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Regularisation by multiplicative noise for reaction-diffusion equations with distributional drift

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Abstract

In this thesis we deal with regularisation by noise phenomena in partial differential equations caused by multiplicative noises. In particular, we study the solvability of stochastic reaction-diffusion equations in the case when the “drift” or “reaction term” is so irregular that it is not even a function, but merely a distribution in the Schwartz-sense. We provide the definition of solution, which is a priori not well-understood. We use Malliavin calculus to obtain estimates related to the density of the solution of the driftless equation, such as quantitative bounds on all Malliavin derivatives of the solution, and a quantitative nondegeneracy result for the first Malliavin derivative. We use these results to prove an “integration by parts” result for computing expectations of functions of the driftless equation. We also show the Lipschitz continuity of the Malliavin derivatives of the driftless equation in the initial condition, and quantify how well the driftless equation approximates the general case with (possibly distributional) drift. We combine the aforementioned results with state-of-the-art stochastic sewing techniques to prove a well-posedness result for the stochastic reaction-diffusion equation with distributional drift, derive stability estimates, and establish the temporal and spatial regularity of the solution. Moreover, we provide bounds on the Hölder norms of the density of solution of the driftless equation, and all of its derivatives.

Intellectual Property

The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

Signed

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Chapter 1

Introduction

1.1 What is an SPDE?

A stochastic partial differential equation (SPDE) is a partial differential equation that is perturbed by some random fluctuations. The simplest and most studied SPDE is the stochastic heat equation

$$(\partial_t - \Delta)v = \xi. \tag{1.1.1}$$

Here $\Delta := \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ denotes the Laplacian in $d \in \mathbb{N}$ dimensions and ξ is a random noise term that represents a random heat source. The solution of the equation (and generally, that of any SPDE) is an infinite dimensional stochastic process. Below we give some other examples of applications of SPDEs. We will use the notation $\nabla := (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$ for the del operator in $d \geq 2$ dimensions, and by convention for $d = 1$ we set $\nabla := \frac{\partial}{\partial x}$.

- The research area of stochastic fluid dynamics is centered around the study of turbulent fluids, which may be modelled by the stochastic Burger's equation

  $$(\partial_t - \Delta)v = v\nabla v + \xi,$$

  or the stochastic Navier-Stokes equation

  $$(\partial_t + v\nabla - \Delta)v + \nabla p = \xi, \qquad \operatorname{div}(v) = 0,$$

  which (just as its deterministic counterpart) remains an area of active research. For a detailed overview of the topic we point the reader to [BF20]

- An important model for surface growth from physics is Kardar-Parisi-Zhang (KPZ) equation

$$(\partial_t - \Delta)h = (\nabla h)^2 + \xi.$$

  The KPZ equation describes the fluctuations of a wide a range of stochastic models, called the KPZ universality class (see [HQ18]).

- The dynamical ϕ_3^4 model is the SPDE

$$(\partial_t - \Delta)\phi = C\phi - \phi^3 + \xi,$$

  which appears in Quantum Field Theory and in the theory of phase transitions (see [Hai16]).

- In the Heath–Jarrow–Morton framework, the price $B(t, T)$ at time t of a zero-coupon bond (paying one unit of currency at the maturity time T) is modelled by

$$B(t, T) = \exp\Big(- \int_0^{T-t} f(t, \theta) d\theta \Big)$$

  where the forward rate $f(t, \cdot)$ is some infinite dimensional stochastic process. In [Con05] the model

$$f(t, \theta) = r(t) + s(t)(Y(\theta) + X_t(\theta))$$

  is established, where the stochastic processes r, s are jointly Markovian, Y is a deterministic shape function, and the deformation map X solves a parabolic SPDE

$$(\partial_t - \partial_\theta - \partial_\theta^2)X_t(\theta) = b_{t,\theta}(X_t(\theta)) + \sigma(X_t(\theta))\xi.$$

- The flow of an ideal gas through a porous medium in the presence of random fluctuations can be described by the stochastic porous media equation (see e.g. [BDPR16])

$$\partial_t v = \Delta(|v|^{m-1}) + \xi,$$

  which is a generalisation of the stochastic heat equation.

Now suppose that instead of modelling the diffusion of heat, we want to model the diffusion of a chemical in a gel. Then we may add a "reaction term" or "drift" $b(v)$ to the stochastic heat equation

(1.1.1), to obtain the stochastic reaction-diffusion equation

$$(\partial_t - \Delta)v = b(v) + \xi. \tag{1.1.2}$$

Note that the noise in above equation is *additive*, i.e. we simply added the noise to the equation. However, in applications (such as the aforementioned one in chemistry) it is often desirable that the magnitude of the noise depends on the state of the solution. Then the following *multiplicative* model is more appropriate (see [Hai21]):

$$(\partial_t - \Delta)v = b(v) + \sigma(v)\xi.$$

This equation is the main topic of this thesis. If the drift b is sufficiently regular, then the existence and uniqueness of solutions is well-known. In the present work we will study the case of irregular (and in fact distributional) drift b.

The question is: how do we formalise the notion of solution for SPDEs? For illustration, we will construct the solution to the additive stochastic heat equation (1.1.1). To this end, consider the partial differential equation (PDE)

$$(\partial_t - \Delta)u(t,x) = F(t,x), \qquad u_0 = 0 \tag{1.1.3}$$

for $(t,x) \in [0,1] \times \mathbb{R}$. In order to construct the solution we recall that the *heat kernel on* $\mathbb{R}$ is given for $(t,x) \in [0,1] \times \mathbb{R}$ by

$$p_t^{\mathbb{R}}(x) := \frac{1}{\sqrt{4\pi t}} \exp\Big(- \frac{|x|^2}{4t}\Big),$$

and for $x, y \in \mathbb{R}$ and $t \in [0,1]$ we denote $p_t^{\mathbb{R}}(x,y) := p_t^{\mathbb{R}}(x-y)$. By Duhamel's formula it is known that the solution to (1.1.3) is given by

$$u(t,x) = \int_0^t \int_{\mathbb{R}} p_{t-r}^{\mathbb{R}}(x,y) F(t,x) dy dr. \tag{1.1.4}$$

Let moreover $W := \{W(t,x) : (t,x) \in [0,1] \times \mathbb{R}\}$ be a *Brownian sheet*, i.e. a centered Gaussian process with covariance function given for $(t,x), (s,y) \in [0,1] \times \mathbb{R}$ by

$$\mathbb{E}\big(W(t,x)W(s,y)\big) = (t \wedge s)(x \wedge y).$$

Although $W(t,x)$ is not differentiable in the t and x variables, we may consider the object

$$\xi(t,x) := \frac{\partial^2 W(t,x)}{\partial t \partial x} \tag{1.1.5}$$

in the generalised sense, which we call "space-time white noise". Informally, we can view ξ as a centered Gaussian process with covariace function

$$\mathbb{E}(\xi(t,x)\xi(s,y)) = \delta(t-s)\delta(x-y)$$

where δ denotes the Dirac-delta. Suppose that we wish to define the solution of (1.1.3) for $F(t,x) := \xi(t,x)$. The main idea is that even though the object $\xi(t,x)$ is not defined pointwise, we still may give meaning to the integral

$$\int_0^t \int_{\mathbb{R}} p_{t-r}^{\mathbb{R}}(x,y) \frac{\partial^2 W(r,y)}{\partial r \partial y} dy dr, \tag{1.1.6}$$

i.e. the right-hand-side of the Duhamel formula (1.1.4), which will allow us to construct the solution to (1.1.3). The idea of constructing a new concept of a stochastic integral to solve the equation is analogous to what we do in the finite dimensional case, where to make sense out of the SDE

$$\frac{dX_t}{dt} = b(X_t) + \sigma(X_t)\frac{dW_t}{dt},$$

we construct the Itô integral $\int_0^1 (\dots) dW_t : L_2(\Omega \times [0,1]) \to L_2(\Omega)$ which allows us to express the solution as

$$X_t = X_0 + \int_0^t b(X_r)dr + \int_0^t \sigma(X_r)dW_r.$$

We proceed with rigorously defining the space-time white noise (1.1.5).

Definition 1.1.1 (Space-time white noise)**.** Let $(H, \langle \cdot, \cdot \rangle_H)$ be a separable Hilbert space. A Gaussian process $\xi := \{\xi(h) : h \in H\}$ is called space-time white noise if

1. $\mathbb{E}\xi(h) = 0$ for all $h \in H$
2. $\mathbb{E}(\xi(h_1)\xi(h_2)) = \langle h_1, h_2 \rangle_H$.

Example 1.1.2. Choose $H = L_2([0,1])$, and for $h \in H$ let $\xi(h) := \int_0^1 h(r) dW_r$. Then for all $h \in H$ we have $\mathbb{E}\xi(h) = 0$, and for $h_1, h_2 \in H$ we have

$$\mathbb{E}(\xi(h_1)\xi(h_2)) = \mathbb{E}\Big(\int_0^1 h_1(r)dW_r \int_0^1 h_2(r')dW_{r'}\Big) = \int_0^1 h_1(r)h_2(r)dr = \langle h_1, h_2\rangle_H.$$

Hence ξ defines space-time white noise with respect to H.

Let $\mathscr{P}$ denote the predictable σ-algebra on $\Omega \times [0,1]$ generated by all left-continuous processes that are adapted to the filtration $(\mathscr{F}_t)_{t\in[0,1]}$ of W. For any metric space $\mathscr{M}$ let $\mathscr{B}(\mathscr{M})$ denote the collection of Borel subsets, and $\mathscr{B}_b(\mathscr{M})$ the collection of bounded Borel subsets. Let $f \in L_2(\Omega \times [0,1] \times \mathbb{R})$ such that $f : \Omega \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R})$-measurable. We proceed with discussing the meaning of the stochastic integral with respect to the space-time white noise ξ corresponding to the separable Hilbert space $H = L_2([0,1] \times \mathbb{R})$. Take the space-time white noise ξ and for $(t, A) \in [0,1] \times \mathscr{B}_b(\mathbb{R})$ define

$$M_t(A) := \xi(\mathbf{1}_{[0,t]\times B}(\cdot)).$$

Then $\{M_t(A) : t \in [0,1], A \in \mathscr{B}_b(\mathbb{R})\}$ is an example of a *Martingale measure*. Roughly speaking, this means that for all $A \in \mathscr{B}_b(\mathbb{R})$ the process $(M_t(A))_{t\in[0,1]}$ is a martingale, and for any disjoint sets $A_1, A_2 \in \mathscr{B}_b(\mathbb{R})$ we have $M_t(A_1 \cup A_2) = M_t(A_1) + M_t(A_2)$, for the formal definition we direct the reader to [Bal18]. According to [Kry99] it was Itô who first considered integration with respect to martingale measures in [Itô51], and his approach to defining the integral with respect to the space-time white noise was popularised by Walsh in [Wal86]. The construction can be summarised as follows: For a simple random field that is defined for $(\omega, t, x) \in \Omega \times [0,1] \times \mathbb{R}$ by

$$f_t(x, \omega) = \mathbf{1}_{[a,b]\times A}(t, x)Y(\omega) \tag{1.1.7}$$

with $0 \le a \le b \le 1$, $A \in \mathscr{B}_b(\mathbb{R})$ and Y a bounded $\mathscr{F}_a$-measurable random variable, the stochastic integral is then defined by

$$\int_0^1 \int_{\mathbb{R}} f_r(y)\xi(dy, dr) := Y\Big(M_b(A) - M_a(A)\Big).$$

The definition is then extended to any $\mathscr{P} \otimes \mathscr{B}(\mathbb{R})$-measurable random field $f \in L_2(\Omega \times [0,1] \times \mathbb{R})$ by approximating f by a sequence of linear combinations of simple random fields of the form (1.1.7). A concise introduction to the above approach can be found in [Bal18]. Below we give a brief description of the analytic approach, which was introduced by Gyöngy and Krylov in [GK81]. The advantage of this approach is that it decomposes the stochastic integral with respect to the space-time noise into a countable sum of Itô integrals, whose properties we are already familiar with. By [Kry99, Section 8.2] the Brownian sheet may be written in the form

$$W(t,x) = \sum_{n\in\mathbb{N}} W_t^n \int_0^x h_n(y)dy$$

where $(h_n)_{n\in\mathbb{N}}$ forms an orthonormal basis for $L_2(\mathbb{R})$ and $(W_t)_{n\in\mathbb{N}}$ is a sequence of independent Wiener processes. Thus recalling the informal definition (1.1.5), in the sense of generalised derivatives we have

$$\xi(t,x) := \frac{\partial^2 W(t,x)}{\partial t \partial x} = \sum_{n\in\mathbb{N}} \frac{dW_t^n}{dt} h_n(x).$$

Heuristically we may perform the following symbolic computation using the above:

$$\begin{aligned}\int_0^1 \int_{\mathbb{R}} f(r,y)\xi(r,y)dydr &= \int_0^1 \int_{\mathbb{R}} f(r,y) \frac{\partial^2 W(r,y)}{\partial y \partial r} dydr \\ &= \int_0^1 \int_{\mathbb{R}} f(r,y) \sum_{n\in\mathbb{N}} \frac{dW_r^n}{dr} h_n(y)dydr \\ &= \sum_{n\in\mathbb{N}} \int_0^1 \int_{\mathbb{R}} f(r,y)h_n(y)dydW_r^n.\end{aligned}$$

The last expression is rigorously defined, and we will use it as definition for the stochastic integral, i.e. we define

$$\int_0^1 \int_{\mathbb{R}} f(r,y)\xi(dy,dr) := \sum_{n\in\mathbb{N}} \int_0^1 \int_{\mathbb{R}} f(r,y)h_n(y)dydW_r^n.$$

From here on we will work on the periodic spatial domain $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (equivalently: the interval $[0,1]$ with the endpoints identified). Analogously to the definition above, for $g \in L_2(\Omega \times [0,1] \times \mathbb{T})$ such that $g : \Omega \times [0,1] \times \mathbb{T} \to \mathbb{R}$ is $\mathscr{P} \otimes \mathscr{B}(\mathbb{T})$-measurable, the stochastic integral is defined for $0 \le s \le t \le 1$ by

$$\int_s^t \int_{\mathbb{T}} g(r,y)\xi(dy,dr) := \sum_{n\in\mathbb{N}} \int_s^t \int_{\mathbb{T}} g(r,y)e_n(y)dydW_r^n$$

where $(e_n)_{n\in\mathbb{N}}$ forms an orthonormal basis for $L_2(\mathbb{T})$. From the expansion above it is clear that

$$\mathbb{E} \int_s^t \int_{\mathbb{T}} f(r,y)\xi(dy,dr) = 0.$$

It is also immediate to see that $(\int_0^t \int_{\mathbb{T}} f(r,y)\xi(dy,dr))_{t\in[0,1]}$ is adapted to the filtration of the white noise. The moments of the stochastic integral can be estimated using the following version of the Burkholder-Davis-Gundhy inequality (see e.g. [LR17]) which states that for $p \in (0, \infty)$ there exists a constant C_p such that for a sequence of adapted processes $(g_n)_{n\in\mathbb{N}}$ and for $T \in [0, 1]$ we have

$$\mathbb{E} \sup_{t\in[0,T]} \Big|\sum_{n\in\mathbb{N}} \int_0^t g_n(r)dW_r^n\Big|^p \leq C_p \mathbb{E}\Big(\int_0^T \sum_{n\in\mathbb{N}} |g_n(r)|^2 dr\Big)^{p/2}.$$

It follows from the above inequality (see e.g. [CHN21]) that for $p \geq 2$ we have

$$\Big\|\int_0^t \int_{\mathbb{T}} f(r, y)\xi(dy, dr)\Big\|_{L_p(\Omega)} \lesssim \Big(\int_0^t \int_{\mathbb{T}} \|f(r, y)\|^2_{L_p(\Omega)} dydr\Big)^{1/2}.$$

In the case where f is deterministic, the distribution of the stochastic integral is explicitly known:

$$\int_0^t \int_{\mathbb{T}} f(r, y)\xi(dy, dr) \sim \mathcal{N}\Big(0, \int_0^t \int_{\mathbb{T}} |f(r, y)|^2 dydr\Big).$$

Example 1.1.3. Let $v_0 \in \mathbb{B}$ and consider the stochastic heat equation

$$(\partial_t - \Delta)v = \xi, \quad v(0, \cdot) = v_0.$$

The solution (the concept of which we will later define rigorously) is given by

$$v(t, x) = P_t v_0(x) + \int_0^t \int_{\mathbb{T}} p_{t-r}(x, y)\xi(dy, dr).$$

with $P_t v_0(x) := \int_{\mathbb{T}} p_t(x, y)v_0(y)dy$, where p_t denotes the periodic heat kernel (see (1.3.10)). Since the integrand of the stochastic integral above is deterministic, we know that

$$v(t, x) \sim \mathcal{N}\Big(P_t v_0(x), \int_0^t \int_{\mathbb{T}} |p_{t-r}(x, y)|^2 dydr\Big).$$

1.2 Notation

Let $H := L_2([0, 1] \times \mathbb{T})$. Let $\xi := \{\xi(h) : h \in H\}$ be space-time white noise on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and suppose that $\mathscr{F}$ is generated by ξ. Let $(\mathscr{F}_t)_{t\in[0,1]}$ be the filtration generated by ξ and augmented by the σ-algebra $\mathcal{N}$ generated by all $\mathbb{P}$-null sets, that is

$$\mathscr{F}_t := \sigma\big(\{\xi(\mathbf{1}_{[0,r)\times B}) : r \in [0, t], B \in \mathscr{B}(\mathbb{T})\}\big) \vee \mathcal{N}$$

where for two σ-algebras $\mathcal{X}, \mathcal{Y}$ we denote $\mathcal{X} \vee \mathcal{Y} := \sigma(\mathcal{X} \cup \mathcal{Y})$. The predictable σ-algebra on $\Omega \times [0,1]$ is denoted by $\mathscr{P}$. The conditional expectation given $\mathscr{F}_t$ is denoted by $\mathbb{E}^t := \mathbb{E}(\cdot|\mathscr{F}_t)$. We use L_p as a shorthand for $L_p(\Omega)$. For a sub-σ-algebra $\mathscr{G} \subset \mathscr{F}$, the conditional L_p-norm is denoted by

$$\| \cdot \|_{L_p|\mathscr{G}} := (\mathbb{E}(| \cdot |^p | \mathscr{G}))^{1/p},$$

and for $p \in [1, \infty)$, $q \in [1, \infty]$ we denote

$$\| \cdot \|_{L^{\mathscr{G}}_{p,q}} := \| \| \cdot \|_{L_p|\mathscr{G}} \|_{L_q}. \tag{1.2.8}$$

Let $A \subseteq \mathbb{T}^d$ and $(B, |\cdot|)$ be a normed space. We denote by $\mathbb{B}(A, B)$ the collection of measurable functions $f : A \to B$ such that

$$\|f\|_{\mathbb{B}(A,B)} := \sup_{x \in A} |f(x)| < \infty.$$

We denote space of continuous functions $f : A \to B$ by $C(A, B)$, and it is also canonically equipped with the $\mathbb{B}$-norm. For $\alpha \in \mathbb{N}$ we denote by $C^\alpha(A, B)$ the space of continuous functions $f : A \to B$ such that for all multi-indices $l = (l_1, \ldots, l_d) \in (\mathbb{Z}_{\geq 0})^d$ with $|l| := \sum_{i=1}^d l_i \leq \alpha$ the derivative $\partial^l f$ is continuous, and

$$\|f\|_{C^\alpha(A,B)} := \sum_{|l| \leq \alpha} \|\partial^l f\|_{\mathbb{B}} < \infty.$$

By convention the above sum includes the term $\|\partial^{(0,\ldots,0)} f\|_{\mathbb{B}}$, where we define $\partial^{(0,\ldots,0)} f := f$. For $\alpha \in (0, 1)$ and $f : A \to B$, the α-Hölder seminorm of f is given by

$$[f]_{C^\alpha(A,B)} := \sup_{\substack{x,y \in A \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha},$$

where the norm in the denominator denotes the l_2 distance in d-dimensions. For $\alpha \in (1, \infty) \setminus \mathbb{Z}$ we then denote by $C^\alpha(A, B)$ the space of all functions such that for all multi-indices $l \in (\mathbb{Z}_{\geq 0})^d$ with $|l| < \alpha$, the derivative $\partial^l f$ exists, and

$$\|f\|_{C^\alpha(A,B)} := \|f\|_{C^{\lfloor \alpha \rfloor}(A,B)} + \sum_{\alpha - 1 \leq |l| < \alpha} [\partial^l f]_{C^{\alpha - |l|}(A,B)} < \infty.$$

The collection of smooth (i.e. infinitely differentiable) and bounded functions with bounded derivatives will be denoted by

$$C^\infty(A,B) := \bigcap_{n=0}^{\infty} C^n(A,B).$$

When no ambiguity can arise, we will simply abbreviate $C^\alpha(A)$ or C^α for $C^\alpha(A,B)$.

In the proofs of lemmas/theorems for two functions f, g we often write $f \lesssim g$ to mean that there exists a constant $N > 0$ such that $f \leq Ng$ and that N depends only on the parameters specified in the corresponding lemma/theorem.

1.3 Mild solutions

We proceed with formalising the concept of solution for the main object of study of the present thesis, i.e. *the multiplicative stochastic reaction diffusion equation* on $[0,1] \times \mathbb{T}$

$$(\partial_t - \Delta)u = b(u) + \sigma(u)\xi, \qquad u|_{t=0} = u_0 \tag{1.3.9}$$

with deterministic initial condition $u_0 \in C(\mathbb{T})$. We begin by defining the solution of (1.3.9) when b and σ are regular functions. The *periodic heat kernel* on $\mathbb{T}$ is defined for $t \in [0,1]$ and $x, y \in \mathbb{T}$ by

$$p_t(x,y) := \sum_{k\in\mathbb{Z}} p_t^{\mathbb{R}}(x - y + k), \tag{1.3.10}$$

and for $f : \mathbb{T} \to \mathbb{R}$ and $(t,x) \in (0,1] \times \mathbb{T}$ we denote $P_t f(x) := \int_{\mathbb{T}} p_t(x,y) f(y) dy$ and $P_0 f := f$. Similarly, the convolution of a map $g : \mathbb{R} \to \mathbb{R}$ with $p_t^{\mathbb{R}}$ is denoted by $P_t^{\mathbb{R}} g$, and we set $P_0^{\mathbb{R}} g := g$.

Definition 1.3.1 (Mild solution)**.** Let $u : \Omega \times [0,1] \times \mathbb{T} \to \mathbb{R}$ be a $\mathscr{P} \otimes \mathscr{B}(\mathbb{T})$-measurable random field such that $u(t,x)$ is continuous in $(t,x) \in [0,1] \times \mathbb{T}$. We say that u is a *mild solution* of (1.3.9) if for each $(t,x) \in [0,1] \times \mathbb{T}$ we have

$$u(t,x) = P_t u_0(x) + \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y) b(u(r,y)) dy dr + \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y) \sigma(u(r,y)) \xi(dy,dr)$$

almost surely.

Remark 1.3.2. Notice that for $\sigma = 0$ this is simply the deterministic reaction-diffusion equation. Moreover for $b = 0$ and for nonzero σ the above equation is the stochastic heat equation. If we set both coefficients to be zero, i.e, $b = \sigma = 0$, we get $u(t,x) = P_t u_0(x)$ which is the solution to the deterministic heat equation.

The well-posedness of (1.3.9) for the case of Lipschitz coefficients is a classic result (see [Wal86]):

Proposition 1.3.3. *Suppose that $b, \sigma \in C^1$. Then there exists a unique mild solution u to (1.3.9).*

The existence part of the lemma above is shown by proving that the Picard iteration scheme $(U_n)_{n\in\mathbb{Z}_{\geq 0}}$ given by

$$\begin{aligned} U_0(x) &= \int_{\mathbb{T}} p_t(x,y)u_0(y)dy, \\ U_{n+1}(t,x) &:= U_0(t,x) + \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y)b(U_n(r,y))dydr \\ &\quad + \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y)\sigma(U_n(r,y))\xi(dy,dr). \end{aligned}$$

converges to a solution (see e.g. [Wal86]). It is instructive to see at how uniqueness is proven, since the techniques used in the proof can be seen as a starting point for the proof of uniqueness for the non-Lipschitz case. We will need the following lemmas:

Lemma 1.3.4. *For every $\gamma \in (1,2]$ there exists a constant $N(\gamma) > 0$ such that for all $t \in [0,1]$,*

$$\int_0^t \int_{\mathbb{T}} p_{t-r}^{\gamma}(x,y)e^{-\lambda(t-r)}dydr \leq \frac{N}{\sqrt{\lambda}}.$$

Proof. The left-hand-side is controlled by

$$\begin{aligned} \int_0^t (t-r)^{-1/2(\gamma-1)}e^{-\lambda(t-r)}dydr \leq \int_0^t \frac{1}{\sqrt{t-r}}e^{-\lambda(t-r)}dr &= \frac{2}{\sqrt{\lambda}}\int_0^{\sqrt{\lambda t}} e^{-\theta^2}d\theta \\ &= \frac{\sqrt{\pi}}{\sqrt{\lambda}}\operatorname{erf}(\sqrt{\lambda t}) \leq \frac{\sqrt{\pi}}{\sqrt{\lambda}} \end{aligned}$$

where we used the change of variables $\theta := \sqrt{\lambda}(t-r)^{1/2}$ and the fact that $|\operatorname{erf}(\cdot)| \leq 1$. □

Lemma 1.3.5 (A Grönwall-type inequality)**.** *Fix $s \geq 0$. Let $C \in \mathbb{B}([s,1],\mathbb{R})$ be a non-decreasing function and let $f : [s,1] \times \mathbb{T} \to [0,\infty)$ be a bounded function. Suppose that there exists $\gamma \in (1,2]$ and $N_0 \geq 0$ such that for all $t \in [s,1]$ and $x \in \mathbb{T}$ we have*

$$f(t,x) \leq C(t) + N_0 \int_s^t \int_{\mathbb{T}} p_{t-r}^{\gamma}(x,y)f(r,y)dydr.$$

Then there exists a constant $N = N(\gamma, N_0)$ such that for all $t \in [s,1]$ we have

$$\sup_{x\in\mathbb{T}} f(t,x) \leq NC(t).$$

Proof. Let $\lambda > 0$ and consider the non-decreasing function of time $m : [s, 1] \to \mathbb{R}$, that is given by

$$m_t := \sup_{s \le r \le t} \sup_{x \in \mathbb{T}} \big(f(r, x) e^{-\lambda r} \big).$$

Then

$$f(t, x) \lesssim C(t) + \int_0^t \int_{\mathbb{T}} p^{\gamma}_{t-r}(x, y) m_r e^{\lambda r} dy dr,$$

where used the definition of m and the fact that $[s, t] \subset [0, t]$. Multiplying both sides by $e^{-\lambda t}$ and noting that $m_r \le m_t$ for $r \le t$ gives

$$f(t, x) e^{-\lambda t} \lesssim C(t) e^{-\lambda t} + m_t \int_0^t \int_{\mathbb{T}} p^{\gamma}_{t-r}(x, y) e^{-\lambda(t-r)} dy dr.$$

Let $T \in [s, 1]$. Using Lemma 1.3.4 to estimate the second term, and taking supremum over $(t, x) \in [s, T] \times \mathbb{T}$ we get

$$m_T \lesssim C(T) + \frac{m_T}{\sqrt{\lambda}}.$$

Choosing λ to be sufficiently large, we get that $m_T \lesssim C(T)$. Since, $T \in [s, 1]$ was arbitrary, the result follows by the definition of m. □

Now we are in position to prove the uniqueness of solutions to (1.3.9) for the case when $b, \sigma \in C^1$.

Proof of uniqueness in the Lipschitz case. Suppose that u_1, u_2 are two solutions. Then

$$\begin{aligned} u_1(t, x) - u_2(t, x) &= \int_0^t \int_{\mathbb{T}} p_{t-r}(x, y)(b(u_1(r, y)) - b(u_2(r, y))) dy dr \\ &\quad + \int_0^t \int_{\mathbb{T}} p_{t-r}(x, y)(\sigma(u_1(r, y)) - \sigma(u_2(r, y))) \xi(dy, dr). \end{aligned}$$

Therefore by the Minkowski inequality, the Burkholder-Davis-Gundhy inequality and the Hölder inequality and by the regularity of b and σ:

$$\begin{aligned} \|u_1(t, x) - u_2(t, x)\|_{L_p} &\lesssim \|b\|_{C^1} \int_0^1 \int_{\mathbb{T}} p_{t-r}(x, y) \|u_1(r, y) - u_2(r, y)\|_{L_p} dy dr \\ &\quad + \|\sigma\|_{C^1} \Big(\int_0^t \int_{\mathbb{T}} p^2_{t-r}(x, y) \|u_1(r, y) - u_2(r, y)\|^2_{L_p} dy dr \Big)^{1/2} \\ &\lesssim \Big(\int_0^t \int_{\mathbb{T}} p^2_{t-r}(x, y) \|u_1(r, y) - u_2(r, y)\|^2_{L_p} dy dr \Big)^{1/2}. \end{aligned}$$

Therefore

$$\|u_1(t,x) - u_2(t,x)\|_{L_p}^2 \lesssim \int_0^t \int_{\mathbb{T}} p_{t-r}^2(x,y)\|u_1(r,y) - u_2(r,y)\|_{L_p}^2 dydr.$$

Hence by Lemma 1.3.5 we have that

$$\sup_{(t,x)\in[0,1]\times\mathbb{T}} \|u_1(t,x) - u_2(t,x)\|_{L_p} = 0$$

and thus for all $(t,x) \in [0,1] \times \mathbb{T}$ we have $u_1(t,x) = u_2(t,x)$ almost surely, hence uniqueness holds. □

The regularity of the solution in the Lipschitz case is also classical. To state the result we introduce the following notation: For a random field $f : \Omega \times [0,1] \times \mathbb{T} \to \mathbb{R}$ and for $\gamma_1, \gamma_2 \in [0,1]$ we say that f is of class $C^{\gamma_1,\gamma_2}([0,1] \times \mathbb{T}, L_p)$ if

$$\|f\|_{C^{\gamma_1,\gamma_2}([0,1]\times\mathbb{T},L_p)} := \sup_{(t,x)\in[0,1]\times\mathbb{T}} \|f(t,x)\|_{L_p} + \sup_{0\le s<t\le 1} \sup_{x,y\in\mathbb{T}} \frac{\|f(t,x) - f(s,y)\|_{L_p}}{|t-s|^{\gamma_1} + |x-y|^{\gamma_2}} < \infty.$$

Proposition 1.3.6. *Suppose that $b, \sigma \in C^1$ and that u solves the* (1.3.9). *Then for all $p \ge 1$ and for any $\varepsilon \in (0, \frac{1}{2})$ we have*

$$u - P.u_0 \in C^{1/4-\varepsilon/2,1/2-\varepsilon}([0,1] \times \mathbb{T}, L_p),$$

so in particular if $u_0 \in C^{1/2-\varepsilon}(\mathbb{T}, \mathbb{R})$ then $u \in C^{1/4-\varepsilon/2,1/2-\varepsilon}([0,1] \times \mathbb{T}, L_p)$.

In the present thesis it will also be shown that this remains true for a much larger class of drifts b (provided that σ is sufficiently smooth and nondegenerate).

1.4 Besov spaces and negative Hölder spaces

The aim of this section is to introduce Besov spaces, to extend Hölder spaces to negative exponents, and to highlight the connection between the two types of spaces. In order to define Besov spaces, first we recall some terminology. An *annulus* in $\mathbb{R}^d$ (with $d \in \mathbb{N}$) is a set of the form $\{x \in \mathbb{R}^d : a < |x| < b\}$ for some $0 < a \le b \le 1$. A function $f : \mathbb{R}^d \to \mathbb{R}$ is called *radial* if $f(x) = f(y)$ for any $x, y \in \mathbb{R}^d$ such that $|x| = |y|$. Let

$$\mathcal{S} := \left\{ f \in C^\infty(\mathbb{R}^d, \mathbb{R}) : \forall a, b \in \mathbb{N}^d, \sup_{x\in\mathbb{R}^d} \left| \prod_{i=1}^d x_i^{a_i} \partial_1^{b_1} \dots \partial_d^{b_d} f(x) \right| < \infty \right\}$$

denote the *Schwartz space of rapidly decreasing functions.* A linear continuous map $\mathcal{S} \to \mathbb{R}$ is called a *tempered distribution*. The space of tempered distributions is denoted by $\mathcal{S}'$.

Definition 1.4.1 (dyadic partition of unity)**.** A sequence of compactly supported infinitely differentiable radial functions $(\eta_j)_{j=-1}^{\infty}$ is called a *dyadic partition of unity* if the following hold:

- $\text{supp}(\eta_{-1})$ is a closed ball (with respect to the l_2-norm on $\mathbb{R}^d$) centered at the origin.
- $\text{supp}(\eta_0)$ is the closure of an annulus.
- $\eta_j(x) = \eta_0(2^{-j}x)$ for all $x \in \mathbb{R}^d$ and for $j \in \mathbb{Z}_{\geq 0}$.
- $\sum_{j=-1}^{\infty} \eta_j(x) = 1$ for all $x \in \mathbb{R}^d$.
- $\sum_{j=-1}^{\infty} |\eta_j(x)|^2 \in [\frac{1}{2}, 1]$ for $x \in \mathbb{R}^d$.
- For any $i, j \in \mathbb{Z}_{\geq -1}$, if $|i - j| \geq 2$ then $\text{supp}(\eta_i) \cap \text{supp}(\eta_j) = \emptyset$.

We say that a compactly supported infinitely differentiable radial function $\eta : \mathbb{R}^d \to [0, 1]$ *generates a dyadic partition of unity* if there exists a partition of unity $(\eta_j)_{j=-1}^{\infty}$ such that $\eta_0 = \eta$.

It is known that there exists a compactly supported infinitely differentiable radial function $\eta : \mathbb{R}^d \to [0, 1]$ that generates a partition of unity $(\eta_j)_{j=-1}^{\infty}$ (see e.g. [BCD11]). Define the *Paley-Littlewood blocks*

$$\Delta_j f := \mathcal{F}^{-1}\Big(\eta_j \mathcal{F}(f)\Big)$$

where $\mathcal{F}$ denotes the Fourier transform, i.e. $\mathcal{F}(f)(x) := \int_{\mathbb{R}^d} f(y)e^{-i2\pi x \cdot y}dy$ and the inverse transform $\mathcal{F}^{-1}$ is given by $\mathcal{F}^{-1}(f) = \int_{\mathbb{R}^d} f(y)e^{i2\pi x \cdot y}dy$.

Definition 1.4.2 (Besov-space)**.** Let $s \in \mathbb{R}$ and $p, q \in [1, \infty]$ We define $\| \cdot \|_{\mathcal{B}^s_{p,q}} : \mathcal{S}' \to [0, \infty]$ for $q < \infty$ by

$$\|f\|_{\mathcal{B}^s_{p,q}} := \Big(\sum_{j=-1}^{\infty} \Big(2^{js} \|\Delta_j f\|_{L_p} \Big)^q \Big)^{1/q}$$

and for $q = \infty$ by

$$\|f\|_{\mathcal{B}^s_{p,\infty}} := \sup_{j \geq -1} 2^{js} \|\Delta_j f\|_{L_p}.$$

We say that a tempered distribution $f \in \mathcal{S}'$ belongs in the Besov space $\mathcal{B}^s_{p,q}$, if $\|f\|_{\mathcal{B}^s_{p,q}} < \infty$.

It turns out that the space $\mathscr{B}^s_{p,q}$ does not depend on our choice of generator η for the dyadic partition of unity. To indicate the domain and codomain, we may use the notation $\mathscr{B}(\mathbb{R}^d, \mathbb{R})$. The following result can be found e.g. in [ABLM24]

Proposition 1.4.3. *Let $\alpha \in (0, \infty) \setminus \mathbb{Z}$. Then*

$$C^\alpha(\mathbb{R}^d, \mathbb{R}) = \mathscr{B}^\alpha_{\infty,\infty}(\mathbb{R}^d, \mathbb{R}).$$

We extend the notion of Hölder-spaces for negative exponents as follows: For $\alpha < 0$ we say that a tempered distribution f is of class $C^\alpha(\mathbb{R}, \mathbb{R})$, if

$$\|f\|_{C^\alpha(\mathbb{R},\mathbb{R})} := \sup_{\varepsilon \in (0,1]} \varepsilon^{-\alpha/2} \|P^{\mathbb{R}}_\varepsilon f\|_{\mathbb{B}(\mathbb{R},\mathbb{R})} < \infty.$$

The following result is taken from [DGL23], and we will use it to relate Besov and Hölder spaces to each other for the case of negative regularity.

Proposition 1.4.4. *Let $\gamma \in \mathbb{R} \setminus \mathbb{Z}$. There exists a constant $N = N(\gamma)$ such that for all $f \in C^\gamma(\mathbb{R})$ we have*

$$\|(1 - \Delta)^{-1} f\|_{C^{\gamma+2}(\mathbb{R})} \le N \|f\|_{C^\gamma(\mathbb{R})}.$$

While for negative exponents the Besov and Hölder spaces might not coincide, they are still essentially the same. To be rigorous, the following relation holds between the two types of spaces:

Lemma 1.4.5. *Let $\alpha \in (-\infty, 0) \setminus \mathbb{Z}$. For any $\varepsilon > 0$ we have*

$$\mathscr{B}^{\alpha+\varepsilon}_{\infty,\infty}(\mathbb{R}, \mathbb{R}) \subset C^\alpha(\mathbb{R}, \mathbb{R}) \subset \mathscr{B}^\alpha_{\infty,\infty}(\mathbb{R}, \mathbb{R})$$

Proof. We begin with proving that $\mathscr{B}^{\alpha+\varepsilon}_{\infty,\infty} \subset C^\alpha$. To this end let $f \in \mathscr{B}^{\alpha+\varepsilon}_{\infty,\infty}$. by [Per20, inequality (3.4)] for $\beta \in \mathbb{R}$ and $\gamma \ge 0$ we have

$$\|P^{\mathbb{R}}_t f\|_{\mathscr{B}^{\beta+\gamma}_{\infty,\infty}} \lesssim t^{-\gamma/2} \|f\|_{\mathscr{B}^\beta_{\infty,\infty}}.$$

Choosing $\beta := \alpha + \varepsilon$ and $\gamma := -\alpha$ (which indeed satisfies $\gamma \ge 0$), we get that

$$\|P^{\mathbb{R}}_t f\|_{\mathscr{B}^\varepsilon_{\infty,\infty}} \lesssim t^{\alpha/2} \|f\|_{\mathscr{B}^{\alpha+\varepsilon}_{\infty,\infty}}.$$

Using Proposition 1.4.3 and the inequality above, we have

$$\|P_t^{\mathbb{R}} f\|_{\mathbb{B}} \le \|P_t^{\mathbb{R}} f\|_{C^\varepsilon} \lesssim \|P_t^{\mathbb{R}} f\|_{\mathscr{B}^{\varepsilon}_{\infty,\infty}} \lesssim t^{\alpha/2} \|f\|_{\mathscr{B}^{\alpha+\varepsilon}_{\infty,\infty}}.$$

Hence multiplying both sides by $t^{-\alpha/2}$, it follows that

$$t^{-\alpha/2} \|P_t^{\mathbb{R}} f\|_{\mathbb{B}} \lesssim \|f\|_{\mathscr{B}^{\alpha+\varepsilon}_{\infty,\infty}}.$$

Now taking the supremum of both sides over $t \in (0,1]$, by the definition of the C^α-norm we get that

$$\|f\|_{C^\alpha} \lesssim \|f\|_{\mathscr{B}^{\alpha+\varepsilon}_{\infty,\infty}}.$$

The above inequality shows that any $f \in \mathscr{B}^{\alpha+\varepsilon}_{\infty,\infty}$ must be also in C^α. Hence it is proven that

$$\mathscr{B}^{\alpha+\varepsilon}_{\infty,\infty} \subset C^\alpha. \tag{1.4.11}$$

We proceed with proving that $C^\alpha \subset \mathscr{B}^{\alpha}_{\infty,\infty}$. It suffices to show that for all $n \in \mathbb{Z}_{\ge 0}$ we have for $\alpha \in (-2n, \infty) \setminus \mathbb{Z}$ that $C^\alpha \subset \mathscr{B}^{\alpha}_{\infty,\infty}$. The statement is known to be true in initial case $n = 0$ by Proposition 1.4.3. Suppose that the statement holds for some $n \in \mathbb{Z}_{\ge 0}$. We aim to show that it also holds for $n + 1$. To this end let $\alpha \in (-2(n+1), \infty) \setminus \mathbb{Z}$ and $g \in C^\alpha$. Define

$$G := \int_0^\infty e^{-t} (P_t^{\mathbb{R}} g) dt = (1 - \Delta)^{-1} g \tag{1.4.12}$$

where the second equality is stated e.g. in [DGL23, Proof of Lemma 2.4]. Then by Proposition 1.4.4 we have

$$G \in C^{\alpha+2}.$$

But $\alpha + 2 \in (-2n, \infty) \setminus \mathbb{Z}$, and thus by the induction hypothesis we have $C^{\alpha+2} \subset \mathscr{B}^{\alpha+2}_{\infty,\infty}$ and thus it follows that

$$G \in \mathscr{B}^{\alpha+2}_{\infty,\infty}. \tag{1.4.13}$$

Hence using that by (1.4.12) we have $(1 - \Delta)G = g$, the triangle inequality, [JIP23, Lemma 1], and

(1.4.13), we get that

$$\|g\|_{\mathscr{B}^{\alpha}_{\infty,\infty}} \leq \|G\|_{\mathscr{B}^{\alpha}_{\infty,\infty}} + \|\Delta G\|_{\mathscr{B}^{\alpha}_{\infty,\infty}} \lesssim \|G\|_{\mathscr{B}^{\alpha+2}_{\infty,\infty}} < \infty,$$

and thus $g \in \mathscr{B}^{\alpha}_{\infty,\infty}$. We have shown that if $g \in C^{\alpha}$, then we must have $g \in \mathscr{B}^{\alpha}_{\infty,\infty}$. This proves that

$$C^{\alpha} \subset \mathscr{B}^{\alpha}_{\infty,\infty}. \tag{1.4.14}$$

Recalling that we assumed that $\alpha \in (-2(n+1), \infty) \setminus \mathbb{Z}$ the induction argument is complete, and thus (1.4.14) holds for any $\alpha \in (-\infty, 0) \setminus \mathbb{Z}$. Hence the proof is finished. □

For $\alpha > -1$ we denote the completion of C^{∞} in the norm $\|\cdot\|_{C^{\alpha}}$ by $C^{\alpha+}$.

Remark 1.4.6. For all $\varepsilon > 0$ we have the inclusions $C^{\alpha+\varepsilon} \subset C^{\alpha+} \subset C^{\alpha}$.

1.5 Introduction to regularisation by noise

Recall that an equation is called *"well-posed"* if for any initial condition there exists a unique solution. If a problem is not well-posed, we call it *"ill-posed"*. While *"regularisation by noise"* is not a rigorously defined term, we give the following informal definition (see e.g. [BDG21]): The phenomenon when the presence of a random forcing makes an ill-posed problem well-posed.

The simplest example is the following. Let $b : \mathbb{R} \to \mathbb{R}$ be a function, $x_0 \in \mathbb{R}$, and consider the ordinary differential equation (ODE)

$$dX_t = b(X_t)dt, \qquad X_0 = x_0.$$

If b is not Lipschitz, then the solution might not be unique, and if b is not continuous then the solution might not exist at all. However perturbing the equation with (possibly multiplicative) Brownian noise, we obtain the stochastic differential equation (SDE)

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = x_0, \tag{1.5.15}$$

which is strongly well-posed even if b is merely bounded and measurable, provided that $\sigma \in C^2$ such that $\sigma \geq \mu$ with some constant $\mu > 0$ (see [Ver80]). The origins of regularisation by noise can be traced back to Zvonkin's seminal work [Zvo74], where he constructed a map $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ (which became known as the Zvonkin transform) such that for each $t \in [0,1]$ the map $f(t,\cdot)$ is a bijection and $f(t,X_t)$ solves an SDE with no drift, and using this he proved the well-posedness of (1.5.15) in 1 dimension for bounded and measurable drift b. In [Por75] Portenko proved the existence of solutions to (1.5.15) in higher dimensions for the case when the drift is the sum of a bounded and an L_p-integrable function. In [Ver80] Veretennikov has used the Zvonkin transform to provide a well-posedness result in higher dimensions for bounded and measurable b . For the case of additive noise, path-by-path uniqueness has been proven by Davie in [Dav07], which means that for almost all Brownian paths W_t there is a unique X_t that solves the equation. In [KR05] Krylov and Röckner show the strong well-posedness of the SDE

$$dX_t = b(t,X_t)dt + dW_t$$

under the condition that $b \in L_q([0,1], L_p(\mathbb{R}^d))$ with $p \geq 2, q > 2$ such that $\frac{2}{q} + \frac{d}{p} < 1$.

While the phenomenon of regularisation by noise may seem strange, there is actually an intuitively clear explanation: the noise pushes the solution out of the points of the domain where b is poorly behaved. Consequentially, the solution will not "get stuck" in a problematic part of the domain. For this, however it is needed that the noise is present, which is why we had to impose the condition that σ is bounded away from zero. Below we will show some examples of regularisation by noise in finite and infinite dimensions.

Example 1.5.1 (Regularisation by noise repairing the uniqueness of an ODE)**.** Since the map $x \mapsto \sqrt{|x|}$ is not Lipschitz-continuous, uniqueness fails for the ODE

$$dX_t = \sqrt{|X_t|}dt, \quad X_0 = 0, \quad t \in [0,1].$$

Indeed, for any $c \in [0,1]$, the function

$$X_t = \frac{(t-c)^2}{4}\mathbf{1}_{(c,\infty)}(t)$$

is a solution, since for $t \in (c,1]$ we have

$$\frac{dX_t}{dt} = \frac{d}{dt}\Big(\frac{(t-c)^2}{4}\Big) = \frac{1}{2}(t-c) = \sqrt{\frac{1}{4}(t-c)^2} = \sqrt{|X_t|}$$

and for $t \in [0,c]$ we have

$$\frac{dX_t}{dt} = \frac{d}{dt}0 = 0 = \sqrt{|X_t|}.$$

However, the equation that is obtained by perturbing our ODE with additive Brownian noise, i.e. the SDE

$$dX_t = \sqrt{|X_t|}dt + dW_t$$

admits a unique strong solution in any spatial domain of the form $[-n, n]$ with $n \in \mathbb{N}$ (until the solution hits the boundary), since on a compact domain $x \mapsto \sqrt{|x|}$ is bounded and measurable (and in fact of $C^{1/2}$ regularity).

The following example is taken from [AG01]

Example 1.5.2 (Regularisation by noise repairing the uniqueness of a PDE)**.** Consider the deterministic reaction-diffusion equation

$$\begin{aligned}(\partial_t - \Delta)u(t,x) = 2\sqrt{\sin(\pi x)u(t,x)} + \pi^2 u(t,x) \quad & \forall (t,x) \in [0,\infty) \times [0,1],\\ u(t,0) = u(t,1) = 0 \quad & \forall t \in [0,\infty),\\ u(0,x) = 0 \quad & \forall x \in [0,1].\end{aligned}$$

Uniqueness fails, since both $u(t,x) = 0$ and $u(t,x) = t^2\sin(\pi x)$ solve the equation. However the addition of a Brownian noise term guarantees the existence of a unique mild solution

Example 1.5.3 (Regularisation by noise repairing the existence of solutions of an ODE and a PDE)**.** Recall that the sign function is given by $\text{sgn}(x) := -\mathbf{1}_{(-\infty,0)}(t) + \mathbf{1}_{(0,\infty)}(t)$ and thus define a square wave by

$$b(x) := \text{sgn}(\sin(x)).$$

Since b is discontinuous, the ODE $dX_t = b(X_t)dt$ does not have a solution. However since b is bounded and measurable, the SDE

$$dX_t = b(X_t)dt + dW_t, \qquad X_0 = 0$$

admits a unique strong solution. Similarly even though the PDE $(\partial_t - \Delta)u = b(u)$ does not have a solution, the SPDE

$$(\partial_t - \Delta)u = b(u) + \xi, \qquad u(0,\cdot) = 0$$

admits a unique mild solution.

If $b \in C^\alpha$ with $\alpha \geq 0$ then it is clear how to define the solution of the SDE (1.5.15), as we can rewrite it as the integral equation

$$X_t = x_0 + \int_0^t b(X_r)dr + \int_0^t \sigma(X_r)dW_r.$$

However if $b \in C^\alpha$ with $\alpha \in (-1, 0)$ then the composition $b(X_t)$ is not defined, since b is not a pointwise defined object, but merely a generalised function in the Schwartz-sense. Hence the usual concept of solution breaks down. To overcome this difficulty, we could say that (X_t) is a *regularised solution* of the above SDE if there is a drift term D_t such that

1. For any sequence $(b^n)_{n\in\mathbb{N}} \subset C^\infty$ such that $b^n \to b$ in C^α we have that

$$\sup_{t\in[0,1]} \Big| \int_0^t b^n(X_t)dt - D_t \Big| \to 0$$

in probability as $n \to \infty$.

2. For all $t \in [0, 1]$ we have

$$X_t = x_0 + D_t + \int_0^t \sigma(X_t)dW_t.$$

The above approach originates from the paper [BC01] by Bass and Chen, where they show the strong well-posedness of the SDE (1.5.15) for $b \in \mathscr{B}^\alpha_{\infty,\infty}$ with $\alpha > -1/2$. The way that Bass and Chen define the solution for distributional drift was generalised to SPDEs in [ABLM24]. The definition that we will use in the present work to characterise the solution of (1.3.9) is consistent with the definition used in [ABLM24].

Remark 1.5.4. For this approach to work we need such a sequence (b^n) to exist, in other words we need $b \in C^{\alpha+}$. However due to the chain of embeddings in Remark 1.4.6, it is equivalent to prove well-posedness for $b \in C^\alpha$ for all $\alpha \in (-1, 0)$ and for $b \in C^{\alpha+}$ for all $\alpha \in (-1, 0)$.

1.6 The literature

In the field of stochastic partial differential equations (SPDEs), the first results on regularisation by noise can be traced back to the works of Gyöngy and Pardoux [GP93a], [GP93b]. Therein, the authors consider SPDEs of the form

$$(\partial_t - \Delta)u = b(u) + \xi, \qquad u|_{t=0} = u_0, \tag{1.6.16}$$

which corresponds to (1.3.9) with $\sigma = 1$. It is well known that the deterministic counterpart of (1.6.16) admits a unique solution provided that b is a Lipschitz continuous function. Without Lipschitz regularity, solutions may not exists or may not be identified uniquely. The situation changes in the presence of noise. It is shown in [GP93a] and [GP93b] that (1.6.16) admits a unique strong solution provided that b is merely the sum of a bounded measurable function and an L_p-integrable function with some $p \geq 2$. Similar results were obtained for SPDEs in an abstract Hilbert-space framework with bounded and measurable drift in [DPFPR13]. In [BM19], Butkovsky and Mytnik show when b is bounded and measurable, path-by-path uniqueness also holds for (1.6.16). For such drift, discrete approximation schemes for the solution of (1.6.16) have been established with an optimal rate in [BDG23], quantifying earlier results from [Gyö98, Gyö99].

Notice that in all the previous results, b is quite irregular, nevertheless it is a function. The first well-posedness result which accommodates distributional drift b is due to Athreya, Butkovsky, Mytnik, and Lê in [ABLM24]. In such case, the composition $b(u)$ is not well-defined a priori and solutions to (1.6.16) are defined in a regularised sense. They show in [ABLM24] that (1.6.16) admits a unique probabilistically strong solution provided that b belongs to the Besov space $\mathscr{B}^{\alpha}_{q,\infty}$ with $\alpha - 1/q \geq -1$, $\alpha > -1$, and $q \in [1, \infty]$. Such Besov space includes bounded measurable functions, L_1-integrable functions, as well as Radon measures. To obtain such results, [ABLM24] establishes Lipschitz regularity for some related singular integrals using the stochastic sewing lemma introduced in [Lê20]. The regularity threshold -1 is in agreement with the finite dimensional analogue [CG16] where it is shown that any SDE driven by additive fractional Brownian motion with Hurst parameter $H \in (0, 1)$ has a unique solution provided that the drift belongs to the Besov–Hölder space $\mathscr{B}^{\alpha}_{\infty,\infty}$ with $\alpha > 1 - \frac{1}{2H}$. The two results are related by setting $H = 1/4$, which is the temporal regularity of the random field solution of (1.6.16) with $b = 0$. Quantitative convergence of discrete approximation schemes under the assumptions of [ABLM24] is also considered by Goudenége, Haress and Richard in [GHR24], extending [BDG23].

All of the aforementioned results concern the additive noise case. For the multiplicative case, much less is known. In [BGP94] the authors show that (1.3.9) has a unique solution when σ is regular and bounded away from 0, and the drift is measurable and satisfies the “one-sided linear growth condition” that $yb(y) \lesssim 1 + y^2$ for $y \in \mathbb{R}$. This was followed by the papers [Gyö95, AG01] where well-posedness is shown for the case of locally bounded/integrable drift respectively. The proofs from these references rely on the Girsanov theorem, L_p-estimates for the density of the driftless equation, and a comparison principle. In particular, their method also uses comparison between the equation and its driftless counterpart, and they use Malliavin calculus to derive estimates for the density $f_{t,x}$ of the solution ϕ of the drifless equation, building on the previous work [PZ93] of Pardoux and Zhang where they used Malliavin calculus to study continuity properties of the density. However the method of [BGP94] is based on the estimation of $\int_0^T \int_{\mathbb{T}} \int_{\mathbb{R}} |f_{t,x}(y)|^p dydxdt$, and to this end they only need to estimate the first two Malliavin derivatives of ϕ. In comparison, in the present thesis we derive estimates for the Hölder norms of all derivatives $f_{t,x}^{(n)}$ of the density. To achieve this, we must derive estimates on all Malliavin derivatives of ϕ. An other improvement of the present thesis is the rate of blowup of the negative moments of the Malliavin matrix. In particular, while in [BGP94] the blowup is of order $t^{-1/2}$ for small times, in the present thesis we are able to get this down to $t^{-1/4}$.

The well-posedness result in this thesis is an analogue of [ABLM24] for the multiplicative noise case. Namely, we show existence and uniqueness for (1.3.9) when σ is regular and bounded away from 0, the drift belongs to the Besov–Hölder space $\mathscr{B}^{\alpha}_{q,\infty}$ with $\alpha > -1$ and $q = \infty$. For simplicity, we do not consider the case when $q < \infty$, which allows us to obtain qualitative stability results and highlight the essential elements of our approach. Similar to [ABLM24], our method also relies on the stochastic sewing lemma from [Lê20] which does not rely on Girsanov theorem nor comparison principles. Therefore, the techniques within could also be applied to equations driven by Lévy noise and to systems of equations. Compare with [ABLM24], while the probabilistic properties of the noise term in the additive case are explicitly understood, this is no longer the case for our multiplicative equation (1.3.9). Therefore, employing the sewing methods in the present thesis is more involved than [ABLM24]. In the sewing arguments in previous works, one approximates a solution using the integral form of the corresponding equation. This works quite well in the additive noise case, [CG16, ABLM24]. It also works quite well in some multiplicative noise cases if the noise is not too irregular, for example equations driven by fractional Brownian motion with Hurst parameter $H > 1/2$ (see [DG24]). However, for $H < 1/2$, this approach leads to suboptimal results. The same is true for the setting of the present thesis. With such an approach, one would only be able to obtain well-posedness when b has positive Hölder regularity (and in that case well-posedness is already known, see [GP93a] and [GP93b]). Instead, in order to cover the whole regime $b \in \mathscr{B}^{\alpha}_{\infty,\infty}$ with $\alpha > -1$, we come up with sewing arguments that employ the flow of the driftless equation. Consequently, we need (and obtain) some regularisation estimates related to the density of the solution to the driftless equation and its derivatives. These estimates are achieved via Malliavin calculus which demands a relatively high regularity from σ. This approach is not equation-specific but rather works as a general principle.

1.7 Formulation and the well-posedness result

We introduce our main assumptions and the concept of solution to (1.3.9) for the case of distributional drift.

Assumption 1.7.1. The function b is of class $C^{\alpha+}$ for some $\alpha \in (-1, 0)$ and the function σ is of class C^4. Moreover, there exists a positive constant μ such that

$$\sigma^2(x) \geq \mu^2 \quad \text{for all } x \in \mathbb{R}.$$

Finally, the initial condition $u_0 : \mathbb{T} \to \mathbb{R}$ is a bounded and continuous deterministic function.

Definition 1.7.2 (Regularised solution)**.** Let $u : \Omega \times [0, 1] \times \mathbb{T} \to \mathbb{R}$ be a $\mathscr{P} \otimes \mathscr{B}(\mathbb{T})$-measurable random field, such that $u(t, x)$ is continuous in $(t, x) \in [0, 1] \times \mathbb{T}$. We say that u is a *regularised solution* of (1.3.9) if there exists a $\mathscr{P} \otimes \mathscr{B}(\mathbb{T})$-measurable random field $D^u : \Omega \times [0, 1] \times \mathbb{T} \to \mathbb{R}$ such that

1. For any sequence $(b^n)_{n\in\mathbb{N}} \subset C^\infty$ such that $b^n \to b$ in C^α, we have that

$$\sup_{(t,x)\in[0,1]\times\mathbb{T}} \left| D^u_t(x) - \int_0^t \int_{\mathbb{T}} p_{t-r}(x, y) b^n(u(r, y)) dy dr \right| \longrightarrow 0 \qquad (1.7.17)$$

   in probability.

2. For each $(t, x) \in [0, 1] \times \mathbb{T}$,

$$u(t, x) = P_t u_0(x) + D^u_t(x) + \int_0^t \int_{\mathbb{T}} p_{t-r}(x, y) \sigma(u(r, y)) \xi(dy, dr) \qquad \text{a.s.} \qquad (1.7.18)$$

Remark 1.7.3. For a given regularised solution u, the random field D^u is uniquely characterised by relation (1.7.17). Furthermore, in the more regular setting when $\alpha \geq 0$, Definition 1.7.2 reduces to the standard notion of a mild solution. In such case by (1.7.17) one has $D^u_t(x) = \int_0^t \int_{\mathbb{T}} p_{t-r}(x, y) b(u(r, y)) dy dr$.

For $(S, T) \in [0, 1]^2$ such that $S \leq T$, let us define the simplices

$$[S, T]^2_{\leq} := \{(s, t) \in [S, T]^2 : s \leq t\} \text{ and } [S, T]^2_{<} := \{(s, t) \in [S, T]^2 : s < t\}.$$

To describe the regularity of the solutions, we introduce the following spaces of random fields.

Definition 1.7.4 (The spaces $\mathscr{V}^\beta_p$, $\mathscr{U}^\beta_p$ and $\mathscr{U}^\beta$)**.** Let $\beta \in [0, 1]$ and $p \in [1, \infty)$. We denote by $\mathscr{V}^\beta_p[0, 1]$ the collection of all $\mathscr{P} \otimes \mathscr{B}(\mathbb{T})$-measurable functions $f : \Omega \times [0, 1] \times \mathbb{T} \to \mathbb{R}$ such that $f \in \mathbb{B}([0, 1] \times \mathbb{T}, L_p)$

and

$$[f]_{\mathscr{V}_p^\gamma[0,1]} := \sup_{x\in\mathbb{T}} \sup_{(s,t)\in[0,1]^2_<} \frac{\|f_t(x) - P_{t-s}f_s(x)\|_{L_{p,\infty}^{\mathscr{F}_s}}}{|t-s|^\gamma} < \infty.$$

For $(S,T) \in [0,1]^2_\leq$, the space $\mathscr{V}_p^\beta[S,T]$ and the corresponding seminorm are defined analogously. We denote by $\mathscr{U}_p^\beta$ the collection of all regularised solutions u of (1.3.9) such that $D^u \in \mathscr{V}_p^\beta[0,1]$. We moreover define

$$\mathscr{U}^\beta := \bigcap_{p=1}^{\infty} \mathscr{U}_p^\beta.$$

We are now in position to state our main theorem.

Theorem 1.7.5 (Well-posedness)**.** *Let Assumption 1.7.1 hold. There exists a regularised solution u to (1.3.9) in the class $\mathscr{U}^{1+\alpha/4}$. Moreover if v is another solution of (1.3.9) in the class $\mathscr{U}_2^\beta$ for some $\beta \geq \frac{1}{2} - \frac{\alpha}{4}$, then $u(t,x) = v(t,x)$ almost surely for all $(t,x) \in [0,1] \times \mathbb{T}$.*

1.8 Overview of methods of proofs

The bulk of the proofs relies on moment estimates for singular integrals which are typically of the form

$$I := \int_0^1 \int_{\mathbb{T}} h(y) f(u(r,y)) dy dr$$

where h is an integrable function, u is a solution to (1.3.9) and f is a distribution with negative Hölder regularity. An effective tool to estimate moments of I, which emerges from [Lê20], is the stochastic sewing lemma. Heuristically, the lemma decomposes I corresponding to partitions of the time interval $[0,1]$ with vanishing mesh size. More precisely, let π be a partition of $[0,1]$, then one writes

$$I = \sum_{[s,t]\in\pi} \int_s^t \int_{\mathbb{T}} h(y) f(u(r,y)) dy dr.$$

On each subinterval $[s,t]$, we approximate the random variable $\int_s^t \int_{\mathbb{T}} (\ldots) dy dr$ by its conditional expectation given $\mathscr{F}_s$, i.e. $\mathbb{E}^s \int_s^t \int_{\mathbb{T}} (\ldots) dy dr$. Because the conditional law of $u(r,y)$ given $\mathscr{F}_s$ is unknown a priori, we further approximate $u(r,y)$ by a random variable, denoted by $\psi^s(r,y)$. There are two desirable properties for these approximations. First, one must recover I when the mesh size of π vanishes, namely

$$I = \lim_{|\pi|\downarrow 0} \sum_{[s,t]\in\pi} \mathbb{E}^s \int_s^t \int_{\mathbb{T}} h(y) f(\psi^s(r,y)) dy dr.$$

Second, the conditional expectation $\mathbb{E}^s f(\psi^s(r,y))$ is well-defined and can be estimated so that for some $p \geq 2$ and $\varepsilon > 0$, one has

$$\|\mathbb{E}^s \int_s^t \int_{\mathbb{T}} h(y) f(\psi^s(r,y)) dy dr\|_{L_p(\Omega)} \lesssim (t-s)^{\frac{1}{2}+\varepsilon} \tag{1.8.19}$$

and

$$\|\mathbb{E}^s \int_a^t \int_{\mathbb{T}} h(y)[f(\psi^s(r,y)) - f(\psi^a(r,y)]\|_{L_p(\Omega)} \lesssim (t-s)^{1+\varepsilon} \tag{1.8.20}$$

for every $s \leq a \leq t$. Under these two properties, the stochastic sewing lemma can be applied, and it provides estimates for the p-th moment of I.

Let us explain how ψ^s is chosen. Relation (1.7.18) provides a natural decomposition of a solution as the sum of a nondegenerate noise and the drift, namely

$$u(t,x) = P_t u_0(x) + D_t^u(x) + V_t(x), \text{ where } V_t(x) = \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y)\sigma(u(r,y))\xi(dy,dr).$$

It follows that for each $s \leq t$,

$$u(t,x) = P_{t-s}u(s,\cdot)(x) + [D_t^u(x) - P_{t-s}D_s^u(x)] + [V_t(x) - P_{t-s}V_s(x)].$$

One could then choose to approximate $u(t,x)$ by the random variable

$$\psi^s(t,x) := P_{t-s}u(s,\cdot)(x) + [V_t(x) - P_{t-s}V_s(x)].$$

The error of this approximation can be quantified by the following estimate

$$\|u(t,x) - \psi^s(t,x)\|_{L_p(\Omega)} \lesssim |t-s|^\gamma \tag{1.8.21}$$

for every $s \leq t$ and for some $\gamma > 0$. The larger the value of γ is, the better the approximation is. We note that

$$[V_t(x) - P_{t-s}V_s(x)] = \int_s^t \int_{\mathbb{T}} p_{t-r}(x,y)\sigma(u(r,y))\xi(dy,dr).$$

In the additive case (i.e. when σ is a constant), $V_t(x) - P_{t-s}V_s(x)$ has a normal distribution and hence, the conditional expectation $\mathbb{E}^s f(\psi^s(t,x))$ can be evaluated precisely. The stochastic sewing method described above can be applied (i.e. achieving (1.8.19) and (1.8.20)) under some suitable regularity assumptions on f and that $\gamma > 1/2 - \alpha/4 \approx 3/4$ for $\alpha \approx -1$ (recall that $-1 < \alpha < 0$ is the regularity of the drift). This is the approach from [ABLM24].

Going toward the multiplicative noise case, one might hope that a similar argument would work. Notice that in this case, the distribution of $V_t(x) - P_{t-s}V_s(x)$ conditionally on $\mathscr{F}_s$ is not known a priori. A naive way to circumvent this issue is to consider

$$\psi^s(t,x) := P_{t-s}u(s,\cdot)(x) + \int_s^t \int_{\mathbb{T}} p_{t-r}(x,y)\sigma(u(s,y))\xi(dy,dr), \tag{1.8.22}$$

which is obtained by freezing the solution in the integrand at time s. In this way, conditionally on $\mathscr{F}_s$, $\psi^s(t,x)$ once again has a normal distribution, which allows for concrete analysis. However, one can not go far with this choice as it is immediate that

$$u(t,x) - \psi^s(t,x) = \int_s^t \int_{\mathbb{T}} p_{t-r}(x,y)\big(\sigma(u(r,y)) - \sigma(u(s,y))\big)\xi(dy,dr),$$

whose moments are (expectedly) of order $|t-s|^{1/2}$ (consisting of two contributions of the same order $1/4$ from the stochastic integral and from the temporal regularity of the solution). The exponent $1/2$ falls short of the required threshold $3/4$ which is necessary in the additive case. This makes the naive approximation (1.8.22) unsuitable for the sewing method under Assumption 1.7.1.

In order to resolve these issues, we introduce the following approximation

$$\psi^s(t,x) := \phi^{u(s,\cdot),s}(t,x), \tag{1.8.23}$$

where $\phi^{z,s}$ denotes the solution to the *driftless equation*

$$(\partial_t - \Delta)\phi^{z,s} = \sigma(\phi^{z,s})\xi, \quad \phi^{z,s}(s,\cdot) = z(\cdot).$$

Observe that when σ is a constant, (1.8.22) and (1.8.23) coincide, but otherwise they are generally different. Indeed, we show in Section 3.1 that the approximation (1.8.23) satisfies the estimate (1.8.21) with $\gamma = 1 + \alpha/4$ which is larger than $1/2 - \alpha/4$ as is required for the application of the sewing method. The distribution of $\psi^s(t,x)$ conditioned on $\mathscr{F}_s$ might not be as explicit as in the additive noise case but nevertheless, one can extract the information which is sufficient to verify (1.8.19) and (1.8.20). This essentially boils down to obtaining estimates related to the density of the solution of the driftless equation and its derivatives, which are achieved by tools from Malliavin calculus (see Section 2.3).

When comparing our method to the existing ones from the literature, we can draw some similarities as well as genuine differences. The works [BGP94, Gyö95, AG01] also utilise estimates on the density of the solution to the driftless equation, however, in a completely different way. In fact, these works use Girsanov theorem to extract relevant and useful a priori estimates for the solution to (1.3.9) from the solution of the driftless equation. Under our main assumption, the Girsanov theorem is not applicable which makes this argument obsolete. Additionally, our uniqueness argument relies on qualitative stability estimates, as opposed to comparison principles in the aforementioned works. Similarly to [ABLM24], we also use stochastic sewing method. However, while [ABLM24] relies on the approximation (1.8.22), we introduce and utilise the better approximation (1.8.23). To the best of the author's knowledge, this is the first time it has been used in the study of regularisation by noise phenomena by sewing methods. Furthermore, because the conditional law of ψ^s is not explicit, additional works have been carried out in order to apply the sewing method successfully.

The driftless equation also appears in [CD22] in the study of regularisation by multiplicative fractional noise for SDEs. In this work, the authors employ a transformation, which is based on the inverse of the flow generated by the driftless equation, to transform the original equation into an additive one. Comparing the results of [CD22] and [DG24] reveals that such transformation is quite demanding and does not lead to results which are in alignment with [CG16]. The connection between (1.3.9) and the driftless equation is well-known, perhaps since the Girsanov theorem. Another instance of such relation appears in [IS01] in a different context. Our work therefore exhibits a new connection between the two equations.

1.9 Stochastic sewing

We begin with introducing increment notation. Let $(S,T) \in [0,1]^2_<$. For any functions $\mathscr{A} : [S,T] \to \mathbb{R}$, $A : [S,T]^2_\leq \to \mathbb{R}$, for any $(s,t) \in [0,1]^2_\leq$ and $a \in [s,t]$, we define $\mathscr{A}_{s,t} := \mathscr{A}_t - \mathscr{A}_s$, and $\delta A_{s,a,t} := A_{s,t} - A_{s,a} - A_{a,t}$.

Before stating the stochastic sewing lemma, we first give a general idea about what it does. Suppose that we are given some $p \geq 2$, $\mathscr{A} : [0,1] \to L_p(\Omega)$ starting from zero, such that $\mathscr{A}_t$ is $\mathscr{F}_t$-measurable, and we want to bound the L_p-norm $\|\mathscr{A}_{s,t}\|_{L_p}$. The idea of stochastic sewing is the following: Instead of trying to bound $\|\mathscr{A}_{s,t}\|_{L_p}$ directly, we construct an object $A_{s,t}$ according to the following criteria:

1. $A_{s,t}$ has to be sufficiently "close" to $\mathscr{A}_{s,t}$ on any time interval $[s,t] \subset [0,1]$.

2. $A_{s,t}$ should be easier to approximate than $\mathscr{A}_{s,t}$.

A stochastic sewing lemma will then enable us to "sew together" the local bounds on $A_{s,t}$ on the small time intervals $[s,t]$ into a bound for $\mathscr{A}_{s,t}$. The most strightforward application is bounding the L_p-norm of

$$\mathscr{A}_{s,t} = \int_s^t f(X(r))dr$$

where $(X_r)_{r\in[0,1]}$ is some adapted stochastic process and f is some irregular function.

The first stochastic sewing lemma is introduced in [Lê20]. To best suit our purpose herein, we state a conditional version of the lemma which applies in settings with $L^{\mathscr{F}_s}_{q,p}$-norms (defined in (1.2.8)). This version is originated from the works [FHL24, ABLM24, Lê23], where the reader can find its proof.

Lemma 1.9.1 (Conditional stochastic sewing lemma)**.** *Let p,q satisfy $2 \le q \le p \le \infty$ with $q < \infty$. Let $(S,T) \in [0,1]^2_\le$ and let $A : [S,T]^2_\le \to L_p(\Omega)$ be a function such that for any $(s,t) \in [S,T]^2_\le$ the random vector $A_{s,t}$ is $\mathscr{F}_t$-measurable. Suppose that for some $\varepsilon_1, \varepsilon_2 > 0$ and C_1, C_2 the bounds*

$$\|A_{s,t}\|_{L^{\mathscr{F}_s}_{q,p}} \le C_1|t-s|^{1/2+\varepsilon_1}, \quad \|\mathbb{E}^s \delta A_{s,a,t}\|_{L_p} \le C_2|t-s|^{1+\varepsilon_2} \tag{1.9.24}$$

hold for all $S \le s \le a \le t \le T$. Then, there exists a unique map $\mathscr{A} : [S,T] \to L_p(\Omega)$ such that $\mathscr{A}_S = 0$, $\mathscr{A}_t$ is $\mathscr{F}_t$-measurable for all $t \in [S,T]$, and the following bounds hold for some constants $K_1, K_2 > 0$:

$$\|\mathscr{A}_{s,t} - A_{s,t}\|_{L^{\mathscr{F}_s}_{q,p}} \le K_1|t-s|^{1/2+\varepsilon_1} \tag{1.9.25}$$

$$\|\mathbb{E}^s(\mathscr{A}_{s,t} - A_{s,t})\|_{L_p} \le K_2|t-s|^{1+\varepsilon_2}. \tag{1.9.26}$$

Furthermore, there exists a constant K depending only on $\varepsilon_1, \varepsilon_2, d, p$ such that $\mathscr{A}$ satisfies the bound

$$\|\mathscr{A}_{s,t}\|_{L^{\mathscr{G}_s}_{q,p}} \le KC_1|t-s|^{1/2+\varepsilon_1} + KC_2|t-s|^{1+\varepsilon_2}$$

for all $(s,t) \in [S,T]^2_\le$.

We will call A *a germ of* the process $\mathscr{A}$. In practice, we mostly take $q = p$ (in which case, $L^{\mathscr{F}_s}_{q,p}$-norm and L_p-norm coincide) and $p = \infty$.

Example 1.9.2 (A simple example of a sewing argument)**.** Let f be a smooth function and W a Wiener process. Suppose that the want to show that $\| \int_s^t f'(W_r)dr\|_{L_p} \lesssim (t-s)^{1/2+\varepsilon/2}$ in such a way that the bound only depends on the C^ε-norm of f for some arbitrarily small $\varepsilon > 0$.

To this end we may define the germ to be the conditional expectation

$$A_{s,t} := \mathbb{E}^s \int_s^t f'(W_r)dr.$$

Then using that W_r can be decomposed to an $\mathscr{F}_s$-measurable term (W_s) and a term ($W_r - W_s$) that is independent from $\mathscr{F}_s$, we get that

$$A_{s,t} = \int_s^t \mathbb{E}^s f(W_r - W_s + W_s)dr = \int_s^t \mathbb{E}\Gamma(W_s)dr$$

with $\Gamma : \mathbb{R} \to \mathbb{R}$ given by

$$\Gamma(x) := \mathbb{E} f'(W_{r-s} + x).$$

But using integration by parts and the fact that $\int_{\mathbb{R}} p^{\mathbb{R}}_{r-s}(y)yf(x)dy = f(x)\mathbb{E}W_{r-s} = 0$, we can see that

$$\begin{aligned}\Gamma(x) &= \int_{\mathbb{R}} p^{\mathbb{R}}_{r-s}(y)f'(y+x)dy \\ &= -\int_{\mathbb{R}} p^{\mathbb{R}}_{r-s}(y)\frac{y}{r-s}f(y+x)dy \\ &= -\frac{1}{r-s}\int_{\mathbb{R}} p^{\mathbb{R}}_{r-s}(y)y(f(y+x) - f(x))dy.\end{aligned}$$

Therefore

$$\begin{aligned}\|\Gamma(x)\|_{L_p} &\lesssim (r-s)^{-1}\int_{\mathbb{R}} p^{\mathbb{R}}_{r-s}(y)|y|\|f\|_{C^\varepsilon}|y|^\varepsilon dy \\ &\lesssim \|f\|_{C^\varepsilon}(r-s)^{-1}\mathbb{E}|W_{r-s}|^{1+\varepsilon} \\ &\lesssim \|f\|_{C^\varepsilon}(r-s)^{-1}(r-s)^{(1+\varepsilon)/2} \\ &\lesssim \|f\|_{C^\varepsilon}(r-s)^{-1/2+\varepsilon/2}\end{aligned}$$

It follows by integrating the above bound that

$$\|A_{s,t}\|_{L_p} \le \|f\|_{C^\varepsilon}(t-s)^{1/2+\varepsilon/2}.$$

We also have

$$\mathbb{E}^s A_{s,a,t} = \mathbb{E}^s(A_{s,t} - A_{s,a} - A_{a,t}) = \mathbb{E}^s\Big(\int_s^t - \int_s^a - \int_a^t\Big)f'(W_r)dr = 0.$$

Note moreover that for $\mathscr{A}_t := \int_0^t f'(W_r)dr$ we have

$$\begin{aligned}\|A_{s,t} - \mathscr{A}_{s,t}\|_{L_p} = \Big\|\mathbb{E}^s \int_s^t f'(W_r)dr - \int_s^t f'(W_r)dr\Big\|_{L_p} &\lesssim \Big\|\int_s^t f'(W_r)dr\Big\|_{L_p} \\ &\lesssim \|f'\|_{\mathbb{B}}(t-s).\end{aligned}$$

(Note that according to the sewing lemma the $\|f'\|_{\mathbb{B}}$ in the last line will not appear in the final bound!) Finally, since $A_{s,t} = \mathbb{E}^s\mathscr{A}_{s,t}$, it follows that

$$\|\mathbb{E}^s(A_{s,t} - \mathscr{A}_{s,t})\|_{L_p} = 0.$$

Hence by the stochastic sewing lemma it follows that $\|\mathscr{A}_{s,t}\|_{L_p} \lesssim \|f\|_{C^\varepsilon}(t-s)^{1/2+\varepsilon/2}$, i.e.

$$\Big\|\int_s^t f'(W_r)dr\Big\|_{L_p} \lesssim \|f\|_{C^\varepsilon}(t-s)^{1/2+\varepsilon/2}.$$

1.10 Malliavin calculus

The Malliavin calculus was first developed and introduced by Paul Malliavin in the seminal work [Mal78]. It is an infinite dimensional differential calculus "with respect to" the white noise. In particular, it makes it possible to differentiate random fields with respect to the space-time white noise, and thus it extends the theory of Sobolev-spaces to the stochastic setting. A great introduction to the topic is [Hai21], and for more detail we recommend the references [Nua06] and [SS04].

We begin with a simple example to explain why and how we will use Malliavin calculus in this thesis. Let $(W_t)_{t\in[0,1]}$ be a Wiener process and f a function. An important tool in regularisation by noise is that using the regularising property of the noise, we can obtain a bound on $\mathbb{E}f'(W_t)$ in terms of $\|f\|_{\mathbb{B}}$. In particular, using integration by parts, we get that

$$\begin{aligned}\mathbb{E}f'(W_t) &= \int_{\mathbb{R}} p_t^{\mathbb{R}}(y)f'(y)dy \\ &= [p_t^{\mathbb{R}}(y)f(y)]_{y\to-\infty}^{y\to\infty} - \int_{\mathbb{R}} \frac{d}{dy}(p_t^{\mathbb{R}}(y))f(y)dy \\ &= \int_{\mathbb{R}} p_t^{\mathbb{R}}(y)\frac{y}{t}f(y)dy,\end{aligned}$$

and therefore using that $W_t \sim \mathcal{N}(0,t)$ and the Hölder inequality yields

$$|\mathbb{E}f'(W_t)| \leq \frac{\|f\|_{\mathbb{B}}}{t}\int_{\mathbb{R}} p_t^{\mathbb{R}}(y)|y|dy = \frac{\|f\|_{\mathbb{B}}}{t}\mathbb{E}|W_t| \lesssim \frac{\|f\|_{\mathbb{B}}}{t}\big(\mathbb{E}|W_t|^2\big)^{1/2} \lesssim \frac{\|f\|_{\mathbb{B}}}{\sqrt{t}}.$$

The above bound is an interesting result, since if instead of $\mathbb{E}f'(W_t)$ we would have $\mathbb{E}f'(x) = f'(x)$ for some deterministic value $x \in \mathbb{R}$, then obtaining such a bound would be impossible, and we would be forced to bound in terms of the (much stronger) C^1-norm of f.

In the above example we used that the density of $W_t \sim \mathcal{N}(0,t)$ is explicitly known. However, often we are in the situation that instead of W_t we are given a more general random variable X for which the density does not have a nice analytic form (e.g. when $X = u(t,x)$ is the solution of some SPDE). In this case we cannot use the above method. To overcome this issue, it is often still possible to use Malliavin calculus, which gives us the tools to construct a random variable G such that

$$\mathbb{E}f'(X) = \mathbb{E}\Big(f(X)G\Big),$$

which will allow us to get a bound of the form

$$|\mathbb{E}f'(X)| \leq \|f\|_{\mathbb{B}}\|G\|_{L_1}.$$

The theory extends to repeated integration by parts as well. Below we introduce the formalism of Malliavin calculus in infinite dimensions and state some key results.

Recall that we set $H := L_2([0,1]\times\mathbb{T})$. Let $\mathcal{W}$ denote the the space of *smooth and cylindrical random variables*, i.e. random variables of the form

$$F = f(\xi(h_1),\ldots,\xi(h_n))$$

for some $n \in \mathbb{N}$, $h_1,\ldots,h_n \in H$, and for some smooth f such that f and its partial derivatives of all orders have polynomial growth. The *Malliavin derivative* of such a random variable is given by

$$\mathscr{D}_{\theta,\zeta}F := \sum_{i=1}^{n} \partial_i f(\xi(h_1),\ldots,\xi(h_n))h_i(\theta,\zeta)$$

for all $(\theta,\zeta) \in [0,1]\times\mathbb{T}$ where ∂_i denotes partial derivative with respect to the i-th argument. For $k \in \mathbb{N}$ we say that a map $g : ([0,1]\times\mathbb{T})^k \to \mathbb{R}$ is of class $H^{\otimes k}$ if $\|g\|_{H^{\otimes k}} := (\int_{([0,1]\times\mathbb{T})^k} |g(\eta)|^2 d\eta)^{1/2} < \infty$, and we set $\|\cdot\|_{L_p(\Omega,H^{\otimes k})} := \|\|\cdot\|_{H^{\otimes k}}\|_{L_p}$. For all $k \in \mathbb{N}$, $p \geq 1$ the iterated Malliavin derivative is defined for $(\theta_1,\zeta_1),\ldots,(\theta_k,\zeta_k) \in [0,1]\times\mathbb{T}$ by $\mathscr{D}^k_{(\theta_1,\zeta_1),\ldots,(\theta_k,\zeta_k)} := \mathscr{D}_{\theta_1,\zeta_1}\ldots\mathscr{D}_{\theta_k,\zeta_k}$, and it is closable as an operator from $L_p(\Omega)$ into $L_p(\Omega;H^{\otimes k})$ (see [Hai21]). By convention, the 0-th Malliavin derivative is the identity map, and $H^{\otimes 0} := \mathbb{R}$. For $k \in \mathbb{Z}_{\geq 0}$ and $p \geq 1$, we denote by $\mathscr{W}^k_p$ the completion of $\mathscr{W}$ with respect to the norm

$$F \mapsto \|F\|_{\mathscr{W}_p^k} := \Big(\mathbb{E}|F|^p + \sum_{i=1}^k \mathbb{E}\|\mathscr{D}^i F\|_{H^{\otimes i}}^p\Big)^{1/p}.$$

We moreover use the notation

$$\mathscr{W}^k := \bigcap_{p\geq 1} \mathscr{W}_p^k.$$

On the class $\mathscr{W}_p^k$ one can also define the $\dot{\mathscr{W}}_p^k$-seminorm by

$$F \mapsto \|F\|_{\dot{\mathscr{W}}_p^k} := \|\|\mathscr{D}^k F\|_{H^{\otimes k}}\|_{L_p}.$$

By convention, we have

$$\|\cdot\|_{\mathscr{W}_p^0} = \|\cdot\|_{\dot{\mathscr{W}}_p^0} = \|\cdot\|_{L_p}.$$

Note that $\|\cdot\|_{\mathscr{W}_p^k}$ and $\sum_{i=0}^K \|\cdot\|_{\dot{\mathscr{W}}_p^i}$ are equivalent norms. The above definitions can be extended for the Hilbert-space valued case as follows. Let V be a separable Hilbert-space, and consider the family $\mathscr{W}(V)$ of random variables of the form

$$F = \sum_{i=1}^m F_i v_i$$

for some $F_1, \ldots, F_m \in \mathscr{W}$, and $v_1, \ldots, v_m \in V$. Recall that for sets A, B and maps $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ the tensor product $f \otimes g : A \times B \to \mathbb{R}$ is defined by $(f \otimes g)(x, y) = f(x)g(y)$. For $k \geq 1$, we define

$$\mathscr{D}^k F := \sum_{j=1}^m \mathscr{D}^k F_j \otimes v_j.$$

Then $\mathscr{D}^k$ is a closable operator from $L_p(\Omega; V)$ into $L_p(\Omega; H^{\otimes k} \otimes V)$ for any $p \geq 1$. We define the space $\mathscr{W}_p^k(V)$ as the completion $\mathscr{W}(V)$ with respect to the norm

$$F \mapsto \|F\|_{\mathscr{W}_p^k(V)} := \Big(\mathbb{E}\|F\|_V^p + \sum_{i=1}^k \mathbb{E}\|\mathscr{D}^i F\|_{H^{\otimes i}\otimes V}^p\Big)^{1/p}.$$

For a random variable $u \in L_2(\Omega; H)$ it is said that $u \in \mathrm{dom}(\delta)$, if there exists a constant $c > 0$ such that

$$\mathbb{E}\langle \mathscr{D}F, u\rangle_H \leq c\|F\|_{L_2}$$

for all $F \in \mathscr{W}_2^1$. If this holds, then $\delta(u)$ denotes the unique element of $L_2(\Omega)$ that satisfies

$$\mathbb{E}(F\delta(u)) = \mathbb{E}\langle \mathscr{D}F, u\rangle_H$$

for any $F \in \mathscr{W}_2^1$. The random variable $\delta(u)$ is called the *Skorokhod integral* (or the *divergence*) of u. If in addition u is adapted, then the Skorokhod integral coincides with the usual stochastic integral, that is for all $t \in [0,1]$ we have

$$\int_0^t \int_{\mathbb{T}} u(r,y)\xi(dy,dr) = \delta(u\mathbf{1}_{[0,t]}).$$

The following result follows from [Nua06, Proposition 2.1.4]

Proposition 1.10.1 (Malliavin integration by parts)**.** *Let $n \in \mathbb{N}$, $u, G_0 \in \mathscr{W}^n$ and let $f : \mathbb{R} \to \mathbb{R}$ be n times differentiable. Suppose moreover that for all $p \in [1,\infty)$, we have $\mathbb{E}\|\mathscr{D}u(t,x)\|_H^{-p} < \infty$. Define iterated Skorokhod integrals recursively for $k \in \{0,\dots,n-1\}$ by*

$$G_{k+1} := \delta\Big(\frac{\mathscr{D}u}{\|\mathscr{D}u\|_H^2} G_k\Big).$$

The following holds:

$$\mathbb{E}\big(\nabla^n f(u) G_0\big) = \mathbb{E}\big(f(u) G_n\big).$$

We also recall the combinatorial notation from [CHN21]. Let $n \in \mathbb{N}$.

- For $1 \le k \le n$, we denote by $\Lambda(n,k)$ the set of partitions of the integer n of length k, that is, if $\lambda \in \Lambda(n,k)$, then $\lambda \in \mathbb{N}^k$, and by writing $\lambda = (\lambda_1,\dots,\lambda_k)$, it satisfies

$$\lambda_1 \ge \cdots \ge \lambda_k \ge 1 \qquad \text{and} \qquad \sum_{i=1}^k \lambda_i = n.$$

- For $\lambda \in \Lambda(n,k)$, we let $\mathscr{P}(n,\lambda)$ be all partitions of n ordered objects $\{\theta_1,\dots,\theta_n\}$, with $\theta_1 \ge \cdots \ge \theta_n$ into k groups $\{\theta_1^1,\dots,\theta_{\lambda_1}^1\},\dots,\{\theta_1^k,\dots,\theta_{\lambda_k}^k\}$, such that within each group the elements are ordered, i.e. $\theta_1^j \ge \cdots \ge \theta_{\lambda_j}^j$ for $1 \le j \le k$. Note that $|\mathscr{P}(n,\lambda)| = \binom{n}{\lambda_1,\dots,\lambda_k} = \frac{n!}{\lambda_1!\dots\lambda_k!}$.

- For a generic element

$$\gamma := ((\theta_1,\zeta_1),\dots,(\theta_n,\zeta_n)) \in ([0,1]\times\mathbb{T})^n,$$

we will denote by $\hat{\gamma}_k$ the element of $([0,1]\times\mathbb{T})^{n-1}$ that is obtained by omitting the k-th entry of γ, i.e.

$$\hat{\gamma}_k := ((\theta_1,\zeta_1),\dots,(\theta_{k-1},\zeta_{k-1}),(\theta_{k+1},\zeta_{k+1}),\dots,(\theta_n,\zeta_n)). \tag{1.10.27}$$

We state some generic estimates on the Malliavin derivatives of functions of random variables which are needed in later sections. The proofs of these results rely purely on elementary principles, such as the chain rule.

Proposition 1.10.2 ([CHN21, Lemma 5.3]). *Suppose that $f \in C^n$ and $\phi \in \mathscr{W}^n$. Then, for almost all $\gamma = ((\theta_1, \zeta_1), \dots, (\theta_n, \zeta_n)) \in ([0,1] \times \mathbb{T})^n$, we have*

$$\mathscr{D}^n_\gamma f(\phi) = \sum_{k=1}^n f^{(k)}(\phi) \sum_{\lambda \in \Lambda(n,k)} \sum_{\mathscr{P}(n,\lambda)} \prod_{j=1}^k \mathscr{D}^{\lambda_j}_{(\theta^j_1, \zeta^j_1), \dots, (\theta^j_{\lambda_j}, \zeta^j_{\lambda_j})} \phi. \tag{1.10.28}$$

Lemma 1.10.3. *Fix some constants $\varepsilon > 0$ and $n \in \mathbb{N}$. For $i \in \{1, \dots, 4\}$ consider random variables $\phi^i \in \mathscr{W}^n$. Suppose that for all $p \in [1, \infty)$ and $k \in \{1, \dots, n-1\}$ there exists a constant $N_0 = N_0(k,p)$ such that*

$$\max_{i \in \{1,\dots,4\}} \|\phi^i\|_{\dot{\mathscr{W}}^k_p} \le N_0 \varepsilon^k. \tag{1.10.29}$$

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is smooth. For all $p \in [1, \infty)$ the following statements hold.

(a) There exists a constant $N = N(n, p, \|f\|_{C^n}) > 0$ such that

$$\|f(\phi^1)\|_{\dot{\mathscr{W}}^n_p} \le N\varepsilon^n + N\|\phi^1\|_{\dot{\mathscr{W}}^n_p}.$$

(b) There exists a constant $N = N(n, p, \|f\|_{C^{n+1}})$ such that

$$\|f(\phi^1) - f(\phi^2)\|_{\dot{\mathscr{W}}^n_p} \le N \sum_{i=0}^{n-1} \varepsilon^{n-i} \|\phi^1 - \phi^2\|_{\dot{\mathscr{W}}^i_{2p}} + N\|\phi^1 - \phi^2\|_{\dot{\mathscr{W}}^n_p}. \tag{1.10.30}$$

(c) Suppose moreover that (1.10.29) also holds for $k = n$. There exists a constant $N = N(n, p, \|f\|_{C^{n+2}})$ such that

$$\begin{aligned}
&\|f(\phi^1) - f(\phi^2) - f(\phi^3) + f(\phi^4)\|_{\dot{\mathscr{W}}^n_p} \\
&\quad\le N \sum_{i+j+k=n} \|\phi^1 - \phi^2\|_{\dot{\mathscr{W}}^i_{4p}} \Big(\|\phi^1 - \phi^3\|_{\dot{\mathscr{W}}^j_{4p}} + \|\phi^2 - \phi^4\|_{\dot{\mathscr{W}}^j_{4p}} \Big) \varepsilon^k \\
&\qquad + N \sum_{i=0}^{n-1} \|\phi^1 - \phi^2 - \phi^3 + \phi^4\|_{\dot{\mathscr{W}}^i_{2p}} \varepsilon^{n-i} + N\|\phi^1 - \phi^2 - \phi^3 + \phi^4\|_{\dot{\mathscr{W}}^n_p}.
\end{aligned}$$

Proof. By (1.10.28) we can see that

$$\|f(\phi^1)\|_{\dot{\mathscr{W}}_p^n} \lesssim \|f\|_{C^1}\|\phi^1\|_{\dot{\mathscr{W}}_p^n} + \|f\|_{C^n}\sum_{k=2}^{n}\sum_{\lambda\in\Lambda(n,k)}\sum_{\mathscr{P}(n,\lambda)}\prod_{j=1}^{k}\|\phi^1\|_{\dot{\mathscr{W}}_{2^k p}^{\lambda_j}}.$$

The second term in this expression can be estimated using (1.10.29) by

$$\sum_{k=2}^{n}\sum_{\lambda\in\Lambda(n,k)}\sum_{\mathscr{P}(n,\lambda)}\prod_{j=1}^{k}\varepsilon^{\lambda_j} \lesssim \varepsilon^n$$

where we used the definition of $\Lambda(n,k)$. This proves point (a).

We proceed by proving point (b). Note that by the Minkowski inequality and the Leibniz rule, by (1.10.29), and by point (a) we get

$$\begin{aligned}
\|f(\phi^1) - f(\phi^2)\|_{\dot{\mathscr{W}}_p^n} &= \Big\|\int_0^1 f'(\theta\phi^1 + (1-\theta)\phi^2)(\phi^1-\phi^2)d\theta\Big\|_{\dot{\mathscr{W}}_p^n}\\
&\lesssim \int_0^1\Big(\sum_{i=0}^{n-1}\|f'(\theta\phi^1+(1-\theta)\phi^2)\|_{\dot{\mathscr{W}}_{2p}^{n-i}}\|\phi^1-\phi^2\|_{\dot{\mathscr{W}}_{2p}^{i}}\\
&\qquad + \|\|f'\big(\theta\phi^1+(1-\theta)\phi^2\big)\mathscr{D}^n(\phi^1-\phi^2)\|_{H^{\otimes n}}\|_{L_p}\Big)d\theta\\
&\lesssim \sum_{i=0}^{n-1}\|f'\|_{C^{n-i}}\varepsilon^{n-i}\|\phi^1-\phi^2\|_{\dot{\mathscr{W}}_{2p}^{i}} + \|f'\|_{\mathbb{B}}\|\phi^1-\phi^2\|_{\dot{\mathscr{W}}_p^n}.
\end{aligned}$$

From here point (b) follows.

Finally, we prove point (c). By Lemma 1.11.6 we have that

$$\begin{aligned}
&\|f(u^1) - f(u^2) - f(u^3) + f(u^4)\|_{\dot{\mathscr{W}}_p^n}\\
&\leq \Big\|\int_0^1\int_0^1(\phi^1-\phi^2)\Big(\theta(\phi^1-\phi^3)+(1-\theta)(\phi^2-\phi^4)\Big)\nabla^2 f(\Theta_1(\theta,\eta))d\eta d\theta\Big\|_{\dot{\mathscr{W}}_p^n}\\
&\quad + \Big\|(\phi^1-\phi^2-\phi^3+\phi^4)\int_0^1\nabla f(\Theta_2(\theta))d\theta\Big\|_{\dot{\mathscr{W}}_p^n}\\
&=: A + B,
\end{aligned}$$

where for each $\theta,\eta\in[0,1]$, the expressions $\Theta_1(\theta,\eta),\Theta_2(\theta)$ are convex combinations of $\phi^1,\dots,\phi^4$. By the Minkowski inequality and by the Hölder inequality, we get we get

$$\begin{aligned}
A \lesssim \int_0^1\int_0^1\sum_{i+j+k=n}\|\phi^1-\phi^2\|_{\dot{\mathscr{W}}_{4p}^{i}}\|\theta(\phi^1-\phi^3)+(1-\theta)(\phi^2-\phi^4)\|_{\dot{\mathscr{W}}_{4p}^{j}}\\
\times\|\nabla^2 f(\Theta_1(\theta,\eta))\|_{\dot{\mathscr{W}}_{4p}^{k}}\,d\eta d\theta.
\end{aligned}$$

By using point (a) for $k \geq 1$, and using the regularity of f for $k = 0$, we can see that $\|\nabla^2 f(\Theta(\theta,\eta))\|_{\dot{\mathscr{W}}^k_{4p}} \lesssim \varepsilon^k$. Therefore we get

$$A \lesssim \sum_{i+j+k=n} \|\phi^1 - \phi^2\|_{\dot{\mathscr{W}}^i_{4p}} \Big(\|\phi^1 - \phi^3\|_{\dot{\mathscr{W}}^j_{4p}} + \|\phi^2 - \phi^4\|_{\dot{\mathscr{W}}^j_{4p}} \Big) \varepsilon^k.$$

Finally,

$$\begin{aligned} B &\lesssim \int_0^1 \sum_{i=0}^{n-1} \|\phi^1 - \phi^2 - \phi^3 + \phi^4\|_{\dot{\mathscr{W}}^i_{2p}} \|\nabla f(\Theta_1(\theta))\|_{\dot{\mathscr{W}}^{n-i}_{2p}} d\theta + \|\phi^1 - \phi^2 - \phi^3 + \phi^4\|_{\dot{\mathscr{W}}^n_p} \|f'\|_{\mathbb{B}} \\ &\lesssim \sum_{i=0}^{n-1} \|\phi^1 - \phi^2 - \phi^2 + \phi^4\|_{\dot{\mathscr{W}}^i_{2p}} \varepsilon^{n-i} + \|\phi^1 - \phi^2 - \phi^3 + \phi^4\|_{\dot{\mathscr{W}}^n_p} \end{aligned}$$

where the last inequality again follows from point (a). Hence the proof is finished. □

Lemma 1.10.4. *Consider constants $\varepsilon, c > 0$, $n \in \mathbb{Z}_{\geq 0}$. Let $X \in \mathscr{W}^{n+1}$, $Y \in \mathscr{W}^n$ with $Y \geq 0$. Suppose that for all $p \in (2,\infty)$ there exists a constant $N_0 = N_0 > 0$ such that for all $k \in \{1,\dots,n+1\}$, $l \in \{0,\dots,n\}$ we have*

$$\|X\|_{\dot{\mathscr{W}}^k_p} \leq N_0 \varepsilon^k, \qquad \|Y\|_{\dot{\mathscr{W}}^l_p} \leq N_0 \varepsilon^{c+l}, \qquad \mathbb{E}[Y^{-p}] \leq N_0 \varepsilon^{-cp}. \tag{1.10.31}$$

Define an H-valued random variable by $w := \frac{\mathscr{D}X}{Y}$. Then, for each $p \in [1,\infty)$, there exists a constant $N = N(N_0, n, p) > 0$ such that

$$\|\mathscr{D}^n w\|_{L_p(\Omega, H^{\otimes(n+1)})} \leq N \varepsilon^{n+1-c}.$$

Proof. Due to Hölder's inequality we may assume that $p > 2$. By (1.10.31) and a simple approximation argument (shifting Y away from 0), for $m \in \{1,\dots,n\}$, $\gamma \in ([0,1]\times\mathbb{T})^m$ we have

$$\mathscr{D}^m_\gamma (Y)^{-1} = \sum_{k=1}^m \frac{(-1)^k k!}{(Y)^{k+1}} \sum_{\lambda \in \Lambda(n,k)} \sum_{\mathscr{P}(n,\lambda)} \prod_{j=1}^k \mathscr{D}^{\lambda_j}_{(\theta^j_1,\zeta^j_1),\dots(\theta^j_{\lambda_j},\zeta^j_{\lambda_j})} Y.$$

Thus by (1.10.31) we have

$$\begin{aligned} \|\mathscr{D}^m (Y^{-1})\|_{L_p(\Omega;H^{\otimes m})} &\lesssim \sum_{k=1}^m \varepsilon^{-c(k+1)} \sum_{\lambda\in\Lambda(m,k)} \sum_{\mathscr{P}(m,\lambda)} \prod_{j=1}^k \varepsilon^{c+\lambda_j} \\ &\lesssim \sum_{k=1}^m \varepsilon^{-c(k+1)} \sum_{\lambda\in\Lambda(m,k)} \sum_{\mathscr{P}(m,\lambda)} \varepsilon^{\sum_{j=1}^k (c+\lambda_j)} \end{aligned}$$

$$\lesssim \sum_{k=1}^{m} \varepsilon^{-c(k+1)+ck+m} \lesssim \varepsilon^{m-c}. \tag{1.10.32}$$

For $\gamma \in ([0,1]\times\mathbb{T})^n$ and for $\eta \in [0,1]\times\mathbb{T}$, using the Leibniz rule, we have

$$\mathscr{D}^n_\gamma w(\eta) = \sum_{\lambda_1+\lambda_2=n} \sum_{(\gamma_1,\gamma_2)\in\mathscr{P}(n,2)} \mathscr{D}^{\lambda_1}_{\gamma_1}(\mathscr{D}_\eta X)\mathscr{D}^{\lambda_2}_{\gamma_2}(Y^{-1}). \tag{1.10.33}$$

Using (1.10.33), (1.10.31) and (1.10.32) we get that

$$\begin{aligned}\|\|\mathscr{D}^n w\|_{H^{\otimes(n+1)}}\|_{L_p} &\lesssim \sum_{\lambda_1+\lambda_2=n} \sum_{(\gamma_1,\gamma_2)\in\mathscr{P}(n,2)} \|\|\mathscr{D}^{\lambda_1+1}X\|_{H^{\otimes(\lambda_1+1)}}\|_{L_{2p}}\|\|\mathscr{D}^{\lambda_2}(Y^{-1})\|_{H^{\otimes\lambda_2}}\|_{L_{2p}} \\ &\lesssim \sum_{=\lambda_1+\lambda_2=n} \sum_{(\gamma_1,\gamma_2)\in\mathscr{P}(n,2)} \varepsilon^{\lambda_1+1}\varepsilon^{\lambda_2-c} \lesssim \varepsilon^{n+1-c}\end{aligned}$$

as required. □

1.11 Useful estimates

Lemma 1.11.1. *Let* $\varepsilon \in (0,1/2)$, $\gamma \in (0,\varepsilon)$, *and define*

$$\delta := \frac{2(\varepsilon-\gamma)}{1-2\gamma}.$$

Then $\delta \in (0,1)$, *and for all* $(t,x),(s,y) \in [0,1]\times\mathbb{T}$, *we have*

$$\left(|t-s|^{1/4-\gamma/2} + |x-y|^{1/2-\gamma}\right)^{1-\delta} \le |t-s|^{1/4-\varepsilon/2} + |x-y|^{1/2-\varepsilon}. \tag{1.11.34}$$

Proof. We begin by noting that since $\varepsilon \in (0,1/2)$, we have $\delta < \frac{2(\varepsilon-\gamma)}{2\varepsilon-2\gamma} = 1$. The positivity of δ also immediately follows from the fact that $0<\gamma<\varepsilon<1/2$. So we have $1-\delta \in (0,1)$, and thus the map $x \mapsto |x|^{1-\delta}$ is subadditive. Hence the left hand side of (1.11.34) is bounded by

$$|t-s|^{(1/4-\gamma/2)(1-\delta)} + |x-y|^{(1/2-\gamma)(1-\delta)}.$$

Now we just need to check that the powers in this expression match the powers on the right hand side of (1.11.34). This is indeed true, since

$$(\frac{1}{4}-\frac{\gamma}{2})(1-\delta) = \frac{1}{4}(1-2\gamma)\left(1-\frac{2(\varepsilon-\gamma)}{1-2\gamma}\right) = \frac{1}{4}(1-2\gamma-2(\varepsilon-\gamma)) = \frac{1}{4}-\frac{\varepsilon}{2},$$

and

$$\Big(\frac{1}{2}-\gamma\Big)(1-\delta)=\frac{1}{2}(1-2\gamma)\Big(1-\frac{2(\varepsilon-\gamma)}{1-2\gamma}\Big)=\frac{1}{2}(1-2\gamma-2(\varepsilon-\gamma))=\frac{1}{2}-\varepsilon,$$

and thus the proof is finished. □

Proposition 1.11.2. *For any* $\gamma\in[0,1]$ *there exists a constant* $N=N(\gamma)>0$ *such that for all* $t\in[0,1]$ *and* $x,\bar{x},y\in\mathbb{T}$ *we have*

$$|p_t(x,y)-p_t(\bar{x},y)|\leq N|x-\bar{x}|^{\gamma}t^{-\gamma/2}\big(p_{2t}(x,y)+p_{2t}(\bar{x},y)\big). \tag{1.11.35}$$

Moreover for any $\gamma,\beta\in[0,1]$ *with* $\alpha\leq\beta$ *there exists a constant* $N=N(\gamma,\beta)>0$ *such that for all* $(s,t)\in[0,1]^2_{\leq}$ *and* $x,\bar{x}\in\mathbb{T}$ *and for all* $f\in C^{\alpha}(\mathbb{T})$ *we have*

$$|P_tf(x)-P_sf(\bar{x})|\leq N\|f\|_{C^{\gamma}}(|x-\bar{x}|^{\beta}+|t-s|^{\beta/2})s^{(\gamma-\beta)/2}. \tag{1.11.36}$$

The first inequality of the lemma above is taken from the proof of [ABLM24, Lemma C2], while the second inequality can be found in [BDG23].

Proposition 1.11.3. *For any* $\varepsilon\in(0,1]$ *there exists a constant* $N=N(\varepsilon)>0$ *such that for all* $(s,t)\in[0,1]^2_{\leq}$*, the following inequalities hold:*

$$\int_{\mathbb{T}}|p_t(x,y)-p_t(\bar{x},y)|dy\leq N|x-\bar{x}|^{\varepsilon}t^{-\varepsilon/2}, \tag{1.11.37}$$

$$\int_0^t\int_{\mathbb{T}}|p_{t-r}(x,y)-p_{t-r}(\bar{x},y)|^2dydr\leq N|x-\bar{x}|^{1-\varepsilon}t^{\varepsilon/2}, \tag{1.11.38}$$

$$\int_0^s\int_{\mathbb{T}}|p_{t-r}(x,y)-p_{s-r}(x,y)|^2dydr\leq N|t-s|^{1/2-\varepsilon/2}. \tag{1.11.39}$$

Proof. The inequality (1.11.37) can be found in Lemma C2 of ([ABLM24]). We proceed with proving (1.11.38). To this end note that by (1.11.35) for $\gamma\in[0,1]$ for all $t\in[0,1]$ and $x,\bar{x}\in\mathbb{T}$ we have

$$\begin{aligned}\int_0^t\int_{\mathbb{T}}|p_{t-r}(x,y)-p_{t-r}(\bar{x},y)|^2dydr&\lesssim\int_0^t\int_{\mathbb{T}}|x-\bar{x}|^{2\gamma}(t-r)^{-\gamma}(p_{2(t-r)}(x,y)+p_{2(t-r)}(\bar{x},y))^2dydr\\&\lesssim|x-\bar{x}|^{2\gamma}\int_0^t(t-r)^{-\gamma-\frac{1}{2}}dr\\&\lesssim|x-\bar{x}|^{2\gamma}t^{\frac{1}{2}-\gamma},\end{aligned}$$

where the penultimate inequality follows from the fact that $\int_{\mathbb{T}}|p_{2(t-r)}(x,y)|^2dy\lesssim(t-r)^{-1/2}$. Now choosing $\gamma := \frac{1}{2} - \frac{\varepsilon}{2} \in [0, \frac{1}{2})$, it follows that

$$\int_0^t \int_{\mathbb{T}} |p_{t-r}(x,y) - p_{t-r}(\bar{x},y)|^2 dy dr \lesssim |x-\bar{x}|^{2(\frac{1}{2}-\frac{\varepsilon}{2})} t^{\frac{1}{2}-(\frac{1}{2}-\frac{\varepsilon}{2})} = |x-\bar{x}|^{1-\varepsilon} t^{\frac{\varepsilon}{2}}$$

as required.

We proceed with proving (1.11.39). To this end for $0 \le r \le s \le t \le 1$ and for $x, y \in \mathbb{T}$ define

$$f_r(y) := p_{t-r}(x,y) - p_{s-r}(x,y).$$

We may write the square of the above in the following form:

$$|f_r(y)|^2 = (p_{t-r}(x,y) - p_{s-r}(x,y)) f_r(y) = p_{t-r}(x,y) f_r(y) - p_{s-r}(x,y) f_r(y).$$

Using the above and (1.11.36), we can see that for $\beta, \gamma \in [0,1]$ we have

$$\int_{\mathbb{T}} |f_r(y)|^2 dy = P_{t-r} f_r(x) - P_{s-r} f_r(x) \lesssim \|f_r\|_{C^\gamma} |t-s|^{\beta/2} (s-r)^{(\gamma-\beta)/2}.$$

Applying this with $\gamma = 0$ and $\beta = 1-\varepsilon \in [0,1)$, and using that by the triangle inequality we have $\|f_r\|_{C^0} \le \|p_{t-r}(x,\cdot)\|_{C^0} + \|p_{s-r}(x,\cdot)\|_{C^0} \lesssim (s-r)^{-1/2}$, we get that

$$\begin{aligned}\int_0^s \int_{\mathbb{T}} |p_{t-r}(x,y) - p_{s-r}(x,y)|^2 dy dr &= \int_0^s \int_{\mathbb{T}} |f_r(y)|^2 dy dr \\ &\lesssim \int_0^s \|f_r\|_{C^0} |t-s|^{\frac{1}{2}-\frac{\varepsilon}{2}} (s-r)^{\frac{\varepsilon}{2}-\frac{1}{2}} dr \\ &\lesssim |t-s|^{\frac{1}{2}-\frac{\varepsilon}{2}} \int_0^s (s-r)^{\frac{\varepsilon}{2}-1} dr \\ &\lesssim |t-s|^{\frac{1}{2}-\frac{\varepsilon}{2}} s^{\varepsilon/2} \lesssim |t-s|^{\frac{1}{2}-\frac{\varepsilon}{2}},\end{aligned}$$

where we used that $s \le 1$. Thus the proof is finished.

□

Lemma 1.11.4 (A commonly used corollary of Hölder's inequality)**.** *Let $\gamma \in (1,3)$, $\delta \in (0,3)$. There exists*

$$p > \frac{3-\delta}{3-\gamma}, \quad \textit{such that} \quad \left(\gamma - \frac{\delta}{p}\right)\frac{p}{p-1} \ge 1, \tag{1.11.40}$$

and a constant $N = N(\delta,\gamma,p) > 0$*, such that for all* $(s,t) \in [0,1]^2_{\leq}$ *we have*

$$\Big(\int_s^t \int_{\mathbb{T}} |p_{t-r}(x,y)|^{\gamma} f(r,y)dydr\Big)^p \\ \leq N(t-s)^{\frac{(3-\gamma)p}{2}+\frac{\delta-3}{2}} \int_s^t \int_{\mathbb{T}} |p_{t-r}(x,y)|^{\delta} f^p(r,y)dydr. \quad (1.11.41)$$

Proof. Note that for any $\gamma \in (1,3)$, $\delta \in (0,3)$ we have $\lim_{p\to\infty}\Big(\gamma - \frac{\delta}{p}\Big)\frac{p}{p-1} = \gamma > 1$, and thus it follows that for sufficiently large p the conditions (1.11.40) hold. By Hölder's inequality, the left-hand-side of (1.11.41) is bounded by

$$\Big(\int_s^t \int_{\mathbb{T}} |p_{t-r}(x,y)|^{(\gamma-\frac{\delta}{p})\frac{p}{p-1}} dydr\Big)^{\frac{p-1}{p}\cdot p} \int_s^t \int_{\mathbb{T}} |p_{t-r}(x,y)|^{\delta} f^p(r,y)dydr.$$

Moreover using the results $\|p_t\|_{\mathbb{B}(\mathbb{T})} \lesssim t^{-1/2}$ and $\|p_t\|_{L_1(\mathbb{T})} = 1$ to interpolate, we can see that the first factor is bounded by

$$\Big(\int_s^t (t-r)^{-\frac{1}{2}\Big((\gamma-\frac{\delta}{p})\frac{p}{p-1}-1\Big)}dr\Big)^{p-1} \lesssim (t-s)^{\Big(-\frac{1}{2}\big((\gamma-\frac{\delta}{p})\frac{p}{p-1}-1\big)+1\Big)(p-1)} = (t-s)^{\frac{(3-\gamma)p}{2}+\frac{\delta-3}{2}},$$

and thus the proof is finished. □

Lemma 1.11.5 (Conditional BDG inequality for stochastic convolutions)**.** *Let* $0 \leq s \leq t$*,* $n \in \mathbb{Z}_{\geq 0}$ *and let* $X : \Omega \times [0,1]\times\mathbb{T} \to H^{\otimes n}$ *be a* $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})$*-measurable* $H^{\otimes n}$*-valued random field. For all* $p \in [2,\infty)$ *there exists a constant* C_p *such that if* $f_t \in L_2([0,t]\times\mathbb{T})$ *for all* $t \in [0,1]$*, then for all* $(s,t) \in [0,1]^2_{\leq}$ *we have*

$$\mathbb{E}^s\Big\|\int_s^t \int_{\mathbb{T}} f_t(r,y)X(r,y)\xi(dy,dr)\Big\|^p_{H^{\otimes n}} \\ \leq C_p\mathbb{E}^s\Big(\int_s^t \int_{\mathbb{T}} f_t^2(r,y)\|X(r,y)\|^2_{H^{\otimes n}}dydr\Big)^{p/2}, \quad (1.11.42)$$

and consequentially

$$\Big\|\Big\|\int_s^t \int_{\mathbb{T}} f_t(r,y)X(r,y)\xi(dy,dr)\Big\|_{H^{\otimes n}}\Big\|^2_{L_p|\mathscr{F}_s} \\ \leq C_p\int_s^t \int_{\mathbb{T}} f_t^2(r,y)\|\|X(r,y)\|_{H^{\otimes n}}\|^2_{L_p|\mathscr{F}_s}dydr. \quad (1.11.43)$$

The inequality (1.11.42) follows from the classic conditional BDG inequality. From (1.11.42) we can see that (1.11.43) holds by the Minkowski inequality

Lemma 1.11.6. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable. Then for $\phi_1, \ldots, \phi_4 \in \mathbb{R}$ we have*

$$\begin{aligned} & f(\phi_1) - f(\phi_2) - f(\phi_3) + f(\phi_4) \\ & = \int_0^1 \int_0^1 (\phi_1 - \phi_2)(\theta(\phi_1 - \phi_3) + (1-\theta)(\phi_2 - \phi_4))\nabla^2 f(\Theta_1(\theta,\eta)) d\eta d\theta \\ & \quad + (\phi_1 - \phi_2 - \phi_3 + \phi_4) \int_0^1 \nabla f(\Theta_2(\theta)) d\theta \end{aligned} \tag{1.11.44}$$

where $\Theta_1(\theta,\eta)$ and $\Theta_2(\theta)$ are the convex combinations of $\phi_1, \ldots, \phi_4$ given by

$$\begin{aligned} \Theta_1(\theta,\eta) &:= \eta(\theta\phi_1 + (1-\theta)\phi_2) + (1-\eta)(\theta\phi_3 + (1-\theta)\phi_4), \\ \Theta_2(\theta) &:= \theta\phi_3 + (1-\theta)\phi_4. \end{aligned}$$

Moreover

$$\begin{aligned} & |f(\phi_1) - f(\phi_2) - f(\phi_3) + f(\phi_4)| \\ & \quad \leq \|f\|_{C^2} |\phi_1 - \phi_2||\phi_1 - \phi_3| + \|f\|_{C^1} |\phi_1 - \phi_2 - \phi_3 + \phi_4|. \end{aligned} \tag{1.11.45}$$

Proof. We begin by proving (1.11.44). Using the notation

$$\phi_{1,2}^\theta := \theta\phi_1 + (1-\theta)\phi_2, \qquad \phi_{3,4}^\theta := \theta\phi_3 + (1-\theta)\phi_4,$$

the expression $f(\phi_1) - f(\phi_2) - f(\phi_3) + f(\phi_4)$ can be rewritten as

$$\begin{aligned} & (\phi_1 - \phi_2) \int_0^1 \nabla f(\phi_{1,2}^\theta) d\theta - (\phi_3 - \phi_4) \int_0^1 \nabla f(\phi_{3,4}^\theta) d\theta \\ & = (\phi_1 - \phi_2) \int_0^1 \left(\nabla f(\phi_{1,2}^\theta) - \nabla f(\phi_{3,4}^\theta) \right) d\theta + (\phi_1 - \phi_2 - \phi_3 + \phi_4) \int_0^1 \nabla f(\phi_{3,4}^\theta) d\theta. \end{aligned}$$

The second term is exactly as desired, and the first term can be written as

$$(\phi_1 - \phi_2) \int_0^1 (\phi_{1,2}^\theta - \phi_{3,4}^\theta) \int_0^1 \nabla^2 f(\eta\phi_{1,2}^\theta + (1-\eta)\phi_{3,4}^\theta) d\eta d\theta$$

which is indeed the first term of the desired expression. Hence (1.11.44) is proven. To prove (1.11.45), we set

$$\delta_{i,j} := \phi_j - \phi_i$$

and we note that $f(\phi_1) - f(\phi_2) - f(\phi_3) + f(\phi_4)$ can be written as

$$\begin{aligned}
&f(\phi_1) - f(\phi_1 + \delta_{1,2}) - f(\phi_3) + f(\phi_3 + \delta_{1,2}) - f(\phi_3 + \delta_{1,2}) + f(\phi_3 + \delta_{3,4}) \\
&= -\delta_{1,2} \int_0^1 \nabla f(\phi_1 + \theta\delta_{1,2}) d\theta + \delta_{1,2} \int_0^1 \nabla f(\phi_3 + \theta\delta_{1,2}) d\theta \\
&\qquad + (\delta_{3,4} - \delta_{1,2}) \int_0^1 \nabla f(\phi_3 + \theta\delta_{3,4} + (1-\theta)\delta_{1,2}) d\theta \\
&= \delta_{1,2}\delta_{1,3} \int_0^1 \int_0^1 \nabla^2 f(\eta\phi_3 + (1-\eta)\phi_1 + \theta\delta_{1,2}) d\theta d\eta \\
&\qquad + (\delta_{3,4} - \delta_{1,2}) \int_0^1 \nabla f(\phi_3 + \theta\delta_{3,4} + (1-\theta)\delta_{1,2}) d\theta.
\end{aligned}$$

Hence (1.11.45) follows as well. □

Lemma 1.11.7 (The $\mathscr{V}_p^\gamma$-bracket is triangular in time)**.** *Let $p \in [1, \infty)$, $\gamma > 0$, and let $f \in \mathscr{V}_p^\gamma$. Then for all $0 \le S \le Q \le T \le 1$ we have*

$$[f]_{\mathscr{V}_p^\gamma[S,T]} \le 2[f]_{\mathscr{V}_p^\gamma[S,Q]} + 2[f]_{\mathscr{V}_p^\gamma[Q,T]}. \tag{1.11.46}$$

Consequently, for any integer $K \ge 2$ we have

$$[f]_{\mathscr{V}_p^\gamma} \lesssim 2^K \sum_{i=0}^{K-1} [f]_{\mathscr{V}_p^\gamma[\frac{i}{K}, \frac{i+1}{K}]}. \tag{1.11.47}$$

Proof. For $(s,t) \in [0,1]^2_\le$ define

$$A(s,t) := \sup_{x \in \mathbb{T}} \|f_t(x) - P_{t-s} f_s(x)\|_{L_{p,\infty}^{\mathscr{F}_s}}.$$

For $(s,t) \in [S,Q]^2_\le \cup [Q,T]^2_\le$, we clearly have

$$A(s,t) \le [f]_{\mathscr{V}_p^\gamma[s,t]} (t-s)^\gamma \le \Big([f]_{\mathscr{V}_p^\gamma[S,Q]} + [f]_{\mathscr{V}_p^\gamma[Q,T]} \Big) (t-s)^\gamma. \tag{1.11.48}$$

We also need to check what happens in the case when $Q \in (s,t)$. Then we write

$$\begin{aligned}
A(s,t) &\le \sup_{x \in \mathbb{T}} \|f_t(x) - P_{t-Q} f_Q(x)\|_{L_{p,\infty}^{\mathscr{F}_s}} + \sup_{x \in \mathbb{T}} \|P_{t-Q} f_Q(x) - P_{t-s} f_s(x)\|_{L_{p,\infty}^{\mathscr{F}_s}} \\
&= B(s,t) + C(s,t).
\end{aligned}$$

Note that as $s \le Q$, we have $\|\cdot\|_{L_p|\mathscr{F}_s} \le \|\|\cdot\|_{L_p|\mathscr{F}_Q}\|_{L_p|\mathscr{F}_s} \le \|\cdot\|_{L^Q_{p,\infty}}$, and thus

$$B(s,t) \le \sup_{x\in\mathbb{T}} \|f_t(x) - P_{t-Q} f_Q(x)\|_{L^Q_{p,\infty}} = A(Q,t).$$

Moreover using that $s \le Q$, we have

$$C(s,t) = \sup_{x\in\mathbb{T}} \|P_{t-Q}\big(f_Q - P_{Q-s} f_s\big)(x)\|_{L^{\mathscr{F}_s}_{p,\infty}} \le \sup_{x\in\mathbb{T}} \|f_Q(x) - P_{Q-s} f_s(x)\|_{L^{\mathscr{F}_s}_{p,\infty}} = A(s,Q).$$

By the above bounds on B and C, we conclude that

$$A(s,t) \le A(s,Q) + A(Q,t) \le \Big([f]_{\mathscr{V}^\gamma_p[S,Q]} + [f]_{\mathscr{V}^\gamma_p[Q,T]}\Big)(t-s)^\gamma \tag{1.11.49}$$

By adding up the bounds (1.11.48) and (1.11.49), we can see that for all $(s,t) \in [S,T]^2_\le$ and $Q \in [S,T]$, we have

$$A(s,t) \le 2\Big([f]_{\mathscr{V}^\gamma_p[S,Q]} + [f]_{\mathscr{V}^\gamma_p[Q,T]}\Big)(t-s)^\gamma,$$

from which the desired result follows. □

Lemma 1.11.8 (The L_p-valued $C^{1/4,1/2}$-norm, and the ${\mathscr{V}_p}^{1/4}$-bracket)**.** *Let $\alpha \in (-1,0)$ and $p \in [1,\infty)$. There exists a constant $N = N(p,\alpha) > 0$ such that for $f \in {\mathscr{V}_p}^{1/4} \cap C^{0,1/2}([0,1]\times\mathbb{T}, L_p)$ we have*

$$\|f\|_{C^{1/4,1/2}([0,1]\times\mathbb{T},L_p)} \le N[f]_{\mathscr{V}_p^{1/4}} + N\|f\|_{C^{0,1/2}([0,1]\times\mathbb{T},L_p)}.$$

Proof. We decompose the space–time Hölder norm to the sup norm, and spatial and temporal seminorms as follows:

$$\begin{aligned}&\|f^n\|_{C^{1/4,1/2}([0,1]\times\mathbb{T},L_p)}\\ &\le \|f^n\|_{\mathbb{B}([0,1]\times\mathbb{T},L_p)} + \sup_{x\in\mathbb{T}}[f^n(\cdot,x)]_{C^{1/4}([0,1])} + \sup_{t\in[0,1]}[f^n(t,\cdot)]_{C^{1/2}(\mathbb{T},L_p)}.\end{aligned} \tag{1.11.50}$$

To bound the temporal seminorm, note that for $(s,t) \in [0,1]^2_\le$ we have

$$\begin{aligned}\|f(t,\cdot) - f(s,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)} &\le \|f(t,\cdot) - P_{t-s}f(s,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)} + \|P_{t-s}f(s,\cdot) - f(s,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}\\ &=: A(s,t) + B(s,t).\end{aligned} \tag{1.11.51}$$

Since $\|\cdot\|_{L_p} \le \|\cdot\|_{L_{p,\infty}^{\mathscr{F}_s}}$, it follows that

$$A(s,t) \le [f]_{\mathscr{V}_p^{1/4}[s,t]}(t-s)^{1/4}.$$

Moreover by a standard heat kernel estimate

$$B(s,t) \lesssim \|f(s,\cdot)\|_{C^{1/2}(\mathbb{T},L_p)}(t-s)^{1/4}.$$

By putting the above bounds on A and B into (1.11.51), we can see that

$$\sup_{x\in\mathbb{T}}[f(\cdot,x)]_{C^{1/4}([0,1],L_p)} \lesssim [f]_{\mathscr{V}_p^{1/4}[s,t]} + \|f\|_{C^{0,1/2}([0,1]\times\mathbb{T},L_p)}.$$

Using this bound on the second term of (1.11.50) finishes the proof. □

Chapter 2

Malliavin calculus for the driftless equation

This chapter is concerned with the solution of the driftless multiplicative stochastic heat equation

$$(\partial_t - \Delta)\phi = \sigma(\phi)\xi, \qquad \phi(0,\cdot)=\phi_0 \tag{2.0.1}$$

and its Malliavin derivatives. Herein, $\phi_0 \in C(\mathbb{T})$ is fixed and the solution $\phi : \Omega \times [0,1] \times \mathbb{T} \to \mathbb{R}$ is a $\mathscr{P} \otimes \mathscr{B}(\mathbb{T})$-measurable random field, a.s. continuous on $[0,1] \times \mathbb{T}$, and satisfies the following equation almost surely

$$\phi(t,x) = P_t\phi_0(x) + \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y)\sigma(\phi(r,y))\xi(dy,dr), \quad \forall (t,x) \in [0,1] \times \mathbb{T}. \tag{2.0.2}$$

2.1 Moment bounds for Malliavin derivatives

We will show the following result.

Lemma 2.1.1. *Let ϕ be the solution of (2.0.1). For any $n \in \mathbb{Z}_{\geq 0}$ and $p \in [1,\infty)$, if in addition $\sigma \in C^n$, then there exists some constant $N = N(n,p,\|\sigma\|_{C^n})$ such that for all $t \in [0,1]$ we have*

$$\sup_{x\in\mathbb{T}} \|\phi(t,x)\|_{\dot{\mathscr{W}}_p^n} \leq N(1 + \mathbf{1}_{n=0}\|u_0\|_{\mathbb{B}(\mathbb{T})})t^{n/4}.$$

To show this, let us recall the following non-quantitative result from [BP98, Proposition 4.3].

Proposition 2.1.2 (Boundedness and Malliavin differentiability)**.** *Let ϕ be the solution of (2.0.1). For any $n \in \mathbb{N}$, $p \in [1,\infty)$, if $\sigma \in C^n$, then we have*

$$\sup_{(t,x)\in[0,1]\times\mathbb{T}} \|\phi(t,x)\|_{\mathscr{W}_p^n} < \infty.$$

We will also need the following result from [CHN21, Lemma 5.6]:

Proposition 2.1.3. *Let ϕ be the solution of (2.0.1), and let $n \in \mathbb{N}, p \in [1,\infty), \sigma \in C^n$. Then for all $(t,x) \in [0,1] \times \mathbb{T}$ and for almost every $\gamma = (\theta_i, \zeta_i)_{i=1}^n \in ([0,1] \times \mathbb{T})^n$, we have*

$$\begin{aligned}\mathscr{D}^n_\gamma \phi(t,x) = \mathbf{1}_{[0,t]}(\theta^*) \sum_{k=1}^n p_{t-\theta_k}(x,\zeta_k) \mathscr{D}^{n-1}_{\hat{\gamma}_k}[\sigma(\phi(\theta_k,\zeta_k))] \\ + \mathbf{1}_{[0,t]}(\theta^*) \int_{\theta^*}^t \int_{\mathbb{T}} p_{t-r}(x,y) \mathscr{D}^n_\gamma[\sigma(\phi(r,y))] \xi(dy,dr),\end{aligned} \tag{2.1.3}$$

where $\hat{\gamma}_k$ is defined by (1.10.27) and $\theta^* := \max_{k\in\{1,\dots,n\}} \theta_k$.

Remark 2.1.4. Note that in the above equation the stochastic integral can be taken over the time interval $[0,t]$ rather than $[\theta^*, t]$, since for $r \le \theta^*$ the Malliavin derivative $\mathscr{D}^n_\gamma \sigma(\phi(r,y))$ is zero (see [SS04, Remark 5.1]).

We can now proceed with the proof of Lemma 2.1.1.

Proof of Lemma 2.1.1. We will prove the result by induction. By the BDG[1] inequality and by the boundedness of σ, the result holds for the case $n = 0$. Suppose that the result holds for the first $(n-1)$ Malliavin derivatives. We aim to show that the result also holds for the n-th Malliavin derivative. Assume without loss of generality that $p \ge 2$. By (2.1.3) and the BDG inequality, we get

$$\begin{aligned}\|\phi(t,x)\|^2_{\dot{\mathscr{W}}^n_p} &\lesssim \|p_{t-\cdot}(x,\cdot)\mathscr{D}^{n-1}\sigma(\phi(\cdot,\cdot))\mathbf{1}_{[0,t]}(\cdot)\|^2_{L_p(\Omega;H^n)} \\ &\quad + \int_0^t \int_{\mathbb{T}} p^2_{t-r}(x,y) \|\sigma(\phi(r,y))\|^2_{\dot{\mathscr{W}}^n_p} dy dr \\ &=: A(t,x) + B(t,x).\end{aligned} \tag{2.1.4}$$

We proceed with proving that

$$A(t,x) \lesssim t^{n/2}. \tag{2.1.5}$$

Indeed, in the $n = 1$ case, we have by the boundedness of σ that

$$A(t,x) = \|p_{t-\cdot}(x,\cdot)\sigma(\phi(\cdot,\cdot))\mathbf{1}_{[0,t]}(\cdot)\|^2_{L_p(\Omega,H)} \lesssim \|p_{t-\cdot}(x,\cdot)\mathbf{1}_{[0,t]}(\cdot)\|^2_H \lesssim t^{1/2}.$$

Moreover in the $n \ge 2$ case, by point (a) of Lemma 1.10.3 and by the induction hypothesis we have for

[1]We call the Burkholder-Davis-Gundy inequality “BDG inequality” for brevity.

$(\theta, \zeta) \in [0,t] \times \mathbb{T}$ that

$$\|\sigma(\phi(\theta,\zeta))\|_{\dot{\mathscr{W}}_p^{n-1}} \lesssim \theta^{(n-1)/4} + \|\phi(\theta,\zeta)\|_{\dot{\mathscr{W}}_p^{n-1}} \lesssim \theta^{(n-1)/4} \lesssim t^{(n-1)/4},$$

and thus using Minkowski's inequality and the above bound, we get that

$$\begin{aligned} A(t,x) &= \big\|\|\|p_{t-\cdot}(x,\cdot)\|\mathscr{D}^{n-1}\sigma(\phi(\cdot,\cdot))\|_{H^{\otimes(n-1)}}\mathbf{1}_{[0,t]}(\cdot)\|_H\big\|_{L_p}^2 \\ &\leq \left\|p_{t-\cdot}(x,\cdot)\|\sigma(\phi(\cdot,\cdot))\|_{\dot{\mathscr{W}}_p^{n-1}}\mathbf{1}_{[0,t]}(\cdot)\right\|_H^2 \\ &\lesssim t^{(n-1)/2}\|p_{t-\cdot}(x,\cdot)\mathbf{1}_{[0,t]}(\cdot)\|_H^2 \lesssim t^{n/2} \end{aligned}$$

as required. Hence (2.1.5) is proven. We now proceed by bounding B. By point (a) of Lemma 1.10.3 and by the induction hypothesis we have

$$\begin{aligned} B(t,x) &\lesssim \int_0^t \int_{\mathbb{T}} p_{t-r}^2(x,y)\Big(r^{n/4} + \|\phi(r,y)\|_{\dot{\mathscr{W}}_p^n}\Big)^2 dydr \\ &\lesssim \int_0^t \int_{\mathbb{T}} p_{t-r}^2(x,y) r^{n/2} dydr + \int_0^t \int_{\mathbb{T}} p_{t-r}^2(x,y)\|\phi(r,y)\|_{\dot{\mathscr{W}}_p^n}^2 dydr \\ &\lesssim t^{(n+1)/2} + \int_0^t \int_{\mathbb{T}} p_{t-r}^2(x,y)\|\phi(r,y)\|_{\dot{\mathscr{W}}_p^n}^2 dydr. \end{aligned} \tag{2.1.6}$$

By (2.1.4), and by our bounds (2.1.5), (2.1.6) on A, B, we conclude that

$$\|\phi(t,x)\|_{\dot{\mathscr{W}}_p^n}^2 \lesssim t^{n/2} + \int_0^t \int_{\mathbb{T}} p_{t-r}^2(x,y)\|\phi(r,y)\|_{\dot{\mathscr{W}}_p^n}^2 dydr.$$

By Proposition 2.1.2 we have that $\sup_{(r,y)} \|\phi(r,y)\|_{\dot{\mathscr{W}}_p^n} < \infty$. Therefore by Lemma 1.3.5, the statement we aim to show also holds for the n-th Malliavin derivative. Thus the proof is finished. □

Lemma 2.1.5. *Let $n \in \mathbb{Z}_{\geq 0}$, $p \in [1,\infty)$, $\sigma \in C^{n+1}$, and let ϕ solve (2.0.1). There exists some constant $N = N(n, p, \|\sigma\|_{C^{n+1}})$ such that for all $t \in (0,1]$ we have*

$$\sup_{x \in \mathbb{T}} \left\| \|\mathscr{D}\phi(t,x)\|_H^2 \right\|_{\dot{\mathscr{W}}_p^n} \leq N t^{(n+2)/4}.$$

Proof. By using the Minkowski inequality, the Leibniz rule, Hölder's inequality, and Lemma 2.1.1, we can see that

$$\|\|\mathscr{D}\phi(t,x)\|_H^2\|_{\dot{\mathscr{W}}_p^n}$$

$$
\begin{aligned}
&= \Big\| \Big\| \mathscr{D}^n \int_0^1 \int_{\mathbb{T}} (\mathscr{D}_{\theta,\zeta}\phi(t,x))^2 d\zeta d\theta \Big\|_{H^{\otimes n}} \Big\|_{L_p} \\
&\leq \Big\| \int_0^1 \int_{\mathbb{T}} \| \mathscr{D}^n \big(\mathscr{D}_{\theta,\zeta}\phi(t,x) \mathscr{D}_{\theta,\zeta}\phi(t,x) \big) \|_{H^{\otimes n}} d\zeta d\theta \Big\|_{L_p} \\
&\leq \sum_{i=0}^{n} \Big\| \int_0^1 \int_{\mathbb{T}} \| \mathscr{D}^i (\mathscr{D}_{\theta,\zeta}\phi(t,x)) \mathscr{D}^{n-i} (\mathscr{D}_{\theta,\zeta}\phi(t,x)) \|_{H^{\otimes n}} d\zeta d\theta \Big\|_{L_p} \\
&= \sum_{i=0}^{n} \Big\| \int_0^1 \int_{\mathbb{T}} \| \mathscr{D}^i \mathscr{D}_{\theta,\zeta}\phi(t,x) \|_{H^{\otimes i}} \| \mathscr{D}^{n-i} \mathscr{D}_{\theta,\zeta}\phi(t,x) \|_{H^{\otimes (n-i)}} d\zeta d\theta \Big\|_{L_p} \\
&\leq \sum_{i=0}^{n} \Big\| \Big(\int_0^1 \int_{\mathbb{T}} \| \mathscr{D}^i \mathscr{D}_{\theta,\zeta}\phi(t,x) \|^2_{H^{\otimes i}} d\zeta d\theta \Big)^{1/2} \Big(\int_0^1 \int_{\mathbb{T}} \| \mathscr{D}^{n-i} \mathscr{D}_{\bar\theta,\bar\zeta}\phi(t,x) \|^2_{H^{\otimes (n-i)}} d\bar\zeta d\bar\theta \Big)^{1/2} \Big\|_{L_p} \\
&= \sum_{i=0}^{n} \Big\| \| \mathscr{D}^{i+1}\phi(t,x) \|_{H^{\otimes (i+1)}} \| \mathscr{D}^{n-i+1}\phi(t,x) \|_{H^{\otimes (n-i+1)}} \Big\|_{L_p} \\
&\leq \sum_{i=0}^{n} \| \phi(t,x) \|_{\dot{\mathscr{W}}^{i+1}_{2p}} \| \phi(t,x) \|_{\dot{\mathscr{W}}^{n-i+1}_{2p}} \lesssim \sum_{i=0}^{n} t^{(i+1)/4} t^{(n-i+1)/4} \lesssim t^{(n+2)/4}
\end{aligned}
$$

as required. ◻

2.2 Lipschitzness in the initial condition

For any $z \in C(\mathbb{T})$, let ϕ^z denote the solution of (2.0.1) with $\phi_0 = z$. For $n \in \mathbb{N}$, $\sigma \in C^n$, $q \in [1, \infty)$, $(z_1, z_2) \in (C(\mathbb{T}))^2$, $(t,x) \in [0,1] \times \mathbb{T}$, we define

$$F^{(2)}_{q,n}(t,x,z_1,z_2) := \| \phi^{z_1}(t,x) - \phi^{z_2}(t,x) \|_{\dot{\mathscr{W}}^n_q}, \tag{2.2.7}$$

$$\Sigma^{(2)}_{q,n}(t,x,z_1,z_2) := \| \sigma(\phi^{z_1}(t,x)) - \sigma(\phi^{z_2}(t,x)) \|_{\dot{\mathscr{W}}^n_q}. \tag{2.2.8}$$

The main result of this section is the following:

Lemma 2.2.1. *Let $n \in \mathbb{Z}_{\geq 0}$ and assume that $\sigma \in C^{n+1}$. For all $q \in [2, \infty)$, $(t,x) \in [0,1] \times \mathbb{T}$, we have that*

$$F^{(2)}_{q,n}(t,x,\cdot,\cdot) \in C((C(\mathbb{T}))^2).$$

Moreover for all $q, p_1 \in [2, \infty)$ and $p_2 \in [2, \infty]$ there exists a constant $N = N(n, p_1, p_2, q, \|\sigma\|_{C^{n+1}})$ such that for any σ-algebra $\mathscr{G} \subset \mathscr{F}$ and any random variables $Z, \bar{Z} \in L_{p_1 \vee p_2}(\Omega, C(\mathbb{T}))$ and for all $(t,x) \in [0,1] \times \mathbb{T}$ we have

$$\big\| F^{(2)}_{q,n}(t,x,Z,\bar{Z}) \big\|_{L^{\mathscr{G}}_{p_1,p_2}} \leq N t^{n/4} \sup_{x \in \mathbb{T}} \| Z(x) - \bar{Z}(x) \|_{L^{\mathscr{G}}_{p_1,p_2}}.$$

Proof. The result will be proven by induction.

Step 1: We show that the statement holds for $n = 0$. For $z, \bar{z} \in C(\mathbb{T})$, we have by (2.0.2) that

$$\phi^z(t,x) - \phi^{\bar{z}}(t,x) = P_t(z-\bar{z})(x) + \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y)(\sigma(\phi^z(r,y)) - \sigma(\phi^{\bar{z}}(r,y))\xi(dy,dr).$$

Therefore, by the BDG inequality we get

$$\begin{aligned} &\|\phi^z(t,x) - \phi^{\bar{z}}(t,x)\|_{L_q}^2 \\ &\qquad \lesssim |P_t(z-\bar{z})(x)|^2 + \int_0^t \int_{\mathbb{T}} p_{t-r}^2(x,y)\|\phi^z(r,y) - \phi^{\bar{z}}(r,y)\|_{L_q}^2 dydr. \end{aligned} \tag{2.2.9}$$

Notice that by the triangle inequality and Proposition 2.1.2, it follows that the norm in the integrand is bounded in (r, y). Hence using Lemma 1.3.5, we conclude that

$$\begin{aligned} F_{q,0}^{(2)}(t,x,z,\bar{z}) = \|\phi^z(t,x) - \phi^{\bar{z}}(t,x)\|_{L_q} &\lesssim \sup_{(t,x)\in[0,1]\times\mathbb{T}} |P_t(z-\bar{z})(x)| \\ &\lesssim \sup_{x\in\mathbb{T}} |z(x) - \bar{z}(x)|. \end{aligned} \tag{2.2.10}$$

By the triangle inequality, it follows that $F_{q,0}^{(2)}(t,x,\cdot,\cdot) \in C((C(\mathbb{T}))^2, \mathbb{R})$. This implies that $F_{q,0}^{(2)}(t,x,Z,\bar{Z})$ is indeed defined as a random variable. We begin by showing that the desired inequality holds for the case when $\|Z\|_{\mathbb{B}}, \|\bar{Z}\|_{\mathbb{B}} < N$ almost surely, for some constant $N < \infty$. By evaluating (2.2.9) at $(z, \bar{z}) = (Z, \bar{Z})$, and then taking the $L_{p_1,p_2}^{\mathscr{G}}$-norm of the square root of both sides, we get

$$\begin{aligned} &\|F_{q,0}^{(2)}(t,x,Z,\bar{Z})\|_{L_{p_1,p_2}^{\mathscr{G}}} \\ &\lesssim \|P_t(Z-\bar{Z})(x)\|_{L_{p_1,p_2}^{\mathscr{G}}} + \left\| \int_0^t \int_{\mathbb{T}} p_{t-r}^2(x,y)|F_{q,0}^{(2)}(r,y,Z,\bar{Z})|^2 dydr \right\|_{L_{\frac{p_1}{2},\frac{p_2}{2}}^{\mathscr{G}}}^{1/2} \\ &\lesssim \sup_{w\in\mathbb{T}} \|Z(w) - \bar{Z}(w)\|_{L_{p_1,p_2}^{\mathscr{G}}} + \left(\int_0^t \int_{\mathbb{T}} p_{t-r}^2(x,y)\|F_{q,0}^{(2)}(r,y,Z,\bar{Z})\|_{L_{p_1,p_2}^{\mathscr{G}}}^2 dydr \right)^{1/2}, \end{aligned}$$

where to obtain the last expression, we used Minkowski's inequality and the assumption that $p_1, p_2 \geq 2$. Hence we may conclude that

$$\begin{aligned} \|F_{q,0}^{(2)}(t,x,Z,\bar{Z})\|_{L_{p_1,p_2}^{\mathscr{G}}}^2 \lesssim \left(\sup_{w\in\mathbb{T}} \|Z(w) - \bar{Z}(w)\|_{L_{p_1,p_2}^{\mathscr{G}}} \right)^2 \\ + \int_0^t \int_{\mathbb{T}} p_{t-r}^2(x,y)\|F_{q,0}^{(2)}(r,y,Z,\bar{Z})\|_{L_{p_1,p_2}^{\mathscr{G}}}^2 dydr. \end{aligned}$$

Note that by (2.2.10) and due to the fact that $\|Z\|_{\mathbb{B}}, \|\bar{Z}\|_{\mathbb{B}} \leq N$, we have $\sup_{r,y} \|F_{q,0}^{(2)}(r,y,Z,\bar{Z})\|_{L_{p_1,p_2}^{\mathscr{G}}} \lesssim$

$2N$. Hence by the above inequality and by Lemma 1.3.5 we get

$$\|F^{(2)}_{q,0}(t,x,Z,\bar{Z})\|_{L^{\mathscr{G}}_{p_1,p_2}} \lesssim \sup_{w\in\mathbb{T}} \|Z(w)-\bar{Z}(w)\|_{L^{\mathscr{G}}_{p_1,p_2}}.$$

We now remove the assumption that $\|Z\|_{\mathbb{B}}, \|\bar{Z}\|_{\mathbb{B}} < N$. Indeed, if this does not hold, then we can construct the sequence of truncations $(Z_N, \bar{Z}_N) := ((N\wedge Z)\vee(-N), (N\wedge\bar{Z})\vee(-N))$. Then using the continuity of $F^{(2)}_{q,0}$ and Fatou's lemma, the above inequality, and the fact that truncation is a Lipschitz operation, we get

$$\begin{aligned}
\|F^{(2)}_{q,0}(t,x,Z,\bar{Z})\|_{L^{\mathscr{G}}_{p,\infty}} &\le \liminf_{N\to\infty} \|F^{(2)}_{q,0}(t,x,Z_N,\bar{Z}_N)\|_{L^{\mathscr{G}}_{p_1,p_2}} \\
&\lesssim \liminf_{N\to\infty} \sup_{w\in\mathbb{T}} \|Z_N(w)-\bar{Z}_N(w)\|_{L^{\mathscr{G}}_{p_1,p_2}} \\
&\le \sup_{w\in\mathbb{T}} \|Z(w)-\bar{Z}(w)\|_{L^{\mathscr{G}}_{p_1,p_2}}.
\end{aligned}$$

This finishes the proof of the case $n=0$.

Step 2: Suppose that the result holds for $F^{(2)}_{q,0},\dots,F^{(2)}_{q,n-1}$ for some $n\in\mathbb{N}$. We aim to show that it also holds for $F^{(2)}_{q,n}$. We first assume that for all $(t,x)\in[0,1]\times\mathbb{T}$, $F^{(2)}_{q,n}(t,x,z,\bar{z})$ is continuous in the $(z,\bar{z})$ variable and later we will show that this is indeed the case. By Proposition 2.1.3, we have

$$\begin{aligned}
F^{(2)}_{q,n}(t,x,z,\bar{z}) &= \|\mathscr{D}^n[\phi^z(t,x)-\phi^{\bar{z}}(t,x)]\|_{L_q(\Omega,H^{\otimes n})} \\
&\lesssim \Big\| p_{t-\cdot}(x,\cdot)\mathscr{D}^{n-1}\big(\sigma(\phi^z(\cdot,\cdot))-\sigma(\phi^{\bar{z}}(\cdot,\cdot))\big)\mathbf{1}_{[0,t]}\Big\|_{L_q(\Omega;H^{\otimes n})} \\
&\quad + \Big\| \int_0^t\int_{\mathbb{T}} p_{t-r}(x,y)\mathscr{D}^n\big(\sigma(\phi^z(r,y))-\sigma(\phi^{\bar{z}}(r,y))\big)\xi(dy,dr)\Big\|_{L_q(\Omega;H^{\otimes n})} \\
&=: A(z,\bar{z})+B(z,\bar{z}). \qquad (2.2.11)
\end{aligned}$$

By Lemma 2.1.1 we may apply point (b) of Lemma 1.10.3 with $\varepsilon = s^{1/4}$, to see that for any $m\le n$ and $(s,y)\in[0,1]\times\mathbb{T}$ we have

$$\begin{aligned}
\Sigma^{(2)}_{q,m}(s,y,Z,\bar{Z}) &= \|\sigma(\phi^z(s,y))-\sigma(\phi^{\bar{z}}(s,y))\|_{\dot{\mathscr{W}}^m_q} \\
&\lesssim \sum_{i=0}^{m-1} s^{(m-i)/4}\|\phi^z(s,y)-\phi^{\bar{z}}(s,y)\|_{\dot{\mathscr{W}}^i_{2q}} + \|\phi^z(s,y)-\phi^{\bar{z}}(s,y)\|_{\dot{\mathscr{W}}^m_q}.
\end{aligned}$$

Thus by using the induction hypothesis and the definition of $F^{(2)}$, we can see that

$$\|\Sigma^{(2)}_{q,m}(s,y,Z,\bar{Z})\|_{L^{\mathscr{G}}_{p_1,p_2}} \lesssim s^{m/4}\sup_{y\in\mathbb{T}}\|Z(y)-\bar{Z}(y)\|_{L^{\mathscr{G}}_{p_1,p_2}} + \|F^{(2)}_{q,m}(s,y,Z,\bar{Z})\|_{L^{\mathscr{G}}_{p_1,p_2}}. \quad (2.2.12)$$

We can bound A by applying this result (with $m := n-1$) as follows:

$$
\begin{aligned}
&\|A(Z,\bar Z)\|_{L^{\mathscr{G}}_{p_1,p_2}} \\
&\lesssim \Big\| p_{t-\cdot}(x,\cdot)\big\|\Sigma^{(2)}_{q,n-1}(\cdot,\cdot,Z,\bar Z)\big\|_{L^{\mathscr{G}}_{p_1,p_2}} \mathbf{1}_{[0,t]}\Big\|_H \\
&\lesssim \Big\| p_{t-\cdot}(x,\cdot) \sup_{(\theta,\zeta)\in[0,t]\times\mathbb{T}} \Big(\theta^{\frac{n-1}{4}} \sup_{w\in\mathbb{T}} \|Z(w)-\bar Z(w)\|_{L^{\mathscr{G}}_{p_1,p_2}} + \|F^{(2)}_{q,n-1}(\theta,\zeta,Z,\bar Z)\|_{L^{\mathscr{G}}_{p_1,p_2}}\Big)\mathbf{1}_{[0,t]}\Big\|_H \\
&\lesssim t^{n/4} \sup_{w\in\mathbb{T}} \|Z(w)-\bar Z(w)\|_{L^{\mathscr{G}}_{p_1,p_2}} \qquad (2.2.13)
\end{aligned}
$$

where for the last inequality we used the induction hypothesis. Now we also bound B. To this end, note that by the BDG inequality we have

$$
B(z,\bar z) \lesssim \Big(\int_0^t \int_{\mathbb{T}} p^2_{t-r}(x,y)\|\sigma(\phi^z(r,y)) - \sigma(\phi^{\bar z}(r,y))\|^2_{\dot{\mathscr{W}}^n_q} dydr\Big)^{1/2},
$$

and thus using (2.2.12) we get

$$
\begin{aligned}
&\|B(Z,\bar Z)\|^2_{L^{\mathscr{G}}_{p_1,p_2}} \\
&\lesssim \int_0^t \int_{\mathbb{T}} p^2_{t-r}(x,y)\big\|\Sigma^{(2)}_{q,n}(r,y,Z,\bar Z)\big\|^2_{L^{\mathscr{G}}_{p_1,p_2}} dydr \\
&\lesssim \int_0^t \int_{\mathbb{T}} p^2_{t-r}(x,y)\Big(t^{n/4}\sup_{w\in\mathbb{T}}\|Z(w)-\bar Z(w)\|_{L^{\mathscr{G}}_{p_1,p_2}} + \|F^{(2)}_{q,n}(r,y,Z,\bar Z)\|_{L^{\mathscr{G}}_{p_1,p_2}}\Big)^2 dydr \\
&\lesssim t^{(n+1)/2}\sup_{w\in\mathbb{T}}\|Z(w)-\bar Z(w)\|^2_{L^{\mathscr{G}}_{p_1,p_2}} + \int_0^t\int_{\mathbb{T}} p^2_{t-r}(x,y)\|F^{(2)}_{q,n}(r,y,Z,\bar Z)\|^2_{L^{\mathscr{G}}_{p_1,p_2}} dydr. \quad (2.2.14)
\end{aligned}
$$

By putting the bounds (2.2.13) and (2.2.14) into (2.2.11) we can see that

$$
\begin{aligned}
\|F^{(2)}_{q,n}(t,x,Z,\bar Z)\|^2_{L^{\mathscr{G}}_{p_1,p_2}} &\lesssim t^{n/2}\sup_{w\in\mathbb{T}}\|Z(w)-\bar Z(w)\|^2_{L^{\mathscr{G}}_{p_1,p_2}} \\
&\quad + \int_0^t\int_{\mathbb{T}} p^2_{t-r}(x,y)\|F^{(2)}_{q,n}(r,y,Z,\bar Z)\|^2_{L^{\mathscr{G}}_{p_1,p_2}} dydr. \qquad (2.2.15)
\end{aligned}
$$

If $(Z,\bar Z) = (z,\bar z) \in (C(\mathbb{T}))^2$ is deterministic, then we may repeat the proof without assuming that $F^{(2)}_{q,n}(t,x,z,\bar z)$ is continuous in $(z,\bar z)$. The above inequality then simply states that

$$
|F^{(2)}_{q,n}(t,x,z,\bar z)|^2 \lesssim t^{n/2}\sup_{w\in\mathbb{T}}|z(w)-\bar z(w)|^2 + \int_0^t\int_{\mathbb{T}} p^2_{t-r}(x,y)|F^{(2)}_{q,n}(r,y,z,\bar z)|^2 dydr
$$

Note that by the definition of $F^{(2)}$ and by Proposition 2.1.2 we have

$$
\sup_{(t,x)\in[0,1]\times\mathbb{T}} |F^{(2)}_{q,n}(t,x,z,\bar z)| \le \sup_{(t,x)\in[0,1]\times\mathbb{T}} \|\phi^z(t,z)\|_{\dot{\mathscr{W}}^n_q} + \sup_{(t,x)\in[0,1]\times\mathbb{T}} \|\phi^{\bar z}(t,z)\|_{\dot{\mathscr{W}}^n_q} < \infty,
$$

so using Lemma 1.3.5 we get

$$\|\phi^{z}(t,x)-\phi^{\bar z}(t,x)\|_{\dot{\mathscr{W}}^n_q}=|F^{(2)}_{q,n}(t,x,z,\bar z)|\lesssim t^{n/4}\sup_{x\in\mathbb{T}}|z(x)-\bar z(x)|\lesssim \sup_{x\in\mathbb{T}}|z(x)-\bar z(x)|.$$

From this it easily follows that for all $(t,x)\in[0,1]\times\mathbb{T}$ the map $F^{(2)}_{q,n}(t,x,\cdot,\cdot)$ is of class $C((C(\mathbb{T}))^2,\mathbb{R})$. Now going back to (2.2.15), if $\|Z\|_{\mathbb{B}},\|\bar Z\|_{\mathbb{B}}\leq N$ for some given $N>0$ then the desired result follows by Lemma 1.3.5. For the general case we can repeat the truncation argument from the $n=0$ case to finish the proof. □

2.3 Nondegeneracy

Throughout the section we assume that $\sigma\in C^1$ such that there exists a constant $\mu>0$ such that for all $x\in\mathbb{R}$ we have $\sigma^2(x)\geq\mu^2$. Let $\phi^1,\dots,\phi^K$ solve the driftless equation (2.0.1) with initial conditions $\phi^1_0,\dots,\phi^K_0$ respectively. Consider the convex combination

$$\Theta(t,x):=\sum_{i=1}^{K}c_i\phi^i(t,x). \tag{2.3.16}$$

with $\sum_{i=1}^K c_i=1$ and $c_1,\dots,c_K\in[0,1]$. For a smooth map g and a nonnegative integer n, we aim to obtain estimates on the expectation of $\nabla^n g(\Theta(t,x))$ which depend only on a Besov–Hölder norm of g with a negative index, see Lemma 2.3.4 below.

The following lemma quantifies Theorem 4.5 in the chapter by Nualart in [DKM+09].

Lemma 2.3.1. *For any $p\in(2,\infty)$ there exists some constant $N=N(p,\|\sigma\|_{C^1},\mu)$, such that for all $t\in[0,1]$ we have*

$$\sup_{x\in\mathbb{T}}\mathbb{E}\|\mathscr{D}\Theta(t,x)\|_H^{-p}\leq Nt^{-p/4}.$$

Proof. By Proposition 2.1.3 we have for $(t,x),(\theta,\zeta)\in[0,1]\times\mathbb{T}$ that

$$\begin{aligned}\mathscr{D}_{\theta,\zeta}\Theta(t,x)&=\mathbf{1}_{[0,t]}(\theta)p_{t-\theta}(x,\zeta)\sum_{i=1}^K c_i\sigma(\phi^i(\theta,\zeta))\\&\quad+\mathbf{1}_{[0,t]}(\theta)\int_0^t\int_{\mathbb{T}}p_{t-r}(x,y)\Big(\sum_{i=1}^K c_i\sigma'(\phi^i(r,y))\mathscr{D}_{\theta,\zeta}\phi^i(r,y)\Big)\xi(dy,dr)\\&=:A(\theta,\zeta)+B(\theta,\zeta).\end{aligned}$$

From this we can see that

$$\int_0^t \int_{\mathbb{T}} |\mathscr{D}_{\theta,\zeta}\Theta(t,x)|^2 d\zeta d\theta \geq \frac{1}{2}\int_{t-\delta}^t \int_{\mathbb{T}} |A(\theta,\zeta)|^2 d\zeta d\theta - \int_{t-\delta}^t \int_{\mathbb{T}} |B(\theta,\zeta)|^2 d\zeta d\theta$$
$$=: I_A^\delta - I_B^\delta.$$

So since $|A(\theta,\zeta)| \geq \mathbf{1}_{[0,t]}(\theta) p_{t-\theta}(x,\zeta)\mu$, and by the properties of the heat kernel, it follows that

$$I_A^\delta = \frac{1}{2}\int_{t-\delta}^t \int_{\mathbb{T}} |A(\theta,\zeta)|^2 d\zeta d\theta \geq \mu^2 \int_{t-\delta}^t \int_{\mathbb{T}} |p_{t-\theta}(x,\zeta)|^2 d\zeta d\theta \geq k\mu^2\delta^{1/2}$$

for some universal constant $k > 0$. Thus

$$I_A^\delta \geq c_0\delta^{1/2} \quad \text{with} \quad c_0 = k\mu^2.$$

Therefore for $\varepsilon \in (0, c_0\delta^{1/2})$ we have

$$\begin{aligned}
\mathbb{P}\Big(\int_0^t \int_{\mathbb{T}} |\mathscr{D}_{\theta,\zeta}\Theta(t,x)|^2 d\zeta d\theta < \varepsilon\Big) &\leq \mathbb{P}(I_A^\delta - I_B^\delta < \varepsilon) \\
&\leq \mathbb{P}(I_B^\delta > c_0\delta^{1/2} - \varepsilon) \\
&\leq (c_0\delta^{1/2} - \varepsilon)^{-p}\mathbb{E}|I_B^\delta|^p
\end{aligned} \tag{2.3.17}$$

where the last inequality holds by Markov's inequality. We will now need to bound the expectation in the last line. Note that

$$\begin{aligned}
\mathbb{E}|I_B^\delta|^p &= \mathbb{E}\Big|\int_{t-\delta}^t \int_{\mathbb{T}} \Big|\int_0^t \int_{\mathbb{T}} p_{t-r}(x,y)\Big(\sum_{i=1}^K c_i\sigma'(\phi^i(r,y))\mathscr{D}_{\theta,\zeta}\phi^i(r,y)\Big)\xi(dy,dr)\Big|^2 d\zeta d\theta\Big|^p \\
&= \mathbb{E}\Big\|\int_0^t \int_{\mathbb{T}} p_{t-r}(x,y)\Big(\sum_{i=1}^K c_i\sigma'(\phi^i(r,y))\mathscr{D}\phi^i(r,y)\Big)\xi(dy,dr)\Big\|_{L_2([t-\delta,t]\times\mathbb{T})}^{2p} \\
&\lesssim \mathbb{E}\Big(\int_{t-\delta}^t \int_{\mathbb{T}} |p_{t-r}(x,y)|^2 \sum_{i=1}^K \|\mathscr{D}\phi^i(r,y)\|_{L_2([t-\delta,t]\times\mathbb{T})}^2 dydr\Big)^p
\end{aligned}$$

where we used the BDG inequality, and the fact that $\|\mathscr{D}\phi^i(r,y)\|_{L_2([t-\delta,t]\times\mathbb{T})} = 0$ for $r < t-\delta$. Noting that $r - \delta < t - \delta$, and that $\mathscr{D}_{\theta,\zeta}\phi^i(r,y) = 0$ for $\theta > r$, we may bound the $L_2([t-\delta,t]\times\mathbb{T})$ norm in the expression by the $L_2([r-\delta,r]\times\mathbb{T})$-norm, and write

$$\mathbb{E}|I_B^\delta|^p \lesssim \sum_{i=1}^K \mathbb{E}\Big(\int_{t-\delta}^t \int_{\mathbb{T}} |p_{t-r}(x,y)|^2 \|\mathscr{D}\phi^i(r,y)\|_{L_2([r-\delta,r]\times\mathbb{T})}^2 dydr\Big)^p$$

$$\leq \delta^{p/2-1}\sum_{i=1}^{K}\int_{t-\delta}^{t}\int_{\mathbb{T}} p_{t-r}(x,y)\mathbb{E}\|\mathscr{D}\phi^i(r,y)\|^{2p}_{L_2([r-\delta,r]\times\mathbb{T})}dydr$$
$$=: \delta^{p/2-1}\sum_{i=1}^{K} G_i \tag{2.3.18}$$

where we used Lemma 1.11.4 with $\gamma = 2$ and $\delta = 1$. To bound G_i, we will need to bound $\|\mathscr{D}\phi^i(r,y)\|_{L_2([r-\delta,r]\times\mathbb{T})}$. To this end, note that for $(\theta,\zeta) \in [0,t]\times\mathbb{T}$ we have

$$\mathscr{D}\phi^i(t,x) = p_{t-\theta}(x,\zeta)\sigma(\phi^i(\theta,\zeta)) + \int_0^t\int_{\mathbb{T}} p_{t-r}(x,y)\sigma'(\phi^i(r,y))\mathscr{D}_{\theta,\zeta}\phi^i(r,y)\xi(dy,dr).$$

Therefore using the BDG inequality, we get

$$\begin{aligned}\mathbb{E}\|\mathscr{D}\phi^i(t,x)\|^q_{L_2([t-\delta,t]\times\mathbb{T})} &\lesssim \|p_{t-\cdot}(x,\cdot)\|^q_{L_2([t-\delta,t]\times\mathbb{T})}\\ &\quad + \mathbb{E}\Big(\int_0^t\int_{\mathbb{T}} |p_{t-r}(x,y)|^2\|\mathscr{D}\phi^i(r,y)\|^2_{L_2([t-\delta,t]\times\mathbb{T})}dydr\Big)^{q/2}\\ &=: \bar{A}+\bar{B}.\end{aligned}$$

We have $\bar{A} \lesssim \delta^{q/4}$. Moreover by applying Lemma 1.11.4 with $\gamma = \delta = 2$ to $\bar{B}$ and noting again that $\|\mathscr{D}\phi^i(r,y)\|_{L_2([t-\delta,t]\times\mathbb{T})} \leq \|\mathscr{D}\phi^i(r,y)\|_{L_2([r-\delta,r]\times\mathbb{T})}$ we get

$$\bar{B} \lesssim t^{(p-1)/2}\int_0^t\int_{\mathbb{T}} |p_{t-r}(x,y)|^2\mathbb{E}\|\mathscr{D}\phi^i(r,y)\|^q_{L_2([r-\delta,r]\times\mathbb{T})}dydr$$

for $q > 2$. Therefore we obtain

$$\mathbb{E}\|\mathscr{D}\phi^i(t,x)\|^q_{L_2([r-\delta,r]\times\mathbb{T})} \lesssim \delta^{q/4} + \int_0^t\int_{\mathbb{T}} |p_{t-r}(x,y)|^2\mathbb{E}\|\mathscr{D}\phi^i(r,y)\|^q_{L_2([r-\delta,r]\times\mathbb{T})}dydr.$$

By Proposition 2.1.2 we can see that the q-th moment in the integrand is bounded in (r,y). Hence by Lemma 1.3.5 it follows that

$$\mathbb{E}\|\mathscr{D}\phi^i(t,x)\|^q_{L_2([r-\delta,r]\times\mathbb{T})} \lesssim \delta^{q/4}.$$

Applying this with $q = 2p$ to bound G_i, we get

$$G_i \lesssim \int_{t-\delta}^{t}\int_{\mathbb{T}} p_{t-r}(x,y)\delta^{p/2}dydr \lesssim \delta^{p/2+1}. \tag{2.3.19}$$

Now putting (2.3.19) into (2.3.18), we get $\mathbb{E}|I_B^\delta|^p \lesssim \delta^p$. Putting this into (2.3.17) we see that for all

$\delta \in [0,t]$ and all $\varepsilon \in (0, c_0\delta^{1/2})$, we have

$$\mathbb{P}(\|\mathscr{D}\Theta(t,x)\|_H^2 < \varepsilon) \lesssim (c_0\delta^{1/2} - \varepsilon)^{-p}\delta^p.$$

So if $\varepsilon \in (0, (c_0/2)\sqrt{t})$, we can choose $\delta(\varepsilon) := \frac{4}{c_0^2}\varepsilon^2$, to get

$$\mathbb{P}(\|\mathscr{D}\Theta(t,x)\|_H^2 < \varepsilon) \lesssim \varepsilon^p.$$

Let $L := (2/c_0)^{p/2}t^{-p/4}$. Notice that if $\gamma > L$, then $\gamma^{-2/p} \in (0, (c_0/2)\sqrt{t})$, and consequently we have

$$\mathbb{P}(\|\mathscr{D}\Theta(t,x)\|_H^2 < \gamma^{-2/p}) \lesssim \gamma^{-2}.$$

Hence, we have

$$\begin{aligned}\mathbb{E}\|\mathscr{D}\Theta(t,x)\|_H^{-p} = \int_0^\infty \mathbb{P}\big(\|\mathscr{D}\Theta\|_H^{-p} \geq \gamma\big)d\gamma &\leq L + \int_L^\infty \mathbb{P}(\|\mathscr{D}\Theta\|_H^2 < \gamma^{-2/p})d\gamma \\ &\lesssim L + \int_L^\infty \gamma^{-2}d\gamma \\ &\lesssim L + L^{-1} \lesssim t^{-p/4},\end{aligned}$$

which finishes the proof. □

For $(t,x) \in [0,1]\times\mathbb{T}$, we consider the H-valued random variables $w_{t,x}$ which are given for all $(\theta,\zeta) \in [0,1]\times\mathbb{T}$ by

$$w_{t,x}(\theta,\zeta) := \frac{\mathscr{D}_{\theta,\zeta}\Theta(t,x)}{\|\mathscr{D}\Theta(t,x)\|_H^2}.$$

For given $n \in \mathbb{N}$ and for a $C([0,1]\times\mathbb{T})$-valued random variable G_0 such that for all $(t,x) \in [0,1]\times\mathbb{T}$ $G_0(t,x) \in \mathscr{W}^n$, we may define iterated Skorokhod integrals for all $k \in \{0,\dots,n-1\}$ and $(t,x) \in [0,1]\times\mathbb{T}$ recursively by

$$G_{k+1}(t,x) = \delta(w_{t,x}G_k(t,x)).$$

Then by Proposition 1.10.1, for any $f \in C^\infty$ we have the integration-by-parts formula

$$\mathbb{E}\big(\nabla^k f(\Theta)G_0\big) = \mathbb{E}\big(f(\Theta)G_k\big)$$

To bound the iterations $(G_k)_{k\in\{0,\dots,n\}}$, we will need the following bounds on w and its Malliavin derivatives.

Lemma 2.3.2. *Let $p \in [1, \infty)$, $n \in \mathbb{Z}_{\geq 0}$, and $\sigma \in C^{n+1}$. Then there exists a constant $N = N(p, n, \|\sigma\|_{C^{n+1}}, \mu)$ such that for all $(t, x) \in [0, 1] \times \mathbb{T}$, we have*

$$\|\mathscr{D}^n w_{t,x}\|_{L_p(\Omega; H^{\otimes(n+1)})} \leq N t^{(n-1)/4}.$$

Proof. Fix $(t, x) \in [0, 1] \times \mathbb{T}$ and let $X := \Theta(t, x)$, $Y := \|\mathscr{D}\Theta(t, x)\|_H^2$ and $w := (\mathscr{D}X)/Y$. We may assume that $p > 2$. By Lemma 2.1.1, Lemma 2.1.5 and Lemma 2.3.1 respectively, we have

$$\|X\|_{\dot{\mathscr{W}}_p^k} \lesssim t^{\frac{k}{4}}, \qquad \|Y\|_{\dot{\mathscr{W}}_p^k} \lesssim t^{\frac{2+k}{4}}, \qquad \mathbb{E}(Y)^{-p} \lesssim t^{-\frac{2p}{4}}.$$

Therefore by Lemma 1.10.4 (with $\varepsilon := t^{1/4}$ and $c = 2$) to obtain

$$\|\mathscr{D}^n w\|_{L_p(\Omega; H^{\otimes(n+1)}} \lesssim (t^{1/4})^{n+1-2} \lesssim t^{(n-1)/4}$$

as required. □

Lemma 2.3.3. *Let $n \in \mathbb{Z}_{\geq 0}$, and $\sigma \in C^n$. Then for each $k, m \in \mathbb{Z}_{\geq 0}$ such that $k + m \leq n$ and for all $p \in [1, \infty)$ there exists a constant $N = N(k, m, p, \|\sigma\|_{C^n}, \mu)$ such that with $q := 2^m p$ we have for all $(t, x) \in [0, 1] \times \mathbb{T}$ that*

$$\|G_m(t, x)\|_{\mathscr{W}_p^k} \leq N t^{-m/4} \|G_0(t, x)\|_{\mathscr{W}_q^{k+m}}.$$

Proof. For notational convenience, fix $(t, x) \in [0, 1] \times \mathbb{T}$, set $w := w_{t,x}$, and for $i = 1, \ldots, n$ set $G_i := G_i(t, x)$. The proof will be done by induction with respect to the m variable. For $m = 0$ the statement is obviously true. Now suppose that the statement is true for some $m \leq n - 1$. That is, we suppose that for all $l \in \mathbb{Z}_{\geq 0}$ such that $l + m \leq n$ we have

$$\|G_m\|_{\mathscr{W}_p^l} \lesssim t^{-m/4} \|G_0\|_{\mathscr{W}_{2^m p}^{l+m}}.$$

We show that the statement is also true for $m + 1$, i.e. that for all $k \in \mathbb{Z}_{\geq 0}$ such that $k + (m + 1) \leq n$, we have

$$\|G_{m+1}\|_{\mathscr{W}_p^k} \lesssim t^{-(m+1)/4} \|G_0\|_{\mathscr{W}_{2^{m+1} p}^{k+m+1}}. \tag{2.3.20}$$

Let $k \in \mathbb{Z}_{\geq 0}$, such that $k+m+1 \leq n$. Since the divergence $\delta : \mathscr{W}_p^{k+1} \to \mathscr{W}_p^k$ is a bounded linear operator (see [Nua06, Proposition 1.5.7 and point 1 of remarks of Chapter 1]), we have

$$\begin{aligned}\|G_{m+1}\|_{\mathscr{W}_p^k} = \|\delta(wG_m)\|_{\mathscr{W}_p^k} &\lesssim \|wG_m\|_{\mathscr{W}_p^{k+1}(H)} \\ &\lesssim \sum_{i=0}^{k+1} \|\mathscr{D}^i(wG_m)\|_{L_p(\Omega;H^{\otimes(i+1)})} \\ &\lesssim \sum_{i=0}^{k+1} \sum_{\lambda_1+\lambda_2=i} \|\mathscr{D}^{\lambda_1} w\|_{L_{2p}(\Omega;H^{\otimes(\lambda_1+1)})} \|\mathscr{D}^{\lambda_2} G_m\|_{L_{2p}(\Omega;H^{\otimes\lambda_2})}. \quad (2.3.21)\end{aligned}$$

By Lemma 2.3.2, and since $\lambda_1 \geq 0$, we have

$$\|\mathscr{D}^{\lambda_1} w\|_{L_{2p}(\Omega;H^{\otimes(\lambda_1+1)})} \lesssim t^{(\lambda_1-1)/4} \lesssim t^{-1/4}. \quad (2.3.22)$$

Moreover since $\lambda_2 + m \leq k+1+m \leq n$, by the induction hypothesis we have

$$\|\mathscr{D}^{\lambda_2} G_m\|_{L_{2p}(\Omega;H^{\otimes\lambda_2})} \leq \|G_m\|_{\mathscr{W}_{2p}^{\lambda_2}} \lesssim t^{-m/4}\|G_0\|_{\mathscr{W}_{2^{m+1}p}^{\lambda_2+m}} \leq t^{-m/4}\|G_0\|_{\mathscr{W}_{2^{m+1}p}^{k+1+m}}. \quad (2.3.23)$$

Now putting (2.3.22) and (2.3.23) into (2.3.21), we get

$$\|G_{m+1}\|_{\mathscr{W}_p^k} \lesssim t^{-1/4} t^{-m/4} \|G_0\|_{\mathscr{W}_{2^{m+1}p}^{k+1+m}}.$$

Hence (2.3.20) holds, and the proof is finished. □

Lemma 2.3.4. *Let $n \in \mathbb{Z}_{\geq 0}$, $\sigma \in C^{n+1}$, $\beta \in (-2,-1) \cup (-1,0)$, and set $q := 2^{n+2}\mathbf{1}_{(-1,0)}(\beta) + 2^{n+3}\mathbf{1}_{(-2,-1)}(\beta)$ and $m := (n+1)\mathbf{1}_{(-1,0)}(\beta) + (n+2)\mathbf{1}_{(-2,-1)}(\beta)$. There exists a constant $N = N(n, \beta, \|\sigma\|_{C^{n+1}}, \mu)$ such that for all $g \in C^\infty$, $(t,x) \in [0,1] \times \mathbb{T}$, we have*

$$\left|\mathbb{E}\big(\nabla^n g(\Theta(t,x))G_0(t,x)\big)\right| \leq N\|g\|_{C^\beta} t^{\frac{\beta-n}{4}} \|G_0(t,x)\|_{\mathscr{W}_q^m}.$$

Proof. Let $f \in C^\infty$ be the solution of $(1-\Delta)f = g$ and let $(t,x) \in [0,1] \times \mathbb{T}$. By Proposition 1.10.1 and by the definition of f, we get

$$\begin{aligned}|\mathbb{E}(\nabla^n g(\Theta(t,x))G_0(t,x))| &= |\mathbb{E}(g(\Theta(t,x))G_n(t,x))| \\ &\leq |\mathbb{E}(f(\Theta(t,x))G_n(t,x))| + |\mathbb{E}(\Delta f(\Theta(t,x))G_n(t,x))| =: A + B.\end{aligned}$$

It follows easily from Lemma 2.3.3 and Proposition 1.4.4 that

$$A \leq \|f\|_{\mathbb{B}} \|G_n(t,x)\|_{L_1} \leq \|f\|_{C^{2+\beta}} t^{-n/4} \|G_0(t,x)\|_{\mathscr{W}^n_{2n}} \lesssim \|g\|_{C^\alpha} t^{-n/4+\beta/4} \|G_0(t,x)\|_{\mathscr{W}^m_q}. \quad (2.3.24)$$

It remains to be shown that the desired bound also holds on B. To this end, we first note that by Jensen's inequality and by the BDG inequality for $\gamma \in (0,1)$ we have

$$\begin{aligned}\Big\|\Big(\Theta(t,x) - \sum_{i=1}^K c_i P_t \phi^i(0,\cdot)(x)\Big)^\gamma\Big\|_{L_2} &\leq \Big\|\sum_{i=1}^K c_i \Big(\phi^i(t,x) - P_t\phi^i(0,\cdot)(x)\Big)\Big\|^\gamma_{L_2} \\ &\leq \sum_{i=1}^K \Big\|\int_0^t \int_{\mathbb{T}} p_{t-r}(x,y)\sigma(\phi^i(r,y))\xi(dy,dr)\Big\|^\gamma_{L_2} \\ &\lesssim t^{\gamma/4}. \end{aligned} \quad (2.3.25)$$

We first consider the case that $\beta \in (-1,0)$. By Proposition 1.10.1, the fact that $\mathbb{E}G_{n+1} = 0$, and (2.3.25) with $\gamma = 1+\beta \in (0,1)$, we get

$$\begin{aligned} B &= |\mathbb{E}(\nabla f(\Theta(t,x)) G_{n+1}(t,x))| \\ &= \Big|\mathbb{E}\Big(\Big(\nabla f(\Theta(t,x)) - \nabla f\Big(\sum_{i=1}^K c_i P_t \phi^i(0,\cdot)(x)\Big)\Big) G_{n+1}(t,x)\Big)\Big| \\ &\lesssim \|\nabla f\|_{C^{1+\beta}} \Big\|\Big(\Theta(t,x) - \sum_{i=1}^K c_i P_t\phi^i(0,\cdot)(x)\Big)^{1+\beta}\Big\|_{L_2} \|G_{n+1}(t,x)\|_{L_2} \\ &\lesssim \|f\|_{C^{2+\beta}} t^{\frac{1+\beta}{4}} \|G_{n+1}(t,x)\|_{L_2} \\ &\lesssim \|g\|_{C^\beta} t^{\frac{\beta-n}{4}} \|G_0(t,x)\|_{\mathscr{W}^{n+1}_q}, \end{aligned} \quad (2.3.26)$$

where for the last inequality we used Proposition 1.4.4 and Lemma 2.3.3. We now also deal with the case when $\beta \in (-2,-1)$. Repeating the same steps with one more iteration of Malliavin integration by parts and with $\gamma = 2+\beta \in (0,1)$, we can see that

$$\begin{aligned} B &= |\mathbb{E}(f(\Theta(t,x)) G_{n+2}(t,x))| \\ &\lesssim \|f\|_{C^{2+\beta}} \Big\|\Big(\Theta(t,x) - \sum_{i=1}^K c_i P_t\phi^i(0,\cdot)(x)\Big)^{2+\beta}\Big\|_{L_2} \|G_{n+2}(t,x)\|_{L_2} \\ &\lesssim \|g\|_{C^\beta} t^{\frac{\beta-n}{4}} \|G_0(t,x)\|_{\mathscr{W}^{n+2}_q}. \end{aligned} \quad (2.3.27)$$

By (2.3.26) and (2.3.27) we can see that for all $\beta \in (-2,-1) \cup (-1,0)$ we have

$$B \lesssim \|g\|_{C^\beta} t^{\frac{\beta-n}{4}} \|G_0(t,x)\|_{\mathscr{W}^m_q}. \quad (2.3.28)$$

By the bounds (2.3.24) and (2.3.28) on A and B respectively, the proof is finished. □

Let $s \geq 0$ and suppose that $Z : \Omega \times \mathbb{T} \to \mathbb{R}$ is an $\mathscr{F}_s \otimes \mathscr{B}(\mathbb{T})$-measurable map, such that $Z(x)$ is continuous in x and that $\sup_{x\in\mathbb{T}} \|Z(x)\|_{L_2} < \infty$. Let $\phi^{Z,s}$ denote the solution of

$$(\partial_t - \Delta)\phi^{Z,s} = \sigma(\phi^{Z,s})\xi \qquad \text{in } (s,1)\times\mathbb{T}, \qquad \phi_s^{Z,s} = Z. \tag{2.3.29}$$

For $(t,x) \in [s,1]\times\mathbb{T}$, the solution satisfies the integral equation

$$\phi^{Z,s}(t,x) = P_{t-s}Z(x) + \int_s^t \int_{\mathbb{T}} p_{t-r}(x,y)\sigma(\phi^{Z,s}(r,y))\xi(dy,dr). \tag{2.3.30}$$

We will moreover use the shorthand

$$\phi^Z(t,x) := \phi^{Z,0}(t,x). \tag{2.3.31}$$

In the next lemma, we show a Markov-type property which will be used often. Recall that $C(\mathbb{T})$ denotes the collection of continuous functions $f : \mathbb{T} \to \mathbb{R}$, and it is equipped with the sup-norm $\|\cdot\|_{\mathbb{B}}$. The topology induced by this norm generates the Borel σ-algebra $\mathscr{B}(C(\mathbb{T}))$ which coincides with the cylindrical σ-algebra. Moreover, recall that since $C(\mathbb{T})$ is separable, the notions of measurable, weakly measurable, and strongly measurable $C(\mathbb{T})$-valued maps on Ω coincide. In addition, a continuous random field $u : \Omega\times\mathbb{T} \to \mathbb{R}$ is actually a $C(\mathbb{T})$-valued random variable.

Lemma 2.3.5. *Let $b, \sigma \in C^1(\mathbb{R})$, $M \in \mathbb{N}$, $(Z_i)_{i=1}^M \subset L_2(\Omega, \mathscr{F}_s, \mathbb{P}; C(\mathbb{T})) \cap \mathbb{B}(\mathbb{T}, L_2(\Omega))$, and let $\phi^{Z,s}$ be the unique solution of (2.3.29). Further, for $p \in [1,\infty)$, $f \in C^1(\mathbb{R}^M)$, $t \in [s,1]$, and $x \in \mathbb{T}$, define $g : (C(\mathbb{T}))^M \to \mathbb{R}$ by*

$$g(z_1,\dots,z_M) := \mathbb{E}f\big(\phi^{z_1}(t-s,x),\dots,\phi^{z_M}(t-s,x)\big).$$

Then, for $i = 1,\dots,M$ and $Z_i \in L_2(\Omega, \mathscr{F}_s, \mathbb{P}; C(\mathbb{T})) \cap \mathbb{B}(\mathbb{T}, L_2(\Omega))$, we have

$$\mathbb{E}^s f\big(\phi^{Z_1,s}(t,x),\dots,\phi^{Z_M,s}(t,x)\big) = g(Z_1,\dots,Z_M). \tag{2.3.32}$$

Proof. Suppose first that the Z_i are simple random variables of the form

$$Z_i = \sum_{k=1}^K h_{k,i}\mathbf{1}_{E_k} \tag{2.3.33}$$

where $K \in \mathbb{N}$, $(h_{k,i})_{k=1}^K \subset C(\mathbb{T})$ and $(E_k)_{k=1}^K \subset \mathscr{F}_s$ is a partition of Ω. In this case, we have for $(t,x) \in [s,1] \times \mathbb{T}$ that

$$\begin{aligned}
&\mathbb{E}^s f(\phi^{Z_1,s}(t,x), \ldots, \phi^{Z_M,s}_{s,t}(t,x)) \\
&= \mathbb{E}^s f\Big(\sum_{k=1}^K \mathbf{1}_{E_k} \phi^{h_{k,1},s}(t,x), \ldots, \sum_{k=1}^K \mathbf{1}_{E_k} \phi^{h_{k,M},s}(t,x)\Big) \\
&= \mathbb{E}^s \sum_{k=1}^K \mathbf{1}_{E_k} f(\phi^{h_{k,1},s}(t,x), \ldots, \phi^{h_{k,M},s}(t,x)) \\
&= \sum_{k=1}^K \mathbf{1}_{E_k} \mathbb{E}^s f(\phi^{h_{k,1},s}(t,x), \ldots, \phi^{h_{k,M},s}(t,x)) \\
&= \sum_{k=1}^K \mathbf{1}_{E_k} \mathbb{E} f(\phi^{h_{k,1}}(t-s,x), \ldots, \phi^{h_{k,M}}(t-s,x)) \\
&= \sum_{k=1}^K \mathbf{1}_{E_k} g(h_{k,1}, \ldots, h_{k,M}) = g(Z_1, \ldots, Z_M),
\end{aligned}$$

which shows (2.3.32). For the general case, since $Z_i \in L_2(\Omega, \mathscr{F}_s, \mathbb{P}; C(\mathbb{T}))$, for $i = 1, \ldots, M$ there exist sequences $(Z_i^n)_{n\in\mathbb{N}}$ of the form (2.3.33) such that $\|Z_i^n - Z_i\|_{\mathbb{B}(\mathbb{T})} \to 0$ almost surely and in $L_2(\Omega)$ as $n \to \infty$. For those $Z_i^n s$ and for $(t,x) \in [s,1] \times \mathbb{T}$ we have

$$\mathbb{E}^s f(\phi^{Z_1^n,s}(t,x), \ldots \phi^{Z_M^n,s}(t,x)) = g(Z_1^n, \ldots, Z_M^n). \tag{2.3.34}$$

It follows from Lemma 2.2.1 that for all $(t,x) \in [s,1] \times \mathbb{T}$, the map $L_2(\Omega; C(\mathbb{T})) \ni Z \mapsto \phi^{Z,s}(t,x) \in L_2(\Omega)$ is Lipschitz. From this, it firstly follows that $\phi^{Z_i^n,s}(t,x) \to \phi^{Z_i,s}(t,x)$ in $L_2(\Omega)$, which by using the Lipschitz continuity of f implies that

$$\mathbb{E}^s f(\phi^{Z_1^n,s}(t,x), \ldots, \phi^{Z_M^n,s}(t,x)) \longrightarrow \mathbb{E}^s f(\phi^{Z_1,s}(t,x), \ldots \phi^{Z_M,s}(t,x))$$

in $L_2(\Omega)$. Secondly, it also follows that the function $g : C(\mathbb{T})^M \to \mathbb{R}$ is continuous. Hence, upon taking the limit in probability with $n \to \infty$ in (2.3.34), the result follows. □

Lemma 2.3.6. *Let $K \in \mathbb{N}$, and for $z_1, \ldots, z_K \in C(\mathbb{T})$, $(s,t) \in [0,1]^2_{\leq}$, $x \in \mathbb{T}$, consider the convex combination*

$$\Phi^{z_1,\ldots,z_K,s}(t,x) := \sum_{i=1}^K c_i \phi^{z_i,s}(t,x).$$

Let $h \in C(\mathbb{R}^K)$*, and for* $q \in [1,\infty)$ *and* $i \in \mathbb{Z}_{\geq 0}$ *define* $H_{q,i} : (C(\mathbb{T}))^K \to \mathbb{R}$ *by*

$$H_{q,i}(z_1,\dots,z_K) := \|h\big(\phi^{z_1}(t-s,x),\dots,\phi^{z_K}(t-s,x)\big)\|_{\dot{\mathscr{W}}_q^i}.$$

Let $Z_1,\dots,Z_K$ *be* $\mathscr{F}_s$*-measurable* $C(\mathbb{T})$*-valued random variables and* $g \in C^\infty(\mathbb{R})$*. For all* $n \in \mathbb{Z}_{\geq 0}$*,* $\beta \in (-2,-1)\cup(-1,0)$ *there exists a constant* $N = N(n,\beta,\|\sigma\|_{C^{n+1}},\mu)$ *such that with* $q := 2^{n+2}\mathbf{1}_{(-1,0)}(\beta)+2^{n+3}\mathbf{1}_{(-2,-1)}(\beta)$ *and* $m := (n+1)\mathbf{1}_{(-1,0)}(\beta)+(n+2)\mathbf{1}_{(-2,-1)}(\beta)$ *we have*

$$\begin{aligned}&\Big|\mathbb{E}^s\Big(\nabla^n g\Big(\Phi^{Z_1,\dots,Z_K,s}(t,x)\Big)h\big(\phi^{Z_1,s}(t,x),\dots,\phi^{Z_K,s}(t,x)\big)\Big)\Big| \\ &\qquad\qquad\leq N\|g\|_{C^\beta}(t-s)^{\frac{\beta-n}{4}}\sum_{i=0}^m H_{q,i}\big(Z_1,\dots,Z_K\big).\end{aligned}$$

Proof. By Lemma 2.3.5 we have

$$\Big|\mathbb{E}^s\Big(\nabla^n g\big(\Phi^{Z_1,\dots,Z_K,s}(t,x)\big)h\big(\phi^{Z_1,s}(t,x),\dots,\phi^{Z_K,s}(t,x)\big)\Big)\Big| = G(Z_1,\dots,Z_K), \tag{2.3.35}$$

where for $z_1,\dots,z_K \in C(\mathbb{T})$ we define

$$G(z_1,\dots,z_K) := \Big|\mathbb{E}\Big(\nabla^n g\Big(\sum_{i=1}^K c_i\phi^{z_i}(t-s,x)\Big)h(\phi^{z_i}(t-s,x),\dots,\phi^{z_K}(t-s,x))\Big)\Big|.$$

By Lemma 2.3.4 we have

$$\begin{aligned}G(z_1,\dots,z_K) &\lesssim \|g\|_{C^\beta}(t-s)^{\frac{\beta-n}{4}}\|h(\phi^{z_1}(t-s,x),\dots,\phi^{z_M}(t-s,x))\|_{\mathscr{W}_q^m} \\ &\lesssim \|g\|_{C^\beta}(t-s)^{\frac{\beta-n}{4}}\sum_{i=0}^m\|h(\phi^{z_1}(t-s,x),\dots,\phi^{z_M}(t-s,x))\|_{\dot{\mathscr{W}}_q^i} \\ &= \|g\|_{C^\beta}(t-s)^{\frac{\beta-n}{4}}\sum_{i=0}^m H_{q,i}(z_1,\dots,z_K).\end{aligned} \tag{2.3.36}$$

Now putting (2.3.36) into (2.3.35), the desired result follows. □

2.4 Estimates on the density

In the previous sections we have proven estimates on the Malliavin derivatives of the solution $\phi(t,x)$ to driftless equation (2.0.1) equation, which allowed us to prove Lemma 2.3.3. In turn this lemma implies estimates on the Hölder norm of the density $f_{t,x} : \mathbb{R} \to [0,\infty)$ of $\phi(t,x)$..

Theorem 2.4.1 (Regularity of density)**.** *Let* $\gamma > 0$*. Suppose that* $\sigma \in C^{\lfloor\gamma\rfloor+2}$ *satisfies* $\sigma \geq \mu$ *for some* *constant $\mu > 0$. For $(t,x) \in (0,1] \times \mathbb{T}$ the solution $\phi(t,x)$ of (2.0.1) admits a density $f_{t,x}$ and there exists a constant $N = N(\|\sigma\|_{C^{\lfloor\gamma\rfloor+2}}, \mu)$, such that for each $(t,x) \in (0,1] \times \mathbb{T}$, the density satisfies*

$$\sup_{x\in\mathbb{T}} \|f_{t,x}\|_{C^\gamma(\mathbb{R},\mathbb{R})} \le Nt^{-(\gamma+1)/4}.$$

Proof. Let $(t,x) \in [0,1] \times \mathbb{T}$, and define iterated Skorokhod integrals by

$$\begin{aligned} G_0 &:= 1, \\ G_{k+1}(t,x) &:= \delta\Big(\frac{\mathscr{D}\phi(t,x)}{\|\mathscr{D}\phi(t,x)\|_{H^2}} G_k(t,x)\Big). \end{aligned}$$

Then by Proposition 1.10.1 for $n \in \mathbb{Z}_{\ge 0}$ and for any n-times differentiable map $g : \mathbb{R} \to \mathbb{R}$ we have that $\phi(t,x)$ and the sequence $(G_n(t,x))_{n\in\mathbb{N}}$ satisfies the integration by parts formula

$$\mathbb{E}\nabla^n g(\phi(t,x)) = \mathbb{E}\Big(g(\phi(t,x))G_n(t,x)\Big).$$

Hence by [SS04, Proposition 2.1/(2.3)] we have that the n-th derivative $\partial^n f_{t,x}(z) := \frac{\partial^n f_{t,x}(z)}{\partial z^n}$ of the density of ϕ is given for $z \in \mathbb{R}$ by

$$\partial^n f_{t,x}(z) = (-1)^n \mathbb{E}\Big(\mathbf{1}_{z\le\phi(t,x)} G_{n+1}(t,x)\Big).$$

Therefore using Lemma 2.3.3 we get

$$|\partial^n f_{t,x}(z)| \lesssim \mathbb{E}\Big(|\mathbf{1}_{z\le\phi(t,x)}||G_{n+1}|\Big) \lesssim \|G_{n+1}\|_{L_1} \lesssim t^{-(n+1)/4}. \tag{2.4.37}$$

Note moreover that for $\beta \in (0,1)$, for $z_1, z_2 \in \mathbb{R}$, we have using the above inequality that

$$\begin{aligned} |\partial^n f_{t,x}(z_1) - \partial^n f_{t,x}(z_2)| &= |\partial_z^n f_{t,x}(z_1) - \partial_z^n f_{t,x}(z_2)|^{1-\beta} |\partial_z^n f_{t,x}(z_1) - \partial_z^n f_{t,x}(z_2)|^\beta \\ &\lesssim \|\partial_z^n f_{t,x}\|_{\mathbb{B}}^{1-\beta} \|\partial_z^n f_{t,x}\|_{C^1}^\beta |z_1 - z_2|^\beta \\ &\lesssim \|f_{t,x}\|_{C^n}^{1-\beta} \|f_{t,x}\|_{C^{n+1}}^\beta |z_1 - z_2|^\beta \\ &\lesssim \big(t^{-\frac{n+1}{4}}\big)^{1-\beta} \big(t^{-\frac{n+2}{4}}\big)^\beta |z_1 - z_2|^\beta = t^{-\frac{n+1-\beta n-\beta+\beta n+2\beta}{4}} |z_1 - z_2|^\beta \\ &= t^{-\frac{n+1+\beta}{4}} |z_1 - z_2|^\beta. \end{aligned}$$

Therefore we get

$$[\partial^n f_{t,x}]_{C^\beta} \lesssim t^{-\frac{n+1+\beta}{4}}. \tag{2.4.38}$$

Finally, the desired estimate follows by applying (2.4.37) and (2.4.38) with $n := \lfloor \gamma \rfloor$ and with $\beta := \gamma - \lfloor \gamma \rfloor$ to obtain

$$\|f_{t,x}\|_{C^\gamma} = \|f_{t,x}\|_{C^{n+\beta}} = \|f_{t,x}\|_{C^n} + [f^n_{t,x}]_{C^\beta} \lesssim t^{-\frac{n+1}{4}} + t^{-\frac{n+1+\beta}{4}} \lesssim t^{-\frac{n+1+\beta}{4}} = t^{-\frac{\gamma+1}{4}},$$

and thus the proof is finished. □

Chapter 3

Well-posedness

3.1 Driftless approximation

In this section we deal with the approximation of the solution $u(t,x)$ by $\phi^{u(s,\cdot),s}(t,x)$. The main results of this section are Lemma 3.1.4 and Lemma 3.1.6.

Lemma 3.1.1 (Boundedness of regularised solutions)**.** *Let u be a regularised solution of (1.3.9) with initial condition $u(0,\cdot) = u_0 \in C(\mathbb{T})$ and let $p \in [2,\infty)$. There exists a constant $N = N(p, \|\sigma\|_{\mathbb{B}})$ such that for all $(t,x) \in [0,1] \times \mathbb{T}$ we have*

$$\|u(t,x)\|_{L_p} \leq N\Big(\|u_0\|_{\mathbb{B}(\mathbb{T})} + \|D^u_t(x)\|_{L_p} + t^{1/4}\Big). \tag{3.1.1}$$

Consequently, if $u \in \mathscr{U}^0_p$, then

$$\sup_{(t,x)\in[0,1]\times\mathbb{T}} \|u(t,x)\|_{L_p} < \infty. \tag{3.1.2}$$

Proof. From (1.7.18) and the BDG inequality we can see that

$$\|u(t,x)\|^2_{L_p} \lesssim \|u_0\|^2_{\mathbb{B}(\mathbb{T})} + \|D^u_t(x)\|^2_{L_p} + \int_0^t \int_{\mathbb{T}} p^2_{t-r}(x,y)\|\sigma\|^2_{\mathbb{B}} dy dr,$$

and thus the inequality (3.1.1) follows. Now suppose that $u \in \mathscr{U}^0_p$. Then noting that $D^u_t = D^u_t - P_{t-0}D^u_0$, we have

$$\sup_{(t,x)\in[0,1]\times\mathbb{T}} \|D^u_t(x)\|_{L_p} \leq [D^u]_{\mathscr{V}^0_p} < \infty.$$

Also, note that by Assumption 1.7.1 the initial condition $u(0,\cdot)$ is bounded. Therefore all terms on the right hand side of (3.1.1) are bounded in (t, x), and thus (3.1.2) follows. □

Recall that the random fields $\phi^{z,s}(t, x)$ and $\phi^{z}(t, x)$ are defined by (2.3.30) and (2.3.31) respectively. Let $\sigma \in C^1$, $\alpha \in (-1, 0)$, $\beta \in [0, 1]$, $p \in [1, \infty)$, and for $i = 1, 2$ let $b^i \in C^\alpha$, and suppose that $u^i \in \mathscr{U}_p^\beta$ are regularised solutions of the stochastic reaction–diffusion equations

$$(\partial_t - \Delta)u^i = b^i(u^i) + \sigma(u^i)\xi. \tag{3.1.3}$$

For $(S, T) \in [0, 1]^2_\leq$ we define the $\mathscr{S}_p^\beta[S, T]$-*bracket* of u^1 and u^2 by

$$[u^1, u^2]_{\mathscr{S}_p^\beta[S,T]} := \sup_{x\in\mathbb{T}} \sup_{(s,t)\in[S,T]^2_\leq} \frac{\|u^1(t, x) - \phi^{u^1(s,\cdot),s}(t, x) - u^2(t, x) + \phi^{u^2(s,\cdot),s}(t, x)\|_{L_p}}{|t - s|^\beta}. \tag{3.1.4}$$

Remark 3.1.2. Note that for all $s \in [0, 1]$, by definition, the random field $u^i(s, x)$ is continuous in x, and by Lemma 3.1.1 we have $\sup_{x\in\mathbb{T}} \|u^i(s, x)\|_{L_p} < \infty$. Therefore the equation (2.3.29) starting from $u^i(s, \cdot)$ does indeed have a unique solution (see e.g. [Wal86]), which is denoted by $\phi^{u^i(s,\cdot),s}(t, x)$. Consequentially, the expression (3.1.4) is well-defined.

For brevity, we will use the convention

$$[u^1, u^2]_{\mathscr{S}_p^\beta} := [u^1, u^2]_{\mathscr{S}_p^\beta[0,1]}.$$

Moreover recalling the definition of the $\mathscr{V}_p^\beta[S, T]$-bracket from Definition 1.7.4, we set

$$[f]_{\mathscr{V}_p^\beta} := [f]_{\mathscr{V}_p^\beta[0,1]}.$$

By the triangle inequality and by Lemma 2.2.1 the following result holds:

Lemma 3.1.3. *Let $\sigma \in C^1$, $b^1, b^2 \in C^\alpha$, $\beta \in (0, 1]$, $p \in [2, \infty)$ and let u^1, u^2 be regularised solutions of (3.1.3) in the class $\mathscr{U}_p^\beta$. There exists some constant $N = N(p, \|\sigma\|_{C^1}, \alpha, \beta)$ such that for all $(s, t) \in [0, 1]^2_\leq$ we have*

$$\|u^1(t, \cdot) - u^2(t, \cdot)\|_{\mathbb{B}(\mathbb{T},L_p)} \leq [u^1, u^2]_{\mathscr{S}_p^\beta[s,t]}(t - s)^\beta + N\|u^1(s, \cdot) - u^2(s, \cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}.$$

Lemma 3.1.4 (Driftless approximation)**.** *Let $\sigma \in C^1$, $\alpha \in (-1, 0)$, $b \in C^\alpha$. Let $p \in [2, \infty)$, $\beta \in [0, 1 + \frac{\alpha}{4}]$ and let u be a regularised solution of (1.3.9) in the class $\mathscr{U}_p^\beta$. Then there exists a constant*

$N = N(p, \|\sigma\|_{C^1}, \alpha, \beta)$ *such that for all* $0 \le s \le t \le 1$, *we have*

$$\sup_{x\in\mathbb{T}} \|u(t,x) - \phi^{u(s,\cdot),s}(t,x)\|_{L^{\mathscr{F}_s}_{p,\infty}} \le N[D^u]_{\mathscr{V}^\beta_p[s,t]}(t-s)^\beta.$$

Proof. By splitting the stochastic integral in (2.0.2) at time s, we have

$$u(t,x) = P_t u(0,\cdot)(x) + D^u_t(x) + \int_0^s \int_{\mathbb{T}} p_{t-r}(x,y)\sigma(u(r,y))\xi(dy,dr) + \int_s^t \int_{\mathbb{T}} p_{t-r}(x,y)\sigma(u(r,y))\xi(dy,dr). \tag{3.1.5}$$

Moreover using (2.3.30) to compute $\phi^{u(s,\cdot),s}(t,x)$ and then (1.7.18) to express $u(s,\cdot)$, we get

$$\begin{aligned}\phi^{u(s,\cdot),s}(t,x) &= P_{t-s}\Big(P_s u(0,\cdot) + D^u_s(\cdot) + \int_0^s \int_{\mathbb{T}} p_{s-r}(\cdot,y)\sigma(u(r,y))\xi(dy,dr)\Big)(x)\\ &\quad + \int_s^t \int_{\mathbb{T}} p_{t-r}(x,y)\sigma(\phi^{u(s,\cdot),s}(r,y))\xi(dy,dr)\\ &= P_t u(0,\cdot)(x) + P_{t-s}D^u_s(x) + \int_0^s \int_{\mathbb{T}} p_{t-r}(x,y)\sigma(u(r,y))\xi(dy,dr)\\ &\quad + \int_s^t \int_{\mathbb{T}} p_{t-r}(x,y)\sigma(\phi^{u(s,\cdot),s}(r,y))\xi(dy,dr)\end{aligned} \tag{3.1.6}$$

where the last equality follows from the semigroup property that $P_{t-s}P_s = P_t$. Comparing (3.1.5) and (3.1.6), we can see that the error of the driftless approximation is given by

$$u(t,x) - \phi^{u(s,\cdot),s}(t,x) = D^u_t(x) - P_{t-s}D^u_s(x) + \int_s^t \int_{\mathbb{T}} p_{t-r}(x,y)\big(\sigma(u(r,y)) - \sigma(\phi^{u(s,\cdot),s}(r,y))\big)\xi(dy,dr).$$

Hence by the conditional BDG inequality (see Lemma 1.11.5), we get

$$\|u(t,x) - \phi^{u(s,\cdot),s}(t,x)\|^2_{L_p|\mathscr{F}_s} \lesssim \|D^u_t(x) - P_{t-s}D^u_s(x)\|^2_{L_p|\mathscr{F}_s} + \int_s^t \int_{\mathbb{T}} p^2_{t-r}(x,y)\big\|\sigma(u(r,y)) - \sigma(\phi^{u(s,\cdot),s}(r,y))\big\|^2_{L_p|\mathscr{F}_s} dydr. \tag{3.1.7}$$

Therefore

$$\|u(t,x) - \phi^{u(s,\cdot),s}(t,x)\|^2_{L^{\mathscr{F}_s}_{p,\infty}} \lesssim [D^u]^2_{\mathscr{V}^\beta_p[s,t]}(t-s)^{2\beta} + \int_s^t \int_{\mathbb{T}} p^2_{t-r}(x,y)\|u(r,y) - \phi^{u(s,\cdot),s}(r,y)\|^2_{L^{\mathscr{F}_s}_{p,\infty}} dydr. \tag{3.1.8}$$

From (3.1.7) and the fact that $u \in \mathscr{U}_p^\beta$, we see that

$$\sup_{(t,x)\in[0,1]\times\mathbb{T}} \|u(t,x) - \phi^{u(s,\cdot),s}(t,x)\|_{L_{p,\infty}^{\mathscr{F}_s}} \lesssim [D^u]_{\mathscr{V}_p^0} + \|\sigma\|_{\mathbb{B}} < \infty.$$

Hence by (3.1.8) and by Lemma 1.3.5, we obtain that

$$\sup_{x\in\mathbb{T}} \|u(t,x) - \phi^{u(s,\cdot),s}(t,x)\|_{L_{p,\infty}^{\mathscr{F}_s}}^2 \lesssim [D^u]_{\mathscr{V}_p^\beta[s,t]}^2 (t-s)^{2\beta},$$

which implies the desired result. □

Assumption 3.1.5. Let $\alpha \in (-1,0)$, $b \in C^\alpha$, $n \in \mathbb{Z}_{\geq 0}$, and let $\sigma \in C^{n+2}$ such that there exists a constant $\mu > 0$ such that for all $x \in \mathbb{R}$, $\sigma^2(x) > \mu^2$. Let $\beta \in [\frac{1}{2}, 1 + \frac{\alpha}{4}]$, and suppose that u^1, u^2 are regularised solutions of (1.3.9) in the class $\mathscr{U}^\beta$,

For $(s,a) \in [0,1]_{\leq}^2$ consider the $(C(\mathbb{T}))^4$-valued $\mathscr{F}_a$-measurable random variable

$$Z := \Big(\phi^{u^1(s,\cdot),s}(a,\cdot), \phi^{u^2(s,\cdot),s}(a,\cdot), u^1(a,\cdot), u^2(a,\cdot)\Big). \tag{3.1.9}$$

Recall the definitions of $F^{(2)}$ and $\Sigma^{(2)}$ from (2.2.7) and (2.2.8). Moreover for $(t,x) \in [0,1] \times \mathbb{T}$ and $z = (z_1, \ldots, z_4) \in (C(\mathbb{T}))^4$, define

$$F_{q,n}^{(4)}(t,x,z) := \|\phi^{z_1}(t,x) - \phi^{z_2}(t,x) - \phi^{z_3}(t,x) + \phi^{z_4}(t,x)\|_{\dot{\mathscr{W}}_q^n}, \tag{3.1.10}$$

$$\Sigma_{q,n}^{(4)}(t,x,z) := \|\sigma(\phi^{z_1}(t,x)) - \sigma(\phi^{z_2}(t,x)) - \sigma(\phi^{z_3}(t,x)) + \sigma(\phi^{z_4}(t,x))\|_{\dot{\mathscr{W}}_q^n}. \tag{3.1.11}$$

By Lemma 2.2.1, it follows that the expression in (3.1.10) is continuous in z. Similarly, by Lemma 2.2.1, the product and chain rule formulas for Malliavin derivatives, it is easy to see that the same holds for the expression in (3.1.11). Our next task is to obtain an estimate on $F^{(4)}$ evaluated at Z. This estimate is given in the next lemma.

Lemma 3.1.6 (Four point estimate)**.** *Let Assumption 3.1.5 hold. Then for all $p \in [2,\infty)$, there exists a constant $N = N(n,p,\|\sigma\|_{C^{n+2}},\alpha,\beta)$ such that for all $(S,T) \in [0,1]_{\leq}^2$, $(s,a) \in [S,T]_{\leq}^2$, $t \in [0,1-a+s]$ and $x \in \mathbb{T}$, we have that*

$$\sup_{x\in\mathbb{T}} \|F_{p,n}^{(4)}(t,\, x, Z)\|_{L_p}$$
$$\leq N(1 + \max_{i\in\{1,2\}} [D^{u^i}]_{\mathscr{V}_{2p}^\beta})\Big([u^1,u^2]_{\mathscr{S}_p^{1/2}[S,a]} + \|u^1(S,\cdot) - u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}\Big)|a-s|^{\frac{1}{2}}$$

where Z is defined by (3.1.9).

To prove the above estimate, we will need the following auxiliary lemma.

Lemma 3.1.7. *Let Assumption 3.1.5 hold. Then for all $p \in [2,\infty)$ there exists a constant $N = N(n, p, \|\sigma\|_{C^{n+2}}, \alpha, \beta)$ such that for all $(s,a) \in [0,1]^2_{\leq}$, $t \in [0, 1-a+s]$ and $x \in \mathbb{T}$ we have*

$$\|\Sigma^{(4)}_{p,n}(t,x,Z)\|_{L_p} \leq N \max_{i\in\{1,2\}} [D^{u^i}]_{\mathscr{V}^\beta_{2p}} \sup_{x\in\mathbb{T}} \|u^1(s,\cdot) - u^2(s,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}(a-s)^\beta$$
$$+ N\mathbf{1}_{n\geq 1} \sum_{i=0}^{n-1} \|F^{(4)}_{2p,i}(t,x,Z)\|_{L_p} + N\|F^{(4)}_{p,n}(t,x,Z)\|_{L_p}.$$

Proof. We begin by proving the result for $n \geq 1$. By point (c) in Lemma 1.10.3 (which we can apply with $\varepsilon = t^{1/4} \in [0,1]$ by Lemma 2.1.1) and by the triangle inequality and Hölder's inequality we have that

$$\begin{aligned}
&\|\Sigma^{(4)}_{p,n}(t,x,Z)\|_{L_p|\mathscr{F}_s} \\
&\lesssim \sum_{i+j\leq n} \big\|F^{(2)}_{4p,i}(t,x,Z_1,Z_2)\big\|_{L_{2p}|\mathscr{F}_s} \Big(\sum_{l\in\{1,2\}} \big\|F^{(2)}_{4p,j}(t,x,Z_l,Z_{l+2})\big\|_{L_{2p}|\mathscr{F}_s} \Big) \\
&\quad + \sum_{i=0}^{n-1} \big\|F^{(4)}_{2p,i}(t,x,Z)\big\|_{L_p|\mathscr{F}_s} + \|F^{(4)}_{p,n}(t,x,Z)\|_{L_p|\mathscr{F}_s} \\
&=: A + B + C.
\end{aligned}$$

We can immediately see that $\|B\|_{L_p}, \|C\|_{L_p}$ can be estimated by the second and third terms of the desired bound. We proceed with showing that $\|A\|_{L_p}$ can be estimated by the first term of the desired bound. To this end, note that by Lemma 2.2.1 and by Lemma 3.1.4 we have for $l = 1,2$, uniformly in $j \in \{0,\ldots,n\}$ that

$$\begin{aligned}
\big\|F^{(2)}_{4p,j}(t,x,Z_l,Z_{l+2})\big\|_{L^{\mathscr{F}_s}_{2p,\infty}} &\lesssim \sup_{x\in\mathbb{T}} \|Z_l(x) - Z_{l+2}(x)\|_{L^{\mathscr{F}_s}_{2p,\infty}} \\
&= \sup_{x\in\mathbb{T}} \|\phi^{u^l(s,\cdot),s}(a,x) - u^l(a,x)\|_{L^{\mathscr{F}_s}_{2p,\infty}} \\
&\lesssim (a-s)^\beta \max_{l\in\{1,2\}} [D^{u^l}]_{\mathscr{V}^\beta_{2p}[s,a]}.
\end{aligned}$$

Therefore

$$A \lesssim (a-s)^\beta \max_{l\in\{1,2\}} [D^{u^l}]_{\mathscr{V}^\beta_{2p}} \sum_{i+j\leq n} \|F^{(2)}_{4p,i}(t,x,Z_1,Z_2)\|_{L_{2p}|\mathscr{F}_s}. \tag{3.1.12}$$

Applying Lemma 2.2.1 with $q = 4p$, $(p_1, p_2) = (2p, p)$, $\mathscr{G} = \mathscr{F}_s$, we get

$$\begin{aligned}\|F^{(2)}_{4p,i}(t,x,Z_1,Z_2)\|_{L^{\mathscr{F}_s}_{2p,p}} &\lesssim t^{i/4}\sup_{x\in\mathbb{T}}\|Z_1(x)-Z_2(x)\|_{L^{\mathscr{G}}_{2p,p}}\\ &= t^{i/4}\sup_{x\in\mathbb{T}}\|\phi^{u^1(s,\cdot),s}(a,x)-\phi^{u^2(s,\cdot),s}(a,x)\|_{L^{\mathscr{G}}_{2p,p}}\\ &= t^{i/4}\sup_{x\in\mathbb{T}}\|F_{2p,0}(a-s,x,u^1(s,\cdot),u^2(s,\cdot))\|_{L_p},\end{aligned}$$

for all $i \in \{0,\dots,n\}$, where the last equality holds by Lemma 2.3.5. So applying Lemma 2.2.1 with $q = 2p$, $p_1 = p_2 = p$, and with an arbitrary sub-σ-algebra $\mathscr{G} \subset \mathscr{F}$, we get from the above inequality that

$$\|F^{(2)}_{4p,i}(t,x,Z_1,Z_2)\|_{L^{\mathscr{F}_s}_{2p,p}} \lesssim \sup_{x\in\mathbb{T}}\|u^1(s,x)-u^2(s,x)\|_{L_p},$$

and the bound is uniform in $i \in \{0,\dots,n\}$. Now taking the L_p-norm on (3.1.12),and applying the above inequality, we get that

$$\begin{aligned}\|A\|_{L_p} &\lesssim (a-s)^{\beta}\max_{l\in\{1,2\}}[D^{u^l}]_{\mathscr{V}^{\beta}_{2p}}\sum_{i+j+k}\|F^{(2)}_{4p,i}(t,x,Z_1,Z_2)\|_{L^{\mathscr{F}_s}_{2p,p}}\\ &\lesssim (a-s)^{\beta}\max_{l\in\{1,2\}}[D^{u^l}]_{\mathscr{V}^{\beta}_{2p}}\sup_{x\in\mathbb{T}}\|u^1(s,x)-u^2(s,x)\|_{L_p}.\end{aligned}$$

which finishes the proof for the $n \geq 1$ case. Finally, for the $n = 0$ case, note that using (1.11.44), we have

$$\begin{aligned}\|\Sigma^{(4)}_{p,n}(t,x,Z)\|_{L_p|\mathscr{F}_s} \leq \|F^{(4)}_{2p,n}(t,x,Z_1,Z_2)\|_{L_{2p}|\mathscr{F}_s}&\sum_{l\in\{1,2\}}\|F^{(2)}_{2p,j}(t,x,Z_l,Z_{l+2})\|_{L_{2p}|\mathscr{F}_s}\\ &+\|F^{(4)}_{p,0}(t,x,Z)\|_{L_p|\mathscr{F}_s}\\ =: \tilde{A}+\tilde{C}.&\end{aligned}$$

By estimating $\|\tilde{A}\|_{L_p}$ and $\|\tilde{C}\|_{L_p}$ the same way as we did for $\|A\|_{L_p}$ and $\|C\|_{L_p}$ respectively, one can show that the desired result also holds for $n = 0$, which finishes the proof. □

We are now in position to prove Lemma 3.1.6.

Proof of Lemma 3.1.6. We begin by confirming that $\sup_{(t,x)\in[0,1]\times\mathbb{T}}\|F^{(4)}_{p,n}(t,x,Z)\|_{L_p} < \infty$. This is indeed true, since by the triangle inequality, Lemma 2.2.1 and Lemma 3.1.4 we have

$$\begin{aligned}\sup_{x\in\mathbb{T}}\|F^{(4)}_{p,n}(t,x,Z)\|_{L_p} &\leq \sum_{i\in\{1,2\}}\|F^{(2)}_{p,n}\big(t,x,\phi^{u^i(s,\cdot),s}(a,\cdot),u^i(a,\cdot)\big)\|_{L_p}\\ &\lesssim \sum_{i\in\{1,2\}}t^{\frac{n}{4}}\sup_{x\in\mathbb{T}}\|\phi^{u^i(s,\cdot),s}(a,x)-u^i(a,x)\|_{L_p}\end{aligned}$$

$$\lesssim t^{\frac{n}{4}}(a-s)^{\beta} \max_{i\in\{1,2\}}[D^{u^i}]_{\mathscr{V}_p^{\beta}}. \tag{3.1.13}$$

The rest of the proof will be done by induction.

Step 1: We prove that the statement holds for $n = 0$. By (2.3.30) we have for $z \in (C(\mathbb{T}))^4$ that

$$(\phi^{z_1} - \phi^{z_2} - \phi^{z_3} + \phi^{z_4})(t,x) = P_t(z_1 - z_2 - z_3 + z_4)(x)$$
$$+ \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y)\big(\sigma(\phi^{z_1}(r,y)) - \sigma(\phi^{z_2}(r,y)) - \sigma(\phi^{z_3}(r,y)) + \sigma(\phi^{z_4}(r,y))\big)\xi(dy,dr).$$

Therefore by the BDG inequality

$$\begin{aligned} &F_{p,0}^{(4)}(t,x,z) \\ &\lesssim P_t(z_1 - z_2 - z_3 + z_4)(x) + \Big(\int_0^t \int_{\mathbb{T}} p_{t-r}^2(x,y)\big|\Sigma_{p,0}^{(4)}(r,y,z)\big|^2 dydr\Big)^{1/2} \\ &=: A(z) + B(z). \end{aligned} \tag{3.1.14}$$

By the definition of the $\mathscr{S}_p^{1/2}$-bracket, we have

$$\begin{aligned} \|A(Z)\|_{L_p} &\le \sup_{x\in\mathbb{T}} \|\phi^{u^1(s,\cdot),s}(a,x) - \phi^{u^2(s,\cdot),s}(a,x) - u^1(a,x) + u^2(a,x)\|_{L_p} \\ &\lesssim [u^1,u^2]_{\mathscr{S}_p^{1/2}[s,a]}(a-s)^{1/2}. \end{aligned}$$

Moreover using Lemma 3.1.7, one can show that

$$\begin{aligned} \|B(Z)\|_{L_p} \lesssim &\max_{i\in\{1,2,\}}[D^{u^i}]_{\mathscr{V}_{2p}^{\beta}} \sup_{x\in\mathbb{T}} \|u^1(s,x) - u^2(s,x)\|_{L_p}(a-s)^{\beta} \\ &+ \Big(\int_0^t \int_{\mathbb{T}} p_{t-r}^2(x,y)\|F_{p,0}^{(4)}(r,y,Z)\|_{L_p}^2 dydr\Big)^{1/2}. \end{aligned}$$

Using the above bounds on A, B, the decomposition (3.1.14), and the assumption that we have $\beta \ge 1/2$, we can see that

$$\begin{aligned} &\|F_{p,0}^{(4)}(t,x,Z)\|_{L_p}^2 \\ &\lesssim \Big|(1 + \max_{i\in\{1,2,\}}[D^{u^1}]_{\mathscr{V}_{2p}^{\beta}})\big([u^1,u^2]_{\mathscr{S}_p^{1/2}[S,a]} + \|u^1(S,\cdot) - u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}\big)(a-s)^{1/2}\Big|^2 \\ &\quad + \int_0^t \int_{\mathbb{T}} p_{t-r}^2(x,y)\|F_{p,0}^{(4)}(r,y,Z)\|_{L_p}^2 dydr. \end{aligned}$$

Since by assumption $u^1, u^2 \in \mathscr{U}_{2p}^{\beta}$, we have $\max_{i\in\{1,2\}}[D^{u^i}]_{\mathscr{V}_{2p}^{\beta}} < \infty$. Therefore by (3.1.13) we have that the norm in the integrand is bounded in (r, y). Hence using Lemma 1.3.5 finishes the proof for the $n = 0$ case.

Step 2: Let $n \in \mathbb{N}$ and suppose that the statement holds for $F^{(4)}_{p,0}, \dots, F^{(4)}_{p,n-1}$ for all $p \geq 2$. We aim to show that then the result also holds for $F^{(4)}_{p,n}$ for all $p \geq 2$. Let $\gamma = (\theta_i, \zeta_i)_{i=1}^n \in ([0,1] \times \mathbb{T})^n$ and $z \in (C(\mathbb{T}))^4$. Then by Proposition 2.1.3 we have that

$$\begin{aligned}
&\mathscr{D}^n_\gamma(\phi^{z_1} - \phi^{z_2} - \phi^{z_3} + \phi^{z_4})(t, x) \\
&= \mathbf{1}_{[0,t]}(\theta^*) \sum_{k=1}^n p_{t-\theta_k}(x, \zeta_k) \\
&\qquad \times \mathscr{D}^{n-1}_{\hat{\gamma}_k}\big[\sigma(\phi^{z_1}(\theta_k, \zeta_k)) - \sigma(\phi^{z_2}(\theta_k, \zeta_k)) - \sigma(\phi^{z_3}(\theta_k, \zeta_k)) + \sigma(\phi^{z_4}(\theta_k, \zeta_k))\big] \\
&+ \mathbf{1}_{[0,t]}(\theta^*) \int_0^t \int_{\mathbb{T}} p_{t-r}(x, y) \\
&\qquad \times \mathscr{D}^n_\gamma\big[\sigma(\phi^{z_1}(r, y)) - \sigma(\phi^{z_2}(r, y)) - \sigma(\phi^{z_3}(r, y)) + \sigma(\phi^{z_4}(r, y))\big]\xi(dy, dr).
\end{aligned}$$

Taking the $\|\cdot\|_{L_p(\Omega; H^{\otimes n})}$-norm of both sides and using the BDG inequality gives

$$\begin{aligned}
&F^{(4)}_{p,n}(t, x, z_1, z_2, z_3, z_4) \\
&\lesssim \|\mathbf{1}_{[0,t]}(\cdot)p_{t-\cdot}(x, \cdot)\mathscr{D}^{n-1}[\sigma(\phi^{z_1}(\cdot,\cdot)) - \sigma(\phi^{z_2}(\cdot,\cdot)) - \sigma(\phi^{z_3}(\cdot,\cdot)) + \sigma(\phi^{z_4}(\cdot,\cdot))]\|_{L_p(\Omega; H^{\otimes n})} \\
&\qquad + \Big(\int_0^t \int_{\mathbb{T}} p^2_{t-r}(x, y)|\Sigma^{(4)}_{p,n}(r, y, z)|^2 dy dr\Big)^{1/2} \\
&=: A(z_1, z_2, z_3, z_4) + B(z_1, z_2, z_3, z_4).
\end{aligned} \tag{3.1.15}$$

We begin by bounding A. Note that

$$\begin{aligned}
&A(z) \lesssim \\
&\|\mathbf{1}_{[0,t]}(\cdot)p_{t-\cdot}(x, \cdot)\|\|\mathscr{D}^{n-1}[\sigma(\phi^{z_1}(\cdot,\cdot)) - \sigma(\phi^{z_2}(\cdot,\cdot)) - \sigma(\phi^{z_3}(\cdot,\cdot)) + \sigma(\phi^{z_4}(\cdot,\cdot))]\|_{H^{\otimes(n-1)}}\|_{L_p}\|_H \\
&= \|\mathbf{1}_{[0,t]}(\cdot)p_{t-\cdot}(x, \cdot)\Sigma^{(4)}_{p,n-1}(\cdot,\cdot, z)\|_H,
\end{aligned}$$

and thus (recalling the definition of Z from (3.1.9)) we have $A(Z) \lesssim \|\mathbf{1}_{[0,t]}(\cdot)p_{t-\cdot}(x, \cdot)\Sigma^{(4)}_{p,n-1}(\cdot,\cdot, Z)\|_H$. Hence using the Minkowski inequality we obtain that

$$\begin{aligned}
\|A(Z)\|_{L_p} &\lesssim \|\mathbf{1}_{[0,t](\cdot)}p_{t-\cdot}(x, \cdot)\|\Sigma^{(4)}_{p,n-1}(\cdot,\cdot, Z)\|_{L_p}\|_H \\
&\leq \|\mathbf{1}_{[0,t]}(\cdot)p_{t-\cdot}(x, \cdot)\|_H \sup_{(\theta,\zeta)\in[0,t]\times\mathbb{T}} \|\Sigma^{(4)}_{p,n-1}(\theta, \zeta, Z)\|_{L_p}.
\end{aligned}$$

To bound the first factor, we note that $\|\mathbf{1}_{[0,t]}(\cdot)p_{t-\cdot}(x, \cdot)\|_H \lesssim t^{1/4} \leq 1$, and to bound the second factor we use that by Lemma 3.1.7 and by the induction hypothesis, for $(\theta,\zeta)\in[0,t]\times\mathbb{T}$ we have that

$$\begin{aligned}
&\|\Sigma^{(4)}_{p,n-1}(\theta,\zeta,Z)\|_{L_p}\\
&\lesssim \max_{i\in\{1,2\}}[D^{u^i}]_{\mathscr{V}^\beta_p}\|u^1(s,\cdot)-u^2(s,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}(a-s)^\beta+\sum_{i=0}^{n-1}\|F^{(4)}_{2p,i}(\theta,\zeta,Z)\|_{L_p}\\
&\lesssim (1+\max_{i\in\{1,2\}}[D^{u^i}]_{\mathscr{V}^\beta_{2p}})\Big([u^1,u^2]_{\mathscr{S}^{1/2}_p[S,a]}+\|u^1(S,\cdot)-u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}\Big)(a-s)^{\frac{1}{2}},
\end{aligned}$$

where we used Lemma 3.1.3 and that by assumption we have $\beta\geq 1/2$. Hence we can see that

$$\|A(Z)\|_{L_p}\lesssim (1+\max_{i\in\{1,2\}}[D^{u^i}]_{\mathscr{V}^\beta_{2p}})\Big([u^1,u^2]_{\mathscr{S}^{1/2}_p[S,a]}+\|u^1(S,\cdot)-u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}\Big)(a-s)^{\frac{1}{2}}.$$

We proceed with bounding B. Note that

$$\|B(Z)\|_{L^{\mathscr{F}_s}_{2,p}}\lesssim\Big(\int_0^t\int_{\mathbb{T}}p^2_{t-r}(x,y)\|\Sigma^{(4)}_{q,n}(r,y,Z)\|^2_{L^{\mathscr{F}_s}_{2,p}}\,dydr\Big)^{1/2}. \tag{3.1.16}$$

By Lemma 3.1.7 and the induction hypothesis we have for all $(r,y)\in[0,t]\times\mathbb{T}$ that

$$\begin{aligned}
&\|\Sigma^{(4)}_{p,n}(r,y,Z)\|_{L_p}\\
&\lesssim \max_{i\in\{1,2\}}[D^{u^i}]_{\mathscr{V}^\beta_{2p}}\|u^1(s,\cdot)-u^2(s,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}(a-s)^\beta\\
&\qquad+\sum_{i=0}^{n-1}\|F^{(4)}_{2p,i}(r,y,Z)\|_{L_p}+\|F^{(4)}_{p,n}(r,y,Z)\|_{L_p}\\
&\lesssim (1+\max_{i\in\{1,2\}}[D^{u^i}]_{\mathscr{V}^\beta_{2p}})\Big([u^1,u^2]_{\mathscr{S}^{1/2}_p[S,a]}+\|u^1(S,\cdot)-u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}\Big)(a-s)^{1/2}\\
&\qquad\quad+\|F^{(4)}_{p,n}(r,y,Z)\|_{L_p}.
\end{aligned}$$

where we again used the assumption that $\beta\geq\frac{1}{2}$. Putting this into (3.1.16), we can see that

$$\begin{aligned}
&\|B(Z)\|_{L_p}\\
&\lesssim (1+\max_{i\in\{1,2\}}[D^{u^i}]_{\mathscr{V}^\beta_{2p}})\Big([u^1,u^2]_{\mathscr{S}^{1/2}_p[S,a]}+\|u^1(S,\cdot)-u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}\Big)(a-s)^{1/2}\\
&\qquad+\Big(\int_0^t\int_{\mathbb{T}}p^2_{t-r}(x,y)\|F^{(4)}_{p,n}(r,y,Z)\|^2_{L_p}\,dydr\Big)^{1/2}.
\end{aligned}$$

By our bounds on A,B we may conclude that

$$\|F^{(4)}_{p,n}(t,x,Z)\|_{L_p}$$

$$\lesssim (1+\max_{i\in\{1,2\}}[D^{u^i}]_{\mathscr{V}^\beta_{2p}})\Big([u^1,u^2]_{\mathscr{S}^{1/2}_p[S,a]}+\|u^1(S,\cdot)-u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}\Big)(a-s)^{1/2}$$
$$+\Big(\int_0^t\int_{\mathbb{T}}p^2_{t-r}(x,y)\|F^{(4)}_{p,n}(r,y,Z)\|^2_{L_p}\,dydr\Big)^{1/2}.$$

By (3.1.13) the norm in the integrand is bounded in (r,y). Hence by Lemma 1.3.5 the proof is finished. □

Lemma 3.1.8 (Four point BDG inequality for driftless approximations)**.** *Let $p\in[2,\infty)$, $\sigma\in C^2$, $\alpha\in(-1,0)$, $\beta\in[0,1+\alpha/4]$ and for $i=1,2$, let $b^i\in C^\alpha$ and suppose that $u^i\in\mathscr{U}^\beta$ are regularised solutions of*

$$(\partial_t-\Delta)u^i=b^i(u^i)+\sigma(u^i)\xi(dy,dr).$$

There exists a constant $N=N(p,\|\sigma\|_{C^2},\varepsilon,\alpha,\beta)$ such that for $(s,t)\in[0,1]^2_\leq$ we have

$$\Big\|\int_s^t\int_{\mathbb{T}}p_{t-r}(x,y)$$
$$\times\Big(\sigma(u^1(r,y))-\sigma(u^2(r,y))-\sigma(\phi^{u^1(s,\cdot),s}(r,y))+\sigma(\phi^{u^2(s,\cdot),s}(r,y))\Big)\xi(dy,dr)\Big\|_{L_p}$$
$$\leq N[D^{u^1}]_{\mathscr{V}^\beta_{2p}}\|u^1(s,\cdot)-u^2(s,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}(t-s)^{\frac14+\beta}$$
$$+N\Big(\int_s^t\int_{\mathbb{T}}p^2_{t-r}(x,y)\big\|u^1(r,y)-u^2(r,y)-\phi^{u^1(s,\cdot),s}(r,y)+\phi^{u^2(s,\cdot),s}(r,y)\big\|^2_{L_p}\,dydr\Big)^{1/2}.$$

Proof. By the BDG inequality and by (1.11.45) in Lemma 1.11.6 we have

$$\Big\|\int_s^t\int_{\mathbb{T}}p_{t-r}(x,y)\Big(\sigma(u^1(r,y))-\sigma(u^2(r,y))$$
$$-\sigma(\phi^{u^1(s,\cdot),s}(r,y))+\sigma(\phi^{u^2(s,\cdot),s}(r,y))\Big)\xi(dy,dr)\Big\|^2_{L_p}$$
$$\lesssim\int_s^t\int_{\mathbb{T}}p^2_{t-r}(x,y)\Big\|\sigma(u^1(r,y))-\sigma(u^2(r,y))$$
$$-\sigma(\phi^{u^1(s,\cdot),s}(r,y))+\sigma(\phi^{u^2(s,\cdot),s}(r,y))\Big\|^2_{L_p}\,dydr$$
$$\lesssim I+J$$

with

$$I:=\int_s^t\int_{\mathbb{T}}p^2_{t-r}(x,y)\Big\|\Big(\phi^{u^1(s,\cdot),s}(r,y)-\phi^{u^2(s,\cdot),s}(r,y)\Big)\Big(\phi^{u^1(s,\cdot),s}(r,y)-u^1(r,y)\Big)\Big\|^2_{L_p}\,dydr$$
$$J:=\int_s^t\int_{\mathbb{T}}p^2_{t-r}(x,y)\|u^1(r,y)-u^2(r,y)-\phi^{u^1(s,\cdot),s}(r,y)+\phi^{u^2(s,\cdot),s}(r,y)\|^2_{L_p}\,dydr.$$

By the tower rule, Hölder's inequality, Lemma 3.1.4, Lemma 2.3.5 and Lemma 2.2.1 (where we recall the definition (2.2.7) of $F^{(2)}$), we can see that

$$
\begin{aligned}
&\Big\|\Big(\phi^{u^1(s,\cdot),s}(r,y)-\phi^{u^2(s,\cdot),s}(r,y)\Big)\Big(\phi^{u^1(s,\cdot),s}(r,y)-u^1(r,y)\Big)\Big\|_{L_p}\\
&\le \Big\|\big\|\phi^{u^1(s,\cdot),s}(r,y)-\phi^{u^2(s,\cdot),s}(r,y)\big\|_{L_{2p}|\mathscr{F}_s}\big\|\phi^{u^1(s,\cdot),s}(r,y)-u^1(r,y)\big\|_{L_{2p}|\mathscr{F}_s}\Big\|_{L_p}\\
&\lesssim [D^{u^1}]_{\mathscr{V}^\beta_{2p}}(t-s)^\beta\Big\|\big\|\phi^{u^1(s,\cdot),s}(r,y)-\phi^{u^2(s,\cdot),s}(r,y)\big\|_{L_{2p}|\mathscr{F}_s}\Big\|_{L_p}\\
&= [D^{u^1}]_{\mathscr{V}^\beta_{2p}}(t-s)^\beta\Big\|F^{(2)}_{2p,0}\big(r-s,y,u^1(s,\cdot),u^2(s,\cdot)\big)\Big\|_{L_p}\\
&\lesssim [D^{u^1}]_{\mathscr{V}^\beta_{2p}}(t-s)^\beta\|u^1(s,\cdot)-u^2(s,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}.
\end{aligned}
$$

Putting this bound into the definition of I, we get that

$$
\begin{aligned}
I &\lesssim [D^{u^1}]^2_{\mathscr{V}^\beta_{2p}}(t-s)^{2\beta}\|u^1(s,\cdot)-u^2(s,\cdot)\|^2_{C(\mathbb{T},L_p)}\int_s^t\int_{\mathbb{T}}p^2_{t-r}(x,y)dydr\\
&\lesssim [D^{u^1}]^2_{\mathscr{V}^\beta_{2p}}(t-s)^{2\beta+\frac12}\|u^1(s,\cdot)-u^2(s,\cdot)\|^2_{\mathbb{B}(\mathbb{T},L_p)}.
\end{aligned}
$$

This bound on I and the definition of J together give the desired bound. □

3.2 Regularisation estimates

Let u, u^1, u^2 be regularised solutions of (1.3.9) with potentially different drift terms, f be a measurable kernel on $(0,1]\times\mathbb{T}$ and g be a smooth function on $\mathbb{R}$. In this section, we obtain quantitative bounds for expressions of the forms $\int_s^t\int_{\mathbb{T}} f_r(y)g(u(r,y))dydr$ and $\int_s^t\int_{\mathbb{T}} f_r(y)(g(u^1(r,y))-g(u^2(r,y)))dydr$ which depend on a Besov–Hölder norm of g with a negative index.

Lemma 3.2.1. *Let Assumption 1.7.1 hold and let u be a regularised solution of (1.3.9). Suppose that $f:(0,1]\times\mathbb{T}\to\mathbb{R}$ is a measurable function such that there exist constants $K>0$ and $\zeta\in[0,\frac14]$ such that for all $t\in(0,1]$ it holds that*

$$\int_{\mathbb{T}}|f_t(y)|dy\le Kt^{-\zeta}.$$

Let $p\in[1,\infty)$. For all $\lambda\in\big(4\zeta-2,-1\big)\cup(-1,0)$ and for all β in the nonempty set $(\frac14-\frac{\lambda}{4}+\zeta,1+\frac{\alpha}{4}]$, if $u\in\mathscr{U}^\beta_2$ then there exists a constant $N=N(p,\|\sigma\|_{C^2},\mu,\lambda,\alpha,\beta,\zeta)$, such that for all $g\in C^\infty$, $(s,t)\in[0,1]^2_\le$ and $\mathscr{G}\in\{\mathscr{F}_s,\{\emptyset,\Omega\}\}$ we have

$$
\begin{aligned}
&\Big\|\int_s^t\int_{\mathbb{T}}f_{t-r}(y)g(u(r,y))dydr\Big\|_{L^{\mathscr{G}}_{p,\infty}}\\
&\quad\le N\|g\|_{C^\lambda}K\Big((t-s)^{1+\frac{\lambda}{4}-\zeta}+[D^u]_{\mathscr{V}^\beta_2[s,t]}(t-s)^{\beta+\frac{\lambda+3}{4}-\zeta}\Big).
\end{aligned}
$$

Proof. We may assume that $p > 2$. For $(S,T) \in [0,1]^2_{\leq}$ and $(s,t) \in [S,T]^2_{\leq}$ we consider the germ

$$A_{s,t} := \mathbb{E}^s \int_s^t \int_{\mathbb{T}} f_{T-r}(y) g(\phi^{u(s,\cdot),s}(r,y))dydr.$$

Then

$$|A_{s,t}| \lesssim \int_s^t \int_{\mathbb{T}} |f_{T-r}(y)||\mathbb{E}^s g(\phi^{u(s,\cdot),s}(r,y))|dydr.$$

Note that by Lemma 2.3.6 (with $n = 0$) we have

$$|\mathbb{E}^s g(\phi^{u(s,\cdot),s}(r,y))| \lesssim \|g\|_{C^\lambda}(r-s)^{\lambda/4}.$$

By the two inequalities above, by the fact that $\|f_{T-r}\|_{L_1(\mathbb{T})} \leq K(T-r)^{-\zeta} \leq K(t-r)^{-\zeta}$ and by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\|A_{s,t}\|_{L^{\mathscr{G}_s}_{p,\infty}} &\lesssim \|g\|_{C^\lambda} \int_s^t (r-s)^{\lambda/4} \int_{\mathbb{T}} |f_{T-r}(y)|dydr \\
&\lesssim \|g\|_{C^\lambda} \int_s^t (r-s)^{\lambda/4}(t-r)^{-\zeta} dr \\
&\lesssim \|g\|_{C^\lambda} K(t-s)^{1+\lambda/4-\zeta}.
\end{aligned}$$

From the assumption that $\lambda > 4\zeta - 2$ it follows that the exponent $1 + \lambda/4 - \zeta$ is greater than $1/2$, and thus the first condition in (1.9.24) is satisfied. Let $a \in [s,t]$. Then

$$\begin{aligned}
|\mathbb{E}^s \delta A_{s,a,t}| &= |\mathbb{E}^s(A_{s,t} - A_{s,a} - A_{a,t})| \\
&= \Big|\mathbb{E}^s \int_a^t \int_{\mathbb{T}} f_{T-r}(y)\mathbb{E}^a\Big(g(\phi^{u(s,\cdot),s}(r,y)) - g(\phi^{u(a,\cdot),a}(r,y))\Big)dydr\Big|.
\end{aligned}$$

By the Fundamental Theorem of Calculus and Lemma 2.3.6 (with $n = 1$), we get that

$$\begin{aligned}
&\Big|\mathbb{E}^a\Big(g(\phi^{u(s,\cdot),s}(r,y)) - g(\phi^{u(a,\cdot),a}(r,y))\Big)\Big| \\
&= \Big|\int_0^1 \mathbb{E}^a\Big(\nabla g\Big(\theta\phi^{u(s,\cdot),s}(r,y) + (1-\theta)\phi^{u(a,\cdot),a}(r,y)\Big)\Big(\phi^{u(s,\cdot),s}(r,y) - \phi^{u(a,\cdot),a}(r,y)\Big)\Big)d\theta\Big| \\
&= \Big|\int_0^1 \mathbb{E}^a\Big(\nabla g\Big(\theta\phi^{\phi^{u(s,\cdot),s}(a,\cdot),\,a}(r,y) + (1-\theta)\phi^{u(a,\cdot),a}(r,y)\Big) \\
&\qquad\qquad \times \Big(\phi^{\phi^{u(s,\cdot),s}(a,\cdot),\,a}(r,y) - \phi^{u(a,\cdot),a}(r,y)\Big)\Big)d\theta\Big| \\
&\lesssim \|g\|_{C^\lambda}(r-a)^{(\lambda-1)/4} \sum_{i=0}^3 F^{(2)}_{16,i}(r-a, y, \phi^{u(s,\cdot),s}(a,\cdot), u(a,\cdot))
\end{aligned}$$

where $F^{(2)}$ is defined by (2.2.7). Therefore using the above result, Lemma 2.2.1 and Lemma 3.1.4, we get

$$\begin{aligned}
&\mathbb{E}^s\Big|\mathbb{E}^a\Big(g(\phi^{u(s,\cdot),s}(r,y)) - g(\phi^{u(a,\cdot),a}(r,y))\Big)\Big| \\
&\lesssim \|g\|_{C^\lambda}(r-a)^{(\lambda-1)/4}\sum_{i=0}^{3}\mathbb{E}^s F^{(2)}_{16,i}\big(r-a, y, \phi^{u(s,\cdot),s}(a,\cdot), u(a,\cdot)\big) \\
&\lesssim \|g\|_{C^\lambda}(r-a)^{(\lambda-1)/4}\sup_{x\in\mathbb{T}}\|\phi^{u(s,\cdot),s}(a,x) - u(a,x)\|_{L^{\mathscr{F}_s}_{2,\infty}} \\
&\lesssim \|g\|_{C^\lambda}(r-a)^{(\lambda-1)/4}[D^u]_{\mathscr{V}^\beta_2[S,T]}(a-s)^\beta.
\end{aligned}$$

By the above inequality, by the assumptions on f and by the fact that $t-a, a-s \le t-s$, we get

$$\begin{aligned}
\|\mathbb{E}^s\delta A_{s,a,t}\|_{L_\infty} &\lesssim \mathbb{E}^s\int_a^t\int_{\mathbb{T}} f_{T-r}(y)\|g\|_{C^\lambda}[D^u]_{\mathscr{V}^\beta_2[S,T]}(r-a)^{(\lambda-1)/4}(a-s)^\beta dydr \\
&\lesssim \|g\|_{C^\lambda}[D^u]_{\mathscr{V}^\beta_2[S,T]}(a-s)^\beta\int_a^t(r-a)^{(\lambda-1)/4}\int_{\mathbb{T}} f_{T-r}(y)dydr \\
&\lesssim \|g\|_{C^\lambda}[D^u]_{\mathscr{V}^\beta_2[S,T]}(a-s)^\beta K\int_a^t(t-r)^{-\zeta}(r-a)^{\lambda/4-1/4}dr \\
&\lesssim \|g\|_{C^\lambda}[D^u]_{\mathscr{V}^\beta_2[S,T]}K(t-s)^{\beta-\zeta+\lambda/4+3/4}.
\end{aligned}$$

By the assumption that $\beta > 1/4-\lambda/4+\zeta$, it follows that the exponent $\beta-\zeta+\lambda/4+3/4$ is greater than 1, and thus the second condition in (1.9.24) is also satisfied. Let

$$\mathscr{A}_{s,t} := \int_s^t\int_{\mathbb{T}} f_{T-r}(y)g(u(r,y))dydr.$$

By the regularity of g and by Lemma 3.1.4 we can easily see that (1.9.25) and (1.9.26) are satisfied. All conditions of Lemma 1.9.1 are satisfied. Consequently, the conclusion follows from Lemma 1.9.1 and the fact that $(S,T) \in [0,1]^2_\le$ was arbitrary. □

Corollary 3.2.2. *Let Assumption 1.7.1 hold and let u be a regularised solution of (1.3.9) and let $p \in [1,\infty)$. Then for all $\lambda \in (-2,-1)\cup(-1,0)$, $\beta \in (\frac{1}{4}-\frac{\lambda}{4}, 1+\frac{\alpha}{4}]$, if $u \in \mathscr{U}^\beta_2$ then there exists a constant $N = N(p, \|\sigma\|_{C^2}, \mu, \lambda, \alpha, \beta)$, such that for all $g \in C^\infty$, $(s,t) \in [0,1]^2_\le$, $x \in \mathbb{T}$, we have*

$$\begin{aligned}
&\Big\|\int_s^t\int_{\mathbb{T}} p_{t-r}(x,y)g(u(r,y))dydr\Big\|_{L^{\mathscr{F}_s}_{p,\infty}} \\
&\le N\|g\|_{C^\lambda}\Big((t-s)^{1+\lambda/4} + [D^u]_{\mathscr{V}^\beta_2[s,t]}(t-s)^{\beta+\frac{\lambda+3}{4}}\Big).
\end{aligned}$$

Proof. Fix $x \in \mathbb{T}$ and for each $(r,y) \in (0,1]\times\mathbb{T}$ define $f_r(y) := p_r(x,y)$. Then f is a measurable function which satisfies $\int_{\mathbb{T}} f_r(y)dy = 1$. Hence, applying Lemma 3.2.1 (with $K := 1$, $\zeta := 0$), we obtain the result. □

Corollary 3.2.3. *Let Assumption 1.7.1 hold and let u be a regularised solution of (1.3.9) and let* $p \in [1, \infty)$*. Then for all* $\lambda \in (-1, 0)$ *and for all* $\beta \in (\frac{1}{2} - \frac{\lambda}{4}, 1 + \frac{\alpha}{4}]$*, if* $u \in \mathscr{U}_2^\beta$ *then there exists a constant* $N = N(p, \|\sigma\|_{C^2}, \mu, \lambda, \beta)$ *such that for all* $g \in C^\infty$*,* $(s,t) \in [0,1]^2_\leq$ *and* $x \in \mathbb{T}$ *we have*

$$\Big\| \int_s^t \int_{\mathbb{T}} \big(p_{t-r}(x,y) - p_{t-r}(\bar{x},y)\big) g(u(r,y)) dy dr \Big\|_{L_{p,\infty}^{\mathscr{F}_s}} \leq N \|g\|_{C^\lambda} (1 + [D^u]_{\mathscr{V}_2^\beta}) |x - \bar{x}|^{1/2}.$$

Proof. Fix $x, \bar{x} \in \mathbb{T}$, and for each $(r,y) \in (0,1] \times \mathbb{T}$, define $f_r(y) := p_r(x,y) - p_r(\bar{x},y)$. Then by (1.11.37), we have $\int_{\mathbb{T}} f_r(y) dy \leq C|x - \bar{x}|^{1/2} r^{-1/4}$ for some constant positive C. Hence applying Lemma 3.2.1 (with $K := C|x - \bar{x}|^{1/2}$ and $\zeta := 1/4$), we obtain the stated estimate. □

Lemma 3.2.4. *Let* $p \in [2, \infty)$*,* $\alpha \in (-1, 0)$*, and let* $\sigma \in C^4$ *such that there exists constant* $\mu > 0$ *such that for all* $x \in \mathbb{R}$ *we have* $\sigma^2(x) \geq \mu^2$*. For* $i = 1, 2$*, let* $b^i \in C^\alpha$ *and let* u^i *be regularised solutions of*

$$(\partial_t - \Delta) u^i = b^i(u^i) + \sigma(u^i)\xi$$

in the class $\mathscr{U}^\beta$ *for some* $\beta \in (\frac{1}{2} - \frac{\alpha}{4}, 1 + \frac{\alpha}{4}]$*. There exists a constant* $N = N(p, \|\sigma\|_{C^4}, \mu, \alpha, \beta)$ *such that for all* $g \in C^\infty$*,* $(s,t) \in [0,1]^2_\leq$ *and* $x \in \mathbb{T}$ *we have*

$$\begin{aligned} &\Big\| \int_s^t \int_{\mathbb{T}} p_{t-r}(x,y) \big(g(u^1(r,y)) - g(u^2(r,y))\big) dy dr \Big\|_{L_p} \\ &\leq N \|g\|_{C^\alpha} (1 + \max_{i \in \{1,2\}} [D^{u^i}]_{\mathscr{V}_{2p}^\beta}) (t-s)^{(3+\alpha)/4} \\ &\qquad \times \Big([u^1, u^2]_{\mathscr{S}_p^{1/2}[s,t]} + \|u^1(s,\cdot) - u^2(s,\cdot)\|_{\mathbb{B}(\mathbb{T}, L_p)} \Big). \end{aligned}$$

Proof. Let $(S,T) \in [0,1]^2_\leq$, $x \in \mathbb{T}$ and for $(s,t) \in [S,T]^2_\leq$ define the germ

$$A_{s,t}(x) := \mathbb{E}^s \int_s^t \int_{\mathbb{T}} p_{T-r}(x,y) \Big(g(\phi^{u^1(s,\cdot),s}(r,y)) - g(\phi^{u^2(s,\cdot),s}(r,y)) \Big) dy dr.$$

We first bound $\|A_{s,t}\|_{L_p}$. Using Lemma 2.3.6(with $n = 1$ and thus $q = 8$) and recalling the definition of $F^{(2)}$ from (2.2.7), we have

$$\begin{aligned} &\Big| \mathbb{E}^s \Big(g(\phi^{u^1(s,\cdot),s}(r,y)) - g(\phi^{u^2(s,\cdot),s}(r,y)) \Big) \Big| \\ &= \Big| \int_0^1 \mathbb{E}^s \Big(\nabla g \Big(\theta \phi^{u^1(s,\cdot),s}(r,y) + (1-\theta) \phi^{u^2(s,\cdot),s}(r,y) \Big) \Big(\phi^{u^1(s,\cdot),s}(r,y) - \phi^{u^2(s,\cdot),s}(r,y) \Big) \Big) d\theta \Big| \end{aligned}$$

$$\lesssim \|g\|_{C^\alpha}(r-s)^{(\alpha-1)/4}\sum_{i=0}^{2} F_{8,i}^{(2)}\big(r-s,y,u^1(s,\cdot),u^2(s,\cdot)\big).$$

We take the L_p-norm on the inequality, and by Lemma 2.2.1 we get

$$\begin{aligned}&\|\mathbb{E}^s(g(\phi^{u^1(s,\cdot),s}(r,y))-g(\phi^{u^2(s,\cdot),s}(r,y))\|_{L_p}\\ &\qquad\lesssim \|g\|_{C^\alpha}(r-s)^{(\alpha-1)/4}\|u^1(s,\cdot)-u^2(s,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}.\end{aligned}$$

Using the definition of A, and the above inequality, we get

$$\begin{aligned}\|A_{s,t}(x)\|_{L_p}&\lesssim \int_s^t\int_{\mathbb{T}} p_{T-r}(x,y)\|g\|_{C^\alpha}(r-s)^{(\alpha-1)/4}\|u^1(s,\cdot)-u^2(s,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}dydr\\ &\lesssim \|g\|_{C^\alpha}\Big([u^1,u^2]_{\mathscr{S}_p^{1/2}[S,T]}+\|u^1(S,\cdot)-u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}\Big)(t-s)^{(3+\alpha)/4}. \quad (3.2.17)\end{aligned}$$

We proceed with an estimate for $\|\mathbb{E}^s\delta A_{s,a,t}\|_{L_p}$ for $a\in[s,t]$. Note that

$$\begin{aligned}|\mathbb{E}^s\delta A_{s,a,t}|&=|\mathbb{E}^s(A_{s,t}-A_{s,a}-A_{a,t})|\\ &=\Big|\mathbb{E}^s\int_a^t\int_{\mathbb{T}} p_{T-r}(x,y)\mathbb{E}^a\Big(g(\phi^{u^1(s,\cdot),s}(r,y))-g(\phi^{u^2(s,\cdot),s}(r,y))\\ &\qquad -g(\phi^{u^1(a,\cdot),a}(r,y))+g(\phi^{u^2(a,\cdot),a}(r,y))\Big)dydr\Big|.\end{aligned}$$

For $(r,y)\in[0,1]\times\mathbb{T}$ and $z\in(C(\mathbb{T}))^4$, we define

$$\Gamma_{r,y}(z):=\Big|\mathbb{E}\big(g(\phi^{z_1}(r-a,y))-g(\phi^{z_2}(r-a,y))-g(\phi^{z_3}(r-a,y))+g(\phi^{z_4}(r-a,y))\big)\Big|.$$

For brevity, we fix $(r,y)\in[s,t]\times\mathbb{T}$ and we set $\Gamma:=\Gamma_{r,y}$, $\phi_i:=\phi_{r-a}^{z_i}(y)$ and $\delta_{i,j}:=\phi_j-\phi_i$. By Lemma 1.11.6 we get

$$\begin{aligned}\Gamma(z)\le&\Big|\int_0^1\int_0^1\mathbb{E}\Big(\delta_{1,2}\big(\theta\delta_{1,3}+(1-\theta)\delta_{2,4}\big)\nabla^2 g(\Theta_1(\theta,\eta))\Big)d\theta d\eta\Big|\\ &+\Big|\int_0^1\mathbb{E}\big((\delta_{3,4}-\delta_{1,2})\nabla g(\Theta_2(\theta))\big)d\theta\Big|\end{aligned}$$

where Θ_1,Θ_2 are convex combinations[1] of $\phi_1,\dots,\phi_4$. Hence using Lemma 2.3.4 (with $n=2$ and $n=1$

[1] In particular:

$$\Theta_1(\theta,\eta):=\eta(\theta\phi_1+(1-\theta)\phi_2)+(1-\eta)(\theta\phi_3+(1-\theta)\phi_4),\qquad \Theta_2(\theta):=\theta\phi_3+(1-\theta)\phi_4,$$

but this is not important for the proof.

for the first and second terms respectively) and recalling the definition of $F^{(4)}$ from (3.1.10) we get that

$$\begin{aligned}\Gamma(z) &\lesssim \|g\|_{C^\alpha}(r-a)^{-1/2+\alpha/4}\int_0^1 \|\delta_{1,2}(\theta\delta_{1,3}+(1-\theta)\delta_{2,4})\|_{\mathscr{W}^3_{16}}d\theta \\ &\quad + \|g\|_{C^\alpha}(r-a)^{-1/4+\alpha/4}\|\delta_{3,4}-\delta_{1,2}\|_{\mathscr{W}^2_8} \\ &\lesssim \|g\|_{C^\alpha}(r-a)^{-1/2+\alpha/4}\sum_{i=0}^{3} F^{(2)}_{32,i}(r-a,y,z_1,z_2) \\ &\qquad \times \Big(F^{(2)}_{32,i}(r-a,y,z_1,z_3)+F^{(2)}_{32,i}(r-a,y,z_2,z_4)\Big) \\ &\quad + \|g\|_{C^\alpha}(r-a)^{-1/4+\alpha/4}\sum_{i=0}^{2} F^{(4)}_{8,i}(r-a,y,z). \end{aligned} \tag{3.2.18}$$

Let

$$Z := (\phi^{u^1(s,\cdot)}_{s,a}, \phi^{u^2(s,\cdot)}_{s,a}, u^1(a,\cdot), u^2(a,\cdot)).$$

By Lemma 2.2.1 and Lemma 3.1.4, we have that for $l=1,2$, $i\in\mathbb{Z}_{\geq 0}$ that

$$\|F^{(2)}_{32,i}(r-a,y,Z_l,Z_{l+2})\|_{L_2|\mathscr{F}_s} \lesssim \sup_{x\in\mathbb{T}}\|Z_l(x)-Z_{l+2}(x)\|_{L^{\mathscr{F}_s}_{2,\infty}} \lesssim [D^{u^l}]_{\mathscr{V}^\beta_p}(t-s)^\beta.$$

Using this, and Lemma 2.2.1, we can see that

$$\begin{aligned} &\|\mathbb{E}^s\Big(F^{(2)}_{32,i}(r-a,y,Z_1,Z_2)F^{(2)}_{32,i}(r-a,y,Z_l,Z_{l+2})\Big)\|_{L_p} \\ &\lesssim \|\|F^{(2)}_{32,i}(r-a,y,Z_1,Z_2)\|_{L_2|\mathscr{F}_s}\|F^{(2)}_{32,i}(r-a,y,Z_l,Z_{l+2})\|_{L_2|\mathscr{F}_s}\|_{L_p} \\ &\lesssim [D^{u^l}]_{\mathscr{V}^\beta_p}(t-s)^\beta\|\|F^{(2)}_{32,i}(r-a,y,Z_1,Z_2)\|_{L_2|\mathscr{F}_s}\|_{L_p} \\ &\lesssim [D^{u^l}]_{\mathscr{V}^\beta_p}(t-s)^\beta\sup_{x\in\mathbb{T}}\|\phi^{u^1(s,\cdot),s}(a,x)-\phi^{u^2(s,\cdot),s}(a,x)\|_{L^{\mathscr{F}_s}_{2,p}} \\ &\lesssim \max_{l\in\{1,2\}}[D^{u^l}]_{\mathscr{V}^\beta_{2p}}(t-s)^\beta\sup_{x\in\mathbb{T}}\|u^1(s,x)-u^2(s,x)\|_{L_p}. \end{aligned} \tag{3.2.19}$$

Note moreover that by Lemma 3.1.6,

$$\begin{aligned} &\sum_{i=0}^{2}\|\mathbb{E}^s F^{(4)}_{8,i}(r-a,y,Z)\|_{L_p} \\ &\qquad \lesssim (1+\max_{l\in\{1,2\}}[D^{u^l}]_{\mathscr{V}^\beta_{2p}})\Big([u^1,u^2]_{\mathscr{S}^{1/2}_p[S,T]}+\|u^1(S,\cdot)-u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}\Big)|t-s|^{1/2}. \end{aligned} \tag{3.2.20}$$

Using (3.2.18), (3.2.19) and (3.2.20), we get that

$$\|\mathbb{E}^s\Gamma(Z)\|_{L_p}$$

$$
\begin{aligned}
&\lesssim \|g\|_{C^\alpha}(r-a)^{-1/2+\alpha/4}\sum_{i=0}^{3}\sum_{l\in\{1,2\}}\|\mathbb{E}^s\Big(F^{(2)}_{32,i}(r-a,y,Z_1,Z_2)F^{(2)}_{32,i}(r-a,y,Z_l,Z_{l+2})\Big)\|_{L_p}\\
&\quad+\|g\|_{C^\alpha}(r-a)^{-1/4+\alpha/4}\sum_{i=0}^{2}\|\mathbb{E}^sF^{(4)}_{8,i}(r-a,y,Z)\|_{L_p}\\
&\lesssim \|g\|_{C^\alpha}(r-a)^{-1/2+\alpha/4}\max_{l\in\{1,2\}}[D^{u^l}]_{\mathscr{V}^\beta_{2p}}(t-s)^\beta\sup_{x\in\mathbb{T}}\|u^1(s,x)-u^2(s,x)\|_{L_p}\\
&\quad+\|g\|_{C^\alpha}(r-a)^{-1/4+\alpha/4}\\
&\qquad\times\sum_{i=0}^{2}(1+\max_{l\in\{1,2\}}[D^{u^l}]_{\mathscr{V}^\beta_{2p}})([u^1,u^2]_{\mathscr{S}^{1/2}_p[S,T]}+\|u^1(S,\cdot)-u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)})(t-s)^{1/2}\\
&\lesssim \|g\|_{C^\alpha}(1+\max_{l\in\{1,2\}}[D^{u^l}]_{\mathscr{V}^\beta_{2p}})([u^1,u^2]_{\mathscr{S}^{1/2}_p[S,T]}+\|u^1(S,\cdot)-u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)})\\
&\quad\times\Big((r-a)^{-1/2+\alpha/4}(t-s)^\beta+(r-a)^{-1/4+\alpha/4}(t-s)^{1/2}\Big).
\end{aligned}
$$

Therefore, by using Lemma 2.3.5 and the above bound, we get

$$
\begin{aligned}
&\|\mathbb{E}^s\delta A_{s,a,t}\|_{L_p}\\
&\lesssim \int_a^t\int_{\mathbb{T}}p_{T-r}(x,y)\|\mathbb{E}^s\Gamma_{r,y}(Z)\|_{L_p}dydr\\
&\lesssim \|g\|_{C^\alpha}(1+\max_{l\in\{1,2\}}[D^{u^l}]_{\mathscr{V}^\beta_{2p}})([u^1,u^2]_{\mathscr{S}^{1/2}_p[S,T]}+\|u^1(S,\cdot)-u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)})\\
&\quad\times\Big((t-s)^\beta\int_a^t\int_{\mathbb{T}}p_{t-r}(x,y)(r-a)^{-1/2+\alpha/4}dydr\\
&\qquad\quad+(t-s)^{1/2}\int_a^t\int_{\mathbb{T}}p_{t-r}(x,y)(r-a)^{-1/4+\alpha/4}dydr\Big)\\
&\lesssim \|g\|_{C^\alpha}(1+\max_{l\in\{1,2\}}[D^{u^l}]_{\mathscr{V}_{2p}})([u^1,u^2]_{\mathscr{S}^{1/2}_p[S,T]}+\|u^1(S,\cdot)-u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)})\\
&\quad\times\Big((t-s)^\beta(t-a)^{1/2+\alpha/4}+(t-s)^{1/2}(t-a)^{3/4+\alpha/4}\Big)\\
&\lesssim \|g\|_{C^\alpha}(1+\max_{l\in\{1,2\}}[D^{u^l}]_{\mathscr{V}_{2p}})([u^1,u^2]_{\mathscr{S}^{1/2}_p[S,T]}+\|u^1(S,\cdot)-u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)})\\
&\quad\times\Big((t-s)^{\beta+1/2+\alpha/4}+(t-s)^{5/4+\alpha/4}\Big).
\end{aligned}
\tag{3.2.21}
$$

Note that $\beta+1/2+\alpha/4$ and $5/4+\alpha/4$ are greater than 1 by the assumptions that $\beta>1/2-\alpha/4$ and that $\alpha>-1$. Consequently, by (3.2.17) and (3.2.21), we have that the condition (1.9.24) is satisfied. In addition, by using Lemma 3.1.4 and the regularity of g, it is straightforward to see that the process

$$
\mathscr{A}_t:=\int_0^t\int_{\mathbb{T}}p_{T-r}(x,y)\big(g(u^1(r,y))-g(u^2(r,y))\big)dydr
$$

satisfies (1.9.25) and (1.9.26). Consequently, the conclusion follows from Lemma 1.9.1 and the fact that $(S,T)\in[0,1]^2_\leq$ was arbitrary. □

Corollary 3.2.5. *Let $p \in [2,\infty)$ and let $\sigma \in C^4$ such that there exists a constant $\mu > 0$ such that $\sigma^2(x) \geq \mu^2$. For $i = 1,2$, let $b^i \in C^\alpha$ and let u^i be regularised solutions of*

$$(\partial_t - \Delta)u^i = b^i(u^i) + \sigma(u^i)\xi$$

in the class $\mathcal{U}^\beta$ for some $\beta \in \big(\frac{1}{2} - \frac{\alpha}{4}, 1 + \frac{\alpha}{4}\big]$. There exists a constant $N = N(p, \|\sigma\|_{C^4}, \mu, \alpha, \beta)$ such that for all $g^1, g^2 \in C^\infty$, $(s,t) \in [0,1]^2_{\leq}$, we have

$$\begin{aligned}
&\Big\| \int_s^t \int_{\mathbb{T}} p_{t-r}(x,y)\big(g^1(u^1(r,y)) - g^2(u^2(r,y))\big)dydr \Big\|_{L_p} \\
&\leq N\big(1 + \max_{i\in\{1,2\}} [D^{u^i}]_{\mathscr{V}^\beta_{2p}}\big)|t-s|^{(3+\alpha)/4} \\
&\qquad \times \Big(\|g^1 - g^2\|_{C^{\alpha-1}} + \|g^2\|_{C^\alpha}\Big([u^1,u^2]_{\mathcal{S}_p^{1/2}[s,t]} + \|u^1(s,\cdot) - u^2(s,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}\Big)\Big).
\end{aligned}$$

Proof. Since $\beta > \frac{1}{2} - \frac{\alpha}{4} = \frac{1}{4} - \frac{\alpha-1}{4}$, we can see that the condition of Corollary 3.2.2 is satisfied with $\lambda = \alpha - 1$. The desired result follows from Corollary 3.2.2 and Lemma 3.2.4 by the triangle inequality. □

3.3 The $\mathcal{S}_p$-bracket of two solutions

Throughout the section we work with the following assumption:

Assumption 3.3.1. Let $\sigma \in C^4$ such that there exists constant $\mu > 0$ such that $\sigma^2(x) \geq \mu^2$. Let $\alpha \in (-1,0)$, $\beta \in (\frac{1}{2} - \frac{\alpha}{4}, 1 + \frac{\alpha}{4}]$, and suppose that for $i = 1,2$ we are given $b^i \in C^{\alpha+}$ and that u^i are regularised solutions of

$$(\partial_t - \Delta)u^i = b^i(u^i) + \sigma(u^i)\xi(dy,dr)$$

in the class $\mathcal{U}^\beta$ with initial conditions $u^i(0,\cdot) = u^i_0 \in C(\mathbb{T})$.

Recall the definition of the $\mathcal{S}_p^{1/2}$-bracket from (3.1.4). Informally, the aim of this section is to show that

$$[u^1,u^2]_{\mathcal{S}_p^{1/2}[0,1]} \lesssim \|u^1_0 - u^2_0\|_{\mathbb{B}(\mathbb{T})} + \|b^1 - b^2\|_{C^{\alpha-1}}.$$

Lemma 3.3.2. *Let Assumption 3.3.1 hold and let $p \in [2,\infty)$. Then $[u^1,u^2]_{\mathcal{S}_p^{1/2}} < \infty$. Moreover there exists a constant $N = N(p,\mu,\|\sigma\|_{C^4},\alpha,\beta)$ such that*

$$\begin{aligned}
&[u^1,u^2]_{\mathcal{S}_p^{1/2}[s,t]} \\
&\leq N\big(1 + \max_{i\in\{1,2\}} [D^{u^i}]_{\mathscr{V}^\beta_{2p}}\big)(1 + \|b^2\|_{C^\alpha})
\end{aligned}$$

$$\times \Big(\|b^1 - b^2\|_{C^{\alpha-1}} + [u^1, u^2]_{\mathcal{S}_p^{1/2}[s,t]} + \|u^1(s,\cdot) - u^2(s,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)} \Big)(t-s)^{(1+\alpha)/4}. \quad (3.3.22)$$

Proof. Let $(S,T) \in [0,1]^2_{\leq}$. We begin by verifying that the $\mathcal{S}_p^{1/2}$-bracket is finite. By the triangle inequality, and by Lemma 3.1.4, we have for $(s,t) \in [S,T]^2_{\leq}$ that

$$\begin{aligned}
&\sup_{x\in\mathbb{T}} \|u^1(t,x) - \phi^{u^1(s,\cdot),s}(t,x) - u^2(t,x) + \phi^{u^2(s,\cdot),s}(t,x)\|_{L_{p,\infty}^{\mathcal{F}_s}} \\
&\leq \max_{i\in\{1,2\}} \sup_{x\in\mathbb{T}} \|u^i(t,x) - \phi^{u^i(s,\cdot),s}(t,x)\|_{L_{p,\infty}^{\mathcal{F}_s}} \\
&\lesssim \max_{i\in\{1,2\}} [D^{u^i}]_{\mathcal{V}_p^\beta} (t-s)^\beta \lesssim \max_{i\{1,2\}} [D^{u^i}]_{\mathcal{V}_p^\beta} (t-s)^{1/2},
\end{aligned}$$

where we used that by assumption we have $\beta \geq \frac{1}{2} - \frac{\alpha}{4} > \frac{1}{2}$. Thus by the fact that $\|\cdot\|_{L_p} = \|\|\cdot\|_{L_p|\mathcal{F}_s}\|_{L_p} \leq \|\cdot\|_{L_{p,\infty}^{\mathcal{F}_s}}$, it follows that

$$[u^1, u^2]_{\mathcal{S}_p^{1/2}} \lesssim \max_{i\in\{1,2\}} [D^{u^i}]_{\mathcal{V}_p^\beta}, \quad (3.3.23)$$

which is finite, since by assumption $u^i \in \mathcal{U}^\beta$. Note that for $(s,t) \in [0,1]^2_{\leq}$, $x \in \mathbb{T}$ we have by (2.3.30) and by (1.7.18) that

$$\begin{aligned}
&u^1(t,x) - u^2(t,x) - \phi^{u^1(s,\cdot),s}(t,x) + \phi^{u^2(s,\cdot),s}(t,x) = \\
&= u^1(t,x) - u^2(t,x) - P_{t-s}\big(u^1(s,\cdot) - u^2(s,\cdot)\big)(x) \\
&\qquad - \int_s^t \int_{\mathbb{T}} p_{t-r}(x,y)\big(\sigma(\phi^{u^1(s,\cdot),s}(r,y)) - \sigma(\phi^{u^2(s,\cdot),s}(r,y)\big)\xi(dy,dr) \\
&= \Big(D_t^{u^1} - D_t^{u^2} - P_{t-s}D_s^{u^1} + P_{t-s}D_s^{u^2}\Big)(x) \\
&\quad + \int_s^t \int_{\mathbb{T}} p_{t-r}(x,y)\big(\sigma(u^1(r,y)) - \sigma(u^2(r,y)) \\
&\qquad\qquad - \sigma(\phi^{u^1(s,\cdot),s}(r,y)) + \sigma(\phi^{u^2(s,\cdot),s}(r,y))\big)\xi(dy,dr) \\
&=: I(t,x) + J(t,x).
\end{aligned}$$

For $i = 1,2$ let $(b^{i,n})_{n\in\mathbb{N}} \subset C^\infty$ with $b^{i,n} \to b^i$ in C^α. Then by Definition 1.7.2 and by Fatou's lemma, we have

$$\begin{aligned}
&\sup_{(t,x)\in[0,1]\times\mathbb{T}} \|I(t,x)\|_{L_p} \\
&\lesssim \liminf_{n\to\infty} \sup_{(t,x)\in[0,1]\times\mathbb{T}} \Big\| \int_s^t \int_{\mathbb{T}} p_{t-r}(x,y)\big(b^{1,n}(u^1(r,y)) - b^{2,n}(u^2(r,y))\big)dydr \Big\|_{L_p}
\end{aligned}$$

So by Corollary 3.2.5 we have

$$\|I(t,x)\|_{L_p} \lesssim |t-s|^{(3+\alpha)/4}(1+\max_{i\in\{1,2\}}[D^{u^i}]_{\mathscr{V}^\beta_{2p}})$$
$$\times\Big(\|b^1-b^2\|_{C^{\alpha-1}}+\|b^2\|_{C^\alpha}\Big([u^1,u^2]_{\mathcal{S}_p^{1/2}[s,t]}+\|u^1(s,\cdot)-u^2(s,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}\Big)\Big).$$

Note moreover that by Lemma 3.1.8 we have

$$\|J(t,x)\|_{L_p}$$
$$\lesssim [D^{u^1}]_{\mathscr{V}^\beta_{2p}}\|u^1(s,\cdot)-u^2(s,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}(t-s)^{\frac14+\beta}$$
$$+\Big(\int_s^t\int_{\mathbb{T}}p_{t-r}(x,y)\big\|u^1(r,y)-u^2(r,y)-\phi^{u^1(s,\cdot),s}(r,y)+\phi^{u^2(s,\cdot),s}(r,y)\big\|^2_{L_p}dydr\Big)^{1/2}.$$

By our bounds on I,J and by the observation that $\frac14+\beta>\frac14+\frac12-\frac{\alpha}{4}>\frac34+\frac{\alpha}{4}$, we conclude that

$$\|u^1(t,x)-u^2(t,x)-\phi^{u^1(s,\cdot),s}(t,x)+\phi^{u^2(s,\cdot),s}(t,x)\|^2_{L_p}\lesssim$$
$$\lesssim(1+\max_{i\in\{1,2\}}[D^{u^i}]_{\mathscr{V}^\beta_{2p}})^2(1+\|b^2\|_{C^\alpha})^2$$
$$\times\Big(\|b^1-b^2\|_{C^{\alpha-1}}+[u^1,u^2]_{\mathcal{S}_p^{1/2}[s,t]}+\|u^1(s,\cdot)-u^2(s,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}\Big)^2(t-s)^{(3+\alpha)/2}$$
$$+\int_s^t\int_{\mathbb{T}}p_{t-r}(x,y)\big\|u^1(r,y)-u^2(r,y)-\phi^{u^1(s,\cdot),s}(r,y)+\phi^{u^2(s,\cdot),s}(r,y)\big\|^2_{L_p}dydr.$$

Note that the norm in the integrand is bounded in (r,y), since it is bounded by $[u^1,u^2]_{\mathcal{S}_p^{1/2}}$, which is finite by (3.3.23). Using Lemma 1.3.5, and Lemma 3.1.3 (where we recall that $S\le s\le t$), we get

$$\|u^1(t,x)-u^2(t,x)-\phi^{u^1(s,\cdot),s}(t,x)+\phi^{u^2(s,\cdot),s}(t,x)\|_{L_p}$$
$$\lesssim(1+\max_{i\in\{1,2\}}[D^{u^i}]_{\mathscr{V}^\beta_{2p}})(1+\|b^2\|_{C^\alpha})$$
$$\times\Big([u^1,u^2]_{\mathcal{S}_p^{1/2}[S,t]}+\|u^1(S,\cdot)-u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}+\|b^1-b^2\|_{C^{\alpha-1}}\Big)(t-s)^{(3+\alpha)/4}.$$

Therefore dividing both sides by $(t-s)^{1/2}$ and taking supremum over $(s,t)\in[S,T]^2_<$, we obtain the desired bound with (S,T) in place of (s,t). Now the desired result follows by the fact that $(S,T)\in[0,1]^2_\le$ was arbitrary. □

Lemma 3.3.3 (Splitting the $\mathcal{S}_p^{1/2}$-bracket)**.** *Let Assumption 3.3.1 hold and let $p\in[2,\infty)$. There exists a* *constant* $N = N(p, \|\sigma\|_{C^4}, \mu, \alpha, \beta)$ *such for all* $(S,T) \in [0,1]^2_{\leq}$ *and* $Q \in [S,T]$ *we have*

$$[u^1, u^2]_{\mathcal{S}_p^{1/2}[S,T]} \leq N(1 + \max_{i\in\{1,2\}} [D^{u^i}]_{\mathcal{V}_{2p}^{\beta}})\Big([u^1,u^2]_{\mathcal{S}_p^{1/2}[S,Q]} + \|u^1(S,\cdot) - u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}\Big) + 2[u^1,u^2]_{\mathcal{S}_p^{1/2}[Q,T]}.$$

Proof. For $(s,t) \in [0,1]^2_{\leq}$, we set

$$A(s,t) := \sup_{x\in\mathbb{T}} \|u^1(t,x) - u^2(t,x) - \phi^{u^1(s,\cdot),s}(t,x) + \phi^{u^1(s,\cdot),s}(t,x)\|_{L_p}.$$

For $(s,t) \in [S,Q]^2_{\leq}$ or $(s,t) \in [Q,T]^2_{\leq}$, we clearly have

$$A(s,t) \leq \big([u^1,u^2]_{\mathcal{S}_p^{1/2}[S,Q]} + [u^1,u^2]_{\mathcal{S}_p^{1/2}[Q,T]}\big)|t-s|^{1/2}. \tag{3.3.24}$$

For $s \leq Q < t$, by using the triangle inequality and keeping in mind the definition of $F^{(4)}_{p,0}$ (see (3.1.10)) we have

$$\begin{aligned} A(s,t) &\leq A(Q,t) \\ &\quad + \sup_{x\in\mathbb{T}} \|\phi^{u^1(Q,\cdot),Q}(t,x) - \phi^{u^2(Q,\cdot),Q}(t,x) - \phi^{u^1(s,\cdot),s}(t,x) + \phi^{u^2(s,\cdot),s}(t,x)\|_{L_p} \\ &= A(Q,t) + \sup_{x\in\mathbb{T}} \big\|F^{(4)}_{p,0}\big(t-Q,\, x,\, \phi^{u^1(s,\cdot),s}(Q,\cdot), \phi^{u^2(s,\cdot),s}(Q,\cdot), u^1(Q,\cdot), u^2(Q,\cdot)\big)\big\|_{L_p} \end{aligned}$$

From this and Lemma 3.1.6, we conclude that for $s \leq Q < t$

$$\begin{aligned} A(s,t) &\leq [u^1,u^2]_{\mathcal{S}_p^{1/2}[Q,T]}|t-s|^{1/2} \\ &\quad + N(1 + \max_{i\in\{1,2\}} [D^{u^i}]_{\mathcal{V}_{2p}^{\beta}})\Big([u^1,u^2]_{\mathcal{S}_p^{1/2}[S,Q]} + \|u^1(S,\cdot) - u^2(S,\cdot)\|_{\mathbb{B}(\mathbb{T},L_p)}\Big)|t-s|^{1/2}. \end{aligned} \tag{3.3.25}$$

By the above combined with (3.3.24), the inequality (3.3.25) holds for any $(s,t) \in [S,T]^2_{\leq}$, from which the claim follows. □

Lemma 3.3.4. *Let Assumption 3.3.1 hold and let* $K \in \mathbb{Z}_{\geq 2}$, $p \in [2,\infty)$. *There exists a constant* $N = N(p, \|\sigma\|_{C^4}, K, \mu, \alpha, \beta)$ *such that with* $M := N(1 + \max_{i\in\{1,2\}} [D^{u^i}]_{\mathcal{V}_{2p}})$ *we have*

$$[u^1,u^2]_{\mathcal{S}_p^{1/2}} \leq (K-1)M^{K-1}\|u^1_0 - u^2_0\|_{\mathbb{B}(\mathbb{T})} + 2\sum_{i=0}^{K-1} M^i [u^1,u^2]_{\mathcal{S}_p[\frac{K-i-1}{K},\frac{K-i}{K}]}.$$

Proof. Let $a_{s,t} := [u^1,u^1]_{\mathcal{S}_p^{1/2}[s,t]}$ and $u^{1,2}_0 := \|u^1_0 - u^2_0\|_{\mathbb{B}(\mathbb{T})}$. We will begin by using induction to show that for all $n \in \{1, \dots, K-1\}$ we have

$$a_{0,1} \le M^n a_{0,\frac{K-n}{K}} + \Big(\sum_{i=1}^{n} M^i\Big) u_0^{1,2} + 2\sum_{i=0}^{n-1} M^i a_{\frac{K-i-1}{K},\frac{K-i}{K}}. \tag{3.3.26}$$

By Lemma 3.3.3 we have that

$$a_{0,1} \le M(a_{0,\frac{K-1}{K}} + u_0^{1,2}) + 2a_{\frac{K-1}{K},1},$$

therefore (3.3.26) holds for the initial case $n = 1$. Now suppose that (3.3.26) holds for some $n \in \mathbb{N}$. We will show that it also holds for $n + 1$. To this end, we first apply the induction hypothesis and then Lemma 3.3.3, to get that

$$\begin{aligned}
a_{0,1} &\le M^n a_{0,\frac{K-n}{K}} + \Big(\sum_{i=1}^{n} M^i\Big) u_0^{1,2} + 2\sum_{i=0}^{n-1} M^i a_{\frac{K-i-1}{K},\frac{K-i}{K}} \\
&\le M^n \Big(M\big(a_{0,\frac{K-n-1}{K}} + u_0^{1,2}\big) + 2a_{\frac{K-n-1}{K},\frac{K-n}{K}} \Big) + \Big(\sum_{i=1}^{n} M^i\Big) u_0^{1,2} + 2\sum_{i=0}^{n-1} M^i a_{\frac{K-i-1}{K},\frac{K-i}{K}} \\
&= M^{n+1} a_{0,\frac{K-n-1}{K}} + M^{n+1} u_0^{1,2} + \Big(\sum_{i=1}^{n} M^i\Big) u_0^{1,2} + 2M^n a_{\frac{K-n-1}{K},\frac{K-n}{K}} + 2\sum_{i=0}^{n-1} M^i a_{\frac{K-i-1}{K},\frac{K-i}{K}} \\
&= M^{n+1} a_{0,\frac{K-n-1}{K}} + \Big(\sum_{i=1}^{n+1} M^i\Big) u_0^{1,2} + 2\sum_{i=0}^{n} M^i a_{\frac{K-i-1}{K},\frac{K-i}{K}}
\end{aligned}$$

as required. Therefore (3.3.26) is proven. Now choosing $n := K - 1$ in (3.3.26), we get

$$\begin{aligned}
a_{0,1} &\le \Big(\sum_{i=1}^{K-1} M^i\Big) u_0^{1,2} + M^{K-1} a_{0,\frac{1}{K}} + 2\sum_{i=0}^{K-2} M^i a_{\frac{K-i-1}{K},\frac{K-i}{K}} \\
&\le (K-1)M^{K-1} u_0^{1,2} + 2\sum_{i=0}^{K-1} M^i a_{\frac{K-i-1}{K},\frac{K-i}{K}}
\end{aligned}$$

as required. □

Lemma 3.3.5. *Let Assumption 3.3.1 hold. For all $p \in [2,\infty)$ there exists a positive constant $K_0 = K_0(\max_{i\in\{1,2\}}[D^{u^i}]_{\mathscr{V}_{2p}^{\beta}}, p, \|\sigma\|_{C^4}, \mu, \alpha, \beta)$ such that if $K \in \mathbb{Z}$ satisfies $K > K_0$, then there exists a constant $M = M(p, \|\sigma\|_{C^1}, \alpha, \beta)$ such that for all $n \in \{0, \dots, K\}$ we have that*

$$[u^1, u^2]_{\mathcal{S}_p^{1/2}[\frac{n}{K},\frac{n+1}{K}]} \le M^K \Big(\|u^1(0,\cdot) - u^2(0,\cdot)\|_{\mathbb{B}(\mathbb{T})} + \|b^1 - b^2\|_{C^{\alpha-1}} \Big).$$

Proof. By Lemma 3.3.2 there exists some $N = N(p, \mu, \|\sigma\|_{C^4}, \alpha, \beta) > 0$ such that for all $K \in \mathbb{N}$ and

$n \in \{0, \dots, K-1\}$ we have

$$\begin{aligned}
&[u^1, u^2]_{\mathcal{S}_p^{1/2}[\frac{n}{K}, \frac{n+1}{K}]} \\
&\le N(1 + \max_{i \in \{1,2\}} [D^{u^i}]_{\mathscr{V}_{2p}^{\beta}})(1 + \|b^2\|_{C^\alpha}) \\
&\qquad \times \Big([u^1, u^2]_{\mathcal{S}_p^{1/2}[\frac{n}{K}, \frac{n+1}{K}]} + \big\|u^1(\frac{n}{K}, \cdot) - u^2(\frac{n}{K}, \cdot)\big\|_{\mathbb{B}(\mathbb{T}, L_p)} + \big\|b^1 - b^2\big\|_{C^{\alpha-1}} \Big) K^{-(1+\alpha)/4}.
\end{aligned}$$

Let $\lceil \cdot \rceil$ denote the ceiling function, and define the constants

$$\begin{aligned}
\tilde{N} &:= N(1 + \max_{i \in \{1,2\}} [D^{u^i}]_{\mathscr{V}_{2p}^{\beta}})(1 + \|b^2\|_{C^\alpha}), \\
K_0 &:= \left\lceil (2\tilde{N})^{\frac{4}{1+\alpha}} \right\rceil.
\end{aligned} \tag{3.3.27}$$

Then for $K > K_0$ we have that

$$[u^1, u^2]_{\mathcal{S}_p^{1/2}[\frac{n}{K}, \frac{n+1}{K}]} \le \big\|u^1(\frac{n}{K}, \cdot) - u^2(\frac{n}{K}, \cdot)\big\|_{\mathbb{B}(\mathbb{T}, L_p)} + \|b^1 - b^2\|_{C^{\alpha-1}}. \tag{3.3.28}$$

In particular, by choosing $n = 0$, we have

$$[u^1, u^2]_{\mathcal{S}_p^{1/2}[0, \frac{1}{K}]} \le \|u_0^1 - u_0^2\|_{\mathbb{B}(\mathbb{T})} + \|b^1 - b^2\|_{C^{\alpha-1}}. \tag{3.3.29}$$

Let

$$a_n := [u^1, u^2]_{\mathcal{S}_p^{1/2}[\frac{n}{K}, \frac{n+1}{K}]} + \big\|u^1(\frac{n}{K}, \cdot) - u^2(\frac{n}{K}, \cdot)\big\|_{\mathbb{B}(\mathbb{T}, L_p)} + \|b^1 - b^2\|_{C^{\alpha-1}}.$$

In the $n = 0$ case we can use (3.3.29) to bound the first term to get

$$a_0 \le 2\left(\|u_0^1 - u_0^2\|_{\mathbb{B}(\mathbb{T})} + \|b^1 - b^2\|_{C^{\alpha-1}} \right). \tag{3.3.30}$$

For the general case $n \in \{1, \dots, K-1\}$ we first use (3.3.28) to get rid of the first term in the definition of a_n, and then we apply Lemma 3.1.3 as follows:

$$\begin{aligned}
a_n &\le 2\Big(\big\|u^1(\frac{n}{K}, \cdot) - u^2(\frac{n}{K}, \cdot)\big\|_{\mathbb{B}(\mathbb{T}, L_p)} + \|b^1 - b^2\|_{C^{\alpha-1}} \Big) \\
&\le M\Big([u^1, u^2]_{\mathcal{S}_p^{1/2}[\frac{n-1}{K}, \frac{n}{K}]} + \big\|u^1(\frac{n-1}{K}, \cdot) - u^2(\frac{n-1}{K}, \cdot)\big\|_{\mathbb{B}(\mathbb{T}, L_p)} + \|b^1 - b^2\|_{C^{\alpha-1}} \Big) \\
&= M a_{n-1}
\end{aligned}$$

for some constant $M = M(p, \|\sigma\|_{C^1}, \alpha, \beta) > 2$. Iterating this result n times and then applying (3.3.30), we get

$$a_n \le M^n a_0 \le M^n 2\big(\|u_0^1 - u_0^2\|_{\mathbb{B}(\mathbb{T})} + \|b^1 - b^2\|_{C^{\alpha-1}}\big)$$
$$\le M^{K-1} M\big(\|u_0^1 - u_0^2\|_{\mathbb{B}(\mathbb{T})} + \|b^1 - b^2\|_{C^{\alpha-1}}\big),$$

which finishes the proof. □

Lemma 3.3.6 ($\mathcal{S}_p^{1/2}$-stability of regularised solutions)**.** *Let Assumption 3.3.1 hold and let $p \in [2,\infty)$. There exists a continuous map (with dependencies as indicated below)*

$$f = f_{p,\|\sigma\|_{C^4},\mu,\alpha,\beta} : [0,\infty)^2 \to [0,\infty)$$

such that $f(x,y)$ is increasing in both the x and y variables, and that the following inequality holds:

$$[u^1,u^2]_{\mathcal{S}_p^{1/2}} \le f\big(\max_{i\in\{1,2\}} \|b^i\|_{C^\alpha}, \max_{i\in\{1,2\}} [D^{u^i}]_{\mathcal{V}_{2p}^\beta}\big)\Big(\|u_0^1 - u_0^2\|_{\mathbb{B}(\mathbb{T})} + \|b^1 - b^2\|_{C^{\alpha-1}}\Big).$$

Proof. Let $K \in \mathbb{Z}$ be sufficiently large so that it satisfies the assumption of Lemma 3.3.5. By (3.3.27) we know that we can choose $K = N_0(1 + \max_{i\in\{1,2\}}[D^{u^i}]_{\mathcal{V}_{2p}^\beta})^{\frac{4}{1+\alpha}}(1 + \|b^2\|_{C^\alpha})^{\frac{4}{1+\alpha}}$ with $N_0 = N_0(p,\mu,\|\sigma\|_{C^4},\alpha,\beta)$. Then there exists a constant $M_1 = M_1(p,\|\sigma\|_{C^1},\alpha,\beta)$ such that

$$[u^1,u^2]_{\mathcal{S}_p^{1/2}[\frac{n}{K},\frac{n+1}{K}]} \le M_1^K\big(\|u_0^1 - u_0^2\|_{\mathbb{B}(\mathbb{T})} + \|b^1 - b^2\|_{C^{\alpha-1}}\big). \tag{3.3.31}$$

Recall moreover that by Lemma 3.3.4 there exists a constant $N_2 = N_2(p,\|\sigma\|_{C^4},K,\mu,\alpha,\beta)$ such that for $M_2 := N_2(1 + \max_{i\in\{1,2\}}[D^{u^i}]_{\mathcal{V}_{2p}^\beta})$ we have

$$[u^1,u^2]_{\mathcal{S}_p^{1/2}} \le (K-1)M_2^{K-1}\|u_0^1 - u_0^2\|_{\mathbb{B}(\mathbb{T})} + 2\sum_{i=0}^{K-1} M_2^i [u^1,u^2]_{\mathcal{S}_p^{1/2}[\frac{K-i-1}{K},\frac{K-i}{K}]}. \tag{3.3.32}$$

By (3.3.31), we get that the second term on the right hand side of (3.3.32) is bounded by

$$2\sum_{i=0}^{K-1} M_2^i M_1^K\big(\|u_0^1 - u_0^2\|_{\mathbb{B}(\mathbb{T})} + \|b^1 - b^2\|_{C^{\alpha-1}}\big)$$
$$\le 2(K-1)(M_1M_2)^K\big(\|u_0^1 - u_0^2\|_{\mathbb{B}(\mathbb{T})} + \|b^1 - b^2\|_{C^{\alpha-1}}\big).$$

Therefore

$$[u^1, u^2]_{\mathscr{S}_p^{1/2}} \le (K-1)\big(M_2^{K-1} + 2(M_1 M_2)^K\big)\big(\|u_0^1 - u_0^2\|_{\mathbb{B}(\mathbb{T})} + \|b^1 - b^2\|_{C^{\alpha-1}}\big),$$

and the desired result follows by the definitions of K, M_1, M_2. □

3.4 The $\mathscr{V}_p$-bracket of the drift and an a priori estimate

The aim of this section is to provide a priori bounds on a regularised solution of (1.3.9) under Assumption 1.7.1.

Lemma 3.4.1. *Let Assumption 1.7.1 hold, let $\beta \in (\frac{1}{4} - \frac{\alpha}{4}, 1 + \frac{\alpha}{4}]$ and assume that u is a regularised solution of (1.3.9) in the class $\mathscr{U}_2^\beta$. Then u is also of class $\mathscr{U}^\beta$. Moreover for all $p \in [2, \infty)$ there exists a constant $N = N(p, \|\sigma\|_{C^4}, \mu, \alpha, \beta) > 0$ such that*

$$[D^u]_{\mathscr{V}_p^\beta} \le N \exp\Big(N\|b\|_{C^\alpha}^{\frac{4}{\alpha+3}}\Big).$$

Proof. Let $(b^n)_{n\in\mathbb{N}} \subset C^\infty$ be a sequence of smooth functions such that $b^n \to b$ in C^α. Then by the definition of D^u (see (1.7.17)), by the conditional Fatou's lemma and the usual Fatou's lemma, for $p \ge 2$ and for $(s,t) \in [0,1]_\le^2$, $x \in \mathbb{T}$ we have that

$$\|D_t^u(x) - P_{t-s} D_s^u(x)\|_{L_{p,\infty}^{\mathscr{F}_s}} \lesssim \liminf_{n\to\infty} \sup_{x\in\mathbb{T}} \Big\| \int_s^t \int_{\mathbb{T}} p_{t-r}(x,y) b^n(u(r,y))\,dy\,dr \Big\|_{L_{p,\infty}^{\mathscr{F}_s}}.$$

Therefore by applying Corollary 3.2.2 we know that

$$\begin{aligned}\|D_t^u(x) - P_{t-s} D_s^u(x)\|_{L_{p,\infty}^{\mathscr{F}_s}} &\lesssim \|b\|_{C^\alpha}\Big((t-s)^{1+\alpha/4} + [D^u]_{\mathscr{V}_2^\beta[s,t]}(t-s)^{\beta+\frac{\alpha+3}{4}}\Big)\\ &\lesssim \|b\|_{C^\alpha}\Big((t-s)^{\beta} + [D^u]_{\mathscr{V}_2^\beta[s,t]}(t-s)^{\beta+\frac{\alpha+3}{4}}\Big),\end{aligned}$$

where we used the assumption that $\beta \le 1 + \alpha/4$. Hence there exists $\tilde{N} = \tilde{N}(p, \|\sigma\|_{C^4}, \mu, \alpha, \beta)$ such that for all $(s,t) \in [0,1]_\le^2$ we have

$$[D^u]_{\mathscr{V}_p^\beta[s,t]} \le \tilde{N}\|b\|_{C^\alpha} + \tilde{N}\|b\|_{C^\alpha}[D^u]_{\mathscr{V}_2^\beta[s,t]}(t-s)^{(\alpha+3)/4}. \tag{3.4.33}$$

Since we assumed that $u \in \mathscr{U}_2^\beta$, we have $[D^u]_{\mathscr{V}_2^\beta} < \infty$, and thus by the inequality (3.4.33) we have $[D^u]_{\mathscr{V}_p^\beta} < \infty$, and thus $u \in \mathscr{U}_p^\beta$. Since $p \ge 2$ was arbitrary, it follows that $u \in \mathscr{U}^\beta$.

Note that on the right hand side of (3.4.33), the $[D^u]_{\mathscr{V}_2^\beta[s,t]}$ may be replaced with $[D^u]_{\mathscr{V}_p^\beta[s,t]}$. Hence choosing sufficiently large $K \in \mathbb{N}$, it follows that

$$\max_{i\in\{0,\dots,K-1\}} [D^u]_{\mathscr{V}_p^\beta[\frac{i}{K},\frac{i+1}{K}]} \lesssim \|b\|_{C^\alpha}. \tag{3.4.34}$$

To this end we may pick $K := \left\lceil (2\tilde{N}\|b\|_{C^\alpha})^{\frac{4}{\alpha+3}} \right\rceil$. Moreover using Lemma 1.11.7 and the inequality (3.4.34), we obtain that

$$\begin{aligned}
[D^u]_{\mathscr{V}_p^\beta[0,1]} &\leq 2^K \sum_{i=0}^{K-1} [D^u]_{\mathscr{V}_p^\beta[\frac{i}{K},\frac{i+1}{K}]} \\
&\lesssim 2^K \sum_{i=0}^{K-1} \|b\|_{C^\alpha} \lesssim K2^K\|b\|_{C^\alpha},
\end{aligned}$$

which finishes the proof . □

Lemma 3.4.2 (The regularity of D^u). *Let Assumption 1.7.1 hold, and let $p \in [2,\infty)$, $\beta \in (\frac{1}{2}-\frac{\alpha}{4}, 1+\frac{\alpha}{4}]$. There exists a constant $N = N(p, \|\sigma\|_{C^4}, \mu, \alpha, \beta) > 0$ such that if u is a regularised solution of class $\mathscr{U}^\beta$, then*

$$\|D^u\|_{C^{\frac{1}{4},\frac{1}{2}}([0,1]\times\mathbb{T},L_p)} \leq N(1+\|b\|_{C^\alpha})(1+[D^u]_{\mathscr{V}_p^\beta}).$$

Proof. Noting that $\|\cdot\|_{L_p} = \|\|\cdot\|_{L_p|\mathscr{F}_s}\|_{L_p} \leq \|\|\cdot\|_{L_p|\mathscr{F}_s}\|_{L_\infty}$ and that from the definition of D^u (see (1.7.17)) we have $D_0^u = 0$, we conclude for all $(t,x) \in [0,1]\times\mathbb{T}$ that

$$\|D_t^u(x)\|_{L_p} \leq \|D_t^u(x) - P_{t-0}D_0^u(x)\|_{L_{p,\infty}^{\mathscr{F}_s}} \leq [D^u]_{\mathscr{V}_p^\beta}. \tag{3.4.35}$$

Let $(b^n)_{n\in\mathbb{N}} \subset C^\infty$ be a sequence of smooth functions such that $b^n \to b$ in C^α. By (1.7.17), Fatou's lemma and Corollary 3.2.3, we can see that for all $x, \bar{x} \in \mathbb{T}$ and $t \in [0,1]$ we have

$$\begin{aligned}
\|D_t^u(x) - D_t^u(\bar{x})\|_{L_p} &\leq \liminf_{n\to\infty} \left\| \int_0^t \int_{\mathbb{T}} (p_{t-r}(x,y) - p_{t-r}(\bar{x},y)) b^n(u(r,y))\,dy\,dr \right\|_{L_p} \\
&\lesssim \|b\|_{C^\alpha}(1+[D^u]_{\mathscr{V}_p^\beta})|x-\bar{x}|^{1/2}.
\end{aligned} \tag{3.4.36}$$

By (3.4.35) and (3.4.36) we can see that

$$\sup_{t\in[0,1]} \|D_t^u\|_{C^{1/2}(\mathbb{T})} \lesssim (1+\|b\|_{C^\alpha})(1+[D^u]_{\mathscr{V}_p^\beta}). \tag{3.4.37}$$

Finally, note that since by assumption we have $\beta > \frac{1}{2} - \frac{\alpha}{4} > \frac{1}{4}$, and thus

$$[D^u]_{\mathscr{V}_p^{1/4}} \leq [D^u]_{\mathscr{V}_p^\beta}. \tag{3.4.38}$$

By (3.4.37) and (3.4.38), the desired bound holds for the $C^{0,\frac{1}{2}}([0,1] \times \mathbb{T}, L_p)$-norm and for the $\mathscr{V}_p^{1/4}$-bracket. Hence by Lemma 1.11.8 the proof is finished. □

Lemma 3.4.3 (An a priori estimate)**.** *Let Assumption 1.7.1 hold, and let* $p \in [2, \infty)$, $\varepsilon \in (0, \frac{1}{2})$, $\beta \in (\frac{1}{2} - \frac{\alpha}{4}, 1 + \frac{\alpha}{4}]$. *There exists a constant* $N = N(p, \|\sigma\|_{C^4}, \mu, \alpha, \beta, \varepsilon) > 0$ *such that if u is a regularised solution of class* $\mathscr{U}^\beta$*, then*

$$\|u - P.u_0(\cdot)\|_{C^{1/4-\varepsilon/2,1/2-\varepsilon}([0,1]\times\mathbb{T},L_p)} \leq N(1 + \|b\|_{C^\alpha})(1 + [D^u]_{\mathscr{V}_p^\beta}).$$

Proof. For $(t, x) \in [0, 1] \times \mathbb{T}$ denote

$$V_t(x) := \int_0^t \int_{\mathbb{T}} p_{t-r}(x, y)\sigma(u(r, y))\xi(dy, dr).$$

By the triangle inequality

$$\|u - Pu_0\|_{C^{1/4-\varepsilon/2,1/2-\varepsilon}([0,1]\times\mathbb{T},L_p)} \leq \|D^u\|_{C^{1/4-\varepsilon/2,1/2-\varepsilon}([0,1]\times\mathbb{T},L_p)} + \|V\|_{C^{1/4-\varepsilon/2,1/2-\varepsilon}([0,1]\times\mathbb{T},L_p)}.$$

But by Lemma 3.4.2, we know that $\|D^u\|_{C^{1/4,1/2}([0,1]\times\mathbb{T},L_p)} \lesssim (1 + \|b\|_{C^\alpha})(1 + [D^u]_{\mathscr{V}_p^\beta})$ and it can be seen from the BDG inequality and by the heat kernel estimates (1.11.38) and (1.11.39) that $\|V\|_{C^{1/4-\varepsilon/2,1/2-\varepsilon}([0,1]\times\mathbb{T},L_p)} \lesssim 1$, and thus the proof is finished. □

3.5 The proof of well-posedness

Theorem 3.5.1 (Uniqueness)**.** *Let Assumption 1.7.1 hold, let* $\beta \in (\frac{1}{2} - \frac{\alpha}{4}, 1 + \frac{\alpha}{4}]$ *and suppose that* u^1, u^2 *are regularised solutions of (1.3.9) in the class* $\mathscr{U}_2^\beta$*. Then* $u^1(t, x) = u^2(t, x)$ *almost surely for all* $(t, x) \in [0, 1] \times \mathbb{T}$*.*

Proof. Since $u^1, u^2 \in \mathscr{U}_2^\beta$, it also follows by Lemma 3.4.1 that $u^1, u^2 \in \mathscr{U}^\beta$. Thus Assumption 3.3.1 satisfied. Therefore by Lemma 3.3.6 we have for $p \in [2, \infty)$ that

$$[u^1, u^2]_{\mathscr{S}_p^{1/2}} \leq 0.$$

So since $u^1(t,\cdot) - u^2(t,\cdot) = u^1(t,\cdot) - u^2(t,\cdot) - \phi^{u^1(0,\cdot),s}(t,\cdot) + \phi^{u^2(0,\cdot),s}(t,\cdot)$, it follows that

$$\sup_{(t,x)\in[0,1]\times\mathbb{T}} \|u^2(t,x) - u^2(t,x)\|_{L_p} = 0,$$

and the desired result follows. □

Let Assumption 1.7.1 hold. The rest of the section is concerned with proving the existence of regularised solutions in the class $\mathscr{U}^{1+\frac{\alpha}{4}}$. Let $(b^n)_{n\in\mathbb{N}} \subset C^\infty$ such that $b^n \to b$ in C^α. Suppose that for all $n \in \mathbb{N}$, u^n is the classical mild solution of the SPDE

$$(\partial_t - \Delta)u^n = b^n(u^n) + \sigma(u^n)\xi, \qquad u^n(0,\cdot)= u(0,\cdot). \tag{3.5.39}$$

We call $(u^n)_{n\in\mathbb{N}}$ the sequence of *approximate solutions*, and for $(t,x) \in [0,1]\times\mathbb{T}$ we define the corresponding *approximate drift term* and *approximate noise term* respectively by

$$D_t^{u^n}(x) := \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y) b^n(u^n(r,y))dydr,$$
$$V_t^{u^n}(x) := \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y) \sigma(u^n(r,y))\xi(dy,dr).$$

By Lemma 3.4.1 we have for all $p \geq 1$, that

$$\sup_{n\in\mathbb{N}} [D^{u^n}]_{\mathscr{V}_p^{1+\alpha/4}} < \infty. \tag{3.5.40}$$

Lemma 3.5.2 (Convergence of the approximate drift and noise terms)**.** *Let Assumption 1.7.1 hold, and let $p \in [1,\infty)$ and $\varepsilon \in (0,\frac{1}{2})$. Then the sequences $(D^{u^n})_{n\in\mathbb{N}}$, $(V^{u^n})_{n\in\mathbb{N}}$ are convergent in $C^{\frac{1}{4}-\frac{\varepsilon}{2},\frac{1}{2}-\varepsilon}([0,1]\times\mathbb{T}, L_p)$.*

Proof. Assume without loss of generality that $p > 2$. By Corollary 3.2.5 (with $\beta = 1 + \frac{\alpha}{4}$) and by Lemma 3.3.6 we have

$$\begin{aligned} &\sup_{(t,x)\in[0,1]\times\mathbb{T}} \|D_t^{u^n}(x) - D_t^{u^m}(x)\|_{L_p} \\ &= \sup_{(t,x)\in[0,1]\times\mathbb{T}} \Big\| \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y) \big(b^n(u^n(r,y)) - b^m(u^m(r,y))\big)dydr \Big\|_{L_p} \\ &\lesssim \|b^n - b^m\|_{C^{\alpha-1}} + [u^n,u^m]_{\mathscr{S}_p^{1/2}[0,1]} \lesssim \|b^n - b^m\|_{C^\alpha} \longrightarrow 0 \end{aligned} \tag{3.5.41}$$

as $n \to \infty$. Moreover by Lemma 3.4.2 (with $\beta = 1 + \frac{\alpha}{4}$) and by (3.5.40), we have that

$$\sup_{n \in \mathbb{N}} \|D^{u^n}\|_{C^{1/4,1/2}([0,1]\times\mathbb{T},L_p)} < \infty. \tag{3.5.42}$$

By (3.5.41), (3.5.42), and by a standard interpolation argument, we can see that $(D^{u^n})_{n\in\mathbb{N}}$ is Cauchy in $C^{\frac{1}{4}-\frac{\varepsilon}{2},\frac{1}{2}-\varepsilon}([0,1]\times\mathbb{T},L_p)$.

We proceed with showing that the same is true for the sequence $(V^n)_{n\in\mathbb{N}}$. To this end note that by the BDG inequality, by the definition of the $\mathscr{S}_p^{1/2}$-bracket, and by Lemma 3.3.6 we have

$$\begin{aligned}
&\sup_{(t,x)\in[0,1]\times\mathbb{T}} \|V_t^{u^n}(x) - V_t^{u^m}(x)\|_{L_p} \\
&= \sup_{(t,x)\in[0,1]\times\mathbb{T}} \Big\| \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y)\big(\sigma(u^n(r,y)) - \sigma(u^m(r,y))\big)\xi(dy,dr)\Big\|_{L_p} \\
&\lesssim t^{1/4}\|u^n - u^m\|_{\mathbb{B}([0,1]\times\mathbb{T},L_p)} \le [u^n,u^m]_{\mathscr{S}_p^{1/2}} \lesssim \|b^n - b^m\|_{C^{\alpha-1}} \longrightarrow 0
\end{aligned} \tag{3.5.43}$$

as $n, m \to \infty$. Let $\gamma \in (0,\varepsilon)$. Using the BDG inequality and the heat kernel estimates (1.11.38), (1.11.39), we can see that for all $n \in \mathbb{N}$, $s,t \in [0,1]$, $x,\bar{x} \in \mathbb{T}$ the following estimates hold:

$$\begin{aligned}
\|V_t^{u^n}(x) - V_t^{u^n}(\bar{x})\|_{L_p}^2 &\lesssim \int_0^t \int_{\mathbb{T}} (p_{t-r}(x,y) - p_{t-r}(\bar{x},y))^2 dy dr \lesssim |x-\bar{x}|^{1-2\gamma}, \\
\|V_t^{u^n}(x) - V_s^{u^n}(x)\|_{L_p}^2 &\lesssim \int_0^s (p_{t-r}(x,y) - p_{s-r}(x,y))^2 dy dr + \int_s^t p_{t-r}^2(x,y) dy dr \\
&\lesssim |t-s|^{1/2-\gamma}.
\end{aligned}$$

Therefore we conclude that

$$\sup_{n \in \mathbb{N}} \|V^{u^n}\|_{C^{\frac{1}{4}-\frac{\gamma}{2},\frac{1}{2}-\gamma}([0,1]\times\mathbb{T},L_p)} < \infty. \tag{3.5.44}$$

By (3.5.43), (3.5.44), and by a standard interpolation argument, we can see that $(V^n)_{n\in\mathbb{N}}$ is also Cauchy in $C^{\frac{1}{4}-\frac{\varepsilon}{2},\frac{1}{2}-\varepsilon}([0,1]\times\mathbb{T},L_p)$, and thus the proof is finished. □

Consistently with the above lemmas, we will thus denote

$$D^{\tilde{u}} := \lim_{n\to\infty} D^{u^n} \quad \text{and} \quad V^{\tilde{u}} := \lim_{n\to\infty} V^{u^n},$$

where the limits are taken pointwise in $(t,x) \in [0,1]\times\mathbb{T}$, in probability. Moreover, it follows that for all

$\varepsilon \in (0,1/2)$ and $p \in [1,\infty)$ we have $D^{\tilde{u}}, V^{\tilde{u}} \in C^{\frac{1}{4}-\frac{\varepsilon}{2},\frac{1}{2}-\varepsilon}([0,1]\times\mathbb{T}, L_p)$ and

$$\lim_{n\to\infty}\Big(\|D^{\tilde{u}} - D^{u^n}\|_{C^{\frac{1}{4}-\frac{\varepsilon}{2},\frac{1}{2}-\varepsilon}([0,1]\times\mathbb{T},L_p)} + \|V^{\tilde{u}} - V^{u^n}\|_{C^{\frac{1}{4}-\frac{\varepsilon}{2},\frac{1}{2}-\varepsilon}([0,1]\times\mathbb{T},L_p)}\Big) = 0. \tag{3.5.45}$$

Moreover for $(t,x) \in [0,1]\times\mathbb{T}$, we define

$$\tilde{u}(t,x) := P_t u_0(x) + D^{\tilde{u}}_t(x) + V^{\tilde{u}}_t(x). \tag{3.5.46}$$

Lemma 3.5.3 ($V^{\tilde{u}}$ is the noise term of $\tilde{u}$)**.** *Let Assumption 1.7.1 hold. For all $(t,x) \in [0,1]\times\mathbb{T}$, we have*

$$V^{\tilde{u}}_t(x) = \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y)\sigma(\tilde{u}(r,y))\xi(dy,dr).$$

Proof. By the definitions of $D^{\tilde{u}}$ and $V^{\tilde{u}}$ (see (3.5.45)), by Fatou's lemma, and by the definition of $\tilde{u}$ (see (3.5.46)) we have for $p \geq 2$ that

$$\begin{aligned}
&\|V^{\tilde{u}}_t(x) - \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y)\sigma(\tilde{u}(r,y))\xi(dy,dr)\|^2_{L_p} \\
&\leq \liminf_{n\to\infty}\Big\| \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y)\sigma(u^n(r,y)) - \sigma(\tilde{u}(r,y))\xi(dy,dr)\Big\|_{L_p} \\
&\lesssim t^{1/4}\liminf_{n\to\infty} \|u^n - \tilde{u}\|_{\mathbb{B}([0,1]\times\mathbb{T},L_p)} \\
&\lesssim \lim_{n\to\infty}\|D^{u^n} - D^{\tilde{u}}\|_{\mathbb{B}([0,1]\times\mathbb{T},L_p)} + \lim_{n\to\infty}\|V^{u^n} - V^{\tilde{u}}\|_{\mathbb{B}([0,1]\times\mathbb{T},L_p)} = 0,
\end{aligned}$$

and thus the proof is finished. □

We proceed with verifying that the definition of $D^{\tilde{u}}$ is not an abuse of notation, i.e. that $D^{\tilde{u}}$ is indeed the drift of $\tilde{u}$ as prescribed in (1.7.17). To this end, we will first need to prove the following lemma.

Lemma 3.5.4. *Let Assumption 1.7.1 hold, and for $n \in \mathbb{N}$ define random fields $f^n : \Omega\times[0,1]\times\mathbb{T} \to \mathbb{R}$ by*

$$f^n(t,x) := D^{\tilde{u}}_t(x) - \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y)b^n(\tilde{u}(r,y))dydr. \tag{3.5.47}$$

Then for any $p \in [1,\infty)$ we have that $\|f^n\|_{C^{\frac{1}{4},\frac{1}{2}}([0,1]\times\mathbb{T},L_p)} \longrightarrow 0$ as $n \to \infty$.

Proof. To bound the sup norm, we note that by Fatou's lemma, Corollary 3.2.2 (with $g = b^m - b^n$) and Lemma 3.4.1, we have that

$$\|f^n\|_{\mathbb{B}([0,1]\times\mathbb{T},L_p)} = \sup_{(t,x)\in[0,T]\times\mathbb{T}}\Big\|D^{\tilde{u}}_t(x) - \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y)b^n(\tilde{u}(r,y))dydr\Big\|_{L_p}$$

$$
\begin{aligned}
&\le \liminf_{m\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{T}} \Big\| \int_0^t \int_{\mathbb{T}} p_{t-r}\big(b^m(u^m(r,y)) - b^n(u^m(r,y))\big)dydr \Big\|_{L_p} \\
&\lesssim \liminf_{m\to\infty} \|b^m - b^n\|_{C^\alpha}(1 + [D^{u^m}]_{\mathscr{V}_p^{1+\alpha/4}})(t-s)^{1+\alpha/4} \lesssim \|b - b^n\|_{C^\alpha},
\end{aligned}
$$

and thus

$$
\lim_{n\to\infty} \|f^n\|_{\mathbb{B}([0,1]\times\mathbb{T},L_p)} = 0. \tag{3.5.48}
$$

Next, we bound the spatial seminorm. Let $x, \bar{x} \in \mathbb{T}$. In the calculation below we will use the definitions of $D^{\tilde{u}}\ \tilde{u}$, f^n (see (3.5.45), (3.5.46), and (3.5.47)) and the continuity of the approximate drifts, Fatou's lemma, Corollary 3.2.3 (with $g(x) = b^m(x) - b^n(x)$) and (3.5.40),

$$
\begin{aligned}
&\sup_{t\in[0,1]} \|f^n(t,x) - f^n(t,\bar{x})\|_{L_p} \\
&= \sup_{t\in[0,1]} \Big\| D_t^{\tilde{u}}(x) - D_t^{\tilde{u}}(\bar{x}) - \int_0^t \int_{\mathbb{T}} (p_{t-r}(x,y) - p_{t-r}(\bar{x},y)) b^n(\tilde{u}(r,y))dydr \Big\|_{L_p} \\
&= \sup_{t\in[0,1]} \liminf_{m\to\infty} \Big\| \int_0^t \int_{\mathbb{T}} \big(p_{t-r}(x,y) - p_{t-r}(\bar{x},y)\big)\big(b^m(u^m(r,y)) - b^n(u^m(r,y))\big)dydr \Big\|_{L_p} \\
&\lesssim \liminf_{m\to\infty} \|b^m - b^n\|_{C^\alpha}(1 + [D^{u^m}]_{\mathscr{V}_{2p}^{1+\alpha/4}})|x-\bar{x}|^{1/2} \lesssim \|b - b^n\|_{C^\alpha}|x-\bar{x}|^{1/2}.
\end{aligned}
$$

Therefore

$$
\lim_{n\to\infty} \sup_{t\in[0,1]} [f^n(t,\cdot)]_{C^{1/2}(\mathbb{T},L_p)} = 0. \tag{3.5.49}
$$

Finally, note that for $s, t \in [0,1]$ we have by Fatou's lemma, Corollary 3.2.2 (with $g = b^m - b^n$), and Lemma 3.4.1, that

$$
\begin{aligned}
&\sup_{x\in\mathbb{T}} \|f^n(t,\cdot) - P_{t-s} f^n(s,x)\|_{L_{p,\infty}^{\mathscr{F}_s}} \\
&= \sup_{x\in\mathbb{T}} \Big\| D_t^{\tilde{u}}(x) - \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y) b^n(\tilde{u}(r,y))dydr \\
&\qquad\qquad - P_{t-s}\Big(D_s^{\tilde{u}}(\cdot) - \int_0^s \int_{\mathbb{T}} p_{t-r}(x,y) b^n(\tilde{u})dydr\Big) \Big\|_{L_{p,\infty}^{\mathscr{F}_s}} \\
&\le \sup_{x\in\mathbb{T}} \liminf_{m\to\infty} \Big\| \int_s^t \int_{\mathbb{T}} p_{t-r}(x,y)(b^m(u^m(r,y)) - b^n(u^m(r,y)))dydr \Big\|_{L_{p,\infty}^{\mathscr{F}_s}} \\
&\lesssim \liminf_{m\to\infty} \|b^m - b^n\|_{C^\alpha}(1 + [D^{u^m}]_{\mathscr{V}_{2p}^{1+\alpha/4}})(t-s)^{1+\alpha/4} \\
&\lesssim \|b - b^n\|_{C^\alpha}(t-s)^{1+\alpha/4}.
\end{aligned}
$$

It follows that

$$\lim_{n\to\infty} [f^n]_{\mathscr{V}_p^{1+\alpha/4}} = 0. \tag{3.5.50}$$

By (3.5.48), (3.5.49), (3.5.50), and by Lemma 1.11.8 the proof is finished. □

Corollary 3.5.5 ($D^{\tilde{u}}$ is the drift of $\tilde{u}$)**.** *Let Assumption 1.7.1 hold. Then the pair $(\tilde{u}, D^{\tilde{u}})$ satisfies the condition (1.7.17) from Definition 1.7.2, that is for any sequence $(b^n)_{n\in\mathbb{N}} \subset C^\infty$ such that $b^n \to b$ in C^α, we have*

$$\sup_{(t,x)\in[0,T]\times\mathbb{T}} \left| D_t^{\tilde{u}}(x) - \int_0^t \int_{\mathbb{T}} p_{t-r}(x,y) b^n(\tilde{u}(r,y)) dy dr \right| \longrightarrow 0$$

in probability as $n \to \infty$.

Theorem 3.5.6 (Existence)**.** *Let Assumption 1.7.1 hold. Then the process $\tilde{u}$ is a regularised solution of (1.3.9) in the class $\mathscr{U}^{1+\alpha/4}$.*

Proof. Since for all $n \in \mathbb{N}$, the random field u^n (which is a classically defined mild solution) is $\mathscr{P} \otimes \mathscr{B}(\mathbb{T})$-measurable, so is the limit $\tilde{u}$. By the definition of $\tilde{u}$ and by Lemma 3.5.2 we have that

$$\tilde{u} - P_{\cdot} u_0 \in C^{1/4-\varepsilon, 1/4-\varepsilon/2}([0,1] \times \mathbb{T}, L_p)$$

for $p \geq 1$ and for any $\varepsilon > 0$. Therefore by Kolmogorov's continuity theorem, the random field $\tilde{u}(t,x) - P_t u(0,\cdot)(x)$ is continuous in (t,x). So noting that $P_t u(0,x)$ is also continuous in (t,x), it follows that $\tilde{u}(t,x)$ is continuous in (t,x). Note moreover that by Corollary 3.5.5, the pair $(\tilde{u}, D^{\tilde{u}})$ satisfies (1.7.17). Finally, we observe that by the definition of $\tilde{u}$ and by Lemma 3.5.3 the integral equation (1.7.18) is satisfied. Therefore it is clear that $\tilde{u}$ is a regularised solution of (1.3.9). Moreover for all $p \geq 1$ we have

$$[D^{\tilde{u}}]_{\mathscr{V}_p^{1+\alpha/4}} \leq \liminf_{n\to\infty} [D^{u^n}]_{\mathscr{V}_p^{1+\alpha/4}} \leq \sup_{n\in\mathbb{N}} [D^{u^n}]_{\mathscr{V}_p^{1+\alpha/4}} < \infty,$$

where the last inequality holds by (3.5.40). Therefore $\tilde{u} \in \mathscr{U}^{1+\alpha/4}$, and the proof is finished. □

Bibliography

[ABLM24] S. Athreya, O. Butkovsky, K. Lê, and L. Mytnik. Well-posedness of stochastic heat equation with distributional drift and skew stochastic heat equation. *Comm. Pure Appl. Math.* **77**, no. 5, (2024), 2708–2777. doi:10.1002/cpa.22157.

[AG01] A. Alabert and I. Gyöngy. On stochastic reaction-diffusion equations with singular force term. *Bernoulli* **7**, no. 1, (2001), 145–164. doi:10.2307/3318606.

[Bal18] R. M. Balan. A gentle introduction to spdes: the random field approach, 2018. URL https://arxiv.org/abs/1812.02812.

[BC01] R. F. Bass and Z.-Q. Chen. Stochastic differential equations for Dirichlet processes. *Probab. Theory Related Fields* **121**, no. 3, (2001), 422–446. doi:10.1007/s004400100151.

[BCD11] H. Bahouri, J.-Y. Chemin, and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, vol. 343 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. doi:10.1007/978-3-642-16830-7.

[BDG21] O. Butkovsky, K. Dareiotis, and M. Gerencsér. Approximation of SDEs: a stochastic sewing approach. *Probab. Theory Related Fields* **181**, no. 4, (2021), 975–1034. doi:10.1007/s00440-021-01080-2.

[BDG23] O. Butkovsky, K. Dareiotis, and M. Gerencsér. Optimal rate of convergence for approximations of SPDEs with nonregular drift. *SIAM J. Numer. Anal.* **61**, no. 2, (2023), 1103–1137. doi:10.1137/21M1454213.

[BDPR16] V. Barbu, G. Da Prato, and M. Röckner. *Stochastic porous media equations*, vol. 2163 of *Lecture Notes in Mathematics*. Springer, [Cham], 2016. doi:10.1007/978-3-319-41069-2.

[BF20] L. A. Bianchi and F. Flandoli. Stochastic Navier-Stokes equations and related models. *Milan J. Math.* **88**, no. 1, (2020), 225–246. doi:10.1007/s00032-020-00312-9.

[BGP94] V. Bally, I. Gyöngy, and E. Pardoux. White noise driven parabolic SPDEs with measurable drift. *J. Funct. Anal.* **120**, no. 2, (1994), 484–510. doi:10.1006/jfan.1994.1040.

[BM19] O. Butkovsky and L. Mytnik. Regularization by noise and flows of solutions for a stochastic heat equation. *Ann. Probab.* **47**, no. 1, (2019), 165–212. doi:10.1214/18-AOP1259.

[BP98] V. Bally and E. Pardoux. Malliavin calculus for white noise driven parabolic SPDEs. *Potential Anal.* **9**, no. 1, (1998), 27–64. doi:10.1023/A:1008686922032.

[CD22] R. Catellier and R. Duboscq. Regularization by noise for rough differential equations driven by gaussian rough paths. *arXiv preprint arXiv:2207.04251* (2022).

[CG16] R. Catellier and M. Gubinelli. Averaging along irregular curves and regularisation of ODEs. *Stochastic Processes and their Applications* **126**, no. 8, (2016), 2323–2366. doi:https://doi.org/10.1016/j.spa.2016.02.002.

[CHN21] L. Chen, Y. Hu, and D. Nualart. Regularity and strict positivity of densities for the nonlinear stochastic heat equation. *Mem. Amer. Math. Soc.* **273**, no. 1340, (2021), v+102. doi:10.1090/memo/1340.

[Con05] R. Cont. Modeling term structure dynamics: an infinite dimensional approach. *Int. J. Theor. Appl. Finance* **8**, no. 3, (2005), 357–380. doi:10.1142/S0219024905003049.

[Dav07] A. M. Davie. Uniqueness of solutions of stochastic differential equations. *Int. Math. Res. Not. IMRN* , no. 24, (2007), Art. ID rnm124, 26. doi:10.1093/imrn/rnm124.

[DG24] K. Dareiotis and M. Gerencsér. Path-by-path regularisation through multiplicative noise in rough, Young, and ordinary differential equations. *Ann. Probab.* **52**, no. 5, (2024), 1864–1902. doi:10.1214/24-aop1686.

[DGL23] K. Dareiotis, M. Gerencsér, and K. Lê. Quantifying a convergence theorem of Gyöngy and Krylov. *Ann. Appl. Probab.* **33**, no. 3, (2023), 2291–2323. doi:10.1214/22-aap1867.

[DKM+09] R. Dalang, D. Khoshnevisan, C. Mueller, D. Nualart, and Y. Xiao. *A minicourse on stochastic partial differential equations*, vol. 1962 of *Lecture Notes in Mathematics*.

Springer-Verlag, Berlin, 2009. Held at the University of Utah, Salt Lake City, UT, May 8–19, 2006.

[DPFPR13] G. Da Prato, F. Flandoli, E. Priola, and M. Röckner. Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift. *Ann. Probab.* **41**, no. 5, (2013), 3306–3344. doi:10.1214/12-AOP763.

[FHL24] P. K. Friz, A. Hocquet, and K. Lê. Rough stochastic differential equations, 2024. URL https://arxiv.org/abs/2106.10340.

[GHR24] L. Goudenège, E. M. Haress, and A. Richard. Numerical approximation of the stochastic heat equation with a distributional reaction term, 2024. URL https://arxiv.org/abs/2405.08201.

[GK81] I. Gyöngy and N. V. Krylov. On stochastic equations with respect to semimartingales. I. *Stochastics* **4**, no. 1, (1980/81), 1–21. doi:10.1080/03610918008833154.

[GP93a] I. Gyöngy and E. Pardoux. On quasi-linear stochastic partial differential equations. *Probab. Theory Related Fields* **94**, no. 4, (1993), 413–425. doi:10.1007/BF01192556.

[GP93b] I. Gyöngy and E. Pardoux. On the regularization effect of space-time white noise on quasi-linear parabolic partial differential equations. *Probab. Theory Related Fields* **97**, no. 1-2, (1993), 211–229. doi:10.1007/BF01199321.

[Gyö95] I. Gyöngy. On non-degenerate quasi-linear stochastic partial differential equations. *Potential Anal.* **4**, no. 2, (1995), 157–171. doi:10.1007/BF01275588.

[Gyö98] I. Gyöngy. Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. I. *Potential Anal.* **9**, no. 1, (1998), 1–25. doi:10.1023/A:1008615012377.

[Gyö99] I. Gyöngy. Lattice approximations for stochastic quasi-linear parabolic partial differential equations driven by space-time white noise. II. *Potential Anal.* **11**, no. 1, (1999), 1–37. doi:10.1023/A:1008699504438.

[Hai16] M. Hairer. Regularity structures and the dynamical Φ^4_3 model. In *Current developments in mathematics 2014*, 1–49. Int. Press, Somerville, MA, 2016.

[Hai21] M. Hairer. Introduction to Malliavin calculus, 2021. URL https://www.hairer.org/notes/Malliavin.pdf.

[HQ18] M. Hairer and J. Quastel. A class of growth models rescaling to KPZ. *Forum Math. Pi* **6**, (2018), e3, 112. doi:10.1017/fmp.2018.2.

[IS01] P. Imkeller and B. Schmalfuss. The conjugacy of stochastic and random differential equations and the existence of global attractors. *J. Dynam. Differential Equations* **13**, no. 2, (2001), 215–249. doi:10.1023/A:1016673307045.

[Itô51] K. Itô. On stochastic differential equations. *Mem. Amer. Math. Soc.* **4**, (1951), 51.

[JIP23] L. M. C. Jáquez, E. Issoglio, and J. Palczewski. Convergence rate of numerical scheme for SDEs with a distributional drift in Besov space, 2023. URL https://arxiv.org/abs/2309.11396.

[KR05] N. V. Krylov and M. Röckner. Strong solutions of stochastic equations with singular time dependent drift. *Probab. Theory Related Fields* **131**, no. 2, (2005), 154–196. doi:10.1007/s00440-004-0361-z.

[Kry99] N. V. Krylov. An analytic approach to SPDEs. In *Stochastic partial differential equations: six perspectives*, vol. 64 of *Math. Surveys Monogr.*, 185–242. Amer. Math. Soc., Providence, RI, 1999. doi:10.1090/surv/064/05.

[Lê20] K. Lê. A stochastic sewing lemma and applications. *Electron. J. Probab.* **25**, (2020), Paper No. 38, 55. doi:10.1214/20-ejp442.

[Lê23] K. Lê. Stochastic sewing in Banach spaces. *Electron. J. Probab.* **28**, (2023), Paper No. 26, 22. doi:10.1214/23-ejp918.

[LR17] S. V. Lototsky and B. L. Rozovsky. *Stochastic partial differential equations*. Universitext. Springer, Cham, 2017. doi:10.1007/978-3-319-58647-2.

[Mal78] P. Malliavin. Stochastic calculus of variation and hypoelliptic operators. In *Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976)*, Wiley-Intersci. Publ., 195–263. John Wiley & Sons, New York-Chichester-Brisbane, 1978.

[Nua06] D. Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second ed., 2006.

[Per20] N. Perkowski. Paracontrolled distributions and singular SPDEs. *Lecture notes* (2020).

[Por75] N. I. Portenko. On the existence of solutions of stochastic differential equations with integrable drift coefficient. *Theory of Probability & Its Applications* **19**, no. 3, (1975), 552–557. doi:10.1137/1119061.

[PZ93] E. Pardoux and T. S. Zhang. Absolute continuity of the law of the solution of a parabolic SPDE. *J. Funct. Anal.* **112**, no. 2, (1993), 447–458. doi:10.1006/jfan.1993.1040.

[SS04] M. Sanz-Solé. A course on Malliavin calculus with applications to stochastic partial differential equations. *Lecture Notes* (2004).

[Ver80] A. J. Veretennikov. Strong solutions and explicit formulas for solutions of stochastic integral equations. *Mat. Sb. (N.S.)* **111(153)**, no. 3, (1980), 434–452, 480.

[Wal86] J. B. Walsh. An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV—1984*, vol. 1180 of *Lecture Notes in Math.*, 265–439. Springer, Berlin, 1986. doi:10.1007/BFb0074920.

[Zvo74] A. K. Zvonkin. A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb. (N.S.)* **93(135)**, (1974), 129–149, 152.