Representations of Quivers and Cherednik Algebras

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Declaration

I confirm that the work submitted is my own, except where work which has formed part of jointly-authored publications has been included. The contributions of both myself and the other authors to this work have been explicitly indicated below. I also confirm that appropriate credit has been given within the thesis where reference has been made to the work of others.

> The exposition in Chapter 1 and elements of Chapters 3, 4, 5 and 6 are based on the paper [CR24]: Oleg Chalykh and Bradley Ryan, *DAHAs of Type $C^\vee C_n$ and Character Varieties*, 2024. At present, a preprint is available on arXiv but a journal submission is expected in the foreseeable future. The main results and proofs were written by myself. The initial idea, the background and suggestions for some technical arguments were communicated by Oleg Chalykh. I wrote the first draft and Oleg Chalykh had editorial input.

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Abstract

We study the spherical subalgebra of the double affine Hecke algebra of type $C^\vee C_n$ and relate it, at the classical level $q = 1$, to a certain character variety of the Riemann sphere with four punctures that we call the Calogero-Moser space. This establishes a conjecture from [EGO06]. As a by-product, we construct a completed phase space for the trigonometric van Diejen system and explicitly integrate the dynamics. We conclude by suggesting how one could quantise the main isomorphism, and discussing some preliminary work that aims to reconcile the Poisson bracket on the Calogero-Moser space with the bracket coming via its interpretation as the moduli space of flat connections on a punctured Riemann surface by Fock and Rosly [FR99].

Contents

Chapter 1

Introduction

The *double affine Hecke algebra* (DAHA) was introduced by Ivan Cherednik, who famously used them to prove several conjectures on Macdonald polynomials [Che95a, Che95b]. Its rational limit, the so-called *rational Cherednik algebras*, was then developed by Pavel Etingof and Victor Ginzburg in their seminal paper [EG02]. Since their introduction, there has been much more literature that studies these classes of algebras. Although some authors call the DAHA the *Cherednik algebra* after its founder, be aware it is often an abuse of nomenclature that this phrase is reserved for the rational limit of the DAHA. This isn't an issue for most of this thesis, because we work with the true DAHA, but the exposition herein will abuse this phrase also.

One of the key insights of [EG02] was that the *spherical subalgebra* of the Cherednik algebra associated to a Coxeter group $W \subseteq \mathrm{GL}(V)$ provides an interesting Poisson deformation of the orbifold T^*V/W. In particular, for the symmetric group $W = S_n$, [EG02] related the spherical subalgebra, at the so-called classical level $t = 0$, to the Calogero-Moser space:

$$\mathcal{M}_n = \{X, Y \in \mathrm{Mat}_{n\times n}(\mathbb{C}) : \mathrm{rank}([X, Y] - \mathbb{1}_n) = 1\} /\!\!/ \mathrm{GL}_n(\mathbb{C}),$$

previously studied by George Wilson [Wil98]. These spaces have their origin in the theory of Hamiltonian reduction and integrable systems [KKS78]. They are smooth $2n$-dimensional affine varieties that can be viewed as completed phase spaces [OP83] for the classical Calogero-Moser system [Cal71, Mos75], at the same time parametrising rational solutions to the KP hierarchy [Wil98]. In order to formulate this result of [EG02], we recall that the Cherednik algebra $\mathsf{H}_{t,c}$ of W contains the group algebra $\mathbb{C}W$, and the spherical subalgebra is defined as $e\mathsf{H}_{t,c}e$, where $e = \frac{1}{|W|}\sum_{w\in W} w$ is the group algebra symmetriser. Recall also that the spherical subalgebra is commutative at the classical level $t = 0$.

Theorem 1.1 (cf. [EG02, Theorem 1.23]) *For $W = S_n$ and $c \neq 0$, we have an isomorphism*

$$\mathrm{Spec}(e\mathsf{H}_{0,c}e) \cong \mathcal{M}_n.$$

Equivalently, one can say that $e\mathsf{H}_{0,c}e$ is isomorphic to the algebra of regular functions $\mathbb{C}[\mathcal{M}_n]$ on the Calogero-Moser space. Also, Theorem 1.1 admits a representation theoretic interpretation [EG02, Theorem 1.24]: every finite-dimensional irreducible representation of $\mathsf{H}_{0,c}$ has dimension $n!$, giving rise to a set-theoretic bijection with the points of Calogero-Moser space, that is

$$\mathrm{Irrep}(\mathsf{H}_{0,c}) \cong \mathcal{M}_n.$$

This motivates studying the varieties $\mathrm{Spec}(e\mathsf{H}_{0,c}e)$ in other analogous situations. For example, Etingof and Ginzburg consider symplectic reflection groups $W = G^n \rtimes S_n$ for a finite subgroup $G \subseteq \mathrm{SU}(2)$ and prove a generalisation of Theorem 1.1 for the corresponding symplectic reflection algebras, cf. [EG02, Theorem 11.16].

Another closely-related result of that kind was obtained by Alexei Oblomkov in [Obl04], where he studied the analogous problem for the DAHA $H_{q,\tau}$ of type GL_n. Recall that this DAHA contains the finite Hecke algebra of S_n, so one defines the spherical subalgebra as $\mathbf{e}H_{q,\tau}\mathbf{e}$, where $\mathbf{e}$ here denotes the so-called *Hecke symmetriser*. The associated smooth affine variety here is

$$CM_\tau = \{X, Y \in \mathrm{GL}_n(\mathbb{C}) : \mathrm{rank}(\tau XYX^{-1}Y^{-1} - \tau^{-1}\mathbb{1}_n) = 1\} /\!\!/ \mathrm{GL}_n(\mathbb{C}).$$

This space was previously identified as a completed phase space for the Ruijsenaars-Schneider system (a relativistic Calogero-Moser system) [FR99], so it is also dubbed a Calogero-Moser space. The spherical subalgebra in this context also becomes commutative at the classical level $q = 1$, and the main result is now stated.

Theorem 1.2 ([Obl04, Theorem 6.1]) *If τ is not a root of unity, we have an isomorphism*

$$\mathrm{Spec}(\mathbf{e}H_{1,\tau}\mathbf{e}) \cong CM_\tau.$$

Again, this means that we have an isomorphism between the spherical subalgebra $\mathbf{e}H_{1,\tau}\mathbf{e}$ and the algebra of regular functions $\mathbb{C}[CM_\tau]$. Oblomkov also establishes the representation theoretic version of Theorem 1.2: the finite-dimensional irreducible representations of $H_{1,\tau}$ have dimension $n!$, giving rise to a set-theoretic bijection with the points of CM_τ [Obl04, Corollary 6.2], that is

$$\mathrm{Irrep}(H_{1,\tau}) \cong CM_\tau.$$

A further generalisation was proposed in [EOR06] (in rank one) and [EGO06] (in higher rank) by way of introducing generalised DAHAs (GDAHAs) associated to star-shaped quivers, prevalently working with those of affine Dynkin type $\widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$. In these cases, [EGO06, Conjecture 5.1.1] postulates that the corresponding spherical subalgebras at the classical level $q = 1$ are isomorphic to certain character varieties of a punctured Riemann sphere. While some important steps towards a proof have been made in the paper itself [EGO06], it remained completely open.

The main result of this thesis is a proof of the aforementioned conjecture for the GDAHA of type $\widetilde{D}_4$, and an interpretation of the corresponding character variety from the point-of-view of integrable systems and Hamiltonian dynamics. In this case, the GDAHA is isomorphic to the DAHA $\mathcal{H}_{q,\boldsymbol{\tau}}$ of type $C^\vee C_n$ as studied in [Sah99, NS00, Sto00]. We have a spherical subalgebra $\mathbf{e}\mathcal{H}\mathbf{e}$ where the idempotent $\mathbf{e}$ is again the Hecke symmetriser. The associated affine variety is

$$\mathcal{C}_n = \{A_i \in [\Lambda_i] : A_1 A_2 A_3 A_4 = \mathbb{1}_{2n}\} /\!\!/ \operatorname{GL}_{2n}(\mathbb{C}),$$

where $[\Lambda_i] \subseteq \operatorname{GL}_{2n}(\mathbb{C})$ are semi-simple conjugacy classes (4.1) defined later, given explicitly in terms of the five non-zero DAHA parameters $\boldsymbol{\tau}$. It turns out that the parameters defining these conjugacy classes are *generic* (in the sense of Definition 2.12), which is sufficient to guarantee that $\mathcal{C}_n$ is a smooth $2n$-dimensional affine variety. Our main result will now be stated.

Theorem 1.3 ([CR24, Theorem 6.3]) *For generic parameters $\boldsymbol{\tau}$, we have an isomorphism*

$$\operatorname{Spec}(\mathbf{e}\mathcal{H}_{1,\boldsymbol{\tau}}\mathbf{e}) \cong \mathcal{C}_n.$$

Indeed, we can reformulate this as an isomorphism between the spherical subalgebra $\mathbf{e}\mathcal{H}_{1,\boldsymbol{\tau}}\mathbf{e}$ and the algebra of regular functions $\mathbb{C}[\mathcal{C}_n]$ on the above character variety. Furthermore, we can again interpret this result from a representation theory viewpoint: the finite-dimensional irreducible representations of $\mathcal{H}_{1,\boldsymbol{\tau}}$ have dimension $2^n n!$, giving rise to a set-theoretic bijection with the points of $\mathcal{C}_n$ [CR24, Corollary 6.4], that is

$$\operatorname{Irrep}(\mathcal{H}_{1,\boldsymbol{\tau}}) \cong \mathcal{C}_n.$$

Our proof relies crucially on the existence of the *Basic Representation* (see Propositions 2.8 and 3.16). This is unavailable for the GDAHAs of type $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, so our approach does not directly carry over and the conjecture [EGO06, Conjecture 5.1.1] in its entirety remains open. Nevertheless, the isomorphism we do establish in Theorem 1.3 has a nice application to the theory of integrable systems. Namely, our main result here (see Theorem 5.23) says that $\mathcal{C}_n$ can be viewed as a completed phase space for the trigonometric van Diejen system, and the corresponding Hamiltonian dynamics on $\mathcal{C}_n$ can be explicitly integrated.

Structure of the Thesis

In Chapter 2, we dedicate some time to introducing the main themes of the thesis: Hecke algebras (for the DAHA), character varieties and quiver varieties (for Calogero-Moser space), affine algebraic varieties (for the statement of the main result) and Poisson structures (for the Hamiltonian dynamics). In Chapter 3, we introduce the DAHA of type $C^\vee C_n$ and discuss its spherical subalgebra, an important duality property and the so-called Basic Representation. In Chapter 4, we define a character variety $\mathcal{C}_n$ associated to this DAHA, analyse a map Φ as constructed in [EGO06] which associates to points of $\mathcal{C}_n$ finite-dimensional representations of the DAHA. In Chapter 5, we introduce coordinates on $\mathcal{C}_n$ and show that Φ restricts to an isomorphism on a suitable coordinate chart. A technical argument then follows to show that two such charts, one of which is obtained from the DAHA duality property, intersect transversally. This then allows us to prove the main result Theorem 1.3 (rather, Theorem 5.17). We then explain a way to quantise this isomorphism (see Proposition 6.2), before applying our main result to the trigonometric van Diejen system. We conclude with Chapter 6, wherein we discuss future avenues of research, where our work fits into the wider context, and a final preliminary venture into how one obtains the dynamics on $\mathcal{C}_n$ via Hamiltonian reduction in the sense of Fock and Rosly.

Chapter 2

Preliminaries

This chapter will motivate the main ingredients of the thesis. We begin with some exposition on double affine Hecke algebras (DAHAs), followed by some basic notions on multiplicative quiver varieties and a brief overview of the structure of a Poisson bracket on a smooth manifold. Each of the numbered sections herein will (roughly) correspond to the chapters presented thereafter.

2.1 Hecke Algebras

We will motivate one of the most important aspects of this thesis by recalling some definitions regarding reflection groups, braid groups and root systems. One of the best places to find a detailed look at Hecke algebras, which we will follow, is the book [Mac03]. Throughout, let V be a (complexified) n-dimensional vector space endowed with the Euclidean inner product $\langle \cdot, \cdot \rangle$.

2.1.1 Root Systems

A *root system* is a finite subset $R \subseteq V$ of non-zero vectors that spans V, is closed under taking reflections in hyperplanes orthogonal to any $\alpha \in R$, and the projection of any root onto some $\alpha \in R$ is a half-integer multiple of α. The *dual root system* $R^\vee$ consists of the coroots $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ for each $\alpha \in R$. Recall also that the *simple roots* $a_1, \ldots, a_n$ form a basis of V such that every $\alpha \in R$ can be written either as an $\mathbb{N}^+$- or $\mathbb{N}^-$-linear combination $\sum_i m_i a_i$ of these roots. This allows us to decompose $R = R_+ \cup R_-$ into the union of *positive roots* and *negative roots*, and $R_- = -R_+$.

The *affine-linear functions* on V are functions $V \to \mathbb{C}$ that are sums of linear functionals and constant functions. The set of affine-linear functions is denoted $\widehat{V}$. By identifying the space of linear functionals with V (via the inner product), we see that $\widehat{V} \cong V \oplus \mathbb{C}\delta$ where $\delta \equiv c$ is constant on V, the exponential of which is $e^c =: q$. The corresponding *affine root system* is

$$\widetilde{R} = \{\alpha + k\delta : \alpha \in R \text{ and } k \in \mathbb{Z}\} \subseteq \widehat{V}.$$

The *highest root* is the unique element $\varphi \in R_+$ whose decomposition as a sum of simple roots $\sum_i m_i a_i$ is such that $\sum_i m_i$ is maximal. In other words, $\varphi + a_i \notin R$ for any $i = 1, \dots, n$. We can extend the set $\{a_1, \dots, a_n\}$ of simple roots in R to a basis of the affine root system $\widetilde{R}$ by adjoining the element

$$a_0 \coloneqq \delta - \varphi.$$

Example 2.1 ([Bou02, Plate III]) Let $V = \mathbb{R}^n$ and (ε_i) denote the canonical basis. We call

$$R = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\} \cup \{\pm 2\varepsilon_i : 1 \leq i \leq n\}$$

the root system of type C_n, and the corresponding subset of positive roots is given by

$$R_+ = \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\} \cup \{2\varepsilon_i : 1 \leq i \leq n\}.$$

The affine root system $\widetilde{R}$ has the following as a basis of simple roots, for $i = 1, \dots, n-1$:

$$a_0 = \delta - 2\varepsilon_1, \qquad a_i = \varepsilon_i - \varepsilon_{i+1}, \qquad a_n = 2\varepsilon_n.$$

The following final example will relate to something we see in Chapter 5, particularly §5.2.

Example 2.2 ([Bou02, Plate IV]) Let $V = \mathbb{R}^n$ and (ε_i) denote the canonical basis. We call

$$R = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\}$$

the root system of type D_n, and the corresponding subset of positive roots is given by

$$R_+ = \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\}.$$

The affine root system $\widetilde{R}$ has the following as a basis of simple roots, for $i = 1, \dots, n-1$:

$$a_0 = \delta - \varepsilon_1 - \varepsilon_2, \qquad a_i = \varepsilon_i - \varepsilon_{i+1}, \qquad a_n = \varepsilon_{n-1} + \varepsilon_n.$$

A root system is *reduced* if the only scalar multiples of $\alpha \in R$ that belong to the root system are $\pm\alpha$. Additionally, a root system is *irreducible* if it cannot be decomposed as $R = R_1 \cup R_2$ such that $\langle \alpha, \beta \rangle = 0$ for $\alpha \in R_1$ and $\beta \in R_2$.

Lemma 2.3 ([Mac03, §1.3]) *Let R be a non-reduced irreducible root system. Then, it determines two reduced irreducible root systems such that $R = R_1 \cup R_2$, namely*

$$R_1 \coloneqq \{\alpha \in R : \tfrac{1}{2}\alpha \notin R\} \qquad \textit{and} \qquad R_2 \coloneqq \{\alpha \in R : 2\alpha \notin R\}.$$

The *type* of a reduced root system is the type of Dynkin diagram that encodes it. In the case of a non-reduced irreducible root system, if the corresponding reduced irreducible root systems R_1 and R_2 as defined in Lemma 2.3 are of respective types X_1 and X_2, we say R is of *type* (X_1, X_2).

Notation 2.4 A root system of particular interest to us throughout the thesis is the non-reduced irreducible system of type $(C_n^\vee, C_n)$, meaning R_1 is of type $C_n^\vee$ and R_2 is of type C_n. Henceforth, we shall adopt the slightly more compact notation $C^\vee C_n$ when referring to this root system.

2.1.2 Weyl Groups

The corresponding *Weyl group* W is that which is generated by reflections s_α, given by

$$s_\alpha(\beta) = \beta - \frac{2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}\alpha = \beta - \langle \alpha^\vee, \beta\rangle\alpha.$$

The corresponding *affine Weyl group* $\widetilde{W}$ consists of the invertible affine transformations of V generated by the reflections $s_{\widetilde{\alpha}}$ for each $\widetilde{\alpha} \in \widetilde{R}$. Explicitly, these are reflections across the hyperplanes $\widetilde{\alpha}(\beta) = 0$, that is $s_{\widetilde{\alpha}}(\beta) = \beta - \widetilde{\alpha}(\beta)\alpha^\vee$. Since $R \subseteq \widetilde{R}$ (set $k = 0$), we have $W \subseteq \widetilde{W}$.

Remark 2.5 In fact, Lemma 2.3 and everything thereafter holds also for an *affine* root system $\widetilde{R}$. Better yet, each of the reduced root systems R_1 and R_2 has the same underlying Weyl group.

A *translation* $t : V \to V$ is an affine-linear transformation of the form $t_\lambda(x) = x + \lambda$ for some fixed $\lambda \in V$ (most authors' convention is to denote the translation only by the shift vector). Any lattice $L \subseteq V$ admits a *translation lattice* $t(L)$ consisting of t_λ for $\lambda \in L$. In particular, let $Q \coloneqq \sum_\alpha \mathbb{Z}\alpha$ be the *root lattice* and $Q^\vee \coloneqq \sum_\alpha \mathbb{Z}\alpha^\vee$ the *coroot lattice*. With this, one identifies

$$\widetilde{W} \cong t(Q^\vee) \rtimes W.$$

The action above is given explicitly by $t_\lambda \bullet v = v - \delta\lambda$. Given the above identification, we could instead replace the coroot lattice $Q^\vee$ with a bigger lattice that admits a W-action. To that end, let $P \coloneqq \sum_i \mathbb{Z}c_i$ be the *weight lattice* and $P^\vee \coloneqq \sum_i \mathbb{Z}b_i$ be the *coweight lattice*, which are defined respectively by $\langle c_i, a_j^\vee\rangle = \delta_{ij}$ and $\langle b_i, a_j\rangle = \delta_{ij}$. The *extended affine Weyl group* is defined as

$$\widehat{W} \coloneqq t(P^\vee) \rtimes W.$$

There is a $\widehat{W}$-action $\widehat{w} \bullet f(x) = f(\widehat{w}^{-1} \bullet x)$ on the ring $\mathbb{C}(V)$ of meromorphic functions, which extends to an action on the space of affine-linear functions $\widehat{V}$. In particular, this can be viewed as an action on the affine root system $\widetilde{R}$ which permutes the affine roots.

The orthogonal reflections $s_i \coloneqq s_{a_i}$ corresponding to the affine simple roots generate the affine Weyl group $\widetilde{W}$. Hence, any $\widetilde{w} \in \widetilde{W}$ can be written as $\widetilde{w} = s_{i_1} \cdots s_{i_k}$. The *length* $\ell(\widetilde{w})$ of this element is the smallest k over all such decompositions. The subgroup of affine Weyl elements of length zero is denoted Ω. Geometrically, the length of an affine Weyl group element $\widetilde{w}$ is the number of walls separating the alcove $C \coloneqq \{x \in V : \langle x, a_i \rangle > 0\}$ from $\widetilde{w}^{-1}C$, see [Mac03, §2.2]. Hence, the group Ω can be thought of as the group of automorphisms on the alcove-separating walls, i.e. on the affine basis $\{a_0, \dots, a_n\}$.

2.1.3 Braid Groups

The *extended affine braid group* $\widehat{\mathfrak{B}}$ is the group generated by elements $T_{\widehat{w}}$ for $\widehat{w} \in \widehat{W}$ subject to the relations $T_{\widehat{v}} T_{\widehat{w}} = T_{\widehat{v}\widehat{w}}$ if $\ell(\widehat{v}) + \ell(\widehat{w}) = \ell(\widehat{v}\widehat{w})$. Let $T_i \coloneqq T_{s_i}$ and write any $\widehat{w} \in \widehat{W}$ in reduced form $\widehat{w} = s_{i_1} \cdots s_{i_\ell} \omega$ where $\omega \in \Omega$. By writing $T_{\widehat{w}} \coloneqq T_{i_1} \cdots T_{i_\ell} T_\omega$, we can think of $\widehat{\mathfrak{B}}$ as being generated by $T_0, \dots, T_n$ and T_ω for $\omega \in \Omega$ subject to the following relations:

(i) $T_i T_j \cdots = T_j T_i \cdots$. ($i \neq j$ with $\operatorname{ord}(s_i s_j)$ factors)

(ii) $T_{\omega_1} T_{\omega_2} = T_{\omega_1 \omega_2}$. ($\omega_1, \omega_2 \in \Omega$)

(iii) $T_\omega T_i = T_j T_\omega$. ($\omega s_i = s_j \omega$)

We then denote by $\widetilde{\mathfrak{B}}$ and $\mathfrak{B}$ the subgroups generated by $T_0, \dots, T_n$ and $T_1, \dots, T_n$, respectively; they are called the *affine braid group* and *finite braid group*. It turns out that the T_ω generate a subgroup of $\widehat{\mathfrak{B}}$ isomorphic to the subgroup Ω, which gives rise to the identification

$$\widehat{\mathfrak{B}} \cong \widetilde{\mathfrak{B}} \rtimes \Omega.$$

Remark 2.6 For a *dominant* coweight $\lambda \in P^\vee$, one where $\langle \lambda, a_i \rangle \geq 0$ for all simple roots a_i, one can define the element $Y^\lambda \coloneqq T_{t(\lambda)}$ associated to the translation by λ. Because any $\lambda \in P^\vee$ admits a decomposition as a difference $\lambda = \mu - \nu$ of dominant coweights, one may define $Y^\lambda \coloneqq Y^\mu (Y^\nu)^{-1}$ now for any $\lambda \in P^\vee$. According to [Mac03, (3.2.4)], one has $T_i^{\pm 1} Y^{s_i \lambda} T_i = Y^\lambda$ where the sign corresponds to $\langle \lambda, a_i \rangle = 1$ or $\langle \lambda, a_i \rangle = 0$, respectively. With this, [Mac03, (3.3.1)] identifies

$$\widehat{\mathfrak{B}} \cong \mathfrak{B} \rtimes \{Y^\lambda : \lambda \in P^\vee\}.$$

2.1.4 Double Affine Hecke Algebras

From [Mac03, §3.4], the *double affine braid group* $\mathscr{B}$ is the group generated by the extended affine braid group $\widehat{\mathfrak{B}}$ and a multiplicative group isomorphic to some lattice in the space of affine-linear functions $\widehat{V}$. Alternatively, the isomorphism from Remark 2.6 allows one to view the double affine braid group as two extended affine braid groups overlapping on a finite braid group:

$$\mathscr{B} \cong \{X^\mu : \mu \in P\} \otimes \mathfrak{B} \otimes \{Y^\lambda : \lambda \in P^\vee\}.$$

Definition 2.7 Let $\tau_0, \ldots, \tau_n \in \mathbb{C}^*$ with $\tau_i = \tau_j$ if s_i and s_j are conjugate in $\widehat{W}$. The affine Hecke algebra $\widetilde{\mathfrak{H}}$ (resp. double affine Hecke algebra $\mathscr{H}$) is the quotient of the group algebra $\mathbb{C}\widetilde{\mathfrak{B}}$ (resp. $\mathbb{C}\mathscr{B}$) of the extended affine braid group (resp. double affine braid group) by the Hecke relations

$$T_i - T_i^{-1} = \tau_i - \tau_i^{-1}.$$

For the non-reduced root system of type $C^\vee C_n$, special attention is needed. Here, we introduce $\tau_i^\vee \coloneqq 1$ for all i in the reduced case, and for all $i = 1, \ldots, n-1$ in type $C^\vee C_n$. In other words, we have introduced two new parameters $\tau_0^\vee$ and $\tau_n^\vee$ for this root system, alongside the three existing parameters τ_0, τ_n and $\tau_1 = \cdots = \tau_{n-1}$. The corresponding DAHA has two elements involving $a_0^\vee \coloneqq a_0/2$ and $a_n^\vee \coloneqq a_n/2$ satisfying Hecke relations with $\tau_0^\vee$ and $\tau_n^\vee$ (cf. [Mac03, (4.3.19)]):

$$T_0^\vee \coloneqq T_0^{-1} X^{-a_0^\vee} \qquad \text{and} \qquad T_n^\vee \coloneqq X^{-a_n^\vee} T_n^{-1}.$$

A final ingredient that is crucial to the proof of the main theorem is now discussed in some generality, before we write it explicitly for the DAHA of interest to us. First, let $\mathscr{D}_q$ be the algebra of q-difference operators on the vector space V, that is the algebra generated by the meromorphic functions $\mathbb{C}(V)$ and the translations $t(P^\vee)$ of the coweight lattice, with the action $t(\lambda) \bullet f(x) = f(x + c\lambda)$. If we extend the parameters by setting $\tau_\alpha \coloneqq \tau_{\widehat{w}(\alpha)}$, then we can adopt similar notation to [Mac03, §4.2] in defining the following the rational expressions for any $\alpha \in R$:

$$b_\alpha(\boldsymbol{X}) \coloneqq \frac{\tau_\alpha - \tau_\alpha^{-1} + (\tau_\alpha^\vee - (\tau_\alpha^\vee)^{-1}) X^{\alpha/2}}{1 - X^\alpha} \tag{2.1}$$

and

$$c_\alpha(\boldsymbol{X}) \coloneqq \frac{\tau_\alpha^{-1} - \tau_\alpha X^\alpha - (\tau_\alpha^\vee - (\tau_\alpha^\vee)^{-1}) X^{\alpha/2}}{1 - X^\alpha}. \tag{2.2}$$

There is significant cancellation by substituting $\tau_\alpha^\vee = 1$, so the situation for reduced irreducible root systems looks simpler. Nevertheless, we can use (2.1) to define a crucial representation of the DAHA that works even in type $C^\vee C_n$, and (2.2) to discuss a localisation thereafter. We will use the notations $b_i(\boldsymbol{X}) \coloneqq b_{a_i}(\boldsymbol{X})$ and $c_i(\boldsymbol{X}) \coloneqq c_{a_i}(\boldsymbol{X})$ when $\alpha = a_i$ is any (affine) simple root.

Proposition 2.8 ([Mac03, (4.7.4)], Basic Representation) *There is an injective homomorphism of* *algebras* $\beta : \mathscr{H} \to \mathscr{D}_q \rtimes \mathbb{C}W$ *given by* $\beta(X^\mu) = X^\mu$, $\beta(T_i) = \tau_i s_i + b_i(\boldsymbol{X})(1 - s_i)$ *and* $\beta(T_\pi) = \pi$.

This representation persists when we specialise to the so-called *classical level* $q = 1$, at which point the algebra of difference operators becomes commutative (a Laurent polynomial algebra). This is stated in [Obl04, Remark 3.1], and proven in the so-called untwisted case (cf. reduced root systems) in [Geh06, Theorem 2.1.10]. A proof, in the so-called twisted case, is discussed in [Sto11, Theorem 3.1]; see also [vDEZ18, §3] which includes the $C^\vee C_n$ case. In *op. cit.*, they view the operators $\tau_\alpha s_\alpha + b_\alpha(\boldsymbol{X})(1 - s_\alpha)$ as elements of the semi-direct product of a localisation of the group algebra $\mathbb{C}P$ of the weight lattice with $\mathbb{C}W$. The aforementioned localisation is by the (ideal generated by the) so-called *Weyl denominator*

$$\delta(\boldsymbol{X}) \coloneqq \prod_{\alpha \in R} (1 - X^\alpha). \tag{2.3}$$

They also remark that the injective Basic Representation can then be lifted to an isomorphism $\mathscr{H}_{\delta_\tau(\boldsymbol{X})} \cong \mathbb{C}P_{\delta_\tau(\boldsymbol{X})} \rtimes \mathbb{C}W$ upon localisation this time by the *τ-deformed Weyl denominator*

$$\delta_\tau(\boldsymbol{X}) \coloneqq \prod_{\alpha \in R} (1 - X^\alpha)(\tau_\alpha^{-1} - \tau_\alpha X^\alpha). \tag{2.4}$$

For the DAHA of type $C^\vee C_n$, the underlying root system is non-reduced and so the τ-deformed Weyl denominator involves the additional parameters $\tau_\alpha^\vee$ such that $\tau_\alpha^\vee = 1$ if $\alpha \in R$ but $\alpha/2 \notin R$, where R is the root system of type C_n; see [Mac03, p. 64] and cf. [Sto20, (9.2.1)]. Explicitly, this yields the following generalisation which reduces to (2.4) upon substituting $\tau_\alpha^\vee = 1$:

$$\delta_\tau(\boldsymbol{X}) = \prod_{\alpha \in R} (1 - X^\alpha)\Big(\tau_\alpha^{-1} - \tau_\alpha X^\alpha - (\tau_\alpha^\vee - (\tau_\alpha^\vee)^{-1}) X^{\alpha/2}\Big). \tag{2.5}$$

Remark 2.9 Some authors, like Macdonald, prefer a slightly different convention when it comes to indexing their roots: (2.1) and (2.2) are transformed by $\alpha \mapsto 2\alpha$. In this case, the additional parameters $\tau_\alpha^\vee = \tau_\alpha$ if $\alpha \in R$ but $2\alpha \notin R$. By substituting this into their version of $b_\alpha(\boldsymbol{X})$ and $c_\alpha(\boldsymbol{X})$, one obtains a common factor in both denominator and numerator that reduces the expressions down; their Weyl denominators (2.3)–(2.5) will vary slightly from ours.

2.2 Character Varieties

In this section, we recall some general facts about character varieties of punctured Riemann surfaces. The primary reference we use is the paper [HLRV11], where more details are provided.

Let $g, k \in \mathbb{N}$ be non-negative integers, and fix conjugacy classes $C_1, \ldots, C_k \subseteq \mathrm{GL}_m(\mathbb{C})$. Using the notation $(X,Y) \coloneqq XYX^{-1}Y^{-1}$ for the group commutator, we will now define the set

$$\mathfrak{R}_{g,k} \coloneqq \{X_1, Y_1, \ldots, X_g, Y_g \in \mathrm{GL}_m(\mathbb{C}), A_i \in C_i : (X_1, Y_1) \cdots (X_g, Y_g) A_1 \cdots A_k = \mathbb{1}_m\}.$$

Describing $\mathfrak{R}_{g,k}$ (in particular, determining whether or not it is non-empty) is a very non-trivial problem, and is difficult even in the genus $g = 0$ case. We will discuss a method for determining said solutions in §2.3, but for now we simply state the named problem.

Problem 2.10 (Multiplicative Deligne-Simpson Problem) *Find the irreducible (no common proper invariant subspace) solutions of this equation, for fixed conjugacy classes $C_1, \ldots, C_k \subseteq \mathrm{GL}_m(\mathbb{C})$:*

$$A_1 \cdots A_k = \mathbb{1}_m, \qquad \text{with } A_i \in C_i,$$

There is a natural $\mathrm{GL}_m(\mathbb{C})$-action on $\mathfrak{R}_{g,k}$ by conjugation. Because the corresponding quotient may be singular, we wish to consider only closed orbits. This is called the *GIT-quotient* (for geometric invariant theory), and is denoted by $/\!/$. We use this to make the following definition.

Definition 2.11 The $\mathrm{GL}_m(\mathbb{C})$-character variety is the GIT-quotient

$$\mathfrak{M}_{g,k} \coloneqq \mathfrak{R}_{g,k} /\!/ \mathrm{GL}_m(\mathbb{C}).$$

The affine variety $\mathfrak{M}_{g,k}$ can be viewed as the moduli space of $\mathrm{GL}_m(\mathbb{C})$-representations of the fundamental group of a genus g Riemann surface $\Sigma_{g,k} = \Sigma_g \setminus \{x_1, \ldots, x_k\}$ with k punctures where we also prescribe the conjugacy classes for the matrix representatives of the loops around each puncture.

There are no assumptions made on the conjugacy classes C_i in Definition 2.11, so the character variety $\mathfrak{M}_{g,k}$ is not smooth in general. It may even be that it is empty. Fortunately, there are some relatively mild assumptions one can make which will guarantee some useful properties of $\mathfrak{M}_{g,k}$, cf. [HLRV11, Definition 2.1.1].

Assume that the conjugacy classes C_i are semi-simple, that is they are represented by diagonal matrices (that encode the eigenvalues, possibly repeated, of A_i). Let us write λ_{ij} with $i = 1, \ldots, k$ and $j = 1, \ldots, d_i$ for the distinct eigenvalues of A_i, and μ_{ij} for their multiplicities. For all i,

$$\mu_{i1} + \cdots + \mu_{i\,d_i} = m.$$

Definition 2.12 Let λ_{ij} be the distinct eigenvalues of the semi-simple matrices $A_i \in \mathrm{GL}_m(\mathbb{C})$, with corresponding multiplicities μ_{ij}. The conjugacy classes C_i represented by A_i are generic if

$$\prod_{i=1}^{k}\prod_{j=1}^{d_i}\lambda_{ij}^{\mu_{ij}} = 1$$

and, for any $1 \le s < m$ and a collection of numbers $\nu_{ij} \le \mu_{ij}$ with $\nu_{i1} + \cdots + \nu_{i\,d_i} = s$ for each i,

$$\prod_{i=1}^{k}\prod_{j=1}^{d_i}\lambda_{ij}^{\nu_{ij}} \neq 1.$$

This genericity condition guarantees that the action of $\mathrm{PGL}_m(\mathbb{C}) \coloneqq \mathrm{GL}_m(\mathbb{C})/\mathbb{C}^*$ on $\mathfrak{R}_{g,k}$ is free.

Theorem 2.13 ([HLRV11, Theorem 2.1.5]) *Let C_i be semi-simple generic conjugacy classes. If non-empty, $\mathfrak{M}_{g,k}$ is a smooth equidimensional variety of dimension*

$$d = 2 + (2g + k - 2)m^2 - \sum_{i,j} \mu_{ij}^2. \tag{2.6}$$

One of the most technical parts in Oblomkov's proof of Theorem 1.2 was his need to establish the irreducibility of CM_τ. The first step was a straightforward argument showing smoothness. The second step is a very difficult technical proof that CM_τ is connected, which suffices to show irreducibility because CM_τ is also smooth. Note that this is a $\mathrm{GL}_n(\mathbb{C})$-character variety $\mathfrak{M}_{1,1}$ of a one-punctured torus. In our case, smoothness follows immediately from Theorem 2.13, provided our eigendata is generic, and irreducibility follows from a hard result by Hausel, Letellier and Rodriguez-Villegas. More on this – with an independent argument using some combinatorics based on a now-proven conjecture of theirs (see [Mel20]) – is discussed in Appendix A.

Theorem 2.14 ([HLRV13, Theorem 1.1.1]) *Let C_i be semi-simple generic conjugacy classes. If non-empty, $\mathfrak{M}_{g,k}$ is connected and, hence, irreducible.*

Recall that irreducibility follows immediately from smoothness and connectedness. Indeed, a point in the intersection of two irreducible components would necessarily be singular, and thus contradict smoothness. In this case, (2.6) is the formula for $\dim(\mathfrak{M}_{g,k})$.

2.3 Multiplicative Quiver Varieties

The pioneering paper [CBS06] introduces an algebra defined on a quiver, from which one can obtain a variety that encodes solutions of the multiplicative Deligne-Simpson problem (Problem 2.10). Some of the general exposition of this section can also be found in the lectures [CB99] by Crawley-Boevey.

2.3.1 Quivers

First and foremost, a *quiver* is simply a directed graph (so-named because it contains arrows). More precisely, it is a pair $Q = (Q_0, Q_1)$ consisting of vertices Q_0 and arrows Q_1 between pairs of vertices. For an arrow $a \in Q_1$, we say its *head* $h(a) \in Q_0$ is the vertex where the arrow terminates, and its *tail* $t(a) \in Q_0$ is the vertex where the arrow originates. In other words, such an arrow can be expressed pictorially as $t(a) \xrightarrow{a} h(a)$.

Definition 2.15 A quiver representation of a quiver Q is an assignment (V_v, f_a) of vector spaces V_v to each vertex v and linear maps $f_a : V_{t(a)} \to V_{h(a)}$ to each arrow $t(a) \xrightarrow{a} h(a)$. Fixing the dimension vector $\mathbf{n} \coloneqq (n_v) = (\dim(V_v))$, the space of quiver representations is precisely

$$\mathrm{Rep}(Q, \mathbf{n}) \cong \prod_{a \in Q_1} \mathrm{Mat}_{n_{h(a)} \times n_{t(a)}}(\mathbb{C}).$$

There is a natural action on the space $\mathrm{Rep}(Q, \mathbf{n})$ by simultaneous conjugation, that is by

$$\mathrm{GL}(\mathbf{n}) \coloneqq \prod_{v \in Q_0} \mathrm{GL}_{n_v}(\mathbb{C}). \tag{2.7}$$

A *path* is a concatenation of arrows that we read right-to-left. This agrees with [CBS06] so that in the quiver representation, the arrows can be replaced by linear maps/matrices without any additional modifications needed; their composition/multiplication makes sense from the get-go. Note that the indices on the vertex and arrow sets are there to indicate that the vertices Q_0 are paths of length zero and the arrows Q_1 are paths of length one.

The *path algebra* is the algebra $\mathbb{C}Q$ generated by the trivial paths e_v and arrows $a \in Q_1$, where multiplication is the concatenation of paths with the following expected relations: $e_v^2 = e_v$ for each $v \in Q_0$, $e_{h(a)}a = a = ae_{t(a)}$ for each $a \in Q_1$ and $e_v e_w = 0$ when $v \neq w$.

Remark 2.16 The category of quiver representations is equivalent to the category of (left) $\mathbb{C}Q$-modules, i.e. the category of representations of the path algebra; see [ARS95, Theorem III.1.5]. Briefly summarising, the quiver representation (V_v, f_a) admits an element of the path algebra $V \coloneqq \bigoplus_{v \in Q_0} V_v$. Conversely, $V \in \mathbb{C}Q$ induces the representation (V_v, f_a) where $V_v \coloneqq e_v V$ and f_a is just multiplication by a, which is contained in $V_{h(a)}$ by the relation $a = e_{h(a)}a$.

Given a quiver Q, one can define its *double* $\overline{Q} = (Q_0, \overline{Q}_1)$ which has the same vertex set but an enlarged arrow set: for every arrow $v \xrightarrow{a} w$, we attach to the picture the opposite arrow $v \xleftarrow{a^*} w$. It is often convenient to extend this slightly to an involution on the set $\overline{Q}_1$ by setting $(a^*)^* = a$.

From this, we obtain an indicator-like function $\varepsilon : \overline{Q}_1 \to \{\pm 1\}$ given by

$$\varepsilon(a) = \begin{cases} 1 & \text{if } a \in Q_1 \\ -1 & \text{if } a^* \in Q_1 \end{cases}.$$

2.3.2 Preprojective Algebras

One of the critical definitions of interest to us here is that of a *preprojective algebra*. These were studied by Gelfand and Ponomarev in [GP79], with deformed versions introduced in [CBH98]. These correspond to an *additive* Deligne-Simpson problem. In the setting of interest to us, we want a multiplicative analogue, namely the algebra introduced by Crawley-Boevey and Shaw.

Definition 2.17 (cf. [CBS06, Definition 1.2]) Suppose Q is a quiver and $\mathbf{q} = (q_v) \in (\mathbb{C}^*)^{|Q_0|}$. The multiplicative preprojective algebra is the algebra $\Lambda^{\mathbf{q}}(Q)$ arising as the localisation of the path algebra $\mathbb{C}\overline{Q}$ of the double by the elements $1 + aa^*$ for $a \in \overline{Q}_1$, modulo the following relation:

$$\prod_{a \in \overline{Q}_1} (1 + aa^*)^{\varepsilon(a)} = \sum_{v \in Q_0} q_v e_v.$$

Some care should be made to choose an ordering on the arrow set $\overline{Q}_1$, and the above product is then taken with respect to said order. However, [CBS06, Theorem 1.4] establishes that $\Lambda^{\mathbf{q}}$ doesn't depend on the choice of ordering (and orientation) up to isomorphism. We have therefore opted only to alert the reader to this detail and neglected explicit meaning of it ourselves. As for the representation theory, one identifies representations of the multiplicative preprojective algebra as a subspace $\mathrm{Rep}(\Lambda^{\mathbf{q}}, \mathbf{n}) \subseteq \mathrm{Rep}(\overline{Q}, \mathbf{n})$ that satisfies analogues of the relations in Definition 2.17:

$$\mathrm{id}_{V_{h(a)}} + f_a \circ f_{a^*} \text{ is invertible for all } a \in \overline{Q}_1$$

and

$$\prod_{\substack{a \in \overline{Q}_1 \\ h(a) = v}} (\mathrm{id}_{V_{h(a)}} + f_a \circ f_{a^*})^{\varepsilon(a)} = q_{h(a)} \, \mathrm{id}_{V_{h(a)}}.$$

2.3.3 Quiver Varieties

There is the natural group action by (2.7) on quiver representation spaces, which we can quotient out (and keep only the closed orbits). But the $\mathrm{GL}(\mathbf{n})$-conjugation action by non-zero scalar multiples of the identity is trivial, so we further quotient by $\{z\mathbb{1} : z \neq 0\} \cong \mathbb{C}^*$ without loosing anything. In other words, we quotient by $\mathrm{PGL}(\mathbf{n}) \coloneqq \mathrm{GL}(\mathbf{n})/\mathbb{C}^*$ which has dimension

$$d \coloneqq -1 + \sum_{v \in Q_0} n_v^2. \tag{2.8}$$

Definition 2.18 The multiplicative quiver variety associated to a quiver Q is the GIT-quotient

$$\mathcal{M}_{\mathbf{q},\mathbf{n}}(Q) = \mathrm{Rep}(\Lambda^{\mathbf{q}}, \mathbf{n}) /\!\!/ \mathrm{PGL}(\mathbf{n}).$$

In the same way that the multiplicative preprojective algebra is a multiplicative analogue of the usual (deformed) preprojective algebra $\Pi^{\boldsymbol{\lambda}}$ from Crawley-Boevey and Holland, the multiplicative quiver varieties are analogues of the varieties corresponding to $\Pi^{\boldsymbol{\lambda}}$, the so-called *Nakajima quiver varieties* introduced in [Nak94, §2].

We wish to say something about the dimension of $\mathcal{M}_{\mathbf{q},\mathbf{n}}(Q)$. First, let's recall the *Ringel form*

$$\langle \cdot, \cdot \rangle : \mathbb{N}^{|Q_0|} \times \mathbb{N}^{|Q_0|} \to \mathbb{N}, \qquad \langle \mathbf{n}, \mathbf{m} \rangle = \sum_{v \in Q_0} n_v m_v - \sum_{a \in Q_1} n_{t(a)} m_{h(a)}, \tag{2.9}$$

The Ringel form applied to two copies of the same tuple gives us the so-called the *Tits form*, i.e.

$$q(\mathbf{n}) \coloneqq \langle \mathbf{n}, \mathbf{n} \rangle. \tag{2.10}$$

A symmetric variant of the Ringel form (2.9) is known as the *Cartan form.* Explicitly, this is

$$(\mathbf{n}, \mathbf{m}) \coloneqq \langle \mathbf{n}, \mathbf{m} \rangle + \langle \mathbf{m}, \mathbf{n} \rangle = 2 \sum_{v \in Q_0} n_v m_v - \sum_{a \in Q_1} \Big(n_{t(a)} m_{h(a)} + m_{t(a)} n_{h(a)} \Big). \tag{2.11}$$

For convenience, we introduce the following given in terms of the Tits form:

$$p(\mathbf{n}) \coloneqq 1 - q(\mathbf{n}). \tag{2.12}$$

The last bit of notation we define, for any $\mathbf{q} = (q_v) \in (\mathbb{C}^*)^{|Q_0|}$ and $\mathbf{n} = (n_v) \in \mathbb{N}^{|Q_0|}$, is

$$\mathbf{q}^{\mathbf{n}} \coloneqq \prod_{v \in Q_0} q_v^{n_v}.$$

The next result will allow us to compute the dimension of a multiplicative quiver variety from the data of the multiplicative preprojective algebra. One can also use [CBS06, Theorems 1.8 and 1.10], and [CF17, Theorem 2.8], in the case that the representations of $\Lambda^{\mathbf{q}}$ are simple.

Proposition 2.19 ([CBS06, Theorem 1.11]) *For all* $\mathbf{q}$*, let any non-trivial decomposition of the dimension vector* $\mathbf{n} = \sum_i \mathbf{n}_i$ *into a sum of positive roots with* $\mathbf{q}^{\mathbf{n}_i} = 1$ *satisfy* $p(\mathbf{n}) > \sum_i p(\mathbf{n}_i)$*. Then, if non-empty,* $\mathrm{Rep}(\Lambda^{\mathbf{q}}, \mathbf{n})$ *is smooth and equidimensional of dimension* $d + 2p(\mathbf{n})$*, with* d *as in* (2.8) *and* p *as in* (2.12)*. The group* $\mathrm{PGL}(\mathbf{n})$ *acts freely on* $\mathrm{Rep}(\Lambda^{\mathbf{q}}, \mathbf{n})$*, which means the dimension of the multiplicative quiver variety* $\mathcal{M}_{\mathbf{q},\mathbf{n}}(Q)$ *is* $d + 2p(\mathbf{n}) - d = 2p(\mathbf{n})$*.*

The assumption that the decomposition of $\mathbf{n}$ into positive roots has $\mathbf{q}^{\mathbf{n}_i} = 1$ will for us be encoded by the genericity condition (Definition 2.12) placed on the eigenvalues of certain matrices that we can extract from a representation of $\Lambda^{\mathbf{q}}$. Namely, there is no such decomposition and thus Proposition 2.19 holds immediately in the cases that we wish to apply it.

Recall that a short exact sequence of modules $0 \to A \to M \to B \to 0$ is *split* if $M \cong A \oplus B$. There is a group $\mathrm{Ext}^1(A, B)$ that measures how many ways there are to form a short exact sequence of the form $0 \to A \to M \to B \to 0$, whose zero element corresponds to the trivial extension $M = A \oplus B$. The final result of this section is an explicit formula for the dimension of this so-called *first Ext group*; this particular lemma is crucial for an argument in §5.1.1.

Lemma 2.20 (cf. [CBS06, Theorem 1.6]) *For finite-dimensional* $\Lambda^{\mathbf{q}}$*-modules* M_i *and* M_j*,*

$$\dim\Big(\mathrm{Ext}^1_{\Lambda^{\mathbf{q}}}(M_i, M_j)\Big) = \dim\big(\mathrm{Hom}_{\Lambda^{\mathbf{q}}}(M_i, M_j)\big) + \dim\big(\mathrm{Hom}_{\Lambda^{\mathbf{q}}}(M_j, M_i)\big) - (\mathbf{n}_i, \mathbf{n}_j),$$

with $(\cdot,\cdot)$ *as in* (2.11)*, and* $\mathbf{n}_i$ *and* $\mathbf{n}_j$ *the associated respective dimension vectors.*

2.3.4 Obtaining Deligne-Simpson Solutions

Definition 2.21 A star-shaped quiver is a tree with one vertex of degree k, called the *central vertex*, and the rest of degrees one and two, forming the k-legs of the star. We denote by d_i the number of vertices on the i^{th} leg (including the central vertex).

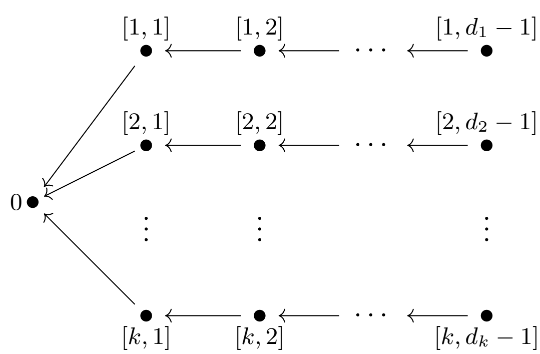


Figure 2.1: The star-shaped quiver.

Suppose $(A_1, \ldots, A_k)$ is a solution of the multiplicative Deligne-Simpson problem (Problem 2.10) with semi-simple conjugacy classes, and denote by λ_{ij} the eigenvalues of A_i. Associated to this is the multiplicative preprojective algebra of a star-shaped quiver. Here, $\mathbf{q} = (q_v)$ is given by

$$q_0 = \prod_{i=1}^{k} \frac{1}{\lambda_{i1}}, \qquad q_{[i,j]} = \frac{\lambda_{ij}}{\lambda_{i\,j+1}}. \tag{2.13}$$

The corresponding representation of $\Lambda^{\mathbf{q}}$ is that with vector spaces

$$V_0 = \mathbb{C}^n, \qquad V_{[i,j]} = \operatorname{im}\big((A_i - \lambda_{i1}) \cdots (A_i - \lambda_{ij})\big),$$

meaning the dimension vector is $\mathbf{n} = (n_0, n_{[i,j]})$ where

$$n_0 = n, \qquad n_{[i,j]} = \operatorname{rank}\big((A_i - \lambda_{i1}) \cdots (A_i - \lambda_{ij})\big). \tag{2.14}$$

Let a_{ij} denote the arrow with tail at the vertex $[i,j]$. The (surjective) linear maps $X_{a_{ij}^*}$ are products that successively kill eigenspaces, and the (injective) linear maps $X_{a_{ij}}$ are inclusions. Conversely, one can extract a Deligne-Simpson solution from such a representation of $\Lambda^{\mathbf{q}}$, namely

$$A_i = \lambda_{i1}(1 + X_{a_{i1}} X_{a_{i1}^*}).$$

A similar construction works more generally for arbitrary conjugacy classes, see [CB04, CBS06].

Proposition 2.22 ([CBS06, Lemma 8.3]) *For a star-shaped quiver Q, there is a simple representation of $\Lambda^{\mathbf{q}}$ if and only if there is an irreducible solution of the associated multiplicative Deligne-Simpson problem.*

For semi-simple generic conjugacy classes, solutions to the associated Deligne-Simpson problem are irreducible. The proof of [CBS06, Lemma 8.3] in that situation implies the following result.

Proposition 2.23 *For generic eigenvalues λ_{ij}, the character variety $\mathfrak{M}_{0,k}$ and the multiplicative quiver variety $\mathcal{M}_{\mathbf{q},\mathbf{n}}(Q)$ are isomorphic. In particular, both are smooth irreducible affine varieties.*

2.4 Affine Algebraic Varieties

We will next recall some of the standard language commonly used in algebraic geometry. This will become important when we prove the main result of the thesis. We offer a rather limited-but-sufficient scope of the algebraic geometry required, cherry-picking from the much better developed work [Har77]. Throughout, we work over the field of complex numbers $K = \mathbb{C}$.

2.4.1 Varieties

The *affine n-space* $\mathbb{A}^n$ is the set of all n-tuples of complex numbers. The *vanishing set* of a subset of polynomials $S \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is the set of points in the affine space where all polynomials in S are simultaneously zero when evaluated at these points, that is

$$V(S) = \{(a_1, \ldots, a_n) \in \mathbb{A}^n : f(a_1, \ldots, a_n) = 0 \text{ for all } f \in S\}.$$

Definition 2.24 An affine algebraic variety is any subset $X \subseteq \mathbb{A}^n$ of the affine n-space that can be expressed as a vanishing set $X = V(S)$ of some subset $S \subseteq \mathbb{C}[x_1, \ldots, x_n]$.

The *coordinate ring* $\mathbb{C}[X]$ of a variety X is the quotient of the polynomial ring in n variables by $I(X) = \{f \in \mathbb{C}[x_1, \ldots, x_n] : f(a_1, \ldots, a_n) = 0 \text{ for all } (a_1, \ldots, a_n) \in X\}$, the so-called *vanishing ideal* of the variety. In other words, the coordinate ring is the restriction of polynomials to X.

Remark 2.25 The coordinate ring is an attribute of (or data defining) a variety. If two varieties are isomorphic, this naturally induces an isomorphism on coordinate rings. This is what we used to re-phrase Theorems 1.1, 1.2 and 1.3. Indeed, the coordinate ring of the variety $\mathrm{Spec}(R)$ is R itself, and the coordinate rings of the corresponding Calogero-Moser spaces are precisely the rings of regular functions on said spaces. Hence, $\mathrm{Spec}(R) \cong M$ implies $R \cong \mathbb{C}[M]$.

2.4.2 Maps Between Varieties

Suppose $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ are affine algebraic varieties. We say that a map $f : X \to Y$ is *regular* if it can be given locally by polynomials, that is f is the restriction to X of some polynomial $F : \mathbb{A}^n \to \mathbb{A}^m$. In this way, we can view the coordinate ring of a variety as a regular map from X to the trivial variety $Y = \mathbb{C}$. It is for this reason that the elements of a coordinate ring $\mathbb{C}[x_1, \ldots, x_n]/I(X)$ are called the *regular functions* on the variety.

Definition 2.26 We can add to the affine n-space a topology called the Zariski topology, where the closed subsets are precisely the vanishing sets $V(S)$ for each $S \subseteq \mathbb{C}[x_1, \ldots, x_n]$. Being more topologically conventional, the open subsets are complements of these vanishing sets, that is the open subsets are complements of the affine algebraic varieties.

Remark 2.27 A generalisation of a variety is a *scheme*, which is a certain type of space isomorphic to the spectrum $\mathrm{Spec}(R)$ of some commutative ring R. Indeed, the affine variety $X = V(S)$ with coordinate ring $\mathbb{C}[x_1, \ldots, x_n]/I(X)$ has associated to it the affine scheme $\mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n]/I(X))$. As a set, recall that $\mathrm{Spec}(R)$ is the set of prime ideals of the ring R. There is a parallel to Definition 2.26 for $\mathrm{Spec}(R)$, whereby we define $V(S) = \{\mathfrak{p} \subseteq R \text{ prime} : \mathfrak{p} \supseteq S\}$ as the set of prime ideals containing the ideal S; these are the closed subsets in the Zariski topology.

A *rational map* between varieties $f : X \dashrightarrow Y$ is a regular map from an open (in the Zariski topology) dense subset $X \supseteq U \to Y$. In other words, a rational map is an equivalence class of pairs (ϕ, U) with $\phi : U \to Y$, where we declare $(\phi, U) \sim (\psi, V)$ if and only if $\phi \equiv \psi$ on the subset $U \cap V$. This generalises the notion of a rational function g/h where $g, h \in \mathbb{C}[x_1, \ldots, x_n]$. Indeed, viewing a rational function as living in the coordinate ring of some variety X, it gives rise to a rational map $f : X \dashrightarrow \mathbb{CP}^1$ which is given by $f(x) = [g(x), h(x)]$ in projective coordinates.

2.5 Poisson Structures and Dynamics

The final preliminary content is really a part of mathematical physics. One of the most complete places to study this background is [Arn89]; we briefly recall some of the important definitions and results to use later when interpreting our main result from a dynamical point-of-view.

2.5.1 Integrability

Let $\mathbb{C}(\mathbf{x})$ be the field of meromorphic functions in the variables $x_1, \ldots, x_n$. A *dynamical system* is a collection of differential equations $\dot{x}_i = F_i(x_1, \ldots, x_n)$, where the dot notation denotes a time derivative. Then, a *vector field* on $\mathbb{C}(\mathbf{x})$ is a linear homogeneous differential operator

$$X_F = \sum_{i=1}^{n} F_i \frac{\partial}{\partial x_i}.$$

These can be used to compute the time evolution of a meromorphic function $f(\mathbf{x})$, namely

$$\dot{f} \coloneqq X_F f.$$

Definition 2.28 A Hamiltonian system in $\mathbb{C}^{2n}$ is a dynamical system of the form

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}. \tag{2.15}$$

The function $H = H(\mathbf{p}, \mathbf{q})$ in Definition 2.28 is called the *Hamiltonian* of the system. For any analytic $f(\mathbf{p}, \mathbf{q})$ on $\mathbb{C}^{2n}$, the dynamics are dictated by the Hamiltonian; its time evolution is

$$\dot{f} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \dot{p}_i + \frac{\partial f}{\partial q_i} \dot{q}_i \right) = \sum_{i=1}^{n} \left(-\frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} \right),$$

Recall that a *Poisson bracket* on a smooth (complex) manifold M is a Lie bracket $\{\cdot, \cdot\} : C^\omega(M) \times C^\omega(M) \to C^\omega(M)$ that satisfies the Leibniz rule, where C^ω denotes the space of complex analytic functions (analogous to the space of smooth functions C^∞ in the case that M is a real manifold).

A smooth manifold equipped with this bracket is called a *Poisson manifold*.

On the phase space $\mathbb{C}^{2n}$, one can view it as a Poisson manifold by introducing the bracket

$$\{f, g\} \coloneqq \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right).$$

Remark 2.29 A Poisson bracket is, in general, degenerate in the sense that there exist functions $f \in C^\omega$ such that, for all $g \in C^\omega$, $\{f, g\} = 0$. These f are called *Casimir functions* (or *Casimirs*).

The space of functions on the phase space $\mathbb{C}^{2n}$ forms an infinite-dimensional algebra called a *Poisson algebra*, whose Poisson centre consists of Casimirs. With respect to a given Hamiltonian, one can define the *Poisson algebra of integrals*

$$\mathcal{F}_H \coloneqq \{f \in C^\omega : \{f, H\} = 0\}.$$

Given that this is an algebra, it is closed under taking functions of its elements. This allows one to fully describe $\mathcal{F}_H$ by only considering so-called *functionally-independent* integrals, which are collections of elements $f_1, \dots, f_k \in \mathcal{F}_H$ such that the Jacobian matrix $(\partial f_i/\partial x_j)$ has rank k.

Theorem 2.30 (Liouville-Arnold Theorem) *If a Hamiltonian system in $\mathbb{C}^{2n}$ has n functionally-independent integrals $f_1, \dots, f_n$ in involution, that is $\{f_i, f_j\} = 0$ for all $i, j = 1, \dots, n$, then it can be solved by quadratures.*

This theorem says that the Hamiltonian equations (2.15) can be explicitly integrated under the hypotheses of Theorem 2.30. One obtains from this the popular definition of integrability for (finite-dimensional) Hamiltonian systems known as *Liouville integrability*.

2.5.2 Deformations of Algebras

Following [Eti09, §3.1], one can obtain a Poisson bracket from a so-called *formal deformation* of a commutative algebra A_0. Namely, consider the ring of formal power series $K = \mathbb{C}[[\hbar_1, \dots, \hbar_n]]$ in n variables, which has maximal ideal $\mathfrak{m} = (\hbar_1, \dots, \hbar_n)$. A formal n-parameter deformation of A_0 is a K-algebra A which is topologically free as a K-module (isomorphic to $V[[\hbar_1, \dots, \hbar_n]]$ for some $\mathbb{C}$-vector space V) alongside an isomorphism

$$\eta_0 : A/\mathfrak{m}A \xrightarrow{\sim} A_0. \tag{2.16}$$

Remark 2.31 One can think of a one-parameter deformation by $\hbar_1 = \hbar$ as the algebra $A_0[[\hbar]]$ whose multiplication $*$ is determined by products in A_0 via following formal power series, where

$\mu_i : A_0 \otimes A_0 \to A_0$ are linear maps, the first of which $\mu_0(a, b) = ab$ is multiplication in A_0:

$$a * b := \sum_{k=0}^{\infty} \mu_k(a, b)\hbar^k.$$

For a one-parameter deformation, the commutator on the corresponding deformed algebra $A_\hbar$ determines the Poisson bracket on the *classical* (non-deformed) algebra A_0. Indeed, for any $a, b \in A_\hbar$, we expand the commutator as $[a, b] = i\hbar\{\eta_0(a), \eta_0(b)\} + \cdots$, and call the elements $\eta_0(a), \eta_0(b) \in A_0$ the *classical limits* of a and b, respectively. It is common to introduce $q = e^{-i\hbar}$ which controls the deformation. In the classical limit $\hbar \to 0$, this corresponds to $q \to 1$. Hence,

$$[a, b] = (q - 1)\{\eta_0(a), \eta_0(b)\} + \cdots . \tag{2.17}$$

Chapter 3

Double Affine Hecke Algebras

The double affine Hecke algebra (DAHA) of a reduced root system first appeared in the paper [Che92] by Ivan Cherednik. A subsequent extension to the non-reduced root system of type $C^\vee C_n$ came about by Sahi in his pioneering paper [Sah99], wherein he constructed this new DAHA to analyse some Macdonald and Koornwinder polynomials. More works [NS00, Sto00] further study the latter. This chapter recapitulates some of the set-up from the aforementioned papers, we follow [EGO06] for the interpretation the DAHA of type $C^\vee C_n$ as a generalised double affine Hecke algebra (GDAHA), and we adapt a construction in [Cha19] to represent some elements of this DAHA by matrices. We conclude by analysing the eigendata of each matrix.

Notation 3.1 In §2.1.4, we introduced notation $\tau_0^\vee, \tau_0, \ldots, \tau_n, \tau_n^\vee$ for the parameters of the DAHA of type $C^\vee C_n$, where $\tau_1 = \cdots = \tau_{n-1}$. Now on, we use the following notation for the parameters:

$$\tau_0 = k_0, \qquad \tau_n = k_n, \qquad \tau_1 = \cdots = \tau_{n-1} = t, \qquad \tau_0^\vee = u_0, \qquad \tau_n^\vee = u_n.$$

3.1 The DAHA and its Spherical Subalgebra

As the name suggests, the DAHA contains two copies of an affine Hecke algebra; of interest to us is the one associated with the root system of type C_n (rather, the affine root system $\widetilde{C}_n$). Namely, the DAHA of [Sah99] contains the affine Hecke algebras $\widetilde{H}_n$ associated with each of C_n and its dual $C_n^\vee$. Let's give an explicit definition of this DAHA that we work with henceforth. This is a particular case of Definition 2.7, made explicit by having now chosen W of type C_n.

Definition 3.2 Let $q, k_0, k_n, t, u_0, u_n \in \mathbb{C}^*$ and write $\boldsymbol{\tau}$ for the parameters (k_0, k_n, t, u_0, u_n). The double affine Hecke algebra of type $C^\vee C_n$ is the associative algebra $\mathcal{H}_{q,\boldsymbol{\tau}}$ over $\mathbb{C}[\boldsymbol{\tau}^{\pm 1}, q^{\pm 1}]$ generated by invertible elements $T_0, \ldots, T_n$ and $X_1, \ldots, X_n$ subject to the following relations:

(i) $T_0 - T_0^{-1} = k_0 - k_0^{-1}$.

(ii) $T_i - T_i^{-1} = t - t^{-1}$. $(i = 1, \ldots, n-1)$

(iii) $T_n - T_n^{-1} = k_n - k_n^{-1}$.

(iv) $[T_i, T_j] = 0$. $(|i - j| > 1)$

(v) $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$.

(vi) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$. $(i = 1, \ldots, n-2)$

(vii) $T_{n-1} T_n T_{n-1} T_n = T_n T_{n-1} T_n T_{n-1}$.

(viii) $[X_i, X_j] = 0$. $(1 \leq i < j \leq n)$

(ix) $[T_i, X_j] = 0$. $(j \neq i, i+1)$

(x) $T_i X_i = X_{i+1} T_i^{-1}$. $(i = 1, \ldots, n-1)$

(xi) $T_0^\vee - (T_0^\vee)^{-1} = u_0 - u_0^{-1}$. $(T_0^\vee \coloneqq q^{-1} T_0^{-1} X_1)$

(xii) $T_n^\vee - (T_n^\vee)^{-1} = u_n - u_n^{-1}$. $(T_n^\vee \coloneqq X_n^{-1} T_n^{-1})$

The DAHA of type $C^\vee C_n$ contains an overt copy of the affine Hecke algebra $\widetilde{H}_n$ of type $\widetilde{C}_n$ as a subalgebra, generated by $T_0, \ldots, T_n$ with parameters k_0, k_n and t. Under the specialisation $k_0 = k_n = t = 1$, this is isomorphic to the group algebra $\mathbb{C}\widetilde{W}$ of the affine Weyl group of type $\widetilde{C}_n$. A similar story persists for the finite Hecke algebra H_n of type C_n, generated by $T_1, \ldots, T_n$ with parameters k_n and t. This is isomorphic to the group algebra $\mathbb{C}W$ under $k_n = t = 1$.

Notation 3.3 As a way to alleviate notation here henceforth, we shall introduce the following:

$$S \coloneqq T_1 \cdots T_{n-1} \qquad \text{and} \qquad S^\dagger \coloneqq T_{n-1} \cdots T_1.$$

Lemma 3.4 *In $\mathcal{H}_{q,\boldsymbol{\tau}}$, we have the relation $qT_0 T_0^\vee S T_n^\vee T_n S^\dagger = 1$.*

Proof: Using $T_0^\vee = q^{-1} T_0^{-1} X_1$ and $T_n^\vee = X_n^{-1} T_n^{-1}$, the relation here becomes $X_1 S X_n^{-1} S^\dagger = 1$, that is $S^\dagger X_1 = X_n S^{-1}$. But this follows by inductively applying relation (x) in Definition 3.2. □

From §2.1.2, recall $w \in W$ admits a reduced decomposition $w = s_{i_1} \cdots s_{i_\ell}$ into simple reflections (that is, where ℓ is minimal). To each simple reflection, one can associate a corresponding Hecke generator $T_i \coloneqq T_{s_i}$, which has corresponding parameter $\tau_i = \tau_{s_i}$. This extends to every $w \in W$ in the sense that $T_w \coloneqq T_{i_1} \cdots T_{i_\ell}$; the corresponding parameter is $\tau_w \coloneqq \tau_{i_1} \cdots \tau_{i_\ell}$. Therefore, the mapping $T_i \mapsto t$ $(i = 1, \ldots, n-1)$ and $T_n \mapsto k_n$ extends to a one-dimensional representation

$$\chi : H_n \to \mathbb{C}, \qquad T_w \mapsto \tau_w.$$

Definition 3.5 (cf. [Mac03, (5.5.7)]) The Hecke symmetriser is the idempotent associated to χ:

$$\mathbf{e} := \frac{1}{\sum_{w\in W} \tau_w^2} \sum_{w\in W} \tau_w T_w.$$

Now, in analogue with the rational Cherednik algebra [EG02] and the DAHA of type GL_n [Obl04], we wish to discuss a certain subalgebra of our DAHA. We will provide a definition and state some structural results in the style of [Obl04, §5] which are proved in an identical way.

Definition 3.6 The spherical subalgebra of the DAHA $\mathcal{H}_{q,\boldsymbol{\tau}}$ is the algebra $\mathbf{e}\mathcal{H}_{q,\boldsymbol{\tau}}\mathbf{e}$.

One of the most important features is that the spherical subalgebra at the classical level $q = 1$ is precisely the centre of the DAHA (see Theorem 5.15), and it admits a useful known quantisation which isn't immediately clear by looking purely at said centre (see §5.5).

3.2 The GDAHA

The GDAHA of [EGO06] is defined using a star-shaped quiver (see Definition 2.21). Recall this is nothing more than a tree with k legs protruding from the central vertex, and that the number of vertices on the i^{th} leg (including the central vertex) is denoted d_i.

Definition 3.7 ([EGO06, Definition 3.2.1]) Let $u_{ij}, t \in \mathbb{C}^*$ for $i = 1, \dots, k$ and $j = 1, \dots, d_i$ and write $\mathbf{u}$ for the parameters (u_{ij}). The generalised double affine Hecke algebra of rank n associated to the star-shaped quiver Q is the associative algebra $\mathscr{H}_n(Q)$ over $\mathbb{C}[\mathbf{u}^{\pm 1}, t^{\pm 1}]$ generated by the invertible elements $U_1, \dots, U_k$ and $T_1, \dots, T_{n-1}$ subject to the following relations:

(i) $T_i - T_i^{-1} = t - t^{-1}$. $\quad (i = 1, \dots, n-1)$

(ii) $[T_i, T_j] = 0$. $\quad (|i-j| > 1)$

(iii) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$. $\quad (i = 1, \dots, n-2)$

(iv) $[U_i, T_j] = 0$. $\quad (i = 1, \dots, k$ and $j = 2, \dots, n-1)$

(v) $[U_i, T_1 U_i T_1] = 0$. $\quad (i = 1, \dots, k)$

(vi) $[U_i, T_1^{-1} U_j T_1] = 0$. $\quad (1 \le i < j \le k)$

(vii) $\prod_{j=1}^{d_i} (U_i - u_{ij}) = 0$. $\quad (i = 1, \dots, k)$

(viii) $(U_1 \cdots U_m) S S^{\dagger} = 1$.

As promised in the introductory paragraph of this chapter, there is an interpretation of the usual DAHA $\mathcal{H}_{q,\boldsymbol{\tau}}$ as a GDAHA. Specifically, we should consider the GDAHA associated to $Q = \widetilde{D}_4$

in order to reconcile Definitions 3.2 and 3.7. Note that $k = 4$ and each $d_i = 2$ for this GDAHA.

Lemma 3.8 ([EGO06, Proposition 3.3.2]) *There is an isomorphism $\varphi : \mathscr{H}_n(\widetilde{D}_4) \to \mathcal{H}_{q,\boldsymbol{\tau}}$ whereby*

$$\varphi(T_i) = T_i, \qquad \varphi(U_1) = qT_0, \qquad \varphi(U_2) = T_0^\vee, \qquad \varphi(U_3) = ST_n^\vee S^{-1}, \qquad \varphi(U_4) = ST_nS^{-1},$$

provided also that the GDAHA parameters are mapped to the DAHA parameters via $t \mapsto t$ and

$$\begin{aligned} &u_{11} \mapsto qk_0, \qquad u_{12} \mapsto -qk_0^{-1}, \qquad u_{21} \mapsto u_0, \qquad u_{22} \mapsto -u_0^{-1}, \\ &u_{31} \mapsto u_n, \qquad u_{32} \mapsto -u_n^{-1}, \qquad u_{41} \mapsto k_n, \qquad u_{42} \mapsto -k_n^{-1}. \end{aligned}$$

Note relation (viii) in Definition 3.7 becomes the Lemma 3.4 relation under the isomorphism φ.

3.3 Duality

Notation 3.9 The collection of n variables $X_1, \ldots, X_n$ is denoted by the shorthand $\boldsymbol{X}$. For an element $\boldsymbol{\lambda} \in \mathbb{Z}^n$, we extend this shorthand to $\boldsymbol{X}^{\boldsymbol{\lambda}}$, which denotes the product $X_1^{\lambda_1} \cdots X_n^{\lambda_n}$.

Apart from the obvious commutative subalgebra $\mathbb{C}[\boldsymbol{X}^{\pm 1}]$ of Laurent polynomials in $X_1, \ldots, X_n$ in $\mathcal{H}_{q,\boldsymbol{\tau}}$, there is another commutative Laurent polynomial subalgebra $\mathbb{C}[\boldsymbol{Y}^{\pm 1}]$ in $Y_1, \ldots, Y_n$, where

$$Y_i \coloneqq T_i \cdots T_{n-1} T_n T_{n-1} \cdots T_1 T_0 T_1^{-1} \cdots T_{i-1}^{-1}. \tag{3.1}$$

Proposition 3.10 ([Che95b, Theorem 1.2], PBW Property) *For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{Z}^n$, the elements $\boldsymbol{X}^{\boldsymbol{\lambda}} T_w \boldsymbol{Y}^{\boldsymbol{\mu}}$ form a basis of $\mathcal{H}_{q,\boldsymbol{\tau}}$, that is every element $h \in \mathcal{H}_{q,\boldsymbol{\tau}}$ admits a unique presentation of the form*

$$h = \sum_{\substack{\boldsymbol{\lambda},\boldsymbol{\mu} \in \mathbb{Z}^n \\ w \in W}} h_{\boldsymbol{\lambda},w,\boldsymbol{\mu}} \boldsymbol{X}^{\boldsymbol{\lambda}} T_w \boldsymbol{Y}^{\boldsymbol{\mu}}, \qquad h_{\boldsymbol{\lambda},w,\boldsymbol{\mu}} \in \mathbb{C}.$$

Sketch of Proof: From [Lus89, Proposition 3.7], we know $\widetilde{H}_n \cong H_n \otimes \mathbb{C}[\boldsymbol{Y}^{\pm 1}]$ as vector spaces. On the other hand, [Mac03, (4.7.4)(ii)] says $\mathcal{H}_{q,\boldsymbol{\tau}} \cong \mathbb{C}[\boldsymbol{X}^{\pm 1}] \otimes \widetilde{H}_n$ as vector spaces. Combining,

$$\mathcal{H}_{q,\boldsymbol{\tau}} \cong \mathbb{C}[\boldsymbol{X}^{\pm 1}] \otimes H_n \otimes \mathbb{C}[\boldsymbol{Y}^{\pm 1}]. \qquad \square$$

The PBW Property indicates that the X_i and Y_i elements of the DAHA should be viewed on equal footing (which is not clear a priori). This is manifested by the so-called *duality*, an isomorphism between two DAHAs with different parameters. This is discussed by Sahi when he introduces the DAHA [Sah99], and Stokman who develops this to obtain not one but two dual DAHAs [Sto00]. However, one of these is an *anti*-isomorphism which, although interesting, would interchange left and right modules; this is not desirable.

The duality that Sahi studies doesn't quite map the DAHA to itself, rather the parameter are inverted, and some are interchanged. The flavour of this map from the perspective of the GDAHA (in light of Lemma 3.8) is that it inverts everything and interchanges U_1 and U_3. Let

$$\begin{aligned} &\mathcal{H}_{q,\boldsymbol{\tau}} && \text{be the DAHA with parameters} && \boldsymbol{\tau} = (k_0, k_n, t, u_0, u_n), \\ &\mathcal{H}_{q^{-1},\boldsymbol{\sigma}} && \text{be the DAHA with parameters} && \boldsymbol{\sigma} = (u_n^{-1}, k_n^{-1}, t^{-1}, u_0^{-1}, k_0^{-1}). \end{aligned}$$

Theorem 3.11 ([Sah99, Theorem 4.2], Duality Isomorphism) *There is a unique isomorphism*

$$\varepsilon : \mathcal{H}_{q,\boldsymbol{\tau}} \to \mathcal{H}_{q^{-1},\boldsymbol{\sigma}} \qquad \text{given by} \qquad \begin{aligned} \varepsilon(T_0) &= S(T_n^\vee)^{-1}S^{-1} \\ \varepsilon(T_i) &= T_i^{-1} \\ \varepsilon(X_i) &= Y_i \end{aligned}$$

for $i = 1, \dots, n$*, where the* j^{th} *parameter in* $\boldsymbol{\tau}$ *maps to the* j^{th} *parameter in* $\boldsymbol{\sigma}$ *for each* $j = 1, \dots, 5$.

Comparing this to [Obl04, §3.5], it is clear that the Duality Isomorphism is the analogue of the Fourier-Cherednik transform that Oblomkov uses when working with the DAHA of type GL_n. The Duality Isomorphism ε will become a key ingredient when we start to consider things from the character variety point-of-view. In particular, once we have constructed some local coordinates, duality allows us to immediately obtain a second coordinate chart (see §5.3).

Remark 3.12 The proof of Proposition 3.10 sheds some light on the "double" in DAHA. Namely, Lusztig's presentation of the affine Hecke algebra $\widetilde{H}_n$ has a dual presentation $\mathbb{C}[\boldsymbol{X}^{\pm 1}] \otimes H_n$ via ε. Therefore, the DAHA of type $C^\vee C_n$ can be thought of as two copies of the affine Hecke algebra of type $\widetilde{C}_n$ that overlap on the finite Hecke algebra of type C_n.

3.4 The Basic Representation

A crucial idea needed to prove Theorem 1.3 is that of the Basic Representation [Sah99, Theorem 3.1], which extends the representation of the affine Hecke algebra $\widetilde{H}_n$ of type $\widetilde{C}_n$ found by Noumi [Nou95] to a representation of the DAHA $\mathcal{H}_{q,\boldsymbol{\tau}}$. This is done by representing the generators of the DAHA by so-called *q-difference-reflection operators.*

First, we consider the ring of *q-difference operators* generated by $X_i^{\pm 1}$ and $P_i^{\pm 1}$ for $i = 1, \dots, n$ subject to $[X_i, X_j] = [P_i, P_j] = 0$ for all i and j, as well as the deformed commutation relation

$$P_i X_j = q^{2\delta_{ij}} X_j P_i. \tag{3.2}$$

Remark 3.13 The DAHA in [Sah99] is defined with square root parameters $\boldsymbol{\tau}^{1/2}$ and $q^{1/2}$. Using either his convention or ours will be, for the most part, simply personal preference. However, his P_i acts by multiplicative q-shift, which corresponds to multiplicative q^2-shift when working with the squared parameters as we do. The change is subtle, but it means the Poisson bracket on the spherical subalgebra $\mathbf{e}\mathcal{H}_{1,\boldsymbol{\tau}}\mathbf{e}$, viewed as a one-parameter deformation of $\mathbf{e}\mathcal{H}_{q,\boldsymbol{\tau}}\mathbf{e}$ in the sense of §2.5.2, is obtained exactly as in (2.17) but instead with coefficient (q^2-1), see (5.16).

The ring of q-difference operators is denoted

$$D_q \coloneqq \mathbb{C}_q[\boldsymbol{X}^{\pm 1}, \boldsymbol{P}^{\pm 1}], \tag{3.3}$$

which becomes the ring of Laurent polynomials in two variables at the classical level $q = 1$. By localising D_q on the (Ore) set $\mathbb{C}[\boldsymbol{X}^{\pm 1}] \setminus \{0\}$ of non-zero Laurent polynomials, we obtain the ring

$$\mathscr{D}_q \coloneqq \mathbb{C}_q(\boldsymbol{X})[\boldsymbol{P}^{\pm 1}]. \tag{3.4}$$

Now, the Weyl group of type C_n consisting of *sign-changing* permutations on n letters

$$W = \mathbb{Z}_2^n \rtimes S_n \tag{3.5}$$

is generated by transpositions $\mathfrak{s}_{ij}$ and sign-reversals $\mathfrak{s}_i$. In general, Weyl group elements $w \in W$ act on functions by $w \bullet f(\boldsymbol{X}) = f(w^{-1} \bullet \boldsymbol{X})$. Here, (3.5) acts naturally on D_q and $\mathscr{D}_q$ by

$$\begin{aligned} \mathfrak{s}_i : X_i \mapsto X_i^{-1}, \quad & P_i \mapsto P_i^{-1}, \\ \mathfrak{s}_{ij} : X_i \mapsto X_j, \quad & P_i \mapsto P_j, \\ \mathfrak{s}_{ij}^{+} = \mathfrak{s}_{ij}\,\mathfrak{s}_i\,\mathfrak{s}_j : X_i \mapsto X_j^{-1}, \quad & P_i \mapsto P_j^{-1}. \end{aligned} \tag{3.6}$$

Definition 3.14 The algebra of q-difference-reflection operators is the semi-direct product

$$\mathscr{D}_q \rtimes \mathbb{C}W.$$

The elements of $\mathscr{D}_q \rtimes \mathbb{C}W$ are finite $\mathbb{C}(\boldsymbol{X})$-linear combinations of the form

$$\sum_{\substack{\boldsymbol{\lambda} \in \mathbb{Z}^n \\ w \in W}} a_{\boldsymbol{\lambda}, w}(\boldsymbol{X}) \boldsymbol{P}^{\boldsymbol{\lambda}} w. \tag{3.7}$$

For $i = 1, \dots, n$, we introduce the notation

$$\begin{aligned} s_0 &= P_1^{-1}\,\mathfrak{s}_1, \\ s_i &= \mathfrak{s}_{i\,i+1}, \\ s_n &= \mathfrak{s}_n\,. \end{aligned} \tag{3.8}$$

It is straightforward to see that $s_0, \dots, s_n$ generate the affine Weyl group $\widetilde{W}$ of type $\widetilde{C}_n$, cf. relations (v)–(vii) in Definition 3.2. As the s_i act on functions via (3.6), there is an isomorphism

$$\mathscr{D}_q \cong \mathbb{C}(\boldsymbol{X}) \rtimes \mathbb{C}\widetilde{W}.$$

Remark 3.15 In the general setting, the above equivalence is only true with the *extended* affine Weyl group $\widehat{W}$ (see §2.1.2) in place of $\widetilde{W}$. The situation in which we find ourselves in the present chapter, where we have fixed W to be the Weyl group (3.5) of type C_n, the objects $\widehat{W}$ and $\widetilde{W}$ coincide and we needn't make a distinction.

Proposition 3.16 ([Sah99, (13)], Basic Representation) *There is an injective homomorphism of algebras $\beta : \mathcal{H}_{q,\boldsymbol{\tau}} \to \mathscr{D}_q \rtimes \mathbb{C}W$ defined as follows for $i = 1, \dots, n-1$:*

$$\begin{aligned} \beta(X_i) &= X_i, \\ \beta(T_0) &= k_0 s_0 + \frac{\overline{k}_0 + q\overline{u}_0 X_1^{-1}}{1 - q^2 X_1^{-2}}(1 - s_0), \\ \beta(T_i) &= t s_i + \frac{\overline{t}}{1 - X_i X_{i+1}^{-1}}(1 - s_i), \\ \beta(T_n) &= k_n s_n + \frac{\overline{k}_n + \overline{u}_n X_n}{1 - X_n^2}(1 - s_n), \end{aligned}$$

where we use $\overline{\cdot}$ to denote the difference between a parameter and its inverse, e.g. $\overline{t} = t - t^{-1}$.

Remark 3.17 For $q \neq 1$, we can view the elements of $\mathscr{D}_q \rtimes \mathbb{C}W$ (and hence the elements of $\mathcal{H}_{q,\boldsymbol{\tau}}$) as operators acting on functions of n variables, with P_i variables acting as multiplicative shifts

$$P_i \bullet f(X_1, X_2, \dots, X_n) = f(X_1, \dots, X_{i-1}, q^2 X_i, X_{i+1}, \dots, X_n) \tag{3.9}$$

and with the s_i elements acting as follows:

$$\begin{aligned} s_0 \bullet f(X_1, X_2, \dots, X_n) &= f(q^2 X_1^{-1}, X_2, \dots, X_n), \\ s_i \bullet f(X_1, X_2, \dots, X_n) &= f(X_1, \dots, X_{i-1}, X_{i+1}, X_i, X_{i+2}, \dots, X_n), \\ s_n \bullet f(X_1, X_2, \dots, X_n) &= f(X_1, \dots, X_{n-1}, X_n^{-1}). \end{aligned} \tag{3.10}$$

We conclude this section by discussing an important corollary of Proposition 3.16 regarding a localisation property of $\mathcal{H}_{q,\boldsymbol{\tau}}$. Let's recall the correct notion of non-commutative localisation.

Definition 3.18 Let R be a ring and $S \subseteq R$ a multiplicative subset. The (right) Ore condition says $rS \cap sR \neq \emptyset$. In other words, for every $r \in R$ and $s \in S$, there exist $r' \in R$ and $s' \in S$ with

$$rs' = sr' \qquad \Leftrightarrow \qquad s^{-1}r = r'(s')^{-1}.$$

In other words, the Ore condition allows one to move localised elements from one side of a ring element to the other (at the expense of perhaps being changing both the localised and ring elements). Of course, there is a natural analogue to Definition 3.18 for left-standing elements.

Notation 3.19 For $S \subseteq R$ a (right) Ore set, the (right) localisation of R by S is denoted $R[S^{-1}]$.

Consider the following multiplicative subset, whose generating elements have $i \neq j$ and $\ell \in \mathbb{Z}$:

$$\mathcal{S} = \left\langle \begin{matrix} 1 - q^\ell X_i^{\pm 1} X_j^{\pm 1}, & 1 - q^\ell t^2 X_i^{\pm 1} X_j^{\pm 1}, & 1 - q^\ell X_i^{\pm 2}, & \\ 1 - q^\ell k_0 u_0 X_i^{\pm 1}, & 1 + q^\ell k_0 u_0^{-1} X_i^{\pm 1}, & 1 - q^\ell k_n u_n X_i^{\pm 1}, & 1 + q^\ell k_n u_n^{-1} X_i^{\pm 1} \end{matrix} \right\rangle. \tag{3.11}$$

Lemma 3.20 *The multiplicative subset $\mathcal{S}$ is a left and right Ore set in $\mathcal{H}_{q,\boldsymbol{\tau}}$.*

Proof: It suffices to prove this on generators $s \in \mathcal{S}$ with each of X_i and T_i from the DAHA. The former is trivial since they commute with every s. As for the latter, we embed the DAHA into the algebra of q-difference-reflection operators via the Basic Representation β. In doing so, T_i acts by s_i from (3.10). Writing the action of s_i on T_i as a superscript, we can see that

$$T_i \underbrace{(s)^{s_i} s}_{s'} = s \underbrace{(s)^{s_i} T_i}_{r'},$$

Since $\mathcal{S}$ is closed under the action of s_i, we indeed have $s' \in \mathcal{S}$, and clearly $r' \in \mathcal{H}_{q,\boldsymbol{\tau}}$. Finally then, the argument for showing that $\mathcal{S}$ is a left Ore argument is completely analogous. □

Proposition 3.21 *The Basic Representation β induces an isomorphism of the Ore localisations*

$$\mathcal{H}_{q,\boldsymbol{\tau}}[\mathcal{S}^{-1}] \cong \mathbb{C}_q[\boldsymbol{X}^{\pm 1}, \boldsymbol{P}^{\pm 1}][\mathcal{S}^{-1}] \rtimes \mathbb{C}W.$$

Proof: The DAHA generators are represented by elements of $\mathbb{C}_q[\boldsymbol{X}^{\pm 1}, \boldsymbol{P}^{\pm 1}][\mathcal{S}^{-1}] \rtimes \mathbb{C}W$ by the same formulae as those appearing in the Basic Representation. Conversely, the Weyl group W is generated by s_i from (3.10) for $i = 1, \ldots, n$. The definition of $\mathcal{S}$ in (3.11) allows us to rearrange the expressions in Proposition 3.16 for each s_i. We see that each X_i is sent to the DAHA generator of the same name and, in light of (3.8), each P_i is ultimately obtained by rearranging the Basic Representation for s_0; we can easily get P_1^{-1} and successively act on it to get the others. □

When $q = 1$, we can replace the Ore set $\mathcal{S}$ with the multiplicative set generated by the elements

$$\delta(\boldsymbol{X}) \coloneqq \prod_{i=1}^{n}(1 - X_i^2)(1 - X_i^{-2}) \prod_{j \neq k}(1 - X_j X_k)(1 - X_j^{-1} X_k)(1 - X_j X_k^{-1})(1 - X_j^{-1} X_k^{-1}) \quad (3.12)$$

and

$$\begin{aligned} \delta_{\boldsymbol{\tau}}(\boldsymbol{X}) \coloneqq & \prod_{i<j}(1 - t^2 X_i X_j)(1 - t^2 X_i^{-1} X_j)(1 - t^2 X_i X_j^{-1})(1 - t^2 X_i^{-1} X_j^{-1}) \\ & \times \prod_{i=1}^{n}(1 - k_0 u_0 X_i)(1 + k_0 u_0^{-1} X_i)(1 - k_n u_n X_i)(1 + k_n u_n^{-1} X_i) \\ & \times \prod_{i=1}^{n}(1 - k_0 u_0 X_i^{-1})(1 + k_0 u_0^{-1} X_i^{-1})(1 - k_n u_n X_i^{-1})(1 + k_n u_n^{-1} X_i^{-1}). \end{aligned} \quad (3.13)$$

The product of (3.12) and (3.13) is the W-invariant form of the τ-deformed Weyl denominator (2.5) for R the affine root system of type C_n. For instance, the factors $(1-k_n u_n X_n)(1+k_n u_n^{-1} X_n)$ appear in the τ-deformed Weyl denominator when $\alpha = a_n = 2\varepsilon_n$ is the n^{th} simple root.

In the sequel, we use subscripts $\delta(\boldsymbol{X})$ and $\delta_{\boldsymbol{\tau}}(\boldsymbol{X})$ for respective localisations by each of (3.12) and (3.13), and a superscript W for the subalgebra comprising of W-invariant elements.

Corollary 3.22 *Let $\mathcal{H} = \mathcal{H}_{1,\boldsymbol{\tau}}$. The Basic Representation β induces isomorphisms of localisations*

$$\mathcal{H}_{\delta(\boldsymbol{X})\delta_{\boldsymbol{\tau}}(\boldsymbol{X})} \cong \mathbb{C}[\boldsymbol{X}^{\pm 1}, \boldsymbol{P}^{\pm 1}]_{\delta(\boldsymbol{X})\delta_{\boldsymbol{\tau}}(\boldsymbol{X})} \rtimes \mathbb{C}W$$

and

$$\mathbf{e}\mathcal{H}_{\delta(\boldsymbol{X})\delta_{\boldsymbol{\tau}}(\boldsymbol{X})}\mathbf{e} \cong \mathbb{C}[\boldsymbol{X}^{\pm 1}, \boldsymbol{P}^{\pm 1}]^W_{\delta(\boldsymbol{X})\delta_{\boldsymbol{\tau}}(\boldsymbol{X})}\mathbf{e}.$$

Proof: The first isomorphism is the classical limit $q = 1$ of Proposition 3.21. The second is obtained from the first by embedding the DAHA into the algebra of q-difference-reflection operators via β, multiplying on the left- and right-hand sides by the Hecke symmetriser $\mathbf{e}$, and finally pushing $\mathbf{e}$ through to the right. The Laurent polynomial ring then becomes the ring of W-invariant Laurent polynomials. □

Chapter 4

Representations of the DAHA

We mentioned in Chapter 1 that the results of Etingof and Ginzburg (Theorem 1.1) and Oblomkov (Theorem 1.2) admit representation-theoretic interpretations. The purpose of this chapter is to explore a similar situation this time associated to the DAHA of type $C^\vee C_n$. In particular, we explicate the relationship between the finite-dimensional irreducible representations of this DAHA and a certain character variety called *Calogero-Moser space* that is suggested in [EGO06].

4.1 Calogero-Moser Space

We begin this chapter by first looking towards the character variety side of the story. Namely, we introduce this section's titular object as a particular specialised character variety: let us fix $C_i = [\Lambda_i] \subseteq \mathrm{GL}_{2n}(\mathbb{C})$ to be the semi-simple conjugacy classes defined by the diagonal matrices

$$\begin{aligned}
\Lambda_1 &= \mathrm{diag}(\underbrace{-k_0^{-1}, \dots, -k_0^{-1}}_{n}, \underbrace{k_0, \dots, k_0}_{n}), \\
\Lambda_2 &= \mathrm{diag}(\underbrace{-u_0^{-1}, \dots, -u_0^{-1}}_{n}, \underbrace{u_0, \dots, u_0}_{n}), \\
\Lambda_3 &= \mathrm{diag}(\underbrace{-u_n^{-1}, \dots, -u_n^{-1}}_{n}, \underbrace{u_n, \dots, u_n}_{n}), \\
\Lambda_4 &= \mathrm{diag}(\underbrace{-k_n^{-1}, \dots, -k_n^{-1}}_{n}, \underbrace{k_n t^{-2}, \dots, k_n t^{-2}}_{n-1}, \underbrace{k_n t^{2n-2}}_{1}).
\end{aligned} \tag{4.1}$$

Definition 4.1 The Calogero-Moser space of type $C^\vee C_n$ is the $\mathrm{GL}_{2n}(\mathbb{C})$-character variety $\mathfrak{M}_{0,4}$ of the four-punctured Riemann sphere with conjugacy classes $C_i = [\Lambda_i]$ defined above by (4.1):

$$\mathcal{C}_n \coloneqq \{A_i \in [\Lambda_i] : A_1 A_2 A_3 A_4 = \mathbb{1}_{2n}\} /\!\!/ \mathrm{GL}_{2n}(\mathbb{C}).$$

Note that the conjugacy classes here make reference to the DAHA parameters. In particular, recall the genericity condition we imposed on the conjugacy classes in Definition 2.12. However, another notion of genericity will be helpful and yield nice results, here at the level of DAHA.

Definition 4.2 We call $\boldsymbol{\tau} = (k_0, k_n, t, u_0, u_n)$ generic if, for every $a, b, c, d, e \in \mathbb{Z}$ not all-zero,

$$k_0^a k_n^b t^c u_0^d u_n^e \neq 1.$$

Proposition 4.3 *For generic $\boldsymbol{\tau}$, the Calogero-Moser space $\mathcal{C}_n$ is non-empty and smooth.*

Proof: Non-emptiness follows from an explicit construction of a coordinate chart (see §4.3). As for smoothness, it suffices to show the eigendata (4.1) is generic, so that the result then follows from Theorem 2.13. Let λ_{ij} be the eigenvalues with multiplicities μ_{ij} of the matrix Λ_i. Then,

$$\prod_{i=1}^{4} \prod_{j=1}^{d_i} \lambda_{ij}^{\mu_{ij}} = 1.$$

The multiplicity of the eigenvalue $\lambda_{41} = k_n t^{2n-2}$ is $\mu_{41} = 1$, so there are only two choices for ν_{41}.

(i) If $\nu_{41} = 1$, then since $s < 2n$, there will always be a surviving t-term or k_n-term. Indeed, $\nu_{42} = n - 1$ kills the t-terms, but we must have $\nu_{43} < n$ and the product will contain k_n.

(ii) If $\nu_{41} = 0$, we have a persistent t-dependence from the penultimate eigenvalue λ_{42}. □

Irreducibility is now readily established from [HLRV11, HLRV13], in particular Theorem 2.14. By following this line of reasoning, we have avoided a complicated technical argument as had to be done in Oblomkov's case [Obl04, §2.3]. Nevertheless, the reader is directed to Appendix A for a combinatorial proof (of connectedness, hence irreducibility in the smooth case) using [Mel20].

Corollary 4.4 *For generic $\boldsymbol{\tau}$, the Calogero-Moser space $\mathcal{C}_n$ is irreducible.*

Moreover, we also know the dimension of Calogero-Moser space; it is given by the formula (2.6):

$$\dim(\mathcal{C}_n) = 2 + (0 + 4 - 2)(2n)^2 - (7n^2 + (n-1)^2 + 1) = 2n.$$

It is sometimes convenient to use the following description of the space $\mathcal{C}_n$. For $V = \mathbb{C}^{2n}$, let $\mathcal{R}_n$ be the set of $X, Y, T \in \mathrm{GL}(V)$ and $(v, w) \in \mathrm{Hom}(\mathbb{C}, V) \oplus \mathrm{Hom}(V, \mathbb{C})$ subject to the relations

$$T - T^{-1} = (u_0 - u_0^{-1})\mathbb{1}_V, \tag{4.2}$$

$$XT^{-1} - TX^{-1} = (k_0 - k_0^{-1})\mathbb{1}_V, \tag{4.3}$$

$$T^{-1}Y^{-1} - YT = (u_n - u_n^{-1})\mathbb{1}_V, \tag{4.4}$$

$$tYTX^{-1} - t^{-1}XT^{-1}Y^{-1} = (k_n t^{-1} - k_n^{-1} t)\mathbb{1}_V + (t - t^{-1})vw, \tag{4.5}$$

$$wv = \frac{t^{2n}-1}{t^2-1}k_n + \frac{1-t^{-2n}}{1-t^{-2}}k_n^{-1}. \tag{4.6}$$

There is a natural action of $g \in \mathrm{GL}(V)$ on $\mathcal{R}_n$, namely

$$g \bullet (X, Y, T, v, w) := (gXg^{-1}, gYg^{-1}, gTg^{-1}, gv, wg^{-1}). \tag{4.7}$$

Proposition 4.5 *Let* $\boldsymbol{\tau} = (k_0, k_n, t, u_0, u_n)$ *be generic in the sense of* Definition 4.2. *Then, the action* (4.7) *on* $\mathcal{R}_n$ *is free and we have an isomorphism of varieties* $\mathcal{C}_n \cong \mathcal{R}_n / \mathrm{GL}(V)$.

Proof: To semi-simple matrices A_1, A_2, A_3, A_4 with the eigenvalues (4.1), associate $X = A_1A_2$, $Y = A_4A_1 = A_3^{-1}A_2^{-1}$ and $T = A_2$. Let $\lambda_1, \ldots, \lambda_{2n}$ be the eigenvalues of A_4 in *reverse* order as they appear in (4.1), with respective corresponding eigenvectors $v_1, \ldots, v_{2n}$. We then take v to be the map $\mathbb{C} \to V$ sending 1 to v_1, and define the map $w : V \to \mathbb{C}$ by

$$w(v_1) = \frac{t^{2n}-1}{t^2-1}k_n + \frac{1-t^{-2n}}{1-t^{-2}}k_n^{-1}, \qquad w(v_i) = 0 \text{ for } i = 2, \ldots, 2n.$$

It is easy to see that such X, Y, T, v, w satisfy the relations (4.2)–(4.6).

As for a map in the opposite direction, we take X, Y, T, v, w satisfying the relations (4.2)–(4.6) and define $A_1 = XT^{-1}$, $A_2 = T$, $A_3 = T^{-1}Y^{-1}$ and $A_4 = YTX^{-1}$. Obviously, $A_1A_2A_3A_4 = \mathbb{1}_V$. From (4.3), A_1 is diagonalisable with eigenvalues from the set $\{k_0, -k_0^{-1}\}$; one can argue similarly for A_2 and A_3, but we still need to establish that the eigenvalues have equal multiplicity. On the other hand, (4.5) and (4.6) will imply that $tA_4 - t^{-1}A_4^{-1}$ is diagonalisable with two eigenvalues: $k_n t^{-1} - k_n^{-1} t$ (of multiplicity $2n-1$) and $k_n t^{2n-1} - k_n^{-1} t^{-2n+1}$ (of multiplicity one). This implies A_4 is also diagonalisable, with $2n-1$ eigenvalues from the set $\{k_n t^{-2}, -k_n^{-1}\}$ and one from $\{k_n t^{2n-2}, -k_n^{-1} t^{-2n}\}$. But the product of the matrix determinants is one, which gives us a relation of the form $k_0^a k_n^b t^c u_0^d u_n^e = 1$ with some $a, b, c, d, e \in \mathbb{Z}$. Since the parameters are generic, the only possibility is that $a = b = c = d = e = 0$. This forces A_1, A_2, A_3 to have two eigenvalues each of multiplicity n. This, in turn, implies that $\det(A_4) = (-1)^n$, and we find that this is only possible if A_4 has eigenvalues as in (4.1). □

In view of Proposition 2.23, there is an interpretation of the Calogero-Moser space from the angle of quiver varieties. In these terms, the character variety $\mathfrak{M}_{0,4}$ corresponds to the multiplicative preprojective algebra of a *framed* affine Dynkin quiver of type $\widetilde{D}_4$, as sketched in Figure 4.1 below. The so-called framed quiver Q^∞ is obtained from $Q = \widetilde{D}_4$ by extending one of its legs by a single vertex (denoted ∞). The corresponding data (2.13) and (2.14) defining the quiver variety in the particular case of Q^∞ is given explicitly below:

$$\begin{aligned} \mathbf{q} &= (q_0, q_1, q_2, q_3, q_4, q_\infty) = (k_0 u_0 u_n k_n, -k_0^{-2}, -u_0^{-2}, -u_n^{-2}, -k_n^{-2} t^2, t^{-2n}), \\ \mathbf{n} &= (n_0, n_1, n_2, n_3, n_4, n_\infty) = (2n, n, n, n, n, 1). \end{aligned} \tag{4.8}$$

Corollary 4.6 *For generic $\boldsymbol{\tau}$ and quiver data (4.8), the Calogero-Moser space $\mathcal{C}_n \cong \mathcal{M}_{\mathbf{q},\mathbf{n}}(Q^\infty)$.*

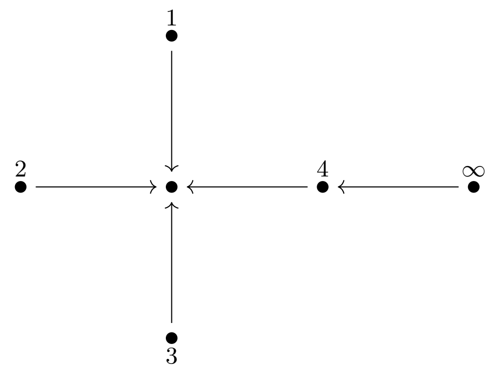


Figure 4.1: The quiver Q^∞ corresponding to the Calogero-Moser space $\mathcal{C}_n$.

4.2 The Etingof-Gan-Oblomkov Map

Notation 4.7 Throughout, we use $\mathcal{H} = \mathcal{H}_{1,\boldsymbol{\tau}}$ for the DAHA at the classical level $q = 1$.

One of the key constructions of [EGO06, §5.2] is a map from the set of irreducible (regular, in a certain sense) representations of the GDAHAs for $Q = \widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ at the classical level $q = 1$ to a suitable character variety $\mathfrak{M}_{0,k}$. In our situation, for $Q = \widetilde{D}_4$, this provides us with a map

$$\Phi : \mathrm{Irrep}'(\mathcal{H}) \to \mathcal{C}_n,$$

where $\mathrm{Irrep}'(\mathcal{H}) \subseteq \mathrm{Irrep}(\mathcal{H})$ is the subset of irreducible representations that restrict to the regular representation of the finite Hecke algebra $H_n \subseteq \mathcal{H}$. In particular, $\dim(V) = 2^n n!$ for all $V \in \mathrm{Irrep}'(\mathcal{H})$. In fact, the main result of the thesis will imply $\mathrm{Irrep}'(\mathcal{H}) = \mathrm{Irrep}(\mathcal{H})$, but this is certainly non-obvious a priori (see Corollary 5.18).

Recall the one-dimensional representation $\chi : H_n \to \mathbb{C}$ that assigns $T_i \mapsto t$ $(i = 1, \ldots, n-1)$ and $T_n \mapsto k_n$. Consider the subalgebra $H_n' \subseteq H_n$ generated by $T_2, \ldots, T_n$, and denote by $\chi' : H_n' \to \mathbb{C}$ the restriction of χ to H_n'. Following [EGO06, §4.3], for a representation $V \in \mathrm{Rep}(H_n)$, consider

$$V' = \{f : \chi' \to V\},$$

the space of homomorphisms between χ' and V, viewed as H_n'-modules. We may consider V' as a subspace of V. More explicitly, let $\mathbf{e}'$ be the Hecke symmetriser of the subalgebra H'_n, that is

$$\mathbf{e}' := \frac{1}{\sum_{w\in W'} \tau_w^2} \sum_{w\in W'} \tau_w T_w, \tag{4.9}$$

where $W' \subseteq W$ is the Weyl group associated to H'_n. We can view $V' = \mathbf{e}'V$, as in the second proof of [EGO06, Lemma 4.3.1]. But H_n has basis $(T_w)_{w\in W}$ by [Mac03, (4.1.3)] and $\mathbf{e}' \in V'$, so it follows that $(\mathbf{e}'w)_{w\in W}$ is a basis for the set of cosets $W'\backslash W$. Hence, the dimension of V' is the number of cosets which, for us, is $|W|/|W'| = 2^n n!/(2^{n-1}(n-1)!)$. Consequently, $\dim(V') = 2n$.

Proposition 4.8 ([EGO06, cf. Proposition 5.2.10]) *For $S, S^\dagger$ as in* Notation 3.3*, let $Z_i \in \mathcal{H}$ be*

$$Z_1 = T_0, \qquad Z_2 = T_0^\vee, \qquad Z_3 = ST_n^\vee S^{-1}, \qquad Z_4 = ST_nS^\dagger. \tag{4.10}$$

(i) *For $V \in \mathrm{Irrep}'(\mathcal{H})$, the elements Z_i commute with H'_n and hence preserve the subspace V'.*

(ii) *For $V \in \mathrm{Irrep}'(\mathcal{H})$, denote by $A_i \in \mathrm{End}_{\mathbb{C}}(V')$ the restriction of Z_i onto the subspace V':*

$$A_i = Z_i\Big|_{V'}. \tag{4.11}$$

Then, the assignment $V \mapsto (A_1, A_2, A_3, A_4)$ defines a map $\Phi :\ \mathrm{Irrep}'(\mathcal{H}) \to \mathcal{C}_n$.

At the quantum level, one can abuse notation and replace Z_1 (4.10) with the element $Z_1 = qT_0$. This will be briefly used in Chapter 6 only, but the reader can keep this quantisation in mind.

Definition 4.9 The map Φ from Proposition 4.8(ii) is henceforth called the EGO map.

Remark 4.10 Our definition of Φ is slightly different from the one in [EGO06]. In *op. cit.*, the authors use the subalgebras H and $H' \subseteq H$ generated respectively by $ST_nS^{-1}, T_1, \dots, T_{n-1}$ and $ST_nS^{-1}, T_1, \dots, T_{n-2}$. These are clearly analogous to the subalgebras H_n and H'_n in our set-up (in fact, their H can also be generated by $T_1, \dots, T_n$ so it is exactly H_n). Their H' is used to define a subspace $V' \subseteq V$ in a similar way. Instead of the above elements Z_i (4.10), they consider

$$\widetilde{U}_i := S^\dagger Z_i (S^\dagger)^{-1}.$$

Their definition of the map Φ then uses the restriction of these $\widetilde{U}_i$ onto V', in contrast with our A_i (4.11). But the proof of [EGO06, Proposition 5.2.10] is easily adaptable to our setting.

4.3 From DAHA to Matrices

Our next task is to make the EGO map more explicit. It will be convenient to use a realisation of $\mathcal{H}_{q,\boldsymbol{\tau}}$ by operator-valued matrices as in [Cha19]. At the classical level $q = 1$, this provides us with a coordinate chart on the Calogero-Moser space $\mathcal{C}_n$; in the language of Problem 2.10, we obtain a family of solutions to the corresponding multiplicative Deligne-Simpson problem.

We begin by considering the vector space

$$M \coloneqq \mathbb{C}W \otimes \mathscr{D}_q. \tag{4.12}$$

We identify $M \cong \mathscr{D}_q \rtimes \mathbb{C}W$, writing its elements as

$$f = \sum_{w\in W} w f_w, \qquad f_w \in \mathscr{D}_q. \tag{4.13}$$

The action of $\mathscr{D}_q \rtimes \mathbb{C}W$ on itself, by left multiplication, provides us with a faithful representation

$$\pi : \mathscr{D}_q \rtimes \mathbb{C}W \to \mathrm{Mat}_{|W|\times|W|}(\mathscr{D}_q). \tag{4.14}$$

Identifying $\mathcal{H}_{q,\boldsymbol{\tau}}$ with its image under the Basic Representation, (4.14) implies a representation

$$\mathcal{H}_{q,\boldsymbol{\tau}} \hookrightarrow \mathrm{Mat}_{|W|\times|W|}(\mathscr{D}_q).$$

Let $W' \subseteq W$ be the Weyl subgroup generated by elements that fix the first index, that is

$$W' \coloneqq \big\langle \mathfrak{s}_i, \mathfrak{s}_{ij} : i, j = 2, \dots, n \big\rangle. \tag{4.15}$$

Associated to W' is the subspace of W'-invariants

$$M' = e'M, \qquad e' \coloneqq \frac{1}{|W'|}\sum_{w\in W'} w. \tag{4.16}$$

We choose $\{\mathfrak{s}_{1i}, \mathfrak{s}_{1i}^{+}\}$ with $i = 1, \dots, n$ as the coset representatives in $W'\backslash W$, where $\mathfrak{s}_{11} \coloneqq \mathrm{id}$ and $\mathfrak{s}_{11}^{+} \coloneqq \mathfrak{s}_1$. Each element $f \in M'$ of this subspace admits a unique presentation of the below form:

$$f = e'\left(\sum_{j=1}^{n} \mathfrak{s}_{1j}\, f_j + \sum_{j=1}^{n} \mathfrak{s}_{1j}^{+}\, f_j^{+}\right), \qquad f_j, f_j^{+} \in \mathscr{D}_q. \tag{4.17}$$

Lemma 4.11 *Let Z denote any of the elements Z_1, Z_2, Z_3, Z_4 in (4.10). Then, Z preserves M'.*

Proof: This is by a standard argument. First, notice that the Basic Representation implies

$$T_i - \tau_i = c_i(\boldsymbol{X})(s_i - 1)$$

for the rational function $c_i(\boldsymbol{X}) = c_{a_i}(\boldsymbol{X})$ as defined using (2.2), cf. [Mac03, (4.3.12)]. Notice that each Z commutes with the generators $T_2, \dots, T_n$ of the Hecke subalgebra H_n'. Indeed, each of these proofs is similar and thus the full details are omitted, but we will proceed with a proof in the case $Z = Z_4$ (this is the more technical of the bunch, but it still boils down to careful application of the DAHA relations). To begin with, for all $i = 3, \dots, n-3$, we see that

$$\begin{aligned}
T_i Z_4 &= T_i S T_n S^\dagger \\
&= T_i T_1 \cdots T_{n-1} T_n T_{n-1} \cdots T_1 \\
&= T_1 \cdots T_{i-2} T_i T_{i-1} T_i T_{i+1} \cdots T_{n-1} T_n T_{n-1} \cdots T_1 \\
&= T_1 \cdots T_{i-2} T_{i-1} T_i T_{i-1} T_{i+1} \cdots T_{n-1} T_n T_{n-1} \cdots T_1 \\
&= T_1 \cdots T_{i-2} T_{i-1} T_i T_{i+1} \cdots T_{n-1} T_n T_{n-1} \cdots T_{i+1} T_{i-1} T_i T_{i-1} T_{i-2} \cdots T_1 \\
&= T_1 \cdots T_{i-2} T_{i-1} T_i T_{i+1} \cdots T_{n-1} T_n T_{n-1} \cdots T_{i+1} T_i T_{i-1} T_i T_{i-2} \cdots T_1 \\
&= T_1 \cdots T_{i-2} T_{i-1} T_i T_{i+1} \cdots T_{n-1} T_n T_{n-1} \cdots T_{i+1} T_i T_{i-1} T_{i-2} \cdots T_1 T_i \\
&= S T_n S^\dagger T_i \\
&= Z_4 T_i.
\end{aligned}$$

The above range of indices is restricted simply due to the particular DAHA relation used to show commutativity. Of course, we must also proceed for $i = 2, n-2, n-1$ (which are near-identical) and $i = n$. This latter case uses another DAHA relation, so we explicate it here:

$$\begin{aligned}
T_n Z_4 &= T_n S T_n S^\dagger \\
&= T_n T_1 \cdots T_{n-1} T_n T_{n-1} \cdots T_1 \\
&= T_1 \cdots T_{n-2} T_n T_{n-1} T_n T_{n-1} T_{n-2} \cdots T_1 \\
&= T_1 \cdots T_{n-2} T_{n-1} T_n T_{n-1} T_n T_{n-2} \cdots T_1 \\
&= T_1 \cdots T_{n-2} T_{n-1} T_n T_{n-1} T_{n-2} \cdots T_1 T_n \\
&= S T_n S^\dagger T_n \\
&= Z_4 T_n.
\end{aligned}$$

Consequently, for $i = 2, \dots, n$ and Z any of the four elements (4.10), we have

$$(s_i - 1)Z = \frac{1}{c_i(\boldsymbol{X})}(T_i - \tau_i)Z = \frac{1}{c_i(\boldsymbol{X})}Z(T_i - \tau_i) = \frac{1}{c_i(\boldsymbol{X})}Z c_i(\boldsymbol{X})(s_i - 1),$$

from which it follows that $(s_i - 1)Ze' = 0$. Hence, if $f \in M$ is W'-invariant, then $(s_i - 1)Zfe' = (s_i - 1)Ze'f = 0$; the element Zf is also W'-invariant. In other words, $Z(M') \subseteq M'$. □

One of the key insights of [Cha19] is that one can explicitly obtain a matrix of a DAHA element that preserves the subspace M'. We will use this to now derive explicit expressions for the $\mathscr{D}_q$-valued matrices A_i representing the action of Z_i on the subspace M', defined by (4.11).

Let Z be an element that preserves $M' = e'M$; it is sufficient to consider W'-invariant elements $Zw = wZ$ for all $w \in W'$. We compute the corresponding matrix by pulling group elements to the left of Zf, where $f \in M'$ has the form in (4.17); this concept is discussed in [Cha19, §4.3]. Our convention is that the matrix has columns corresponding to the rational functions $f_1, \dots, f_n, f_1^+, \dots, f_n^+$ and rows corresponding to the cosets $\mathfrak{s}_{11}, \dots, \mathfrak{s}_{1n}, \mathfrak{s}_{11}^+, \dots, \mathfrak{s}_{1n}^+$.

Proposition 4.12 *Consider an arbitrary W'-invariant element of the form*

$$Z \coloneqq A + B\,\mathfrak{s}_1 + \sum_{i\neq 1}^{n} C_i\,\mathfrak{s}_{1i} + \sum_{i\neq 1}^{n} C_i^+\,\mathfrak{s}_{1i}^+\,. \tag{4.18}$$

Then, for a general element $f \in M'$ written in the form (4.17), we see that

$$\begin{aligned} Zf = e' \sum_{j=1}^{n} &\left(\mathfrak{s}_{1j}(A)^{\mathfrak{s}_{1j}} + \mathfrak{s}_{1j}^+(B)^{\mathfrak{s}_{1j}\,\mathfrak{s}_1} + \sum_{i\neq j}^{n} \mathfrak{s}_{1i}(C_i)^{\mathfrak{s}_{1j}\,\mathfrak{s}_{1i}} + \sum_{i\neq j}^{n} \mathfrak{s}_{1i}^+(C_i^+)^{\mathfrak{s}_{1j}\,\mathfrak{s}_{1i}^+} \right) f_j \\ &+ e' \sum_{j=1}^{n} \left(\mathfrak{s}_{1j}^+(A)^{\mathfrak{s}_{1j}^+} + \mathfrak{s}_{1j}(B)^{\mathfrak{s}_{1j}^+\,\mathfrak{s}_1} + \sum_{i\neq j}^{n} \mathfrak{s}_{1i}(C_i)^{\mathfrak{s}_{1j}^+\,\mathfrak{s}_{1i}} + \sum_{i\neq j}^{n} \mathfrak{s}_{1i}^+(C_i^+)^{\mathfrak{s}_{1j}^+\,\mathfrak{s}_{1i}^+} \right) f_j^+ . \end{aligned}$$

Proof: This is a careful-but-simple calculation like [Cha19, (4.16)]. Writing $f \in M'$ as in (4.17),

$$\begin{aligned} Zf &= Ze' \left(\sum_{j=1}^{n} \mathfrak{s}_{1j}\, f_j + \sum_{j=1}^{n} \mathfrak{s}_{1j}^+\, f_j^+ \right) \\ &= e'Z \left(\sum_{j=1}^{n} \mathfrak{s}_{1j}\, f_j + \sum_{j=1}^{n} \mathfrak{s}_{1j}^+\, f_j^+ \right) \\ &= e' \sum_{j=1}^{n} \left(A + B\,\mathfrak{s}_1 + \sum_{i\neq 1}^{n} C_i\,\mathfrak{s}_{1i} + \sum_{i\neq 1}^{n} C_i^+\,\mathfrak{s}_{1i}^+ \right) \mathfrak{s}_{1j}\, f_j \\ &\quad + e' \sum_{j=1}^{n} \left(A + B\,\mathfrak{s}_1 + \sum_{i\neq 1}^{n} C_i\,\mathfrak{s}_{1i} + \sum_{i\neq 1}^{n} C_i^+\,\mathfrak{s}_{1i}^+ \right) \mathfrak{s}_{1j}^+\, f_j^+ \\ &= e' \sum_{j=1}^{n} \left(A\,\mathfrak{s}_{1j} + B\,\mathfrak{s}_1\,\mathfrak{s}_{1j} + \sum_{i\neq 1}^{n} C_i\,\mathfrak{s}_{1i}\,\mathfrak{s}_{1j} + \sum_{i\neq 1}^{n} C_i^+\,\mathfrak{s}_{1i}^+\,\mathfrak{s}_{1j} \right) f_j \\ &\quad + e' \sum_{j=1}^{n} \left(A\,\mathfrak{s}_{1j}^+ + B\,\mathfrak{s}_1\,\mathfrak{s}_{1j}^+ + \sum_{i\neq 1}^{n} C_i\,\mathfrak{s}_{1i}\,\mathfrak{s}_{1j}^+ + \sum_{i\neq 1}^{n} C_i^+\,\mathfrak{s}_{1i}^+\,\mathfrak{s}_{1j}^+ \right) f_j^+ \end{aligned}$$

$$= e' \sum_{j=1}^{n} \left(\mathfrak{s}_{1j} (A)^{\mathfrak{s}_{1j}} + \mathfrak{s}_1 \, \mathfrak{s}_{1j} (B)^{\mathfrak{s}_{1j} \, \mathfrak{s}_1} + \sum_{i \neq j}^{n} \mathfrak{s}_{1i} \, \mathfrak{s}_{1j} (C_i)^{\mathfrak{s}_{1j} \, \mathfrak{s}_{1i}} + \sum_{i \neq j}^{n} \mathfrak{s}_{1i}^{+} \, \mathfrak{s}_{1j} (C_i^{+})^{\mathfrak{s}_{1j} \, \mathfrak{s}_{1i}^{+}} \right) f_j$$
$$+ e' \sum_{j=1}^{n} \left(\mathfrak{s}_{1j}^{+} (A)^{\mathfrak{s}_{1j}^{+}} + \mathfrak{s}_1 \, \mathfrak{s}_{1j}^{+} (B)^{\mathfrak{s}_{1j}^{+} \, \mathfrak{s}_1} + \sum_{i \neq j}^{n} \mathfrak{s}_{1i} \, \mathfrak{s}_{1j}^{+} (C_i)^{\mathfrak{s}_{1j}^{+} \, \mathfrak{s}_{1i}} + \sum_{i \neq j}^{n} \mathfrak{s}_{1i}^{+} \, \mathfrak{s}_{1j}^{+} (C_i^{+})^{\mathfrak{s}_{1j}^{+} \, \mathfrak{s}_{1i}^{+}} \right) f_j^{+}.$$

The final step is to write any products of coset representatives as one representative by absorbing into e' anything that preserves M', e.g. $e' \, \mathfrak{s}_{1i} \, \mathfrak{s}_{1j} = e' \, \mathfrak{s}_{ij} \, \mathfrak{s}_{1i} = e' \, \mathfrak{s}_{1i}$. This completes the proof. □

This demonstrates that any element of $\mathscr{D}_q \rtimes \mathbb{C}W$ preserving M' can, therefore, be represented by a $\mathscr{D}_q$-valued matrix of size $2n \times 2n$. We shall soon see some examples of this. In particular, in light of Lemma 4.11, we can determine the matrices corresponding to each of the DAHA elements Z_1, Z_2, Z_3, Z_4 by writing them in the form (4.18). The task of finding A_1 and A_2 turns out to be simple, in view of the Basic Representation. The other matrices are trickier, but A_4 is (essentially) done in [Cha19, Proposition 4.3] and A_3 can be found by analogy.

Before we apply Proposition 4.12 to the DAHA elements (4.10), we first try some simple examples to get a grasp on how it is done. But also, the first of these particular examples will be helpful when determining the matrix representing A_4 from a closely-related result by Chalykh.

Example 4.13 We will compute the matrices representing each of X_1 and P_1. Writing X_1 in the form (4.18), we have $A = X_1$ and $B = C_i = C_i^{+} = 0$. Following Proposition 4.12 then, the only non-zero coefficients in Zf are $(A)^{\mathfrak{s}_{1j}} = X_j$ and $(A)^{\mathfrak{s}_{1j}^{+}} = X_j^{-1}$. Thus, for X_1, we obtain

$$X \coloneqq \operatorname{diag}(X_1, \ldots, X_n, X_1^{-1}, \ldots, X_n^{-1}).$$

An identical line of reasoning allows us to conclude that the matrix representing P_1 is

$$P \coloneqq \operatorname{diag}(P_1, \ldots, P_n, P_1^{-1}, \ldots, P_n^{-1}).$$

Example 4.14 We will compute the matrix representing the inversion element $\mathfrak{s}_1$ from (3.6). This is already written in the form (4.18), albeit very degenerately because the coefficients are $A = 0$, $B = 1$ and $C_i = C_i^{+} = 0$. Following Proposition 4.12 then, the only non-zero coefficients in Zf are $(B)^{\mathfrak{s}_{1j} \, \mathfrak{s}_1} = 1$ and $(B)^{\mathfrak{s}_{1j}^{+} \, \mathfrak{s}_1} = 1$. Consequently, we obtain the following block-diagonal matrix:

$$\begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}.$$

Example 4.15 We will compute the matrix representing the affine element $s_0 \in \widetilde{W}$ from (3.10).

Recall from (3.8) that we can write $s_0 = P_1^{-1}\,\mathfrak{s}_1$. This is quite similar to Example 4.14, but here the coefficients in the corresponding expression (4.18) are $A = 0$, $B = P_1^{-1}$ and $C_i = C_i^+ = 0$. Hence, by the above proposition, the only surviving coefficients in the expression for Zf are $(B)^{\mathfrak{s}_{1j}\,\mathfrak{s}_1} = P_j$ and $(B)^{\mathfrak{s}_{1j}^+\,\mathfrak{s}_1} = P_j^{-1}$. Alternatively, we can multiply the matrices of P_1^{-1} (cf. Example 4.13) and $\mathfrak{s}_1$. Either way, the corresponding matrix is again block-diagonal:

$$\begin{pmatrix} 0 & \mathrm{diag}(P_1^{-1},\ldots,P_n^{-1}) \\ \mathrm{diag}(P_1,\ldots,P_n) & 0 \end{pmatrix}.$$

Theorem 4.16 *The matrix representing $Z_1 = qT_0$ is that whose ij^{th} entry is*

$$(A_1)_{ij} = \begin{cases} \dfrac{q\overline{k}_0 + q^2\overline{u}_0 X_i^{-1}}{1 - q^2 X_i^{-2}} & \text{if } i = j \\ qk_0 P_i^{-1} - \dfrac{q\overline{k}_0 + q^2\overline{u}_0 X_i^{-1}}{1 - q^2 X_i^{-2}} P_i^{-1} & \text{if } i - j = \pm n \\ 0 & \text{otherwise} \end{cases},$$

and the matrix representing $Z_2 = T_0^\vee$ is that whose ij^{th} entry is

$$(A_2)_{ij} = \begin{cases} -\dfrac{q^{-1}\overline{k}_0 X_i + q^{-2}\overline{u}_0 X_i^2}{1 - q^{-2} X_i^2} & \text{if } i = j \\ q^{-1}k_0 P_i^{-1} X_i^{-1} - \dfrac{q^{-1}\overline{k}_0 + \overline{u}_0 X_i^{-1}}{1 - q^2 X_i^{-2}} P_i^{-1} X_i^{-1} & \text{if } i - j = \pm n \\ 0 & \text{otherwise} \end{cases},$$

extending the indices from $\{1,\ldots,n\}$ to $\{1,\ldots,2n\}$ by setting $X_{n+i} \coloneqq X_i^{-1}$ and $P_{n+i} \coloneqq P_i^{-1}$.

Proof: The proofs of each part of this result are very similar, so we only provide details for the first. Throughout, we do not distinguish T_i and $\beta(T_i)$. Namely, we assume that the DAHA is embedded in $\mathscr{D}_q \rtimes \mathbb{C}W$. This allows us to use the Basic Representation, alongside (3.8), to write

$$T_0 = a_0(X_1) + b_0(X_1)P_1^{-1}\,\mathfrak{s}_1,$$

for $a_0, b_0 \in \mathbb{C}_q(\boldsymbol{X}^{\pm 1})$. Since Z_1 preserves M' by Lemma 4.11, we apply Proposition 4.12 where

$$A = a_0(X_1), \qquad B = b_0(X_1)P_1^{-1}, \qquad C_i = 0, \qquad C_i^+ = 0. \qquad \square$$

As we indicated earlier, the above matrices are pretty easy to find. However, $Z_3 = ST_n^\vee S^{-1}$ and $Z_4 = ST_n S^\dagger$ require some more careful work. Fortunately, the matrix for $Z_4\,\mathfrak{s}_1$ is determined in

[Cha19], so we can easily obtain from this the matrix for Z_4. Indeed, the matrix representing $\mathfrak{s}_1$ was calculated in Example 4.14, so multiplying by its inverse ($\mathfrak{s}_1$ itself) will yield the desired matrix. Therefore, we will only discuss the matrix for Z_3 in depth.

The key technique employed in [Cha19, Lemma 4.1] that we borrow is to split Z_3 into three parts, each of which we can restrict to M'. However, the elements S and $S^\dagger$ do not preserve M', so Proposition 4.12 cannot be used yet. Instead, we insert the following cyclic permutation $c := \mathfrak{s}_{12}\,\mathfrak{s}_{23}\cdots\mathfrak{s}_{n-1\,n}$ into our DAHA elements so that Z_3 and Z_4 are expressed as follows:

$$\begin{aligned} Z_3 &= Sc^{-1}(cT_n^\vee c^{-1})cS^{-1}, \\ Z_4 &= Sc^{-1}(cT_n c^{-1})cS^\dagger. \end{aligned} \tag{4.19}$$

The task at present is to restrict each of the factors to M'. Due to the presence of the cyclic permutation, it turns out that Sc^{-1} (and therefore its inverse cS^{-1}) and $cS^\dagger$ still don't preserve M'. However, we can work with a slightly larger subspace than the usual M', namely

$$M'' = e''M, \qquad e'' := \frac{1}{|S_{n-1}|}\sum_{w\in S_{n-1}} w. \tag{4.20}$$

Note that we are symmetrising over the *honest* permutations of the indices $\{2,\ldots,n\}$ instead of those that also account for arbitrary sign changes. Namely, this is the group algebra symmetriser corresponding to the subgroup S_{n-1} in the Weyl subgroup W' from (4.15). Consequently, we see that $M' \subseteq M''$. If we can restrict the factors of (4.19) to M'', then it descends to M' for free.

Lemma 4.17 *The elements Sc^{-1} (and therefore its inverse cS^{-1}) and $cS^\dagger$ preserve M''.*

Proof: This is similar in spirit to the proof of Lemma 4.11. Namely, we show these elements commute with T_i for $i = 2,\ldots,n-1$, carefully using the DAHA relations. For $i = 3,\ldots,n-2$,

$$\begin{aligned} T_iSc^{-1} &= T_iT_1\cdots T_{n-1}c \\ &= T_1\cdots T_{i-2}T_iT_{i-1}T_iT_{i+1}\cdots T_{n-1}c^{-1} \\ &= T_1\cdots T_{i-2}T_{i-1}T_iT_{i-1}T_{i+1}\cdots T_{n-1}c^{-1} \\ &= ST_{i-1}c^{-1} \\ &= Sc^{-1}T_i. \end{aligned}$$

The cases $i = 2$ and $i = n-1$ are basically the same. Note a straight substitution above would break our notation, e.g. $i-2$ doesn't make sense when $i=2$, but the arguments run parallel:

$$\begin{aligned} T_2Sc^{-1} &= T_2T_1\cdots T_{n-1}c \\ &= T_1T_2T_1T_3\cdots T_{n-1}c^{-1} \\ &= T_1T_2T_3\cdots T_{n-1}T_1c^{-1} \\ &= ST_1c^{-1} \\ &= Sc^{-1}T_2, \end{aligned} \qquad \begin{aligned} T_{n-1}Sc^{-1} &= T_{n-1}T_1\cdots T_{n-1}c \\ &= T_1\cdots T_{n-3}T_{n-1}T_{n-2}T_{n-1}c^{-1} \\ &= T_1\cdots T_{n-3}T_{n-2}T_{n-1}T_{n-2}c^{-1} \\ &= ST_{n-2}c^{-1} \\ &= Sc^{-1}T_{n-1}. \end{aligned}$$

As for the second case, the argument is very similar. Indeed, for $i=3,\ldots,n-2$, we have

$$\begin{aligned} T_icS^\dagger &= T_icT_{n-1}\cdots T_1 \\ &= cT_{i-1}T_{n-1}\cdots T_1 \\ &= cT_{n-1}\cdots T_{i+1}T_{i-1}T_iT_{i-1}T_{i-2}\cdots T_1 \\ &= cT_{n-1}\cdots T_{i+1}T_iT_{i-1}T_iT_{i-2}\cdots T_1 \\ &= cS^\dagger T_i. \end{aligned}$$

The situation is the same as above; identical proofs work for $i=2$ and $i=n-1$, but a straight-up substitution fails due to the notation. On this occasion, we omit the explicit calculations. □

Remark 4.18 In the above proof, and in general when concerned with the preservation of M'', imagine the T_i are embedded in $\mathscr{D}_q \rtimes \mathbb{C}W$ via the Basic Representation β (see Proposition 3.16). Doing this reveals that T_n involves $s_n = \mathfrak{s}_n$, which is an inversion and hence not absorbed by e''. This explains why we needn't consider T_nSc^{-1}; attempting such an argument, we conclude that these two elements do not commute, but this is expected.

Using the Basic Representation, for $i=1,\ldots,n-1$, we can write the DAHA generators T_i as

$$T_i = b_{i\,i+1} + a_{i\,i+1}\,\mathfrak{s}_{i\,i+1}, \tag{4.21}$$

where

$$a_{ij} = \frac{t^{-1} - tX_iX_j^{-1}}{1 - X_iX_j^{-1}} \qquad \text{and} \qquad b_{ij} = \frac{\bar{t}}{1 - X_iX_j^{-1}}. \tag{4.22}$$

Notation 4.19 The above rational functions have i^{th} index with positive power and j^{th} index with a negative power. We adopt the following convention henceforth: indices ${}_{ij}$ denote variables $X_iX_j^{-1}$, indices ${}^{+}_{ij}$ denote variables X_iX_j and indices ${}^{-}_{ij}$ denote variables $X_i^{-1}X_j^{-1}$. For (4.22),

$$a^+_{ij} = \frac{t^{-1} - tX_iX_j}{1 - X_iX_j}, \qquad a^-_{ij} = \frac{t^{-1} - tX_i^{-1}X_j^{-1}}{1 - X_i^{-1}X_j^{-1}}, \qquad b^+_{ij} = \frac{\bar{t}}{1 - X_iX_j}, \qquad b^-_{ij} = \frac{\bar{t}}{1 - X_i^{-1}X_j^{-1}}.$$

The Hecke relations $T_i^{-1} = T_i - \bar{t}$ for $i = 1, \ldots, n$ allow us to write T_i^{-1} in analogy with (4.21):

$$T_i^{-1} = \beta_{i\,i+1} + \alpha_{i\,i+1}\,\mathfrak{s}_{i\,i+1}, \tag{4.23}$$

where

$$\alpha_{ij} = \frac{t^{-1} - tX_iX_j^{-1}}{1 - X_iX_j^{-1}} \qquad \text{and} \qquad \beta_{ij} = \frac{\bar{t}X_iX_j^{-1}}{1 - X_iX_j^{-1}}. \tag{4.24}$$

Remark 4.20 Comparing the rational functions (4.22) and (4.24), it is clear $\alpha_{ij} = a_{ij}$, and still somewhat noticeable that $\beta_{ij} = b_{ij} - \bar{t} = -b_{ji}$. We could simply stick with the notation in (4.22) alone, but it is sometimes convenient to track whether or not we are talking about T_i vs. T_i^{-1} by differentiating the Latin and Greek letters; this is sometimes done henceforth.

Lemma 4.21 *The elements Sc^{-1}, cS^{-1} and $cS^\dagger$ can be written using* (4.21) *and* (4.23) *as follows:*

(i) $Sc^{-1} = (a_{12} + b_{12}\,\mathfrak{s}_{12}) \cdots (a_{1\,n-1} + b_{1\,n-1}\,\mathfrak{s}_{1\,n-1})(a_{1n} + b_{1n}\,\mathfrak{s}_{1n})$.

(ii) $cS^{-1} = (\alpha_{n1} + \beta_{n1}\,\mathfrak{s}_{1n})(\alpha_{n-1\,1} + \beta_{n-1\,1}\,\mathfrak{s}_{1\,n-1}) \cdots (\alpha_{21} + \beta_{21}\,\mathfrak{s}_{12})$.

(iii) $cS^\dagger = (a_{n1} + b_{n1}\,\mathfrak{s}_{1n})(a_{n-1\,1} + b_{n-1\,1}\,\mathfrak{s}_{n-1\,1}) \cdots (a_{21} + b_{21}\,\mathfrak{s}_{21})$.

Proof: The proofs are all similar, so we proceed only with Sc^{-1}. The strategy is to express $S = T_1 \cdots T_{n-1}$ as a product of factors of the form (4.21), pull out a factor of $\mathfrak{s}_{i\,i+1}$ from the right-hand side of each of these and push them all the way through to cancel the c^{-1}. Indeed,

$$\begin{aligned} Sc^{-1} &= T_1T_2 \cdots T_{n-1}c^{-1} \\ &= (b_{12} + a_{12}\,\mathfrak{s}_{12})(b_{23} + a_{23}\,\mathfrak{s}_{23}) \cdots (b_{n-1\,n} + a_{n-1\,n}\,\mathfrak{s}_{n-1\,n})c^{-1} \\ &= (a_{12} + b_{12}\,\mathfrak{s}_{12})\,\mathfrak{s}_{12}(a_{23} + b_{23}\,\mathfrak{s}_{23})\,\mathfrak{s}_{23} \cdots (a_{n-1\,n} + b_{n-1\,n}\,\mathfrak{s}_{n-1\,n})\,\mathfrak{s}_{n-1\,n}\,c^{-1} \\ &= (a_{12} + b_{12}\,\mathfrak{s}_{12}) \cdots (a_{1\,n-1} + b_{1\,n-1}\,\mathfrak{s}_{1\,n-1})(a_{1n} + b_{1n}\,\mathfrak{s}_{1n})cc^{-1}. \end{aligned}$$ □

Proposition 4.22 *The restrictions of the following elements onto M' are given as follows:*

(i) Sc^{-1} *has the form* $A + \sum_{i\neq 1}^{n} B_i\,\mathfrak{s}_{1i}$, *where* $A = \prod_{k\neq 1}^{n} a_{1k}$ *and* $B_i = b_{1i} \prod_{k\neq 1,i}^{n} a_{ik}$.

(ii) cS^{-1} *has the form* $C + \sum_{i\neq 1}^{n} D_i\,\mathfrak{s}_{1i}$, *where* $C = \prod_{k\neq 1}^{n} \alpha_{k1}$ *and* $D_i = \beta_{i1} \prod_{k\neq 1,i}^{n} \alpha_{ki}$.

(iii) $cS^\dagger$ *has the form* $E + \sum_{i\neq 1}^{n} F_i\,\mathfrak{s}_{1i}$, *where* $E = \prod_{k\neq 1}^{n} a_{k1}$ *and* $F_i = b_{i1} \prod_{k\neq 1,i}^{n} a_{ki}$.

Proof: (i) Looking at Lemma 4.21(i), the term corresponding to the trivial coset comes from selecting the first summand in each factor, that is by selecting a_{1k} for each $k = 2, \ldots, n$, giving the expression for A. Next, B_2 (the coefficient of the representative of the coset $\mathfrak{s}_{12}$) is determined by selecting $b_{12}\,\mathfrak{s}_{12}$ from the first factor and a_{1k} for each $k = 3, \dots, n$ from each of the remaining factors, and pushing $\mathfrak{s}_{12}$ through. Hence, we obtain

$$B_2 = b_{12} \prod_{k \neq 1,2}^{n} a_{2k}.$$

It remains to note that $\mathfrak{s}_{ik}\Big(A + \sum_{i\neq 1}^n B_i\,\mathfrak{s}_{1i}\Big)e' = \Big(A + \sum_{i\neq 1}^n B_i\,\mathfrak{s}_{1i}\Big)e'$, which means B_i can be obtained from B_2 via the action of $\mathfrak{s}_{2i}$. This gives the formula presented in the statement.

(ii) The proof is near-identical to the first case (i). Looking at Lemma 4.21(ii), we can determine C straightforwardly. The main difference is we first obtain D_n (instead of D_2, as one may initially expect), and then hit it with the action of $\mathfrak{s}_{ni}$ to obtain the expression for $D_i = (D_n)^{\mathfrak{s}_{ni}}$.

(iii) This is very similar to (ii), wherein we can determine E quickly from Lemma 4.21(iii) and then find F_n just as easily. We conclude by obtaining via the $\mathfrak{s}_{ni}$-action, that is $F_i = (F_n)^{\mathfrak{s}_{ni}}$. □

Remark 4.23 Recall from §2.1 the DAHA parameters τ_i can be expanded to the corresponding (affine) root system by setting $\tau_\alpha \coloneqq \tau_{w(\alpha)}$ for an (affine) Weyl element w. Therefore, we can extend the indices of $\beta(T_i)$ from Proposition 2.8 to any (affine) root. This is called the *$\mathcal{R}$-matrix*

$$\mathcal{R}(\alpha) \coloneqq \tau_\alpha s_\alpha + b_\alpha(\boldsymbol{X})(1 - s_\alpha),$$

where the s_α is the orthogonal reflection with respect to the hyperplane $\alpha(x) = 0$. Their name comes from the fact that they solve a version of the Yang-Baxter equation. In particular, if one considers the $\mathcal{R}$-matrix associated to an affine simple root a_i for $i = 0, \dots, n$, then we see that

$$\mathcal{R}(a_i) = T_i\,\mathfrak{s}_{i\,i+1} \qquad \Leftrightarrow \qquad T_i = \mathcal{R}(a_i)\,\mathfrak{s}_{i\,i+1}\,.$$

This is the language preferred in [Cha19, §3.2]. It also alleviates some notation. Indeed,

$$\mathcal{R}_{ij} \coloneqq \mathcal{R}(\varepsilon_i - \varepsilon_j) = a_{ij} + b_{ij}\,\mathfrak{s}_{ij} \qquad \text{and} \qquad \mathcal{R}^+_{ij} \coloneqq \mathcal{R}(\varepsilon_i + \varepsilon_j) = a^+_{ij} + b^+_{ij}\,\mathfrak{s}^+_{ij},$$

where the rational functions are those from (4.22) and Notation 4.19. For $i = 1, \dots, n-1$,

$$T_i = \mathcal{R}_{i\,i+1}\,\mathfrak{s}_{i\,i+1},$$

which we substitute into the Hecke relation to determine the inverse $\mathcal{R}$-matrix representing T_i:

$$\begin{aligned}
(T_i - t)(T_i + t^{-1}) = 0 \quad &\Leftrightarrow \quad (\mathcal{R}_{i\,i+1}\,\mathfrak{s}_{i\,i+1} - t)(\mathcal{R}_{i\,i+1}\,\mathfrak{s}_{i\,i+1} + t^{-1}) = 0 \\
&\Leftrightarrow \quad \mathcal{R}_{i\,i+1}\,\mathcal{R}_{i+1\,i} - 1 + (t^{-1}\,\mathcal{R}_{i\,i+1} - t\,\mathcal{R}_{i\,i+1})\,\mathfrak{s}_{i\,i+1} = 0 \\
&\Leftrightarrow \quad \mathcal{R}_{i\,i+1}(\mathcal{R}_{i+1\,i} - \bar{t}\,\mathfrak{s}_{i\,i+1}) = 1.
\end{aligned}$$

In fact, this can be generalised to any two indices, giving us the following expression:

$$\mathcal{R}_{ij}^{-1} = (\mathcal{R}_{ij})^{\mathfrak{s}_{ij}} - \bar{t}\,\mathfrak{s}_{ij},$$

Proceeding similarly for $\mathcal{R}_{ij}^{+}$, we express these inverses in terms of the functions from (4.24):

$$\mathcal{R}_{ij}^{-1} = \alpha_{ji} + \beta_{ji}\,\mathfrak{s}_{ij} \qquad \text{and} \qquad (\mathcal{R}_{ij}^{+})^{-1} = \alpha_{ij}^{+} + \beta_{ij}^{+}\,\mathfrak{s}_{ij}^{+}\,.$$

Lemma 4.24 *We have the following $\mathcal{R}$-matrix expressions of the factors in* (4.19)*:*

$$Sc^{-1} = \mathcal{R}_{12}\,\mathcal{R}_{13}\cdots\mathcal{R}_{1n}, \qquad cS^{-1} = \mathcal{R}_{1n}^{-1}\,\mathcal{R}_{1\,n-1}^{-1}\cdots\mathcal{R}_{12}^{-1}, \qquad cS^{\dagger} = \mathcal{R}_{n1}\,\mathcal{R}_{n-1\,1}\cdots\mathcal{R}_{21}\,.$$

Proof: This is an easy change of notation, in light of the expressions from Remark 4.23. □

The $\mathcal{R}$-matrix digression now comes to a close. Next, there are two remaining factors in each of (4.19) that we are yet to discuss: $cT_n^{\vee}c^{-1}$ and cT_nc^{-1}. Recalling $T_n^{\vee} = X_n^{-1}T_n^{-1}$ from Definition 3.2, we now introduce yet more notation, this time specific to the element T_n. To that end, let

$$T_n = b_n + a_n\,\mathfrak{s}_n,$$

where, for $i = n$,

$$a_i = \frac{k_n^{-1} - k_nX_i^2 - \overline{u}_nX_i}{1 - X_i^2} \qquad \text{and} \qquad b_i = \frac{\overline{k}_n + \overline{u}_n}{1 - X_i^2}. \tag{4.25}$$

Note the need for only a single index. In parallel with Notation 4.19, we see that $a_i = a_i^{+}$ and $b_i = b_i^{+}$. Consequently, the only additional notation we may require is following shorthand:

$$a_i^{-} := \frac{k_n^{-1} - k_nX_i^{-2} - \overline{u}_nX_i^{-1}}{1 - X_i^{-2}} \qquad \text{and} \qquad b_i^{-} := \frac{\overline{k}_n + \overline{u}_n}{1 - X_i^{-2}}.$$

As before, we can also write the inverse of T_n using expressions similar to those in (4.25). Namely,

$$T_n^{-1} = \beta_n + \alpha_n\,\mathfrak{s}_n,$$

where, for $i = n$,

$$\alpha_i = \frac{k_n^{-1} - k_nX_i^2 - \overline{u}_nX_i}{1 - X_i^2} \qquad \text{and} \qquad \beta_i = \frac{\overline{k}_nX_i^2 + \overline{u}_nX_i}{1 - X_i^2}. \tag{4.26}$$

We can now carefully compute the expressions of the middle factors using this notation. The

A_4-factor is easiest because we can push c through to the right-hand side, which leaves us with

$$cT_n c^{-1} = b_1 + a_1\,\mathfrak{s}_1\,. \tag{4.27}$$

It isn't much extra work for the A_3-factor; using the fact that $T_n^\vee = X_n^{-1}T_n^{-1}$, we can substitute the expression for T_n^{-1} and push through c to produce $cT_n^\vee c^{-1} = X_1^{-1}(\beta_1+\alpha_1\,\mathfrak{s}_1)$. We can relabel $X_1^{-1}\beta_1$ and $X_1^{-1}\alpha_1$ as new Greek letters, and multiply by a suitable power of X_1 so that all such terms have positive powers (so we can still follow the convention of Notation 4.19). Thus, we get

$$cT_n^\vee c^{-1} = \delta_1 + \gamma_1\,\mathfrak{s}_1, \tag{4.28}$$

where

$$\gamma_i := \frac{k_n^{-1} - k_n X_i^2 - \overline{u}_n X_i}{X_i(1-X_i^2)} \qquad \text{and} \qquad \delta_i = \frac{\overline{k}_n X_i + \overline{u}_n}{1-X_i^2}. \tag{4.29}$$

Combining (4.28) with Proposition 4.22, the restrictions of Z_3 and Z_4 from (4.19) to M' can now be written more explicitly. Note that this involves the cosets $\mathfrak{s}_{1i}$ and $\mathfrak{s}_1$. However, the general element (4.18) involves also the cosets $\mathfrak{s}_{1i}^+$. Therefore, it is convenient to input $\mathfrak{s}_1\,\mathfrak{s}_1 = \mathrm{id}$ between the second and third factors. Pulling one into the second factor and pushing the other through the third factor produces

$$Z_3\Big|_{M'} = \left(A + \sum_{i\neq 1}^{n} B_i\,\mathfrak{s}_{1i}\right)(\gamma_1 + \delta_1\,\mathfrak{s}_1)\left((C)^{\mathfrak{s}_1} + \sum_{i\neq 1}^{n}(D_i)^{\mathfrak{s}_1}\,\mathfrak{s}_{1i}^+\right)\mathfrak{s}_1 \tag{4.30}$$

and

$$Z_4\Big|_{M'} = \left(A + \sum_{i\neq 1}^{n} B_i\,\mathfrak{s}_{1i}\right)(a_1 + b_1\,\mathfrak{s}_1)\left((E)^{\mathfrak{s}_1} + \sum_{i\neq 1}^{n}(F_i)^{\mathfrak{s}_1}\,\mathfrak{s}_{1i}^+\right)\mathfrak{s}_1\,. \tag{4.31}$$

Lemma 4.25 *The element (4.30) can be written as* $U + V\,\mathfrak{s}_1 + \sum_{i\neq 1}^{n}(W_i\,\mathfrak{s}_{1i} + W_i^+\,\mathfrak{s}_{1i}^+)$, *where*

$$U = X_1^{-1}k_n^{-1}t^{2-2n} - V - \sum_{i\neq 1}^{n}(W_i + W_i^+), \qquad V = \gamma_1 \prod_{k\neq 1}^{n} a_{1k}\alpha_{1k}^+,$$

$$W_i = \gamma_i^- b_{1i}^+\alpha_{1i} \prod_{k\neq 1,i}^{n} a_{ki}^-\alpha_{ki}, \qquad W_i^+ = \gamma_i b_{1i}\alpha_{1i}^+ \prod_{k\neq 1,i}^{n} a_{ik}\alpha_{ik}^+,$$

and the element (4.31) *can be written as* $\mathcal{A}+\mathcal{B}\,\mathfrak{s}_1+\sum_{i\neq 1}^{n}(\mathcal{C}_i\,\mathfrak{s}_{1i}+\mathcal{C}_i^+\,\mathfrak{s}_{1i}^+)$*, where*

$$\mathcal{A}=k_nt^{2n-2}-\mathcal{B}-\sum_{i\neq 1}^{n}(\mathcal{C}_i+\mathcal{C}_i^+),\qquad \mathcal{B}=a_1\prod_{k\neq 1}^{n}a_{1k}a_{1k}^+,$$

$$\mathcal{C}_i=a_i^-b_{1i}^+a_{1i}\prod_{k\neq 1,i}^{n}a_{ki}^-a_{ki},\qquad \mathcal{C}_i^+=a_ib_{1i}a_{1i}^+\prod_{k\neq 1,i}^{n}a_{ik}a_{ik}^+.$$

Proof: The structure of these proofs is very much the same, and a close relative of the latter is done in [Cha19, Lemma 4.2]. We therefore provide the proof of the former restriction. Let's initially focus only on the first three factors in brackets, that is temporarily ignoring the $\mathfrak{s}_1$ at the end. If the lemma is to be believed, these three factors can be written as

$$V+U\,\mathfrak{s}_1+\sum_{i\neq 1}^{n}(W_i^+\,\mathfrak{s}_{1i}+W_i\,\mathfrak{s}_{1i}^+),$$

after which incorporating the final factor $\mathfrak{s}_1$ on the right will give the result. It suffices to focus on just these, then. We see that the trivial coset is represented if and only if we select A, γ_1, $(C)^{\mathfrak{s}_1}$ in the factors of (4.30), which immediately implies the formula for V. Next, the only way to represent the $\mathfrak{s}_{1i}$-coset is to select $B_i\,\mathfrak{s}_{1i}$, γ_1, $(C)^{\mathfrak{s}_1}$ in the factors of (4.30). Pushing the $\mathfrak{s}_{1i}$ through then gives the expression for W_i^+. We showed in Lemma 4.11 that Z_3 preserves M', meaning it is invariant under the action of the Weyl subgroup W'. Therefore, $W_i=(W_i^+)^{\mathfrak{s}_i}$. Finally, let $e\in\mathbb{C}W$ be the full symmetriser. On one hand, applying the operator e to the restriction, all group elements are absorbed and we are left with

$$\left(V+U\,\mathfrak{s}_1+\sum_{i\neq 1}^{n}(W_i^+\,\mathfrak{s}_{1i}+W_i\,\mathfrak{s}_{1i}^+)\right)e=\left(V+U+\sum_{i\neq 1}^{n}(W_i^++W_i)\right)e.$$

On the other hand, $T_ie=\tau_ie$ for each i (from which the Hecke relation implies $T_i^{-1}e=\tau_i^{-1}e$) and $T_iX_{i+1}^{-1}=X_i^{-1}T_i^{-1}$, which is an equivalent form of the cross relation (ix) in Definition 3.2. Repeatedly applying this fact, it is then straightforward to see that $(ST_n^\vee S^{-1})e$ is precisely

$$(T_1\cdots T_{n-1}X_n^{-1}T_n^{-1}T_{n-1}^{-1}\cdots T_1^{-1})e=(X_1^{-1}T_1^{-1}\cdots T_{n-1}^{-1}T_n^{-1}T_{n-1}^{-1}\cdots T_1^{-1})e=X_1^{-1}k_n^{-1}t^{2-2n}e.$$

Combining these two avenues of thought, we conclude that

$$V+U+\sum_{i\neq 1}^{n}(W_i^++W_i)=X_1^{-1}k_n^{-1}t^{2-2n}.\qquad\square$$

We are now able to reap the rewards of this technical lemma and find the remaining matrices.

Theorem 4.26 *The matrix representing $Z_3 = ST_n^{\vee}S^{-1}$ is that whose ij^{th} entry is*

$$(A_3)_{ij} = \begin{cases} \gamma_j^- \sideset{}{^{\diamond}}\prod_{k=1}^{2n} a_{jk}^- & \text{if } i-j=\pm n \\ \gamma_j^- b_{ij}^+ a_{ij} \sideset{}{^{\diamond}}\prod_{k=1}^{2n} a_{jk}^- & \text{if } i-j\neq 0,\pm n \\ X_i^{-1}k_n^{-1}t^{2-2n} - \sum_{k\neq i}(A_3)_{ik} & \text{if } i=j \end{cases},$$

and the matrix representing $Z_4 = ST_nS^{\dagger}$ is that whose ij^{th} entry is

$$(A_4)_{ij} = \begin{cases} a_j^- \sideset{}{^{\diamond}}\prod_{k=1}^{2n} a_{jk}^- & \text{if } i-j=\pm n \\ a_j^- b_{ij}^+ a_{ij} \sideset{}{^{\diamond}}\prod_{k=1}^{2n} a_{jk}^- & \text{if } i-j\neq 0,\pm n \\ k_n t^{2n-2} - \sum_{k\neq i}(A_4)_{ik} & \text{if } i=j \end{cases}$$

extending the indices from $\{1,\dots,n\}$ to $\{1,\dots,2n\}$ by setting $X_{n+i} \coloneqq X_i^{-1}$. The symbol $\sideset{}{^{\diamond}}\prod$ means we take a product but exclude the values of k for which $k-i=0,\pm n$ and $k-j=0,\pm n$.

Proof: This is a consequence of using Proposition 4.12, with Z being each of the expressions in Lemma 4.25. The latter is again closely related to a result by Chalykh, namely [Cha19, Proposition 4.3], so we omit the easy translation between that work and this. Instead, we proceed with a full derivation of A_3. Let's begin by reminding the reader of Remark 4.20, wherein we explain that $\alpha = a$; this replacement is henceforth made ubiquitously when referring back to the content of Lemma 4.25. We first work with the off-diagonal entries ($j = i \pm n$), encoded by

$$(V)^{\mathfrak{s}_{1j}\,\mathfrak{s}_1} = \left(\gamma_1^- \prod_{k\neq 1}^{n} a_{1k}^- a_{k1}\right)^{\mathfrak{s}_{1j}} = \gamma_j^- \prod_{k\neq j}^{n} a_{jk}^- a_{kj}$$

and

$$(V)^{\mathfrak{s}_{1j}^+\,\mathfrak{s}_1} = \left(\gamma_1^- \prod_{k\neq 1}^{n} a_{1k}^- a_{k1}\right)^{\mathfrak{s}_{1j}^+} = \gamma_j \prod_{k\neq j}^{n} a_{jk} a_{kj}^+.$$

Note that these belong to the off-diagonals in the lower-left and upper-right blocks, respectively. But because we extend the indices by setting $\varepsilon_{n+j} \coloneqq -\varepsilon_j$ and the upper-right block has columns indexed by $n+j$, changing the index amounts to applying $\mathfrak{s}_j$. Explicitly, this produces for us

$$(V)^{\mathfrak{s}_{1j}^+\,\mathfrak{s}_1} \overset{\mathfrak{s}_j}{\longmapsto} \gamma_j^- \prod_{k\neq j}^{n} a_{jk}^- a_{kj}. \tag{4.32}$$

Hence, all off-diagonal entries of A_3 are described by (4.32). But notice this can be re-written as

$$\gamma_j^- \prod_{\substack{k=1\\k\neq j}}^{n} a_{jk}^- \prod_{\substack{k=1\\k\neq j}}^{n} a_{kj} = \gamma_j^- \prod_{\substack{k=1\\k\neq j}}^{n} a_{jk}^- \prod_{\substack{k=n+1\\k\neq j\pm n}}^{2n} a_{jk}^- = \gamma_j^- \prod_{\substack{k=1\\k-j\neq 0,\pm n}}^{2n} a_{jk}^- =: \gamma_j^- \prod_{k=1}^{2n}{}^{\diamond}\, a_{jk}^-$$

by again using the index change $\varepsilon_{n+j} \coloneqq -\varepsilon_j$. We now move onto the non-diagonal entries of the matrix ($i \neq j$ and $i \neq j \pm n$). Proceeding as for the off-diagonals, these entries are encoded by

$$(W_i)^{\mathfrak{s}_{1j}\,\mathfrak{s}_{1i}} = \left(\gamma_1^- b_{1i}^+ a_{i1} \prod_{k\neq 1,i}^{n} a_{k1}^- a_{k1}\right)^{\mathfrak{s}_{1j}} = \gamma_j^- b_{ji}^+ a_{ij} \prod_{k\neq i,j}^{n} a_{kj}^- a_{kj},$$

$$(W_i)^{\mathfrak{s}_{1j}^+\,\mathfrak{s}_{1i}} = \left(\gamma_1^- b_{1i}^+ a_{i1} \prod_{k\neq 1,i}^{n} a_{k1}^- a_{k1}\right)^{\mathfrak{s}_{1j}^+} = \gamma_j b_{ij} a_{ij}^+ \prod_{k\neq i,j}^{n} a_{jk} a_{kj}^+,$$

$$(W_i^+)^{\mathfrak{s}_{1j}\,\mathfrak{s}_{1i}^+} = \left(\gamma_1^- b_{1i} a_{1i}^- \prod_{k\neq 1,i}^{n} a_{1k}^- a_{k1}\right)^{\mathfrak{s}_{1j}} = \gamma_j^- b_{ji} a_{ji}^- \prod_{k\neq i,j}^{n} a_{jk}^- a_{kj}$$

and

$$(W_i^+)^{\mathfrak{s}_{1j}^+\,\mathfrak{s}_{1i}^+} = \left(\gamma_1^- b_{1i} a_{1i}^- \prod_{k\neq 1,i}^{n} a_{1k}^- a_{k1}\right)^{\mathfrak{s}_{1j}^+} = \gamma_j b_{ji}^- a_{ji} \prod_{k\neq i,j}^{n} a_{jk} a_{kj}^+.$$

These belong in the upper-left, upper-right, lower-left and lower-right blocks, respectively. We again use the change of indices to see if there is a common formula describing all such entries. The upper-right is indexed by $j+n$, so changing the index amounts to applying $\mathfrak{s}_j$. Similarly, the lower-left is indexed by $i+n$, so changing the index amounts to applying $\mathfrak{s}_i$; the lower-right is indexed by $i+n$ **and** $j+n$, so changing indices amounts to applying $\mathfrak{s}_i\,\mathfrak{s}_j$. Hence, we obtain

$$\begin{aligned}
(W_i)^{\mathfrak{s}_{1j}^+\,\mathfrak{s}_{1i}} &\overset{\mathfrak{s}_j}{\longmapsto} \gamma_j^- b_{ij}^+ a_{ij} \prod_{k\neq i,j}^{n} a_{jk}^- a_{kj},\\
(W_i^+)^{\mathfrak{s}_{1j}\,\mathfrak{s}_{1i}^+} &\overset{\mathfrak{s}_i}{\longmapsto} \gamma_j^- b_{ji}^+ a_{ij} \prod_{k\neq i,j}^{n} a_{jk}^- a_{kj},\\
(W_i^+)^{\mathfrak{s}_{1j}^+\,\mathfrak{s}_{1i}^+} &\overset{\mathfrak{s}_i\,\mathfrak{s}_j}{\longmapsto} \gamma_j^- b_{ji}^+ a_{ij} \prod_{k\neq i,j}^{n} a_{jk}^- a_{kj}.
\end{aligned} \tag{4.33}$$

By the index symmetry of the rational functions with superscripts $^{\pm}$, the three expressions in

(4.33) coincide (and equal the expression for the upper-left entries). Hence, all non-diagonal entries of A_3 are described by this common expression. But notice this can be re-written as

$$\gamma_j^- b_{ij}^+ a_{ij} \prod_{\substack{k=1\\k\neq i,j}}^{n} a_{jk}^- \prod_{\substack{k=1\\k\neq i,j}}^{n} a_{kj} = \gamma_j^- b_{ij}^+ a_{ij} \prod_{\substack{k=1\\k\neq i,j}}^{n} a_{jk}^- \prod_{\substack{k=n+1\\k\neq i,j\pm n}}^{2n} a_{kj}^- = \gamma_j^- b_{ij}^+ a_{ij} \prod_{\substack{k=1\\k-i\neq 0,\pm n\\k-j\neq 0,\pm n}}^{2n} a_{jk}^- =: \gamma_j^- b_{ij}^+ a_{ij} \prod_{k=1}^{2n}{}^{\diamond}\, a_{jk}^-.$$

The final entries to consider are those on the main diagonal ($i = j$), which are encoded by

$$(U)^{\mathfrak{s}_{1j}} = X_j^{-1} k_n^{-1} t^{2-2n} - (V)^{\mathfrak{s}_{1j}} - \sum_{i\neq j}^{n} \Big((W_i)^{\mathfrak{s}_{1j}} + (W_i^+)^{\mathfrak{s}_{1j}} \Big)$$

and

$$(U)^{\mathfrak{s}_{1j}^+} = X_j k_n^{-1} t^{2-2n} - (V)^{\mathfrak{s}_{1j}^+} - \sum_{i\neq j}^{n} \Big((W_i)^{\mathfrak{s}_{1j}^+} + (W_i^+)^{\mathfrak{s}_{1j}^+} \Big).$$

These are the entries in the upper-left and lower-right blocks. But since the lower-right block is indexed by $i + n$ **and** $j + n$, changing indices amounts to applying $\mathfrak{s}_i\, \mathfrak{s}_j$. Explicitly, this produces

$$(U)^{\mathfrak{s}_{1j}^+} \xmapsto{\mathfrak{s}_i\, \mathfrak{s}_j} X_j^{-1} k_n^{-1} t^{2-2n} - (V)^{\mathfrak{s}_{1j}} - \sum_{i\neq j}^{n} \Big((W_i^+)^{\mathfrak{s}_{1j}^+} + (W_i)^{\mathfrak{s}_{1j}} \Big). \tag{4.34}$$

Hence, all diagonal entries of A_3 are described by this common expression. Now, it is not difficult to see that $(V)^{\mathfrak{s}_{1j}}$ is described by (4.32), and both of $(W_i)^{\mathfrak{s}_{1j}}$ and $(W_i)^{\mathfrak{s}_{1j}^+}$ are described by (4.33). Be aware that these equalities are true **except** with i and j interchanged (but this matters not since $i = j$ here). Lastly, (4.34) can be re-written in the form as stated. □

Corollary 4.27 *The product of the matrices $qA_1A_2A_3A_4 = \mathbb{1}_{2n}$.*

Proof: This is clear by applying the representation (4.14) to the relation in Lemma 3.4. □

One can give a cleaner alternate proof that A_3 has the form stated in Theorem 4.26, in light of Corollary 4.27. Indeed, $Z_1Z_2 = qT_0T_0^\vee = X_1$, whose matrix we derived in Example 4.13. Thus,

$$A_1A_2A_3A_4 = XA_3A_4 = \mathbb{1}_{2n} \qquad \Rightarrow \qquad A_3 = X^{-1}A_4^{-1}.$$

We can use Proposition 4.22 and (4.27) to write an analogue of (4.31) but this time for Z_4^{-1}.

Notation 4.28 Recall that $\alpha = a$ are the same rational functions, noted in Remark 4.20, but it is often convenient to track whether we are talking about the restriction of an element onto M' vs. its inverse onto M'. Thus, we use blackboard bold letters to denote replacing the functions

a's and b's with their Greek counterparts α's and β's. For convenience, here they are explicitly:

$$A = \prod_{k\neq 1}^{n} a_{1k} \quad \text{and} \quad B_i = b_{1i} \prod_{k\neq 1,i}^{n} a_{ik} \qquad \text{become} \qquad \mathbb{A} = \prod_{k\neq 1}^{n} \alpha_{1k} \quad \text{and} \quad \mathbb{B}_i = \beta_{1i} \prod_{k\neq 1,i}^{n} \alpha_{ik},$$

$$E = \prod_{k\neq 1}^{n} a_{k1} \quad \text{and} \quad F_i = b_{i1} \prod_{k\neq 1,i}^{n} a_{ki} \qquad \text{become} \qquad \mathbb{E} = \prod_{k\neq 1}^{n} \alpha_{k1} \quad \text{and} \quad \mathbb{F}_i = \beta_{i1} \prod_{k\neq 1,i}^{n} \alpha_{ki}.$$

Therefore, one can check (using the $\mathcal{R}$-matrix description from Remark 4.23, for instance) that

$$Z_4^{-1}\Big|_{M'} = \left(\mathbb{A} + \sum_{i\neq 1}^{n} \mathbb{B}_i\, \mathfrak{s}_{1i}\right)(\alpha_1 + \beta_1\, \mathfrak{s}_1)\left((\mathbb{E})^{\mathfrak{s}_1} + \sum_{i\neq 1}^{n} (\mathbb{F}_i)^{\mathfrak{s}_1}\, \mathfrak{s}_{1i}^{+}\right)\mathfrak{s}_1\,. \tag{4.35}$$

Lemma 4.29 *The element (4.35) can be written as* $\mathcal{U} + \mathcal{V}\,\mathfrak{s}_1 + \sum_{i\neq 1}^{n}(\mathcal{W}_i\,\mathfrak{s}_{1i} + \mathcal{W}_i^{+}\,\mathfrak{s}_{1i}^{+})$*, where*

$$\mathcal{U} = k_n^{-1} t^{2-2n} - \mathcal{V} - \sum_{i\neq 1}^{n}(\mathcal{W}_i + \mathcal{W}_i^{+}), \qquad \mathcal{V} = \alpha_1 \prod_{k\neq 1}^{n} \alpha_{1k}\alpha_{1k}^{+},$$

$$\mathcal{W}_i = \alpha_i^{-}\beta_{1i}^{+}\alpha_{1i} \prod_{k\neq 1,i}^{n} \alpha_{ki}^{-}\alpha_{ki}, \qquad \mathcal{W}_i^{+} = \alpha_i\beta_{1i}\alpha_{1i}^{+} \prod_{k\neq 1,i}^{n} \alpha_{ik}\alpha_{ik}^{+}.$$

Proof: The structure is identical to that of the proof of Lemma 4.25. As before, we initially focus only on the first three factors and temporarily ignore the $\mathfrak{s}_1$ at the right-hand end. Our goal is therefore to prove that the first three factors can be written as

$$\mathcal{V} + \mathcal{U}\,\mathfrak{s}_1 + \sum_{i\neq 1}^{n}(\mathcal{W}_i^{+}\,\mathfrak{s}_{1i} + \mathcal{W}_i\,\mathfrak{s}_{1i}^{+}),$$

after which incorporating the final factor $\mathfrak{s}_1$ on the right will give the result. To this end, we see that the trivial cost is represented if and only if we select $\mathbb{A}, \alpha_1, (\mathbb{E})^{\mathfrak{s}_1}$, which immediately implies the formula for $\mathcal{V}$. Next, the only way to represent the $\mathfrak{s}_{1i}$-coset is to select $\mathbb{B}_i\,\mathfrak{s}_{1i}, \alpha_1, (\mathbb{E})^{\mathfrak{s}_1}$; pushing the $\mathfrak{s}_{1i}$ through then gives the expression for $\mathcal{W}_i^{+}$. Because (4.35) is invariant under the action of the Weyl subgroup W', we have $\mathcal{W}_i = (\mathcal{W}_i^{+})^{\mathfrak{s}_i}$. Finally, let $e \in \mathbb{C}W$ be the full symmetriser. Applying this to the restriction, every group element is absorbed, which means

$$\left(\mathcal{V} + \mathcal{U}\,\mathfrak{s}_1 + \sum_{i\neq 1}^{n}(\mathcal{W}_i^{+}\,\mathfrak{s}_{1i} + \mathcal{W}_i\,\mathfrak{s}_{1i}^{+})\right)e = \left(\mathcal{V} + \mathcal{U} + \sum_{i\neq 1}^{n}(\mathcal{W}_i^{+} + \mathcal{W}_i)\right)e.$$

On the other hand, recall that $T_i e = \tau_i e$ where τ_i is the parameter corresponding to this Hecke element. Applying this to Z_4^{-1}, we obtain $(ST_nS^\dagger)^{-1}e = (t^{1-n}k_n^{-1}t^{1-n})e = k_n^{-1}t^{2-2n}e$. Hence,

$$\mathcal{V} + \mathcal{U} + \sum_{i \neq 1}^{n} (\mathcal{W}_i^+ + \mathcal{W}_i) = k_n^{-1}t^{2-2n}. \qquad \square$$

Corollary 4.30 *The matrix representing $Z_4^{-1} = (ST_nS^\dagger)^{-1}$ is that whose ij^{th} entry is*

$$(A_4^{-1})_{ij} = \begin{cases} \alpha_j^- \mathop{\prod\hspace{-0.9em}\diamond\hspace{0.4em}}\limits_{k=1}^{2n} \alpha_{jk}^- & \text{if } i-j = \pm n \\ \alpha_j^- \beta_{ij}^+ \alpha_{ij} \mathop{\prod\hspace{-0.9em}\diamond\hspace{0.4em}}\limits_{k=1}^{2n} \alpha_{jk}^- & \text{if } i-j \neq 0, \pm n \\ k_n^{-1}t^{2-2n} - \sum\limits_{k \neq i} (A_4^{-1})_{ik} & \text{if } i = j \end{cases}$$

extending the indices from $\{1, \ldots, n\}$ to $\{1, \ldots, 2n\}$ by setting $X_{n+i} := X_i^{-1}$. The symbol $\prod\hspace{-0.9em}\diamond\hspace{0.4em}$ means we take a product but exclude the values of k for which $k - i = 0, \pm n$ and $k - j = 0, \pm n$.

Sketch of Proof: Follow the same strategy as in the proof of Theorem 4.26. $\square$

Thus, $(A_3)_{ij} = (X^{-1}A_4^{-1})_{ij} = (X^{-1})_{i\ell}(A_4^{-1})_{\ell j}$. But $X^{-1} = \mathrm{diag}(X_1^{-1}, \ldots, X_n^{-1}, X_1, \ldots, X_n)$ as a trivial corollary of Example 4.13, from which we can write its entries $(X^{-1})_{i\ell} = \delta_{i\ell}X_i^{-1}$ with the usual extended index convention. Given this is non-zero if and only if $i = \ell$, we conclude that

$$(A_3)_{ij} = \begin{cases} X_i^{-1}\alpha_j^- \mathop{\prod\hspace{-0.9em}\diamond\hspace{0.4em}}\limits_{k=1}^{2n} \alpha_{jk}^- & \text{if } i-j = \pm n \\ X_i^{-1}\alpha_j^- \beta_{ij}^+ \alpha_{ij} \mathop{\prod\hspace{-0.9em}\diamond\hspace{0.4em}}\limits_{k=1}^{2n} \alpha_{jk}^- & \text{if } i-j \neq 0, \pm n \\ X_i^{-1}k_n^{-1}t^{2-2n} - \sum\limits_{k \neq i} X_i^{-1}(A_4^{-1})_{ik} & \text{if } i = j \end{cases}$$

$$= \begin{cases} X_j a_j^- \mathop{\prod\hspace{-0.9em}\diamond\hspace{0.4em}}\limits_{k=1}^{2n} a_{jk}^- & \text{if } i-j = \pm n \\ X_j a_j^- b_{ij}^+ a_{ij} \mathop{\prod\hspace{-0.9em}\diamond\hspace{0.4em}}\limits_{k=1}^{2n} a_{jk}^- & \text{if } i-j \neq 0, \pm n \\ X_i^{-1}k_n^{-1}t^{2-2n} - \sum\limits_{k \neq i} (X_i^{-1})_{ii}(A_4^{-1})_{ik} & \text{if } i = j \end{cases}$$

$$= \begin{cases} \gamma_j^- \prod_{k=1}^{2n} a_{jk}^- & \text{if } i-j=\pm n \\ \gamma_j^- b_{ij}^+ a_{ij} \prod_{k=1}^{2n} a_{jk}^- & \text{if } i-j\neq 0,\pm n, \\ X_i^{-1}k_n^{-1}t^{2-2n} - \sum_{k\neq i}(A_3^{-1})_{ik} & \text{if } i=j \end{cases}$$

which agrees verbatim with Theorem 4.26. Note we have again used $\alpha = a$ (see Remark 4.20), $X_i^{-1} = X_j$ when $i-j=\pm n$, and also the identities $X_j a_j^- = \gamma_j^-$ and $X_i^{-1}\beta_{ij}^+ = X_j b_{ij}^+$, which are obvious when you view them explicitly using the expressions (4.22), (4.24), (4.25) and (4.29).

We now want to ensure that the conjugacy classes containing each of the classical ($q=1$) matrices A_1, A_2, A_3 and A_4 calculated in Theorems 4.16 and 4.26 are the correct ones as defined in (4.1). It turns out to be a straightforward task for A_1 and A_2 in light of the faithfulness of the matrix representation (4.14), and A_3 given a result by Stokman. Indeed, A_1, A_2 and A_3 not only satisfy the product relation $qA_1A_2A_3A_4 = \mathbb{1}_{2n}$ (Corollary 4.27), but also the respective quadratic relations of T_0, $T_0^\vee$ and $T_n^\vee$ (the latter of which is due to [Sto04, Proposition 6.6]), i.e.

$$A_1 - A_1^{-1} = \overline{k}_0\mathbb{1}_{2n}, \qquad A_2 - A_2^{-1} = \overline{u}_0\mathbb{1}_{2n}, \qquad A_3 - A_3^{-1} = \overline{u}_n\mathbb{1}_{2n}. \tag{4.36}$$

For A_4, the Hecke relation is deformed as explained below; a similar proof for a slightly different element is done in [EGO06, §4.3] and is readily adapted to our situation, cf. Remark 4.10.

Recall that H_n' is the Hecke subalgebra associated to W' with symmetriser $\mathbf{e}'$ defined in (4.9).

Lemma 4.31 (cf. [Mac03, (5.5.14), (5.5.15)]) *Let R be a root system and W the associated Weyl group, with R' and W' the subsystem and subgroup stabilised by the first coordinate, respectively. For R_+ and R'_+ the respective positive roots and $c_\alpha(\boldsymbol{X})$ defined as in (2.2), we have the following:*

(i) $\mathbf{e}\mathbf{e}' = \mathbf{e}$.

(ii) $\mathbf{e} = ec_+(\boldsymbol{X})$ *where* $c_+(\boldsymbol{X}) := \prod_{\alpha\in R_+} c_\alpha(\boldsymbol{X}^{-1})$ *and* $e \in \mathbb{C}W$ *is the usual symmetriser.*

(iii) $\mathbf{e}' = e'c'_+(\boldsymbol{X})$ *where* $c'_+(\boldsymbol{X}) := \prod_{\alpha\in R'_+} c_\alpha(\boldsymbol{X}^{-1})$ *and* $e' \in \mathbb{C}W'$ *is the usual symmetriser.*

In other words, $e'M = \mathbf{e}'M$ and so $\mathbf{e}'$ acts on M' by identity. As for the action of $\mathbf{e}$ on M', this was calculated in [Cha19] and the proof is reproduced here for the convenience of the reader.

Lemma 4.32 ([Cha19, §3.8 and §4.4]) *Consider the column and row vectors* $\mathbf{v}, \mathbf{w}$ *given by*

$$\mathbf{v} = (1,\dots,1)^T, \qquad \mathbf{w} = (\phi_1,\dots,\phi_{2n}), \qquad \phi_i = a_i^- \prod_{k\neq i}^{n} a_{ik}^- a_{ki}.$$

The Hecke symmetriser $\mathbf{e}$ *acts on* $M' = e'M$ *by the rank-one matrix* $\gamma^{-1}\mathbf{vw}$*, where*

$$\gamma = \frac{t^{2n}-1}{t^2-1}k_n + \frac{1-t^{-2n}}{1-t^{-2}}k_n^{-1}. \tag{4.37}$$

Proof: We first follow the proof of [Cha19, Proposition 3.3]. From Lemma 4.31, we see that

$$\mathbf{e}e' = \mathbf{e}e'\frac{1}{c'_+(\boldsymbol{X})} = \mathbf{e}\frac{1}{c'_+(\boldsymbol{X})} = ec_+(\boldsymbol{X})\frac{1}{c'_+(\boldsymbol{X})} = e\prod_{\alpha\in R_+\setminus R'_+} c_\alpha(\boldsymbol{X}^{-1}) =: e\phi(\boldsymbol{X}).$$

In our situation, R and W are of type C_n, so recall from Example 2.1 that the subset of positive roots is $R_+ = \{\varepsilon_i \pm \varepsilon_j\} \cup \{2\varepsilon_i\}$ where $1 \leq i < j \leq n$. Hence, $\alpha \in R_+ \setminus R'_+$ has the form $\alpha = \varepsilon_1 \pm \varepsilon_k$ $(k = 2, \ldots, n)$ or $\alpha = 2\varepsilon_1$. As a consequence, the above product can be made explicit. Indeed,

$$\phi(\boldsymbol{X}) = c_{2\varepsilon_1}(\boldsymbol{X}^{-1})\prod_{k\neq 1}^{n} c_{\varepsilon_1+\varepsilon_k}(\boldsymbol{X}^{-1})c_{\varepsilon_1-\varepsilon_k}(\boldsymbol{X}^{-1}) = a_1^- \prod_{k\neq 1}^{n} a_{1k}^- a_{k1}.$$

Using the matrix representation à la Proposition 4.12, with arbitrary $f \in M'$ (4.17), we have

$$\phi f = \phi e'\left(\sum_{i=1}^{n} \mathfrak{s}_{1i}\, f_i + \sum_{i=1}^{n} \mathfrak{s}_{1i}^+\, f_i^+\right) = e'\left(\sum_{i=1}^{n} \mathfrak{s}_{1i}(\phi)^{\mathfrak{s}_{1i}} f_i + \sum_{i=1}^{n} \mathfrak{s}_{1i}^+(\phi)^{\mathfrak{s}_{1i}^+} f_i^+\right).$$

In other words, ϕ acts as the diagonal matrix $\operatorname{diag}(\phi_1, \ldots, \phi_n, \phi_1^+, \ldots, \phi_n^+)$ where $\phi_i := (\phi)^{\mathfrak{s}_{1i}}$ and $\phi_i^+ := (\phi)^{\mathfrak{s}_{1i}^+}$. But the group symmetriser e acts as the matrix of all-ones, from which it follows that the Hecke symmetriser $\mathbf{e}$ acts on M' by $\mathbf{vw}$ up to a constant factor. Hence, $\mathbf{vw} = \gamma\mathbf{e}$ on M' for some $\gamma \in \mathbb{C}^*$. To calculate this constant, we use the idempotence property $\mathbf{e}^2 = \mathbf{e}$, from which we see that $\gamma = \mathbf{wv}$. To determine this constant, we can take the variables $X_i \to \infty$ within a particular Weyl chamber. This yields

$$\mathbf{wv} = \frac{t^{2n}-1}{t^2-1}k_n + \frac{1-t^{-2n}}{1-t^{-2}}k_n^{-1}, \tag{4.38}$$

as stated. □

Lemma 4.33 *The restriction of* Z_4 *onto* M' *satisfies the equation*

$$tZ_4 - t^{-1}Z_4^{-1} = (k_n t^{-1} - k_n^{-1}t) + \sum_{i=1}^{n}(\mathfrak{s}_{1i} + \mathfrak{s}_{1i}^+)\bar{t}a_1^- \prod_{k\neq 1}^{n} a_{1k}^- a_{k1}, \tag{4.39}$$

where we again consider $\mathfrak{s}_{11} = \mathrm{id}$ *and* $\mathfrak{s}_{11}^+ = \mathfrak{s}_1$.

Proof: We use the restrictions of Z_4 and Z_4^{-1} onto M' from Lemmata 4.25 and 4.29, and continue with the notation therein. The strategy is to compare coset coefficients. For the $\mathfrak{s}_1$-coset,

$$(t\mathcal{A} - t^{-1}\mathcal{U})\,\mathfrak{s}_1 = \left(ta_1 \prod_{k\neq 1}^{n} a_{1k}a_{1k}^{+} - t^{-1}\alpha_1 \prod_{k\neq 1}^{n} \alpha_{1k}\alpha_{1k}^{+} \right) \mathfrak{s}_1$$
$$= \bar{t}a_1 \prod_{k\neq 1}^{n} a_{1k}a_{1k}^{+}\,\mathfrak{s}_1,$$

using the fact that the α's and a's coincide (Remark 4.20). So that it stands on the correct side as in the statement, we pull $\mathfrak{s}_1$ all the way through to the left. Using index symmetry, we obtain

$$\mathfrak{s}_1\,\bar{t}a_1^{-} \prod_{k\neq 1}^{n} a_{k1}a_{k1}^{-}.$$

Slightly more involved, we now work with the $\mathfrak{s}_{1j}$-coset. Here then, we see that

$$(t\mathcal{C}_j^{+} - t^{-1}\mathcal{W}_j^{+})\,\mathfrak{s}_{1j} = \left(ta_j^{-}b_{1j}^{+}a_{1j} \prod_{k\neq 1,j}^{n} a_{kj}^{-}a_{kj} - t^{-1}\alpha_j^{-}\beta_{1j}^{+}\alpha_{1j} \prod_{k\neq 1,j}^{n} \alpha_{kj}^{-}\alpha_{kj} \right) \mathfrak{s}_{1j}$$
$$= \left(a_j^{-}(tb_{1j}^{+} - t^{-1}b_{1j}^{+} + t^{-1}\bar{t})a_{1j} \prod_{k\neq 1,j}^{n} a_{kj}^{-}a_{kj} \right) \mathfrak{s}_{1j}$$
$$= \left(\bar{t}a_j^{-}a_{1j}^{-}a_{1j} \prod_{k\neq 1,j}^{n} a_{kj}^{-}a_{kj} \right) \mathfrak{s}_{1j}$$
$$= \left(\bar{t}a_j^{-} \prod_{k\neq j}^{n} a_{kj}^{-}a_{kj} \right) \mathfrak{s}_{1j}\,.$$

We again used that the α's and a's coincide, but also that β's coincide with $b-\bar{t}$. From the Basic Representation, it is possible to see that $\bar{t}(b_{1j}^{+} + t^{-1}) = \bar{t}a_{1j}^{-}$; this is what we used when moving from the second equality to the third. Again pulling $\mathfrak{s}_{1j}$ all the way through to the left yields

$$\mathfrak{s}_{1j}\,\bar{t}a_1^{-} \prod_{k\neq 1}^{n} a_{k1}a_{k1}^{-}.$$

Similarly for the $\mathfrak{s}_{1j}^{+}$-coset, we have

$$(t\mathcal{C}_j - t^{-1}\mathcal{W}_j)\,\mathfrak{s}_{1j}^{+} = \left(ta_jb_{1j}a_{1j}^{+} \prod_{k\neq 1,j}^{n} a_{jk}a_{jk}^{+} - t^{-1}\alpha_j\beta_{1j}\alpha_{1j}^{+} \prod_{k\neq 1,j}^{n} \alpha_{jk}\alpha_{jk}^{+} \right) \mathfrak{s}_{1j}^{+}$$

$$= \left(a_j (t b_{1j} - t^{-1} b_{1j} + t^{-1}\bar{t}) a_{1j}^+ \prod_{k \neq 1,j}^{n} a_{jk} a_{jk}^+ \right) \mathfrak{s}_{1j}^+$$
$$= \left(\bar{t} a_j a_{j1} a_{1j}^+ \prod_{k \neq 1,j}^{n} a_{jk} a_{jk}^+ \right) \mathfrak{s}_{1j}^+$$
$$= \left(\bar{t} a_j \prod_{k \neq j}^{n} a_{jk} a_{jk}^+ \right) \mathfrak{s}_{1j}^+ .$$

The difference here compared to that of the previous coset is that we use $\bar{t}(b_{1j} + t^{-1}) = \bar{t} a_{j1}$, which we one again see from the Basic Representation. Finally, pulling $\mathfrak{s}_{1j}^+$ to the left gives us

$$\mathfrak{s}_{1j}^+ \bar{t} a_1^- \prod_{k \neq 1}^{n} a_{k1} a_{k1}^- .$$

The remaining term is the identity, and it turns out to be somewhat more involved. First of all, we can simplify things using what we have already calculated for the other cosets:

$$\begin{aligned} t\mathcal{B} - t^{-1}\mathcal{V} &= t\left(k_n t^{2n-2} - \mathcal{A} - \sum_{j \neq 1}^{n} (\mathcal{C}_j + \mathcal{C}_j^+) \right) - t^{-1} \left(k_n^{-1} t^{2-2n} - \mathcal{U} - \sum_{j \neq 1}^{n} (\mathcal{W}_j + \mathcal{W}_j^+) \right) \\ &= k_n t^{2n-1} - k_n^{-1} t^{1-2n} - (t\mathcal{A} - t^{-1}\mathcal{U}) - \sum_{j \neq 1}^{n} (t\mathcal{C}_j - t^{-1}\mathcal{W}_j) - \sum_{j \neq 1}^{n} (t\mathcal{C}_j^+ - t^{-1}\mathcal{W}_j^+) \\ &= k_n t^{2n-1} - k_n^{-1} t^{1-2n} - \bar{t} a_1 \prod_{k \neq 1}^{n} a_{1k} a_{1k}^+ - \sum_{j \neq 1}^{n} \bar{t} a_j \prod_{k \neq j}^{n} a_{jk} a_{jk}^+ - \sum_{j \neq 1}^{n} \bar{t} a_j^- \prod_{k \neq j}^{n} a_{kj}^- a_{kj} . \end{aligned}$$

Based on the statement we are after, it suffices to show that subtracting $(k_n t^{-1} - k_n^{-1} t)$ from the above will give us $\bar{t} a_1^- \prod_{k \neq 1}^{n} a_{1k}^- a_{k1}$. This is equivalent to verifying the following equality:

$$k_n t^{2n-1} - k_n^{-1} t^{1-2n} + k_n^{-1} t - k_n t^{-1} \tag{4.40}$$
$$\|$$
$$\bar{t} \left(a_1^- \prod_{k \neq 1}^{n} a_{1k}^- a_{k1} + a_1 \prod_{k \neq 1}^{n} a_{1k} a_{1k}^+ + \sum_{j \neq 1}^{n} \left(a_j \prod_{k \neq j}^{n} a_{jk} a_{jk}^+ + a_j^- \prod_{k \neq j}^{n} a_{kj}^- a_{kj} \right) \right)$$
$$\|$$
$$\bar{t} \sum_{j=1}^{n} \left(a_j \prod_{k \neq j}^{n} a_{jk} a_{jk}^+ + a_j^- \prod_{k \neq j}^{n} a_{kj}^- a_{kj} \right).$$

Note that the bracket on the 'right'-hand side of the equality is invariant under the action of $\mathfrak{s}_\ell$

and $\mathfrak{s}_{1\ell}$ for all $\ell = 1, \ldots, n$. This guarantees there are no poles and so that rational function is actually a polynomial. Moreover, the degrees in the numerator and denominator agree, since the expression is 'homogeneous'. Therefore, the polynomial has degree zero; it is constant. It remains to show that it agrees with the constant on the 'left'-hand side. Let's consider an asymptotics argument where each $X_\ell \to \infty$ given $X_1 \gg \cdots \gg X_n \gg 1$. For all $i, j = 1, \ldots, n$, we have these:

(i) $a_i \to k_n$,

(ii) $a_i^- \to k_n^{-1}$,

(iii) $a_{ij} \to t^{-1}$ **if** $X_i \ll X_j$,

(iv) $a_{ij} \to t$ **if** $X_i \gg X_j$,

(v) $a_{ij}^+ \to t$,

(vi) $a_{ij}^- \to t^{-1}$.

Note that any order on the variables can be used, but it will slightly alter the way in which we apply (iii) and (iv) from above. For fixed j and asymptotic order $X_1 \gg \cdots \gg X_n$, we need only worry about the factors $\prod a_{jk}$ and $\prod a_{kj}$ in each summand since the asymptotics of the other factors are independent of their indices. Well, (iii) and (iv) from above imply that

$$\prod_{k\neq 1}^{n} a_{jk} = (t^{-1})^{j-1} \cdot t^{n-j} \qquad \text{and} \qquad \prod_{k\neq 1}^{n} a_{kj} = t^{j-1} \cdot (t^{-1})^{n-j}.$$

The 'right'-hand side of the equality, under the above asymptotics, thus behaves as the constant

$$\bar{t} \sum_{j=1}^{n} \Big(k_n t^{2(n-j)} + k_n^{-1} t^{-2(n-j)} \Big).$$

If we now expand $\bar{t} = t - t^{-1}$ into the sum, the resulting expression exhibits some telescoping:

$$\sum_{j=1}^{n} \Big(k_n t^{2(n-j)+1} - k_n t^{2(n-j)-1} + k_n^{-1} t^{-2(n-j)+1} - k_n^{-1} t^{-2(n-j)-1} \Big).$$

Indeed, working through term-by-term, we see it is equal to precisely what we wanted, namely

$$k_n t^{2n-1} - k_n^{-1} t^{1-2n} + k_n^{-1} t - k_n t^{-1}. \qquad \square$$

Remark 4.34 The above proof is independent of the chosen asymptotic order on (rather, the Weyl chamber within which live) the variables $X_1, \ldots, X_n$. Indeed, selecting a different ordering will change the behaviour of the a_{ij} in the limit $X_\ell \to \infty$; the difference is the exponents in $\prod a_{jk}$ and $\prod a_{kj}$, but these still simplify and the sum again telescopes to the same constant (4.40).

Corollary 4.35 *In $\mathcal{H}_{q,\boldsymbol{\tau}}$, the element $Z_4 = S T_n S^\dagger$ satisfies the relation*

$$(tZ_4 - t^{-1} Z_4^{-1} - k_n t^{-1} + k_n^{-1} t)\mathbf{e}' = (k_n t^{2n-1} - k_n^{-1} t^{1-2n} + k_n^{-1} t - k_n t^{-1})\mathbf{e}. \tag{4.41}$$

Proof: Act by both sides on $M' = e'M = \mathbf{e}'M$, so that it follows immediately from Lemma 4.33. Note that the sum of cosets in (4.39) is equivalent to the matrix of all-ones, that is the matrix representing the Hecke symmetriser $\mathbf{e}$. This is, at the level of matrices, equivalent to

$$tA_4 - t^{-1}A_4^{-1} - (k_n t^{-1} - k_n^{-1} t)\mathbb{1}_{2n} = (t - t^{-1})\mathbf{vw}. \tag{4.42}$$

We know $\mathbf{vw} = \gamma\mathbf{e}$ on M', and $\bar{t}\gamma$ is the coefficient of the right-hand side of (4.41). Because $\mathcal{H}_{q,\boldsymbol{\tau}}$ acts faithfully on M, the relation (4.41) holds in totality. □

Corollary 4.36 *At the classical level $q = 1$, the quadruple (A_1, A_2, A_3, A_4), with matrices A_i from* Theorems 4.16 *and* 4.26*, represents a point on the Calogero-Moser space. This gives a coordinate chart on $\mathcal{C}_n$ with $2n$ coordinates X_i, P_i.*

Proof: This is proved using the description of the Calogero-Moser space à la Proposition 4.5. Indeed, set $X = A_1A_2$, $Y = A_4A_1$ and $T = A_2$. In doing so, equations (4.36), (4.38) and (4.42) imply the relations (4.2)–(4.6) and thus we have a chart. □

One can also see that the action by a finite Weyl group element $w \in W$ on $(\boldsymbol{P}, \boldsymbol{X})$ is equivalent to conjugating (A_1, A_2, A_3, A_4) by the permutation matrix representing w. In other words, the action on coordinates permutes the matrix entries in the expected way. Therefore, we have a well-defined map

$$\Upsilon : \mathcal{U}/W \to \mathcal{C}_n, \qquad (\boldsymbol{P}, \boldsymbol{X}) \mapsto (A_1, A_2, A_3, A_4), \tag{4.43}$$

where

$$\mathcal{U} := (\mathbb{C}^*)^n \times ((\mathbb{C}^*)^n \setminus D), \qquad \text{with } D = \{\boldsymbol{X} \in (\mathbb{C}^*)^n : \delta(\boldsymbol{X}) = 0\}. \tag{4.44}$$

4.4 The EGO Map on a Chart

Let us explain how the work done in §4.3 is related to the EGO map from Definition 4.9. At the classical level $q = 1$, it is clear that the Basic Representation β induces an injective map $\mathcal{H} \hookrightarrow \mathbb{C}[\boldsymbol{X}^{\pm 1}, \boldsymbol{P}^{\pm 1}]_{\delta(\boldsymbol{X})} \rtimes \mathbb{C}W$. Thus, one can construct a family of irreducible representations of $\mathcal{H}$ from irreducible representations of $\mathbb{C}[\boldsymbol{X}^{\pm 1}, \boldsymbol{P}^{\pm 1}]_{\delta(\boldsymbol{X})} \rtimes \mathbb{C}W$ by restriction, cf. [Obl04, §3.3].

Notation 4.37 To alleviate some of the notation, let $R := \mathbb{C}[\boldsymbol{X}^{\pm 1}, \boldsymbol{P}^{\pm 1}]_{\delta(\boldsymbol{X})}$ and $\widehat{R} := R \rtimes \mathbb{C}W$.

Now, pick a point $(\boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathcal{U}$ (4.44) and define a one-dimensional representation $\chi_{\boldsymbol{\mu},\boldsymbol{\nu}}$ of R by

$$\chi_{\boldsymbol{\mu},\boldsymbol{\nu}} : f(\boldsymbol{X}, \boldsymbol{P}) \mapsto f(\boldsymbol{\nu}, \boldsymbol{\mu}),$$

from which we can induce a finite-dimensional module

$$V_{\boldsymbol{\mu},\boldsymbol{\nu}} \coloneqq \operatorname{Ind}_R^{\widehat{R}} \chi_{\boldsymbol{\mu},\boldsymbol{\nu}} = \widehat{R} \otimes_R \chi_{\boldsymbol{\mu},\boldsymbol{\nu}}. \tag{4.45}$$

Proposition 4.38 *Assume $\delta_{\boldsymbol{\tau}}(\boldsymbol{\nu}) \neq 0$. Then, viewed as an $\mathcal{H}$-module, $V_{\boldsymbol{\mu},\boldsymbol{\nu}}$ belongs to* $\operatorname{Irrep}'(\mathcal{H})$ *and its image under the EGO map is represented by the tuple (A_1, A_2, A_3, A_4), where the A_i are the matrices from* Theorems 4.16 *and* Theorems 4.26 *with $q = 1$ under the specialisation $P_i = \mu_i$ and $X_i = \nu_i$.*

Proof: Since $\delta(\boldsymbol{\nu})\delta_{\boldsymbol{\tau}}(\boldsymbol{\nu}) \neq 0$, one can use Corollary 3.22 to view $V_{\boldsymbol{\mu},\boldsymbol{\nu}}$ as a module over

$$\mathcal{H}_{\delta(\boldsymbol{X})\delta_{\boldsymbol{\tau}}(\boldsymbol{X})} \cong \mathbb{C}[\boldsymbol{X}^{\pm 1}, \boldsymbol{P}^{\pm 1}]_{\delta(\boldsymbol{X})\delta_{\boldsymbol{\tau}}(\boldsymbol{X})} \rtimes \mathbb{C}W.$$

It is clearly irreducible as a $(\mathbb{C}[\boldsymbol{X}^{\pm 1}, \boldsymbol{P}^{\pm 1}]_{\delta(\boldsymbol{X})\delta_{\boldsymbol{\tau}}(\boldsymbol{X})} \rtimes \mathbb{C}W)$-module, and thus also as an $\mathcal{H}$-module (cf. [Obl04, Proposition 3.3]). It is isomorphic to $\mathbb{C}W$ as a W-module, by construction. Hence, $V_{\boldsymbol{\mu},\boldsymbol{\nu}}$ is isomorphic to the regular representation as an H_n-module by a deformation argument (our genericity assumption from Definition 4.2 on the parameters implies that H_n is semi-simple). This establishes that $V_{\boldsymbol{\mu},\boldsymbol{\nu}} \in \operatorname{Irrep}'(\mathcal{H})$.

To apply the EGO map to the module $V_{\boldsymbol{\mu},\boldsymbol{\nu}}$, one needs to consider the subspace V' on which the Hecke subalgebra H_n' acts according to the character χ', see §4.2. This means that $(T_i - \tau_i)v = 0$ for $v \in V'$ and $i = 2, \ldots, n$. We claim that $V' = e'V_{\mu,\nu}$, where e' is the symmetriser (4.16). Indeed, $T_i - \tau_i = c_i(\mathbf{X})(s_i - 1)$, where $c_i(\boldsymbol{X})$ is invertible on $V_{\boldsymbol{\mu},\boldsymbol{\nu}}$ due to the condition $\delta(\boldsymbol{\nu})\delta_{\boldsymbol{\tau}}(\boldsymbol{\nu}) \neq 0$. Hence, $(s_i - 1)v = 0$ for $v \in V'$ and $i = 2, \ldots, n$. This establishes that $V' = e'V_{\boldsymbol{\mu},\boldsymbol{\nu}}$.

By definition, a basis of $V_{\boldsymbol{\mu},\boldsymbol{\nu}}$ is given by elements $w \otimes 1$ where $w \in W$ and with 1 denoting a basis vector in $\chi_{\boldsymbol{\mu},\boldsymbol{\nu}}$. Elements of V' can then be written similarly to elements (4.17) of M', i.e.

$$v = e'\left(\sum_{i=1}^{n} \mathfrak{s}_{1i} \otimes f_i + \sum_{i=1}^{n} \mathfrak{s}_{1i}^{+} \otimes f_i^{+}\right), \qquad f_i, f_i^{+} \in \mathbb{C}. \tag{4.46}$$

Any element of $\mathbb{C}[\boldsymbol{X}^{\pm 1}, \boldsymbol{P}^{\pm 1}]_{\delta(\boldsymbol{X})\delta_{\boldsymbol{\tau}}(\boldsymbol{X})} \rtimes \mathbb{C}W$ preserving V' can be represented by a matrix of size $2n \times 2n$. For example, the action of X_1 and P_1 on V' are given by the respective matrices

$$X \coloneqq \operatorname{diag}(\nu_1, \ldots, \nu_n, \nu_1^{-1}, \ldots, \nu_n^{-1}), \qquad P \coloneqq \operatorname{diag}(\mu_1, \ldots, \mu_n, \mu_1^{-1}, \ldots, \mu_n^{-1}).$$

Clearly, we are in the same setting as in §4.3, with the only difference being that X_i and P_i are specialised to ν_i and μ_i, respectively. Hence, the action of Z_1, Z_2, Z_3 and Z_4 on V' is found by specialising the formulae in Theorems 4.16 and 4.26. □

Remark 4.39 The $\mathcal{H}$ module $V_{\boldsymbol{\mu},\boldsymbol{\nu}}$ admits the following interpretation, cf. [Obl04, Lemma 6.1]. Assuming that $\delta(\boldsymbol{\nu})\delta_{\boldsymbol{\tau}}(\boldsymbol{\nu}) \neq 0$, one can consider a one-dimensional representation $\chi_{\boldsymbol{\mu},\boldsymbol{\nu}}$ of the (commutative) W-invariant algebra $\mathbb{C}[\boldsymbol{X}^{\pm 1}, \boldsymbol{P}^{\pm 1}]^W_{\delta(\boldsymbol{X})\delta_{\boldsymbol{\tau}}(\boldsymbol{X})}\mathsf{e}$ defined by $f(\boldsymbol{X}, \boldsymbol{P})\mathsf{e} \mapsto f(\boldsymbol{\nu}, \boldsymbol{\mu})$. We can further restrict $\chi_{\boldsymbol{\mu},\boldsymbol{\nu}}$ to the spherical subalgebra $\mathsf{e}\mathcal{H}\mathsf{e}$ using Corollary 3.22. Hence,

$$V_{\boldsymbol{\mu},\boldsymbol{\nu}} \cong \mathcal{H}\mathsf{e} \otimes_{\mathsf{e}\mathcal{H}\mathsf{e}} \chi_{\boldsymbol{\mu},\boldsymbol{\nu}}. \tag{4.47}$$

Chapter 5

Calogero-Moser Coordinates

We have obtained $2n$ coordinates on $\mathcal{C}_n$ coming from the double affine Hecke algebra (DAHA), in particular, from the map Υ defined in (4.43). The goal of this chapter is to restrict Υ to a suitable subset upon which it is injective and identify the image of said subset explicitly as a subset of the Calogero-Moser space; this gives us our first coordinate chart on Calogero-Moser space. From this, we obtain a second chart by interpreting the Duality Isomorphism ε at the level of character varieties, and finish by proving the main result (Theorem 1.3).

Without further ado, let us define the following subset of $\mathcal{U}$ from (4.44):

$$\mathcal{U}_{\boldsymbol{\tau}} := (\mathbb{C}^*)^n \times ((\mathbb{C}^*)^n \setminus (D \cup D_{\boldsymbol{\tau}})), \qquad \text{with } D_{\boldsymbol{\tau}} = \{\boldsymbol{X} \in (\mathbb{C}^*)^n : \delta_{\boldsymbol{\tau}}(\boldsymbol{X}) = 0\}. \tag{5.1}$$

Proposition 5.1 *The map Υ is injective on the subset $\mathcal{U}_{\boldsymbol{\tau}}/W$.*

Proof: Let $\Upsilon(\boldsymbol{P}, \boldsymbol{X}) = (A_1, A_2, A_3, A_4)$ and $\Upsilon(\boldsymbol{P}', \boldsymbol{X}') = (A_1', A_2', A_3', A_4')$ be such that

$$\Upsilon(\boldsymbol{P}, \boldsymbol{X}) = \Upsilon(\boldsymbol{P}', \boldsymbol{X}') \qquad \text{and} \qquad (A_1, A_2, A_3, A_4) = g(A_1', A_2', A_3', A_4')g^{-1}, \tag{5.2}$$

for some $g \in \mathrm{GL}_{2n}(\mathbb{C})$. From the definitions of A_1 and A_2, the matrix $X = A_1 A_2$ represents the action of X_1 on M', so it has the diagonal form as calculated in Example 4.13. Therefore, up to the action by W, we may assume $\boldsymbol{X} = \boldsymbol{X}'$, which then forces g to be diagonal and implies both $A_3 = A_3'$ and $A_4 = A_4'$. Combining this with the last entries of the tuples in (5.2), we have $gA_4g^{-1} = A_4$ (with g diagonal). However, the condition $\delta_{\boldsymbol{\tau}}(\boldsymbol{X}) \neq 0$ implies that all the off-diagonal entries of A_4 are non-zero. This forces g to be a multiple of the identity. Consequently, $A_1 = A_1'$. Because the rational functions a_i (4.25) are non-zero on $\mathcal{U}_{\boldsymbol{\tau}}$, it follows that $\boldsymbol{P} = \boldsymbol{P}'$. $\square$

By Corollary 4.36, having an injective algebraic map $\Upsilon : \mathcal{U}_{\boldsymbol{\tau}}/W \to \mathcal{C}_n$, we would like to characterise its image. Both sides have dimension $2n$, so irreducibility of $\mathcal{C}_n$ tells us that the image of Υ is automatically dense in $\mathcal{C}_n$. Now, at every point of $\mathcal{U}_{\boldsymbol{\tau}}$, recall that the matrix $X = A_1A_2$ is the diagonal matrix calculated in Example 4.13. In particular, its eigenvalues are paired-off into reciprocals. We next demonstrate that this fact holds globally.

Lemma 5.2 *For any $(A_1, A_2, A_3, A_4) \in \mathcal{C}_n$, $X = A_1A_2$ satisfies $\operatorname{tr} X^k = \operatorname{tr} X^{-k}$ for all $k \in \mathbb{Z}$, and its eigenvalues appear in pairs (x_i, x_i^{-1}) for each $i = 1, \dots, n$. Hence, there is a global map*

$$\mathcal{C}_n \to (\mathbb{C}^*)^n/W, \qquad (A_1, A_2, A_3, A_4) \mapsto (x_1, \dots, x_n).$$

Proof: We first need to show that $\det(X - \lambda\mathbb{1}_{2n}) = \det\bigl(X^{-1} - \lambda\mathbb{1}_{2n}\bigr)$. By the above, this is true on a dense subset of $\mathcal{C}_n$ (namely, on the image of Υ), hence it is true everywhere. This means that the eigenvalues of X, distinct from ±1, appear in pairs (x_i, x_i^{-1}). We claim that each of the eigenvalues ±1 has even multiplicity, and so a pair $(1, 1)$ or $(-1, -1)$ can be viewed as special case of (x_i, x_i^{-1}) with $x_i = \pm1$. To see this, perturb the point (A_1, A_2, A_3, A_4) so that X has eigenvalues different from ±1. This means that eigenvalues of X pair-off into reciprocals, which obviously remains true under degeneration of some of the x_i into ±1. □

In other words, the eigenvalues of $X = A_1A_2$ are always of the form $(x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1})$, which defines a point $\mathbf{x} \coloneqq (x_1, \dots, x_n) \in (\mathbb{C}^*)^n$, but only up to the action of W. Our next goal is to establish the following result.

Proposition 5.3 *The map Υ (4.43) defines an isomorphism $\mathcal{U}_{\boldsymbol{\tau}}/W \cong \mathcal{C}_n^{\boldsymbol{\tau}}$, where*

$$\mathcal{C}_n^{\boldsymbol{\tau}} \coloneqq \{(A_1, A_2, A_3, A_4) \in \mathcal{C}_n : \delta(\mathbf{x})\delta_{\boldsymbol{\tau}}(\mathbf{x}) \neq 0\}. \tag{5.3}$$

In view of Proposition 5.1, we need only show Υ is onto. Given a point $(A_1, A_2, A_3, A_4) \in \mathcal{C}_n^{\boldsymbol{\tau}}$, we show that it is represented by the matrices from Theorems 4.16 and 4.26 in a suitable basis (at the classical level $q = 1$). Unlike in [Obl04], we cannot easily solve the matrix equations determining the A_i, so we instead use the interpretation involving the Deligne-Simpson problem and the representation of a multiplicative preprojective algebra (see §2.3.3).

5.1 Two Deligne-Simpson Problems

The condition $\delta(\mathbf{x}) \neq 0$ guarantees that $X = A_1A_2$ is diagonalisable, so we readily assume that

$$X = \operatorname{diag}(x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}), \qquad \text{with } \delta(\mathbf{x})\delta_{\boldsymbol{\tau}}(\mathbf{x}) \neq 0. \tag{5.4}$$

The problem of determining the matrices A_1, A_2, A_3 and A_4 then splits into two separate problems, by considering a so-called *pair of pants* decomposition of the four-punctured sphere.

Problem 5.4 *Let X be the matrix (5.4) and $C_i \subseteq \mathrm{GL}_{2n}(\mathbb{C})$ the conjugacy classes in (4.1).*

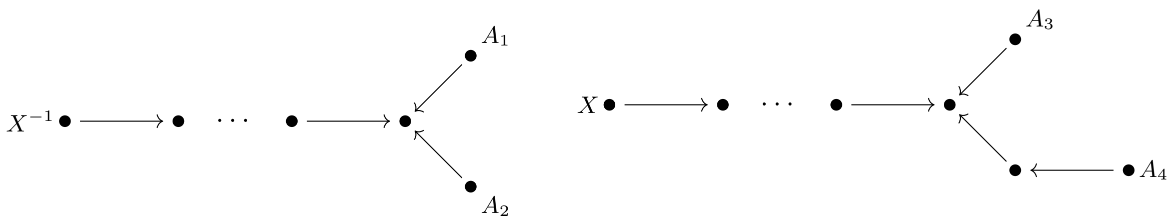
(i) *Find $A_1 \in C_1$ and $A_2 \in C_2$ with*

$$A_1 A_2 X^{-1} = \mathbb{1}_{2n}.$$

(ii) *Find $A_3 \in C_3$ and $A_4 \in C_4$ with*

$$X A_3 A_4 = \mathbb{1}_{2n}.$$

Each of Problem 5.4 is a Deligne-Simpson problem on the three-punctured sphere $\Sigma_{0,3}$. The corresponding star-shaped quivers are shown below in Figure 5.1.



(a) The quiver Q_1 for Problem 5.4(i).

(b) The quiver Q_2 for Problem 5.4(ii).

Figure 5.1: The quivers associated with Problem 5.4.

The dimension vectors of the quivers Q_1 and Q_2, reading vertices left-to-right, are

$$\mathbf{n}_1 = (1, \ldots, 2n-1, 2n, n, n) \qquad \text{and} \qquad \mathbf{n}_2 = (1, \ldots, 2n-1, 2n, n, n, 1). \tag{5.5}$$

We know already that each of these problems has a solution. As we shall explain, the solution is unique up to conjugation, and the corresponding multiplicative quiver variety in each case consists of a point. Additional n coordinates $p_1, \ldots, p_n$ arise corresponding to the different ways of "glueing" the two solutions together. The following result will be used.

Theorem 5.5 ([CBS06, Lemma 1.5 and Theorem 1.8]) *The dimension of a simple $\Lambda^{\mathbf{q}}$-module is a positive root for the corresponding quiver Q. Moreover, if α is the dimension of a $\Lambda^{\mathbf{q}}$-module, then α admits a decomposition into a sum of positive roots $\alpha = \beta + \gamma + \cdots$ with $\mathbf{q}^\beta = \mathbf{q}^\gamma = \cdots = 1$.*

5.1.1 Problem 5.4(i)

The quiver $Q = Q_1$ is a finite Dynkin quiver of type D_{2n+2}. We consider the multiplicative preprojective algebra $\Lambda^{\mathbf{q}}$ with the parameters $\mathbf{q}$ (2.13) determined from the eigenvalues of X, A_1 and A_2. We use the eigenvalues of X in the following order going from left-to-right along Q: $x_n^{-1}, \ldots, x_1^{-1}, x_n, \ldots, x_1$. We are interested in $\Lambda^{\mathbf{q}}$-modules of dimension

$$\alpha = \mathbf{n}_1 = (1, \ldots, 2n-1, 2n, n, n).$$

By Theorem 5.5, α must be a sum of positive roots of Q. Since the arrows of Q are represented by injective/surjective maps, the support of each summand $\beta, \gamma, \ldots$ should include the central node. This gives us the following possibilities for the summands in Figure 5.2, cf. Example 2.2.

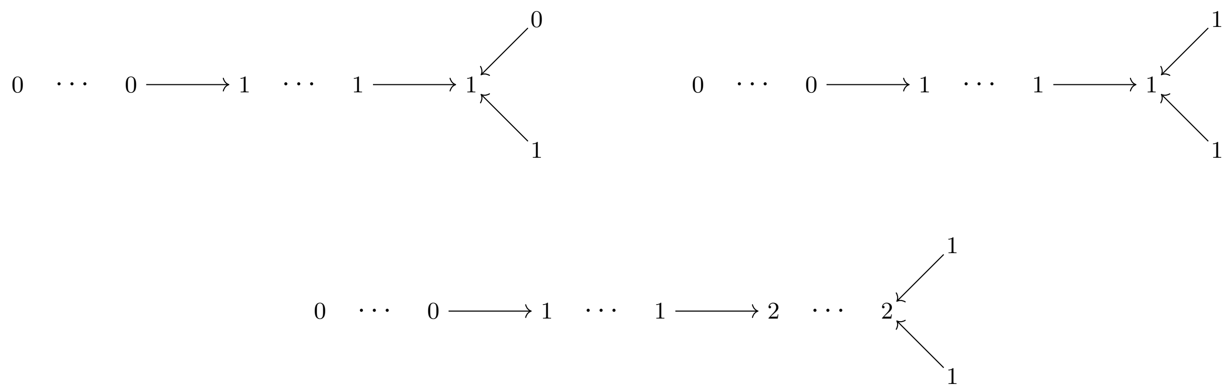


Figure 5.2: Positive roots supported at the central node.

The first two types of summand can be ruled out because $\mathbf{q}^\beta \neq 1$ in these cases. For example, in the first case, we have a one-dimensional subspace at the central node which has to be an eigenspace for X^{-1}, A_1 and A_2 with respective eigenvalues $x_i^{\pm 1}$, k_0 and u_0. But $A_1 A_2 X^{-1} = \mathbb{1}_{2n}$, so it follows that $x_i^{\pm 1} k_0 u_0 = 1$, a contradiction to the assumption $\delta_{\boldsymbol{\tau}}(\mathbf{x}) \neq 0$ made in (5.4).

Consequently, each positive root summand $\beta, \gamma, \ldots$ should be of the final type. In this case, we have a two-dimensional subspace at the central node. As above, this corresponds to X^{-1} having eigenvalues $x_i^{\pm 1}$, $x_j^{\pm 1}$ for some i, j. The condition $\mathbf{q}^\beta = 1$ then forces these eigenvalues to be reciprocal to one another. Hence, β must be one of $e_{1\,n+1}, e_{2\,n+2}, \ldots, e_{n\,2n}$, where e_{ij} is the $2n$-tuple associated with the last positive root in Figure 5.2, that is

$$e_{ij} = (0, \ldots, 0, \underbrace{1}_{i^{\text{th}}}, \ldots, 1, \underbrace{2}_{j^{\text{th}}}, \ldots, 2, 1, 1).$$

Hence, the only allowed decomposition of $\alpha = \mathbf{n}_1$ into a sum of simple roots is

$$\alpha = e_{1\,n+1} + e_{2\,n+2} + \cdots + e_{n\,2n}. \tag{5.6}$$

As the next lemma shows, this determines the $\Lambda^{\mathbf{q}}$-module uniquely.

Lemma 5.6 (i) *For each root $\beta_i := e_{i\,n+i}$, there is a unique $\Lambda^{\mathbf{q}}$ module M_i of dimension β_i.*

(ii) *If M is a $\Lambda^{\mathbf{q}}$-module of dimension $\alpha = \mathbf{n}_1$, then M is isomorphic to the direct sum of M_i for $i = 1, \dots, n$. Hence, up to isomorphism, there is one $\Lambda^{\mathbf{q}}$-module of dimension $\alpha = \mathbf{n}_1$.*

Proof: (i) Existence and uniqueness of M_i follow from [CBS06, Theorem 1.9], using the fact that Q is of finite Dynkin type and thus its roots are real. Alternatively, the problem of constructing M_i can be interpreted as the description of the monodromy of the classical hypergeometric equation, where uniqueness (so-called *rigidity* in *op. cit.*) as well as the existence are well know.

(ii) From (5.6), it follows that the composition series of M consists of the modules $M_1, \dots, M_n$. By using Lemma 2.20, one can check that $\mathrm{Ext}^1_{\Lambda^{\mathbf{q}}}(M_i, M_j) = 0$ for $i \neq j$. It suffices to confirm that the Cartan form (β_i, β_j) between any of the summands of α is zero. Let $i < j$ without loss of generality and use $(\cdot)_k$ to denote the k^{th} entry of the vector. By (2.11), the Cartan form is

$$2\sum_{k=1}^{2n+2}(\beta_i)_k(\beta_j)_k - \sum_{k=1}^{2n}\big((\beta_i)_k(\beta_j)_{k+1} + (\beta_j)_k(\beta_i)_{k+1}\big) - (\beta_i)_{2n}(\beta_j)_{2n+2} - (\beta_j)_{2n}(\beta_i)_{2n+2}.$$

Because $i < j$, the first non-zero term occurs at the j^{th} place. Explicitly then, we work with

$$\begin{array}{rcccccccccccccccccc}
 & & {\scriptstyle 1} & & {\scriptstyle i-1} & {\scriptstyle i} & & {\scriptstyle j-1} & {\scriptstyle j} & & {\scriptstyle n+i-1} & {\scriptstyle n+i} & & {\scriptstyle n+j-1} & {\scriptstyle n+j} & & {\scriptstyle 2n} & {\scriptstyle 2n+1} & {\scriptstyle 2n+2} \\
\beta_i & = & (0, & \dots, & 0, & 1, & \dots, & 1, & 1, & \dots, & 1, & 2, & \dots, & 2, & 2, & \dots, & 2, & 1, & 1), \\
\beta_j & = & (0, & \dots, & 0, & 0, & \dots, & 0, & 1, & \dots, & 1, & 1, & \dots, & 1, & 2, & \dots, & 2, & 1, & 1).
\end{array}$$

Carefully substituting the corresponding entries into the Cartan form expression, we obtain

$$\begin{aligned}
(\beta_i, \beta_j) &= 2\big((n+i-j) + 2(j-i) + 4(n-j+1) + 2\big) \\
&\qquad - \big((n+i-j+1) + 2(j-i-1) + 4(n-j+1) + 4\big) \\
&\qquad\quad - \big((n+i-j-1) + 2(j-i+1) + 4(n-j) + 4\big) \\
&= 2(5n-i-3j+6) - (5n-i-3j+7) - (5n-i-3j+5) \\
&= 0.
\end{aligned}$$

But each $\mathrm{Hom}_{\Lambda^{\mathbf{q}}}(M_i, M_j) = 0$ due to Schur's Lemma. Hence, substituting this along with the zero Cartan form into Lemma 2.20, we confirm that M splits into the direct sum of the M_i. □

5.1.2 Problem 5.4(ii)

The quiver $Q = Q_2$ is not of finite Dynkin type, but we can use the fact it becomes Dynkin after the removal of the extending vertex (the framing). We again begin by considering the multiplicative preprojective algebra $\Lambda^{\mathbf{q}}$ with the parameters $\mathbf{q}$ determined from the eigenvalues of X, A_3 and A_4. We are interested in $\Lambda^{\mathbf{q}}$-modules of dimension

$$\alpha = \mathbf{n}_2 = (1, \ldots, 2n-1, 2n, n, n, 1).$$

Lemma 5.7 *Any $\Lambda^{\mathbf{q}}$-module of dimension $\alpha = \mathbf{n}_2$ is simple.*

Proof: If if were not simple, α would be a sum $\beta + \gamma + \cdots$ of roots of Q with $\mathbf{q}^\beta = \mathbf{q}^\gamma = \cdots = 1$. At least one of them, β say, must have a zero entry corresponding to the extended vertex. Hence, it is a root of the finite Dynkin graph of type D_{2n+2}. The same analysis as for Problem 5.4(i) can then be carried out; the difference is that the eigenvalues for A_4 have powers of t, preventing $\mathbf{q}^\beta = 1$, a contradiction. □

By [CBS06, Theorem 1.8], $\alpha = \mathbf{n}_2$ is a root of Q_2. The dimension of the multiplicative quiver variety can be found using [CBS06, Theorem 1.10]. Namely, it is $2p(\alpha)$, where $p(\alpha) \coloneqq 1 - q(\alpha)$ and $q(\alpha)$ being the Tits form. By direct calculation, $q(\alpha) = 1$ and hence $p(\alpha) = 0$. Therefore, the variety $\mathcal{M}_{\mathbf{q},\alpha}(Q_2)$ is zero-dimensional. This is the so-called *rigid* case of the Deligne-Simpson problem, in which case its solution is unique; see [CB04, Theorem 1.5].

5.2 The First Chart

We can now use the quiver interpretation in order to prove Proposition 5.3. Recall that we must show every point in $\mathcal{C}_n^{\boldsymbol{\tau}}$ (5.3) is represented by (A_1, A_2, A_3, A_4) in accordance with the formulae for the matrices in Theorems 4.16 and 4.26, at the classical level $q = 1$.

Proof of Proposition 5.3: Let us first set all $P_i = 1$ and denote the corresponding matrices $A_i^\bullet$. From the construction of these matrices, we know that $A_1^\bullet A_2^\bullet = X$ and $A_3^\bullet A_4^\bullet = X^{-1}$, where X is the matrix from Example 4.13. This gives us solutions to Problems 5.4(i) and (ii). Assuming $\delta_{\boldsymbol{\tau}}(\boldsymbol{X}) \neq 0$, each solution to these Deligne-Simpson problems is unique up to conjugation by a matrix that leaves X unchanged (i.e. up to conjugation by diagonal matrices). This gives rise to two conjugation matrices C and D, and thus the general solution of the main Deligne-Simpson problem, given X, is

$$(A_1, A_2, A_3, A_4) = (CA_1^\bullet C^{-1}, CA_2^\bullet C^{-1}, DA_3^\bullet D^{-1}, DA_4^\bullet D^{-1}).$$

Using simultaneous conjugation on this solution, we can set $D = \mathbb{1}$, meaning $A_3^\bullet = A_3$ and $A_4^\bullet = A_4$ are already in the desired form. On the other hand, $A_1^\bullet$ and $A_2^\bullet$ have most of their entries equal to zero. One can use their formulae from Theorem 4.16 (with all $P_i = 1$) and verify that they commute with diagonal matrices of the form $C = \operatorname{diag}(c_1, \dots, c_n, c_{n+1}, \dots, c_{2n})$. However, conjugating each of the matrices $A_1^\bullet$ and $A_2^\bullet$ by this C, we see that there are only n degrees of freedom because the diagonal entries pair-up: the off-diagonal blocks involve $c_i c_{n+i}^{-1}$ or its inverse. Therefore, we can define $P_i := c_i c_{n+i}^{-1}$ and consider $C = \operatorname{diag}(1, \dots, 1, P_1, \dots, P_n)$; conjugating $A_1^\bullet$ and $A_2^\bullet$ by these such C results in precisely A_1 and A_2 from Theorem 4.16. So, any point in $\mathcal{C}_n^\tau$ with given X satisfying (5.4) is represented by the A_i we calculated in §4.3. $\square$

5.3 The Second Chart

Recall the Duality Isomorphism ε from Theorem 3.11. It acts on the elements (4.10) by

$$\varepsilon(Z_1) = (Z_3')^{-1}, \qquad \varepsilon(Z_2) = (Z_2')^{-1}, \qquad \varepsilon(Z_3) = (Z_1')^{-1}, \qquad \varepsilon(Z_4) = (Z_4')^{-1}. \tag{5.7}$$

Here, the images are understood as elements of the DAHA $\mathcal{H}' := \mathcal{H}_{q^{-1},\boldsymbol{\sigma}}$. At the classical level $q = 1$, this induces an isomorphism of the corresponding character varieties, given by

$$\mathcal{E} : (A_1, A_2, A_3, A_4) \mapsto (A_1', A_2', A_3', A_4') := (A_3^{-1}, A_2^{-1}, A_1^{-1}, A_4^{-1}), \tag{5.8}$$

or, in the alternative form from Proposition 4.5, given by

$$\mathcal{E} : (X, Y, T, v, w) \mapsto (X', Y', T', v', w') := (Y, X, T^{-1}, v, w). \tag{5.9}$$

Notation 5.8 Use ε' and $\mathcal{E}'$ for the duality maps applied to the DAHA/character variety for $\mathcal{H}'$.

The maps (5.7) and (5.8) are involutions in the sense that $\varepsilon' \circ \varepsilon = \mathcal{E}' \circ \mathcal{E} = \mathrm{id}$. Obviously, we have $\mathcal{E}' \circ \Phi = \Phi \circ \varepsilon$, where Φ is the EGO map from Definition 4.9. We can use $\mathcal{E}$ to construct a second coordinate chart on $\mathcal{C}_n$ by transferring coordinates from the corresponding variety $\mathcal{C}_n'$. In this chart, the matrix $Y = A_3^{-1} A_2^{-1} = A_4 A_1$ is put into diagonal form

$$Y = \operatorname{diag}(y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}), \qquad \text{with } \delta(\mathbf{y})\delta_{\boldsymbol{\sigma}}(\mathbf{y}) \neq 0, \tag{5.10}$$

so $y_1, \dots, y_n$ give n of the coordinates, with the remaining n coordinates read-off of A_2 or A_3.

We can transport everything across to this second chart and proceed as we did in §4.3 to compute matrices, and §5.1 for the interpretation via multiplicative Deligne-Simpson problems.

Let us briefly outline how we can obtain matrices representing Z_2' and Z_3' explicitly (in a manor similar to that of Theorem 4.16). First, the Duality Isomorphism ε is an involution of the DAHA, so we know the images of the inverse elements directly from Theorem 3.11. This is an important observation which we use below.

Proposition 5.9 *The matrix representing $Z_2' = (T_0^\vee)'$ is that whose ij^{th} entry is*

$$(A_2')_{ij} = \begin{cases} \dfrac{q\overline{u}_n Y_i + \overline{u}_0}{1 - q^2 Y_i^2} & \text{if } i = j \\ q u_n^{-1} Q_i^{-1} Y_i^{-1} + \dfrac{q\overline{u}_n + \overline{u}_0 Y_i^{-1}}{1 - q^{-2} Y_i^{-2}} Q_i^{-1} Y_i^{-1} & \text{if } i - j = \pm n \\ 0 & \text{otherwise} \end{cases},$$

and the matrix representing $Z_3' = (S T_n^\vee S^{-1})'$ is that whose ij^{th} entry is

$$(A_3')_{ij} = \begin{cases} -\dfrac{q^{-1}\overline{u}_n Y_i^{-2} + \overline{u}_0 Y_i^{-1}}{1 - q^{-2} Y_i^{-2}} & \text{if } i = j \\ q^{-1} u_n^{-1} Q_i^{-1} + \dfrac{q^{-1}\overline{u}_n + q^{-2}\overline{u}_0 Y_i^{-1}}{1 - q^{-2} Y_i^{-2}} Q_i^{-1} & \text{if } i - j = \pm n \\ 0 & \text{otherwise} \end{cases},$$

extending the indices from $\{1, \ldots, n\}$ to $\{1, \ldots, 2n\}$ by setting $Y_{n+i} \coloneqq Y_i^{-1}$ and $Q_{n+i} \coloneqq Q_i^{-1}$.

Sketch of Proof: The matrices in question are readily computed from (5.7) by applying the Basic Representation β' this time of the dual DAHA $\mathcal{H}_{q^{-1},\boldsymbol{\sigma}}$ (defined in §3.3). We have chosen to use $\boldsymbol{Q}$-variables in the dual setting in analogy with the $\boldsymbol{P}$-variables in the usual story. The proof is then near-identical to that of Theorem 4.16. □

Finally, the arguments involving quivers is now essentially a replacement of notation from that which we had before. In particular, the pair of Deligne-Simpson problems corresponding to this second chart is stated as follows, cf. Problem 5.4.

Problem 5.10 *Let Y be the matrix (5.10) and $C_i \subseteq \mathrm{GL}_{2n}(\mathbb{C})$ the conjugacy classes in (4.1).*

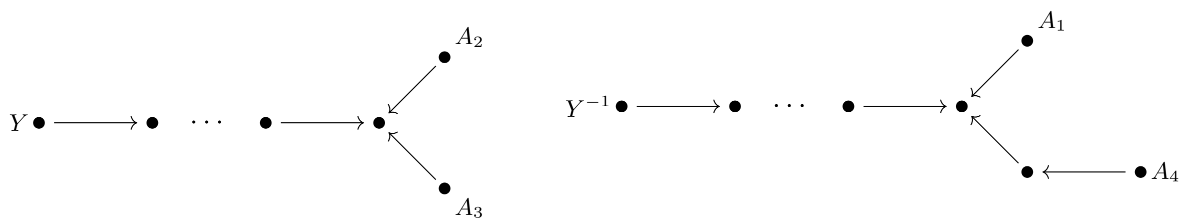
(i) *Find $A_2 \in C_2$ and $A_3 \in C_3$ with*

$$A_2 A_3 Y = \mathbb{1}_{2n}.$$

(ii) *Find $A_1 \in C_1$ and $A_4 \in C_4$ with*

$$A_1 Y^{-1} A_4 = \mathbb{1}_{2n}.$$

As with the first coordinate chart, the Deligne-Simpson problems here are again problems on the three-punctured sphere $\Sigma_{0,3}$. They correspond to the star-shaped quivers in Figure 5.3 below.



(a) The quiver for Problem 5.10(i).

(b) The quiver for Problem 5.10(ii).

Figure 5.3: The quivers associated with Problem 5.10.

This dual analysis ultimately leads to the establishment of the following result, completely similar to the way it is done in §5.2 using the quiver ingredients, cf. Proposition 5.3.

Proposition 5.11 *The map* Υ (4.43) *defines an isomorphism* $\mathcal{U}_{\boldsymbol{\sigma}}/W \cong \mathcal{C}_n^{\boldsymbol{\sigma}}$*, where*

$$\mathcal{C}_n^{\boldsymbol{\sigma}} \coloneqq \{(A_1, A_2, A_3, A_4) \in \mathcal{C}_n : \delta(\mathbf{y})\delta_{\boldsymbol{\sigma}}(\mathbf{y}) \neq 0\}. \tag{5.11}$$

5.4 The Main Result

Proofs of our main results can now be obtained by repeating the arguments from [Obl04]. The reader is reminded of the notation $\mathcal{H} = \mathcal{H}_{1,\boldsymbol{\tau}}$ adopted in Notation 4.7, which we again use, as well as the crucial properties of the spherical subalgebra we reference in Theorem 5.15.

Notation 5.12 Throughout, $\mathcal{Z} \coloneqq Z(\mathcal{H})$ is the centre of the DAHA of type $C^\vee C_n$ at $q = 1$.

We will summarise some of the general results about the spherical subalgebra below that will be needed for the proof of the main theorem. But this involves some definitions from commutative algebra and algebraic geometry. Therefore, we first briefly remind the reader of some key definitions that crop up in the theory of algebraic varieties. We can use standard texts once again, say [Har77, §II.8] or [Eis95, §18].

Indeed, let R be a ring and M an R-module. A *regular sequence* for M is a collection of elements $r_1, \ldots, r_k \in R$ such that r_i is not a zero divisor in the quotient $M/\langle r_1, \ldots, r_{i-1}\rangle$. In the case that R is a local ring (that is it has a unique maximal ideal, say $\mathfrak{m}$), the *depth* of R is the maximal length of a regular sequence for which every $r_i \in \mathfrak{m}$. A local Noetherian ring R is *Cohen-Macaulay* if $\operatorname{depth}(R) = \dim(R)$, where the latter denotes the Krull dimension. The condition is used also to define a Cohen-Macaulay *module*, that is one where $\operatorname{depth}(M) = \dim(M)$.

Definition 5.13 A variety V is *Cohen-Macaulay* if all of its local rings $\mathcal{O}_{x,V}$ are Cohen-Macaulay.

Similar to Definition 5.13, a variety V is *normal* if all of its local rings $\mathcal{O}_{x,V}$ are integrally closed domains, meaning that $\mathcal{O}_{x,V}$ is an integral domain with self-closure in its field of fractions.

Theorem 5.14 (cf. [DG65, Theorem 5.8.6], Serre's Normality Criterion) *A variety V is normal if and only if V is regular in codimension one, and the regular maps on $V \setminus Y$ extend to Y for any subvariety Y whose codimension is at least two.*

These conditions are respectively referred to as (R1) and (S2) in the literature. The former states $V_\mathfrak{p}$ is *regular* for all $\mathrm{hgt}(\mathfrak{p}) \leq 1$, meaning the minimum number of generators of its maximal ideal is equal to its Krull dimension $\dim(V_\mathfrak{p})$. The latter condition states that $V_\mathfrak{p}$ has depth at least two for all prime ideals $\mathfrak{p}$ with $\mathrm{hgt}(\mathfrak{p}) \geq 2$.

We also require the following result that is lifted from Oblomkov's paper; his proof is done in sufficient generality to work for any DAHA, not just the type GL_n DAHA he usually works with.

Theorem 5.15 ([Obl04, Theorem 5.1]) *The following are true:*

(i) *The spherical subalgebra $\mathbf{e}\mathcal{H}\mathbf{e}$ is commutative.*

(ii) *The variety $M = \mathrm{Spec}(\mathbf{e}\mathcal{H}\mathbf{e})$ is irreducible, normal and Cohen-Macaulay.*

(iii) *The right $\mathbf{e}\mathcal{H}\mathbf{e}$-module $\mathcal{H}\mathbf{e}$ is Cohen-Macaulay.*

(iv) *The left $\mathcal{H}$-action on $\mathcal{H}\mathbf{e}$ induces an isomorphism of algebras $\mathcal{H} \cong \mathrm{End}_{\mathbf{e}\mathcal{H}\mathbf{e}}(\mathcal{H}\mathbf{e})$.*

(v) *The centre $\mathcal{Z} \xrightarrow{\sim} \mathbf{e}\mathcal{H}\mathbf{e}$ by the Satake isomorphism $\eta : z \mapsto z\mathbf{e}$. Hence, $M = \mathrm{Spec}(\mathcal{Z})$.*

Analogously to [Obl04, Lemma 5.1] (cf. Corollary 3.22), we have

$$\mathcal{Z}_{\delta(\boldsymbol{X})\delta_{\boldsymbol{\tau}}(\boldsymbol{X})} \cong \mathbb{C}[\boldsymbol{X}^{\pm 1}, \boldsymbol{P}^{\pm 1}]^W_{\delta(\boldsymbol{X})\delta_{\boldsymbol{\tau}}(\boldsymbol{X})}.$$

Equivalently, we have an isomorphism $\mathrm{Spec}(\mathcal{Z}_{\delta(\boldsymbol{X})\delta_{\boldsymbol{\tau}}(\boldsymbol{X})}) \cong \mathcal{U}_{\boldsymbol{\tau}}/W$. We obtain from this the fact

$$\Upsilon : \mathrm{Spec}(\mathcal{Z}_{\delta(\boldsymbol{X})\delta_{\boldsymbol{\tau}}(\boldsymbol{X})}) \xrightarrow{\sim} \mathcal{C}_n^{\boldsymbol{\tau}} \tag{5.12}$$

by combining it with Υ (4.43) and Proposition 5.3. Consequently, we have a rational map

$$\Upsilon : \mathrm{Spec}(\mathcal{Z}) \dashrightarrow \mathcal{C}_n. \tag{5.13}$$

One final result we require for the proof of the main theorem is the following.

Lemma 5.16 *In $\mathrm{Spec}(\mathcal{Z})$, any hypersurfaces $f(\boldsymbol{X}) = 0$ and $g(\boldsymbol{Y}) = 0$ intersect transversally.*

Proof: Because we are in the (spectrum of the) centre, $f \in \mathbb{C}[\boldsymbol{X}^{\pm 1}]^W$ and $g \in \mathbb{C}[\boldsymbol{Y}^{\pm 1}]^W$ are W-invariant. By the Basic Representation (at the classical level $q = 1$), we can embed the centre $\mathcal{Z} \hookrightarrow \mathbb{C}(\boldsymbol{X})[\boldsymbol{P}^{\pm 1}]^W$. Consider the prime factorisation of $f(\boldsymbol{X})$ in $\mathcal{Z}$; any factor is of the form

$f'(\boldsymbol{X}) \in \mathbb{C}(\boldsymbol{X})^W$ because there is no algebraic relation between the $\boldsymbol{X}$- and $\boldsymbol{P}$-variables, i.e. the field $\mathbb{C}(\boldsymbol{X}, \boldsymbol{P})$ is a purely transcendental extension of $\mathbb{C}(\boldsymbol{X})$ with the obvious transcendence base $\{P_1, \dots, P_n\}$. But by the Duality Isomorphism, we can argue identically with the dual Basic Representation that any prime factor of $g(\boldsymbol{Y})$ in $\mathcal{Z}$ has the form $g'(\boldsymbol{Y}) \in \mathbb{C}(\boldsymbol{Y})^W$. Finally, it remains to show that each hypersurface has no common factor. Indeed, write $f' = p/q$ for $p, q \in \mathbb{C}[\boldsymbol{X}^{\pm 1}]^W$ and $g' = r/s$ for $r, s \in \mathbb{C}[\boldsymbol{Y}^{\pm 1}]^W$. If there existed a common factor $f'(\boldsymbol{X}) = g'(\boldsymbol{Y})$, this is equivalent to

$$\frac{p(\boldsymbol{X})}{q(\boldsymbol{X})} = \frac{r(\boldsymbol{Y})}{s(\boldsymbol{Y})} \qquad \Rightarrow \qquad p(\boldsymbol{X})s(\boldsymbol{Y}) = q(\boldsymbol{X})r(\boldsymbol{Y}).$$

This uses commutativity; these polynomials are W-invariant and so belong to $\mathcal{Z}$. By the PBW Property (Proposition 3.10), $p(\boldsymbol{X}) = q(\boldsymbol{X})$ and $r(\boldsymbol{Y}) = s(\boldsymbol{Y})$, contradicting primality. □

Theorem 5.17 (cf. [Obl04, Theorem 6.1]) *For generic $\boldsymbol{\tau}$, the rational map Υ (5.13) is a regular isomorphism of algebraic varieties. In particular, $M = \mathrm{Spec}(\mathcal{Z}) = \mathrm{Spec}(\mathbf{e}\mathcal{H}\mathbf{e})$ is smooth.*

Proof: Using duality in the senses of (5.7) and (5.8), we obtain from (5.12) an isomorphism

$$\Upsilon' : \mathrm{Spec}(\mathcal{Z}_{\delta(\boldsymbol{Y})\delta_{\boldsymbol{\sigma}}(\boldsymbol{Y})}) \xrightarrow{\sim} \mathcal{C}_n^{\boldsymbol{\sigma}}$$

for a suitable open subset $\mathcal{C}_n^{\boldsymbol{\sigma}} \subseteq \mathcal{C}_n$. The maps Υ and Υ' agree on the intersection

$$\mathrm{Spec}(\mathcal{Z}_{\delta(\boldsymbol{X})\delta_{\boldsymbol{\tau}}(\boldsymbol{X})}) \cap \mathrm{Spec}(\mathcal{Z}_{\delta(\boldsymbol{Y})\delta_{\boldsymbol{\sigma}}(\boldsymbol{Y})}). \tag{5.14}$$

Indeed, for a point z in (5.14), the corresponding $\mathcal{H}$-modules (4.45) are completely determined by the one-dimensional character χ_z representing this point z, cf. Remark 4.39. As a result, Υ is regular on (5.14). But $\delta(\boldsymbol{X})\delta_{\boldsymbol{\tau}}(\boldsymbol{X}) = 0$ and $\delta(\boldsymbol{Y})\delta_{\boldsymbol{\sigma}}(\boldsymbol{Y}) = 0$ are transversal in $\mathrm{Spec}(\mathcal{Z})$, by Lemma 5.16, from which it follows that Υ is regular everywhere except a subset of codimension two. We know that $\mathrm{Spec}(\mathcal{Z})$ is normal (by Theorem 5.15(ii)), so Υ extends to a regular map on the whole variety by Serre's Normality Criterion. Since the Calogero-Moser space is irreducible (by Corollary 4.4), it follows that Υ is dominant, and thus we can use [Sha13, Theorem 2.21] to guarantee a birational inverse. □

As an immediate consequence, by the same arguments as in [Obl04], we arrive at the following.

Corollary 5.18 (cf. [Obl04, Corollaries 6.1 and 6.2]) (i) *$\mathcal{H}\mathbf{e}$ is a projective $\mathbf{e}\mathcal{H}\mathbf{e}$-module.*

(ii) *$\mathcal{H} = \mathrm{End}(E)$, where E is a vector bundle over $\mathrm{Spec}(\mathcal{Z})$, i.e. $\mathcal{H}$ is an Azumaya algebra.*

(iii) *Every irreducible representation of $\mathcal{H}$ is of the form $V_z = \mathcal{H}\mathbf{e} \otimes_{\mathbf{e}\mathcal{H}\mathbf{e}} \chi_z$ for $z \in \mathrm{Spec}(\mathcal{Z})$.*

(iv) *V_z has dimension $2^n n!$, and is a regular representation of the finite Hecke algebra H_n.*

5.5 Application to the Trigonometric van Diejen System

The spherical subalgebra $\mathbf{e}\mathcal{H}_{1,\boldsymbol{\tau}}\mathbf{e}$ of the DAHA of type $C^\vee C_n$ at the classical level $q = 1$ admits a one-parameter deformation, namely $\mathbf{e}\mathcal{H}_{q,\boldsymbol{\tau}}\mathbf{e}$. This gives rise to a Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$. We can use the isomorphism Υ established in the main result (Theorem 1.3) to carry this bracket across to the Calogero-Moser space $\mathcal{C}_n$. We can then apply our results to study the trigonometric van Diejen integrable system [vD95]. In particular, we show that $\mathcal{C}_n$ is a completed phase space for the trigonometric van Diejen system (which is a $C^\vee C$-analogue of the Ruijsenaars-Schneider system) and obtain explicit log-canonical coordinates on the character variety.

The elements $\mathbf{w}Y^k\mathbf{v}$ belong to the algebra of quantum integrals of the trigonometric van Diejen system, with $\mathbf{w}$ and $\mathbf{v}$ from Lemma 4.32, cf. [Cha19, Proposition 4.5]. At the classical level $q = 1$, the algebra of integrals is generated by $\operatorname{tr} Y^k$. It is difficult to work with these Hamiltonians. As such, we instead work with $\operatorname{tr} X^k$ in $(\boldsymbol{X}, \boldsymbol{P})$ coordinates; showing that they are in involution and computing the dynamics is more straightforward. We can then apply duality (5.8) to conclude analogously for $\operatorname{tr} Y^k$.

The q-difference Macdonald operators are the quantum Hamiltonians of a relativistic version of the trigonometric Calogero-Moser system (in type GL_n) [Rui87]. Per [Sto20], these pairwise-commuting operators arise as the image of a symmetric combination of $\boldsymbol{Y}$-variables under $\operatorname{sym} \circ \beta$, where β is the Basic Representation and the *symmetrising map* $\operatorname{sym} : \mathscr{D}_q \rtimes \mathbb{C}W \to \mathscr{D}_q$ associates to the q-difference-reflection operator $F = \sum_{w\in W} f_w w$ the q-difference operator

$$F_{\mathrm{sym}} \coloneqq \sum_{w\in W} f_w.$$

In type $C^\vee C_n$, one uses q-difference Koornwinder operators (of which the Macdonald operators are a particular case), see [Nou95, Sto04]. Noumi identified $(Y_1 + \cdots + Y_n + Y_1^{-1} + \cdots + Y_n^{-1})_{\mathrm{sym}}$ with a linear combination of a scalar with the q-difference Koornwinder operator

$$\sum_{i=1}^{n} \Big(\Phi_i(\mathbf{x})(t(\varepsilon_i) - 1) + \Phi_i(\mathbf{x}^{-1})(t(-\varepsilon_i) - 1) \Big), \tag{5.15}$$

where $t(\lambda)$ are translations (see §2.1.2) and

$$\Phi_i(\mathbf{x}) = k_0^{-1}k_n^{-1}t^{2n-2}\frac{(1-ax_i)(1-bx_i)(1-cx_i)(1-dx_i)}{(1-x_i^2)(1-q^2x_i^2)}\prod_{j\neq i}\frac{(1-t^2x_ix_j)(1-t^2x_ix_j^{-1})}{(1-x_ix_j)(1-x_ix_j^{-1})},$$

with constants $(a, b, c, d) = (qk_0u_0, -qk_0u_0^{-1}, k_nu_n, -k_nu_n^{-1})$. If one passes to the classical level, this corresponds to $\operatorname{tr} Y$. Indeed, it is possible to use the expressions of the (classical) matrices A_4 and A_1 from Theorems 4.26 and 4.16 to calculate this explicitly; one obtains (5.15) at $q = 1$.

Remark 5.19 This is slightly easier to see if we use the expression for A_1 from [CR24, Proposition 4.8], in particular the factorised functions (4.13) and (4.14) immediately preceding *loc. cit.*.

5.5.1 Poisson Bracket

Recall from §2.5.2 that the Poisson bracket on a commutative algebra A_0 can be read from the commutator on its one-parameter deformation $A_\hbar$. In light of Remark 3.13 then, (2.17) becomes

$$[a, b] = (q^2 - 1)\{\eta_0(a), \eta_0(b)\} + \cdots . \tag{5.16}$$

Proposition 5.20 (i) *The coordinates X_i, P_i on $\mathcal{C}_n$ are log-canonical, that is*

$$\{P_i, X_j\} = \delta_{ij} P_i X_j, \qquad \{P_i, P_j\} = \{X_i, X_j\} = 0.$$

(ii) *The duality map $\mathcal{E}$ (5.8) is a Poisson anti-automorphism, so $\mathcal{E}_*\{\,\cdot\,,\,\cdot\,\}_{\mathcal{C}_n} = -\{\,\cdot\,,\,\cdot\,\}_{\mathcal{C}'_n}$.*

(iii) *The Poisson bracket on $\mathcal{C}_n$ is non-degenerate, so $\mathcal{C}_n$ is symplectic.*

Proof: (i) This follows because the coordinates X_i, P_i correspond to the generators of the algebra $\mathbb{C}[\boldsymbol{X}^{\pm 1}, \boldsymbol{P}^{\pm 1}]$, whose Poisson bracket comes from the quantised algebra $\mathbb{C}_q[\boldsymbol{X}^{\pm 1}, \boldsymbol{P}^{\pm 1}]$. Indeed, $[P_i, P_j] = [X_i, X_j] = 0$ implies the trivial brackets. The deformed commutation relation implies

$$[P_i, X_j] = P_i X_j - X_j P_i = (q^{2\delta_{ij}} - 1) X_j P_i = (q^2 - 1)\delta_{ij} X_j P_i,$$

from which the bracket of the classical (commuting) variables as stated follows from (5.16).

(ii) The duality map $\mathcal{E}$ is induced by the algebra automorphism ε. It therefore respects the Poisson structure on the spherical subalgebra, with the change of sign for the bracket caused by the fact that ε interchanges $q \leftrightarrow q^{-1}$.

(iii) For the non-degeneracy, we observe in (i) that the bracket is log-canonical and thus non-degenerate in both coordinate charts. Therefore, the bracket is non-degenerate except on a subset of codimension two, and hence non-degenerate globally by Hartog's Extension Theorem. □

5.5.2 The First Integrable System

We now work in the first chart (X_i, P_i) and consider the functions $h_k := \operatorname{tr} X^k$ with $X = A_1 A_2$,

$$h_k = \sum_{i=1}^{n} (X_i^k + X_i^{-k}). \tag{5.17}$$

It is clear that these Hamiltonians are in involution, that is $\{h_k, h_l\} = 0$ for all $k, l = 1, \ldots, n$.

Lemma 5.21 *For the Hamiltonians* (5.17) *and* $i = 1, \dots, n$, *we have* $\{X_i, h_k\} = 0$ *and*

$$\{P_i^{\pm 1}, h_k\} = \pm k P_i^{\pm 1}(X_i^k - X_i^{-k}).$$

Proof: View the coordinates $P_i = e^p$ and $X_i = e^q$. The definition of the Poisson bracket tells us $\{P_i^{\pm 1}, X_i^k\} = \{e^{\pm p}, e^{kq}\} = \pm e^{\pm p} \cdot k e^{kq} = \pm k P_i^{\pm 1} X_i^k$. Log-canonicity (Proposition 5.20) implies that all other brackets are zero, and we obtain the stated result via linearity of the bracket. □

This means that the Hamiltonian dynamics governed by h_k is separated in $(\boldsymbol{X}, \boldsymbol{P})$ coordinates:

$$X_i(t) = X_i(0), \qquad P_i(t) = e^{kt(X_i^k - X_i^{-k})} P_i(0).$$

We describe the corresponding dynamics in invariant terms, globally on $\mathcal{C}_n$. It is convenient to do so on the representation variety associated to the conjugacy classes defined by (4.1), that is

$$\mathfrak{R}_n := \mathfrak{R}_{0,4} = \{A_i \in C_i : A_1 A_2 A_3 A_4 = \mathbb{1}_{2n}\}.$$

Recall $\mathcal{C}_n = \mathfrak{R}_n /\!\!/ \mathrm{GL}_{2n}(\mathbb{C})$. Introduce the following $\mathrm{GL}_{2n}(\mathbb{C})$-invariant vector field on $\mathfrak{R}_n$:

$$\dot{A}_1 = -k(A_1 X^k - X^k A_1), \qquad \dot{A}_2 = -k(A_2 X^k - X^k A_2), \qquad \dot{A}_3 = 0, \qquad \dot{A}_4 = 0. \tag{5.18}$$

This can be easily integrated, giving $X = A_1 A_2$ constant and

$$A_1(t) = e^{ktX^k} A_1(0) e^{-ktX^k}, \quad A_2(t) = e^{ktX^k} A_2(0) e^{-ktX^k}, \quad A_3(t) = A_3(0), \quad A_4(t) = A_4(0).$$

The dynamics is called *complete* if the trajectories don't diverge to infinity in finite time given any initial condition, so $A_i(t)$ are well-defined for all $t \in \mathbb{R}$. The GIT-quotient ensures that everything works upon taking a quotient, and that the trajectories on $\mathcal{C}_n$ are projections of those from the auxiliary space $\mathfrak{R}_n$.

Proposition 5.22 *The Hamiltonian dynamics on* $\mathcal{C}_n$ *governed by* $h_k = \operatorname{tr} X^k$ *can be obtained by projecting the dynamics* (5.18) *onto* $\mathcal{C}_n$. *The dynamics is complete on* $\mathcal{C}_n$.

Proof: This can be confirmed by a straightforward calculation in coordinates. Indeed, the fact $\dot{A}_3 = \dot{A}_4 = 0$ is clear from Theorem 4.26; there is no $\boldsymbol{P}$-dependence whatsoever, meaning the entry-wise brackets are zero and thus $\{A_3, h_k\} = \{A_4, h_k\} = 0$, confirming the final two equations in (5.18) on this chart. As for $\dot{A}_1$ and $\dot{A}_2$, we argue only for the former since the latter is essentially identical. Recall from Theorem 4.16 that the only $\boldsymbol{P}$-dependence occurs in the ij^{th} entries where $i - j = \pm n$. The idea is to first use Lemma 5.21 and apply the Poisson bracket entry-wise, and then to compare this to $A_1X^k - X^kA_1$. Indeed,

$$(A_1X^k)_{ij} = \begin{cases} \dfrac{\overline{k}_0 + \overline{u}_0 X_i^{-1}}{1 - X_i^{-2}} X_i^k & \text{if } i = j \\ k_0 P_i^{-1} X_i^k - \dfrac{\overline{k}_0 + \overline{u}_0 X_i^{-1}}{1 - X_i^{-2}} P_i^{-1} X_i^k & \text{if } i - j = \pm n \\ 0 & \text{otherwise} \end{cases}$$

and

$$(X^kA_1)_{ij} = \begin{cases} \dfrac{\overline{k}_0 + \overline{u}_0 X_i^{-1}}{1 - X_i^{-2}} X_i^k & \text{if } i = j \\ k_0 P_i^{-1} X_i^{-k} - \dfrac{\overline{k}_0 X_i + \overline{u}_0}{1 - X_i^{-2}} P_i^{-1} X_i^{-k} & \text{if } i - j = \pm n \\ 0 & \text{otherwise} \end{cases}.$$

Consequently, the difference in question is now easy to see:

$$(A_1X^k - X^kA_1)_{ij} = \begin{cases} 0 & \text{if } i = j \\ \left(k_0 - \dfrac{\overline{k}_0 + \overline{u}_0 X_i^{-1}}{1 - X_i^{-2}}\right) P_i^{-1}(X_i^k - X_i^{-k}) & \text{if } i - j = \pm n \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} -\dfrac{1}{k}\left\{\dfrac{\overline{k}_0 + \overline{u}_0 X_i^{-1}}{1 - X_i^{-2}}, h_k\right\} & \text{if } i = j \\ -\dfrac{1}{k}\left\{k_0 P_i^{-1} - \dfrac{\overline{k}_0 + \overline{u}_0 X_i^{-1}}{1 - X_i^{-2}} P_i^{-1}, h_k\right\} & \text{if } i - j = \pm n \\ -\dfrac{1}{k}\{0, h_k\} & \text{otherwise} \end{cases}.$$

by the Leibniz rule. It is now clear that the formula for $\dot{A}_1$ in (5.18) holds on the coordinate chart. By analytic continuation, the result is valid globally on $\mathcal{C}_n$. Because the dynamics is obviously complete on the auxiliary space $\mathfrak{R}_n$, the completeness descends to $\mathcal{C}_n$. □

5.5.3 The Second Integrable System

Applying the duality map $\mathcal{E}$, one can interchange the roles of $X = A_1A_2$ and $Y = A_4A_1$. The latter matrix Y has been identified in [Cha19, Corollary 4.4] as a Lax matrix for the van Diejen system. In particular, its conserved quantities are given by $H_k \coloneqq \operatorname{tr} Y^k$, so we will take them as

Hamiltonians of this system. The corresponding dynamics on $\mathfrak{R}_n$ takes the form

$$\dot{A}_1 = 0, \qquad \dot{A}_2 = -k(Y^k A_2 - A_2 Y^k), \qquad \dot{A}_3 = -k(Y^k A_3 - A_3 Y^k), \qquad \dot{A}_4 = 0, \qquad (5.19)$$

which integrates to give $Y = A_4 A_1$ constant and

$$A_1(t) = A_1(0), \quad A_2(t) = e^{-ktY^k} A_2(0) e^{ktY^k}, \quad A_3(t) = e^{-ktY^k} A_3(0) e^{ktY^k}, \quad A_4(t) = A_4(0).$$

Theorem 5.23 *The Hamiltonian dynamics on $\mathcal{C}_n$ governed by $H_k = \operatorname{tr} Y^k$ can be obtained by projecting the dynamics (5.19) onto $\mathcal{C}_n$. The dynamics is complete on $\mathcal{C}_n$.*

The second coordinate chart on $\mathcal{C}_n$ provides the action-angle variables for the van Diejen system. The action variables are determined by the eigenvalues of Y, and the angle variables, Q_i, are the dual counterparts of P_i. Correspondingly, the functions $h_k = \operatorname{tr} X^k$ in terms of these action-angle coordinates assume the form of the van Diejen Hamiltonians with dual parameters. This picture is analogous to the Ruijsenaars duality well-known in the GL_n-case [Rui88, FGNR00, FK11], and it is a non-trivial manifestation of the duality for DAHAs. In a special limiting case of the five-parameter van Diejen system, such duality was established in [FM17, FM19].

Chapter 6

Further Work

This PhD thesis has established an isomorphism between a character variety of the four-punctured sphere with prescribed generic semi-simple conjugacy classes (the so-called Calogero-Moser space) and the spherical subalgebra of the DAHA of type $C^\vee C_n$, settling part of a conjecture by Etingof-Gan-Oblomkov [EGO06]. However, there are still a number of interesting related problems that remain open.

One such is an extension of our result to the so-called *spin case*, that is where we alter the representation of the affine $\widetilde{D}_4$ quiver, and interpret this from an integrable systems point-of-view. This would yield a new integrable system: a spin generalisation of Koornwinder-van Diejen system. Also, it should be possible to realise a quantised version (see §6.1) of the isomorphism $\mathbf{e}\mathcal{H}\mathbf{e} \cong \mathbb{C}[\mathcal{C}_n]$ from the point-of-view of quantised quiver varieties and quantum Hamiltonian reduction [ELOR08, Jor14, Wen24]. The aforementioned spin analogue may also be quantisable.

Another intriguing question is whether there are further generalisations for affine $\widetilde{D}_m$ quivers (with $m > 4$). Some recent results by Braverman-Finkelberg-Nakajima [BFN19] on quantised Coulomb branches of $3d$ $\mathcal{N} = 4$ gauge theories suggest that such a generalisation might exist. One approach may be to start with attempts to generalise the quiver variety by combining $\widetilde{D}_4$ and $\widetilde{A}$-type quivers in a new way. This would provide a (yet missing) quiver interpretation of Braverman-Finkelberg-Nakajima's results.

Finally, one can begin looking at the generalised DAHA associated to affine $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$ with the intention of proving the entirety of [EGO06, Conjecture 5.1.1]. This is a challenging task because the corresponding DAHAs are far less understood. However, it may be that the recent work by D. Dal Martello and M. Mazzocco [DMM24] can be harnessed to deal with these more obscure algebras (at least, perhaps, the GDAHA associated with $\widetilde{E}_6$).

6.1 Quantisation

Our main result is an isomorphism between affine varieties, so recall this induces an isomorphism between coordinate rings $\mathbf{e}\mathcal{H}\mathbf{e} \cong \mathbb{C}[\mathcal{C}_n]$. Our calculations of the matrices A_i in §4.3 were done at the quantum level, so it stands to reason that this should generalise this isomorphism to $q \neq 1$; this is the content of Proposition 6.2. But we first continue to work at the classical level $q = 1$ for the next auxiliary result, viewing $\mathcal{C}_n$ using the alternate form from Proposition 4.5 with relations (4.2)–(4.6).

Lemma 6.1 *The algebra $\mathbb{C}[\mathcal{C}_n]$ of regular functions on the Calogero-Moser space is, as an algebra, generated by wX^mY^nv and wX^mY^nTv with $m, n \in \mathbb{Z}$.*

Sketch of Proof: By definition, $\mathbb{C}[\mathcal{C}_n]$ is generated by traces of words $a \in \mathbb{C}\langle A_1^{\pm 1}, A_2^{\pm 1}, A_3^{\pm 1}, A_4^{\pm 1}\rangle$. Using the alternative form of the character variety in Proposition 4.5, it is equivalent to say this algebra is generated by traces of words $a \in \mathbb{C}\langle X^{\pm 1}, Y^{\pm 1}, T^{\pm 1}\rangle$. To show that this is equally generated by $wa(X, Y, T)v$ then, let the *degree* of such a word be defined by counting the total number of $X^{\pm 1}$, $Y^{\pm 1}$ and $T^{\pm 1}$. But we can swap generators of a word by using the relations (4.2)–(4.6), modulo words of smaller degree. This defines a filtration on the set of such words. It suffices to consider X^mY^nT and X^mY^n with $m, n \in \{0, \pm 1\}$. For example, by (4.2) and (4.3),

$$\begin{aligned}
\operatorname{tr}(X^uY^vT) &= \operatorname{tr}(XX^{u-1}Y^vT) \\
&= \operatorname{tr}(X^{u-1}Y^vTX) \\
&= \operatorname{tr}(X^{u-1}Y^v(T^{-1} + \overline{u}_0)X) \\
&= \operatorname{tr}(X^{u-1}Y^vT^{-1}X) + \cdots \\
&= \operatorname{tr}(X^{u-1}Y^v(TX^{-1} + \overline{k}_0)) + \cdots \\
&= \operatorname{tr}(X^{u-1}Y^vTX^{-1}) + \cdots \\
&= \operatorname{tr}(X^{u-2}Y^vT) + \cdots .
\end{aligned}$$

Similar is true of the Y-generators, and this is why we need only consider powers in $\{0, \pm 1\}$. Note that T satisfies a Hecke relation (4.2), so we can automatically replace T by its inverse and a constant. The statement follows by showing that the traces $\operatorname{tr}(X^mY^nT)$ and $\operatorname{tr}(X^mY^n)$ are themselves generated by wX^mY^nTv and wX^mY^nv, respectively; there are 18 total cases. □

If we work at the quantum level with q arbitrary, consider the $\mathscr{D}_q$-valued matrices A_i defined by Theorems 4.16 and 4.26, as well as $X = A_1A_2$, $Y = A_4A_1$, $T = A_2$, $\mathbf{v}$ and $\mathbf{w}$ as in Lemma 4.32.

Proposition 6.2 *The elements $\mathbf{w}b(X, Y, T)\mathbf{v}$ form a subalgebra of $\mathscr{D}_q^W$ isomorphic to $\mathbf{e}\mathcal{H}_{q,\boldsymbol{\tau}}\mathbf{e}$, with arbitrary non-commutative polynomials $b \in \mathbb{C}\langle X^{\pm 1}, Y^{\pm 1}, T^{\pm 1}\rangle$.*

Proof: Recall the elements $Z_i \in \mathcal{H}_{q,\boldsymbol{\tau}}$ from (4.10). For $a \in \mathbb{C}\langle Z_1^{\pm 1}, Z_2^{\pm 1}, Z_3^{\pm 1}, Z_4^{\pm 1}\rangle$, the elements

$$\mathbf{e}a(Z_1, Z_2, Z_3, Z_4)\mathbf{e}$$

clearly belong to the spherical subalgebra $\mathbf{e}\mathcal{H}_{q,\boldsymbol{\tau}}\mathbf{e}$. Moreover, the relation (4.41) implies that these elements form a subalgebra of $\mathbf{e}\mathcal{H}_{q,\boldsymbol{\tau}}\mathbf{e}$, which we denote $A_{q,\boldsymbol{\tau}}$. The matrix representation π (4.14), combined with Lemmata 4.11 and 4.32, defines an isomorphic subalgebra $B_{q,\boldsymbol{\tau}} \subseteq \mathscr{D}_q^W$:

$$A_{q,\boldsymbol{\tau}} \xrightarrow{\sim} B_{q,\boldsymbol{\tau}}, \qquad \mathbf{e}a(Z_1, Z_2, Z_3, Z_4)\mathbf{e} \mapsto \gamma^{-1}\mathbf{w}a(A_1, A_2, A_3, A_4)\mathbf{v},$$

where γ is the constant (4.37). At the classical level $q = 1$, we have $B_{1,\boldsymbol{\tau}} \cong \mathbf{e}\mathcal{H}_{1,\boldsymbol{\tau}}\mathbf{e}$ by Lemma 6.1 combined with Theorem 5.17. Hence, they remain isomorphic for arbitrary q. Finally, any element of the form $\mathbf{w}a(A_1, A_2, A_3, A_4)\mathbf{v}$ can be transformed into $\mathbf{w}b(X, Y, T)\mathbf{v}$ by using the relation $qA_1A_2A_3A_4 = 1$ from Corollary 4.27. □

Corollary 6.3 *The spherical subalgebra* $\mathbf{e}\mathcal{H}_{q,\boldsymbol{\tau}}\mathbf{e}$ *is, as an algebra, generated by* $\mathbf{e}X_1^aY_1^b\mathbf{e}$ *and* $\mathbf{e}X_1^aY_1^bT_0^\vee\mathbf{e}$ *with* $a, b \in \mathbb{Z}$. *Its copy in* $\mathscr{D}_q^W$ *is generated by* $\mathbf{w}X^aY^b\mathbf{v}$ *and* $\mathbf{w}X^aY^bT\mathbf{v}$ *with* $a, b \in \mathbb{Z}$.

6.2 Hamiltonian Reduction

One of the areas we started to study, but are yet to wholly complete, was an interpretation of the Poisson structure on the Calogero-Moser space via Hamiltonian reduction. In general, a character variety of a Riemann surface can be seen as a result of (infinite-dimensional) Hamiltonian reduction, performed on the moduli space of smooth connections [AB83], and so it has a canonical Poisson structure, cf. [Gol84, Gol86]. Fock and Rosly [FR99] explained how to obtain the same space by a finite-dimensional reduction, modelling flat connections by combinatorial connections on graphs embedded into the surface.

Applying this to the one-punctured torus, they obtained the variety CM_τ appearing in Theorem 1.2 and were able to interpret it as a completed phase space for the Ruijsenaars-Schneider system, see [FR99, Appendix] (the observation that the Ruijsenaars-Schneider system naturally arises on the moduli space of flat connections on the one-punctured torus goes back to [GN95]). Carrying out the same approach for the four-punctured sphere would be technically difficult, but we can fortunately arrive at a similar interpretation without much hard work, thanks to the results at hand. Below, we freely use the terminology and notation from [FR99], so the reader should consult that paper for the details.

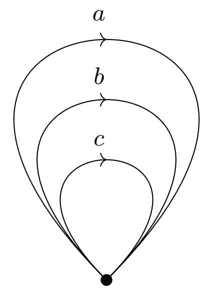


Figure 6.1: A graph corresponding to the four-punctured sphere.

Let us consider the space of G-valued graph connections on the graph in Figure 6.1, representing the four-punctured sphere. It has three edges, to each of which we associate an element of G. Hence, the moduli space of graph connections in this case is $\mathcal{A}^l = G \times G \times G$. The graph has one vertex, so the gauge group $\mathcal{G}^l = G$ acts on $\mathcal{A}^l$ by simultaneous conjugation.

According to [FR99], the choice of a Poisson structure on $\mathcal{A}^l$ is based on a choice of a classical r-matrix. We work with the group $G = \mathrm{GL}_m(\mathbb{C})$ for $m = 2n$, and choose the standard r-matrix

$$r = \sum_{i<j}^{m} E_{ij} \otimes E_{ji} + \frac{1}{2} \sum_{i=1}^{m} E_{ii} \otimes E_{ii}.$$

Remark 6.4 We can express $r = r_a + t$ as the sum of a skew-symmetric and symmetric part, for

$$r_a = \frac{1}{2}(r - r_{21}) \qquad \text{and} \qquad t = \frac{1}{2}(r + r_{21}),$$

where $r_{21} = r^\omega$ is obtained via the action of the tensor swap $\omega : x \otimes y \mapsto y \otimes x$. It is often common to see $r = r_{12}$ in the literature, to further emphasise the order of the tensor factors.

This defines a Poisson bivector on $\mathcal{A}^l$, by [FR99, (16)], which then induces a Poisson bracket on $\mathcal{A}^l/\mathcal{G}^l$. By [FR99, Proposition 5], the resulting Poisson manifold is isomorphic to the moduli space $\mathcal{M}$ of flat connections on the four-punctured sphere, with the Atiyah-Bott Poisson structure. Symplectic leaves correspond to fixing the conjugacy classes of holonomies around the punctures, so we choose the conjugacy classes C_i defined by (4.1), resulting in our character variety of the Calogero-Moser space $\mathcal{C}_n$. In terms of $(\mathbf{A}, \mathbf{B}, \mathbf{C}) \in \mathcal{A}^l$, this means imposing the constraints

$$\mathbf{A} \in C_1, \qquad \mathbf{A}^{-1}\mathbf{B} \in C_2, \qquad \mathbf{B}^{-1}\mathbf{C} \in C_3, \qquad \mathbf{C}^{-1} \in C_4.$$

Therefore, a priori, we have two Poisson brackets on $\mathcal{C}_n$: the Atiyah-Bott bracket $\{\,\cdot\,,\,\cdot\,\}_{\mathrm{AB}}$ and the bracket coming from the DAHA $\{\,\cdot\,,\,\cdot\,\}_{\mathrm{DAHA}}$. We claim that these brackets are the same.

Before the proof, we write the Poisson brackets on $\mathcal{A}^l$, which are analogues of [FR99, (A2)–(A4)]:

$$\{\mathbf{A},\mathbf{A}\} = r_a(\mathbf{A}\otimes\mathbf{A}) + (\mathbf{A}\otimes\mathbf{A})r_a + (1\otimes\mathbf{A})r_{21}(\mathbf{A}\otimes 1) - (\mathbf{A}\otimes 1)r(1\otimes\mathbf{A}), \tag{6.1}$$

$$\{\mathbf{A},\mathbf{B}\} = r(\mathbf{A}\otimes\mathbf{B}) - (\mathbf{A}\otimes\mathbf{B})r_{21} + (1\otimes\mathbf{B})r_{21}(\mathbf{A}\otimes 1) - (\mathbf{A}\otimes 1)r(1\otimes\mathbf{B}), \tag{6.2}$$

with $\{\mathbf{B},\mathbf{B}\}$, $\{\mathbf{C},\mathbf{C}\}$ completely similar to (6.1), and $\{\mathbf{A},\mathbf{C}\}$, $\{\mathbf{B},\mathbf{C}\}$ completely similar to (6.2). Any other brackets are found by using anti-commutativity and the Leibniz rule.

Proposition 6.5 *The isomorphism* $\mathbf{e}\mathcal{H}\mathbf{e} \cong \mathbb{C}[\mathcal{C}_n]$ *is a Poisson map, that is it identifies the natural Poisson bracket on the spherical subalgebra with the Atiyah-Bott bracket on the character variety.*

Proof: Working on $\mathcal{A}^l$ with the Fock-Rosly bracket, we take $\operatorname{tr}\mathbf{B}^k$ and calculate its brackets with $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$. This is similar to the way [FR99, (A6)–(A8)] are derived, and the result is

$$\{\mathbf{A},\operatorname{tr}\mathbf{B}^k\} = -k(\mathbf{A}\mathbf{B}^k - \mathbf{B}^k\mathbf{A}), \qquad \{\mathbf{B},\operatorname{tr}\mathbf{B}^k\} = 0, \qquad \{\mathbf{C},\operatorname{tr}\mathbf{B}^k\} = 0. \tag{6.3}$$

Upon the identification $\mathbf{A} = A_1$, $\mathbf{B} = A_1A_2 = X$ and $\mathbf{C} = A_4^{-1}$, the vector field $\{\,\cdot\,,\operatorname{tr}\mathbf{B}^k\}$ is the same as in (5.18). Projecting onto $\mathcal{C}_n \cong \mathcal{A}^l/\mathcal{G}^l$, it becomes clear that $\operatorname{tr}X^k$ defines the same vector field with respect to each of the Poisson brackets, which is to say

$$\{\,\cdot\,,\operatorname{tr}X^k\}_{\mathrm{AB}} = \{\,\cdot\,,\operatorname{tr}X^k\}_{\mathrm{DAHA}}.$$

Similarly, for $Y = \mathbf{C}^{-1}\mathbf{A}$, we calculate brackets between $\operatorname{tr}Y^k$ and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and notice

$$\{\,\cdot\,,\operatorname{tr}Y^k\}_{\mathrm{AB}} = \{\,\cdot\,,\operatorname{tr}Y^k\}_{\mathrm{DAHA}}.$$

We see now that $\operatorname{tr}X^k$ and $\operatorname{tr}Y^k$ are in the kernel of $\{\,\cdot\,,\,\cdot\,\} \coloneqq \{\,\cdot\,,\,\cdot\,\}_{\mathrm{AB}} - \{\,\cdot\,,\,\cdot\,\}_{\mathrm{DAHA}}$. But the $2n$ functions $(\operatorname{tr}X^k,\operatorname{tr}Y^k)_{k=1,\ldots,n}$ are functionally independent generically on $\mathcal{C}_n$, so $\{\,\cdot\,,\,\cdot\,\} \equiv 0$. □

Corollary 6.6 *The coordinates* X_i, P_i *on* $\mathcal{C}_n$ *are log-canonical with respect to the Atiyah-Bott bracket. The spherical subalgebra* $\mathbf{e}\mathcal{H}_{q,\boldsymbol{\tau}}\mathbf{e}$ *provides a quantisation of the character variety* $\mathcal{C}_n$*, equipped with the Atiyah-Bott bracket.*

We conclude that the symplectic leaf $\mathcal{C}_n$ of the moduli space $\mathcal{M}$ of flat $\mathrm{GL}_{2n}(\mathbb{C})$-connections on the four-punctured sphere (chosen by specifying the conjugacy classes) serves as a completed phase space for the van Diejen system.

Remark 6.7 The variety $\mathcal{C}_n$, as a multiplicative quiver variety, seems obtainable by quasi-Hamiltonian reduction by the result of Van den Bergh [VdB08]; see also [MT14]. We expect the resulting Poisson bracket to coincide with the Atiyah-Bott bracket.

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Appendix A

Irreducibility of Calogero-Moser Space

The celebrated result [Mel20, Theorem 7.12] formulates the Poincaré polynomial of a $\mathrm{GL}_m(\mathbb{C})$-character variety of a punctured Riemann surface. Connectedness, and thus irreducibility in the smooth case, follows by proving that the constant term of this polynomial is one. In this appendix, we detail an asymptotics argument different-yet-akin to the proof of the general result (see Theorem 2.14) given in [HLRV13]. Note that our work is not dependant on this section.

Recall that the k^{th} *Betti number* of a topological space counts the number of k-dimensional holes. More formally, it is the rank of the k^{th} *homology group* $H_k(X) = \ker(\partial_k)/\operatorname{im}(\partial_{k+1})$, where $\partial_k : C_k \to C_{k-1}$ is a homomorphism between Abelian groups for each $k \in \mathbb{N}$. In words, we say that this group contains equivalence classes of *cycles* that don't appear as the boundary of a $(k+1)$-dimensional submanifold. From these Betti numbers, we can write down a generating function called the *Poincaré polynomial*. Our task is to prove that the zeroth Betti number (the number of connected components) of Calogero-Moser space $\mathcal{C}_n$ is one.

To this end, the paper [Mel20] of Anton Mellit provides a proof of a conjecture by Hausel, Letellier and Rodriguez-Villegas, which gives an explicit expression for the Poincaré polynomial of $\mathfrak{M}_{g,k}$ in the semi-simple generic case. We will state his theorem only for the $\mathfrak{M}_{0,4}$ character variety (the case of higher genus has some additional complexity that we need not worry about).

Notation A.1 Throughout, $\mathbf{X} = (x_1, x_2, \dots)$ denotes a sequence of infinitely-many variables. The ring of symmetric functions in these variables is then denoted $\mathrm{Sym}[\mathbf{X}]$, an element of which is denoted $f[\mathbf{X}]$. We then use subscripts $\mathbf{X}_i$ to distinguish different collections of infinitely-many variables. The square brackets indicates a *plethystic substitution* (see Definition A.6).

Finally, because the character variety $\mathfrak{M}_{0,4}$ is smooth (see Theorem 2.13), we have access to de Rham's Theorem and the main theorem we use in this section is stated for cohomology.

Theorem A.2 ([Mel20, cf. Theorem 7.12], Mellit's Theorem) *Let $\mathfrak{M}_{0,4}$ have semi-simple generic conjugacy classes. Then, its Poincaré polynomial $\sum_{i=0}^{4n}(-1)^i q^{\frac{i}{2}} \dim(H^i(\mathfrak{M}_{0,4},\mathbb{C}))$ is given by*

$$P(\mathfrak{M}_{0,4}, q) = q^n \left\langle \mathbb{H}^{\mathrm{HLV}}_{0,4}[\underline{\mathbf{X}}; q^{-1}, 1, T]\Big|_{T^{2n}}, \prod_{i=1}^{4} h_{\mu^i}[\mathbf{X}_i] \right\rangle,$$

where μ^i is the tuple of multiplicities for the i^{th} conjugacy class and $\underline{\mathbf{X}}$ denotes $\mathbf{X}_1, \dots, \mathbf{X}_4$.

A.1 Symmetric Functions

Much of the understanding of Mellit's Theorem comes via symmetric function theory, for which [Mac95] is a brilliant reference that we will rely on. Before we start to dissect Theorem A.2, let's first recall some important symmetric polynomials.

Definition A.3 Let $\mathbf{X} = (x_1, x_2, \dots)$ be a collection of infinitely-many variables and $n \in \mathbb{Z}^+$.

(i) An elementary symmetric function is the sum of distinct products of n variables:

$$e_n[\mathbf{X}] = \sum_{k_1 < \dots < k_n} x_{k_1} \cdots x_{k_n}.$$

(ii) A homogeneous symmetric function is the sum of distinct monomials of degree n:

$$h_n[\mathbf{X}] = \sum_{k_1 \le \dots \le k_n} x_{k_1} \cdots x_{k_n}.$$

(iii) A power-sum symmetric function is the sum of distinct powers of n:

$$p_n[\mathbf{X}] = \sum_k x_k^n.$$

Recall a *partition* of a positive integer $m \in \mathbb{Z}^+$ is a tuple $\mu = (\mu_1, \dots, \mu_\ell)$ of positive integers in descending order with $\mu_1 + \dots + \mu_\ell = m$. The positive integer m is called its *size*, denoted $|\mu|$. The set of all such partitions is denoted $\mathcal{P}_m$. The corresponding *partition statistic* is defined as

$$n(\mu) = \sum_{i=1}^{\ell} \mu_i (i-1). \tag{A.1}$$

We can extend the notion of a partition by considering μ as having infinitely-many zeros after the final positive entry, i.e. $\mu_{\ell+1} = \mu_{\ell+2} = \dots = 0$. In this way, we can extend the definitions of the functions in Definition A.3 to ones on partitions, not just integers. Well, for $\mu \in \mathcal{P}_m$, define

$$e_\mu := \prod_{i=1}^{\ell} e_{\mu_i}[\mathbf{X}], \qquad h_\mu := \prod_{i=1}^{\ell} h_{\mu_i}[\mathbf{X}], \qquad p_\mu := \prod_{i=1}^{\ell} p_{\mu_i}[\mathbf{X}]. \tag{A.2}$$

Definition A.4 Let $\mathbf{X} = (x_1, x_2, \dots)$ be a collection of infinitely-many variables and $\mu \in \mathcal{P}_m$. The monomial symmetric function is the sum of all monomials whose exponents are the parts in μ:

$$m_\mu = \sum_{\sigma \in S_\infty} \prod_k x_k^{\mu_{\sigma(k)}},$$

that is where we sum over all permutations of the parts in the partition $\mu = (\mu_1, \dots, \mu_\ell, 0, 0, \dots)$.

It turns out that any symmetric function can be written uniquely in terms of the $\{h_\mu\}$, and also in terms of the $\{m_\nu\}$, where $\mu, \nu \in \mathcal{P}$ are partitions. In other words, these two collections form bases of $\mathrm{Sym}[\mathbf{X}]$ viewed as a $\mathbb{Z}$-module. It turns out that these two bases are *dual* to one another.

Definition A.5 ([Mac95, (4.5)]) The Hall pairing is a scalar product defined on $\mathrm{Sym}[\mathbf{X}]$ by

$$\langle h_\mu, m_\nu \rangle = \delta_{\mu\nu},$$

where $\mu, \nu \in \mathcal{P}$ are partitions and the right-hand side is the Kronecker delta.

As is done in [HLRV11, (2.3.1)], we extend the Hall pairing definition as to cover a collection $\underline{\mathbf{X}}$ of infinitely-many variables. Indeed, let $\mathrm{Sym}[\mathbf{X}_1, \dots, \mathbf{X}_k] = \mathrm{Sym}[\mathbf{X}_1] \otimes \cdots \otimes \mathrm{Sym}[\mathbf{X}_k]$ and declare

$$\langle f_1[\mathbf{X}_1] \cdots f_k[\mathbf{X}_k], g_1[\mathbf{X}_1] \cdots g_k[\mathbf{X}_k] \rangle := \langle f_1[\mathbf{X}_1], g_1[\mathbf{X}_1] \rangle \cdots \langle f_k[\mathbf{X}_k], g_k[\mathbf{X}_k] \rangle. \tag{A.3}$$

Looking at the content of Theorem A.2, we have described the inner product and the homogeneous symmetric functions $h_{\mu_{ij}}$ appearing in the right-hand input. It remains to introduce the generating function $\mathbb{H}_{0,4}^{\mathrm{HLV}}$, which has a complicated definition in terms of an operation on symmetric functions called the *plethystic logarithm*. Although this is rather difficult to use explicitly, we can use some work in [HLRV11] to get a better combinatorial grasp on things. We provide some narrative on the general idea, but refer the reader to [HLRV11, §2.3.3] for more details.

Definition A.6 Let $f \in \mathrm{Sym}[\mathbf{X}]$ and $M = m_1 + m_2 + \cdots$ be a formal sum of monomials. The plethystic substitution $f[A]$ is the formal series obtained by writing f in the basis of power sum symmetric functions p_n and making the substitution $p_n = x_1^n + x_2^n + \cdots \mapsto m_1^n + m_2^n + \cdots$, i.e.

$$f = \sum_{n=1}^{\infty} a_n p_n(x_1, x_2, \dots) \mapsto \sum_{n=1}^{\infty} a_n p_n(m_1, m_2, \dots) =: f[M].$$

As discussed in Notation A.1, we denote our symmetric functions using square brackets $f[\mathbf{X}]$. This is now consistent with Definition A.6 in the following way: if we view $\mathbf{X}$ as denoting the formal sum $x_1 + x_2 + \cdots$ of all our variables, then the plethystic substitution of $\mathbf{X}$ into f is

$$f[\mathbf{X}] = f(x_1, x_2, \dots).$$

We now refer to [HLRV11, (2.3.3)] for the discussion on the *plethystic exponential* and its inverse, the *plethystic logarithm*. The former PExp is a map which takes a formal power series in one variable with coefficients in Sym[$\mathbf{X}$] *without* constant term to one with a constant term, whose inverse PLog is a map in the opposite direction. We will see a simple example of each below.

Example A.7 (cf. [Mel18, §2]) Let $f[\mathbf{X}] = \mathbf{X}$, meaning $f(x_1, x_2, \dots) = x_1 + x_2 + \cdots$. Then,

$$\mathrm{PExp}[\mathbf{X}] = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} p_n[\mathbf{X}]\right),$$

where p_n is the power-sum symmetric function from Definition A.3. Its inverse is

$$\mathrm{PLog}[1 + \mathbf{X}] = \sum_{n=1}^{\infty} \frac{\ddot{\mu}(n)}{n} p_n[\log(1 + \mathbf{X})],$$

where $\ddot{\mu}(n)$ is the *Möbius function* (the umlaut is used to distinguish it from a partition μ); recall this is zero if n is not square-free and, in the square-free case, either ± 1 depending on whether the number of distinct prime factors is even or odd, respectively.

A.2 The HLRV Kernel

We are close to being able to introduce $\mathbb{H}_{0,4}^{\mathrm{HLV}}$, but the final piece of the set-up we require is some theory developed by Garsia and Haiman [GH96], namely the concept of (*modified*) *Macdonald polynomials*. These are also rather complicated, but controlled combinatorially by partitions. With this in mind, we first recall an important (partial) order on the set $\mathcal{P}_m$ of partitions of m.

Definition A.8 Let $\lambda, \mu \in \mathcal{P}_m$. The dominance (partial) ordering on $\mathcal{P}_m$ is where we declare $\lambda \preceq \mu$ if, for every $k \geq 1$, the sum of the first k parts of λ is at most the sum of the first k parts of μ:

$$\lambda_1 + \cdots + \lambda_k \leq \mu_1 + \cdots + \mu_k.$$

At this point, we will recall a pictorial representation of partitions via so-called *Young diagrams*. Namely, a partition $\mu = (\mu_1, \dots, \mu_\ell)$ is represented by ℓ-rows of boxes in the English style, that is the first (top) row contains μ_1 boxes, the second row contains μ_2 boxes, and so forth.

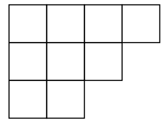

Figure A.1: The Young diagram representing $\mu = (4, 3, 2)$.

Definition A.9 Let $\mu \in \mathcal{P}_m$ be a partition. The dual partition is the unique $\mu' \in \mathcal{P}_m$ obtained by taking the transpose of its Young diagram and extracting from it the corresponding partition.

In other words, the dual to $\mu = (4, 3, 2)$ is precisely $\mu' = (3, 3, 2, 1)$. Indeed, the transpose of the Young diagram in Figure A.1 means we obtain the partition whose parts are now represented by the columns of the original diagram from left-to-right. Before we move on, let's remind the reader that the *arm-length* a of a Young cell is defined to be the number of cells to its right, and the *leg-length* l is defined to be the number of cells below it.

Definition A.10 ([Hai99, Proposition 2.6]) Let $M_{\preceq \mu} := \{m_\lambda : \lambda \preceq \mu\}$ be the subspace spanned by monomial symmetric functions with respect to the dominance order $\preceq$. The (modified) Macdonald polynomials $\widetilde{H}_\mu$ are the unique polynomials that satisfy the following:

(i) $\widetilde{H}_\mu[(t-1)\mathbf{X}] \in M_{\preceq \mu}$.

(ii) $\widetilde{H}_\mu[(q-1)\mathbf{X}] \in M_{\preceq \mu'}$.

(iii) $\widetilde{H}_\mu[1] = 1$.

Remark A.11 There is a technicality we haven't overtly mentioned in Definition A.10, namely the involvement of the parameters q and t. When we define the Macdonald polynomials, we are actually working now with symmetric functions in infinitely-many variables $\mathbf{X} = (x_1, x_2, \dots)$ whose coefficients lie in the field of rational functions $\mathbb{Q}(q, t)$. In fact, $\{\widetilde{H}_\mu\}_{\mu \in \mathcal{P}}$ is a basis of $\mathbb{Q}(q, t) \otimes \mathrm{Sym}[\mathbf{X}]$, cf. [Mac95, Chapter IV, (4.7)]. Consequently, we denote $\widetilde{H}_\mu = \widetilde{H}_\mu[\mathbf{X}; q, t]$.

One can use the Hall pairing to define a new scalar product called the *Macdonald pairing*. This is done in [Mel20, Porposition 2.7], for example, wherein Mellit explains that the Macdonald polynomials are orthogonal with respect to this new pairing. More interestingly, there is an explicit combinatorial formula for the Macdonald pairing of $\widetilde{H}_\mu$ with itself given in terms of arm-lengths and leg-lengths of the Young diagram representing μ.

Proposition A.12 ([Mel20, Proposition 2.7]) *The Macdonald polynomials* $\widetilde{H}_\mu$ *from* Definition A.10 *are orthogonal with respect to the* Macdonald pairing*, which is defined below via the Hall pairing:*

$$\langle F[\mathbf{X}], G[\mathbf{X}] \rangle_{q,t} := \langle F[\mathbf{X}], G[(q-1)(1-t)\mathbf{X}] \rangle.$$

Moreover, because the Hall pairing is non-degenerate, one can obtain the explicit formula

$$\langle \widetilde{H}_\mu, \widetilde{H}_\mu \rangle_{q,t} = \prod_{a,l} (q^{a+1} - t^l)(q^a - t^{l+1}),$$

where the product is over the arm-lengths a and leg-lengths l of the Young cells of $\mu \in \mathcal{P}$.

We have finally established everything necessary to meaningfully discuss $\mathbb{H}^{\mathrm{HLV}}_{0,4}$. Recall that we mentioned the involvement of the plethystic logarithm, which is still somewhat complicated. Although the next definition involves PLog, we will soon extend the notion of a partition to that of a *type*, which will allow us to remove all explicit mention of the plethystics. First, to alleviate notation, we call the reciprocal of the following Macdonald pairing the *deformed hook product*:

$$\mathcal{H}_\mu \coloneqq \frac{1}{\langle \widetilde{H}_\mu, \widetilde{H}_\mu \rangle_{q,t}} = \prod_{a,l} \frac{1}{(q^{a+1} - t^l)(q^a - t^{l+1})}. \tag{A.4}$$

Definition A.13 The HLV kernel is the quantity appearing in the left input of Theorem A.2, i.e.

$$\mathbb{H}^{\mathrm{HLV}}_{0,k} = (q-1)(1-t)\,\mathrm{PLog}\left(\sum_{\mu \in \mathcal{P}} \mathcal{H}_\mu \prod_{i=1}^{k} \widetilde{H}_\mu[\mathbf{X}_i; q, t] T^{|\mu|} \right).$$

A.3 From Partitions to Types

To circumvent the plethystic logarithm issue in Definition A.13, we will use an important notion introduced in [HLRV11, §2.3.1]. First, recall that the *lexicographic ordering* where we declare $\lambda <_{\mathrm{lex}} \mu$ if there exists i such that $\lambda_j = \mu_j$ for all $j < i$ but then $\lambda_i < \mu_i$. This is a total ordering and is implied by the dominance ordering. Now, following [HLRV11, §2.3.1], consider the set of pairs $(d, \mu) \in \mathbb{Z}^+ \times \mathcal{P}^*$ of integers and non-zero partitions, on which there is a total ordering $\geq$:

- If $d > e$, then $(d, \lambda) > (e, \mu)$.
- If $|\lambda| > |\mu|$, then $(d, \lambda) > (d, \mu)$.
- If $|\lambda| = |\mu|$ but $\lambda >_{\mathrm{lex}} \mu$, then $(d, \lambda) \geq (d, \mu)$.

Definition A.14 A type is a sequence of pairs $\omega \coloneqq (d_1, \mu^1) \cdots (d_\ell, \mu^\ell)$ where $(d_i, \mu^i) \geq (d_{i+1}, \mu^{i+1})$ with respect to the above total order for each i, and upper subscripts are used to avoid confusion with the parts of a partition. The set of types is denoted $\mathcal{T}$. We call the d_i the degrees of ω and ℓ the length of ω. The size of the type ω is defined as $|\omega| \coloneqq \sum_{i=1}^{\ell} d_i |\mu^i|$.

Let's now discuss how we can extend a symmetric function defined on partitions to one defined on types. There is a blueprint for exactly this in [HLRV11, §2.3.2]. Indeed, let $A_\mu[\mathbf{X}_1, \ldots, \mathbf{X}_n; q, t]$

be defined on partitions where $A_{(0)} = 1$. This extends to a type ω of length ℓ as follows:

$$A_\omega[\mathbf{X}_1, \ldots, \mathbf{X}_n; q, t] := \prod_{i=1}^{\ell} A_{\mu^i}[\mathbf{X}_1^{d_i}, \ldots, \mathbf{X}_n^{d_i}; q^{d_i}, t^{d_i}]. \tag{A.5}$$

Definition A.15 Let $\omega = (d_1, \mu^1) \cdots (d_\ell, \mu^\ell)$ be a type. The type coefficient is the number

$$C_\omega := \begin{cases} \dfrac{\mu(d)}{d}(-1)^{\ell-1} \dfrac{(\ell-1)!}{\prod_\mu \mathrm{mult}_\omega(d,\mu)!} & \text{if } d_i = d \text{ for all } i \\ 0 & \text{otherwise} \end{cases},$$

where $\ddot{\mu}(d)$ is the *Möbius function* and $\mathrm{mult}_\omega(d, \mu)$ is the number of appearances of (d, μ) in ω.

It turns out this is sufficient to avoid the plethystics woven into the statement of Theorem A.2, in particular the plethystic logarithm in the HLV kernel from Definition A.13. Indeed, [HLRV11, (2.3.9)] expresses the plethystic logarithm of a sum indexed by partitions now as a sum indexed by types. This is almost a manual replacement of μ by ω, albeit with an appearance by C_ω:

$$\mathrm{PLog}\left(\sum_{\mu \in \mathcal{P}} A_\mu T^{|\mu|}\right) = \sum_{\omega \in \mathcal{T}} C_\omega A_\omega T^{|\omega|}. \tag{A.6}$$

We will now apply (A.6) to the right-hand side of Mellit's Theorem. But first, we will define

$$\mathbb{H}_{0,4}(q,t) := \left\langle \mathbb{H}_{0,4}^{\mathrm{HLV}}[\underline{\mathbf{X}}, q, t, T]\Big|_{T^{2n}}, \prod_{i=1}^{4} h_{\mu^i}[\mathbf{X}_i] \right\rangle. \tag{A.7}$$

Note the $\big|_{T^{2n}}$ means we extract from the series expansion in T the coefficient of T^{2n}. Looking at (A.6), this amounts to considering partitions/types of size $2n$. With this in mind, and using the multiplicativity of the extended Hall pairing established in (A.3), we can re-write (A.7) as

$$\mathbb{H}_{0,4}(q,t) = \sum_{|\omega|=2n} (q-1)(1-t) C_\omega \mathcal{H}_\omega \prod_{i=1}^{4} \left\langle \widetilde{H}_\omega[\mathbf{X}_i; q, t], h_{\mu^i}[\mathbf{X}_i] \right\rangle. \tag{A.8}$$

Notation A.16 To simplify notation, we will henceforth refer to (A.8) in the following way:

$$\mathbb{H}_{0,4}(q,t) =: \sum_{|\omega|=2n} \mathbb{H}_{0,4}^{\omega}(q,t).$$

Notice that the right-hand side of the expression in Theorem A.2 is $q^n \mathbb{H}_{0,4}(q^{-1}, 1)$. Recall we are interested in the constant term (the zeroth Betti number/the number of connected components). This is the q^{-n}-term in $\mathbb{H}_{0,4}(q^{-1}, 1)$, which means it is the q^n-term in $\mathbb{H}_{0,4}(q, 1)$. Our task is to isolate this term from the summation in ω above in (A.8) and show that its coefficient is one.

A.4 Kostka Coefficients

The first step on our journey to better understanding the Hall pairing, rather the Macdonald polynomials, within each $\mathbb{H}^{\omega}_{0,4}(q, t)$ is to first recall yet more combinatorics regarding the Young diagram representation of a partition.

Definition A.17 Let $\mu \in \mathcal{P}_m$ be a partition. A Young tableaux T of shape μ is a filling of the Young cells in the associated Young diagram of μ with integers from the set $\{1, 2, \dots, m\}$.

(i) T is semi-standard if the filling (allowing repetition) is *weakly* increasing along the rows and *strictly* increasing down the columns. The set of such tableaus is denoted $\mathrm{SSYT}(\mu)$.

(ii) T is standard if the filling (without repetition) is *strictly* increasing both along the rows and down the columns. The set of such tableaus is denoted $\mathrm{SYT}(\mu)$.

1	2	6	6
2	3	7	
4	8		

Figure A.2: A semi-standard Young tableaux of shape $\mu = (4, 3, 2)$.

Definition A.18 Let $\lambda \in \mathcal{P}$ be a partition and $\mu \in \mathbb{N}^k$ a tuple of integers. The Kostka number $K_{\lambda\mu}$ is the number of semi-standard Young tableaux of shape λ with *weight* μ, meaning the filling of the tableaux is done with μ_i copies of i for each $i = 1, \dots, k$.

At the moment, the second index μ is rather general. However, it is somewhat clear that there are no semi-standard Young tableaux of shape $\mu \in \mathcal{P}_m$ if the weight λ has more than m parts (because then we are trying to fit more labels in than there are Young cells). In fact, we can rapidly reduce the number of considerations with the following basic lemma.

Lemma A.19 ([Sta99, Proposition 7.10.5]) *The Kostka number $K_{\lambda\mu} \neq 0$ if and only if $\lambda, \mu \in \mathcal{P}_m$ are partitions of the same size and the former dominates the latter, meaning $\lambda \succeq \mu$.*

It turns out that the Kostka numbers generate, in the monomial basis, an important type of symmetric function. This will be a crucial ingredient for getting a grasp on the workings of both the Macdonald polynomials and the Hall pairing. Let's now define the aforementioned functions.

Definition A.20 Let $\mathbf{X} = (x_1, x_2, \dots)$ be a collection of infinitely-many variables and $\mu \in \mathcal{P}$. The Schur symmetric function is the sum of monomials in which the power k_i of x_i is the number of appearances of i in the filling of the Young tableaux μ in a semi-standard way, that is

$$s_\mu[\mathbf{X}] = \sum_{T \in \mathrm{SSYT}(\mu)} x_1^{k_1} x_2^{k_2} \cdots .$$

The key fact [GP92, (I.2)] is that the Schur functions decompose in the monomial basis as

$$s_\nu = \sum_\eta K_{\nu\eta} m_\eta. \tag{A.9}$$

But from [Hai01, (7)], one sees that we can write the Macdonald polynomials in the Schur basis:

$$\widetilde{H}_\mu = \sum_\nu \widetilde{K}_{\nu\mu}(q,t) s_\nu. \tag{A.10}$$

Remark A.21 Macdonald defines so-called *q,t-Kostka polynomials* $K_{\nu\mu}(q,t)$ in [Mac95, Chapter VI, (8.11)] as the coefficients appearing in the Schur basis expansion of a so-called integral form. The coefficients appearing in the above expansion (A.10) are *modified q,t-Kostka polynomials*. We often drop the "modified" because we rarely consider the usual polynomials, but the relation between the two is made explicit in [Hai01, p. 5], for example, and will be rather useful later:

$$\widetilde{K}_{\nu\mu}(q,t) = t^{n(\mu)} K_{\nu\mu}(q,t), \tag{A.11}$$

where $n(\mu)$ is the partition statistic from (A.1). The reader should note this neat relationship:

$$K_{\nu\mu} = K_{\nu\mu}(0,1) = \widetilde{K}_{\nu\mu}(0,1).$$

It is absolutely non-obvious that the q,t-Kostka polynomials (i) are Laurent polynomials, and (ii) have non-negative integer coefficients. This was conjectured in [GH96], the *Macdonald Positivity Conjecture*, and proven a few years later by Haiman as a corollary of [Hai01, Proposition 3.7.3].

Observe that we can combine (A.9) with (A.10) to obtain a decomposition of $\widetilde{H}_\mu$ in the monomial basis. This will then allow us to apply the Hall pairing. The final comment we make before proceeding with some auxiliary calculations is this: recall that the μ^i in Theorem A.2 are the multiplicities of the eigenvalues in each conjugacy class. In the context of the Calogero-Moser space $\mathcal{C}_n$, this eigendata is that from (4.1). Explicitly, the partitions we work with are

$$\mu^1 = (n,n), \qquad \mu^2 = (n,n), \qquad \mu^3 = (n,n), \qquad \mu^4 = (n, n-1, 1).$$

Lemma A.22 *We have the following behaviours of the Kostka numbers for $\nu \in \mathcal{P}_{2n}$:*

(i) $K_{\nu(n,n)} \neq 0$ *if and only if* $\nu = (a,b)$, *with* $b = 0$ *allowed.*

(ii) $K_{\nu(n,n-1,1)} \neq 0$ *if and only if* $\nu = (a,b,1)$ *or* $\nu = (\alpha,\beta)$, *with* $\beta = 0$ *allowed.*

Proof: Per Lemma A.19, we consider partitions ν that dominate each of (n,n) and $(n,n-1,1)$.

(i) The partitions that dominate (n,n) are precisely those with two parts and the first part larger than n; this is captured by $\nu = (a,b)$ with allowing $b = 0$, i.e. $\nu = (2n)$.

(ii) The partitions that dominate $(n,n-1,1)$ are precisely those with either last part one and first part larger than n or with two parts and the first part at least n; this is captured by $\nu = (a,b,1)$ and $\nu = (\alpha,\beta)$ with allowing $\beta = 0$, i.e. $\nu = (2n)$. □

Corollary A.23 *We have the following values of the Kostka numbers:*

(i) $$K_{\nu(n,n)} = \begin{cases} 1, & \text{if } \nu = (a,b) \text{ or } (2n) \\ 0, & \text{otherwise} \end{cases}.$$

(ii) $$K_{\nu(n,n-1,1)} = \begin{cases} 1, & \text{if } \nu = (a,b,1) \text{ or } (n,n) \text{ or } (2n) \\ 2, & \text{if } \nu = (a,b) \neq (n,n) \\ 0, & \text{otherwise} \end{cases}.$$

Proof: We here use the interpretation that $K_{\nu\mu}$ counts semi-standard Young tableaux of shape ν and weight μ, the latter of which tells us to fill it with μ_1 ones, μ_2 twos, μ_3 threes, and so on. We know from Lemma A.22 the non-zeroness of $K_{\nu(n,n)}$ and $K_{\nu(n,n-1,1)}$.

(i) The only non-zero $K_{\nu(n,n)}$ occur if we can semi-standardly fill a Young diagram of a partition of $2n$ with n ones and n twos. Let $\nu = (a,b) \in \mathcal{P}_{2n}$ with $b = 0$ allowed. To ensure that the columns are strictly increasing, the second row will always consist only of twos. This means that there are still n ones and $n - b$ twos to fit into the tableaux. However, there is only one way to do this and still adhere to semi-standardness: fill the top row with the ones followed by the remaining twos. This is the only way to keep the rows weakly increasing. Because this is the unique semi-standard Young tableaux of shape (a,b) and weight (n,n), it follows that $K_{\nu(n,n)} = 1$ in this case.

(ii) The only non-zero $K_{\nu(n,n-1,1)}$ occur if we can semi-standardly fill a Young diagram of a partition of $2n$ with n ones, $n-1$ twos and one three. There are four cases, from which we obtain the result.

- Let $\nu = (2n)$. There is obviously only one way to fill this, so $K_{(2n)(n,n-1,1)} = 1$.

- Let $\nu = (n, n)$. This is similar to the first case, but because the tableaux is rectangular, this forces the three in the right-most cell of the second row. The filling is then just ones along the top and twos in the remaining second row cells. As there is no true choice here, we have $K_{(n,n)(n,n-1,1)} = 1$.
- Let $\nu = (a, b, 1) \in \mathcal{P}_{2n}$. We must have the three in the lone third-row cell (otherwise the first column would not be strictly increasing). Hence, the first cell in the second row must be a two, and therefore so are all cells therein. Again then, there is no choice: we fill as much of the first row as possible with ones and continue with twos and this is it. Consequently, $K_{(a,b,1)(n,n-1,1)} = 1$.
- Let $\nu = (a, b) \in \mathcal{P}_{2n}$ with a and $b \neq 0$ not both n; we have $a > b$. Because the first row is longer than the second, the ones can only go in the top (otherwise this would contradict strictly increasing columns). There are two ways to complete the filling, where either the three appears at the end of the first row (and it has no cell below it since $a > b$), as in Figure A.3(a), or it appears at the end of the second row, as in Figure A.3(b). This demonstrates that $K_{(a,b)(n,n-1,1)} = 2$ and concludes the proof. □

(a) Three in the first row. **(b)** Three in the second row.

Figure A.3: The semi-standard Young tableaux counted by $K_{(a,b)(n,n-1,1)}$ with $a > b$.

Definition A.24 Let $\mu \in \mathcal{P}$ be a partition and T be a standard Young tableaux of shape μ. An entry $k \in T$ is called a descent if $k + 1 \in T$ appears in any row strictly below it. The set of descents is denoted $\mathrm{Des}(T)$. The major index of the tableaux is the sum of all descents, that is

$$\mathrm{maj}(T) = \sum_{k \in \mathrm{Des}(T)} k.$$

We can now prove an important stepping stone en route to the goal of showing that the Calogero-Moser space is connected, i.e. the Poincaré polynomial of $\mathcal{C}_n$ has constant term equal to one.

Proposition A.25 *If $\omega = (1, 2n)$, then the q^n-term appears in $\mathbb{H}^{\omega}_{0,4}$ with coefficient* 1.

Proof: One can write the Macdonald polynomials in the monomial basis by combining (A.10) and (A.9). The Hall pairing guarantees the only surviving terms are those arising from $m_{(n,n)}$

and $m_{(n,n-1,1)}$. Combining these observations, this means we need only concern ourselves with

$$\text{(i)} \quad \sum_{\nu} \widetilde{K}_{\nu\mu}(q,t) K_{\nu(n,n)} \qquad \text{and} \qquad \text{(ii)} \quad \sum_{\nu} \widetilde{K}_{\nu\mu}(q,t) K_{\nu(n,n-1,1)}. \tag{A.12}$$

As we assume $\omega = (1, 2n)$, the partition of interest here is $\mu = (2n)$. From [Mac95, p. 362],

$$K_{\nu(m)}(q,t) = \sum_{T \in \mathrm{SYT}(\nu)} q^{\mathrm{maj}(T)}. \tag{A.13}$$

In particular, (A.13) implies that $K_{\nu(2n)}(q,t)$ is a polynomial in q alone. Because the partition statistic $n((m)) = 0$ for the trivial partition, it follows from (A.11) that $\widetilde{K}_{\nu(2n)}(q,t)$ is also a polynomial in q alone. It remains to demonstrate that $\mathbb{H}^{\omega}_{0,4}$ has highest term q^n. For the moment, we care not about the value of the Kostka numbers in (A.12) but rather only their non-zeroness. By Lemma A.22 then, we consider the following cases:

(i) The Hall inner product is non-zero if and only if $\nu = (a, b)$; we are interested in

$$\widetilde{K}_{(a,b)(2n)}(q,t).$$

Looking at (A.13), the maximum major index comes from the standard Young tableaux T of shape (n, n) whose columns are labelled $[1, 2], [3, 4], \ldots, [2n-1, 2n]$. Thus, the major index is the sum of the odd numbers from 1 to $2n$, meaning precisely that $\mathrm{maj}(T) = n^2$.

(ii) The Hall inner product is non-zero if and only if $\nu = (a, b, 1)$ or (α, β); we are interested in

$$\widetilde{K}_{(a,b,1)(2n)}(q,t) + \widetilde{K}_{(\alpha,\beta)(2n)}(q,t).$$

The second summand above is dealt with by (i), the q-term with the largest power is q^{n^2}. As for the first, the maximum major index comes from the standard Young tableaux T of shape $(n, n-1, 1)$ whose columns are labelled $[1, 3, 2n], [2, 5], \ldots, [2n-4, 2n-1], [2n-2]$. Thus, the major index is one less than the sum of the even numbers from 2 to $2n$, that is $\mathrm{maj}(T) = n^2 + n - 1$. Note that this is always at least the power of the second summand.

Notice that the hook product (A.4) for the type $\omega = (1, 2n)$ simplifies as each leg-length $l = 0$:

$$\mathcal{H}_{\omega} = \mathcal{H}_{(2n)} = \prod_{a=0}^{2n-1} \frac{1}{(q^{a+1}-1)(q^a - t)}.$$

The highest power of q in this denominator is $\sum_{a=0}^{2n-1}(2a+1) = 4n^2$. Recall from (A.8) that

$$\mathbb{H}^{\omega}_{0,4} = (q-1)(1-t) C_{\omega} \mathcal{H}_{\omega} \left\langle \widetilde{H}_{\omega}, h_{(n,n)} \right\rangle^3 \left\langle \widetilde{H}_{\omega}, h_{(n,n-1,1)} \right\rangle. \tag{A.14}$$

We have $C_\omega = 1$ directly from Definition A.15. Considering the asymptotics when $q \gg t$,

$$\mathbb{H}^{\omega}_{0,4} \sim q\frac{(q^{n^2})^3(q^{n^2+n-1})}{q^{4n^2}} = \frac{q^{4n^2+n}}{q^{4n^2}} = q^n.$$

Finally, combining the expressions for the Hall inner products in (i) and (ii) above, and recalling that the q-power of the first summand in (ii) dominates that of the second, it is an immediate consequence of Corollary A.23 that the coefficient of q^n is one. □

Our next target is to prove the converse of Proposition A.25; this is a bit more broad in the sense that it appears we must consider arbitrary types $|\omega| = 2n$ of the correct size. If we can show that none of them produce asymptotics leading to a q^n-term, we have proved the result we are after. The full set-up is again very combinatorial, but let's first state a simple lemma.

Lemma A.26 *Let $d \in \mathbb{Z}^+$ and $\mu \in \mathcal{P}$. Then, $m_\mu[\mathbf{X}^d] = m_{d\mu}[\mathbf{X}]$ where $d\mu := (d\mu_1, \dots, d\mu_\ell)$.*

Proof: Let $\mathbf{X} = x_1 + x_2 + \cdots$. From the definition of the monomial symmetric function, we get

$$m_\mu[\mathbf{X}^d] = \sum_{\alpha\sim\mu}\left(x_1^d x_2^d \cdots\right)^\alpha = \sum_{\alpha\sim\mu}(x_1x_2\cdots)^{d\alpha} = m_{d\mu}[\mathbf{X}],$$

where $\alpha \in \mathbb{N}^\ell$ and $\alpha \sim \mu$ if and only if α is a rearrangement of $\mu = (\mu_1, \dots, \mu_\ell)$ as an ℓ-tuple. □

We now prove an important consequence of Lemma A.26 which will dramatically reduce the number of types we must consider when proving the converse of Proposition A.25. Namely, it turns out that we need only consider types concentrated in degree $d = 1$.

Corollary A.27 *For $\omega = (d_1, \mu^1)\cdots(d_\ell, \mu^\ell)$ with some $d_k \neq 1$, we have $\left\langle \widetilde{H}_\omega, h_{(n,n-1,1)}\right\rangle = 0$.*

Proof: Per (A.5), the Macdonald polynomial indexed by the type ω is the product

$$\widetilde{H}_\omega[\mathbf{X}; q, t] = \prod_{i=1}^{\ell} \widetilde{H}_{\mu^i}[\mathbf{X}^{d_i}; q^{d_i}, t^{d_i}].$$

The (extended) Hall pairing is multiplicative per (A.3); it suffices to consider only the k^{th} factor, that is where $d_k \neq 1$. Expanding $\widetilde{H}_{\mu^k}$ in the monomial basis, as in the proof of Proposition A.25,

$$\begin{aligned}\widetilde{H}_{\mu^k}[\mathbf{X}^{d_i}; q^{d_i}, t^{d_i}] &= \sum_\eta\sum_\nu \widetilde{K}_{\nu\mu^k}(q^{d_k}, t^{d_k})K_{\nu\eta}m_\eta[\mathbf{X}^{d_k}] \\ &= \sum_\eta\sum_\nu \widetilde{K}_{\nu\mu^k}(q^{d_k}, t^{d_k})K_{\nu\eta}m_{d_k\eta}[\mathbf{X}],\end{aligned}$$

by Lemma A.26. The only way the Hall pairing with $h_{(n,n-1,1)}$ is non-zero is if $d_k\eta = (n, n-1, 1)$. But because $d_k \in \mathbb{Z}^+$, this occurs if and only if $\eta = (n, n-1, 1)$ and $d_k = 1$, a contradiction. □

A.5 Statistics from Young Diagrams

There is a neat combinatorial formula for the Macdonald polynomials provided in [HHL05] which will give us a good handle on their asymptotic behaviour. In order to understand this though, we must first introduce yet more combinatorial notions. Throughout, $\phi : \mu \to \mathbb{Z}^+$ is a so-called *filling function* which assigns a positive integer to each Young cell of μ.

Definition A.28 A descent of ϕ is a Young cell y whose filling is strictly larger than the Young cell x immediately above, that is $\phi(y) > \phi(x)$. The set of descents of ϕ is denoted $\mathrm{Des}(\phi)$.

This notion of a descent is *not* the same as that in Definition A.24, so one should be vigilant when encountering this word (but we henceforth only refer to descents as in Definition A.28).

Definition A.29 For two Young cells $u, v \in \mu$, we say u attacks v if either of the following occur:

(i) u is strictly to the left of v in the same row.

(ii) u is strictly to the right of v in the row immediately below that which contains v.

(a) Attacking cells in the same row. **(b)** Attacking cells in neighbouring rows.

Figure A.4: Two types of cell attacks in a Young diagram.

In other words, every cell attacks its arm and gets attacked by the arm of the cell below it. Now, the *Hanunó'o reading order* of a Young diagram of $\mu \in \mathcal{P}$ is the total ordering on Young cells given by reading them row-by-row, left-to-right, bottom-to-top (so-named after the Hanunó'o script which is famous for being written upwards). For two Young cells $u, v \in \mu$, we denote this $u \prec v$. If we let $u = (i, j)$ and $v = (a, b)$ indexed by the row and column, respectively, then

$$(i, j) \prec (a, b) \qquad \Leftrightarrow \qquad (-i, j) <_{\mathrm{lex}} (a, b).$$

Definition A.30 An inversion of ϕ is a pair of Young cells (u, v) where u attacks v, the fillings satisfy $\phi(u) > \phi(v)$ and $u \prec v$. The set $\mathrm{Inv}(\phi)$ is the set of pairs of cells that are inversions of ϕ.

This definition as made in [HHL05, (11)] is easier because they use the French style, where the Young diagram has its longest part on the bottom; this would be like drawing $\mu = (\mu_\ell, \ldots, \mu_1)$ in the English convention. The order they put on their Young diagrams is then the standard reading order. The next definition is crucial for the Macdonald polynomial formula we will use.

Definition A.31 Let $\phi : \mu \to \mathbb{Z}^+$ be a filling function of the Young diagram of a partition $\mu \in \mathcal{P}$.

(i) The major statistic of ϕ is as follows, where $l(u)$ is the usual leg-length of Young cell u:

$$\mathrm{maj}(\phi) := \left|\mathrm{Des}(\phi)\right| + \sum_{u \in \mathrm{Des}(\phi)} l(u).$$

(ii) The inversion statistic of ϕ is as follows, where $a(u)$ is the usual arm-length of Young cell u:

$$\mathrm{inv}(\phi) := \left|\mathrm{Inv}(\phi)\right| - \sum_{u \in \mathrm{Des}(\phi)} a(u).$$

It is not clear a priori that the inversion statistic $\mathrm{inv}(\phi)$ is non-negative for any filling ϕ, but this turns out to be the case; a short justification is provided by Haglund in [Hag04, Remark 2].

Example A.32 Consider the Young diagram of $\mu = (4, 3, 2)$ with a filling as in Figure A.5 below.

$$\begin{array}{|c|c|c|c|} \hline 5 & 3 & 7 & 4 \\ \hline 1 & 2 & 9 \\ \cline{1-3} 6 & 8 \\ \cline{1-2} \end{array}$$

Figure A.5: A filling ϕ of the Young diagram of $\mu = (4, 3, 2)$.

Per Definition A.28, the set of descents of ϕ in this example is

$$\mathrm{Des}(\phi) = \left\{ \boxed{6}, \boxed{8}, \boxed{9} \right\}.$$

Next, here is a list of all attacking pairs, where (u, v) means that cell u attacks cell v:

$$\left(\boxed{5},\boxed{3}\right), \left(\boxed{5},\boxed{7}\right), \left(\boxed{5},\boxed{4}\right), \left(\boxed{3},\boxed{7}\right), \left(\boxed{3},\boxed{4}\right), \left(\boxed{7},\boxed{4}\right), \left(\boxed{1},\boxed{2}\right),$$
$$\left(\boxed{1},\boxed{9}\right), \left(\boxed{2},\boxed{9}\right), \left(\boxed{6},\boxed{8}\right), \left(\boxed{2},\boxed{5}\right), \left(\boxed{9},\boxed{5}\right), \left(\boxed{9},\boxed{3}\right), \left(\boxed{8},\boxed{1}\right).$$

Per Definition A.30, we can pick from the above list the inversions of ϕ and form the set of them:

$$\mathrm{Inv}(\phi) = \left\{ \left(\boxed{5},\boxed{3}\right), \left(\boxed{5},\boxed{4}\right), \left(\boxed{7},\boxed{4}\right), \left(\boxed{9},\boxed{5}\right), \left(\boxed{9},\boxed{3}\right), \left(\boxed{8},\boxed{1}\right) \right\}.$$

We can now immediately compute the major and inversion statistics from Definition A.31:

$$\mathrm{maj}(\phi) = 3 + (0 + 0 + 0) = 3 \qquad \text{and} \qquad \mathrm{inv}(\phi) = 6 - (1 + 0 + 0) = 5.$$

We can now state a crucial fact we will use when proving the converse of Proposition A.25.

Proposition A.33 ([HHL05, Theorem 2.2]) *The Macdonald polynomial $\widetilde{H}_\mu$ is given by*

$$\widetilde{H}_\mu[\mathbf{X}; q, t] = \sum_\phi q^{\mathrm{inv}(\phi)} t^{\mathrm{maj}(\phi)} \mathbf{X}^\phi,$$

where we sum over all fillings and $\mathbf{X}^\phi = \prod_{u \in \mu} x_{\phi(u)}$ is a monomial in at most $|\mu|$ variables.

From Proposition A.33, it appears we must study $\mathrm{inv}(\phi)$ to access the behaviour of the q-powers. But the minus makes it more troublesome than $\mathrm{maj}(\phi)$. Fortunately, there is a q, t-*symmetry* satisfied by the Macdonald polynomials which allows us to interchange these two indeterminates at the expense of replacing the partition μ with its dual μ' (see [CHM+22, Remark 3.2]), that is

$$\widetilde{H}_\mu[\mathbf{X}; q, t] = \widetilde{H}_{\mu'}[\mathbf{X}; t, q]. \tag{A.15}$$

Combining (A.15) with Proposition A.33 then, we arrive at the following useful expression:

$$\widetilde{H}_{\mu'}[\mathbf{X}; t, q] = \sum_\phi t^{\mathrm{inv}(\phi)} q^{\mathrm{maj}(\phi)} \mathbf{X}^\phi. \tag{A.16}$$

A.6 Asymptotics

Recall from Mellit's Theorem (Theorem A.2) that the Poincaré polynomial arises after taking the Hall pairing of the HLV kernel with each of $h_{(n,n)}$ and $h_{(n,n-1,1)}$. From Definition A.5, the only surviving Macdonald polynomial coefficients are those of $m_{(n,n)}$ and $m_{(n,n-1,1)}$. So, we are concerned with fillings where (i) $\mathbf{X}^\phi = m_{(n,n)}$, and (ii) $\mathbf{X}^\phi = m_{(n,n-1,1)}$, meaning the following:

(i) ϕ is a filling by two letters of n copies and n copies, say $\phi = 1^n 2^n$.

(ii) ϕ is a filling by three letters of n copies, $n-1$ copies and 1 copy, say $\phi = 1^n 2^{n-1} 3$.

The next auxiliary result will be an application of (A.16) to find upper bounds on the q-powers of the above-mentioned Hall pairings. Where we use the notation $\sim$ to denote that two objects behave the same asymptotically, we use the notation $\lesssim$ to denote an asymptotic upper bound.

Proposition A.34 *Let $\mu = (\mu_1, \ldots, \mu_\ell) \in \mathcal{P}_{2n}$. Then, we have the following:*

(i) $\langle \widetilde{H}_\mu, h_{(n,n)} \rangle \lesssim \begin{cases} q^{\frac{1}{4}(\mu_1^2+\cdots+\mu_\ell^2)}, & \textit{if } \mu_1 \textit{ is even} \\ q^{\frac{1}{4}(\mu_1^2+\cdots+\mu_\ell^2)-\frac{1}{4}}, & \textit{if } \mu_1 \textit{ is odd} \end{cases}.$

(ii) $\langle \widetilde{H}_\mu, h_{(n,n-1,1)} \rangle \lesssim \begin{cases} q^{\frac{1}{4}(\mu_1^2+\cdots+\mu_\ell^2)+\frac{1}{2}\mu_1-1}, & \textit{if } \mu_1 \textit{ is even} \\ q^{\frac{1}{4}(\mu_1^2+\cdots+\mu_\ell^2)+\frac{1}{2}\mu_1-\frac{3}{4}}, & \textit{if } \mu_1 \textit{ is odd} \end{cases}.$

Proof: From Definition A.31, we can maximise the major statistic by ensuring that there are as many descents of μ' as possible, as high up as they can go. This amounts to *vertical* fillings by (i) dominoes $[1,2]$, or (ii) dominoes $[1,2]$ with one triomino $[1,2,3]$. So if we now use (A.16), this means we transpose the Young diagram and now we fill μ with *horizontal* dominoes with the same labels as above. Note that descents are now counted as neighbouring cells where the right is greater than the left. We then replace $l(u)$ by $a(u)$ in Definition A.31(i). Strictly speaking, we are defining an equivalent statistic by

$$\operatorname{maj}'(\phi) := \left|\operatorname{Des}'(\phi)\right| + \sum_{u \in \operatorname{Des}'(\phi)} a(u), \tag{A.17}$$

where $\operatorname{Des}'(\phi)$ is the set of filled cells whose left-neighbour's filling is less. The filling considered in (A.17) is that of the transpose of the partition; we abuse notation and omit the prime, because $\operatorname{maj}(\phi_\mu) = \operatorname{maj}'(\phi_{\mu'})$. Note that for any $\mu \in \mathcal{P}$, the arm-length of a Young cell u in the ij^{th} place (as usual, labelled akin to positions in a matrix) is given by the simple formula

$$a(u_{ij}) = \mu_i - j. \tag{A.18}$$

(i) We consider a filling $\phi = 1^n 2^n$ and focus on maximising the major statistic for a single row μ_i; the maximum is obtained if we fill by as many horizontal dominoes $\boxed{1}\boxed{2}$ starting from the far-left and continuing until we run out of (at least) one of the filling numbers; we complete the filling by putting what's left in what remains; see Figure A.6 below.

1	2	1	2	1	2	1	$\cdots$

Figure A.6: Maximising the major statistic for row μ_i when $\mathbf{X}^\phi = h_{(n,n)}$.

We clearly must separate into cases depending on the parity of μ_i. But regardless, note that $\operatorname{maj}(\phi)$ is calculated by summing the number of descents (shaded in grey) and their arms; each descent is in an even-indexed column. Thus, we conclude that

$$\left|\mathrm{Des}(\phi)\right| = \begin{cases} \frac{\mu_i}{2}, & \text{if } \mu_i \text{ is even} \\ \\ \frac{\mu_i-1}{2}, & \text{if } \mu_i \text{ is odd} \end{cases}$$

and

$$\sum_{u\in\mathrm{Des}(\phi)} a(u) = \begin{cases} \displaystyle\sum_{k=1}^{\mu_i/2}(\mu_i - 2k), & \text{if } \mu_i \text{ is even} \\ \\ \displaystyle\sum_{k=1}^{(\mu_i-1)/2}(\mu_i - 2k), & \text{if } \mu_i \text{ is odd} \end{cases}$$

$$= \begin{cases} \frac{\mu_i^2}{4} - \frac{\mu_i}{2}, & \text{if } \mu_i \text{ is even} \\ \\ \frac{\mu_i^2}{4} - \frac{\mu_i}{2} + \frac{1}{4}, & \text{if } \mu_i \text{ is odd} \end{cases}.$$

Summing the above gives us the options for the major statistic depending on parity:

$$\mathrm{maj}(\phi_{\mu_i}) = \begin{cases} \frac{\mu_i^2}{4}, & \text{if } \mu_i \text{ is even} \\ \\ \frac{\mu_i^2-1}{4}, & \text{if } \mu_i \text{ is odd} \end{cases}. \tag{A.19}$$

(ii) We consider a filling $\phi = 1^n 2^{n-1} 3$ and focus on maximising the major statistic for a single row μ_i; the maximum is obtained if we fill by starting with $\boxed{1|2|3}$ in the far-left of *row one* μ_1 and continuing with as many horizontal dominoes $\boxed{1|2}$ henceforth, and continuing until we run out of (at least) one of the filling numbers; we complete the filling by putting what's left in what remains; see Figure A.7 below.

1	2	3	1	2	1	2	1	···

Figure A.7: Maximising the major statistic for row μ_1 when $\mathbf{X}^\phi = h_{(n,n-1,1)}$.

We again have a parity dichotomy. In this case, the first descent is in the second column and all subsequent descents in odd-indexed columns (third and onwards). Hence, we get

$$\left|\mathrm{Des}(\phi)\right| = \begin{cases} \frac{\mu_1}{2}, & \text{if } \mu_1 \text{ is even} \\ \frac{\mu_1+1}{2}, & \text{if } \mu_1 \text{ is odd} \end{cases}$$

and

$$\sum_{u \in \mathrm{Des}(\phi)} a(u) = \begin{cases} \left(\sum_{k=1}^{(\mu_1/2)-1} \mu_1 - (2k+1) \right) + \mu_1 - 2, & \text{if } \mu_1 \text{ is even} \\ \left(\sum_{k=1}^{(\mu_1-1)/2} \mu_1 - (2k+1) \right) + \mu_1 - 2, & \text{if } \mu_1 \text{ is odd} \end{cases}$$

$$= \begin{cases} \frac{\mu_1^2}{4} - 1, & \text{if } \mu_1 \text{ is even} \\ \frac{\mu_1^2-5}{4}, & \text{if } \mu_1 \text{ is odd} \end{cases}.$$

Again, summing the above gives us the options for the major statistic depending on parity:

$$\mathrm{maj}(\phi_{\mu_1}) = \begin{cases} \frac{\mu_1^2}{4} + \frac{\mu_1}{2} - 1, & \text{if } \mu_1 \text{ is even} \\ \frac{\mu_1^2}{4} + \frac{\mu_1}{2} - \frac{3}{4}, & \text{if } \mu_1 \text{ is odd} \end{cases}. \tag{A.20}$$

To find the major statistic of the entire partition $\phi = \phi_\mu$ (and therein the q-power of the relevant Hall pairing), it suffices to sum over the major statistics for each row $\mathrm{maj}(\phi_{\mu_i})$. However, one must take care with the case $\phi = 1^n 2^{n-1} 3$; note that filling the first row obeys (ii) above, but every subsequent row is filled with only ones and twos, so this obeys (i) above. We therefore separate into cases depending on the parity of the first row μ_1.

(a) If μ_1 is even, we use the even case from (A.20) and, for subsequent rows, we notice that the even case for (A.19) is strictly larger than the odd case. In all, we assume that every part is even (since this is the 'worst-case scenario'). Hence, the q-power of $\mathrm{maj}(\phi)$ is *at most*

$$\begin{aligned} \frac{1}{4}\mu_1^2 + \frac{1}{2}\mu_1 - 1 + \sum_{i=2}^{\ell} \frac{\mu_i^2}{4} &= \frac{1}{4}(\mu_1^2 + \cdots + \mu_\ell^2) + \frac{1}{2}\mu_1 - 1 \qquad && \text{if } \phi = 1^n 2^{n-1} 3, \\ \sum_{i=1}^{\ell} \frac{\mu_i^2}{4} &= \frac{1}{4}(\mu_1^2 + \cdots + \mu_\ell^2) && \text{if } \phi = 1^n 2^n. \end{aligned}$$

(b) If μ_1 is odd, we use the odd case from (A.20) and, for subsequent rows, we still use the even case for (A.19) as it is is strictly larger. In all, we assume that every part except μ_1 is even (since this is the 'worst case scenario'). Hence, the q-power of $\mathrm{maj}(\phi)$ is *at most*

$$\begin{aligned} \frac{1}{4}\mu_1^2+\frac{1}{2}\mu_1-\frac{3}{4}+\sum_{i=2}^{\ell}\frac{\mu_i^2}{4} &= \frac{1}{4}(\mu_1^2+\cdots+\mu_\ell^2)+\frac{1}{2}\mu_1-\frac{3}{4} && \text{if } \phi = 1^n2^{n-1}3,\\ \frac{1}{4}\mu_1^2-\frac{1}{4}+\sum_{i=2}^{\ell}\frac{\mu_i^2}{4} &= \frac{1}{4}(\mu_1^2+\cdots+\mu_\ell^2)-\frac{1}{4} && \text{if } \phi = 1^n2^n. \end{aligned}$$

The result as stated is immediate from interpreting these cases as Hall pairings with the relevant homogeneous symmetric polynomial; this is really a consequence of (A.16). □

The converse of Proposition A.25 has us consider types of size $2n$ concentrated in degree one (a consequence of Lemma A.26). There are obviously a lot of them, so we will prove the following technical lemma to greatly reduce the amount of required considerations for ω. In words, if $\omega = (1,\mu^1)\cdots(1,\mu^\ell)$ is our type, we can build from the constituent partitions a brand new partition by taking all the parts amongst $\mu^1,\dots,\mu^\ell$ and arranging them from largest-to-smallest.

Lemma A.35 *Let $\mu^1,\dots,\mu^\ell \in \mathcal{P}$ be partitions. Then, the partition $\eta = (\eta_1,\dots,\eta_m)$ defined by*

$$\eta_1 := \max_{i,j}\{\mu_i^j\} \qquad \textit{and} \qquad \eta_k := \max_{i,j}\Big(\{\mu_i^j\}\setminus\{\eta_1,\dots,\eta_{k-1}\}\Big)$$

has $\mathcal{H}_\eta \sim \mathcal{H}_\omega$, for $\omega = (1,\mu^1)\cdots(1,\mu^\ell)$ the type concentrated in degree one built from the μ^j.

Proof: Looking at (A.4), it is clear that the deformed hook product behaves asymptotically as

$$\mathcal{H}_\mu \sim q^{-\sum_{u\in\mu}(2a(u)+1)}. \tag{A.21}$$

But the arm-length $a(u)$ is calculated row-wise, and the partition η obtained as above is nothing more than stacking rows of Young cells from amongst the μ^j. Therefore, we can conclude that

$$\mathcal{H}_\eta \sim q^{-\sum_{u\in\eta}(2a(u)+1)} = \prod_j q^{-\sum_{u\in\mu^j}(2a(u)+1)} \sim \mathcal{H}_\omega.$$ □

Theorem A.36 *The q^n-term appears in $\mathbb{H}^\omega_{0,4}$ if and only if $\omega = (1,2n)$, and with coefficient 1.*

Proof: (⇐) This is Proposition A.25.

(⇒) Since $C_\omega = 0$ unless ω is concentrated in degree d, and as a result of Corollary A.27, it is sufficient to consider only types of the form $\omega = (1,\mu^1)\cdots(1,\mu^\ell)$ where $|\omega| = 2n$; this means that $\left|\mu^1\right| + \cdots + \left|\mu^\ell\right| = 2n$ by the definition of the size of a type. The strategy of the proof is this: if we have a type of length $\ell = 1$, we will show that the optimal partition (in the sense of maximising the q-power) is $\mu_1 = (2n)$. We then argue that for a type of length $\ell \geq 2$, we can form a single partition whose q-power is no less than that of the original type. Recall (A.14), i.e.

$$\mathbb{H}^{\omega}_{0,4} = (q-1)(1-t)C_\omega \mathcal{H}_\omega \left\langle \widetilde{H}_\omega, h_{(n,n)} \right\rangle^3 \left\langle \widetilde{H}_\omega, h_{(n,n-1,1)} \right\rangle.$$

Suppose $\omega = (1, \mu)$ is the type of interest. Because its length $\ell = 1$, we can replace ω simply by μ in the formula above. The asymptotics of the deformed hook product (A.4) are given in the previous proof, see (A.21). We simplify this using the arm-length formula (A.18). Indeed then,

$$\sum_{u\in\mu} a(u) = \sum_{i=1}^{\ell}\sum_{j=1}^{\mu_i}(\mu_i - j) = \sum_{i=1}^{\ell}\left(\mu_i^2 - \frac{1}{2}\mu_i(\mu_i+1)\right) = \frac{1}{2}(\mu_1^2 + \cdots + \mu_\ell^2) - \frac{1}{2}|\mu|.$$

Therefore, we have refined the asymptotic expression for the deformed hook product to

$$\mathcal{H}_\mu \sim q^{-(\mu_1^2+\cdots+\mu_\ell^2)}.$$

Using this along with Proposition A.34 and (A.14), we conclude that

$$\mathbb{H}^{\mu}_{0,4} \lesssim \begin{cases} q^{1+\frac{3}{4}(\mu_1^2+\cdots+\mu_\ell^2)+\frac{1}{4}(\mu_1^2+\cdots+\mu_\ell^2)+\frac{1}{2}\mu_1-1-(\mu_1^2+\cdots+\mu_\ell^2)}, & \text{if } \mu_1 \text{ is even} \\ q^{1+\frac{3}{4}(\mu_1^2+\cdots+\mu_\ell^2)-\frac{3}{4}+\frac{1}{4}(\mu_1^2+\cdots+\mu_\ell^2)+\frac{1}{2}\mu_1-\frac{3}{4}-(\mu_1^2+\cdots+\mu_\ell^2)}, & \text{if } \mu_1 \text{ is odd} \end{cases}$$

$$= \begin{cases} q^{\frac{1}{2}\mu_1}, & \text{if } \mu_1 \text{ is even} \\ q^{\frac{1}{2}(\mu_1-1)}, & \text{if } \mu_1 \text{ is odd} \end{cases}.$$

We see that this is maximised when μ_1 is even and is equal to the entire length of the partition, that is $\mu_1 = 2n$. This occurs if and only if $\mu = (2n)$; this is the optimal partition amongst all types with length one. Next, suppose $\ell \geq 2$, so the general form of the type is $\omega = (1,\mu^1)\cdots(1,\mu^\ell)$. Since the sum of the Young cells in totality is still $2n$, we can fill the partitions concurrently with (i) n ones and n twos or (ii) n ones, $n-1$ twos and one three; maximising over each part of each partition will produce the largest possible $\mathrm{maj}(\phi_\omega)$ and thus largest q-power of the respective Hall pairing. But this is equivalent to defining a new partition as in Lemma A.35; geometrically, η amounts to splitting the Young diagrams of the μ^j into rows and stacking them top-to-bottom from longest-to-shortest. The resulting partition $\eta \in \mathcal{P}_{2n}$ is such that

$$\mathrm{maj}(\phi_\eta) = \mathrm{maj}(\phi_\omega) \qquad \text{and} \qquad \mathcal{H}_\eta \sim \mathcal{H}_\omega.$$

The latter is precisely the statement of Lemma A.35 and the former is obvious because we maximise the major statistic row-by-row. However, we know that the largest q-power obtainable from a single partition is q^n when it is $(2n)$; since $\ell \geq 2$, it must be that $\eta_1 \leq 2n-1$ and so the largest q-power of ω is strictly less than n via the argument above. □

Corollary A.37 *For generic conjugacy classes, the Calogero-Moser space $\mathcal{C}_n$ is connected.*

Proof: We show that the constant term (the number of connected components) in the Poincaré polynomial from Theorem A.2 is one. This polynomial is precisely $q^n\mathbb{H}_{0,4}(q^{-1},1)$ in the language of (A.7). We can decompose $\mathbb{H}_{0,4}(q,1)$ into a sum over types of size $2n$, see (A.8) and Notation A.16. We interpret Theorem A.36 as saying that only the summand $\mathbb{H}^{\omega}_{0,4}(q,t)$ where $\omega = (1,(2n))$ has a q^n-term, and with coefficient one. Therefore, $\mathbb{H}_{0,4}(q,1)$ has a single q^n-term with coefficient one, meaning $\mathbb{H}_{0,4}(q^{-1},1)$ has a single q^{-n}-term with coefficient one, and consequently the Poincaré polynomial $q^n\mathbb{H}_{0,4}(q^{-1},1)$ has constant part exactly one. □