Eccentric Sets

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Declarations

I confirm that the work submitted is my own, except where work which has formed part of jointly authored publications has been included. My contribution and the other authors to this work has been explicitly indicated below. I confirm that appropriate credit has been given within the thesis where reference has been made to the work of others.

The results appearing in Section 3.6 of Chapter 3 are based on joint work with Asaf Karagila [KR24], in which the contributions of all authors to the work were equal.

All results are reproduced in this thesis with the permission of all coauthors.

Figure 2.1 appears in [RSW24] and is reproduced with the permission of the authors.

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For the hedgehogs and pigeons.

Riddles: they either delight or torment. Their delight lies in solutions. Answers provide bright moments of comprehension perfectly suited for children who still inhabit a world where solutions are readily available. Implicit in the riddle's form is a promise that the rest of the world resolves just as easily. And so riddles comfort the child's mind which spins wildly before the onslaught of so much information and so many subsequent questions.

The adult world, however, produces riddles of a different variety. They do not have answers and are often called enigmas or paradoxes. Still the old hint of the riddle's form corrupts these questions by re-echoing the most fundamental lesson: there must be an answer. From there comes torment.

Edith Skourja's "Riddles Without" in *Riddles Within*, ed. Amon Whitten (Chicago: Sphinx Press, 1994), p. 17–57.

Mark D. Danielewski, *House of Leaves*

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Thank you.

Eccentric Sets

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Abstract

This thesis is an anthology of papers that I have written during my time as a PhD student, split into three main chapters of mathematical content that each represent a facet of my research.

Chapter 3. *From* [KR24; Rya24c]. I begin with ZF set theory without the axiom of choice (whereas Chapters 4 and 5 use full ZFC). In this setting it is possible that there is a set X such that, for some ordinal α, there is a surjection $X \to \alpha$ but no injection $\alpha \to X$. Such *eccentric* sets (for 'tis their name) inform us about the structure of their universe of sets, and thus I present the *Hartogs–Lindenbaum spectrum*, a documentation of eccentricity within a universe. I show bounds for the Hartogs–Lindenbaum spectrum in models of SVC before constructing a model of ZF with maximal Hartogs–Lindenbaum spectrum (that is, the universe is as eccentric as possible).

Chapter 4. *From* [Rya24a]. Within this chapter I examine *maximal θ-independent families* (where θ is a cardinal): 'large' collections $\mathcal{A} \subseteq \mathscr{P}(X)$ for some X such that, in some sense, the elements of $\mathcal{A}$ are 'independent' or 'random'. While maximal $\aleph_0$-independent families are guaranteed to exist by Zorn's Lemma, this dramatically fails for $\theta > \aleph_0$, instead requiring the presence of large cardinals. I exhibit a method of constructing proper classes of maximal θ-independent families by forcing over models with large cardinals.

Chapter 5. *From* [Rya24b]. I end with inspiration from model theory. An important concept in model theory (and computer science, as it happens) is VC dimension, a measurement of 'shattering'. VC dimension was originally defined in a finitary manner, as model theory is wont to do, so I extend the definition to allow nuanced infinite dimensions, presenting *string dimension*. This gives rise to ideals of low-dimensional sets, and I investigate the covering numbers of these ideals.

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Chapter 1

Introduction

> "This was his daily prayer; crouched on his knees, burbling numbers with the persistence of waves against the shore, and never a drop closer."
>
> Lara Welch, *Feast of Saints*

SET THEORY IS OFTEN CALLED A 'FOUNDATION' OF MATHEMATICS, something which can, given enough effort, simulate any kind of mathematical tool, argument, or question that one may wish to express.[1] It is also, strangely, only deals with very basic objects: sets. A set is merely a collection of other things,[2] which we[3] refer to as its *elements*. A set does not order its elements, does not contain multiple copies of any elements, and is uniquely determined by its elements. If $X = \{0, 1\}$,[4] then the set $\{1, 0\}$ is precisely the same set as X, as is the set $\{0, 0, 1\}$. X is the unique set among all sets containing exactly the number 0, the number 1, and nothing else.

Despite these limitations, set theory finds a way to describe every other

[1] Of course, there *are* many forms of mathematical expression that set theory alone handles sub-optimally, but those usually fall into the realm of self-reference.

[2] Which, eventually, mathematicians decided are *also* sets.

[3] Indeed. You, too, now refer to curious objects found in sets as the elements of that set, and henceforth I shall be following the mathematical convention of plural pronouns in the text.

[4] These curly brackets are used to describe the elements of a set. In the case of X here, there are only two elements: 0 and 1. However, we may also describe a set more indirectly by the properties of its elements: $\{n \in \mathbb{N} \mid \frac{n}{2} \text{ has no remainder}\}$ is the set of every natural number n such that $\frac{n}{2}$ is a whole number, so this is the set of even numbers. We may also go the other way, such as $\{n^2 \mid n \in \mathbb{N}\}$ being the set $\{0^2, 1^2, 2^2, \ldots\}$ of square numbers. Note that in this text we take 0 to be a natural number. This is just because doing so makes considerably more sense for this work.

mathematical object that we could normally want: natural numbers, integers, functions, real numbers, Turing machines, hypercubes (and therefore a good description of Graham's number), geometric surfaces, and many many other concepts.[5] There is an overview of the exact encodings that will be useful to this text in Chapter 2, but the specific details are not very interesting.[7]

One could reasonably ask in this case why one need study set theory at all, as we seem to have only talked about how it is an esoteric collection of ideas that can, with effort emulate the normal mathematics that most mathematicians are already comfortable with.[9] For this, we must acknowledge the spectre of independence.

1.1 Independence

One of the great shocks of modern mathematics were Gödel's *incompleteness theorems* [Göd31], which can be thought of as saying that any sufficiently advanced system of mathematical rules is *incomplete*.[10] That is, there will be a mathematical statement that the system of rules can neither prove nor disprove.

Incomplete systems of rules are no stranger to us. Suppose that, perhaps for reasons of law, we are required to define what a 'clock' is,[12] and come up with a list of rules such that an object will satisfy all of those rules if and only if it is a clock. Then there are certain deductions that we can make using this list of rules. For example, we can say that if an object obeys these rules and is working properly, then it will change at least once a second; or we could say

[5] *Ex congeriēs quodlibet* ('from sets, anything')[6] is an enticing slogan.

[6] Though *set* as we mean it here has no true translation into Latin.

[7] And *should not* be very interesting. If set theory is doing its job, then an encoding (and *any* encoding) of a mathematical object should be able to be thought of as being exactly the same as that object.[8] Therefore, the set that is said to 'encode' the number π is no more exciting than π itself (which is a bad example, since π is a very exciting number).

[8] There is an interesting conversation about objects-in-intension and objects-in-extension to be had, but this text is not the place.

[9] Indeed I am sadly restraining myself from delving into a discussion on the origins and history of Zermelo–Fraenkel (ZF) set theory, its axioms, and its reception.

[10] We also crucially require that the system of rules is semidecidable (which simply means that there is a computer program that can list every rule)[11] and assume that the system of rules is not inconsistent (so the rules do not contradict one another).

[11] In mathematics we sometimes have an infinite set T of these rules, so what we really mean here is that there is a computer program that will start listing rules $r_1, r_2, r_3, \ldots$, listing *only* rules in T, and eventually listing *every* rule in T.

[12] Please put aside the ontology for a moment and live in a world in which this is a well-defined concept. Perhaps invoking Wittgenstein [Wit53] will help.

that any object obeying these rules exhibits gravitational force on other objects in the universe. However, not every statement can be settled by this list of rules. For example, if our statement is "the object has hands", then there certainly can be no proof of this statement from the rules that we have listed alone, since any digital clock without hands must obey every one of our rules. Similarly, we cannot *disprove* the statement due to the existence of analogue clocks. Therefore, our system of rules is *incomplete*.

Even so, the idea of *mathematical* incompleteness is a far cry from the usual comfort of maths, in which we think of the concepts that we engage in as being immutable, part of the universe. How can there be a statement about natural numbers that we can neither prove nor disprove? This is part of where set theory shines (at least, since the 60s). By Cohen's method of *forcing* [Coh63] we are able to take objects (which we often call *universes*) that obey our set of rules (ZF set theory) and modify them, while not violating any of the rules of ZF. In this way, we can inspect what changed. Are there more real numbers than before? Are infinite games determined now? This is a thesis of independence, showing how certain mathematical statements may be true in one universe but false in the next.

1.2 Summary of results

For the benefit of readers who may be mostly interested in knowing the results within, rather than patiently wading through paragraphs of analogies regarding mathematical concepts, let us try to list, somewhat concisely, the main results of each chapter. These lists are incomplete, instead aiming to highlight the main ideas and results within. Explanations of the symbols and concepts used here are found either in the global preliminaries Chapter 2 or within the respective chapter itself.

Chapter 3: The Hartogs–Lindenbaum Spectrum

Denoting by Ord the class of ordinals,[13] the *Hartogs number* of a set X, denoted $\aleph(X)$, is the least $\alpha \in \mathrm{Ord}$ such that there is no injection $\alpha \to X$, and the *Lindenbaum number* of X, denoted $\aleph^*(X)$, is the least $\alpha \in \mathrm{Ord}$ such that $\alpha \neq 0$

[13]That is ordinals-as-sets, per the von Neumann interpretation [Neu23], which we introduce in Section 2.2.

and there is no surjection $X \to \alpha$.[14] When invoking the axiom of choice AC,[15] one has that $\aleph(X) = \aleph^*(X)$ for every set X. However, the statement "for all X, $\aleph(X) = \aleph^*(X)$" does not *imply* AC. Indeed a result of Pincus [Pel78][16] shows that "for all X, $\aleph(X) = \aleph^*(X)$" is in fact equivalent to $\mathsf{AC}_{\mathsf{WO}}$ (well-orderable families of non-empty sets have a choice function). By extending the proof that Pincus used to obtain this result we construct many more equivalent statements.

Theorem (Theorem A)**.** *The following are equivalent:*

1. *For all* X, $\aleph(X) = \aleph^*(X)$*;*
2. *there is* κ *such that for all* X, $\aleph^*(X) \geqslant \kappa \implies \aleph(X) = \aleph^*(X)$*;*
3. *there is* κ *such that for all* X, $\aleph(X) \geqslant \kappa \implies \aleph(X) = \aleph^*(X)$*;*
4. $\mathsf{AC}_{\mathsf{WO}}$*;*
5. *for all* X, $\aleph(X)$ *is a successor; and*
6. *for all* X, $\aleph(X)$ *is regular.*

The second part of Chapter 3, Section 3.5, is dedicated to the *presence* of eccentric sets.[17] In particular, we try and find bounds for the constructions that allow us to take an eccentric set and 'lift' it to one of higher Hartogs number.

Theorem (Propositions 3.5.9 and 3.5.10 and Theorem 3.5.11)**.** *Let* B *be such that:* $\aleph(B)$ *is singular or a limit cardinal; or* $\aleph(B) < \aleph^*(B)$*. Then there is are cardinals* Ω *and* κ *such that, for all* $\lambda \geqslant \Omega$ *with* $\mathrm{cf}(\lambda) = \mathrm{cf}(\aleph(B))$*, there is a set* X *with* $\aleph(X) = \lambda$ *and* $\max\{\kappa, \lambda^+\} \leqslant \aleph^*(X) \leqslant \max\{\kappa^+, \lambda^+\}$*.*[18]

If $\aleph(B)$ *is singular or a limit cardinal then* $\Omega \leqslant \sup\{\aleph(\alpha^\alpha) \mid \alpha < \aleph(B)\}$ *and* $\kappa = \aleph^*\left(B^{\underline{<\aleph(B)}}\right)$*. If* $\aleph(B)$ *is regular and* $\aleph(B) < \aleph^*(B)$ *then* $\Omega = \aleph(B)$ *and* $\kappa = \aleph^*(B)$*.*

A natural question that arises is that of the *spectrum* of Hartogs and Lindenbaum values of models of ZF. We define the *Hartogs–Lindenbaum*

[14] Hartogs and Lindenbaum numbers (from [Har15] and [LT26] respectively) are discussed in more depth in Section 3.1. Injections and surjections are defined in Section 2.1.
[15] See Section 2.1.3.
[16] Pelc attributes the result to Pincus in this paper.
[17] An *eccentric* set is a set X such that $\aleph(X) < \aleph^*(X)$.
[18] This result can be improved to $\aleph^*(X) = \max\{\kappa, \lambda^+\}$, see Corollary 3.4.5.

spectrum of $M \vDash \mathsf{ZF}$, denoted $\mathrm{Spec}_{\aleph}(M)$, to be the class of all pairs of cardinals $\langle\lambda,\kappa\rangle$ such that, for some set X, $\aleph(X)=\lambda$ and $\aleph^*(X)=\kappa$. While in general the spectrum of a model of ZF can be quite wild (see Theorem C), by taking on the additional assumption of SVC,[19] we find a much more tame structure.

Theorem (Theorem B). *Let $M \vDash \mathsf{SVC}$. Then there is a cardinal ϕ, cardinals $\psi \leqslant \chi_0 \leqslant \Omega$, a cardinal $\psi^* \geqslant \psi$, a cardinal $\chi \in [\chi_0, \chi_0^+]$, and a set $C \subseteq [\phi, \chi_0)$ such that*

$$\mathrm{Spec}_{\aleph}(M) = \bigcup \begin{cases} \mathrm{SC} = \{\langle\lambda^+,\lambda^+\rangle \mid \lambda \in \mathrm{Card}\} \\ \mathfrak{D} \subseteq \{\langle\lambda,\kappa\rangle \mid \psi \leqslant \lambda \leqslant \kappa \leqslant \chi, \psi^* \leqslant \kappa\} \\ \mathfrak{C} \subseteq \{\langle\lambda,\lambda^+\rangle \mid \mathrm{cf}(\lambda) \in C, \lambda < \Omega\} \\ \mathfrak{U} = \{\langle\lambda,\lambda^+\rangle \mid \mathrm{cf}(\lambda) \in C, \lambda \geqslant \Omega\}.^{20} \end{cases}$$

Here $\mathrm{SC}, \mathfrak{D}, \mathfrak{C}$, and $\mathfrak{U}$ simply denote different parts of the spectrum: SC is the pairs of successor cardinals present in every spectrum; $\mathfrak{D}$ is the chaotic part of the spectrum[21] in which any combination of Hartogs and Lindenbaum numbers could happen, though only up to a bounded Lindenbaum number; $\mathfrak{C}$ is the irregular oblate cardinals,[22] which may or may not appear at various points in this bounded region; and $\mathfrak{U}$ is the class of regularly appearing oblate cardinals. This is telling us that in models of SVC, the only eccentric sets that appear past a certain bound (χ) are from oblate cardinals with cofinality in C, and past another bound (Ω) every cardinal with cofinality in C is oblate.

Finally, we look into effecting eccentric sets, both one at a time and, later, all at once. We achieve this by using *symmetric extensions*, a way of extending a model M of ZF into a larger model N of ZF by means of a "symmetric system" (see Section 2.4 and in particular Section 2.4.1).

Theorem (Theorem 3.6.2). *Let $\lambda \leqslant \kappa$ be infinite cardinals. There is a symmetric system $\langle\mathbb{P},\mathscr{G},\mathscr{F}\rangle$ and a $\mathbb{P}$-name $\dot{X} \in \mathsf{HS}_{\mathscr{F}}$ such that*

$$\mathbb{1}_{\mathbb{P}} \Vdash^{\mathsf{HS}} \text{“}\aleph(\dot{X}) = \check{\lambda} \textit{ and } \aleph^*(\dot{X}) = \check{\kappa}\text{”}.$$

[19] *Small violations of choice*, "there is a set S such that for all X there is an ordinal η and a surjection $f\colon S\times\eta\to X$". We give a brief history and more thorough treatment to SVC in Section 3.2.1.

[20] Note that $\mathfrak{C}$ is a *subset* of the possible 'irregular oblate cardinals'. There may be X such that $\aleph(X)=\lambda$ and $\aleph^*(X)=\lambda^+$ for $\lambda<\Omega$ and $\mathrm{cf}(\lambda)\in C$, but we cannot be certain. On the other hand, $\mathfrak{D}$ is the *entire class* of 'oblate cardinals' described. That is, if $\lambda\geqslant\Omega$ and $\mathrm{cf}(\lambda)\in C$ then we can guarantee that there is X such that $\aleph(X)=\lambda$ and $\aleph^*(X)=\lambda^+$.

[21] $\mathfrak{D}$ of course is short for "dracones", as in *hic sunt dracones* ('here be dragons').

[22] An *oblate cardinal* is a cardinal λ such that for some set X, $\aleph(X)=\lambda$ and $\aleph^*(X)=\lambda^+$.

Practically, this means that in any model M of ZF one can extend M to a new model N of ZF such that there is a set $X \in N$ with $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$ in N. By performing this process simultaneously for all pairs of cardinals $\langle\lambda, \kappa\rangle$,[23] we obtain a model in which *every* pair of cardinals $\langle\lambda, \kappa\rangle$ has an associated eccentric set.[24]

Theorem (Theorem C)**.** *ZF is equiconsistent with ZF+"for all infinite cardinals $\lambda \leqslant \kappa$ there is a set X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$".*

Chapter 4: Maximal Independent Families

Maximal independent families are a curious combinatorial construction. An *independent family* is a set $\mathcal{A} \subseteq \mathscr{P}(X)$ such that, for all finite partial $p\colon \mathcal{A} \to 2$, $\bigcap\{A \mid p(A) = 1\} \cap \bigcap\{X \setminus A \mid p(A) = 0\} \neq \emptyset$. In ZFC (ZF with the axiom of choice) it is quite easy to construct these objects and, using Zorn's lemma [Kur22; Zor35],[25] one can extend an independent family to one that cannot be further increased (a *maximal* independent family). However, the situation dramatically changes when one replaces 'finite' in the definition with, say, 'countable' (a *σ-independent family*). In [Kun83] Kunen explores this matter and shows that the presence of a single maximal σ-independent family entails an inner model with a measurable cardinal,[26] a dramatically stronger assumption than ZFC alone. Not only this, but from a measurable cardinal one can take a forcing extension to produce a maximal σ-independent family. We slightly improve upon these techniques and use them, alongside appropriate large cardinal assumptions, to generate proper classes of maximal θ-independent families, for various uncountable θ.[27]

The statements of the following results are somewhat necessarily dense. $\mathrm{Add}(A, B)$ is described in Definition 2.4.2, Easton supports in Definition 2.4.8, Mitchell rank ($o(\kappa)$) in Definition 4.2.1, θ-strong compactness in Definition 4.2.2, and general forcing preliminaries in Section 2.4.

Proposition (Proposition 4.4.5)**.** *Let κ be measurable with normal measure $\mathcal{U}$, $2^\kappa = \kappa^+$, and $A \in \mathcal{U}$ be a set of regular cardinals. Let G be V-generic for the Easton-support iteration $\mathbb{P} = \ast_{\alpha \in A} \mathrm{Add}(\alpha, \alpha^+)$. Then in $V[G]$ there is a*

[23] This is a slight understatement of the process, but hopefully conveys the correct intuition.
[24] Rather, the pairs $\langle\lambda, \kappa\rangle$ with $\aleph_0 \leqslant \lambda \leqslant \kappa$.
[25] A theorem that is equivalent to AC under the additional assumption of ZF.
[26] Inner models and measurable cardinals are defined in Sections 2.4 and 2.6 respectively.
[27] A θ-independent family replaces 'finite' with 'cardinality less than θ' in the definition of independent family.

maximal κ-independent family $\mathcal{A} \subseteq \mathscr{P}(\kappa)$. Furthermore, if $\mathcal{V} \in V$ is a normal measure on κ with $A \notin \mathcal{V}$ then there is a normal measure $\hat{\mathcal{V}} \supseteq \mathcal{V}$ on κ in $V[G]$.

Theorem (Theorem D)**.** *Let κ be θ^+-strongly compact for some uncountable regular $\theta < \kappa$, with $2^{<\kappa} = \kappa$, and let G be V-generic for $\mathrm{Add}(\theta, \kappa)$. In $V[G]$, for all $\lambda \geqslant \kappa$ with $\mathrm{cf}(\lambda) \geqslant \kappa$, there is a maximal θ-independent family $\mathcal{A} \subseteq \mathscr{P}(\lambda)$.*

Theorem (Theorem E)**.** *Let V be a model of* ZFC+GCH*. Then there is a class-length forcing iteration $\mathbb{P}$ preserving* ZFC+GCH *such that, if $G \subseteq \mathbb{P}$ is V-generic, then whenever κ is a measurable cardinal in V there is a maximal κ-independent family $\mathcal{A} \subseteq \mathscr{P}(\kappa)$ in $V[G]$. Furthermore, whenever κ is a measurable cardinal in V, $o(\kappa)^V = 1 + o(\kappa)^{V[G]}$, and whenever κ is non-measurable in V it remains non-measurable in $V[G]$.*

Chapter 5: String Dimension

In this chapter we explore the idea of a *cardinal characteristic*, some well-defined cardinal value that (usually) ZFC alone does not solve. For example, the cardinality of the real numbers—often denoted $\mathfrak{c}$ for *continuum*—is an infinite 'aleph number', $\aleph_\alpha$ for some ordinal α. However, ZFC alone cannot deduce the exact value of α, or even if it is finite, or countable![28] In the case of this chapter, we define an analogue to VC dimension that allows for infinitary shattering: the *string dimension* of $F \subseteq 2^\kappa$ is the least δ such that for all $A \subseteq \kappa$, if $|A| = |\delta|$, then $\{f \restriction A \mid f \in F\} \neq 2^A$.[29] We then consider the question of how many low-string-dimension subsets of 2^κ it takes to 'cover' 2^κ. More precisely, we define $\mathfrak{sd}(\delta, \kappa)$ to be the least size of a family $\mathcal{A} \subseteq \mathscr{P}(2^\kappa)$ that covers 2^κ (that is, $\bigcup \mathcal{A} = 2^\kappa$), and for all $F \in \mathcal{A}$, the string dimension of F is strictly less than δ. While the results within this chapter are mostly smaller, building up together to give us a good understanding of string dimension, one curiosity is that for strong limit cardinals[30] the characteristic $\mathfrak{sd}$ is often determined.

Theorem (Theorem 5.4.4)**.** *If κ is a strong limit, and δ is least such that $2^\delta \geqslant \mathrm{cf}(\kappa)$, then $\mathfrak{sd}(\delta, \kappa) = 2^\kappa$.*

[28]That is, the so-called 'continuum hypothesis' is very independent of ZFC.

[29]Alternatively, one can imagine this from a 'shattering' point of view: A family $\mathcal{F} \subseteq \mathscr{P}(X)$ *shatters* $Y \subseteq X$ if $\{A \cap Y \mid A \in \mathcal{F}\} = \mathscr{P}(Y)$. The string dimension of $\mathcal{F}$ is the least δ such that $\mathcal{F}$ shatters no subset of X of cardinality δ. Here we translate between $f \in 2^X$ and $A \subseteq X$ by the conversion $f \mapsto \{x \in X \mid f(x) = 1\}$, discussed in more detail in Section 2.1.

[30]Cardinals κ such that, for all $\alpha < \kappa$, $2^\alpha < \kappa$.

Using the various techniques from throughout the chapter, we eventually construct Figure 5.6 (reproduced in this section), which documents the known consistency for statements such as "$\mathfrak{sd}(\aleph_1, \aleph_1) < \mathfrak{sd}(\aleph_0, \aleph_1)$".

Con ↓<→	$\mathfrak{sd}(\aleph_1, \aleph_0)$	$\mathfrak{c}$	$\mathrm{cf}(\mathfrak{c})$	$\mathrm{cf}\left(2^{\aleph_1}\right)$	$\mathfrak{sd}(\aleph_2, \aleph_1)$	$\mathfrak{sd}(\aleph_1, \aleph_1)$	$\mathfrak{sd}(\aleph_0, \aleph_1)$
$\mathfrak{sd}(\aleph_1, \aleph_0)$		Yes	Yes	Yes	Yes	Yes	Yes
$\mathfrak{c}$	No		No	Yes	Yes	Yes	Yes
$\mathrm{cf}(\mathfrak{c})$	No	Yes		Yes	Yes	Yes	Yes
$\mathrm{cf}\left(2^{\aleph_1}\right)$	?	Yes	Yes		No	Yes	Yes
$\mathfrak{sd}(\aleph_2, \aleph_1)$	?	Yes	Yes	Yes		Yes	Yes
$\mathfrak{sd}(\aleph_1, \aleph_1)$	No	Yes	Yes	Yes	No		Yes
$\mathfrak{sd}(\aleph_0, \aleph_1)$	No	No	No	?	No	No	

Figure 5.6: Known consistencies regarding $\mathfrak{sd}(\delta, \kappa)$ with $\kappa = \aleph_0$ or $\aleph_1$.

In this investigation it also became helpful to properly understand κ-finality. One may wish that in, say, a forcing iteration of length γ, the γth stage of the iteration does not add any real numbers, despite each previous stage doing so. We say that a forcing iteration is *κ-final* to mean that every sequence of length less than κ composed of ground-model elements was added at an intermediate stage.[31] By adapting the definition of distributivity, we come up with a notion of pseudodistributivity and show that it exactly characterises this property.

Theorem (Theorem F)**.** *$\mathbb{P}$ is κ-final if and only if it is κ-pseudodistributive.*

This allowed for some very general characterisations of when certain forcing iterations (or products viewed as iterations) would increase or decrease the cardinal characteristics $\mathfrak{sd}(\delta, \kappa)$, though this technique can also be applied to many cardinal characteristics derived from combinatorial descriptions.

1.3 Eccentricity

One of my[32] earliest projects was working with Asaf Karagila on [KR24], in which we demonstrate how to introduce a new set X into the universe such that its *Hartogs number* $\aleph(X)$ and *Lindenbaum number*[33] $\aleph^*(X)$ were equal to some pre-determined values λ and κ. What you must believe for the moment is that most of the time[34] any set X will have $\aleph(X) = \aleph^*(X)$, and so introducing

[31] So in the case of not adding reals, ω_1-finality suffices.
[32] Let us return briefly to personal pronouns for this sappy soliloquy.
[33] Introduced properly in Chapter 3.
[34] By which I mean 'in ZFC'.

some new Y such that $\aleph(Y) < \aleph^*(Y)$ is an oddity. In my mind, I pictured the 'normal' sets, those X with $\aleph(X) = \aleph^*(X)$, as circular, a ring without deformity. The new sets, on the other hand, those Y with $\aleph(Y) < \aleph^*(Y)$, I imagined as distorted in some way, more akin to an ellipse. Therefore, it seemed natural to me that this property that we are injecting into our universe when we undertake these constructions is *eccentricity*. The more eccentric the ellipse, the further from that circle it became, the further from the 'expected' behaviour of sets that we see.

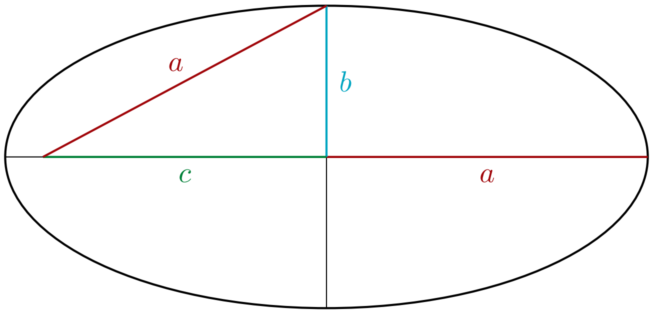

Figure 1.1: The eccentricity of an ellipse can be calculated as c divided by a. As eccentricity approaches 0, the ellipse becomes a circle.

This is, of course, not our typical use of the word eccentric and this nomenclature, of course, tickled me to no end. By the time of [Rya24c], my second paper, I had carved out this name and it was accepted[35] in the corpus of mathematics. This is one joy of mathematical research, the ability to directly enter the stream of data and place oneself among it. Even if my impacts remain small, they carry with them a part of myself.

Therefore, we enter into this thesis with fascination and hope, carving out a space among the quantum foam of the universe—our universe, *any* universe—from which to plot a course. After all, this is my daily prayer; crouched on my knees, burbling numbers with the persistence of waves against the shore, and never a drop closer.

[35] Or at least published.

Chapter 2

Preliminaries

> "One of my brothers was devoted to what he called the smallest infinity. The most realistic impossibility—so close that one could almost count it."
>
> Lara Welch, *Feast of Saints*

We work in ZF and, in Chapters 4 and 5, we work in ZFC, though we are each encouraged to attempt to place this work in an appropriate metalogical context. Unless otherwise specified, we shall generally work in a fixed universe of sets, denoted V, with membership relation $\in$. Given a set X, we denote by $\mathscr{P}(X)$ the power set of X, $\{Y \mid Y \subseteq X\}$, and we denote by $\bigcup X$ the union of X: $\{Z \mid (\exists Y) Z \in Y \wedge Y \in X\}$. We generally use angular brackets to denote tuples, so $\langle a, b\rangle$ is the tuple (or ordered pair, or sequence) of a and b.[36] This notation extends to indexed tuples, so $\langle a_i \mid i \in I\rangle$ is the tuple with a_i in the ith position. Given sets X, Y, we denote by $X \times Y$ the set of tuples $\langle x, y\rangle$ such that $x \in X$ and $y \in Y$. When $X = Y$, we typically write X^2 instead of $X \times X$. For finite[37] tuples $\vec{a} = \langle a_i \mid i = 0, \ldots, n-1\rangle$ and $\vec{b} = \langle b_j \mid j = 0, \ldots, m-1\rangle$, we denote by $\vec{a} \frown \vec{b}$ the *concatenation* of $\vec{a}$ and $\vec{b}$,

[36] We shall arbitrarily take the Kuratowski encoding of binary tuples from [Kur21], so $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$. Thus the first element of $\langle a, b\rangle$ is the unique element of $\bigcap\langle a, b\rangle$, and the second element of $\langle a, b\rangle$ is the unique element appearing in exactly one member of $\langle a, b\rangle$.

[37] In fact this concept extends naturally to any linearly ordered collection of tuples each indexed by a linear order. A *linear order* is a tuple $\langle L, \leqslant\rangle$ where L is a set and $\leqslant \subseteq L^2$ such that, if we write $a \leqslant b$ to mean $\langle a, b\rangle \in \leqslant$, the following hold: For all $a \in L$, $a \leqslant a$; if $a, b \in L$ are distinct then exactly one of $a \leqslant b$ or $b \leqslant a$ holds; and if $a, b, c \in L$, $a \leqslant b$, and $b \leqslant c$, then $a \leqslant c$. One is invited to imagine what it means to concatenate two linear orders.

the tuple $\langle c_k \mid k = 0, \dots, n+m-1\rangle$ given by

$$c_k = \begin{cases} a_k & \text{if } k < n \\ b_{k-n} & \text{if } k \geqslant n. \end{cases}$$

2.1 Functions

By a *(total) function* $f\colon X \to Y$ we mean a set $f \subseteq X \times Y$ such that for all $x \in X$ there is *unique* $y \in Y$ such that $\langle x, y\rangle \in f$. By a *partial function* $f\colon X \to Y$ we mean a set $f \subseteq X \times Y$ such that for all $x \in X$ there is *at most one* $y \in Y$ such that $\langle x, y\rangle \in f$. By the *domain* of a function f we mean $\{x \mid (\exists y)\langle x, y\rangle \in f\}$, denoted $\operatorname{dom}(f)$. Note that a partial function $f\colon X \to Y$ is just a total function $\operatorname{dom}(f) \to Y$. For $x \in \operatorname{dom}(f)$ we write $f(x)$ for the unique element of Y such that $\langle x, f(x)\rangle \in f$, and may write $x \mapsto y$ to mean $y = f(x)$.[38] A function f is an *injection* if for all $x, x' \in X$, $f(x) = f(x') \implies x = x'$; a *surjection* if for all $y \in Y$ there is $x \in X$ such that $f(x) = y$; and a *bijection* if it is an injection and a surjection. Given a function $f\colon X \to Y$ and a set A, we denote by $f \restriction A$ the restriction of f to A. Hence, $f \restriction A$ is the function $X \cap A \to Y$ given by $f \cap (A \times Y)$. We denote by $f``A$ the *pointwise image* of $A$ under $f$. That is, $f``A = \{f(a) \mid a \in A \cap X\}$. We denote by $f^{-1}(A)$ the *preimage* of the set A, that is $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$. Given sets X and Y, we write:

1. Y^X for the set of functions $X \to Y$;
2. $|X| \leqslant |Y|$ to mean there is an injection $X \to Y$;
3. $|X| \leqslant^* |Y|$ to mean there is a partial surjection $Y \to X$;[39]
4. $|X| < |Y|$ to mean $|X| \leqslant |Y|$ and $|Y| \nleqslant |X|$;
5. $|X| <^* |Y|$ to mean $|X| \leqslant^* |Y|$ and $|Y| \nleqslant^* |X|$;

[38] Indeed, we sometimes use this notation to define a function. If the domain of f is understood, say $\operatorname{dom}(f) = A$, then we write $x \mapsto \mathtt{P}_x$ to mean that $f = \{\langle x, \mathtt{P}_x\rangle \mid x \in A\}$, where $\mathtt{P}$ is some process of obtaining a set after being given x. For example, we might define the successor function on the natural numbers by writing '$x \mapsto x+1$', rather than $\{\langle x, x+1\rangle \mid x \in \mathbb{N}\}$.

[39] Equivalently, either $X = \emptyset$ or there is a (total) surjection $Y \to X$. $\emptyset$ here denotes the 'empty set,' the unique set with no elements.

6. $|X| = |Y|$ to mean $|X| \leqslant |Y|$ and $|Y| \leqslant |X|$;[40]

7. $|X| =^* |Y|$ to mean $|X| \leqslant^* |Y|$ and $|Y| \leqslant^* |X|$;

8. $\mathscr{P}_X(Y)$ to mean $\{A \subseteq Y \mid |A| < |X|\}$; and

9. $[X]^Y$ to mean $\{A \subseteq Y \mid |A| = |Y|\}$.

We shall sometimes say that X and Y are *equipotent* or *have the same cardinality* to mean $|X| = |Y|$.

We sometimes identify $\{0,1\}^X$ and $\mathscr{P}(X)$ by identifying the function $f\colon X \to \{0,1\}$ with the set $\{x \in X \mid f(x) = 1\}$.

2.1.1 Reals

Usually there is a strong consensus on what a "real number" is in mathematics: a (possibly infinite) collection of digits 0–9 with a decimal point somewhere in the middle and possibly a 'minus symbol' $-$ at the front.[42] However, for myriad fascinating reasons that we shall not divulge, within this text we somewhat loosely use "real number" (or usually just a *real*) to mean one of a few different things. While we may say "real number" to mean one of those plus-or-minus decimal point objects, we may also say "real number" to mean:

- a function $\mathbb{N} \to \mathbb{N}$;[43]
- a function $\mathbb{N} \to \{0,1\}$; or
- a subset of $\mathbb{N}$.

Usually the symbol $\mathbb{R}$ is reserved for the set of real numbers as we typically understand them. The most important takeaway when dealing with this is that $\mathbb{N}^{\mathbb{N}}$, $\{0,1\}^{\mathbb{N}}$, and $\mathscr{P}(\mathbb{N})$ have structure that makes them naturally similar to real numbers, and that $|\mathbb{R}| = |\mathbb{N}^{\mathbb{N}}| = |\{0,1\}^{\mathbb{N}}| = |\mathscr{P}(\mathbb{N})|$.

[40] By the Cantor–Bernstein theorem,[41] $|X| = |Y|$ if and only if there is a bijection $X \to Y$.

[41] Stated in [Can87] without proof (it seems likely that Cantor's original (unwritten) proof relied on the linear ordering of cardinality under $\leqslant$, which was shown to be equivalent to the axiom of choice in [Har15]). An early published proof can be found in [Zer08], and a more modern proof in English can be found in [Jec03, Theorem 3.2].

[42] Except $0.999\ldots = 1.000\ldots$, so I suppose we have to say that those are "the same" in some sense, and $-0 = 0$, and $1.0 = 1$, and... well, let us say that problems arise with this definition. Fortunately, mathematicians have come up with a much more robust way of defining real numbers that does away with much of this semantic difficulty, but that is beyond the scope of this text.

[43] Where $\mathbb{N}$ is the set of natural numbers, 0, 1, 2, etc. We identify $\mathbb{N}$ with ω, defined later.

2.1.2 Relations

By a (binary) *relation* on a set X, we mean a set $R \subseteq X^2$, where we interpret R as the set of pairs $\langle x, y\rangle$ such that x is related to y (according to R). When R is understood as a relation on X, rather than a subset of X^2, we will often write $x\ R\ y$ in place of $\langle x, y\rangle \in R$. This is especially common for relations such as '$<$', as we almost always write $x < y$ instead of $\langle x, y\rangle \in <$. Other common notation includes $R(x, y)$ to mean $\langle x, y\rangle \in R$. More generally, we sometimes consider relations between two non-equal sets, so $R \subseteq X \times Y$, and in this case we will still write $x\ R\ y$ to mean $\langle x, y\rangle \in R$. We also sometimes consider relations on more than two sets, such as $R \subseteq X \times Y \times Z$,[44] or even $R \subseteq \prod A$ (see Section 2.2) for an arbitrary collection A of sets. If R is a relation on sets $X_1, \ldots, X_n$, we say that R has *arity* n. There is usually no 'interfix' notation for relations of arity greater than two.[45] A binary relation R on X is *well-founded* if all non-empty $Y \subseteq X$ have an R*-minimal element*. That is, there is $x \in Y$ such that for all $y \in Y$, $y\ R\ x$ does not hold.

2.1.3 The axiom of choice

For a set X, we denote by $\prod X$ the collection of functions $c\colon X \to \bigcup X$ such that, for all $x \in X$, $f(x) \in x$. For all X, $c \in \prod X$ is called a *choice function*, and the *axiom of choice* AC is the statement that for all sets X, if $\emptyset \notin X$, then $\prod X \neq \emptyset$. Consequences of the axiom of choice are discussed further in Section 3.2.

2.2 Ordinals

A set X is *transitive* if for all $Y \in X$ and all $Z \in Y$, $Z \in X$. By the *transitive closure* of X we mean the smallest transitive class containing X, so $\bigcap\{Y \mid X \subseteq Y \text{ and } Y \text{ is transitive}\}$. It is a theorem of ZF that the transitive closure of X is always a set (in particular, we do not need to contend with the empty intersection problem). A *well-order* is a linear order $\langle L, \leqslant\rangle$ such that every non-empty subset of L has a $\leqslant$-least element.[46] By an *ordinal* we mean

[44] While it is technically an abuse of notation, we generally identify $X \times (Y \times Z)$ with $(X \times Y) \times Z$, as we can easily translate between the two by the function $\langle x, \langle y, z\rangle\rangle \mapsto \langle\langle x, y\rangle, z\rangle$.

[45] One interesting exception is the independence relation $\mathop{\smile\!\!\!\!\!\!|}$ in model theory. Common notation for 'A is independent from B over C' (that is, $\langle A, B, C\rangle \in \mathop{\smile\!\!\!\!\!\!|}$) is $A \mathop{\smile\!\!\!\!\!\!|}_C B$.

[46] That is, for all non-empty $P \subseteq L$ there is $a \in P$ such that for all $b \in P$, $a \leqslant b$. If we define $<$ by $a < b$ if $a \leqslant b$ and $a \neq b$ then this is equivalent to $<$ being a well-founded relation

a transitive set X that is well-ordered by $\in$.[47] We denote by Ord the class of ordinals. Ordinals are usually denoted by Greek letters, particularly those early in the alphabet. In the case of finite ordinals, we identify the (unique) ordinal of length n with the natural number n. In particular, $0 = \emptyset$, and we often use that $n = \{0, \ldots, n-1\}$.[48] We endow Ord with the relations $<$ and $\leqslant$, defined by $\alpha < \beta$ if $\alpha \in \beta$, and $\alpha \leqslant \beta$ if $\alpha \in \beta$ or $\alpha = \beta$. Given a set X of ordinals, the *supremum* of this set, $\sup X$, is $\bigcup X$. Note that $\sup X$ is indeed an ordinal and is genuinely the supremum of X in order $\langle \mathrm{Ord}, \leqslant \rangle$. We define ω to be the least infinite ordinal, so $\omega = \sup\{n \mid n \text{ is a finite ordinal}\}$.[49] An ordinal α is a *limit ordinal* if $\sup \alpha = \alpha$. Otherwise, α is a *successor ordinal*.

Fact 2.2.1. *The class relations* $\langle \mathrm{Ord}, < \rangle$ *and* $\langle V, \in \rangle$ *are well-founded.*

In particular, inductive definitions for ordinals are well-defined. Loosely speaking, this means that if we want to define a function or process f on ordinals then we can define $f(\alpha)$ using $f \restriction \alpha$ without worrying about circular logic. Let us use this to define *ordinal arithmetic*:

Given an ordinal α, we define $\alpha + 0 = \alpha$ and $\alpha + 1 = \{\alpha\} \cup \alpha$. Note that $\alpha + 1$ is also an ordinal. For $\beta > 1$ we define $\alpha + \beta = \sup\{(\alpha + \gamma) + 1 \mid \gamma < \beta\}$. We also define $\alpha \times \beta = \sup\{(\alpha \times \gamma) + \alpha \mid \gamma < \beta\}$ (we sometimes write $\alpha\beta$ instead of $\alpha \times \beta$). While ordinal exponentiation can also be defined, we do not use it in this text. Note that α is a successor ordinal if and only if there is β such that $\alpha = \beta + 1$.

Given two ordinals α and β, we use interval notation to mean intervals in Ord. That is,

$$\begin{aligned}
(\alpha, \beta) &= \{\gamma \in \mathrm{Ord} \mid \alpha < \gamma < \beta\}\\
[\alpha, \beta] &= \{\gamma \in \mathrm{Ord} \mid \alpha \leqslant \gamma \leqslant \beta\}\\
(\alpha, \beta] &= \{\gamma \in \mathrm{Ord} \mid \alpha < \gamma \leqslant \beta\}\\
[\alpha, \beta) &= \{\gamma \in \mathrm{Ord} \mid \alpha \leqslant \gamma < \beta\}.
\end{aligned}$$

Note that these intervals can be empty. For example, for any α, $[\alpha, \alpha) = \emptyset$.

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on L.

[47] Rather, X is well-ordered by the relation $\in \restriction X \coloneqq \{\langle x, y \rangle \in X^2 \mid x \in y\}$. Thinking of $\in$ as the collection $\{\langle x, y \rangle \mid x \in y\}$ (importantly this is not a *set*, but rather a collection of sets that we call a *class*), we could write the relation $\in \restriction X$ as $\in \cap X^2$.

[48] For example, we shall henceforth say $2^{\mathbb{N}}$ instead of $\{0,1\}^{\mathbb{N}}$.

[49] If you are worried that $\{n \mid n \text{ is a finite ordinal}\}$ may not be a set, have no fear. We simply declare that it is as an axiom.

Given a set X of ordinals, we denote by $\mathrm{ot}(X)$ the *order type* of X. That is, the unique ordinal α such that $\langle \alpha, \in \rangle \cong \langle X, < \rangle$.[50] Given ordinals α and β, we denote by $[\alpha]^{(\beta)}$ the set $\{X \subseteq \alpha \mid \mathrm{ot}(X) = \beta\}$.[51]

A set $X \subseteq \alpha$ is *cofinal in* α if for all $\beta < \alpha$ there is $\gamma \in X$ such that $\beta \leqslant \gamma$. By the *cofinality* of α we mean $\min\{\mathrm{ot}(X) \mid X \subseteq \alpha \text{ is cofinal}\}$, denoted $\mathrm{cf}(\alpha)$. By a *cofinal sequence in* α, we mean a sequence $\langle \beta_\gamma \mid \gamma < \delta \rangle$ such that:

1. $\{\beta_\gamma \mid \gamma < \delta\}$ is cofinal in α; and
2. for all $\gamma < \gamma' < \delta$, $\beta_\gamma < \beta_{\gamma'}$.

Note that any cofinal subset of α begets a cofinal sequence in α by enumerating its elements.

Given that $\langle V, \in \rangle$ is well-founded, we may define *set rank* inductively by

$$\mathrm{rk}(X) = \sup\{\mathrm{rk}(Y) + 1 \mid Y \in X\}.$$

We denote by V_α the set $\{X \mid \mathrm{rk}(X) < \alpha\}$. Equivalently, we can define V_α by $V_\alpha = \bigcup\{\mathscr{P}(V_\beta) \mid \beta < \alpha\}$.

2.2.1 Mostowski collapse

Let E be a binary relation with (potentially class-sized) domain A. The structure $\langle A, E \rangle$ is said to be: *extensional* if, for all $x, y \in A$, $x = y$ if and only if $(\forall z) E(z, x) \leftrightarrow E(z, y)$; *well-founded* if for all non-empty $X \subseteq A$ there is $x \in X$ such that, for all $y \in X$, $\neg E(y, x)$; and *set-like* if for all $x \in A$, the *a priori* class $\{y \in A \mid E(y, x)\}$ is in fact a set. If $\langle A, E \rangle$ is an extensional, well-founded, and set-like structure then there is a unique transitive class B such that $\langle B, \in \rangle \cong \langle A, E \rangle$. We refer to B as the *Mostowski collapse* of $\langle A, E \rangle$.

2.3 Cardinals

Given a set X, we denote by $|X|$ its cardinal number. In the abstract, we would like $|X|$ to behave as the equivalence classes of the equipotence relation: $|X| = \{Y \mid |X| = |Y|\}$.[52] However, this is necessarily a proper class (unless $X = \emptyset$) and so for referring to the

[50] Meaning $\langle \alpha, \in \rangle$ and $\langle X, < \rangle$ are isomorphic as linear orders.
[51] Note that this is distinct from $[\alpha]^\beta$, meaning $\{X \subseteq \alpha \mid |X| = |\beta|\}$.
[52] Or, alternatively, we would like $|X|$ to be some canonically chosen set such that X and $|X|$ are equipotent. This is possible in the case of ZFC, but cannot be done in general in ZF.

object $|X|$ in-universe, we break into two cases. If X can be well-ordered, then $|X| = \min\{\alpha \in \mathrm{Ord} \mid |X| = |\alpha|\}$. Otherwise, $|X|$ is the *Scott cardinal* of X, the set $\{Y \in V_\alpha \mid |X| = |Y|\}$, where α is least such that the set is non-empty. By a *cardinal* we mean a well-ordered cardinal. That is, an ordinal α such that $|\alpha| = \alpha$. We denote by Card the class of cardinals. We typically denote cardinals by Greek letters, particularly κ and λ. When comparing the cardinalities of sets and cardinals, we may omit the 'absolute value' notation from the cardinal and simply write, for example, $|X| \leqslant \lambda$ instead of $|X| \leqslant |\lambda|$.

Given two cardinals λ and κ, we denote by $\lambda + \kappa$ the cardinality of the set $\{0\} \times \lambda \cup \{1\} \times \kappa$, and by $\lambda \times \kappa$ (or $\lambda\kappa$) the cardinality of the set $\lambda \times \kappa$.[53] If both λ and κ are finite then this coincides with the usual arithmetic on finite numbers. Otherwise, $\lambda + \kappa = \lambda \times \kappa = \max\{\lambda, \kappa\}$. We use λ^κ to denote the cardinality of the set λ^κ. We denote by λ^+ the least cardinal greater than λ. That is, $\lambda^+ = \min\{\kappa \in \mathrm{Card} \mid |\lambda| < |\kappa|\}$.[54] If α is an ordinal, then $\lambda^{+\alpha}$ is defined inductively to be $\sup\{(\lambda^{+\beta})^+ \mid \beta < \alpha\}$, and $\aleph_\alpha \coloneqq (\omega)^{+\alpha}$. If α is finite and small, we sometimes write α many +s to indicate $^{+\alpha}$, so $\lambda^{++} = \lambda^{+2}$, $\lambda^{+++} = \lambda^{+3}$, etc. When trying to emphasise the ordinal aspect of $\aleph_\alpha$, we may denote $\aleph_\alpha$ by ω_α. Given a cardinal λ, we inductively define $\beth_\alpha(\lambda)$ to be

$$\sup\left(\{\lambda\} \cup \left\{2^{\beth_\gamma(\lambda)} \,\middle|\, \gamma < \alpha\right\}\right).$$

Remark. Much of cardinal arithmetic is an implicit abuse of notation. When two objects α and β are understood to be *ordinals*, we use $+$ and $\times$ to refer to ordinal arithmetic. Conversely, if α and β are understood to be *cardinals*, we use $+$ and $\times$ to refer to cardinal arithmetic. When there is room for ambiguity, we shall explicitly call out which form of arithmetic is being used.

When λ and κ are understood as cardinals (rather than as ordinals), we denote by (λ, κ), $[\lambda, \kappa]$, etc. the intervals restricted to cardinals. So for example $(\lambda, \kappa) = \{\mu \in \mathrm{Card} \mid \lambda < \mu < \kappa\}$.

Given a set X, we write $[X]^\lambda$ to mean $\{A \subseteq X \mid |A| = \lambda\}$ as usual, and write $[X]^{<\lambda}$ to mean $\{A \subseteq X \mid |A| < \lambda\}$. That is, $[X]^{<\lambda} = \mathscr{P}_\lambda(X)$.

When working in nested universes of sets $V \subseteq W$, where V is transitive in W,[55] we may use superscripts to denote certain objects as 'computed' in a

[53] This abuse of notation (and indeed the upcoming abuse) is not generally a source of confusion, though we are careful to be more precise whenever confusion may arise.

[54] This object is well-defined and exists for all λ.

[55] So the set membership relation $\in_V$ is just $\in_W \cap V \times V$ and, for all $x \in V$, if $y \in W$ and $y \in x$, then $y \in V$.

given universe. For example, $\mathscr{P}(X)^V$ denotes the power set of X *in* V. That is, $\mathscr{P}(X)^V = \{A \subseteq X \mid A \in V\}$, which may be a strict subset of $\mathscr{P}(X)^W$.

2.4 Forcing

Forcing is a fundamental technique that permeates all corners of this thesis. One can imagine it thus: we are given a universe (we set theorists use 'universe' to mean a collection of 'sets' obeying certain rules, but you are free to imagine any complicated collection of stuff, such as the universe that we live in, your local government office, etc.) and, within that universe, a collection of promises. We shall denote the universe by V, and the promises by $\mathbb{P}$. These promises are generally about some new structure that we want to build, and represent the interests of many different parties. Sometimes, one promise will entirely satisfy another: If p is the promise "this new road must have at least two lanes" and q is the promise "this new road must have at least three lanes", then q is *stronger* than p. On the other hand, we cannot satisfy every promise: There may be two promises p and q[56] that are *incompatible*, written $p \perp q$. However, by a miracle of mathematics,[57] we may find a brand new 'generic' structure G that satisfies many of the promises in $\mathbb{P}$.[58] However, we could not find G in the current universe V. Rather, we had to introduce G from outside of the universe, thus extending our new universe to what we call $V[G]$.

Let us re-introduce forcing for the more mathematically savvy reader, before our final, most technical preliminaries on the subject. Cohen's *forcing* was a watershed idea that fundamentally altered how we are able to do set theory. A major issue of early set theory was how difficult it was to manipulate models. While Gödel [Göd40] found great success in exhibiting the *constructible universe* L, an *inner model*,[59] and similar methods could be used to find other inner models, there were no techniques for *extending* models. Indeed, it was not even known if "$V = L$"[60] was a consequence of ZF. In [Coh63], Cohen lays out the forcing technique and shows the independence of four great open problems of set theory. Namely, he shows that if ZF is consistent, then there

[56] For example, p is "this new road must have *exactly* two lanes" and q is "this new road must have *exactly* three lanes".

[57] The Baire categoricity theorem.

[58] For some well-defined notion of 'many of the promises'.

[59] Assume that we have *some* model V of $\mathsf{ZF(C)}$, with set membership relation $\in$. Then an *inner model* of V is a transitive subcollection $M \subseteq V$.

[60] That is, "there are no non-constructible sets". [Jec03, Chapter 13] gives an approachable overview, though we would be remiss not to mention [Kan09, Chapter 3].

are models of ZF satisfying each of the following:

1. There is a non-constructible set, yet the continuum hypothesis[61] holds;
2. $\mathbb{R}$ has no well-ordering;
3. AC holds, but CH fails; and
4. the axiom of countable choice for two-element sets fails.[62]

While this impact was immediate and of great importance, the even greater looming influence was the technique itself. Cohen had devised a way to extend a model V of ZFC by introducing a new set G, which we denote $V[G]$. For the construction to work, Cohen required that the new set G be 'generic' in some sense, which has since been formalised and refined into the modern conceptualisation. What we call forcing today is a sleek machine that, despite relentless study, continues to surprise in its power. The abstract notions of products, iterations, and morphisms are all well-understood, and many properties[63] have arisen in the abstract study of forcing. It is truly a remarkable tool.

Let us return to the serious business of preliminaries. By a *notion of forcing* (or just a *forcing*) we mean a preorder[64] $\langle \mathbb{P}, \leqslant_{\mathbb{P}} \rangle$ with maximum element denoted $\mathbb{1}_{\mathbb{P}}$. In many cases, we will omit instances of the subscript $\mathbb{P}$ when clear from context. We call the elements of $\mathbb{P}$ *conditions* and, for two conditions p, q, we write $q \leqslant p$ to mean that q *extends* p (or q is *stronger* than p).[65] In particular, we force downwards. Two conditions p, p' are said to be *compatible*, written $p \parallel_{\mathbb{P}} p'$, if they have a common extension in $\mathbb{P}$. Otherwise, we say that p and q are *incompatible*, denoted $p \perp_{\mathbb{P}} q$. By a $\mathbb{P}$-*name*, we mean a set of tuples $\langle p, \dot{x} \rangle$, where $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name. More formally, we define $V_\alpha^{\mathbb{P}}$ inductively for $\alpha \in \mathrm{Ord}$ by

$$V_\alpha^{\mathbb{P}} = \bigcup_{\beta<\alpha} \mathscr{P}(\mathbb{P} \times V_\beta^{\mathbb{P}}).$$

[61] The continuum hypothesis (CH) says that the real numbers have cardinality $\aleph_1$ or, in the language of cardinal arithmetic, $2^{\aleph_0} = \aleph_1$. Some prefer to define CH as "if $X \subseteq \mathbb{R}$ then $|X| \leqslant |\mathbb{N}|$ or $|X| = |\mathbb{R}|$", which is a weaker statement in ZF with interesting implications.

[62] That is, there is a set X such that $|X| = \aleph_0$, and every element of X has cardinality 2, but there is no choice function for X. This can be thought of as "you cannot pick one sock from an infinite pile of pairs of socks", though the same statement fails with pairs of shoes because one could, say, always pick the left shoe.

[63] Such as chain conditions, distributivity, centredness, etc.

[64] A binary relation $\leqslant$ on a set X is a *preorder* if it is *reflexive* (for all $x \in X$, $x \leqslant x$) and *transitive* (for all $x, y, z \in X$, if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$).

[65] We follow Goldstern's alphabet convention: if † and ‡ are letters representing elements of $\mathbb{P}$, with ‡ coming after † alphabetically, then † $\leqslant$ ‡ only if † = ‡.

Then a $\mathbb{P}$-name is an element of $V^{\mathbb{P}} = \bigcup_{\alpha\in\mathrm{Ord}} V^{\mathbb{P}}_\alpha$.

A set $A \subseteq \mathbb{P}$ is an *antichain* if all conditions in A are pairwise incompatible. $D \subseteq \mathbb{P}$ is *dense* if for all $p \in \mathbb{P}$ there is $q \in D$ such that $q \leqslant p$. $O \subseteq \mathbb{P}$ is *open* if for all $p \in O$ and all $q \leqslant p$, $q \in O$. $F \subseteq \mathbb{P}$ is a *filter* if for all $q \in F$ and $p \geqslant q$, $p \in F$, and for all $p, p' \in F$ there is $q \in F$ such that $q \leqslant p, p'$. A *V-generic filter* of $\mathbb{P}$ is a filter $G \subseteq \mathbb{P}$ such that for all dense $D \subseteq \mathbb{P}$ with $D \in V$, $G \cap D \neq \emptyset$. For $\dot{x}$ a $\mathbb{P}$-name and G a V-generic filter, the *interpretation* of $\dot{x}$ by G is inductively defined as

$$\dot{x}^G = \left\{\dot{y}^G \;\middle|\; (\exists p \in G)\langle p, \dot{y}\rangle \in \dot{x}\right\}.$$

The *forcing extension* of V by G is then $V[G] = \{\dot{x}^G \mid \dot{x} \in V^{\mathbb{P}}\}$.[66] Given a set of $\mathbb{P}$-names X we define the *bullet name* of X, denoted $X^\bullet$, to be $\{\langle \mathbb{1}_{\mathbb{P}}, \dot{x}\rangle \mid \dot{x} \in X\}$. Then $X^\bullet$ is always interpreted as the set $\{\dot{x}^G \mid \dot{x} \in X\}$ in the extension $V[G]$. This notation extends to tuples, functions, etc. with ground model domains. For $X \in V$ we define the *check name* of X, denoted $\check{X}$, to be $\{\check{x} \mid x \in X\}^\bullet$. That is, $\check{X}^G = X \in V[G]$ for all V-generic $G \subseteq \mathbb{P}$. When it would be unwieldy to place a check above the symbol(s) representing a set, we may put the check to the right of the set instead, for example using $\langle 1, 2\rangle^\vee$ for the check name of the tuple $\langle 1, 2\rangle$, or $(2^{\omega_1})^\vee$ for the set of all *ground model* functions $\omega_1 \to 2$. We may alternatively use the bullet notation to define a canonical name of a definable object. For example, $\mathscr{P}(\check{X})^\bullet$ is the canonical name for the power set of X in the forcing extension.

For each formula $\varphi(x_0, \ldots, x_{n-1})$ in the language of set theory with no parameters there is a definable relation $\Vdash_{\mathbb{P}} \varphi(x_0, \ldots, x_{n-1})$ between $\mathbb{P}$ and $(V^{\mathbb{P}})^n$ such that $p \Vdash_{\mathbb{P}} \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ if and only if for all V-generic filters G with $p \in G$, $V[G] \vDash \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$.

A forcing $\mathbb{P}$ is:

1. κ-c.c., or has the *κ-chain condition*, if every antichain in $\mathbb{P}$ has cardinality less than κ.[67]

2. *κ-distributive* if for all $\gamma < \kappa$ and all V-generic G, $(V^\gamma)^{V[G]} = (V^\gamma)^V$.

3. *κ-closed* if for all $\gamma < \kappa$ and all descending chains $\{p_\alpha \mid \alpha < \gamma\}$, so $\alpha < \beta$ implies that $p_\alpha \geqslant p_\beta$, there is $q \in \mathbb{P}$ such that for all $\alpha < \gamma$, $q \leqslant p_\alpha$. We

[66] Note that we sometimes use 'set builder' notation for proper classes. As long as we do not try to pass off these classes as sets, no issues shall arise from this.

[67] We generally write c.c.c. (*countable chain condition*) instead of $\aleph_1$-c.c.

shall write σ-closed for $\aleph_1$-closed.

Fact 2.4.1 ([Jec03, Theorem 15.6, Exercise 15.5])**.** *A forcing $\mathbb{P}$ is κ-distributive if and only if the intersection of fewer than κ-many open dense sets is open dense.*

Definition 2.4.2 (Cohen forcing)**.** Given non-empty sets A and B we denote by $\mathrm{Add}(B, A)$ the notion of forcing given by partial functions $p\colon A \times B \to 2$ such that $|p| < |B|$.[68] Note that if κ is a cardinal and A is non-empty then $\mathrm{Add}(\kappa, A)$ is $\mathrm{cf}(\kappa)$-closed and has the $(\kappa^{<\kappa})^+$-c.c. Note that $\mathrm{Add}(B, A) \cong \mathrm{Add}(B', A')$ if and only if $|B| = |B'|$ $|A \times B| = |A' \times B'|$.

Definition 2.4.3 (Product forcing)**.** Given a collection $\{\langle \mathbb{P}_i, \leqslant_i\rangle \mid i \in I\}$ of notions of forcing, we say that $\langle \mathbb{P}, \leqslant_{\mathbb{P}}\rangle$ is a *product forcing of* $\{\mathbb{P}_i \mid i \in I\}$ if:

1. Each condition $p \in \mathbb{P}$ is an element of the set-theoretic product $\prod_{i\in I} \mathbb{P}_i$. That is, p is a function with domain I such that $p(i) \in \mathbb{P}_i$ for all $i \in I$. For $p \in \mathbb{P}$ we define the *support* of p to be $\mathrm{supp}(p) = \{i \in I \mid p(i) \neq \mathbb{1}_{\mathbb{P}_i}\}$.
2. For all $p \in \prod_{i\in I} \mathbb{P}_i$ such that $\mathrm{supp}(p)$ is finite, $p \in \mathbb{P}$.
3. $q \leqslant_{\mathbb{P}} p$ if and only if for all $i \in I$, $q(i) \leqslant_i p(i)$.
4. For all p, $p' \in \mathbb{P}$, if $\mathrm{supp}(p) \cap \mathrm{supp}(p') = \emptyset$ then the function q given by $q(i) = \min\{p(i), p'(i)\}$ is a condition in $\mathbb{P}$.
5. For all $p \in \mathbb{P}$ and $I_0 \subseteq I$ the function $p \restriction I_0$ is a condition in $\mathbb{P}$, where $p \restriction I_0(i) = p(i)$ if $i \in I_0$ and $\mathbb{1}_{\mathbb{P}_i}$ otherwise.

We shall write $\mathbb{P} = \prod_{i\in I} \mathbb{P}_i$ to mean that $\mathbb{P}$ is a product forcing of $\{\mathbb{P}_i \mid i \in I\}$. Given $\mathbb{P} = \prod_{i\in I} \mathbb{P}_i$ and $I_0 \subseteq I$, we denote by $\mathbb{P} \restriction I_0$ the notion of forcing given by $\{p \restriction I_0 \mid p \in \mathbb{P}\} = \{p \in \mathbb{P} \mid \mathrm{supp}(p) \subseteq I_0\}$, with $q \leqslant_{\mathbb{P}\restriction I_0} p$ if $q \leqslant_{\mathbb{P}} p$. In particular, if I is an ordinal and $\alpha \in I$ then $\mathbb{P} \restriction \alpha$ refers to the forcing $\mathbb{P} \restriction \{\beta \mid \beta < \alpha\}$.

In some cases we may consider partial functions p, in which case we identify "$i \notin \mathrm{dom}(p)$" with "$p(i) = \mathbb{1}_{\mathbb{P}_i}$". In light of this identification, note that $\mathbb{P} \restriction I_0$ is isomorphic to a product forcing of $\{\mathbb{P}_i \mid i \in I_0\}$.

[68] When $|A| \geqslant |B|$ we denote by $\mathrm{Fn}(A, 2, B)$ the notion of forcing with conditions that are partial functions $p\colon A \to 2$ such that $|p| < |B|$. In this case $\mathrm{Fn}(A, 2, B)$ is isomorphic to $\mathrm{Add}(B, A)$. While $\mathrm{Fn}(A, 2, B)$ is not used in this paper, it is used in [Kun83], to which we sometimes refer in Chapter 4.

By the λ-*support product* $\prod_{i\in I}\mathbb{P}_i$ we mean the product forcing given by those functions p such that $|\mathrm{supp}(p)| < \lambda$. By the *finite-support product* we mean the $\aleph_0$-support product, so $\mathbb{P}$ is exactly those conditions of finite support. By the *full-support product* $\prod_{i\in I}\mathbb{P}_i$ we mean the product forcing with no restriction on the support of the conditions. For families $\{\mathbb{P}_\alpha \mid \alpha < \gamma\}$ of notions of forcing indexed by an ordinal, we say that $\mathbb{P} = \prod_{\alpha<\gamma}\mathbb{P}_\alpha$ is of *bounded support* if for all $p \in \mathbb{P}$ there is $\alpha < \gamma$ such that $p = p \restriction \alpha$.

Note that when I is finite every product forcing of $\{\langle\mathbb{P}_i, \leqslant_i\rangle \mid i \in I\}$ is the full-support product by Item (2). In the case that I is small, we often write out the product using $\times$ notation, so for example $\prod_{i\in\{0,1\}}\mathbb{P}_i$ would be written $\mathbb{P}_0 \times \mathbb{P}_1$.

The following proposition is due to Solovay [Sol70], but can also be found in [Jec03, Lemma 15.9].

Proposition 2.4.4 (The Product Lemma, [Sol70])**.** *Let* $\mathbb{P} = \mathbb{P}_0 \times \mathbb{P}_1$*.* $G \subseteq \mathbb{P}$ *is* V*-generic if and only if* $G = G_0 \times G_1$*, where* $G_0 \subseteq \mathbb{P}_0$ *is* V*-generic and* $G_1 \subseteq \mathbb{P}_1$ *is* $V[G_0]$*-generic. Furthermore,* $V[G] = V[G_0][G_1]$*.*

As an immediate corollary, we have that whenever G is V-generic for $\mathbb{P} = \prod_{i\in I}\mathbb{P}_i$ and $I_0 \subseteq I$, since $\mathbb{P} = (\mathbb{P}\restriction I_0) \times (\mathbb{P}\restriction (I \setminus I_0))$, $G\restriction I_0 = G \cap (\mathbb{P}\restriction I_0)$ is V-generic for $\mathbb{P}\restriction I_0$. Furthermore, given V-generic $G_0 \subseteq \mathbb{P}\restriction I_0$ and $p \in \mathbb{P}$, if $p\restriction I_0 \in G_0$ then there is V-generic $G \subseteq \mathbb{P}$ such that $p \in G \supseteq G_0$ (by forcing with $\{q \in \mathbb{P}_1 \mid q \leqslant p\}$ in $V[G_0]$).

Fact 2.4.5. *Let* $\mathbb{P} = \prod_{i\in I}\mathbb{P}_i$*, and let* $I_0 \subseteq I$*. Suppose that* $\dot{X}, \dot{Y}$ *are* $\mathbb{P}\restriction I_0$*-names,* $p \in \mathbb{P}$*, and* $p \Vdash \dot{X} = \dot{Y}$*. Then* $p\restriction I_0 \Vdash \dot{X} = \dot{Y}$*. The same is true of the formula* $\dot{X} \in \dot{Y}$*.*[69]

Definition 2.4.6 (Forcing iteration)**.** Let $\mathbb{P}$ be a notion of forcing and $\dot{\mathbb{Q}}$ a $\mathbb{P}$-name such that $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\dot{\mathbb{Q}}$ is a notion of forcing". Then we denote by $\mathbb{P} * \dot{\mathbb{Q}}$ the notion of forcing with conditions of the form $\langle p, \dot{q}\rangle$ with $p \in \mathbb{P}$ and $\mathbb{1}_{\mathbb{P}} \Vdash \dot{q} \in \dot{\mathbb{Q}}$, and order given by $\langle p', \dot{q}'\rangle \leqslant \langle p, \dot{q}\rangle$ if $p' \leqslant p$ and $p' \Vdash \dot{q}' \leqslant \dot{q}$. By Scott's trick,[70] this can be represented as a set in the ground model, rather than a proper class of names.

[69]This is essentially an absoluteness argument: If $p \Vdash \dot{X} = \dot{Y}$ and G_0 is V-generic for $\mathbb{P}\restriction I_0$ with $p\restriction I_0 \in G_0$, then let G be V-generic for $\mathbb{P}$ such that $p \in G \supseteq G_0$. Then $V[G] \vDash \dot{X}^G = \dot{Y}^G$. However, since $\dot{X}$ and $\dot{Y}$ are $\mathbb{P}\restriction I_0$-names, $\dot{X}^G = \dot{X}^{G_0}$ and $\dot{Y}^G = \dot{Y}^{G_0}$. Furthermore, $V[G_0]$ is a transitive submodel of $V[G]$, and so $V[G_0] \vDash \dot{X}^{G_0} = \dot{Y}^{G_0}$. Hence if G is any V-generic filter for $\mathbb{P}$ with $p\restriction I_0 \in G$, $\dot{X}^{G\restriction I_0} = \dot{Y}^{G\restriction I_0}$, and so $\dot{X}^G = \dot{Y}^G$. That is, $p\restriction I_0 \Vdash \dot{X} = \dot{Y}$.

[70]See Abstract 626t by Dana Scott in [Kle55].

A collection $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma,\ \beta < \gamma \rangle$ is a *(forcing) iteration* if:

1. $\mathbb{P}_0 = \{\mathbb{1}\}$;
2. for all $\alpha < \gamma$, $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash$ "$\dot{\mathbb{Q}}_\alpha$ is a notion of forcing";
3. for all $\alpha \leqslant \gamma$, the conditions in $\mathbb{P}_\alpha$ are functions with domain α such that, for all $\beta < \alpha$, $\mathbb{1}_{\mathbb{P}_\beta} \Vdash p(\beta) \in \dot{\mathbb{Q}}_\beta$;
4. for all $\alpha \leqslant \gamma$, the ordering on $\mathbb{P}_\alpha$ is given by $q \leqslant p$ if and only if, for all $\beta < \alpha$, $q \upharpoonright \beta \leqslant_{\mathbb{P}_\beta} p \upharpoonright \beta$, and $q \upharpoonright \beta \Vdash_{\mathbb{P}_\beta} q(\beta) \leqslant p(\beta)$; and
5. whenever $\beta < \alpha \leqslant \gamma$, $q \in \mathbb{P}_\beta$ and $p \in \mathbb{P}_\alpha$ are such that $q \leqslant_{\mathbb{P}_\beta} p \upharpoonright \beta$, the condition $q \cup (p \upharpoonright (\alpha \setminus \beta)) \in \mathbb{P}_\alpha$.

Note that at successor stages we have $\mathbb{P}_{\alpha+1} \cong \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. If G is V-generic for $\mathbb{P}_\gamma$, and $\alpha < \gamma$, we shall denote by $G \upharpoonright \alpha$ the restriction of each function $p \in G$ to the domain α. This is a V-generic filter for $\mathbb{P}_\alpha$, and as such G provides a chain $\langle V[G \upharpoonright \alpha] \mid \alpha \leqslant \gamma \rangle$ of models of ZFC.

Given a condition $p \in \mathbb{P}_\gamma$, and $\alpha \leqslant \gamma$, we denote by $p \upharpoonright \alpha$ the condition in $\mathbb{P}_\alpha$ given by that restriction. We shall denote by $\mathbb{P}_\gamma/\alpha$ the canonical $\mathbb{P}_\alpha$-name for the notion of forcing in $V[G \upharpoonright \alpha]$ given by the final $\gamma \setminus \alpha$ iterands. Similarly, we shall denote by p/α the canonical name for the $\mathbb{P}_\gamma/\alpha$-condition that is the final $\gamma \setminus \alpha$ co-ordinates of p:

$$p/\alpha = \left\{ \langle \mathbb{1}_{\mathbb{P}_\alpha}, \langle \check{\delta}, p(\delta) \rangle^\bullet \rangle \;\middle|\; \alpha \leqslant \delta < \gamma \right\}.$$

This allows us to view $\mathbb{P}_\gamma$ as the iteration $\mathbb{P}_\alpha * \mathbb{P}_\gamma/\alpha$. As in the case of products, we have the following.

Fact 2.4.7. *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ be an iteration and suppose that $\dot{X}, \dot{Y}$ are $\mathbb{P}_\alpha$-names. For $p \in \mathbb{P}_\gamma$, if $p \Vdash \dot{X} = \dot{Y}$ then $p \upharpoonright \alpha \Vdash \dot{X} = \dot{Y}$. The same is true of the formula $\dot{X} \in \dot{Y}$.*

Note that any well-ordered product forcing $\mathbb{P} = \prod_{\alpha<\gamma} \mathbb{P}_\alpha$ can be viewed as an iteration via $\langle \mathbb{P} \upharpoonright \alpha,\ \check{\mathbb{P}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$, and hence many results regarding forcing iterations descend directly to products, such as Fact 2.4.5 being a descent of Fact 2.4.7.

Definition 2.4.8. Given a forcing iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \delta \rangle$ of limit length δ, the *inverse limit* of the system is the notion of forcing $\mathbb{P}$ with conditions given by functions p with domain δ such that, for all $\gamma < \delta$, $p \restriction \gamma \in \mathbb{P}_\gamma$. The ordering is given by $q \leqslant p$ if for all $\gamma < \delta$, $q \restriction \gamma \leqslant_\gamma p \restriction \gamma$. The *direct limit* of the system is the notion of forcing $\mathbb{P}$ with conditions given by functions p with domain γ for some $\gamma < \delta$ such that $p \in \mathbb{P}_\gamma$. The ordering is given by $q \leqslant p$ if $\gamma = \mathrm{dom}(p) \leqslant \mathrm{dom}(q)$ and $q \restriction \gamma \leqslant_\gamma p$. By an *Easton support iteration* we mean that for all limit ordinals α, $\mathbb{P}_\alpha$ is taken as an inverse limit if α is singular, and a direct limit otherwise.

Definition 2.4.9. Given two notions of forcing $\mathbb{P}$ and $\mathbb{Q}$, a function $\psi\colon \mathbb{P} \to \mathbb{Q}$ is a *dense embedding* if:

1. For all $p, p' \in \mathbb{P}$, $p \leqslant_{\mathbb{P}} p'$ if and only if $\psi(p) \leqslant_{\mathbb{Q}} \psi(p')$; and
2. for all $q \in \mathbb{Q}$ there is $p \in \mathbb{P}$ such that $\psi(p) \leqslant_{\mathbb{Q}} q$.

Note that in this case we also have $p \perp_{\mathbb{P}} p'$ if and only if $\psi(p) \perp_{\mathbb{Q}} \psi(p')$. If there is a dense embedding $\mathbb{P} \to \mathbb{Q}$ then we say that $\mathbb{P}$ and $\mathbb{Q}$ are *equivalent*.

Fact 2.4.10. *If $\mathbb{P}$ and $\mathbb{Q}$ are equivalent then for all V-generic filters $G \subseteq \mathbb{P}$ and $H \subseteq \mathbb{Q}$, there are $H' \in V[G]$ and $G' \in V[H]$ such that $V[G] = V[H']$, $V[H] = V[G']$, G' is V-generic for $\mathbb{P}$, and H' is V-generic for $\mathbb{Q}$.*

We shall often make the additional assumption that $\mathbb{P}$ is *separative*: for all distinct $p, q \in \mathbb{P}$, either $q \leqslant p$ and $p \nleqslant q$; or there is $r \leqslant q$ such that $r \perp p$. This may be done as the separative quotient of any notion of forcing is an equivalent notion of forcing.

Given a separative notion of forcing $\mathbb{P}$, there is a unique (up to isomorphism) complete Boolean algebra $B(\mathbb{P})$ such that $\mathbb{P}$ densely embeds into $B(\mathbb{P})$.[71] If $\mathbb{B}$ is a complete Boolean algebra, and $\varphi(\dot{x}_0, \dots, \dot{x}_{n-1})$ is a formula with parameters in $V^{\mathbb{B}}$, then there is unique $p \in \mathbb{B}$ such that, for all $q \in \mathbb{B}$, $q \Vdash \varphi(\dot{x}_0, \dots, \dot{x}_{n-1})$ if and only if $q \leqslant p$. We denote this condition by $\|\varphi(\dot{x}_0, \dots, \dot{x}_{n-1})\|$.

2.4.1 Symmetric extensions

While symmetric extensions do not appear outside of Chapter 3, it behoves us to mention them in tandem with forcing. It is key to the role of forcing that if $V \vDash \mathsf{ZFC}$, and G is V-generic for some notion of forcing $\mathbb{P} \in V$, then $V[G] \vDash \mathsf{ZFC}$. However, this demands additional techniques for trying to establish results

[71] In fact, for forcing we need to omit the bottom element $\mathbb{0}$ of the complete Boolean algebra, though we may refer to it by, for example, writing $p \wedge p' = \mathbb{0}$ to mean $p \perp_{\mathbb{B}} p'$.

that are inconsistent with AC. Symmetric extensions extend the technique of forcing in this very way by constructing an intermediate model between V and $V[G]$ that is a model of ZF.

Given a notion of forcing $\mathbb{P}$, we shall denote by $\mathrm{Aut}(\mathbb{P})$ the collection of automorphisms of $\mathbb{P}$. Let $\mathbb{P}$ be a notion of forcing and $\pi \in \mathrm{Aut}(\mathbb{P})$. Then π extends naturally to act on $\mathbb{P}$-names by recursion: $\pi\dot{x} = \{\langle \pi p, \pi\dot{y}\rangle \mid \langle p, \dot{y}\rangle \in \dot{x}\}$.

Such automorphisms extend to the forcing relation in the following way, proved in [Jec03, Lemma 14.37].

Lemma 2.4.11 (The Symmetry Lemma)**.** *Let $\mathbb{P}$ be a forcing, $\pi \in \mathrm{Aut}(\mathbb{P})$, and $\dot{x}$ a $\mathbb{P}$-name. Then $p \Vdash \varphi(\dot{x})$ if and only if $\pi p \Vdash \varphi(\pi\dot{x})$.*

Note in particular that for all $\pi \in \mathrm{Aut}(\mathbb{P})$ we have $\pi\mathbb{1} = \mathbb{1}$. Therefore, $\pi\check{x} = \check{x}$ for all ground model sets x, and $\pi\{\dot{x}_i \mid i \in I\}^\bullet = \{\pi\dot{x}_i \mid i \in I\}^\bullet$, similarly extending to tuples, functions, etc.

Given a group $\mathscr{G}$, a *filter of subgroups* of $\mathscr{G}$ is a non-empty set $\mathscr{F}$ of subgroups of $\mathscr{G}$ that is closed under supergroups and finite intersections. We say that $\mathscr{F}$ is *normal* if whenever $H \in \mathscr{F}$ and $\pi \in \mathscr{G}$, then $\pi H \pi^{-1} \in \mathscr{F}$.

A *symmetric system* is a triple $\langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ such that $\mathbb{P}$ is a notion of forcing, $\mathscr{G}$ is a group of automorphisms of $\mathbb{P}$, and $\mathscr{F}$ is a normal filter of subgroups of $\mathscr{G}$. Given such a symmetric system, we say that a $\mathbb{P}$-name $\dot{x}$ is *$\mathscr{F}$-symmetric* if $\mathrm{sym}_{\mathscr{G}}(\dot{x}) = \{\pi \in \mathscr{G} \mid \pi\dot{x} = \dot{x}\} \in \mathscr{F}$. $\dot{x}$ is *hereditarily $\mathscr{F}$-symmetric* if this notion holds for every $\mathbb{P}$-name hereditarily appearing in $\dot{x}$. We denote by $\mathsf{HS}_{\mathscr{F}}$ the class of hereditarily $\mathscr{F}$-symmetric names. When clear from context, we will omit subscripts and simply write $\mathrm{sym}(\dot{x})$ or HS. The following theorem, [Jec03, Lemma 15.51], is then key to the study of symmetric extensions.

Theorem 2.4.12. *Let $\langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ be a symmetric system, $G \subseteq \mathbb{P}$ a V-generic filter, and let M denote the class $\mathsf{HS}^G_{\mathscr{F}} = \{\dot{x}^G \mid \dot{x} \in \mathsf{HS}_{\mathscr{F}}\}$. Then M is a transitive model of* ZF *such that $V \subseteq M \subseteq V[G]$.*

Finally, we have a forcing relation for symmetric extensions $\Vdash^{\mathsf{HS}}$ defined by relativising the forcing relation $\Vdash$ to the class HS. This relation has the same properties and behaviour of the standard forcing relation $\Vdash$. Moreover, when $\pi \in \mathscr{G}$, the Symmetry Lemma holds for $\Vdash^{\mathsf{HS}}$.

We will sometimes define the 'automorphism group' $\mathscr{G}$ to be a group acting on $\mathbb{P}$. However, even when this action is not faithful the theory of symmetric extensions holds, and indeed if we must we can work with the quotient of $\mathscr{G}$ by its $\mathbb{P}$-action kernel.

Example (Cohen's first model)**.** Let V be a model of ZFC[72] and, working in V, let $\mathbb{P} = \mathrm{Add}(\omega, \omega)$, the forcing given by finite partial functions $p\colon \omega \times \omega \to 2$ and ordered by $q \leqslant p$ if $q \supseteq p$. The group $\mathscr{G}$ is the finitary permutations of ω, those bijections $\pi\colon \omega \to \omega$ such that $\{n < \omega \mid \pi n \neq n\}$ is finite. We define a group action of $\mathscr{G}$ on $\mathbb{P}$ by $\pi p(\pi n, m) = p(n, m)$. For $E \in [\omega]^{<\omega}$, let $\mathrm{fix}(E) \leqslant \mathscr{G}$ be the subgroup $\{\pi \in \mathscr{G} \mid \pi \upharpoonright E = \mathrm{id}\}$, and let $\mathscr{F}$ be the filter of subgroups of $\mathscr{G}$ generated by the $\mathrm{fix}(E)$ as E varies over $[\omega]^{<\omega}$. Let G be a V-generic filter for $\mathbb{P}$ and $M = \mathsf{HS}^G$ be the symmetric extension of V by this symmetric system. Note that, since $\mathbb{P}$ is c.c.c., V, M, and $V[G]$ will agree on the cardinalities and cofinalities of ordinals, and V and $V[G]$ will agree on the cardinalities of all sets in V.

For each $n < \omega$, let $\dot{a}_n$ be the $\mathbb{P}$-name $\{\langle p, \check{m}\rangle \mid p(n, m) = 1\}$ and note that for all $\pi \in \mathscr{G}$, $\pi \dot{a}_n = \dot{a}_{\pi n}$. Let $\dot{A} = \{\dot{a}_n \mid n < \omega\}^\bullet$, so $\pi \dot{A} = \dot{A}$ for all $\pi \in \mathscr{G}$. Let A be the realisation of the name $\dot{A}$ in M. Note that $V[G] \vDash |A| = \aleph_0$ witnessed by, say, $\{\langle \check{n}, \dot{a}_n\rangle^\bullet \mid n < \omega\}^\bullet$.

Lemma 2.4.13. *In M, A is infinite, but there is no injection $\omega \to A$.*[73]

Proof. Suppose that $\dot{f} \in \mathsf{HS}$ and $p \in \mathbb{P}$ were such that $p \Vdash \dot{f}\colon \check{\omega} \to \dot{A}$. Let $E \in [\omega]^{<\omega}$ be such that $\mathrm{fix}(E) \leqslant \mathrm{sym}(\dot{f})$, let $n \notin E$, and let $q \leqslant p$ be such that, for some $m < \omega$, $q \Vdash \dot{f}(\check{m}) = \dot{a}_n$. If no such q exists, then $p \Vdash \dot{f}``\check{\omega} \subseteq \{\dot{a}_n \mid n \in E\}^\bullet$, and so certainly $p \Vdash$ "$\dot{f}$ is not an injection". Let $n' \notin E \cup \{k < \omega \mid (\exists k' < \omega)\langle k, k'\rangle \in \mathrm{dom}(q)\}$ (which must exist as $\mathrm{dom}(q)$ is finite), and let π be the transposition $(n\ n')$. Then $\pi \in \mathrm{fix}(E) \leqslant \mathrm{sym}(\dot{f})$, so $\pi q \Vdash \dot{f}(\check{m}) = \dot{a}_{n'}$. However, $q \parallel \pi q$, so

$$q \cup \pi q \Vdash \dot{f}(\check{m}) = \dot{a}_n \wedge \dot{f}(\check{m}) = \dot{a}_{n'} \neq \dot{a}_n,$$

contradicting that $p \Vdash$ "$\dot{f}$ is a function". □

2.4.2 Wreath products

We may exhibit groups of automorphisms $\mathscr{G}$ as permutation groups with an action on the notion of forcing. By a *permutation group* (of the set X) we mean a subgroup of S_X, the group of bijections $X \to X$. If $\pi \in S_X$, then by the *support* of π, written $\mathrm{supp}(\pi)$, we mean the set $\{x \in X \mid \pi(x) \neq x\}$. Given

[72]Traditionally we take $V = L$, though none of our work requires this.
[73]Sets A such that $|\omega| \nleqslant |A|$ are called *Dedekind-finite*.

an infinite cardinal λ we denote by $S_X^{<\lambda}$ the subgroup of S_X of permutations π such that $|\mathrm{supp}(\pi)| < \lambda$.

Definition 2.4.14 (Wreath product)**.** Given two permutation groups $G \leqslant S_X$ and $H \leqslant S_Y$, the *wreath product* of G and H, denoted $G \wr H$, is the subgroup of permutations $\pi \in S_{X\times Y}$ which have the following property:

> There is $\pi^* \in G$ and a sequence $\langle \pi_x \mid x \in X\rangle \in H^X$ such that for all $\langle x, y\rangle \in X \times Y$, $\pi(x, y) = \langle \pi^*(x), \pi_x(y)\rangle$.

That is, π first permutes each column $\{x\} \times Y$ according to some $\pi_x \in H$, and then acts on the X-co-ordinate of $X \times Y$, permuting its columns via some $\pi^* \in G$.

Given $\pi \in G \wr H$, we will use the notation π^* and π_x to mean the elements of G and H respectively from the definition. Note that if $\pi, \sigma \in G \wr H$, then $(\pi\sigma)^* = \pi^*\sigma^*$.

Note also that $\{\mathrm{id}\} \wr S_Y \leqslant S_{X\times Y}$ is the group of all $\pi \in S_{X\times Y}$ such that for all $\langle x, y\rangle \in X \times Y$, $\pi(x, y) \in \{x\} \times Y$.

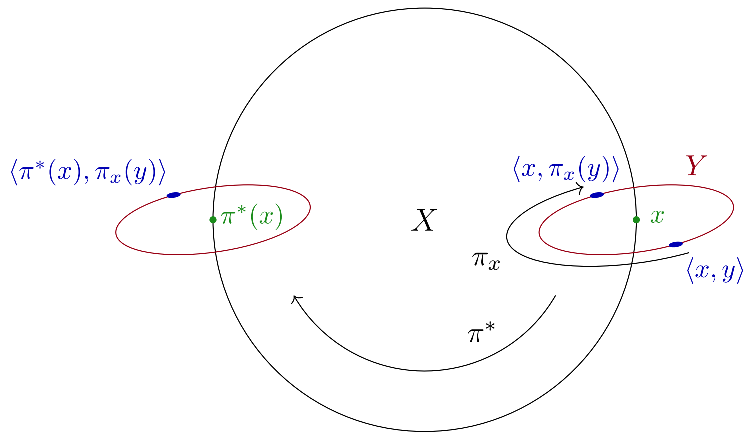

Figure 2.1: The action of a wreath product. The co-ordinate $\langle x, y\rangle$ is transformed by π first according to π_x into $\langle x, \pi_x(y)\rangle$, and then according to π^* into $\langle \pi^*(x), \pi_x(y)\rangle$.

2.5 Topology

Definition 2.5.1. A *tree* is a partially ordered set $\langle T, \leqslant\rangle$ with minimum element such that, for all $t \in T$, $\{s \in T \mid s < t\}$ is well-ordered by $\leqslant$. A *binary tree* is a set $T \subseteq 2^{<\alpha}$, where α is an ordinal, such that whenever $t \in T$ and

$s \subseteq t$, $s \in T$. A set $b \subseteq T$ is a *branch* if it is a $\subseteq$-chain, and it is *maximal* if for any branch $c \subseteq T$, if $b \subseteq c$ then $b = c$. It is additionally *cofinal* if it has order type α. Given a tree $T \subseteq 2^{<\alpha}$, we denote by $[T]$ its *closure*, the set $\{x \in 2^\alpha \mid (\forall \beta < \alpha) x \restriction \beta \in T\}$. We say that a set $C \subseteq 2^\alpha$ is *closed* if and only if there is a tree $T \subseteq 2^{<\alpha}$ such that $C = [T]$. We shall only deal with trees that have no non-cofinal maximal branches, and in this case we shall say that the tree is of *height* α. A tree T of height ω is *perfect* if for all $s \in T$ there is $t \supseteq s$ such that $t ^\frown \langle 0 \rangle \in T$ and $t ^\frown \langle 1 \rangle \in T$.

The topology of trees is explored further in Section 5.3.1.

Fact 2.5.2. *If T is perfect then $|[T]| = \mathfrak{c}$. If $C \subseteq 2^\omega$ is closed and uncountable then there is a perfect tree T such that $[T] \subseteq C$. In particular, a closed subset of 2^ω is either countable or has cardinality $\mathfrak{c}$.*

2.6 Ideals and filters

$\mathcal{I} \subseteq \mathscr{P}(X)$ is an *ideal* (on X) if $\mathcal{I} \cap \{\emptyset, X\} = \{\emptyset\}$ and $\mathcal{I}$ is closed under subsets of elements and finite unions of elements. We say that $\mathcal{I}$ is: λ-*complete* if $\mathcal{I}$ is closed under unions of fewer than λ many elements;[74] λ-*saturated* if, for all $\{A_\alpha \mid \alpha < \lambda\} \subseteq \mathscr{P}(X) \setminus \mathcal{I}$, there are $\alpha < \beta < \lambda$ such that $A_\alpha \cap A_\beta \notin \mathcal{I}$; *non-trivial* if $[X]^{<\omega} \subseteq \mathcal{I}$; and *prime* if for all $A \subseteq X$, $A \in \mathcal{I}$ or $X \setminus A \in \mathcal{I}$.

$\mathcal{F} \subseteq \mathscr{P}(X)$ is a *filter* (on X) if $\mathcal{F}^* := \{X \setminus A \mid A \in \mathcal{F}\}$ is an ideal on X. We say that $\mathcal{F}$ is λ-*complete* if $\mathcal{F}^*$ is λ-complete (as an ideal), and an *ultrafilter* if $\mathcal{F}^*$ is prime. An ultrafilter $\mathcal{U}$ on a set $S \subseteq \mathscr{P}(X)$ is: *non-principal* (or *free*) if for all $x \in X$, $\{x\} \notin \mathcal{U}$; and *fine* if for all $x \in X$, $\{A \in S \mid x \in A\} \in \mathcal{U}$. We shall say that an ultrafilter $\mathcal{U}$ on a cardinal κ is a *measure* if it is non-principal and κ-complete, and that a measure $\mathcal{U}$ on κ is *normal* if for all $A \in \mathcal{U}$ and $f \in \prod A$ there is $B \in \mathcal{U}$ such that $f \restriction B$ is constant. A cardinal κ is *measurable* if there is a measure on κ.

Given two models $V \subseteq W$ of ZFC and an ideal $\mathcal{I} \in V$ on X, we denote by $\langle \mathcal{I} \rangle^W$ the ideal generated by $\mathcal{I}$ in the extension:

$$\langle \mathcal{I} \rangle^W = \left\{ A \in \mathscr{P}(X)^W \;\middle|\; (\exists Y \in \mathcal{I})\, A \subseteq Y \right\}.$$

[74] We usually write *σ-complete* instead of $\aleph_1$-complete.

Chapter 3

The Hartogs–Lindenbaum Spectrum

> "Favarial was built above the surface of a deep body of fresh water the size of a small sea. The avenues and plazas stood supported by pylons that dug deep into the lake's murky floor, the buildings on great spans supported by those avenues and plazas. Unless one stood at the side of a roadway and deliberately looked over the edge, one might never notice the lake at all."
>
> Ari Marmell, *Agents of Artifice*

AN IMPORTANT CONSEQUENCE OF THE AXIOM OF CHOICE IS THE *partition principle*, which says that if there is a surjection $Y \to X$ then there is an injection $X \to Y$.[76] However, in the absence of choice this principle may well fail,[77] even if X is well-orderable. We call sets Y such that, for some $\alpha \in \mathrm{Ord}$, $|\alpha| \leqslant^* |Y|$ but $|\alpha| \nleqslant |Y|$ *eccentric*, and will describe them in finer detail through the language of *Hartogs* and *Lindenbaum numbers*. This chapter charts the eccentricity of a universe of sets and describes

Acknowledgements. I would like to thank Andrew Brooke-Taylor and Asaf Karagila. Both provided valuable feedback on both papers that constitute this chapter.[75] I would also like to thank Carla Simons for her help and encouragement in the process of writing my first paper during my PhD.

[75] Indeed, Asaf Karagila is a co-author of [KR24], the paper that forms the basis of Section 3.6.

[76] In fact, with the axiom of choice one can even say that every surjection $g\colon Y \to X$ *splits*, so there is an injection $f\colon X \to Y$ such that $gf = \mathrm{id}$.

[77] It is, at time of writing, an open question if the partition principle is equivalent to the axiom of choice.

what sorts of eccentricity cannot, and must, appear.

3.1 Introduction

The axiom of choice affords the working mathematician an elegant and straightforward classification of size for all sets: for all X there is a least ordinal α such that X has the same cardinality as α. Indeed, grappling with how to compare sizes of infinite objects (or appropriate abstractions of this notion) is an inalienable aspect of modern mathematical foundations. Even without the axiom of choice, it is still understood that if Y is a superset of X, then Y is still 'at least as big as X' in some way. Taking this further, we are able to compare the cardinalities of sets through functions that map between them. If there is a surjection from X onto ω, then we can still describe X as being 'at least as big as ω'. Even when we lose the straightforward description of cardinality classes that is obtained from the axiom of choice, we can still consider comparisons between sets and ordinals in this way.

Definition 3.1.1. Let X be a set. The *Hartogs number* of X is

$$\aleph(X) \coloneqq \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant |X|\}.$$

The *Lindenbaum number* of X is

$$\aleph^*(X) \coloneqq \min\{\alpha \in \mathrm{Ord} \mid |\alpha| \nleqslant^* |X|\}.$$

If X is well-orderable, then $\aleph(X) = \aleph^*(X) = |X|^+$, and so in the case that AC holds, these descriptions of sets tell us no more than cardinality already did. However, if the axiom of choice does not hold and X is not well-orderable then such descriptions still provide insight into the cardinality of X.

The existence of $\aleph(X)$ is a theorem of ZF by Hartogs's lemma: *"For every arbitrary set M, there always exists a well-ordered set L such that $|L| \nless |M|$".*[78] The existence of $\aleph^*(X)$ is similarly a theorem of ZF by a lemma of Lindenbaum's theorem: *"The axiom of choice is equivalent to the following propositions: [...] For all A, B, either $|A| \leqslant^* |B|$ or $|B| \leqslant^* |A|$".*[79]

[78] [Har15, p. 442], trans. Hope Duncan and Calliope Ryan-Smith. A proof in English can be found in [Gol96, Theorem 8.18].

[79] [LT26, Théorème 82.A_6], trans. Calliope Ryan-Smith. The first published proof is [Sie47], and a proof in English can be found in [Sie65, Chapter XVI, Section 3, Theorem 1].

It is important to note that for all X both $\aleph(X)$ and $\aleph^*(X)$ must be cardinal numbers, with $\aleph(X) \leqslant \aleph^*(X)$. As observed, if we assume AC then $\aleph(X) = \aleph^*(X)$ for all sets X, but the statement "$(\forall X)\aleph(X) = \aleph^*(X)$" is in general weaker than AC, and is in fact equivalent to the axiom of choice only for well-ordered families of non-empty sets (see Theorem 3.3.1 and Theorem B). This equivalence uses a powerful construction that takes those sets X with $\aleph(X) \neq \aleph^*(X)$ and 'transfers' this property to sets of larger Hartogs and Lindenbaum number. We refer to sets X such that $\aleph(X) \neq \aleph^*(X)$ as *eccentric*, and cardinals λ such that there is X with $\aleph^*(X) = \aleph(X)^+ = \lambda^+$ as *oblate*. In this paper, we shall fine-tune this construction to produce many more equivalent statements.

Theorem (Theorem A)**.** *The following are equivalent:*

1. *For all X, $\aleph(X) = \aleph^*(X)$;*
2. *there is κ such that for all X, $\aleph^*(X) \geqslant \kappa \implies \aleph(X) = \aleph^*(X)$;*
3. *there is κ such that for all X, $\aleph(X) \geqslant \kappa \implies \aleph(X) = \aleph^*(X)$;*
4. $\mathsf{AC}_{\mathsf{WO}}$*;*
5. *for all X, $\aleph(X)$ is a successor; and*
6. *for all X, $\aleph(X)$ is regular.*

While we would like to include the statements "for all X, $\aleph^*(X)$ is a successor" and "for all X, $\aleph^*(X)$ is regular" in our theorem,[80] this does not hold: in Cohen's first model, $\mathsf{AC}_{\mathsf{WO}}$ fails, but $\aleph^*(X)$ is a regular successor for all X (see Corollary 3.5.7).

In light of Theorem A and that $\mathsf{AC}_{\mathsf{WO}}$ is not a consequence of ZF, it is quite possible to build models eccentric sets are found. Let us produce a classification tool for such objects.

Definition 3.1.2 (Hartogs–Lindenbaum Spectrum)**.** Given a model M of ZF, the *Hartogs–Lindenbaum spectrum* (or simply *spectrum*) of M is the class

$$\mathrm{Spec}_{\aleph}(M) := \{\langle\lambda, \kappa\rangle \mid (\exists X)\aleph(X) = \lambda, \aleph^*(X) = \kappa\}.$$

[80]For symmetry, if nothing else.

This paper explores the possible spectra of models of ZF that arise as symmetric extensions of models of AC. This behaviour is captured internally to a model by *small violations of choice*, or SVC, which we expand upon in Section 3.2.1. In this setting, the Hartogs–Lindenbaum spectrum is broken down into four parts.

Theorem (Theorem B)**.** *Let* $M \vDash \mathsf{SVC}$*. There are cardinals* $\phi \leqslant \psi \leqslant \chi_0 \leqslant \Omega$*, a cardinal* $\psi^* \geqslant \psi$*, a cardinal* $\chi \in [\chi_0, \chi_0^+]$*, and a set* $C \subseteq [\phi, \chi_0)$ *such that*

$$\mathrm{Spec}_{\aleph}(M) = \bigcup \begin{cases} \mathrm{SC} = \{\langle \lambda^+, \lambda^+ \rangle \mid \lambda \in \mathrm{Card}\} \\ \mathfrak{D} \subseteq \{\langle \lambda, \kappa \rangle \quad \mid \psi \leqslant \lambda \leqslant \kappa \leqslant \chi, \psi^* \leqslant \kappa\} \\ \mathfrak{C} \subseteq \{\langle \lambda, \lambda^+ \rangle \quad \mid \mathrm{cf}(\lambda) \in C, \lambda < \Omega\} \\ \mathfrak{U} = \{\langle \lambda, \lambda^+ \rangle \quad \mid \mathrm{cf}(\lambda) \in C, \lambda \geqslant \Omega\}. \end{cases}$$

So there is:

- A necessary core to the spectrum $\{\langle \lambda^+, \lambda^+ \rangle \mid \lambda \in \mathrm{Card}\}$ that every model of ZF contains, since $\aleph(\lambda) = \aleph^*(\lambda) = \lambda^+$ for all cardinals λ;
- a bounded, chaotic part of the spectrum containing those $\langle \lambda, \kappa \rangle$ that have no restrictions other than $\psi \leqslant \lambda \leqslant \kappa \leqslant \chi$ and $\psi^* \leqslant \kappa$;
- a bounded but potentially irregular part of the spectrum in which the only values are $\langle \lambda, \lambda^+ \rangle$ for those λ such that $\mathrm{cf}(\lambda) \in C$, with $\lambda < \Omega$; and
- and an unbounded, controlled tail of the spectrum containing precisely $\langle \lambda, \lambda^+ \rangle$ for all λ such that $\mathrm{cf}(\lambda) \in C$.

Notice in particular that outside of a bounded initial segment, eccentricity in the spectrum is entirely contingent on well-behaved oblate cardinals of a certain bounded cofinality.

Contrasting this imposition on the spectrum, we later construct symmetric systems that install eccentric sets of arbitrary Hartogs and Lindenbaum number,[81] followed by defining a class-sized product of these systems to install *all* possible Hartogs–Lindenbaum values.

Theorem (Theorem C)**.** ZF *is equiconsistent with* $\mathsf{ZF}+$*"for all infinite cardinals* $\lambda \leqslant \kappa$ *there is a set* X *such that* $\aleph(X) = \lambda \leqslant \kappa = \aleph^*(X)$*".*

[81] With the caveat that we must have $\aleph(X) \leqslant \aleph^*(X)$, and if $\aleph(X)$ is finite then trivially $\aleph(X) = \aleph^*(X) \neq 0$.

3.2 Preliminaries

Given sets A, B, we denote by $B^{\underline{A}}$ the set of injections $A \to B$. Given an ordinal α, we define $B^{\underline{<\alpha}} = \bigcup_{\beta<\alpha} B^{\underline{\beta}}$, the set of injections $\beta \to B$ for any $\beta < \alpha$.

3.2.1 Choice-like axioms

Throughout, we shall use several axioms that can be considered to be partial fulfilments of the full strength of AC. Recall that if X is a set of non-empty sets, a *choice function* for X is a function $c\colon X \to \bigcup X$ such that for all $x \in X$, $f(x) \in x$. We denote by $\prod X$ the set of choice functions for X.

The axiom of choice AC_X

For any set X, we shall denote by AC_X the statement that all families of non-empty sets indexed by X admit a choice function. If α is an ordinal, we shall denote by $\mathsf{AC}_{<\alpha}$ the statement that all families of non-empty sets indexed by some $\beta < \alpha$ admit a choice function. Finally, by $\mathsf{AC}_{\mathsf{WO}}$, we mean the statement that all well-orderable families of non-empty sets admit a choice function; equivalently, this can be written $(\forall\alpha \in \mathrm{Ord})\mathsf{AC}_\alpha$, or $(\forall\alpha \in \mathrm{Ord})\mathsf{AC}_{\aleph_\alpha}$. When the subscript is omitted, we mean the full axiom of choice: every family of non-empty sets admits a choice function, $(\forall X)\mathsf{AC}_X$.

The principle of dependent choice

Recall that a partially ordered set $\langle T, \leqslant\rangle$ is a *tree* if T has a minimum element and for all $t \in T$ the set $\{s \in T \mid s \leqslant t\}$ is well-ordered by $\leqslant$. Given a cardinal λ, we say that T is λ-*closed* if all $\leqslant$-chains in T of length less than λ have an upper bound. DC_λ, the *principle of dependent choice* (for λ), is the statement that every λ-closed tree has a maximal element or a chain of order type λ. When the subscript is omitted, we mean DC_ω, and we use $\mathsf{DC}_{<\lambda}$ to mean $(\forall\mu < \lambda)\mathsf{DC}_\mu$. While DC_λ does imply AC_λ, we do not have the reverse implication.[82]

Note that if T is a λ-closed tree of height λ then any chain of length λ can be extended by downwards closure to a cofinal branch.

[82] We prove DC_λ implies AC_λ in Proposition 3.2.3. That $\mathsf{AC}_{\mathsf{WO}}$ does not imply $\mathsf{DC}_{\aleph_1}$ was first proved in [Pin69], or can be found in [Jec73, Theorem 8.9], and that AC_λ does not imply DC for any λ was first proved in [Jen67], or can be found in [Jec73, Theorem 8.12].

Comparability and dual comparability

For a set X, we shall denote by W_X the *axiom of comparability (to X)*: for all Y, either $|Y| \leqslant |X|$ or $|X| \leqslant |Y|$. Similarly, we shall define the *dual axiom of comparability (to X)*, W^*_X, to be the statement that for all Y, either $|Y| \leqslant^* |X|$ or $|X| \leqslant^* |Y|$. The following observation, Proposition 3.2.1, is immediate.

Proposition 3.2.1. W_λ *is equivalent to the statement "for all X, either X is well-orderable or $\aleph(X) \geqslant \lambda^+$". Likewise, W^*_λ is equivalent to the statement "for all X, either X is well-orderable or $\aleph^*(X) \geqslant \lambda^+$".*

Definition 3.2.2. Since $(\forall\lambda)\mathsf{DC}_\lambda$, $(\forall\lambda)\mathsf{W}_\lambda$, and $(\forall\lambda)\mathsf{W}^*_\lambda$ are all equivalent to AC, whenever M is a model of $\mathsf{ZF} + \neg\mathsf{AC}$ we shall denote by λ_{DC} (respectively λ_{W}, λ^*_{W}) the least cardinal λ such that DC_λ (respectively W_λ, W^*_λ) does not hold. Similarly, whenever M is a model of $\mathsf{ZF} + \neg\mathsf{AC}_{\mathsf{WO}}$ we shall denote by λ_{AC} the least cardinal λ such that AC_λ does not hold.

Items (1) and (2) of Proposition 3.2.3 are due to Lévy [Lév64], and Item (3) is immediate.

Proposition 3.2.3. *For all cardinals λ,*

1. DC_λ *implies* W_λ*;*
2. DC_λ *implies* AC_λ*; and*
3. W_λ *implies* W^*_λ*.*

Hence, $\lambda_{\mathsf{DC}} \leqslant \lambda_{\mathsf{W}} \leqslant \lambda_{\mathsf{W}^*}$, and $\lambda_{\mathsf{DC}} \leqslant \lambda_{\mathsf{AC}}$.

Proof. We repeat the proof of Items (1) and (2) found in [Jec73, Theorem 8.1].

(1). Let X be a set such that $|X| \not< \lambda$. Let $T = X^{\underline{<\lambda}}$, ordered by $\subseteq$. Then T is a λ-closed tree of height λ, with upper bounds of chains given by unions. Thus, T has a maximal element or a cofinal branch. However, if $f\colon \alpha \to X$ is in T and $x \notin f``\alpha$, then f extends to $f \cup \{\langle\alpha, x\rangle\} \in T$, so for f to be maximal it must be a bijection, contradicting that $|X| \not< \lambda$. Hence, T has a chain of order type λ. The union of such a chain is an injection $\lambda \to X$, so $\lambda \leqslant |X|$ as required.

(2). Let $X = \{X_\alpha \mid \alpha < \lambda\}$ be a family of non-empty sets. Let T be the set of partial choice functions with domains of the form $Y_\gamma := \{X_\alpha \mid \alpha < \gamma\}$ for some $\gamma < \lambda$, ordered by $\subseteq$. Note that for any chain $C \subseteq T$, $\bigcup C$ is a partial choice function for X with domain of the form Y_γ for some $\gamma \leqslant \lambda$. Hence, if

C is a chain of length less than λ with no upper bound in T, it must be that $\bigcup C$ is a total choice function, the object that we are searching for. So we may instead assume that T is λ-closed and thus has a maximal element or a chain of order type λ. Any choice function on Y_γ can be extended to a choice function on $Y_{\gamma+1}$, so T has no maximal elements. Instead, there is a chain of order type λ in T, the union of which defines a choice function on $Y_\lambda = X$ as required.

(3). This follows from $|X| \leqslant |Y|$ implying $|X| \leqslant^* |Y|$. □

Small violations of choice

In [Bla79], Blass introduces a choice-like axiom called *small violations of choice*, also written SVC. At its inception, it was defined by setting $\mathsf{SVC}(S)$ to be "for all X there is an ordinal η and a surjection $f\colon \eta \times S \to X$", where S is a set (known as the *seed*). We then use SVC to mean $(\exists S)\mathsf{SVC}(S)$. However, this is equivalent to several other statements.

Fact 3.2.4 ([Bla79; Usu21])**.** *The following are equivalent:*

1. $M \vDash \mathsf{SVC}$*;*
2. $M \vDash$ "*There is a set A such that for all X there is an ordinal η and an injection $f\colon X \to A \times \eta$*"*;*
3. *there is an inner model $V \subseteq M$ with $V \vDash \mathsf{ZFC}$ and a symmetric system $\langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle \in V$ such that $M = \mathsf{HS}^G_{\mathscr{F}}$ for some V-generic $G \subseteq \mathbb{P}$;*
4. *there is an inner model $V \subseteq M$ such that $V \vDash \mathsf{ZFC}$ and there is $x \in M$ such that $M = V(x)$; and*
5. *there is a notion of forcing $\mathbb{P} \in M$ such that $\mathbb{1}_{\mathbb{P}} \Vdash \mathsf{AC}$.*

Definition 3.2.5 (Injective Seed)**.** By $\mathsf{SVC}^+(A)$ we mean the injective form of SVC: for all X there is an ordinal η and an injection $f\colon X \to A \times \eta$. We refer to A as an *injective seed*.

While we could write SVC^+ to mean $(\exists A)\mathsf{SVC}^+(A)$, we note that SVC^+ is equivalent to SVC.[83]

Proposition 3.2.6. $\mathsf{ZF} \vdash \mathsf{SVC} \longleftrightarrow \mathsf{SVC}^+$.

[83] The proof does rely on power set, though, so the distinction between SVC and SVC^+ may be of interest to anyone working with ZF^-.

Proof. ($\Longleftarrow$) If A is an injective seed then A is a seed.

($\Longrightarrow$) Let S be a seed. We claim that $\mathscr{P}(S)$ is an injective seed. Indeed, given X, let $f\colon \eta \times S \to X$ be a surjection. For $x \in X$, let α_x be $\min\{\alpha < \eta \mid (\exists s \in S) f(\alpha, s) = x\}$. Since f is a surjection, $x \mapsto \alpha_x$ is well-defined. Then define $g\colon X \to \mathscr{P}(S) \times \eta$ via $g(x) = \langle \{s \in S \mid f(\alpha_x, s) = x\}, \alpha_s \rangle$. If $g(x) = g(y)$ then $\alpha_x = \alpha_y$, and thus for all s such that $f(\alpha_x, s) = x$, we have $f(\alpha_y, s) = y$. However, $f(\alpha_x, s) = f(\alpha_y, s)$, so $x = y$ as required. $\square$

3.3 $\mathsf{AC}_{\mathsf{WO}}$

$\mathsf{AC}_{\mathsf{WO}}$, the axiom of choice for all well-orderable families of non-empty sets, is known to be equivalent to the statement $(\forall X)\aleph(X) = \aleph^*(X)$, and the proof makes use of the idea of transferring eccentricity upwards. This concept is best explained by proving the theorem.

Theorem 3.3.1 ([Pel78])**.** $\mathsf{AC}_{\mathsf{WO}}$ *is equivalent to* $(\forall X)\aleph(X) = \aleph^*(X)$.

Proof. ($\implies$). Let X be a set. We always have that $\aleph(X) \leqslant \aleph^*(X)$, so it is sufficient to prove that $\aleph^*(X) \leqslant \aleph(X)$. Let $\lambda < \aleph^*(X)$, and $f\colon X \to \lambda$ be a surjection. Since f is a surjection, if we set $C = \{f^{-1}(\{\alpha\}) \mid \alpha < \lambda\}$ then C is a well-ordered family of non-empty sets, and so by $\mathsf{AC}_{\mathsf{WO}}$, there is a choice function $c\colon \lambda \to X$. However, c must be an injection since $f^{-1}(\alpha) \cap f^{-1}(\beta) = \emptyset$ whenever $\alpha \neq \beta$.

($\impliedby$). We shall prove $\mathsf{AC}_{\aleph_\delta}$ for all ordinals δ by induction. Suppose that we have established $\mathsf{AC}_{<\aleph_\delta}$ (indeed, this is a theorem of ZF for $\delta = 0$), and let $X = \{X_\alpha \mid \alpha < \aleph_\delta\}$ where $X_\alpha \neq \emptyset$ for all $\alpha < \aleph_\delta$. By induction, for all $\alpha < \aleph_\delta$, $Y_\alpha := \prod_{\gamma<\alpha} X_\gamma \neq \emptyset$. Define by induction on $\alpha < \aleph_\delta$ the cardinal κ_α and the set D_α in the following way:

$$\kappa_\alpha := \aleph\left(\bigcup\{D_\beta \mid \beta < \alpha\}\right) \text{ and } D_\alpha := Y_\alpha \times \kappa_\alpha.$$

Let $D = \bigcup\{D_\alpha \mid \alpha < \aleph_\delta\}$ and $\lambda = \sup\{\kappa_\alpha \mid \alpha < \aleph_\delta\}$. By projection to its second co-ordinate, there is a surjection $D \to \lambda$, and so $\aleph^*(D) \geqslant \lambda^+$. By assumption, we must also have that $\aleph(D) \geqslant \lambda^+$. Let $f\colon \lambda \to D$ be an injection.

For all γ, let $f(\gamma) = \langle y_\gamma, \delta_\gamma\rangle \in Y_{\varepsilon_\gamma} \times \kappa_{\varepsilon_\gamma}$ for some $\varepsilon_\gamma < \aleph_\delta$. From the fact that $\lambda > \aleph(\bigcup_{\beta<\alpha} D_\beta)$ for all $\alpha < \delta$, it cannot be the case that $f``\lambda \subseteq \bigcup_{\beta<\alpha} D_\beta$ for any $\alpha < \delta$. Hence, for all α, there is γ such that $\alpha < \varepsilon_\gamma$ and hence $\alpha \in \operatorname{dom}(y_\gamma)$. Let $c(\alpha) = y_\gamma(\alpha) \in X_\alpha$, where γ is least such that $\alpha \in \operatorname{dom}(y_\gamma)$. Then $c \in \prod X$ is a choice function as desired. □

The method here is similar to that of [Lév64, Theorem 16], in which an essentially identical construction is used in order to show the equivalence between a certain well-ordered choice principle and the non-existence of sets X such that $\aleph(X) = \lambda$ but $|X| > \mu$ for all $\mu < \lambda$.[84]

[84] With some effort, one can view [Lév64, Theorem 16] as a local version of the theorem that $\mathsf{AC}_{\mathsf{WO}}$ is equivalent to "for all X, $\aleph(X)$ is a successor". Indeed, Lévy's axiom $C(\alpha)$ that says "for all functions F with domain α such that, for all $\beta < \alpha$, $F(\beta) \neq \emptyset$ and $\prod F``\beta \neq \emptyset$, $\prod F``\alpha \neq \emptyset$" is a natural perspective for a local version of Theorem A.

Inspired by the proof of Theorem 3.3.1, we produce a general framework for taking a set X and producing a set D of larger Lindenbaum number with some control over the Hartogs and Lindenbaum numbers produced.

Definition 3.3.2. Let $\delta > 0$ a limit ordinal, $X = \{X_\alpha \mid \alpha < \delta\}$ be such that for all $\alpha < \delta$, $Y_\alpha := \prod_{\beta<\alpha} X_\beta \neq \emptyset$, and κ be a cardinal. Inductively define the cardinals κ_α and sets D_α for $\alpha < \delta$ as follows:

$$\kappa_\alpha := \aleph\left(\bigcup\{D_\beta \mid \beta < \alpha\}\right) + \kappa \text{ and } D_\alpha := Y_\alpha \times \kappa_\alpha.$$

We then define the *upwards transfer construction* $D = D(X, \kappa)$ as $\bigcup_{\alpha<\delta} D_\alpha$ and $\lambda = \lambda(X, \kappa)$ as $\sup\{\kappa_\alpha \mid \alpha < \delta\}$. Observe that $\lambda > \kappa$ and that λ is a limit cardinal.

Proposition 3.3.3. *$\aleph(D) \geqslant \lambda$, $\aleph^*(D) \geqslant \lambda^+$, and if $\aleph(D) \geqslant \lambda^+$ then $\prod_{\alpha<\delta} X_\alpha$ is non-empty.*

Proof. Let $\mu < \lambda$. Then there is $\alpha < \delta$ such that $\mu < \kappa_\alpha$. Hence, by fixing $y \in Y_\alpha$, the function $\gamma \mapsto \langle y, \gamma\rangle$ is an injection $\mu \to D$. Therefore, $\aleph(D) \geqslant \lambda$. On the other hand, by projection to the second co-ordinate we have $\aleph^*(D) \geqslant \lambda^+$.

Finally, suppose that $\aleph(D) \geqslant \lambda^+$, so there is an injection $f\colon \lambda \to D$. Note that since $\lambda > \aleph(\bigcup_{\beta<\alpha} D_\beta)$ for all $\alpha < \delta$, we cannot have that $f``\lambda \subseteq \bigcup_{\beta<\alpha} D_\beta$ for any $\alpha < \delta$. Hence, $f``\lambda$ intersects D_α for unboundedly many α and so, by projection to the first co-ordinate and the well-order of $f``\lambda$, we may select some $y_\alpha \in Y_\alpha$ for unboundedly many α. Putting these partial choice functions together yields $c \in \prod X$ as desired, as in the proof of Theorem 3.3.1. □

With upwards transfer construction in hand, we may produce a great many new statements that are all equivalent to $\mathsf{AC_{WO}}$ through the general framework of Theorem 3.3.1.

Theorem A. *The following are equivalent:*

1. *For all X, $\aleph(X) = \aleph^*(X)$;*
2. *there is κ such that for all X, $\aleph^*(X) \geqslant \kappa \implies \aleph(X) = \aleph^*(X)$;*
3. *there is κ such that for all X, $\aleph(X) \geqslant \kappa \implies \aleph(X) = \aleph^*(X)$;*
4. $\mathsf{AC_{WO}}$*;*
5. *for all X, $\aleph(X)$ is a successor; and*

6. *for all X, $\aleph(X)$ is regular.*

Proof. ((1) $\implies$ (2)). Immediate.

((2) $\implies$ (3)). Immediate from $\aleph^*(X) \geqslant \aleph(X)$ for all X.

((4) $\iff$ (1)). See Theorem 3.3.1.

((4) $\implies$ (5)). Let X be such that $\aleph(X) \geqslant \lambda$ for a limit cardinal λ. For all $\mu < \lambda$, recall that $X^{\underline{\mu}}$ denotes the set of injections $\mu \to X$, and use $\mathsf{AC}_{\mathsf{WO}}$ to find $F \in \prod_{\mu<\lambda} X^{\underline{\mu}}$. Define the sequence $\langle x_\alpha \mid \alpha < \eta \rangle$ by concatenating the sequences $\langle F(\mu)(\beta) \mid \beta < \mu \rangle$ and removing duplicate entries. This must be an injection $\eta \to X$ for some ordinal η, and we claim that $\eta \geqslant \lambda$, proving that $\aleph(X) \geqslant \lambda^+$ as required. If instead $\eta < \lambda$, then letting $\eta < \mu < \lambda$ we find that $F(\mu)``\mu \subseteq \{x_\alpha \mid \alpha < \eta\}$, so $F(\mu)$ is not an injection, a contradiction. Hence $\eta \geqslant \lambda$ as required.

((4) $\implies$ (6)). By $\mathsf{AC}_{\mathsf{WO}}$, all successor cardinals are regular.[85] Also by Condition (4), we deduce Condition (5), so $\aleph(X)$ is a successor for all X. Therefore, $\aleph(X)$ is regular for all X.

(Each of (3), (5), and (6) implies (4)). Since the proofs of these three implications are almost identical, they have been packaged here.

We shall prove $\mathsf{AC}_{\aleph_\delta}$ by induction on δ. Suppose that we have $\mathsf{AC}_{<\aleph_\delta}$ (which is a theorem of ZF in the case of $\delta = 0$), and let $X = \{X_\alpha \mid \alpha < \aleph_\delta\}$ where $X_\alpha \neq \emptyset$ for all $\alpha < \aleph_\delta$. By induction, $\prod_{\beta<\alpha} X_\beta \neq \emptyset$ for all $\alpha < \aleph_\delta$, so setting $D = D(X, \kappa + \aleph_\delta)$ and $\lambda = \lambda(X, \kappa + \aleph_\delta)$ we get that $\aleph(D) \geqslant \lambda > \kappa + \aleph_\delta$. For each of the Conditions (3), (5), and (6) we may use Proposition 3.3.3 to prove that $\prod X \neq \emptyset$ by showing that $\aleph(D) \geqslant \lambda^+$.

First, assume Condition (3). Then $\aleph(D) \geqslant \kappa$, so $\aleph(D) = \aleph^*(D) \geqslant \lambda^+$, and thus $\prod X \neq \emptyset$.

If we instead assume Condition (5), then since λ is a limit cardinal we have $\aleph(D) \geqslant \lambda^+$, and thus $\prod X \neq \emptyset$.

Finally, if we assume Condition (6), then since $\lambda = \sup\{\kappa_\alpha \mid \alpha < \aleph_\delta\}$ and $\lambda > \aleph_\delta$, we get that λ is a singular cardinal, so $\aleph(D) \geqslant \lambda^+$, and thus $\prod X \neq \emptyset$.

Hence, in each case, we can conclude Condition (4). □

Note that, by combining the techniques exhibited in the proof of Theorem A, one can produce a vast collection of conditions that are equivalent to $\mathsf{AC}_{\mathsf{WO}}$.

[85] If $\alpha < \kappa^+$ and $f\colon \alpha \to \kappa^+$ is a strictly increasing sequence, then by $\mathsf{AC}_{\mathsf{WO}}$ we may simultaneously pick injections $f(\beta) \to \kappa$ for all $\beta < \alpha$ and an injection $\alpha \to \kappa$, so we may inject $f``\alpha$ into $\kappa \times \kappa = \kappa$.

For example, $\mathsf{AC}_{\mathsf{WO}}$ is equivalent to the statement "there is κ such that for all X, if $\aleph^*(X) \geqslant \kappa$ then either $\aleph(X)$ is a successor or $\aleph(X)$ is regular".

3.4 Eccentric arithmetic

Before we unpack the Hartogs–Lindenbaum spectrum, it will be helpful to prove some preliminary facts about the cardinal arithmetic of Hartogs and Lindenbaum numbers. In time, these will be used to construct 'lifts' of eccentric sets, similar to the upwards transfer construct of Definition 3.3.2.

Proposition 3.4.1. *If $B \neq \emptyset$ is Dedekind-finite, then for all $1 < n < \omega$, $|B^{\underline{n}}| < |B^n|$. If B is Dedekind-infinite, then for all $\alpha < \aleph(B)$, $|B^{\underline{\alpha}}| = |B^\alpha|$.*

Proof. Firstly, let $1 < \aleph(B) \leqslant \omega$ and $n < \omega$. Assume that $|B| \geqslant n$, as otherwise $|B^{\underline{n}}| = 0 < |B^n|$ is immediate, so let $b_0, \dots, b_{n-1} \in B$ be an arbitrary sequence of distinct elements of B. Towards a contradiction, let $F\colon B^n \to B^{\underline{n}}$ be an injection. We shall definably extend $b_0, \dots, b_{n-1}$ to a sequence $\langle b_i \mid i < \omega\rangle \in B^\omega$ of distinct elements, contradicting that $\aleph(B) \leqslant \aleph_0$. Suppose that we have already defined $b_0, \dots, b_{m-1}$ distinct some $m \geqslant n$. By, say, iterated use of the the Cantor pairing function,[86] we have a canonical way of ordering m^n, and since $|\{b_0, \dots, b_{m-1}\}^{\underline{n}}| = n!\binom{m}{n} < m^n$,[87] it is not the case that for all $\langle i_0, \dots, i_{n-1}\rangle \in m^n$, $F(\langle b_{i_j} \mid j < n\rangle)``n \subseteq \{b_0, \dots, b_{m-1}\}$. So take $\langle i_0, \dots, i_{n-1}\rangle$ to be least[88] such that $F(\langle b_{i_j} \mid j < n\rangle)``n \cap \{b_0, \dots, b_{m-1}\} \neq \emptyset$ and define $b_m = F(\langle b_{i_j} \mid j < n\rangle)(k)$, where k is taken to be the least such that $F(\langle b_{i_j} \mid j < n\rangle)(k) \notin \{b_0, \dots, b_{m-1}\}$.

Now let B be Dedekind-infinite and $\alpha < \aleph(B)$. Then $\alpha \times \alpha \times \omega < \aleph(B)$, so let $f\colon \alpha \times \alpha \times \omega \to B$ be an injection. We shall define a function $F\colon B^\alpha \to B^{\underline{\alpha}}$ and show that it is injective. For $g\colon \alpha \to B$ and $\beta < \alpha$, we set

$$F(g)(\beta) = \begin{cases} g(\beta) & g(\beta) \notin g``\beta \cup f``(\alpha \times \alpha \times \omega) \\ f(\gamma, \delta, n+1) & g(\beta) = f(\gamma, \delta, n) \notin g``\beta \\ f(\gamma, \beta, 0) & \gamma < \beta \text{ is least such that } g(\beta) = g(\gamma). \end{cases}$$

This is perhaps better illuminated by an algorithmic description: First reserve the set $A = f``(\alpha \times \alpha \times \omega)$. Given β, if $g(\beta)$ is distinct from all prior $g(\alpha)$ and

[86] $\pi(i,j) = \frac{1}{2}(i+j)(i+j+1) + i$.
[87] True for $n \geqslant 2$.
[88] In our canonical linear ordering.

$g(\beta) \notin A$, then we set $F(g)(\beta) = g(\beta)$. If instead, $g(\beta) = g(\gamma)$, where $\gamma < \beta$ is the earliest that $g(\beta)$ appears, then we set $F(g)(\beta) = f(\gamma, \beta, 0)$. Now, for any $g(\beta) \in A$, we increase the ω co-ordinate by one, so if $g(\beta) = f(\gamma, \delta, n)$ then we set $F(g)(\beta) = f(\gamma, \delta, n+1)$.

In this way, given $F(g)$, one can recover g: if $F(g)(\beta) \notin A$ then $g(\beta)$ is $F(g)(\beta)$; if $F(g)(\beta) = f(\gamma, \beta, 0)$, then $F(g)(\beta) = g(\gamma) = F(g)(\gamma)$; and if $F(g)(\beta) = f(\gamma, \delta, n+1)$ then $g(\beta) = f(\gamma, \delta, n)$. Thus, F is an injection as required. □

Corollary 3.4.2. *For all infinite B and $0 < \alpha < \aleph(B)$, $\aleph(B^{\underline{\alpha}}) = \aleph(B^\alpha)$ and $\aleph^*(B^{\underline{\alpha}}) = \aleph^*(B^\alpha)$.*

Proof. For Dedekind-infinite B, $|B^{\underline{\alpha}}| = |B^\alpha|$. For Dedekind-finite B, $\alpha > 0$ is finite, so $\aleph(B^\alpha) = \aleph(B)^\alpha = \aleph(B)$. On the other hand, $0 < \alpha$, so $|B| \leqslant |B^{\underline{\alpha}}|$ and thus $\aleph(B) \leqslant \aleph(B^{\underline{\alpha}}) \leqslant \aleph(B^\alpha) = \aleph(B)$. To establish the result for $\aleph^*$, we do not have productivity of Lindenbaum numbers, so an alternative strategy is required.

For the rest of the proof, let $X_{\overline{Y}}$ denote the set of surjections $X \to Y$ and let $n = \alpha < \omega$.[89] Given any function $f : n \to B$, we may characterise f via an injection $k_f : m \to B$, where $m = |f``n| \leqslant n$, and a surjection $e_f : n \to m$, where $k_f(i)$ is the ith unique element of the sequence $\langle f(j) \mid j < n \rangle$, and e_f is the unique function such that $f(j) = k_f(e_f(j))$ for all $j < n$. Since we can recover f from this data, $f \mapsto \langle k_f, e_f \rangle$ provides a bijection between B^n and $\bigcup_{1 \leqslant m \leqslant n} B^{\underline{m}} \times n_{\overline{m}}$. Therefore, noting that $n_{\overline{m}}$ is finite and that, if at least one of A or B are infinite then $\aleph^*(A + B) = \aleph^*(A) + \aleph^*(B)$,[90]

$$\begin{aligned}
\aleph^*(B^n) &= \aleph^*\left(\bigcup_{m=1}^{n} (B^{\underline{m}} \times n_{\overline{m}})\right) \\
&= \sum_{m=1}^{n} \aleph^*\left(B^{\underline{m}} \times n_{\overline{m}}\right) \\
&= \sum_{m=1}^{n} \aleph^*\left(B^{\underline{m}}\right) \\
&\leqslant n \times \aleph^*\left(B^{\underline{n}}\right) \\
&= \aleph^*\left(B^{\underline{n}}\right).
\end{aligned}$$

[89] We make this change of variable only to ease the syntactical intuition that certain objects are finite in the remainder of the proof.

[90] In particular, if A is infinite and F is finite then $\aleph^*(A \times F) = \aleph^*(A)$.

We certainly have that $|B^{\underline{n}}| \leqslant |B^n|$, so we conclude that $\aleph^*(B^n) = \aleph^*(B^{\underline{n}})$. □

Proposition 3.4.3. *For all B,*

$$\sup\{\aleph(B^\alpha) \mid \alpha < \aleph(B)\} = \sup\{\aleph(\alpha^\alpha) \mid \alpha < \aleph(B)\}.$$

Proof. Let $\mu = \aleph(B)$. If $\mu = \aleph_0$ then $\aleph(B^\alpha) = \aleph_0$ for all $\alpha < \aleph_0$, and so $\sup\{\aleph(B^\alpha) \mid \alpha < \aleph_0\} = \sup\{\aleph(\alpha^\alpha) \mid \alpha < \aleph_0\} = \aleph_0$, so let us assume instead that B is Dedekind-infinite.

($\geqslant$). If $|\alpha| \leqslant |B|$ then $|\alpha^\alpha| \leqslant |B^\alpha|$, so certainly $\aleph(\alpha^\alpha) \leqslant \aleph(B^\alpha)$ for all $\alpha < \mu$.

($\leqslant$). We shall show that for all infinite $\alpha < \mu$,

$$\aleph(B^\alpha) \leqslant \sup\{\aleph(\kappa^\kappa) \mid \alpha \leqslant \kappa < \mu\}.$$

Since B is Dedekind-infinite, $|B^\alpha| = |B^{\underline{\alpha}}|$, so let $f\colon \lambda \to B^{\underline{\alpha}}$ be an injection for some $\lambda < \aleph(B^\alpha)$. By lexicographic ordering, $\langle f(\beta,\gamma) \mid \beta < \lambda, \gamma < \alpha\rangle$ is a well-ordered subset of B with order type κ for some $\alpha \leqslant \kappa < \mu$. By this identification, f induces an injection $\hat{f}\colon \lambda \to \kappa^{\underline{\alpha}}$, and so (since κ is Dedekind-infinite) $\lambda < \aleph(\kappa^\alpha)$. That is, if $\lambda < \aleph(B^\alpha)$ then there exists $\kappa < \mu$ such that $\lambda < \aleph(\kappa^\alpha)$, so $\aleph(B^\alpha) \leqslant \sup\{\aleph(\kappa^\alpha) \mid \alpha \leqslant \kappa < \mu\}$. It follows that

$$\begin{aligned}\sup\{\aleph(B^\alpha) \mid \alpha < \mu\} &\leqslant \sup\{\aleph(\kappa^\alpha) \mid \alpha \leqslant \kappa < \mu\} \\ &= \sup\{\aleph(\kappa^\kappa) \mid \kappa < \mu\}.\end{aligned}$$ □

Proposition 3.4.4. *Let $A = \bigcup_{\alpha<\lambda} A_\alpha$, where the A_α are pairwise disjoint and non-empty. Let $\chi = \sup\{\aleph^*(A_\alpha) \mid \alpha < \lambda\}$. Then*

$$\max\{\chi, \lambda^+\} \leqslant \aleph^*(A) \leqslant \max\{\chi^+, \lambda^+\}.$$

If $\aleph^*(A) > \chi$ then $\mathrm{cf}(\chi) \leqslant \lambda$.

Proof. Certainly $\lambda < \aleph^*(X)$ by considering $\{\langle a, \alpha\rangle \mid a \in A_\alpha, \alpha < \lambda\}$, and for all $\eta < \chi$ there is $\alpha < \lambda$ such that $\eta \leqslant \aleph^*(A_\alpha) \leqslant \aleph^*(A)$, so $\chi \leqslant \aleph^*(A)$.

Let $f\colon A \to \mu$ be a surjection. For all $\alpha < \lambda$, let $\beta_\alpha = \mathrm{ot}(f``A_\alpha)$, noting that $\beta_\alpha < \aleph^*(A_\alpha) \leqslant \chi$ and hence $\sup\{\beta_\alpha \mid \alpha < \lambda\} \leqslant \chi$. Then f gives us a surjection $\chi \times \lambda \to \mu$, and so either $\mu \leqslant \chi$ or $\mu \leqslant \lambda$. That is, $\aleph^*(A) \leqslant \max\{\chi^+, \lambda^+\}$ as required.

Note that if $f\colon A \to \chi$ is a surjection then there cannot be $\alpha < \lambda$ such that $|f``A_\alpha| = \chi$, so if $\chi < \aleph^*(A)$ then $\mathrm{cf}(\chi) \leqslant \lambda$. □

Corollary 3.4.5. *For all infinite A and cardinals λ,*

$$\max\{\aleph^*(A), \lambda^+\} \leqslant \aleph^*(A \times \lambda) \leqslant \max\{\aleph^*(A)^+, \lambda^+\}.$$

Proof. $A \times \lambda = \bigcup_{\alpha<\lambda}(A \times \{\alpha\})$, so the bound follows by Proposition 3.4.4, noting that $\chi = \aleph^*(A)$.[91] □

Proposition 3.4.6. *Let $A = \bigcup_{b\in B} A_b$ be an infinite set, where the A_b are pairwise disjoint and non-empty. Let $\chi = \sup\{\aleph^*(A_b) \mid b \in B\}$. Then $\max\{\aleph^*(B), \chi\} \leqslant \aleph^*(A) \leqslant \max\{\aleph^*(B)^+, \chi^+\}$.*

Proof. Since the A_b are pairwise disjoint and non-empty, $|B| \leqslant^* |A|$, and so $\aleph^*(B) \leqslant \aleph^*(A)$. Similarly, for all $b \in B$, $|A_b| \leqslant |A|$ and so $\aleph^*(A_b) \leqslant \aleph^*(A)$. Hence, $\chi \leqslant \aleph^*(A)$.

Let $f\colon A \to \mu$ be a surjection. Then, for $b \in B$, let $\beta_b = \mathrm{ot}(f``A_b)$, so $\beta_b < \aleph^*(A_b) \leqslant \chi$ and $\beta = \sup\{\beta_b \mid b \in B\}$. Then f descends to a surjection $g\colon B \times \beta \to \mu$. Thus, by Corollary 3.4.5, $\mu < \max\{\aleph^*(B)^+, \beta^+\}$. That is, $\aleph^*(A) \leqslant \max\{\aleph^*(B)^+, \chi^+\}$ as required. □

Corollary 3.4.7. *For all non-empty A and B,*

$$\aleph^*(A) \times \aleph^*(B) \leqslant \aleph^*(A \times B) \leqslant \aleph^*(A)^+ \times \aleph^*(B)^+.$$

In particular, $\aleph^*(A^2) \leqslant \aleph^*(A)^+$.

Proof. Let $C = \bigcup_{b\in B} A \times \{b\}$ in Proposition 3.4.6. □

Lemma 3.4.8. *Let δ be an ordinal and $I = \{\alpha \leqslant \delta \mid |\alpha| = |\delta|\}$. There is a set $\langle k_\alpha \mid \alpha \in I\rangle$ such that, for all α, k_α is a bijection $\alpha \to \delta$.*

Proof. Let $k\colon \delta \to |\delta|$ be a bijection. Then, for all $\alpha \in I$, $k \restriction \alpha$ is an injection $\alpha \to |\delta|$. Since $|\alpha| = |\delta|$, the Mostowski collapse of $k``\alpha$ induces a bijection $\pi_\alpha\colon k``\alpha \to |\delta|$. Hence $k^{-1} \circ \pi_\alpha \circ k \restriction \alpha$ is a bijection $\alpha \to \delta$ as required. □

[91] There is a proof, due to Yunhe Peng, that in fact $\aleph^*(A \times \lambda) = \max\{\aleph^*(A), \lambda^+\}$ when one of A or λ is infinite. This was privately communicated by Peng, though the case $\lambda = \omega$ is published in [PS24, Lemma 3.6]. To avoid scuttlebutt as much as possible, we shall refrain from using this result and merely point out that it would strengthen both Proposition 3.5.9 and Proposition 3.5.10 to "$\aleph^*(X) = \max\{\kappa, \lambda^+\}$".

3.5 Spectra

Theorem A gives the very strong conclusion that $\mathsf{AC_{WO}}$ is not just equivalent to $(\forall X)\aleph(X) = \aleph^*(X)$, but also that $\aleph(X)$ is a successor cardinal for all X. In this way, $\mathsf{AC_{WO}}$ gives us that the class of all pairs $\langle\aleph(X), \aleph^*(X)\rangle$ is minimal, that is just $\{\langle\lambda^+, \lambda^+\rangle \mid \lambda \in \mathrm{Card}\}$. However, it is possible to violate $\mathsf{AC_{WO}}$, and we inspect here the various ways in which this can be violated in models of SVC.

In Section 3.6 we show that it is consistent with ZF to have a model M of maximal spectrum, so

$$\mathrm{Spec}_\aleph(M) = \{\langle\lambda, \kappa\rangle \mid \aleph_0 \leqslant \lambda \leqslant \kappa\} \cup \{\langle n+1, n+1\rangle \mid n < \omega\}.$$

However, this is achieved with a class-sized product of symmetric extensions and, by Theorem B, cannot be exhibited as a set-generic symmetric extension (of a model of ZFC). We shall show that, in the case that M is a set-generic symmetric extension of a model of ZFC, $\mathrm{Spec}_\aleph(M)$ is eventually strongly controlled. For example, there is a cardinal Ω such that for all X, if $\aleph(X) \geqslant \Omega$, then $\aleph^*(X) \leqslant \aleph(X)^+$.

3.5.1 The spectrum of Cohen's first model

Let us first establish the methods that will be employed in our favourite test model of $\mathsf{ZF} + \mathsf{SVC} + \neg\mathsf{AC}$: Cohen's first model.[92] In fact, Cohen's first model is not even a model of AC_ω, witnessed by a set A such that $\aleph(A) = \aleph_0$, and so is certainly not a model of $\mathsf{AC_{WO}}$.[93] Letting M be Cohen's first model for this section, recall that M forms part of a chain of transitive submodels $V \subseteq M \subseteq V[G]$, where $V \vDash \mathsf{ZFC}$ and G is a V-generic filer for $\mathrm{Add}(\omega, \omega)$. Since $\mathrm{Add}(\omega, \omega)$ is c.c.c., V and $V[G]$ (and hence M) agree on the cardinality and cofinality of ordinals.

Fact 3.5.1 ([Jec73, Section 5.5])**.** *In M there is a set A such that:*

1. $\aleph(A) = \aleph_0$*;*
2. $V[G] \vDash |A| = \aleph_0$*;*

[92] Specifics of the construction of M can be found at Section 2.4.1, but we only need a few preliminary facts.

[93] By [Jec73, Section 2.4.1], if $\mathsf{AC}_{\aleph_0}$ holds then every infinite set has a countably infinite subset. See also Lemma 3.5.17.

3. $\mathsf{SVC}^+([A]^{<\omega})$*; and*

4. *for every infinite* X *there is a surjection* $f\colon X \to \omega$*.*

An immediate corollary of this fact is the following.

Corollary 3.5.2. *In* M*,* $\aleph^*(A) = \aleph_1$*.*

Proof. By Item (4) of Fact 3.5.1, there is a surjection $A \to \omega$ in M, and so $\aleph^*(A) \geqslant \aleph_1$. However, by Item (2) there is no surjection $A \to \aleph_1$ in $V[G]$ and so certainly no such surjection in M. Hence, $\aleph^*(A) = \aleph_1$. □

We shall use these facts alongside the techniques laid out in Section 3.3 to produce a complete picture of the spectrum of Cohen's first model.

Theorem 3.5.3. $\mathrm{Spec}_\aleph(M) = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda \in \mathrm{Card}\} \cup \{\langle\lambda, \lambda^+\rangle \mid \mathrm{cf}(\lambda) = \aleph_0\}$.

To translate this to the notation of Theorem B, we have that $\phi = \psi = \aleph_0$, $\chi_0 = \chi = \Omega = \psi^* = \aleph_1$, and $C = \{\aleph_0\}$. Hence,

$$\mathrm{Spec}_\aleph(M) = \bigcup \begin{cases} \mathrm{SC} = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda \in \mathrm{Card}\} \\ \mathfrak{D} \subseteq \{\langle\lambda, \kappa\rangle \quad \mid \aleph_0 \leqslant \lambda \leqslant \kappa \leqslant \aleph_1, \aleph_1 \leqslant \kappa\} \\ \mathfrak{C} \subseteq \{\langle\lambda, \lambda^+\rangle \quad \mid \mathrm{cf}(\lambda) = \aleph_0, \lambda < \aleph_1\} \\ \mathfrak{U} = \{\langle\lambda, \lambda^+\rangle \quad \mid \mathrm{cf}(\lambda) = \aleph_0, \lambda \geqslant \aleph_1\}. \end{cases}$$

Therefore, both $\mathfrak{D}$ and $\mathfrak{C}$ are subsets of $\{\langle\aleph_0, \aleph_1\rangle\}$. However, we have already seen by Fact 3.5.1 and Corollary 3.5.2 that $\aleph(A) = \aleph_0$ and $\aleph^*(A) = \aleph_1$, and thus $\mathfrak{D} = \mathfrak{C} = \{\langle\aleph_0, \aleph_1\rangle\}$.

The upper bound

We begin with an upper bound for the spectrum of M.

Lemma 3.5.4. *For all* X *and all* μ *with* $\aleph(X) \leqslant \mu < \aleph^*(X)$*,* $\mathrm{cf}(\mu) \leqslant \aleph_0$*.*

Proof. We may assume that $X \subseteq [A]^{<\omega} \times \eta$ for some ordinal η. For each $a \in [A]^{<\omega}$, let $X_a = X \cap (\{a\} \times \eta)$, so each X_a is well-orderable. Let $\lambda = \aleph(X)$, $\kappa = \aleph^*(X)$, and $\mu \in [\lambda, \kappa)$. Since $\mu < \kappa$, there is a surjection $f\colon X \to \mu$. Hence, in $V[G]$, we have that $\mu = \bigcup_{a \in [A]^{<\omega}} f``X_a$, so μ is the union of countably many sets (note that A is countable in $V[G]$, and so $[A]^{<\omega}$ is as well). If μ has uncountable cofinality then there is $a \in [A]^{<\omega}$ such that $V[G] \vDash |X_a| \geqslant \mu$. Since M and $V[G]$ agree on the cardinalities of sets of ordinals, and X_a is well-orderable, we have that $|X_a| \geqslant \mu$ in M as well. However, this contradicts $\mu \geqslant \aleph(X)$. Hence we must have $\mathrm{cf}(\mu) = \aleph_0$. □

Lemma 3.5.5. *For all X, $\aleph^*(X)$ is a successor.*

Proof. As before, we may assume that $X \subseteq [A]^{<\omega} \times \eta$ for some ordinal η, and again we shall denote by X_a the set $X \cap (\{a\} \times \eta)$ for all $a \in [A]^{<\omega}$. Let κ be a limit cardinal such that $\aleph^*(X) \geqslant \kappa$. Then we must show that $\aleph^*(X) > \kappa$, that is we must show that there is a surjection $X \to \kappa$. Let $\mu < \kappa$ be infinite. Since κ is a limit, we have that $\mu^+ < \kappa$, and μ^+ is regular in both M and $V[G]$. Since $\mu^+ < \kappa$, there is a surjection $f\colon X \to \mu^+$, so in $V[G]$ we have $\mu^+ = \bigcup\{f``X_a \mid a \in [A]^{<\omega}\}$, a countable union. Since $\mathrm{cf}(\mu^+) > \aleph_0$, there is $a \in [A]^{<\omega}$ such that $|f``X_a| \geqslant \mu^+$ in $V[G]$, and so $|X_a| \geqslant \mu^+$ in $V[G]$. However, X_a is well-orderable, and so $|X_a| \geqslant \mu^+$ in M as well. Therefore, for all $\mu < \kappa$, there is $a \in [A]^{<\omega}$ such that $|X_a| > \mu$. Hence the projection of X onto its second co-ordinate is a surjection onto a subset of η of cardinality at least κ, and so this can be turned into a surjection $X \to \kappa$. □

Corollary 3.5.6. *If $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$, then one of the following holds:*

(a) *$\kappa = \lambda$ are successors; or*

(b) *$\kappa = \lambda^+$ and $\mathrm{cf}(\lambda) = \aleph_0$.*

Corollary 3.5.7. *In* ZF*, both "for all X, $\aleph^*(X)$ is regular" and "for all X, $\aleph^*(X)$ is a successor" are strictly weaker than* $\mathsf{AC}_{\mathsf{WO}}$*.*

Proof. Suppose that $\kappa = \lambda$. Since κ is a successor, λ must be as well. Suppose instead that $\kappa > \lambda$. Then by Lemma 3.5.4, for all $\mu \in [\lambda, \kappa)$, $\mathrm{cf}(\mu) = \aleph_0$. However, $\mathrm{cf}(\lambda^+) = \lambda^+ > \aleph_0$ in M, so $\lambda^+ \notin [\lambda, \kappa)$. That is, $\kappa = \lambda^+$. □

The lower bound

To complete Theorem 3.5.3, we must show the lower bound for the spectrum.

Lemma 3.5.8. *For all λ such that $\mathrm{cf}(\lambda) = \aleph_0$ there is a set X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.*

Proof. By Corollary 3.5.6, it will be sufficient to prove that there is a set X such that $\aleph(X) = \lambda$. By Fact 3.5.1, we may assume that $\lambda > \aleph_0$. Let $\langle \delta_n \mid n < \omega \rangle$ be a cofinal sequence in λ, and set $X = \bigcup_{n<\omega} A^{\underline{n}} \times \delta_n$. We certainly have that $\aleph(X) \geqslant \lambda$, as for all $n < \omega$ there is an injection $f_n\colon \delta_n \to X$ by taking arbitrary

$c \in A^{\underline{n}}$ and setting $f_n(\alpha) = \langle c, \alpha\rangle$. Suppose now that there were an injection $f\colon \lambda \to X$. Note that in $V[G]$, $|A^{\underline{n}}| = \aleph_0$, and so $|\bigcup_{m<n} A^{\underline{m}} \times \delta_m| < \lambda$ for all $n < \omega$. Therefore, $f``\lambda$ is not a subset of $\bigcup_{m<n} A^{\underline{m}} \times \delta_m$ for any $n < \omega$, so $f``\lambda$ produces a well-ordered collection of injections $m \to A$ for arbitrarily large m. Putting these injections together we construct an injection $\omega \to A$, contradicting $\aleph(A) = \aleph_0$. Hence no injection $\lambda \to X$ exists, so $\aleph(X) = \lambda$ as desired. □

Combining Corollary 3.5.6 and Lemma 3.5.8, we immediately obtain Theorem 3.5.3:

$$\mathrm{Spec}_{\aleph}(M) = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda \in \mathrm{Card}\} \cup \{\langle\lambda, \lambda^+\rangle \mid \mathrm{cf}(\lambda) = \aleph_0\}.$$

3.5.2 Lifting eccentricity

In Lemma 3.5.8 we saw an example of how to 'lift' the Dedekind-finite set A to an eccentric set X of higher Hartogs number (of countable cofinality) along the lines of the upwards transfer construction. We can generalise this method to any set B such that $\aleph(B)$ is a limit cardinal or a singular cardinal (even if B is not eccentric), which we demonstrate in Proposition 3.5.9. Another method can also be employed in the case that $\aleph(B)$ is regular and $\aleph(B) < \aleph^*(B)$, even if $\aleph(B)$ is not a limit (Proposition 3.5.10). These methods work in ZF, even without SVC, though it becomes much easier to control the constructions in the case that SVC holds, as we will explore in Section 3.5.3.

Proposition 3.5.9. *Let B be such that $\aleph(B) = \mu$ is singular or a limit, and $\lambda \geqslant \sup\{\aleph(\alpha^\alpha) \mid \alpha < \mu\}$ be such that $\mathrm{cf}(\lambda) = \mathrm{cf}(\mu)$. Then there is a set X such that $\aleph(X) = \lambda$ and, setting $\kappa = \aleph^*(B^{\underline{\leq\mu}})$,*

$$\max\{\kappa, \lambda^+\} \leqslant \aleph^*(X) \leqslant \max\{\kappa^+, \lambda^+\}.$$

Proof. Let $\langle\delta_\alpha \mid \alpha < \mu\rangle$ be a cofinal sequence in λ with $\delta_0 > 0$. For each $\alpha < \mu$, let $X_\alpha = B^{\underline{\alpha}} \times \delta_\alpha$, and let $X = \bigcup_{\alpha<\mu} X_\alpha$. Since $\aleph(B) = \mu$, for all $\alpha < \mu$, $B^{\underline{\alpha}} \neq \emptyset$, and so δ_α embeds into $X_\alpha \subseteq X$. Hence, $\aleph(X) \geqslant \lambda$. Projection to the second co-ordinate begets a surjection $X \to \lambda$, and thus $\aleph^*(X) > \lambda$.

Let $\beta < \mu$. We shall show that $|\bigcup_{\alpha<\beta} X_\alpha| \leqslant |X_\beta \times \beta|$:

We start by defining an injection $F_0\colon \bigcup_{\alpha<|\beta|} X_\alpha \to X_\beta \times |\beta|$. Let $c \in B^{\underline{\beta}}$. Then, for all $\alpha < |\beta|$, we have a definable embedding $e_\alpha\colon B^{\underline{\alpha}} \to B^{\underline{\beta}}$ where $e_\alpha(i)$ is given by concatenating the injections i and c, minus any duplicates.

Since $|\alpha| < |\beta|$, the sequence $i``\alpha \frown c``\beta$ will still have order type at least β after removing duplicates. $e_\alpha(i) \restriction \alpha = i$, so e_α is indeed an injection. Hence we obtain

$$F_0 \colon \bigcup\{X_\alpha \mid |\alpha| < |\beta|\} \to B^{\underline{\beta}} \times \delta_\beta \times |\beta|$$

given by, for $\langle f, \gamma\rangle \in X_\alpha$, $F_0(f, \gamma) = \langle e_\alpha(f), \gamma, \alpha\rangle$.

By Lemma 3.4.8, let $\langle k_\alpha \mid |\alpha| = |\beta|\rangle$ be a sequence such that, for all α, k_α is a bijection $\beta \to \alpha$. Then we have an injection

$$F_1 \colon \bigcup\{X_\alpha \mid |\beta| \leqslant \alpha < \beta\} \to B^{\underline{\beta}} \times \delta_\beta \times (\beta \setminus |\beta|)$$

given by, for $\langle f, \gamma\rangle \in X_\alpha$, $F_1(f, \gamma) = \langle f k_\alpha, \gamma, \alpha\rangle$.

Hence, $|\bigcup_{\alpha<\beta} X_\alpha| \leqslant |X_\beta \times \beta|$ as required, and so

$$\begin{aligned}\aleph\left(\bigcup\{X_\alpha \mid \alpha < \beta\}\right) &\leqslant \aleph(X_\beta \times \beta)\\ &= \aleph(B^{\beta}) \times |\delta_\beta|^+ \times |\beta|^+\\ &\leqslant \lambda,\end{aligned}$$

recalling from Proposition 3.4.3 that

$$\begin{aligned}\aleph(B^{\beta}) &\leqslant \sup\{\aleph(B^{\alpha}) \mid \alpha < \mu\}\\ &= \sup\{\aleph(\alpha^{\alpha}) \mid \alpha < \mu\}\\ &\leqslant \lambda.\end{aligned}$$

Hence, if $f \colon \lambda \to X$ is an injection, then $f``\lambda$ intersects X_α for unboundedly many α. Given such a well-ordered sequence, if μ is a limit then one can reconstruct an injection $\mu \to B$, contradicting that $\aleph(B) = \mu$. The case that μ is a singular successor is somewhat more complicated.

Claim 3.5.9.1. *If $\mu = \chi^+$ is singular, then there is no injection $\lambda \to X$.*

Proof of Claim. By our prior work, any such injection would induce an injection $h \colon \mathcal{C} \to B^{\underline{\leq \mu}}$, where $\mathcal{C} \subseteq \mu$ is unbounded, and for each $\alpha \in \mathcal{C}$, $h(\alpha)$ is an injection $\alpha \to B$. By passing to a cofinal subsequence, we may assume that $|\mathcal{C}| < \mu$. Consider $I = \{h(\alpha)(\beta) \mid \beta < \alpha, \alpha \in \mathcal{C}\}$. By lexicographic ordering, I is well-orderable, and thus $|I| = \chi$, say $I = \{x_\gamma \mid \gamma < \chi\}$. Hence we have a uniform sequence of injections $i_\alpha \colon \alpha \to \chi$ for $\alpha \in \mathcal{C}$ where $i_\alpha(\beta) = \gamma$ if $h(\alpha)(\beta) = x_\gamma$. By taking a Mostowski collapse, we obtain bijections $k_\alpha \colon \alpha \to \chi$

for all $\alpha \in \mathcal{C}$ such that $|\alpha| = |\eta|$. Hence, we obtain a surjection $g \colon \eta \times \mathcal{C} \to \mu$ by $g(\beta, \alpha) = k_\alpha^{-1}(\beta)$, contradicting that $\mu > |\eta \times \mathcal{C}| = \eta$. $\dashv$

Therefore, $\aleph(X) = \lambda$ as required. Note that

$$B^{\leq\mu} \times \{0\} \subseteq X \subseteq B^{\leq\mu} \times \lambda,$$

so, setting $\kappa = \aleph^*(B^{\leq\mu})$,

$$\begin{aligned} \kappa &\leqslant \aleph^*(X) \\ &\leqslant \aleph^*(B^{\leq\mu} \times \lambda) \\ &\leqslant \max\{\kappa^+, \lambda^+\}. \end{aligned}$$

However, we showed earlier that $\lambda < \aleph^*(X)$, and thus

$$\max\{\kappa, \lambda^+\} \leqslant \aleph^*(X) \leqslant \max\{\kappa^+, \lambda^+\}. \qquad \square$$

If $\mu = \chi^+$ is a singular successor, $B = \chi$ works for the statement of Proposition 3.5.9. On the other hand, if $\mu = \aleph_0$ then the requirement on λ is merely $\lambda \geqslant \aleph_0$.

Proposition 3.5.10. *Let B be a set, $\mu \in [\aleph(B), \aleph^*(B))$ be a regular cardinal, and $\lambda \geqslant \mu$ be such that $\mathrm{cf}(\lambda) = \mu$. Then there is a set X such that $\aleph(X) = \lambda$ and, setting $\kappa = \aleph^*(B)$,*

$$\max\{\kappa, \lambda^+\} \leqslant \aleph^*(X) \leqslant \max\{\kappa^+, \lambda^+\}.$$

Proof. Let $f \colon B \to \mu$ be a surjection and $\langle \delta_\alpha \mid \alpha < \mu \rangle$ be a cofinal sequence in λ with $\delta_0 > 0$. For each $\alpha < \mu$ let $X_\alpha = f^{-1}(\{\alpha\}) \times \delta_\alpha$, and let $X = \bigcup_{\alpha<\mu} X_\alpha$. Since δ_α embeds into $X_\alpha \subseteq X$, $\aleph(X) \geqslant \lambda$. Note that for all $\beta < \mu$,

$$\bigcup_{\alpha<\beta} X_\beta \subseteq B \times \sup\{\delta_\alpha \mid \alpha < \beta\} \subseteq \ B \times \delta_\beta.$$

Hence $\aleph(\bigcup_{\alpha<\beta} X_\alpha) \leqslant \mu \times |\delta_\beta|^+ \leqslant \lambda$. Therefore, any injection $\lambda \to X$ must have image contained in an unbounded collection of the X_α, and such a well-ordered collection of elements could be stitched together to produce an injection $\mathrm{cf}(\mu) = \mu \to B$, contradicting $\aleph(B) = \mu$. Hence $\aleph(X) = \lambda$. On the other hand, projection begets a surjection $X \to \lambda$, so $\aleph^*(X) > \lambda$.

Note that $B \times \{0\} \subseteq X \subseteq B \times \lambda$ so, setting $\kappa = \aleph^*(B)$,

$$\begin{aligned}\kappa &\leqslant \aleph^*(X) \\ &\leqslant \aleph^*(B \times \lambda) \\ &\leqslant \max\{\kappa^+, \lambda^+\}.\end{aligned}$$

However, we have already shown that $\lambda < \aleph^*(X)$, and thus

$$\max\{\kappa, \lambda^+\} \leqslant \aleph^*(X) \leqslant \max\{\kappa^+, \lambda^+\}. \qquad \square$$

The requirement that μ is regular cannot be removed from Proposition 3.5.10 in general. Consider the model from Theorem 3.6.2 to add X such that $\aleph(X) = \aleph_1$ and $\aleph^*(X) = \aleph_{\omega+1}$. Then, as noted in the proof of Theorem 3.6.2, the symmetric extension will satisfy DC and thus there is no A such that $\aleph(A) = \aleph_\omega$ (see Lemma 3.5.17).

Remark. Proposition 3.5.10 positively answers [Rya24c, Question 5.3]: *"Let μ be weakly inaccessible and suppose that for some set X, $\aleph(X) < \mu < \aleph^*(X)$. Must there exist Y such that $\aleph(Y) = \mu$?"*

By combining Propositions 3.5.9 and 3.5.10, we obtain a generic lifting theorem.

Theorem 3.5.11. *If there exists B such that $\aleph(B)$ is a limit, is singular, or such that $\aleph(B) < \aleph^*(B)$, then there is a cardinal Ω such that, for all $\lambda \geqslant \Omega$, if $\mathrm{cf}(\lambda) = \mathrm{cf}(\aleph(B))$, then there is X such that $\aleph(X) = \lambda < \aleph^*(X)$.*

Our best bounds for Ω are currently $\sup\{\aleph(\alpha^\alpha) \mid \alpha < \aleph(B)\}$ when $\aleph(B)$ is singular or a limit and $\aleph(B)$ when $\aleph(B)$ is regular but $\aleph(B) < \aleph^*(B)$.[94]

3.5.3 The spectrum of a model of SVC

When dealing with Cohen's first model, having an outer model of ZFC that agrees on the cardinalities and cofinalities of ordinals was an important fact that appeared in almost every proof of Section 3.5.1. Fortunately, this is not unique to Cohen's first model, and is very close to the conclusions that can be drawn from SVC. Let M be a model of $\mathsf{SVC} + \neg\mathsf{AC}$, witnessed by an inner

[94]In particular, if we begin in a model of $\mathsf{ZFC} + \mathsf{GCH}$ and take the symmetric extension described in Theorem 3.6.2 with $\mu \geqslant \kappa > \lambda$ regular, then by GCH in the outer and inner models, $\Omega = \aleph(B)$ in all cases.

model V of ZFC, a symmetric system $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$, and seed[95] A. Let $V[G]$ be the outer forcing extension of M, so $V \subseteq M \subseteq V[G]$ for V-generic $G \subseteq \mathbb{P}$, and $M = \mathsf{HS}^G_{\mathscr{F}}$. For a set $X \in M$, denote by $\lceil X \rceil$ the least ordinal δ such that $V[G] \vDash$ "$|X| = |\delta|$ and δ is a cardinal". Unlike in Cohen's model, we may not have that $\lceil \kappa \rceil = \kappa$ for all cardinals $\kappa \in M$, since $\mathbb{P}$ may collapse some cardinals, but by appealing to large enough cardinals we are able to overcome this obstacle.

Fact 3.5.12 ([Jec03, Theorem 15.3])**.** *There is a cardinal τ ($\tau = |\mathbb{P}|^+$ is always sufficient) such that for all ordinals $\alpha, \beta \geqslant \tau$, $V \vDash |\alpha| = |\beta|$ if and only if $V[G] \vDash |\alpha| = |\beta|$. Furthermore, if $\eta \geqslant \tau$ is a cardinal then $V \vDash \mathrm{cf}(\alpha) = \eta$ if and only if $V[G] \vDash \mathrm{cf}(\alpha) = \eta$.*

The following is an immediate corollary of Fact 3.5.12.

Corollary 3.5.13. *There is a cardinal τ such that V, M, and $V[G]$ agree on cardinalities above τ and cofinalities greater than τ.*

Let τ be the least cardinal satisfying the conditions of Corollary 3.5.13. Finally, fix a seed A for M and let $\nu = \lceil A \rceil$.

Throughout this section, all sets and statements about sets are taken in the context of M unless stated otherwise.

An upper bound

We first aim to create an upper bound on the Hartogs–Lindenbaum spectrum of M by showing scenarios in which combinations of Hartogs and Lindenbaum numbers are not possible. Once this has been established, the technology for building the lower bound will be quite automatic, as we have already laid out all of the groundwork in Section 3.5.2.

The proof of Proposition 3.5.14 is similar to the proof of Lemma 3.5.4, but is adapted to the use of a 'surjective' seed.

Proposition 3.5.14. *Suppose that $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$. Then for all $\mu \in [\lambda, \kappa)$, $\mathrm{cf}(\mu) < \max\{\nu^+, \tau\}$.*

Proof. Let η be such that there is a surjection $g \colon A \times \eta \to X$. For each $a \in A$, let $X_a = g``(\{a\} \times \eta)$, noting that each $X_a$ is well-orderable. Let $\mu < \kappa$, so there is a surjection $f \colon X \to \mu$. Then $\mu = \bigcup_{a \in A} f``X_a$, so if $\mathrm{cf}(\mu)^M \geqslant \max\{\nu^+, \tau\}$

[95] Not necessarily an injective seed, as was the case in Section 3.5.1.

then $\mathrm{cf}(\mu)^{V[G]} = \mathrm{cf}(\mu)^M > \nu$, and hence there is $a \in A$ such that $|f``X_a| \geqslant \mu$. Since X_a is well-orderable in M, $|\mu| \leqslant^* |X_a|$ implies that $|\mu| \leqslant |X_a| \leqslant |X|$, and so $\mu < \aleph(X)$. □

Corollary 3.5.15. *If $\aleph(X) = \lambda$ and $\lambda^+ \geqslant \max\{\nu^+, \tau\}$ then $\aleph^*(X) \leqslant \lambda^+$.*

Proof. Since $\lambda^+ \geqslant \tau$ and $V[G] \vDash \mathrm{cf}(\lambda^+) = \lambda^+$ we have $M \vDash \mathrm{cf}(\lambda^+) = \lambda^+$, and in particular $M \vDash \mathrm{cf}(\lambda^+) \geqslant \max\{\nu^+, \tau\}$. Therefore, by Proposition 3.5.14, $\lambda^+ \notin [\lambda, \aleph^*(X))$, so we must have that $\aleph^*(X) \leqslant \lambda^+$. □

Lemma 3.5.16. *For all X, if $\aleph^*(X) > \max\{\tau, \nu\}$ then $\aleph^*(X)$ is a successor cardinal.*

Proof. Let $\kappa > \max\{\tau, \nu\}$ be a limit cardinal and suppose that X is such that $\aleph^*(X) \geqslant \kappa$. We shall show that $\aleph^*(X) \geqslant \kappa^+$.

Let $\eta \in \mathrm{Ord}$ be least such that there is a surjection $g\colon A \times \eta \to X$. We may assume without loss of generality that for each $\alpha < \eta$ there is $a \in A$ such that $g(a, \alpha) \notin g``(A \times \alpha)$. Hence the map $x \mapsto \min\{\alpha < \eta \mid (\exists a \in A) g(a, \alpha) = x\}$ is a surjection $X \to \eta$. We aim to show that $\eta \geqslant \kappa$.

Let $\mu \in (\max\{\tau, \nu\}, \kappa)$, so there is a surjection $f\colon X \to \mu$. In $V[G]$, $|A \times \eta| = \nu \times \eta \geqslant \mu > \max\{\tau, \nu\}$. Since $\eta > \tau$, we get that $|A \times \eta|^{V[G]} = \eta$, and so $\eta \geqslant \mu$. $\kappa > \max\{\tau, \nu\}$ is a limit cardinal, so $\eta \geqslant \mu$ for all $\mu \in (\max\{\tau, \nu\}, \kappa)$ implies that $\eta \geqslant \kappa$ as required. □

Note that Lemma 3.5.17 does not make use of our SVC assumption and indeed holds in ZF.

Lemma 3.5.17. *Assume AC_μ, and let $\lambda > \mu$ be such that $\mathrm{cf}(\lambda) = \mu$. Then for all X, $\aleph(X) \neq \lambda$.*

Proof. Firstly, by AC_μ we have that λ is a limit cardinal. If $\lambda = \eta^+$, say, and $\langle \delta_\alpha \mid \alpha < \mu \rangle$ is a cofinal sequence in λ such that $|\delta_\alpha| = \eta$ for all α, we can use AC_μ to simultaneously pick bijections $\delta_\alpha \to \eta$ for all α, so $|\lambda^+| \leqslant \mu \times \eta$, a contradiction. Suppose that $\aleph(X) \geqslant \lambda$, and let $\langle \lambda_\alpha \mid \alpha < \mu \rangle$ be a cofinal sequence of cardinals[96] in λ. For all $\alpha < \mu$, let $A_\alpha = X^{\underline{\lambda_\alpha}}$, noting that $A_\alpha \neq \emptyset$ for all α since $\aleph(X) \geqslant \lambda$. By AC_μ, we may pick $c \in \prod_{\alpha < \mu} A_\alpha$. Ordering $\{c(\alpha, \beta) \mid \alpha < \mu, \beta < \lambda_\alpha\}$ lexicographically and removing duplicates, we obtain an injection $f\colon \kappa \to X$ for some κ, and $\kappa \geqslant \lambda_\alpha$ for all $\alpha < \mu$, so $\kappa \geqslant \lambda$ and hence $\aleph(X) > \lambda$ as required. □

[96] It is possible to stipulate that it is a sequence of cardinals because λ is a limit.

Recall that λ_{AC} is the least cardinal λ such that AC_λ does not hold.

Proposition 3.5.18. *If $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$, then one of the following holds:*

(a) $\kappa \leqslant \max\{\nu^+, \tau^+\}$;

(b) $\kappa = \lambda$ *are successors; or*

(c) $\kappa = \lambda^+$ *and* $\lambda_{\mathsf{AC}} \leqslant \mathrm{cf}(\lambda) < \max\{\nu^+, \tau\}$.

Proof. If $\kappa \leqslant \max\{\nu^+, \tau^+\}$ then we are in Case (a), so assume otherwise. By Lemma 3.5.16, κ is a successor cardinal. If $\lambda = \kappa$, then we are in Case (b), so assume that $\lambda < \kappa$. Suppose for contradiction that $\kappa > \lambda^+$. Then setting $\mu = \max\{\lambda^+, \nu^+, \tau^+\}$, $\mu \in [\lambda, \kappa)$, so by Proposition 3.5.14 $\mathrm{cf}(\mu) < \max\{\nu^+, \tau\}$. However, $V[G] \vDash \mathrm{cf}(\mu) = \mu \geqslant \tau$, since μ is a successor cardinal, and so we have $M \vDash \mathrm{cf}(\mu) = \mu \geqslant \max\{\nu^+, \tau\}$, contradicting that $\mathrm{cf}(\mu) < \max\{\nu^+, \tau\}$. Therefore, $\kappa = \lambda^+$. Finally, we have that $\lambda_{\mathsf{AC}} \leqslant \mathrm{cf}(\lambda) < \max\{\nu^+, \tau\}$ by Proposition 3.5.14 and Lemma 3.5.17. □

The underlying pattern of Proposition 3.5.18 is that after the chaos of $\aleph^*(X) \leqslant \max\{\nu^+, \tau^+\}$ and the inevitability of $\aleph(X) = \aleph^*(X) = \lambda^+$, all that we have are sets X with $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$ for some cardinal λ. Indeed, this scenario is the only one in which we may have an eccentric set of arbitrarily large Hartogs or Lindenbaum number.

Definition 3.5.19. An *oblate cardinal* is a cardinal λ such that there is a set X with $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.

As noted, the only candidates for oblate cardinals $\lambda \geqslant \max\{\nu^+, \tau\}$ are singular, with $\lambda_{\mathsf{AC}} \leqslant \mathrm{cf}(\lambda) < \max\{\nu^+, \tau\}$. However, this does not tell us which of those cardinals will be oblate. Shortly we will use our framework for an upwards transfer of eccentricity from Section 3.5.2 to find a large number of oblate cardinals.

Recall that λ_{W} is the least cardinal λ such that W_λ does not hold, and that λ^*_{W} is defined analogously for W^*_λ. Combining Propositions 3.2.1 and 3.5.18, we produce an upper bound of $\mathrm{Spec}_\aleph(M)$ in three parts:

1. The successors, $\mathrm{SC} = \{\langle\lambda^+, \lambda^+\rangle \mid \lambda \in \mathrm{Card}\}$. In fact, we always have that $\mathrm{SC} \subseteq \mathrm{Spec}_\aleph(M)$ since $\aleph(\lambda) = \aleph^*(\lambda) = \lambda^+$ for all cardinals λ.

2. The 'bounded chaos', those $\langle\lambda,\kappa\rangle$ where $\lambda_{\mathsf{W}} \leqslant \lambda \leqslant \kappa \leqslant \max\{\nu^+,\tau^+\}$ and $\lambda^*_{\mathsf{W}} \leqslant \kappa$. If we have $\aleph(X)^+ < \aleph^*(X)$ for some set X, then it must appear here.

3. The oblate cardinals, those $\langle\lambda,\lambda^+\rangle$ with $\mathrm{cf}(\lambda) \in [\lambda_{\mathsf{AC}}, \max\{\nu^+,\tau\})$.

A lower bound

Having just seen the upper bound of the spectrum of a model of SVC, we now wish to exhibit a lower bound, which we shall construct entirely through controlling oblate cardinals. By Theorem A, if any eccentric set exists then we must have eccentric sets of arbitrarily large Hartogs or Lindenbaum number. However, by Proposition 3.5.18, if $\aleph(X) \geqslant \max\{\nu^+,\tau\}$ and X is eccentric, then we must have that $\aleph(X)$ is an oblate cardinal. Therefore, there must be a proper class of oblate cardinals. In this section we will use our methods from Section 3.5.2 to lift oblate cardinals to larger oblate cardinals with the same cofinality, under the guise of SVC.

Lemma 3.5.20. *Let B be such that $\aleph(B) = \mu$ a limit or singular cardinal, and λ such that $\lambda \geqslant \sup\{\aleph(\alpha^\alpha) \mid \alpha < \mu\}$ and $\mathrm{cf}(\lambda) = \mathrm{cf}(\mu)$. Then there is a set X such that $\aleph(X) = \lambda$ and $\aleph^*(X) > \lambda$. Furthermore, if $\lambda^+ \geqslant \max\{\nu^+,\tau\}$ then $\aleph^*(X) = \lambda^+$.*

Proof. By Proposition 3.5.9, there is a set X such that $\aleph(X) = \lambda < \aleph^*(X)$. If $\lambda^+ \geqslant \max\{\nu^+,\tau\}$ then, by Corollary 3.5.15, $\aleph^*(X) \leqslant \lambda^+$. □

Lemma 3.5.21. *Let B be a set, $\mu \in [\aleph(B), \aleph^*(B))$ be a regular cardinal, and $\lambda \geqslant \mu$ be such that $\mathrm{cf}(\lambda) = \mu$. Then there is X such that $\aleph(X) = \lambda < \aleph^*(X)$. Furthermore, if $\lambda^+ \geqslant \max\{\nu^+,\tau\}$ or $\lambda^+ \geqslant \aleph^*(B)$, then $\aleph^*(X) = \lambda^+$.*

Proof. By Proposition 3.5.10 there is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) > \lambda$. If $\lambda^+ \geqslant \max\{\nu^+,\tau\}$, by Corollary 3.5.15, $\aleph^*(X) \leqslant \lambda^+$. If $\lambda^+ \geqslant \aleph^*(B)$, then by Proposition 3.5.10, $\aleph^*(X) = \lambda^+$. □

Corollary 3.5.22. *If B is such that $\aleph(B) = \mu$ and $\aleph^*(B) > \mu$ then for all large enough λ with $\mathrm{cf}(\lambda) = \mathrm{cf}(\mu)$ there is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.*

Proof. If μ is regular then use Lemma 3.5.21. Otherwise, use Lemma 3.5.20. □

Proposition 3.5.23. *There is a set $C \subseteq [\lambda_{\mathsf{AC}}, \max\{\nu^+,\tau\})$ and a cardinal Ω such that for all $\lambda \geqslant \Omega$, λ is an oblate cardinal if and only if $\mathrm{cf}(\lambda) \in C$.*

Proof. Firstly, by Proposition 3.5.14 and Lemma 3.5.17, if λ is an oblate cardinal then $\mathrm{cf}(\lambda) \in [\lambda_{\mathsf{AC}}, \max\{\nu^+, \tau\})$. For each $\mu \in [\lambda_{\mathsf{AC}}, \max\{\nu^+, \tau\})$, if there is an oblate cardinal λ with $\mathrm{cf}(\lambda) = \mu$, then Corollary 3.5.22 guarantees that there is a cardinal Ω_μ such that each $\eta \geqslant \Omega_\mu$ with $\mathrm{cf}(\eta) = \mu$ is oblate. Let $C = \{\mu \in [\lambda_{\mathsf{AC}}, \max\{\nu^+, \tau\}) \mid (\exists \lambda \in \mathrm{Card})\, \mathrm{cf}(\lambda) = \mu \text{ and } \lambda \text{ is oblate}\}$. Then, setting $\Omega = \sup\{\Omega_\mu \mid \mu \in C\}$, we have that for all $\lambda \geqslant \Omega$, λ is oblate if and only if $\mathrm{cf}(\lambda) \in C$. □

In fact, we get more than Proposition 3.5.23. By Corollary 3.5.22, C is precisely the set $\{\mathrm{cf}(\aleph(X)) \mid \aleph(X) < \aleph^*(X)\}$. Finally, putting together Propositions 3.2.1, 3.5.18 and 3.5.23, we obtain Theorem B.

Theorem B. *Let $M \vDash \mathsf{SVC}$. Then there is a cardinal ϕ, cardinals $\psi \leqslant \chi_0 \leqslant \Omega$, a cardinal $\psi^* \geqslant \psi$, a cardinal $\chi \in [\chi_0, \chi_0^+]$, and a set $C \subseteq [\phi, \chi_0)$ such that*

$$\mathrm{Spec}_{\aleph}(M) = \bigcup \begin{cases} \mathrm{SC} = \{\langle \lambda^+, \lambda^+ \rangle \mid \lambda \in \mathrm{Card}\} \\ \mathfrak{D} \subseteq \{\langle \lambda, \kappa \rangle \mid \psi \leqslant \lambda \leqslant \kappa \leqslant \chi, \psi^* \leqslant \kappa\} \\ \mathfrak{C} \subseteq \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) \in C, \lambda < \Omega\} \\ \mathfrak{U} = \{\langle \lambda, \lambda^+ \rangle \mid \mathrm{cf}(\lambda) \in C, \lambda \geqslant \Omega\}. \end{cases}$$ [97]

In the notation of this section, we have $\phi = \lambda_{\mathsf{AC}}$, $\psi = \lambda_{\mathsf{W}}$, $\psi^* = \lambda^*_{\mathsf{W}}$, $\chi_0 = \max\{\nu^+, \tau\}$, and $\chi = \max\{\nu^+, \tau^+\}$.

The techniques used in Section 3.5.1 to produce cleaner, stricter bounds only rely on the total agreement of cardinalities and cofinalities of ordinals between Cohen's model and the outer model. Any other model of SVC in which this occurs will have similarly tight control over the spectrum, assuming that one can find a seed (which is no small feat).

3.6 Constructing eccentric sets

Up to this point we have been broadly focused on the limitations of the Hartogs–Lindenbaum spectrum. Despite the presence of lifting and classes of oblate cardinals, our concentration on SVC has allowed us to have a fine structural understanding of the spectrum in the case of set-generic extensions of models of

[97] Note that $\mathfrak{C}$ is a *subset* of the possible irregular oblate cardinals. There may be X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$ for $\lambda < \Omega$ and $\mathrm{cf}(\lambda) \in C$, but we cannot be certain. On the other hand, $\mathfrak{D}$ is the *entire class* of oblate cardinals described. That is, if $\lambda \geqslant \Omega$ and $\mathrm{cf}(\lambda) \in C$ then we can guarantee that there is X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \lambda^+$.

ZFC. However, we do not always have to concern ourselves with this restriction. We shall spend the remainder of the chapter constructing, for a given pair $\langle\lambda,\kappa\rangle$ of infinite cardinals, a symmetric extension in which there is X such that $\aleph(X)=\lambda$ and $\aleph^*(X)=\kappa$ in a way that preserves an arbitrarily large initial segment of the universe. By combining a proper class of these constructions, appropriately preserving larger and larger segments of the universe in a product symmetric system, we shall produce a class-sized symmetric system forcing that the Hartogs–Lindenbaum spectrum is maximal.

Theorem 3.6.1 (Theorem C)**.** ZF *is equiconsistent with* ZF+*"for all infinite cardinals $\lambda\leqslant\kappa$ there is a set X such that $\aleph(X)=\lambda$ and $\aleph^*(X)=\kappa$".*

That is, we shall produce a model M of ZF such that

$$\operatorname{Spec}_{\aleph}(M)=\{\langle\lambda,\kappa\rangle\mid\aleph_0\leqslant\lambda\leqslant\kappa\}\cup\{\langle n+1,n+1\rangle\mid n<\omega\}.$$

3.6.1 Creating an eccentric set

Let us spend some time constructing a single set X such that $\aleph(X)=\lambda$ and $\aleph^*(X)=\kappa$, so that a large product of this construction can be used in Section 3.6.2 to prove Theorem C. Indeed, other constructions exists for adding eccentric sets of specified Hartogs and Lindenbaum numbers, but our method is well-suited for being part of a product.

Theorem 3.6.2. *Let $\lambda\leqslant\kappa$ be infinite cardinals. There is a symmetric system $\langle\mathbb{P},\mathscr{G},\mathscr{F}\rangle$ and a $\mathbb{P}$-name $\dot{X}\in\mathsf{HS}_{\mathscr{F}}$ such that*

$$\mathbb{1}_{\mathbb{P}}\Vdash^{\mathsf{HS}}\text{“}\aleph(\dot{X})=\check{\lambda}\text{ and }\aleph^*(\dot{X})=\check{\kappa}\text{”}.$$

Proof. Let μ be an infinite cardinal, and let $\mathbb{P}=\operatorname{Add}(\mu,\kappa\times\lambda\times\mu)$. That is, the conditions of $\mathbb{P}$ are partial functions $p\colon\kappa\times\lambda\times\mu\times\mu\to 2$ such that $|\operatorname{dom}(p)|<\mu$, with $q\leqslant p$ if $q\supseteq p$.

For $p\in\mathbb{P}$ and $A\subseteq\kappa\times\lambda\times\mu$, we will write $p\restriction A$ to mean $p\restriction A\times\mu$, and for $B\subseteq\kappa\times\lambda$ we will write $p\restriction B$ to mean $p\restriction B\times\mu\times\mu$. Furthermore, we shall write $\operatorname{supp}(p)$ to mean the projection of the domain of p to its first three co-ordinates, so $\operatorname{supp}(p)\subseteq\kappa\times\lambda\times\mu$.

We define the following $\mathbb{P}$-names:

1. $\dot{y}_{\alpha,\beta,\gamma}:=\{\langle p,\check{\delta}\rangle\mid p\in\mathbb{P},\delta<\mu,p(\alpha,\beta,\gamma,\delta)=1\}$;
2. $\dot{x}_{\alpha,\beta}:=\{\dot{y}_{\alpha,\beta,\gamma}\mid\gamma<\mu\}^\bullet$; and

3. $\dot{X} \coloneqq \{\dot{x}_{\alpha,\beta} \mid \langle\alpha,\beta\rangle \in \kappa\times\lambda\}^\bullet$.

In the extension, $\dot{X}$ will be the name for the set X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$.

Let $\mathscr{G} = S^{<\lambda}_{\kappa\times\lambda} \wr S_\mu$.[98] That is to say, $\mathscr{G}$ is the group of permutations π in the wreath product $S_{\kappa\times\lambda} \wr S_\mu$ such that $\pi^* \in S_{\kappa\times\lambda}$ fixes all but fewer than λ-many elements of $\kappa\times\lambda$. $\mathscr{G}$ acts on $\mathbb{P}$ via $\pi p(\pi(\alpha,\beta,\gamma),\delta) = p(\alpha,\beta,\gamma,\delta)$. Note that, for $\pi \in \mathscr{G}$,

$$\begin{aligned}\pi\dot{y}_{\alpha,\beta,\gamma} &= \{\langle\pi p, \pi\check{\delta}\rangle \mid p \in \mathbb{P}, \delta < \mu, p(\alpha,\beta,\gamma,\delta) = 1\}\\ &= \{\langle\pi p, \check{\delta}\rangle \mid p \in \mathbb{P}, \delta < \mu, \pi p(\pi(\alpha,\beta,\gamma),\delta) = 1\}\\ &= \{\langle p, \check{\delta}\rangle \mid p \in \mathbb{P}, \delta < \mu, p(\pi(\alpha,\beta,\gamma),\delta) = 1\}\\ &= \dot{y}_{\pi(\alpha,\beta,\gamma)}.\end{aligned}$$

Similar verification shows that $\pi\dot{x}_{\alpha,\beta} = \dot{x}_{\pi^*(\alpha,\beta)}$ and $\pi\dot{X} = \dot{X}$. When we have defined the filter of subgroups $\mathscr{F}$ (a task we shall begin upon the conclusion of this sentence), it will be clear from these calculations that these names are hereditarily $\mathscr{F}$-symmetric.

For $I \in [\kappa]^{<\kappa}$, $J \in [I\times\lambda]^{<\lambda}$, and $K \in [J\times\mu]^{<\lambda}$, let $H_{I,J,K}$ be the subgroup of $\mathscr{G}$ given by those π such that:

1. $\pi^* \restriction I\times\lambda \in \{\mathrm{id}\} \wr S_\lambda$;
2. $\pi^* \restriction J = \mathrm{id}$; and
3. $\pi \restriction K = \mathrm{id}$.

That is, we are taking those $\pi \in S^{<\lambda}_{\kappa\times\lambda} \wr S_\mu$ such that π^* fixes setwise the columns $\{\alpha\}\times\lambda$ for α in the set I and fixes pointwise the set J. We then further require that π fixes pointwise the set K.

Let $\mathscr{F}$ be the filter of subgroups of $\mathscr{G}$ generated by groups of the form $H_{I,J,K}$ for $I \in [\kappa]^{<\kappa}$, $J \in [I\times\lambda]^{<\lambda}$, and $K \in [J\times\mu]^{<\lambda}$.[99] We shall refer to a triple I, J, K as being 'appropriate' to mean that it satisfies these conditions.

By prior calculations, $\pi\dot{y}_{\alpha,\beta,\gamma} = \dot{y}_{\alpha,\beta,\gamma}$ whenever $\pi(\alpha,\beta,\gamma) = \langle\alpha,\beta,\gamma\rangle$, so $\mathrm{sym}(\dot{y}_{\alpha,\beta,\gamma}) \geqslant H_{\{\alpha\},\{\langle\alpha,\beta\rangle\},\{\langle\alpha,\beta,\gamma\rangle\}} \in \mathscr{F}$ for all. We similarly obtain that for all α and β, $\mathrm{sym}(\dot{x}_{\langle\alpha,\beta\rangle}) \geqslant H_{\{\alpha\},\{\langle\alpha,\beta\rangle\},\emptyset} \in \mathscr{F}$, and $\mathrm{sym}(\dot{X}) = \mathscr{G} \in \mathscr{F}$.

[98] Recall the definition of the wreath product Definition 2.4.14.

[99] Since $\mathbb{P}$ is $\mathrm{cf}(\mu)$-closed and $\mathscr{F}$ is $\mathrm{cf}(\lambda)$-complete, $\mathsf{DC}_{<\min\{\mathrm{cf}(\mu),\mathrm{cf}(\lambda)\}}$ holds in the symmetric extension. A proof can be found in [Kar19b, Lemma 1].

Claim 3.6.2.1. *$\mathscr{F}$ is normal. Hence, $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ is a symmetric system.*

Proof of Claim. Note that for appropriate I, J, K and I', J', K',

$$H_{I,J,K} \cap H_{I',J',K'} = H_{I \cup I', J \cup J', K \cup K'}$$

and $I \cup I', J \cup J', K \cup K'$ is appropriate. Therefore, for all $H \leqslant \mathscr{G}$, $H \in \mathscr{F}$ if and only if there is appropriate I, J, K such that $H \geqslant H_{I,J,K}$. Hence, to show that $\mathscr{F}$ is normal, it is sufficient to show that for all appropriate I, J, K and all $\pi \in \mathscr{G}$, there is appropriate I', J', K' such that $\pi H_{I,J,K} \pi^{-1} \geqslant H_{I',J',K'}$, or equivalently that $H_{I,J,K} \geqslant \pi^{-1} H_{I',J',K'} \pi$. Given such I, J, K and π, we define

$$\begin{aligned} K' &= \pi``K \\ J' &= \mathrm{Proj}(K') \cup \mathrm{supp}(\pi^*) \cup J \\ I' &= \mathrm{Proj}(J') \cup I, \end{aligned}$$

where Proj is projection away from the last co-ordinate in both cases. Note that $|K'| = |K| < \lambda$, $|J'| \leqslant |J| + |\mathrm{supp}(\pi^*)| + |K'| < \lambda$, and $|I'| \leqslant |I| + |J'| < \kappa$, and that the inclusion of the projections in the definitions of I', J', K' means that I', J', K' is appropriate. We claim that $H_{I',J',K'}$ is the required group. Let $\sigma \in H_{I',J',K'}$, then we must show that $\pi^{-1}\sigma\pi \in H_{I,J,K}$.

For all $\langle \alpha, \beta, \gamma \rangle \in K$, $\pi(\alpha, \beta, \gamma) \in K'$, so $\sigma(\pi(\alpha, \beta, \gamma)) = \pi(\alpha, \beta, \gamma)$ and hence $\pi^{-1}\sigma\pi(\alpha, \beta, \gamma) = \langle \alpha, \beta, \gamma \rangle$ as required.

We now note that, since $\mathrm{supp}(\pi^*) \subseteq J'$, $\mathrm{supp}(\pi^*) \cap \mathrm{supp}(\sigma^*) = \emptyset$, and hence $(\pi^{-1}\sigma\pi)^* = \sigma^*$. Combined with $J \subseteq J'$ and $I \subseteq I'$, we have that $\pi^{-1}\sigma\pi \in H_{I,J,K}$. ⊣

Claim 3.6.2.2. *Let $q \in \mathbb{P}$, $H = H_{I,J,K} \in \mathscr{F}$, and $\langle \alpha, \beta \rangle, \langle \alpha', \beta' \rangle \in \kappa \times \lambda$. The following are equivalent:*

1. *There is $\pi \in H$ such that $\pi q \parallel q$ and π^* is the transposition $(\langle \alpha, \beta \rangle\ \langle \alpha', \beta' \rangle)$.*
2. $\{\alpha, \alpha'\} \cap I \neq \emptyset \implies \alpha = \alpha'$, *and*
   $\{\langle \alpha, \beta \rangle, \langle \alpha', \beta' \rangle\} \cap J \neq \emptyset \implies \langle \alpha, \beta \rangle = \langle \alpha', \beta' \rangle$.

Proof of Claim. ((1) $\implies$ (2)). By the definition of H, if there is such a $\pi \in H$ then Condition (2) must be satisfied.

((2) $\implies$ (1)). If $\langle \alpha, \beta \rangle$ and $\langle \alpha', \beta' \rangle$ satisfy Condition (2), then any $\pi \in \mathscr{G}$ such that $\pi^* = (\langle \alpha, \beta \rangle\ \langle \alpha', \beta' \rangle)$ is a candidate for an element of H

(as $|\mathrm{supp}(\pi^*)| = 2 < \lambda$). If $\langle\alpha,\beta\rangle = \langle\alpha',\beta'\rangle$ then we may take $\pi = \mathrm{id}$, so assume otherwise.[100] Then we let $A = \{\gamma < \mu \mid \langle\alpha,\beta,\gamma\rangle \in \mathrm{supp}(q)\}$ and $B = \{\gamma' < \mu \mid \langle\alpha',\beta',\gamma'\rangle \in \mathrm{supp}(q)\}$. Since $|A|, |B| < \mu$, there is a permutation σ of μ such that $\sigma``A \cap B = \emptyset$ and $A \cap \sigma``B = \emptyset$. Therefore, setting $\pi_{\alpha,\beta} = \pi_{\alpha',\beta'} = \sigma$ we will have that

$$\mathrm{supp}(q \restriction \{\langle\alpha,\beta\rangle\}) \cap \mathrm{supp}(\pi q \restriction \{\langle\alpha,\beta\rangle\}) = \sigma``A \cap B = \emptyset$$

and

$$\mathrm{supp}(q \restriction \{\langle\alpha',\beta'\rangle\}) \cap \mathrm{supp}(\pi q \restriction \{\langle\alpha',\beta'\rangle\}) = A \cap \sigma``B = \emptyset.$$

Hence $q \restriction \{\langle\alpha,\beta\rangle, \langle\alpha',\beta'\rangle\} \parallel \pi q \restriction \{\langle\alpha,\beta\rangle, \langle\alpha',\beta'\rangle\}$, and for all other $\langle\alpha'',\beta''\rangle$ we have $q \restriction \langle\alpha'',\beta''\rangle = \pi q \restriction \langle\alpha'',\beta''\rangle$. That is, $\pi q \parallel q$. ⊣

The remainder of the proof shall be spent showing that the name $\dot{X}$ does give us the object that we are searching for; that is, $\mathbb{1} \Vdash^{\mathsf{HS}} \aleph(\dot{X}) = \check{\lambda}$ and $\mathbb{1} \Vdash^{\mathsf{HS}} \aleph^*(\dot{X}) = \check{\kappa}$. We shall first prove the inequalities $\mathbb{1} \Vdash^{\mathsf{HS}} \aleph(\dot{X}) \geqslant \check{\lambda}$ and $\mathbb{1} \Vdash^{\mathsf{HS}} \aleph^*(\dot{X}) \geqslant \check{\kappa}$, and then prove that they can be sharpened to equalities.

Towards the inequalities, for any $\alpha, \eta < \kappa$, let

$$\iota_{\alpha,\eta} := \begin{cases} \alpha & \alpha < \eta \\ 0 & \text{Otherwise,} \end{cases}$$

and consider the name $\dot{e}_\eta := \{\langle \dot{x}_{\alpha,\beta}, \check{\iota}_{\alpha,\eta}\rangle^\bullet \mid \langle\alpha,\beta\rangle \in \kappa \times \lambda\}^\bullet$. Routine verification shows $\mathrm{sym}(\dot{e}_\eta) \geqslant H_{\eta,\emptyset,\emptyset}$, so $\dot{e}_\eta \in \mathsf{HS}$. $\mathbb{1} \Vdash$ "$\dot{e}_\eta\colon \dot{X} \to \check{\eta}$ is a surjection", and thus $\mathbb{1} \Vdash^{\mathsf{HS}} \aleph^*(\dot{X}) \geqslant \check{\kappa}$.

Similarly, for any $\eta < \lambda$ and any $\alpha \in \kappa$, let $\dot{m}_\eta := \{\langle \check{\beta}, \dot{x}_{\alpha,\beta}\rangle^\bullet \mid \beta < \eta\}^\bullet$. Routine verification shows that $\mathrm{sym}(\dot{m}_\eta) \geqslant H_{\{\alpha\},\{\alpha\}\times\eta,\emptyset}$, so $\dot{m}_\eta \in \mathsf{HS}$ as well. Furthermore, $\mathbb{1} \Vdash$ "$\dot{m}_\eta\colon \check{\eta} \to \dot{X}$ is an injection", and thus $\mathbb{1} \Vdash^{\mathsf{HS}} \aleph(\dot{X}) \geqslant \check{\lambda}$.

It remains to show that these inequalities are, in fact, equalities, starting with $\aleph^*$. Suppose that $\dot{f} \in \mathsf{HS}$, with $H = H_{I,J,K} \leqslant \mathrm{sym}(\dot{f})$, and $p \Vdash \dot{f}\colon \dot{X} \to \check{\kappa}$. Furthermore suppose that for some $q \leqslant p$ and $\langle\alpha,\beta\rangle \in \kappa \times \lambda$ there is η such that $q \Vdash \dot{f}(\dot{x}_{\alpha,\beta}) = \check{\eta}$.

By Claim 3.6.2.2, if $\alpha \notin I$ then for any $\alpha' \notin I$ and any $\beta' \in \lambda$ there is $\pi \in H$ such that $\pi^*(\alpha,\beta) = \langle\alpha',\beta'\rangle$ and $\pi q \parallel q$. Then $\pi q \Vdash \dot{f}(\dot{x}_{\alpha',\beta'}) = \check{\eta}$, so

[100] In particular, $\langle\alpha,\beta\rangle, \langle\alpha',\beta'\rangle \notin J$ and so $\langle\alpha,\beta,\gamma\rangle, \langle\alpha',\beta',\gamma'\rangle \notin K$ for all γ, γ', and so we are free to choose $\pi_{\langle\alpha,\beta\rangle}$ and $\pi_{\langle\alpha',\beta'\rangle} \in S_\mu$.

$q \cup \pi q \leqslant q$ forces that $\dot{f}(\dot{x}_{\alpha,\beta}) = \dot{f}(\dot{x}_{\alpha',\beta'})$. Hence p forces that $\dot{f}$ is constant outside of $I \times \lambda$.

If instead $\alpha \in I$ but $\langle \alpha, \beta \rangle \notin J$ then again by Claim 3.6.2.2, for any $\beta' \in \lambda$ such that $\langle \alpha, \beta' \rangle \notin J$ there is $\pi \in H$ such that $\pi^*(\alpha, \beta) = \langle \alpha, \beta' \rangle$ and $\pi q \parallel q$. Once again $\pi q \Vdash \dot{f}(\dot{x}_{\alpha',\beta'}) = \check{\eta}$, and so p forces that in $(I \times \lambda) \setminus J$, the value of $\dot{f}(\dot{x}_{\alpha,\beta})$ depends only on α. This means that $\dot{f}$ takes at most $|J| + |I| + 1 < \kappa$ different values, so cannot be a surjection. Thus $\mathbb{1} \Vdash^{\mathsf{HS}} \aleph^*(\dot{X}) = \check{\kappa}$.

Finally, suppose that $\dot{f} \in \mathsf{HS}$ and $p \Vdash \dot{f} \colon \check{\lambda} \to \dot{X}$. Let $H = H_{I,J,K} \leqslant \operatorname{sym}(\dot{f})$. We shall show that $p \Vdash \dot{f}``\check{\lambda} \subseteq \{\dot{x}_{\alpha,\beta} \mid \langle \alpha, \beta \rangle \in J\}^\bullet$, and hence $\dot{f}$ cannot be injective.

Suppose otherwise, that for some $q \leqslant p$, $\langle \alpha, \beta \rangle \notin J$, and $\eta < \lambda$ we have that $q \Vdash \dot{f}(\check{\eta}) = \dot{x}_{\alpha,\beta}$. Since $\langle \alpha, \beta \rangle \notin J$, for any $\beta' \in \lambda$ such that $\langle \alpha, \beta' \rangle \notin J$ there is $\pi \in H$ such that $\pi^*(\alpha, \beta) = \langle \alpha, \beta' \rangle$ and $\pi q \parallel q$. Since $|J| < \lambda$ we may take $\beta' \neq \beta$, and so $\pi q \Vdash \dot{f}(\check{\eta}) = \pi \dot{x}_{\alpha,\beta} = \dot{x}_{\alpha,\beta'}$. Therefore, $\pi q \cup q \Vdash \dot{x}_{\alpha,\beta} = \dot{f}(\check{\eta}) = \dot{x}_{\alpha,\beta'}$, contradicting our assumption that $\beta' \neq \beta$. Thus our assertion is proved and $\mathbb{1} \Vdash^{\mathsf{HS}} \aleph(\dot{X}) = \check{\lambda}$. □

3.6.2 A maximal spectrum

We have now produced enough technology to prove Theorem C, that ZF is equiconsistent with ZF + "For all pairs of infinite cardinals $\lambda \leqslant \kappa$, there is a set X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$". The structure shall be similar to the treatment of class products of symmetric extensions found in, for example, [Kar18].

We begin in a model V of $\mathsf{ZFC}+\mathsf{GCH}$, and will inductively define a symmetric system $\langle \mathbb{P}_\alpha, \mathscr{G}_\alpha, \mathscr{F}_\alpha \rangle$ for each $\alpha \in \mathrm{Ord}$. Each such system will be precisely of the form described in Theorem 3.6.2, and so to fully define each system we need only define the parameters $\lambda_\alpha, \kappa_\alpha$, and μ_α. First, let $\{\langle \lambda_\alpha, \kappa_\alpha \rangle \mid \alpha \in \mathrm{Ord}\}$ be an enumeration of all pairs of cardinals $\langle \lambda, \kappa \rangle$ with $\aleph_0 \leqslant \lambda \leqslant \kappa$, using (for example) the Gödel pairing function.[101] Then we take μ_α to be the least cardinal satisfying the following conditions:

1. μ_α is regular;

2. for all $\beta < \alpha$, $\mu_\beta < \mu_\alpha$;

[101] The Gödel pairing function definably orders Ord^2 by setting $\langle \alpha_0, \beta_0 \rangle < \langle \alpha_1, \beta_1 \rangle$ if: $\max(\alpha_0, \beta_0) < \max(\alpha_1, \beta_1)$; $\max(\alpha_0, \beta_0) = \max(\alpha_1, \beta_1)$ and $\alpha_0 < \alpha_1$; or finally if $\max(\alpha_0, \beta_0) = \max(\alpha_1, \beta_1)$, $\alpha_0 = \alpha_1$, and $\beta_0 < \beta_1$.

3. for all $\beta \leqslant \alpha$, $\kappa_\beta < \mu_\alpha$;
4. setting $\mathbb{Q}$ to be the finite-support product $\prod_{\beta<\alpha} \mathbb{P}_\beta$, $|\mathbb{Q}| < \mu_\alpha$;
5. for all $\beta < \alpha$, $\mathbb{1}_\mathbb{Q} \Vdash_\mathbb{Q} |\dot{V}_{\mu_\beta^+}| < \check{\mu}_\alpha$; and
6. $\aleph_\alpha < \mu_\alpha$.

Let $\mathbb{P}$ be the finite-support product of all $\mathbb{P}_\alpha$, $\mathscr{G}$ the finite-support product of all $\mathscr{G}_\alpha$, and $\mathscr{F}$ the finite-support product of all $\mathscr{F}_\alpha$. For $E \subseteq \mathrm{Ord}$, we denote by $\mathbb{P} \upharpoonright E$ (respectively $\mathscr{G} \upharpoonright E, \mathscr{F} \upharpoonright E$) the restriction of $\mathbb{P}$ (respectively $\mathscr{G}, \mathscr{F}$) to the co-ordinates found in E. Since any $\mathbb{P}$-name $\dot{x}$ is a set, it is a $\mathbb{P} \upharpoonright \alpha$-name for some α, and so $\dot{x}$ is hereditarily $\mathscr{F}$-symmetric if and only if it is hereditarily $\mathscr{F} \upharpoonright \alpha$-symmetric for some α. Therefore, setting $\mathsf{HS} = \mathsf{HS}_\mathscr{F}$, $\mathsf{HS}_\alpha = \mathsf{HS}_{\mathscr{F} \upharpoonright \alpha}$, and letting G be V-generic for $\mathbb{P}$, we get that

$$\bigcup_{\alpha \in \mathrm{Ord}} \mathsf{HS}_\alpha^{G \upharpoonright \alpha} = \bigcup_{\alpha \in \mathrm{Ord}} \mathsf{HS}_\alpha^G = \left(\bigcup_{\alpha \in \mathrm{Ord}} \mathsf{HS}_\alpha \right)^G = \mathsf{HS}^G.$$

Let $M = \mathsf{HS}^G$ and $M_\alpha = \mathsf{HS}_\alpha^{G \upharpoonright \alpha}$.[102] Then we have that $M = \bigcup_{\alpha \in \mathrm{Ord}} M_\alpha$. We wish to prove that $M \vDash \mathsf{ZF}$, and shall use the following theorem, [Kar19a, Theorem 9.2].

Theorem 3.6.3. *Let $\langle \mathbb{P}_\alpha, \mathscr{G}_\alpha, \mathscr{F}_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ be a finite-support product of symmetric extensions of homogeneous systems. Suppose that for each η there is α^* such that for all $\alpha \geqslant \alpha^*$, the* αth *symmetric extension does not add new sets of rank at most η. Then no sets of rank at most η are added by limit steps either. In particular, the end model satisfies* ZF*.*

The conditions of the theorem are also desirable for our construction. We shall show that for all α there is a hereditarily symmetric name $\dot{X}_\alpha$ such that $\dot{X}_\alpha \in \mathsf{HS}_{\alpha+1}$ and $M_{\alpha+1} \vDash$ "$\aleph(\dot{X}_\alpha^G) = \lambda_\alpha$ and $\aleph^*(\dot{X}_\alpha^G) = \kappa_\alpha$". In this case, if we can preserve a large enough initial segment of $M_{\alpha+1}$ for the rest of the iteration, then $\dot{X}_\alpha^G$ will still have this property in M.

We shall require the following fact, that is proved in [Kar18, Lemma 2.3].

Lemma 3.6.4. *Let κ be a regular cardinal, $\mathbb{P}$ a κ-c.c. forcing, and $\mathbb{Q}$ a κ-distributive forcing. If $\mathbb{1}_\mathbb{Q} \Vdash$ "$\check{\mathbb{P}}$ is $\check{\kappa}$-c.c.", then $\mathbb{1}_\mathbb{P} \Vdash$ "$\check{\mathbb{Q}}$ is $\check{\kappa}$-distributive".*

[102] Note that M_α is not denoting $V_\alpha^M = \{x \in M \mid \mathrm{rk}(x) < \alpha\} = \mathscr{P}^\alpha(\emptyset)^M$.

Lemma 3.6.5. *Let $\delta < \beta < \alpha$. Then M_β and M_α agree on sets of rank less than μ_δ^+.*

Proof. It is sufficient to prove that for all $\beta \in \mathrm{Ord}$, M_β and $M_{\beta+1}$ agree on sets of rank less than μ_δ; this is the successor stage for an induction on α of the statement of this proposition, and Theorem 3.6.3 provides the induction at the limit stage. Towards this end, let $\delta < \beta \in \mathrm{Ord}$ and $N = V[G \upharpoonright \beta]$. We shall show that $\mathbb{P}_\beta$ adds no sets of rank less than μ_δ^+ to N, and since $M_\beta \subseteq N$ and $M_{\beta+1} \subseteq N[G(\beta)]$, the claim is proved.

Let $\kappa = |V^N_{\mu_\delta^+}|$. Then, by the definition of μ_β, $\kappa < \mu_\beta$, and so it is sufficient to prove that $\mathbb{P}_\beta$ is μ_β-distributive. $\mathbb{P}_\beta = \mathrm{Add}(\mu_\beta, \kappa_\beta \times \lambda_\beta \times \mu_\beta)^V$, and so certainly in V it is μ_β^+-distributive. Furthermore, by definition, $|\mathbb{P} \upharpoonright \beta| < \mu_\beta$, so $\mathbb{P} \upharpoonright \beta$ is μ_β-c.c., and indeed $\mathbb{P}_\beta \Vdash$ "$\check{\mathbb{P}} \upharpoonright \beta$ is $\check{\mu}_\beta$-c.c.". Hence, by Lemma 3.6.4, $\mathbb{P} \upharpoonright \beta \Vdash$ "$\check{\mathbb{P}}_\beta$ is μ_β-distributive", as required. □

Lemma 3.6.6. *For all $\alpha \in \mathrm{Ord}$,*

$$M_{\alpha+1} \vDash (\exists X_\alpha)(\aleph(X_\alpha) = \lambda_\alpha \leqslant \kappa_\alpha = \aleph^*(X_\alpha)).$$

Proof. The proof follows Theorem 3.6.2. Indeed, letting N be the symmetric extensions of V by the iterand $\langle \mathbb{P}_\alpha, \mathscr{G}_\alpha, \mathscr{F}_\alpha \rangle$, we have that $V \subseteq N \subseteq M_\alpha$ are transitive models and, by Theorem 3.6.2, there is $X_\alpha \in N$ such that $N \vDash \aleph(X_\alpha) = \lambda_\alpha \wedge \aleph^*(X_\alpha) = \kappa_\alpha$. To finish, we need only show that this is preserved into M_α. However, the technique from Theorem 3.6.2 for showing that X_α has the correct Hartogs and Lindenbaum number in N works for M_α, as the homogeneity argument can be performed on only the αth co-ordinate without issue. □

With some care over the construction of X_α, we may now prove Theorem C.

Theorem C. ZF *is equiconsistent with* ZF+*"for all infinite cardinals $\lambda \leqslant \kappa$ there is a set X such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$".*

Proof. By Lemma 3.6.5 and Theorem 3.6.3, $M \vDash \mathsf{ZF}$. By Lemma 3.6.6, for all $\alpha \in \mathrm{Ord}$, $M_{\alpha+1} \vDash$ "$\aleph(X_\alpha) = \lambda_\alpha$ and $\aleph^*(X_\alpha) = \kappa_\alpha$". Note that, for all $\alpha \in \mathrm{Ord}$, the set X_α is constructed as an element of $\mathscr{P}^3(\mu_\alpha)$. Since $\mu_\alpha > \kappa_\alpha \geqslant \lambda_\alpha$, it must be the case that any function $\lambda_\alpha \to X_\alpha$ or $X_\alpha \to \kappa_\alpha$ must have rank less than μ_α^+. Hence, by Lemma 3.6.5, we have that for all $\alpha < \beta$, $M_\beta \vDash$ "$\aleph(X_\alpha) = \lambda_\alpha$ and $\aleph^*(X_\alpha) = \kappa_\alpha$". This, combined with Theorem 3.6.3, shows that for all $\alpha \in \mathrm{Ord}$, $M \vDash$ "$\aleph(X_\alpha) = \lambda_\alpha$ and $\aleph^*(X_\alpha) = \kappa_\alpha$". □

Chapter 4

Maximal Independent Families

I was not permitted to die.
I was a promise.
"You were a ghost."
What an honour it is to walk, so that all below may live.

Lara Welch, *Feast of Saints*

INFINITY HAS A STRANGENESS ABOUT IT THAT TENDS TO WORM its way into some very surprising places. This chapter focuses on *(maximal) θ-independent families*, which are collections of sets $\mathcal{A} \subseteq \mathscr{P}(X)$ such that the elements of $\mathcal{A}$ are, in a sense, 'random' on the 'scale of θ'. Using Zorn's lemma, one can show that any $\aleph_0$-independent family can be extended to a *maximal* $\aleph_0$-independent family. On the other hand, taking θ any larger no longer necessitates such objects; even the existence of a single maximal $\aleph_1$-independent family entails an inner model with a measurable cardinal. This shows off a general pattern that sometimes can be found in set theory, where a nice finitary object may be well-behaved,[103] but a natural infinitary analogue of the object can only be shown to exist using large cardinal assumptions.

In this chapter we extend a method of Kunen's for constructing maximal θ-independent families from large cardinals, selecting large cardinal assumptions that allow us to have a proper class of such families.

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[103]Where "good behaviour" here usually refers to existence.

4.1 Introduction

Definition 4.1.1. For a regular cardinal θ and a set X such that $|X| \geqslant \theta$, a *θ-independent family on X* is $\mathcal{A} \subseteq \mathscr{P}(X)$ such that $|\mathcal{A}| \geqslant \theta$ and, for all partial functions $p\colon \mathcal{A} \to 2$ with $|p| < \theta$,

$$\mathcal{A}^p := \bigcap\{A \mid p(A) = 1\} \cap \bigcap\{X \setminus A \mid p(A) = 0\} \neq \emptyset.$$

For $\theta = \aleph_0$ we usually say *independent* instead of $\aleph_0$-independent, and for $\theta = \aleph_1$ we usually say *σ-independent* instead of $\aleph_1$-independent.

$\mathcal{A}$ is *maximal θ-independent* if, for all θ-independent $\mathcal{A}' \supseteq \mathcal{A}$ on X, $\mathcal{A}' = \mathcal{A}$.

The existence of a maximal σ-independent family entails an inner model with a measurable cardinal, and the construction can be 'reversed': by forcing over a model with a measurable cardinal one can obtain a model in which there is a maximal σ-independent family. The proof of this can be extended to larger cardinal properties, something that was known at the time: In [Kun83], Kunen comments that a single strongly compact cardinal κ would beget, in an appropriate forcing extension, maximal σ-independent families $\mathcal{A} \subseteq \mathscr{P}(\lambda)$ for all λ such that $\mathrm{cf}(\lambda) \geqslant \kappa$. We shall prove this result whilst reducing the consistency strength requirement to κ being merely $\aleph_1$-strongly compact, and generalise the setting to θ-independence.

Theorem (Theorem D)**.** *Let κ be a strong limit and θ^+-strongly compact for some regular, uncountable $\theta < \kappa$, and let G be V-generic for $\mathrm{Add}(\theta, \kappa)$. In $V[G]$, for all cardinals $\lambda \geqslant \kappa$ with $\mathrm{cf}(\lambda) \geqslant \kappa$, there is a maximal θ-independent family $\mathcal{A} \subseteq \mathscr{P}(\lambda)$.*

We also extend the technique to the case that there is a proper class of measurable cardinals, iterating the process to induce a model in which, for all ground-model measurable cardinals κ, there is a maximal κ-independent family $\mathcal{A} \subseteq \mathscr{P}(\kappa)$ in the forcing extension. An analysis of the iteration also shows that the Mitchell rank of cardinals is very nearly preserved.

Theorem (Theorem E)**.** *Let V be a model of $\mathsf{ZFC}+\mathsf{GCH}$. There is a class-length forcing iteration $\mathbb{P}$ preserving $\mathsf{ZFC}+\mathsf{GCH}$ such that, if $G \subseteq \mathbb{P}$ is V-generic, then whenever κ is a measurable cardinal in V there is a maximal κ-independent family $\mathcal{A} \subseteq \mathscr{P}(\kappa)$ in $V[G]$. Furthermore, whenever κ is a measurable cardinal in V, $o(\kappa)^V = 1 + o(\kappa)^{V[G]}$, and whenever κ is non-measurable in V it remains non-measurable in $V[G]$.*

4.2 Preliminaries

4.2.1 Elementary embeddings

Given a σ-complete ultrafilter $\mathcal{U}$ on an infinite set X and functions $f, g\colon X \to V$ we say that $f =_{\mathcal{U}} g$ if $\{x \in X \mid f(x) = g(x)\} \in \mathcal{U}$, and denote by $[f]_{\mathcal{U}}$ the $=_{\mathcal{U}}$-equivalence class of f.[104] We then endow these classes with the relation $\in_{\mathcal{U}}$ given by $[f]_{\mathcal{U}} \in_{\mathcal{U}} [g]_{\mathcal{U}}$ if $\{x \in X \mid f(x) \in g(x)\} \in \mathcal{U}$. Finally, we identify this ultrapower construction

$$\mathrm{Ult}(V,\mathcal{U}) = (\{[f]_{\mathcal{U}} \mid f\colon X \to V\}, \in_{\mathcal{U}})$$

and the Mostowski collapse M of this structure, going as far as to say $a = [f]_{\mathcal{U}}$ to mean that a is the element of M associated with $[f]_{\mathcal{U}}$ under the collapse. We then refer to the elementary embedding $j_{\mathcal{U}}\colon V \to M$ given by $j_{\mathcal{U}}(a) = [c_a]_{\mathcal{U}}$, where c_a is the constant function $X \to \{a\}$, as the associated *ultrapower embedding* of $\mathcal{U}$. We say that a transitive inner model M is λ-*closed* to mean that $M^{<\lambda} \subseteq M$. We may say that M is λ-closed *in* V to emphasise that we specifically mean $M^{<\lambda} \cap V = M \cap V^{<\lambda}$. The *critical point* of an elementary embedding $j\colon V \to M$, where $M \subseteq V$ is a transitive class, is denoted $\mathrm{crit}(j)$ and defined to be $\min\{\alpha \mid \alpha < j(\alpha)\}$.[105] If $j = j_{\mathcal{U}}$ for some $\mathcal{U} \in V$ then $\mathrm{crit}(j_{\mathcal{U}})$ is a measurable cardinal in V (and hence a regular strong limit).

Definition 4.2.1 (The Mitchell order)**.** Given normal measures[106] $\mathcal{U}$ and $\mathcal{V}$ on κ, say that $\mathcal{V} \lhd \mathcal{U}$ if $\mathcal{V} \in \mathrm{Ult}(V,\mathcal{U})$. The relation $\lhd$ here is called the *Mitchell order*. This order was introduced and proved to be well-founded by Mitchell in [Mit74], so we may therefore endow such measures with their *Mitchell rank* $o(\mathcal{U})$, the order type of $\{\mathcal{V} \mid \mathcal{V} \lhd \mathcal{U}\}$. Similarly we define the Mitchell rank of a cardinal κ to be the height of the tree induced by $\lhd$, denoted $o(\kappa)$. In particular, our convention is that $o(\kappa) > 0$ if and only if κ is measurable. Also, $o(\mathcal{U}) = \alpha$ if and only if $\{\lambda < \kappa \mid o(\lambda) = \alpha\} \in \mathcal{U}$.[107]

[104] In fact, for formality, let us denote by $(f)_{\mathcal{U}}$ the class $\{g\colon X \to V \mid f =_{\mathcal{U}} g\}$, and then let $[f]_{\mathcal{U}} = (f)_{\mathcal{U}} \cap V_\alpha$, where α is least such that this is non-empty. This example of Scott's trick is well-behaved and allows us to treat $f \mapsto [f]_{\mathcal{U}}$ as a definable class function.

[105] If $\mathrm{crit}(j)$ is undefined then way may say $\mathrm{crit}(j) = \infty$, but we never encounter this case in this text.

[106] Recall that a *normal measure* on κ is a κ-complete ultrafilter $\mathcal{U}$ such that for all $A \in \mathcal{U}$ and $f \in \prod A$ there is $B \in \mathcal{U}$ such that $f \upharpoonright B$ is constant.

[107] This is easiest to prove by noting $j_{\mathcal{U}}(\langle o(\alpha) \mid \alpha < \kappa\rangle)(\kappa) = o(\mathcal{U})$ (from [Mit83]).

θ-strongly compact cardinals

We also require a large cardinal property that was introduced in [BM14].

Definition 4.2.2. For $\theta \leqslant \kappa$ we say that κ is *θ-strongly compact* if every κ-complete filter on an arbitrary set X can be extended to a θ-complete ultrafilter on X.

Note that sometimes in the literature a "θ-strongly compact" cardinal refers to a cardinal $\kappa \leqslant \theta$ such that there is a κ-complete fine ultrafilter on $\mathscr{P}_\kappa(\theta)$. We shall not make use of this other definition.

Theorem 4.2.3 ([BM14, Theorem 4.7])**.** *The following are equivalent:*

1. *κ is θ-strongly compact.*
2. *For all $\alpha \geqslant \kappa$ there is an elementary embedding $j\colon V \to M$, where M is a transitive inner model of* ZFC*, such that* $\mathrm{crit}(j) \geqslant \theta$ *and, for some $D \in M$, $j``\alpha \subseteq D$ and $M \vDash |D| < j(\kappa)$.*
3. *For all $\alpha \geqslant \kappa$ there is a fine θ-complete ultrafilter on $\mathscr{P}_\kappa(\alpha)$.*

Fact. *If $\mathcal{U}$ is a fine θ-complete ultrafilter on $\mathscr{P}_\kappa(\alpha)$ then $j_{\mathcal{U}}$ satisfies Item* (2) *with $D = [\mathrm{id}]_{\mathcal{U}}$.*

Corollary 4.2.4, remarked upon in [BM14], is immediate but important.

Corollary 4.2.4. *Let κ be θ-strongly compact.*

(i) *$\mu = \min\{\lambda \geqslant \theta \mid \lambda$ is measurable$\}$ is well-defined, $\mu \leqslant \kappa$, and κ is μ-strongly compact.*

(ii) *For all $\lambda \geqslant \kappa$, λ is θ-strongly compact. In particular, there is a θ-strongly compact cardinal λ such that $2^{<\lambda} = \lambda$.*

Proof. Item (i). Firstly, by Item (2) in Theorem 4.2.3 with $\alpha = \kappa^+$, we obtain $j\colon V \to M$ with $\theta \leqslant \mathrm{crit}(j)$. Since there is an injection $D \to j(\kappa)$ in M with $j``\kappa^+ \subseteq D$, we must have $\kappa^+ \leqslant j(\kappa)$, so $\mathrm{crit}(j) \leqslant \kappa$. Hence $\mathrm{crit}(j)$ is measurable with $\theta \leqslant \mathrm{crit}(j) \leqslant \kappa$.

Let μ be the least measurable cardinal with $\mu \geqslant \theta$. Then for each $\alpha \geqslant \kappa$ there is $j\colon V \to M$ such that $\mathrm{crit}(j) \geqslant \theta$ and, for some $D \in M$, $j``\alpha \subseteq D$ and $M \vDash |D| < j(\kappa)$. However, in this case we must also have $\mathrm{crit}(j) \geqslant \mu$. Hence κ is μ-strongly compact.

Item (ii). For all $\lambda \geqslant \kappa$, any λ-complete filter $\mathcal{F}$ on any set X is also κ-complete. Hence, by the definition of κ being θ-strongly compact, the filter $\mathcal{F}$ extends to a θ-complete ultrafilter $\mathcal{U}$ on X. Setting $\lambda = \beth_\omega(\kappa)$ we have $2^{<\lambda} = \lambda$ and λ is θ-strongly compact. $\square$

4.2.2 Closure points and elementary embeddings

In Section 4.4.2 we shall wish to consider which elementary embeddings are found in a forcing extension. In our instance we are making use of an Easton-support iteration that, as is often the case for Easton-support iterations, admits no elementary embeddings that do not extend a ground-model embedding. To show this, we will make use of *closure points*, as explored in [Ham03].

Definition 4.2.5. A notion of forcing has a *closure point* at δ if it can be factored as $\mathbb{P}_0 * \dot{\mathbb{P}}_1$, where $\mathbb{P}_0$ is atomless, $|\mathbb{P}_0| \leqslant \delta$, and $\mathbb{1} \Vdash_{\mathbb{P}_0}$ "$\dot{\mathbb{P}}_1$ is $\leqslant\delta$ strategically closed".[108]

The following result is a combination of Lemma 13 and Theorem 10 in [Ham03]. While Lemma 13 and Theorem 10 in [Ham03] are more powerful than what we present here, Proposition 4.2.6 is all that we will need.

Proposition 4.2.6 (Hamkins)**.** *If $\mathbb{P}$ has a closure point at δ, $G \subseteq \mathbb{P}$ is V-generic, and $\mathcal{U} \in V[G]$ is a normal measure on $\kappa > \delta$, then $\mathcal{U} \cap V \in V$ is a normal measure on κ.*

[108] We shall not define strategic closure here. A δ^+-closed notion of forcing is $\leqslant\delta$ strategically closed, and we will only ever use this case. A definition can be found in [FK10, Chapter 12, Definition 5.15].

4.3 Hammers

We have two main tools at our disposal that work together to produce models in which there is a θ-independent family on a set X. Let us begin with Lemma 4.3.1, a method for showing that maximal θ-independent families exist contingent on the presence of a particular ideal $\mathcal{I}$. The proof and result are essentially due to Kunen in the form of [Kun83, Lemma 2.1], but we have softened the requirements.

Lemma 4.3.1. *Let θ be a regular uncountable cardinal, X a set with $|X| \geqslant \theta$, and $\mathcal{I}$ a θ-complete ideal over X such that $\mathrm{Add}(\theta, 2^X)$ densely embeds into $\mathscr{P}(X)/\mathcal{I}$. Then there is a maximal θ-independent family $\mathcal{A} \subseteq \mathscr{P}(X)$.*

Proof. Let $\mathbb{P} = \mathrm{Add}(\theta, 2^X)$ and $\psi\colon \mathbb{P} \to \mathscr{P}(X)/\mathcal{I}$ be a dense embedding. For $A \subseteq X$, let $[A]$ be the $\mathcal{I}$-equivalence class containing A. For all $f \in 2^X$ and $\delta < \theta$ choose $A_{f,\delta} \subseteq X$ such that $[A_{f,\delta}] = \psi(\{\langle f, \delta, 1\rangle\})$ and define $\varphi\colon \mathbb{P} \to \mathscr{P}(X)$ by

$$\varphi(p) = \bigcap\{A_{f,\delta} \mid p(f,\delta) = 1\} \cap \bigcap\{X \setminus A_{f,\delta} \mid p(f,\delta) = 0\}. \qquad (*)$$

Claim 4.3.1.1. *For all $p \in \mathbb{P}$, $\psi(p) = [\varphi(p)]$.*

Proof of Claim. We shall first show that $\psi(\{\langle f, \delta, 0\rangle\}) = [X \setminus A_{f,\delta}]$. Let $B_{f,\delta}$ be chosen so that $\psi(\{\langle f, \delta, 0\rangle\}) = [B_{f,\delta}]$. Since $\{\langle f, \delta, 0\rangle\} \perp \{\langle f, \delta, 1\rangle\}$, we have that $A_{f,\delta} \cap B_{f,\delta} \in \mathcal{I}$. For all $A \notin \mathcal{I}$ there is $p \in \mathbb{P}$ such that $\psi(p) \leqslant [A]$, and either $p \parallel \{\langle f, \delta, 0\rangle\}$ or $p \parallel \{\langle f, \delta, 1\rangle\}$. Hence, setting $[C] = \psi(p)$, we have that $C \cap A_{f,\delta} \notin \mathcal{I}$ or $C \cap B_{f,\delta} \notin \mathcal{I}$. In particular, letting $D = X \setminus (A_{f,\delta} \cup B_{f,\delta})$, we have $D \cap A_{f,\delta} = D \cap B_{f,\delta} = \emptyset \in \mathcal{I}$, so $D \in \mathcal{I}$. That is, $[A_{f,\delta} \cup B_{f,\delta}] = [X]$ and so $[B_{f,\delta}] = [X \setminus A_{f,\delta}]$ as required.

Therefore, for all $p \in \mathbb{P}$, $\psi(p) \leqslant [A_{f,\delta}]$ whenever $p(f,\delta) = 1$ and similarly $\psi(p) \leqslant [X \setminus A_{f,\delta}]$ whenever $p(f,\delta) = 0$. Given that $|p| < \theta$ and $\mathcal{I}$ is θ-complete, this means that $\psi(p) \leqslant [\varphi(p)]$. Setting $\psi(p) = [A]$, if $[A] < [\varphi(p)]$ then $\varphi(p) \setminus A \notin \mathcal{I}$ and so there is $q \in \mathbb{P}$ such that $\psi(q) \leqslant [\varphi(p) \setminus A]$. In particular, $\psi(q) \leqslant [A_{f,\delta}] = \psi(\{\langle f, \delta, 1\rangle\})$ whenever $p(f,\delta) = 1$ and similarly $\psi(q) \leqslant [X \setminus A_{f,\delta}] = \psi(\{\langle f, \delta, 0\rangle\})$ whenever $p(f,\delta) = 0$. That is, $q \leqslant p$ and thus $\psi(q) \leqslant [A]$. However, this cannot be the case since $[\varphi(p) \setminus A] \perp [A]$. ⊣

Let $\mathcal{A} = \{A_{f,\delta} \mid f \in 2^X,\ \delta < \theta\}$. Then for all $p \in \mathbb{P}$, $[\varphi(p)] = \psi(p) \neq [\emptyset]$, so $\varphi(p) \notin \mathcal{I}$ and thus $\mathcal{A}$ is θ-independent. Furthermore, for all $A \notin \mathcal{I}$ there is $p \in \mathbb{P}$ such that $\psi(p) = [\varphi(p)] \leqslant [A]$, and thus for all $A \subseteq X$ there is $p \in \mathbb{P}$

such that $[\varphi(p)] \leqslant [A]$ or $[\varphi(p)] \leqslant [X \setminus A]$. However, this is not quite true maximality, as we would require that for all $A \subseteq X$ there is $p \in \mathbb{P}$ such that $\varphi(p) \subseteq A$ or $\varphi(p) \subseteq X \setminus A$. To achieve this we alter $\mathcal{A}$ slightly.

Enumerate $\mathcal{I}$ as $\{C_f \mid f \in 2^X\}$ (with repeat entries if necessary) and define $A'_{f,\delta} = A_{f,\delta} \setminus C_f$. Since the representatives $A_{f,\delta} \in \psi(\{\langle f, \delta, 1\rangle\})$ were chosen arbitrarily, Claim 4.3.1.1 still holds for $\mathcal{A}' = \{A'_{f,\delta} \mid f \in 2^X,\ \delta < \theta\}$, where we define φ' analogously to φ in Equation ($*$). Hence, if $A \notin \mathcal{I}$ then there is $p \in \mathbb{P}$ such that $[\varphi'(p)] \leqslant [A]$, so $\varphi'(p) \setminus A = C_f \in \mathcal{I}$. Since $|p| < \theta$ there is $\delta < \theta$ such that $\langle f, \delta\rangle \notin \operatorname{dom}(p)$, and hence $\varphi'(p \cup \{\langle f, \delta, 1\rangle\}) \subseteq \varphi'(p) \setminus C_f \subseteq A$ as required. □

Remark. The statement of Lemma 4.3.1 is, on the surface, a strengthening of Kunen's result, as we have removed two requirements (both $2^{<\theta} = \theta$ and that $\mathcal{I}$ is θ^+-saturated) and weakened further requirements (we only need $\mathcal{I}$ to be θ-complete, rather than $|X|$-complete, and only demand that there is a dense embedding of $\operatorname{Add}(\theta, 2^X)$ into $\mathscr{P}(X)/\mathcal{I}$, rather than an isomorphism). This weakening is partially illusory. The proof of Lemma 4.3.1 in [Kun83] makes little use of some of these extraneous assumptions, and some of these requirements that we have altered are consequences: by Theorem 4.4.2 it will be the case that $2^{<\theta} = \theta$ and, since $\operatorname{Add}(\theta, 2^X)$ densely embeds into $\mathscr{P}(X)/\mathcal{I}$, we recover that $\mathcal{I}$ is θ^+-saturated by the chain condition.

Our second tool is an old technique present in [Kun83] (among many other places) for obtaining ideals $\mathcal{I}$ on X such that $\mathscr{P}(X)/\mathcal{I}$ is a complete Boolean algebra isomorphic to a desired notion of forcing. This method is closely tied to the idea of lifting elementary embeddings: if $j\colon V \to M$ is an elementary embedding and G is V-generic, then one can lift the elementary embedding to $\hat{\jmath}\colon V[G] \to M[j(G)]$, where $j(G)$ is an appropriate M-generic filter. However, if j was definable in V and $j(G) \notin V[G]$, then we may be unable to lift the embedding definably in $V[G]$. Theorem 4.3.2 can be understood intuitively as the idea that if $j = j_{\mathcal{U}}$ is an ultrapower embedding and $G \subseteq \mathbb{P}$ is V-generic, then $\mathcal{I} = \langle \mathcal{U}^* \rangle^{V[G]}$ will be a prime ideal only if j lifts to $V[G]$, and if it does not then $\mathscr{P}(X)/\mathcal{I}$ is the extra amount of forcing required to successfully lift the embedding: $\mathbb{P} * \mathscr{P}(X)/\mathcal{I} \cong j(\mathbb{P})$.

This technique has been the subject of much refinement, culminating in Foreman's *Duality Theorem*, from [For13], which bring precipitous ideals and a more refined definition of $\mathcal{I}$ into the fold. We do not need quite the level of complexity that the Duality Theorem affords, and so we shall present a specialised version that is localised to prime ideals, the scenario that we have described. For the case that $\dot{\mathbb{R}}$ is forced to be trivial, one can follow the technique of [Kun83] in which the special case of $\mathbb{P} = \mathrm{Add}(\omega_1, \kappa)$ was applied[109] to see a proof.

Theorem 4.3.2. *Let $\mathcal{U}$ be a σ-complete ultrafilter on a set X with ultrapower embedding $j = j_{\mathcal{U}} \colon V \to M$. Let $\mathbb{P}$ be a forcing such that $j(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{Q}} * \dot{\mathbb{R}}$ by an isomorphism π satisfying $\pi(j(p)) = \langle p, \mathbb{1}, \mathbb{1}\rangle$. Suppose that, for all V-generic $G * H \subseteq \mathbb{P} * \dot{\mathbb{Q}}$, there is $M[G * H]$-generic $F \subseteq \dot{\mathbb{R}}^{G*H}$ with $F \in V[G * H]$. Then there is a $\mathbb{P}$-name for an ideal $\dot{\mathcal{I}}$ on X such that $\mathbb{1} \Vdash_{\mathbb{P}} \mathscr{P}(\check{X})^{\bullet}/\dot{\mathcal{I}} \cong B(\dot{\mathbb{Q}})$.*

In the special case that $\dot{\mathbb{R}}$ is forced to be the trivial forcing, $\mathcal{I}$ is in fact $\langle \mathcal{U}^* \rangle^{V[G]}$, the ideal generated by $\mathcal{U}^*$ in the extension. Hence, if $\mathbb{P}$ is θ-distributive and $\mathcal{U}$ is θ-complete then $\mathcal{I}$ will be θ-complete as well.

4.4 Nails

Let us now apply our tools to produce examples of maximal θ-independent families, beginning with Theorem 4.4.1. Our presentation of this result is a slight extension of Kunen's original method to allow for general regular uncountable θ.

Theorem 4.4.1 ([Kun83, Theorem 2])**.** *Let κ be a measurable cardinal, $\theta < \kappa$ be uncountable and regular, and G be V-generic for $\mathrm{Add}(\theta, \kappa)$. Then there is a maximal θ-independent family $\mathcal{A} \subseteq \mathscr{P}(2^\theta)$ in $V[G]$.*

Proof. Let κ have measure $\mathcal{U}$ and ultrapower embedding $j = j_{\mathcal{U}} \colon V \to M$, so $\mathrm{crit}(j) = \kappa$. Then in $V[G]$ we have $\theta^{<\theta} = \theta$, $2^\theta = \kappa$, and

$$\begin{aligned} j(\mathrm{Add}(\theta, \kappa)) &= \mathrm{Add}(\theta, j(\kappa)) \\ &\cong \mathrm{Add}(\theta, j``\kappa) \times \mathrm{Add}(\theta, j(\kappa) \setminus j``\kappa) \\ &\cong \mathrm{Add}(\theta, \kappa) \times \mathrm{Add}(\theta, j(\kappa) \setminus \kappa) \times \{\mathbb{1}\}. \end{aligned}$$

Since each $p \in \mathrm{Add}(\theta, \kappa)$ is such that $|p| < \theta$, $j(p) = j``p = p$. Furthermore, since M is κ^+-closed, $\mathrm{Add}(\theta, j(\kappa))^M = \mathrm{Add}(\theta, j(\kappa))^V$. By Theorem 4.3.2, setting $\mathcal{I} = \langle \mathcal{U}^* \rangle^{V[G]}$, we have $\mathscr{P}(\kappa)/\mathcal{I} \cong B(\mathrm{Add}(\theta, j(\kappa) \setminus \kappa))$. Note here that since $\mathrm{Add}(\theta, \kappa)$ is θ-closed, $\mathrm{Add}(\theta, X)^V = \mathrm{Add}(\theta, X)^{V[G]}$ for all $X \in V$ and so

[109] Rather, the special case $\mathbb{P} = \mathrm{Fn}(\kappa, 2, \omega_1)$, but these are isomorphic.

the isomorphism class of $\mathrm{Add}(\theta, X)$ (in either V or $V[G]$) depends only on the cardinality of X. Furthermore, since $\mathrm{Add}(\theta, \kappa)$ is θ-closed, $\mathcal{I}$ is θ-complete.

Finally, since κ is measurable, $2^\kappa < j(\kappa) < (2^\kappa)^+$ and so $\mathrm{Add}(\theta, j(\kappa) \setminus \kappa)$ is isomorphic to $\mathrm{Add}(\theta, 2^\kappa)$. Therefore, by Lemma 4.3.1, there is a maximal θ-independent family $\mathcal{A} \subseteq \mathscr{P}(\kappa) = \mathscr{P}(2^\theta)$ in $V[G]$. □

Hence, if ZFC plus the existence of a measurable cardinal is consistent, then so is ZFC plus the existence of a maximal σ-independent family on 2^{ω_1}. Furthering this, Kunen recovers the consistency of a measurable cardinal from the consistency of a maximal θ-independent family.

Theorem 4.4.2 ([Kun83, Theorem 1])**.** *Let θ be an uncountable regular cardinal such that there is a maximal θ-independent family $\mathcal{A} \subseteq \mathscr{P}(\lambda)$. Then $2^{<\theta} = \theta$ and, for some κ such that $\sup\{(2^\alpha)^+ \mid \alpha < \theta\} \leqslant \kappa \leqslant \min\{\lambda, 2^\theta\}$, there is a non-trivial θ^+-saturated κ-complete ideal over κ.*

The full proof may be found in [Kun83], with the roles of κ and λ swapped, but we shall sketch it here.

Sketch proof. We say that maximal θ-independent $\mathcal{A} \subseteq \mathscr{P}(\lambda)$ is *globally maximal* if, setting P to be the set of partial functions $p\colon \mathcal{A} \to 2$ with $|p| < \theta$,

$$(\forall p \in P)(\forall X \subseteq \mathcal{A}^p)(\exists q \supseteq p)(\mathcal{A}^q \subseteq X \vee \mathcal{A}^q \cap X = \emptyset).$$

Fact. *There is $p \in P$ such that $\mathcal{A}/p = \{A \cap \mathcal{A}^p \mid A \in \mathcal{A} \setminus \mathrm{dom}(p)\}$ is globally maximal θ-independent.*

Hence, replacing λ by some $\lambda' < \lambda$ if necessary, we may assume that $\mathcal{A}$ is globally maximal θ-independent on λ. Let

$$\mathcal{I}_{\mathcal{A}} = \{X \subseteq \lambda \mid (\forall p \in P)\mathcal{A}^p \nsubseteq X\}.$$

Then $\mathcal{I}_{\mathcal{A}}$ is θ^+-saturated and $(2^\alpha)^+$-complete for all $\alpha < \theta$. Setting κ be least such that $\mathcal{I}_{\mathcal{A}}$ is not κ^+-complete, we can refine $\mathcal{I}_{\mathcal{A}}$ to a κ-complete θ^+-saturated ideal on κ. We immediately have $\sup\{(2^\alpha)^+ \mid \alpha < \theta\} \leqslant \kappa \leqslant \lambda$. On the other hand, if $\mathcal{A}_0 \in [\mathcal{A}]^\theta$ then $P_0 = \{\mathcal{A}^p \mid p\colon \mathcal{A}_0 \to 2 \text{ is a total function}\} \subseteq \mathcal{I}_{\mathcal{A}}$, but $\bigcup P_0 = \lambda \notin \mathcal{I}_{\mathcal{A}}$ and so $\kappa \leqslant 2^\theta$. □

Remark. The maximal θ-independent families constructed by Lemma 4.3.1 are globally maximal θ-independent. Furthermore, when we later construct maximal κ-independent families on κ, the bounds directly give us that there is a κ-complete and κ^+-saturated ideal on κ.

Corollary 4.4.3. *If there is a maximal θ-independent family for some uncountable regular θ then there is an inner model containing a measurable cardinal.*

Proof. By [Kun70, Section 11],[110] if κ carries a κ-complete, κ^+-saturated ideal then there is an inner model in which κ is measurable. Since $\kappa \geqslant \theta$, θ^+-saturated implies κ^+-saturated. □

In fact, [Kun70, Theorem 11.13] can be extended to any finite collection of saturated ideals on increasing cardinals, as noted in [Sch22].

Lemma 4.4.4 (Schlutzenberg)**.** *If $\kappa_0 < \cdots < \kappa_{n-1}$ are uncountable regular cardinals such that for all $i < n$ there is a normal κ_i-complete, κ_i^+-saturated ideal $\mathcal{I}_i \subseteq \mathscr{P}(\kappa_i)$, then in $L[\mathcal{I}_0, \ldots, \mathcal{I}_{n-1}]$, κ_i is measurable for all i.*

It follows from Theorem 4.4.2 and Lemma 4.4.4 that if there is a maximal θ-independent family on λ and a maximal θ'-independent family on λ' such that $\min\{\lambda, 2^\theta\} < \sup\{(2^\alpha)^+ \mid \alpha < \theta'\}$, then there is an inner model with two measurable cardinals, and indeed this pattern holds for all finite collections of such families. Therefore, a corollary of Theorem D is that the consistency of an $\aleph_1$-strongly compact cardinal implies the consistency of any finite number of measurable cardinals (though this is already known).

Kunen briefly sketches how to obtain a maximal κ-independent family on inaccessible κ, starting with a model in which κ is measurable. This requires a slightly more delicate use of Theorem 4.3.2 to obtain the result. We have also included additional content regarding lifting normal measures, which will be useful when proving Theorem E.

Proposition 4.4.5. *Let κ be measurable with normal measure $\mathcal{U}$, $2^\kappa = \kappa^+$, and $A \in \mathcal{U}$ be a set of regular cardinals. Let G be V-generic for the Easton-support iteration $\mathbb{P} = \ast_{\alpha \in A} \operatorname{Add}(\alpha, \alpha^+)$. Then in $V[G]$ there is a maximal κ-independent family $\mathcal{A} \subseteq \mathscr{P}(\kappa)$. Furthermore, if $\mathcal{V} \in V$ is a normal measure on κ such that $A \notin \mathcal{V}$ then there is a normal measure $\hat{\mathcal{V}} \supseteq \mathcal{V}$ on κ in $V[G]$.*

[110] [Kun70, Section 11] is a short section showing that if $\mathcal{I}$ is a normal, κ-complete, κ^+-saturated non-trivial ideal on κ then κ is measurable in $L[\mathcal{I}]$. Note that, by [Sol71], the existence of any κ-complete, κ^+-saturated, non-trivial ideal on κ implies the existence of a normal one.

Proof. Let $j = j_{\mathcal{U}} \colon V \to M$, and $H \subseteq \mathrm{Add}(\kappa, \kappa^+)$ be $V[G]$-generic. Note that $j(\kappa) < (2^\kappa)^+ = \kappa^{++}$. Furthermore, M is κ^+-closed, and this is preserved by the forcing (as $\mathbb{P} * \mathrm{Add}(\kappa, \kappa^+) \in M$), so $M[G * H]$ is also κ^+-closed.

Let $\dot{\mathbb{R}} = j(\mathbb{P})/(\mathbb{P} * \mathrm{Add}(\kappa, \kappa^+))$, noting that due to the Easton support we truly have $j(\mathbb{P}) \cong \mathbb{P} * \mathrm{Add}(\kappa, \kappa^+) * \dot{\mathbb{R}}$ as required in Theorem 4.3.2. Let

$$\mathbb{R} = \dot{\mathbb{R}}^{G*H} = \mathop{\ast}_{\alpha \in j(A) \setminus \kappa^+} \mathrm{Add}(\alpha, (\alpha^+)^M)^M.$$

$|\mathbb{P}|^V = \kappa$, so $|j(\mathbb{P})|^M = j(\kappa)$, and hence $|\mathbb{R}|^{V[G]} = \kappa^+$. Furthermore, each iterand of $\mathbb{R}$ is α-closed according to M for some $\alpha \geqslant \kappa^+$. Since $M[G * H]$ is κ^+-closed, this means that each iterand of $\mathbb{R}$ is κ^+-closed (in $V[G * H]$) and, since it is an iteration of length $j(\kappa) \geqslant \kappa^+$, $\mathbb{R}$ itself is κ^+-closed. However, $\mathbb{P}$ has only κ-many maximal antichains: each iterand is of cardinality less than κ and so has fewer than κ-many antichains. Hence $M[G*H] \vDash$ "$\mathbb{R}$ has only $j(\kappa)$-many maximal antichains". Since $j(\kappa) < \kappa^{++}$ we can build an $M[G*H]$-generic filter $F \subseteq \mathbb{R}$ in $V[G*H]$. Hence, by Theorem 4.3.2, in $V[G]$ there is an ideal $\mathcal{I}$ on κ such that $\mathscr{P}(\kappa)/\mathcal{I} \cong B(\mathrm{Add}(\kappa, \kappa^+))^{V[G]}$. This ideal can be expressed as

$$\mathcal{I} := \left\{ \dot{A}^G \subseteq \kappa \;\middle|\; \mathbb{1} \Vdash_{\mathbb{P} * \mathrm{Add}(\kappa,\kappa^+) * \dot{\mathbb{R}}/\dot{F}} \check{\kappa} \notin j(\dot{A}) \right\},$$

where $\dot{F}$ is a $\mathbb{P} * \mathrm{Add}(\kappa, \kappa^+)$-name for an $M[G * H]$-generic ideal $F \subseteq \mathbb{R}$. In this case, if $\{\dot{A}_\alpha \mid \alpha < \gamma\} \subseteq \mathcal{I}$ for some $\gamma < \kappa$ then, since $\mathrm{crit}(j) = \kappa$, $j(\bigcup \dot{A}_\alpha) = \bigcup j(\dot{A}_\alpha)$, and so $\bigcup \dot{A}_\alpha^G \in \mathcal{I}$. Hence, $\mathcal{I}$ is κ-complete as required and so, by Lemma 4.3.1, there is a maximal κ-independent family on κ in $V[G]$.

Conversely, let $\mathcal{V} \in V$ be a normal measure on κ such that $A \notin \mathcal{V}$, with ultrapower embedding $i \colon V \to N$. Then $i(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{R}}$ (without the $\mathrm{Add}(\kappa, \kappa^+)$ iterand), and so $F \in V[G]$. Thus i lifts to $\hat{\imath} \colon V[G] \to M[G * F]$ in $V[G]$ and obtain normal measure $\hat{\mathcal{V}} = \{B \subseteq \kappa \mid \kappa \in \hat{\imath}(B)\}$ on κ extending $\mathcal{V}$. □

4.4.1 A θ^+-strongly compact cardinal

These techniques are ripe for transfer to other large cardinal properties. In the following we shall find that κ being θ^+-strongly compact[111] for uncountable regular θ is sufficient to produce the ultrapower embeddings $j \colon V \to M$ that give rise to a proper class of λ such that there is a maximal θ-independent families $\mathcal{A} \subseteq \mathscr{P}(\lambda)$. The transfer is not entirely clean, as we additionally

[111] Note that, for fixed θ, there is $\mu > \theta$ such that κ is μ-strongly compact if and only if κ is θ^+-strongly compact.

require that $2^{<\kappa} = \kappa$, but as noted in Corollary 4.2.4 this does not increase the consistency strength of the assumption.

Theorem D. *Let κ be θ^+-strongly compact for some uncountable regular $\theta < \kappa$, with $2^{<\kappa} = \kappa$, and let G be V-generic for $\mathrm{Add}(\theta, \kappa)$. In $V[G]$, for all $\lambda \geqslant \kappa$ with $\mathrm{cf}(\lambda) \geqslant \kappa$, there is a maximal θ-independent family $\mathcal{A} \subseteq \mathscr{P}(\lambda)$.*

Proof. Let λ be such that $\mathrm{cf}(\lambda) \geqslant \kappa$. We wish to use Lemma 4.3.1 to show that there is a maximal θ-independent family $\mathcal{A} \subseteq \mathscr{P}(X)$, where $X = \mathscr{P}_\kappa(\lambda)^V$ (noting that $|X| = \lambda$). We therefore require a θ-complete ideal $\mathcal{I}$ over X such that $B(\mathrm{Add}(\theta, 2^X))$ is isomorphic to $\mathscr{P}(X)/\mathcal{I}$ in $V[G]$, which we shall obtain through Theorem 4.3.2.

Let $\mathcal{U} \in V$ be a fine θ-complete ultrafilter on X and $j = j_{\mathcal{U}} \colon V \to M$. Since $\kappa \geqslant \mathrm{crit}(j) > \theta$,

$$\begin{aligned} j(\mathrm{Add}(\theta, \kappa)) &= \mathrm{Add}(\theta, j(\kappa)) \\ &\cong \mathrm{Add}(\theta, j``\kappa) \times \mathrm{Add}(\theta, j(\kappa) \setminus j``\kappa) \\ &\cong \mathrm{Add}(\theta, \kappa) \times \mathrm{Add}(\theta, j(\kappa) \setminus \kappa) \times \{\mathbb{1}\}. \end{aligned}$$

Furthermore, each $p \in \mathrm{Add}(\theta, \kappa)$ is such that $|p| < |\theta|$, and thus $j(p) = j``p$, so the isomorphism extends $j(p) \mapsto \langle p, \mathbb{1}, \mathbb{1}\rangle$ as required. Setting $\mathcal{I} = \langle \mathcal{U}^* \rangle^{V[G]}$, we have $B(\mathrm{Add}(\theta, j(\kappa) \setminus \kappa)^V) \cong \mathscr{P}(X)/\mathcal{I}$ in $V[G]$ by Theorem 4.3.2. To finish we therefore need only show that

$$\mathrm{Add}(\theta, j(\kappa) \setminus \kappa)^V \cong \mathrm{Add}(\theta, 2^X)^{V[G]}.$$

$\mathrm{Add}(\theta, \kappa)$ is θ-closed so, for all $Y \in V$, $\mathrm{Add}(\theta, Y)^V = \mathrm{Add}(\theta, Y)^{V[G]}$ and so it is sufficient to prove that $|j(\kappa) \setminus \kappa| = |(2^\lambda)^{V[G]}|$.

$\mathrm{Add}(\theta, \kappa)$ is $(\theta^{<\theta})^+$-c.c. and $\theta^{<\theta} \leqslant \theta^{<\mathrm{crit}(j)} = \mathrm{crit}(j) \leqslant \kappa$, so $\mathrm{Add}(\theta, \kappa)$ is κ^+-c.c. By standard techniques,[112] $(2^\lambda)^{V[G]} \leqslant (|\mathrm{Add}(\theta, \kappa)|^{\kappa \times \lambda})^V$. Since $|\mathrm{Add}(\theta, \kappa)| \leqslant \kappa^\theta \leqslant \lambda^\lambda$, we get that $|(2^\lambda)^{V[G]}| \leqslant |(2^\lambda)^V|$. On the other hand, certainly $(2^\lambda)^V \subseteq (2^\lambda)^{V[G]}$ so we conclude that $|(2^\lambda)^V| = |(2^\lambda)^{V[G]}|$. It is therefore sufficient to show that $|2^\lambda| = |j(\kappa) \setminus \kappa|$ in V. To that end, we work in V for the remainder of the proof.

Since $2^\lambda > \kappa$ it is sufficient to show that $2^\lambda \leqslant j(\kappa) < (2^\lambda)^+$. Let $D = [\mathrm{id}]_{\mathcal{U}}$ in M. By the fineness of $\mathcal{U}$, $j``\lambda \subseteq D$ and $M \vDash |D| < j(\kappa)$. By elementarity,

[112] One could adapt the proof of [Jec03, Lemma 15.1] to incorporate chain conditions, for example.

$M \vDash (\forall\gamma < j(\kappa))2^\gamma \leqslant j(\kappa)$, and hence $M \vDash |\mathscr{P}(D)^M| \leqslant j(\kappa)$.

$2^\lambda \leqslant |\mathscr{P}(D)^M|$ as follows: Consider the function $f\colon \mathscr{P}(\lambda) \to \mathscr{P}(D)^M$ given by $f(A) = j(A) \cap D$. Since $j``\lambda \subseteq D$ we have that if $f(A) = f(B)$ then $j``A = j``B$ and so $A = B$. Hence f is an injection and $2^\lambda \leqslant |j(\kappa)|$.[113]

On the other hand, $j(\kappa) = \{[f]_\mathcal{U} \mid f\colon X \to \kappa\}$ and so $j(\kappa) < (\kappa^\lambda)^+ = (2^\lambda)^+$. Thus $2^\lambda \leqslant j(\kappa) < (2^\lambda)^+$ as required. □

4.4.2 A class of measurable cardinals

Assume GCH and suppose that $\kappa < \lambda$ are the two smallest measurable cardinals. By [LS67], if G is V-generic for some $\mathbb{P}$, where $|\mathbb{P}| < \lambda$, then λ is still measurable in $V[G]$. Hence, as in Proposition 4.4.5, if we force with $\ast_{\alpha\in A} \operatorname{Add}(\alpha, \alpha^+)$, where $A = \{\alpha < \kappa \mid \alpha \text{ is regular}\}$, then there will be a maximal κ-independent family $\mathcal{A} \subseteq \mathscr{P}(\kappa)$ in the forcing extension. Furthermore, this forcing has cardinality κ and so λ will still be measurable, and GCH will still hold. If we were to repeat this, say letting $\mathbb{P}' = \ast_{\alpha\in A'} \operatorname{Add}(\alpha, \alpha^+)$ in the forcing extension, where $A' = \{\alpha < \lambda \mid \kappa < \alpha \wedge \alpha \text{ is regular}\}$, then again Proposition 4.4.5 shows that in a new forcing extension by $\mathbb{P}'$ there is a maximal λ-independent family $\mathcal{A}' \subseteq \mathscr{P}(\lambda)$. However, since $\mathbb{P}'$ is κ^+-closed, no new subsets of κ nor sequences of length κ in $\mathcal{A}$ have been added, so $\mathcal{A}$ is still maximal κ-independent in the second forcing extension. One may reasonably expect that we can continue iterating this procedure to produce a (potentially class-size) forcing extension $V[G]$ such that, whenever κ is measurable in V, there is a maximal κ-independent family on κ.

The naïve approach to this argument has us construct the Easton-support iteration $\ast_{\alpha\in A} \operatorname{Add}(\alpha, \alpha^+)$, where A is the class of all regular non-measurable cardinals. We would then hope to use Proposition 4.4.5 to show that if $G \subseteq \mathbb{P}$ is V-generic and $\mathcal{U}$ is a normal measure on κ then we can construct a maximal κ-independent family on κ in $V[G]$. While this may work, one must be aware of a few potential issues. Firstly, we shall also end up excluding successors of measurable cardinals from A. In doing so, the tail after iterating up to stage κ is κ^{++}-closed, and so no subsets of $\mathscr{P}(\kappa)$ are added, simplifying some arguments. Since we shall only be dealing with normal measures, this will have no adverse impact on the argument.

Additionally, one must be careful of the condition $A \in \mathcal{U}$ stipulated in

[113] This method is similar to [JP79, Lemma 3.3.2], but could be older. We are grateful for Goldberg's help in [Gol23] for this result.

Proposition 4.4.5. If $\mathcal{U}$ was such that $\{\alpha < \kappa \mid \alpha \text{ is measurable}\} \in \mathcal{U}$, then $\kappa \notin j_{\mathcal{U}}(A \cap \kappa)$ and so $j_{\mathcal{U}}(\mathbb{P}_\kappa)$ is not isomorphic to $\mathbb{P}_\kappa * \mathrm{Add}(\kappa, \kappa^+) * \dot{\mathbb{R}}$ as desired. Instead $j_{\mathcal{U}} \colon V \to M$ may be lifted to $\hat{\jmath} \colon V[G \restriction \kappa] \to M[G \restriction \kappa * F]$ in $V[G \restriction \kappa]$ (and then can be lifted to $\tilde{\jmath} \colon V[G] \to M[j(G)]$ in $V[G]$ by the closure of $\mathbb{P}/\mathbb{P}_\kappa$). Therefore we must be sure to use $\mathcal{U}$ with $A \cap \kappa \in \mathcal{U}$ in our argument, which is to say $o(\mathcal{U}) = 0$. Fortunately, such such measures always exist by the well-foundedness of $\lhd$.

Continuing along our lifting argument, if $\mathcal{U} \in V$ is a normal measure on κ and $o(\mathcal{U})^V > 0$ then there is a normal measure $\hat{\mathcal{U}} \supseteq \mathcal{U}$ in $V[G]$. This allows us to show that if $o(\kappa)^V > \alpha$ then $o(\kappa)^{V[G]} \geqslant \alpha$. Though not all Mitchell ranks are preserved (we shall see that if $o(\kappa)^V = 1$ then $o(\kappa)^{V[G]} = 0$), the reduction shall be 'minimal': a closure point argument à la Proposition 4.2.6 gives us that if $\mathcal{U} \in V[G]$ is a normal measure in the forcing extension then $\mathcal{U} \cap V \in V$ is a normal measure in V. Hence if $o(\kappa)^V > 0$ then $o(\kappa)^V = 1 + o(\kappa)^{V[G]}$ exactly. That is, $o(\kappa)^V = o(\kappa)^{V[G]} - 1$ if $o(\kappa)^V$ is positive and finite, and otherwise $o(\kappa)^V = o(\kappa)^{V[G]}$. This operation warrants some ad-hoc notation. For $\alpha \in \mathrm{Ord}$, let

$$^{-}\alpha := \begin{cases} 0 & \alpha = 0 \\ \alpha - 1 & 0 < \alpha < \omega \\ \alpha & \omega \leqslant \alpha. \end{cases}$$

Our suggested interpretation of this operation is that, given some well-founded relation $\langle X, \prec \rangle$, we may produce a new relation $\langle {}^{-}X, \prec \rangle$ by setting ${}^{-}X$ to be those $x \in X$ that are not minimal with respect to $\prec$. Then if α is the height of $\prec$ on X, ${}^{-}\alpha$ is the height of $\prec$ restricted to ${}^{-}X$.

The only other consideration is GCH. However, this is easy to force while preserving the Mitchell rank of all cardinals, such as with the Easton-support iteration $\ast_{o(\kappa)>0} \mathrm{Add}(\kappa^+, 1)$.

Theorem E. *Let V be a model of* ZFC + GCH*. Then there is a class-length forcing iteration $\mathbb{P}$ preserving* ZFC+GCH *such that, if $G \subseteq \mathbb{P}$ is V-generic, then whenever κ is a measurable cardinal in V there is a maximal κ-independent family $\mathcal{A} \subseteq \mathscr{P}(\kappa)$ in $V[G]$. Furthermore, whenever κ is a measurable cardinal in V, $o(\kappa)^V = 1 + o(\kappa)^{V[G]}$, and whenever κ is non-measurable in V it remains non-measurable in $V[G]$.*

Proof. Let us first define our iteration system $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha \in \mathrm{Ord} \rangle$. Let $\mathbb{P}_n = \{\mathbb{1}\}$ for $n < \omega$ and $\dot{\mathbb{Q}}_\omega = \mathbb{P}_\omega = \mathrm{Add}(\omega, 1)$. For all $\alpha > \omega$, let $\dot{\mathbb{Q}}_\alpha = \{\mathbb{1}\}^\bullet$ if α is not a cardinal, is singular, is measurable, or is the successor of a measurable cardinal. Otherwise, let $\dot{\mathbb{Q}}_\alpha$ be a $\mathbb{P}_\alpha$-name for $\mathrm{Add}(\alpha, \alpha^+)$ in the extension. That is, letting A be the class of uncountable, non-measurable, regular cardinals that are not successors of measurable cardinals, $\mathbb{P} = \mathrm{Add}(\omega, 1) * \ast_{\alpha \in A} \mathrm{Add}(\alpha, \alpha^+)$. We iterate this with Easton support: at limit stage α, if α is regular, let $\mathbb{P}_\alpha$ be the direct limit of all $\mathbb{P}_\beta$ for $\beta < \alpha$, otherwise let $\mathbb{P}_\alpha$ be the inverse limit. Let $\mathbb{P}$ be the direct limit of all $\mathbb{P}_\alpha$. We treat the ωth stage differently here to apply a closure point argument subsequently.

Note that for all regular α, $\mathbb{P} = \mathbb{P}_\alpha * (\mathbb{P}/\mathbb{P}_\alpha)$, where $\mathbb{P}_\alpha$ is α^+-c.c. and $\mathbb{P}/\mathbb{P}_\alpha$ is forced to be α-closed. By a standard application of forcing techniques we have that $\mathbb{P}$ is tame and thus will preserve ZFC.[114] For measurable κ, we also have that $\mathbb{P} = \mathbb{P}_\kappa * (\mathbb{P}/\mathbb{P}_\kappa)$ with $\mathbb{P}/\mathbb{P}_\kappa$ forced to be κ^{++}-closed. Therefore, if $\mathbb{P}_\kappa$ adds a maximal κ-independent family $\mathcal{A} \subseteq \mathscr{P}(\kappa)$ then, after forcing with $\mathbb{P}/\mathbb{P}_\kappa$, $\mathcal{A}$ will still be maximal κ-independent. It therefore remains to show that $\mathbb{P}_\kappa$ does indeed add such a family, in the manner of Proposition 4.4.5. However, after taking care to pick a normal measure $\mathcal{U} \in V$ with $o(\mathcal{U})^V = 0$, we may apply Proposition 4.4.5 without modification.

The rest of the proof will be spent showing that, for all κ, $o(\kappa)^{V[G]} = {}^{-}o(\kappa)^V$. Note that $\mathbb{P}_{\kappa^{++}} = \mathbb{P}_\omega * (\mathbb{P}_{\kappa^{++}}/\mathbb{P}_\omega)$, where $\mathbb{P}_\omega = \mathrm{Add}(\omega, 1)$ and $\mathbb{P}_{\kappa^{++}}/\mathbb{P}_\omega$ is σ-closed. That is, $\mathbb{P}_{\kappa^{++}}$ has a closure point at ω. Therefore, if $\mathcal{U} \in V[G \restriction \kappa^{++}]$ is a normal measure on κ then, by Proposition 4.2.6, $\mathcal{U} \cap V \in V$ is a normal measure on κ in V. Since $\mathbb{P}/\mathbb{P}_{\kappa^{++}}$ is κ^{++}-closed, any normal measure on κ in $V[G]$ must have already been present in $V[G \restriction \kappa^{++}]$, so in particular if $o(\kappa)^V = 0$ then $o(\kappa)^{V[G]} = 0$. Consequently, if $o(\kappa)^{V[G]} > 0$ then $o(\kappa)^V > 0$, so $\mathbb{P}_{\kappa^{++}} = \mathbb{P}_\kappa$. Having established this, the following claim will be helpful for our lifting arguments.

Claim 4.4.5.1. *If $\mathcal{U} \in V[G]$ is a normal measure on κ, then*

$$C_\kappa := \{\lambda < \kappa \mid o(\lambda)^V > 0\} \in \mathcal{U}.$$

Proof of Claim. By prior calculations let us work in $V[G \restriction \kappa]$ and let

$$j = j_{\mathcal{U}} \colon V[G \restriction \kappa] \to N = M[j(G \restriction \kappa)]$$

[114][Jec03, Chapter 15] provides a comprehensive overview of preservation of ZFC using class products. [Fri00] has a deep treatment of class length forcing iterations.

be the associated ultrapower embedding. Note that

$$j(\mathbb{P}_\kappa) = \mathrm{Add}(\omega, 1) * \mathop{\ast}_{\alpha \in j(A \cap \kappa)} \mathrm{Add}(\alpha, (\alpha^+)^N)^N.$$

By the κ^+-closure of N in $V[G \restriction \kappa]$, if $\kappa \in j(A)$ then $\mathrm{Add}(\kappa, \kappa^+)^{V[G \restriction \kappa]}$ is an iterand of $j(\mathbb{P}_\kappa)$ and we can extract from $j(G)$ a $V[G \restriction \kappa]$-generic filter for $\mathrm{Add}(\kappa, \kappa^+)^{V[G \restriction \kappa]}$. However, j is definable in $V[G \restriction \kappa]$ and so certainly such an object cannot exist in $V[G \restriction \kappa]$. Hence, $\kappa \notin j(A)$ and so $\kappa \setminus A \in \mathcal{U}$. Since $\mathcal{U}$ is normal, we also have that $\{\lambda^+ \mid \lambda < \kappa, o(\lambda)^V > 0\} \notin \mathcal{U}$ and $\{\lambda < \kappa \mid \mathrm{cf}(\lambda) < \lambda\} \notin \mathcal{U}$, so $C_\kappa \in \mathcal{U}$. ⊣

We can now show that $o(\kappa)^{V[G]} = {}^-o(\kappa)^V$ exactly. We shall do this by induction, so suppose that for all $\lambda < \kappa$, $o(\lambda)^{V[G]} = {}^-o(\lambda)^V$. As we have shown that $o(\kappa)^V = 0$ implies $o(\kappa)^{V[G]} = 0$, let us assume that $o(\kappa)^V > 0$.

($o(\kappa)^{V[G]} \leqslant {}^-o(\kappa)^V$). Suppose that $o(\kappa)^{V[G]} > {}^-o(\kappa)^V$, witnessed by normal measure $\mathcal{U} \in V[G]$ such that $o(\mathcal{U})^{V[G]} = {}^-o(\kappa)^V$. By Claim 4.4.5.1, $C_\kappa \in \mathcal{U} \cap V$, and hence $o(\mathcal{U} \cap V)^V > 0$ and $o(\kappa)^V > 1$. In particular, for any α, if ${}^-\alpha = {}^-o(\kappa)^V$ then $\alpha = o(\kappa)^V$. Therefore,

$$\begin{aligned}\{\lambda < \kappa \mid o(\lambda)^{V[G]} = {}^-o(\kappa)^V\} &= \{\lambda < \kappa \mid {}^-o(\lambda)^V = {}^-o(\kappa)^V\} \\ &= \{\lambda < \kappa \mid o(\lambda)^V = o(\kappa)^V\} \\ &\in \mathcal{U} \cap V,\end{aligned}$$

and so $o(\mathcal{U} \cap V)^V = o(\kappa)^V$, a contradiction.

($o(\kappa)^{V[G]} \geqslant {}^-o(\kappa)^V$). By Proposition 4.4.5, if $\mathcal{U} \in V$ is a normal measure on κ such that $A \cap \kappa \notin \mathcal{U}$ (i.e. $o(\mathcal{U})^V > 0$), there is $\hat{\mathcal{U}} \supseteq \mathcal{U}$ a normal measure on κ in $V[G \restriction \kappa]$. Furthermore, since $\mathbb{P}/\mathbb{P}_\kappa$ is κ^{++}-closed, $\hat{\mathcal{U}}$ is still a normal measure on κ in $V[G]$. Since $o(\kappa)^{V[G]} \geqslant 0$ by definition, let us assume that $o(\kappa)^V > 1$ and prove that $o(\kappa)^{V[G]} \geqslant {}^-o(\kappa)^V$. If $\mathcal{U} \in V$ is such that $o(\mathcal{U})^V > 0$ then

$$\{\lambda < \kappa \mid o(\lambda)^V = o(\mathcal{U})^V\} = \{\lambda < \kappa \mid o(\lambda)^{V[G]} = {}^-o(\mathcal{U})^V\} \in \hat{\mathcal{U}}.$$

Hence, for all $\alpha < o(\kappa)^V$, ${}^-\alpha < o(\kappa)^{V[G]}$, so ${}^-o(\kappa)^V \leqslant o(\kappa)^{V[G]}$ as required. □

Note that this result on the Mitchell rank may not be reversible. Let $\mathcal{U}, \mathcal{V} \in V$ be any two normal measures on some κ with $o(\kappa)^V = 1$, and $A \in \mathcal{U} \setminus \mathcal{V}$ a set of regular cardinals. Then forcing with the Easton-support iteration

$\ast_{\alpha\in A}\operatorname{Add}(\alpha,\alpha^+)$ will produce a maximal κ-independent family $\mathcal{A}\subseteq\mathscr{P}(\kappa)$ thanks to $\mathcal{U}$, but $\hat{\mathcal{V}}$ will witness that κ is measurable in the forcing extension. However, there need not be an inner model witnessing $o(\kappa)>1$.

Chapter 5

String Dimension

> "Number One. Imagine a sphere, in which the exterior surface is red and the interior surface is white. Maintaining the continuity of the sphere, mentally distort the sphere so that the external surface is on the inside, and the internal on the outside–"
>
> Marina and Sergey Dyachenko, *Vita Nostra*

MATHEMATICS IS A SUBJECT OF CONSTANT CONFLUENCE. IDEAS developed in two separate fields on two separate continents can be found to be hiding the same technology under the hood, suffusing each respective subject with research from the other. Geometry underlies analysis underlies set theory underlies geometry, and the mathematicians keep pacing Escher's infinite staircase. It is an enormous philosophical conundrum that such discoveries keep occurring[115] and an incredible joy each time.

Vladimir Vapnik and Alexey Červonenkis developed a theory of computational learning that connected error rates of various stochastic processes to an idea of some $\mathcal{F} \subseteq \mathscr{P}(X)$ 'picking out' subsets of $S \subseteq X$. This same notion eventually wormed its way into model theory, where models that can definably exhibit this behaviour for arbitrarily large S are declared *wild*.[116] In extending

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[115] For those that do not wish to dismiss the question by stating that the reason is obvious, one way or another.

[116] Or *not tame*.

this property to even more arbitrary, infinitary, settings, defining the *string dimension* of $\mathcal{F} \subseteq \mathscr{P}(X)$, cardinal characteristics naturally became involved.

This chapter primarily focuses on the following question: "Given a set X and a cardinal δ, what is the smallest possible size of a collection $\mathcal{C} \subseteq \mathscr{P}(\mathscr{P}(X))$ such that $\bigcup \mathcal{C} = \mathscr{P}(X)$ and each $\mathcal{F} \in \mathcal{C}$ has string dimension less than δ?"

5.1 Introduction

VC dimension and VC classes arose from an inspection of machine learning algorithms in [VČ64] and [VČ68] by Vapnik and Červonenkis, in which the authors show that for a particular class of decision rules the rate of possible error falls into precisely one of two camps: exponential, or polynomial. In the latter case, a constant d is attributed to the rules that describes the greatest value at which the error rate $e(n)$ is equal to 2^n for all $n < d$, after which point the error rate is bounded by n^d. This constant would become known as the *VC dimension* of the class of rules, and such classes would be known as *VC classes*.

Definition 5.1.1 (VC class)**.** Let X be an infinite set and $\mathcal{F} \subseteq \mathscr{P}(X)$. For $A \subseteq X$, let $\mathcal{F} \cap A = \{Y \cap A \mid Y \in \mathcal{F}\}$. Then, for $n < \omega$, let

$$f_{\mathcal{F}}(n) = \max\{|\mathcal{F} \cap A| \mid A \in [X]^n\}.$$

We say that $\mathcal{F}$ is a *VC class* if there is a number $d < \omega$ such that for all $n > d$, $f_{\mathcal{F}}(n) < 2^n$. The least such d for which this holds is the *VC dimension* of $\mathcal{F}$, and it turns out that for all $n \geqslant d$, $f_{\mathcal{F}}(n) \leqslant n^d$ (see [VČ68, Theorem 1] or [Dri98, Chapter 5, Theorem 1.2]).[117]

We say that $\mathcal{F}$ *shatters* A if for all $B \subseteq A$ there is $Y \in \mathcal{F}$ such that $Y \cap A = B$. Equivalently, $\mathcal{F} \cap A = \mathscr{P}(A)$. Note then that $f_{\mathcal{F}}(n) = 2^n$ if and only if $\mathcal{F}$ shatters some set of cardinality n.

The concept of VC classes of sets proved incredibly robust and came to heavily influence classification theory in model theory. Instead of considering arbitrary collections of sets $\mathcal{F} \subseteq \mathscr{P}(X)$, one takes a first-order structure M and a first-order formula $\varphi(x; y)$ in the language of M. From these one considers the class $M[\varphi] = \{\varphi(M; a) \mid a \in M^y\}$ of solutions to $\varphi(x; a)$ as a varies over

[117] In fact, we can be even more precise. Either we have that $f_{\mathcal{F}}(d) = 2^d$ or, for all $n \geqslant d$, $f_{\mathcal{F}}(n) \leqslant \sum_{i<d} \binom{n}{i}$.

M^y.[118] This manifests as a subset of $\mathscr{P}(M^x)$ (notation defined in Section 5.2.1) and, hence, $M[\varphi]$ may or may not be a VC class. A formula $\varphi(x;y)$ is then said to have the *independence property*[119] if $M[\varphi]$ is not a VC class, whereas $\varphi(x;y)$ is *NIP* (*not the independence property*) otherwise. If $M[\varphi]$ is NIP for all formulae φ, then M itself is said to be NIP.

Suppose that $M[\varphi]$ is not a VC class. Then for all $n < \omega$, there is a collection of x-tuples $\langle a_i \mid i < n\rangle$ and a collection of y-tuples $\langle b_I \mid I \subseteq n\rangle$ such that $M \vDash \varphi(a_i; b_I)$ if and only if $i \in I$. Hence by the compactness theorem for first-order logic, if $\varphi(x;y)$ has infinite VC dimension, then for all infinite cardinals κ there is an elementary extension $N \succeq M$ and sequences $\langle a_\alpha \mid \alpha < \kappa\rangle$, $\langle b_I \mid I \subseteq \kappa\rangle$ such that $N \vDash \varphi(a_\alpha; b_I)$ if and only if $\alpha \in I$. It is therefore immaterial in the study of first-order theories to distinguish between various 'sizes' of infinite VC dimension.

This conceptualisation of VC dimension through structures and formulae need not be restricted to first-order logic, and indeed it is not. For example, the notion of VC dimension for positive logic has been developed in [DM24], and VC dimension is applied to the generalised idea of definable sets in [Dri98], amongst others. However, these are derived from roots deeply embedded in finite combinatorics and their definitions exhibit this. In contexts of classes of structures that do not obey a compactness theorem, or pure classes of sets, it may be important to distinguish between various classes that are not VC classes in the finitary sense, but do still fail to shatter in some way. Our notion of *string dimension* is a highly general form of VC dimension, with which we derive a cardinal $\mathfrak{sd}(\delta,\kappa)$—where $\kappa^+ \leqslant \mathfrak{sd}(\delta,\kappa) \leqslant 2^\kappa$—that measures how many classes of bounded string dimension are required to cover 2^κ. In Section 5.5.2 we shall investigate the possible values of $\mathfrak{sd}(\delta,\kappa)$ for $\kappa = \aleph_0$ or $\aleph_1$, the relations that can be built between them, and some possible constellations of these cardinals. In the case that κ is a strong limit we obtain the following maximality result.

Theorem (Theorem 5.4.4)**.** *If κ is a strong limit and δ is the least cardinal such that $2^\delta \geqslant \mathrm{cf}(\kappa)$ then $\mathfrak{sd}(\delta,\kappa) = 2^\kappa$. In particular, $\mathfrak{sd}(\aleph_0,\aleph_0) = 2^{\aleph_0}$.*

We also classify a helpful feature of some forcing iterations, which we call *finality*. When making forcing arguments for cardinal characteristics, one

[118] The notation $M[\varphi]$ is non-standard but we will abandon it after this section.

[119] As model theorists are wont to do, the 'wildness' condition here has been given the name, and the 'tameness' property is viewed (and labelled) as the negation of the wildness property.

often uses an iteration or product forcing and then argues that, say, every real number in the final model was added at an intermediate stage. For example, the ω-length finite-support 'iteration' of $\mathrm{Add}(\omega, 1)$ adds a new Cohen real at each stage, but also adds a new Cohen real in the final model that does not appear at any intermediate (finite) step. On the other hand, if $\mathbb{P}$ is the Cohen forcing that introduces ω_1-many new subsets of ω to V, then enumerating those new subsets as $\langle c_\alpha \mid \alpha < \omega_1 \rangle$ we have that

$$(2^\omega)^{V[\langle c_\alpha | \alpha < \omega_1 \rangle]} = \bigcup_{\beta < \omega_1} (2^\omega)^{V[\langle c_\alpha | \alpha < \beta \rangle]}.$$

Generalising this property, we say that an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ of notions of forcing is *κ-final* if for all V-generic $G \subseteq \mathbb{P}_\gamma$ and all $\lambda < \kappa$,

$$(V^\lambda)^{V[G]} = \bigcup_{\alpha < \gamma} (V^\lambda)^{V[G \restriction \alpha]}.$$

Such a property is generally argued for by using chain conditions, but we provide exact conditions on this behaviour in a way that is analogous to distributivity. Therefore, we may exhibit the chain condition style arguments as special cases of finality, as per the following.

Proposition 5.1.2. *Let $\mathbb{P} = \prod_{\alpha < \gamma} \mathbb{P}_\alpha$ be a product forcing of bounded support; that is, $p \in \mathbb{P}$ only if there is $\delta < \gamma$ such that $p = p \restriction \delta$. If $\mathbb{P}$ has the $\mathrm{cf}(\gamma)$-chain condition then $\mathbb{P}$ is $\mathrm{cf}(\gamma)$-final.*

This result is further extended to iterations in Proposition 5.5.3.

5.2 Preliminaries

When F is a set of functions we take $F \restriction A$ to be the pointwise restriction $\{f \restriction A \mid f \in F\}$. Using the usual correspondence between $\mathscr{P}(X)$ and 2^X (the bijection $2^X \to \mathscr{P}(X)$ given by $f \mapsto f^{-1}(\{1\})$) note that if $F \subseteq 2^X$ corresponds to $\mathcal{F} \subseteq \mathscr{P}(X)$, then $F \restriction A$ corresponds to $\{Y \cap A \mid Y \in \mathcal{F}\}$.

5.2.1 Model theory

When we denote a formula by $\varphi(x)$ or $\varphi(x; y)$, the variables x, y, etc. may be taken to be tuples of variables and we assume that these constitute all free variables appearing in φ. If M is a first-order structure and x is a variable or tuple from M, then M^x denotes the set of all tuples from M with the same length as x. If $\varphi(x; y)$ is an $\mathcal{L}$-formula, A is a set of x-tuples, and $b \in M^y$, then we define $\varphi(A; b)$ to be the set $\{a \in A \mid M \vDash \varphi(a; b)\}$.

VC dimension

VC dimension as a classification tool in model theory is entirely standard, and we follow standard conventions here. Throughout this section we shall fix a first-order language $\mathcal{L}$ and an $\mathcal{L}$-structure M.

Definition 5.2.1 (VC dimension)**.** Let $\varphi(x; y)$ be an $\mathcal{L}$-formula (possibly with parameters from M). We say that $\varphi(x; y)$ has *VC dimension* $\geqslant n$ if there are sequences $\langle a_i \mid i < n\rangle$ and $\langle b_I \mid I \subseteq n\rangle$ from M^x and M^y respectively such that, for all $i < n$ and $I \subseteq n$, $M \vDash \varphi(a_i; b_I)$ if and only if $i \in I$.

If there is a natural number n such that $\varphi(x; y)$ has VC dimension $\geqslant n$, but does not have VC dimension $\geqslant n + 1$, then we say that $\varphi(x; y)$ has VC dimension n. Otherwise, we say that $\varphi(x; y)$ has VC dimension ∞, or infinite VC dimension.

Note that the statement "$\varphi(x; y)$ has VC dimension n" is a first-order formula. In fact, if $\varphi(x; y; z)$ is the formula φ without parameters, then for all $c \in M^z$, if $\varphi(x; y; c)$ has VC dimension $\geqslant n$, then so does $\varphi(x; y^\frown z)$. Therefore if M is such that no formula $\varphi(x; y)$ has infinite VC dimension then this is true for all elementarily equivalent models, and we say that M is *NIP*, or *dependent*. By this same argument, being NIP is a property of a complete first-order theory.[120] We say that M is *IP*, or *independent*, if it is not NIP.

NIP theories have powerful structural decompositions that are explored in, for example, [Sim15], as well as many other places. Results using NIP often use it as a generalisation of stability and it can be seen as a way to ensure that a structure is not too "random".[121]

Example (Graph regularity)**.** Let $\langle G, E\rangle$ be a finite graph, A, B be subsets of G, and $\varepsilon > 0$. Defining $E(A, B)$ to be $E \cap (A \times B)$ (the set of edges between A and B) we say that $\langle A, B\rangle$ is *ε-regular* if $|E(A, B)| < \varepsilon|A||B|$ or $|A||B| - |E(A, B)| < \varepsilon|A||B|$. That is, A and B are either 'almost a clique' or 'almost independent'. A powerful classical result is the Szemerédi regularity

[120] In fact one needs a slightly stronger argument than this. See [Sim15] for a thorough treatment of this topic.

[121] Indeed the *Random Graph* (see [Ack37]) is an archetypal IP structure.

lemma from [Sze78] stating that for every $\varepsilon > 0$ there is $N < \omega$ such that *any* finite graph $\langle G, E\rangle$ may be decomposed into $G = \bigcup_{i<m} G_i$ (for some $m \leqslant N$) so that $||G_i| - |G_j|| \leqslant 1$ for all $i, j < m$ and all but at most εm^2 of the pairs $\langle G_i, G_j\rangle$ are ε-regular.

When we impose additional structural requirements on these graphs then the results may become much stronger. Malliaris and Shelah extend this in [MS14] to the case of k-stable graphs (those omitting the k-sized half-graph) to remove the requirement for at most εm^2 exceptional pairs (so *all* pairs are ε-regular).

Conversely, the weaker requirement that $\langle G, E\rangle$ has finite VC dimension is explored by Chernikov and Starchenko in [CS21] to dramatically reduce the bound on N in Szemerédi's regularity lemma. In [Gow97] Gowers shows that a lower bound for N in Szemerédi's regularity lemma is at least a tower of twos[122] of height on the order of $\varepsilon^{-1/16}$, while the Chernikov–Starchenko bound in the NIP case is only polynomial in ε^{-1}.

Further sub-classifications of NIP have also been explored for this, such as the regularity lemma for distal structures found in [CS18].

A final classical result is in the realm of algebraic structures: the Baldwin–Saxl theorem from [BS76].

Theorem (Baldwin–Saxl)**.** *Let G be a group definable in an NIP theory T. Let $\{H_a \mid a \in A\}$ be a uniformly definable family of subgroups of G. Then there is an integer N such that for any finite intersection $\bigcap_{a\in A} H_a$, there is a subset $A_0 \in [A]^N$ with $\bigcap_{a\in A} H_a = \bigcap_{a\in A_0} H_a$.*

5.2.2 Cardinal characteristics

The field of cardinal characteristics can be considered to be the investigation of topological or combinatorial characteristics or invariants of ideals of sets, particularly ideals on Polish spaces or generalised Polish spaces. We briefly recall definitions and properties of ideals. An *ideal* on a set X is a collection $\mathcal{I} \subseteq \mathscr{P}(X)$ of subsets of X such that: $\emptyset \in \mathcal{I}$; if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$; and if $A \subseteq B \in \mathcal{I}$ then $A \in \mathcal{I}$.

By a *σ-ideal* we mean a σ-complete ideal, so one that is closed under countable unions. Given a collection $\mathcal{A} \subseteq \mathscr{P}(X)$, we shall denote by $\langle \mathcal{A}\rangle$ the

[122] A *tower of twos of height* $n+1$ is 2^k, where k is a tower of twos of height n. A tower of twos of height 0 is 1.

ideal on X generated by $\mathcal{A}$:

$$\langle\mathcal{A}\rangle = \left\{B \subseteq X \;\middle|\; (\exists A_0, \ldots, A_{n-1} \in \mathcal{A})\, B \subseteq \bigcup\{A_i \mid i < n\}\right\}.$$

Similarly, we shall denote by $\langle\mathcal{A}\rangle_\sigma$ the σ-ideal on X generated by $\mathcal{A}$:

$$\langle\mathcal{A}\rangle_\sigma = \left\{B \subseteq X \;\middle|\; (\exists\mathcal{B} \in [\mathcal{A}]^{\leqslant\omega})\, B \subseteq \bigcup\mathcal{B}\right\}.$$

Given a σ-ideal $\mathcal{I}$ on an infinite set X, we may define various combinatorial characteristics of the ideal in the following way.

$$\mathbf{cof}(\mathcal{I}) = \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I},\ (\forall B \in \mathcal{I})(\exists A \in \mathcal{A})B \subseteq A\} \qquad \text{(Cofinality)}$$
$$\mathbf{cov}(\mathcal{I}) = \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I},\ \bigcup\mathcal{A} = X\} \qquad \text{(Covering)}$$
$$\mathbf{non}(\mathcal{I}) = \min\{|A| \mid A \subseteq X,\ A \notin \mathcal{I}\} \qquad \text{(Uniformity)}$$
$$\mathbf{add}(\mathcal{I}) = \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I},\ \bigcup\mathcal{A} \notin \mathcal{I}\} \qquad \text{(Additivity)}$$

It is easy to prove that these cardinals obey the relationships outline in Figure 5.1, where $A \to B$ indicates that $\mathsf{ZFC} \vdash |A| \leqslant |B|$.

$$\begin{array}{ccccc}
 & & \mathbf{cov}(\mathcal{I}) & \longrightarrow & \mathbf{cof}(\mathcal{I}) & \longrightarrow & 2^{|X|} \\
 & & \uparrow & & \uparrow & & \\
\aleph_1 & \longrightarrow & \mathbf{add}(\mathcal{I}) & \longrightarrow & \mathbf{non}(\mathcal{I}) & &
\end{array}$$

Figure 5.1: The combinatorial cardinal characteristics of $\mathcal{I}$.

$$\begin{array}{ccccccccccc}
 & & \mathbf{cov}(\mathcal{N}) & \longrightarrow & \mathbf{non}(\mathcal{M}) & \longrightarrow & \mathbf{cof}(\mathcal{M}) & \longrightarrow & \mathbf{cof}(\mathcal{N}) & \longrightarrow & 2^{\aleph_0} \\
 & & \Big\uparrow & & \uparrow & & \uparrow & & \Big\uparrow & & \\
 & & & & \mathfrak{b} & \longrightarrow & \mathfrak{d} & & & & \\
 & & & & \uparrow & & \uparrow & & & & \\
\aleph_1 & \longrightarrow & \mathbf{add}(\mathcal{N}) & \longrightarrow & \mathbf{add}(\mathcal{M}) & \longrightarrow & \mathbf{cov}(\mathcal{M}) & \longrightarrow & \mathbf{non}(\mathcal{N}) & &
\end{array}$$

Figure 5.2: Cichoń's diagram.

The equalities $\mathbf{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathbf{cov}(\mathcal{M})\}$ and $\mathbf{cof}(\mathcal{M}) = \max\{\mathbf{non}(\mathcal{M}), \mathfrak{d}\}$ are not labelled.

Traditionally one considers ideals on the set of real numbers (or on subsets of other Polish spaces[123]) with a basis of Borel sets, and so we have that $\mathbf{cof}(\mathcal{I})$ is further bounded by $\mathfrak{c} = 2^{\aleph_0}$. "Is $\aleph_1 < \mathfrak{c}$?" is already independent of ZFC and these cardinal characteristics only exacerbate this independence. For example, setting $\mathcal{M}$ to be the collection of meagre subsets of the real numbers, it is possible to have $\mathbf{cov}(\mathcal{M}) > \mathbf{non}(\mathcal{M})$ or $\mathbf{cov}(\mathcal{M}) < \mathbf{non}(\mathcal{M})$. Another traditional ideal to consider is $\mathcal{N}$, the collection of Lebesgue null sets of real numbers. Alongside two additional cardinal characteristics $\mathfrak{b}$ and $\mathfrak{d}$,[124] we obtain Figure 5.2: Cichoń's diagram.[125]

[123] A *Polish space* is a separable completely metrisable topological space. Generally in set theory we only concern ourselves with $\mathbb{R}$, 2^ω, and ω^ω.

[124] Given functions $f, g\colon \omega \to \omega$, we say that g *dominates* f, written $f \leqslant^* g$, if there is $N < \omega$ such that for all $n \geqslant N$, $f(n) \leqslant g(n)$. We say that $A \subseteq \omega^\omega$ is *unbounded* if there is no g dominating all $f \in A$ simultaneously, and we say that A is *dominating* if for all $f \in \omega^\omega$ there is $g \in A$ dominating f. $\mathfrak{b}$ (respectively $\mathfrak{d}$) is defined to be the least cardinality of an unbounded (respectively dominating) set. These cardinals are not made use of in this text, and for a more thorough introduction to them one should go to [FK10, Chapter 6].

[125] Astonishingly, Cichoń's diagram is known to be "complete", in that there are no provable (from ZFC) inequalities between entries that are not already present in Figure 5.2. Furthermore, it is possible to construct models of ZFC in which the ten "independent" entries (all but $\mathbf{add}(\mathcal{M})$ and $\mathbf{cof}(\mathcal{M})$) are distinct from one another without the use of large cardinals (see [Gol+22]).

5.3 String dimension

In the context of model theory, a formula $\varphi(x;y)$ is said to *shatter* a set $A \subseteq M^x$ if there is a sequence $\langle b_I \mid I \subseteq A\rangle$ of y-tuples such that $M \vDash \varphi(a;b_I)$ if and only if $a \in I$. This is in practice identical to the definition of shattering used in the introduction: $\varphi(x;y)$ shatters A if and only if $\{\varphi(M^x;b) \mid b \in M^y\}$ shatters A. Equivalently, if $\{\varphi(A;b) \mid b \in M^y\} = \mathscr{P}(A)$. Therefore φ has VC dimension at least n in M if and only if φ shatters some set of x-tuples of cardinality at least n. This idea can easily be extended to infinite sets of x-tuples, and even to pure sets devoid of structure.

Definition 5.3.1 (Shattering and string dimension)**.** Let X be a set and $F \subseteq 2^X$. Then we say that F *shatters* $A \subseteq X$ if $F \restriction A = 2^A$. The *string dimension* of F, denoted $\mathrm{d}_{\mathfrak{s}}(F)$, is the least cardinal δ such that F shatters no $A \in [X]^\delta$.

Note that while the definition of string dimension is almost the same as VC dimension, there are some important differences. Most obviously, string dimension takes into account the shattering of infinite sets and the cardinality of those sets.

Secondly, the string dimension measures the failure to shatter, whereas VC dimension measures successful shattering. Hence, in the case that the VC dimension of F is finite, say equal to n, then $\mathrm{d}_{\mathfrak{s}}(F) = n+1$. In the case that $\mathrm{d}_{\mathfrak{s}}(F) = \delta$ for a limit cardinal δ, then F shatters sets of unbounded cardinality below δ, but no set of cardinality δ. This avoids the cumbersome notation $\mathrm{d}_{\mathfrak{s}}(F) =$ "$<\delta$".

Finally, we have departed from considering families $\mathcal{F} \subseteq \mathscr{P}(X)$ and shall instead consider sets $F \subseteq 2^X$. This will greatly aid in notational clarity and succinctness when dealing with these objects in the future.

Proposition 5.3.2. *Let X be a set and λ, μ be cardinals. Suppose that for all $\alpha < \mu$, $F_\alpha \subseteq 2^X$. If $F = \bigcup_{\alpha<\mu} F_\alpha$ shatters a set of size $\mu \times \lambda$ then one of the F_α shatters a set of size λ.*

Proof. Without loss of generality, we identify the set that F shatters with $\mu \times \lambda$ and consider it a subset of X. Suppose that $F_\alpha \restriction \{\alpha\} \times \lambda = 2^{\{\alpha\}\times\lambda}$ for some $\alpha < \mu$. Then we are done. On the other hand, if this never happens then for all $\alpha < \mu$ there is $f_\alpha \colon \{\alpha\} \times \lambda \to 2$ such that $f_\alpha \notin F_\alpha \restriction \{\alpha\} \times \lambda$. However, F shatters $\mu \times \lambda$, so there is $g \in F$ such that $g \restriction \mu \times \lambda = \bigcup_{\alpha<\mu} f_\alpha$. Then $g \in F_\beta$, say, and $g \restriction \{\beta\} \times \lambda = f_\beta$, contradicting that $f_\beta \notin F_\beta \restriction \{\beta\} \times \lambda$. □

Proposition 5.3.2 allows us to consider ideals of sets of bounded string dimension with confidence.

Definition 5.3.3 (String ideal)**.** Let δ, κ be cardinals. The *string ideal* $\mathcal{I}_{\mathfrak{s}}(\delta, \kappa)$ is the σ-ideal generated by the set

$$\mathcal{S}(\delta, \kappa) := \{F \subseteq 2^\kappa \mid \mathrm{d}_{\mathfrak{s}}(F) < \delta\}.$$

Proposition 5.3.4. *For all infinite cardinals δ, κ, $\mathcal{S}(\delta, \kappa)$ is* $\mathrm{cf}(\delta)$*-complete. In particular, if δ is infinite then $\mathcal{S}(\delta, \kappa)$ is an ideal and if δ has uncountable cofinality then $\mathcal{S}(\delta, \kappa) = \mathcal{I}_{\mathfrak{s}}(\delta, \kappa)$.*

Proof. Let $\mu < \mathrm{cf}(\delta)$ and $\{F_\alpha \mid \alpha < \mu\} \subseteq \mathcal{S}(\delta, \kappa)$. We treat the case that δ is a limit and a successor separately.

First, suppose that δ is a successor, say $\delta = \eta^+$, so $\mathrm{cf}(\delta) = \delta$ and we are aiming to show that $F = \bigcup\{F_\alpha \mid \alpha < \mu\}$ shatters no set of cardinality η. Since $\delta > \mu$ we have that $\eta \geqslant \mu$ and so, by Proposition 5.3.2, if F shatters a set of cardinality $\eta = \mu \times \eta$ then there is $\alpha < \mu$ such that F_α shatters a set of cardinality η, contradicting that $F_\alpha \in \mathcal{S}(\eta^+, \kappa)$.

On the other hand, for limit cardinal δ, let $\langle \delta_\beta \mid \beta < \mathrm{cf}(\delta)\rangle$ be a cofinal sequence of cardinals in δ such that $\delta_0 = \mu$. Then we wish to demonstrate that $F = \bigcup\{F_\alpha \mid \alpha < \mu\}$ has string dimension less than δ, so $\mathrm{d}_{\mathfrak{s}}(F) \leqslant \delta_\beta$ for some $\beta < \mathrm{cf}(\delta)$. Towards a contradiction, assume otherwise. By Proposition 5.3.2, for all $\beta < \mathrm{cf}(\delta)$, F shatters a set of cardinality $\delta_\beta = \delta_\beta \times \mu$ and so there is $\alpha_\beta < \mu$ such that F_{α_β} shatters a set of cardinality δ_β as well. Since $\mu < \mathrm{cf}(\delta)$ and $\mathrm{cf}(\delta)$ is regular, there is $\alpha < \mu$ such that for cofinally many $\beta < \mathrm{cf}(\delta)$, F_α shatters a set of cardinality δ_β. Hence, $\mathrm{d}_{\mathfrak{s}}(F_\alpha) \geqslant \delta$, contradicting $F_\alpha \in \mathcal{S}(\delta, \kappa)$. □

5.3.1 Topology

Since the structure induced by shattering and dimension of subsets of 2^X depend only on the cardinality of X, we shall in general be only looking at 2^κ for cardinals κ. In this case, we will endow 2^κ with the *bounded-support product topology*. Namely, basic open subsets of 2^κ will be of the form $[t] = \{x \in 2^\kappa \mid x \supseteq t\}$ where $t\colon \alpha \to 2$ for some $\alpha < \kappa$ (though there is no harm in allowing $\mathrm{dom}(t) \subseteq \alpha$ for some $\alpha < \kappa$ instead). In this case, closed subsets of 2^κ are precisely the sets of cofinal branches of subtrees of $2^{<\kappa}$.

For $F \subseteq 2^\kappa$ we shall denote by $\overline{F}$ the closure of F in this topology. Namely,

$$\overline{F} = \{x \in 2^\kappa \mid (\forall \alpha < \kappa)(\exists y \in F) x \upharpoonright \alpha = y \upharpoonright \alpha\}.$$

Proposition 5.3.5. *If $F \subseteq 2^\kappa$ and $\mathrm{d}_{\mathfrak{s}}(F) < \mathrm{cf}(\kappa)$ then $\mathrm{d}_{\mathfrak{s}}(\overline{F}) = \mathrm{d}_{\mathfrak{s}}(F)$.*

Proof. We always have that $\mathrm{d}_{\mathfrak{s}}(F) \leqslant \mathrm{d}_{\mathfrak{s}}(\overline{F})$ since $F \subseteq \overline{F}$, so it suffices to prove the other direction. Indeed we shall actually prove the slightly stronger statement that for all $A \subseteq \alpha < \kappa$, $\overline{F} \upharpoonright A = F \upharpoonright A$. However, this is clear from the definition of $\overline{F}$ since for all $f \in \overline{F} \upharpoonright A$ there is $g \in \overline{F}$ such that $g \upharpoonright A = f$, and since $g \in \overline{F}$ there is $g' \in F$ such that $g' \upharpoonright \alpha = g \upharpoonright \alpha$, and so $g' \upharpoonright A = f \in F \upharpoonright A$. □

In particular, Proposition 5.3.5 means that the ideal $\mathcal{I}_{\mathfrak{s}}(\delta, \kappa)$ is generated by closed sets whenever $\delta \leqslant \mathrm{cf}(\kappa)$ and indeed is closed under the topological closure operation. This fails in the case of $\delta = \mathrm{cf}(\kappa)^+$ since, for example,

$$\left\{f : \omega \to 2 \;\middle|\; f^{-1}(\{1\}) \text{ is finite}\right\} \subseteq 2^\omega$$

has string dimension $\aleph_0$, but its closure is the full space 2^ω which has string dimension $\aleph_1$.

Definition 5.3.6. A set $F \subseteq 2^\kappa$ is *nowhere dense* if for all $s \in 2^{<\kappa}$ there is $t \in 2^{<\kappa}$ such that $t \supseteq s$ and $[t] \cap F = \emptyset$.

Proposition 5.3.7. *If $F \subseteq 2^\kappa$ and $\mathrm{d}_{\mathfrak{s}}(F) < \kappa$ then F is nowhere dense.*

Proof. Suppose that F is somewhere dense. That is, there is $s \in 2^{<\kappa}$ such that for all $t \supseteq s$, $F \cap [t] \neq \emptyset$. Let $\delta < \kappa$ and set $A = (\mathrm{dom}(s) + \delta) \setminus \mathrm{dom}(s)$,[126] noting that $\mathrm{dom}(s) + \delta < \kappa$ is bounded. For each $y \in 2^A$ there is $x_y \in F \cap [s ^\frown y]$ and in this case $x_y \upharpoonright A = y$. This witnesses that $F \upharpoonright A = 2^A$ and hence $\mathrm{d}_{\mathfrak{s}}(F) \geqslant \delta$. Therefore $\mathrm{d}_{\mathfrak{s}}(F) \geqslant \kappa$. □

While we are using the bounded-support product topology so that closed sets are generated by trees (for Proposition 5.3.5) the proof of Proposition 5.3.7 would have worked just as well in the κ-support product topology in which basic open sets are of the form $[t]$ for partial $t : \kappa \to 2$ with $|\mathrm{dom}(t)| < \kappa$.

Corollary 5.3.8. *For all $\delta \leqslant \kappa$, $\mathcal{I}_{\mathfrak{s}}(\delta, \kappa) \subseteq \mathcal{M}(\kappa, \kappa)$, the ideal of κ-meagre subsets of 2^κ (in either the bounded-support product or the κ-support product topologies).*

[126] Here "$\mathrm{dom}(s) + \delta$" refers to the ordinal sum of $\mathrm{dom}(s)$ and δ.

Remark. When $\mathfrak{sd}$ has been defined, a consequence of Corollary 5.3.8 will be that $\mathfrak{sd}(\kappa,\kappa) \geqslant \mathbf{cov}(\mathcal{M}(\kappa,\kappa))$.

Definition 5.3.9. The *(Lebesgue) null ideal* $\mathcal{N}$ is the σ-ideal of Lebesgue measure zero subsets of 2^ω.

Proposition 5.3.10. *$\mathcal{I}_{\mathfrak{s}}(\aleph_0,\aleph_0) \subseteq \mathcal{N}$.*

Proof. We shall require the following fact, [Dri98, Chapter 5, Theorem 1.2], translated into our notation.

Fact 5.3.11 ([Dri98])**.** *Let X be an infinite set and $F \subseteq 2^X$. Define*

$$f_F(n) := \max(|F \restriction A| \mid A \in [X]^n).$$

Then either $f_F(n) = 2^n$ for all $n < \omega$, or there is $d < \omega$ such that $f_F(n) \leqslant n^d$ for all $n \geqslant d$.

Note that since $\mathcal{I}_{\mathfrak{s}}(\aleph_0,\aleph_0)$ is generated by $\mathcal{S}(\aleph_0,\aleph_0)$ as a σ-ideal, it suffices to prove that each $F \in \mathcal{S}(\aleph_0,\aleph_0)$ has Lebesgue measure zero. Let $F \in \mathcal{S}(\aleph_0,\aleph_0)$, and let $N = \mathrm{d}_{\mathfrak{s}}(F)$. By Fact 5.3.11, there is $d < \omega$ such that for all sufficiently large n, $f_F(n) \leqslant n^d$. Hence, for all sufficiently large n there is $O_n \subseteq 2^n$ such that $|O_n| \leqslant n^d$ and for all $x \in F$, $x \restriction n \in O_n$. Therefore $F \subseteq \bigcup\{[t] \mid t \in O_n\}$ for all sufficiently large n. The Lebesgue measure of $[t]$ is 2^{-n}, so the Lebesgue measure of F is at most $n^d 2^{-n}$ for all n. Since $\lim_{n\to\infty} n^d 2^{-n} = 0$, F has Lebesgue measure zero. □

5.4 Cardinal characteristics of $\mathcal{I}_{\mathfrak{s}}(\delta,\kappa)$

Having defined the ideals $\mathcal{I}_{\mathfrak{s}}(\delta,\kappa)$, it becomes natural to consider the cardinal characteristics associated with them. Recall the definition of the covering number of a σ-ideal $\mathcal{I}$ on underlying set X:

$$\mathbf{cov}(\mathcal{I}) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I}, \bigcup\mathcal{A} = X\}.$$

Definition 5.4.1. For infinite cardinals δ and κ, with $\delta \leqslant \kappa^+$, we define the cardinal $\mathfrak{sd}(\delta,\kappa)$ to be $\mathbf{cov}(\mathcal{I}_{\mathfrak{s}}(\delta,\kappa))$. That is, $\mathfrak{sd}(\delta,\kappa)$ is the least size of a family of subsets of 2^κ such that their union is 2^κ, but each subset in the family has string dimension less than δ.

By noting that for $\delta \leqslant \chi \leqslant \kappa^+$, $\mathcal{I}_{\mathfrak{s}}(\delta,\kappa) \subseteq \mathcal{I}_{\mathfrak{s}}(\chi,\kappa)$ we immediately obtain that $\mathfrak{sd}(\delta,\kappa) \geqslant \mathfrak{sd}(\chi,\kappa)$. Similarly, if $\kappa \leqslant \eta$ then any covering of 2^η by elements of $\mathcal{I}_{\mathfrak{s}}(\delta,\eta)$ descends to a covering of 2^κ by elements of $\mathcal{I}_{\mathfrak{s}}(\delta,\kappa)$, and hence $\mathfrak{sd}(\delta,\kappa) \leqslant \mathfrak{sd}(\delta,\eta)$.

In the specific case of $\mathfrak{sd}(\kappa^+,\kappa)$, note that if $F \restriction X = 2^X$ then $|F| \geqslant |2^X|$, and hence if $|F| < 2^\kappa$, $F \in \mathcal{I}_{\mathfrak{s}}(\kappa^+,\kappa)$.[127]

Enumerate 2^κ as $\{x_\alpha \mid \alpha < 2^\kappa\}$, let $\{\alpha_\gamma \mid \gamma < \mathrm{cf}(2^\kappa)\}$ be a cofinal sequence in 2^κ, and set $F_\gamma = \{x_\alpha \mid \alpha < \alpha_\gamma\}$. Then $\{F_\gamma \mid \gamma < \mathrm{cf}(2^\kappa)\}$ is a covering of 2^κ by elements of $\mathcal{I}_{\mathfrak{s}}(\kappa^+,\kappa)$. Therefore $\mathfrak{sd}(\kappa^+,\kappa) \leqslant \mathrm{cf}(2^\kappa)$.

By Proposition 5.3.4, $\mathcal{I}_{\mathfrak{s}}(\kappa^+,\kappa)$ is κ^+-complete, so $\mathfrak{sd}(\kappa^+,\kappa) \geqslant \kappa^+$ and indeed $\mathfrak{sd}(\delta,\kappa) \geqslant \kappa^+$ for all $\delta \leqslant \kappa^+$. On the other end of the spectrum, for all $x \in 2^\kappa$ we have that $\{x\} \in \mathcal{I}_{\mathfrak{s}}(\aleph_0,\kappa)$, so $\mathfrak{sd}(\aleph_0,\kappa) \leqslant 2^\kappa$ and indeed $\mathfrak{sd}(\delta,\kappa) \leqslant 2^\kappa$ for all $\delta \leqslant \kappa^+$.

Compiling all of these results, we obtain Figure 5.3, where $A \to B$ indicates that $\mathsf{ZFC} \vdash |A| \leqslant |B|$.

Remark. In the definition of $\mathfrak{sd}(\delta,\kappa)$ we excluded the case that κ is finite as this reduces to a combinatorial problem with a determined finite solution. We have also excluded the case that δ is finite as $\mathfrak{sd}(n,\kappa) = \mathfrak{sd}(\aleph_0,\kappa)$ for all finite n. Finally, we have excluded the case that $\delta > \kappa^+$ since $2^\kappa \in \mathcal{I}_{\mathfrak{s}}(\kappa^{++},\kappa)$ and hence $\mathfrak{sd}(\kappa^{++},\kappa) = 1$.

Example. By Proposition 5.3.5, $\mathcal{S}(\aleph_1,\aleph_0)$ is σ-closed and thus equal to $\mathcal{I}_{\mathfrak{s}}(\aleph_1,\aleph_0)$. Therefore $\mathcal{I}_{\mathfrak{s}}(\aleph_1,\aleph_0)$ may be simply described as

$$\left\{F \subseteq 2^\omega \;\middle|\; (\forall X \in [\omega]^\omega) F \restriction X \neq 2^X\right\}.$$

This ideal is first defined in [Cic+93] as $\mathfrak{P}_2$. Note that if $F \subseteq 2^\omega$ is such that $|F| < \mathfrak{c}$, then certainly $F \in \mathcal{I}_{\mathfrak{s}}(\aleph_1,\aleph_0)$. Therefore $\mathbf{non}(\mathcal{I}_{\mathfrak{s}}(\aleph_1,\aleph_0)) = \mathfrak{c}$ and, as with every such ideal, $\mathbf{cov}(\mathcal{I}_{\mathfrak{s}}(\aleph_1,\aleph_0)) \leqslant \mathrm{cf}(\mathfrak{c})$ by the same argument as the start of Section 5.4.

The cardinal $\mathfrak{sd}(\aleph_1,\aleph_0)$ is not determined by ZFC (unlike $\mathfrak{sd}(\aleph_0,\aleph_0)$, among others, as we will see later). Indeed, as shown in [Cic+93], $\mathfrak{sd}(\aleph_1,\aleph_0)$ (what they call $\mathbf{cov}(\mathfrak{P}_2)$) is consistently strictly larger than all traditional cardinal characteristics of the continuum.[128] The only upper bound that one can obtain is $\mathfrak{sd}(\aleph_1,\aleph_0) \leqslant \mathbf{cof}(\mathcal{N})^+$, and this cannot be improved.

[127] Indeed $[2^\kappa]^{<|2^\delta|} \subseteq \mathcal{I}_{\mathfrak{s}}(\delta,\kappa)$ for all δ and κ.

[128] Those appearing in Cichoń's Diagram (Figure 5.2) as well as $\mathfrak{a}, \mathfrak{e}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{m}, \mathfrak{p}, \mathfrak{r}, \mathfrak{s}$, and $\mathfrak{u}$. See [FK10, Chapter 6] for definitions.

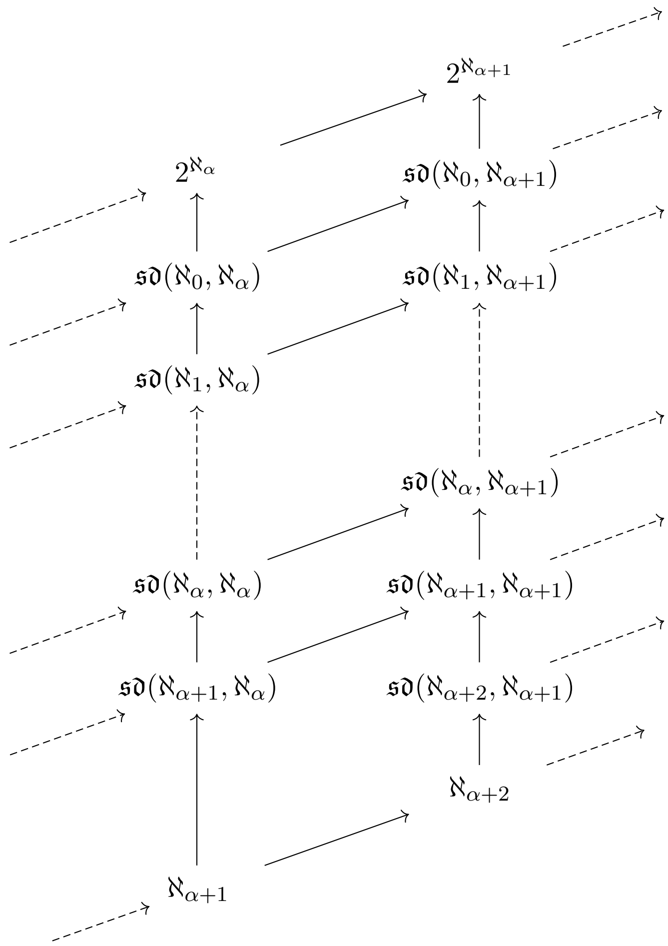

Figure 5.3: Relationships between values of $\mathfrak{sd}(\delta, \kappa)$ in ZFC. Not pictured is that $\mathfrak{sd}(\kappa^+, \kappa) \leqslant \mathrm{cf}(2^\kappa) \leqslant 2^\kappa$ for all κ.

5.4.1 $\mathfrak{sd}$ for strong limit cardinals

In this section we will exhibit a general construction for a class of trees $\langle T_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ such that $|[T_\alpha]| = 2^{|\alpha|}$, and it is in some sense "difficult" to have large subsets of $[T_\alpha]$ that are of low string dimension. In the case that $\alpha = \kappa$ is a strong limit cardinal (that is, for all $\beta < \kappa$, $2^{|\beta|} < \kappa$), we can show that the height of T_κ is κ. In this case we obtain a maximality result for $\mathfrak{sd}(\delta, \kappa)$ for certain values of δ.

We begin by treating $\mathfrak{sd}(\aleph_0, \aleph_0)$ to show off the initial stages of the construction.

Proposition 5.4.2. $\mathfrak{sd}(\aleph_0, \aleph_0) = \mathfrak{c}$. *In fact, there is a perfect set $X \subseteq 2^\omega$ such* *that* $\mathcal{I}_{\mathfrak{s}}(\aleph_0, \aleph_0) \cap \mathscr{P}(X) = [X]^{<\omega}$.

Proof. We shall construct a perfect tree $T \subseteq 2^{<\omega}$ such that whenever $A \subseteq [T]$ has cardinality at least 2^n, A shatters a set of size n. Hence, if $A \subseteq [T]$ is infinite then $\mathrm{d}_{\mathfrak{s}}(A) \geqslant \omega$. Therefore any covering of 2^ω (and by extension of $[T]$) by finite-dimensional sets must be of cardinality at least $\mathfrak{c}$.

We shall construct binary trees T_n for $n < \omega$ of finite height such that T_n has 2^n cofinal branches all of the same length, and T_{n+1} only *end-extends* T_n (that is, if $s \in T_{n+1}$ and $|s| < \mathrm{ht}(T_n)$ then $s \in T_n$). This construction will be such that for all $n > 0$ and $k < n$, if $A \subseteq [T_n]$ has cardinality at least 2^k, then A shatters a set of size k.

Let $T_0 = \emptyset$, so T_0 has one cofinal branch as required. Given T_n, enumerate the cofinal branches as $\langle b_j \mid j < 2^n \rangle$, and then for each $m \leqslant n$, enumerate $[2^n \times 2]^{2^m}$ as $\{I_{k,m} \mid k < N_m\}$, where $N_m = |[2^n \times 2]^{2^m}|$. For each $k < N_m$, let $\varphi_{k,m} \colon I_{k,m} \to 2^k$ be an arbitrary bijection. Then, for each $\langle j, i \rangle \in 2^n \times 2$ and $k < N_m$, set $\partial_{k,m}(j, i)$ to be $\varphi_{k,m}(j, i)$ if $\langle j, i \rangle \in I_{k,m}$ and 0^k otherwise.[129] Setting $\prod$ to be concatenation here, we define T_{n+1} to be the tree with cofinal branches precisely of the form

$$c_{j,i} = b_j {}^\frown \langle i \rangle {}^\frown \prod_{m \leqslant n} \left(\prod_{k < N_m} \partial_{k,m}(j, i) \right)$$

for $\langle j, i \rangle \in 2^n \times 2$. Note that T_{n+1} has exactly 2^{n+1} cofinal branches of equal finite height as desired. Since we split each cofinal branch of T_n once to produce T_{n+1}, the heights of the T_n are unbounded as n goes to ω. Furthermore, by construction, whenever $A \subseteq [T_{n+1}]$ has cardinality at least 2^m, we have that A shatters a set of size m. Let $T = T_\omega = \bigcup_{n<\omega} T_n$. Then whenever $A \subseteq [T]$ has cardinality at least 2^m, there is a level $n < \omega$ such that $|A \restriction n| \geqslant 2^m$. Then these branches pass through T_{n+1} from the first splitting and so will shatter a set of size m. □

This construction is the easiest to produce, but we can easily extend it to arbitrary ordinals.

The general construction

We shall inductively build a class of trees $\langle T_\alpha \mid \alpha \in \mathrm{Ord} \rangle$ such that the following hold:

[129] 0^k here refers to the k-length tuple of zeroes, not the set of functions $k \to 0$.

1. the only maximal branches of T_α are cofinal;
2. setting λ_α to be the cardinality of $[T_\alpha]$ we have $\lambda_\alpha = 2^{|\alpha|}$;
3. setting χ_α to be the height of T_α, $\alpha \leqslant \chi_\alpha < (2^{2^{|\alpha|}})^+$; and
4. whenever $A \subseteq [T_{\alpha+1}]$ is of cardinality at least $2^{|\beta|}$ for $\beta \leqslant \alpha$, A shatters a set of size at least $|\beta|$.

Begin with the same construction of T_n for $n < \omega$, and set $T_\omega = \bigcup\{T_n \mid n < \omega\}$. Given T_α, enumerate its cofinal branches as $\langle b_\gamma \mid \gamma < 2^{|\alpha|}\rangle$ and, for $\beta \leqslant \alpha$, enumerate $[2^{|\alpha|} \times 2]^{2^{|\beta|}}$ as $\{I_{\varepsilon,\beta} \mid \varepsilon < 2^{2^{|\alpha|}}\}$. For each $\varepsilon < 2^{2^{|\alpha|}}$, let $\varphi_{\varepsilon,\beta} \colon I_\varepsilon \to 2^\beta$ be an arbitrary bijection. Then, for each $\langle \gamma, i\rangle \in 2^{|\alpha|} \times 2$ and $\varepsilon < 2^{2^{|\alpha|}}$, let

$$\partial_{\varepsilon,\beta}(\gamma, i) = \begin{cases} \varphi_{\varepsilon,\beta}(\gamma, i) & \text{if } \langle \gamma, i\rangle \in I_{\varepsilon,\beta} \\ 0^\beta & \text{otherwise.} \end{cases}$$

Setting $\prod$ to be concatenation once again, let $T_{\alpha+1}$ be the tree with cofinal branches precisely of the form

$$c_{\gamma,i} = b_\gamma \frown \langle i\rangle \frown \prod_{\beta \leqslant \alpha} \left(\prod_{\varepsilon < 2^{2^{|\alpha|}}} \partial_{\varepsilon,\beta}(\gamma, i) \right)$$

for each $\langle \gamma, i\rangle \in 2^{|\alpha|} \times 2$. Note that $\lambda_{\alpha+1} = 2^{|\alpha+1|}$ as required. Given that $\chi_\alpha \geqslant \alpha$ and each cofinal branch of $T_{\alpha+1}$ is strictly longer than those in T_α, $\chi_{\alpha+1} \geqslant \alpha + 1$ as required. On the other hand,

$$\chi_{\alpha+1} = \chi_\alpha + 1 + \left(\alpha \times 2^{2^{|\alpha|}} \times \alpha\right),$$

and since $|\chi_\alpha| \leqslant 2^{2^{|\alpha|}}$ and $(2^{2^{|\alpha|}})^+$ is regular, $|\chi_{\alpha+1}| \leqslant 2^{2^{|\alpha+1|}}$ as required.

If $\alpha > \omega$ is a limit cardinal, then let $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$.

Definition 5.4.3. Given a cardinal κ, let $\log(\kappa)$ be the least cardinal λ such that $2^\lambda \geqslant \kappa$. In particular, $\log(\kappa) \leqslant \kappa$, and $\log(\kappa) = \kappa$ if and only if κ is a strong limit.

Theorem 5.4.4. *If κ is a strong limit, then $\mathfrak{sd}(\log(\mathrm{cf}(\kappa)), \kappa) = 2^\kappa$.*

Proof. Using the notation of our calculations in this section, we have that $\chi_\kappa \geqslant \kappa$ and $\chi_\kappa \leqslant \sup\{\chi_\alpha \mid \alpha < \kappa\}$. However, for all $\alpha < \kappa$, $\chi_\alpha < (2^{2^{|\alpha|}})^+ < \kappa$. Hence, $\chi_\kappa = \kappa$, so $T_\kappa \subseteq 2^{<\kappa}$ is a tree with 2^κ cofinal branches. Suppose that

$A \subseteq [T_\kappa]$ is such that $|A| \geqslant \kappa$. For $\delta < \log(\mathrm{cf}(\kappa))$, let $A' \subseteq A$ be of cardinality $2^\delta < \mathrm{cf}(\kappa)$, enumerated as $\{x_\alpha \mid \alpha < 2^\delta\}$. Then for $\alpha < \beta < 2^\delta$, define

$$\gamma_{\alpha,\beta} = \min\{\gamma < \kappa \mid x_\alpha(\gamma) \neq x_\beta(\gamma)\} < \kappa.$$

Since $2^\delta < \mathrm{cf}(\kappa)$, $\sup\{\gamma_{\alpha,\beta} \mid \alpha < \beta < 2^\delta\}$ is less than κ, and so $|A' \restriction \gamma| = 2^\delta$ for some $\gamma < \kappa$. Suppose that $\chi_\alpha < \gamma \leqslant \chi_{\alpha+1}$. Then by the construction of $T_{\alpha+2}$, A' will shatter a set of size at least δ. Hence, for all $\delta < \log(\mathrm{cf}(\kappa))$, A shatters a set of size δ. That is, $\mathrm{d}_{\mathfrak{s}}(A) \geqslant \log(\mathrm{cf}(\kappa))$ and so $A \notin \mathcal{I}_{\mathfrak{s}}(\log(\mathrm{cf}(\kappa)), \kappa)$. Therefore, to cover $[T_\kappa]$ in sets of dimension less than $\log(\mathrm{cf}(\delta))$, each set will need to be of cardinality less than κ. $\mathrm{cf}(2^\kappa) \geqslant \kappa^+$, so 2^κ such sets will be needed. Similarly, any covering of 2^κ by sets of dimension less than $\log(\mathrm{cf}(\kappa))$ will descend to a covering of $[T_\kappa]$, so the same restrictions apply. □

Corollary 5.4.5. *If κ is strongly inaccessible, then $\mathfrak{sd}(\kappa, \kappa) = 2^\kappa$ and, as a result, $\mathfrak{sd}(\delta, \kappa) = 2^\kappa$ for all $\delta \leqslant \kappa$.*

5.5 Forcing

We wish to explore how notions of forcing will affect the value of $\mathfrak{sd}(\delta, \kappa)$. One robust method for reducing $\mathfrak{sd}(\delta, \kappa)$ is to construct a chain of forcing extensions $\langle M_\alpha \mid \alpha \leqslant \gamma \rangle$ for some small γ such that, in M_γ, $\{(2^\kappa)^{M_\alpha} \mid \alpha < \gamma\}$ is a covering of 2^κ by elements of $\mathcal{I}_{\mathfrak{s}}(\delta, \kappa)$ and hence $\mathfrak{sd}(\delta, \kappa)^{M_\gamma} \leqslant \mathrm{cf}(\gamma)$. This can be achieved if we meet the following criteria: for all $\alpha < \gamma$, $(2^\kappa)^{M_\alpha} \in \mathcal{I}_{\mathfrak{s}}(\delta, \kappa)^{M_{\alpha+1}}$; for all $\alpha < \gamma$, $\mathcal{I}_{\mathfrak{s}}(\delta, \kappa)^{M_\alpha} \subseteq \mathcal{I}_{\mathfrak{s}}(\delta, \kappa)^{M_\gamma}$; and $(2^\kappa)^{M_\gamma} = \bigcup\{(2^\kappa)^{M_\alpha} \mid \alpha < \gamma\}$.

We can also use similar concepts to increase the value of $\mathfrak{sd}(\delta, \kappa)$ in some scenarios, though the strategy is more difficult to describe heuristically. Instead of guaranteeing that $(2^\kappa)^{M_\alpha} \in \mathcal{I}_{\mathfrak{s}}(\delta, \kappa)^{M_\gamma}$, we enforce that any description of a subset of $\mathcal{I}_{\mathfrak{s}}(\delta, \kappa)^{M_\gamma}$ of cardinality less than τ is already present in an intermediate M_α, and then use $M_{\alpha+1}$ to guarantee that this subset of $\mathcal{I}_{\mathfrak{s}}(\delta, \kappa)^{M_\gamma}$ cannot cover $(2^\kappa)^{M_\gamma}$. In this case we must have that $\mathfrak{sd}(\delta, \kappa)^{M_\gamma}$ is at least τ, as any smaller candidate is destroyed.

In both these cases we are using variations of *finality* and the *New Set–New Function property*. The former guarantees that new functions $\kappa \to 2$ are not added to M_γ, or that new subsets of $\mathcal{I}_{\mathfrak{s}}(\delta, \kappa)$ are not added to M_γ, and the latter (introduced in [Cic+93]) controls if $(2^\kappa)^{M_\alpha} \in \mathcal{I}_{\mathfrak{s}}(\delta, \kappa)^{M_\gamma}$.

5.5.1 Finality

In forcing constructions it can be very helpful to be able to control when sequences of ordinals are added to a model. For example, one may hope that a product forcing $\mathbb{P} = \prod_{\alpha<\gamma} \mathbb{P}_\alpha$ adds no new real numbers, or indeed any sequences of length ω, at limit stages. When this occurs, one can produce refined models of ZFC in which cardinal characteristics are kept at precise values by, say, adding new real numbers at successor stages to eliminate old meagre or null sets. We shall generalise this concept and produce an exact criterion for it to hold.

Definition 5.5.1 (Finality)**.** Let $\langle M_\alpha \mid \alpha \leqslant \gamma\rangle$ be a chain of models of ZF with the same ordinals, and let κ be such an ordinal. We say that the sequence is *κ-final* if for all $\eta < \kappa$, and all $b\colon \eta \to M_0$ in M_γ, there is $\alpha < \gamma$ such that $b \in M_\alpha$. That is,

$$(M_0^{<\kappa})^{M_\gamma} = \bigcup\Big\{ (M_0^{<\kappa})^{M_\alpha} \;\Big|\; \alpha < \gamma\Big\}.$$

An iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma\rangle$ is κ-final if for all V-generic $G \subseteq \mathbb{P}_\gamma$, $\langle V[G \restriction \alpha] \mid \alpha \leqslant \gamma\rangle$ is κ-final.

In most cases, we will only care about the most basic utility of κ-finality: that 2^κ, or perhaps κ^κ, has no new elements added in the final limit of an iteration or product. However, it turns out that if $(\kappa \times \mathbb{P})^\kappa$ has no new elements added at stage γ, then κ^+-finality is obtained for free.

Definition 5.5.2 (Pseudodistributive)**.** Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma\rangle$ be a forcing iteration, let $\mathbb{P} = \mathbb{P}_\gamma$, and let $\mathcal{A} = \{A_\alpha \mid \alpha < \eta\}$ be a collection of maximal antichains in $\mathbb{P}$. A *pseudorefinement* of $\mathcal{A}$ is a maximal antichain A in $\mathbb{P}$ such that for all $q \in A$ there is $\delta = \delta_q < \gamma$ such that for all $\alpha < \eta$, $p \in A_\alpha$, and $r \leqslant p, q$, we have $r \restriction \delta \Vdash q/\delta \leqslant p/\delta$.[130] That is, whenever $r_0 \leqslant p \restriction \delta, q \restriction \delta$, either $r_0 \Vdash q/\delta \leqslant p/\delta$ or, for some $s_0 \leqslant r_0$, $s_0 \Vdash q/\delta \perp p/\delta$.

We call $\mathbb{P}$ *κ-pseudodistributive* if for all $\eta < \kappa$ and all collections $\mathcal{A}$ of η-many maximal antichains in $\mathbb{P}$, there is a pseudorefinement for $\mathcal{A}$.

For the rest of the section, $\mathbb{P}$ will always refer to the final stage of a forcing iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma\rangle$ of length γ, so $\mathbb{P} = \mathbb{P}_\gamma$. We shall denote by $\mathbb{P}/\delta$ the canonical $\mathbb{P}_\delta$-name for the final $\gamma \setminus \delta$ co-ordinates of $\mathbb{P}$. If a chosen

[130] Recall that p/δ refers to the canonical $\mathbb{P}_\delta$-name for the final $\gamma \setminus \delta$ co-ordinates of $\mathbb{P}$.

value of δ is implicit, $p_0 \in \mathbb{P}_\delta$, and $\mathbb{1}_{\mathbb{P}_\delta} \Vdash \dot{p}_1 \in \mathbb{P}/\delta$, then we may denote by $\langle p_0, \dot{p}_1 \rangle$ the condition $p \in \mathbb{P}$ such that $p \restriction \delta = p_0$ and $\mathbb{1}_{\mathbb{P}_\delta} \Vdash p/\delta = \dot{p}_1$, viewing $\mathbb{P}$ as the iteration $\mathbb{P}_\delta * \mathbb{P}/\delta$.

Theorem F. $\mathbb{P}$ *is κ-final if and only if it is κ-pseudodistributive.*

Proof. ($\Longrightarrow$). Let $\eta < \kappa$ and $\mathcal{A} = \{A_\alpha \mid \alpha < \eta\}$ be a collection of maximal antichains in $\mathbb{P}$. Let $\dot{f} = \{\langle p, \langle \check{\alpha}, \check{p} \rangle^\bullet \rangle \mid \alpha < \eta, p \in A_\alpha\}$ so that for all V-generic $G \subseteq \mathbb{P}$, $\dot{f}^G(\alpha)$ is the unique element of $G \cap A_\alpha$. In particular, $\mathbb{1} \Vdash \dot{f} \colon \check{\eta} \to \check{\mathbb{P}}$ and hence $\mathbb{1} \Vdash (\exists \delta < \gamma)(\exists g \in V^{\mathbb{P}_\delta})\dot{f} = g$. Let A be a maximal antichain in $\mathbb{P}$ such that $q \in A$ only if there is $\delta_q < \gamma$ and a $\mathbb{P}_{\delta_q}$-name $\dot{g}_q$ such that $q \Vdash \dot{g}_q = \dot{f}$. We shall show that A is our desired pseudorefinement of $\mathcal{A}$.

Let $q \in A$ and set $\delta = \delta_q$. Suppose, for a contradiction, that there is $\alpha < \eta$, $p \in A_\alpha$, and $r \leqslant p, q$ such that $r \restriction \delta \nVdash q/\delta \leqslant p/\delta$. By separativity, extending $r \restriction \delta$ to r_0 if necessary, let $\dot{r}_1$ be a $\mathbb{P}_\delta$-name such that

$$r_0 \Vdash \text{“}\dot{r}_1 \leqslant q/\delta \text{ and } \dot{r}_1 \perp p/\delta\text{”}.$$

Let $p' \in A_\alpha$ be such that $p' \parallel \langle r_0, \dot{r}_1 \rangle$, witnessed by s, and note that we must have $p' \neq p$. Then $s \leqslant \langle r_0, \dot{r}_1 \rangle \leqslant \langle q \restriction \delta, q/\delta \rangle = q$, and so $s \Vdash \dot{f} = \dot{g}_q$. However, $s \leqslant p'$, so $s \Vdash \dot{f}(\check{\alpha}) = \check{p}'$. Hence, $s \Vdash \dot{g}_q(\check{\alpha}) = \check{p}'$ and in fact, since $\dot{g}_q$ is a $\mathbb{P}_\delta$-name, $s \restriction \delta \Vdash \dot{g}_q(\check{\alpha}) = \check{p}'$. On the other hand, $r \leqslant p, q$, so $r \Vdash \dot{f}(\check{\alpha}) = \dot{g}_q(\check{\alpha}) = \check{p}$ and so $r \restriction \delta \Vdash \dot{g}_q(\check{\alpha}) = \check{p}$, contradicting that $s \restriction \delta \leqslant r \restriction \delta$.

($\Longleftarrow$). Let $\eta < \kappa$ and $\dot{f}$ be a $\mathbb{P}$-name for a function $\check{\eta} \to \check{X}$ for some $X \in V$. For each $\alpha < \eta$, let A_α be a maximal antichain in $\mathbb{P}$ such that $p \in A_\alpha$ only if p decides the value of $\dot{f}(\check{\alpha})$ and let A be a pseudorefinement of $\{A_\alpha \mid \alpha < \eta\}$. Fix $q \in A$, set $\delta = \delta_q$, and define the $\mathbb{P}_\delta$-name $\dot{g}_q$ as follows:

$$\dot{g}_q = \Big\{ \langle r \restriction \delta, \langle \check{\alpha}, \check{x} \rangle^\bullet \rangle \;\Big|\; p \in A_\alpha,\ p \Vdash \dot{f}(\check{\alpha}) = \check{x},\ r \leqslant p, q \Big\}.$$

We now need only show that $q \Vdash \dot{f} = \dot{g}_q$ as then, by the predensity of A, we will have that $\mathbb{1}_\mathbb{P} \Vdash (\exists \delta < \gamma)(\exists g \in V^{\mathbb{P}_\delta})\dot{f} = g$ as required. Suppose that for some $q' \leqslant q$ and $x, x' \in X$, $q' \Vdash \dot{f}(\check{\alpha}) = \check{x} \wedge \dot{g}_q(\check{\alpha}) = \check{x}'$.

Let $p \in A_\alpha$ be such that $p \parallel q'$, witnessed by $r \leqslant p, q'$, and note in particular that $p \parallel q$. Since p and q' are compatible, p decided the value of $\dot{f}(\check{\alpha})$, and q' forced that $\dot{f}(\check{\alpha}) = \check{x}$, we must have that $p \Vdash \dot{f}(\check{\alpha}) = \check{x}$. Hence $\langle r \restriction \delta, \langle \check{\alpha}, \check{x} \rangle^\bullet \rangle \in \dot{g}_q$ and thus r (and indeed $r \restriction \delta$) forces that $\dot{g}_q(\check{\alpha}) = \check{x}$. On the other hand, $r \leqslant q'$ so $r \restriction \delta \Vdash \dot{g}_q(\check{\alpha}) = \check{x}'$. To consider how it could be that $r \restriction \delta \Vdash \dot{g}_q(\check{\alpha}) = \check{x}$, suppose that $p' \in A_\alpha$ is such that $p' \Vdash \dot{f}(\check{\alpha}) = \check{x}'$ and

$r' \leqslant p', q$ witnesses $\langle r' \restriction \delta, \langle \check{\alpha}, \check{x}' \rangle^\bullet \rangle \in \dot{g}_q$, with $r \restriction \delta \parallel r' \restriction \delta$. Let $s_0 \leqslant r \restriction \delta, r' \restriction \delta$. Then $\langle s_0, r'/\delta \rangle \leqslant p', q$, so $s_0 \Vdash q/\delta \leqslant p'/\delta$. Similarly, $\langle s_0, r/\delta \rangle \leqslant p, q$, so $s_0 \Vdash q/\delta \leqslant p/\delta$. Hence $\langle s_0, q/\delta \rangle \leqslant p, p'$, so $p = p'$ and $x = x'$. □

Remark. In the proof of Theorem F we implicitly use that the notion of forcing $\mathbb{P}$ is *separative*. That is, for all $p, p' \in \mathbb{P}$, if $p' \nleqslant p$ then there is $q \leqslant p'$ such that $q \perp p$. While one may always quotient a notion of forcing by its inseparable[131] elements to produce a separative preorder, it is sometimes easier to allow inseparable elements. In this case, the proof works perfectly well by changing the condition in Definition 5.5.2 to "whenever $r \leqslant p, q$, $r \restriction \delta$ forces that $q/\delta \leqslant p/\delta$ or there is an extension of $r \restriction \delta$ forcing that q/δ and p/δ are inseparable".

Remark. Note that it only makes sense to consider the finality of an iteration of limit length, as $\mathbb{P} * \dot{\mathbb{Q}}$ is κ-final if and only if $\dot{\mathbb{Q}}$ adds no functions $\eta \to V$ for all $\eta < \kappa$. That is, $\mathbb{1}_\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is $\check{\kappa}$-distributive". Henceforth we shall assume that any iteration is of limit length.

In many cases we will be able to produce a pseudorefinement of a collection of maximal antichains in which the δ is fixed. In this case, whenever there is a $\mathbb{P}$-name for a function $\eta \to V$ we will be able to determine in the ground model some upper bound δ for where the function appears. A typical example of this is when considering bounded-support iterations with a chain condition, such as the following Proposition 5.5.3.

Proposition 5.5.3. *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$ be a forcing iteration such that $\mathbb{P} = \mathbb{P}_\gamma$ has the $\mathrm{cf}(\gamma)$-chain condition and, for all $p \in \mathbb{P}$, there is $\alpha < \gamma$ such that $\mathbb{1}_\mathbb{P} \Vdash \{\check{\beta} \mid p(\beta) \neq \mathbb{1}_{\dot{\mathbb{Q}}_\beta}\}^\bullet \subseteq \check{\alpha}$. Then $\mathbb{P}$ is $\mathrm{cf}(\gamma)$-final.*

Proof. Let $\eta < \mathrm{cf}(\gamma)$, and $\{A_\alpha \mid \alpha < \eta\}$ a collection of maximal antichains in $\mathbb{P}$. By the chain condition, for each $\alpha < \eta$ there is $\delta_\alpha < \gamma$ such that for all $p \in A_\alpha$ and $\beta > \delta_\alpha$, $\mathbb{1}_\mathbb{P} \Vdash p(\beta) = \mathbb{1}_{\dot{\mathbb{Q}}_\beta}$. Furthermore, since $\eta < \mathrm{cf}(\gamma)$, $\delta = \sup\{\delta_\alpha \mid \alpha < \eta\} < \gamma$. Then $\{\mathbb{1}_\mathbb{P}\}$ is a pseudorefinement of $\{A_\alpha \mid \alpha < \eta\}$; if $p \in A_\alpha$ then $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash p/\delta = \mathbb{1}_\mathbb{P}/\delta$, so certainly for all $r \leqslant p, \mathbb{1}_\mathbb{P}$ we have $r \Vdash p/\delta \leqslant \mathbb{1}_\mathbb{P}/\delta$. □

In the case of product forcing, the statement of Proposition 5.5.3 becomes "if $\mathbb{P} = \prod_{\alpha<\gamma} \mathbb{P}_\alpha$ is of bounded support and has the $\mathrm{cf}(\gamma)$-chain condition, then $\mathbb{P}$ is $\mathrm{cf}(\gamma)$-final".

[131] Two conditions are inseparable if they witness that a preorder is not separative.

Example. Another trivial instance of finality is when $\mathbb{P}$ is distributive: if $\mathbb{P}$ adds no functions $\eta \to V$ for any $\eta < \kappa$ then certainly $\mathbb{P}$ is κ-final. However, chain conditions and distributivity are not the only ways to obtain finality.

Lemma 5.5.4. *For a forcing iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma \rangle$, $\mathbb{P}_\gamma$ is κ-final if and only if, for some (equivalently all) $\alpha < \gamma$, $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash$ "$\mathbb{P}_\gamma/\alpha$ is $\check{\kappa}$-final".*

In particular, if $\mathbb{P} = \prod_{\alpha<\gamma} \mathbb{P}_\alpha$ is κ-final and $\mathbb{R}$ is any notion of forcing, then $\mathbb{R} \times \mathbb{P}$, viewed as the product $\mathbb{R} \times \prod_{\alpha<\gamma} \mathbb{P}_\alpha$, is also κ-final.

Note that for any notions of forcing $\mathbb{P}$ and $\mathbb{Q}$, if $\mathbb{P}$ is not κ-c.c. then $\mathbb{P} \times \mathbb{Q}$ is also not κ-c.c. Similarly, if $\mathbb{P}$ is not κ-distributive then $\mathbb{P} \times \mathbb{Q}$ is also not κ-distributive. Hence, let $\mathbb{P}$ be $\mathrm{Add}(\omega_1, 1)$, so $\mathbb{P}$ is not c.c.c. but is σ-distributive, and let $\mathbb{Q}$ be $\mathrm{Add}(\omega, \omega_1)$, so $\mathbb{Q}$ is c.c.c but is not σ-distributive. Since $\mathbb{Q}$ is ω_1-final when viewed as the finite-support product $\prod_{\alpha<\omega_1} \mathrm{Add}(\omega, 1)$ by Proposition 5.5.7, $\mathbb{P} \times \mathbb{Q}$ is neither c.c.c. nor σ-distributive but is still ω_1-final when viewed as the finite-support product $\mathrm{Add}(\omega_1, 1) \times \prod_{\alpha<\omega_1} \mathrm{Add}(\omega, 1)$.

Since we are looking at chains of models given by forcing, we actually get an even stronger conclusion than the definition of finality for free in the following Proposition 5.5.5.

Proposition 5.5.5. *Let $\mathbb{P}$ be a forcing iteration of length γ that is κ-final. Then for all V-generic $G \subseteq \mathbb{P}$, all $\eta < \kappa$, and all $\alpha < \gamma$,*

$$(V[G \restriction \alpha]^\eta)^{V[G]} = \bigcup \left\{ (V[G \restriction \alpha]^\eta)^{V[G \restriction \beta]} \;\middle|\; \beta < \gamma \right\}.$$

That is, $V[G]$ contains no functions $\eta \to V[G \restriction \alpha]$ for any $\alpha < \gamma$ that were not already present in some prior $V[G \restriction \beta]$.

Proof. Let $f \colon \eta \to X$ be a function in $V[G]$, where $X \in V[G \restriction \alpha]$. Let $\dot{X}$ be a $\mathbb{P}_\alpha$-name for X, and define the function g with domain η via demanding that $g(\beta)$ is a $\mathbb{P}_\alpha$-name for $f(\beta)$. Then g is a function $\eta \to V$, so there is $\delta < \gamma$ such that $g \in V[G \restriction \delta]$. Taking $\delta \geqslant \alpha$ without loss of generality, we may determine $f(\beta)$ in $V[G \restriction \delta]$ by noting that $f(\beta) = g(\beta)^{G \restriction \alpha}$. □

5.5.2 Cohen forcing

Before we expand on the New Set–New Function property, it would behove us to show off how basic notions of forcing follow the constructions laid out at the beginning of Section 5.5. To this end, we shall spend some time exploring how Cohen forcing affects the value of $\mathfrak{sd}(\delta, \kappa)$, implementing our strategy in this case.

Fact 5.5.6. *The following hold for* $\mathrm{Add}(\chi, \lambda)$ *when* χ *is regular and* $\lambda > 0$*:*

1. $\mathrm{Add}(\chi, \lambda)$ *is* χ*-closed, and so adds no sequences of ground-model elements of length less than* χ*. In particular, if* $\kappa < \chi$ *then* $\mathrm{Add}(\chi, \lambda)$ *adds no elements to* 2^κ*;*
2. $\mathrm{Add}(\chi, \lambda)$ *is* $(\chi^{<\chi})^+$*-c.c.; and*
3. $\mathrm{Add}(\chi, \lambda)$ *forces that* $\chi^{<\chi} = \chi$*.*

Let χ, λ, and λ' be cardinals, with χ regular. Since $\mathrm{Add}(\chi, \lambda)$ is χ-closed and conditions in $\mathrm{Add}(\chi, \lambda')$ are sequences of length less than χ, whenever G is V-generic for $\mathrm{Add}(\chi, \lambda)$ then $\mathrm{Add}(\chi, \lambda')^V = \mathrm{Add}(\chi, \lambda')^{V[G]}$. Hence, one can view $\mathrm{Add}(\chi, \lambda)$ as the χ-support product or iteration of $\mathrm{Add}(\chi, 1)$ with itself λ-many times. In fact, if $\langle \alpha_\beta \mid \beta < \gamma \rangle$ is a collection of ordinals such that $\lambda \leqslant \sum_{\beta<\gamma} \alpha_\beta < \lambda^+$ then we can view $\mathrm{Add}(\chi, \lambda)$ as the forcing iteration generated by $\langle \mathrm{Add}(\chi, \alpha_\beta) \mid \beta < \gamma \rangle$. This perspective can be helpful for calculations in forcing extensions by Cohen forcing.

Proposition 5.5.7. *Suppose that* χ *is a regular cardinal and* λ *is such that* $\chi^{<\chi} < \mathrm{cf}(\lambda)$*. Then* $\mathrm{Add}(\chi, \lambda)$ *is* $\mathrm{cf}(\lambda)$*-final when viewed as the* χ*-support product* $\prod_{\alpha<\lambda} \mathrm{Add}(\chi, 1)$*.*

Proof. $\chi^{<\chi} < \mathrm{cf}(\lambda)$, so certainly $\chi < \lambda$ and hence, when viewed as the product $\prod_{\alpha<\lambda} \mathrm{Add}(\chi, 1)$, the forcing $\mathrm{Add}(\chi, \lambda)$ is of bounded support. Furthermore, $\mathrm{Add}(\chi, \lambda)$ is $(\chi^{<\chi})^+$-c.c. so, by Proposition 5.5.3, if $\chi^{<\chi} < \mathrm{cf}(\lambda)$ then $\mathrm{Add}(\chi, \lambda)$ is $\mathrm{cf}(\lambda)$-final. □

Note that in Proposition 5.5.8 λ need not be a cardinal; the proof works equally well if λ is replaced by an ordinal and we view $\mathrm{Add}(\chi, \lambda)$ as an ordinal-length product.

Proposition 5.5.8 formalises the rough argument made at the beginning of Section 5.5 to show that $\mathrm{Add}(\chi, \lambda)$ can create an upper bound to $\mathfrak{sd}(\delta, \kappa)$ for certain values of δ and κ.

Proposition 5.5.8. *Let* $\lambda > \kappa \geqslant \chi^{<\chi}$ *and* $\kappa \geqslant \delta \geqslant \chi$*, with* χ *and* δ *regular. Then* $\mathrm{Add}(\chi, \lambda)$ *forces that* $\mathfrak{sd}(\delta^+, \kappa) = \kappa^+$*.*

Proof. Since $\lambda \geqslant \kappa^+$, we may relabel $\mathrm{Add}(\chi,\lambda)$ to $\mathrm{Add}(\chi,\lambda+\kappa^+)$, where $\lambda+\kappa^+$ here represents the ordinal sum of λ and κ^+. Let G be V-generic for $\mathrm{Add}(\chi,\lambda+\kappa^+)$ and set H_0 to be the first λ co-ordinates of G, H to be the final κ^+ co-ordinates of G, and $W = V[H_0]$. Then H is W-generic for $\mathrm{Add}(\chi,\kappa^+)$, and $V[G] = W[H]$. Note that κ^+ is not collapsed by $\mathrm{Add}(\chi,\lambda)$ due to the $(\chi^{<\chi})^+$-c.c. and $\kappa \geqslant \chi^{<\chi}$, so $\mathrm{Add}(\chi,(\kappa^+)^V) = \mathrm{Add}(\chi,(\kappa^+)^{V[H_0]})$. We shall show that for all $\alpha < \kappa^+$, $(2^\kappa)^{W[H\restriction\alpha]} \in \mathcal{I}_{\mathfrak{s}}(\delta^+,\kappa)^{W[H]}$. This, combined with Proposition 5.5.7, shows that $\{(2^\kappa)^{W[H\restriction\alpha]} \mid \alpha < \kappa^+\}$ is a covering of $(2^\kappa)^{W[H]}$ by elements of $\mathcal{I}_{\mathfrak{s}}(\delta^+,\kappa)^{W[H]}$.

Towards a contradiction, let $\alpha < \kappa^+$ and $X \in ([\kappa]^\delta)^{W[H]}$ be such that $(2^\kappa)^{W[H\restriction\alpha]} \restriction X = \left(2^X\right)^{W[H]}$. Then there is $\beta < \kappa^+$, which we shall take without loss of generality to be at least α, such that $X \in W[H\restriction\beta]$. Therefore,

$$\begin{aligned}\left(2^X\right)^{W[H]} &= (2^\kappa)^{W[H\restriction\alpha]} \restriction X\\ &\subseteq \left(2^X\right)^{W[H\restriction\beta]}\\ &\subseteq \left(2^X\right)^{W[H]},\end{aligned}$$

and so $\left(2^X\right)^{W[H\restriction\beta]} = \left(2^X\right)^{W[H]}$. However, $|X| \geqslant \chi$ so there is $x \in \left(2^X\right)^{W[H]}$ such that $x \notin W[H\restriction\beta]$. For example, let $\varphi\colon \delta \to X$ be a bijection in $W[H]$. By Proposition 5.5.5 and $\delta \leqslant \kappa$, there is $\beta' \geqslant \beta$ such that $\varphi \in W[H\restriction\beta']$. Let x be the generic real $\bigcup(G(\beta'))$, so $x\colon \chi \to 2$ and $x \notin W[H\restriction\beta']$. Let $y = x \frown 0^{\delta\setminus\chi}$ and $z = y \circ \varphi$. Then $z\colon X \to 2$ but $z \notin W[H\restriction\beta']$ as then $z \circ \varphi^{-1} = y \in W[H\restriction\beta']$ and thus $y\restriction\chi = x \in W[H\restriction\beta']$, a contradiction. Hence, $(2^\kappa)^{W[H\restriction\alpha]} \in \mathcal{I}_{\mathfrak{s}}(\delta^+,\kappa)^{W[H]}$ as desired. □

Proposition 5.5.9. *Let κ be regular, $\delta < \kappa$, and $\kappa^{<\kappa} < \lambda$. Then* $\mathrm{Add}(\kappa,\lambda)$ *forces that $\mathfrak{sd}(\delta^+,\kappa) \geqslant \lambda$.*

Proof. The bulk of the proof rests on the following claim.

Claim 5.5.9.1. *Let κ be regular, $\delta < \kappa$, and γ a limit ordinal such that $\kappa^{<\kappa} < \mathrm{cf}(\gamma)$. Then* $\mathrm{Add}(\kappa,\gamma)$ *forces that $\mathfrak{sd}(\delta^+,\kappa) \geqslant \mathrm{cf}(\gamma)$.*

Proof of Claim. Let G be V-generic for $\mathrm{Add}(\kappa,\gamma)$. By Proposition 5.5.7, $\mathrm{Add}(\kappa,\gamma)$ is $\mathrm{cf}(\gamma)$-final. Since $\delta < \kappa$ and $\mathrm{Add}(\kappa,\gamma)$ is κ-closed, we have $([\kappa]^\delta)^V = ([\kappa]^\delta)^{V[G]}$ and indeed $|[\kappa]^\delta| \leqslant \kappa^{<\kappa} < \mathrm{cf}(\gamma)$ in both V and $V[G]$.

Note that if $X \in ([\kappa]^\delta)^{V[G]}$ then, since δ is a cardinal in $V[G]$, X has order type at least δ. Furthermore, if $F \subseteq 2^\kappa$ and $F \restriction X = 2^X$, then for all $Y \subseteq X$,

$F \restriction Y = 2^Y$. Hence, when we look for $X \in [\kappa]^\delta$ such that $F \restriction X = 2^X$, we may without loss of generality only consider $X \in [\kappa]^{(\delta)}$, the set of $X \subseteq \kappa$ of order type δ. Note that $([\kappa]^{(\delta)})^V = ([\kappa]^{(\delta)})^{V[G]}$, just as in the case of $[\kappa]^\delta$. For X a set of ordinals, let $\varphi_X : X \to \mathrm{ot}(X)$ be the unique order-preserving bijection.

Suppose, for a contradiction, that in $V[G]$ we have $\mathfrak{sd}(\delta^+, \kappa) = \tau < \mathrm{cf}(\gamma)$, witnessed by a covering $F = \{F_\alpha \mid \alpha < \tau\}$. Then there is $h : \tau \times [\kappa]^{(\delta)} \to 2^\delta$ such that for each $X \in [\kappa]^{(\delta)}$ and $\alpha < \tau$, $h(\alpha, X) \circ \varphi_X \notin F_\alpha \restriction X$. That is, $h(\alpha, \cdot)$ acts as a witness that $F_\alpha \in \mathcal{I}_{\mathfrak{s}}(\delta^+, \kappa)$. Replacing $[\kappa]^{(\delta)}$ by its cardinality in V, say $|[\kappa]^{(\delta)}|^V = \chi < \mathrm{cf}(\gamma)$, we have a function $h : \tau \times \chi \to V$. By $\mathrm{cf}(\gamma)$-finality, there is $\beta < \gamma$ such that $h \in V[G \restriction \beta]$. We shall now show that, setting $x : \kappa \to 2$ to be $\bigcup G(\beta)$, the βth generic subset of κ added by G, that for all $\alpha < \eta$ there is $X \in [\kappa]^{(\delta)}$ such that $x \restriction X = h(\alpha, X) \circ \varphi_X$. Hence, $x \notin \bigcup F$, contradicting that $\bigcup F = 2^\kappa$.

Formally, x is the realisation of the $\mathrm{Add}(\kappa, \gamma)$-name

$$\dot{x} = \{\langle p, \langle \check{\iota}, \check{\varepsilon} \rangle^\bullet \rangle \mid p \in \mathrm{Add}(\kappa, \gamma),\, p(\beta, \iota) = \varepsilon\},$$

so whenever $p \in \mathrm{Add}(\kappa, \gamma)$ and $\langle \beta, \iota \rangle \notin \mathrm{supp}(p)$, we can extend p to decide the value of $\dot{x}(\check{\iota})$ as we desire.

Working in $V[G \restriction \beta]$, let $p \in \mathrm{Add}(\kappa, \gamma)/\beta = \mathrm{Add}(\kappa, \gamma \setminus \beta)$ and $\alpha < \eta$. Since $|p| < \kappa$, there is $X \in [\kappa]^{(\delta)}$ such that $\mathrm{supp}(p) \cap (\{\beta\} \times X) = \emptyset$. Extend p to $q = p \cup \{\langle \langle \beta, \iota \rangle, h(\alpha, X)(\iota) \rangle \mid \iota \in X\}$. By density, we have that there must be some $X \in [\kappa]^{(\delta)}$ such that $x \restriction X = h(\alpha, X) \circ \varphi_X$. Since α was arbitrary, this holds for all $\alpha < \eta$ as desired. $\dashv$

We may now conclude the proof. If λ is regular then the result follows directly from Claim 5.5.9.1, so suppose that λ is singular. Since λ is singular, it is a limit cardinal and, since $\kappa^{<\kappa} < \lambda$, we have that λ is the supremum of the cardinals τ such that $\kappa^{<\kappa} < \tau < \lambda$. For each such τ, note that $\mathrm{Add}(\kappa, \lambda)$ is isomorphic to $\mathrm{Add}(\kappa, \lambda + \tau^+)$, and so by Claim 5.5.9.1, $\mathrm{Add}(\kappa, \lambda)$ forces that $\mathfrak{sd}(\delta^+, \kappa)$ is at least $\mathrm{cf}(\tau^+) = \tau^+$. Therefore, $\mathrm{Add}(\kappa, \lambda)$ forces that $\mathfrak{sd}(\delta^+, \kappa)$ is at least $\sup\{\tau^+ \mid \kappa^{<\kappa} < \tau < \lambda\} = \lambda$ as required. $\square$

Example (The constellation of $\mathfrak{sd}(\delta, \kappa)$ for $\kappa \leqslant \aleph_1$). We may use the tools that we have developed to consider some possible constellations of $\mathfrak{sd}(\delta, \kappa)$ for $\kappa = \aleph_0$ or $\aleph_1$. Inspired by Figure 5.3 and Proposition 5.4.2, the left hand side of Figure 5.4 shows the constellation of what we know in ZFC about these characteristics before any forcing.

Let $\mathbb{P} = \mathrm{Add}(\omega_1, \kappa) \times \mathrm{Add}(\omega, \lambda)$, where we choose κ and λ such that $\mathbb{1}_{\mathbb{P}}$ forces that $2^{\aleph_1} = \kappa$ and $2^{\aleph_0} = \lambda$. That is, $\kappa^{\omega_1^\omega} = \kappa \geqslant \lambda$, $\mathrm{cf}(\kappa) > \omega_1$, and $\mathrm{cf}(\lambda) > \omega$. Let $G \times H$ be V-generic for $\mathbb{P}$.

We first inspect $V[G]$, the forcing extension generated by $\mathrm{Add}(\omega_1, \kappa)$. Note that in $V[G]$ we have CH, since $\mathrm{Add}(\omega_1, \kappa)$ forces that $\aleph_1^{\aleph_0} = \aleph_1$. Hence

$$\aleph_1 = \mathfrak{sd}(\aleph_1, \aleph_0) = \mathfrak{sd}(\aleph_0, \aleph_0) = \mathfrak{c}.$$

Furthermore, by Proposition 5.5.8 (and CH) we have $\mathfrak{sd}(\aleph_2, \aleph_1) = \aleph_2$. Finally, by Proposition 5.5.9, $\mathfrak{sd}(\aleph_1, \aleph_1) \geqslant \kappa$. Combining these results, we obtain the right hand side of Figure 5.4.

In $V[G][H]$, the extension generated by further forcing with $\mathrm{Add}(\omega, \lambda)$ in $V[G]$, we have that $\mathfrak{sd}(\aleph_1, \aleph_0) = \aleph_1$ and $\mathfrak{sd}(\aleph_1, \aleph_1) = \mathfrak{sd}(\aleph_2, \aleph_1) = \aleph_2$ by Proposition 5.5.8. Hence we obtain Figure 5.5. We do not know the value that $\mathfrak{sd}(\aleph_0, \aleph_1)$ takes in this model, other than the bounds $\mathfrak{c} \leqslant \mathfrak{sd}(\aleph_0, \aleph_1) \leqslant 2^{\aleph_1}$. By inspecting these constellations we produce a summary of the consistency of statements of the form $|A| < |B|$ for various objects A and B, summarised in Figure 5.6.

5.5.3 New Set–New Function

The New Set–New Function property is introduced in [Cic+93] as a way to explain the behaviour of large classes of forcing on $\mathfrak{sd}(\aleph_1, \aleph_0)$ ($\mathbf{cov}(\mathfrak{P}_2)$, as it is denoted in [Cic+93]). We wish to expand upon this concept so that we can replicate the arguments of Propositions 5.5.8 and 5.5.9 for a wider class of forcings. This will complete the final half of the rough argument outline at the start of Section 5.5: When is $(2^\kappa)^M \in \mathcal{I}_{\mathfrak{s}}(\delta, \kappa)^N$, for $M \subseteq N$?

Definition 5.5.10. Let V be a model of ZF and let $\delta, \kappa \in V$ be ordinals.[132] We say that a notion of forcing $\mathbb{P} \in V$ has the *New Set–New Function property* for $\langle \delta, \kappa \rangle$, written $\mathrm{NSNF}(\delta, \kappa)$, if for all V-generic $G \subseteq \mathbb{P}$, $(2^\kappa)^V \in \mathcal{I}_{\mathfrak{s}}(\delta, \kappa)^{V[G]}$. Equivalently, for all $X \in ([\kappa]^{(\delta)})^{V[G]}$ there is $x \in (2^X)^{V[G]}$ such that for all $y \in (2^\kappa)^V$, $x \nsubseteq y$, so there is $\alpha \in X$ such that $x(\alpha) \neq y(\alpha)$.

In fact, one may define the New Set–New Function property for arbitrary nested models $M \subseteq N$ of ZF. We say that $M \subseteq N$ has the New Set–New Function property for $\langle \delta, \kappa \rangle$ if $(2^\kappa)^M \in \mathcal{I}_{\mathfrak{s}}(\delta, \kappa)^N$.

[132] Though usually one will only care about the case that δ and κ are cardinals.

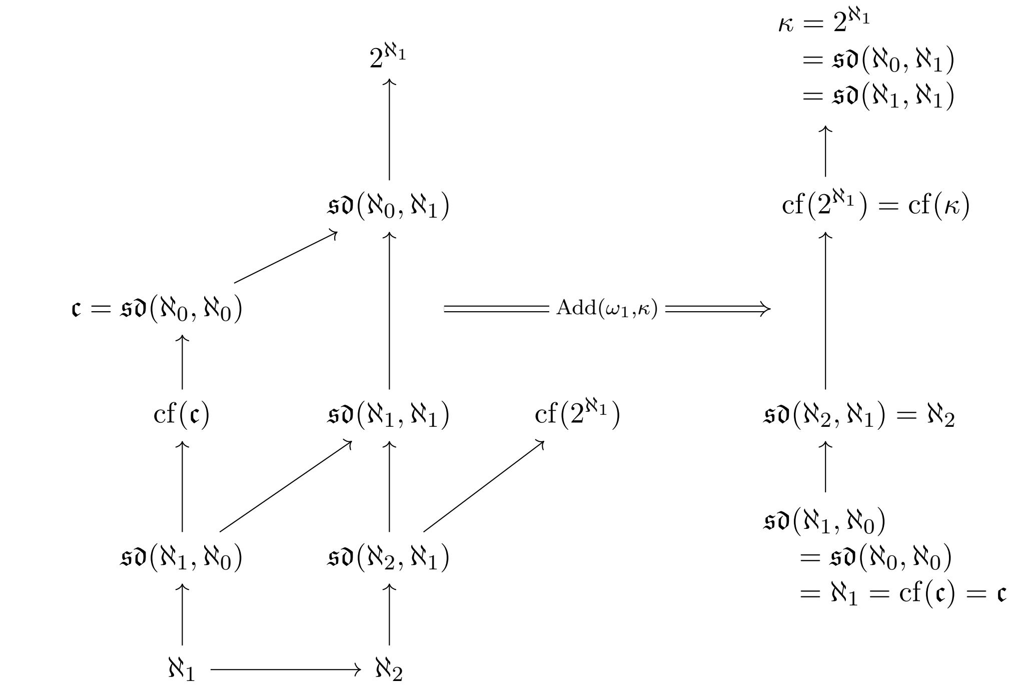

Figure 5.4: Constellation of $\mathfrak{sd}(\delta, \tau)$ for $\tau = \aleph_0$ or $\aleph_1$ before and after forcing with $\mathrm{Add}(\omega_1, \kappa)$.

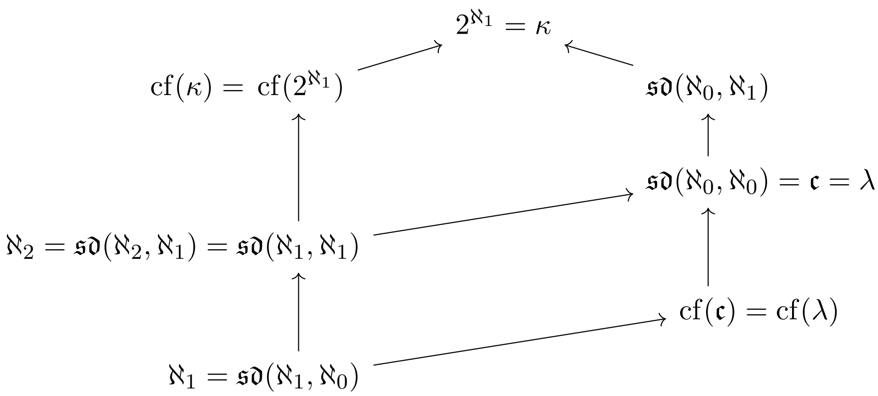

Figure 5.5: Constellation of $\mathfrak{sd}(\delta, \tau)$ for $\tau = \aleph_0$ or $\aleph_1$ after forcing with $\mathrm{Add}(\omega_1, \kappa) \times \mathrm{Add}(\omega, \lambda)$.

Con ↓<→	$\mathfrak{sd}(\aleph_1, \aleph_0)$	$\mathfrak{c}$	$\mathrm{cf}(\mathfrak{c})$	$\mathrm{cf}\left(2^{\aleph_1}\right)$	$\mathfrak{sd}(\aleph_2, \aleph_1)$	$\mathfrak{sd}(\aleph_1, \aleph_1)$	$\mathfrak{sd}(\aleph_0, \aleph_1)$
$\mathfrak{sd}(\aleph_1, \aleph_0)$		Yes	Yes	Yes	Yes	Yes	Yes
$\mathfrak{c}$	No		No	Yes	Yes	Yes	Yes
$\mathrm{cf}(\mathfrak{c})$	No	Yes		Yes	Yes	Yes	Yes
$\mathrm{cf}\left(2^{\aleph_1}\right)$	?	Yes	Yes		No	Yes	Yes
$\mathfrak{sd}(\aleph_2, \aleph_1)$	?	Yes	Yes	Yes		Yes	Yes
$\mathfrak{sd}(\aleph_1, \aleph_1)$	No	Yes	Yes	Yes	No		Yes
$\mathfrak{sd}(\aleph_0, \aleph_1)$	No	No	No	?	No	No	

Figure 5.6: Known consistencies regarding $\mathfrak{sd}(\delta, \kappa)$ with $\kappa = \aleph_0$ or $\aleph_1$. A box in row A and column B is labelled "Yes" if it is known to be consistent with ZFC that $|A| < |B|$, and it is labelled "No" if it is known that ZFC proves $|B| \leqslant |A|$. What has been filled out thus far is obtained from our example models, inspecting Figures 5.4 and 5.5.

Note that in [Cic+93], where NSNF was originally defined, the phrase "NSNF" was used to mean what we call "NSNF(ω, ω)", while we have no analogue for what is referred to as "NSNF(ω)" in that paper.

Proposition 5.5.11 ([Cic+93, Lemma 4.5])**.** *Let $\mathbb{B}$ be a complete Boolean algebra. The following are equivalent:*

1. $\mathbb{B}$ *has the* NSNF(ω, ω) *property.*
2. *For all $\langle u_n \mid n < \omega \rangle \in \mathbb{B}^\omega$, if $\prod_{n<\omega} \sum_{k>n} u_k = \mathbb{1}$ then there is a decomposition $u_n = u_n^0 \vee u_n^1$ for each $n < \omega$, where $u_n^0 \wedge u_n^1 = \mathbb{0}$, such that $\prod_{g \in 2^\omega} \sum_{n<\omega} u_n^{g(n)} = \mathbb{1}$.*

Proof. We shall only prove the *only if* direction, as the other is similar.

Let $\dot{X} = \{\langle u_n, \check{n} \rangle \mid n < \omega\}$, and note that the condition $\prod_{n<\omega} \sum_{k>n} u_k = \mathbb{1}$ is precisely saying that $\mathbb{1} \Vdash |\dot{X}| = \check{\omega}$. By NSNF$(\omega, \omega)$, there is a name $\dot{x}$ for a function $\dot{X} \to \check{2}$ such that for all $y \in 2^\omega$, $\mathbb{1} \Vdash (\exists n < \omega) \dot{x}(n) \neq \check{y}(n)$. For $i < 2$, let $u_n^i = \| \dot{x}(\check{n}) = \check{i} \|$, that is $u_n^i = \sum \{u \in \mathbb{B} \mid u \Vdash \dot{x}(\check{n}) = \check{i}\}$, and note that in this case $u_n = u_n^0 \vee u_n^1$ and $u_n^0 \wedge u_n^1 = \mathbb{0}$. Given $y \in 2^\omega$, the condition $\mathbb{1} \Vdash (\exists n < \omega) \dot{x}(n) \neq \check{y}(n)$ is precisely saying that $\sum_{n<\omega} u_n^{1-y(i)} = \mathbb{1}$. Therefore, letting $g = 1 - y$, $\sum_{n<\omega} u_n^{g(n)} = \mathbb{1}$ as well. Since y (and therefore g) was arbitrary, we indeed recover that $\prod_{g \in 2^\omega} \sum_{n<\omega} u_n^{g(n)} = \mathbb{1}$. □

The proof of Proposition 5.5.11 can quite easily extend to NSNF(δ, κ) for any $\delta \leqslant \kappa$ if we can guarantee that $\mathbb{B}$ will not collapse δ or add new sets to $[\kappa]^{<\delta}$ (for example, if $\mathbb{B}$ is δ-distributive).

Proposition 5.5.12. *Suppose that $\mathbb{B}$ is a δ-distributive complete Boolean algebra. Then $\mathbb{B}$ has the* NSNF(δ,κ) *property if and only if for all sequences* $\langle u_\alpha \mid \alpha < \kappa\rangle \in \mathbb{B}^\kappa$, *if* $\prod_{A\in[\kappa]^{<\delta}} \sum_{\alpha\notin A} u_\alpha = \mathbb{1}$, *then there is a decomposition* $u_\alpha = u_\alpha^0 \vee u_\alpha^1$ *for all* $\alpha < \kappa$, *where* $u_\alpha^0 \wedge u_\alpha^1 = \mathbb{0}$, *such that* $\prod_{g\in 2^\kappa} \sum_{\alpha<\kappa} u_\alpha^{g(\alpha)} = \mathbb{1}$.

With the technology of NSNF, we can automate some arguments for decreasing $\mathfrak{sd}(\delta,\kappa)$.

Proposition 5.5.13. *Let* $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leqslant \gamma, \beta < \gamma\rangle$ *be a* κ^+*-final iteration such that:*

1. *For all* $\alpha < \gamma$ *there is* $\alpha' \in (\alpha,\gamma)$ *such that* $\mathbb{1}_\alpha \Vdash$ *"*$\mathbb{P}_{\alpha'}/\alpha$ *has the* NSNF(δ,κ) *property"; and*
2. *for all* $\beta < \gamma$ *there is* $\beta' \in (\beta,\gamma)$ *such that* $\mathbb{1}_\beta \Vdash$ *"*$\mathbb{P}_{\beta'}/\beta$ *adds a new function* $\delta \to 2$*".*

Then $\mathbb{1}_\gamma \Vdash \mathfrak{sd}(\delta^+,\kappa) \leqslant \mathrm{cf}(\gamma)$.

Proof. Let G be V-generic for $\mathbb{P}_\gamma$. We shall first show that for all $\alpha < \gamma$,

$$\mathcal{I}_\mathfrak{s}(\delta^+,\kappa)^{V[G\restriction\alpha]} \subseteq \mathcal{I}_\mathfrak{s}(\delta^+,\kappa)^{V[G]}.$$

Suppose otherwise, so there is $F \in \mathcal{I}_\mathfrak{s}(\delta^+,\kappa)^{V[G\restriction\alpha]}$ such that $F \notin \mathcal{I}_\mathfrak{s}(\delta^+,\kappa)^{V[G]}$ and hence there is $X \in ([\kappa]^{(\delta)})^{V[G]}$ such that $F\restriction X = (2^X)^{V[G]}$. Since $\mathbb{P}_\gamma$ is κ^+-final there is $\beta < \gamma$ such that $X \in V[G\restriction\beta]$. Therefore,

$$\left(2^X\right)^{V[G]} = F\restriction X \subseteq \left(2^X\right)^{V[G\restriction\beta]} \subseteq \left(2^X\right)^{V[G]},$$

and thus $(2^X)^{V[G\restriction\beta]} = (2^X)^{V[G]}$. Since the order type of X is δ we similarly get that $(2^\delta)^{V[G\restriction\beta]} = (2^\delta)^{V[G]}$. However, this contradicts our assumption.

By assumption, for all $\alpha < \gamma$ there is $\alpha' < \gamma$ such that $(2^\kappa)^{V[G\restriction\alpha]}$ is an element of $\mathcal{I}_\mathfrak{s}(\delta^+,\kappa)^{V[G\restriction\alpha']}$, and so $(2^\kappa)^{V[G\restriction\alpha]} \in \mathcal{I}_\mathfrak{s}(\delta^+,\kappa)^{V[G]}$. By κ^+-finality, letting $\mathcal{C} \subseteq \gamma$ be a cofinal sequence in γ we have that $\{(2^\kappa)^{V[G\restriction\alpha]} \mid \alpha \in \mathcal{C}\}$ is a covering of $(2^\kappa)^{V[G]}$ by elements of $\mathcal{I}_\mathfrak{s}(\delta^+,\kappa)^{V[G]}$ of cardinality $\mathrm{cf}(\gamma)$ as required. □

From the perspective of this result, we can view Proposition 5.5.8 as a corollary that uses the κ^+-finality of $\mathrm{Add}(\chi,\lambda+\kappa^+)$ and the NSNF(δ^+,κ) property inherent to $\mathrm{Add}(\chi,1)$ for appropriate values of χ, κ, λ, and δ.

5.5.4 Highly distributive forcing

We finally briefly consider two instances of forcing that change very little about $\mathfrak{sd}(\delta,\kappa)$ by having a high level of distributivity.

Lemma 5.5.14. *$(2^\kappa)^+$-distributive forcings do not change $\mathfrak{sd}(\delta,\kappa)$.*

Proof. Let $\mathbb{P}$ be $(2^\kappa)^+$-distributive. Then no new subsets of 2^κ nor any subsets of κ are added by $\mathbb{P}$, and so $\mathrm{d}_{\mathfrak{s}}(F)$ is unchanged by $\mathbb{P}$ for all F. Finally, any covering of 2^κ by elements of $\mathfrak{sd}(\delta,\kappa)$ will be of cardinality at most 2^κ and so no new sequences of this form may be added by $\mathbb{P}$. □

Lemma 5.5.15. *Suppose that $[\kappa]^{<\delta} < \chi$ and $\mathbb{P}$ is χ-distributive. Setting $\eta = \mathfrak{sd}(\delta,\kappa)$ in the ground model, $\mathbb{P}$ forces that $\min\{\chi,\eta\} \leqslant \mathfrak{sd}(\delta,\kappa) \leqslant \eta$.*

In particular, if $\chi \leqslant \eta$ then $\mathbb{P}$ does not change $\mathfrak{sd}(\delta,\kappa)$, and if $\mathbb{P}$ collapses η to χ then $\mathbb{P}$ forces that $\mathfrak{sd}(\delta,\kappa) = \chi$.

Proof. Since $\chi > [\kappa]^{<\delta} \geqslant \kappa$, we have that $\mathbb{P}$ does not add new elements to 2^κ. Furthermore, $\delta < \chi$, so δ is not collapsed and $[\kappa]^{<\delta}$ is similarly unchanged. Hence, if $\mathcal{C} \subseteq \mathcal{I}_{\mathfrak{s}}(\delta,\kappa)$ is a covering of 2^κ in the ground model, it is still a covering of 2^κ by elements of $\mathcal{I}_{\mathfrak{s}}(\delta,\kappa)$ in the forcing extension. Therefore $\mathbb{P}$ cannot increase $\mathfrak{sd}(\delta,\kappa)$.

Let G be V-generic for $\mathbb{P}$, and suppose that $\mathcal{C} = \{F_\alpha \mid \alpha < \gamma\} \subseteq \mathcal{I}_{\mathfrak{s}}(\delta,\kappa)^{V[G]}$ is a covering of 2^κ, where $\gamma < \chi$. Then it suffices to prove that $\gamma \geqslant \eta = \mathfrak{sd}(\delta,\kappa)^V$. For $\alpha < \gamma$ and $X \in [\kappa]^{\mathrm{d}_{\mathfrak{s}}(F_\alpha)}$, let $h(\alpha,X) \in 2^X$ be such that $h(\alpha,X) \notin F_\alpha \upharpoonright X$. Then h is a sequence of length at most $\gamma \times [\kappa]^{<\delta} < \chi$, and so $h \in V$. For $\alpha < \gamma$, let

$$F'_\alpha = \left\{x \in (2^\kappa)^V \;\middle|\; \left(\forall X \in [\kappa]^{\mathrm{d}_{\mathfrak{s}}(F_\alpha)}\right) x \upharpoonright X \neq h(\alpha,X)\right\} \supseteq F_\alpha \cap V.$$

Then $\{F'_\alpha \mid \alpha < \gamma\} \in V$ and is a covering of 2^κ by elements of $\mathcal{I}_{\mathfrak{s}}(\delta,\kappa)$ (witnessed by h). Hence, $\gamma \geqslant \mathfrak{sd}(\delta,\kappa)^V$ as required. □

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