Distality to and from Combinatorics

Ho Wang Mervyn Tong

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To Sam, who still
speaks through his faith

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One question I often get asked is what it looks like to do a PhD in maths. My response usually includes reading papers, standing in front of a blackboard, and banging my head against a wall, but there is much behind the scenes that I neglect to mention, many people that have made this journey worth its while.

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I was once asked which chapter in my thesis I would name after my fiancée. My answer was Chapter 2, *~~Brenda~~ Preliminaries*, because she has been the background support for all of my work. I will address her shortly, but this analogy is true of many more people.

I must begin with **my parents**. Mum and dad, you saw that I was drawn to maths at a very young age, and you did everything you could to help me fulfil my passion and potential. I resisted being sent to the UK before university, but I ended up growing so much not just as a mathematician but as a person. You made many sacrifices for me and put my educational needs above your own; I thank you, sincerely, for always having my best interests at heart.

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Finally, I give thanks and praise to **the Lord my God**, the Maker of heaven and earth. Lord, You have woven intricate patterns into fabric of pure maths, and I have been blessed with the privilege of discovering and assimilating them. Thank You for creating us in Your image to comprehend such weird and wonderful matters, and for giving me understanding of just the right things to complete each of my projects. How true are these words that David speaks of You:

O Lord, our Lord,
how majestic is your name in all the earth!
You have set your glory above the heavens.
Out of the mouth of babies and infants,
you have established strength because of your foes,
to still the enemy and the avenger.

When I look at your heavens, the work of your fingers,
the moon and the stars, which you have set in place,
what is man that you are mindful of him,
and the son of man that you care for him?

Yet you have made him a little lower than the heavenly beings
and crowned him with glory and honor.
You have given him dominion over the works of your hands;
you have put all things under his feet,
all sheep and oxen,
and also the beasts of the field,
the birds of the heavens, and the fish of the sea,
whatever passes along the paths of the seas.

O Lord, our Lord,
how majestic is your name in all the earth!

Psalm 8 (ESV)

Abstract

In this thesis, we demonstrate the intimate connection between the model-theoretic notion of *distality* and concepts from *combinatorics*: developments in distality both lead *to* and come *from* those in combinatorics.

Chapter 3 demonstrates the *from* direction. We prove that expansions of Presburger arithmetic by a predicate $R \subseteq \mathbb{N}$ are distal when R satisfies certain arithmetic combinatorial properties. We do so by constructing distal decompositions (or strong honest definitions), a form of cell decomposition with desirable combinatorial properties.

Chapter 4 demonstrates the *to* direction. We prove that relations definable in a distal structure have better bounds for the Zarankiewicz problem, a classical problem in extremal combinatorics. In fact, we prove that these bounds are enjoyed by any relation satisfying an improved version of Szemerédi regularity lemma, a classical theorem in extremal combinatorics. Thus, motivated by distality, we discover an interaction between two areas of extremal combinatorics.

Chapter 5 demonstrates both the *to* and the *from* directions. We show that the developments of higher-arity distality and higher-arity (hypergraph) regularity lemmas inform one another. The centrepiece of the chapter is a homogeneous hypergraph regularity lemma that we derive for structures satisfying higher-arity distality. In the quest for this, we develop strong honest definitions for higher-arity distality, whose efficacy is supported by the regularity lemma.

Keywords. Distality, strong honest definitions, distal cell decompositions, o-minimality, expansions of Presburger arithmetic, sparse predicates, Zarankiewicz problem, regularity lemma, higher-arity distality, hypergraphs.

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Contents

Chapter 1

Introduction

In this chapter, we give an overview of the themes and main results of this thesis, as well as directions for future research.

This thesis explores the interplay between *distality* and *combinatorics*. Rather than just connecting these two words with the conjunction *and* to title the thesis, we thought that the prepositions *to* and *from* better conveyed our thesis (that is, our argument): that the symbiosis between distality and combinatorics is a mutualistic one. While the combinatorial applicability of distality (the *to* direction) is gaining increasing traction, the utility of combinatorial considerations for research in distality (the *from* direction) is, in our opinion, undervalued, and we hope to contribute to the development of both directions with this thesis.

Let us zoom out momentarily. Applying model-theoretic results to other fields of mathematics is, of course, not a new idea. Model theory (in the mainstream) is the study of first-order logical structures, and many objects of interest in other fields of mathematics are definable in first-order logical structures in a natural language. A great triumph of model theory is that logical properties of first-order structures are often intimately connected with properties of interest in other fields of mathematics. Notable examples include the logical property of *o-minimality*, which is widely considered a front-runner in Grothendieck's quest for a tame topology, and that of *stability*, which generalises coset-likeness in group theory and continuity in functional analysis.

These logical properties are known to model theorists as *dividing lines*, as they divide the 'universe' of first-order structures into those that have the property (or often its negation) — and are therefore *tame* — and those that do not have the property — and are therefore *wild*. *Distality* is one such dividing line (or, as some might argue, a special case of the dividing line *NIP*), with which we shall get very comfortable over the next few chapters. We shall see that distal structures have very good combinatorial properties, so much so that the authors of [9] postulate that 'distal structures provide the most general natural setting for investigating questions in "generalised incidence combinatorics"'.

This thesis strengthens the connection between distality and combinatorics, while also exploring dividing lines adjacent to distality and their interactions with combinatorics.

1.1 Main achievements

We now summarise the main achievements of this thesis.

In Chapter 3, *Distality from Combinatorics*, we recover distality from combinatorial data. A common theme in model-theoretic research is to take a tame structure M, a predicate $R \subseteq M$, and study whether the structure (M, R) is also tame. We apply this theme to the structure $M = (\mathbb{Z}, <, +)$, Presburger arithmetic. The structure $(\mathbb{Z}, <, +)$ is known to be distal, and we seek predicates $R \subseteq \mathbb{Z}$ such that $(\mathbb{Z}, <, +, R)$ is still distal. We prove the following.

Theorem A (Theorem 3.4.8)**.** *Let $R \subseteq \mathbb{N}$ be congruence-periodic and sparse. Then $(\mathbb{Z}, <, +, R)$ is distal.*

Examples of such $R \subseteq \mathbb{N}$ include $\{d^n : n \in \mathbb{N}\}$ for any $d \in \mathbb{N}_{\geq 2}$, the set of Fibonacci numbers, and $\{n! : n \in \mathbb{N}\}$. The definitions of *congruence-periodic* and *sparse*, for which we refer the reader to Chapter 3, are arithmetic combinatorial in nature. In particular, although the original formulation of sparsity is more agreeable to a logician, in Theorem 3.2.19 we prove that sparsity is equivalent to *regularity* (where previously only one implication was known), a notion defined in terms of recurrence relations.

We prove Theorem A by constructing *strong honest definitions* or *distal decompositions*. Distal decompositions are cell decompositions with desirable combinatorial properties, and a structure is distal if and only if every formula $\phi(x; y)$ has a distal decomposition. Proofs of distality by constructing distal decompositions are rare due to their technical nature, but are combinatorially superior: by analysing the distal decompositions constructed, one can obtain combinatorial information about definable sets in the structure.

In Chapter 4, *Distality to Combinatorics*, we recover combinatorial interactions from a distality assumption. The chapter has two combinatorial protagonists, the first of which is the (k-graph) *Zarankiewicz problem*, a classical problem in extremal graph theory which asks for the maximum number of edges a k-partite k-graph can have if it is $K_{u,\dots,u}$-free, that is, it omits the complete hypergraph $K_{u,\dots,u}$. The best known upper bound (the *Zarankiewicz bound*) in general, proved in [17], is $O_u(n^{k-1/u^{k-1}})$, where n is the size of each vertex class. It is proved in [52] that when the hypergraph is defined by a semialgebraic relation $E(x_1, \dots, x_k)$, the Zarankiewicz bound can be improved to $O_{u,E}(F^0_{\bar{d}}(n_1, \dots, n_k))$, where $n_1, \dots, n_k$ are the sizes of the vertex classes, $\bar{d} = (|x_1|, \dots, |x_k|)$, and the function $F^0_{\bar{d}}$ is defined in Definition 4.4.2. This is asymptotically smaller than the previous bound.

Generalising from semialgebraic relations to those definable in a distal structure, it is proved in [9] that when $k = 2$ and the graph relation $E(x_1, x_2)$ is definable in a distal structure, the Zarankiewicz bound can be similarly improved. Our goal was to extend this to $k \geq 3$ — to show that k-partite k-graphs definable in a distal structure have similarly improved Zarankiewicz bounds — but we found something better. Here, we encounter our second combinatorial protagonist: the (k-graph) *Szemerédi regularity lemma*, a classical result in extremal graph theory that allows every k-graph to be partitioned into a bounded number of uniform pieces. It is known that k-graphs definable in a distal structure satisfy an improved version of this theorem called the *distal regularity lemma*, in which the sizes of the partitions are polynomial in the reciprocal of the error, and the uniform pieces are homogeneous (see Definition 4.2.3). Collecting the degrees of the polynomials into a *strong distal regularity tuple* $\bar{c}$, we state the main result of Chapter 4. We refer the reader to Definitions 4.2.3 and 4.4.2 for the relevant definitions.

Theorem B (Theorem 4.4.5)**.** *Let $E(x_1, ..., x_k)$ be a relation on a set M, with strong distal regularity tuple $\bar{c} = (c_1, ..., c_k) \in \mathbb{R}^k_{\geq 1}$ and coefficient λ. For all finite $P_i \subseteq M^{x_i}$ with $n_i := |P_i|$, if $E(P_1, ..., P_k)$ is $K_{u,...,u}$-free, then for all $\varepsilon > 0$,*

$$|E(P_1, ..., P_k)| \ll_{u,\bar{c},\lambda,\varepsilon} F^{\varepsilon}_{\bar{c}}(n_1, ..., n_k).$$

The slogan for Theorem B is that k-graphs satisfying the distal regularity lemma have better Zarankiewicz bounds. That is, motivated by distality considerations, we find an interaction between our two combinatorial protagonists.

Given a relation/hypergraph E that satisfies the distal regularity lemma, Theorem B provides a recipe for computing explicit (that is, numerical) Zarankiewicz bounds for E, namely, by computing strong distal regularity tuples for E. This is not the only means to compute explicit Zarankiewicz bounds. Indeed, to conclude Chapter 4, we use a different approach to compute such bounds for certain 3-graphs definable in an o-minimal structure, and we state an abridged version of this result below. (Note that o-minimal structures are distal, so such graphs satisfy the distal regularity lemma).

Theorem C (Theorem 4.6.2, abridged)**.** *Let M be an o-minimal L-structure expanding an ordered field. Let $\phi(x_1, x_2, x_3; y) \in L$ with $|x_1| = |x_2| = |x_3| = 2$. For all $b \in M^y$ and finite $P_i \subseteq M^{x_i}$ with $|P_1| = |P_2| = |P_3| =: n$, if $\phi(P_1, P_2, P_3; b)$ is $K_{u,u,u}$-free, then*

$$|\phi(P_1, P_2, P_3; b)| \ll_{\phi,u} n^{2.44}.$$

This extends the special case where M is the real ordered field, where a bound of $O_{\phi,u}(n^{2.4})$ is known [52, Theorem 1.6].

In the eponymous Chapter 5, *Distality to and from Combinatorics*, we develop the theories of higher-arity distality and hypergraph regularity by using each to inform the other. We mentioned above that k-graphs definable in a distal structure satisfy a homogeneous version of the k-graph Szemerédi regularity lemma. Combinatorial intuition tells us that the most general context in which k-graphs satisfy such a homogeneous regularity lemma should not be distality, which can be seen as a binary notion, but rather a k-ary generalisation such as the notion of (strong) k-distality in the literature.

We prove that this is indeed the case, by developing the theory of k-strong honest definitions for a formula $\phi(x_1, ..., x_k; y)$. Recall that a theory is distal if and only if every formula $\phi(x; y)$ has a strong honest definition. The appropriate generalisation of strong honest definitions to higher arity is not clear from the literature. Motivated by our quest for homogeneous regularity lemmas, we crystallise the definition of k-strong honest definitions and prove the following.

Theorem D (Theorem 5.4.12)**.** *Let T be an NIP L-theory and let $k \in \mathbb{N}^+$. Then T is strongly k-distal if and only if every $\phi(x_1, ..., x_k; y) \in L$ has a k-strong honest definition.*

The main result of Chapter 5, in abridged form, is the following hypergraph regularity lemma. For the full statement, we refer the reader to Theorem 5.5.9.

Theorem E (Theorem 5.1.8)**.** *Let $k \geq 2$. Let M be an NIP L-structure, and let $\phi(x_1, ..., x_{k-1}; x_k) \in L(M)$ have a $(k-1)$-strong honest definition, with $|x_1| = \cdots = |x_k| =: d$. Then, for all $\delta > 0$, there is a natural number $K \leq poly_\phi(\delta^{-1})$ and a formula $\theta(x_1, ..., x_{k-1}, z) \in L$ such that the following holds.*

Let $V \subseteq M^d$ be M-definable, and let $\mu(x_1)$ be a global measure, generically stable over M. Then there is a partition $V^{k-1} = V_1 \sqcup \cdots \sqcup V_K$, where each $V_i = \theta(x_1, ..., x_{k-1}, c)$ for some $c \in M^z$, inducing the partition

$$\mathcal{Q} := \left\{ \left\{ v = (v_1, ..., v_k) \in V^k : v_{\neq i} \in V_{j_i} \text{ for all } i \in [k] \right\} : j_1, ..., j_k \in [K] \right\}$$

of V^k, such that $\sum_{Q \in \mathcal{Q} \text{ not } \phi\text{-homogeneous}} \mu^{(k)}(Q) \leq \delta \mu(V)^k$.

1.2 Directions for future research

We now discuss some open problems that build on and extend the work of this thesis. Some of these are essentially converses to the main results discussed above where, having found a connection from distality *to* combinatorics, say, we now seek the corresponding connection to distality *from* combinatorics.

In Theorem A, we prove that $(\mathbb{Z}, <, +, R)$ is distal when $R \subseteq \mathbb{N}$ satisfies certain arithmetic combinatorial properties. It is natural to seek a converse: to recover combinatorial properties of $R \subseteq \mathbb{N}$ from the distality of $(\mathbb{Z}, <, +, R)$. More generally, we pose the following problem.

Problem A (Problem 3.1.3)**.** *Characterise the class of predicates $R \subseteq \mathbb{N}$ such that $(\mathbb{Z}, <, +, R)$ is distal.*

It is our hope that this might provide an answer to the question, posed in [57, Question 11.16], of whether a non-distal NIP expansion of $(\mathbb{N}, <)$ exists.

In Theorem B, we derive improved Zarankiewicz bounds for relations satisfying the distal regularity lemma. There are numerous ways to extend and strengthen this result. One natural goal is to make the bounds in this theorem explicit, which is tantamount to the following problem.

Problem B (Problem 4.1.3)**.** *Compute (strong) distal regularity tuples for relations satisfying the distal regularity lemma, such as those definable in a distal structure.*

Theorem B acts as an advert for model theory to the combinatorial world since, motivated by distality, we found a variant of the Szemerédi regularity lemma that gives rise to improved Zarankiewicz bounds. We therefore pose the following problem, in hope and expectation that model-theoretic considerations will continue to contribute to a solution.

Problem C (Problem 4.1.4)**.** *Which other variants of the Szemerédi regularity lemma give rise to improved Zarankiewicz bounds?*

In Theorem C, we derive explicit Zarankiewicz bounds for certain 3-graphs definable in an o-minimal structure. Our result is restricted to 3-graphs because our proof makes use of machinery that currently only exists in the binary setting (namely, a cutting lemma). Thus, we pose the following problem, not only for its own sake, but also because finding a solution would likely require the development of higher-arity versions of such machinery.

Problem D (Problem 4.6.7)**.** *Fix an o-minimal expansion M of an ordered field. Find explicit Zarankiewicz bounds for relations $\phi(x_1, ..., x_k; y)$ definable in M, where $k \geq 2$.*

We now turn to questions concerning (strong) k-distality. Theorems D and E, and much of Chapter 5, require a global NIP assumption. Since there are (strongly) k-distal structures that are not NIP, we pose the following problem in hopes of strengthening our results.

Problem E (Problems 5.4.15, 5.5.10)**.** *Can the NIP assumption be removed from Theorems D and E (and other results in Chapter 5)?*

The precise relationship between k-distality and strong k-distality is not known. Unsurprisingly, the latter implies the former, but the converse is open. Much of Chapter 5, especially Theorem D, applies to strongly k-distal structures. We wonder if they also apply to k-distal structures.

Problem F (Problem 5.4.16)**.** *Can the assumption of strong k-distality be replaced by k-distality in Theorem D? If not, do k-distal theories admit a (necessarily weaker) version of k-strong honest definitions?*

The applicability of Theorem E is currently limited by the fact that, for $k \geq 2$, there are very few examples of NIP strongly k-distal structures that are not strongly $(k-1)$-distal. In fact, there are none in the literature for $k \geq 3$, and in all known examples of NIP strongly 2-distal structures that are not distal, there are no non-degenerate ternary relations to which we can apply Theorem E. This represents a big gap in the literature which we hope to fill.

Problem G (Problems 5.3.11, 5.3.12)**.** *Find interesting examples of NIP strongly k-distal structures.*

Much of Chapters 4 and 5 concern regularity lemmas that hold in NIP strongly k-distal structures (or simply distal structures when $k = 1$), which are results in the direction from distality *to* combinatorics. The converse is very interesting indeed. One weakness of distality (and its higher-arity counterparts), when compared to NIP and stability, is that there is no fruitful *local* definition of distality, that is, a notion of a distal formula. One main achievement of this thesis is that we have developed the theory of NIP strongly k-distal regularity lemmas, which can be turned into a local definition: we can investigate the properties of a formula that satisfies the NIP strongly k-distal regularity lemma.

In particular, we can ask how much k-distality can be recovered from such a formula. For instance, we pose the following problem.

Problem H (Problem 5.5.14)**.** *Let $\phi(x_0, ..., x_k)$ be a relation on a set M that satisfies the NIP strongly k-distal regularity lemma. Must (M, ϕ) admit an expansion that is NIP strongly k-distal? What if we assume (M, ϕ) is NIP?*

Note that, by Theorem 4.5.1, the answer to the first part of the question is negative when $k = 1$ and (M, ϕ) is not assumed to be NIP, but we are not aware of the answer in other cases.

Instead of recovering the full strength of NIP strong k-distality from the regularity lemma, we can also ask which properties of NIP strongly k-distal structures are already implied by the satisfaction of the regularity lemma. This is the spirit of Theorem B, where we show that satisfying the distal regularity lemma is sufficient for improved Zarankiewicz bounds. We seek similar results.

Problem I (Problem 5.5.15)**.** *Let $\phi(x_0, ..., x_k)$ be a relation on a set M that satisfies the NIP strongly k-distal regularity lemma. Investigate the (combinatorial) properties of ϕ.*

Much of the analysis of Chapter 5 depends on the k-strong honest definitions that we develop for formulas $\phi(x_1, ..., x_k; y)$. In particular, recall Theorem D, where we show that an NIP theory is strongly k-distal if and only if every formula $\phi(x_1, ..., x_k; y)$ has a k-strong honest definition. For reasons to be explained in Sections 5.6 and 5.7, one of which will be alluded to below, we believe it is more natural to define (dual) k-strong honest definitions for formulas $\phi(x; y_1, ..., y_k)$ instead. Unfortunately, we are unable to prove an analogue of Theorem D for dual k-strong honest definitions; we state this as a conjecture.

Problem J (Conjecture 5.6.2)**.** *Decide the following conjecture.*

Let T be an NIP L-theory and let $k \in \mathbb{N}^+$. If T is strongly k-distal, then every $\phi(x; y_1, ..., y_k) \in L$ has a dual k-strong honest definition.

Note that Proposition 5.6.5 establishes a partial converse to this conjecture, where we slightly strengthen the notion of dual k-strong honest definitions.

One comparative strength of dual k-strong honest definitions is that they give rise to a form of cell decompositions with good geometric properties, analogous to distal cell decompositions. Unfortunately, as Problem J is undecided, we do not know if these cell decompositions always exist in NIP strongly k-distal theories.

Problem K (Problem 5.7.5)**.** *Find an analogue of distal cell decompositions for (NIP) strongly k-distal theories.*

Chapter 2

Preliminaries

In this chapter, we lay out some conventions used throughout the thesis and review some key definitions and results from the literature.

This thesis is divided into five *chapters* (numbered 1, 2, 3, 4, 5). Each chapter is divided into *sections* (numbered 1.1, say), and some sections are divided into *subsections* (numbered 1.1.1, say).

2.1 Notation and basic definitions

In this thesis, all logical structures are first-order.

Unless otherwise stated, arguments in a formula are *tuples* of variables. If M is a structure and x is a tuple of variables, then we write $M^x := M^{|x|}$.

Let M be an L-structure and $\phi(x_1, ..., x_k)$ be an L-formula. For $A_i \subseteq M^{x_i}$, write $\phi(A_1, ..., A_k)$ for the set $\{(a_1, ..., a_k) \in A_1 \times \cdots \times A_k : M \models \phi(a_1, ..., a_k)\}$, and for $b_k \in M^{x_k}$, write $\phi(A_1, ..., A_{k-1}, b_k)$ for the set $\{(a_1, ..., a_{k-1}) \in A_1 \times \cdots \times A_{k-1} : M \models \phi(a_1, ..., a_{k-1}, b_k)\}$.

We sometimes partition the variables in a formula using a semicolon rather than a comma to indicate contextual distinction between the variables.

We often conflate a formula with the set it defines.

2.1.1 Saturation

Given a cardinal κ, a structure M is *κ-saturated* if, for all $A \subseteq M$ with $|A| < \kappa$, every type over A is satisfiable in M. We will often work in a *sufficiently saturated* structure $\mathbb{M}$: this is a structure that is κ-saturated for some κ sufficiently large for our purposes. In this case, a subset $A \subseteq \mathbb{M}$ is *small* if $|A| < \kappa$.

2.1.2 Indexing

For $k \in \mathbb{N}^+$, $[k] := \{1, ..., k\}$.

Let $c = (c_1, ..., c_k)$ be a k-tuple. For $I \subseteq [k]$ enumerated in increasing order by $i_1, ..., i_l$, let c_I denote the l-tuple $(c_{i_1}, ..., c_{i_l})$. For $i \in [k]$, $c_{\neq i} := c_{[k]\setminus\{i\}}$.

Let X be a set, and let $k \in \mathbb{N}$. Write $\binom{X}{k} := \{A \subseteq X : |A| = k\}$.

If y is an n-tuple with entries in a set Y (that is, $y \in Y^n$), we sometimes simply write $y \in Y$, but $X \subseteq Y$ always means $X \subseteq Y^1$.

2.1.3 Covers and partitions

Given a set X, a collection $(X_1, ..., X_k)$ of subsets of X is said to *cover* X if $X = X_1 \cup \cdots \cup X_k$. If, additionally, $X_1, ..., X_k$ are pairwise disjoint, we say that they *partition* X; we will often write this partition as $X = X_1 \sqcup \cdots \sqcup X_k$. A partition $X_1 \sqcup \cdots \sqcup X_k$ of a finite set X is said to be an *equipartition* if, for all $i, j \in [k]$, $||X_i| - |X_j|| \leq 1$. A cover $X = X_1 \cup \cdots \cup X_k$ *refines* a cover $X = Y_1 \cup \cdots \cup Y_l$ if, for all $i \in [k]$, there is $j \in [l]$ such that $X_i \subseteq Y_j$.

2.1.4 Asymptotics

Let D, E be sets and $f(x, y), g(x, y) : D \times E \to \mathbb{R}_{\geq 0}$. Write $f(x, y) = O_x(g(x, y))$, $f(x, y) \ll_x g(x, y)$, or $g(x, y) \gg_x f(x, y)$ if there is $C = C(x) : D \to \mathbb{R}_{\geq 0}$ such that $f(x, y) \leq Cg(x, y)$ for all $x \in D$ and $y \in E$.

Let $h(y) : E \to \mathbb{R}_{\geq 0}$. Write $f(x, y) \leq \mathrm{poly}_x(h(y))$ if there is $C = C(x) : D \to \mathbb{R}_{\geq 0}$ such that $f(x, y) \leq Ch(y)^C$ for all $x \in D$ and $y \in E$.

Suppose $D = (0, r)$ for some $r \in \mathbb{R}^+ \cup \{\infty\}$. Write $f(x, y) = o_{x\to 0}(g(x, y))$ if there is $C = C(x) : D \to \mathbb{R}_{\geq 0}$ such that $\lim_{x\to 0} C(x) = 0$ and $f(x, y) \leq Cg(x, y)$ for all $x \in D$ and $y \in E$.

2.2 VC-dimension

Throughout this section, fix a set system $(X, \mathcal{S})$, that is, X is a set and $\mathcal{S} \subseteq \mathcal{P}(X)$. For $A \subseteq X$, write $\mathcal{S} \cap A := \{S \cap A : S \in \mathcal{S}\}$, and say that $\mathcal{S}$ *shatters* A if $\mathcal{S} \cap A = \mathcal{P}(A)$.

Definition 2.2.1. The *shatter function* of $\mathcal{S}$ is the function $\pi_{\mathcal{S}} : \mathbb{N} \to \mathbb{N}$ such that $\pi_{\mathcal{S}}(n) = \max\{|\mathcal{S} \cap A| : A \subseteq X, |A| = n\}$.

For all $n \in \mathbb{N}$, $\pi_{\mathcal{S}}(n) \leq 2^n$, with equality if and only if there is $A \subseteq X$ of size n such that $\mathcal{S}$ shatters A.

Definition 2.2.2. The *VC-dimension* of $\mathcal{S}$ is

$$\mathrm{VC}(\mathcal{S}) := \max\{|A| : A \subseteq X, \mathcal{S} \text{ shatters } A\} = \max\{n \in \mathbb{N} : \pi_{\mathcal{S}}(n) = 2^n\}$$

if this maximum exists, and ∞ otherwise.

The following theorem, often known as the Sauer–Shelah Lemma, presents a striking dichotomy. A proof can be found in [42].

Theorem 2.2.3. *If* $\mathrm{VC}(\mathcal{S}) \leq d$*, then* $\pi_{\mathcal{S}}(n) \leq \sum_{i=0}^{d} \binom{n}{i}$*. In particular, either* $\pi_{\mathcal{S}}(n) = 2^n$ *for all* $n \in \mathbb{N}$*, or* $\pi_{\mathcal{S}}$ *is bounded by a polynomial, that is, there is* $d \in \mathbb{N}$ *such that* $\pi_{\mathcal{S}}(n) = O(n^d)$ *for all* $n \in \mathbb{N}$*.*

Theorem 2.2.3 says that $\mathrm{VC}(\mathcal{S}) = \infty$ if and only if $\pi_{\mathcal{S}}(n) = 2^n$ for all $n \in \mathbb{N}$, and $\mathrm{VC}(\mathcal{S}) < \infty$ if and only if $\pi_{\mathcal{S}}$ is bounded by a polynomial.

The *dual* of $(X, \mathcal{S})$ is the set system $(\mathcal{S}, \mathcal{S}^*)$, where

$$\mathcal{S}^* := \{\{S \in \mathcal{S} : x \in S\} : x \in X\}.$$

Define the *dual shatter function* of $\mathcal{S}$ to be $\pi^*_{\mathcal{S}} := \pi_{\mathcal{S}^*}$, and the *VC-codimension* or *dual VC-dimension* of $\mathcal{S}$ to be $\mathrm{VC}^*(\mathcal{S}) := \mathrm{VC}(\mathcal{S}^*)$.

Let us give an alternative, equivalent account of these dual notions. Given a subset $S \subseteq X$, write $S^1 := S$ and $S^0 := X \setminus S$.

Definition 2.2.4. Let $\emptyset \neq \mathcal{S}_0 \subseteq \mathcal{S}$. Given $\varepsilon \in \{0,1\}^{\mathcal{S}_0}$ (that is, ε is a function $\mathcal{S}_0 \to \{0,1\}$), let $A_\varepsilon := \bigcap_{S \in \mathcal{S}_0} S^{\varepsilon(S)}$. If $A_\varepsilon \neq \emptyset$, say that A_ε is a *Boolean atom* of $\mathcal{S}_0$. Write $\mathrm{BA}(\mathcal{S}_0)$ for the set of Boolean atoms of $\mathcal{S}_0$.

It is clear that $\mathrm{BA}(\mathcal{S}_0)$ is a cover of X. Given distinct $\varepsilon, \varepsilon' \in \{0,1\}^{\mathcal{S}_0}$, we have that $A_\varepsilon \cap A'_\varepsilon = \emptyset$. Thus, $\mathrm{BA}(\mathcal{S}_0)$ is a partition of X, and

$$|\mathrm{BA}(\mathcal{S}_0)| = |\{\varepsilon \in \{0,1\}^{\mathcal{S}_0} : A_\varepsilon \neq \emptyset\}|.$$

Lemma 2.2.5. *For all $n \in \mathbb{N}$, $\pi^*_{\mathcal{S}}(n) = \max\{|BA(\mathcal{S}_0)| : \mathcal{S}_0 \subseteq \mathcal{S}, |\mathcal{S}_0| = n\}$.*

Proof. For finite $\mathcal{S}_0 \subseteq \mathcal{S}$ and $\varepsilon \in \{0,1\}^{\mathcal{S}_0}$,

$$A_\varepsilon \neq \emptyset \Leftrightarrow \exists x \in X \bigwedge_{S \in \mathcal{S}_0} x \in S^{\varepsilon(S)} \Leftrightarrow \{S \in \mathcal{S}_0 : \varepsilon(S) = 1\} \in \mathcal{S}^* \cap \mathcal{S}_0.$$

Thus, for finite $\mathcal{S}_0 \subseteq \mathcal{S}$, there is a bijection between $\mathrm{BA}(\mathcal{S}_0)$ and $\mathcal{S}^* \cap \mathcal{S}_0$. □

Definition 2.2.6. Say that the set systems $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ are *isomorphic*, written $(X, \mathcal{S}) \cong (Y, \mathcal{T})$ or $\mathcal{S} \cong \mathcal{T}$, if there is a bijection $f : X \to Y$ such that $\mathcal{T} = \{\{f(x) : x \in S\} : S \in \mathcal{S}\}$.

It is clear that isomorphic set systems have the same (dual) shatter function and thus (dual) VC-dimension.

Consider the double dual $(\mathcal{S}^*, \mathcal{S}^{**})$ of the set system $(X, \mathcal{S})$. The map $X \to \mathcal{S}^*, x \mapsto \{S \in \mathcal{S} : x \in S\}$ would be an isomorphism of set systems if it were a bijection. However, it fails to be a bijection when there are distinct $x, x' \in X$ such that for all $S \in \mathcal{S}$, $x \in S$ if and only if $x' \in S$.

Definition 2.2.7. Define an equivalence relation $\sim$ on X by declaring $x \sim x'$ if, for all $S \in \mathcal{S}$, $x \in S$ if and only if $x' \in S$. Define the *skeleton* of $(X, \mathcal{S})$ to be the set system $(X/\sim, \mathcal{S}/\sim)$, where $\mathcal{S}/\sim := \{\{[x] : x \in S\} : S \in \mathcal{S}\}$.

Now indeed we have $(X/\sim, \mathcal{S}/\sim) \cong (\mathcal{S}^*, \mathcal{S}^{**})$, witnessed by the bijection $X/\sim \to \mathcal{S}^*, [x] \mapsto \{S \in \mathcal{S} : x \in S\}$.

Lemma 2.2.8. *We have* $\pi_{\mathcal{S}} = \pi_{\mathcal{S}/\sim} = \pi_{\mathcal{S}^{**}}$ *and* $\pi^*_{\mathcal{S}} = \pi^*_{\mathcal{S}/\sim} = \pi^*_{\mathcal{S}^{**}}$.

Proof. Since $\mathcal{S}/\sim \cong \mathcal{S}^{**}$, it suffices to prove that $\pi_{\mathcal{S}} = \pi_{\mathcal{S}/\sim}$ and $\pi^*_{\mathcal{S}} = \pi^*_{\mathcal{S}/\sim}$. The latter holds since $\mathcal{S}^* \cong (\mathcal{S}/\sim)^*$, witnessed by the bijection $\mathcal{S} \to \mathcal{S}/\sim$, $S \mapsto \{[x] : x \in S\}$. For the former, observe that for all finite $A \subseteq X$, we have $|\mathcal{S} \cap A| = |\mathcal{S} \cap A_0|$ for any set $A_0 \subseteq A$ of representatives for $A/\sim$. □

The following result is folklore.

Proposition 2.2.9. *If* $\mathrm{VC}(\mathcal{S}) \leq k$, *then* $\mathrm{VC}^*(\mathcal{S}) < 2^{k+1}$.

Proof. Suppose $\mathrm{VC}^*(\mathcal{S}) \geq 2^{k+1}$. Then, there is a subset of $\mathcal{S}$ of size 2^{k+1}, say $\{S_I : I \subseteq [k+1]\}$, shattered by $\mathcal{S}^*$. Thus, there are distinct $x_1, ..., x_{k+1} \in X$ such that, for all $I \subseteq [k+1]$ and $i \in [k+1]$, $S_I \in \{S \in \mathcal{S} : x_i \in S\}$ if and only if $i \in I$, that is, $x_i \in S_I$ if and only if $i \in I$. Thus, $\{x_1, ..., x_{k+1}\}$ is shattered by $\mathcal{S}$, and so $\mathrm{VC}(\mathcal{S}) \geq k + 1$. □

Corollary 2.2.10. $\mathrm{VC}(\mathcal{S}) < \infty$ *if and only if* $\mathrm{VC}^*(\mathcal{S}) < \infty$.

Proof. Combine Proposition 2.2.9 and Lemma 2.2.8. □

We finish this section by showing that taking the restriction of a set system does not increase the VC-(co)dimension.

Lemma 2.2.11. *Let* $Y \subseteq X$ *and consider the set system* $(Y, \mathcal{S} \cap Y)$. *We have that* $\mathrm{VC}(\mathcal{S} \cap Y) \leq \mathrm{VC}(\mathcal{S})$ *and* $\mathrm{VC}^*(\mathcal{S} \cap Y) \leq \mathrm{VC}^*(\mathcal{S})$.

Proof. If $A \subseteq Y$ is shattered by $\mathcal{S} \cap Y$, then A is shattered by $\mathcal{S}$. Thus, we have $\mathrm{VC}(\mathcal{S} \cap Y) \leq \mathrm{VC}(\mathcal{S})$.

For the dual, suppose $S_1, ..., S_n \in \mathcal{S}$ are such that $\{S_1 \cap Y, ..., S_n \cap Y\}$ is a set of size n that is shattered by $(\mathcal{S} \cap Y)^*$. That is, there is $(y_I \in Y : I \subseteq [n])$ such that, for all $i \in [n]$, $y_I \in S_i$ if and only if $i \in I$. Then $\{S_1, ..., S_n\}$ is a set of size n that is shattered by $\mathcal{S}$. Thus, we have $\mathrm{VC}^*(\mathcal{S} \cap Y) \leq \mathrm{VC}^*(\mathcal{S})$. □

2.3 NIP

Throughout this section, fix a complete L-theory T, a sufficiently saturated model $\mathbb{M} \models T$, and $\phi(x; y) \in L$.

Definition 2.3.1. The *shatter function* π_ϕ (respectively *dual shatter function* π^*_ϕ, *VC-dimension* $\mathrm{VC}(\phi)$, *dual VC-dimension*, and *VC-codimension* $\mathrm{VC}^*(\phi)$) of ϕ (with respect to T) is defined to be that of the set system $(M^x, \{\phi(x; b) : b \in M^y\})$ for any $M \models T$.

It is straightforward to check that, since T is complete, the definition above is truly independent of $M \models T$.

The dual of the set system $(M^x, \{\phi(x; b) : b \in M^y\})$ is isomorphic to the skeleton of the set system $(M^y, \{\phi(a; y) : a \in M^x\})$. Thus, writing $\phi^*(y; x) := \phi(x; y)$, we have $\pi^*_\phi = \pi_{\phi^*}$ and $\mathrm{VC}^*(\phi) = \mathrm{VC}(\phi^*)$ by Lemma 2.2.8. As in the previous section, we can give an alternative account of the dual shatter function and dual VC-dimension using Boolean atoms. Write $\phi^1 := \phi$ and $\phi^0 := \neg\phi$.

Definition 2.3.2. Let $a \in \mathbb{M}^x$, and let $B \subseteq \mathbb{M}^y$ be small. The *ϕ-type of a over B* is

$$\mathrm{tp}_\phi(a/B) := \{\phi^\varepsilon(x; b) : b \in B, \varepsilon \in \{0, 1\}, \models \phi^\varepsilon(a; b)\},$$

and we let $S_\phi(B) := \{\mathrm{tp}_\phi(a/B) : a \in \mathbb{M}^x\}$ be the set of ϕ-types over B.

The set $S_\phi(B)$ is the collection of those sets $\{\phi^{\varepsilon(b)}(x; b) : b \in B\}$, where $\varepsilon : B \to \{0, 1\}$, that are consistent (equivalently, by compactness, finitely satisfiable in any model containing B). If B is finite, then $S_\phi(B) = \{\mathrm{tp}_\phi(a/B) : a \in M^x\}$ for any $M \models T$ such that $B \subseteq M^y$.

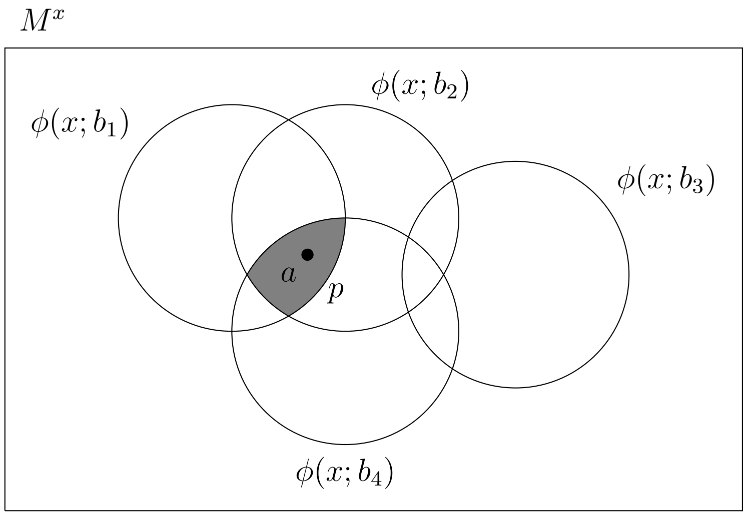

Figure 2.1: The ϕ-types over B

It is often helpful to conflate a ϕ-type over B, say $\{\phi^{\varepsilon(b)}(x;b) : b \in B\}$, with the set of its realisations, that is, $\bigcap_{b\in B} \phi^{\varepsilon(b)}(x;b)$. This way, a ϕ-type over B is precisely a Boolean atom of $\{\phi(x;b) : b \in B\}$. By Lemma 2.2.5, for all $n \in \mathbb{N}$,

$$\pi^*_\phi(n) = \max\{|S_\phi(B)| : B \subseteq \mathbb{M}^y, |B| = n\} = \max\{|S_\phi(B)| : B \subseteq M^y, |B| = n\}$$

for any $M \models T$. Furthermore, $S_\phi(B)$ forms a partition of $\mathbb{M}^x$ and if B is finite, then $S_\phi(B)$ forms a partition of M^x for any $M \models T$ such that $B \subseteq M^y$.

Fixing $M \models T$ and $B = \{b_1, b_2, b_3, b_4\} \subseteq M^y$, Figure 2.1 illustrates $S_\phi(B)$ as a partition of M^x. The sets $\{\phi(x;b) : b \in B\}$ form a Venn diagram in the universe M^x, whose non-empty regions are precisely the ϕ-types over B, forming the partition $S_\phi(B)$ of M^x. Taking the shaded ϕ-type p as an example, for any $a \in p$, we have that $p = \mathrm{tp}_\phi(a/B)$ is the ϕ-type of a over B, and every $a' \in p$ has the same ϕ-type over B: for all $b \in B$, $\models \phi(a;b)$ if and only if $\models \phi(a';b)$.

Definition 2.3.3. Say that ϕ has the *independence property (IP)* if $\mathrm{VC}(\phi) = \infty$, and *not the independence property (NIP)* otherwise.

In an egregious abuse of grammar consistent (equivalently, satisfiable) with the literature, we say that ϕ *is*, rather than *has*, IP or NIP. By Corollary 2.2.10 and Theorem 2.2.3, the conditions that ϕ is NIP, ϕ^* is NIP, π_ϕ is bounded by a polynomial, and π^*_ϕ is bounded by a polynomial are all equivalent.

Definition 2.3.4. Say that T is *NIP* if every $\psi(x;y) \in L$ is NIP, in which case also say that every model of T is *NIP*.

2.4 Types

We follow the treatment in [48]. Throughout this section, fix a complete L-theory T and a sufficiently saturated model $\mathbb{M} \models T$. If $B \subseteq \mathbb{M}$ and a, a' are tuples from $\mathbb{M}$ of the same length, write $a \equiv_B a'$ to mean $\mathrm{tp}(a/B) = \mathrm{tp}(a'/B)$. For a type $q \in S(B)$ and $B_0 \subseteq B$, write $q|B_0 := q \cap L(B_0)$.

Henceforth in this section, all parameter sets are small.

Definition 2.4.1. Let $B \subseteq \mathbb{M}$. A type $p(x) \in S(\mathbb{M})$ is *invariant over* B or B*-invariant* if, for all $\phi(x;y) \in L$ and $d, d' \in \mathbb{M}$ with $d \equiv_B d'$, $\phi(x;d) \in p$ if and only if $\phi(x;d') \in p$.

Definition 2.4.2. Let $p(x), q(y) \in S(\mathbb{M})$ with p invariant over B. Define the product $(p \otimes q)(x,y) \in S(\mathbb{M})$ as follows. For $\phi(x,y) \in L(B')$ with $B \subseteq B'$, $\phi(x,y) \in p \otimes q$ if and only if $\phi(x,d) \in p$ for some/all $d \in \mathbb{M}$ such that $d \models q|B'$.

The following fact is straightforward to prove, and can be found as Facts 2.19 and 2.20 of [48].

Fact 2.4.3. *Let* $p(x), q(y), r(z) \in S(\mathbb{M})$ *with* p, q *invariant over* B*.*

(i) The $\otimes$ *operation is associative:* $(p \otimes q) \otimes r = p \otimes (q \otimes r)$*.*

(ii) The type $p \otimes q$ *is invariant over* B*.*

In light of (i), we make the following definition.

Definition 2.4.4. Let $p(x) \in S(\mathbb{M})$ be invariant over B. For $n \in \mathbb{N}$, we define $p^{(n)}(x_1, ..., x_n) := p(x_1) \otimes \cdots \otimes p(x_n) \in S(\mathbb{M})$. Define $p^{(\omega)}(x_1, x_2, ...) := \bigcup_{n \in \mathbb{N}^+} p^{(n)}(x_1, ..., x_n)$.

We define a few properties of types.

Definition 2.4.5. Let $p(x) \in S(\mathbb{M})$ and $B \subseteq \mathbb{M}$.

(i) Say that p is *finitely satisfiable over* B if, for all $\phi(x) \in p$, there is $b \in B$ such that $\models \phi(b)$.

(ii) Say that p is *definable over* B if, for all $\phi(x; y) \in L$, there is $\psi(y) \in L(B)$ such that for all $d \in \mathbb{M}$, $\phi(x; d) \in p$ if and only if $\models \psi(d)$.

(iii) Say that p is *generically stable over* B if it is finitely satisfiable and definable over B.

The following fact is straightforward to prove, and can be found as Examples 2.16 and 2.17 of [48].

Fact 2.4.6. *Let $p(x) \in S(\mathbb{M})$ be finitely satisfiable or definable over B. Then p is invariant over B.*

We give some examples of generically stable types.

Example 2.4.7 (Realised types)**.** Let $B \subseteq \mathbb{M}$ and $a \in B$. It is straightforward to see that $\mathrm{tp}(a/\mathbb{M})$ is generically stable over B.

The following two examples are taken from [48, Example 2.31].

Example 2.4.8. Let $T = \mathrm{Th}(\mathbb{Q}, R_0, R_1, ...)$, where $\mathbb{Q} \models R_n(x, y)$ if and only if $x < y < x+n$. Let $p(x)$ be the unique global type extending $\{\neg R_n(x, a), \neg R_n(a, x) : a \in \mathbb{M}, n \in \mathbb{N}\}$. Then $p(x)$ is generically stable over a small model.

Example 2.4.9. Let T be the theory of the two-sorted structure (V, R), where R is a real closed field equipped with the ordered field structure, V is an infinite-dimensional R-vector space equipped with the group structure, and there is a binary function symbol $R \times V \to V$ for scalar multiplication. Let $p(x)$ be the unique global type in the sort V extending

$$\{x \notin W : W \text{ is an } \mathbb{M}\text{-definable proper vector subspace}\}.$$

Then $p(x)$ is generically stable over a small model.

2.5 Indiscernible sequences

We follow the treatment in [48]. Throughout this section, fix a complete L-theory T and $\mathbb{M} \models T$ sufficiently saturated. In this section, all parameter sets are small.

Henceforth, when a sequence $(I, <)$ is defined, it is implicit that the entries of I are tuples from $\mathbb{M}$, all of the same length, and $(I, <)$ is a linear order. If $S \subseteq \mathbb{M}$ is such that $a \in S$ for all $a \in I$, we say that I is *in* S.

Definition 2.5.1. Let $(I, <)$ be a sequence and $B \subseteq \mathbb{M}$. Say that I is *indiscernible over* B or B*-indiscernible* if, for all $n \in \mathbb{N}$ and $a_1 < \cdots < a_n$ and $a'_1 < \cdots < a'_n$ in I, we have $a_1 \cdots a_n \equiv_B a'_1 \cdots a'_n$. When $B = \emptyset$, say that I is *indiscernible*.

Definition 2.5.2. Let $(I, <)$ be a sequence and $B \subseteq \mathbb{M}$. The *Ehrenfeucht–Mostowski type (EM-type) of* I *over* B, denoted $\mathrm{tp}^{\mathrm{EM}}(I/B)$, is

$$\{\phi(x_1, ..., x_n) \in L(B) : \mathbb{M} \models \phi(a_1, ..., a_n) \text{ for all } a_1 < \cdots < a_n \text{ in } I\}.$$

If $(I, <)$ is indiscernible over B, then for all $n \in \mathbb{N}$, $\{\phi(x_1, ..., x_n) \in \mathrm{tp}^{\mathrm{EM}}(I/B)\}$ is a complete type over B.

Definition 2.5.3. Let $p(x) \in S(\mathbb{M})$ be invariant over B, and let $(I, <)$ be a sequence with $|a| = |x|$ for all $a \in I$. Say that I is a *Morley sequence of* p *over* B if $\mathrm{tp}^{\mathrm{EM}}(I/B) = p^{(\omega)}|B$, in which case we write $I \models p^{(\omega)}|B$.

Note that such a sequence is necessarily indiscernible over B. Indeed, for all $n \in \mathbb{N}^+$ and $a_1 < \cdots < a_n$ in I, $\mathrm{tp}(a_1, ..., a_n/B) = p^{(n)}|B$.

2.6 Keisler measures (NIP)

We follow the treatment in [48]. Throughout this section, fix a complete L-theory T and a sufficiently saturated model $\mathbb{M} \models T$; **we assume that T is NIP.**

For $B \subseteq \mathbb{M}$ and x a tuple of variables, write $\mathcal{L}_x(B)$ be the Boolean algebra of B-definable subsets of $\mathbb{M}^x$. We will often represent an element of $\mathcal{L}_x(B)$ by an

$L(B)$-formula defining it.

Definition 2.6.1. Let x be a tuple of variables and $B \subseteq \mathbb{M}$. A function $\mu(x) : \mathcal{L}_x(B) \to [0,1]$ is a *(Keisler) measure over* B if it is a finitely additive probability measure, that is:

(i) $\mu(x = x) = 1$;

(ii) For all $\phi(x) \in \mathcal{L}_x(B)$, $\mu(\neg\phi(x)) = 1 - \mu(\phi(x))$;

(iii) For all disjoint $\phi_1(x), ..., \phi_k(x) \in \mathcal{L}_x(B)$, $\mu\left(\bigvee_{i=1}^k \phi_i(x)\right) = \sum_{i=1}^k \mu(\phi_i(x))$.

If $B = \mathbb{M}$, say that μ is a *global (Keisler) measure.*

Remark 2.6.2. In measure-theoretic literature, probability measures are often assumed to be σ-additive (that is, countably additive) functions on σ-algebras. Following Keisler's original paper [31] on Keisler measures and subsequent model-theoretic literature, we use the term *probability measure* to mean a (finitely additive) function on a Boolean algebra which respects Boolean operations in the sense of Definition 2.6.1.

Keisler measures are generalisations of types. Indeed, every $p(x) \in S_x(B)$ induces a measure over B that sends $\phi(x)$ to 1 if $\phi \in p$ and 0 otherwise.

The following fact says that every Keisler measure over B extends uniquely to a regular Borel probability measure $\tilde{\mu}$ on $S_x(B)$, that is, a σ-additive regular probability measure on the set of Borel subsets of $S_x(B)$. By *regular*, we mean that if $X \subseteq S_x(B)$ is Borel, then

$$\inf\{\tilde{\mu}(U) : U \supseteq X \text{ is open}\} = \sup\{\tilde{\mu}(F) : F \subseteq X \text{ is closed}\}.$$

Recall that $S_x(B)$ has a basis of clopen sets given by $\{[\phi(x)] : \phi(x) \in L(B)\}$, where $[\phi(x)] = \{p \in S_x(B) : \phi \in p\}$.

Fact 2.6.3. *Let $\mu(x)$ be a Keisler measure over B. Then there is a unique regular Borel probability measure $\tilde{\mu}$ on $S_x(B)$ such that $\tilde{\mu}([\phi(x)]) = \mu(\phi(x))$ for all $\phi(x) \in L(B)$.*

Proof. See [48, Section 7.1]. More details can be found in the online version [49, Section 7.1] of this source. □

Thus, abusing notation, by a Keisler measure $\mu(x)$ over B we will often mean the unique associated regular Borel measure on $S_x(B)$.

The definitions introduced for types in the previous section can be extended to Keisler measures. Henceforth in this section, all parameter sets are small.

Definition 2.6.4. Let $\mu(x)$ be a global measure, and let $B \subseteq \mathbb{M}$.

(i) Say that μ is *invariant over* B or B*-invariant* if, for all $\phi(x; y) \in L$ and $d, d' \in \mathbb{M}$ such that $d \equiv_B d'$, $\mu(\phi(x; d)) = \mu(\phi(x; d'))$.

(ii) Say that μ is *finitely satisfiable over* B if, for all $\phi(x) \in L(\mathbb{M})$ such that $\mu(\phi(x)) > 0$, there is $b \in B$ such that $\models \phi(b)$.

(iii) Say that μ is *definable over* B if μ is B-invariant and, for all $\phi(x; y) \in L$ and $r \in [0, 1]$, the set

$$\{q \in S_y(B) : \mu(\phi(x; d)) < r \text{ for all/some } d \in \mathbb{M} \text{ such that } d \models q\}$$

is an open subset of $S_y(B)$.

(iv) Say that μ is *generically stable over* B if it is finitely satisfiable and definable over B.

The following fact is straightforward to prove, and can be found in Section 7.4 of [48].

Fact 2.6.5. *Let $\mu(x)$ be a global measure, and let $B \subseteq \mathbb{M}$. If μ is finitely satisfiable over B, then μ is B-invariant.*

We record the following 'closure property' of generically stable measures.

Proposition 2.6.6. *Let $\mu(x)$ be a global measure, generically stable over $B \subseteq \mathbb{M}$. Let $\mu'(x, y)$ be the global measure defined by $\mu'(\phi(x, y)) := \mu(\phi(x, b))$ for some fixed $b \in B$. Then μ' is generically stable over B.*

This is a straightforward consequence of the following lemma.

Lemma 2.6.7. *In items (i) and (iii) of Definition 2.6.4, the clause '$\phi(x) \in L$' can be replaced by '$\phi(x) \in L(B)$'.*

Proof. For item (i), this is obvious. For item (iii), suppose μ is definable over B, and let $\phi(x; y) \in L(B)$ and $r \in [0, 1]$. Thus, there is $\psi(x; y, z) \in L$ and $b \in B$ such that $\phi(x; y) = \psi(x; y, b)$. We wish to show that the set W, given by

$$\{q \in S_y(B) : \mu(\psi(x; d, b)) < r \text{ for all/some } d \in \mathbb{M} \text{ such that } d \models q\},$$

is an open subset of $S_y(B)$.

Since μ is definable over B, the set X, given by

$$\{p \in S_{yz}(B) : \mu(\psi(x; d, e)) < r \text{ for all/some } (d, e) \in \mathbb{M} \text{ such that } (d, e) \models p\},$$

is an open subset of $S_{yz}(B)$, and thus so is the set $X \cap [z = b]$, which equals

$$\{p \in S_{yz}(B) \cap [z = b] : \mu(\psi(x; d, b)) < r \text{ for all/some } d \in \mathbb{M} \text{ such that } (d, b) \models p\}.$$

Thus, we have $X \cap [z = b] = \bigcup_{i \in I} [\theta_i(y, z)]$ for some formulas $\theta_i(y, z) \in L(B)$. It is straightforward to check that $W = \bigcup_{i \in I} [\theta_i(y, b)]$, and so W is an open subset of $S_y(B)$. □

We now define products of measures.

Definition 2.6.8. Let $\mu(x), \lambda(y)$ be global measures, with μ invariant over some $M \models T$. The *product* $(\mu \otimes \lambda)(x, y)$ is the global measure such that, for all $\phi(x, y; b) \in L(\mathbb{M})$,

$$(\mu \otimes \lambda)(\phi(x, y; b)) = \int_{S_y(N)} f \, \mathrm{d}\lambda|_N,$$

where N is any small model containing $M \cup \{b\}$ and $f : S_y(N) \to [0, 1]$ sends $q \in S_y(N)$ to $\mu(\phi(x, d; b))$ for any $d \models q$.

For this to be well-defined, it must be independent of the choice of N, and the function f must be measurable. Proofs of these facts (which rely on the fact that T is NIP) can be found in Section 7.4 of [48].

The following fact is taken from Section 7.4 of [48].

Fact 2.6.9. *(i) The product operation is associative.*

(ii) If $\mu(x)$ and $\lambda(y)$ are global measures, generically stable over a small model M, then $(\mu \otimes \lambda)(x, y)$ is generically stable over M.

In light of (i), given a global measure $\mu(x)$ and $k \in \mathbb{N}^+$, we define the global measure $\mu^{(k)}(x_1, ..., x_k) := \mu(x_1) \otimes \cdots \otimes \mu(x_k)$.

Although Definition 2.6.4 makes sense in an arbitrary theory, in NIP theories, generically stable measures μ admit an extremely useful alternative formulation, via an 'ε-net theorem'. Roughly speaking, this says that for every definable family $\mathcal{A}$, the μ-measure of every $A \in \mathcal{A}$ can be uniformly approximated by sampling the membership relation $x \in A$.

To state this formally, we first fix the following notation.

Definition 2.6.10. Let $k \in \mathbb{N}$ and $A \subseteq \mathbb{M}^k$. For $n \in \mathbb{N}^+$ and $a_1, ..., a_n \in \mathbb{M}^k$, let

$$\mathrm{Av}(\{a_1, ..., a_n\}; A) := \mathrm{Av}(a_1, ..., a_n; A) := \frac{\#\{i \in [n] : a_i \in A\}}{n}.$$

Theorem 2.6.11. *(T is NIP.) Let $M \models T$. The following are equivalent for a global measure $\mu(x)$.*

(i) The measure μ is generically stable over M.

(ii) Let $\phi(x; y) \in L$ and $\varepsilon \in (0, 1]$. Then there are $a_1, ..., a_n \in M^x$ such that, for all $b \in \mathbb{M}$,

$$\left|\mathrm{Av}(a_1, ..., a_n; \phi(x; b)) - \mu(\phi(x; b))\right| < \varepsilon.$$

(iii) The statement (ii) with $\mathbb{M}$ replaced by M.

Moreover, in (ii) and (iii), we may assume $n = O_{\mathrm{VC}(\phi)}(\varepsilon^{-2} \log(2\varepsilon^{-1}))$; in particular, n can be chosen independently of μ and M.

Proof. See [48, Theorem 7.29]. The 'moreover' statement is a combination of [48, Lemma 7.24] and Lemma 2.2.3. □

We use this theorem to show that an ultraproduct of generically stable measures is generically stable (in the NIP theory T).

Definition 2.6.12. Let $(r_i : i \in I)$ be a sequence of constants in $[0, 1]$, and let $\mathcal{U}$ be an ultrafilter on I. The *ultralimit* $\lim_{\mathcal{U}} r_i$ is the unique $r \in [0, 1]$ such that, for all $\varepsilon > 0$, $\{i \in I : |r_i - r| < \varepsilon\} \in \mathcal{U}$.

Definition 2.6.13. Let $(\mathbb{M}_i : i \in I)$ be a sequence of sufficiently saturated L-structures, and for $i \in I$ let $\mu_i(x)$ be a global measure (over $\mathbb{M}_i$). Let $\mathcal{U}$ be an ultrafilter on I, and let $\mathbb{M} := \prod_{\mathcal{U}} \mathbb{M}_i$. The *ultraproduct* $(\prod_{\mathcal{U}} \mu_i)(x)$ of $(\mu_i : i \in I)$ is the following global measure (over $\mathbb{M}$). Let $\phi(x; b) \in L(\mathbb{M})$, and let $(b_i : i \in I)$ be a representative for b. Declare $(\prod_{\mathcal{U}} \mu_i)(\phi(x; b)) := \lim_{\mathcal{U}} \mu_i(\phi(x; b_i))$.

Proposition 2.6.14. *(T is NIP.) Let $(M_i : i \in I)$ be a sequence of models of T, and for $i \in I$ let $\mu_i(x)$ be a global measure generically stable over M_i. Let $\mathcal{U}$ be an ultrafilter on I. Then the ultraproduct measure $\mu := \prod_{\mathcal{U}} \mu_i$ is generically stable over $M := \prod_{\mathcal{U}} M_i$.*

Proof. We follow [50, Corollary 1.3]. Let $\phi(x; y) \in L$ and $\varepsilon \in (0, 1]$, and let $n \in \mathbb{N}$ be given by Theorem 2.6.11. For all $i \in I$, there are $a^i_1, ..., a^i_n \in M_i$ such that, for all $b_i \in M_i$,

$$\left|\mathrm{Av}(a^i_1, ..., a^i_n; \phi(x; b_i)) - \mu_i(\phi(x; b_i))\right| < \varepsilon/2.$$

For $k \in [n]$, let $a_k := (a^i_k : i \in I) \in M$. Fix $b \in M$, and let $(b_i : i \in I)$ be a representative for b. Then $\{i \in I : |\mu_i(\phi(x; b_i)) - \mu(\phi(x; b))| < \varepsilon/2\} \in \mathcal{U}$, and so

$$\left\{i \in I : \left|\mathrm{Av}(a^i_1, ..., a^i_n; \phi(x; b_i)) - \mu(\phi(x; b))\right| < \varepsilon\right\} \in \mathcal{U}.$$

By Łoś's Theorem, $\left|\mathrm{Av}(a_1, ..., a_n; \phi(x; b)) - \mu(\phi(x; b))\right| < \varepsilon$ as required. □

We give three examples of generically stable Keisler measures, taken from [48, Example 7.32].

Example 2.6.15 (Average types)**.** Let $M \models T$. Let $(p_n(x) : n \in \mathbb{N})$ be a sequence of global types, generically stable over some $A \subseteq \mathbb{M}$, and let $(c_n : n \in \mathbb{N})$ be a sequence of constants in $[0,1]$ such that $\sum_n c_n = 1$. Let $\mu(x)$ be the measure $\sum_n c_n p_n$ over M, that is, $\mu(\phi(x)) = \sum_n c_n \cdot \mathbb{1}(\phi \in p_n)$. Then μ is generically stable over A — see [48, Example 7.32].

Example 2.6.16 (Counting measures)**.** Let $a_1, ..., a_n \in \mathbb{M}^x$. The *(normalised) finite counting measure* supported on $\{a_1, ..., a_n\}$ is the global measure $\mu(x)$ such that $\mu(A) = \frac{1}{n}\#\{i \in [n] : a_i \in A\}$ for each definable $A \subseteq \mathbb{M}^x$. This measure is generically stable over $\{a_1, ..., a_n\}$. Indeed, for $i \in [n]$ let $p_i := \mathrm{tp}(a_i/\mathbb{M})$, which is generically stable over $\{a_i\}$. Then $\mu = \sum_{i \in [n]} \frac{1}{n} p_i$, and so μ is generically stable over $\{a_1, ..., a_n\}$ by the previous example.

We now define *pseudofinite counting measures* as ultraproducts of finite counting measures. Suppose $\mathbb{M} = \prod_{\mathcal{U}} \mathbb{M}_i$, where $(\mathbb{M}_i : i \in I)$ is a sequence of models of T and $\mathcal{U}$ is an ultrafilter on I. For $i \in I$, let $\mu_i(x)$ be a finite counting measure with support in $\mathbb{M}_i$. Say that the ultraproduct $\prod_{\mathcal{U}} \mu_i$ is a *pseudofinite counting measure*. By Proposition 2.6.14, since T is NIP, $\prod_{\mathcal{U}} \mu_i$ is generically stable.

Example 2.6.17 (Average measures)**.** Let $M \models T$, and let $I = (a_i : i \in [0,1])$ be an indiscernible sequence in M^x. The *average measure* $\mu(x)$ of I is such that, for all $\phi(x) \in L(\mathbb{M})$, $\mu(\phi) = \lambda(\{i \in [0,1] : \mathbb{M} \models \phi(a_i)\})$, where λ is the Lebesgue measure on $\mathbb{R}$. Then μ is well-defined and generically stable over M — see [48, Example 7.32].

2.7 Distality

Part of this section, especially the content on strong honest definitions, is presented (with minor differences) in our paper [54].

Throughout this section, fix a complete L-theory T and a sufficiently saturated model $\mathbb{M} \models T$. In this section, all parameter sets are small. It is time to introduce the main character of this thesis.

Definition 2.7.1. Say that T (and any $M \models T$) is *distal* if the following holds.

Let I_0, I_1, I_2 be (dense) infinite sequences without endpoints, whose elements are n-tuples. Let $a_0, a_1 \in \mathbb{M}^n$ be such that $I_0 + a_0 + I_1 + I_2$ and $I_0 + I_1 + a_1 + I_2$ are indiscernible. Then $I_0 + a_0 + I_1 + a_1 + I_2$ is indiscernible.

Note that, by compactness, the inclusion or exclusion of 'dense' in the statement above does not change the definition of distality.

Example 2.7.2. Examples of distal theories/structures include the theory DLO of densely linearly ordered (without endpoints) structures, o-minimal structures, and Presburger arithmetic $(\mathbb{Z}, <, +)$. Recall that a linearly ordered structure $(M, <, ...)$ is *o-minimal* if every M-definable subset of M is a finite union of points and intervals.

Many equivalent definitions of distality are now known, but Definition 2.7.1 was the original formulation by Simon in [47], except that there it is also assumed that T is NIP. It turns out that this assumption is superfluous: distality (according to Definition 2.7.1) implies NIP. It is unclear whether this is well-known within the model-theoretic community. In the literature, alternative formulations of distality are often favoured over Definition 2.7.1; some of these formulations obviously imply NIP, and some were shown to be equivalent to Definition 2.7.1 under the assumption of NIP. The first proof in the literature (of which we are aware) of the fact that distality implies NIP appears in [55, Corollary 6.8], where it is credited to Chernikov.

We now state two alternative formulations of distality that are often favoured over Definition 2.7.1. The following formulation is introduced by Simon in [47, Lemma 2.7], where the equivalence with Definition 2.7.1 is also established (under the assumption of NIP, which can be removed as argued above). Simon calls this the *external characterisation* of distality, and we shall call Definition 2.7.1 the *internal characterisation* of distality.

Theorem 2.7.3. *The theory T is distal if and only if the following holds.*

Let I_0, I_1 be (dense) infinite sequences without endpoints, whose elements are n-tuples. Let $a \in \mathbb{M}^n$ and $B \subseteq \mathbb{M}$. If $I_0 + a + I_1$ is indiscernible and $I_0 + I_1$ is B-indiscernible, then $I_0 + a + I_1$ is B-indiscernible.

To state the next formulation of distality, we make the following definition.

Definition 2.7.4. Let $\phi(x; y) \in L$. A formula $\psi(x; z) \in L$ is a *strong honest definition* for ϕ if the following holds.

Let $B \subseteq M \models T$ with $|B| \geq 2$, and let $a \in M^x$. Let $(M', B') \succcurlyeq (M, B)$ be $|M|^+$-saturated. Then there is $c \in (B')^z$ such that $a \models \psi(x; c)$ and $\psi(x; c) \vdash \mathrm{tp}_\phi(a/B^y)$.

The final clause will often be written as $a \models \psi(x; c) \vdash \mathrm{tp}_\phi(a/B^y)$. This definition admits a finitary formulation that is often more useful (and agreeable to non-model theorists).

Lemma 2.7.5. *Let $\phi(x; y) \in L$. For $\psi(x; z) \in L$, the following are equivalent:*

(i) The formula ψ is a strong honest definition for ϕ;

(ii) Let $B \subseteq M \models T$ with $2 \leq |B| < \infty$, and let $a \in M^x$. Then there is $c \in B^z$ such that $a \models \psi(x; c) \vdash \mathrm{tp}_\phi(a/B^y)$.

Proof. That (i) implies (ii) is immediate. For the converse, given a, B, B' as in the statement of (i), observe that

$$\{\psi(a; z)\} \cup \left\{\forall x \left(\psi(x; z) \to \big(\phi(x; b) \leftrightarrow \phi(a; b)\big)\right) : b \in B^y\right\}$$

is finitely satisfiable in B, so is satisfiable in B' by compactness and saturation. $\square$

Remark 2.7.6. In light of (ii) in Lemma 2.7.5, the tuple z can be taken to be copies of y, say, $z = (y_1, ..., y_k)$ where $|y_1| = \cdots = |y_k| = |y|$. Then, (ii) can be restated equivalently as follows.

Let $M \models T$, $B \subseteq M^y$ with $2 \leq |B| < \infty$, and let $a \in M^x$. Then there is $c \in B^k$ such that $a \models \psi(x; c) \vdash \mathrm{tp}_\phi(a/B)$.

The following theorem, our third characterisation of distality, is due to Chernikov and Simon [10, Theorem 21].

Theorem 2.7.7. *The theory T is distal if and only if every formula $\phi(x;y) \in L$ has a strong honest definition.*

This characterisation of distality forms the basis of much of the work in this thesis. As we shall see, it is very well suited to combinatorial applications. Perhaps the first sign of combinatorial promise is that strong honest definitions give rise to a desirable form of cell decomposition. The following definition is taken from [9]. For a set X, let $\mathcal{P}_{\text{fin}}(X)$ denote the set of finite subsets of X.

Definition 2.7.8. Fix $\phi(x;y) \in L$ and $M \models T$. An *abstract (cell) decomposition* for ϕ is a function $\mathcal{F} : \mathcal{P}_{\text{fin}}(M^y) \to \mathcal{P}(\mathcal{P}(M^x))$ such that, for all finite $B \subseteq M^y$, $\mathcal{F}(B)$ is a cover of M^x which refines the partition of M^x given by the set $S_\phi(B)$ of ϕ-types over B.

Say that such $\mathcal{F}$ is a *distal (cell) decomposition* for ϕ if there is a formula $\psi(x; y_1, ..., y_k)$ such that, for all finite $B \subseteq M^y$ with $|B| \geq 2$ and $F \in \mathcal{F}(B)$, there are $b_1, ..., b_k \in B$ such that $F = \psi(x; b_1, ..., b_k)$. In this case, say that $\mathcal{F}$ is *defined* by ψ.

Let $\phi(x;y) \in L$ and $M \models T$, and suppose $\psi(x; y_1, ..., y_k)$ is a strong honest definition for ϕ (see Remark 2.7.6). Then ϕ has a distal decomposition defined by ψ. Indeed, given $B \subseteq M^y$ with $|B| \geq 2$, by Lemma 2.7.5, we may set $\mathcal{F}(B)$ to be a subset of $\{\psi(x;c) : c \in B^k\}$ that forms a cover of M^x refining $S_\phi(B)$.

Figure 2.2 illustrates this decomposition. Here, $B = \{b_1, b_2, b_3, b_4\}$, and the regions of the Venn diagram form the partition $S_\phi(B)$ of M^x. The distal decomposition defined by ψ refines $S_\phi(B)$: every ϕ-type over B, such as the grey shaded region, can be written as a union of cells of the form $\psi(x; b'_1, ..., b'_k)$ for $b'_i \in B$.

The key here is that ψ works for any finite $B \subseteq M^y$ with $|B| \geq 2$. A ϕ-type over B can always be defined by a relation of the form $\bigwedge_{b \in B} \phi^{\varepsilon(b)}(x;b)$ for some $\varepsilon : B \to \{0,1\}$, but this relation has $|B|$-many parameters. In contrast, the relation ψ has a fixed number k of parameters.

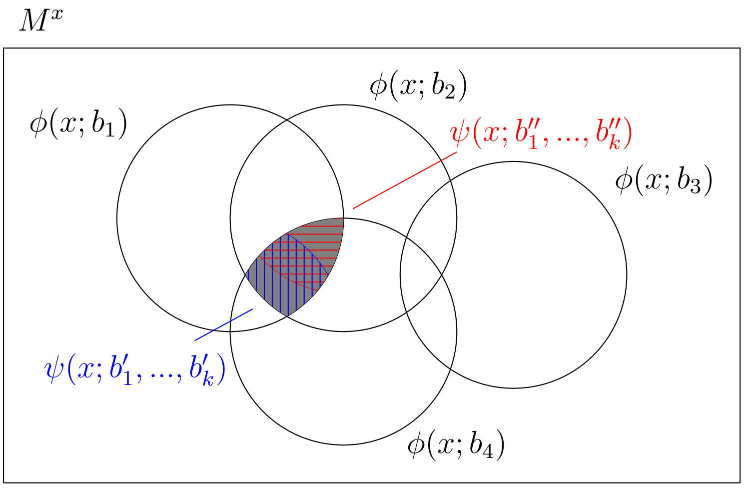

Figure 2.2: Part of the distal decomposition for ϕ

Let us address the awkward condition that $|B| \geq 2$ in Definition 2.7.4 and the subsequent exposition. In short, we require $|B| \geq 2$ so that we may apply standard coding tricks. As an important example, when constructing a strong honest definition for $\phi(x;y)$, it is often convenient to partition M^x into finitely many pieces and use a different formula for each piece.

Proposition 2.7.9. *Fix $\phi(x;y) \in L$ and $M \models T$. Then ϕ has a strong honest definition if and only if for some $n \in \mathbb{N}$ there are formulas $(\psi_i(x;y_1,...,y_k) : i \in [n])$ such that for all $a \in M^x$ and $B \subseteq M^y$ with $2 \leq |B| < \infty$, there is $c \in B^k$ and $i \in [n]$ such that $a \models \psi_i(x;c) \vdash \mathrm{tp}_\phi(a/B)$.*

Proof. The forward direction is immediate. Let $(\psi_i(x;y_1,...,y_k) : i \in [n])$ witness the antecedent of the backward direction. Let

$$\theta(x;y_{i,1},...,y_{i,k},u_i,v_i : i \in [n]) := \bigvee_{i=1}^{n} \left(u_i = v_i \wedge \psi_i(x;y_{i,1},...,y_{i,k})\right),$$

where u_i, v_i are tuples of variables of length $|y|$. We claim that this is a strong honest definition for ϕ. Fix $a \in M^x$ and $B \subseteq M^y$ with $2 \leq |B| < \infty$. There is $c \in B^k$ and $j \in [n]$ such that $a \models \psi_j(x;c) \vdash \mathrm{tp}_\phi(a/B)$. Pick $u_1, v_1, ..., u_n, v_n \in B$ such that $u_i = v_i$ if and only if $i = j$; this is possible since $|B| \geq 2$. Then $a \models \theta(x;c,u_i,v_i : i \in [n])$ since $a \models \psi_j(x;c)$, and $\theta(x;c,u_i,v_i : i \in [n]) \vdash \psi_j(x;c)$ since $u_i \neq v_i$ for all $i \neq j$. But now $\psi_j(x;c) \vdash \mathrm{tp}_\phi(a/B)$. $\square$

Call such $(\psi_i : i \in [n])$ a *system of strong honest definitions* for ϕ.

We turn our attention to closure properties for the existence of strong honest definitions. The following lemma is straightforward to prove.

Lemma 2.7.10. *Let $\phi_1(x; y), \phi_2(x; y) \in L$ respectively have strong honest definitions $\psi_1(x; y_1, ..., y_k), \psi_2(x; y_1, ..., y_l)$.*

(i) The formula $\neg\phi_1(x; y)$ has strong honest definition $\psi_1(x; y_1, ..., y_k)$.

(ii) The formula $\phi_1 \wedge \phi_2(x; y)$ has strong honest definition

$$\psi_1(x; y_1, ..., y_k) \wedge \psi_2(x; y_{k+1}, ..., y_{k+l}).$$

The following fact is [3, Proposition 1.9].

Fact 2.7.11. *The theory T is distal if and only if every formula $\phi(x; y) \in L$ with $|x| = 1$ has a strong honest definition.*

Corollary 2.7.12. *Suppose T has quantifier elimination. Then T is distal if and only if every atomic $\phi(x; y) \in L$ with $|x| = 1$ has a strong honest definition.*

To state the next closure property, we make the following definition.

Definition 2.7.13. Let $\phi(x; y)$ be an L-formula with $m := |x|$ and $n := |y|$. Say that an L-formula $\theta(u; v)$ is a *descendant* of ϕ if

$$\theta(u; v) = \phi(f_1(u), ..., f_m(u); g_1(v), ..., g_n(v))$$

for some L-definable functions $f_1, ..., f_m$ of arity $|u|$ and $g_1, ..., g_n$ of arity $|v|$.

Note that the descendant relation is reflexive and transitive.

Lemma 2.7.14. *Fix an L-structure M with at least two $\emptyset$-definable elements. If an L-formula $\phi(x; y)$ has a strong honest definition, so does any descendant of ϕ.*

Proof. Let $\alpha, \beta \in M$ be distinct $\emptyset$-definable elements. Let $\phi(x; y)$ be an L-formula with $m := |x|$ and $n := |y|$, and suppose it has a strong honest definition

$\psi(x; y^{(1)}, ..., y^{(k)})$. Let $\theta(u; v) = \phi(f_1(u), ..., f_m(u); g_1(v), ..., g_n(v))$ be a descendant of ϕ, for some L-definable functions $f_1, ..., f_m$ and $g_1, ..., g_n$.

For $I, J \subseteq [k]$ disjoint, let $\zeta_{IJ}(u; v^{(i)} : i \in [k] \setminus (I \cup J))$ be the formula $\psi(f_1(u), ..., f_m(u); h_1^{(1)}, ..., h_n^{(1)}, ..., h_1^{(k)}, ..., h_n^{(k)})$, where $h_j^{(i)}$ is α if $i \in I$, β if $i \in J$, and $g_j(v^{(i)})$ otherwise. We claim that $\{\zeta_{IJ} : I, J \subseteq [k] \text{ disjoint}\}$ is a system of strong honest definitions for θ.

Indeed, let $a \in M^u$ and $B \subseteq M^v$ with $2 \leq |B| < \infty$. Let $\bar{B} := \{(g_1(v), ..., g_n(v)) : v \in B\}$, and let $\hat{B} := \bar{B} \cup \{(\alpha, ..., \alpha), (\beta, ..., \beta)\} \subseteq M^n$. Since ψ is a strong honest definition for ϕ and $2 \leq |\hat{B}| < \infty$, there is $c = (c^{(1)}, ..., c^{(k)}) \in \hat{B}^k$ such that $(f_1(a), ..., f_m(a)) \models \psi(x; c)$ and

$$\psi(x; c) \vdash \mathrm{tp}_\phi(f_1(a), ..., f_m(a)/\hat{B}) \supseteq \mathrm{tp}_\phi(f_1(a), ..., f_m(a)/\bar{B}).$$

Let $I := \{i \in [k] : c^{(i)} = (\alpha, ..., \alpha)\}$ and $J := \{i \in [k] : c^{(i)} = (\beta, ..., \beta)\}$. Then, there is a tuple $(w^{(i)} : i \in [k] \setminus (I \cup J))$ from B such that

$$\psi(f_1(u), ..., f_m(u); c) = \zeta_{IJ}(u; w^{(i)} : i \in [k] \setminus (I \cup J)),$$

whence $a \models \zeta_{IJ}(u; w^{(i)} : i \in [k] \setminus (I \cup J)) \vdash \mathrm{tp}_\theta(a/B)$. □

Remark 2.7.15. In the proof above, if the function $v \mapsto (g_1(v), ..., g_n(v))$ were injective, then the formula $\zeta(u; v^{(i)} : i \in [k])$ given by

$$\psi(f_1(u), ..., f_m(u); g_1(v^{(1)}), ..., g_n(v^{(1)}), ..., g_1(v^{(k)}), ..., g_n(v^{(k)}))$$

would have sufficed as a strong honest definition for θ.

Example 2.7.16. As an example of strong honest definitions, we prove that Presburger arithmetic is distal by constructing a system of strong honest definitions for every relevant formula.

It is well known (see, for example, [13]) that Presburger arithmetic admits quantifier elimination in the language $L_{\mathrm{Pres}} := (<, +, -, 0, 1, (\cdot \equiv_m 0)_{m \in \mathbb{N}^+})$, where $\cdot \equiv_m 0$ is a unary relation symbol interpreted as divisibility by m. We write $x \equiv_m y$ to mean $x - y \equiv_m 0$.

By Corollary 2.7.12, it suffices to construct a strong honest definition for every atomic L_{Pres}-formula $\phi(x;y)$ with $|x| = 1$. These have the form $f(x,y) = 0$, $f(x,y) < 0$, or $f(x,y) \equiv_m 0$, where f is a $\mathbb{Z}$-affine function (that is, a $\mathbb{Z}$-linear combination of its arguments plus an integer constant). We can ignore formulas of the form $f(x,y) = 0$, since $f(x,y) = 0 \leftrightarrow f(x,y) < 1 \wedge -f(x,y) < 1$. By Lemma 2.7.14, it suffices to construct strong honest definitions for $\phi(x;y) := x < y$ and $\psi_m(x;y) := x \equiv_m y$.

The formula $\phi(x;y)$ admits a system of strong honest definitions given by $\{x < y, x = y, y < x, y < x < y'\}$, where $|y'| = |y|$; in what follows, we will understand $-\infty < x < y$ to mean $x < y$ and $y < x < +\infty$ to mean $y < x$. Indeed, let $a \in \mathbb{Z}$ and $B \subseteq \mathbb{Z}$ with $2 \leq |B| < \infty$. Enumerate B as $\{b_1, ..., b_n\}$, where $b_1 < \cdots < b_n$. If there is $1 \leq i \leq n$ such that $a = b_i$, then $a \models x = b_i \vdash \mathrm{tp}_\phi(a/B)$. Otherwise, there is $0 \leq i \leq n$ such that $b_i < a < b_{i+1}$ (where $b_0 := -\infty$ and $b_{n+1} := +\infty$), whence $a \models b_i < x < b_{i+1} \vdash \mathrm{tp}_\phi(a/B)$.

The formula $\psi_m(x;y)$ admits a system of strong honest definitions given by $\{x \equiv_m i : 0 \leq i < m\}$. Indeed, let $B \subseteq \mathbb{Z}$ with $2 \leq |B| < \infty$. Given $a \in \mathbb{Z}$, there is $0 \leq i < m$ such that $a \equiv_m i$, whence $a \models x \equiv_m i \vdash \mathrm{tp}_{\psi_m}(a/\mathbb{Z})$.

Chapter 3

Distality *from* Combinatorics: Expansions of Presburger Arithmetic

In this chapter, we recover distality from combinatorial data. Specifically, we prove that the structure $(\mathbb{Z}, <, +, R)$ is distal for all congruence-periodic sparse predicates $R \subseteq \mathbb{N}$, by constructing a strong honest definition for every formula $\phi(x; y)$ with $|x| = 1$, providing a rare example of concrete distal decompositions.

This chapter is presented (with minor differences) in our paper [54]. We thank Pantelis Eleftheriou for providing numerous helpful suggestions on the content and structure of this paper, as well as Pablo Andújar Guerrero and Aris Papadopoulos for fruitful discussions on distality. We would also like to thank the referee for our paper for their helpful comments and corrections. *Soli Deo gloria.*

3.1 Introduction

One of the most important threads of model-theoretic research is identifying and studying *dividing lines* in the universe of structures: properties $\mathcal{P}$ such that structures with $\mathcal{P}$ are 'tame' and 'well-behaved' in some sense.

Two dividing lines that have attracted much interest, not just in model theory but also in fields such as combinatorics and machine learning, are *stability* and NIP. Distality was introduced by Simon to characterise NIP structures that are 'purely unstable'. Indeed, stability and distality can be viewed as two opposite ends of the NIP spectrum: no infinite structure satisfies both simultaneously. However, a stable structure can admit a distal expansion, and this is (a special case of) the subject of curiosity among many model theorists, phrased in [3] as the following problem.

Problem 3.1.1. *Which NIP structures admit distal expansions?*

The reason (or one such reason) this is a question of interest is precisely the fact that distal structures have nice structural properties. As noted in Section 2.7, a structure M is distal if and only if every formula $\phi(x; y)$ in its theory has a strong honest definition (or a distal cell decomposition). Informally, this means that given a finite set $B \subseteq M^y$, there is a decomposition of M^x, uniformly definable from B, into finitely many cells, such that the truth value of $\phi(x; b)$ is constant on each cell for all $b \in B$.

Cell decompositions in general have proved useful for deriving various results, particularly of a combinatorial nature, and distal decompositions are no exception. Many results that hold in the real field, where we have semialgebraic cell decomposition, that were found to generalise to o-minimal structures, where we have o-minimal cell decomposition, turn out to also generalise to distal structures, where we have distal decomposition (recall that o-minimal structures are distal; in fact, o-minimal cell decomposition is a special case of distal decomposition). A notable example concerns the 'strong Erdős–Hajnal property'. It was shown in [1] that every definable relation over the real field has the strong Erdős–Hajnal property. This was later generalised in [5] to every definable, topologically closed relation in any o-minimal expansion of a real-closed field. Finally, it was shown in [11] that a structure is distal *if and only if* every relation in its theory satisfies the definable strong Erdős–Hajnal property.

Such results support the view that distality is an excellent context for certain flavours of combinatorics. Indeed, recall the postulate in [9] that 'distal structures provide the most general natural setting for investigating questions in "[generalised] incidence combinatorics"'.

The main result of this chapter thus fits nicely into the context described above.

Main Theorem (Theorem 3.4.8)**.** *Let $R \subseteq \mathbb{N}$ be a congruence-periodic sparse predicate. Then the structure $(\mathbb{Z}, <, +, R)$ is distal.*

Note that, by [15, Corollary 2.20], such structures $(\mathbb{Z}, <, +, R)$ have dp-rank $\geq \omega$, so our main theorem completely classifies these structures on the model-theoretic map of the universe.

Here, *congruence-periodic* means that, for all $m \in \mathbb{N}^+$, the increasing sequence by which R is enumerated is eventually periodic modulo m. *Sparsity* will be defined in Definition 3.2.2, but for now we content ourselves by noting that sparse predicates include such examples as $d^{\mathbb{N}} := \{d^n : n \in \mathbb{N}\}$ for any $d \in \mathbb{N}_{\geq 2}$, the set of Fibonacci numbers, and $\{n! : n \in \mathbb{N}\}$.

We now give an overview of how this result extends and builds on results in the extant literature. In [34], Lambotte and Point prove that $(\mathbb{Z}, +, <, R)$ is NIP for all congruence-periodic sparse predicates $R \subseteq \mathbb{N}$, so our result is a strengthening of theirs. They also define the notion of a *regular* predicate, show that regular predicates are sparse, allowing them to apply their result to congruence-periodic regular predicates. It turns out that the converse holds: sparse predicates are regular, which we prove in Theorem 3.2.19 as a result of independent interest, providing an equivalent, more intuitive definition of sparsity.

In the same paper, they also prove that $(\mathbb{Z}, +, R)$ is superstable for all regular predicates $R \subseteq \mathbb{N}$. So, if additionally R is congruence-periodic, then our result shows that $(\mathbb{Z}, +, R)$ admits a distal expansion, namely, $(\mathbb{Z}, +, <, R)$. This provides a large class of examples of stable structures with distal expansions, which should provide intuition towards an answer to Problem 3.1.1. We note that examples of NIP structures *without* distal expansions are far scarcer, and so far the only known method of proving that a structure does not have a distal expansion is to exhibit a formula without the strong Erdős–Hajnal property (see [11]). It is our hope that our more direct proof of distality may provide new methods and insights to that end.

To our knowledge, no examples of $R \subseteq \mathbb{N}$ are known such that $(\mathbb{Z}, <, +, R)$ is NIP but not distal. As discussed above, distality is a desirable strengthening of NIP, so it would be pleasant if NIP sufficed for distality for such structures. We therefore pose the following problem.

Problem 3.1.2. *Is there $R \subseteq \mathbb{N}$ such that $(\mathbb{Z}, <, +, R)$ is NIP but not distal?*

In fact, even the existence of a non-distal NIP expansion of $(\mathbb{N}, <)$ appears to be unknown — see [57, Question 11.16]. More broadly, we would like to understand the following problem.

Problem 3.1.3. *Characterise the class of predicates $R \subseteq \mathbb{N}$ such that $(\mathbb{Z}, <, +, R)$ is distal.*

A natural first step to understanding this problem is to investigate the following problem.

Problem 3.1.4. *Let $R \subseteq \mathbb{N}$ be sparse but not necessarily congruence-periodic. Must the structure $(\mathbb{Z}, <, +, R)$ be distal?*

Congruence-periodicity is used in an essential way in our proof, so we expect that a substantial change in approach would be required to provide a positive answer to this question. Note that there are sparse predicates which are not congruence-periodic — see Corollary 3.2.20.

We had previously wondered whether every non-distal structure of the form $(\mathbb{Z}, <, +, R)$ interprets arithmetic, but $R = 2^{\mathbb{N}} \cup 3^{\mathbb{N}}$ serves as a counterexample[1]. Indeed, the resulting structure does not interpret arithmetic [43] and is IP (hence non-distal). A proof of the latter is given in [27] (where, in fact, 2-IP is claimed), but in personal communication with the authors an error was found; they have nonetheless supplied an alternative argument that the structure is (1-)IP.

[1]We thank Gabriel Conant for bringing this to our attention.

Our original motivation for proving the main theorem was to answer a question of Michael Benedikt (personal communication), who asked whether the structure $(\mathbb{Z}, <, +, 2^{\mathbb{N}})$ was distal. His motivation was to know whether the structure has so-called *Restricted Quantifier Collapse (RQC)*, a property satisfied by all distal structures [8]. In personal communication, he informed us that he is also interested in obtaining better VC bounds for formulas in this structure (coauthoring [7] to that end), and that a constructive proof of distality could help in this endeavour. Our proof is nothing but constructive.

3.1.1 Strategy of our proof and structure of the chapter

The proof of our main theorem, Theorem 3.4.8, comprises most of the chapter. In Section 3.2, we define and motivate the terminology used in our main theorem, and state and prove basic facts about sparse predicates that are either useful for our proof or of independent interest. Our proof begins in earnest in Section 3.3.

Let us describe the strategy of the proof. Perhaps its most noteworthy feature, and what distinguishes it from most other proofs of distality, is that we prove that the structure is distal by giving explicit strong honest definitions (hence, distal decompositions) for 'representative' formulas of the theory. Most proofs of distality in the literature go via the definitions of distality using indiscernible sequences (given in Definition 2.7.1 and Theorem 2.7.3), which offers no information on the structure or complexity (such as 'distal density') of the distal decomposition, which is itself a subject of interest, such as in [2]. As phrased in [3], 'occasionally [the characterisation of distality via strong honest definitions] is more useful since it ultimately gives more information about definable sets, and obtaining bounds on the complexity of strong honest definitions is important for combinatorial applications'.

The first stage of the proof is thus to characterise 'representative' formulas of the theory, which is the goal of Section 3.3. The main result in that section is Theorem 3.3.6, where we show that to prove the distality of our structure, it suffices to construct strong honest definitions for suitable so-called (F_n) formulas (where $n \in \mathbb{N}^+$), to be defined in Definition 3.3.5. We prove this by first showing that every formula $\phi(x; y)$ with $|x| = 1$ is (essentially) equivalent to a Boolean combination of so-called (E_n) formulas (Proposition 3.3.7), and then showing that every (E_n) formula is (essentially) equivalent to a Boolean combination of suitable (F_n) formulas and (E_{n-1}) formulas (Corollary 3.3.11). By induction on $n \in \mathbb{N}^+$, this gives an explicit recipe for writing every formula $\phi(x; y)$ with $|x| = 1$ as (essentially) a Boolean combination of suitable (F_n) formulas. This is summarised precisely at the end of Section 3.3.

Constructing strong honest definitions for (F_n) formulas is the goal of Section 3.4 of the chapter. The broad strategy is to induct on $n \in \mathbb{N}^+$. Theorem 3.4.3, which produces new strong honest definitions from existing ones, is a stronger version of the base case $n = 1$ (Corollary 3.4.4), and is also a key ingredient in the inductive step (Theorem 3.4.6). Morally, the base case is $n = 0$ (see Corollary 3.4.4), where the formula is a formula of Presburger arithmetic, hence admitting a strong honest definition since Presburger arithmetic is distal; Corollary 3.4.4 bootstraps this strong honest definition to construct ones for (F_1) formulas using Theorem 3.4.3. Thus, the proof strategy can be described as 'generating strong honest definitions in $(\mathbb{Z}, <, +, R)$ from ones in the distal structure $(\mathbb{Z}, <, +)$', which may prove a useful viewpoint for similar applications in the future.

We thus give a recipe to construct explicit strong honest definitions, and thus distal decompositions, for all formulas $\phi(x; y)$ with $|x| = 1$. However, we make no comment on the structure of these distal decompositions, as the complexity of our construction renders such analysis a separate project. In particular, we make no claim on the 'optimality' of our decomposition, to which little credence is lent by the length of our construction anyway. The objective of this chapter is to provide a rare example of concrete distal decompositions, which the reader may analyse for aspects of distal decompositions in which they are interested.

3.2 Sparsity

This section defines and discusses the notion of sparsity of a predicate.

Sparse predicates were introduced by Semenov in [44]. For an infinite predicate $R \subseteq \mathbb{N}$ enumerated by the increasing sequence $(r_n : n \in \mathbb{N})$, let $\sigma : R \to R$ denote the successor function, that is, $\sigma(r_n) = r_{n+1}$ for all $n \in \mathbb{N}$. By an *operator* on R we mean a function $R \to \mathbb{Z}$ of the form $a_n\sigma^n + \cdots + a_0\sigma^0$, where $a_n, ..., a_0 \in \mathbb{Z}$ and σ^0 is the identity function. For operators A and B, write

$$\begin{cases} A =_R B & \text{if } Az = Bz \text{ for all } z \in R, \\ A >_R B & \text{if } Az > Bz \text{ for cofinitely many } z \in R, \\ A <_R B & \text{if } Az < Bz \text{ for cofinitely many } z \in R. \end{cases}$$

The subscript R is dropped where obvious from context. We also use σ^{-1} to denote the predecessor function, where we define $\sigma^{-1}(\min R) := \min R$.

Example 3.2.1. Consider the predicate $d^{\mathbb{N}} := \{d^n : n \in \mathbb{N}\}$ for some fixed $d \in \mathbb{N}_{\geq 2}$, and let A be an operator on $d^{\mathbb{N}}$, say of the form $a_n\sigma^n + \cdots + a_0\sigma^0$ where $a_n, ..., a_0 \in \mathbb{Z}$. Then, for all $z \in d^{\mathbb{N}}$ we have $Az = (a_nd^n + \cdots + a_0d^0)z$, so the action of A on $d^{\mathbb{N}}$ is multiplication by the constant $a_nd^n + \cdots + a_0d^0$.

Definition 3.2.2 [44, §3]**.** Say that an infinite predicate $R \subseteq \mathbb{N}$ is *sparse* if every operator A on R satisfies the following:

(S1) $A =_R 0$, $A >_R 0$, or $A <_R 0$;

(S2) If $A >_R 0$, then there exists $\Delta \in \mathbb{N}$ such that $A\sigma^{\Delta}z > z$ for all $z \in R$.

Example 3.2.3. Consider again the predicate $d^{\mathbb{N}} = \{d^n : n \in \mathbb{N}\}$ for some fixed $d \in \mathbb{N}_{\geq 2}$. By Example 3.2.1, every operator A on $d^{\mathbb{N}}$ acts as multiplication by a constant $\lambda_A \in \mathbb{Z}$. Thus, (S1) is clearly satisfied. Furthermore, $A >_R 0$ if and only if $\lambda_A > 0$, in which case $A\sigma z = \lambda_A dz > z$ for all $z \in d^{\mathbb{N}}$, so (S2) is also satisfied and $d^{\mathbb{N}}$ is sparse.

Other examples of sparse predicates, given by Semenov in [44, §3], include the set of Fibonacci numbers, $\{n! : n \in \mathbb{N}\}$, and $\{\lfloor e^n \rfloor : n \in \mathbb{N}\}$.

On the other hand, for all $f \in \mathbb{N}[x]$, the predicate $f(\mathbb{N}) = \{f(n) : n \in \mathbb{N}\}$ is not sparse. Indeed, let $f \in \mathbb{N}[x]$; assume without loss of generality that $\deg f \geq 1$. Let A be the operator $\sigma^1 - \sigma^0$, so $A >_R 0$ since f is strictly increasing. There is $g \in \mathbb{N}[x]$ with $\deg g < \deg f$ such that $Af(n) = f(n+1) - f(n) = g(n)$ for all $n \in \mathbb{N}$. Hence, for all $\Delta \in \mathbb{N}$, $A\sigma^\Delta f(n) = Af(n+\Delta) = g(n+\Delta) < f(n)$ for sufficiently large $n \in \mathbb{N}$.

Remark 3.2.4. It may be tempting to conjecture from these examples and non-examples that $R = (r_n : n \in \mathbb{N}) \subseteq \mathbb{N}$ is sparse if and only if $r_{n+1}/r_n \to \theta$ for some $\theta \in \mathbb{R}_{>1} \cup \{\infty\}$. This is sadly false; in fact, the class of sparse predicates is not very rigid at all. As an example, fixing $d \in \mathbb{N}_{\geq 2}$, recall that $d^{\mathbb{N}} = \{d^n : n \in \mathbb{N}\}$ is sparse. However, $T := \{d^n+1 : n \in \mathbb{N}\}$ is not sparse, even though $(d^{n+1}+1)/(d^n+1) \to d$. Indeed, the operator A given by $-\sigma^1 + d\sigma^0$ is the constant function with image $\{d-1\}$, so $A >_T 0$, but for all $\Delta \in \mathbb{N}$, $A\sigma^\Delta z < z$ for cofinitely many $z \in T$.

Thus, the condition $r_{n+1}/r_n \to \theta > 1$ emphatically fails to be *sufficient* for the sparsity of R. However, it transpires to be *necessary*, and more can be said — see Subsection 3.2.3.

3.2.1 Basic properties

In this subsection, we state and prove some basic results about sparse predicates. For the rest of this subsection, fix a sparse predicate $R \subseteq \mathbb{N}$.

For $\mathbf{A} = (A_1, ..., A_n)$ an n-tuple of operators and $z = (z_1, ..., z_n) \in R^n$, we will write $\mathbf{A} \cdot z$ for the dot product of $\mathbf{A}$ and z: that is, $\mathbf{A} \cdot z = A_1 z_1 + \cdots + A_n z_n$. Among others, our main goal in this subsection is to show that if $\mathbf{A}$ is an n-tuple of non-zero operators, then $z \mapsto \mathbf{A} \cdot z$ defines an injective function on a natural subset of R^n (Lemma 3.2.9).

Lemma 3.2.5 [44, Lemma 2]**.** *Let A, B be operators with $A \neq_R 0$. Then, for $\Delta \in \mathbb{N}$ sufficiently large, $|A\sigma^\Delta z| > Bz$ for all $z \in R$.*

Definition 3.2.6. Let $\tilde{R} \subseteq R$. For $n, \Delta \in \mathbb{N}$, write

$$\tilde{R}^n_\Delta := \{(z_1, ..., z_n) \in \tilde{R}^n : z_i \geq \sigma^\Delta z_{i+1} \text{ for all } 1 \leq i \leq n\},$$

where $z_{n+1} := \min \tilde{R}$.

Lemma 3.2.7. *Let $n \in \mathbb{N}^+$, $\boldsymbol{A}$ be an n-tuple of operators such that $A_1 \neq_R 0$, and $\varepsilon > 0$. Then, for all $\Delta \in \mathbb{N}$ sufficiently large and $z \in R^n_\Delta$, we have*

$$(1 - \varepsilon)|A_1 z_1| < |\boldsymbol{A} \cdot z| < (1 + \varepsilon)|A_1 z_1|\,,$$

and $\boldsymbol{A} \cdot z$ has the same sign as $A_1 z_1$.

Proof. By Lemma 3.2.5, there is $\Lambda \in \mathbb{N}$ such that for all $z_2, ..., z_n \in R$,

$$\big|(A_2, ..., A_n) \cdot (z_2, ..., z_n)\big| \leq |A_2 z_2| + \cdots + |A_n z_n| < \sigma^\Lambda z_2 + \cdots + \sigma^\Lambda z_n,$$

whence for all $\Delta \in \mathbb{N}$ and $z \in R^n_\Delta$, $\big|(A_2, ..., A_n) \cdot (z_2, ..., z_n)\big| < n\sigma^{-\Delta+\Lambda}(z_1)$. Thus, by Lemma 3.2.5, for all $\Delta \in \mathbb{N}$ sufficiently large and $z \in R^n_\Delta$, we have $\big|(A_2, ..., A_n) \cdot (z_2, ..., z_n)\big| < \varepsilon |A_1 z_1|$. □

Lemma 3.2.8. *Let A be an operator. If $A >_R 0$ (respectively, $A <_R 0$) then there is $r \in \mathbb{Q}_{>1}$ such that $A\sigma z > rAz$ (respectively, $A\sigma z < rAz$) for cofinitely many $z \in R$. In particular, the function $R \to \mathbb{Z}, z \mapsto Az$ is eventually strictly increasing (respectively, decreasing).*

Proof. We first prove the lemma assuming $A >_R 0$. By Lemma 3.2.5, there is $\Delta \in \mathbb{N}$ such that $A\sigma^\Delta z > 2Az$ for all $z \in R$. Fix $r \in \mathbb{Q}_{>1}$ such that $r^\Delta < 2$; write $r = p/q$ for $p, q \in \mathbb{N}^+$. Let B be the operator defined by $Bz = qA\sigma z - pAz$. If $B \leq_R 0$ then $A\sigma z \leq rAz$ for cofinitely many $z \in R$, whence $A\sigma^\Delta z \leq r^\Delta Az < 2Az$ for cofinitely many $z \in R$, a contradiction. By (S1), we must thus have that $B >_R 0$, whence $A\sigma z > rAz$ for cofinitely many $z \in R$.

If $A <_R 0$, apply the lemma to $-A >_R 0$. □

Here and henceforth, given an n-tuple $\nu = (\nu_1, ..., \nu_n)$ and $1 \leq i \leq n$, we let $\nu_{>i}$ denote $(\nu_{i+1}, ..., \nu_n)$, $\nu_{\geq i}$ denote $(\nu_i, ..., \nu_n)$, and so on.

Lemma 3.2.9. *Let $n \in \mathbb{N}^+$, $\boldsymbol{A}$ be an n-tuple of operators, and $\Delta \in \mathbb{N}$ be sufficiently large.*

Let $z, w \in R^n_\Delta$ be such that $i := \min\{e \in [n] : z_e \neq w_e, A_e \neq 0\}$ is well-defined, and suppose $z_i > w_i$. Then $\boldsymbol{A} \cdot z > \boldsymbol{A} \cdot w$ if $A_i > 0$, and $\boldsymbol{A} \cdot z < \boldsymbol{A} \cdot w$ if $A_i < 0$.

In particular, if $\boldsymbol{A}$ is a tuple of non-zero operators, then $z \mapsto \boldsymbol{A} \cdot z$ defines an injective function on R^n_Δ.

Proof. We first prove the lemma assuming $A_i > 0$. By Lemma 3.2.8, there is $r \in \mathbb{Q}_{>1}$ such that $A_i \sigma x > r A_i x$ for sufficiently large $x \in R$, say for $x \geq \sigma^\Delta(\min R)$, taking $\Delta \in \mathbb{N}$ to be sufficiently large. Let $k \in \mathbb{N}^+$ be such that $r > 1 + 1/k$. By Lemma 3.2.7, taking $\Delta \in \mathbb{N}$ to be sufficiently large, we have

$$\begin{aligned}\mathbf{A}_{\geq i} \cdot z_{\geq i} > \left(1 - \frac{1}{4k}\right) A_i z_i > \left(1 - \frac{1}{4k}\right)\left(1 + \frac{1}{k}\right) A_i w_i \geq \left(1 + \frac{1}{2k}\right) A_i w_i \\ > \mathbf{A}_{\geq i} \cdot w_{\geq i},\end{aligned}$$

where the second inequality follows from the fact that $A_i \sigma x > r A_i x$ for all $x \geq \sigma^\Delta(\min R)$ and $w_i \geq \sigma^\Delta(\min R)$ since $w \in R^n_\Delta$. But now

$$\begin{aligned}\mathbf{A} \cdot z = \mathbf{A}_{<i} \cdot z_{<i} + \mathbf{A}_{\geq i} \cdot z_{\geq i} = \mathbf{A}_{<i} \cdot w_{<i} + \mathbf{A}_{\geq i} \cdot z_{\geq i} > \mathbf{A}_{<i} \cdot w_{<i} + \mathbf{A}_{\geq i} \cdot w_{\geq i} \\ = \mathbf{A} \cdot w.\end{aligned}$$

If $A_i < 0$, apply the lemma to $-A$. □

Remark 3.2.10. In this chapter, we frequently consider tuples $z \in R^n_\Delta$ for some sufficiently large $\Delta \in \mathbb{N}$ rather than $z \in R^n$. The reason for this is that, as shown in the preceding lemmas, R^n_Δ is much better-behaved than R^n. We illustrate this by considering Lemma 3.2.9 for the sparse predicate $R = 2^{\mathbb{N}}$.

As shown in Example 3.2.1, in this context an operator is multiplication by a constant, so let us consider the 3-tuple of operators $\mathbf{A} = (1, 2, 4)$, where 4 denotes multiplication by 4, and so on. By Lemma 3.2.9, if $\Delta \in \mathbb{N}$ is sufficiently large, then the function $z \mapsto \mathbf{A} \cdot z$ is injective on $(2^{\mathbb{N}})^3_\Delta$. In other words, if $x = z_1 + 2z_2 + 4z_3$ for some $z \in (2^{\mathbb{N}})^3_\Delta$, then we can read off z_1, z_2, and z_3 uniquely from x. The following example illustrates the *necessity* of Δ being sufficiently large:

$$96 = 1(32) + 2(16) + 4(8) = 1(64) + 2(8) + 4(4).$$

Meanwhile, the *sufficiency* of Δ being sufficiently large ($\Delta \geq 2$) is clear from the uniqueness of binary expansions, and this is a special case of Lemma 3.2.9.

3.2.2 The $P_\Delta(\cdot; \mathbf{A}, \tilde{R})$ and $Q_\Delta(\cdot; \mathbf{A}, \tilde{R})$ functions

In this subsection, we introduce two functions that are crucial for the rest of the chapter. Throughout this subsection, fix a sparse predicate $R \subseteq \mathbb{N}$, enumerated by the increasing sequence $(r_n : n \in \mathbb{N})$.

Definition 3.2.11. Let $d \in \mathbb{N}^+$, and let $\tilde{R} \subseteq R$ be definable in $(\mathbb{Z}, <, +, R)$. Write $\tilde{R} \subseteq^d R$ if there is $N \subseteq \mathbb{N}$ such that $\tilde{R} := \{r_{N+dt} : t \in \mathbb{N}\}$.

This definition is motivated by the following lemma. For $m, d \in \mathbb{N}^+$, say that R is *eventually periodic* mod m with *minimum period* d if there is $N \in \mathbb{N}^+$ such that $(r_n \mod m : n \geq N)$ is periodic with period d, and for all $d' < d$ and $N \in \mathbb{N}$, there is $n \geq N$ such that $r_n \not\equiv_m r_{n+d'}$ (recall the notation that $x \equiv_m y :\Leftrightarrow x \equiv y \mod m$). Such R is not definable in $(\mathbb{Z}, <, +)$ in general.

Lemma 3.2.12. *Let $m, d \in \mathbb{N}^+$, and suppose R is eventually periodic mod m with minimum period d. Then, for all $N \in \mathbb{N}$, the set $\tilde{R} := \{r_{N+dt} : t \in \mathbb{N}\} \subseteq R$ is definable in $(\mathbb{Z}, <, +, R)$, and thus $\tilde{R} \subseteq^d R$.*

Proof. Up to excluding finitely many elements from $\tilde{R}$ (which does not affect the definability of $\tilde{R}$), we may assume that $(r_n : n \geq N)$ is periodic mod m with minimum period d. Then, for $z \in R$,

$$z \in \tilde{R} \Leftrightarrow z \geq r_N \wedge \bigwedge_{p=0}^{d-1} \sigma^p z \equiv_m \sigma^p r_N.$$

The relation $x \equiv_m y$ is definable in $(\mathbb{Z}, <, +)$, and so $\tilde{R}$ is definable in $(\mathbb{Z}, <, +, R)$. $\square$

Definition 3.2.13. Let $n \in \mathbb{N}^+$, $\mathbf{A}$ be an n-tuple of non-zero operators, and $\Delta \in \mathbb{N}$ be sufficiently large such that the function $z \mapsto \mathbf{A} \cdot z$ is injective on R^n_Δ. For $S \subseteq R^n_\Delta$, write $\mathbf{A} \cdot S := \{\mathbf{A} \cdot z : z \in S\}$. For $\emptyset \neq S \subseteq R^n_\Delta$ such that $\mathbf{A} \cdot S$ is bounded below, let

$$\min_{\mathbf{A}} S := \text{ the unique } z \in S \text{ such that } \mathbf{A} \cdot z = \min \mathbf{A} \cdot S.$$

Similarly, for $\emptyset \neq S \subseteq R^n_\Delta$ such that $\mathbf{A} \cdot S$ is bounded above, let

$$\max_{\mathbf{A}} S := \text{ the unique } z \in S \text{ such that } \mathbf{A} \cdot z = \max \mathbf{A} \cdot S.$$

Definition 3.2.14. Let $d, n \in \mathbb{N}^+$, $\tilde{R} \subseteq^d R$, $\mathbf{A}$ be an n-tuple of non-zero operators, and $\Delta \in d\mathbb{N}$ be sufficiently large such that the function $z \mapsto \mathbf{A} \cdot z$ is injective on R^n_Δ. For $x \in \mathbb{Z}$, let

$$P_\Delta(x; \mathbf{A}, \tilde{R}) := \begin{cases} \max_{\mathbf{A}}\{z \in \tilde{R}^n_\Delta : \mathbf{A} \cdot z < x\} & \text{if } x > \inf \mathbf{A} \cdot \tilde{R}^n_\Delta, \\ \min_{\mathbf{A}} \tilde{R}^n_\Delta & \text{otherwise,} \end{cases}$$

$$Q_\Delta(x; \mathbf{A}, \tilde{R}) := \begin{cases} \min_{\mathbf{A}}\{z \in \tilde{R}^n_\Delta : \mathbf{A} \cdot z \geq x\} & \text{if } x \leq \sup \mathbf{A} \cdot \tilde{R}^n_\Delta, \\ \max_{\mathbf{A}} \tilde{R}^n_\Delta & \text{otherwise.} \end{cases}$$

For $1 \leq i \leq n$, write $P^i_\Delta(x; \mathbf{A}, \tilde{R})$ for $P_\Delta(x; \mathbf{A}, \tilde{R})_i$ and write $Q^i_\Delta(x; \mathbf{A}, \tilde{R})$ for $Q_\Delta(x; \mathbf{A}, \tilde{R})_i$. The parameter $\tilde{R}$ is dropped where obvious from context.

Remark 3.2.15. (i) In other words, if $x > \inf \mathbf{A} \cdot \tilde{R}^n_\Delta$, then $P_\Delta(x; \mathbf{A}, \tilde{R})$ is the element $z \in \tilde{R}^n_\Delta$ maximising $\mathbf{A} \cdot z$ subject to $\mathbf{A} \cdot z < x$. Similarly, if $x \leq \sup \mathbf{A} \cdot \tilde{R}^n_\Delta$, then $Q_\Delta(x; \mathbf{A}, \tilde{R})$ is the element $z \in \tilde{R}^n_\Delta$ minimising $\mathbf{A} \cdot z$ subject to $\mathbf{A} \cdot z \geq x$.

(ii) If $x \leq \inf \mathbf{A} \cdot \tilde{R}^n_\Delta$, then $A_1 >_R 0$ (as otherwise $\inf \mathbf{A} \cdot \tilde{R}^n_\Delta = -\infty$). In this case, by Lemma 3.2.9, $P_\Delta(x;\mathbf{A},\tilde{R}) = \min_{\mathbf{A}} \tilde{R}^n_\Delta$ is the lexicographically minimal element of $\tilde{R}^n_\Delta$, namely,

$$(\sigma^{(n-i)\Delta}(\min \tilde{R}) : 0 \le i < n).$$

Similarly, if $x > \sup \mathbf{A} \cdot \tilde{R}^n_\Delta$, then $A_1 <_R 0$ and

$$Q_\Delta(x;\mathbf{A},\tilde{R}) = \max_{\mathbf{A}} \tilde{R}^n_\Delta = (\sigma^{(n-i)\Delta}(\min \tilde{R}) : 0 \le i < n).$$

Example 3.2.16. As in Remark 3.2.10, consider the example $R = 2^{\mathbb{N}}$ and $\mathbf{A} = (1,2,4)$. Let $\Delta = 2$; it is easy to verify that $z \mapsto \mathbf{A}\cdot z$ is injective on R^3_2. The first four elements of $\mathbf{A}\cdot R^3_2$ are

$$1(16)+2(4)+4(1) = 28, \quad 1(32)+2(4)+4(1) = 44,$$
$$1(32)+2(8)+4(1) = 52, \quad 1(32)+2(8)+4(2) = 56.$$

Since $44 < 47 \le 52$, we have $P_2(47;\mathbf{A},R) = (32,4,1)$ and $Q_2(47;\mathbf{A},R) = (32,8,1)$. Moreover, for all $x \le 28 = \inf \mathbf{A}\cdot R^3_2$, we have $P_2(x;\mathbf{A},R) = (16,4,1)$.

The following lemma establishes some basic properties of $P_\Delta(\cdot;\mathbf{A},\tilde{R})$ and $Q_\Delta(\cdot;\mathbf{A},\tilde{R})$. The proofs are rather straightforward but we include them to provide more intuition on these functions.

Lemma 3.2.17. *Let $\tilde{R} \subseteq^d R$ for some $d \in \mathbb{N}^+$. Let $n \in \mathbb{N}^+$, $\boldsymbol{A}$ be an n-tuple of non-zero operators, and $\Delta \in d\mathbb{N}$ be sufficiently large. Then the following hold.*

(i) For all $x \in \mathbb{Z}$, $x > \boldsymbol{A}\cdot P_\Delta(x;\boldsymbol{A},\tilde{R})$ if and only if $x > \inf \boldsymbol{A}\cdot \tilde{R}^n_\Delta$, and $x \le \boldsymbol{A}\cdot Q_\Delta(x;\boldsymbol{A},\tilde{R})$ if and only if $x \le \sup \boldsymbol{A}\cdot\tilde{R}^n_\Delta$.

(ii) For all $x \in \mathbb{Z}$, $Q^1_\Delta(x;\boldsymbol{A},\tilde{R}) = \sigma^{\varepsilon d} P^1_\Delta(x;\boldsymbol{A},\tilde{R})$ for some $\varepsilon \in \{-1,0,1\}$.

Proof. We first prove (i). If $x \le \inf \mathbf{A}\cdot\tilde{R}^n_\Delta$, then we have $x \le \mathbf{A}\cdot P_\Delta(x;\mathbf{A})$ since $P_\Delta(x;\mathbf{A}) \in \tilde{R}^n_\Delta$. If $x > \inf \mathbf{A}\cdot\tilde{R}^n_\Delta$, then we have $x > \mathbf{A}\cdot P_\Delta(x;\mathbf{A})$ since $P_\Delta(x;\mathbf{A}) \in \{z \in \tilde{R}^n_\Delta : \mathbf{A}\cdot z < x\}$. The statement for $Q_\Delta(\cdot;\mathbf{A})$ can be proven similarly.

We now prove (ii). If $x \leq \inf \mathbf{A} \cdot \tilde{R}^n_\Delta$, then

$$Q_\Delta(x; \mathbf{A}) = \min_{\mathbf{A}}\{z \in \tilde{R}^n_\Delta : \mathbf{A} \cdot z \geq x\} = \min_{\mathbf{A}} \tilde{R}^n_\Delta = P_\Delta(x; \mathbf{A}).$$

Similarly, if $x > \sup \mathbf{A} \cdot \tilde{R}^n_\Delta$, then $Q_\Delta(x; \mathbf{A}) = P_\Delta(x; \mathbf{A})$, so consider the case where $\inf \mathbf{A} \cdot \tilde{R}^n_\Delta < x \leq \sup \mathbf{A} \cdot \tilde{R}^n_\Delta$. Then by definition and part (i) we have that $\mathbf{A} \cdot P_\Delta(x; \mathbf{A}) < x \leq \mathbf{A} \cdot Q_\Delta(x; \mathbf{A})$, and there is no $z \in \tilde{R}^n_\Delta$ such that $\mathbf{A} \cdot P_\Delta(x; \mathbf{A}) < \mathbf{A} \cdot z < \mathbf{A} \cdot Q_\Delta(x; \mathbf{A})$. By Lemma 3.2.9, we are done. □

3.2.3 Sparsity as regularity

We conclude this section by proving that the notion of a sparse predicate coincides with that of a *regular* predicate, defined by Lambotte and Point in [34] and recalled below.

Definition 3.2.18 [34]**.** Let $R \subseteq \mathbb{N}$ be enumerated by the increasing sequence $(r_n : n \in \mathbb{N})$. Say that R is *regular* if $r_{n+1}/r_n \to \theta \in \mathbb{R}_{>1} \cup \{\infty\}$ and, if θ is algebraic over $\mathbb{Q}$ with minimal polynomial $f(x)$, then the operator $f(\sigma) =_R 0$, that is, if $f(x) = \sum_{i=0}^{k} a_i x^i$ then for all $n \in \mathbb{N}$ we have

$$\sum_{i=0}^{k} a_i r_{n+i} = 0.$$

Call θ the *limit ratio* of R.

Lambotte and Point prove that regular predicates are sparse [34, Lemma 2.26]. It turns out that these notions coincide.

Theorem 3.2.19. *Let $R \subseteq \mathbb{N}$. Then R is sparse if and only if R is regular.*

Proof. It suffices to prove the forward direction. Let R be a sparse predicate, enumerated by the increasing sequence $(r_n : n \in \mathbb{N})$. If r_{n+1}/r_n does not converge or diverge, then $\liminf_{n\to\infty} r_{n+1}/r_n \neq \limsup_{n\to\infty} r_{n+1}/r_n$, and so there is some $p \in \mathbb{Q}_{>1}$ such that $\{n \in \mathbb{N} : r_{n+1}/r_n > p\}$ and $\{n \in \mathbb{N} : r_{n+1}/r_n < p\}$ are both infinite. But now, writing $p = a/b$ for $a, b \in \mathbb{N}^+$, the operator A given by

$z \mapsto b\sigma z - az$ satisfies that $Az > 0$ for infinitely many $z \in R$ and $Az < 0$ for infinitely many $z \in R$, a contradiction to (S1).

Thus, $r_{n+1}/r_n \to \theta$ for some $\theta \in \mathbb{R}_{\geq 1} \cup \{\infty\}$. By Lemma 3.2.8 applied to the identity operator, there is $q \in \mathbb{Q}_{>1}$ such that $r_{n+1}/r_n > q$ for all sufficiently large n, so $\theta \neq 1$. Suppose θ is algebraic over $\mathbb{Q}$ with minimum polynomial $f(x) = \sum_{i=0}^{k} a_i x^i$. Towards a contradiction, suppose $f(\sigma) \neq_R 0$. Let $g := f$ if $f(\sigma) >_R 0$, and $g := -f$ if $f(\sigma) <_R 0$. Then, $g(\sigma) >_R 0$, so by (S2), there is $\Delta \in \mathbb{N}$ such that $g(\sigma) r_{n+\Delta} > r_n$ for all $n \in \mathbb{N}$. But

$$\frac{g(\sigma) r_n}{r_n} = \pm \sum_{i=0}^{k} a_i \left(\frac{r_{n+i}}{r_n}\right) \to \pm \sum_{i=0}^{k} a_i \theta^i = 0,$$

and $r_{n+\Delta}/r_n \to \theta^\Delta$, so

$$\frac{g(\sigma) r_{n+\Delta}}{r_n} = \frac{g(\sigma) r_{n+\Delta}}{r_{n+\Delta}} \frac{r_{n+\Delta}}{r_n} \to 0,$$

contradicting the fact that $g(\sigma) r_{n+\Delta} > r_n$ for all $n \in \mathbb{N}$. □

We find that the notion of regularity gives better intuition for what a sparse (equivalently, regular) predicate looks like. In particular, we have the following.

Corollary 3.2.20. *Let $\theta \in \mathbb{R}_{>1} \cup \{\infty\}$ be such that θ is not algebraic over $\mathbb{Q}$. Then there is a sparse (equivalently, regular) predicate $R \subseteq \mathbb{N}$ with limit ratio θ that is not congruence-periodic.*

Proof. For all functions $\varepsilon : \mathbb{N} \to \{0, 1\}$, the predicate $R_\varepsilon \subseteq \mathbb{N}$ enumerated by $(\lfloor \theta^n \rfloor + \varepsilon(n) : n \in \mathbb{N})$ is sparse with limit ratio θ, where for $\theta = \infty$ we define $\theta^n := n!$. It is straightforward to observe that there is $\varepsilon : \mathbb{N} \to \{0, 1\}$ such that R_ε is not eventually periodic mod 2. □

3.3 Reduction to representative formulas

The goal of this section is to find formulas for which constructing strong honest definitions is sufficient for the distality of the structure; this is achieved in

Theorem 3.3.6.

We begin by establishing a 'normal form' for formulas in $(\mathbb{Z}, <, +, R)$, where $R \subseteq \mathbb{N}$ is sparse and congruence-periodic. (Recall that R is *congruence-periodic* if, for all $m \in \mathbb{N}^+$, R is eventually periodic mod m.) The following fact is due to Semenov.

Fact 3.3.1 [44, Theorem 3]**.** *Let $R \subseteq \mathbb{N}$ be sparse. Modulo $(\mathbb{Z}, <, +, R)$, every formula $\phi(x)$ with $|x| = 1$ is equivalent to a disjunction of formulas of the form*

$$\exists z \in R^n \left(\bigwedge_{j=1}^{k} f_j(x) > \boldsymbol{A}^{(j)} \cdot z \wedge \bigwedge_{p=1}^{l} g_p(x) \equiv_{m_p} \boldsymbol{B}^{(p)} \cdot z \wedge \psi(z) \right),$$

where $m_p \in \mathbb{N}^+$, $f_j(x), g_p(x)$ are $\mathbb{Z}$-affine functions, $\boldsymbol{A}^{(j)}, \boldsymbol{B}^{(p)}$ are n-tuples of operators, and $\psi(z)$ is a formula in $(R, <, \sigma, (\cdot \equiv_m c)_{c,m \in \mathbb{N}^+})$.

We will show that this normal form can be simplified if $R \subseteq \mathbb{N}$ is also congruence-periodic.

Theorem 3.3.2. *Let $R \subseteq \mathbb{N}$ be sparse and congruence-periodic. Then, modulo $(\mathbb{Z}, <, +, R)$, every formula $\phi(x)$ with $|x| = 1$ is equivalent to a disjunction of formulas of the form*

$$\exists z \in R^n \left(\bigwedge_{j=1}^{k} f_j(x) > \boldsymbol{A}^{(j)} \cdot z \wedge \bigwedge_{p=1}^{l} g_p(x) \equiv_{m_p} \boldsymbol{B}^{(p)} \cdot z \right),$$

where $m_p \in \mathbb{N}^+$, $f_j(x), g_p(x)$ are $\mathbb{Z}$-affine functions, and $\boldsymbol{A}^{(j)}, \boldsymbol{B}^{(p)}$ are n-tuples of operators.

The key to our proof is the following lemma, which states that if $R \subseteq \mathbb{N}$ is congruence-periodic, then the structure $(R, <, \sigma, (\cdot \equiv_m c)_{c,m \in \mathbb{N}^+})$ has quantifier elimination after expanding by a constant for $\min R$.

Lemma 3.3.3. *Let $R \subseteq \mathbb{N}$ be congruence-periodic. Then the theory $T :=$ Th$(R, <, \sigma, (\cdot \equiv_m c)_{c,m \in \mathbb{N}^+}, r_0)$ has quantifier elimination, where r_0 is a constant interpreted as* $\min R$.

Proof. Fix $\phi(x, \bar{y})$, a conjunction of atomic and negated atomic formulas involving x, where x is a singleton variable. It suffices to prove the following. Let $R_1, R_2 \models T$ have common substructure B, and let $\bar{b}$ be a tuple from B of length $|\bar{y}|$ such that $R_1 \models \exists x\ \phi(x, \bar{b})$. Then, $R_2 \models \exists x\ \phi(x, \bar{b})$.

Atomic and negated atomic formulas involving x have one of the following forms, for $i, j \in \mathbb{N}$, $\Box \in \{=, \neq, <, \leq, >, \geq\}$, and $c, m \in \mathbb{N}^+$:

(i) $\sigma^i x \Box \sigma^j x$, which is equivalent to $\top$ or $\bot$;

(ii) $\sigma^i y \Box \sigma^j x$, where y is a variable or r_0, which is equivalent to $\sigma^{i+k} y \Box \sigma^{j+k} x$ for all $k \in \mathbb{N}$;

(iii) $\sigma^i x \equiv_m c$;

(iv) $\sigma^i x \not\equiv_m c$, which is equivalent to $\bigvee_{b=1, b\neq c}^{m} \sigma^i x \equiv_m b$.

By the Chinese Remainder Theorem, we may assume that all congruences in $\phi(x, \bar{y})$ have the same modulus. Moreover, observe that $\sigma^i y = \sigma^j x$ is equivalent to $\sigma^i y < \sigma^{j+1} x < \sigma^{i+2} y$ and $\sigma^i y \neq \sigma^j x$ is equivalent to $\sigma^i y < \sigma^j x \vee \sigma^j x < \sigma^i y$. Thus, we may assume that $\phi(x, \bar{y}) = \phi(x, (y_{i,0}, y_{i,1})_{0 \leq i \leq l})$ is of the form

$$\bigwedge_{i=0}^{l} y_{i,0} < \sigma^k x < y_{i,1} \wedge \bigwedge_{j=0}^{k'} \sigma^j x \equiv_m c_j,$$

where $k, k', l \in \mathbb{N}$ and $m, c_0, ..., c_{k'} \in \mathbb{N}^+$. We may assume $k = k'$: if $k < k'$, then $y_{i,0} < \sigma^k x < y_{i,1}$ is equivalent to $\sigma^{k'-k} y_{i,0} < \sigma^{k'} x < \sigma^{k'-k} y_{i,1}$, and if $k' < k$, then ϕ is equivalent to

$$\bigvee_{1 \leq c_{k'+1}, ..., c_k \leq m} \left(\bigwedge_{i=0}^{l} y_{i,0} < \sigma^k x < y_{i,1} \wedge \bigwedge_{j=0}^{k} \sigma^j x \equiv_m c_j \right).$$

Let $R_1, R_2 \models T$ have common substructure B, and let $\bar{b}$ be a tuple from B of length $|\bar{y}|$ such that $R_1 \models \exists x\ \phi(x, \bar{b})$. We wish to show that $R_2 \models \exists x\ \phi(x, \bar{b})$.

Since B is linearly ordered, without loss of generality, $\phi(x, \bar{b})$ is equivalent to

$$b_1 < \sigma^k x < b_2 \wedge \bigwedge_{j=0}^{k} \sigma^j x \equiv_m c_j.$$

For $i \in \{1, 2\}$ and $n \in \mathbb{N}$, let $r_n^i := (\sigma^n r_0)^{R_i}$. Let $N \in \mathbb{N}$ be such that $(r_n^1 : n \geq N)$ is periodic mod m, with minimum period d. Then the fact that $R_1 \models \exists x\, \phi(x, \bar{b})$ is witnessed by some

$$x \in \{r_n^1 : 0 \leq n \leq N + d\} \cup \{\sigma^n b_1 : 1 \leq n \leq d\}.$$

Indeed, for all $x \in R_1$, if $x > r_N^1$ then there is $1 \leq n \leq d$ such that $\sigma^j x \equiv_m \sigma^j r_{N+n}^1$ for all $0 \leq j \leq k$. Thus, if $\{r_n^1 : 0 \leq n \leq N + d\}$ does not contain a witness, then $b_1 \geq r_N^1$. But now $\{\sigma^n b_1 : 1 \leq n \leq d\}$ contains a witness, since for all $x \in R_1$, if $x > b_1$ then there is $1 \leq n \leq d$ such that $\sigma^j x \equiv_m \sigma^j \sigma^n b_1$ for all $0 \leq j \leq k$.

Thus, we have that $R_2 \models \exists x\, \phi(x, \bar{b})$, witnessed by some

$$x \in \{r_n^2 : 0 \leq n \leq N + d\} \cup \{\sigma^n b_1 : 1 \leq n \leq d\}. \qquad \square$$

We now prove Theorem 3.3.2.

Proof. Combine Fact 3.3.1 and Lemma 3.3.3. We show that if $\psi(z)$ is a formula in $(R, <, \sigma, (\cdot \equiv_m c)_{c,m\in\mathbb{N}^+})$, then ψ is equivalent in $(\mathbb{Z}, <, +, R)$ to $\bigvee_i \bigwedge_j \theta_{ij}$ for some θ_{ij} each of the form $C > \mathbf{A}\cdot z$ or $C \equiv_m \mathbf{B}\cdot z$, where $C \in \mathbb{Z}$, $\mathbf{A}, \mathbf{B}$ are n-tuples of operators, and $m \in \mathbb{N}^+$. (Note that elements of $\mathbb{Z}$ are $\mathbb{Z}$-affine functions.) By Lemma 3.3.3, it suffices to assume that $\psi(z)$ is atomic or negated atomic.

By a similar analysis to that in the proof of Lemma 3.3.3, ψ is equivalent in $(\mathbb{Z}, <, +, R)$ to a conjunction or disjunction of formulas, each taking one of the following forms, where $i, j \in \mathbb{N}$, $p, q \in [n]$, and $c, m \in \mathbb{N}^+$:

(i) $\sigma^i z_p < \sigma^j z_q$, which is equivalent to $0 > (\sigma^i, -\sigma^j) \cdot (z_p, z_q)$;

(ii) $\pm\sigma^i r_0 > \pm\sigma^j z_q$;

(iii) $c \equiv_m \sigma^j z_q$.

Since $\sigma^j z_q = (0, ..., 0, \sigma^j, 0, ..., 0) \cdot z$, and so on, all the formulas above are in the required form. □

For the rest of the chapter, fix a congruence-periodic sparse predicate $R \subseteq \mathbb{N}$.

Our goal is to write the formulas in Theorem 3.3.2 as Boolean combinations of formulas for which we can construct strong honest definitions; this is achieved in Theorem 3.3.6.

Let $\mathcal{L}^0 := (<, +)$ and $\mathcal{L} := (<, +, R)$.

Definition 3.3.4. Let $\phi(x; y)$ be an $\mathcal{L}$-formula with $|x| = 1$. Say that $\phi(x; y)$ is a *basic* formula if it is a Boolean combination of formulas not involving x and descendants of $\mathcal{L}^0$-formulas.

Note that basic formulas have strong honest definitions by Example 2.7.16, Lemma 2.7.14, and the fact that formulas not involving x have $\top$ as a strong honest definition.

For $n \in \mathbb{N}^+$ and $1 \leq i \leq n$, let $\mathbf{F}^i$ be the 'i^{th} standard n-tuple of operators' (where n is assumed to be obvious from context): for $1 \leq j \leq n$,

$$\mathbf{F}^i_j = \begin{cases} \text{the identity function} & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

We now fix notation for some formulas of particularly desirable forms.

Definition 3.3.5. Let $d, n \in \mathbb{N}^+$, $\tilde{R} \subseteq^d R$, and $\phi(x; ...)$ be an $\mathcal{L}$-formula with $|x| = 1$.

Let y be a tuple of variables. Say that $\phi = \phi(x; y)$ is of the form $(E_n; \tilde{R})$, or just (E_n), if

$$\phi(x; y) = \exists z \in \tilde{R}_0^n \bigwedge_{j=1}^{k} f_j(x, y) > \mathbf{A}^{(j)} \cdot z,$$

where $f_1(x, y), ..., f_k(x, y)$ are $\mathbb{Z}$-affine functions, and $\mathbf{A}^{(1)}, ..., \mathbf{A}^{(k)}$ are n-tuples of operators.

Let $\Delta \in d\mathbb{N}$, y_1, y_2 be singleton variables, and $\mathbf{A}, \mathbf{B}$ be n-tuples of operators.

Say that $\phi = \phi(x; y_1, y_2)$ is of the form $(F_n; \mathbf{A}, \mathbf{B}, \tilde{R}, \Delta)$, or just (F_n), if $\mathbf{A}$ is a tuple of non-zero operators and

$$\phi(x; y_1, y_2) = tx - y_2 < \mathbf{B} \cdot P_\Delta(x - y_1; \mathbf{A}, \tilde{R}),$$

where $t \in \{0, 1\}$ with $t = 1$ unless $\mathbf{B} = \mathbf{F}^i$ for some $1 \leq i \leq n$.

Let u, v be n-tuples of variables, and let $T_{\tilde{R}}(u, v)$ be the formula saying that $u_1, v_1, ..., u_n, v_n \in \tilde{R}$. Say that $\phi = \phi(x; y_1, y_2, u, v)$ is of the form $(G_n; \mathbf{A}, \mathbf{B}, \tilde{R}, \Delta)$, or just (G_n), if either

$$\phi(x; y_1, y_2, u, v) = T_{\tilde{R}}(u, v) \wedge \exists z \in \tilde{R}^n_\Delta \left(y_1 + \mathbf{A} \cdot z < x < y_2 + \mathbf{B} \cdot z \wedge \bigwedge_{i=1}^n u_i \leq z_i \leq v_i \right),$$

or ϕ is obtained from the formula above by deleting some of the u_i (equivalently, setting $u_i = -\infty$) and/or deleting some of the v_i (equivalently, setting $v_i = +\infty$).

It will be convenient to extend the definition of (E_n) formulas to $n = 0$; that is, $\phi(x; y)$ with $|x| = 1$ is of the form (E_0) if

$$\phi(x; y) = \bigwedge_{j=1}^k f_j(x, y) > 0,$$

where $f_1(x, y), ..., f_k(x, y)$ are $\mathbb{Z}$-affine functions. Such formulas are basic.

Our goal is to prove the following theorem.

Theorem 3.3.6. *The following criterion is sufficient for the distality of the structure $(\mathbb{Z}, <, +, R)$.*

Let $d, n \in \mathbb{N}^+$, $\tilde{R} \subseteq^d R$, $\boldsymbol{A}$ be an n-tuple of non-zero operators, and $\boldsymbol{B}$ be an n-tuple of operators. Then, for all sufficiently large $\Delta \in d\mathbb{N}$, every $(F_n; \boldsymbol{A}, \boldsymbol{B}, \tilde{R}, \Delta)$ formula has a strong honest definition.

We prove this in three steps. We first show that every $\mathcal{L}$-formula $\phi(x; y)$ with $|x| = 1$ is equivalent to a Boolean combination of basic formulas and descendants of (E_n) formulas (Proposition 3.3.7). We then show that every (E_n) formula is equivalent to a Boolean combination of basic formulas and descendants of (E_{n-1}) or (G_n) formulas (Proposition 3.3.8). Finally, we show that every (G_n) formula is equivalent to a Boolean combination of basic formulas, (E_{n-1}) formulas, and descendants of (F_n) formulas (Proposition 3.3.10).

Our first checkpoint is the following proposition.

Proposition 3.3.7. *Modulo $(\mathbb{Z}, <, +, R)$, every formula $\phi(x; y)$ with $|x| = 1$ is equivalent to a Boolean combination of basic formulas and descendants of (E_n) formulas.*

Proof. By Theorem 3.3.2, every partitioned $\mathcal{L}$-formula $\phi(x; y)$ with $|x| = 1$ is equivalent to a disjunction of formulas of the form

$$\exists z \in R^n \left(\bigwedge_{j=1}^{k} f_j(x, y) > \mathbf{A}^{(j)} \cdot z \wedge \bigwedge_{p=1}^{l} g_p(x, y) \equiv_{m_p} \mathbf{B}^{(p)} \cdot z \right),$$

where $m_p \in \mathbb{N}^+$, $f_j(x, y), g_p(x, y)$ are $\mathbb{Z}$-affine functions, and $\mathbf{A}^{(j)}, \mathbf{B}^{(p)}$ are n-tuples of operators. By the Chinese Remainder Theorem, it suffices to assume that there is $m \in \mathbb{N}^+$ such that $m = m_p$ for all $1 \leq p \leq l$.

It suffices to show that every such formula is equivalent to a Boolean combination of basic formulas and descendants of (E_s) formulas for some $s \in \mathbb{N}$. We do so by induction on $n \in \mathbb{N}$. When $n = 0$, the formula is a basic formula. Now let $n \geq 1$, and let

$$\phi(x, y) := \exists z \in R^n \left(\bigwedge_{j=1}^{k} f_j(x, y) > \mathbf{A}^{(j)} \cdot z \wedge \bigwedge_{p=1}^{l} g_p(x, y) \equiv_{m} \mathbf{B}^{(p)} \cdot z \right),$$

where $m \in \mathbb{N}^+$, $f_j(x, y), g_p(x, y)$ are $\mathbb{Z}$-affine functions, and $\mathbf{A}^{(j)}, \mathbf{B}^{(p)}$ are n-tuples of operators.

Let $(r_n : n \in \mathbb{N})$ be an increasing enumeration of R. Since R is congruence-periodic, there are $d, N \in \mathbb{N}$ such that $(r_n : n \geq N)$ is periodic mod m with minimum period d. Observe that $\phi(x; y)$ is equivalent to $\phi_0(x; y) \vee \phi_1(x; y)$, where $\phi_0(x;y)$ is the formula

$$\bigvee_{i=1}^{n}\bigvee_{\alpha=0}^{N-1}\exists z\in R^n\left(z_i=r_\alpha\wedge\bigwedge_{j=1}^{k}f_j(x,y)>\mathbf{A}^{(j)}\cdot z\wedge\bigwedge_{p=1}^{l}g_p(x,y)\equiv_m\mathbf{B}^{(p)}\cdot z\right)$$

and $\phi_1(x;y)$ is the formula

$$\exists z\in R^n\left(\bigwedge_{i=1}^{n}z_i\geq r_N\wedge\bigwedge_{j=1}^{k}f_j(x,y)>\mathbf{A}^{(j)}\cdot z\wedge\bigwedge_{p=1}^{l}g_p(x,y)\equiv_m\mathbf{B}^{(p)}\cdot z\right).$$

Consider $\phi_0(x;y)$. Replacing z_i with r_α in the $(i,\alpha)^{\text{th}}$ disjunct, $\phi_0(x;y)$ is equivalent to a disjunction of formulas of the form

$$\exists w\in R^{n-1}\left(\bigwedge_{j=1}^{k}f'_j(x,y)>\mathbf{A}'^{(j)}\cdot w\wedge\bigwedge_{p=1}^{l}g'_p(x,y)\equiv_m\mathbf{B}'^{(p)}\cdot w\right),$$

where $f'_j(x,y),g'_p(x,y)$ are $\mathbb{Z}$-affine functions and $\mathbf{A}'^{(j)},\mathbf{B}'^{(p)}$ are $(n-1)$-tuples of operators. By the induction hypothesis, such formulas are equivalent to a Boolean combination of basic formulas and descendants of (E_s) formulas for some $s\in\mathbb{N}$.

Consider $\phi_1(x;y)$. Let $\tilde{R}:=\{r_{N+dt}:t\in\mathbb{N}\}$. By Lemma 3.2.12, $\tilde{R}\subseteq^d R$. For $1\leq p\leq l$ and $0\leq h_1,...,h_n<d$, let $0\leq b^{(p)}_{h_1,...,h_n}<m$ be such that

$$\mathbf{B}^{(p)}\cdot(r_{N+h_1},...,r_{N+h_n})\equiv_m b^{(p)}_{h_1,...,h_n}.$$

Now $\phi_1(x;y)$ is equivalent to the disjunction over $0\leq h_1,...,h_n<d$ of

$$\bigwedge_{p=1}^{l}g_p(x,y)\equiv_m b^{(p)}_{h_1,...,h_n}\wedge\exists z\in\tilde{R}^n\bigwedge_{j=1}^{k}f_j(x,y)>\mathbf{A}^{(j)}\cdot(\sigma^{h_1}z_1,...,\sigma^{h_n}z_n).$$

But now, for all $0 \leq h_1, ..., h_n < d$,

$$\exists z \in \tilde{R}^n \bigwedge_{j=1}^{k} f_j(x,y) > \mathbf{A}^{(j)} \cdot (\sigma^{h_1} z_1, ..., \sigma^{h_n} z_n)$$

$$\Leftrightarrow \bigvee_{\tau \in \mathrm{Sym}(n)} \exists z \in \tilde{R}^n \left(\bigwedge_{i=1}^{n-1} z_{\tau(i)} \geq z_{\tau(i+1)} \wedge \bigwedge_{j=1}^{k} f_j(x,y) > \mathbf{A}^{(j)} \cdot (\sigma^{h_1} z_1, ..., \sigma^{h_n} z_n) \right)$$

$$\Leftrightarrow \bigvee_{\tau \in \mathrm{Sym}(n)} \exists z \in \tilde{R}_0^n \bigwedge_{j=1}^{k} f_j(x,y) > \mathbf{A}^{(j)} \cdot (\sigma^{h_1} z_{\tau^{-1}(1)}, ..., \sigma^{h_n} z_{\tau^{-1}(n)}),$$

where $\mathrm{Sym}(n)$ is the set of bijections from $[n]$ to $[n]$, so $\phi_1(x;y)$ is equivalent to a Boolean combination of basic formulas and $(E_n; \tilde{R})$ formulas. □

Our next checkpoint is the following proposition.

Proposition 3.3.8. *Let $d, n \in \mathbb{N}^+$, $\tilde{R} \subseteq^d R$, and $\phi(x;y)$ be an $(E_n; \tilde{R})$ formula. Then there is a finite collection $\mathcal{G}_\phi$ of pairs $(\boldsymbol{A}, \boldsymbol{B})$, where $\boldsymbol{A}, \boldsymbol{B}$ are n-tuples of operators, satisfying the following.*

For all $\Delta \in d\mathbb{N}$ sufficiently large, ϕ is equivalent to a Boolean combination of basic formulas and descendants of $(E_{n-1}; \tilde{R})$ or $(G_n; \boldsymbol{A}, \boldsymbol{B}, \tilde{R}, \Delta)$ formulas for $(\boldsymbol{A}, \boldsymbol{B}) \in \mathcal{G}_\phi$.

Towards this checkpoint, we prove the following technical lemma.

Lemma 3.3.9. *Let $d, n \in \mathbb{N}^+$, $\tilde{R} \subseteq^d R$, $\boldsymbol{A}^{(1)}, ..., \boldsymbol{A}^{(k)}$ be n-tuples of operators, and $\Delta \in d\mathbb{N}$ be sufficiently large. Then there are $1 \leq i_1, ..., i_r \leq n$, an $\mathcal{L}^0$-formula θ, and $\mathcal{L}$-definable functions $f_1, ..., f_r, u_1, ..., u_n, v_1, ..., v_n$ such that each u_i (respectively, v_i) either takes values in $\tilde{R}$ or is the constant $(-\infty)$-valued (respectively, $(+\infty)$-valued) function, satisfying that for all $y \in \mathbb{Z}^k$ and $z \in \tilde{R}^n_\Delta$, $\bigwedge_{j=1}^{k} y_j > \boldsymbol{A}^{(j)} \cdot z$ if and only if*

$$\theta(y) \wedge \left(\left(\bigwedge_{j=1}^{k} y_j > \boldsymbol{A}^{(j)} \cdot z \wedge \bigvee_{s=1}^{r} z_{i_s} = f_s(y) \right) \vee \left(\bigwedge_{i=1}^{n} u_i(y) \leq z_i \leq v_i(y) \right) \right).$$

Proof. Let $H_0 := \{j \in [k] : A_i^{(j)} =_R 0 \text{ for all } i \in [n]\}$, and for $i \in [n]$, let

$$H_i^+ := \{j \in [k] : A_i^{(j)} >_R 0, A_e^{(j)} =_R 0 \text{ for all } e < i\},$$
$$H_i^- := \{j \in [k] : A_i^{(j)} <_R 0, A_e^{(j)} =_R 0 \text{ for all } e < i\},$$

and write $H_i := H_i^- \cup H_i^+$. Then $H_0, (H_i^+, H_i^- : i \in [n])$ is a partition of $[k]$.

Let $i \in [n]$. For all $j \in H_i^+$, define the function $f_j : \mathbb{Z} \to \tilde{R}$ by

$$f_j(y) := \begin{cases} \max\{w \in \tilde{R} : \exists z \in \tilde{R}^n_\Delta(\mathbf{A}^{(j)} \cdot z < y \wedge z_i = w)\} & \text{if well-defined,} \\ \min \tilde{R} & \text{otherwise.} \end{cases}$$

By Lemma 3.2.9, for all $j \in H_i^+$, $y \in \mathbb{Z}^k$, and $z \in \tilde{R}^n_\Delta$, if $z_i < f_j(y_j)$ then $y_j > \mathbf{A}^{(j)} \cdot z$, and if $z_i > f_j(y_j)$ then $y_j < \mathbf{A}^{(j)} \cdot z$; thus,

$$y_j > \mathbf{A}^{(j)} \cdot z \Leftrightarrow (y_j > \mathbf{A}^{(j)} \cdot z \wedge z_i = f_j(y_j)) \vee z_i \leq \sigma^{-d} f_j(y_j).$$

Similarly, for all $j \in H_i^-$, defining the function $f_j : \mathbb{Z} \to \tilde{R}$ by

$$f_j(y) := \min\{w \in \tilde{R} : \exists z \in \tilde{R}^n_\Delta(\mathbf{A}^{(j)} \cdot z < y \wedge z_i = w)\},$$

we have that, for all $y \in \mathbb{Z}^k$ and $z \in \tilde{R}^n_\Delta$,

$$y_j > \mathbf{A}^{(j)} \cdot z \Leftrightarrow (y_j > \mathbf{A}^{(j)} \cdot z \wedge z_i = f_j(y_j)) \vee z_i \geq \sigma^{d} f_j(y_j).$$

For all $i \in [n]$, define $u_i(y) := \sup\{\sigma^d f_j(y) : j \in H_i^-\}$ and $v_i(y) := \inf\{\sigma^{-d} f_j(y) : j \in H_i^+\}$. Now, if $y_j > \mathbf{A}^{(j)} \cdot z$ for all $j \in [k] \setminus H_0$, then either $z_i = f_j(y_j)$ for some $i \in [n]$ and $j \in H_i$, or $u_i(y) \leq z_i \leq v_i(y)$ for all $i \in [n]$. Conversely, if $u_i(y) \leq z_i \leq v_i(y)$ for all $i \in [n]$, then $y_j > \mathbf{A}^{(j)} \cdot z$ for all $j \in [k] \setminus H_0$. Thus, for all $y \in \mathbb{Z}^k$ and $z \in \tilde{R}^n_\Delta$, $\bigwedge_{j=1}^k y_j > \mathbf{A}^{(j)} \cdot z$ if and only if

$$\bigwedge_{j \in H_0} y_j > 0 \wedge \left(\left(\bigwedge_{j=1}^k y_j > \mathbf{A}^{(j)} \cdot z \wedge \bigvee_{i=1}^n \bigvee_{j \in H_i} z_i = f_j(y_j) \right) \vee \bigwedge_{i=1}^n u_i(y) \leq z_i \leq v_i(y) \right)$$

as required. □

Proof of Proposition 3.3.8. Let

$$\phi(x;y) = \exists z \in \tilde{R}_0^n \bigwedge_{j=1}^{k} f_j(x,y) > \mathbf{A}^{(j)} \cdot z,$$

where $|x| = 1$, $f_1(x,y), ..., f_k(x,y)$ are $\mathbb{Z}$-affine functions, and $\mathbf{A}^{(1)}, ..., \mathbf{A}^{(k)}$ are n-tuples of operators. We claim that $\mathcal{G}_\phi := \{(\mathbf{A}^{(j)}, -\mathbf{A}^{(l)}) : 1 \le j, l \le k\}$ witnesses the proposition.

For all $\Delta \in d\mathbb{N}$, $\phi(x;y)$ is equivalent to the disjunction of

$$\phi'_\Delta(x;y) := \exists z \in \tilde{R}_\Delta^n \bigwedge_{j=1}^{k} f_j(x,y) > \mathbf{A}^{(j)} \cdot z$$

and

$$\bigvee_{i=1}^{n} \bigvee_{\alpha=0}^{\Delta-1} \exists z \in \tilde{R}_0^n \left(z_i = \sigma^\alpha z_{i+1} \wedge \bigwedge_{j=1}^{k} f_j(x;y) > \mathbf{A}^{(j)} \cdot z \right),$$

where $z_{n+1} := \min \tilde{R}$. Replacing z_i with $\sigma^\alpha z_{i+1}$ in the $(i,\alpha)^{\text{th}}$ disjunct, it is clear that each disjunct is equivalent to

$$\exists w \in \tilde{R}_0^{n-1} \bigwedge_{j=1}^{k} f_j(x,y) > \mathbf{B}^{(j)} \cdot w,$$

for some $(n-1)$-tuples $\mathbf{B}^{(1)}, ..., \mathbf{B}^{(k)}$ of operators, which is an $(E_{n-1}; \tilde{R})$ formula.

Consider $\phi'_\Delta(x;y)$. By multiplying both sides of the inequalities in $\phi'_\Delta(x;y)$, we may assume without loss of generality that there are some $K \in \mathbb{N}^+$ and $0 \le p \le q \le k$ such that, for $1 \le j \le k$,

$$\text{the coefficient of } x \text{ in } f_j = \begin{cases} K & \text{if } j \le p, \\ -K & \text{if } p < j \le q, \\ 0 & \text{if } q < j. \end{cases}$$

For $1 \le j \le k$, let $g_j(y) := f_j(0,y)$. Then $\bigwedge_{j=1}^{k} f_j(x,y) > \mathbf{A}^{(j)} \cdot z$ is (equivalent to)

$$\bigwedge_{j=1}^{p} -g_j(y) + \mathbf{A}^{(j)} \cdot z < Kx \wedge \bigwedge_{j=p+1}^{q} Kx < g_j(y) - \mathbf{A}^{(j)} \cdot z \wedge \bigwedge_{j=q+1}^{k} g_j(y) > \mathbf{A}^{(j)} \cdot z.$$

If $0 = p = q$, then $\phi'_\Delta(x; y)$ is a basic formula. If $0 = p < q$, then for all $\Delta \in d\mathbb{N}$, $\phi'_\Delta(x; y)$ is equivalent to

$$Kx < \sup \left\{ \inf_{p+1 \leq j \leq q} (g_j(y) - \mathbf{A}^{(j)} \cdot z) : z \in \tilde{R}^n_\Delta, \bigwedge_{j=q+1}^{k} g_i(y) > \mathbf{A}^{(i)} \cdot z \right\},$$

which is a basic formula. The case where $0 < p = q$ is similar, so let us assume $0 < p < q$. Now $\bigwedge_{j=1}^{p} -g_j(y) + \mathbf{A}^{(j)} \cdot z < Kx$ is equivalent to

$$\bigvee_{j=1}^{p} \Big(- g_j(y) + \mathbf{A}^{(j)} \cdot z < Kx \wedge \bigwedge_{\substack{i=1 \\ i \neq j}}^{p} -g_j(y) + \mathbf{A}^{(j)} \cdot z \geq -g_i(y) + \mathbf{A}^{(i)} \cdot z \Big),$$

and $\bigwedge_{j=p+1}^{q} Kx < g_j(y) - \mathbf{A}^{(j)} \cdot z$ is equivalent to

$$\bigvee_{j=p+1}^{q} \Big(Kx < g_j(y) - \mathbf{A}^{(j)} \cdot z \wedge \bigwedge_{\substack{i=p+1 \\ i \neq j}}^{q} g_j(y) - \mathbf{A}^{(j)} \cdot z \leq g_i(y) - \mathbf{A}^{(i)} \cdot z \Big).$$

Thus, for all $\Delta \in d\mathbb{N}$, $\phi'_\Delta(x; y)$ is equivalent to

$$\bigvee_{j=1}^{p} \bigvee_{l=p+1}^{q} \exists z \in \tilde{R}^n_\Delta \Big(- g_j(y) + \mathbf{A}^{(j)} \cdot z < Kx < g_l(y) - \mathbf{A}^{(l)} \cdot z \wedge h_{jl}(y, z) \Big),$$

where $h_{jl}(y, z)$ is

$$\bigwedge_{\substack{i=1 \\ i \neq j}}^{p} g_i(y) - g_j(y) \geq (\mathbf{A}^{(i)} - \mathbf{A}^{(j)}) \cdot z$$
$$\wedge \bigwedge_{\substack{i=p+1 \\ i \neq l}}^{q} g_i(y) - g_l(y) \geq (\mathbf{A}^{(i)} - \mathbf{A}^{(l)}) \cdot z \wedge \bigwedge_{i=q+1}^{k} g_i(y) > \mathbf{A}^{(i)} \cdot z.$$

Apply Lemma 3.3.9 to each $h_{jl}(y,z)$, assuming $\Delta \in d\mathbb{N}$ is sufficiently large. For all $1 \le j \le p < l \le q$, there are $1 \le i_1^{jl}, ..., i_{r(j,l)}^{jl} \le n$, an $\mathcal{L}^0$-formula θ_{jl}, and $\mathcal{L}$-definable functions $f_1^{jl}, ..., f_{r(j,l)}^{jl}, u_1^{jl}, ..., u_n^{jl}, v_1^{jl}, ..., v_n^{jl}$ such that each u_i^{jl} (respectively v_i^{jl}) either takes values in $\tilde{R}$ or is the constant $(-\infty)$-valued (respectively, $(+\infty)$-valued) function, satisfying that for all $y \in \mathbb{Z}^k$ and $z \in \tilde{R}^n_\Delta$, $h_{jl}(y,z)$ if and only if

$$\theta_{jl}(y) \wedge \left(\left(h_{jl}(y,z) \wedge \bigvee_{s=1}^{r(j,l)} z_{i_s^{jl}} = f_s^{jl}(y) \right) \vee \left(\bigwedge_{i=1}^{n} u_i^{jl}(y) \le z_i \le v_i^{jl}(y) \right) \right).$$

Then, $\phi'_\Delta(x;y)$ is equivalent to the disjunction of

$$\bigvee_{j=1}^{p} \bigvee_{l=p+1}^{q} \bigvee_{s=1}^{r(j,l)} \Big(\theta_{jl}(y) \wedge \exists z \in \tilde{R}^n_\Delta \Big(-g_j(y) + \mathbf{A}^{(j)} \cdot z < Kx < g_l(y) - \mathbf{A}^{(l)} \cdot z \wedge h_{jl}(y,z) \wedge z_{i_s^{jl}} = f_s^{jl}(y) \Big) \Big),$$

which is equivalent to a Boolean combination of basic formulas and descendants of $(E_{n-1}; \tilde{R})$ formulas (since $z_{i_s^{jl}} = f_s^{jl}(y)$ in the $(j,l,s)^{\text{th}}$ disjunct), and

$$\bigvee_{j=1}^{p} \bigvee_{l=p+1}^{q} \Big(\theta_{jl}(y) \wedge \exists z \in \tilde{R}^n_\Delta \Big(-g_j(y) + \mathbf{A}^{(j)} \cdot z < Kx < g_l(y) - \mathbf{A}^{(l)} \cdot z \wedge \bigwedge_{i=1}^{n} u_i^{jl}(y) \le z_i \le v_i^{jl}(y) \Big) \Big),$$

which is equivalent to a Boolean combination of basic formulas and descendants of $(G_n; \mathbf{A}, \mathbf{B}, \tilde{R}, \Delta)$ formulas for $(\mathbf{A}, \mathbf{B}) \in \mathcal{G}_\phi$. □

Our final checkpoint is the following proposition.

Proposition 3.3.10. *Let $n \in \mathbb{N}^+$ and $\boldsymbol{A}, \boldsymbol{B}$ be n-tuples of operators. Then there is a finite collection $\mathcal{F}_{\boldsymbol{A},\boldsymbol{B}}$ of tuples $(\boldsymbol{I}, \boldsymbol{J})$, where $\boldsymbol{I}$ is an n-tuple of non-zero operators and $\boldsymbol{J}$ is an n-tuple of operators, satisfying the following.*

Let $\tilde{R} \subseteq^d R$ for some $d \in \mathbb{N}^+$. If $\Delta \in d\mathbb{N}$ is sufficiently large, then every $(G_n; \boldsymbol{A}, \boldsymbol{B}, \tilde{R}, \Delta)$ formula is equivalent to a Boolean combination of basic formu-

las, $(E_{n-1}; \tilde{R})$ *formulas, and descendants of* $(F_n; \boldsymbol{I}, \boldsymbol{J}, \tilde{R}, \Delta)$ *formulas for* $(\boldsymbol{I}, \boldsymbol{J}) \in \mathcal{F}_{\boldsymbol{A},\boldsymbol{B}}$*.*

Before proving this, we record a corollary.

Corollary 3.3.11. *Let* $d, n \in \mathbb{N}^+$*,* $\tilde{R} \subseteq^d R$*, and* $\phi(x; y)$ *be an* $(E_n; \tilde{R})$ *formula. Then there is a finite collection* $\mathcal{F}_\phi$ *of tuples* $(\boldsymbol{I}, \boldsymbol{J})$*, where* $\boldsymbol{I}$ *is an* n*-tuple of non-zero operators and* $\boldsymbol{J}$ *is an* n*-tuple of operators, satisfying the following.*

If $\Delta \in d\mathbb{N}$ *is sufficiently large, then* $\phi(x; y)$ *is equivalent to a Boolean combination of basic formulas and descendants of* $(E_{n-1}; \tilde{R})$ *formulas or* $(F_n; \boldsymbol{I}, \boldsymbol{J}, \tilde{R}, \Delta)$ *formulas for* $(\boldsymbol{I}, \boldsymbol{J}) \in \mathcal{F}_\phi$*.*

Proof. For $\mathcal{G}_\phi$ from Proposition 3.3.8, let $\mathcal{F}_\phi := \bigcup_{(\mathbf{A},\mathbf{B}) \in \mathcal{G}_\phi} \mathcal{F}_{\mathbf{A},\mathbf{B}}$ for $\mathcal{F}_{\mathbf{A},\mathbf{B}}$ from Proposition 3.3.10. □

Towards proving Proposition 3.3.10, we prove the following lemma.

Lemma 3.3.12. *Let* $d, n \in \mathbb{N}^+$*,* $\tilde{R} \subseteq^d R$*,* $\boldsymbol{A}, \boldsymbol{B}$ *be* n*-tuples of operators, and* $\Delta \in d\mathbb{N}$ *be sufficiently large. Then for all* $x, y_1, y_2 \in \mathbb{Z}$*,* $u_i \in \tilde{R} \cup \{-\infty\}$*, and* $v_i \in \tilde{R} \cup \{+\infty\}$*, if*

$$\exists z \in \tilde{R}^n_\Delta \left(y_1 + \boldsymbol{A} \cdot z < x < y_2 + \boldsymbol{B} \cdot z \wedge \bigwedge_{i=1}^{n} u_i \leq z_i \leq v_i \right), \tag{1}$$

then either $v_1 = +\infty \wedge (A_1 =_R 0 <_R B_1 \vee A_1 <_R 0 =_R B_1)$ *or there is a witness* $z \in \tilde{R}^n_\Delta$ *satisfying one of the following:*

(i) $z_i = \sigma^\Delta z_{i+1}$ *for some* $1 \leq i \leq n$*, where* $z_{n+1} := \min \tilde{R}$*;*

(ii) $z_i \in \{u_i, v_i\}$ *for some* $1 \leq i \leq n$*;*

(iii) $A_i, B_i \neq_R 0$ *for all* $1 \leq i \leq n$*, and* $z = P_\Delta(x - y_1; \boldsymbol{A})$ *or* $z = P_\Delta(y_2 - x; -\boldsymbol{B})$*.*

This lemma has a rather intuitive interpretation: if (1) holds then, barring some edge cases, z can be chosen to satisfy (iii), that is, to maximise $y_1 + \mathbf{A} \cdot z$ subject to $y_1 + \mathbf{A} \cdot z < x$ — namely, $z = P_\Delta(x - y_1; \mathbf{A})$ — or minimise $y_2 + \mathbf{B} \cdot z$ subject to $x < y_2 + \mathbf{B} \cdot z$ — namely, $z = P_\Delta(y_2 - x; -\mathbf{B})$.

Proof of Lemma 3.3.12. Suppose $v_1 \neq +\infty \vee \neg(A_1 = 0 < B_1 \vee A_1 < 0 = B_1)$. We first show that if $A_i = 0$ or $B_i = 0$ for some $1 \leq i \leq n$, then there is a witness $z \in \tilde{R}^n_\Delta$ satisfying (i) or (ii).

Suppose $A_i = 0$ for some $1 \leq i \leq n$; fix the minimal such i. Suppose there is no witness to (1) satisfying (i) or (ii). Pick a witness $z \in \tilde{R}^n_\Delta$ that minimises

$$\begin{cases} \min\{z_{i-1}, v_i\}/z_i & \text{if } B_i > 0 \\ z_i/\max\{z_{i+1}, u_i\} & \text{if } B_i \leq 0 \end{cases},$$

where $z_0 := +\infty$ and $z_{n+1} := \min \tilde{R}$. Let w be the n-tuple obtained from z by replacing z_i with $\sigma^d z_i$ if $B_i > 0$ and with $\sigma^{-d} z_i$ if $B_i \leq 0$. Since z does not satisfy (i) or (ii), we have that $w \in \tilde{R}^n_\Delta$ and $u_i \leq w_i \leq v_i$. But $\mathbf{B} \cdot z \leq \mathbf{B} \cdot w$ by Lemma 3.2.9, so $y_1 + \mathbf{A} \cdot w = y_1 + \mathbf{A} \cdot z < x < y_2 + \mathbf{B} \cdot z \leq y_2 + \mathbf{B} \cdot w$, whence w is a witness to (1), contradicting our choice of z.

The case where $B_i = 0$ for some $1 \leq i \leq n$ is similar, so henceforth suppose $A_i, B_i \neq 0$ for all $1 \leq i \leq n$, and suppose there is no witness to (1) satisfying (i), (ii), or (iii). By Lemma 3.2.9, we may assume that the function $z \mapsto \mathbf{A} \cdot z$ is injective on $\tilde{R}^n_\Delta$. Now any witness $z \in \tilde{R}^n_\Delta$ to (1) satisfies $\mathbf{A} \cdot z < x - y_1$ and so $\mathbf{A} \cdot z \leq \mathbf{A} \cdot P_\Delta(x - y_1; \mathbf{A})$, and the inequality is strict since z does not satisfy (iii). Fix a witness $z \in \tilde{R}^n_\Delta$ to (1) that maximises $\mathbf{A} \cdot z$.

Let w be the n-tuple obtained from z by replacing z_n with $\sigma^d z_n$ if $A_n > 0$ and with $\sigma^{-d} z_n$ if $A_n < 0$. Since z does not satisfy (i) or (ii), we have that $w \in \tilde{R}^n_\Delta$ and $u_n \leq w_n \leq v_n$. By Lemma 3.2.9, there is no $r \in \tilde{R}^n_\Delta$ such that $\mathbf{A} \cdot r$ lies strictly between $\mathbf{A} \cdot z$ and $\mathbf{A} \cdot w$. Recalling that $\mathbf{A} \cdot z < \mathbf{A} \cdot P_\Delta(x - y_1; \mathbf{A})$, this shows that $\mathbf{A} \cdot w \leq \mathbf{A} \cdot P_\Delta(x - y_1; \mathbf{A})$.

By a similar argument, $\mathbf{B} \cdot w \geq \mathbf{B} \cdot P_\Delta(y_2 - x; -\mathbf{B})$. Thus,

$$y_1 + \mathbf{A} \cdot w \leq y_1 + \mathbf{A} \cdot P_\Delta(x - y_1; \mathbf{A}) < x < y_2 + \mathbf{B} \cdot P_\Delta(y_2 - x; -\mathbf{B}) \leq y_2 + \mathbf{B} \cdot w,$$

so w is a witness to (1). By Lemma 3.2.9, $\mathbf{A} \cdot z < \mathbf{A} \cdot w$, contradicting our choice of z. □

Proof of Proposition 3.3.10. Let

$$\mathcal{F}_{\mathbf{A},\mathbf{B}} := \{(\mathbf{A},\mathbf{B}),(-\mathbf{B},-\mathbf{A})\} \cup \{(\mathbf{A},\mathbf{F}^i),(-\mathbf{B},\mathbf{F}^i) : 1 \le i \le n\}$$

if $\mathbf{A},\mathbf{B}$ are tuples of non-zero operators, and let $\mathcal{F}_{\mathbf{A},\mathbf{B}} := \emptyset$ otherwise (recall that $\mathbf{F}^i$ was defined as the i^{th} standard tuple of operators). We claim that this witnesses the proposition.

Let $\tilde{R} \subseteq^d R$ for some $d \in \mathbb{N}^+$, and let $\Delta \in d\mathbb{N}$ be sufficiently large as in Lemma 3.3.12. Let $\phi(x;y,u,v)$ be a $(G_n;\mathbf{A},\mathbf{B},\tilde{R},\Delta)$ formula, say

$$\phi(x;y,u,v) = T_{\tilde{R}}(u,v) \wedge \exists z \in \tilde{R}^n_\Delta \left(y_1 + \mathbf{A}\cdot z < x < y_2 + \mathbf{B}\cdot z \wedge \bigwedge_{i=1}^n u_i \le z_i \le v_i \right),$$

where some of the u_i (respectively, v_i) may be $-\infty$ (respectively, $+\infty$). Write $T(u,v)$ for $T_{\tilde{R}}(u,v)$.

If $v_1 = +\infty$ and $A_1 =_R 0 <_R B_1$, then $\phi(x;y,u,v)$ is equivalent to

$$\zeta(x;y,u,v) := T(u,v) \wedge \exists z \in \tilde{R}^n_\Delta \left(y_1 + \mathbf{A}\cdot z < x \wedge \bigwedge_{i=1}^n u_i \le z_i \le v_i \right).$$

Indeed, clearly ϕ implies ζ, and if $z \in \tilde{R}^n_\Delta$ witnesses ζ, then for all/some sufficiently large $a \in \tilde{R}$, we have $w := (a, z_{>1}) \in \tilde{R}^n_\Delta$ and

$$y_1 + \mathbf{A}\cdot w = y_1 + \mathbf{A}\cdot z < x < y_2 + \mathbf{B}\cdot w \wedge \bigwedge_{i=1}^n u_i \le z_i \le w_i < v_i.$$

But ζ is equivalent to

$$T(u,v) \wedge x > y_1 + \inf\left\{ \mathbf{A}\cdot z : z \in \tilde{R}^n_\Delta, \bigwedge_{i=1}^n u_i \le z_i \le v_i \right\},$$

which is a basic formula. Thus, if $v_1 = +\infty$ and $A_1 =_R 0 <_R B_1$, then ϕ is equivalent to a basic formula. A similar situation arises if $v_1 = +\infty$ and $A_1 <_R 0 =_R B_1$, so henceforth suppose neither case holds. Let $\bar{\phi}(x;y,u,v,z)$ be the formula

$$y_1 + \mathbf{A} \cdot z < x < y_2 + \mathbf{B} \cdot z \wedge \bigwedge_{i=1}^{n} u_i \le z_i \le v_i.$$

For $1 \le i \le n$, let

$$\alpha_i(x; y, u, v) := T(u, v) \wedge \exists z \in \tilde{R}^n_\Delta \left(z_i = \sigma^\Delta z_{i+1} \wedge \bar{\phi}(x; y, u, v, z)\right),$$
$$\beta_i(x; y, u, v) := T(u, v) \wedge \exists z \in \tilde{R}^n_\Delta \left(z_i = u_i \wedge \bar{\phi}(x; y, u, v, z)\right),$$
$$\gamma_i(x; y, u, v) := T(u, v) \wedge \exists z \in \tilde{R}^n_\Delta \left(z_i = v_i \wedge \bar{\phi}(x; y, u, v, z)\right),$$

where $z_{n+1} := \min \tilde{R}$. Furthermore, if $\mathbf{A}$ and $\mathbf{B}$ are tuples of non-zero operators then let

$$\theta(x; y, u, v) := T(u, v) \wedge x - y_1 > \inf \mathbf{A} \cdot \tilde{R}^n_\Delta$$
$$\wedge\, x < y_2 + \mathbf{B} \cdot P_\Delta(x - y_1; \mathbf{A}) \wedge \bigwedge_{i=1}^{n} u_i \le P^i_\Delta(x - y_1; \mathbf{A}) \le v_i,$$
$$\xi(x; y, u, v) := T(u, v) \wedge y_2 - x > \inf(-\mathbf{B} \cdot \tilde{R}^n_\Delta)$$
$$\wedge\, x > y_1 + \mathbf{A} \cdot P_\Delta(y_2 - x; -\mathbf{B}) \wedge \bigwedge_{i=1}^{n} u_i \le P^i_\Delta(y_2 - x; -\mathbf{B}) \le v_i.$$

By Lemma 3.3.12 (and Lemma 3.2.17), $\phi(x; y, u, v)$ is equivalent to

$$\begin{cases} \theta \vee \xi \vee \bigvee_{i=1}^{n} (\alpha_i \vee \beta_i \vee \gamma_i) & \text{if } \mathbf{A}, \mathbf{B} \text{ are tuples of non-zero operators,} \\ \bigvee_{i=1}^{n} (\alpha_i \vee \beta_i \vee \gamma_i) & \text{otherwise.} \end{cases}$$

Observe that θ is a Boolean combinations of basic formulas and descendants of $(F_n; \mathbf{I}, \mathbf{J}, \tilde{R}, \Delta)$ formulas for $(\mathbf{I}, \mathbf{J}) \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$, since, for all $1 \le i \le n$,

$$u_i \le P^i_\Delta(x - y_1; \mathbf{A}) \le v_i \leftrightarrow u_i - 1 < \mathbf{F}^i \cdot P_\Delta(x - y_1; \mathbf{A}) \wedge \neg(v_i < \mathbf{F}^i \cdot P_\Delta(x - y_1; \mathbf{A})).$$

But this is also true for ξ, since, for example, $x > y_1 + \mathbf{A} \cdot P_\Delta(y_2 - x; -\mathbf{B})$ is a descendant of $-x > -y_2 + \mathbf{A} \cdot P_\Delta(-y_1 + x; -\mathbf{B})$, which is equivalent to $x - y_2 < -\mathbf{A} \cdot P_\Delta(x - y_1; -\mathbf{B})$.

For all $1 \leq i \leq n$, α_i, β_i, and γ_i are equivalent to the conjunction of $T(u,v)$, which is a basic formula, and an $(E_{n-1}, \tilde{R})$ formula, by substituting z_i with $\sigma^{\Delta} z_{i+1}$, u_i, or v_i as appropriate.

Thus, ϕ is equivalent to a Boolean combination of basic formulas, $(E_{n-1}; \tilde{R})$ formulas, and descendants of $(F_n; \mathbf{I}, \mathbf{J}, \tilde{R}, \Delta)$ formulas for $(\mathbf{I}, \mathbf{J}) \in \mathcal{F}_{\mathbf{A},\mathbf{B}}$. □

We are now ready to prove Theorem 3.3.6.

Proof of Theorem 3.3.6. Assume the criterion holds. By Proposition 3.3.7, it suffices to prove that every (E_n) formula has a strong honest definition. We do so by induction on $n \in \mathbb{N}$. An (E_0) formula is a basic formula, so suppose $n \geq 1$.

Let ϕ be an $(E_n; \tilde{R})$ formula, where $\tilde{R} \subseteq^d R$ for some $d \in \mathbb{N}^+$. Let $\mathcal{F}_\phi$ be as in Corollary 3.3.11. Then, for all $\Delta \in d\mathbb{N}$ sufficiently large, ϕ is equivalent to a Boolean combination of basic formulas and descendants of $(E_{n-1}; \tilde{R})$ or $(F_n; \mathbf{I}, \mathbf{J}, \tilde{R}, \Delta)$ formulas for $(\mathbf{I}, \mathbf{J}) \in \mathcal{F}_\phi$. By the induction hypothesis, every $(E_{n-1}; \tilde{R})$ formula has a strong honest definition. Since $\mathcal{F}_\phi$ is finite, by the criterion, for all $\Delta \in d\mathbb{N}$ sufficiently large, every $(F_n; \mathbf{I}, \mathbf{J}, \tilde{R}, \Delta)$ formula for $(\mathbf{I}, \mathbf{J}) \in \mathcal{F}_\phi$ has a strong honest definition. Thus, ϕ is a Boolean combination of formulas with strong honest definitions. □

The rest of the chapter is thus devoted to establishing the sufficiency criterion in Theorem 3.3.6, by constructing strong honest definitions for $(F_n; \mathbf{A}, \mathbf{B}, \tilde{R}, \Delta)$ formulas with Δ sufficiently large. Note that this then gives a strong honest definition for *every* $\mathcal{L}$-formula $\phi(x; y)$ with $|x| = 1$, since we have exhibited a way to write every such formula as a Boolean combination of basic formulas and descendants of $(F_n; \mathbf{A}, \mathbf{B}, \tilde{R}, \Delta)$ formulas with Δ sufficiently large. Indeed, by Proposition 3.3.7, every $\mathcal{L}$-formula $\phi(x; y)$ with $|x| = 1$ is equivalent to a Boolean combination of basic formulas and descendants of (E_n) formulas. Example 2.7.16 gives strong honest definitions for basic formulas, and the proof of Corollary 3.3.11 describes an algorithm for writing every $(E_n; \tilde{R})$ formula as a Boolean combination of descendants of $(E_{n-1}; \tilde{R})$ formulas and descendants of $(F_n; \mathbf{A}, \mathbf{B}, \tilde{R}, \Delta)$ formulas with Δ sufficiently large.

3.4 Main construction

Recall that $R \subseteq \mathbb{N}$ is our fixed congruence-periodic sparse predicate. In this section, we show that every $(F_n; \mathbf{A}, \mathbf{B}, \tilde{R}, \Delta)$ formula with Δ sufficiently large has a strong honest definition.

The broad strategy is to induct on n. Theorem 3.4.3 can be seen as a stronger version of the $n = 1$ case, and Theorem 3.4.6 handles the inductive step.

The following lemma transpires to be surprisingly useful.

Lemma 3.4.1. *Let $d, n \in \mathbb{N}^+$, $\tilde{R} \subseteq^d R$, and $\boldsymbol{A}$ be an n-tuple of non-zero operators with $A_1 >_R 0$ (respectively, $A_1 <_R 0$). Then there is $\Lambda \in \mathbb{N}$ such that the following holds.*

Let $\Delta \in d\mathbb{N}$ be sufficiently large, and let $s, t, x \in \mathbb{Z}$ be such that $s \leq t \leq x$ (respectively, $s \geq t \geq x$). Then there is $0 \leq \alpha \leq \Lambda$ such that $P^1_\Delta(x - s; \boldsymbol{A}) = \sigma^\alpha P^1_\Delta(x - t; \boldsymbol{A})$ or $P^1_\Delta(x - s; \boldsymbol{A}) = \sigma^\alpha P^1_\Delta(t - s; \boldsymbol{A})$.

Let us give an intuitive interpretation of this lemma. Assuming $A_1 > 0$ for the purpose of this discussion, the lemma simply says that if $s \leq t \leq x$, then $x - s$ is 'close' (with respect to the function $P^1_\Delta(\cdot; \mathbf{A})$) to either $x - t$ or $t - s$.

Proof of Lemma 3.4.1. By Lemma 3.2.5, we can fix $\Lambda \in \mathbb{N}$ such that $|A_1 \sigma^\Lambda r| > |8 A_1 \sigma^d r|$ for all $r \in R$. Let $\Delta \in d\mathbb{N}$ be sufficiently large, and let $s, t, x \in \mathbb{Z}$ be such that $s \leq t \leq x$ if $A_1 >_R 0$ and $s \geq t \geq x$ if $A_1 <_R 0$. Let $w := P_\Delta(x - t; \mathbf{A})$ and $z := P_\Delta(t - s; \mathbf{A})$.

First suppose $A_1 >_R 0$. Then

$$t - s \leq \mathbf{A} \cdot Q_\Delta(t - s; \mathbf{A}) < 2 A_1 Q^1_\Delta(t - s; \mathbf{A}) \leq 2 A_1 \sigma^d z_1,$$

where the first and last inequalities are by Lemma 3.2.17 and the second inequality is by Lemma 3.2.7. Similarly, $x - t < 2 A_1 \sigma^d w_1$. But now

$$x - s = (x - t) + (t - s) < 4 A_1 \sigma^d \max\{z_1, w_1\} < \frac{1}{2} A_1 \sigma^\Lambda \max\{z_1, w_1\},$$

so, by Lemma 3.2.7, $x \leq s + \mathbf{A} \cdot u$ for all $u \in \tilde{R}^n_\Delta$ with $u_1 \geq \sigma^\Lambda \max\{z_1, w_1\}$. Thus, $P^1_\Delta(x - s; \mathbf{A}) < \sigma^\Lambda \max\{z_1, w_1\}$. But $x \geq t \geq s$, so $\mathbf{A} \cdot P_\Delta(x - s; \mathbf{A}) \geq \max\{\mathbf{A} \cdot z, \mathbf{A} \cdot w\}$, and thus $P^1_\Delta(x - s; \mathbf{A}) \geq \max\{z_1, w_1\}$ by Lemma 3.2.9.

Now suppose $A_1 <_R 0$. Then $t - s > A_1 z_1$ and $x - t > A_1 w_1$ by Lemma 3.2.17, whence

$$x - s = (x - t) + (t - s) > 2A_1 \max\{z_1, w_1\} > \frac{1}{4} A_1 \sigma^\Lambda \max\{z_1, w_1\},$$

so, by Lemma 3.2.7, $x > s + \mathbf{A} \cdot u$ for all/some $u \in \tilde{R}^n_\Delta$ with $u_1 = \sigma^\Lambda \max\{z_1, w_1\}$. Thus, $P^1_\Delta(x - s; \mathbf{A}) \leq \sigma^\Lambda \max\{z_1, w_1\}$. But $x \leq t \leq s$, so $\mathbf{A} \cdot P_\Delta(x - s; \mathbf{A}) \leq \min\{\mathbf{A} \cdot z, \mathbf{A} \cdot w\}$, and thus $P^1_\Delta(x - s; \mathbf{A}) \geq \max\{z_1, w_1\}$ by Lemma 3.2.9. □

Lemma 3.4.2. *Let $d, n \in \mathbb{N}^+$, $\tilde{R} \subseteq^d R$, $\boldsymbol{A}$ be an n-tuple of non-zero operators, and $\Delta \in d\mathbb{N}$ be sufficiently large. Then the formula $\phi(x; y) := P^1_\Delta(x - y_1; \boldsymbol{A}) = y_2$ has a strong honest definition, given by the conjunction of strong honest definitions for the basic formulas $\phi_1(x; y)$ and $\phi_2(x; y)$ defined as follows. The formula ϕ_1 is given by*

$$\begin{cases} x - y_1 \leq \min\{\boldsymbol{A} \cdot z : z \in \tilde{R}^n_\Delta, z_1 = \sigma^d N\} & \text{if } A_1 >_R 0, \\ x - y_1 > \min\{\boldsymbol{A} \cdot z : z \in \tilde{R}^n_\Delta, z_1 = N\} & \text{if } A_1 <_R 0, \end{cases}$$

where $N := \sigma^{n\Delta}(\min \tilde{R})$, and the formula $\phi_2(x; y)$ is given by

$$\min\{\boldsymbol{A} \cdot z : z \in \tilde{R}^n_\Delta, z_1 = y_2\} < x - y_1 \leq \min\{\boldsymbol{A} \cdot z : z \in \tilde{R}^n_\Delta, z_1 = \sigma^{\varepsilon d} y_2\},$$

where $\varepsilon := 1$ if $A_1 >_R 0$ and $\varepsilon := -1$ if $A_1 <_R 0$.

Proof. Observe that

$$\phi(x; y) \leftrightarrow (y_2 = N \wedge \phi_1(x; y)) \vee (y_2 \in \tilde{R} \wedge y_2 > N \wedge \phi_2(x; y)).$$

Now apply Lemma 2.7.10. □

In the following theorem, we construct strong honest definitions for a class of formulas that includes all $(F_1; \mathbf{A}, \mathbf{B}, \tilde{R}, \Delta)$ formulas with Δ sufficiently large (this inclusion is spelt out in Corollary 3.4.4). We use the notation $I \sqcup J \subseteq [k]$ to mean that $I, J \subseteq [k]$ are disjoint.

Theorem 3.4.3. *Let $\theta(x; y)$ be a formula with $|x| = 1$, and suppose the formulas $\theta(x; y)$ and $\theta'(x; w, y) := \theta(x - w; y)$ both have strong honest definitions, where $|w| = 1$. Let $\gamma(x; y^{(1)}, ..., y^{(k)})$ be a strong honest definition for θ.*

Let $d, n \in \mathbb{N}^+$, $\tilde{R} \subseteq^d R$, $\boldsymbol{A}$ be an n-tuple of non-zero operators, and let $\Lambda \in \mathbb{N}$ be as in Lemma 3.4.1. Let $\Delta \in d\mathbb{N}$ be sufficiently large, $t \in \mathbb{Z}$, and f be an $\mathcal{L}$-definable function of arity 1. Then the formula

$$\phi(x; w, y) := \theta\left(tx - f(P^1_\Delta(x - w; \boldsymbol{A})); y\right)$$

has a system of strong honest definitions

$$\{\zeta_{I_0 J_0 \cdots I_\Lambda J_\Lambda K} : I_\alpha \sqcup J_\alpha \subseteq [k] \text{ for all } 0 \leq \alpha \leq \Lambda, K \subseteq \{0, ..., \Lambda\}\},$$

where $\zeta_{I_0 J_0 \cdots I_\Lambda J_\Lambda K}(x; ...)$ is given by the conjunction of the following:

(i) A strong honest definition $\zeta_1(x; ...)$ for the basic formula $\phi_1(x; w, y) := x \leq w$;

(ii) A strong honest definition $\zeta_2(x; ...)$ for the formula $\phi_2(x; w, y) := \theta(tx - f(\sigma^{n\Delta}(\min \tilde{R})); y)$, which exists since the formula is a descendant of θ;

(iii) For each $0 \leq \alpha \leq \Lambda$, a strong honest definition $\zeta_3^\alpha(x; ...)$ for the formula $\phi_3^\alpha(x; w, y, w', y') := \theta'(tx; f(\sigma^\alpha P^1_\Delta(w' - w; \boldsymbol{A})), y)$, which exists since the formula is a descendant of θ';

(iv) For each $0 \leq \alpha \leq \Lambda$, a strong honest definition $\zeta_4^\alpha(x; ...)$ for the formula $\phi_4^\alpha(x; w, y, w', y') := P^1_\Delta(x - w; \boldsymbol{A}) = \sigma^\alpha P^1_\Delta(w' - w; \boldsymbol{A})$, which exists by Lemma 3.4.2 (and Lemma 2.7.14);

(v) *For each* $0 \leq \alpha \leq \Lambda$, *the formula*

$$\zeta^{\alpha}_{I_\alpha J_\alpha}(x; w, y^{(i)} : i \in [k] \backslash (I_\alpha \cup J_\alpha)) := \gamma(tx - f(\sigma^\alpha P^1_\Delta(x-w; \boldsymbol{A})); \hat{y}^{(1)}, ..., \hat{y}^{(k)}),$$

where, for $1 \leq i \leq k$,

$$\hat{y}^{(i)} := \begin{cases} (0, ..., 0) & \text{if } i \in I_\alpha, \\ (1, ..., 1) & \text{if } i \in J_\alpha, \\ y^{(i)} & \text{otherwise}; \end{cases}$$

(vi) *The formula*

$$\bigwedge_{\alpha \in K} P^1_\Delta(x - w_\alpha; \boldsymbol{A}) = P^1_\Delta(x - w'_\alpha; \boldsymbol{A}) = \sigma^\alpha P^1_\Delta(x - w''_\alpha; \boldsymbol{A}).$$

Let us first describe the idea of the proof, assuming $A_1 >_R 0$ for the purpose of this discussion. We wish to replace $P^1_\Delta(x - w; \mathbf{A})$ in $\phi(x; w, y)$ with a more tractable expression; we can do so by Lemma 3.4.1, which gives us $\Lambda \in \mathbb{N}$ satisfying the following.

Let $x_0 \in \mathbb{Z}$ and $S \subseteq \mathbb{Z}^{1+|y|}$ with $2 \leq |S| < \infty$. Here and henceforth, when it is written that $(b, a) \in S$, it is understood that $|b| = 1$ and $|a| = |y|$. Let $u := \max(\{b : (b, a) \in S, x_0 > b\} \cup \{\min_{(b,a) \in S} b\})$. For all $(b, a) \in S$, if $b > u$ then $P^1_\Delta(x_0 - b; \mathbf{A}) = \sigma^{n\Delta}(\min \tilde{R})$, and if $b \leq u$ then either

(i) $P^1_\Delta(x_0 - b; \mathbf{A}) = \sigma^\alpha P^1_\Delta(x_0 - u; \mathbf{A})$ for some $0 \leq \alpha \leq \Lambda$; or

(ii) $P^1_\Delta(x_0 - b; \mathbf{A}) = \sigma^\alpha P^1_\Delta(u - b; \mathbf{A})$ for some $0 \leq \alpha \leq \Lambda$.

In each of these cases, replacing $P^1_\Delta(x_0 - b; \mathbf{A})$ with the respective expression gives a formula for which we have strong honest definitions.

Proof of Theorem 3.4.3. By Lemma 3.2.7, we may assume $\Delta \in d\mathbb{N}$ is sufficiently large that $\min \mathbf{A} \cdot \tilde{R}^n_\Delta > 0$ if $A_1 >_R 0$ and $\max \mathbf{A} \cdot \tilde{R}^n_\Delta < 0$ if $A_1 <_R 0$.

Fix $x_0 \in \mathbb{Z}$ and $S \subseteq \mathbb{Z}^{1+|y|}$ with $2 \leq |S| < \infty$. Write $\pi_1(S) := \{b : \exists a \ (b,a) \in S\}$ and $\pi_2(S) := \{a : \exists b \ (b,a) \in S\}$. Fix $(b_0, a_0) \in S$ such that

$$b_0 = \begin{cases} \min \pi_1(S) & \text{if } A_1 >_R 0, \\ \max \pi_1(S) & \text{if } A_1 <_R 0. \end{cases}$$

Define

$$u := \begin{cases} \max(\{b \in \pi_1(S) : x_0 > b\} \cup \{b_0\}) & \text{if } A_1 >_R 0, \\ \min(\{b \in \pi_1(S) : x_0 \leq b\} \cup \{b_0\}) & \text{if } A_1 <_R 0. \end{cases}$$

For $i \in \{1, 2\}$, let $c_i \in S^{<\omega}$ be such that $x_0 \models \zeta_i(x; c_i) \vdash \mathrm{tp}_{\phi_i}(x_0/S)$. For $i \in \{3, 4\}$ and $0 \leq \alpha \leq \Lambda$, let $c_i^\alpha \in (S^2)^{<\omega}$ be such that $x_0 \models \zeta_i^\alpha(x; c_i^\alpha)$ and $\zeta_i^\alpha(x; c_i^\alpha) \vdash \mathrm{tp}_{\phi_i^\alpha}(x_0/S^2)$.

Let $T := \pi_2(S) \cup \{(0, ..., 0), (1, ..., 1)\} \subseteq \mathbb{Z}^y$. Then $|T| \geq 2$, so for $0 \leq \alpha \leq \Lambda$, there is $e^\alpha \in T^k$ such that

$$tx_0 - f(\sigma^\alpha P^1_\Delta(x_0 - u; \mathbf{A})) \models \gamma(x; e^\alpha) \vdash \mathrm{tp}_\theta(tx_0 - f(\sigma^\alpha P^1_\Delta(x_0 - u; \mathbf{A}))/T).$$

There are disjoint $I_\alpha, J_\alpha \subseteq [k]$ and $c^\alpha \in \pi_2(S)^{<\omega}$ such that

$$\gamma(tx - f(\sigma^\alpha P^1_\Delta(x - u; \mathbf{A})); e^\alpha) = \zeta^\alpha_{I_\alpha J_\alpha}(x; u, c^\alpha),$$

whence $x_0 \models \zeta^\alpha_{I_\alpha J_\alpha}(x; u, c^\alpha)$.

For $0 \leq \alpha \leq \Lambda$, let $S_\alpha := \left\{b \in \pi_1(S) : P^1_\Delta(x_0 - b; \mathbf{A}) = \sigma^\alpha P^1_\Delta(x_0 - u; \mathbf{A})\right\} \subseteq \mathbb{Z}$, and if $S_\alpha \neq \emptyset$, let $l^{(\alpha)} := \min S_\alpha$ and $r^{(\alpha)} := \max S_\alpha$.

Then we have that

$$\begin{aligned} x_0 \models \bigwedge_{i=1}^{2} \zeta_i(x; c_i) \wedge \bigwedge_{i=3}^{4} \bigwedge_{\alpha=0}^{\Lambda} \zeta_i^\alpha(x; c_i^\alpha) \wedge \bigwedge_{\alpha=0}^{\Lambda} \zeta^\alpha_{I_\alpha J_\alpha}(x; u, c^\alpha) \\ \wedge \bigwedge_{\substack{\alpha=0 \\ S_\alpha \neq \emptyset}}^{\Lambda} P^1_\Delta(x - l^{(\alpha)}; \mathbf{A}) = P^1_\Delta(x - r^{(\alpha)}; \mathbf{A}) = \sigma^\alpha P^1_\Delta(x - u; \mathbf{A}), \end{aligned}$$

and we claim that this formula, which is an instance of $\zeta_{I_0 J_0 \cdots I_\Lambda J_\Lambda K}$ for $K := \{0 \leq \alpha \leq \Lambda : S_\alpha \neq \emptyset\}$, entails $\mathrm{tp}_\phi(x_0/S)$.

Indeed, suppose $x_1 \in \mathbb{Z}$ satisfies this formula, and let $(b', a') \in S$. We wish to show that $\phi(x_0; b', a')$ if and only if $\phi(x_1; b', a')$. Since $x_0, x_1 \models \zeta_1(x; c_1)$, we have that for $i \in \{0, 1\}$,

$$u = \begin{cases} \max(\{b \in \pi_1(S) : x_i > b\} \cup \{b_0\}) & \text{if } A_1 >_R 0, \\ \min(\{b \in \pi_1(S) : x_i \leq b\} \cup \{b_0\}) & \text{if } A_1 <_R 0. \end{cases}$$

Suppose $b' > u$ and $A_1 >_R 0$. Then, for $i \in \{0, 1\}$, we have $x_i - b' \leq 0 < \min \mathbf{A} \cdot \tilde{R}^n_\Delta$ and so $P^1_\Delta(x_i - b'; \mathbf{A}) = \sigma^{n\Delta}(\min \tilde{R})$ by Remark 3.2.15. Thus, for $i \in \{0, 1\}$, we have $\phi(x_i; b', a') \Leftrightarrow \phi_2(x_i; b', a')$. But now, since $x_0, x_1 \models \zeta_2(x; c_2)$, we have $\phi_2(x_0; b', a') \Leftrightarrow \phi_2(x_1; b', a')$, whence $\phi(x_0; b', a') \Leftrightarrow \phi(x_1; b', a')$.

The case where $b' < u$ and $A_1 <_R 0$ is similar, so henceforth suppose either ($b' \leq u$ and $A_1 >_R 0$) or ($b' \geq u$ and $A_1 <_R 0$). By Lemma 3.4.1, we have either

(i) That $P^1_\Delta(x_0 - b'; \mathbf{A}) = \sigma^\alpha P^1_\Delta(x_0 - u; \mathbf{A})$ for some $0 \leq \alpha \leq \Lambda$; or

(ii) That $P^1_\Delta(x_0 - b'; \mathbf{A}) = \sigma^\alpha P^1_\Delta(u - b'; \mathbf{A})$ for some $0 \leq \alpha \leq \Lambda$.

If $0 \leq \alpha \leq \Lambda$ is such that $P^1_\Delta(x_0 - b'; \mathbf{A}) = \sigma^\alpha P^1_\Delta(u - b'; \mathbf{A})$, then since $x_0, x_1 \models \zeta^\alpha_4(x; c^\alpha_4)$, we have $P^1_\Delta(x_0 - b'; \mathbf{A}) = P^1_\Delta(x_1 - b'; \mathbf{A}) = \sigma^\alpha P^1_\Delta(u - b'; \mathbf{A})$. Thus, for $i \in \{0, 1\}$, we have

$$\phi(x_i; b', a') \Leftrightarrow \theta\left(tx_i - f(\sigma^\alpha P^1_\Delta(u - b'; \mathbf{A})); a'\right),$$

and so

$$\phi(x_i; b', a') \Leftrightarrow \theta'\left(tx_i; f(\sigma^\alpha P^1_\Delta(u - b'; \mathbf{A})), a'\right).$$

But now, since $x_0, x_1 \models \zeta^\alpha_3(x; c^\alpha_3)$, we have

$$\theta'\left(tx_0; f(\sigma^\alpha P^1_\Delta(u - b'; \mathbf{A})), a'\right) \Leftrightarrow \theta'\left(tx_1; f(\sigma^\alpha P^1_\Delta(u - b'; \mathbf{A})), a'\right),$$

whence $\phi(x_0; b', a') \Leftrightarrow \phi(x_1; b', a')$.

Suppose instead that we have $P^1_\Delta(x_0 - b'; \mathbf{A}) = \sigma^\alpha P^1_\Delta(x_0 - u; \mathbf{A})$ for some $0 \leq \alpha \leq \Lambda$, and so $l^{(\alpha)} \leq b' \leq r^{(\alpha)}$. But now

$$x_0, x_1 \models P^1_\Delta(x - l^{(\alpha)}; \mathbf{A}) = P^1_\Delta(x - r^{(\alpha)}; \mathbf{A}) = \sigma^\alpha P^1_\Delta(x - u; \mathbf{A}),$$

so by Lemma 3.2.9 we must have $x_0, x_1 \models P^1_\Delta(x - b'; \mathbf{A}) = \sigma^\alpha P^1_\Delta(x - u; \mathbf{A})$. Thus, for $i \in \{0, 1\}$,

$$\phi(x_i; b', a') \Leftrightarrow \theta\left(tx_i - f(\sigma^\alpha P^1_\Delta(x_i - u; \mathbf{A})); a'\right).$$

But now, since $x_0, x_1 \models \zeta^\alpha_{I_\alpha J_\alpha}(x; u, c^\alpha)$, we have

$$x_0, x_1 \models \gamma(tx - f(\sigma^\alpha P^1_\Delta(x - u; \mathbf{A})); e^\alpha)$$

and so

$$\theta\left(tx_0 - f(\sigma^\alpha P^1_\Delta(x_0 - u; \mathbf{A})); a'\right) \Leftrightarrow \theta\left(tx_1 - f(\sigma^\alpha P^1_\Delta(x_1 - u; \mathbf{A})); a'\right),$$

whence $\phi(x_0; b', a') \Leftrightarrow \phi(x_1; b', a')$, which finishes the proof. □

Corollary 3.4.4. *Let $\tilde{R} \subseteq^d R$ for some $d \in \mathbb{N}^+$. Let $t \in \mathbb{Z}$, $\boldsymbol{A}$ be a tuple of non-zero operators, f be an $\mathcal{L}$-definable function of arity 1, and $\Box \in \{<, >\}$. Let $\Delta \in d\mathbb{N}$ be sufficiently large. Then the formula $\phi(x; y) := tx - y_2 \Box f(P^1_\Delta(x - y_1; \boldsymbol{A}))$ has a strong honest definition. In particular, given operators A, B with $A \neq_R 0$, every $(F_1; A, B, \Delta, \tilde{R})$ formula with $\Delta \in d\mathbb{N}$ sufficiently large has a strong honest definition.*

Proof. This follows directly from Theorem 3.4.3 since, for $\theta(x; y_2) := x \Box y_2$,

$$\phi(x; y) = \theta\left(tx - f(P^1_\Delta(x - y_1; \mathbf{A})); y_2\right),$$

and the formulas $\theta(x; y_2)$ and $\theta'(x; w, y_2) := \theta(x - w; y_2)$ have strong honest definitions by Example 2.7.16. □

Having shown that every suitable (F_1) formula has a strong honest definition, we proceed to show this for (F_n) formulas by induction on $n \in \mathbb{N}^+$. The following lemma is crucial for the inductive step. Recall that, given an n-tuple $\nu = (\nu_1, ..., \nu_n)$, we let $\nu_{>1}$ denote $(\nu_2, ..., \nu_n)$.

Lemma 3.4.5. *Let $d, n \in \mathbb{N}^+$ with $n \geq 2$, $\tilde{R} \subseteq^d R$, $\boldsymbol{A}$ be an n-tuple of non-zero operators, and $\Delta \in d\mathbb{N}$ be sufficiently large. Let $a \in \mathbb{Z}$ be such that*

$$a > \inf \boldsymbol{A} \cdot \tilde{R}^n_\Delta \wedge a \leq \max\{\boldsymbol{A} \cdot z : z \in \tilde{R}^n_\Delta, z_1 = P^1_\Delta(a; \boldsymbol{A})\}.$$

Then $P_\Delta\left(a - A_1 P^1_\Delta(a; \boldsymbol{A}); \boldsymbol{A}_{>1}\right) = P^{>1}_\Delta(a; \boldsymbol{A})$.

Proof. Let $u = P_\Delta(a; \mathbf{A})$. Then

$$\mathbf{A}_{>1} \cdot u_{>1} + A_1 P^1_\Delta(a; \mathbf{A}) = \mathbf{A} \cdot P_\Delta(a; \mathbf{A}) < a,$$

and so $\mathbf{A}_{>1} \cdot u_{>1} < a - A_1 P^1_\Delta(a; \mathbf{A})$. Thus, to show that

$$u_{>1} = P_\Delta\left(a - A_1 P^1_\Delta(a; \mathbf{A}); \mathbf{A}_{>1}\right),$$

it suffices to show that there is no $w \in \tilde{R}^{n-1}_\Delta$ such that

$$\mathbf{A}_{>1} \cdot u_{>1} < \mathbf{A}_{>1} \cdot w < a - A_1 P^1_\Delta(a; \mathbf{A}).$$

Towards a contradiction, suppose such a $w \in \tilde{R}^{n-1}_\Delta$ existed, so

$$\mathbf{A} \cdot u < \mathbf{A} \cdot (P^1_\Delta(a; \mathbf{A}), w) < a.$$

By definition of $u = P_\Delta(a; \mathbf{A})$, we must have that $(P^1_\Delta(a; \mathbf{A}), w) \notin \tilde{R}^n_\Delta$, and so $w_1 > \sigma^{-\Delta} P^1_\Delta(a; \mathbf{A})$. Recalling the relevant notation from Definition 3.2.13, let

$$v := \max_{\mathbf{A}}\{z \in \tilde{R}^n_\Delta : z_1 = P^1_\Delta(a; \mathbf{A})\},$$

so by assumption, we have

$$\mathbf{A} \cdot u < \mathbf{A} \cdot (P^1_\Delta(a; \mathbf{A}), w) < a \leq \mathbf{A} \cdot v.$$

But now, since $u_1 = v_1 = P^1_\Delta(a; \mathbf{A})$, we have $\mathbf{A}_{>1} \cdot u_{>1} < \mathbf{A}_{>1} \cdot w < \mathbf{A}_{>1} \cdot v_{>1}$, so by Lemma 3.2.9,

$$w_1 \leq \max\{u_2, v_2\} \leq \sigma^{-\Delta} \max\{u_1, v_1\} = \sigma^{-\Delta} P^1_\Delta(a; \mathbf{A}),$$

which is a contradiction. □

The following theorem describes how a strong honest definition for a (F_n) formula can be obtained from one for a (F_{n-1}) formula.

Theorem 3.4.6. *Let $d, n \in \mathbb{N}^+$ with $n \geq 2$, $\tilde{R} \subseteq^d R$, $\boldsymbol{A}$ be an n-tuple of non-zero operators, and $\boldsymbol{B}$ be an n-tuple of operators. Let $t \in \{0, 1\}$ with $t = 1$ unless $\boldsymbol{B} = \boldsymbol{F}^i$ for some $1 \leq i \leq n$. Suppose that, for all $\Delta \in d\mathbb{N}$ sufficiently large, the formula*

$$\theta(x; y_1, y_2) := tx - y_2 < \boldsymbol{B}_{>1} \cdot P_\Delta(x - y_1; \boldsymbol{A}_{>1}, \tilde{R})$$

has a strong honest definition. Then, for all $\Delta \in d\mathbb{N}$ sufficiently large, the formula

$$\phi(x; y_1, y_2) := tx - y_2 < \boldsymbol{B} \cdot P_\Delta(x - y_1; \boldsymbol{A}, \tilde{R})$$

has a strong honest definition, given by a conjunction of copies of strong honest definitions for

$$\begin{cases} \phi_0 & \text{if } \boldsymbol{B} = \boldsymbol{F}^1, \\ \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, (\phi_6^\alpha, \phi_7^\alpha : -\Delta \leq \alpha \leq \Delta) & \text{if } \boldsymbol{B} \neq \boldsymbol{F}^1, A_1 \neq_R B_1, \text{ and } t = 1, \\ \phi_1, \phi_2, \phi_3, \phi_4, \phi_8 & \text{otherwise,} \end{cases}$$

where

$$\phi_0(x;y_1,y_2) := tx - y_2 < P^1_\Delta(x-y_1;\boldsymbol{A}),$$
$$\phi_1(x;y_1,y_2) := \inf \boldsymbol{A}\cdot \tilde{R}^n_\Delta < x-y_1,$$
$$\phi_2(x;y_1,y_2) := tx - y_2 < \boldsymbol{B}\cdot(\sigma^{(n-i)\Delta}(\min \tilde{R}) : 0 \le i < n),$$
$$\phi_3(x;y_1,y_2) := x - y_1 > \max\{\boldsymbol{A}\cdot z : z\in \tilde{R}^n_\Delta, z_1 = P^1_\Delta(x-y_1;\boldsymbol{A})\},$$
$$\phi_4(x;y_1,y_2) := tx - y_2 < \boldsymbol{B}\cdot \max_{\boldsymbol{A}}\{z\in \tilde{R}^n_\Delta : z_1 = P^1_\Delta(x-y_1;\boldsymbol{A})\},$$
$$\phi_5(x;y_1,y_2) := \begin{cases} P^1_\Delta(x-y_1;\boldsymbol{A}) > \sigma^\Delta P_\Delta(y_1-y_2;B_1-A_1) & \text{if } B_1 - A_1 >_R 0,\\ P^1_\Delta(x-y_1;\boldsymbol{A}) < \sigma^{-\Delta} P_\Delta(y_1-y_2;B_1-A_1) & \text{if } B_1 - A_1 <_R 0,\end{cases}$$
$$\phi_6^\alpha(x;y_1,y_2) := P^1_\Delta(x-y_1;\boldsymbol{A}) = \sigma^\alpha P_\Delta(y_1-y_2;B_1-A_1),$$
$$\phi_7^\alpha(x;y_1,y_2) := tx - y_2 - B_1\sigma^\alpha P_\Delta(y_1-y_2;B_1-A_1)$$
$$< \boldsymbol{B}_{>1}\cdot P_\Delta(x-y_1-A_1\sigma^\alpha P_\Delta(y_1-y_2;B_1-A_1);\boldsymbol{A}_{>1}),$$
$$\phi_8(x;y_1,y_2) := \theta(x - A_1P^1_\Delta(x-y_1;\boldsymbol{A});y_1,y_2).$$

From this we immediately obtain the sufficiency criterion in Theorem 3.3.6 as a corollary.

Corollary 3.4.7. *Let $d,n\in\mathbb{N}^+$, $\tilde{R}\subseteq^d R$, $\boldsymbol{A}$ be an n-tuple of non-zero operators, and $\boldsymbol{B}$ be an n-tuple of operators. Then, for all sufficiently large $\Delta\in d\mathbb{N}$, every $(F_n;\boldsymbol{A},\boldsymbol{B},\tilde{R},\Delta)$ formula has a strong honest definition.*

Proof. Induct on $n\in\mathbb{N}^+$, with Corollary 3.4.4 as the base case $n=1$ and Theorem 3.4.6 as the inductive step. □

Before proving Theorem 3.4.6, let us first justify that the formulas $\phi_0,\dots,\phi_8$ indeed have strong honest definitions, assuming that $\Delta\in d\mathbb{N}$ is sufficiently large.

The formulas ϕ_0, ϕ_3, ϕ_4, and ϕ_5 have strong honest definitions by Corollary 3.4.4 and Lemma 2.7.14, applied with $\Delta\in d\mathbb{N}$ sufficiently large. As an example, to show that ϕ_3 has a strong honest definition (assuming $\Delta\in d\mathbb{N}$ is sufficiently large), one applies Corollary 3.4.4 with $t=1$, $\square$ as $>$, and f mapping $u\mapsto \max\{\mathbf{A}\cdot z : z\in\tilde{R}^n_\Delta, z_1 = u\}$ if $u\in R$ and $u\mapsto 0$ otherwise.

The formulas ϕ_1 and ϕ_2 are basic formulas, so have strong honest definitions.

For $-\Delta \leq \alpha \leq \Delta$, the formula ϕ_6^α has a strong honest definition by Lemmas 3.4.2 and 2.7.14, since it is a descendant of the formula $P_\Delta^1(x - y_1; \mathbf{A}) = y_2$.

For $-\Delta \leq \alpha \leq \Delta$, the formula ϕ_7^α has a strong honest definition by Lemma 2.7.14, since it is a descendant of the formula $\theta(x; y_1, y_2)$, which is assumed to have a strong honest definition.

Finally, consider the formula ϕ_8. It is a descendant of the formula

$$\phi_8'(x; w, y_1, y_2) := \theta(x - A_1 P_\Delta^1(x - w; \mathbf{A}); y_1, y_2),$$

so by Lemma 2.7.14 it suffices to show that ϕ_8' has a strong honest definition. Now the formula

$$\theta'(x; w, y_1, y_2) := \theta(x - w; y_1, y_2) = \theta(x; w + y_1, tw + y_2)$$

is a descendant of θ, which is assumed to have a strong honest definition, and hence so does θ' by Lemma 2.7.14. Thus, the formula ϕ_8' has a strong honest definition by Theorem 3.4.3, applied with $t = 1$ and f mapping $u \mapsto A_1 u$ if $u \in R$ and $u \mapsto 0$ otherwise.

Thus, Theorem 3.4.6 is well-formulated; let us prove it.

Proof of Theorem 3.4.6. Let $\Delta \in d\mathbb{N}$ be sufficiently large such that the function $z \mapsto \mathbf{A} \cdot z$ is injective on $\tilde{R}_\Delta^n$, $\theta(x; y_1, y_2)$ has a strong honest definition, and all the strong honest definitions exist that are claimed to exist in the statement of the theorem. We will show that $\phi(x; y_1, y_2)$ is a Boolean combination of copies of

$$\begin{cases} \phi_0 & \text{if } \mathbf{B} = \mathbf{F}^1, \\ \phi_1, \phi_2, \phi_3, \phi_4, \phi_5, (\phi_6^\alpha, \phi_7^\alpha : -\Delta \leq \alpha \leq \Delta) & \text{if } \mathbf{B} \neq \mathbf{F}^1, A_1 \neq_R B_1, \text{ and } t = 1, \\ \phi_1, \phi_2, \phi_3, \phi_4, \phi_8 & \text{otherwise,} \end{cases}$$

which suffices by Lemma 2.7.10.

If $\mathbf{B} = \mathbf{F}^1$ then $\phi(x; y) \leftrightarrow \phi_0(x; y)$, so henceforth suppose $\mathbf{B} \neq \mathbf{F}^1$. We will show that

$$\phi \leftrightarrow (\neg\phi_1 \wedge \phi_2) \vee \big((\phi_1 \wedge \phi_3 \wedge \phi_4) \vee (\phi_1 \wedge \neg\phi_3 \wedge \phi)\big)\,. \tag{2}$$

If $\neg\phi_1(x;y)$ holds then $x - y_1 \leq \inf \mathbf{A} \cdot \tilde{R}^n_\Delta$, so by Remark 3.2.15 we have $\phi(x;y) \leftrightarrow \phi_2(x;y)$. Henceforth condition on $\phi_1(x;y)$, whence by Lemma 3.2.17,

$$\mathbf{A} \cdot P_\Delta(x - y_1; \mathbf{A}) < x - y_1. \tag{3}$$

If $\phi_3(x;y)$ holds, then $P_\Delta(x - y_1; \mathbf{A}) = \max_{\mathbf{A}}\{z \in \tilde{R}^n_\Delta : z_1 = P^1_\Delta(x - y_1; \mathbf{A})\}$ and so $\phi(x;y) \leftrightarrow \phi_4(x;y)$. Thus, (2) is shown. It suffices now to condition on $\phi_1 \wedge \neg\phi_3$ and show that ϕ is equivalent to a Boolean combination of copies of

$$\begin{cases} \phi_5, (\phi_6^\alpha, \phi_7^\alpha : -\Delta \leq \alpha \leq \Delta) & \text{if } A_1 \neq_R B_1 \text{ and } t = 1, \\ \phi_8 & \text{otherwise.} \end{cases}$$

Henceforth condition on $\phi_1(x;y) \wedge \neg\phi_3(x;y)$. Note then that, assuming $\Delta \in d\mathbb{N}$ is sufficiently large, Lemma 3.4.5 implies

$$P^{>1}_\Delta(x - y_1; \mathbf{A}) = P_\Delta\left(x - y_1 - A_1 P^1_\Delta(x - y_1; \mathbf{A}); \mathbf{A}_{>1}\right). \tag{4}$$

We now split into two cases: $A_1 \neq_R B_1 \wedge t = 1$, and $(A_1 =_R B_1 \wedge t = 1) \vee (\mathbf{B} = \mathbf{F}^i \wedge t = 0)$.

Case 1: $A_1 \neq_R B_1 \wedge t = 1$. We will show that

$$\phi \leftrightarrow \phi_5 \vee \bigvee_{\alpha=-\Delta}^{\Delta} (\phi_6^\alpha \wedge \phi_7^\alpha).$$

Let $\varepsilon := 1$ if $B_1 - A_1 >_R 0$, and $\varepsilon := -1$ if $B_1 - A_1 <_R 0$.

Firstly, suppose $\phi_\perp(x;y)$ holds, where

$$\phi_\perp(x;y) := \begin{cases} P^1_\Delta(x-y_1;\mathbf{A}) < \sigma^{-\Delta} P_\Delta(y_1-y_2;B_1-A_1) & \text{if } B_1-A_1 >_R 0, \\ P^1_\Delta(x-y_1;\mathbf{A}) > \sigma^{\Delta} P_\Delta(y_1-y_2;B_1-A_1) & \text{if } B_1-A_1 <_R 0. \end{cases}$$

In particular, $\inf(B_1-A_1)\tilde{R}^1_\Delta < y_1 - y_2$ by Remark 3.2.15, whence, for $\Delta \in d\mathbb{N}$ sufficiently large,

$$\begin{aligned} y_1 - y_2 &> (B_1-A_1)P_\Delta(y_1-y_2;B_1-A_1) && \text{by Lemma 3.2.17} \\ &> (B_1-A_1)\sigma^{\varepsilon\Delta}P^1_\Delta(x-y_1;\mathbf{A}) && \text{by } \phi_\perp(x;y) \\ &> 2^\varepsilon(B_1-A_1)P^1_\Delta(x-y_1;\mathbf{A}) && \text{by Lemma 3.2.5} \\ &> (\mathbf{B}-\mathbf{A})\cdot P_\Delta(x-y_1;\mathbf{A}) && \text{by Lemma 3.2.7.} \end{aligned}$$

Recalling from (3) that $x - y_1 > \mathbf{A}\cdot P_\Delta(x-y_1;\mathbf{A})$, we have that

$$x - y_2 = x - y_1 + y_1 - y_2 > \mathbf{B}\cdot P_\Delta(x-y_1;\mathbf{A}),$$

and so $\phi(x;y) \leftrightarrow \perp$.

Next, suppose $\phi_5(x;y)$ holds. In particular, $\sup(B_1-A_1)\tilde{R}^1_\Delta \geq y_1 - y_2$ by Remark 3.2.15, whence, for $\Delta \in d\mathbb{N}$ sufficiently large,

$$\begin{aligned} y_1 - y_2 &\leq (B_1-A_1)Q_\Delta(y_1-y_2;B_1-A_1) && \text{by Lemma 3.2.17} \\ &\leq (B_1-A_1)\sigma^{\varepsilon d}P_\Delta(y_1-y_2;B_1-A_1) && \text{by Lemma 3.2.17} \\ &< (B_1-A_1)\sigma^{\varepsilon(d-\Delta)}P^1_\Delta(x-y_1;\mathbf{A}) && \text{by } \phi_5(x;y). \end{aligned}$$

Using $\neg\phi_3(x;y)$, for $\Delta \in d\mathbb{N}$ sufficiently large, we have

$$x - y_1 \leq \max\{\mathbf{A}\cdot z : z \in \tilde{R}^n_\Delta, z_1 = P^1_\Delta(x-y_1;\mathbf{A})\} < (A_1 + \sigma^{-\lfloor \Delta/2 \rfloor})P^1_\Delta(x-y_1;\mathbf{A})$$

by Lemma 3.2.5. Thus, for $\Delta \in d\mathbb{N}$ sufficiently large,

$$\begin{aligned} x - y_2 &= x - y_1 + y_1 - y_2 \\ &< (A_1 + \sigma^{-\lfloor \Delta/2 \rfloor}) P^1_\Delta(x - y_1; \mathbf{A}) + (B_1 - A_1)\sigma^{\varepsilon(d-\Delta)} P^1_\Delta(x - y_1; \mathbf{A}) \\ &< (B_1 - \sigma^{-\lfloor \Delta/2 \rfloor}) P^1_\Delta(x - y_1; \mathbf{A}) && \text{by Lemma 3.2.5} \\ &< \mathbf{B} \cdot P_\Delta(x - y_1; \mathbf{A}) && \text{by Lemma 3.2.5,} \end{aligned}$$

and so $\phi(x; y) \leftrightarrow \top$.

Finally, suppose neither $\phi_\perp(x; y)$ nor $\phi_5(x; y)$ holds. Then there is $-\Delta \leq \alpha \leq \Delta$ such that $\phi_6^\alpha(x; y)$ holds. Conditioning on such $\phi_6^\alpha(x; y)$, we have

$$\begin{aligned} &\phi(x; y) \\ &\leftrightarrow x - y_2 - B_1 P^1_\Delta(x - y_1; \mathbf{A}) < \mathbf{B}_{>1} \cdot P^{>1}_\Delta(x - y_1; \mathbf{A}) \\ &\leftrightarrow x - y_2 - B_1 P^1_\Delta(x - y_1; \mathbf{A}) < \mathbf{B}_{>1} \cdot P_\Delta\left(x - y_1 - A_1 P^1_\Delta(x - y_1; \mathbf{A}); \mathbf{A}_{>1}\right) \end{aligned}$$

by (4). But this is equivalent to $\phi_7^\alpha(x; y)$, since $\phi_6^\alpha(x; y)$ holds. Thus,

$$\phi \leftrightarrow \phi_5 \vee \bigvee_{\alpha=-\Delta}^{\Delta} (\phi_6^\alpha \wedge \phi_7^\alpha).$$

Case 2: $(A_1 =_R B_1 \wedge t = 1)$ or $(\mathbf{B} = \mathbf{F}^i \wedge t = 0)$. We will show that $\phi \leftrightarrow \phi_8$. Recall that we have assumed $\mathbf{B} \neq \mathbf{F}^1$; in particular, $B_1 =_R tA_1$. We have

$$\begin{aligned} &\phi(x; y) \\ &\leftrightarrow tx - B_1 P^1_\Delta(x - y_1; \mathbf{A}) - y_2 < \mathbf{B}_{>1} \cdot P^{>1}_\Delta(x - y_1; \mathbf{A}) \\ &\leftrightarrow tx - B_1 P^1_\Delta(x - y_1; \mathbf{A}) - y_2 < \mathbf{B}_{>1} \cdot P_\Delta\left(x - y_1 - A_1 P^1_\Delta(x - y_1; \mathbf{A}); \mathbf{A}_{>1}\right) \end{aligned}$$

by (4). But this is equivalent to $\phi_8(x; y)$: since $B_1 = tA_1$, we have that $tx - B_1 P^1_\Delta(x - y_1; \mathbf{A}) = t(x - A_1 P^1_\Delta(x - y_1; \mathbf{A}))$. □

Theorem 3.4.8. *The structure $(\mathbb{Z}, <, +, R)$ is distal.*

Proof. Combine Theorem 3.3.6 and Corollary 3.4.7. □

Chapter 4

Distality *to* Combinatorics: Regularity Lemma and Zarankiewicz Bounds

In this chapter, we recover combinatorial interactions from a distality assumption. Specifically, we establish a connection between regularity lemmas and Zarankiewicz bounds that is satisfied by relations definable in a distal structure (and others). Since Kővári, Sós, and Turán proved upper bounds for the Zarankiewicz problem in 1954, much work has been undertaken to improve these bounds, and some have done so by restricting to particular classes of graphs. In 2017, Fox, Pach, Sheffer, Suk, and Zahl proved better bounds for semialgebraic binary relations, and this work was extended by Do in the following year to arbitrary semialgebraic relations. In this chapter, we show that Zarankiewicz bounds in the shape of Do's are enjoyed by all relations satisfying the distal regularity lemma, an improved version of the Szemerédi regularity lemma satisfied by relations definable in distal structures.

With the exception of Section 4.6, this chapter is presented (with minor differences) in our preprint [53]. We thank Pantelis Eleftheriou for his consistent guidance and mentorship, and for suggesting this problem to us. *Soli Deo gloria.*

4.1 Introduction

A classical problem in graph theory is the Zarankiewicz problem, which asks for the maximum number of edges a bipartite graph with n vertices in each class can have if it omits $K_{u,u}$, the complete bipartite graph with u vertices in each class. In 1954, Kővári, Sós, and Turán [32] gave an upper bound of $O_u(n^{2-1/u})$. Remarkably, this remains the tightest known upper bound, although sharpness has only been proven for $u \in \{2,3\}$. In 2017, Fox, Pach, Sheffer, Suk, and Zahl [19] observed that this bound can be improved if the graph is semialgebraic.

Theorem 4.1.1 (Fox–Pach–Sheffer–Suk–Zahl [19, Theorem 1.1]). *Let $E(x,y)$ be a semialgebraic relation on $\mathbb{R}$ with description complexity at most t. Let $d_1 := |x|$ and $d_2 := |y|$. Then, for all finite $P \subseteq \mathbb{R}^x$ and $Q \subseteq \mathbb{R}^y$ with $m := |P|$ and $n := |Q|$, if $E(P,Q)$ is $K_{u,u}$-free, then for all $\varepsilon > 0$ we have*

$$|E(P,Q)| \ll_{u,d_1,d_2,t,\varepsilon} \begin{cases} m^{\frac{2}{3}}n^{\frac{2}{3}} + m + n & \text{if } d_1 = d_2 = 2, \\ m^{\frac{d_2(d_1-1)}{d_1d_2-1}+\varepsilon} n^{\frac{d_1(d_2-1)}{d_1d_2-1}} + m + n & \text{otherwise.} \end{cases}$$

The graph theorist naturally asks if these results can be generalised to k-partite k-uniform hypergraphs (henceforth, a *k-graph* is a k-uniform hypergraph). Erdős led the way in 1964 [17], generalising the result of Kővári et al: a $K_{u,\dots,u}$-free k-partite k-graph with n vertices in each class has $O_u(n^{k-1/u^{k-1}})$ edges. In 2018, Do [14] generalised Theorem 4.1.1, improving Erdős' bounds for semialgebraic k-partite k-graphs.

Theorem 4.1.2 (Do [14, Theorem 1.7]). *Let $E(x_1,\dots,x_k)$ be a semialgebraic relation on $\mathbb{R}$ with description complexity at most t. Let $d_i := |x_i|$. Then, for all finite $P_i \subseteq \mathbb{R}^{x_i}$ with $n_i := |P_i|$, if $E(P_1,\dots,P_k)$ is $K_{u,\dots,u}$-free, then for all $\varepsilon > 0$ we have*

$$|E(P_1,\dots,P_k)| \ll_{u,\bar{d},t,\varepsilon} F^{\varepsilon}_{\bar{d}}(n_1,\dots,n_k),$$

where $\bar{d} := (d_1,\dots,d_k)$.

The function $F^\varepsilon_{\bar{d}}$ will be defined in Definition 4.4.2, but for now we merely note that, when $d_1 = \cdots = d_k =: d$ and $n_1 = \cdots = n_k =: n$,

$$F^\varepsilon_{\bar{d}}(n_1, ..., n_k) \ll_k n^{k - \frac{k}{(k-1)d+1} + k\varepsilon}.$$

In this chapter, we prove an analogue of Theorem 4.1.2 for a much larger class of relations, namely, relations *satisfying the distal regularity lemma*. That is, they satisfy an improved version of the Szemerédi regularity lemma, in which the sizes of the partitions are polynomial in the reciprocal of the error, and the good cells are not just regular but homogeneous (that is, a clique or an anti-clique); see Definition 4.2.3. Collecting the degrees of the polynomials into a *strong distal regularity tuple* $\bar{c}$, we state our main theorem.

Main Theorem (Theorem 4.4.5)**.** *Let $E(x_1, ..., x_k)$ be a relation on a set M, with strong distal regularity tuple $\bar{c} = (c_1, ..., c_k) \in \mathbb{R}^k_{\geq 1}$ and coefficient λ. For all finite $P_i \subseteq M^{x_i}$ with $n_i := |P_i|$, if $E(P_1, ..., P_k)$ is $K_{u,...,u}$-free, then for all $\varepsilon > 0$,*

$$|E(P_1, ..., P_k)| \ll_{u,\bar{c},\lambda,\varepsilon} F^\varepsilon_{\bar{c}}(n_1, ..., n_k).$$

Here, the function $F^\varepsilon_{\bar{c}}$ is precisely the function $F^\varepsilon_{\bar{d}}$ appearing in Theorem 4.1.2, but with $\bar{c}$ in place of $\bar{d}$ (as a tuple of dummy variables). The definition of the *coefficient* is unimportant for this discussion, so we refer the reader to Definition 4.2.3 for it.

The distal regularity lemma is so named because it is satisfied by all relations definable in distal structures. Thus, our main theorem joins a parade of combinatorial properties that have been shown to hold in distal structures in the last decade. It supports the postulate by Chernikov, Galvin, and Starchenko in [9] that 'distal structures provide the most general natural setting for investigating questions in "generalised incidence combinatorics"', where they proved an analogue of Theorem 4.1.1 for binary relations definable in a distal structure. It also motivates the following problem.

Problem 4.1.3. *Compute (strong) distal regularity tuples for relations satisfying the distal regularity lemma, such as those definable in a distal structure.*

It is worth pointing out that our main theorem applies to all relations satisfying the distal regularity lemma, which form a strictly larger class of relations than those definable in a distal structure — see Theorem 4.5.1. However, this does not constitute a refutation of the postulate by Chernikov, Galvin, and Starchenko, as distal structures are still the most general structures in the literature in which all definable relations satisfy the distal regularity lemma.

This chapter presents regularity lemmas as a means of obtaining Zarankiewicz bounds, an approach also adopted in [29]. Improvements on the Szemerédi regularity lemma have been made in various contexts, such as for stable graphs [35] and for graphs with bounded VC-dimension [21]. Following our main theorem, it is natural to pose the following problem.

Problem 4.1.4. *Which other variants of the Szemerédi regularity lemma give rise to improved Zarankiewicz bounds?*

4.1.1 The semialgebraic case

Recently, Tidor and Yu [52] proved that if $E(x_1, ..., x_k)$ is a semialgebraic relation on $\mathbb{R}$, then $(|x_1|, ..., |x_k|)$ is a distal regularity tuple for E, so $(|x_1|+1, ..., |x_k|+1)$ is a strong distal regularity tuple for E, where the corresponding coefficient is a function of the description complexity of E. We refer the reader to Definition 4.2.3 for a precise definition of (strong) distal regularity tuples, but here we emphasise that the word 'strong' refers to requiring equipartitions in the distal regularity lemma.

Thus, if the assumption in our main theorem can be weakened so that $\bar{c}$ is only required to be a distal regularity tuple (that is, the corresponding partitions need not be equipartitions), their result can be combined with ours to recover Theorems 4.1.1 and 4.1.2. In Section 4.3, we show that this assumption can indeed be so weakened when E is a binary relation, thus recovering Theorem

4.1.1. We would like to do likewise for an arbitrary relation, and so we pose the following problem.

Problem 4.1.5. *For an arbitrary relation $E(x_1, ..., x_k)$, can the assumption in our main theorem be weakened so that $\bar{c}$ is only required to be a distal regularity tuple?*

Another way to recover Theorem 4.1.2 would be to resolve the following problem positively.

Problem 4.1.6. *For a semialgebraic relation $E(x_1, ..., x_k)$ on $\mathbb{R}$, is $(|x_1|, ..., |x_k|)$ a strong distal regularity tuple for E?*

We note however that, in [52], Tidor and Yu also proved infinitesimally improved versions of Theorems 4.1.1 and 4.1.2 — removing the ε from the bounds — which our present methods are not able to achieve.

4.1.2 Structure of the chapter

In Section 4.2, we introduce the notion of (strong) distal regularity tuples and prove some of their basic properties. In Section 4.3, we prove a stronger version of the main theorem in the case where the relation is binary, and in Section 4.4, we prove the theorem in full. Finally, in Section 4.5, we discuss the context to which the theorem can be applied.

4.1.3 Notation and basic definitions

In this chapter, we often consider relations as set-theoretic objects rather than definable sets in some structure. We lay out some notation and definitions below, some of which are borrowed from first-order logic.

If $x_1, ..., x_k, y$ are variables, write $y = y(x_1, ..., x_k)$ to mean that y is a function of $x_1, ..., x_k$.

Relations

Let M be a set and $k \in \mathbb{N}^+$. For tuples of variables $x_1, ..., x_k$, a *relation* $E(x_1, ..., x_k)$ on M is a subset of $M^{|x_1|} \times \cdots \times M^{|x_k|}$, or equivalently, a k-partite k-graph on vertex sets $M^{|x_1|}, ..., M^{|x_k|}$. We will often drop the absolute value signs and write M^{x_1} for $M^{|x_1|}$, and so on.

For $a_i \in M^{x_i}$, $E(a_1, ..., a_k)$ is defined to mean $(a_1, ..., a_k) \in E(x_1, ..., x_k)$. For $P_i \subseteq M^{x_i}$ and $b_i \in M^{x_i}$,

$$E(P_1, ..., P_k) := \{(a_1, ..., a_k) \in P_1 \times \cdots \times P_k : E(a_1, ..., a_k)\},$$
$$E(b_1, P_2, ..., P_k) := \{(a_2, ..., a_k) \in P_2 \times \cdots \times P_k : E(b_1, a_2, ..., a_k)\},$$

and we similarly define $E(P_1, ..., P_{i-1}, b_i, P_{i+1}, ..., P_k)$ for all $i \in [k]$. Say that $P_1 \times \cdots \times P_k$ is *E-homogeneous* if $E(P_1, ..., P_k) = P_1 \times \cdots \times P_k$ or $\emptyset$.

Hölder's inequality

Hölder's inequality is the following classical theorem.

Theorem 4.1.7. *Let $a_1, ..., a_n, b_1, ..., b_n, p, q \in \mathbb{R}_{\geq 0}$ be such that $p + q = 1$. Then*

$$\sum_{i=1}^{n} a_i^p b_i^q \leq \left(\sum_{i=1}^{n} a_i\right)^p \left(\sum_{i=1}^{n} b_i\right)^q.$$

4.2 (Strong) distal regularity tuples

We begin by defining the notion of regularity for a bipartite graph.

Definition 4.2.1. Let M be a set, and let $E(x, y)$ be a relation on M. For finite $A \subseteq M^x$ and $B \subseteq M^y$, write

$$d(A, B) := \frac{|E(A, B)|}{|A||B|}.$$

Let $P \subseteq M^x$ and $Q \subseteq M^y$ be finite. For $\delta > 0$, say that the bipartite graph $E(P, Q)$ is *δ-regular* if, for all $A \subseteq P$ and $B \subseteq Q$ with $|A| \geq \delta|P|$ and $|B| \geq \delta|Q|$,

$|d(A, B) - d(P, Q)| \leq \delta$.

In 1978, Szemerédi proved the following celebrated regularity lemma.

Theorem 4.2.2 (Szemerédi, 1978 [51]). *Let M be a set, and let $E(x, y)$ be a relation on M. For all $\delta > 0$, there is $K \in \mathbb{N}$ such that the following holds.*

Let $P \subseteq M^x$ and $Q \subseteq M^y$ be finite. Then there are (equi)partitions $P = A_1 \sqcup \cdots \sqcup A_K$ and $Q = B_1 \sqcup \cdots \sqcup B_K$, and an index set $\Sigma \subseteq [K]^2$ of 'bad cells', such that

(a) 'Meagre bad cells': $\sum_{(i,j)\in\Sigma} |A_i \times B_j| \leq \delta |P \times Q|$; and

(b) 'δ-regular good cells': for all $(i, j) \in [K]^2 \setminus \Sigma$, $E(A_i, B_j)$ is δ-regular.

Szemerédi's proof shows that K can be bounded above by an exponential tower with height a polynomial in $1/\delta$. Hopes of improving this enormous bound in general were quashed in 1997 when Gowers [23] constructed graphs necessitating K of this size.

However, various results have arisen since then that establish better bounds for K in certain contexts, along with additional improvements on the regularity partition. Notably, in 2016, Fox, Pach, and Suk [20] showed that when E is semialgebraic, not only is K upper bounded by a polynomial in $1/\delta$, but also item (b) in Theorem 4.2.2 can be replaced by the condition that for all $(i, j) \in [K]^2 \setminus \Sigma$, $A_i \times B_j$ is E-homogeneous, a very strong form of regularity. In 2018, Chernikov and Starchenko [11] weakened the semialgebraicity assumption and showed that this holds if E is definable in a distal structure, leading to the nomenclature *distal regularity lemma*.

The results of Fox–Pach–Suk and Chernikov–Starchenko hold for relations of arbitrary arity (that is, for hypergraphs as well as graphs). We will state this result formally in an a priori roundabout way, by putting the spotlight on the degree of the polynomial in $1/\delta$ that upper bounds K — this will be important later on.

Definition 4.2.3. Let M be a set, and let $E(x_1, ..., x_k)$ be a relation on M. Let $c_1, ..., c_k \in \mathbb{R}_{\geq 0}$, and write $\bar{c} := (c_1, ..., c_k)$.

Say that $\bar{c}$ is a *distal regularity tuple (respectively, strong distal regularity tuple)* for E if there is a *coefficient* $\lambda > 1$ satisfying the following: for all $\delta > 0$ and finite sets $P_i \subseteq M^{x_i}$ with $n_i := |P_i|$, there are partitions (respectively, equipartitions) $P_i = A^i_1 \sqcup \cdots \sqcup A^i_{K_i}$ and an index set $\Sigma \subseteq [K_1] \times \cdots \times [K_k]$ of 'bad cells' such that

(a) 'Meagre bad cells': $\sum_{(j_1,...,j_k)\in\Sigma} |A^1_{j_1} \times \cdots \times A^k_{j_k}| \leq \lambda\delta n_1 \cdots n_k$;

(b) 'Homogeneous good cells': for all $(j_1, ..., j_k) \in [K_1] \times \cdots \times [K_k] \setminus \Sigma$, $A^1_{j_1} \times \cdots \times A^k_{j_k}$ is E-homogeneous; and

(c) 'Polynomially (in δ^{-1}) many cells' For all $i \in [k]$, $K_i \leq \lambda\delta^{-c_i}$.

Say that E *satisfies the distal regularity lemma* if there is a distal regularity tuple for E.

Remark 4.2.4. For the reader that is familiar with hypergraph regularity, alarm bells may be ringing. This notion of hypergraph regularity appears to merely be a stronger version of what is known as *weak hypergraph regularity*, which has been rendered mostly obsolete due to its combinatorial limitations. We will explain this in much more detail in Chapter 5, but briefly, the weak hypergraph regularity lemma says that a hypergraph $P_1 \times \cdots \times P_k$ can be decomposed into a bounded number of boxes $A^1_{j_1} \times \cdots \times A^k_{j_k}$, most of which are δ-regular (for the obvious generalisation of δ-regularity to hypergraphs). Combinatorialists have observed that, in general, δ-regular boxes are not uniform enough on which to do combinatorics, and a *strong hypergraph regularity* result was eventually developed where hypergraphs are decomposed into 'simplicial complexes' rather than boxes (and the notion of uniformity is more refined than δ-regularity).

Nonetheless, although distal regularity is a version of weak hypergraph regularity, the good cells are not just δ-regular but homogeneous. Homogeneity is the strongest possible form of uniformity, and so the combinatorial limitations that plague the weak regularity lemma do not apply to the distal regularity lemma at all. In fact, distal regularity can be seen as a stronger version of strong hypergraph regularity — see Chapter 5.

It is immediate that strong distal regularity tuples are distal regularity tuples (with the same coefficient), and the following lemma establishes a converse.

Lemma 4.2.5. *Let M be a set, and let $E(x_1, ..., x_k)$ be a relation on M. Let $c_1, ..., c_k \in \mathbb{R}_{\geq 0}$. Writing $\bar{c} := (c_1, ..., c_k)$, suppose $\bar{c}$ is a distal regularity tuple for E with coefficient λ. Then $(c_1 + 1, ..., c_k + 1)$ is a strong distal regularity tuple for E with coefficient $(k + 2)\lambda$.*

Proof. Suppose $\bar{c}$ is a distal regularity tuple for E with coefficient λ. Let $\delta \in (0, \lambda^{-1})$ and let $P_i \subseteq M^{x_i}$ be finite with $n_i := |P_i|$. Fix partitions $P_i = A^i_1 \sqcup \cdots \sqcup A^i_{K_i}$ and an index set of 'bad cells' $\Sigma \subseteq [K_1] \times \cdots \times [K_k]$ as in the definition of $\bar{c}$ as a distal regularity tuple for E.

For each $i \in [k]$, define an equipartition $P_i = B^i_1 \sqcup \cdots \sqcup B^i_{L_i}$ as follows. Let $N_i := \lceil \frac{1}{2}\delta^{c_i+1} n_i \rceil$. For each $j \in [K_i]$, partition A^i_j into a maximal number of parts of size N_i and a part S^i_j of size less than N_i. This gives a new partition $P_i = S^i_1 \sqcup \cdots \sqcup S^i_{K_i} \sqcup T^i_1 \sqcup \cdots \sqcup T^i_{K'_i}$ where, for all $j' \in [K'_i]$, $|T^i_{j'}| = N_i$ and there is a unique $j \in [K_i]$ such that $T^i_{j'} \subseteq A^i_j$. Observe that $K'_i \leq n_i/N_i \leq 2\delta^{-(c_i+1)}$.

By moving elements of $T^i_1, ..., T^i_{K'_i}$ to $S^i_1, ..., S^i_{K_i}$ as much as necessary, we obtain an equipartition $P_i = \bar{S}^i_1 \sqcup \cdots \sqcup \bar{S}^i_{K_i} \sqcup \bar{T}^i_1 \sqcup \cdots \sqcup \bar{T}^i_{K'_i}$ where $|\bar{S}^i_j| \leq N_i - 1$ for all $j \in [K_i]$ and, for all $j' \in [K'_i]$, there is a unique $j \in [K_i]$ such that $\bar{T}^i_{j'} \subseteq A^i_j$. Rename $\bar{S}^i_1, ..., \bar{S}^i_{K_i}$ as $B^i_1, ..., B^i_{K_i}$ and $\bar{T}^i_1, ..., \bar{T}^i_{K'_i}$ as $B^i_{K_i+1}, ..., B^i_{L_i}$. Observe that $L_i = K_i + K'_i \leq (\lambda + 2)\delta^{-(c_i+1)}$, and $\sum_{j=1}^{K_i} |B^i_j| \leq K_i(N_i - 1) \leq \lambda\delta n_i$.

For all $(j_1, ..., j_k) \in ([L_1] \setminus [K_1]) \times \cdots \times ([L_k] \setminus [K_k])$, there is a unique tuple $(i_1, ..., i_k) \in [K_1] \times \cdots \times [K_k]$ such that $B^1_{j_1} \times \cdots \times B^k_{j_k} \subseteq A^1_{i_1} \times \cdots \times A^k_{i_k}$; write $(i_1, ..., i_k) = \pi(j_1, ..., j_k)$. Now set

$$\Lambda := \bigcup_{i=1}^{k} [L_1] \times \cdots \times [L_{i-1}] \times [K_i] \times [L_{i+1}] \times \cdots \times [L_k] \cup \Lambda_0,$$

where $\Lambda_0 := \{(j_1, ..., j_k) \in ([L_1] \setminus [K_1]) \times \cdots \times ([L_k] \setminus [K_k]) : \pi(j_1, ..., j_k) \in \Sigma\}$. We claim that the partitions $P_i = B^i_1 \sqcup \cdots \sqcup B^i_{L_i}$, together with the index set Λ of bad cells, witness that $(c_1 + 1, ..., c_k + 1)$ is a strong distal regularity tuple for E with coefficient $(k + 2)\lambda$.

We already have that, for all $i \in [k]$, $L_i \leq (\lambda + 2)\delta^{-(c_i+1)} \leq (k + 2)\lambda\delta^{-(c_i+1)}$. For all $(j_1, ..., j_k) \in [L_1] \times \cdots \times [L_k] \setminus \Lambda \subseteq ([L_1] \setminus [K_1]) \times \cdots \times ([L_k] \setminus [K_k])$, we have $\pi(j_1, ..., j_k) \notin \Sigma$; that is, $B^1_{j_1} \times \cdots \times B^k_{j_k}$ is contained in an E-homogeneous cell, so is itself E-homogeneous. Therefore, it remains to show that

$$\sum_{(j_1,...,j_k)\in\Lambda} |B^1_{j_1} \times \cdots \times B^k_{j_k}| \leq (k + 2)\lambda\delta n_1 \cdots n_k.$$

Firstly, $\bigsqcup_{(j_1,...,j_k)\in\Lambda_0} B^1_{j_1} \times \cdots \times B^k_{j_k} \subseteq \bigsqcup_{(i_1,...,i_k)\in\Sigma} A^1_{i_1} \times \cdots \times A^k_{i_k}$, so the set on the left has size at most $\lambda\delta n_1 \cdots n_k$. Now

$$\begin{aligned}
&\sum_{(j_1,...,j_k)\in\Lambda} |B^1_{j_1} \times \cdots \times B^k_{j_k}| \\
&= \sum_{i=1}^{k} n_1 \cdots n_{i-1}n_{i+1} \cdots n_k \sum_{j=1}^{K_i} |B^i_j| + \sum_{(j_1,...,j_k)\in\Lambda_0} |B^1_{j_1} \times \cdots \times B^k_{j_k}| \\
&\leq \sum_{i=1}^{k} n_1 \cdots n_{i-1}n_{i+1} \cdots n_k(\lambda\delta n_i) + \lambda\delta n_1 \cdots n_k \\
&= (k + 1)\lambda\delta n_1 \cdots n_k.
\end{aligned}$$

□

Thus, a relation E satisfies the distal regularity lemma if and only if there is a strong distal regularity tuple for E.

The results of Fox–Pach–Suk and Chernikov–Starchenko can now be stated as follows. We use $\mathbb{R}$ to denote the structure of the real ordered field.

Theorem 4.2.6 (Chernikov–Starchenko, 2018 [11]; $M = \mathbb{R}$: Fox–Pach–Suk, 2016 [20]). *Let $\phi(x_1, ..., x_k; y)$ be a relation definable in a distal structure M. Then there are $\bar{c} \in \mathbb{R}^k_{\geq 0}$ and $\lambda > 1$ such that, for all $b \in M^y$, the relation $E(x_1, ..., x_k) := \phi(x_1, ..., x_k; b)$ on M has distal regularity tuple $\bar{c}$ with coefficient λ.*

By Lemma 4.2.5, 'distal regularity tuple' can be replaced with 'strong distal regularity tuple' in the statement above.

We will now forget about the context of $M = \mathbb{R}$ or M as a distal structure, and derive Zarankiewicz bounds for all relations with a (strong) distal regularity tuple, that is, all relations that satisfy the distal regularity lemma.

We finish this section by proving a preliminary Zarankiewicz bound for a relation $E(x_1, ..., x_k)$ with a distal regularity tuple, morally inducting on k.

Lemma 4.2.7. *Let $E(x_1, ..., x_k)$ be a relation on a set M with distal regularity tuple $\bar{c} = (c_1, ..., c_k) \in \mathbb{R}^k_{\geq 1}$ and coefficient λ. Suppose that, for all $i \in [k]$, $F_i : \mathbb{N}^{k-1} \to \mathbb{R}$ is a function satisfying the following.*

Let $u \in \mathbb{N}^+$, and let $a_1, ..., a_u \in M^{x_i}$ be distinct. For all $j \in [k] \setminus \{i\}$, let $P_j \subseteq M^{x_j}$ with $n_j := |P_j|$. If the $(k-1)$-graph $\bigcap_{e=1}^u E(P_1, ..., P_{i-1}, a_e, P_{i+1}, ..., P_k)$ is $K_{u,...,u}$-free, then its size is $O_u(F_i(n_1, ..., n_{i-1}, n_{i+1}, ..., n_k))$.

(i) Let $\gamma \geq 0$, and suppose that for all finite $P_i \subseteq M^{x_i}$ with $n_i := |P_i|$, if $E(P_1, ..., P_k)$ is $K_{u,...,u}$-free, then

$$|E(P_1, ..., P_k)| \ll_{u,c_1,\lambda} n_1^{1-\gamma} n_2 \cdots n_k + n_1 F_1(n_2, ..., n_k).$$

Then the statement above holds with γ replaced by $\frac{1}{1+c_1(1-\gamma)}$.

(ii) For all $i \in [k]$, let $P_i \subseteq M^{x_i}$ be finite with $n_i := |P_i|$. If $E(P_1, ..., P_k)$ is $K_{u,...,u}$-free, then for all $i \in [k]$ and $\varepsilon > 0$,

$$|E(P_1, ..., P_k)| \ll_{u,c_i,\lambda,\varepsilon} n_1 \cdots n_k n_i^{-\frac{1}{c_i}+\varepsilon} + n_i F_i(n_1, ..., n_{i-1}, n_{i+1}, ..., n_k).$$

Proof. (i) Let $\delta = n_1^{-\frac{1}{1+c_1(1-\gamma)}}$. With this value of δ, partition $P_i = A^i_1 \sqcup \cdots \sqcup A^i_{K_i}$ for each $i \in [k]$ as in the definition of $\bar{c}$ as a distal regularity tuple for E, with $\Sigma \subseteq [K_1] \times \cdots \times [K_k]$ the index set of bad cells. Let $T := \bigcup_{(j_1,...,j_k) \in \Sigma} A^1_{j_1} \times \cdots \times A^k_{j_k}$. Without loss of generality, let $0 \leq L \leq K_1$ be such that, for all $1 \leq j \leq K_1$, $|A^1_j| \geq u$ if and only if $j > L$.

Let $j > L$. Then $|A_j^1| \geq u$, so let $a_1, ..., a_u \in A_j^1$ be distinct. Then

$$E(A_j^1, P_2, ..., P_k) \setminus T \subseteq A_j^1 \times \bigcap_{e=1}^{u} E(a_e, P_2, ..., P_k).$$

Since $E(P_1, ..., P_k)$ is $K_{u,...,u}$-free, the $(k-1)$-graph $\bigcap_{e=1}^{u} E(a_e, P_2, ..., P_k)$ is $K_{u,...,u}$-free, and so by assumption

$$|E(A_j^1, P_2, ..., P_k) \setminus T| \ll_u |A_j^1| F_1(n_2, ..., n_k).$$

Let $H_1 := \bigcup_{j=1}^{L} A_j^1$, so $|H_1| \leq Lu \leq K_1 u \leq \lambda\delta^{-c_1} u$. By assumption,

$$\begin{aligned}|E(H_1, P_2, ..., P_k)| &\ll_{u,c_1,\lambda} (\lambda\delta^{-c_1}u)^{1-\gamma} n_2 \cdots n_k + n_1 F_1(n_2, ..., n_k) \\ &\ll_{u,\lambda} \delta^{-c_1(1-\gamma)} n_2 \cdots n_k + n_1 F_1(n_2, ..., n_k).\end{aligned}$$

Thus,

$$\begin{aligned}&|E(P_1, ..., P_k)| \\ &\leq |T| + |E(H_1, P_2, ..., P_k)| + \sum_{j=L+1}^{K_1} |E(A_j^1, P_2, ..., P_k) \setminus T| \\ &\ll_{u,c_1,\lambda} \delta n_1 \cdots n_k + \delta^{-c_1(1-\gamma)} n_2 \cdots n_k + n_1 F_1(n_2, ..., n_k) + \sum_{j=L+1}^{K_1} |A_j^1| F_1(n_2, ..., n_k) \\ &\leq 2n_1^{1-\frac{1}{1+c_1(1-\gamma)}} n_2 \cdots n_k + 2n_1 F_1(n_2, ..., n_k).\end{aligned}$$

(ii) By symmetry, we may assume that $i = 1$. Let $f : [0, \frac{1}{c_1}] \to [\frac{1}{c_1+1}, \frac{1}{c_1}]$ be given by $\gamma \mapsto \frac{1}{1+c_1(1-\gamma)}$. The statement in (i) holds for $\gamma = 0$, so it suffices to show that $f^n(0) \to \frac{1}{c_1}$ as $n \to \infty$. Note that for all $\gamma \in [0, \frac{1}{c_1}]$ we have $\gamma \leq f(\gamma)$, since $(c_1\gamma - 1)(\gamma - 1) \geq 0$, which rearranges to $\gamma(1 + c_1(1-\gamma)) \leq 1$. Thus, $(f^n(0))_n$ is an increasing sequence in $[\frac{1}{c_1+1}, \frac{1}{c_1}]$, and so it converges to some limit $L \in [\frac{1}{c_1+1}, \frac{1}{c_1}]$. But then $L = \frac{1}{1+c_1(1-L)}$, which rearranges to $(c_1L - 1)(L - 1) = 0$, and so $L = \frac{1}{c_1}$ since $c_1 \geq 1$. □

Remark 4.2.8. In Lemma 4.2.7, when $k = 2$, F_i can be chosen to be the constant

1-valued function. Indeed, if a 1-graph is K_u-free, then its size is at most $u-1 = O_u(1)$.

Remark 4.2.9. After the preparation of our preprint [53] on which this chapter is based, we were made aware of a Turán-type argument in [52, Corollary 5.1] which allows one to remove the ε from the bound in Lemma 4.2.7(ii) as long as $c_i > 1$. Even so, bootstrapping this infinitesimally improved bound via our methods does not allow us to remove the ε in our main theorem (Theorem 4.4.5) or its binary counterpart (Theorem 4.3.1), so we retain the statement and proof of Lemma 4.2.7 as written to provide a different perspective and proof method.

4.3 Binary relations

We will first consider binary relations, for two reasons. Firstly, for binary relations, our main theorem holds under a weaker assumption — namely, $\bar{c}$ is only required to be a distal regularity tuple, not a strong distal regularity tuple. Secondly, the exposition is much cleaner for binary relations, and so will hopefully illuminate the proof strategy for arbitrary relations.

Theorem 4.3.1. *Let $E(x, y)$ be a relation on a set M, with distal regularity tuple $\bar{c} = (c_1, c_2) \in \mathbb{R}^2_{\geq 1}$ and coefficient λ. Then, for all finite $P \subseteq M^x$ and $Q \subseteq M^y$ with $m := |P|$ and $n := |Q|$, if $E(P, Q)$ is $K_{u,u}$-free, then for all $\varepsilon > 0$ we have*

$$|E(P, Q)| \ll_{u,\bar{c},\lambda,\varepsilon} m^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon} n^{\frac{c_1(c_2-1)}{c_1c_2-1}+\varepsilon} + m + n,$$

where, if $c_1 = c_2 = 1$, we define $\frac{c_2-1}{c_1c_2-1}, \frac{c_1-1}{c_1c_2-1}$ to be $\lim_{\delta\to 0} \frac{(1+\delta)-1}{(1+\delta)^2-1} = \frac{1}{2}$.

Proof. We will show that, for all $\varepsilon > 0$, there are constants $\alpha = \alpha(u, \bar{c}, \lambda, \varepsilon)$ and $\beta = \beta(u, \bar{c}, \lambda, \varepsilon)$ such that, for all finite $P \subseteq M^x$ and $Q \subseteq M^y$ with $m := |P|$ and $n := |Q|$, if $E(P, Q)$ is $K_{u,u}$-free, then

$$|E(P, Q)| \leq \alpha m^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon} n^{\frac{c_1(c_2-1)}{c_1c_2-1}+\varepsilon} + \beta(m + n). \tag{1}$$

The dependency between constants will be as follows:

(i) δ is sufficiently small in terms of $\bar{c}$, λ, and ε;

(ii) m_0 is sufficiently large in terms of u, $\bar{c}$, λ, ε, and δ;

(iii) β is sufficiently large in terms of m_0, u, $\bar{c}$, λ, and ε;

(iv) α is sufficiently large in terms of m_0, β, u, $\bar{c}$, λ, and δ.

By Lemma 4.2.7(ii) and Remark 4.2.8, $\left|E(P,Q)\right| \ll_{u,\bar{c},\lambda,\varepsilon} m^{1-\frac{1-\nu}{c_1}}n + m$ and $\left|E(P,Q)\right| \ll_{u,\bar{c},\lambda,\varepsilon} mn^{1-\frac{1-\nu}{c_2}} + n$, where $\nu \in (0,\frac{1}{2})$ is chosen such that

$$\left(\frac{1}{1-\nu} - 1\right) \max\left(\frac{c_2(c_1-1)}{c_1c_2-1}, \frac{c_1(c_2-1)}{c_1c_2-1}\right) \leq \varepsilon.$$

Note then in particular that $\frac{c_1-1}{c_1c_2-1}\frac{c_2}{1-\nu} = \frac{1}{1-\nu}\frac{c_2(c_1-1)}{c_1c_2-1} \leq \frac{c_2(c_1-1)}{c_1c_2-1} + \varepsilon$, and similarly $\frac{c_2-1}{c_1c_2-1}\frac{c_1}{1-\nu} = \frac{1}{1-\nu}\frac{c_1(c_2-1)}{c_1c_2-1} \leq \frac{c_1(c_2-1)}{c_1c_2-1} + \varepsilon$.

If $m \leq n^{\frac{1-\nu}{c_2}}$ and β is sufficiently large in terms of u, $\bar{c}$, λ, and ε, then $\left|E(P,Q)\right| \leq \beta n$ since $\left|E(P,Q)\right| \ll_{u,\bar{c},\lambda,\varepsilon} mn^{1-\frac{1-\nu}{c_2}} + n$. Therefore, for the rest of the proof we assume that $n < m^{\frac{c_2}{1-\nu}}$, which implies

$$n = n^{\frac{c_1-1}{c_1c_2-1}} n^{\frac{c_1(c_2-1)}{c_1c_2-1}} \leq m^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon} n^{\frac{c_1(c_2-1)}{c_1c_2-1}} \leq m^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon} n^{\frac{c_1(c_2-1)}{c_1c_2-1}+\varepsilon}. \tag{2}$$

Similarly, for the rest of the proof we assume that $m < n^{\frac{c_1}{1-\nu}}$, which implies

$$m = m^{\frac{c_2-1}{c_2c_1-1}} m^{\frac{c_2(c_1-1)}{c_2c_1-1}} \leq n^{\frac{c_1(c_2-1)}{c_2c_1-1}+\varepsilon} m^{\frac{c_2(c_1-1)}{c_2c_1-1}} \leq m^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon} n^{\frac{c_1(c_2-1)}{c_1c_2-1}+\varepsilon}. \tag{3}$$

Let $\alpha = \alpha(u,\bar{c},\lambda,\varepsilon)$ and $\beta = \beta(u,\bar{c},\lambda,\varepsilon)$ be sufficiently large, to be chosen later. We show by induction on $m+n$ that (1) holds.

Let $m_0 := m_0(u,\bar{c},\lambda,\varepsilon,\delta)$ to be chosen later. If $m+n < m_0$, then (1) holds by choosing values for α and β that are sufficiently large in terms of m_0. Thus, henceforth assume that $m+n \geq m_0$, and suppose that (1) holds when $|P|+|Q| < m+n$. If $m < m_0$, then $|E(P,Q)| < m_0 n \leq \beta n$ assuming $\beta \geq m_0$, so henceforth suppose $m \geq m_0$.

For $\delta := \delta(\bar{c},\lambda,\varepsilon) < 1$ to be chosen later, partition $P = A_1 \sqcup \cdots \sqcup A_{K_1}$ and $Q = B_1 \sqcup \cdots \sqcup B_{K_2}$ as in the definition of $\bar{c}$ as a distal regularity tuple, with

$\Sigma \subseteq [K_1] \times [K_2]$ the index set of bad cells. By refining the partition and replacing λ with $\lambda + 2$ if necessary, we can assume that $|A_i| \le \delta^{c_1} m$ for all $i \in [K_1]$.

For $i \in [K_1]$, let $\Sigma_i := \{j \in [K_2] : (i,j) \in \Sigma\}$. Without loss of generality, let $0 \le L \le L' \le K_1$ be such that:

(i) For all $i \in [K_1]$, $|A_i| \ge u$ if and only if $i > L$; and

(ii) For all $i \in [K_1] \setminus [L]$, $\sum_{j \in \Sigma_i} \left|B_j\right| \le \delta^{1-\varepsilon} n$ if and only if $i > L'$.

Partition P into $H_1 := \bigcup_{i=1}^{L} A_i$, $H_2 := \bigcup_{i=L+1}^{L'} A_i$, and $H_3 := \bigcup_{i=L'+1}^{K_1} A_i$. We will bound $|E(P,Q)|$ by bounding $\left|E(H_1,Q)\right|$, $\left|E(H_2,Q)\right|$, and $\left|E(H_3,Q)\right|$.

Consider $E(H_1,Q)$. Note that $|H_1| \le Lu \le K_1 u \le \lambda\delta^{-c_1} u$. Choosing $m_0 > \lambda\delta^{-c_1} u$, we have $m \ge m_0 > \lambda\delta^{-c_1} u \ge |H_1|$. By the induction hypothesis,

$$\begin{aligned}\left|E(H_1,Q)\right| &\le \alpha(\lambda\delta^{-c_1}u)^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon} n^{\frac{c_1(c_2-1)}{c_1c_2-1}+\varepsilon} + \beta(\lambda\delta^{-c_1}u + n)\\ &\le \frac{\alpha}{4} m^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon} n^{\frac{c_1(c_2-1)}{c_1c_2-1}+\varepsilon} + \beta(m+n)\end{aligned}$$

for m_0 sufficiently large in terms of u, $\bar{c}$, λ, ε, and δ, such that

$$m_0^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon} > 4(\lambda\delta^{-c_1}u)^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon}.$$

Consider $E(H_2,Q)$. By definition, $\bigcup_{i=L+1}^{L'} \bigcup_{j\in\Sigma_i} A_i \times B_j \subseteq \bigcup_{(i,j)\in\Sigma} A_i \times B_j$. The set on the right has size at most $\lambda\delta mn$, and for all $L+1 \le i \le L'$ we have $\sum_{j\in\Sigma_i}\left|B_j\right| > \delta^{1-\varepsilon} n$. Thus,

$$|H_2| = \left|\bigcup_{i=L+1}^{L'} A_i\right| < \frac{\lambda\delta mn}{\delta^{1-\varepsilon} n} = \lambda\delta^{\varepsilon} m.$$

In particular, assuming δ is sufficiently small in terms of λ and ε, we have $|H_2| < m$, so by the induction hypothesis,

$$\begin{aligned}\left|E(H_2,Q)\right| &< \alpha(\lambda\delta^{\varepsilon}m)^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon} n^{\frac{c_1(c_2-1)}{c_1c_2-1}+\varepsilon} + \beta(\lambda\delta^{\varepsilon}m + n)\\ &\le \frac{\alpha}{4} m^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon} n^{\frac{c_1(c_2-1)}{c_1c_2-1}+\varepsilon} + \beta(m+n)\end{aligned}$$

for δ sufficiently small in terms of λ and ε.

Next, consider $E(H_3, Q)$. We will bound its size by partitioning it into two:

$$E(H_3, Q) = \left(\bigcup_{i=L'+1}^{K_1} \bigcup_{j \in [K_2] \setminus \Sigma_i} E(A_i, B_j) \right) \sqcup \left(\bigcup_{i=L'+1}^{K_1} \bigcup_{j \in \Sigma_i} E(A_i, B_j) \right).$$

Fix $L' + 1 \leq i \leq K_1$. For $j \in [K_2] \setminus \Sigma_i$, $E(A_i, B_j) = A_i \times B_j$ or $\emptyset$. Since $|A_i| \geq u$ and $E(A_i, Q)$ is $K_{u,u}$-free, we have $\left| E(A_i, \bigcup_{j \in [K_2] \setminus \Sigma_i} B_j) \right| \leq (u-1)|A_i|$. Hence,

$$\sum_{i=L'+1}^{K_1} \sum_{j \in [K_2] \setminus \Sigma_i} \left| E(A_i, B_j) \right| \leq (u-1)m.$$

Now, $\sum_{j \in \Sigma_i} \left| B_j \right| \leq \delta^{1-\varepsilon} n$ by definition. Recall also that $|A_i| \leq \delta^{c_1} m$. In particular, $|A_i| < m$, so by the induction hypothesis,

$$\begin{aligned}
&\left| E\left(A_i, \bigcup_{j \in \Sigma_i} B_j \right) \right| \\
&\leq \alpha (\delta^{c_1} m)^{\frac{c_2(c_1-1)}{c_1 c_2 - 1} + \varepsilon} (\delta^{1-\varepsilon} n)^{\frac{c_1(c_2-1)}{c_1 c_2 - 1} + \varepsilon} + \beta(\delta^{c_1} m + \delta^{1-\varepsilon} n) \\
&= \alpha \delta^{c_1 + \varepsilon(c_1 - \frac{c_1(c_2-1)}{c_1 c_2 - 1} + 1 - \varepsilon)} m^{\frac{c_2(c_1-1)}{c_1 c_2 - 1} + \varepsilon} n^{\frac{c_1(c_2-1)}{c_1 c_2 - 1} + \varepsilon} + \beta(\delta^{c_1} m + \delta^{1-\varepsilon} n) \\
&\leq \frac{\alpha}{5} \delta^{c_1} m^{\frac{c_2(c_1-1)}{c_1 c_2 - 1} + \varepsilon} n^{\frac{c_1(c_2-1)}{c_1 c_2 - 1} + \varepsilon} + \beta(m+n)
\end{aligned}$$

for δ sufficiently small in terms of $\bar{c}$ and ε. Thus,

$$\begin{aligned}
\sum_{i=L'+1}^{K_1} \sum_{j \in \Sigma_i} \left| E(A_i, B_j) \right| &\leq \lambda \delta^{-c_1} \frac{\alpha}{5} \delta^{c_1} m^{\frac{c_2(c_1-1)}{c_1 c_2 - 1} + \varepsilon} n^{\frac{c_1(c_2-1)}{c_1 c_2 - 1} + \varepsilon} + \lambda \delta^{-c_1} \beta(m+n) \\
&\leq \frac{\alpha}{4} m^{\frac{c_2(c_1-1)}{c_1 c_2 - 1} + \varepsilon} n^{\frac{c_1(c_2-1)}{c_1 c_2 - 1} + \varepsilon} + \lambda \delta^{-c_1} \beta(m+n)
\end{aligned}$$

for α sufficiently large in terms of λ.

Putting all this together,

$$\begin{aligned}
&|E(P,Q)| \\
&= \big|E(H_1,Q)\big| + \big|E(H_2,Q)\big| + \sum_{i=L'+1}^{K_1} \sum_{j\in[K_2]\setminus\Sigma_i} \big|E(A_i,B_j)\big| + \sum_{i=L'+1}^{K_1} \sum_{j\in\Sigma_i} \big|E(A_i,B_j)\big| \\
&\leq \frac{3\alpha}{4} m^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon} n^{\frac{c_1(c_2-1)}{c_1c_2-1}+\varepsilon} + (2\beta + \lambda\delta^{-c_1}\beta + u - 1)(m+n) \\
&\leq \alpha m^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon} n^{\frac{c_1(c_2-1)}{c_1c_2-1}+\varepsilon},
\end{aligned}$$

where the last inequality was obtained from (2) and (3), choosing α to be sufficiently large in terms of β, u, $\bar{c}$, λ, and δ. Thus, (1) holds as claimed. □

Remark 4.3.2. It is straightforward to observe that the bound in Theorem 4.3.1 can be infinitesimally improved to, say, $m^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon} n^{\frac{c_1(c_2-1)}{c_1c_2-1}} + m + n$. Indeed, if $m \leq n^{\frac{1}{2c_1}}$ then $|E(P,Q)| \ll_{u,\bar{c},\lambda} n$ by Lemma 4.2.7. Assuming therefore, without loss of generality, that $n < m^{2c_1}$, by Theorem 4.3.1,

$$|E(P,Q)| \ll_{u,\bar{c},\lambda,\varepsilon} m^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon} n^{\frac{c_1(c_2-1)}{c_1c_2-1}} m^{2c_1\varepsilon} + m + n,$$

and so $|E(P,Q)| \ll_{u,\bar{c},\lambda,\varepsilon} m^{\frac{c_2(c_1-1)}{c_1c_2-1}+\varepsilon} n^{\frac{c_1(c_2-1)}{c_1c_2-1}} + m + n$.

4.4 The general case

We now proceed with the proof of the main theorem for an arbitrary relation.

Lemma 4.4.1. *Let $E(x_1,...,x_k)$ be a relation on a set M, with strong distal regularity tuple $\bar{c} = (c_1,...,c_k)$ and coefficient λ. For all $u \in \mathbb{N}^+$ and distinct $a_1,...,a_u \in M^{x_1}$, the relation*

$$R(x_2,...,x_k) := \bigwedge_{e=1}^{u} E(a_e,x_2,...,x_k)$$

has strong distal regularity tuple $(c_2,...,c_k)$ with coefficient $u\lambda$.

Proof. Let $a_1,...,a_u \in M^{x_1}$ be distinct, and let $P_1 := \{a_1,...,a_u\}$. For $2 \leq i \leq k$, let $P_i \subseteq M^{x_i}$ be finite with $n_i := |P_i|$.

Let $\delta \in (0, u^{-1})$. With this value of δ, obtain equipartitions $P_i = A^i_1 \sqcup \cdots \sqcup A^i_{K_i}$ as in the definition of $\bar{c}$ as a strong distal regularity tuple for E, and let $\Sigma \subseteq [K_1] \times \cdots \times [K_k]$ be the index set of bad cells. Since $u \leq \delta^{-c_1}$, we can assume without loss of generality that the partition of P_1 is a partition into singletons, and $A^1_j = \{a_j\}$ for all $j \in [u]$.

Henceforth, a tuple $(j_2, ..., j_k)$ is understood to be taken from $[K_2] \times \cdots \times [K_k]$. Let

$$\Sigma' := \{(j_2, ..., j_k) : \exists j_1 \in [u]\ (j_1, ..., j_k) \in \Sigma\}.$$

We claim that the equipartitions $P_i = A^i_1 \sqcup \cdots \sqcup A^i_{K_i}$ (for $2 \leq i \leq k$) and the index set Σ' of bad cells are such that

(i) $\sum_{(j_2,...,j_k) \in \Sigma'} |A^2_{j_2} \times \cdots \times A^k_{j_k}| \leq u\lambda\delta n_2 \cdots n_k$;

(ii) For all $(j_2, ..., j_k) \notin \Sigma'$, $A^2_{j_2} \times \cdots \times A^k_{j_k}$ is R-homogeneous;

(iii) $K_i \leq \lambda\delta^{-c_i}$ for all $2 \leq i \leq k$.

To see that (i) holds, observe that

$$\sum_{(j_2,...,j_k) \in \Sigma'} |A^2_{j_2} \times \cdots \times A^k_{j_k}| \leq \sum_{(j_1,...,j_k) \in \Sigma} |A^1_{j_1} \times \cdots \times A^k_{j_k}| \leq \lambda\delta u n_2 \cdots n_k.$$

To see that (ii) holds, let $(j_2, ..., j_k) \notin \Sigma'$ and $(b_2, ..., b_k) \in A^2_{j_2} \times \cdots \times A^k_{j_k}$. Then

$$R(b_2, ..., b_k) \Leftrightarrow \bigwedge_{e=1}^{u} E(a_e, b_2, ..., b_k) \Leftrightarrow \bigwedge_{e=1}^{u} E(A^1_e, A^2_{j_2}, ..., A^k_{j_k}) = A^1_e \times A^2_{j_2} \times \cdots \times A^k_{j_k},$$

where the last equivalence follows from the fact that, for all $e \in [u]$, $A^1_e = \{a_e\}$ and $(e, j_2, ..., j_k) \notin \Sigma$. Thus, $A^2_{j_2} \times \cdots \times A^k_{j_k}$ is R-homogeneous.

Finally, (iii) holds by the choice of our original partition. Thus, $(c_2, ..., c_k)$ is a strong distal regularity tuple for $R(x_2, ..., x_k)$ with coefficient $u\lambda$. □

The following functions appeared in [14].

Definition 4.4.2. For $\bar{c} = (c_1, ..., c_k) \in \mathbb{R}^k_{\geq 1}$, let $E_{\bar{c}} : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$ be the function sending $\bar{n} = (n_1, ..., n_k) \in \mathbb{R}^k_{\geq 0}$ to $E_{\bar{c}}(\bar{n}) := \prod_{i=1}^k n_i^{\gamma_i(\bar{c})}$, where

$$\gamma_i(\bar{c}) := 1 - \frac{\frac{1}{c_i - 1}}{k - 1 + \sum_{j=1}^k \frac{1}{c_j - 1}}.$$

Note that, when $k = 1$, $E_{\bar{c}}$ is the constant 1-valued function. For $\varepsilon \in \mathbb{R}_{>0}$, if $k \geq 2$ then let $F^\varepsilon_{\bar{c}} : \mathbb{R}^k_{\geq 0} \to \mathbb{R}$ be the function sending $\bar{n} = (n_1, ..., n_k) \in \mathbb{R}^k_{\geq 0}$ to

$$F^\varepsilon_{\bar{c}}(\bar{n}) := \sum_{I \subseteq [k], |I| \geq 2} E_{\bar{c}_I}(\bar{n}_I) \prod_{i \in I} n_i^\varepsilon \prod_{i \notin I} n_i + \sum_{j=1}^k \prod_{i \neq j} n_i,$$

and if $k = 1$ then let $F^\varepsilon_{\bar{c}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be the constant 1-valued function. (Recall that the notation of $\bar{c}_I, \bar{n}_I$ was defined in Subsection 2.1.2.)

As written, the exponents in $E_{\bar{c}}(\bar{n})$ are not well-defined when $c_j = 1$ for some $j \in [k]$. In this case, we circumvent this problem by declaring, for all $i \in [k]$,

$$\gamma_i(\bar{c}) := 1 - \lim_{\delta \to 0} \frac{\frac{1}{c_i + \delta - 1}}{k - 1 + \sum_{j=1}^k \frac{1}{c_j + \delta - 1}} = 1 - \frac{\mathbb{1}(c_i = 1)}{|\{j \in [k] : c_j = 1\}|}.$$

Henceforth, all issues that arise when $c_j = 1$ for some $j \in [k]$ can and will be resolved by taking limits like so.

Note that, when $k = 2$,

$$\begin{aligned} F^\varepsilon_{\bar{c}}(m, n) &= m^{1 - \frac{\frac{1}{c_1 - 1}}{1 + \frac{1}{c_1 - 1} + \frac{1}{c_2 - 1}} + \varepsilon} n^{1 - \frac{\frac{1}{c_2 - 1}}{1 + \frac{1}{c_1 - 1} + \frac{1}{c_2 - 1}} + \varepsilon} + m + n \\ &= m^{1 - \frac{c_2 - 1}{c_1 c_2 - 1} + \varepsilon} n^{1 - \frac{c_1 - 1}{c_1 c_2 - 1} + \varepsilon} + m + n \\ &= m^{\frac{c_2(c_1 - 1)}{c_1 c_2 - 1} + \varepsilon} n^{\frac{c_1(c_2 - 1)}{c_1 c_2 - 1} + \varepsilon} + m + n, \end{aligned}$$

so $F^\varepsilon_{\bar{c}}(m, n)$ is the bound appearing in Theorem 4.3.1.

Remark 4.4.3. It is straightforward to observe that, when $k \geq 2$,

$$k F^\varepsilon_{\bar{c}}(\bar{n}) \geq E_{\bar{c}}(\bar{n}) \prod_{i=1}^k n_i^\varepsilon + \sum_{i=1}^k n_i F^\varepsilon_{\bar{c}_{\neq i}}(\bar{n}_{\neq i}).$$

The following lemma says that (in most cases) the dominant term in $F_{\bar{c}}^{\varepsilon}(\bar{n})$ is $E_{\bar{c}}(\bar{n})\prod_{i=1}^{k} n_i^{\varepsilon}$.

Lemma 4.4.4. *Let $k \geq 2$, $\bar{c} = (c_1, ..., c_k) \in \mathbb{R}_{\geq 1}^k$, $\varepsilon > 0$, and $\bar{n} = (n_1, ..., n_k) \in \mathbb{R}_{\geq 0}^k$. Suppose that, for all $i \in [k]$, $n_1 \cdots n_k n_i^{-1/c_i+\varepsilon} \geq n_i F_{\bar{c}_{\neq i}}^{\varepsilon}(\bar{n}_{\neq i})$.*

Then $F_{\bar{c}}^{\varepsilon}(\bar{n}) \ll_{\bar{c},\varepsilon} E_{\bar{c}}(\bar{n})\prod_{i=1}^{k} n_i^{\varepsilon}$, and so $n_i F_{\bar{c}_{\neq i}}^{\varepsilon}(\bar{n}_{\neq i}) \ll_{\bar{c},\varepsilon} E_{\bar{c}}(\bar{n})\prod_{i=1}^{k} n_i^{\varepsilon}$ for all $i \in [k]$ by Remark 4.4.3.

Proof. Our proof mimics, in part, the proof of [14, Lemma 2.10]. To show that $F_{\bar{c}}^{\varepsilon}(\bar{n}) \ll_{\bar{c},\varepsilon} E_{\bar{c}}(\bar{n})\prod_{i=1}^{k} n_i^{\varepsilon}$, it suffices to show that, for all $\emptyset \neq I \subseteq [k]$,

$$E_{\bar{c}}(\bar{n}) \gg_{\bar{c},\varepsilon} E_{\bar{c}_I}(\bar{n}_I)\prod_{i \notin I} n_i^{1-\varepsilon}. \tag{4}$$

We prove this by downward induction on $|I| \in [k]$ via the following claim.

Claim 4.4.4.1. *Let $J \subseteq [k]$ with $|J| \geq 2$. Let $j \in J$, and write $I := J \setminus \{j\}$. For all $\varepsilon > 0$, if $n_j^{-1/c_j+\varepsilon}\prod_{i \in J} n_i \geq n_j E_{\bar{c}_I}(\bar{n}_I)$ then $E_{\bar{c}_J}(\bar{n}_J) \geq n_j^{1-\varepsilon} E_{\bar{c}_I}(\bar{n}_I)$.*

Proof of Claim. Let $\varepsilon > 0$, and suppose $n_j^{-1/c_j+\varepsilon}\prod_{i \in J} n_i \geq n_j E_{\bar{c}_I}(\bar{n}_I)$. Then

$$\prod_{i \in I} n_i^{\frac{\frac{1}{c_i-1}}{|J|-2+\sum_{l \in I}\frac{1}{c_l-1}}} \geq n_j^{\frac{1}{c_j}-\varepsilon}.$$

The i^{th} exponent on the left equals $\frac{c_j-1}{c_j}\frac{1}{c_i-1}\left(\frac{|J|-1+\sum_{l \in J}\frac{1}{c_l-1}}{|J|-2+\sum_{l \in I}\frac{1}{c_l-1}} - 1\right)$, and so

$$\prod_{i \in I} n_i^{\frac{1}{c_i-1}\left(\frac{1}{|J|-2+\sum_{l \in I}\frac{1}{c_l-1}} - \frac{1}{|J|-1+\sum_{l \in J}\frac{1}{c_l-1}}\right)} \geq n_j^{\frac{\frac{1}{c_j-1}}{|J|-1+\sum_{l \in J}\frac{1}{c_l-1}}-\nu\varepsilon}$$

for $\nu := \frac{c_j/(c_j-1)}{|J|-1+\sum_{l \in J}\frac{1}{c_l-1}} \in [0,1]$. Rearranging, we have

$$E_{\bar{c}_J}(\bar{n}_J) \geq n_j^{1-\nu\varepsilon} E_{\bar{c}_I}(\bar{n}_I) \geq n_j^{1-\varepsilon} E_{\bar{c}_I}(\bar{n}_I). \qquad \dashv$$

We now prove (4) by downward induction on $|I| \in [k]$. Since k is finite, we may update the implied constant in each step of the induction. When $|I| = k$, we have $I = [k]$ so (4) holds trivially. Now suppose $|I| < k$, so we may fix $j \notin I$; write $J := I \cup \{j\}$. Since $|I| \geq 1$, we have $|J| \geq 2$. By the induction hypothesis, $E_{\bar{c}}(\bar{n}) \gg_{\bar{c},\varepsilon} E_{\bar{c}_J}(\bar{n}_J) \prod_{i \notin J} n_i^{1-\varepsilon}$.

By assumption, $n_1 \cdots n_k n_j^{-1/c_j+\varepsilon} \geq n_j F^{\varepsilon}_{\bar{c}_{\neq j}}(\bar{n}_{\neq j}) \geq E_{\bar{c}_I}(\bar{n}_I) \prod_{i \notin I} n_i$, which rearranges to $n_j^{-1/c_j+\varepsilon} \prod_{i \in J} n_i \geq n_j E_{\bar{c}_I}(\bar{n}_I)$, and hence $E_{\bar{c}_J}(\bar{n}_J) \geq n_j^{1-\varepsilon} E_{\bar{c}_I}(\bar{n}_I)$ by Claim 4.4.4.1. Thus, as required, we have

$$E_{\bar{c}}(\bar{n}) \gg_{\bar{c},\varepsilon} E_{\bar{c}_J}(\bar{n}_J) \prod_{i \notin J} n_i^{1-\varepsilon} \geq E_{\bar{c}_I}(\bar{n}_I) \prod_{i \notin I} n_i^{1-\varepsilon}. \qquad \square$$

We are ready to prove our main result.

Theorem 4.4.5. *Let $E(x_1, ..., x_k)$ be a relation on a set M, with strong distal regularity tuple $\bar{c} = (c_1, ..., c_k) \in \mathbb{R}^k_{\geq 1}$ and coefficient λ. For all finite $P_i \subseteq M^{x_i}$ with $n_i := |P_i|$, if $E(P_1, ..., P_k)$ is $K_{u,...,u}$-free, then for all $\varepsilon > 0$,*

$$|E(P_1, ..., P_k)| \ll_{u,\bar{c},\lambda,\varepsilon} F^{\varepsilon}_{\bar{c}}(n_1, ..., n_k).$$

Proof. We will do a double induction: first on k, and then on $n_1 + \cdots + n_k$. When $k = 1$ this is trivial. Let $k \geq 2$, and suppose for all $l < k$ that the statement holds. Writing $\gamma_j := \gamma_j(\bar{c})$ for $j \in [k]$, there is some $j \in [k]$ such that $\gamma_j < 1$, so, permuting $x_1, ..., x_k$ if necessary, we may assume that $\gamma_1 < 1$. Let $\varepsilon > 0$. We will show that there are $\alpha = \alpha(u, \bar{c}, \lambda, \varepsilon)$ and $\beta = \beta(u, \bar{c}, \lambda, \varepsilon)$ such that, for all finite $P_i \subseteq M^{x_i}$ with $n_i := |P_i|$, writing $\bar{n} := (n_1, ..., n_k)$, if $E(P_1, ..., P_k)$ is $K_{u,...,u}$-free, then

$$|E(P_1, ..., P_k)| \leq \alpha E_{\bar{c}}(\bar{n}) \prod_{i=1}^{k} n_i^{\varepsilon} + \beta \sum_{i=1}^{k} n_i F^{\varepsilon}_{\bar{c}_{\neq i}}(\bar{n}_{\neq i}). \qquad (5)$$

By Remark 4.4.3, the right hand side is at most $k \max(\alpha, \beta) F^{\varepsilon}_{\bar{c}}(\bar{n})$, so this is sufficient. The dependency between constants will be as follows:

(i) τ is sufficiently large in terms of $\bar{c}$ and ε;

(ii) δ is sufficiently small in terms of $\bar{c}$, λ, and ε;

(iii) m_0 is sufficiently large in terms of u, $\bar{c}$, λ, ε, and δ;

(iv) β is sufficiently large in terms of m_0, u, $\bar{c}$, λ, and ε;

(v) α is sufficiently large in terms of m_0, β, $\bar{c}$, λ, δ, and τ.

Suppose there is $i \in [k]$ such that $n_1 \cdots n_k n_i^{-1/c_i+\varepsilon} < n_i F^{\varepsilon}_{\bar{c}_{\neq i}}(\bar{n}_{\neq i})$. Then, by Lemma 4.2.7(ii), Lemma 4.4.1, and the induction hypothesis, if β is sufficiently large in terms of u, $\bar{c}$, λ, and ε, then

$$|E(P_1, ..., P_k)| \leq \frac{\beta}{2}\left(n_1 \cdots n_k n_i^{-\frac{1}{c_i}+\varepsilon} + n_i F^{\varepsilon}_{\bar{c}_{\neq i}}(\bar{n}_{\neq i})\right) < \beta n_i F^{\varepsilon}_{\bar{c}_{\neq i}}(\bar{n}_{\neq i}).$$

Therefore, henceforth we suppose $n_1 \cdots n_k n_i^{-1/c_i+\varepsilon} \geq n_i F^{\varepsilon}_{\bar{c}_{\neq i}}(\bar{n}_{\neq i})$ for all $i \in [k]$, whence by Lemma 4.4.4 there is $\tau = \tau(\bar{c}, \varepsilon)$ such that, for all $i \in [k]$,

$$n_i F^{\varepsilon}_{\bar{c}_{\neq i}}(\bar{n}_{\neq i}) \leq \tau E_{\bar{c}}(\bar{n}) \prod_{i=1}^{k} n_i^{\varepsilon}. \tag{6}$$

Let $\alpha = \alpha(u, \bar{c}, \lambda, \varepsilon)$ and $\beta = \beta(u, \bar{c}, \lambda, \varepsilon)$ be sufficiently large, to be chosen later. We will show by induction on $n_1 + \cdots + n_k$ that (5) holds.

Let $m_0 \in \mathbb{N}$ such that $m_0 > \lambda\delta^{-c_i}(u+1)$ for all $i \in [k]$. If $i \in [k]$ is such that $n_i < m_0$, then for all $i \neq j \in [k]$,

$$|E(P_1, ..., P_k)| < m_0 n_1 \cdots n_k n_i^{-1} \leq \beta n_1 \cdots n_k n_i^{-1} \leq \beta n_j F^{\varepsilon}_{\bar{c}_{\neq j}}(n_{\neq j}),$$

assuming $\beta \geq m_0$. Thus, (5) holds when $n_1 + \cdots + n_k < k m_0$, and we henceforth suppose $n_i \geq m_0$ for all $i \in [k]$.

For $\delta = \delta(\bar{c}, \lambda, \varepsilon) < \frac{1}{4}$ to be chosen later, obtain equipartitions $P_i = A^i_1 \sqcup \cdots \sqcup A^i_{K_i}$ as in the definition of $\bar{c}$ as a strong distal regularity tuple, with $\Sigma \subseteq [K_1] \times \cdots \times [K_k]$ the index set of bad cells. By refining the partitions and replacing λ with 2λ if necessary, we may assume that $1 \leq |A^i_j| \leq \delta^{c_i} n_i$ for all $i \in [k]$ and $j \in [K_i]$.

Henceforth, a tuple $(j_i, ..., j_k)$ is understood to be taken from $[K_i] \times \cdots \times [K_k]$. Let $I_1 := \sum_{(j_1,...,j_k)\notin\Sigma} |E(A^1_{j_1}, ..., A^k_{j_k})|$ and $I_2 := \sum_{(j_1,...,j_k)\in\Sigma} |E(A^1_{j_1}, ..., A^k_{j_k})|$, so that $\left|E(P_1, ..., P_k)\right| = I_1 + I_2$. We bound I_1 and I_2.

First, consider I_1. For $j_1 \in [K_1]$, let $\Sigma_{j_1} := \{(j_2, ..., j_k) : (j_1, ..., j_k) \in \Sigma\}$, so we have $I_1 = \sum_{j_1=1}^{K_1} \sum_{(j_2,...,j_k)\notin\Sigma_{j_1}} |E(A^1_{j_1}, ..., A^k_{j_k})|$.

Fix $j_1 \in [K_1]$. We have that $|A^1_{j_1}| \geq n_1/K_1 - 1 \geq m_0/(\lambda\delta^{-c_1}) - 1 > u$, so we can fix distinct $a_1, ..., a_u \in A^1_{j_1}$. For $(j_2, ..., j_k) \notin \Sigma_{j_1}$, $E(A^1_{j_1}, ..., A^k_{j_k}) = A^1_{j_1} \times \cdots \times A^k_{j_k}$ or $\emptyset$, and thus

$$\bigcup_{(j_2,...,j_k)\notin\Sigma_{j_1}} E(A^1_{j_1}, ..., A^k_{j_k}) \subseteq A^1_{j_1} \times \bigcap_{e=1}^{u} E(a_e, P_2, ..., P_k).$$

Now $\bigcap_{e=1}^{u} E(a_e, P_2, ..., P_k)$ is the induced $(k-1)$-subgraph on $P_2 \times \cdots \times P_k$ of the relation $R(x_2, ..., x_k) := \bigwedge_{e=1}^{u} E(a_e, x_2, ..., x_k)$ on M. By Lemma 4.4.1, $(c_2, ..., c_k)$ is a strong distal regularity tuple for R with coefficient $u\lambda$. By the induction hypothesis,

$$\left|\bigcap_{e=1}^{u} E(a_e, P_2, ..., P_k)\right| \ll_{u,\bar{c},\lambda,\varepsilon} F^{\varepsilon}_{\bar{c}_{\neq 1}}(\bar{n}_{\neq 1}).$$

Choosing β sufficiently large in terms of u, $\bar{c}$, λ, and ε, we can assume that the implied constant is at most β. Then

$$I_1 \leq \sum_{j_1=1}^{K_1} \beta |A^1_{j_1}| F^{\varepsilon}_{\bar{c}_{\neq 1}}(\bar{n}_{\neq 1}) \leq \beta n_1 F^{\varepsilon}_{\bar{c}_{\neq 1}}(\bar{n}_{\neq 1}).$$

Next, consider I_2. For each $(j_2, ..., j_k)$, let $B_{j_2,...,j_k} := \bigcup_{\substack{1\leq j_1\leq K_1 \\ (j_2,...,j_k)\in\Sigma_{j_1}}} A^1_{j_1}$, so we have $I_2 = \sum_{(j_2,...,j_k)} \left|E(B_{j_2,...,j_k}, A^2_{j_2}, ..., A^k_{j_k})\right|$.

For each $(j_2, ..., j_k)$, let $s_{j_2,...,j_k} := |B_{j_2,...,j_k}|$. Observe that

$$\sum_{(j_2,...,j_k)} s_{j_2,...,j_k} = \sum_{j_1=1}^{K_1} |A^1_{j_1}||\Sigma_{j_1}| \leq \delta^{c_1} n_1 \sum_{j_1=1}^{K_1} |\Sigma_{j_1}| \leq \delta^{c_1} n_1 |\Sigma|$$

and

$$|\Sigma| \leq \frac{\sum_{(j_1,...,j_k)\in\Sigma} |A^1_{j_1} \times \cdots \times A^k_{j_k}|}{\min_{(j_1,...,j_k)\in\Sigma} |A^1_{j_1} \times \cdots \times A^k_{j_k}|} \leq \frac{\lambda\delta n_1 \cdots n_k}{\prod_{i=1}^{k} \frac{n_i}{2K_i}} \leq \frac{\lambda\delta n_1 \cdots n_k}{\prod_{i=1}^{k} \frac{1}{2\lambda}\delta^{c_i} n_i}$$
$$= 2^k \lambda^{k+1} \delta^{1-(c_1+\cdots+c_k)},$$

so $\sum_{(j_2,\dots,j_k)} s_{j_2,\dots,j_k} \leq 2^k\lambda^{k+1}\delta^{1-(c_2+\cdots+c_k)}n_1$. By the induction hypothesis,

$$\begin{aligned} I_2 &= \sum_{(j_2,\dots,j_k)} \left| E(B_{j_2,\dots,j_k}, A^2_{j_2}, \dots, A^k_{j_k}) \right| \\ &\leq \alpha \sum_{(j_2,\dots,j_k)} E_{\bar{c}}(s_{j_2,\dots,j_k}, \delta^{c_2}n_2, \dots, \delta^{c_k}n_k) s^{\varepsilon}_{j_2,\dots,j_k} \prod_{i=2}^{k} (\delta^{c_i}n_i)^{\varepsilon} \\ &\qquad\qquad + \beta \sum_{(j_2,\dots,j_k)} \sum_{i=1}^{k} n_i F^{\varepsilon}_{\bar{c}_{\neq i}}(\bar{n}_{\neq i}) \\ &\leq \alpha \prod_{i=2}^{k} (\delta^{c_i}n_i)^{\gamma_i+\varepsilon} \sum_{(j_2,\dots,j_k)} s^{\gamma_1+\varepsilon}_{j_2,\dots,j_k} + \beta\lambda^{k-1}\delta^{-(c_2+\cdots+c_k)} \sum_{i=1}^{k} n_i F^{\varepsilon}_{\bar{c}_{\neq i}}(\bar{n}_{\neq i}). \end{aligned}$$

Recall that $\gamma_1 < 1$; without loss of generality assume that $\varepsilon < 1-\gamma_1$. By Hölder's inequality,

$$\begin{aligned} \sum_{(j_2,\dots,j_k)} s^{\gamma_1+\varepsilon}_{j_2,\dots,j_k} &\leq \left(\sum_{(j_2,\dots,j_k)} s_{j_2,\dots,j_k} \right)^{\gamma_1+\varepsilon} (\lambda^{k-1}\delta^{-(c_2+\cdots+c_k)})^{1-\gamma_1-\varepsilon} \\ &\leq (2^k\lambda^{k+1}\delta^{1-(c_2+\cdots+c_k)}n_1)^{\gamma_1+\varepsilon} (\lambda^{k-1}\delta^{-(c_2+\cdots+c_k)})^{1-\gamma_1-\varepsilon} \\ &\leq 2^k\lambda^{2k}\delta^{\gamma_1+\varepsilon-(c_2+\cdots+c_k)} n_1^{\gamma_1+\varepsilon}. \end{aligned}$$

Therefore,

$$\prod_{i=2}^{k} (\delta^{c_i}n_i)^{\gamma_i+\varepsilon} \sum_{(j_2,\dots,j_k)} s^{\gamma_1+\varepsilon}_{j_2,\dots,j_k} \leq 2^k\lambda^{2k}\delta^{\gamma_1+\varepsilon-\sum_{i=2}^{k} c_i(1-\gamma_i-\varepsilon)} \prod_{i=1}^{k} n_i^{\gamma_i+\varepsilon}.$$

Since $\gamma_1 = \sum_{i=2}^{k} c_i(1-\gamma_i)$, the exponent of δ evaluates to $(1+c_2+\cdots+c_k)\varepsilon$, and so

$$\prod_{i=2}^{k} (\delta^{c_i}n_i)^{\gamma_i+\varepsilon} \sum_{(j_2,\dots,j_k)} s^{\gamma_1+\varepsilon}_{j_2,\dots,j_k} \leq 2^k\lambda^{2k}\delta^{(1+c_2+\cdots+c_k)\varepsilon} E_{\bar{c}}(\bar{n}) \prod_{i=1}^{k} n_i^{\varepsilon} \leq \frac{1}{2} E_{\bar{c}}(\bar{n}) \prod_{i=1}^{k} n_i^{\varepsilon},$$

for δ sufficiently small in terms of $\bar{c}$, λ, and ε. Thus,

$$I_2 \leq \frac{\alpha}{2} E_{\bar{c}}(\bar{n}) \prod_{i=1}^{k} n_i^{\varepsilon} + \beta\lambda^{k-1}\delta^{-(c_2+\cdots+c_k)} \sum_{i=1}^{k} n_i F^{\varepsilon}_{\bar{c}_{\neq i}}(\bar{n}_{\neq i}).$$

Putting all this together,

$$
\begin{aligned}
|E(P_1, ..., P_k)| &= I_1 + I_2 \\
&\leq \frac{\alpha}{2} E_{\bar{c}}(\bar{n}) \prod_{i=1}^{k} n_i^{\varepsilon} + \beta(1 + \lambda^{k-1}\delta^{-(c_2+\cdots+c_k)}) \sum_{i=1}^{k} n_i F^{\varepsilon}_{\bar{c}_{\neq i}}(\bar{n}_{\neq i}) \\
&\leq \left(\frac{\alpha}{2} + k\tau\beta(1 + \lambda^{k-1}\delta^{-(c_2+\cdots+c_k)})\right) E_{\bar{c}}(\bar{n}) \prod_{i=1}^{k} n_i^{\varepsilon} \qquad \text{by (6)} \\
&\leq \alpha E_{\bar{c}}(\bar{n}) \prod_{i=1}^{k} n_i^{\varepsilon},
\end{aligned}
$$

for α sufficiently large in terms of β, $\bar{c}$, λ, δ, and τ. □

Remark 4.4.6. Similarly to Remark 4.3.2, it is not hard to see that Theorem 4.4.5 remains true if we remove all but one of the occurrences of ε in each summand of $F^{\varepsilon}_{\bar{c}}(\bar{n})$, but we will not demonstrate this in detail. Do makes a similar remark [14, Remark 1.9(ii)].

4.5 Context for distal regularity lemma

By Theorem 4.2.6, relations definable in a distal structure satisfy the distal regularity lemma. We had previously wondered if the converse holds: are distal structures the only source of relations satisfying the distal regularity lemma? That is, if $\phi(x_1, ..., x_k)$ is a relation on a set M satisfying the distal regularity lemma, must the structure (M, ϕ) admit a distal expansion? The answer is no: we are grateful to Martin Bays for suggesting the following counterexample to us in personal communication.

Theorem 4.5.1. *Let K be a finitely generated extension of $\mathbb{F}_p$, such as $K = \mathbb{F}_p(t)$, and let $\phi(x, y; m, c) := (y = mx + c)$ be the point-line incidence relation. Then ϕ satisfies the distal regularity lemma as a relation on K, but the structure (K, ϕ) does not admit a distal expansion.*

Proof. We first argue that ϕ satisfies the distal regularity lemma as a relation on K. By [6, Lemma 4.1], K admits a valuation v with finite residue field, so we may view K as a structure $\mathcal{K}$ over the language $\{+, \times, \leq\}$ of valued fields, where $x \leq y :\Leftrightarrow v(x) \leq v(y)$ in $\mathcal{K}$. Let $\mathcal{K}^*$ be an algebraically closed valued field such that $\mathcal{K} \subseteq \mathcal{K}^*$; it is folklore that $\mathcal{K}^*$ is NIP.

By [6, Theorem 5.6], as a relation on K, ϕ has a strong honest definition $\psi(x, y; m_1, c_1, ..., m_k, c_k)$ definable in $\mathcal{K}^*$, in the following sense: for all finite $B \subseteq K^2$ with $|B| \geq 2$ and $a \in K^2$, there are $b_1, ..., b_k \in B$ such that $\mathcal{K}^* \models \psi(a; b_1, ..., b_k)$ and, for all $a' \in (\mathcal{K}^*)^2$, if $\mathcal{K}^* \models \psi(a'; b_1, ..., b_k)$ then $\mathcal{K}^* \models \phi(a; b') \leftrightarrow \phi(a'; b')$ for all $b' \in B$. The proof of [9, Lemma 3.6] now gives a 'cutting lemma' for ϕ as follows. For all finite $B \subseteq K^2$ with $|B| \geq 2$ and $\delta \in (0, 1)$, there is a cover $\mathcal{F} \subseteq \{\psi((\mathcal{K}^*)^2; b_1, ..., b_k) : b_i \in B\}$ of $(\mathcal{K}^*)^2$, such that $|\mathcal{F}| \leq \text{poly}_{\phi,\psi}(\delta^{-1})$ and for all $F \in \mathcal{F}$,

$$\#\{b \in B : F \subseteq \phi((\mathcal{K}^*)^2; b) \text{ or } F \subseteq \neg\phi((\mathcal{K}^*)^2; b)\} \geq (1 - \delta)|B|.$$

Let $P, Q \subseteq K^2$ be finite with $|Q| \geq 2$ and $\delta \in (0, 1)$; we give appropriate partitions of P and Q to show that ϕ satisfies the distal regularity lemma. Applying the cutting lemma above with $B = Q$, we obtain a cover $\mathcal{F} \subseteq \{\psi((\mathcal{K}^*)^2; b_1, ..., b_k) : b_i \in Q\}$ of $(\mathcal{K}^*)^2$. For all $F \in \mathcal{F}$ and $\sigma \in \{0, 1\}$, let $D_F^\sigma := \{d \in (\mathcal{K}^*)^2 : F \subseteq \phi^\sigma((\mathcal{K}^*)^2; d)\}$. Let $\mathcal{G}$ be the set of Boolean atoms of $\{D_F^\sigma : F \in \mathcal{F}, \sigma \in \{0, 1\}\}$, so $\mathcal{G}$ is a partition of $(\mathcal{K}^*)^2$. Since $\mathcal{K}^*$ is NIP, $|\mathcal{G}| \leq \text{poly}_{\phi,\psi,\mathcal{K}^*}(|\mathcal{F}|)$, and so $|\mathcal{G}| \leq \text{poly}_{\phi,\psi,\mathcal{K}^*}(\delta^{-1})$. Let $\mathcal{F}_0$ be any partition of $(\mathcal{K}^*)^2$ refining $\mathcal{F}$ such that $|\mathcal{F}_0| \leq |\mathcal{F}| \leq \text{poly}_{\phi,\psi}(\delta^{-1})$. The reader is invited to check that the partitions $\mathcal{F}_0 \cap P$ and $\mathcal{G} \cap Q$ are such that $\sum |F \times G| \leq \delta|P||Q|$, where the sum ranges over all $(F, G) \in (\mathcal{F}_0 \cap P) \times (\mathcal{G} \cap Q)$ such that $F \times G$ is not ϕ-homogeneous.

It remains to argue that (K, ϕ) does not admit a distal expansion. Let $\overline{\mathcal{K}} = (K, +, \times)$ be the field structure on K. Now $K \not\supseteq \mathbb{F}_p^{\text{alg}}$ since every subextension of a finitely generated field extension is finitely generated (see, for example, [28, Theorem 24.9]), so $\overline{\mathcal{K}}$ is not NIP by [30, Corollary 4.5]. Thus, there is a formula ψ in $\overline{\mathcal{K}}$ that is not NIP. Now, the field operations $+$ and $\times$ are definable in (K, ϕ): indeed, 0 and 1 are $\emptyset$-definable in (K, ϕ), and for all $p, q, r \in K$, $p + q = r \Leftrightarrow \phi(p, r; 1, q)$ and $p \times q = r \Leftrightarrow \phi(p, r; q, 0)$. Thus, ψ is definable in

(K, ϕ) and all of its expansions. We conclude that every expansion of (K, ϕ) is not NIP and hence not distal. □

In the previous example, the nonexistence of a distal expansion for (K, ϕ) was due to the fact that (K, ϕ) was not NIP. We ask if this is the only obstruction.

Problem 4.5.2. *Let $\phi(x_1, ..., x_k)$ be a relation on a set M satisfying the distal regularity lemma, such that (M, ϕ) is NIP. Must (M, ϕ) admit a distal expansion?*

To our knowledge, the following problem is still open.

Problem 4.5.3. *Let M be a structure in which every relation satisfies the distal regularity lemma. Must M be distal (or admit a distal expansion)?*

4.6 Explicit bounds for some o-minimal 3-graphs

To compute explicit Zarankiewicz bounds from our main theorem, one needs to compute (strong) distal regularity tuples. In this section, we compute explicit Zarankiewicz bounds in a different way, for a special class of graphs satisfying the distal regularity lemma: certain 3-graphs definable in o-minimal structures.

Throughout this section, fix a language L. We have the following fact about certain 2-graphs definable in o-minimal structures.

Fact 4.6.1 [9, Theorem 5.14]. *Let M be an o-minimal L-structure expanding an ordered field. Let $\phi(x_1, x_2; y) \in L$ with $|x_1| = |x_2| = 2$. For all $b \in M^y$ and finite $P_i \subseteq M^{x_i}$ with $n_i := |P_i|$, if $\phi(P_1, P_2; b)$ is $K_{u,u}$-free, then*

$$|\phi(P_1, P_2; b)| \ll_{\phi,u} (n_1 n_2)^{\frac{2}{3}} + n_1 + n_2;$$

in particular, if $n_1 = n_2 =: n$, then $|\phi(P_1, P_2; b)| \ll_{\phi,u} n^{\frac{4}{3}}$.

We prove a corresponding statement for 3-graphs.

Theorem 4.6.2. *Let M be an o-minimal L-structure expanding an ordered field. Let $\phi(x_1, x_2, x_3; y) \in L$ with $|x_1| = |x_2| = |x_3| = 2$. For all $b \in M^y$ and finite*

$P_i \subseteq M^{x_i}$ with $n_i := |P_i|$, if $\phi(P_1, P_2, P_3; b)$ is $K_{u,u,u}$-free, then

$$|\phi(P_1, P_2, P_3; b)| \ll_{\phi,u} \sum_{(\gamma_1,\gamma_2,\gamma_3)\in\Gamma} n_1^{\gamma_1} n_2^{\gamma_2} n_3^{\gamma_3},$$

for $\Gamma := \{(\frac{22}{25}, \frac{39}{50}, \frac{39}{50}), (\frac{41}{25}, -\frac{4}{25}, \frac{21}{25}), (\frac{62}{75}, \frac{24}{25}, \frac{47}{75}), (\frac{22}{25}, \frac{46}{75}, \frac{71}{75}), (\frac{4}{5}, \frac{4}{5}, \frac{4}{5}), (1, \frac{2}{3}, \frac{2}{3}), (\frac{2}{3}, 1, \frac{2}{3}), (\frac{2}{3}, \frac{2}{3}, 1), (0,1,1), (1,0,1), (1,1,0)\}$*; in particular, if $n_1 = n_2 = n_3 =: n$, then*

$$|\phi(P_1, P_2, P_3; b)| \ll_{\phi,u} n^{2.44}.$$

By the improvement of Tidor–Yu [52] on Theorem 4.1.2 (see the end of Subsection 4.1.1), if M is the real ordered field, and hence ϕ is semialgebraic, then the bound in Theorem 4.6.2 can be improved to $O_{\phi,u}(n^{2.4})$. It is natural to ask if these bounds can be reconciled.

Problem 4.6.3. *Can the bound in Theorem 4.6.2 be improved to $O_{\phi,u}(n^{2.4})$?*

Towards proving Theorem 4.6.2, we need the following facts.

Fact 4.6.4 [9, Theorems 3.2, 4.1]**.** *Let M be an o-minimal L-structure expanding an ordered field. Let $\phi(x; y)$ be a formula with $|x| = 2$. Then $\pi^*_\phi(n) = O(n^2)$, and there is a formula $\psi(x; z)$ such that for all finite $Q \subseteq M^y$ and $r \geq 1$, there is a cover $\mathcal{C}$ of M^x of size $O_\psi(r^2)$, such that every $C \in \mathcal{C}$ is an instance of ψ with*

$$\#\{q \in Q : \phi(M^x; q) \text{ crosses } C\} \leq \frac{|Q|}{r}.$$

This is known as a 'cutting lemma'. Here, a set X *crosses* a set C if $C \cap X \neq \emptyset$ and $C \not\subseteq X$.

Fact 4.6.5 [19, Observation 2.6]**.** *Let M be any L-structure. Let $\phi(x; y) \in L$ with $\pi^*_\phi(t) \leq ct^d$ for all $t \in \mathbb{N}$. For all $u \in \mathbb{N}$, there is $c' = c'(c, d, u) \in \mathbb{N}$ such that the following holds.*

Let $P \subseteq M^x, Q \subseteq M^y$ with $m := |P|, n := |Q| \geq u$. Let $\mathcal{F}$ be the set system

$\{\phi(P;q) : q \in Q\}$ on P. Then there are distinct $p_1, ..., p_u \in P$ such that

$$\#\{F \in \mathcal{F} : F \text{ crosses } \{p_1, ..., p_u\}\} \leq c' m^{-\frac{1}{d}} n.$$

We prove a weaker bound which we shall bootstrap to prove Theorem 4.6.2.

Lemma 4.6.6. *Let M be an o-minimal L-structure expanding an ordered field. Let $\phi(x_1, x_2, x_3; y) \in L$ with $|x_1| = |x_2| = |x_3| = 2$. For all $b \in M^y$ and finite $P_i \subseteq M^{x_i}$ with $n_i := |P_i|$, if $\phi(P_1, P_2, P_3; b)$ is $K_{u,u,u}$-free, then,*

(i) $|\phi(P_1, P_2, P_3; b)| \ll_{\phi,u} n_1^{\frac{1}{2}} n_2 n_3 + n_1(n_2 n_3)^{\frac{2}{3}} + n_1 n_2 + n_1 n_3$.

(ii) $|\phi(P_1, P_2, P_3; b)| \ll_{\phi,u} n_1(n_2 n_3)^{\frac{3}{4}} + n_2(n_3 n_1)^{\frac{2}{3}} + n_3(n_1 n_2)^{\frac{2}{3}} + n_1 n_2 + n_2 n_3 + n_3 n_1$.

Proof. Note that (i) implies (ii). Indeed, (i) implies, by symmetry, that

$$|\phi(P_1, P_2, P_3; b)| \ll_{\phi,u} n_2^{\frac{1}{2}} n_3 n_1 + n_2(n_3 n_1)^{\frac{2}{3}} + n_2 n_3 + n_2 n_1,$$
$$|\phi(P_1, P_2, P_3; b)| \ll_{\phi,u} n_3^{\frac{1}{2}} n_1 n_2 + n_3(n_1 n_2)^{\frac{2}{3}} + n_3 n_1 + n_3 n_2.$$

Since $n_1(n_2 n_3)^{\frac{3}{4}}$ is the multiplicative average of $n_2^{\frac{1}{2}} n_3 n_1$ and $n_3^{\frac{1}{2}} n_1 n_2$, (ii) follows.

It remains to prove (i). We follow the proof strategy of [19, Theorem 2.1] and [14, Proposition 4.1]. Write $\phi'(x_1; x_2, x_3) := \phi(x_1, x_2, x_3; b)$. By Fact 4.6.1, up to increasing the bound by $O_{\phi,u}(n_1(n_2 n_3)^{\frac{2}{3}} + n_1 n_2 + n_1 n_3)$, we may assume that $\phi'(p; P_2, P_3)$ contains $K_{u,u}$ for all $p \in P_1$. Now, up to increasing the bound by a factor of u, we may assume that for all distinct $p, p' \in P_1$, we have $\phi'(p; P_2, P_3) \neq \phi'(p'; P_2, P_3)$. (Indeed, since $\phi'(P_1; P_2, P_3)$ is $K_{u,u,u}$-free, there cannot be distinct $p_0, ..., p_{u-1} \in P_1$ such that $\phi'(p_0; P_2, P_3) = \phi'(p_i; P_2, P_3)$ for all $i \in [u-1]$.)

Claim 4.6.6.1. *For all $P' \subseteq P_1$ with $m := |P'| \geq u$, there is $a \in P'$ such that $|\phi'(a; P_2, P_3)| \ll_{\phi,u} m^{-\frac{1}{2}} n_2 n_3 + (n_2 n_3)^{\frac{2}{3}} + n_2 + n_3$.*

Proof of Claim. Let $P' \subseteq P_1$ with $m := |P'| \geq u$. By Fact 4.6.4, $\pi^*_{\phi'}(n) = O_\phi(n^2)$.

Thus, by Fact 4.6.5, there are distinct $a_1, ..., a_u \in P'$ such that

$$\#\{(p_2, p_3) \in P_2 \times P_3 : \phi'(P_1; p_2, p_3) \text{ crosses } \{a_1, ..., a_u\}\}| \ll_{\phi,u} m^{-\frac{1}{2}} n_2 n_3.$$

For all $(p_2, p_3) \in \phi'(a_1; P_2, P_3)$, $\phi'(P_1; p_2, p_3)$ either crosses or contains $\{a_1, ..., a_u\}$. Applying Fact 4.6.1 to the formula $\bigwedge_{i\in[u]} \phi'(a_i; x_2, x_3)$, since $\phi'(P_1, P_2, P_3)$ is $K_{u,u,u}$-free,

$$\#\{(p_2, p_3) \in P_2 \times P_3 : \phi'(P_1; p_2, p_3) \supseteq \{a_1, ..., a_u\}\} \ll_{\phi,u} (n_2 n_3)^{\frac{2}{3}} + n_2 + n_3.$$

Thus, $|\phi'(a_1; P_2, P_3)| \ll_{\phi,u} m^{-\frac{1}{2}} n_2 n_3 + (n_2 n_3)^{\frac{2}{3}} + n_2 + n_3$. ⊣

Iterate Claim 4.6.6.1, beginning with $P' = P_1$ and removing $a \in P'$ until $|P'| < u$. This gives

$$\begin{aligned} |\phi'(P_1, P_2, P_3)| &\ll_{\phi,u} (u-1) n_2 n_3 + \sum_{m=u}^{n_1} (m^{-\frac{1}{2}} n_2 n_3 + (n_2 n_3)^{\frac{2}{3}} + n_2 + n_3) \\ &\ll_u n_1^{\frac{1}{2}} n_2 n_3 + n_1((n_2 n_3)^{\frac{2}{3}} + n_2 + n_3). \end{aligned}$$ □

Before proving Theorem 4.6.2, let us discuss our proof strategy, which builds on the one for [9, Fact 5.14]. The slogan is that we shall apply the cutting lemma, Fact 4.6.4, twice. Let $\phi'(x_1; x_2, x_3) := \phi(x_1, x_2, x_3; b)$. To bound $|\phi'(P_1; P_2, P_3)|$, we find some definable $P' \subseteq P_1$ for which $|\phi'(P'; P_2, P_3)|$ is small. We bound $|\phi'(P'; P_2, P_3)|$ by splitting it into $|\phi'(P'; E(P_2, P_3))|$ and $|\phi'(P'; \neg E(P_2, P_3))|$, where $E(x_2, x_3)$ is the formula saying that $\phi'(M^{x_1}; x_2, x_3)$ crosses P'. Fact 4.6.1 gives a bound for $|\phi'(P'; \neg E(P_2, P_3))|$. The cutting lemma gives a bound for $|E(P_2, P_3)|$, which we bootstrap to bound $|\phi'(P'; E(P_2, P_3))|$. We do so by finding a definable partition $\overline{\mathcal{D}}$ of P_2 and a set $W_D \subseteq P_3$ for each $D \in \overline{\mathcal{D}}$, such that $E(P_2, P_3) \subseteq \bigcup_{D\in\overline{\mathcal{D}}} D \times W_D$ and for all $D \in \overline{\mathcal{D}}$, 'most' $p_3 \in W_D$ are such that $E(M^{x_2}, p_3)$ crosses D. By another application of the cutting lemma, we can insist that $|W_D|$ is small for all $D \in \mathcal{D}$, and so bound $|\phi'(P'; E(P_2, P_3))| \leq \sum_{D\in\overline{\mathcal{D}}} |\phi'(P'; D, W_D)|$.

Proof of Theorem 4.6.2. Write $\phi'(x_1, x_2, x_3) := \phi(x_1, x_2, x_3; b)$. By Lemma 4.6.6,

$$|\phi'(P_1, P_2, P_3)| \ll_{\phi,u} n_1(n_2n_3)^{\frac{3}{4}} + n_2(n_3n_1)^{\frac{2}{3}} + n_3(n_1n_2)^{\frac{2}{3}} + n_1n_2 + n_2n_3 + n_3n_1.$$

Therefore, if $n_1 \leq (n_2n_3)^{\frac{1}{4}}$ then

$$|\phi'(P_1, P_2, P_3)| \ll_{\phi,u} n_2(n_3n_1)^{\frac{2}{3}} + n_3(n_1n_2)^{\frac{2}{3}} + n_1n_2 + n_2n_3 + n_3n_1, \tag{7}$$

and we are done. Suppose instead that $n_1 > (n_2n_3)^{\frac{1}{4}}$, so

$$r := n_1^{\frac{12}{25}}(n_2n_3)^{-\frac{3}{25}} > 1.$$

By Fact 4.6.4, there is a formula $\psi(x_1; z)$, chosen only in terms of ϕ, and a cover $\mathcal{C}$ of M^{x_1}, such that every $C \in \mathcal{C}$ is an instance of ψ with

$$|\{(p_2, p_3) \in P_2 \times P_3 : \phi'(M^{x_1}; p_2, p_3) \text{ crosses } C\}| \leq \frac{n_2n_3}{r},$$

and $|\mathcal{C}| \leq \alpha r^2$ for some $\alpha = \alpha(\psi) = \alpha(\phi)$. Then, there is $C \in \mathcal{C}$ such that

$$|C \cap P_1| \geq n_1(\alpha r^2)^{-1} = \alpha^{-1} n_1^{\frac{1}{25}}(n_2n_3)^{\frac{6}{25}}.$$

Let $P' \subseteq C \cap P_1$ be such that $|P'| = \lceil \alpha^{-1} n_1^{\frac{1}{25}}(n_2n_3)^{\frac{6}{25}} \rceil$. Assuming without loss of generality that $n_1n_2n_3$ is sufficiently large, we have $|P'| \geq u$.

Let $\theta(x_2, x_3; y, z)$ be the formula '$\phi(M^{x_1}, x_2, x_3; y)$ crosses $\psi(M^{x_1}; z)$', that is,

$$\exists x_1, x_1' \left(\psi(x_1; z) \wedge \psi(x_1'; z) \wedge \phi(x_1, x_2, x_3; y) \wedge \neg\phi(x_1', x_2, x_3; y)\right).$$

Let $E(x_2, x_3)$ be the formula '$\phi'(M^{x_1}; x_2, x_3)$ crosses C', which is an instance of θ, and note that

$$|E(P_2, P_3)| \leq \frac{n_2n_3}{r}.$$

By Fact 4.6.4, there is a cover $\mathcal{D}$ of M^{x_2} with $|\mathcal{D}| = O_{\psi,\phi}(r^2) = O_\phi(r^2)$, such that for all $D \in \mathcal{D}$, we have

$$|\{p_3 \in P_3 : E(M^{x_2}, p_3) \text{ crosses } D| \leq \frac{n_3}{r}.$$

We wish to bound

$$|\phi'(P'; P_2, P_3)| = |\phi'(P'; E(P_2, P_3))| + |\phi'(P'; P_2 \times P_3 \setminus E(P_2, P_3))|. \tag{8}$$

We bound the two summands separately, beginning with the latter. Since $|P'| \geq u$, we can fix distinct $a_1, ..., a_u \in P'$. Observe that

$$\phi'(P'; P_2 \times P_3 \setminus E(P_2, P_3)) \subseteq P' \times \{q \in P_2 \times P_3 : \phi'(M^{x_1}; q) \supseteq \{a_1, ..., a_u\}\}.$$

By a similar argument to that used to prove Claim 4.6.6.1, we have

$$|\{q \in P_2 \times P_3 : \phi'(M^{x_1}; q) \supseteq \{a_1, ..., a_u\}\}| \ll_{\phi,u} (n_2 n_3)^{\frac{2}{3}} + n_2 + n_3,$$

and so

$$|\phi'(P'; P_2 \times P_3 \setminus E(P_2, P_3))| \ll_{\phi,u} |P'| \left((n_2 n_3)^{\frac{2}{3}} + n_2 + n_3 \right). \tag{9}$$

We proceed to bound $|\phi'(P'; E(P_2, P_3))|$. Let $\overline{\mathcal{D}}$ be any partition of P_2 refining the cover $\mathcal{D}$ such that $|\overline{\mathcal{D}}| \leq |\mathcal{D}| = O_\phi(r^2)$. For $D \in \overline{\mathcal{D}}$, let

$$U_D := \{p_3 \in P_3 : E(D, p_3) = D\},$$
$$V_D := \{p_3 \in P_3 : E(M^{x_2}, p_3) \text{ crosses } D\}.$$

For all $D \in \overline{\mathcal{D}}$, there is $D' \in \mathcal{D}$ such that $D \subseteq D'$, whence

$$|V_D| \leq |p_3 \in P_3 : E(M^{x_2}, p_3) \text{ crosses } D'| \leq \frac{n_3}{r},$$

and so $\sum_{D \in \overline{\mathcal{D}}} |D \times V_D| \leq n_2 n_3 / r$. Moreover, $\bigcup_{D \in \overline{\mathcal{D}}} D \times U_D \subseteq E(P_2, P_3)$, so by the disjointness of $\overline{\mathcal{D}}$ we have $\sum_{D \in \overline{\mathcal{D}}} |D \times U_D| \leq n_2 n_3 / r$. Writing $W_D := U_D \cup V_D$

for $D \in \overline{\mathcal{D}}$, we have

$$\sum_{D \in \overline{\mathcal{D}}} |D \times W_D| \leq 2n_2n_3/r.$$

Now observe that $E(P_2, P_3) \subseteq \bigcup_{D \in \overline{\mathcal{D}}} D \times W_D$. Indeed, given $(p_2, p_3) \in E(P_2, P_3)$, there is $D \in \overline{\mathcal{D}}$ such that $p_2 \in D$, and either $E(D, p_3) = D$ or $E(M^{x_2}, p_3)$ crosses D, whence $p_3 \in U_D \cup V_D = W_D$.

Let $\overline{\mathcal{D}}_1 := \{D \in \overline{\mathcal{D}} : |D| < n_2/r^{\frac{7}{3}}\}$ and $\overline{\mathcal{D}}_2 := \overline{\mathcal{D}} \setminus \overline{\mathcal{D}}_1$. Now

$$|\phi'(P'; E(P_2, P_3))| \leq \sum_{D \in \overline{\mathcal{D}}_1} |\phi'(P'; D, W_D)| + \sum_{D \in \overline{\mathcal{D}}_2} |\phi'(P'; D, W_D)|. \tag{10}$$

We bound the two summands separately. Firstly, we have

$$\sum_{D \in \overline{\mathcal{D}}_1} |\phi'(P'; D, W_D)| \leq \sum_{D \in \overline{\mathcal{D}}_1} |\phi'(P'; D, P_3)| = \left|\phi'(P'; \bigcup \overline{\mathcal{D}}_1, P_3)\right|.$$

By Lemma 4.6.6 and the fact that $|\bigcup \overline{\mathcal{D}}_1| < (n_2/r^{7/3})|\overline{\mathcal{D}}| \ll_\phi n_2/r^{\frac{1}{3}}$,

$$\sum_{D \in \overline{\mathcal{D}}_1} |\phi'(P'; D, W_D)|$$
$$\ll_\phi |P'| \left(\frac{n_2n_3}{r^{\frac{1}{3}}}\right)^{\frac{3}{4}} + |P'|^{\frac{2}{3}} \left(\frac{n_2n_3^{\frac{2}{3}}}{r^{\frac{1}{3}}} + \frac{n_2^{\frac{2}{3}}n_3}{r^{\frac{2}{9}}}\right) + |P'| \left(\frac{n_2}{r^{\frac{1}{3}}} + n_3\right) + \frac{n_2n_3}{r^{\frac{1}{3}}}. \tag{11}$$

By another application of Lemma 4.6.6,

$$\sum_{D \in \overline{\mathcal{D}}_2} |\phi'(P'; D, W_D)| \ll_{\phi,u} I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &:= |P'| \sum_{D \in \overline{\mathcal{D}}_2} |D \times W_D|^{\frac{3}{4}}, \\
I_2 &:= |P'|^{\frac{2}{3}} \sum_{D \in \overline{\mathcal{D}}_2} (|D||W_D|^{\frac{2}{3}} + |D|^{\frac{2}{3}}|W_D|), \\
I_3 &:= |P'| \sum_{D \in \overline{\mathcal{D}}_2} (|D| + |W_D|), \\
I_4 &:= \sum_{D \in \overline{\mathcal{D}}_2} |D \times W_D|.
\end{aligned}$$

We bound the summands separately. Firstly,

$$I_4 \leq \sum_{D \in \overline{\mathcal{D}}} |D \times W_D| \ll \frac{n_2 n_3}{r}.$$

To bound I_1, we apply Hölder's inequality to obtain

$$I_1 \leq |P'| \left(\sum_{D \in \overline{\mathcal{D}}_2} |D \times W_D| \right)^{\frac{3}{4}} |\overline{\mathcal{D}}_2|^{\frac{1}{4}} \ll_\phi |P'| \left(\frac{n_2 n_3}{r} \right)^{\frac{3}{4}} (r^2)^{\frac{1}{4}} = |P'| \frac{(n_2 n_3)^{\frac{3}{4}}}{r^{\frac{1}{4}}}.$$

We now bound I_2 and I_3. First observe that, since $|D| \geq n_2/r^{\frac{7}{3}}$ for all $D \in \overline{\mathcal{D}}_2$, we have

$$\sum_{D \in \overline{\mathcal{D}}_2} |W_D| \leq \left(n_2/r^{\frac{7}{3}} \right)^{-1} \sum_{D \in \overline{\mathcal{D}}_2} |D||W_D| \ll r^{\frac{4}{3}} n_3.$$

Thus,

$$I_3 \ll |P'| \left(n_2 + r^{\frac{4}{3}} n_3 \right),$$

and by Hölder's inequality,

$$\begin{aligned}
I_2 &\leq |P'|^{\frac{2}{3}} \left(\sum_{D \in \overline{\mathcal{D}}_2} |D \times W_D| \right)^{\frac{2}{3}} \left(\left(\sum_{D \in \overline{\mathcal{D}}_2} |D| \right)^{\frac{1}{3}} + \left(\sum_{D \in \overline{\mathcal{D}}_2} |W_D| \right)^{\frac{1}{3}} \right) \\
&\ll |P'|^{\frac{2}{3}} \left(\frac{n_2 n_3}{r} \right)^{\frac{2}{3}} \left(n_2^{\frac{1}{3}} + r^{\frac{4}{9}} n_3^{\frac{1}{3}} \right) \\
&= |P'|^{\frac{2}{3}} \left(\frac{n_2 n_3^{\frac{2}{3}}}{r^{\frac{2}{3}}} + \frac{n_2^{\frac{2}{3}} n_3}{r^{\frac{2}{9}}} \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{D \in \overline{\mathcal{D}}_2} |\phi'(P'; D, W_D)| \\
&\ll_{\phi,u} |P'| \left(\frac{(n_2 n_3)^{\frac{3}{4}}}{r^{\frac{1}{4}}} + n_2 + r^{\frac{4}{3}} n_3 \right) + |P'|^{\frac{2}{3}} \left(\frac{n_2 n_3^{\frac{2}{3}}}{r^{\frac{2}{3}}} + \frac{n_2^{\frac{2}{3}} n_3}{r^{\frac{2}{9}}} \right) + \frac{n_2 n_3}{r}.
\end{aligned}$$

Combining with (10) and (11), we have

$$\begin{aligned}
&|\phi'(P'; E(P_2, P_3))| \\
&\ll_{\phi,u} |P'| \left(\frac{(n_2 n_3)^{\frac{3}{4}}}{r^{\frac{1}{4}}} + n_2 + r^{\frac{4}{3}} n_3 \right) + |P'|^{\frac{2}{3}} \left(\frac{n_2 n_3^{\frac{2}{3}}}{r^{\frac{1}{3}}} + \frac{n_2^{\frac{2}{3}} n_3}{r^{\frac{2}{9}}} \right) + \frac{n_2 n_3}{r^{\frac{1}{3}}}.
\end{aligned}$$

Combining with (8) and (9), we have

$$\begin{aligned}
&|\phi'(P'; P_2, P_3)| \\
&\ll_{\phi,u} |P'| \left(\frac{(n_2 n_3)^{\frac{3}{4}}}{r^{\frac{1}{4}}} + n_2 + r^{\frac{4}{3}} n_3 + (n_2 n_3)^{\frac{2}{3}} \right) + |P'|^{\frac{2}{3}} \left(\frac{n_2 n_3^{\frac{2}{3}}}{r^{\frac{1}{3}}} + \frac{n_2^{\frac{2}{3}} n_3}{r^{\frac{2}{9}}} \right) + \frac{n_2 n_3}{r^{\frac{1}{3}}}.
\end{aligned}$$

Thus, there is $p \in P'$ such that

$$\begin{aligned}
&|\phi'(p; P_2, P_3)| \\
&\ll_{\phi,u} \frac{(n_2n_3)^{\frac{3}{4}}}{r^{\frac{1}{4}}} + n_2 + r^{\frac{4}{3}}n_3 + (n_2n_3)^{\frac{2}{3}} + |P'|^{-\frac{1}{3}}\left(\frac{n_2n_3^{\frac{2}{3}}}{r^{\frac{1}{3}}} + \frac{n_2^{\frac{2}{3}}n_3}{r^{\frac{2}{9}}}\right) + \frac{n_2n_3}{r^{\frac{1}{3}}|P'|} \\
&\ll_{\phi} n_1^{-\frac{3}{25}}(n_2n_3)^{\frac{39}{50}} + n_2 + n_1^{\frac{16}{25}}n_2^{-\frac{4}{25}}n_3^{\frac{21}{25}} + (n_2n_3)^{\frac{2}{3}} + n_1^{-\frac{13}{75}}n_2^{\frac{24}{25}}n_3^{\frac{47}{75}} + n_1^{-\frac{3}{25}}n_2^{\frac{46}{75}}n_3^{\frac{71}{75}} \\
&\qquad + n_1^{-\frac{1}{5}}(n_2n_3)^{\frac{4}{5}}.
\end{aligned}$$

Remove p from P_1 and iterate this process until at most $(n_2n_3)^{\frac{1}{4}}$ elements remain in P_1. Combining with (7), we have

$$\begin{aligned}
&|\phi'(P_1; P_2, P_3)| \\
&\ll_{\phi,u} n_1^{\frac{22}{25}}(n_2n_3)^{\frac{39}{50}} + n_1n_2 + n_1^{\frac{41}{25}}n_2^{-\frac{4}{25}}n_3^{\frac{21}{25}} + n_1(n_2n_3)^{\frac{2}{3}} + n_1^{\frac{62}{75}}n_2^{\frac{24}{25}}n_3^{\frac{47}{75}} \\
&\qquad + n_1^{\frac{22}{25}}n_2^{\frac{46}{75}}n_3^{\frac{71}{75}} + (n_1n_2n_3)^{\frac{4}{5}} + n_2(n_3n_1)^{\frac{2}{3}} + n_3(n_1n_2)^{\frac{2}{3}} + n_2n_3 + n_3n_1
\end{aligned}$$

as required. □

Fix an o-minimal expansion M of an ordered field. It is natural to generalise the context of Theorem 4.6.2 and pose the following problem.

Problem 4.6.7. *Find explicit Zarankiewicz bounds for relations $\phi(x_1, ..., x_k; y)$ definable in M, where $k \geq 2$.*

Let us first address the case $k \leq 3$ (and $|x_i|$ are arbitrary). Let $t : \mathbb{N}^+ \to \mathbb{N}^+$ be such that $t(n) = n$ if $n \leq 2$ and $t(n) = 2n - 2$ if $n \geq 3$. For all $\varepsilon > 0$, let $d_\varepsilon : \mathbb{N}^+ \to \mathbb{R}^+$ be such that $d_\varepsilon(n) = n$ if $n \leq 2$ and $d_\varepsilon(n) = n + \varepsilon$ if $n \geq 3$.

By [4, Theorem 6.1], [9, Theorems 4.1, 5.7], and [2, Theorem 1.1], we have the following.

- In Fact 4.6.1, if we remove the condition that $|x_1| = |x_2| = 2$, the statement holds with the bound replaced by $O_{\phi,u,\varepsilon}\left(n_1^{\frac{(t-1)d_\varepsilon}{td_\varepsilon - 1}} n_2^{\frac{t(d_\varepsilon - 1)}{td_\varepsilon - 1}} + n_1 + n_2\right)$, where $d_\varepsilon := d_\varepsilon(|x_2|)$ and $t := t(|x_1|)$. (This handles the case $k = 2$.)

- In Fact 4.6.4, if we remove the condition that $|x| = 2$, the statement holds with $O(n^2)$ replaced by $O_\varepsilon(n^{d_\varepsilon(|x|)})$ and $O_\psi(r^2)$ replaced by $O_\psi(r^{t(|x|)})$.

Thus, the proof of Theorem 4.6.2 can be replicated to produce a Zarankiewicz bound for $\phi(x_1, x_2, x_3; y)$, for arbitrary $|x_i|$.

What if $k \geq 4$? Suppose we wanted to replicate the proof of Theorem 4.6.2 to produce a Zarankiewicz bound for $\phi(x_1, ..., x_4; y)$. Let us suppress the parameter variable y. The proof strategy of Theorem 4.6.2, where we obtained a Zarankiewicz bound for the ternary formula $\phi(x_1, x_2, x_3)$, was to apply Fact 4.6.4, a cutting lemma for binary formulas, along with Fact 4.6.1, a Zarankiewicz bound for binary formulas. One would hope that a Zarankiewicz bound for 4-ary formulas can be obtained by applying a cutting lemma for ternary formulas along with Theorem 4.6.2, a Zarankiewicz bound for ternary formulas.

The reader that wishes to pursue this approach needs to clear two obstacles. Firstly, a cutting lemma for ternary formulas does not exist in the literature, and it is not clear what the statement should be. Secondly, the exponents in the bound in Theorem 4.6.2 are terribly asymmetric. Even if a ternary cutting lemma were to emerge, it would be sensible to seek a more symmetric and manageable bound before bootstrapping it to prove a bound for 4-ary formulas.

Chapter 5

Distality *to and from* Combinatorics: Climbing the Arity Ladder

In this chapter, we develop the theories of higher-arity distality and hypergraph regularity by using each to inform the other. Specifically, we develop k-strong honest definitions for NIP strongly k-distal structures, giving rise to a regularity lemma for hypergraphs definable in such structures.

- *Distality to combinatorics.* This expands our understanding of (model-theoretic) contexts for efficient regularity lemmas. In the current literature, distal structures are the most general structures in which definable hypergraphs admit homogeneous regularity lemmas, and we extend this to NIP strongly k-distal structures.

- *Distality from combinatorics.* Although there is work in the literature on higher-arity generalisations of strong honest definitions, it was unclear what the precise formulation should be. We develop such a formulation, whose efficacy is supported by our regularity lemma.

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5.1 Introduction

In Chapter 4, we introduced (hyper)graph regularity lemmas. We saw Szemerédi's regularity lemma, which says that every graph can be decomposed into a bounded number of boxes, most of which are quasirandom. We restate this here for not necessarily bipartite graphs.

Theorem 5.1.1 (Szemerédi, 1978 [51])**.** *For all $\delta > 0$, there is $K \in \mathbb{N}$ such that the following holds.*

Let $G = (V, E)$ be a finite graph. Then there is a partition $V = V_1 \sqcup \cdots \sqcup V_K$ such that

$$\sum_{(V_i, V_j) \text{ not } \delta\text{-regular}} |V_i \times V_j| \leq \delta |V|^2.$$

We emphasise the fact that V^2 is partitioned into *boxes* $V_i \times V_j$, most of which are quasirandom (δ-regular).

Generalising Szemerédi's regularity lemma to k-uniform hypergraphs is a surprisingly complicated task. Partitioning V^k into boxes $V_{i_1} \times \cdots \times V_{i_k}$, with the obvious k-uniform generalisation of δ-regularity, does produce a regularity lemma (work of Chung [12]), but it is limited in applicability; this notion of regularity is often referred to as *weak hypergraph regularity*. The quest for *strong hypergraph regularity* — finding the correct shape (not a box) of the partition pieces and the correct associated notion of quasirandomness — was highly non-trivial. Flagship achievements to that end include work of Gowers [25] and Nagle, Rödl, Schacht, and Skokan [38, 39], but there are many other significant contributions, for an account of which we refer the reader to [37]. In this chapter, we follow Gowers' work; we describe it here briefly, with more exposition to follow in the next section.

A partition of V^k into boxes $V_{i_1} \times \cdots \times V_{i_k}$ is induced by a partition of the vertex set $V = V_1 \sqcup \cdots \sqcup V_K$. It turns out that, for strong hypergraph regularity, we should partition not only the vertex set V but also $V^2, ..., V^{k-1}$, and use these partitions to build our partition of V^k; we will say, for each $l \in [k-1]$, that the partition of V^l is *level* l of the partition of V^k. Specialising to $k = 3$ as an example, if we have partitions $\mathcal{P}_1$ of V and $\mathcal{P}_2$ of V^2, then V^3 is partitioned into the pieces

$$\{(v_1, v_2, v_3) \in V^3 : v_i \in P_i \text{ for all } i \in [3] \text{ and } (v_i, v_j) \in P_{ij} \text{ for all } 1 \leq i < j \leq 3\},$$

where $P_i \in \mathcal{P}_1$ for all $i \in [3]$ and $P_{ij} \in \mathcal{P}_2$ for all $1 \leq i < j \leq 3$. For reasons that will be explained in the next section, these pieces can be thought of as 2-dimensional *simplicial complexes*, and for general k we will refer to the partition pieces as $(k-1)$-dimensional *simplicial complexes*.

We have already seen a hypergraph regularity lemma in Chapter 4. Indeed, we saw that hypergraphs definable in distal structures satisfy the *distal regularity lemma*, in which the notion of quasirandomness is strongest possible: homogeneity. We restate this here for not necessarily partite hypergraphs.

Theorem 5.1.2 (Chernikov–Starchenko, 2018 [11]). *Let T be a distal L-theory, $M \models T$, and let $\phi(x_1, ..., x_k) \in L(M)$ with $|x_1| = \cdots = |x_k| =: d$. Then, for each $\delta > 0$, there is a natural number $K \leq poly_\phi(\delta^{-1})$ such that the following holds.*

Let $V \subseteq M^d$ be finite. Then there is a partition $V = V_1 \sqcup \cdots \sqcup V_K$ such that

$$\sum_{(V_{i_1},...,V_{i_k}) \text{ not } \phi\text{-homogeneous}} |V_{i_1} \times \cdots \times V_{i_k}| \leq \delta |V|^k.$$

In fact, Chernikov–Starchenko [11] prove more than this. As a basis of comparison for later on, we state their result in more generality and strength.

Theorem 5.1.3 (Chernikov–Starchenko, 2018 [11]). *Let T be a distal L-theory, $M, \mathbb{M} \models T$ with $\mathbb{M}$ sufficiently saturated, and let $\phi(x_1, ..., x_k) \in L(M)$ with $|x_1| = \cdots = |x_k| =: d$. Then, for each $\delta > 0$, there is a natural number*

$K \leq poly_\phi(\delta^{-1})$ and a formula $\theta(x_1, z) \in L$ such that the following holds.

Let $V \subseteq \mathbb{M}^d$ be M-definable, and let $\mu(x_1)$ be a global measure, generically stable over M. Then there is a partition $V = V_1 \sqcup \cdots \sqcup V_K$, where each $V_i = \theta(x_1, c)$ for some $c \in M^z$, such that

$$\sum_{(V_{i_1},\ldots,V_{i_k}) \text{ not } \phi\text{-homogeneous}} \mu^{(k)}(V_{i_1} \times \cdots \times V_{i_k}) \leq \delta\mu(V)^k.$$

Given what we just said about how a partition into boxes yields a weak rather than a strong hypergraph regularity lemma, the reader under the (correct) impression that distal regularity is a very strong form of hypergraph regularity is entitled to be confused by the distal regularity lemma, where the partition pieces are, in fact, boxes. The issue is that, in the weak hypergraph regularity lemma, when we decompose a general k-uniform hypergraph into boxes, we can only ask for most of these boxes to be *δ-regular*, which is too weak a notion of quasirandomness for combinatorial arguments such as counting arguments to work (see the next section for more details). However, in the distal regularity lemma, the k-uniform hypergraph can be decomposed into boxes, most of which are *homogeneous*, which is certainly a strong enough notion of quasirandomness.

Nonetheless, it is not helpful to think of distal regularity as a strong version of weak regularity, since the combinatorial arguments that do not work with weak regularity work (very well) with distal regularity. Rather, one should think of distal regularity as a strong version of strong regularity, where the $(k-1)$-dimensional simplicial complexes take on the special form of boxes. Note that boxes are indeed simplicial complexes. Returning to the example of $k = 3$ above, if level 2 of the partition is trivial (that is, the partition $\mathcal{P}_2$ of V^2 is the trivial partition), then each simplicial complex is a box: it has the form

$$\{(v_1, v_2, v_3) \in V^3 : v_i \in P_i \text{ for all } i \in [3]\} = P_1 \times P_2 \times P_3$$

for some $P_i \in \mathcal{P}_1$. For general $k \geq 3$, a partition into simplicial complexes is a partition into boxes in the special case that levels $2, \ldots, k-1$ are trivial (that is, the partitions of $V^2, ..., V^{k-1}$ are trivial).

In summary, for a k-uniform hypergraph (V, E) satisfying the distal regularity lemma, V^k can be partitioned into a bounded number of boxes — a special case of $(k-1)$-dimensional simplical complexes — most of which are E-homogeneous. It is now natural to pose the following problem.

Problem 5.1.4. *For which k-uniform hypergraphs (V, E) can V^k be partitioned into a bounded number of $(k-1)$-dimensional simplical complexes, most of which are E-homogeneous?*

We call such a partition a *homogeneous regularity partition*, and a regularity lemma giving such a partition a *homogeneous regularity lemma*.

When $k = 2$, that is, in the case of graphs, distal structures are the most general known model-theoretic context for graphs with such a partition, since a 1-dimensional simplicial complex is just a box. Distality, when characterised by strong honest definitions, can be seen as a binary notion. It thus makes sense to find answers to Problem 5.1.4 for $k \geq 3$ using a $(k-1)$-ary generalisation of distality.

These were introduced by Walker [55], who introduced two notions of k-ary distality for each $k \in \mathbb{N}^+$. Recall from Section 2.7 that distality has an internal and an external characterisation which are equivalent (Theorem 2.7.3). Walker generalised the internal characterisation to *k-distality* and the external characterisation to *strong k-distality*, such that 1-distality, strong 1-distality, and distality are equivalent, and (strong) k-distality implies (strong) $(k+1)$-distality. As the name suggests, strong k-distality implies k-distality. There is no literature on the converse — in particular, Walker [55] could not decide it.

The distal regularity lemma, Theorem 5.1.3, was proved using strong honest definitions. Recall that an L-theory is (1-)distal if and only if every formula $\phi(x; y) \in L$ has a strong honest definition $\psi(x; z) \in L$: for all $B \subseteq M \models T$ with $2 \leq |B| < \infty$ and $a \in M$, there is $c \in B$ such that for all $b \in B$,

$$a \models \psi(x; c) \vdash \phi(x; b) \leftrightarrow \phi(a; b).$$

Here, we have continued the abuse of notation that if y is an n-tuple with entries in a set Y (that is, $y \in Y^n$), we sometimes simply write $y \in Y$, but $X \subseteq Y$ always means $X \subseteq Y^1$.

In order to find a homogeneous regularity lemma for (strongly) k-distal theories, we define k-strong honest definitions for formulas $\phi(x_1, ..., x_k; y)$, such that an NIP theory is strongly k-distal if and only if such k-strong honest definitions exist. This is done in Definition 5.4.9, of which we give a preview now. Recall that, for a tuple $v = (v_1, ..., v_k)$ and $i \in [k]$, $v_{\neq i} := (v_1, ..., v_{i-1}, v_{i+1}, ..., v_k)$. Given $\phi(x_1, ..., x_k; y) \in L$ with $x := (x_1, ..., x_k)$, a *k-strong honest definition* for ϕ is a $(k+1)$-tuple of L-formulas $(\psi_i(x_{\neq i}, y, z_i) : i \in [k])^\frown(\psi_{k+1}(x, z_{k+1}))$ such that the following holds.

There is $N \in \mathbb{N}$ such that, for all $B \subseteq M \models T$ with $2 \leq |B| < \infty$ and $a = (a_1, ..., a_k) \in M$, there are $c_1^{(j)}, ..., c_{k+1}^{(j)} \in B$ for $j \in [N]$ such that for all $b \in B$, there is $j \in [N]$ such that

$$a \models \psi_{k+1}(x, c_{k+1}^{(j)}) \wedge \bigwedge_{i=1}^{k} \psi_i(x_{\neq i}, b, c_i^{(j)}) \vdash \phi(x; b) \leftrightarrow \phi(a; b).$$

The following is Theorem 5.4.12.

Theorem 5.1.5. *Let T be NIP. Then T is strongly k-distal if and only if every $\phi(x_1, ..., x_k; y) \in L$ has a k-strong honest definition.*

The reader may be concerned that the formulas $\psi_1, ..., \psi_k$ in a k-strong honest definition for $\phi(x_1, ..., x_k; y)$ involve the y-variable as well as the x-variables. Crucially, however, each of $\psi_1, ..., \psi_k$ involves exactly $k-1$ of the x-variables (and y). The intuition is that, for $b \in M$, in order to understand how $x_1, ..., x_k$ interact with b (with respect to ϕ), it is enough to understand how any $k-1$ of the x_i's interacts with b (with respect to $\psi_1, ..., \psi_k$) and how $x_1, ..., x_k$ interact (with respect to ψ_{k+1}). In other words, the interaction of the $k+1$ variables $x_1, ..., x_k, y$ is locally controlled by the interactions of k of those variables.

Observe the resemblance this bears with our discourse on partitions into simplicial complexes. A regularity lemma for a $(k+1)$-uniform hypergraph (V, E)

(V, E) gives a partition V^{k+1} into k-dimensional simplicial complexes that is induced by partitions of $V^1, ..., V^k$. In other words, the behaviour of $E \subseteq V^{k+1}$ is locally controlled by the partitions of $V^1, ..., V^k$.

We now come to the main result of the chapter. The reader can find it in full generality and strength as Theorem 5.5.9, but here we first state an abridged form that fits more directly with the narrative so far. This is a restriction of Corollary 5.5.12.

Theorem 5.1.6. *Let $k \geq 2$. Let M be an L-structure that is NIP, and let $\phi(x_1, ..., x_{k-1}; x_k) \in L(M)$ have a $(k-1)$-strong honest definition, with $|x_1| = \cdots = |x_k| =: d$. Then, for all $\delta > 0$, there is a natural number $K \leq poly_\phi(\delta^{-1})$ such that the following holds.*

For all finite $V \subseteq M^d$, there is a partition $V^{k-1} = V_1 \sqcup \cdots \sqcup V_K$ inducing the partition $\mathcal{Q}$ of V^k given by

$$\left\{\left\{v = (v_1, ..., v_k) \in V^k : v_{\neq i} \in V_{j_i} \text{ for all } i \in [k]\right\} : j_1, ..., j_k \in [K]\right\},$$

such that $\sum_{Q \in \mathcal{Q} \text{ not } \phi\text{-homogeneous}} |Q| \leq \delta |V|^k$.

In particular, the partition pieces of V^k are simplicial complexes, induced by the $(k-1)^{\text{th}}$ level, that is, the partition of V^{k-1}.

Remark 5.1.7. We previously mentioned that a partition of V^k into $(k-1)$-dimensional simplicial complexes is induced by partitions of $V^1, ..., V^{k-1}$. Theorem 5.1.6 did not explicitly involve partitions of $V^1, ..., V^{k-2}$. However, this is not an interesting observation, since any partition of $V^1, ..., V^{k-2}$ can be absorbed into the partition of V^{k-1}. We illustrate this using the example before Theorem 5.1.2, where $k = 3$. There, the partitions $\mathcal{P}_1$ of V and $\mathcal{P}_2$ of V^2 induced a partition, say $\mathcal{Q}$, of V^3. However, we can 'absorb' $\mathcal{P}_1$ into $\mathcal{P}_2$ by forming the following partition of V^2:

$$\left\{\{(v_1, v_2) \in V^2 : v_1 \in P_1, v_2 \in P_2, (v_1, v_2) \in P_{12}\} : P_1, P_2 \in \mathcal{P}_1, P_{12} \in \mathcal{P}_2\right\}.$$

This induces a partition of V^3 as in Theorem 5.1.6, which is precisely $\mathcal{Q}$.

This absorption technique does not work for the general hypergraph regularity lemma, since the definition of quasirandomness is relative to the separate levels (see [24]). However, for homogeneous regularity lemmas, a simplicial complex Q is defined to be ϕ-homogeneous if $\phi(Q) = Q$ or $\emptyset$; there is no reference to the levels used to define Q, so we may as well apply the absorption technique and only partition V^{k-1}.

We now state the main result in more generality and strength. This is a restriction of Corollary 5.5.11.

Theorem 5.1.8. *Let $k \geq 2$. Let M be an L-structure that is NIP, and let $\phi(x_1, ..., x_{k-1}; x_k) \in L(M)$ have a $(k-1)$-strong honest definition, with $|x_1| = \cdots = |x_k| =: d$. Then, for all $\delta > 0$, there is a natural number $K \leq poly_\phi(\delta^{-1})$ and a formula $\theta(x_1, ..., x_{k-1}, z) \in L$ such that the following holds.*

Let $V \subseteq M^d$ be M-definable, and let $\mu(x_1)$ be a global measure, generically stable over M. Then there is a partition $V^{k-1} = V_1 \sqcup \cdots \sqcup V_K$, where each $V_i = \theta(x_1, ..., x_{k-1}, c)$ for some $c \in M^z$, inducing the partition

$$\mathcal{Q} := \left\{ \left\{ v = (v_1, ..., v_k) \in V^k : v_{\neq i} \in V_{j_i} \text{ for all } i \in [k] \right\} : j_1, ..., j_k \in [K] \right\}$$

of V^k, such that $\sum_{Q \in \mathcal{Q} \text{ not } \phi\text{-homogeneous}} \mu^{(k)}(Q) \leq \delta \mu(V)^k$.

There is yet another statement that is stronger and more general, for which we refer the reader to Theorem 5.5.9.

5.1.1 Structure of the chapter

Since this chapter is on hypergraph regularity lemmas and higher-arity distality, it is right that we begin with expositions of the two subjects; these respectively constitute Sections 5.2 and 5.3. In Section 5.4, we define k-strong honest definitions, and show that an NIP theory is strongly k-distal if and only if every formula $\phi(x_1, ..., x_k; y)$ has a k-strong honest definition. In Section 5.5, we state and prove our regularity lemma for formulas $\phi(x_1, ..., x_k; y)$ with a k-strong honest definition in an NIP theory.

In Section 5.6, we remark on the 'dual' setup where we define k-strong honest definitions for formulas $\phi(x; y_1, ..., y_k)$ instead of $\phi(x_1, ..., x_k; y)$. In particular, although we believe this approach to have good motivation (which we describe), we are unable to prove that dual k-strong honest definitions exist in an NIP strongly k-distal theory; we state this as a conjecture and prove a partial converse. In Section 5.7, we highlight the geometric prowess of dual k-strong honest definitions by describing the analogue of distal cell decompositions that they induce, and use this to show that certain pairs (M, P) where M is o-minimal and $P \subseteq M$ do not have dual k-strong honest definitions.

5.1.2 Basic notation

We lay out some basic notation used in the rest of this chapter.

Let $k, l \in \mathbb{N}^+$. A *k-uniform hypergraph*, or a *k-graph*, is a pair $H = (V, E)$ where V is the set of *vertices* and $E \subseteq \binom{V}{k}$ is the set of *hyperedges*, that is, E consists of subsets of V of size k. We sometimes consider hyperedges as tuples rather than sets. If the hyperedge relation E is not specified, we sometimes denote it by H.

A k-graph $H = (V, E)$ is *l-partite* if there is a partition $V_1 \sqcup \cdots \sqcup V_l$ of V such that, for all $e \in E$ and $i \in [l]$, $|e \cap V_i| \leq 1$; in this case, we write $H = (V_1 \sqcup \cdots \sqcup V_l, E)$, and if $k = l$, we write $H = E(V_1, ..., V_k)$ and view it as a subset of $V_1 \times \cdots \times V_k$. We sometimes define a k-partite k-graph $E(V_1, ..., V_k)$ where $V_1, ..., V_k$ are not necessarily disjoint; in that case, the vertex sets are taken to be disjoint copies of $V_1, ..., V_k$.

For $q, r \in \mathbb{R}$ and $\delta \geq 0$, write $q \approx_\delta r$ to mean $|q - r| \leq \delta$.

5.2 Hypergraph regularity

This section is an exposition of hypergraph regularity lemmas. Motivated by applications, we will see why one possible generalisation of Szemerédi's regularity lemma is not very fruitful, and go on to describe a better generalisation, in preparation for our main result (Theorem 5.5.9). In this section, all (hyper)graphs are finite.

In Chapter 4, we defined what it means for a bipartite graph to be regular, and stated Szemerédi's regularity lemma for bipartite graphs. We now reformulate everything in terms of graphs, as is standard in combinatorial literature (although the formulations are equivalent — see Remark 5.2.3).

Definition 5.2.1. Let $E(V_1, V_2)$ be a bipartite graph. For $W \subseteq V_1 \times V_2$, the *relative density* of $E(V_1, V_2)$ in W is

$$d_W(V_1, V_2) := \frac{|E(V_1, V_2) \cap W|}{|W|}.$$

The *density* of $E(V_1, V_2)$ is $d(V_1, V_2) := d_{V_1 \times V_2}(V_1, V_2)$.

For $\delta > 0$, say that $E(V_1, V_2)$ (or (V_1, V_2)) is *δ-regular* if, for all $A_i \subseteq V_i$ with $|A_i| \geq \delta |V_i|$, $d_{A_1 \times A_2}(V_1, V_2) \approx_\delta d(V_1, V_2)$.

This is a notion of *quasirandomness*: if $E(V_1, V_2)$ is δ-regular, then it 'looks like a random graph' since it has roughly the same density everywhere.

Szemerédi's regularity lemma for graphs reads as follows.

Theorem 5.2.2 (Szemerédi, 1978 [51]). *For all $\delta > 0$, there is $K \in \mathbb{N}$ such that the following holds.*

Let $G = (V, E)$ be a graph. Then there is an equipartition $V = V_1 \sqcup \cdots \sqcup V_K$ such that

$$\sum_{E(V_i, V_j) \text{ not } \delta\text{-regular}} |V_i \times V_j| \leq \delta |V|^2.$$

Remark 5.2.3. It is easy to deduce the bipartite Szemerédi's regularity lemma, as stated in Theorem 4.2.2, from the version in Theorem 5.2.2. Conversely, given a graph $G = (V, E)$, form an auxiliary bipartite graph with vertex set $V \sqcup V$, where $x \sim y$ if and only if $(x, y) \in E$. Applying Theorem 4.2.2 to the auxiliary graph gives Theorem 5.2.2 for G.

A notable application of Szemerédi's regularity lemmas is *(graph) removal lemmas*, of which the following *triangle removal lemma* is an important example. The *triangle* is the complete graph on 3 vertices, and a graph is *triangle-free* if it has no triangles.

Theorem 5.2.4 (Ruzsa–Szemerédi, 1978 [41])**.** *For all $c > 0$, there is $a > 0$ such that the following holds. If G is a graph on n vertices with fewer than an^3 triangles, then it can be made triangle-free by removing at most cn^2 edges.*

In order to deduce this from Szemerédi's regularity lemma, one needs the following *triangle counting lemma*. Roughly speaking, it says that if your graph is δ-regular then it contains approximately (in terms of δ) the correct number of triangles, namely, the number of triangles your graph would contain if it were truly random; this is to be expected for a good notion of quasirandomness. We will use the little-o notation as defined in Subsection 2.1.4.

Proposition 5.2.5. *Let G be a 3-partite graph on vertex sets V_1, V_2, V_3 with $n_i := |V_i|$, $p := d(V_1, V_2)$, $q := d(V_2, V_3)$, and $r := d(V_3, V_1)$. For $\delta > 0$, if $E(V_1, V_2)$, $E(V_2, V_3)$, $E(V_3, V_1)$ are all δ-regular, then the number of triangles in G differs from $pqrn_1n_2n_3$ by $o_{\delta\to 0}(n_1n_2n_3)$.*

Proof. The proof is standard and left as an exercise. □

How do we generalise the above to hypergraphs? Perhaps the most obvious approach is to make the following definition.

Definition 5.2.6. Let $E(V_1, ..., V_k)$ be a k-partite k-graph. Let $W \subseteq V_1 \times \cdots \times V_k$. The *relative density* of $E(V_1, ..., V_k)$ in W is

$$d_W(V_1, ..., V_k) := \frac{|E(V_1, ..., V_k) \cap W|}{|W|}.$$

The *density* of $E(V_1, ..., V_k)$ is $d(V_1, ..., V_k) := d_{V_1 \times \cdots \times V_k}(V_1, ..., V_k)$.

For $\delta > 0$, say that $E(V_1, ..., V_k)$ (or $(V_1, ..., V_k)$) is δ*-regular* if, for all $A_i \subseteq V_i$ with $|A_i| \geq \delta |V_i|$, $d_{A_1 \times \cdots \times A_k}(V_1, ..., V_k) \approx_\delta d(V_1, ..., V_k)$.

With this definition, there is indeed a version of Szemerédi's regularity lemma for hypergraphs.

Theorem 5.2.7 (Chung, 1991 [12])**.** *For all $k \in \mathbb{N}^+$ and $\delta > 0$, there is $K \in \mathbb{N}$ such that the following holds.*

Let $H = (V, E)$ be a k-uniform hypergraph. Then there is an equipartition $V = V_1 \sqcup \cdots \sqcup V_K$ such that

$$\sum_{E(V_{i_1}, ..., V_{i_k}) \text{ not } \delta\text{-regular}} |V_{i_1} \times \cdots \times V_{i_k}| \leq \delta |V|^k.$$

We would likewise want a k-uniform hypergraph version of the triangle removal lemma. The natural generalisation of the triangle is the k-simplex: the complete k-uniform hypergraph on $k+1$ vertices, and indeed we have the following k-simplex removal lemma, achieved independently by Gowers [25] and Nagle, Rödl, Schacht, and Skokan [38–40].

Theorem 5.2.8. *For all $c > 0$, there is $a > 0$ such that the following holds. If G is a k-uniform hypergraph on n vertices with fewer than an^{k+1} k-simplices, then it can be made k-simplex-free by removing at most cn^k hyperedges.*

Unfortunately, this cannot be deduced from Theorem 5.2.7, because there is no k-simplex counting lemma for δ-regular hypergraphs. In other words, δ-regularity is not strong enough for counting k-simplices when $k \geq 3$. We illustrate this for $k = 3$, where we refer to the 3-simplex as the tetrahedron.

The following example is taken (with minor tweaks) from [33] but is considered combinatorial folklore. First, a definition.

Definition 5.2.9. Let $G = (V, E)$ be a graph. Write $\Delta(G)$ for the set of triangles in G, that is,

$$\Delta(G) := \left\{ \{v_1, v_2, v_3\} \in \binom{V}{3} : (v_i, v_j) \in E \text{ for all } 1 \leq i < j \leq 3 \right\}.$$

Note that $\Delta(G)$ is a 3-uniform hypergraph on V.

Example 5.2.10 (δ-regularity insufficient for counting tetrahedra)**.** Let V_1, V_2, V_3, V_4 be sets of size n. Let H_1 be the random 4-partite 3-uniform hypergraph on $V_1 \sqcup \cdots \sqcup V_4$, where hyperedges occur with probability 1/8. Let G_2 be the random 4-partite graph on $V_1 \sqcup \cdots \sqcup V_4$, where edges occur with probability 1/2, and let $H_2 = \Delta(G_2)$, a 4-partite 3-uniform hypergraph on $V_1 \sqcup \cdots \sqcup V_4$. Then, with high probability, for all $\delta > 0$, every triple of vertex sets in both H_1 and H_2 is δ-regular with density $\approx_\delta 1/8$. However, H_1 (which is 'truly random') is expected to contain $(1/8^4)n^4$ tetrahedra, while H_2 is expected to contain $(1/2^6)n^4$ tetrahedra.

In fact, not only is δ-regularity insufficient to guarantee roughly the correct number of tetrahedra, it is insufficient to guarantee the *existence* of tetrahedra at all. The example we use to illustrate this is again taken from [33], who attribute originality to [18]. First, a definition.

Definition 5.2.11. An *l-partite tournament* is a directed l-partite graph with exactly one (directed) edge between any two vertices from distinct vertex sets. Say that vertices x, y, z *span a cyclically oriented triangle* if either $(x, y), (y, z), (z, x)$ are all edges or $(x, z), (z, y), (y, x)$ are all edges.

Example 5.2.12 (δ-regularity is insufficient for existence of tetrahedra)**.** Let V_1, V_2, V_3, V_4 be sets of size n. Let G be the random 4-partite tournament on $V_1 \sqcup \cdots \sqcup V_4$ where, for x, y vertices from distinct vertex sets, (x, y) is an edge (as opposed to (y, x)) with probability 1/2. Let H be the associated 4-partite 3-uniform (undirected) hypergraph on $V_1 \sqcup \cdots \sqcup V_4$, such that (x, y, z) is a hyperedge in H if and only if x, y, z span a cyclically oriented triangle in G. Then, with high probability, for all $\delta > 0$, every triple of vertex sets in H is δ-regular with density $\approx_\delta 1/4$. However, H contains no tetrahedra: it is impossible to have four vertices in G, any three of which span a cyclically oriented triangle!

All of this is to say that δ-regularity is not a good notion of quasirandomness for k-uniform hypergraphs when $k \geq 3$. What, then, is the correct notion? The answer to this question is surprisingly complicated, and a complete exposition is outside the scope of this thesis. We give an abridged account and refer the reader to [24] for the full answer.

As alluded to in the introduction, the problem is not so much that we need a better notion of quasirandomness per se, but that our partition pieces have the wrong shape. We have been considering what it means for a hypergraph $H = (V, E)$ to be quasirandom inside a box $V_1 \times \cdots \times V_k$, where $V_i \subseteq V$, but it is time to move beyond boxes.

Definition 5.2.13. A k-dimensional *simplicial complex* is a set Σ of sets of size at most $k+1$, such that if $B \in \Sigma$ and $A \subseteq B$ then $A \in \Sigma$.

We focus again on the case $k = 3$. Let $G = (V_1 \sqcup V_2 \sqcup V_3, E)$ be a 3-partite graph. Then, $\Delta(G) \cup E \cup (V_1 \cup V_2 \cup V_3) \cup \{\emptyset\}$ is a 2-dimensional simplicial complex, which we will just denote by $\Delta(G)$. It turns out that we should define what it means for a 3-uniform hypergraph to be quasirandom inside such a simplicial complex $\Delta(G)$, instead of a box, with G itself being quasirandom.

Of course, we need to settle what it means for G to be quasirandom and H to be quasirandom inside $\Delta(G)$. The former is straightforward: δ-regularity is a good notion of quasirandomness for graphs, so it is the one we use. The latter is far more complicated, and we refer the reader to [24] for the definition. For now, we will simply say (without definition) that H is η-*quasirandom* inside $\Delta(G)$, where $\eta > 0$ is a parameter.

Remark 5.2.14. The reader is urged not to worry about the definition of η-*quasirandom*, but rather to focus on the big picture that we are defining the quasirandomness of H relative to a simplicial complex $\Sigma = \Delta(G)$. The remainder of this section is comprehensible with this definition as a black box, and in the contexts we will consider in later sections, H will be *homogeneous* inside Σ, that is, $H \cap \Sigma = \Sigma$ or $\emptyset$, and homogeneity is a (very strong) special case of η-quasirandomness.

Definition 5.2.15. Let $H = (V, E)$ be a 3-uniform hypergraph, and let G be a 3-partite graph on vertex sets $V_1, V_2, V_3 \subseteq V$. For $\delta, \eta > 0$, say that (G, H) (or $(\Delta(G), H)$) is (δ, η)-*quasirandom* if $G(V_i, V_j)$ is δ-regular for all $1 \leq i < j \leq 3$ and H is η-quasirandom inside $\Delta(G)$.

For $1 \leq i < j \leq 3$, write $d_{ij} := d(V_i, V_j)$ for the density of $G(V_i, V_j)$. Say that (G, H) (or $(\Delta(G), H)$) is η-*quasirandom* if it is (δ, η)-quasirandom for $\delta = (2^{-40}\eta(d_{12}d_{13}d_{23})^{32})^{16}$.

The exact form of the expression $\delta = (2^{-40}\eta(d_{12}d_{13}d_{23})^{32})^{16}$ is not important for this exposition; it suffices to keep in mind that δ is small in terms of η, d_{12}, d_{13}, and d_{23}.

This works out to be the correct notion of quasirandomness for a 3-uniform hypergraph: there is an associated regularity lemma and tetrahedron counting lemma, the combination of which gives a tetrahedron removal lemma (Theorem 5.2.8 for $k = 3$). We first expound the regularity lemma.

Given a 3-uniform hypergraphs $H = (V, E)$, we wish to partition V^3 into simplicial complexes $\Delta(G)$ such that (G, H) is η-quasirandom with few exceptions. To do so, we partition the vertex set V, say $V = V_1 \sqcup \cdots \sqcup V_K$, and we also partition V^2 into bipartite graphs, say $V^2 = G_1 \sqcup \cdots \sqcup G_L$. Our simplicial complexes are then given by

$$\mathcal{Q} := \{\Delta(G_p(V_i, V_j) \cup G_q(V_i, V_k) \cup G_r(V_j, V_k)) : i, j, k \in [K], p, q, r \in [L]\}.$$

Note that every $(x, y, z) \in V^3$ belongs to exactly one simplicial complex, so $\mathcal{Q}$ is a partition of V^3. Indeed, given $(x, y, z) \in V^3$, there are unique $i, j, k \in [K]$ such that $x \in V_i$, $y \in V_j$, and $z \in V_k$, and then unique $p, q, r \in [L]$ such that $(x, y) \in G_p$, $(x, z) \in G_q$, and $(y, z) \in G_r$. We now state the regularity lemma.

Theorem 5.2.16. *For all $\delta, \eta > 0$ there are $K, L \in \mathbb{N}$ such that the following holds.*

Let $H = (V, E)$ be a 3-uniform hypergraph. Then there is a partition of V into sets $V_1, ..., V_K$ and a partition of V^2 into bipartite graphs $G_1, ..., G_L$ inducing the partition

$$\mathcal{Q} := \{\Delta(G_p(V_i, V_j) \cup G_q(V_i, V_k) \cup G_r(V_j, V_k)) : i, j, k \in [K], p, q, r \in [L]\}$$

of V^3, such that $\sum_{Q \in \mathcal{Q} : (Q,H) \text{ not } \eta\text{-quasirandom}} |Q| \leq \delta |V|^3$.

Proof. See [24, Theorem 8.10]. □

We now seek a counting lemma for η-quasirandom pairs (G, H). We wish to count the number of tetrahedra (say) of H contained in $\Delta(G)$, so we may as well assume that $H \subseteq \Delta(G)$. Let H be a 4-partite 3-uniform hypergraph on vertex sets $V_1, ..., V_4$, and let G be a 4-partite graph on vertex sets $V_1, ..., V_4$ such that $H \subseteq \Delta(G)$. If G is truly random and H sits truly randomly inside $\Delta(G)$, how many tetrahedra do we expect H to contain? For $1 \leq i < j \leq 4$, write d_{ij} for the density of $G(V_i, V_j)$. For $1 \leq i < j < k \leq 4$, write $G_{ijk} := G(V_i \cup V_j \cup V_k)$, and write d_{ijk} for the relative density of $H(V_i, V_j, V_k)$ in $\Delta(G_{ijk})$. Let $v_i \in V_i$ for all $i \in [4]$. Then $(v_1, ..., v_4)$ forms a tetrahedron if and only if $(v_i, v_j, v_k) \in H$ for all $1 \leq i < j < k \leq 4$. If each edge of $G(V_i, V_j)$ occurs randomly with probability d_{ij}, and each hyperedge of $H(V_i, V_j, V_k) \subseteq \Delta(G_{ijk})$ occurs randomly with probability d_{ijk}, then the probability that $(v_1, ..., v_4)$ forms a tetrahedron in H is

$$\begin{aligned} &\prod_{1 \leq i < j \leq 4} \mathbb{P}((v_i, v_j) \in G) \prod_{1 \leq i < j < k \leq 4} \mathbb{P}((v_i, v_j, v_k) \in H \mid (v_i, v_j, v_k) \in \Delta(G)) \\ &= \prod_{1 \leq i < j \leq 4} d_{ij} \prod_{1 \leq i < j < k \leq 4} d_{ijk}. \end{aligned}$$

Thus, the expected number of tetrahedra is $\prod_{1 \leq i < j \leq 4} d_{ij} \prod_{1 \leq i < j < k \leq 4} d_{ijk} \prod_{1 \leq i \leq 4} |V_i|$. The counting lemma says that, if $(G(V_i \cup V_j \cup V_k), H)$ is η-quasirandom for all $1 \leq i < j < k \leq 4$, then the number of tetrahedra in H is approximately (in terms of η) this number.

Theorem 5.2.17. *Let $V_1, ..., V_4$ be sets with $n_i := |V_i|$. Let G be a 4-partite graph on vertex sets $V_1, ..., V_4$ with $d_{ij} := d(V_i, V_j)$ for all $1 \leq i < j \leq 4$. Let $\eta > 0$, and let $H \subseteq \Delta(G)$ be a 4-partite 3-uniform hypergraph on vertex sets $V_1, ..., V_4$. For $1 \leq i < j < k \leq 4$, let $G_{ijk} := G(V_i \cup V_j \cup V_k)$, and suppose (G_{ijk}, H) is η-quasirandom and $H(V_i, V_j, V_k)$ has relative density d_{ijk} in $\Delta(G_{ijk})$. Then the number of tetrahedra in H differs from $\prod_{1\leq i<j\leq 4} d_{ij} \prod_{1\leq i<j<k\leq 4} d_{ijk} \prod_{1\leq i\leq 4} n_i$ by at most $o_{\eta\to 0}(\prod_{1\leq i<j\leq 4} d_{ij} \prod_{1\leq i\leq 4} n_i)$.*

Proof. See [24, Theorem 6.8]. □

The previous two theorems can be combined to give the tetrahedron removal lemma (Theorem 5.2.8 for $k = 3$).

The main takeaway from this exposition is that, in a regularity lemma for 3-uniform hypergraphs $H = (V, E)$, V^3 should be partitioned into 2-dimensional simplicial complexes $\Delta(G)$. For k-uniform hypergraphs $H = (V, E)$, V^k should be partitioned into $(k-1)$-dimensional simplicial complexes. That is, there are partitions $\mathcal{P}_i$ of V^i for $i \in [k-1]$, and each partition piece of V^k has the form

$$\{v = (v_1, ..., v_k) \in V^k : v_J \in P_J \text{ for all } \emptyset \neq J \subsetneq [k]\},$$

where $v_J := (v_j : j \in J)$ and $P_J \in \mathcal{P}_{|J|}$ for all $\emptyset \neq J \subsetneq [k]$.

5.3 Higher-arity distality

In this section, we state the definitions of k-distality and strong k-distality from [55], and give some basic properties and examples. Throughout this section, fix a complete L-theory T, and let $\mathbb{M} \models T$ be sufficiently saturated.

Let us recall the internal and external characterisations of distality. As stated in Theorem 2.7.3, these are equivalent.

Definition 5.3.1 (Internal characterisation of distality)**.** Say that T (and any $M \models T$) is *distal* if the following holds.

Let I_0, I_1, I_2 be (dense) infinite sequences without endpoints, whose elements are n-tuples. Let $a_0, a_1 \in \mathbb{M}^n$ such that $I_0 + a_0 + I_1 + I_2$ and $I_0 + I_1 + a_1 + I_2$ are indiscernible. Then $I_0 + a_0 + I_1 + a_1 + I_2$ is indiscernible.

Definition 5.3.2 (External characterisation of distality)**.** Say that T (and any $M \models T$) is *distal* if the following holds.

Let I_0, I_1 be (dense) infinite sequences without endpoints, whose elements are n-tuples. Let $a \in \mathbb{M}^n$ and $B \subseteq \mathbb{M}$ such that $I_0 + a + I_1$ is indiscernible and $I_0 + I_1$ is B-indiscernible. Then $I_0 + a + I_1$ is B-indiscernible.

Walker [55] generalised these two definitions as follows.

Definition 5.3.3. Let $k \in \mathbb{N}^+$. Say that T (and any $M \models T$) is k*-distal* if the following holds.

Let I_0, ..., I_{k+1} be (dense) infinite sequences without endpoints, whose elements are n-tuples. Let $a_0, ..., a_k \in \mathbb{M}^n$ such that, for all $0 \leq j \leq k$,

$$I_0 + a_0 + \cdots + I_{j-1} + a_{j-1} + I_j + I_{j+1} + a_{j+1} + \cdots + I_k + a_k + I_{k+1}$$

is indiscernible. Then $I_0 + a_0 + \cdots + I_k + a_k + I_{k+1}$ is indiscernible.

Definition 5.3.4. Let $k \in \mathbb{N}^+$. Say that T (and any $M \models T$) is *strongly* k*-distal* if the following holds.

Let I_0, I_1 be (dense) infinite sequences without endpoints, whose elements are n-tuples. Let $a \in \mathbb{M}^n$ and $B_1, ..., B_k \subseteq \mathbb{M}$ such that $I_0 + I_1$ is $B_1 \cdots B_k$-indiscernible and $I_0 + a + I_1$ is $B_1 \cdots B_{j-1} B_{j+1} \cdots B_k$-indiscernible for all $j \in [k]$. Then $I_0 + a + I_1$ is $B_1 \cdots B_k$-indiscernible.

Here, $B_1 \cdots B_k := B_1 \cup \cdots \cup B_k$, and so on. Note that both k-distality and strong k-distality say that the interaction of $k + 1$ objects can be controlled by the interactions of k-sized subsets of those objects. Indeed, k-distality says that if any k of $a_0, ..., a_k$ can be inserted to make an indiscernible sequence, then all $k + 1$ of them can be, and strong k-distality says that if $I_0 + I_1$ is indiscernible with respect to any k of $a, B_1, ..., B_k$, then it is indiscernible with respect to all $k+1$ of them.

It is straightforward to see that (strong) k-distality implies (strong) $(k+1)$-distality. Furthermore, as expected, strong k-distality implies k-distality.

Proposition 5.3.5. *Let $k \in \mathbb{N}^+$. If T is strongly k-distal, then it is k-distal.*

Proof. Let $I_0, ..., I_{k+1}$ and $a_0, ..., a_k$ be as in the hypothesis of Definition 5.3.3. For $j \in [k]$, let $B_j := I_{j-1}a_{j-1}$. Then $I_k + a_k + I_{k+1}$ is $B_1 \cdots B_{j-1}B_{j+1} \cdots B_k$-indiscernible for all $j \in [k]$, and $I_k + I_{k+1}$ is $B_1 \cdots B_k$-indiscernible. By strong k-distality, $I_k + a_k + I_{k+1}$ is $B_1 \cdots B_k$-indiscernible, that is, $I_0a_0 \cdots I_{k-1}a_{k-1}$-indiscernible. But now, since $I_0 + a_0 + \cdots + I_{k-1} + a_{k-1} + I_k$ is indiscernible, we have $I_0 + a_0 + \cdots + I_k + a_k + I_{k+1}$ is indiscernible as required. □

Say that a $(k+1)$-ary relation $\phi(y_1, ..., y_{k+1})$ is *degenerate* if it is equivalent in T to a Boolean combination of k-ary relations $\psi(y_1, ..., y_{i-1}, y_{i+1}, ..., y_{k+1})$. The following proposition provides a trivial source of (strongly) k-distal theories.

Proposition 5.3.6. *Let $k \in \mathbb{N}^+$. If every $(k+1)$-ary relation is degenerate, then T is (strongly) k-distal.*

Proof. Let I_0, I_1 and $a, B_1, ..., B_k$ be as in the hypothesis of Definition 5.3.4. Let $b_i \in B_i$ for $i \in [k]$, and let $\phi(x, y_1, ..., y_k) \in L$. Fixing any $a' \in I_0$, we show that $\phi(a, b_1, ..., b_k)$ is equivalent to $\phi(a', b_1, ..., b_k)$.

Since ϕ is degenerate, it is equivalent in T to a Boolean combination of some $\psi_1(x, y_1, ..., y_k), ..., \psi_l(x, y_1, ..., y_k)$, where each ψ_j either has no x-dependence or no y_i-dependence for some $i \in [k]$. Write $\tau(\psi_1, ..., \psi_l)$ for this Boolean combination. Then $\phi(a, b_1, ..., b_k)$ is equivalent to $\tau(\psi_1(a, b_1, ..., b_k), ..., \psi_l(a, b_1, ..., b_k))$, which is in turn equivalent to $\tau(\psi_1(a', b_1, ..., b_k), ..., \psi_l(a', b_1, ..., b_k))$ by indiscernibility. But this is equivalent to $\phi(a', b_1, ..., b_k)$ as required. □

Say that a (strongly) k-distal theory is *trivially* (strongly) k-distal if every $(k+1)$-ary relation is degenerate, and *non-trivially* (strongly) k-distal otherwise.

We give some examples, due to Walker [55], of where structures sit in the (strongly) k-distal hierarchy. For $k \in \mathbb{N}^+$, say that a theory is *strictly* (strongly) $(k+1)$-distal if it is (strongly) $(k+1)$-distal but not (strongly) k-distal.

Example 5.3.7. Let $k \geq 2$, and let RG_k be the $\{E\}$-theory of the random k-uniform hypergraph. That is, $(V, E) \models \mathrm{RG}_k$ if and only if it is an infinite k-uniform hypergraph such that, if $A, B \subseteq \binom{V}{k-1}$ are finite and disjoint, then there is $v \in V$ such that $E(a_1, ..., a_{k-1}, v)$ for all $\{a_1, ..., a_{k-1}\} \in A$ and $\neg E(b_1, ..., b_{k-1}, v)$ for all $\{b_1, ..., b_{k-1}\} \in B$. Then RG_k is strictly (strongly) k-distal.

There are k-partite and ordered k-partite versions of RG_k, both of which are strictly (strongly) k-distal — see [55].

Let RG_ω be the $\{E_2, E_3, ...\}$-theory such that $(V, E_2, E_3, ...) \models \mathrm{RG}_\omega$ if and only if, for all $k \geq 2$, (V, E_k) is an infinite k-uniform hypergraph, and if $A_r, B_r \subseteq \binom{V}{r}$ are finite and disjoint for all $r \in [k-1]$, then there is $v \in V$ such that for all $r \in [k-1]$, $E_{r+1}(a_1, ..., a_r, v)$ for all $\{a_1, ..., a_r\} \in A_r$ and $\neg E_{r+1}(b_1, ..., b_r, v)$ for all $\{b_1, ..., b_r\} \in B_r$. Then, for all $k \in \mathbb{N}^+$, RG_ω is not (strongly) k-distal.

The previous examples were IP. Let us give some NIP examples (still due to Walker [55]).

Example 5.3.8. Fix $t \in \mathbb{N}^+ \cup \{\infty\}$. Let T be the $\{R\}$-theory asserting that R is an equivalence relation with infinitely many equivalence classes, each of size t. Then T is stable and strictly (strongly) 2-distal.

Let T^* be the $\{R, <\}$-theory asserting, in addition to the above, that $<$ is a linear order without endpoints such that each equivalence class is dense in the domain. Then T^* is NIP, unstable, and strictly (strongly) 2-distal.

Example 5.3.9. For all $k \in \mathbb{N}^+$, the theory ACF (respectively, ACVF) of algebraically closed fields (respectively, valued fields) is not (strongly) k-distal. Specifying the characteristic does not change this fact.

In each of the examples above that are strictly (strongly) k-distal for some $k \geq 2$, they are trivially so: every $(k+1)$-ary relation is degenerate. The following construction by Goode [22, §2] is known to be 2-distal, and we show that it is non-trivially 2-distal.

Example 5.3.10. Let $M = (A_1, A_2, f : A_1 \times A_2 \to A_2)$ be a two-sorted structure in which A_1, A_2 are infinite sets and f describes a free action of A_1 on A_2, that is, the induced action on A_2 of the free group $F(A_1)$ generated by A_1 is free. By [22, §2], M is stable and 'trivial for freedom'. By [55, Theorem 8.16], a stable structure is trivial for freedom if and only if it is 2-distal, so M is (strictly) 2-distal. It is not known which $k \in \mathbb{N}^+$ (if it exists) is such that M is strictly strongly k-distal.

We claim that the ternary relation $f(x, y) = z$ is not degenerate, and so M is non-trivially 2-distal. To see this, first observe that every binary relation involving one variable from each sort is equivalent to either $\top$ or $\bot$. Indeed, since $F(A_1)$ acts freely on A_2, the orbits of this action are isomorphic copies of $F(A_1)$. Hence, given $u, u' \in A_1$ and $v, v' \in A_2$, it is easy to construct an automorphism of M sending (u, v) to (u', v'), and so $\mathrm{tp}(u, v/\emptyset) = \mathrm{tp}(u', v'/\emptyset)$.

Thus, if the ternary relation $f(x, y) = z$ were degenerate, it would be equivalent to a formula $\phi(y, z)$, which is clearly absurd. We conclude that $f(x, y) = z$ is not degenerate, and hence M is non-trivially 2-distal.

Apart from similar examples given in [22, §2], we are not aware of other non-trivially (strongly) k-distal structures for $k \geq 2$.

Problem 5.3.11. *Let $k \geq 2$. Find examples of non-trivially (strongly) k-distal theories.*

We are also interested in the following problem, posed in [55, Question 5.2].

Problem 5.3.12. *Let $k \geq 3$. Is there an NIP theory that is strictly (strongly) k-distal?*

We remark that Example 5.3.10 may provide an example of an NIP structure that is strictly strongly k-distal for some $k \geq 3$.

We finish this section by describing the relationship between the (strongly) k-distal hierarchy and the NIP$_k$ hierarchy. The following definition is due to Shelah in [46, Definition 2.4] and [45, Section 5(H)].

Definition 5.3.13. Let $\phi(x; y_1, ..., y_k) \in L$. Say that ϕ is *IP$_k$* if, for all $n \in \mathbb{N}^+$, there are $B_1 \subseteq \mathbb{M}^{y_1}$, ..., $B_k \subseteq \mathbb{M}^{y_k}$ of size n such that ϕ^* shatters $B_1 \times \cdots \times B_k$, that is, for all $S \subseteq B_1 \times \cdots \times B_k$, there is $a \in \mathbb{M}^x$ such that $\phi(a; B_1 \times \cdots \times B_k) = S$. Say that ϕ is *NIP$_k$* if it is not IP$_k$.

Say that T is NIP$_k$ if every formula $\phi(x; y_1, ..., y_k) \in L$ is NIP$_k$.

The following theorem is [55, Proposition 6.7], attributed to Chernikov.

Theorem 5.3.14. *Let $k \in \mathbb{N}^+$. If T is k-distal, then T is NIP$_k$.*

5.4 Higher-arity strong honest definitions

In this section, we derive k-strong honest definitions for $(k+1)$-ary formulas. Not only is it a key tool for the proof of our main result — a regularity lemma for NIP strongly k-distal structures — it is also a result of independent interest. Just as strong honest definitions have proved crucial in the development of distality, it is our hope that k-strong honest definitions will take on the same role in the development of k-distality.

Throughout this section, fix a complete L-theory T, and let $\mathbb{M} \models T$ be sufficiently saturated. We reiterate our abuse of notation that if y is an n-tuple with entries in a set Y (that is, $y \in Y^n$), we sometimes simply write $y \in Y$, but $X \subseteq Y$ always means $X \subseteq Y^1$.

Recall the definition of strong honest definitions for a binary formula.

Definition 5.4.1. Let $\phi(x; y) \in L$. A formula $\psi(x; z) \in L$ is a *strong honest definition* for ϕ if the following holds.

Let $B \subseteq M \models T$ with $2 \leq |B| < \infty$, and let $a \in M$. Then there is $c \in B$ such that, for all $b \in B$,

$$a \models \psi(x; c) \vdash \phi(x; b) \leftrightarrow \phi(a; b).$$

Recall also that T is distal if and only if every formula $\phi(x;y) \in L$ has a strong honest definition. In fact, we have the following.

Theorem 5.4.2. *The following are equivalent.*

(i) The theory T is distal.

(ii) Every formula $\phi(x;y) \in L$ has a strong honest definition.

(iii) Let $\phi(x;y) \in L$, $B \subseteq M \models T$ with $|B| \geq 2$, and $a \in M$. There is $\psi(x;z) \in L$ such that, for all finite $\bar{B} \subseteq B$, there is $c \in B$ such that, for all $b \in \bar{B}$,

$$a \models \psi(x;c) \vdash \phi(x;b) \leftrightarrow \phi(a;b).$$

Proof. That (i) is equivalent to (ii) is, modulo a compactness argument, [10, Theorem 21]. The proof can be used almost verbatim to show that (i) is equivalent to (iii). □

Statement (iii) gives a 'non-uniform' strong honest definition: one that depends not only on the formula $\phi(x;y)$ but also on the parameters a and B.

We wish to define k-strong honest definitions for $(k+1)$-ary formulas, where $k \in \mathbb{N}^+$, and use them to characterise k-distality. Walker proves the following result that makes a significant step towards this goal. Recall that, for a tuple $a = (a_1, ..., a_k)$ and $i \in [k]$, $a_{\neq i} := (a_1, ..., a_{i-1}, a_{i+1}, ..., a_k)$.

Theorem 5.4.3 [56, Theorem 9.18]**.** *Let $k \in \mathbb{N}^+$. The following are equivalent.*

(i) The theory T is strongly k-distal.

(ii) Let $\phi(x;y) \in L$ with $x = (x_1, ..., x_k)$, $B \subseteq M \models T$ with $|B| \geq 2$, and $a = (a_1, ..., a_k) \in M$. Then there is $\psi(x;z) \in L$ such that, for all finite $\bar{B} \subseteq B$, there is $c \in B$ such that, for all $b \in \bar{B}$,

$$a \models \{\psi(x;c)\} \cup \bigcup_{i=1}^{k} \mathrm{tp}(a_{\neq i}/B) \vdash \phi(x;b) \leftrightarrow \phi(a;b). \tag{1}$$

When $k = 1$, (1) simplifies to

$$a \models \psi(x; c) \vdash \phi(x; b) \leftrightarrow \phi(a; b);$$

that is, ψ is precisely a 'non-uniform' strong honest definition for (ϕ, a, B), as in statement (iii) of Theorem 5.4.2. It may therefore be tempting to define, for arbitrary k, ψ to be a 'non-uniform' k-strong honest definition for (ϕ, a, B).

This turns out to be unfruitful. A k-strong honest definition for ϕ should work to refine ϕ-types, but in (1), this is achieved not by ψ alone but by $\{\psi(x; c)\} \cup \bigcup_{i=1}^{k} \mathrm{tp}(a_{\neq i}/B)$. Now, we would like our k-strong honest definition to be a formula rather than a type. By compactness, we know that $\{\psi(x; c)\} \cup \bigcup_{i=1}^{k} \mathrm{tp}(a_{\neq i}/B)$ can be replaced by a finite subset in (1). That is, for all finite $\bar{B}$, there are $\psi_i(x_{\neq i}; c_i) \in \mathrm{tp}(a_{\neq i}/B)$ such that (1) can be replaced by

$$a \models \psi(x; c) \wedge \bigwedge_{i=1}^{k} \psi_i(x_{\neq i}; c_i) \vdash \phi(x; b) \leftrightarrow \phi(a; b).$$

It appears as if we have our 'non-uniform' k-strong honest definition $\psi \wedge \bigwedge_{i=1}^{k} \psi_i$, but the reader must not forget that the choice of the ψ_i here depends on $\bar{B} \subseteq B$. To remove this dependence, we need to do some work. Our first goal is the following theorem.

Theorem 5.4.4. *Let $k \in \mathbb{N}^+$. The following are equivalent.*

(i) The theory T is strongly k-distal.

(ii) Let $\phi(x_1, ..., x_k; y) \in L$, $B \subseteq M \models T$ with $|B| \geq 2$, and $a = (a_1, ..., a_k) \in M$. Write $x := (x_1, ..., x_k)$. Then there are $\psi_i(x_{\neq i}, y, z_i) \in L$ for $i \in [k]$, $\psi_{k+1}(x; z_{k+1}) \in L$, and $N \in \mathbb{N}$, such that for all finite $\bar{B} \subseteq B$, there are $c_1^{(j)}, ..., c_{k+1}^{(j)} \in B$ for $j \in [N]$, such that for all $b \in \bar{B}$, there is $j \in [N]$ with

$$a \models \psi_{k+1}(x, c_{k+1}^{(j)}) \wedge \bigwedge_{i=1}^{k} \psi_i(x_{\neq i}, b, c_i^{(j)}) \vdash \phi(x; b) \leftrightarrow \phi(a; b).$$

(iii) Let $\phi(x_1, ..., x_k; y) \in L$, $B \subseteq M \models T$ with $|B| \geq 2$, and $a = (a_1, ..., a_k) \in M$.

Write $x := (x_1, ..., x_k)$. *Let* $(M', B') \succcurlyeq (M, B)$ *be* $|M|^+$*-saturated. Then there are* $\psi_i(x_{\neq i}, y, z_i) \in L$ *for* $i \in [k]$, $\psi_{k+1}(x; z_{k+1}) \in L$, $N \in \mathbb{N}$, *and* $c_1^{(j)}, ..., c_{k+1}^{(j)} \in B'$ *for* $j \in [N]$, *such that for all* $b \in B$, *there is* $j \in [N]$ *with*

$$a \models \psi_{k+1}(x, c_{k+1}^{(j)}) \wedge \bigwedge_{i=1}^{k} \psi_i(x_{\neq i}, b, c_i^{(j)}) \vdash \phi(x; b) \leftrightarrow \phi(a; b).$$

Note that, in (ii) and (iii) of Theorem 5.4.4, $(\psi_1, ..., \psi_{k+1})$ acts as a 'non-uniform' k-strong honest definition for (ϕ, a, B); recall that 'non-uniformity' refers to its dependence on a and B. After proving Theorem 5.4.4, we will bootstrap it to generate 'uniform' k-strong honest definitions for ϕ — ones that depend only on ϕ and not a or B — under an extra NIP assumption. These will be defined precisely in Definition 5.4.9.

Remark 5.4.5. In (ii) and (iii) of Theorem 5.4.4, the awkward parameter $N \in \mathbb{N}$ arises from a coding process, when we construct $\psi_1, ..., \psi_k$ each as a code for multiple formulas. We are not able to obtain a statement without such N. We will comment on this further after defining 'uniform' k-strong honest definitions (which will also make reference to such N) in Definition 5.4.9.

Towards proving Theorem 5.4.4, we appeal to the following result of Walker. For a type $q \in S(A)$ and $A_0 \subseteq A$, write $q|A_0 := q \cap L(A_0)$.

Lemma 5.4.6 [56, Lemma 9.12]**.** *Suppose* T *is strongly* k*-distal. Let* $B \subseteq M \models T$ *with* $|B| \geq 2$, *and let* $a = (a_1, ..., a_k) \in M$. *Let* $p := \operatorname{tp}(a/\mathbb{M})$, *and for all* $i \in [k]$, *let* $p_{\neq i} := \operatorname{tp}(a_{\neq i}/\mathbb{M})$. *Let* $(M', B') \succcurlyeq (M, B)$ *be* $|M|^+$*-saturated. Then, for all* $q \in S(\mathbb{M})$ *finitely satisfiable over* B,

$$p|B' \cup \bigcup_{i=1}^{k} (p_{\neq i} \otimes q)|B' \vdash (p \otimes q)|B'.$$

We require the following lemmas about finitely satisfiable types. For a tuple of variables y and $B \subseteq B' \subseteq \mathbb{M}$, where B is small but B' is not necessarily small, write $S_y^{\mathrm{fs}}(B', B) := \{p(y) \in S_y(B') : p \text{ is finitely satisfiable over } B\}$.

Lemma 5.4.7. *Let y be a tuple of variables. Let $p(y)$ be a partial type that is finitely satisfiable over a small set $B \subseteq \mathbb{M}$. Then p extends to a (complete) global type that is finitely satisfiable over B. Thus, if $B \subseteq B'$ for some not necessarily small set $B' \subseteq \mathbb{M}$, then*

$$S_y^{fs}(B'; B) = \{q|B' : q \in S_y^{fs}(\mathbb{M}, B)\}.$$

Proof. We follow the argument in [48, Section 2.2]. Since p is finitely satisfiable over B, we can extend the set $\{\phi(B) : \phi(y) \in p\}$ to an ultrafilter $\mathcal{U}$ on B^y. Then p extends to the global type $\{\phi(y) \in L(\mathbb{M}) : \phi(B) \in \mathcal{U}\}$, which is finitely satisfiable over B. □

Lemma 5.4.8. *Let x, y be tuples of variables. Let $a \in \mathbb{M}^x$ and $B \subseteq B' \subseteq \mathbb{M}$, where B is small but B' is not necessarily small. Let $p(x) = \mathrm{tp}(a/\mathbb{M})$. Then*

(i) If $q \in S_y(\mathbb{M})$ is B-invariant, then $p \otimes q = q \otimes p = \{\phi(x, y) : \phi(a, y) \in q\}$.

(ii) The set $S_{a,y}^{fs}(B'; B) := \{(p \otimes q)|B' : q \in S_y^{fs}(\mathbb{M}; B)\}$ is closed in $S_{x,y}(B')$.

Proof. (i) It suffices to show that $p \otimes q = \{\phi(x, y) : \phi(a, y) \in q\}$. Let $\phi(x, y) \in L(C)$, where $\{a\} \cup B \subseteq C$, and let $b \models q|C$. Then

$$\phi(x, y) \in p \otimes q \Leftrightarrow \phi(x, b) \in p \Leftrightarrow \mathbb{M} \models \phi(a, b) \Leftrightarrow \phi(a, y) \in q.$$

(ii) Without loss of generality, suppose $B \neq \emptyset$. It suffices to show that for $r \in S_{x,y}(B')$, $r \in S_{a,y}^{\mathrm{fs}}(B'; B)$ if and only if whenever $\phi(x, y) \in L(B')$ is such that $\{\phi(a, y)\}$ is not finitely satisfiable over B, then $\neg\phi(x, y) \in r$.

Suppose $r \in S_{a,y}^{\mathrm{fs}}(B'; B)$, so we have that $r = (p \otimes q)|B'$ for some $q \in S_y^{\mathrm{fs}}(\mathbb{M}; B)$. If $\phi(x, y) \in L(B')$ is such that $\phi(x, y) \in r$, then $\phi(a, y) \in q$ and so $\{\phi(a, y)\}$ is finitely satisfiable over B. Conversely, suppose whenever $\phi(x, y) \in L(B')$ is such that $\{\phi(a, y)\}$ is not finitely satisfiable over B, then $\neg\phi(x, y) \in r$. Then $r(a, y)$ is finitely satisfiable over B, so extends to some $q \in S_y^{\mathrm{fs}}(\mathbb{M}; B)$ by Lemma 5.4.7. But then $r = (p \otimes q)|B'$: these are complete types such that if $\phi(x, y) \in r$, then $\phi(a, y) \in r(a, y) \subseteq q$, and so $\phi(x, y) \in p \otimes q$. □

We are now ready to prove Theorem 5.4.4.

Proof of Theorem 5.4.4. Firstly, we argue that (i) implies (iii). Suppose T is strongly k-distal. Let $\phi(x_1, ..., x_k; y) \in L$, $B \subseteq M \models T$ with $|B| \geq 2$, and $a = (a_1, ..., a_k) \in M$. Write $x := (x_1, ..., x_k)$. Let $(M', B') \succcurlyeq (M, B)$ be $|M|^+$-saturated. Let $p := \mathrm{tp}(a/\mathbb{M})$, and for all $i \in [k]$, let $p_{\neq i} := \mathrm{tp}(a_{\neq i}/\mathbb{M})$.

Let $q \in S_y(\mathbb{M})$ be finitely satisfiable over B. By Lemma 5.4.6, there is $\varepsilon_q \in \{0, 1\}$ such that

$$r^q := p|B' \cup \bigcup_{i=1}^{k} (p_{\neq i} \otimes q)|B' \vdash \phi^{\varepsilon_q}(x; y).$$

By compactness, there are $\psi^q_{k+1}(x, c^q_{k+1}) \in p|B'$ and $\psi^q_i(x_{\neq i}, y, c^q_i) \in (p_{\neq i} \otimes q)|B'$ for $i \in [k]$ such that

$$\psi^q := \psi^q_{k+1}(x, c^q_{k+1}) \wedge \bigwedge_{i=1}^{k} \psi^q_i(x_{\neq i}, y, c^q_i) \vdash \phi^{\varepsilon_q}(x; y).$$

Now, $\{[\psi^q] : q \in S^{\mathrm{fs}}_y(\mathbb{M}; B)\}$ is an open cover for $S^{\mathrm{fs}}_{a,y}(B'; B)$. By Lemma 5.4.8, $S^{\mathrm{fs}}_{a,y}(B'; B)$ is a closed, hence compact, subset of $S_{x,y}(B')$, so the open cover above has a finite subcover $\{[\psi^q] : q \in Q\}$.

For all $b \in B$, we have $\mathrm{tp}(b/\mathbb{M}) \in S^{\mathrm{fs}}_y(\mathbb{M}; B)$, so there is $q(b) \in Q$ such that $\psi^{q(b)} \in p \otimes \mathrm{tp}(b/\mathbb{M}) = \mathrm{tp}(a, b/\mathbb{M})$, whence

$$a \models \psi^{q(b)}_{k+1}(x, c^{q(b)}_{k+1}) \wedge \bigwedge_{i=1}^{k} \psi^{q(b)}_i(x_{\neq i}, b, c^{q(b)}_i) \vdash \phi^{\varepsilon_{q(b)}}(x; b);$$

in particular, $\models \phi^{\varepsilon_{q(b)}}(a; b)$, and so

$$a \models \psi^{q(b)}_{k+1}(x, c^{q(b)}_{k+1}) \wedge \bigwedge_{i=1}^{k} \psi^{q(b)}_i(x_{\neq i}, b, c^{q(b)}_i) \vdash \phi(x; b) \leftrightarrow \phi(a; b).$$

For all $i \in [k+1]$, we can code $(\psi^q_i : q \in Q)$ into a single formula as follows: for all $b \in B$,

$$a \models \bigvee_{q \in Q} \left(\psi_{k+1}^q(x, c_{k+1}^q) \wedge u_{k+1}^q = v_{k+1}^q\right) \wedge \bigwedge_{i=1}^{k} \bigvee_{q \in Q} \left(\psi_i^q(x_{\neq i}, b, c_i^q) \wedge u_i^q = v_i^q\right)$$
$$\vdash \phi(x; b) \leftrightarrow \phi(a; b),$$

for any $u_1^q, ..., u_{k+1}^q, v_1^q, ..., v_{k+1}^q \in B'$ such that for all $i \in [k+1]$, $u_i^q = v_i^q$ if and only if $q = q(b)$; such u^q, v^q exist since $|B| \geq 2$. Therefore, (iii) holds.

Next, we argue that (iii) implies (ii). Our argument expands that in [10, Corollary 9]. Suppose (iii) holds. Let $\phi(x_1, ..., x_k; y) \in L$, $B \subseteq M \models T$ with $|B| \geq 2$, and $a \in M$. Let $(M', B') \succcurlyeq (M, B)$ be any $|M|^+$-saturated elementary extension, and let $\psi_1, ..., \psi_{k+1}$ and N be given by (iii). Then, for all finite $\bar{B} \subseteq B$, (M', B') satisfies the first-order formula saying that there are $c_1^{(j)}, ..., c_{k+1}^{(j)} \in B'$ for $j \in [N]$ satisfying the conclusion of (ii). Since $(M', B') \succcurlyeq (M, B)$ is an elementary extension, (M, B) satisfies the same formula with B' replaced by B throughout, so (ii) holds.

Finally, we argue that (ii) implies (i). By Theorem 5.4.3, it suffices to show that (ii) implies statement (ii) of Theorem 5.4.3. Let $\phi(x_1, ..., x_k; y) \in L$, $B \subseteq M \models T$ with $|B| \geq 2$, and $a = (a_1, ..., a_k) \in M$. Let $\psi_{k+1}(x; z_{k+1})$ and N be given by (ii), and let $\psi(x; z^{(1)}, ..., z^{(N)}) := \bigvee_{j=1}^{N} \psi_{k+1}(x; z^{(j)})$. Then, for all finite $\bar{B} \subseteq B$, there are $c^{(j)} \in B$ for $j \in [N]$ such that, for all $b \in \bar{B}$,

$$a \models \{\psi(x; c^{(1)}, ..., c^{(N)})\} \cup \bigcup_{i=1}^{k} \mathrm{tp}(a_{\neq i}/B) \vdash \phi(x; b) \leftrightarrow \phi(a; b)$$

as required. □

Our next goal is to bootstrap Theorem 5.4.4 to generate 'uniform' k-strong honest definitions. It is now clear what these should look like.

Definition 5.4.9. Let $\phi(x_1, ..., x_k; y) \in L$; write $x := (x_1, ..., x_k)$. Let $N \in \mathbb{N}$. A $(k+1)$-tuple of L-formulas $(\psi_i(x_{\neq i}, y, z_i) : i \in [k])^\frown(\psi_{k+1}(x, z_{k+1}))$ is a *k-strong honest definition* for ϕ of *degree* N if the following holds.

Let $B \subseteq M \models T$ with $2 \leq |B| < \infty$ and $a = (a_1, ..., a_k) \in M$. Then there are $c_1^{(j)}, ..., c_{k+1}^{(j)} \in B$ for $j \in [N]$ such that for all $b \in B$, there is $j \in [N]$ with

$$a \models \psi_{k+1}(x, c_{k+1}^{(j)}) \wedge \bigwedge_{i=1}^{k} \psi_i(x_{\neq i}, b, c_i^{(j)}) \vdash \phi(x; b) \leftrightarrow \phi(a; b).$$

Remark 5.4.10. By compactness, $(\psi_i(x_{\neq i}, y, z_i) : i \in [k])^\frown(\psi_{k+1}(x, z_{k+1}))$ is a *k-strong honest definition* for ϕ of *degree* N if and only if the following holds.

Let $B \subseteq M \models T$ with $|B| \geq 2$ and $a \in M$. Let $(M', B') \succcurlyeq (M, B)$ be $|M|^+$-saturated. Then there are $c_1^{(j)}, ..., c_{k+1}^{(j)} \in B'$ for $j \in [N]$ such that for all $b \in B$, there is $j \in [N]$ with

$$a \models \psi_{k+1}(x, c_{k+1}^{(j)}) \wedge \bigwedge_{i=1}^{k} \psi_i(x_{\neq i}, b, c_i^{(j)}) \vdash \phi(x; b) \leftrightarrow \phi(a; b).$$

Note that a 1-strong honest definition of degree 1 is a strong honest definition, and if $\psi(x, z)$ is a 1-strong honest definition of degree $N > 1$, then $\bigwedge_{j=1}^{N} \psi(x, z_j)$ is a (1-)strong honest definition (of degree 1). Hence, when defining strong honest definitions, we did not need to make reference to degrees. Sadly, when $k \geq 2$, this trick does not work for k-strong honest definitions, and we do not know whether the reference to degrees can be eliminated.

Problem 5.4.11. *If $\phi(x_1, ..., x_k; y) \in L$ has a k-strong honest definition, does it have a k-strong honest definition of degree 1?*

As we shall see, the reference to degrees does not seem to affect the efficacy of k-strong honest definitions, it merely makes the proofs more awkward.

The main result of this section is as follows.

Theorem 5.4.12. *Let T be NIP and let $k \in \mathbb{N}^+$. The following are equivalent.*

(i) The theory T is strongly k-distal.

(ii) Every $\phi(x_1, ..., x_k; y) \in L$ has a k-strong honest definition.

Our proof bootstraps the 'non-uniform' Theorem 5.4.4, following the strategy in [10, Theorem 21]. The ingredient that necessitates NIP is the following fact.

Fact 5.4.13 [36, Theorem 4; (p,q)-theorem]**.** *For all $p \geq q \in \mathbb{N}^+$, there is $K = K(p,q) \in \mathbb{N}^+$ such that the following holds.*

Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a finite family with $\mathrm{VC}^*(\mathcal{F}) \leq q$, and suppose $\mathcal{F}$ has the (p,q)-property: if $\mathcal{F}_0 \subseteq \mathcal{F}$ has size p, then there is a q-element subset of $\mathcal{F}_0$ with non-empty intersection. Then there is $Y \subseteq X$ of size at most $K(p,q)$ such that $F \cap Y \neq \emptyset$ for all $F \in \mathcal{F}$.

That K only depends on p and q in Fact 5.4.13 is not stated explicitly in [36, Theorem 4]; see [10, Remark 7] for an argument to this end.

We prove a simple compactness lemma.

Lemma 5.4.14. *Let T be strongly k-distal and $\phi(x_1, ..., x_k; y) \in L$. Write $x := (x_1, ..., x_k)$. For all $(k+1)$-tuples of L-formulas $\Psi := (\psi_i(x_{\neq i}, y, z_i) : i \in [k])^\frown(\psi_{k+1}(x, z_{k+1}))$ and $N \in \mathbb{N}$, fix $m_{\Psi,N} \in \mathbb{N}$. Then there are $N_1, ..., N_H \in \mathbb{N}$ and $\Psi^{(h)} := (\psi_i^{(h)}(x_{\neq i}, y, z_i^{(h)}) : i \in [k])^\frown(\psi_{k+1}^{(h)}(x, z_{k+1}^{(h)}))$ for $h \in [H]$ such that the following holds.*

Let $B \subseteq M \models T$ with $|B| \geq 2$, and let $a \in M$. Then there is $h \in [H]$ such that, for all $\bar{B} \subseteq B$ of size at most $m_{\Psi^{(h)},N_h}$, there are $c_1^{(j)}, ..., c_{k+1}^{(j)} \in B$ for $j \in [N_h]$ such that for all $b \in \bar{B}$, there is $j \in [N_h]$ with

$$a \models \psi_{k+1}^{(h)}(x, c_{k+1}^{(j)}) \wedge \bigwedge_{i=1}^{k} \psi_i^{(h)}(x_{\neq i}, b, c_i^{(j)}) \vdash \phi(x; b) \leftrightarrow \phi(a; b).$$

Proof. Let P be a new unary predicate. Let T' be the theory in the language $L' := L \cup \{P, a\}$ saying that if $(M, B, a) \models T'$, then $M \models T$, $|B| \geq 2$, and for every $(k+1)$-tuple of L-formulas $\Psi := (\psi_i(x_{\neq i}, y, z_i) : i \in [k])^\frown(\psi_{k+1}(x, z_{k+1}))$ and $N \in \mathbb{N}$, there is $\bar{B} \subseteq B$ of size at most $m_{\Psi,N}$, for which there are no $c_1^{(j)}, ..., c_{k+1}^{(j)} \in B$ for $j \in [N]$ such that for all $b \in \bar{B}$, there is $j \in [N]$ with

$$a \models \psi_{k+1}(x, c_{k+1}^{(j)}) \wedge \bigwedge_{i=1}^{k} \psi_i(x_{\neq i}, b, c_i^{(j)}) \vdash \phi(x; b) \leftrightarrow \phi(a; b).$$

By Theorem 5.4.4, T' is inconsistent. □

We now prove Theorem 5.4.12.

Proof of Theorem 5.4.12. That (ii) implies (i) follows immediately from Theorem 5.4.4, so we prove that (i) implies (ii).

Suppose T is strongly k-distal. Let $\phi(x_1, ..., x_k; y) \in L$, $x := (x_1, ..., x_k)$, and $d := |y|$. For each $(k+1)$-tuple of L-formulas $\Psi := (\psi_i(x_{\neq i}, y, z_i) : i \in [k])^\frown(\psi_{k+1}(x, z_{k+1}))$ and $N \in \mathbb{N}$, let $\theta_{\Psi,N}(z^{(1)}, ..., z^{(N)}; x, y)$ be the following formula, where for all $j \in [N]$ we have $z^{(j)} = (z_1^{(j)}, ..., z_{k+1}^{(j)})$:

$$\bigvee_{j=1}^{N} \left(\psi_{k+1}(x, z_{k+1}^{(j)}) \wedge \bigwedge_{i=1}^{k} \psi_i(x_{\neq i}, y, z_i^{(j)}) \right.$$

$$\left. \wedge \, \forall x' \left(\left(\psi_{k+1}(x', z_{k+1}^{(j)}) \wedge \bigwedge_{i=1}^{k} \psi_i(x'_{\neq i}, y, z_i^{(j)}) \right) \to (\phi(x'; y) \leftrightarrow \phi(x; y)) \right) \right);$$

let $m_{\Psi,N} := d \cdot \mathrm{VC}^*(\theta_{\Psi,N}) \in \mathbb{N}$. By standard coding tricks, we may apply Lemma 5.4.14 under the assumption that $H = 1$. (Otherwise, the following proof produces $(\psi_1^{(h)}, ..., \psi_{k+1}^{(h)})_{h \in [H]}$ such that, for all a and B, there is $h \in [H]$ such that $(\psi_1^{(h)}, ..., \psi_{k+1}^{(h)})$ works; we then code, for each $i \in [k+1]$, the formulas $(\psi_i^{(h)} : h \in [H])$ into a single formula ψ_i such that, for all a and B, $(\psi_1, ..., \psi_{k+1})$ works.)

Applying Lemma 5.4.14 with $H = 1$, we obtain $\Psi^{(1)} := (\psi_i^{(1)}(x_{\neq i}, y, z_i^{(1)}) : i \in [k])^\frown(\psi_{k+1}^{(1)}(x, z_{k+1}^{(1)}))$ — from which we shall henceforth drop the superscripts — and $N_1 =: N \in \mathbb{N}$. Let $e := |z_1| + \cdots + |z_{k+1}|$. Let $B \subseteq M \models T$ with $2 \leq |B| < \infty$, and let $a \in M^x$. For $b \in B^y$, $\theta_{\Psi,N}(B^{eN}; a, b)$ is the set

$$\left\{ (c^{(1)}, ..., c^{(N)}) \in B^{eN} : \right.$$

$$\left. a \models \psi_{k+1}(x, c_{k+1}^{(j)}) \wedge \bigwedge_{i=1}^{k} \psi_i(x_{\neq i}, b, c_i^{(j)}) \vdash \phi(x; b) \leftrightarrow \phi(a; b) \text{ for some } j \in [N] \right\}.$$

Observe that the family $\mathcal{F} := \{\theta_{\Psi,N}(B^{eN}; a, b) : b \in B^y\} \subseteq \mathcal{P}(B^{eN})$ has the $(m_{\Psi,N}/d, m_{\Psi,N}/d)$-property, that is, any subset of $\mathcal{F}$ of size $m_{\Psi,N}/d$ has non-empty intersection. Indeed, given $b_1, ..., b_{m_{\Psi,N}/d} \in B^y$, there is $\bar{B} \subseteq B$ of size at most $m_{\Psi,N}$ such that $b_1, ..., b_{m_{\Psi,N}/d} \in \bar{B}^y$, and our choice of Ψ (given by Lemma 5.4.14) is precisely such that there is $(c^{(1)}, ..., c^{(N)}) \in \bigcap_{i\in[m_{\Psi,N}/d]} \theta_{\Psi,N}(B^{eN}; a, b_i)$.

By Lemma 2.2.11, $\mathrm{VC}^*(\mathcal{F}) \leq \mathrm{VC}^*(\theta_{\Psi,N}) = m_{\Psi,N}/d$. By the (p,q)-theorem (Fact 5.4.13), there is $Y \subseteq B^{eN}$ of size at most $K = K(m_{\Psi,N}/d, m_{\Psi,N}/d) \in \mathbb{N}$, such that $F \cap Y \neq \emptyset$ for all $F \in \mathcal{F}$. That is, there are $c^{(1)}, ..., c^{(KN)} \in B^e$ such that for all $b \in B^y$, there is $j \in [KN]$ with

$$a \models \psi_{k+1}(x, c^{(j)}_{k+1}) \wedge \bigwedge_{i=1}^{k} \psi_i(x_{\neq i}, b, c^{(j)}_i) \vdash \phi(x; b) \leftrightarrow \phi(a; b).$$

Since the above holds for all $B \subseteq M \models T$ with $2 \leq |B| < \infty$ and $a \in M$, we conclude that $(\psi_1, ..., \psi_{k+1})$ is a k-strong honest definition for ϕ of degree KN. □

We have shown that, under a global NIP assumption, the existence of k-strong honest definitions characterises strong k-distality. Since there are strongly k-distal theories that are not NIP, it is natural to pose the following problem.

Problem 5.4.15. *Can the NIP assumption be removed from Theorem 5.4.12?*

Since (strongly) k-distal theories are NIP$_k$ (Theorem 5.3.14), one may hope that all uses of NIP can be replaced with uses of NIP$_k$. However, this requires an NIP$_k$ version of the (p,q)-theorem, which is yet to be developed. Even the statement of such a theorem is not obvious.

Since it is open whether k-distality is equivalent to strong k-distality, we pose the following problem.

Problem 5.4.16. *Can the assumption of strong k-distality be replaced by k-distality in Theorem 5.4.12? If not, do k-distal theories admit a (necessarily weaker) version of k-strong honest definitions?*

The regularity lemma we shall derive in the next section is for all hypergraphs defined by a formula $\phi(x_1, ..., x_k; y)$ with a k-strong honest definition in an NIP theory. In particular, we do not require the full strength of strong k-distality, since we only require the formula in question to have a k-strong honest definition, rather than all formulas.

5.5 Regularity lemma

We finally come to our *pièce de résistance*, a regularity lemma for formulas $\phi(x_1, ..., x_k; x_{k+1})$ with a k-strong honest definition in an NIP theory.

Remark 5.5.1. We had previously indexed the variables in ϕ as $x_1, ..., x_k, y$, which emphasises the different roles of the x- and y-variables in the k-strong honest definition. In this section, our main result is a regularity lemma for the $(k+1)$-uniform hypergraph $\phi(x_1, ..., x_{k+1})$, where, a priori, none of the variables $x_1, ..., x_{k+1}$ are special. Thus, it is sensible to index the variables as $x_1, ..., x_{k+1}$ (note, however, that x_{k+1} still plays a special role in the proof). This has the added bonus of cleaner presentation. In particular, writing $x := (x_1, ..., x_{k+1})$, a k-strong honest definition for ϕ has the form $(\psi_i(x_{\neq i}, z_i) : i \in [k+1])$.

5.5.1 Main proof

Throughout this subsection, we fix the following.

- An NIP L-theory T and models $M, \mathbb{M} \models T$ with $\mathbb{M}$ sufficiently saturated.
- A formula $\phi(x_1, ..., x_k; x_{k+1}) \in L$; write $x := (x_1, ..., x_{k+1})$.
- A k-strong honest definition $(\psi_i(x_{\neq i}, z_i) : i \in [k+1])$ for ϕ of degree N.

Our goal is the following theorem.

Theorem 5.5.2. *(T is NIP.) For all $\delta \in (0, 1]$, there are $\theta_i(x_{\neq i}, z_i) \in L$ for $i \in [k+1]$ and a natural number $K \leq poly_{\phi,\psi_1,...,\psi_{k+1},N}(\delta^{-1})$ such that the following holds.*

Let $\mu(x_{\neq k+1})$ and $\nu(x_{k+1})$ be Keisler measures, with $\nu(x_{k+1})$ generically stable over M, and let $\omega(x) := \nu(x_{k+1}) \otimes \mu(x_{\neq k+1})$. Then there are partitions $\mathcal{P}_i$ of $\mathbb{M}^{x_{\neq i}}$ for $i \in [k+1]$, each of size at most K, such that:

(i) For all $i \in [k+1]$ and $P_i \in \mathcal{P}_i$, there is $c_i \in M^{z_i}$ such that $P_i = \theta_i(x_{\neq i}, c_i)$;

(ii) $\sum \omega(P_1 \wedge \cdots \wedge P_{k+1}) \leq \delta$, where the sum ranges over all $(P_1, ..., P_{k+1}) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_{k+1}$ such that $P_1 \wedge \cdots \wedge P_{k+1}$ is not ϕ-homogeneous.

In (ii), the notation of $P_1 \wedge \cdots \wedge P_{k+1}$ can be understood by conflating P_i with the formula that defines it, given by (i). That is,

$$P_1 \wedge \cdots \wedge P_{k+1} = \{x \in \mathbb{M} : x_{\neq i} \in P_i \text{ for all } i \in [k+1]\}.$$

Our proof strategy follows that of [11, Theorem 5.8], but it is more efficient — we will discuss this after the proof.

Definition 5.5.3. Let $B \subseteq \mathbb{M}$ not necessarily be small. A *B-definable cell* is a set $\gamma \subseteq \mathbb{M}^x$ of the form $\psi_1(x_{\neq 1}, c_1) \wedge \cdots \wedge \psi_{k+1}(x_{\neq k+1}, c_{k+1})$, where $c := (c_1, ..., c_{k+1}) \in B$; write γ_c for this set. Write $\mathcal{G}_B$ for the set of all B-definable cells.

For $a, c^{(1)}, ..., c^{(N)} \in \mathbb{M}$, let $F_{a,c^{(1)},...,c^{(N)}}$ be the set

$$\{b \in \mathbb{M} : a \models (x_{\neq k+1}, b) \in \gamma_{c^{(j)}} \vdash \phi(x_{\neq k+1}; b) \leftrightarrow \phi(a; b) \text{ for some } j \in [N]\}.$$

A tuple $\Gamma = (\gamma_{c^{(1)}}, ..., \gamma_{c^{(N)}}) \in \mathcal{G}_B^N$ of B-definable cells is *B-complete* if there is $a \in \mathbb{M}$ such that $B^{|x_{k+1}|} \subseteq F_{a,c^{(1)},...,c^{(N)}}$, in which case we say that Γ is B-complete *with respect to a*. For $\Gamma = (\gamma_{c^{(1)}}, ..., \gamma_{c^{(N)}}) \in \mathcal{G}_B^N$, we write $\gamma \in \Gamma$ to mean $\gamma = \gamma_{c^{(j)}}$ for some $j \in [N]$.

Remark 5.5.4. Since $(\psi_i(x_{\neq i}, z_i) : i \in [k+1])$ is a k-strong honest definition for ϕ of degree N, for all $a \in \mathbb{M}$ and $B \subseteq \mathbb{M}$ with $2 \leq |B| < \infty$, there are $c^{(1)}, ..., c^{(N)} \in B$ such that $(\gamma_{c^{(1)}}, ..., \gamma_{c^{(N)}}) \in \mathcal{G}_B^N$ is B-complete with respect to a.

We prove the following 'cutting lemma'.

Proposition 5.5.5. *For all $r \geq 1$, there is a finite set $B \subseteq M$ with $2 \leq |B| = O_{\phi,\psi_1,...,\psi_{k+1},N}(r^2 \log 2r)$ such that the following holds.*

Let $\nu(x_{k+1})$ be a Keisler measure, generically stable over M. For $a \in \mathbb{M}$ and $c^{(1)}, ..., c^{(N)} \in B$, if $(\gamma_{c^{(1)}}, ..., \gamma_{c^{(N)}}) \in \mathcal{G}_B^N$ is B-complete with respect to a, then $\nu(F_{a,c^{(1)},...,c^{(N)}}) \geq 1 - \frac{1}{r}$.

Proof. Let $d := |x_{k+1}|$. Applying Proposition 2.6.11 to the definable family $\mathcal{F} := \{F_{a,c^{(1)},\dots,c^{(N)}} : a, c^{(1)}, \dots, c^{(N)} \in \mathbb{M}\}$, there is $S \subseteq M^d$ of size $O_{\phi,\psi_1,\dots,\psi_{k+1},N}(r^2 \log 2r)$, such that, for all $F \in \mathcal{F}$, $|\nu(F) - \mathrm{Av}(S;F)| \leq \frac{1}{r}$.

Choose $B \subseteq M$ such that B contains all of singletons appearing in S and $2 \leq |B| \leq d|S| + 2$. If $(\gamma_{c^{(1)}}, \dots, \gamma_{c^{(N)}}) \in \mathcal{G}_B^N$ is B-complete with respect to a, then $S \subseteq B^d \subseteq F_{a,c^{(1)},\dots,c^{(N)}}$, that is, $\mathrm{Av}(S; F_{a,c^{(1)},\dots,c^{(N)}}) = 1$, and so $\nu(F_{a,c^{(1)},\dots,c^{(N)}}) \geq 1 - \frac{1}{r}$. □

Definition 5.5.6. Let $Z \subseteq \mathbb{M}^x$, $a \in \mathbb{M}^{x_{\neq k+1}}$, and $b \in \mathbb{M}^{x_{k+1}}$. Write $Z|_a := \{b' \in \mathbb{M}^{x_{k+1}} : (a, b') \in Z\}$ and $Z|_b := \{a' \in \mathbb{M}^{x_{\neq k+1}} : (a', b) \in Z\}$.

We are now ready to prove Theorem 5.5.2.

Proof of Theorem 5.5.2. Apply Proposition 5.5.5 with $r := \frac{1}{\delta} \geq 1$ to obtain $B \subseteq M$ with $2 \leq |B| = O_{\phi,\psi_1,\dots,\psi_{k+1},N}(\delta^{-2} \log 2\delta^{-1}) = O_{\phi,\psi_1,\dots,\psi_{k+1},N}(\delta^{-3})$. We have $|\mathcal{G}_B| \leq |B|^l$ for $l := |z_1| + \cdots + |z_{k+1}|$.

For $\gamma \in \mathcal{G}_B$, let $D_\gamma = \{b \in \mathbb{M}^{x_{k+1}} : \gamma|_b \subseteq \phi(x_{\neq k+1}; b) \text{ or } \gamma|_b \subseteq \neg\phi(x_{\neq k+1}; b)\}$. Let $G := \bigvee_{\gamma \in \mathcal{G}_B} \gamma \wedge D_\gamma$. We claim that $\omega(G) \geq 1 - \delta$. It suffices to show that, for all $a \in \mathbb{M}^{x_{\neq k+1}}$, $\nu\left(G|_a\right) \geq 1 - \delta$.

Fix $a \in \mathbb{M}^{x_{\neq k+1}}$. By Remark 5.5.4, there is $\Gamma = (\gamma_{c^{(1)}}, \dots, \gamma_{c^{(N)}}) \in \mathcal{G}_B^N$ which is B-complete with respect to a. It suffices to show that $(\bigvee_{\gamma \in \Gamma} \gamma \wedge D_\gamma)|_a \supseteq F_{a,c^{(1)},\dots,c^{(N)}}$, as $\nu(F_{a,c^{(1)},\dots,c^{(N)}}) \geq 1 - \delta$ by our choice of $B \subseteq M$ from Proposition 5.5.5. So, suppose $b \in F_{a,c^{(1)},\dots,c^{(N)}}$. Then, there is $\gamma \in \Gamma$ such that

$$a \models (x_{\neq k+1}, b) \in \gamma \vdash \phi(x_{\neq k+1}; b) \leftrightarrow \phi(a; b).$$

In particular, $(a, b) \in \gamma$, so it suffices to show that $b \in D_\gamma$. For all $a' \in \gamma|_b$,

$$a' \models (x_{\neq k+1}, b) \in \gamma \vdash \phi(x_{\neq k+1}; b) \leftrightarrow \phi(a; b),$$

and so $\models \phi(a'; b) \leftrightarrow \phi(a; b)$. Thus, $\gamma|_b \subseteq \phi^\sigma(x_{\neq k+1}; b)$ for the unique $\sigma \in \{0, 1\}$ satisfying $\models \phi^\sigma(a; b)$, and so $b \in D_\gamma$ as required. We have shown that $\omega(G) \geq 1 - \delta$.

For $\gamma \in \mathcal{G}_B$ and $\sigma \in \{0,1\}$, let $D_\gamma^\sigma := \{b \in \mathbb{M}^{x_{k+1}} : \gamma|_b \subseteq \phi^\sigma(x_{\neq k+1}; b)\}$, so that $D_\gamma = D_\gamma^0 \sqcup D_\gamma^1$. Let the partition $\mathcal{P}_1$ of $\mathbb{M}^{x_{\neq 1}}$ be the set of Boolean atoms of $\{\psi_1(x_{\neq 1}, c_1) : c_1 \in B\} \cup \{D_\gamma^\sigma : \gamma \in \mathcal{G}_B, \sigma \in \{0,1\}\}$, where D_γ^σ is identified with the definable set $\{(x_2, ..., x_{k+1}) : x_{k+1} \in D_\gamma^\sigma\}$. For $i \in [k+1] \setminus \{1\}$, let the partition $\mathcal{P}_i$ of $\mathbb{M}^{x_{\neq i}}$ be the set of Boolean atoms of $\{\psi_i(x_{\neq i}, c_i) : c_i \in B\}$. Since $|\mathcal{G}_B| \leq |B|^l$ and M is NIP, for all $i \in [k+1]$ we have that

$$|\mathcal{P}_i| \leq \mathrm{poly}_{\phi,\psi_1,...,\psi_{k+1},N}(|B|) \leq \mathrm{poly}_{\phi,\psi_1,...,\psi_{k+1},N}(\delta^{-1}).$$

It is clear that there are L-formulas $\theta_i(x_{\neq i}, z_i)$ for $i \in [k+1]$, which are functions of $\phi, \psi_1, ..., \psi_{k+1}, N$, and δ, such that (i) holds. To see that (ii) holds, recall that $\omega(G) \geq 1 - \delta$ where $G = \bigvee_{\gamma \in \mathcal{G}_B} \gamma \wedge D_\gamma = \bigvee_{\gamma \in \mathcal{G}_B} \bigvee_{\sigma \in \{0,1\}} \gamma \wedge D_\gamma^\sigma$. For all $\gamma \in \mathcal{G}_B$ and $\sigma \in \{0,1\}$, $\gamma \wedge D_\gamma^\sigma$ is ϕ-homogeneous (indeed, $\gamma \wedge D_\gamma^\sigma \subseteq \phi^\sigma(x)$), and by the definition of Boolean atoms, $\gamma \wedge D_\gamma^\sigma$ is a union of sets of the form $P_1 \wedge \cdots \wedge P_{k+1}$ where $P_i \in \mathcal{P}_i$. Therefore, the union of all ϕ-homogeneous sets of the form $P_1 \wedge \cdots \wedge P_{k+1}$ contains G, and has ω-measure at least $1 - \delta$. This shows that (ii) holds. □

As mentioned before, our proof strategy follows that of [11, Theorem 5.8]. There, they also prove a cutting lemma, which they use to prove that the hypergraph satisfies the definable 'strong Erdős–Hajnal property' [11, Proposition 4.4], before bootstrapping it into a regularity lemma.

Let us state an abridged form of [11, Proposition 4.4].

Proposition 5.5.7. *Let $\chi(x_1, ..., x_{k+1})$ be a relation definable in a distal structure $\mathcal{M}$, and suppose $|x_1| = \cdots = |x_{k+1}|$. For all $\alpha \in (0,1]$, there are $\varepsilon > 0$ and $\theta_i(x_i, z_i) \in L$ for $i \in [k+1]$ such that the following holds.*

Let $\nu(x_{k+1})$ be a Keisler measure, generically stable over $\mathcal{M}$. If $\nu^{(k+1)}(\chi) \geq \alpha$, then there are $c_1, ..., c_{k+1} \in \mathcal{M}$ such that $\bigwedge_{i \in [k+1]} \theta_i(x_i, c_i)$ is contained in χ and has $\nu^{(k+1)}$-measure at least ε.

In our proof, we observe that such an intermediate step is not necessary: once we have a cutting lemma, we can directly define the appropriate partitions to give us the desired regularity lemma. Note that an analogue of the definable strong Erdős–Hajnal property for our relation ϕ can then easily be *deduced* from our regularity lemma.

Corollary 5.5.8. *(T is NIP, ϕ as before.) Suppose $|x_1| = \cdots = |x_{k+1}|$. For all $\alpha \in (0, 1]$, there are $\varepsilon > 0$ and $\theta_i(x_{\neq i}, z_i) \in L$ for $i \in [k+1]$ such that the following holds.*

Let $\nu(x_{k+1})$ be a Keisler measure, generically stable over M. If $\nu^{(k+1)}(\phi) \geq \alpha$, then there are $c_1, ..., c_{k+1} \in M$ such that $\bigwedge_{i\in[k+1]} \theta_i(x_{\neq i}, c_i)$ is contained in ϕ and has $\nu^{(k+1)}$-measure at least ε.

Proof. Applying Theorem 5.5.2 with $\delta = \alpha/2$, we have that

$$\sum \nu^{(k+1)}(P_1 \wedge \cdots \wedge P_{k+1}) \geq \alpha - \delta = \alpha/2,$$

where the sum ranges over all $(P_1, ..., P_{k+1}) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_{k+1}$ such that $P_1 \wedge \cdots \wedge P_{k+1} \subseteq \phi$. Now $|\mathcal{P}_i| \leq K \leq \text{poly}_{\phi,\psi_1,...,\psi_{k+1},N}(\delta^{-1})$ for all $i \in [k+1]$, so one of these tuples $(P_1, ..., P_{k+1})$ is such that $\nu^{(k+1)}(P_1 \wedge \cdots \wedge P_{k+1}) \geq \alpha/(2K^{k+1})$. □

5.5.2 Main result

Throughout this subsection, we fix an NIP L-theory T and models $M, \mathbb{M} \models T$ with $\mathbb{M}$ sufficiently saturated.

We can make Theorem 5.5.2 uniform, in the sense that if $\phi(x_1, ..., x_{k+1}) = \phi'(x_1, ..., x_{k+1}, e)$ for some $e \in M$, then θ_i and K can be chosen independently of e. The following theorem is the most general formulation of our regularity lemma in this chapter.

Theorem 5.5.9. *(T is NIP.) Let $\phi'(x_1, ..., x_k; (x_{k+1}, u)) \in L$ have k-strong honest definition $(\psi_1, ..., \psi_{k+1})$ of degree N. For all $\delta \in (0, 1]$, there are $\theta_i(x_{\neq i}, z_i, u) \in L$ for $i \in [k+1]$, where θ_{k+1} has no u-dependence, and a natural number $K \leq$ poly$_{\phi',\psi_1,...,\psi_{k+1},N}(\delta^{-1})$, such that the following holds.*

Let $\phi(x_1, ..., x_{k+1}) := \phi'(x_1, ..., x_k; (x_{k+1}, e))$ for some $e \in M$. Let $\mu(x_{\neq k+1})$ and $\nu(x_{k+1})$ be Keisler measures, with $\nu(x_{k+1})$ generically stable over M, and let

$\omega(x) := \nu(x_{k+1}) \otimes \mu(x_{\neq k+1})$. *Then there are partitions* $\mathcal{P}_i$ *of* $\mathbb{M}^{x_{\neq i}}$ *for* $i \in [k+1]$, *each of size at most* K, *such that:*

(i) For all $i \in [k+1]$ *and* $P_i \in \mathcal{P}_i$, *there is* $c_i \in M^{z_i}$ *such that* $P_i = \theta_i(x_{\neq i}, c_i, e)$*;*

(ii) $\sum \omega(P_1 \wedge \cdots \wedge P_{k+1}) \leq \delta$, *where the sum ranges over all* $(P_1, ..., P_{k+1}) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_{k+1}$ *such that* $P_1 \wedge \cdots \wedge P_{k+1}$ *is not* ϕ*-homogeneous.*

Proof. Apply Theorem 5.5.2 to ϕ' with the Keisler measures $\mu(x_{\neq k+1})$ and $\nu'(x_{k+1}, u)$, where $\nu'(\chi(x_{k+1}, u)) := \nu(\chi(x_{k+1}, e))$ for all $\chi(x_{k+1}, u) \in L(\mathbb{M})$. By Proposition 2.6.6, ν' is generically stable over M. □

We would like to remove the NIP assumption from our main theorem.

Problem 5.5.10. *Must Theorem 5.5.9 hold if* T *is not necessarily NIP?*

We record the special case of Theorem 5.5.9 where $|x_1| = \cdots = |x_{k+1}|$.

Corollary 5.5.11. *(*T *is NIP.) Let* $\phi'(x_1, ..., x_k; (x_{k+1}, u)) \in L$ *have* k*-strong honest definition* $(\psi_1, ..., \psi_{k+1})$ *of degree* N*, and suppose* $|x_1| = \cdots = |x_{k+1}| =: d$*. For all* $\delta \in (0, 1]$*, there is* $\theta(x_1, ..., x_k, z, u) \in L$ *and a natural number* $K \leq poly_{\phi', \psi_1, ..., \psi_{k+1}, N}(\delta^{-1})$ *such that the following holds.*

Let $\phi(x_1, ..., x_{k+1}) := \phi'(x_1, ..., x_k; (x_{k+1}, e))$ *for some* $e \in M$*. Let* $V \subseteq \mathbb{M}^d$ *be* M*-definable, and let* $\nu(x_{k+1})$ *be a global measure, generically stable over* M*. Then there is a partition* $\mathcal{P}$ *of* V^k *of size at most* K *such that:*

(i) For all $P \in \mathcal{P}$*, there is* $c \in M^z$ *such that* $P = \theta(x_1, ..., x_k, c, e) \cap V^k$*;*

(ii) $\sum \nu^{(k+1)}(P_1 \wedge \cdots \wedge P_{k+1}) \leq \delta \nu(V)^{k+1}$*, where the sum ranges over all* $(P_1, ..., P_{k+1}) \in \mathcal{P}^{k+1}$ *such that* $P_1 \wedge \cdots \wedge P_{k+1}$ *is not* ϕ*-homogeneous.*

Proof. Without loss of generality, suppose $\nu(V) > 0$. Apply Theorem 5.5.9 with $\nu|_V(x_{k+1})$ and $\mu(x_1, ..., x_k) := \nu|_V(x_1) \otimes \cdots \otimes \nu|_V(x_k)$, where $\nu|_V$ is the relativisation of ν to V, given by $\nu|_V(Z) := \nu(V \cap Z)/\nu(V)$ for all definable Z. It is easy to see that $\nu|_V$ is generically stable over M, and hence so is μ by Fact 2.6.9. (Note that the formulas $\theta_1, ..., \theta_{k+1}$ given by Theorem 5.5.9 can easily be coded into one formula θ.) □

Since finite counting measures are generically stable (Example 2.6.16), we have the following statement for finite hypergraphs. We formulate this in a manner more consistent with our earlier combinatorial discourse.

Corollary 5.5.12. *(T is NIP.) Let $\phi'(x_1, ..., x_k; (x_{k+1}, u)) \in L$ have k-strong honest definition $(\psi_1, ..., \psi_{k+1})$ of degree N, and suppose $|x_1| = \cdots = |x_{k+1}| =: d$. For all $\delta \in (0, 1]$, there is $\theta(x_1, ..., x_k, z, u) \in L$ and a natural number $K \leq poly_{\phi',\psi_1,...,\psi_{k+1},N}(\delta^{-1})$ such that the following holds.*

Let $\phi(x_1, ..., x_{k+1}) := \phi'(x_1, ..., x_k; (x_{k+1}, e))$ for some $e \in \mathbb{M}$. Let $V \subseteq \mathbb{M}^d$ be finite. Then there is a partition $\mathcal{P}$ of V^k of size at most K such that:

(i) For all $P \in \mathcal{P}$, there is $c \in M^z$ such that $P = \theta(x_1, ..., x_k, c, e) \cap V^k$;

(ii) The induced partition $\mathcal{Q}$ of V^{k+1}, given by

$$\left\{\left\{w = (w_1, ..., w_{k+1}) \in V^{k+1} : w_{\neq i} \in P_i \textit{ for all } i \in [k+1]\right\} : P_1, ..., P_{k+1} \in \mathcal{P}\right\},$$

is such that $\sum_{Q \in \mathcal{Q} \text{ not } \phi\text{-homogeneous}} |Q| \leq \delta |V|^{k+1}$.

5.5.3 Future work: recovering k-distality

By Theorem 5.4.12, the regularity lemma Corollary 5.5.12 applies to all relations definable in an NIP strongly k-distal structure. As in Section 4.5, we can ask if every relation ϕ on a set M satisfying this regularity lemma (without the definable data) is such that (M, ϕ) admits an expansion that is NIP strongly k-distal.

Definition 5.5.13. Let $\phi(x_1, ..., x_{k+1})$ be a relation on a set M. Say that ϕ *satisfies the NIP strongly k-distal regularity lemma* if the following holds.

For all $\delta \in (0, 1]$, there is a natural number $K \leq \text{poly}_\phi(\delta^{-1})$ such that for all finite $V \subseteq M^d$, there is a partition $\mathcal{P}$ of V^k of size at most K inducing a partition $\mathcal{Q}$ of V^{k+1}, given by

$$\left\{\left\{w = (w_1, ..., w_{k+1}) \in V^{k+1} : w_{\neq i} \in P_i \text{ for all } i \in [k+1]\right\} : P_1, ..., P_{k+1} \in \mathcal{P}\right\},$$

such that $\sum_{Q \in \mathcal{Q} \text{ not } \phi\text{-homogeneous}} |Q| \leq \delta |V|^{k+1}$.

Problem 5.5.14. *Let $\phi(x_1, ..., x_{k+1})$ be a relation on a set M that satisfies the NIP strongly k-distal regularity lemma. Must (M, ϕ) admit an expansion that is NIP strongly k-distal? What if we assume that (M, ϕ) is NIP?*

Note that, by Theorem 4.5.1, when $k = 1$ and (M, ϕ) is not assumed to be NIP, the answer to the first part of the question is negative.

In Chapter 4, we showed that a formula that satisfies the distal regularity lemma already enjoys a particular property of (formulas definable in) distal structures, namely, improved Zarankiewicz bounds. We can ask if a similar phenomenon occurs with the NIP k-distal regularity lemma.

Problem 5.5.15. *Let $\phi(x_1, ..., x_{k+1})$ be a relation on a set M that satisfies the NIP strongly k-distal regularity lemma. Investigate the (combinatorial) properties of ϕ.*

5.6 Dual setup

Throughout this section, fix a complete L-theory T, and let $\mathbb{M} \models T$ be sufficiently saturated.

So far, we have worked with k-strong honest definitions for formulas $\phi(x_1, ..., x_k; y)$. We had previously attempted to define 'dual' k-strong honest definitions for formulas $\phi(x; y_1, ..., y_k)$, to better align with the intuition of NIP$_k$ that we have a k-dimensional box of parameters (as inputs for the k parameter variables $y_1, ..., y_k$). We will also see in Section 5.7 that this dual setup has better geometric properties. However, we are not able to prove that dual k-strong honest definitions exist in an NIP strongly k-distal theory. In this section, we introduce dual k-strong honest definitions, state their existence in an NIP strongly k-distal theory as a conjecture, and prove a partial converse.

Definition 5.6.1. Let $\phi(x; y_1, ..., y_k) \in L$; write $y := (y_1, ..., y_k)$. Let $N \in \mathbb{N}$. A k-tuple of L-formulas $(\psi_i(x, y_{\neq i}, z_i) : i \in [k])$ is a *dual k-strong honest definition* for ϕ of *degree N* if the following holds.

Let $B \subseteq \mathbb{M}$ with $2 \leq |B| < \infty$, and let $a \in \mathbb{M}$. Then there are $c^{(1)}, ..., c^{(N)} \in B$ such that for all $b = (b_1, ..., b_k) \in B$, there is $j \in [N]$ with

$$a \models \bigwedge_{i=1}^{k} \psi_i(x, b_{\neq i}, c_i^{(j)}) \vdash \phi(x; b) \leftrightarrow \phi(a; b).$$

Conjecture 5.6.2. *If T is NIP and strongly k-distal, then every $\phi(x; y_1, ..., y_k) \in L$ has a dual k-strong honest definition.*

We can prove a partial converse. To do so, we need to slightly strengthen our notion of dual k-strong honest definitions for a formula $\phi(x; y_1, ..., y_k)$, to allow $y_1, ..., y_k$ to range over possibly distinct sets $B_1, ..., B_k$. To state this strengthened notion, we fix some notation.

Definition 5.6.3. For sets $B_1, ..., B_k$ and $d_1, ..., d_k \in \mathbb{N}$, write $\Omega_{d_1,...,d_k}(B_1, ..., B_k)$ for the set

$$\left\{ (c_1, ..., c_k) : c_i \in \left(\bigcup_{j \in [k] \setminus \{i\}} B_j \right)^{d_i} \text{ for } i \in [k-1],\ c_k \in \left(\bigcup_{j \in [k]} B_j \right)^{d_k} \right\}.$$

When we write $c \in \Omega_{d_1,...,d_k}(B_1, ..., B_k)$, it is understood that $c = (c_1, ..., c_k)$ where $|c_i| = d_i$. The parameters $d_1, ..., d_k$ are omitted where understood from context.

Definition 5.6.4. Let $\phi(x; y_1, ..., y_k) \in L$; write $y := (y_1, ..., y_k)$. Let $N \in \mathbb{N}$. A k-tuple of L-formulas $(\psi_i(x, y_{\neq i}, z_i) : i \in [k])$ is a *dual k-stronger honest definition* for ϕ of *degree N* if the following holds.

Let $B_1, ..., B_k \subseteq \mathbb{M}$ with $2 \leq |B_i| < \infty$, and let $a \in \mathbb{M}$. Then there are $c^{(1)}, ..., c^{(N)} \in \Omega_{|z_1|,...,|z_k|}(B_1, ..., B_k)$ such that for all $b_i \in B_i$, writing $b := (b_1, ..., b_k)$, there is $j \in [N]$ such that

$$a \models \bigwedge_{i=1}^{k} \psi_i(x, b_{\neq i}, c_i^{(j)}) \vdash \phi(x; b) \leftrightarrow \phi(a; b).$$

If one can prove that dual k-strong honest definitions exist in an NIP strongly k-distal theory, we expect the proof to be adaptable without much difficulty to prove the existence of dual k-stronger honest definitions. That is, we expect Conjecture 5.6.2 to have the same resolution if dual k-strong honest definitions are replaced by dual k-stronger honest definitions. Regardless, with this stronger notion in place, we are ready to prove a partial converse to Conjecture 5.6.2.

Proposition 5.6.5. *If every $\phi(x; y_1, ..., y_k) \in L$ has a dual k-stronger honest definition, then T is strongly k-distal.*

Proof. If $(I, <)$ is a sequence and $a_1, ..., a_m,\ b_1, ..., b_m \in I$, write $(a_1, ..., a_m) \cong (b_1, ..., b_m)$ if, for all $i, j \in [m]$, $a_i < a_j$ if and only if $b_i < b_j$. Suppose $(I, <)$ has entries in $\mathbb{M}^n$ for some $n \in \mathbb{N}$. If $(I, <)$ is B-indiscernible and $(a_1, ..., a_m) \cong (b_1, ..., b_m)$, then $\mathrm{tp}(a_1, ..., a_m/B) = \mathrm{tp}(b_1, ..., b_m/B)$.

Let I_0, I_1 be dense infinite sequences without endpoints, whose entries lie in $\mathbb{M}^n$. Let $a \in \mathbb{M}^n$ and $B_1, ..., B_k \subseteq \mathbb{M}$, such that I_0+a+I_1 is $B_1 \cdots B_{j-1}B_{j+1} \cdots B_k$-indiscernible for all $j \in [k]$ and $I_0 + I_1$ is $B_1 \cdots B_k$-indiscernible. Let $\phi(y_1, ..., y_k, x_0, ..., x_{2m}) \in L$, $b_i \in B_i$, $d_0 < \cdots < d_m \in I_0$ (that is, $(d_0, ..., d_m)$ is a subsequence of I_0), and $d_{m+1} < \cdots < d_{2m} \in I_1$, such that $\models \phi(b_1, ..., b_k, d_0, ..., d_{2m})$. We show that

$$\models \phi(b_1, ..., b_k, d_0, ..., d_{m-1}, a, d_{m+1}, ..., d_{2m}),$$

which would prove that I_0+a+I_1 is $B_1 \cdots B_k$-indiscernible. Write $y := (y_1, ..., y_k)$ and $b := (b_1, ..., b_k)$.

Let E be the set of singletons appearing in I_0, and for $i \in [k] \setminus \{1\}$, let B_i be the union of E with the set of singletons appearing in b_i. By assumption, $\phi(y_1; y_2, ..., y_k, (x_0, ..., x_{2m}))$ has a dual k-stronger honest definition, treating $(x_0, ..., x_{2m})$ as one tuple of variables, and we apply this fact to the parameter sets $B_2, ..., B_k, E$ and the tuple $b_1 \in \mathbb{M}$. We obtain $\psi_i(y_{\neq i}, x_0, ..., x_{2m}, z) \in L$ for $i \in [k] \setminus \{1\}$, $\psi(y, z) \in L$, and $N \in \mathbb{N}$, such that for all finite $J_0 \subseteq I_0$, there are $e^{(1)}, ..., e^{(N)} \in I_0$ such that, for all $e_0 < \cdots < e_{2m} \in J_0$, there is $j \in [N]$ such that

$$b_1 \models \psi(y_1, b_{\neq 1}, e^{(j)}) \wedge \bigwedge_{i=2}^{k} \psi_i(y_1, b_{\neq 1,i}, e_0, ..., e_{2m}, e^{(j)})$$
$$\vdash \phi(y_1, b_{\neq 1}, e_0, ..., e_{2m}) \leftrightarrow \phi(b_1, b_{\neq 1}, e_0, ..., e_{2m}),$$

and so

$$b_1 \models \psi(y_1, b_{\neq 1}, e^{(j)}) \wedge \bigwedge_{i=2}^{k} \psi_i(y_1, b_{\neq 1,i}, e_0, ..., e_{2m}, e^{(j)}) \vdash \phi(y_1, b_{\neq 1}, e_0, ..., e_{2m})$$

since $\models \phi(b_1, ..., b_k, e_0, ..., e_{2m})$ by $b_1 \cdots b_k$-indiscernibility of $I_0 + I_1$.

Choosing J_0 to be sufficiently large, we may choose $e_0, ..., e_{2m}$ to be distinct from $e^{(1)}, ..., e^{(N)}$. By $b_2 \cdots b_k$-indiscernibility of I_0+a+I_1, for some/all $d \in I_0+I_1$ such that $(e_0, ..., e_{2m}, e^{(j)}) \cong (d_0, ..., d_{m-1}, a, d_{m+1}, ..., d_{2m}, d)$, we have that

$$\psi(y_1, b_{\neq 1}, d) \wedge \bigwedge_{i=2}^{k} \psi_i(y_1, b_{\neq 1,i}, d_0, ..., d_{m-1}, a, d_{m+1}, ..., d_{2m}, d)$$
$$\vdash \phi(y_1, b_{\neq 1}, d_0, ..., d_{m-1}, a, d_{m+1}, ..., d_{2m}).$$

But now $I_0 + I_1$ is $b_1 \cdots b_k$-indiscernible and $I_0 + a + I_1$ is $b_{\neq i}$-indiscernible for all $i \in [k]$, so

$$b_1 \models \psi(y_1, b_{\neq 1}, d) \wedge \bigwedge_{i=2}^{k} \psi_i(y_1, b_{\neq 1,i}, d_0, ..., d_{m-1}, a, d_{m+1}, ..., d_{2m}, d),$$

and we conclude that $\models \phi(b_1, ..., b_k, d_0, ..., d_{m-1}, a, d_{m+1}, ..., d_{2m})$. □

Remark 5.6.6. Observe that the previous proof goes through under the weaker assumption that 'non-uniform' dual k-stronger honest definitions exist: that is, for all ϕ, $B_1, ..., B_k$, and a, there is $(\psi_1, ..., \psi_k)$ satisfying the conclusion of Definition 5.6.4.

Why are we concerning ourselves with the dual setup? Other than the intuition of NIP_k discussed at the start of this section, dual k-strong honest definitions also give rise to a useful form of cell decompositions with desirable geometric properties, which is the subject of the next section.

5.7 Higher-arity distal cell decompositions

In this section, we describe the analogue of distal cell decompositions that arises from dual k-strong honest definitions. Throughout this section, fix a complete L-theory T, and let $\mathbb{M} \models T$ be sufficiently saturated.

Let $M \models T$. Recall from Definition 2.7.8 that a strong honest definition $\psi(x;z) \in L$ for a binary formula $\phi(x;y) \in L$ induces a *distal (cell) decomposition* for ϕ. That is, for all finite $B \subseteq M$ of size at least 2, there is a cover $\mathcal{F}(B)$ of M^x, such that for all $F \in \mathcal{F}(B)$,

(i) There is $c \in B$ such that $F = \psi(x;c)$; and

(ii) For all $b \in B$, we have either $F \subseteq \phi(x;b)$ or $F \subseteq \neg\phi(x;b)$.

Note that, in Definition 2.7.8, we have $B \subseteq M^y$ rather than $B \subseteq M$, but these formulations are essentially equivalent.

Suppose now the formula $\phi(x; y_1, ..., y_k) \in L$ has a dual k-strong honest definition $(\psi_i(x, y_{\neq i}, z_i) : i \in [k])$ of degree N. Then, for all $B_1, ..., B_k \subseteq M$ with $2 \leq |B_i| < \infty$, there is a cover $\mathcal{F}(B_1, ..., B_k)$ of M^x, such that for all $F \in \mathcal{F}(B_1, ..., B_k)$, writing $b \in B$ to mean $b = (b_1, ..., b_k)$ for some $b_i \in B_i$,

(i) There are $c^{(1)}, ..., c^{(N)} \in \bigcup_{i \in [k]} B_i$ such that

$$F = \bigwedge_{b \in B} \bigwedge_{i \in [k]} \psi_i(x, b_{\neq i}, c_i^{(j(b))})$$

for some $j(b) \in [N]$; and

(ii) For all $b \in B$, we have either $F \subseteq \phi(x;b)$ or $F \subseteq \neg\phi(x;b)$.

Note that (i) implies that there are $c^{(1)}, ..., c^{(N)} \in \bigcup_{i \in [k]} B_i$ such that

$$F = \bigwedge_{i \in [k]} \bigwedge_{\substack{(b_1, ..., b_{i-1}, b_{i+1}, ..., b_k) \\ b_e \in B_e}} \bigwedge_{j \in J(b_1, ..., b_{i-1}, b_{i+1}, ..., b_k)} \psi_i(x, b_1, ..., b_{i-1}, b_{i+1}, ..., b_k, c_i^{(j)})$$

for some $J(b_1, ..., b_{i-1}, b_{i+1}, ..., b_k) \subseteq [N]$.

What have we achieved? Suppose $|B_1| = \cdots = |B_k| =: n$. Then the set $S_\phi(B_1 \times \cdots \times B_k)$ of ϕ-types over $B_1 \times \cdots \times B_k$ forms a partition of M^x, where each piece has the form $\bigwedge_{b \in B} \phi(x;b)^{\varepsilon_b}$ for some $\varepsilon_b \in \{0,1\}$. In particular, each piece is the intersection of n^k definable sets, defined using up to n^k parameters. The cover $\mathcal{F}(B_1, ..., B_k)$ of M^x refines $S_\phi(B_1 \times \cdots \times B_k)$, and each piece of the cover is the intersection of $O(n^{k-1})$ definable sets, defined using $O(n^{k-1})$ parameters.

This drop in 'dimension' provides a way to show the non-existence of a dual k-strong honest definition: if the set of ϕ-types over $B_1 \times \cdots \times B_k$ is a 'k-dimensional' object which cannot be described with $O(n^{k-1})$ parameters, then a dual k-strong honest definition cannot exist.

We give an example of such an argument.

Definition 5.7.1. Fix an $\mathcal{L}$-structure $\mathcal{M}$. For $A \subseteq \mathcal{M}$, the *$\mathcal{L}$-definable closure of A*, written $\mathrm{dcl}_\mathcal{L}(A)$, is

$$\{f(a_1, ..., a_n) : a_i \in A, f \text{ is a function that is } \mathcal{L}\text{-definable without parameters}\}.$$

Given $A, B \subseteq \mathcal{M}$, say that A is *$\mathrm{dcl}_\mathcal{L}$-independent over B* if for all $a \in A$, $a \notin \mathrm{dcl}_\mathcal{L}((A \setminus \{a\}) \cup B)$. Omit $\mathcal{L}$ throughout when it is obvious from context.

Henceforth in this section, let M be an o-minimal expansion of an ordered abelian group in the language L_0, let $L = L_0 \cup \{P\}$ where P is a new unary predicate, and let T be the L-theory of (M, P), where one of the following holds:

(i) (M, P) is a *dense pair*, that is, $P(M)$ is a proper elementary L_0-substructure of M that is dense in M;

(ii) $M = (\mathbb{R}, <, +, \times)$ and $P(M)$ is a dense subgroup of $(\mathbb{R}^\times, \times)$ such that $P(M) = -P(M)$ and $P(M) \cap \mathbb{R}^+$ has the *Mann property*: for all $a_1, ..., a_n \in \mathbb{Q}^\times$, there are finitely many tuples $(p_1, ..., p_n) \in (P(M) \cap \mathbb{R}^+)^n$ such that $a_1 p_1 + \cdots + a_n p_n = 1$ and $\sum_{i \in I} a_i p_i \neq 0$ for all $\emptyset \neq I \subseteq [n]$;

(iii) $P(M)$ is dense in M and dcl_{L_0}-independent.

In [26], Hieronymi and Nell prove that T is not distal and give references for the fact that T is NIP. Their proof uses the external characterisation of distality given in Theorem 2.7.3. In an unpublished note, Pantelis Eleftheriou and Aris Papadopoulos prove the same fact by showing the non-existence of strong honest definitions for the formula $\phi(x; y) := x \in y + P$. Here, we generalise their argument to show the non-existence of k-strong honest definitions for the formula $\phi(x; y_1, ..., y_k) := x \in y_1 + \cdots + y_k + P$. We are grateful for their permission to include our generalisation of their argument.

Henceforth in this section, unless otherwise specified, definability is with parameters and in the language L. We use P and $P(M)$ interchangeably.

Definition 5.7.2. Let $X \subseteq M^n$ be definable. Say that X is *large* if there is $m \in \mathbb{N}^+$ and a definable function $f : M^{mn} \to M$ such that $f(X^m)$ contains an open interval in M. Say that X is *small* if it is not large.

Let $Z \subseteq M^n$ be definable. Say that X is *small in* Z if $X \cap Z$ is small, and *co-small in* Z if $Z \setminus X$ is small.

Fact 5.7.3. *The following hold for the structure* (M, P)*.*

(F1) The set P *is small.*

(F2) A finite union of small sets is small.

(F3) Let $X \subseteq M$ *be* A*-definable, where* $A \subseteq M$*. Then there are* A*-definable elements* $a_1 \leq \cdots \leq a_m$ *in* M *such that, writing* $a_0 := -\infty$ *and* $a_{m+1} := +\infty$*, for all* $i \in [m + 1]$ *we have that* X *is either small or co-small in* $[a_{i-1}, a_i]$*. In particular, if* X *is large, then it is co-small in one of these intervals.*

(F4) An A*-definable set* $X \subseteq M^n$ *is small if and only if there is an* $L_0(A)$*-definable function* $f : M^m \to M^n$ *such that* $X \subseteq f(P^m)$*.*

Proof. For statement (F1), see [16, Section 2] for references. Statements (F2), (F3), and (F4) respectively follow from Corollary 3.15, Lemma 3.3, and Lemma 3.11 of [16]. □

Theorem 5.7.4. *Let $k \in \mathbb{N}^+$. The formula $\phi(x; y_1, ..., y_k) := x \in y_1 + \cdots + y_k + P$ does not have a dual k-strong honest definition.*

Proof. Write $y := (y_1, ..., y_k)$. Suppose for a contradiction that $(\psi_i(x, y_{\neq i}, z_i) : i \in [k])$ is a dual k-strong honest definition for ϕ of degree N. Let $n > \max\{kN|z_i| : i \in [k]\}$ be a natural number. By (F1) and compactness, we can find $B \subseteq M$ consisting of kn elements dcl_L-independent over P, and we let $B = B_1 \sqcup \cdots \sqcup B_k$ be any equipartition of B, so $|B_i| = n$ for all $i \in [k]$. By the discussion above, there are $F^1, ..., F^m \subseteq M$ such that:

- $M = \bigcup_{r \in [m]} F^r$.
- For all $r \in [m]$, there are $c^{(1)}, ..., c^{(N)} \in \bigcup_{i \in [k]} B_i$ such that $F^r = \bigcap_{i \in [k]} F_i^r$, where

$$F_i^r := \bigcap_{\substack{(b_1, ..., b_{i-1}, b_{i+1}, ..., b_k) \\ b_e \in B_e}} \bigcap_{j \in J(b_1, ..., b_{i-1}, b_{i+1}, ..., b_k)} \psi_i(x, b_1, ..., b_{i-1}, b_{i+1}, ..., b_k, c_i^{(j)})$$

  for some $J(b_1, ..., b_{i-1}, b_{i+1}, ..., b_k) \subseteq [N]$.
- For all $b_i \in B_i$ and $r \in [m]$, we have either $F^r \subseteq (b_1 + \cdots + b_k + P)$ or $F^r \subseteq M \setminus (b_1 + \cdots + b_k + P)$.

Now consider the set:

$$M \setminus \left(\bigcup_{b_i \in B_i} (b_1 + \cdots + b_k + P) \right) = \bigcap_{b_i \in B_i} (M \setminus (b_1 + \cdots + b_k + P)).$$

There are $r_1, ..., r_t \in [m]$ such that

$$\bigcup_{j \in [t]} F^{r_j} = \bigcap_{b_i \in B_i} (M \setminus (b_1 + \cdots + b_k + P)).$$

This is a finite intersection of co-small sets, which is large by (F2). Again by (F2), one of the sets $F^{r_j} =: F$ is large. For $i \in [k]$, let $F_i := F_i^{r_j}$, so that $F = \bigcap_{i \in [k]} F_i$, and hence each F_i is large. For each $i \in [k]$, there is $\bar{B}_i \subseteq B_i$ with $|\bar{B}_i| \leq N|z_i|$, such that F_i is $(\bar{B}_i \cup \bigcup_{i \neq j \in [k]} B_j)$-definable.

Since $F_1, ..., F_k$ are large with large intersection, by (F3), there are intervals $I_1, ..., I_k$ with $I := I_1 \cap \cdots \cap I_k \neq \emptyset$ such that for all $i \in [k]$, F_i is co-small in I_i and I_i is $(\bar{B}_i \cup \bigcup_{i \neq j \in [k]} B_j)$-definable. By (F4), for all $i \in [k]$, there is a $(\bar{B}_i \cup \bigcup_{i \neq j \in [k]} B_j)$-definable function $f_i : M^{l_i} \to M$ such that $I_i \setminus f_i(P^{l_i}) \subseteq F_i$. Since $F = \bigcap_{i \in [k]} F_i$,

$$I \setminus \bigcup_{i \in [k]} f_i(P^{l_i}) \subseteq F \subseteq \bigcap_{b_i \in B_i} (M \setminus (b_1 + \cdots + b_k + P)).$$

Let $b_i \in B_i$ for all $i \in [k]$. Then, $I \setminus \bigcup_{i \in k} f_i(P^{l_i}) \subseteq I \setminus (b_1 + \cdots + b_k + P)$, and thus $I \cap (b_1 + \cdots + b_k + P) \subseteq \bigcup_{i \in k} f_i(P^{l_i})$. Since P is dense in M, we have that $I \cap (b_1 + \cdots + b_k + P) \neq \emptyset$. Hence, there is some $p \in P$ such that $b_1 + \cdots + b_k + p \in \bigcup_{i \in [k]} f_i(P^{l_i})$.

Since this holds for all $b_i \in B_i$, by the pigeonhole principle we can fix $i \in [k]$ such that

$$\#\{(b_1, ..., b_k) \in B_1 \times \cdots \times B_k : b_1 + \cdots + b_k + p \in f_i(P^{l_i}) \text{ for some } p \in P\} \geq n^k/k.$$

Observe that the projection of this set onto B_i has size at least n/k. Thus, as f_i is $(\bar{B}_i \cup \bigcup_{i \neq j \in [k]} B_j)$-definable,

$$\# \left\{ b_i \in B_i : b_i \in \operatorname{dcl} \left(P \cup \bar{B}_i \cup \bigcup_{i \neq j \in [k]} B_j \right) \right\} \geq n/k.$$

Since $n/k > N|z_i| \geq |\bar{B}_i|$, this contradicts the fact that B_i is dcl-independent over $P \cup \bigcup_{i \neq j \in [k]} B_j$. □

The proof of Theorem 5.7.4 demonstrates the geometric efficacy of our dual k-strong honest definitions: it was able to capture the fact that $x \in y_1 + \cdots + y_k + P$ is a 'k-dimensional' object which cannot be described by a '$(k-1)$-dimensional' decomposition of M.

Since we have not been able to show that dual k-strong honest definitions always exist in an NIP strongly k-distal theory, our proof does not show that

T is not strongly k-distal. Of course, we know that k-strong honest definitions always exist in an NIP strongly k-distal theory, so one may attempt to adapt the proof above to show that, say, the formula $x_1 + \cdots + x_k \in y + P$ does not have a k-strong honest definition. However, it is unclear to us how k-strong honest definitions can be used to generate a similar form of cell decompositions that capture the geometric intuition described above. For now, NIP strongly k-distal structures do not have such cell decompositions to call their own, and the quest for these continues.

Problem 5.7.5. *Find an analogue of distal cell decompositions for (NIP) strongly k-distal theories.*

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