter

Now let $x_0 \in \mathcal{X}_f(d_1)$ denote an initial state reached via *closed-loop*, and let $x_i = f^i\big(x_0, 0, \{w\}_0^{i-1}\big)$ denote a state reached after i subsequent steps of open-loop operation. The main result depicts a sufficient condition under which $x_i \in \Gamma_N^\beta$. It uses the following lemma.

Lemma 6.13. *For all* $x_0 \in \mathcal{X}_f(d_1)$*,*

$$V_N^0(x_0) \leq \xi V_f(x_0) + \tau \bar{w}_{\lambda_f}, \tag{6.44a}$$

where

$$1 \leq \xi := \beta - (\beta - 1)\big(1 - \gamma^N\big) \leq \beta, \tag{6.44b}$$

$$\tau := \beta N - (\beta - 1)\left(N - 1 - \frac{\gamma - \gamma^N}{1 - \gamma}\right). \tag{6.44c}$$

The latter inequality is strict if $\beta > 1$.

Proof. See Appendix 6.A. □

Theorem 6.3. *If*

$$i \leq \left\lfloor \ln\left(\frac{(\zeta-1)(\beta d_1 + N d_2) + \bar{w}_{\lambda_V}}{(\zeta-1)\big(\xi d_1 + \tau\bar{w}_{\lambda_f}\big) + \bar{w}_{\lambda_V}}\right) \Big/ \ln\zeta \right\rfloor, \tag{6.45}$$

then $x_i \in \Gamma_N^{\beta}$.

Proof. Since $x_0 \in \mathcal{X}_f(d_1)$,

$$V_N^0(x_0) \leq \xi V_f(x_0) + \tau\bar{w}_{\lambda_f} \leq \xi d_1 + \tau\bar{w}_{\lambda_f},$$

and by Lemma 6.6,

$$V_N^0(x_i) \leq \zeta^i V_N^0(x_0) + \bar{w}_{\lambda_V}\frac{\zeta^i - 1}{\zeta - 1}.$$

By comparison, it follows that

$$V_N^0(x_i) \leq \zeta^i\big(\xi V_f(x_0) + \tau\bar{w}_{\lambda_f}\big) + \bar{w}_{\lambda_V}\frac{\zeta^i - 1}{\zeta - 1} \leq \zeta^i\big(\xi d_1 + \tau\bar{w}_{\lambda_f}\big) + \bar{w}_{\lambda_V}\frac{\zeta^i - 1}{\zeta - 1}.$$

Therefore, $x_i \in \Gamma_N^{\beta}$ is ensured if

$$\zeta^i\big(\xi d_1 + \tau\bar{w}_{\lambda_f}\big) + \bar{w}_{\lambda_V}\frac{\zeta^i - 1}{\zeta - 1} \leq \beta d_1 + N d_2, \tag{6.46}$$

which, with simple rearrangement, completes the proof. □

6.4.3.4 Case 2b - buffered control: bound on i given $x_0 \in \mathcal{X}_f(d_1)$ and the open-loop cost value parameter

Now let $x_0 \in \mathcal{X}_f(d_1)$ denote an initial state reached via *buffered control*, and let $x_i = f^i\big(x_0, 0, \{w\}_0^{i-1}\big)$ denote a state reached after i subsequent steps of open-loop operation. The main result here depicts a sufficient condition under which $x_i \in \Gamma_N^{\beta}$. It uses the following lemma.

Lemma 6.14. *For all $x_0 \in \mathcal{X}_f(d_1)$,*

$$V_N^0(x_0) \leq \xi V_f(x_0) + (\beta + N)\bar{w}_{\lambda_f}, \tag{6.47a}$$

where

$$1 \leq \xi := \beta - (\beta - 1)\big(1 - \gamma^N\big) \leq \beta. \tag{6.47b}$$

The latter inequality is strict if $\beta > 1$.

Proof. See Appendix 6.B. □

Theorem 6.4. *If*

$$i \leq \left\lfloor \ln\!\left(\frac{(\zeta - 1)(\beta d_1 + N d_2) + \bar{w}_{\lambda_V}}{(\zeta - 1)\big(\xi d_1 + (\beta + N)\bar{w}_{\lambda_f}\big) + \bar{w}_{\lambda_V}} \right) \Big/ \ln \zeta \right\rfloor, \tag{6.48}$$

then $x_i \in \Gamma_N^\beta$.

Proof. Similar to Theorem 6.3. □

In the fours cases we have presented bounds on the number of consecutive packet losses in terms of the constants depicted in Lemma 6.2 and Lemma 6.3 (which is known to exist) and Lemma 6.6 (which is assumed to exist). However, they needed to be estimated by analysing the behaviour of the *value function* along closed-loop and open-loop system trajectories with and without the buffer.

Unlike the analysis in Chapter 5, we now account for disturbances in the open-loop trajectories. This makes the problem challenging since the disturbances accumulate at each time step. In what follows we present the analysis in which the state x_0 has arrived $\mu\Gamma_N^\beta$.

6.4.3.5 Case 1a - closed-loop control: bound on i given $x_0 \in \mu\Gamma_N^\beta$, and the open-loop stage-cost and terminal-cost parameters

First, the lemma that links the value function at x_i, the stage cost and terminal cost function at x_0 under the assumption of open-loop operation in the interim.

Lemma 6.15. *Suppose Assumption 6.9 holds. For all $x_0 \in \mu\Gamma_N^\beta$,*

$$V_N^0(x_i) \leq \Psi\ell(x_0, u_0) + \Phi V_f(x_0) + \theta\bar{w}_{\eta_\ell} + \varphi\bar{w}_{\lambda_f}, \tag{6.49a}$$

where

$$\Psi := \sigma^i \frac{\sigma^N - 1}{\sigma - 1}, \quad \Phi := \beta\rho^{N+i}, \tag{6.49b}$$

$$\theta := \frac{\sigma^i - 1}{\sigma - 1} + \frac{1}{\sigma - 1}\left(\frac{\sigma^N - \sigma}{\sigma - 1} - (N-1)\right), \quad \varphi := \beta\frac{\rho^{N+i} - 1}{\rho - 1}. \tag{6.49c}$$

Proof. See Appendix 6.C. □

Theorem 6.5. *By Lemma 6.7 with $\sigma, \rho > 1$. If*

$$\beta\rho^{N+i}d_1 + \sigma^i\frac{\sigma^N - 1}{\sigma - 1}\ell(x_0, u_0) + \theta\bar{w}_{\eta_\ell} + \varphi\bar{w}_{\lambda_f} \leq \beta d_1 + Nd_2 \tag{6.50}$$

is satisfied, then $x_i \in \Gamma_N^\beta$.

Proof. By (6.49a), for any $x_0 \in \mu\Gamma_N^\beta$,

$$V_N^0(x_i) \leq \Psi\ell(x_0, u_0) + \Phi V_f(x_0) + \theta\bar{w}_{\eta_\ell} + \varphi\bar{w}_{\lambda_f} \leq \Psi\ell(x_0, u_0) + \Phi d_1 + \theta\bar{w}_{\eta_\ell} + \varphi\bar{w}_{\lambda_f},$$

since $x_0 \in \Gamma_N^\beta$ and $V_f(x_0) \leq d_1$. Therefore, $x_i \in \Gamma_N^\beta$ if

$$\Phi d_1 + \Psi\ell(x_0, u_0) + \theta\bar{w}_{\eta_\ell} + \varphi\bar{w}_{\lambda_f} \leq \beta d_1 + Nd_2. \tag{6.51}$$

Rearrangement establishes the result. □

The problem with (6.50) is that it does not provide an explicit bound on i. However, the expression here can be analysed to provide a lower bound on the i that satisfies the condition.

Theorem 6.6. *The i that satisfies* (6.50) *is bounded as*

$$i \geq \left\lfloor \ln\left(\frac{\beta d_1 + Nd_2 - \Upsilon}{\beta\rho^N d_1 + \Theta}\right) \Big/ \ln\rho \right\rfloor =: i^*, \tag{6.52a}$$

where

$$\Upsilon := \beta\frac{\bar{w}_{\lambda_f}}{\rho - 1} + \frac{\bar{w}_{\eta_\ell}}{\sigma - 1}\left(\frac{\sigma^N - \sigma}{\sigma - 1} - N\right) \tag{6.52b}$$

$$\Theta := \beta\frac{\bar{w}_{\lambda_f}\rho^N}{\rho - 1}. \tag{6.52c}$$

Proof. See Appendix 6.D. □

This gives a conservative result that says the system is stable if the number of consecutive packet losses $i \leq i^*$.

We must remark that if there are no disturbances, *i.e.* $\bar{w}_{\lambda_f} = 0$ and $\bar{w}_{\eta_\ell} = 0$, we recover the bound on i for the nominal case in Theorem 5.4.

6.5 Stability analysis of linear systems

As we did analyse in Chapter 5, the stability analysis of linear systems is done as a specialization of the analysis for nonlinear system. The results are more restrictive since it is easy to compute or estimate the various constants and parameters involved in the different bounds.

6.5.1 Problem formulation

Consider that $f(x, vu) = Ax + vBu$, that is

$$x_{k+1} = Ax_k + v_k Bu_k + w_k \tag{6.53}$$

is linear a time-invariant system with additive process noise.

Assumption 6.11. *The matrices A and B are known and the pair (A, B) is stabilizable.*

Consider the setting depicted in Fig. 6.4, where the system with additive noise is connected to a controller via a partially lossy UDP-like communication channel, and the communication between the controller and system actuator is subject to random packet losses.

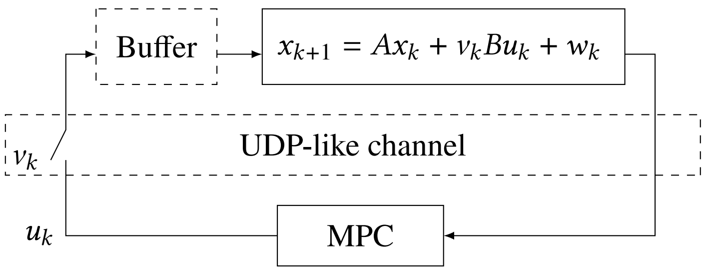


Figure 6.4: A system with additive noise in connection with a controller via a UDP-like channel. The C-A channel is subject to random packet losses.

Some definitions, assumptions and lemmas for the nominal *linear* systems in Chapter 5 remain valid the analysis presented in this section and can be adopted without repetition. Thus, the following are referenced accordingly:

- The controller formulation in Section 5.5.2.

- The assumptions and lemmas from the stability without terminal constraint set conditions in Section 5.5.3.
- The Definitions 5.5 and 5.6, and Lemmas 5.8 and 5.9 from the stability analysis for linear NCS conditions in Section 5.5.4.

Then, based on these, we can now formulate the following.

6.5.2 Case 1a - closed-loop control: bound on i given $x_0 \in \mu\Gamma_N^\beta$, and the linear open-loop stage-cost and terminal-cost parameters

Let $x_0 \in \mu\Gamma_N^\beta$, $\mu \in (0,1]$, be an initial state reached via closed-loop control, and let $x_i = A^i x_0 + \{w\}_0^{i-1}$ be a state reached after i subsequent steps of open-loop operation. In what follows, we show the conditions under which x_i is contained in Γ_N^β.

Theorem 6.7. *The i that satisfies* (6.50)*, for the linear case, is bounded as*

$$i \geq \left\lfloor \ln\left(\frac{\beta d_1 + N d_2 - \bar{\Upsilon}}{\beta \bar{\rho}^N d_1 + \bar{\Theta}}\right) \Big/ \ln \bar{\rho} \right\rfloor =: i^*, \tag{6.54a}$$

where

$$\bar{\Upsilon} := \beta \frac{\bar{w}_{\lambda_f}}{\bar{\rho} - 1} + \frac{\bar{w}_{\eta_\ell}}{\bar{\sigma} - 1}\left(\frac{\bar{\sigma}^N - \bar{\sigma}}{\bar{\sigma} - 1} - N\right) \tag{6.54b}$$

$$\bar{\Theta} := \beta \frac{\bar{w}_{\lambda_f} \bar{\rho}^N}{\bar{\rho} - 1}. \tag{6.54c}$$

This gives a conservative result that says the linear system is stable if the number of consecutive packet losses $i \leq i^*$.

6.6 Numerical example 1: nonlinear system

Consider the nonlinear system in Section 5.6. With additive noise, the system $x_{k+1} = f(x_k, u_k) + w_k$ is

$$x_{k+1} = r \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} x_k + \epsilon \|x_k\|_2^2 + v_k \begin{bmatrix} 1.5 \\ 0 \end{bmatrix} u_k + w_k,$$

where $r = 1.05$, $\theta = 0.12$, $\epsilon = 0.0001$, w_k is bounded in $\mathcal{W} = \{w \colon \|w\|_2^2 \leq 0.09\}$, and the input constraint set $\mathcal{U} = \{u \colon |u| \leq 1\}$.

The controller is designed as $\ell(x,u) = x^\top Q x + u^\top R u$, $V_f(x) = x^\top Q_f x$ and $\kappa_f(x)$, with $R = 1$, $\beta = 1$, horizon $N = 3$, and

$$Q = \begin{bmatrix} 0.41 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_f = \begin{bmatrix} 0.99 & 1.50 \\ 1.50 & 9.56 \end{bmatrix},$$

where Q_f is a local approximation solution that satisfies Assumption 5.10 given a local approximation of $\kappa_f(x) \approx K_f x$ around the origin.

The controller satisfies Definition 6.1 with $d_1 = 3.03$ and $d_2 = 1.23$; therefore, for $N = 3$ and $\beta = 1$,

$$\Gamma_3^1 = \{x \colon V_N^0(x) \leq 6.73\}.$$

6.6.1 Case 1a - closed-loop control: $x_0 \in \mu\Gamma_N^\beta$

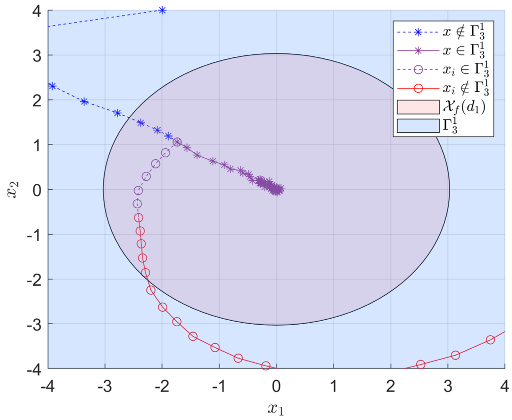
Let $x = \begin{bmatrix} -2 & 4 \end{bmatrix}^\top$ denote the initial state not in Γ_3^1. After closed-loop operation without packet dropouts, x reaches the state $x_0 = \begin{bmatrix} -1.74 & 1.06 \end{bmatrix}^\top \in \Gamma_3^1$, and it successfully converges around the origin. However, supposing that all i subsequent packets are instead dropped ($\nu_=0$), for $1 \leq i \leq 5$, the states x_1 to x_5 in open-loop operation remain in Γ_3^1 but x_6 leaves the set, see Fig. 6.5.

By reviewing the closed-loop trajectory we obtain $\mu = 0.71$, and reviewing the open-loop trajectory gives us $\zeta = 1.054$ (Assumption 6.8). Theorem 6.1 predicts $i^* = \lfloor 5.76 \rfloor = 5$, given $\bar{w}_{\lambda_V} = 0.04$, meaning that x_i, for $i = 5$ steps, is guaranteed to remain within Γ_3^1. Thus, the main result for the nonlinear Case 1a is verified, see Fig. 6.5.

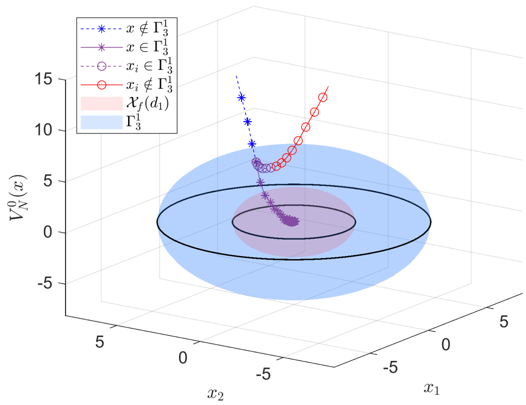
6.6.2 Case 1b - closed-loop control: $x_0 \in \mathcal{X}_f(d_1)$

For Case 1b, under closed-loop operation without packet dropouts, x reaches the state $x_0 = \begin{bmatrix} -1.39 & 0.76 \end{bmatrix}^\top \in \mathcal{X}_f(d_1)$ and converges successfully around the origin. Nevertheless, supposing that all i subsequent packets are instead dropped ($\nu_k = 0$), for $1 \leq i \leq 12$, the states x_1 to x_{12} in open-loop operation remain in Γ_3^1 but x_{13} leaves the set, see Fig. 6.6.

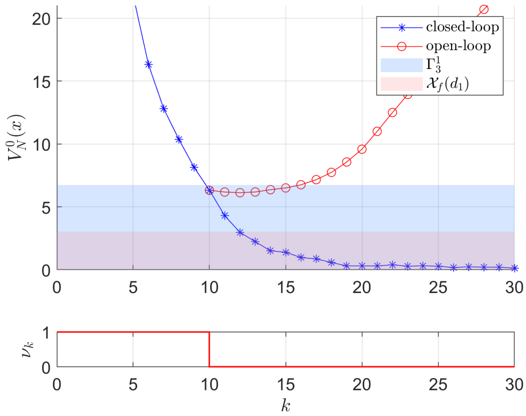
Theorem 6.3 predicts $i^* = \lfloor 12.27 \rfloor = 12$, given $\bar{w}_{\lambda_f} = 0.05$ and $\bar{w}_{\lambda_V} = 0.04$, $\gamma = 0.66$, $\rho = 1.01$



(a) Phase-portrait



(b) Phase-portrait vs optimal cost value



(c) Optimal cost value

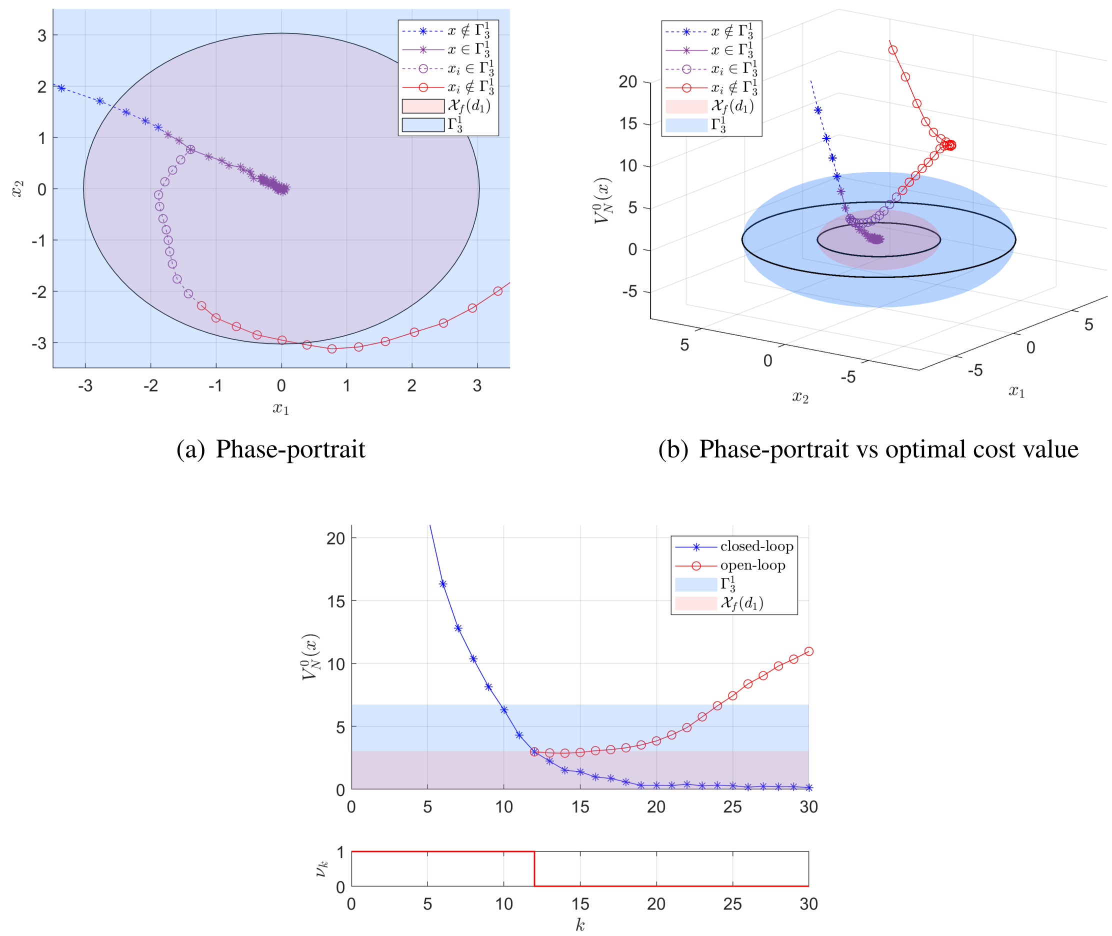
Figure 6.5: Case 1a - nonlinear system: phase-portrait and optimal cost value under closed-loop control.

and $\sigma = 1.02$, meaning that x_i, for $i = 12$ steps, is guaranteed to remain within Γ_3^1. Thus, the main result for the nonlinear Case 1b is verified, see Fig. 6.6.

6.6.3 Case 2a - buffered control: $x_0 \in \mu^* \Gamma_N^\beta$

For the buffered control Cases, we first evaluate the behaviour of the trajectory while there are packet dropouts before reaching the Γ_3^1. In Fig. 6.7, from time step $k = 4$ to $k = 6$, the system experiments $i = 3$ consecutive packet dropouts and the buffer enters in action. However, the size of the buffer is $N = 3$, and it only transfers x to $x_0 = \begin{bmatrix} -2.51 & 1.39 \end{bmatrix}^\top \in \Gamma_3^1$. Supposing that all i subsequent packets are dropped, for $1 \le i \le 5$, the states x_1 to x_5 in open-loop operation remain in Γ_3^1 but x_6 leaves the set.

Design choice of the buffer size: The approximate minimum size of the buffer such that x



(a) Phase-portrait

(b) Phase-portrait vs optimal cost value

(c) Optimal cost value

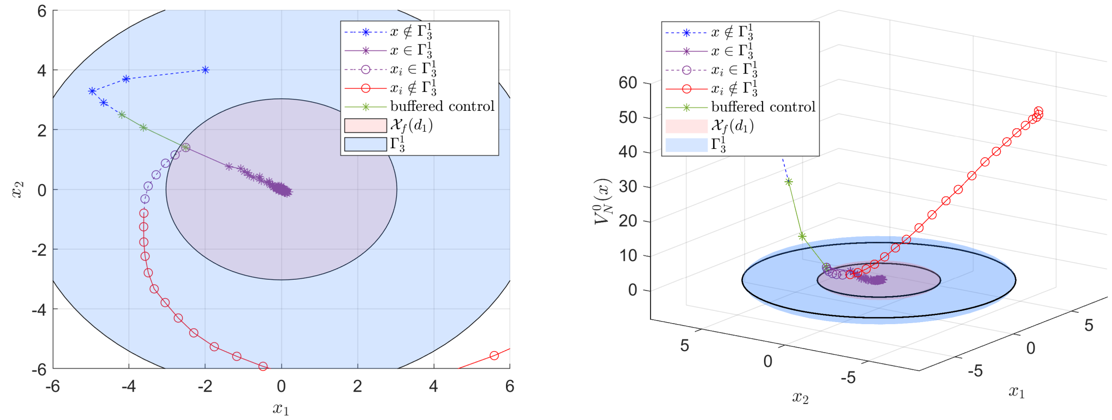
Figure 6.6: Case 1b - nonlinear system: phase-portrait and optimal cost value under closed-loop control.

reaches Γ_N^{β} is $N_b \geq N + i^* = 8$. In Fig. 6.8, x under buffered control operation is transferred to $x_0 = \begin{bmatrix} -2.09 & 1.16 \end{bmatrix}^\top \in \Gamma_8^1$ despite consecutive packet dropouts, verifying the design choice of the buffer size.

Now we analyse the open-loop trajectory and verify the bound for Case 2a. Let $N = 8$ and knowing the controller satisfies Definition 6.1 with $d_1 = 3.03$ and $d_2 = 1.16$; then, for $\beta = 1$,

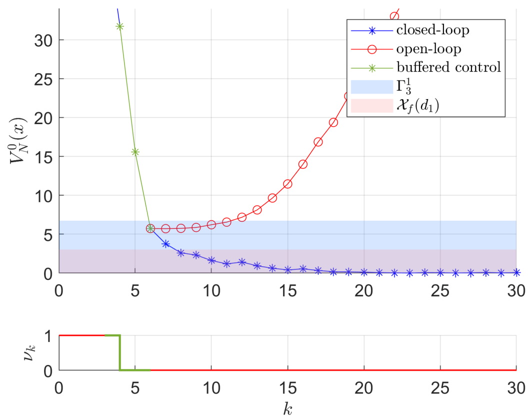
$$\Gamma_8^1 = \left\{x \colon V_N^0(x) \leq 12.30\right\}.$$

Let $x = \begin{bmatrix} -2 & 4 \end{bmatrix}^\top$ denote the initial state not in Γ_8^1. After buffered control operation for $i = 8$ consecutive packet losses, x reaches the state $x_0 = \begin{bmatrix} -2.09 & 1.16 \end{bmatrix}^\top \in \Gamma_8^1$. However, supposing that all i subsequent packets are still dropped, for $1 \leq i \leq 7$, the states x_1 to x_7 in open-loop



(a) Phase-portrait

(b) Phase-portrait vs optimal cost value



(c) Optimal cost value

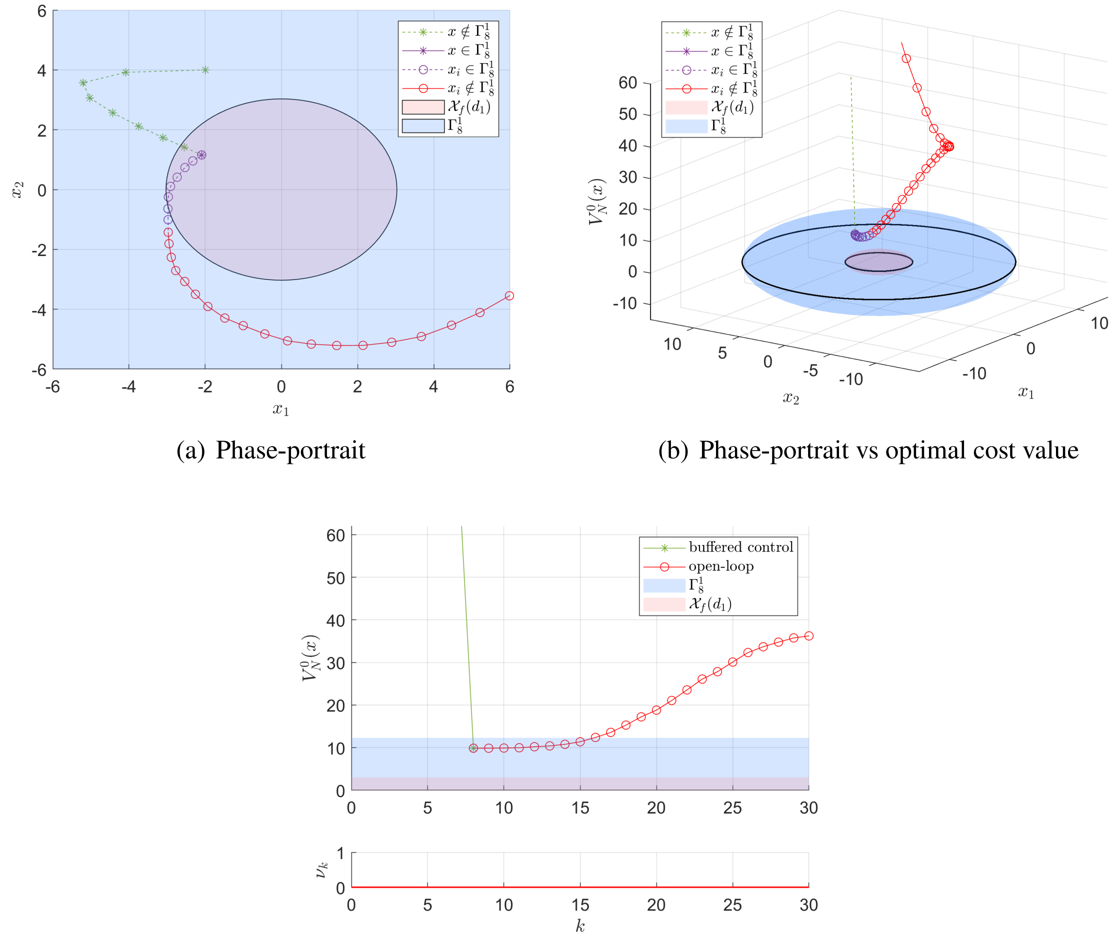
Figure 6.7: Case 2a - nonlinear system: phase-portrait and optimal cost value under closed-loop and buffered control.

operation remain in Γ_8^1 but x_8 leaves the set, see Fig. 6.8.

By reviewing the buffered control trajectory we obtain $\mu^* = 0.74$, and reviewing the open-loop trajectory gives us $\zeta = 1.037$ (Assumption 6.8). Theorem 6.2 predicts $i^* = \lfloor 7.52 \rfloor = 7$, given $\bar{w}_{\lambda V} = 0.04$, meaning that x_i, for $i = 7$ steps, is guaranteed to remain within Γ_8^1. Thus, the main result for the nonlinear Case 2a is verified, see Fig. 6.8.

6.6.4 Case 2b - buffered control: $x_0 \in \mathcal{X}_f(d_1)$

Evaluating the behaviour of the trajectory while there are packet dropouts before reaching the $\mathcal{X}_f(d_1)$. In Fig. 6.9, from time step $k = 7$ to $k = 9$, the system experiments $i = 3$ consecutive packet dropouts and the buffer enters in action. However, the size of the buffer is $N = 3$, and it only transfers x to $x_0 = \begin{bmatrix} -1.68 & 0.94 \end{bmatrix}^\top \in \mathcal{X}_f(d_1)$. Supposing that all i subsequent packets are



(a) Phase-portrait

(b) Phase-portrait vs optimal cost value

(c) Optimal cost value

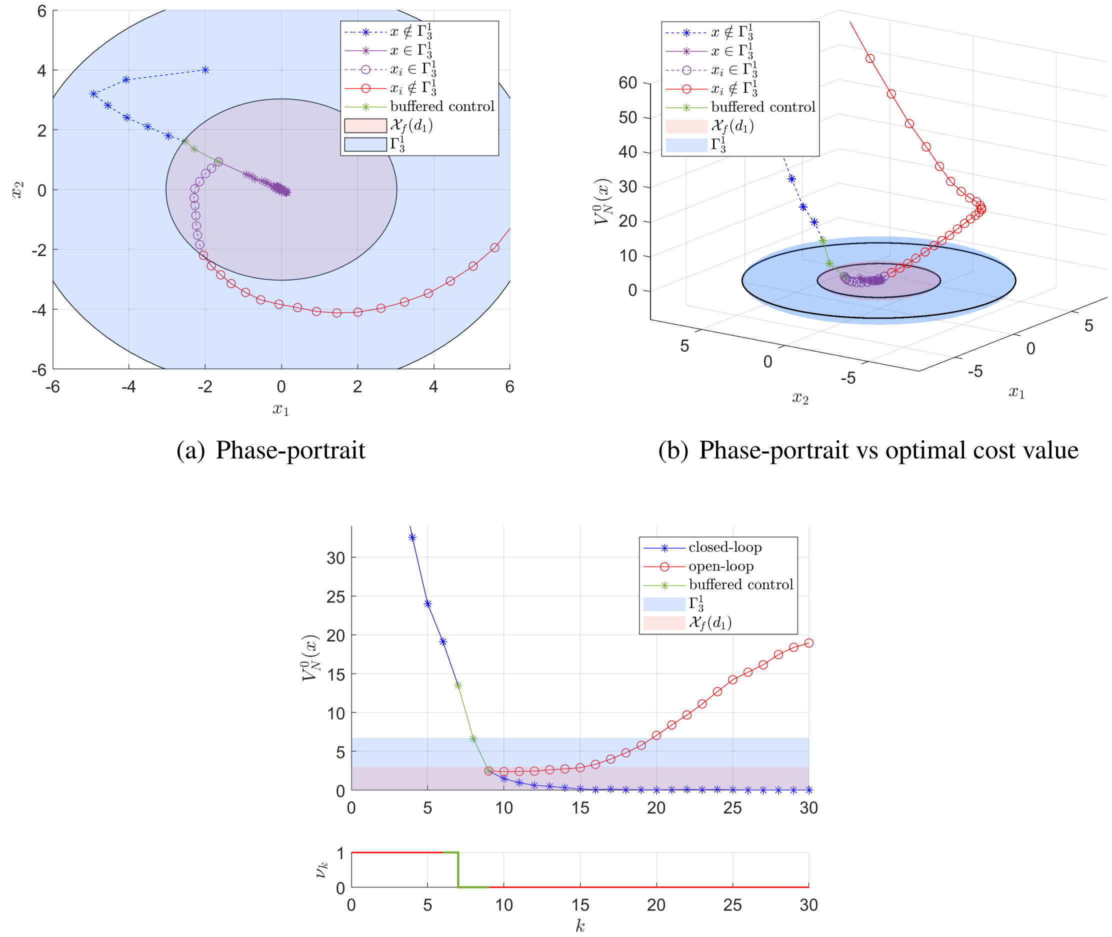
Figure 6.8: Case 2a - nonlinear system: phase-portrait and optimal cost value under buffered control only.

dropped, for $1 \leq i \leq 10$, the states x_1 to x_{10} in open-loop operation remain in Γ_3^1 but x_{11} leaves the set.

Design choice of the buffer size: The approximate minimum size of the buffer such that x reaches Γ_N^β is $N_b \geq N + i^* = 11$. In Fig. 6.10, x under buffered control operation is transferred to $x_0 = \begin{bmatrix} -1.13 & 0.62 \end{bmatrix}^\top \in \mathcal{X}_f(d_1)$ despite consecutive packet dropouts, verifying the design choice of the buffer size.

Now we analyse the open-loop trajectory and verify the bound for Case 2b. Let $N = 11$ and knowing the controller satisfies Definition 6.1 with $d_1 = 3.03$ and $d_2 = 1.13$; then, for $\beta = 1$,

$$\Gamma_{11}^1 = \{x \colon V_N^0(x) \leq 15.49\}.$$



(a) Phase-portrait

(b) Phase-portrait vs optimal cost value

(c) Optimal cost value

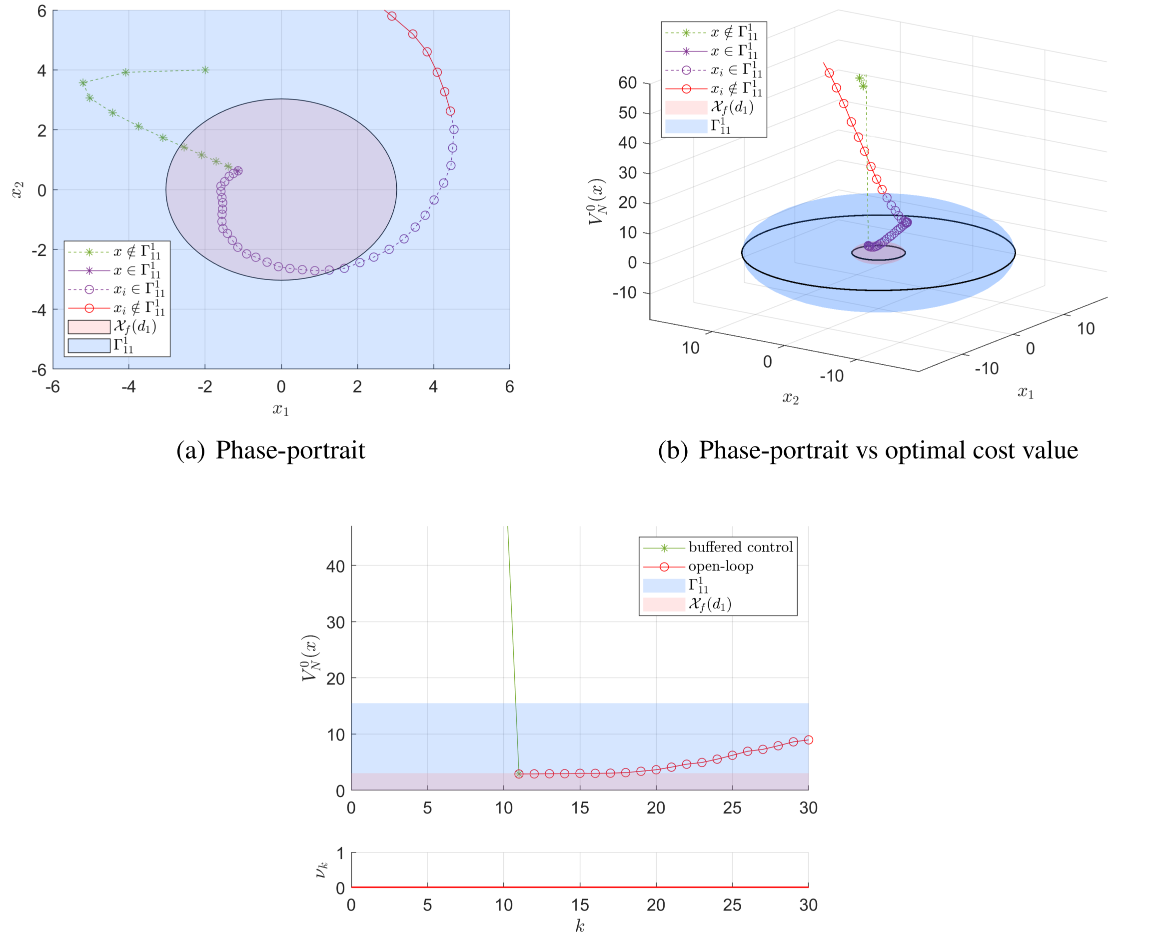
Figure 6.9: Case 2b - nonlinear system: phase-portrait and optimal cost value under closed-loop and buffered control.

Under buffered control operation, x reaches the state $x_0 = \begin{bmatrix} -1.13 & 0.62 \end{bmatrix}^\top \in \mathcal{X}_f(d_1)$ after $i = 11$ consecutive packet losses. However, supposing that all i subsequent packets are still dropped, for $1 \leq i \leq 34$, the states x_1 to x_{34} in open-loop operation remain in Γ_{11}^1 but x_{35} leaves the set, see Fig. 6.10.

Theorem 6.4 predicts $i^* = \lfloor 31.61 \rfloor = 34$, given $\bar{w}_{\lambda_f} = 0.05$ and $\bar{w}_{\lambda_V} = 0.04$, and $\gamma = 0.66$, meaning that x_i, for $i = 34$ steps, is guaranteed to remain within Γ_{11}^1. Thus, the main result for the nonlinear Case 2b is verified, see Fig. 6.10.

6.6.5 Case 1a - closed-loop control: $x_0 \in \mu\Gamma_N^\beta$ given σ and ρ

Alternately, by Section 6.4.3.5 we can approximate the result in Section 6.6.1 with the open-loop stage-cost σ and terminal-cost ρ parameters. Theorem 6.6 predicts $i^* = \lfloor 5.36 \rfloor = 5$, given



(a) Phase-portrait

(b) Phase-portrait vs optimal cost value

(c) Optimal cost value

Figure 6.10: Case 2b - nonlinear system: phase-portrait and optimal cost value under buffered control only.

$\bar{w}_{\lambda_f} = 0.03$, $\bar{w}_{\eta_V} = 0.06$, $\rho = 1.01$ and $\sigma = 1.02$, meaning that x_i, for $i = 5$ steps, is guaranteed to remain within Γ^1_3, see Fig. 6.5.

6.7 Numerical example 2: linear system

Consider the linear system used in Section 5.7. With additive noise, the systems is $x_{k+1} = Ax_k + \nu_k B u_k + w_k$,

$$A = \begin{bmatrix} 1.04 & -0.13 \\ 0.13 & 1.04 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

w_k is bounded in $\mathcal{W} = \left\{w \colon \|w\|_2^2 \le 0.09\right\}$, and the input constraint set $\mathcal{U} = \{u \colon |u| \le 1\}$.

The controller has $\ell(x,u) = x^\intercal Q x + u^\intercal R u$, $V_f(x) = x^\intercal Q_f x$ and $\kappa_f(x) = K_f x$, with $R = 1$,

$K_f = \begin{bmatrix} -0.71 & -0.77 \end{bmatrix}$, $\beta = 1$, prediction horizon $N = 3$, and

$$Q = \begin{bmatrix} 0.41 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_f = \begin{bmatrix} 1.39 & 1.93 \\ 1.93 & 10.25 \end{bmatrix}.$$

The controller satisfies Assumptions 5.10 and 5.4 with $d_1 = 2.73$ and $d_2 = 1.25$; therefore, with $N = 3$ and $\beta = 1$,

$$\Gamma_3^1 = \{x \colon V_N^0(x) \leq 6.49\}.$$

6.7.1 Case 1a - closed-loop control: $x_0 \in \mu\Gamma_N^\beta$ given $\overline{\sigma}$ and $\overline{\rho}$

Let $x = \begin{bmatrix} 2 & -4 \end{bmatrix}^\top$ denote the initial state not in Γ_3^1. Under closed-loop operation without packet dropouts and subject to uncertainty, x reaches the $x_0 = \begin{bmatrix} 1.55 & -0.91 \end{bmatrix}^\top \in \Gamma_3^1$, and eventually x_0 successfully converges around the origin under closed-loop control. However, supposing that all i subsequent packets are instead dropped, for $1 \leq i \leq 8$, the states x_1 to x_8 remain in Γ_3^1 but x_9 leaves the set, see Fig. 6.11.

The constants in Lemmas 5.7–5.9 are evaluated such that $\bar{\gamma} = 0.83$, $\bar{\rho} = 1.10$ and $\bar{\sigma} = 1.10$, and since we know x_0, we have that $\gamma = 0.69$, $\rho = 1.03$ and $\sigma = 1.04$, confirming $\bar{\gamma} > \gamma$, $\bar{\rho} > \rho$ and $\bar{\sigma} > \sigma$. From Lemma 6.7 we have that $\bar{w}_{\lambda_f} = 0.04$ and $\bar{w}_{\eta_\ell} = 0.49$.

Theorem 6.7 predicts $i^* = \lfloor 8.04 \rfloor = 8$, meaning that x_i, for $i = 8$ steps, is guaranteed to remain within Γ_3^1. Thus, the main results for Case 1a—and its conservativeness—are verified, see Fig. 6.11.

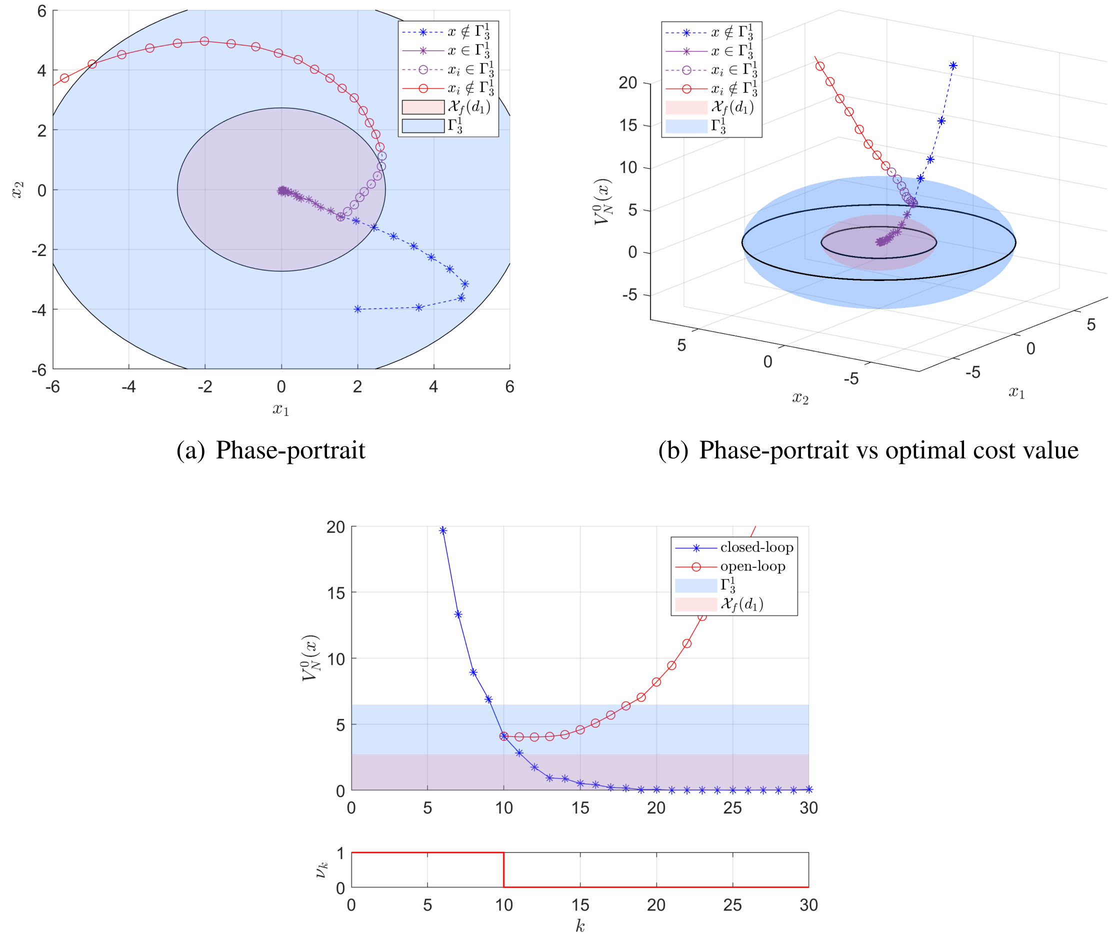
6.8 Numerical example 3: linear system

Consider the linear the system $x_{k+1} = Ax_k + v_k B u_k + w_k$, with

$$A = \begin{bmatrix} 1.04 & -0.12 \\ 0.12 & 1.04 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

w_k is bounded in $\mathcal{W} = \{w \colon \|w\|_2^2 \leq 0.09\}$, and the input constraint set $\mathcal{U} = \{u \colon |u| \leq 1\}$.

The controller is designed with $R = 1$, $K_f = \begin{bmatrix} -0.76 & -0.71 \end{bmatrix}$, $\beta = 1.1$, prediction horizon



(a) Phase-portrait

(b) Phase-portrait vs optimal cost value

(c) Optimal cost value

Figure 6.11: Case 1a - linear system: phase-portrait and optimal cost value under closed-loop control.

$N = 3$, and

$$Q = \begin{bmatrix} 0.7 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q_f = \begin{bmatrix} 1.75 & 2.11 \\ 2.11 & 12.26 \end{bmatrix}.$$

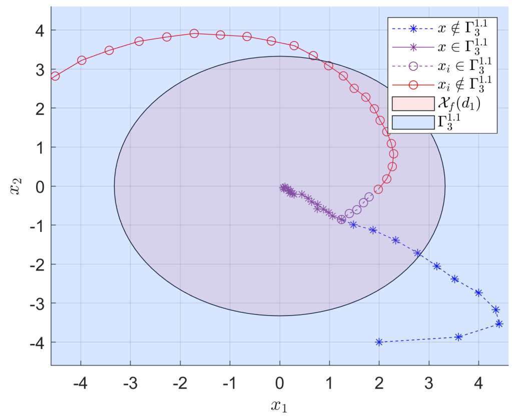
By the Assumptions 5.10 and 5.4 we have $d_1 = 3.02$ and $d_2 = 1.19$; therefore, with $N = 3$ and $\beta = 1.1$, the set

$$\Gamma_3^{1.1} = \{x\colon V_N^0(x) \leq 6.88\}.$$

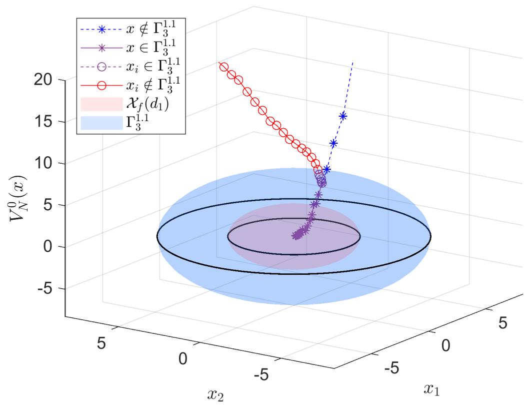
6.8.1 Case 1a - closed-loop control: $x_0 \in \mu\Gamma_N^{\beta}$ given $\overline{\sigma}$ and $\overline{\rho}$

Let the initial state $x = \begin{bmatrix} 2 & -4 \end{bmatrix}^\top$ not in $\Gamma_3^{1.1}$. Under closed-loop operation without packet dropouts and subject to uncertainty, x has reached the $x_0 = \begin{bmatrix} 1.24 & -0.86 \end{bmatrix}^\top \in \Gamma_3^{1.1}$, and eventually x_0 successfully converges around the origin. Nevertheless, supposing that all i

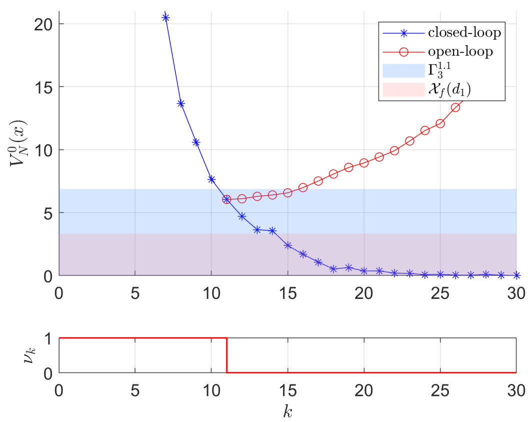
subsequent packets are instead dropped, for $1 \leq i \leq 4$, the states x_1 to x_4 remain in $\Gamma_3^{1.1}$ but x_5 leaves the set, see Fig. 6.12.



(a) Phase-portrait



(b) Phase-portrait vs optimal cost value



(c) Optimal cost value

Figure 6.12: Case 1a - linear system: phase-portrait and optimal cost value under closed-loop control.

From Lemmas 5.7–5.9 we have that $\bar{\gamma} = 0.87$, $\bar{\rho} = 1.09$ and $\bar{\sigma} = 1.09$, and with x_0 we calculate $\gamma = 0.73$, $\rho = 1.03$ and $\sigma = 1.02$, confirming $\bar{\gamma} > \gamma$, $\bar{\rho} > \rho$ and $\bar{\sigma} > \sigma$. Applying Lemma 6.7 give us $\bar{w}_{\lambda_f} = 0.03$ and $\bar{w}_{\eta_\ell} = 0.06$.

Theorem 6.7 predicts $i^* = \lfloor 4.27 \rfloor = 4$, meaning that x_i, for $i = 4$ steps, is guaranteed to remain within $\Gamma_3^{1.1}$. Thus, the main results for Case 1a are verified, see Fig. 6.12.

6.9 Conclusions

This chapter investigated the robust stability of a discrete-time nonlinear system controlled by MPC under input constraints, random packet losses in the C–A channel, and bounded additive disturbances. As in the previous chapter, the MPC formulation employs a terminal cost function that satisfies a local CLF condition, but no explicit terminal constraints. To mitigate packet losses, a buffer stores the nominal optimal control sequence and applies to the plant during dropouts, while the system is simultaneously affected by disturbances.

We showed the conditions under which the system state remains within the RoA despite the combined effects of the uncertainty and consecutive packet losses. In contrast to the nominal case studied in Chapter 5, the presence of additive disturbances limits the action of the buffer: at best, buffered control does not guarantee the transfer of an initial state into the terminal region associated with the CLF terminal cost. However, the analysis can be done under the restrictive assumption that the buffer transfers an initial state into the terminal region if and only if the disturbance is sufficiently small.

The nonlinear analysis was structured around four distinct cases, defined by the initial location of the state (in the RoA or in the terminal region) and by the control mode (closed-loop or buffered vs open-loop). These cases were tested numerically by three examples. In the numerical example 1, for each case, the evolution of the value function is analysed along both closed-loop and open-loop trajectories. The simulation results show that, if the bounded additive disturbance is sufficiently small, the derived upper bounds on the number of consecutive packet dropouts that can be tolerated before the state leaves the RoA while maintaining robust stability are verified. In the numerical examples 2 and 3, two discrete LTI systems were used to verify the specialized case shown in the linear stability section. These bounds are approximated using the open-loop stage-cost and terminal-cost parameters, meaning that the result is more conservative but easy to compute.

Moreover, like in Chapter 5, the results of the simulations suggest that the minimum buffer size required to transfer an initial state to the RoA or the terminal region, is approximately equal to the sum of the prediction horizon and the bound on the number of consecutive packet losses in open-loop operation before the state leaves the RoA. However, in the presence of additive disturbances, this guideline becomes more conservative and should be interpreted strictly as a design heuristic rather than a definitive rule. Further theoretical analysis is required to characterize buffer-sizing strategies when the buffer length is decoupled from the prediction horizon and when disturbance levels are non-negligible.

Table 6.3 summarizes the main technical results, assumptions, setting, associated estimated and computable parameters, and differences with Chapter 5.

Aspect	Main Assumptions / Setting	Key Technical Results
RoA Γ_N^β and $\mathcal{X}_f(d_1)$	Implicitly defined by MPC design (N, Q, R, Q_f) and $\beta V_f(x)$	Robust invariance of the Γ_N^β and $\mathcal{X}_f(d_1)$ are guaranteed only under bounded disturbances.
Limitation vs. Chapter 5	Presence of disturbances	Unlike the nominal case, buffer cannot guarantee reachability of the state to the terminal region.
Key robustness result	Combined disturbances and packet losses	Explicit conditions ensure the state remains within the RoA despite uncertainty and consecutive packet dropouts.
Stability mechanism	Value-function-based analysis	Stability is maintained only if the number of consecutive packet losses is bounded and the disturbance is sufficiently small.
Nonlinear case analysis	Four-case decomposition	Analysis distinguishes initial state location (RoA vs. terminal region) and control mode (closed-loop/buffered vs. open-loop).
Nonlinear closed-loop behaviour	Nonlinear dynamics, MPC feedback applied	Value function decrease is characterized by **estimated parameters** μ, μ^*, γ, ξ (Example 1).
Nonlinear open-loop behaviour	Buffer exhausted, open-loop evolution	Value function growth is characterized by **estimated parameters** ζ, σ, ρ (Example 1).
Linear case analysis	Assumptions, definitions and lemmas from Chapter 5	**Computable parameters**: closed-loop $\bar{\gamma} > \gamma$, and open-loop $\bar{\rho} > \rho, \bar{\sigma} > \sigma$ (Example 2 and 3).
Uncertainty bounds	Small disturbance magnitude	Uncertain **estimated parameters** in closed-loop $\tau, \bar{w}_{\lambda_V}, \bar{w}_{\lambda_f}$, and open-loop $\bar{w}_{\eta_\ell}, \bar{w}_{\lambda_f}$ behaviours.
Conservatism	Analytical upper bounds	Linear bounds are conservative but informative for tuning horizon length and buffer size (Example 2 and 3).
Design guideline	Empirical observation from simulations	Minimum buffer size ≈ prediction horizon + open-loop packet-loss bound.

Table 6.3: Summary of technical results.

6.A Proof of Lemma 6.13

First, let us define the following.

Lemma 6.16 (Sum of $V_f(\cdot)$ in closed-loop)**.** *Suppose $x_j \in \mathcal{X}_f(d_1)$, then*

$$\sum_{j=i}^{N+i-1} V_f(x_j) \leq \frac{1-\gamma^N}{1-\gamma} V_f(x_i) + \frac{\bar{w}_{\lambda_f}}{1-\gamma}\left(N - 1 - \frac{\gamma - \gamma^N}{1-\gamma}\right). \tag{6.55}$$

Proof. From (6.16a)

$$\sum_{j=i}^{N+i-1} V_f(x_j) = V_f(x_i) + V_f(x_{i+1}) + \dots \tag{6.56a}$$

$$\leq V_f(x_i) + \gamma V_f(x_i) + \lambda_f \|w_i\|_2^2 + \dots \tag{6.56b}$$

grouping the finite sequences

$$\leq \sum_{j=0}^{N-1} \gamma^i V_f(x_i) + \bar{w}_{\lambda_f} \sum_{j=1}^{N-1} \frac{1-\gamma^j}{1-\gamma}, \tag{6.56c}$$

and solving the finites series completes the proof. □

We need one more lemma referring to the disturbance sequence. The proof is obtained by applying (6.16c).

Lemma 6.17. *Let $\mathbf{w} := \{w_j, w_{j+1}, \dots\} \in \mathcal{W}$ be a disturbance sequence, $j \in \mathbb{N}_{\geq 0}$, then for $i \geq j$*

$$\sum_{j=i}^{N+i-1} \lambda_f \|w_j\|_2^2 \leq N\bar{w}_{\lambda_f}, \tag{6.57a}$$

$$\sum_{j=i}^{N+i-1} \lambda_f \|w_{j-1}\|_2^2 \leq (N + i - 1 - (i-1))\bar{w}_{\lambda_f} = N\bar{w}_{\lambda_f}, \tag{6.57b}$$

Now, consider a state $x_i \in \mathcal{X}_f(d_1)$ and controls $u_j = \kappa_f(x_j) \in \mathcal{U}$, $j \in \mathbb{N}_{[i,N+i-1]}$, the cost value for $N + i$ steps in closed-loop with disturbance is

$$V_N^0(x_i) \leq \sum_{j=i}^{N+i-1} \ell(x_j, u_j) + \beta V_f(x_{N+i}), \tag{6.58a}$$

$$= \sum_{j=i}^{N+i-1} \Big(\ell(x_j, u_j) + \beta V_f(x_{j+1}) - \beta V_f(x_j)\Big) + \beta V_f(x_i), \tag{6.58b}$$

using (6.16b)

$$\leq \sum_{j=i}^{N+i-1} \Big(V_f(x_j) - V_f(x_{j+1}) + \lambda_f \|w_j\|_2^2 + \beta V_f(x_{j+1}) - \beta V_f(x_j)\Big) + \beta V_f(x_i), \tag{6.58c}$$

ordering

$$= \sum_{j=i}^{N+i-1} \Big((1-\beta)\big(V_f(x_j) - V_f(x_{j+1})\big) + \lambda_f \|w_j\|_2^2\Big) + \beta V_f(x_i), \tag{6.58d}$$

owing to (6.16a)

$$= \sum_{j=i}^{N+i-1} \Big((1-\beta)\big((1-\gamma)V_f(x_j) - \lambda_f \|w_j\|_2^2\big) + \lambda_f \|w_j\|_2^2\Big) + \beta V_f(x_i), \tag{6.58e}$$

ordering

$$= (1-\beta)(1-\gamma) \sum_{j=i}^{N+i-1} V_f(\bar{x}_j) + \beta V_f(x_i) + \sum_{j=i}^{N+i-1} \Big((1-\beta)\lambda_f \|w_{j-1}\|_2^2 + \beta \lambda_f \|w_j\|_2^2\Big), \tag{6.58f}$$

using (6.16a) and Lemma 6.17

$$\leq (1-\beta)(1-\gamma) \sum_{j=i}^{N+i-1} V_f(x_j) + \beta V_f(x_i) + \beta \sum_{j=i}^{N+i-1} \lambda_f \|w_j\|, \tag{6.58g}$$

applying Lemma 6.16

$$\leq (1-\beta)\left((1-\gamma^N)V_f(x_i) + \bar{w}_{\lambda_f}\left(N - 1 - \frac{\gamma - \gamma^N}{1-\gamma}\right)\right) + \beta V_f(x_i) + \beta N \bar{w}_{\lambda_f}, \tag{6.58h}$$

reordering completes the proof.

6.B Proof of Lemma 6.14

First, let us define the following lemma. The proof is obtained by applying (6.14a) and (6.15b).

Lemma 6.18 ($V_f(\cdot)$ in closed-loop)**.** *Suppose that $x_j \in \mathcal{X}_f(d_1)$, from Lemma 6.4, $x_{j+1} = f(\bar{x}_j, \kappa_f(\bar{x}_j)) + w_j \in \mathcal{X}_f(d_1)$, and $\bar{x}_{j+1} = f(\bar{x}_j, \kappa_f(\bar{x}_j)) \in \mathcal{X}_f(d_1)$, for some $j \in \mathbb{N}_{\geq 0}$. Then,*

$$V_f(x_{j+1}) \leq V_f(\bar{x}_{j+1}) + \lambda_f \|w_j\|_2^2 \tag{6.59a}$$

$$\leq \gamma V_f(\bar{x}_j) + \lambda_f \|w_j\|_2^2, \tag{6.59b}$$

Now, let us consider a nominal state $\bar{x}_{i|N} = \bar{x}_i \in \mathcal{X}_f(d_1)$ and nominal controls $\bar{u}_{j|N} = \bar{u}_j = \kappa_f(\bar{x}_j) \in \mathcal{U}$, $j \in \mathbb{N}_{[i,N+i-1]}$, obtained and stored in the buffer at time step N. In this case, the buffer supplies the nominal input sequence to the plant, meaning that the cost value is evaluated on the nominal trajectories of the states and nominal inputs.

However, the disturbance is still present but there is no closed-loop, which means that we can decouple the disturbance sequence and analyse the cost value with the disturbance sequence in parallel. After $N + i$ steps the value function is

$$V_N^0(\bar{x}_i) \leq \sum_{j=i}^{N+i-1} \ell(x_j, u_j) + \beta V_f(x_{N+i}) \tag{6.60a}$$

$$= \sum_{j=i}^{N+i-1} \Big(\ell(x_j, u_j) + \beta V_f(x_{j+1}) - \beta V_f(x_j)\Big) + \beta V_f(x_i), \tag{6.60b}$$

applying (6.15a) such that $\ell(x, \bar{u}) \leq \ell(\bar{x}, \bar{u}) + \lambda_f \|w_{-1}\|$

$$\leq \sum_{j=N}^{N+i-1} \Big(\ell(\bar{x}_j, \bar{u}_j) + \lambda_f \|w_{j-1}\|_2^2 + \beta V_f(x_{j+1}) - \beta V_f(x_j)\Big) + \beta V_f(x_i), \tag{6.60c}$$

using (6.16b)

$$\begin{aligned} \leq \sum_{j=i}^{N+i-1} \Big(& V_f(x_j) - V_f(x_{j+1}) + \lambda_f \|w_j\|_2^2 - \lambda_f \|w_{j-1}\|_2^2 + \lambda_f \|w_{j-1}\|_2^2 \\ &+ \beta V_f(x_{j+1}) - \beta V_f(x_j)\Big) + \beta V_f(x_i) \end{aligned} \tag{6.60d}$$

$$= \sum_{j=i}^{N+i-1} \Big((1-\beta)\big(V_f(x_j) - V_f(x_{j+1})\big) + \lambda_f \|w_j\|_2^2\Big) + \beta V_f(x_i), \tag{6.60e}$$

owing to (6.59b)

$$= \sum_{j=i}^{N+i-1} \Big((1-\beta)\big((1-\gamma)V_f(\bar{x}_j) + \lambda_f\|w_{j-1}\|_2^2 - \lambda_f\|w_j\|_2^2\big) + \lambda_f\|w_j\|_2^2\Big) + \beta V_f(x_i), \tag{6.60f}$$

ordering

$$= (1-\beta)(1-\gamma)\sum_{j=i}^{N+i-1} V_f(\bar{x}_j) + \beta V_f(x_i) + \sum_{j=i}^{N+i-1}\Big((1-\beta)\lambda_f\|w_{j-1}\|_2^2 + \beta\lambda_f\|w_j\|_2^2\Big), \tag{6.60g}$$

using (6.59a)

$$\begin{aligned} &\leq (1-\beta)(1-\gamma)\sum_{j=0}^{N-1}\gamma^j V_f(\bar{x}_i) + \beta\big(V_f(\bar{x}_i) + \lambda_f\|w_{i-1}\|_2^2\big) \\ &\quad + \sum_{j=i}^{N+i-1}\Big((1-\beta)\lambda_f\|w_{j-1}\|_2^2 + \beta\lambda_f\|w_j\|_2^2\Big), \end{aligned} \tag{6.60h}$$

and knowing that $\sum_{j=m}^{n} z^j = \frac{z^m - z^{n+1}}{1-z}$, for $m \in \mathbb{N}_{[0,n]}$ and $|z| < 1$,

$$\begin{aligned} &\leq \left(\beta - (1-\gamma)(\beta-1)\frac{1-\gamma^N}{1-\gamma}\right)V_f(\bar{x}_i) + \beta\lambda_f\|w_{i-1}\|_2^2 \\ &\quad + \sum_{j=i}^{N+i-1}\Big((1-\beta)\lambda_f\|w_{j-1}\|_2^2 + \beta\lambda_f\|w_j\|_2^2\Big), \end{aligned} \tag{6.60i}$$

from Lemma 6.17

$$\leq \big(\beta - (\beta-1)\big(1-\gamma^N\big)\big)V_f(\bar{x}_i) + (\beta+N)\bar{w}_{\lambda_f}, \tag{6.60j}$$

and considering that $\bar{x}_i \in \mathcal{X}_f(d_1) \implies x_0 \in \mathcal{X}_f(d_1)$, completes the proof. □

6.C Proof of Lemma 6.15

First, let us define the following lemmas.

Lemma 6.19 (Sum of $\ell(\cdot,\cdot)$ in open-loop)**.** *Suppose* $x_j \in \Gamma_N^\beta$, $u_j \in \mathcal{U}$, $x_{j+1} = f(x_j, 0) + w_j$,

$$\sum_{j=i}^{N+i-1} \ell(x_j, 0) \leq \frac{\sigma^N - 1}{\sigma - 1}\ell(x_i, 0) + \frac{\bar{w}_{\eta_\ell}}{\sigma - 1}\left(\frac{\sigma^N - \sigma}{\sigma - 1} - (N-1)\right). \tag{6.61}$$

Proof. Knowing that $\ell(x_{i+1}, 0) \leq \sigma\ell(x_i, 0) + \eta_\ell \|w_i\|_2^2$,

$$\sum_{j=i}^{N+i-1} \ell(x_j, 0) = \ell(x_i, 0) + \ell(x_{i+1}, 0) + \dots \tag{6.62a}$$

$$\leq \ell(x_i, 0) + \sigma\ell(x_i, 0) + \eta_\ell \|w_i\|_2^2 + \dots \tag{6.62b}$$

grouping the finite sequences

$$\leq \sum_{j=0}^{N-1} \sigma^i \ell(x_i, 0) + \bar{w}_{\eta_\ell} \sum_{j=1}^{N-1} \frac{\sigma^j - 1}{\sigma - 1}, \tag{6.62c}$$

and solving the finites series completes the proof. □

Lemma 6.20 ($\ell(\cdot,\cdot)$ in open-loop)**.** *Suppose* $x_j \in \Gamma_N^\beta$, $u_j \in \mathcal{U}$, $x_{j+1} = f(x_j, 0) + w_j$,

$$\ell(x_i, 0) \leq \sigma^i \ell(x_0, u_0) + \bar{w}_{\eta_\ell} \frac{\sigma^i - 1}{\sigma - 1}. \tag{6.63}$$

Proof. From Lemma 6.7, knowing that $\ell(x_{i+1}, 0) \leq \sigma\ell(x_i, 0) + \eta_\ell \|w_i\|_2^2$,

$$\ell(x_i, 0) \leq \sigma\ell(x_{i-1}, 0) + \bar{w}_{\eta_\ell}, \tag{6.64a}$$

$$\leq \sigma\big(\sigma\ell(x_{i-2}, 0) + \bar{w}_{\eta_\ell}\big) + \bar{w}_{\eta_\ell},$$

$$\leq \quad \vdots$$

$$\leq \sigma^i \ell(x_0, u_0) + \bar{w}_{\eta_\ell} \sum_{j=0}^{i-1} \sigma^j, \tag{6.64b}$$

and solving the finites series completes the proof. □

Lemma 6.21 ($V_f(\cdot)$ in open-loop)**.** *Suppose $x_j \in \mathcal{X}_f(d_1)$, $x_{j+1} = f(x_j, 0) + w_j$. Then,*

$$V_f(x_{N+i}) \leq \rho^N V_f(x_i) + \bar{w}_{\lambda_f} \frac{\rho^N - 1}{\rho - 1}. \tag{6.65}$$

Proof. Applying (6.28b)

$$\begin{aligned} V_f(x_{N+i}) &\leq \rho V_f(x_{N+i-1}) + \bar{w}_{\lambda_f}, && (6.66a)\\ &\leq \rho\big(\rho V_f(x_{N+i-2} + \bar{w}_{\lambda_f}) + \bar{w}_{\lambda_f}\big), \\ &\leq \quad \vdots \\ &\leq \rho^N V_f(x_i) + \bar{w}_{\lambda_f} \sum_{j=i}^{N+i-1} \rho^j, && (6.66b) \end{aligned}$$

and solving the finites series completes the proof. □

Now, let us consider $x_i \in \Gamma_N^\beta$, after $N + i$ steps for i consecutive packet losses the value function is

$$V_N^0(x_i) \leq \sum_{j=i}^{N+i-1} \ell(x_j, 0) + \beta V_f(x_{N+i}), \tag{6.67a}$$

applying Lemma 6.19 and Lemma 6.21

$$\leq \frac{\sigma^N - 1}{\sigma - 1}\ell(x_i, 0) + \frac{\bar{w}_{\eta_\ell}}{\sigma - 1}\left(\frac{\sigma^N - \sigma}{\sigma - 1} - (N-1)\right) + \beta\left(\rho^N V_f(x_i) + \bar{w}_{\lambda_f}\frac{\rho^N - 1}{\rho - 1}\right). \tag{6.67b}$$

The latter represents the value function at x_i in open-loop operation. Since our task is to approximate the growth of the cost value in Theorem 6.1 induced by ζ and the uncertainty starting from $x_0 \in \mu\Gamma_N^\beta$ up to x_i by applying Lemma 6.7 for the stage cost and terminal cost, then from Lemma 6.20 and Lemma 6.21 we have that

$$\begin{aligned} &\leq \frac{\sigma^N - 1}{\sigma - 1}\ell(x_0, u_0) + \bar{w}_{\eta_\ell}\frac{\sigma^i - 1}{\sigma - 1} + \frac{\bar{w}_{\eta_\ell}}{\sigma - 1}\left(\frac{\sigma^N - \sigma}{\sigma - 1} - (N-1)\right) \\ &\quad + \beta\left(\rho^{N+i} V_f(x_0) + \bar{w}_{\lambda_f}\frac{\rho^{N+i} - 1}{\rho - 1}\right), \end{aligned} \tag{6.67c}$$

reordering completes the proof. □

6.D Proof of Theorem 6.6

Rearranging (6.50) for i leads to

$$\sigma^i\left(1+\frac{b\rho^i}{a\sigma^i}\right)\leq\frac{c}{a} \tag{6.68a}$$

$$\ln\left(\sigma^i\right)+\ln\left(1+\frac{b\rho^i}{a\sigma^i}\right)\leq\ln\left(\frac{c}{a}\right), \tag{6.68b}$$

where

$$a=\frac{\sigma^N-1}{\sigma-1}\ell(x_0,u_0)+\frac{\bar{w}_{\eta_\ell}}{\sigma-1},\quad b=\beta\rho^N d_1+\beta\frac{\bar{w}_{\lambda_f}\rho^N}{\rho-1}, \tag{6.68c}$$

$$c=\beta\left(d_1-\frac{\bar{w}_{\lambda_f}}{\rho-1}\right)+Nd_2-\frac{\bar{w}_{\eta_\ell}}{\sigma-1}\left(\frac{\sigma^N-\sigma}{\sigma-1}-N\right), \tag{6.68d}$$

knowing that $\ln\left(\frac{x}{y}\right)\leq\ln\left(1+\frac{x}{y}\right)$, if $x>y$, let i^* denote the lower bound of i, *i.e.* $i^*\leq i$ for which (6.50) holds, then

$$\ln\left(\sigma^{i^*}\right)+\ln\left(\frac{b\rho^{i^*}}{a\sigma^{i^*}}\right)-\ln\left(\frac{c}{a}\right)\leq\ln\left(\sigma^i\right)+\ln\left(1+\frac{b\rho^i}{a\sigma^i}\right)-\ln\left(\frac{c}{a}\right)\leq 0 \tag{6.68e}$$

$$\ln\left(\sigma^{i^*}\right)+\ln\left(\frac{b\rho^{i^*}}{a\sigma^{i^*}}\right)-\ln\left(\frac{c}{a}\right)\leq 0, \tag{6.68f}$$

reordering we have that

$$i^*\ln(\rho)\leq\ln\left(\frac{c}{a}\right)-\ln\left(\frac{b}{a}\right), \tag{6.68g}$$

leads to

$$i^*\leq\frac{\ln\left(\frac{c}{b}\right)}{\ln\rho}, \tag{6.68h}$$

which completes the proof. □

Chapter 7

Conclusions and future work

Contents

As discussed in Chapter 2, there is significant research on resilient techniques based on MPC control, with some utilizing the classic formulation and others using modified versions. While some of these approaches perform well, they rely on several assumptions such as the applicability of the Separation Principle or global stability, which may be impossible to satisfy in practice. The contributions of the thesis are in the domains of Separation Principle for MPC-controlled systems, cybersecurity in NCS, stochastic constrained MPC systems, robust control, and Lyapunov-based stability analysis. Therefore, in the following concluding remarks, the main results addressing these challenges along with future work recommendations are described in detail.

7.1 Input-constrained output-feedback stochastic MPC

7.1.1 Reverse Separation Principle

The literature (Section 2.2.2) showed that, for LQG-controlled system under random packet losses, the Separation Principle *holds* for TCP-like protocol but *does not hold* for UDP-like protocol. However, in Chapter 4, we demonstrated that for input-constrained MPC-controlled system under the same filtering process, the Separation Principle *does not hold* for TCP-like but *holds* for UDP-like protocols.

This result depicted a kind of reverse Separation Principle, in which the UDP-MPC cost function enjoys monotonic decrease but the TCP-MPC cost function does not enjoy the same monotonicity. The consequence was that, in the TCP case there is no guarantee that the TCP-MPC cost function remains in the RoA, while such a guarantee exists for the UDP-MPC cost function.

The next contribution of Chapter 4 was the theoretical analysis that showed a trade-off between the estimation errors and prediction errors for both protocols (Section 4.4.2.2). This information asymmetry is due to the implicit optimal control effect over both communication protocols: if the control increases the estimation error in the UDP case, then, it reduces the prediction error in the TCP case by the same margin, and vice versa.

7.1.2 Effects of design parameters on the control law

Knowing that the implicit control law of the problem setting embeds the effect of the probability of successful packet delivery $\bar{\nu}$, Section 4.5 analysed the explicit PWA input-constrained control law of the problem showing the influence of $\bar{\nu}$, the scalar β that enlarges the RoA by penalizing the terminal cost, and the prediction horizon N. Numerical examples showed that increasing β shrinks the polyhedral partitions of the critical regions while preserving the convergence around the origin. In contrast, $\bar{\nu}$ has a bigger impact by shrinking ever more the partitions and making convergence of the MPC-UDP around the origin more difficult.

7.1.3 Directions for future work

- A primary direction for future research is to integrate formally state constraints into the problem formulation. Certainly, it is more challenging, but the analysis would benefit from the recursive feasibility guarantees. The first approach is to parametrize the control input in order to reduce the accumulated covariance in the stochastic value function (as remarked in Section 4.3.2), keeping the original problem setting and subsequently performing the stability and recursive feasibility analyses. The other approach would be to guarantee feasibility by softening the constraints using slack variables; under certain assumptions, this formulation can also ensure closed-loop stability subject to uncertainties [114].
- Since the KF used in the current formulation does not account for constraints, the use of Moving Horizon Estimation (MHE) technique–considered as the dual of MPC–is recommended. The MHE fully incorporates input and state constraints through a cost function defined over moving horizon of the most recent N measurements [13]. Therefore, the benefit of the combination of both, MPC and MHE, would probably give *stronger* stability guarantees and *improved* monotonicity conditions for both TCP-like and UDP-like protocols.
- The section on the effects of the design parameters would benefit from a theoretical analysis showing the conditions under which β, $\bar{v}$ and N explicitly guarantee stability and monotonicity on the derived PWA and PWQ functions. Also, the analysis of the explicit solution could be strengthened from the study of set-invariance methods [12].

7.2 Nominal stability of State-Feedback MPC under consecutive packet losses

As discussed in the introduction of Chapter 5, relaxing the problem formulation of Chapter 4 allowed us to focus the stability analysis from the controller's viewpoint. The simplified problem was now a *nominal* MPC-controlled discrete-time nonlinear system subject to input constraints and random packet losses in the C-A channel under UDP-like protocol, *i.e.* no ACK of packet dropouts to the controller. To mitigate the effect of packet losses, it was introduced a buffer mechanism. Thus, in the problem setting, it was studied the worst-case scenario of consecutive packet losses.

7.2.1 Stability without terminal constraints and the limitation of the buffer mechanism

Since, in general, the *global* CLF for the terminal cost is impossible to meet for constrained systems, the nominal MPC formulation considers a *local* CLF terminal cost. Thus, the combination of stability analysis of the value function and the use of technical results on MPC stability analysis without terminal constraints, *i.e.* the definition of the RoA based on the terminal cost penalization, prediction horizon, and the lower-bound of the stage cost and upper-bound of the terminal cost functions, helped to perform the posteriori analysis.

Moreover, in the worst-case scenario of consecutive packet losses, it was demonstrated that the use of a buffer can, at best, facilitates the transfer of the initial state to the terminal region associated with the CLF-based terminal cost. This resulted on two cases: (i) the state has reached a subset of the RoA, and (ii) the state has reached the terminal region.

7.2.2 Stability guarantees for nonlinear and linear systems

For both cases, the number of consecutive packet losses the nonlinear system can tolerate while maintaining stability are upper bounded by expressions dependent on system and controller parameters obtained numerically, *i.e.* closed-loop and open-loop parameters obtained by analysing the behaviour of the trajectories of the value functions along the closed-loop and open-loop operations (with and without packet losses).

The specialized analysis in the linear case tried to overcome the previous limitation. The upper-bound expressions on the consecutive packet dropouts before losing stability are subject to the stage-cost and terminal-cost open-loop parameters obtained from the system and controller parameters without analysing the cost value function numerically. The resulting bounds, however, confirm their inherent conservatism.

The numerical examples illustrated that, despite the conservatism of these bounds for both nonlinear and linear systems, they can be used to fine-tune the controller's design parameters, especially when the buffer alone is insufficient to guarantee stability under consecutive packet losses. The suggestion of the minimum buffer size is approximately equal to the sum of the prediction horizon and the obtained bounds, must be interpreted as a tuning guideline rather than a definitive rule, since further theoretical analysis is required.

7.2.3 Directions for future work

- One of the first future works will be the decoupling of the buffer size. Since its value depends on the prediction horizon, the decoupling proposition could consider the case when the buffer size exceeds than the prediction horizon. It will be interesting to analyse its implication on the stability analysis of the cost value function for both cases.
- The minimum buffer size approximation–which was revealed by the numerical results–needs further theoretical analysis. Future work will focus on formally characterizing the relationship between the buffer size, prediction horizon, and upper bounds on consecutive packet losses.
- One of the drawbacks of the upper bounds in the nonlinear case is their sensitivity to closed-loop and open-loop parameters, which are obtained by analysing the evolution of the value function along the trajectories. If the nonlinear system dynamics in closed-loop and open-loop do not exhibit exponential decay or growth, the parameters may be estimated inaccurately, leading to invalid upper bounds. Therefore, a rigorous theoretical sensitivity analysis needs to be investigated.
- As it was suggested in Chapter 4, the introduction of state constraints would benefit the analysis. The worst-case scenario of consecutive packet losses will require recursive feasibility guarantees that–in principle–will relate the aforementioned closed-loop and open-loop parameters, leading to (probably) more conservative results.
- The analysis did not require stochastic stability conditions, *i.e.* analysing the expectation of the value function. Therefore, it would be interesting to perform stochastic stability analysis framework, considering assumptions and definitions on stochastic Lyapunov-like functions studied in the literature [115].

7.3 Robust stability of State-Feedback MPC under consecutive packet losses

Extending the results of Chapter 5, this chapter employed the same input-constrained nominal MPC controller formulation but integrated the bounded additive disturbances in the problem setting. This represented a key advancement over Chapter 5, the analysis was challenging since the nominal MPC in the loop was aiming to regulate the nonlinear uncertain system. Nevertheless, the stability conditions without terminal constraints and the limitations of the buffer mechanism were still valid, but the presence of disturbances gave four different cases.

These cases were defined by two factors: (i) whether the initial state lies in the RoA or in the terminal region, and (ii) whether the plant operates purely in closed-loop mode or uses buffered control before packet losses occur.

Accordingly, the four cases considered were:

i) Closed-loop control with the initial state in the RoA.

ii) Closed-loop control with the initial state in the terminal region.

iii) Buffered control with the initial state in the RoA.

iv) Buffered control with the initial state in the terminal region.

This case-based classification extended the analysis of Chapter 5 by explicitly accounting for uncertainty-dependent trajectories.

7.3.1 Robust stability guarantees for nonlinear and linear systems

For each of these cases, we identified robust stability conditions under which the state remains within the RoA despite bounded disturbances and consecutive packet losses. Moreover, the upper bounds on the number of consecutive packet losses that can be tolerated while maintaining robust stability were derived.

In the nonlinear case, the evolution of the value function was analysed along both closed-loop and open-loop trajectories considering the disturbances in order to obtain the open-loop and closed-loop parameters. The simulation results confirmed that, if the bounded additive disturbance is sufficiently small, the proposed upper bounds on packet losses are respected, thereby validating the robustness of the theoretical analysis.

For the linear case, the results verified the approximated upper-bounds using the open-loop stage-cost and terminal-cost parameters, which are easy to compute but more conservative.

Similar to Chapter 5, despite their conservatism, the bounds can be used to fine-tune the controller design parameters, especially when the buffer mechanism alone is insufficient to guarantee stability under consecutive packet losses and disturbances. The suggested minimum buffer size for the robust case is even restrictive in the presence of the disturbances. This highlights a fundamental limitation of buffering strategies under uncertainty and constitutes a key insight beyond the nominal analysis of Chapter 5.

7.3.2 Directions for future research

- The integration of *multiplicative disturbances* constitutes a natural extension of the current framework and may reveal additional stability and performance trade-offs. In particular, such disturbances would allow the analysis to capture parametric uncertainties and gain variations while preserving the buffer mitigation strategy.
- The problem setting can incorporate the TCP-like protocol such that performance and stability conditions for both, TCP and UDP cases, can be compared. This extension would allow the assessment of the benefits and limitations of ACK-based communications schemes [60].
- The Bernoulli packet loss model can be replaced by the Markov packet loss model. Such extension would generalize both the problem formulation and the resulting analysis, providing a more realistic representation of network-induced effects commonly observed in practice. Some works discussed in Chapter 2, could be used as guidelines *e.g.* [42].
- In general, the introduction of FDI attacks will improve considerably the main results because DoS and FDI attacks working simultaneously can be very dangerous. Works on this topic in Chapter 2 may serve as useful guidelines.
- Finally, the current linear-system setting should be extended to fully characterize all four disturbance and packet-loss cases identified in the analysis. This would allow complete theoretical results, facilitating a clearer comparison between nominal and robust scenarios.

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