Algebraic structures for compositions of subsets of the unit square, Representation Theory and Congruences

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Abstract

In this thesis we first define a certain magma. This magma is an attempt to pass into mathematical form some aspects of human 'pictorial (and geometric) thinking' (the elements are mathematical versions of 'pictures'). We then generalise this magma in a natural way to 'higher dimensions'. And then we study aspects of the representation theory and structure of these magmas. In particular we investigate associative quotients, by a variety of means. And also sub-magmas that pass closer to some classical mathematical structures such as braid groups.

Our core definition is for the magma itself: 3.1.20. But while everything depends on this, it is relatively straightforward. Our main results are 4.3.19, 5.4.22 and 6.2.2.

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Chapter 1

Introduction

Let $[0, 1] \subset \mathbb{R}$ denote the unit interval, and $[0, 1]^d$ the unit d-dimenionsal box. In this thesis, we define a 'stack/shrink' composition $\boxtimes$ on the power set $\mathcal{P}([0, 1]^d)$ for each $d \in \mathbb{N}$ (and some other power sets), focusing in particular on $d = 2$. We show that the initial composition $\boxtimes$ is closed but not associative, so for each d we have a magma - in particular $\mathfrak{M} = (\mathcal{P}([0, 1]^2), \boxtimes)$. See Proposition 3.1.20 and 3.2.4.

We look for subsets of these magmas closed under the composition. See Proposition 4.1.5 for example, but in particular guided by the idea of braid-like structures, for example in cases $d = 2, 3$. See proposition 4.3.19. (In general this seems a hard problem, and it is not our aim here to solve completely. But the results like 4.3.19 indicate interesting first steps.) The construction itself can be found in Chapter 4 onwards.

We then investigate congruence relations on this magma (and its submagmas) with associative (even unital) quotient. See Theorem 5.4.22 and 6.2.2 and 6.2.5. The construction itself can be found in Chapters 5 and 6 onwards.

1.1 Thesis Introduction

This work is about so-called 'passport photo' magmas. It is motivated in part by the magmas and 'magmoids' introduced in Torzewska et al's paper [TFMM23] on motion groupoids.

The basic passport photo (pp) magma is $\mathcal{P}([0, 1]^2)$ made into a magma by the operation, denoted $\boxtimes$, of stacking elements and then shrinking the stack.

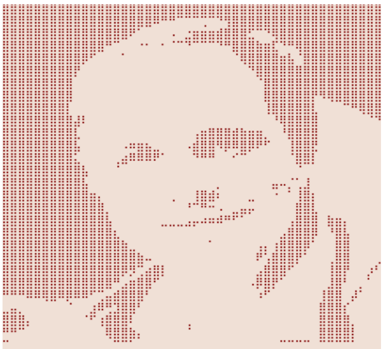


Figure 1.1: An element of $\mathcal{P}([0,1]^2)$, that is, a subset of $[0,1]^2$.

This magma has many interesting substructures and quotient structures.

Roughly speaking, an element of $\mathcal{P}([0,1]^2)$ is a black-and-white picture (see figure 1.1 for example - for an element $p \in \mathcal{P}([0,1]^2)$ we think of black in position x in $[0,1]^2$ if $x \in p$, and white otherwise). And there are many algebraic structures that have picture realisations of elements (braid groups [CF63, KT08, Kas12] and partition algebras such as Temperley–Lieb algebras [Mar94, Ban13, DM06], for example), and where composition is represented by stacking pictures. In many cases these realisations make strong implicit use of the properties of $[0,1]^2$ as an Euclidean metric topological space. In the construction of structures derived from the pp magma, then, a role can be given to paths in the space and its subspaces $a \in \mathcal{P}([0,1]^2)$. (Or, more precisely, to the question of existence of paths between given points in a. The details of parameterisation of paths are generally less important in practice.)
A key property here is that if $a, b \in \mathcal{P}([0,1]^2)$ and f is a path (from $f(0)$ to f(1)) in a then f has an image in the stack-shrink composition $a \boxtimes b$ that is a path in $a \boxtimes b$ from the image in $a \boxtimes b$ of $f(0)$ to the image of $f(1)$.
Another key property is that if a and b above both contain paths, and these paths have suitable endpoints, then $a \boxtimes b$ may contain a path that joins the individual paths together.

These properties are often taken for granted in pictures of braids for example (where the paths represent strings in the braid). The idea for this thesis is to not take such things for granted, but instead to look at them carefully. (The motivation is that mathematical braids, which live in a space such as $\mathbb{R}^d$, are models of physical structures where neither the real string nor the physical space is exactly the same as its mathematical model. For example a real string is made of molecules and hence atoms, and in this sense is not strictly continuous.)

Remark: Here we are not talking about 'braid diagrams', where a braid in 3d is projected into 2d but drawn with broken rather than crossing lines when the *projection* forces a 'crossing'. In our setting the 'picture' of a braid in 3d would belong to our $d = 3$ case (think more of a 3d printer than a conventional printer!); and our $d = 2$ case corresponds to braids that really lie only in the plane, without projection. Note that in the 2d case there is not really enough space for strings to braid in the sense of tangling. But our main interest here is in the mathematical structure of the strings themselves rather than ambient topological properties (see Chapter 4).

As noted, most pictures that arise as representations of algebraic structures are pictures of embedded manifolds - typically paths (see e.g. [KT08]). But by considering underlying sets such as $\mathcal{P}([0,1]^2)$ we allow for much more general kinds of pictures. This generalisation raises many questions and possibilities. And it is these aspects that we try to start to address in this thesis.

Let us now discuss in more detail the main motivations, main aims, and the context of this work.

As noted above, this project is essentially motivated by interest in questions arising in, and from, Torzewska et al's motion groupoids paper [TFMM23]. That paper introduces and studies motion groupoids and mapping class groupoids, but (with Physics applications in mind) starting from more general magmas and magmoids. These arise from the non-associative 'stack-shrink' composition of certain subsets of a suitable ambient interval - typically $[0,1]^d$ for some d. In Torzewska's case the subsets are the images of subsets of $[0,1]^{d-1}$ - typically submanifolds such as points and loops - under 'time-evolution' into the dth dimension. Thus elements of $\mathcal{P}([0,1]^{d-1})$ are generalised 'particles' (or field configurations with particle content, in the sense that $\mathcal{P}([0,1]^{d-1})$ is equivalent to the set of functions $\hom([0,1]^{d-1}, \{0,1\})$, which Torzewska interprets in the Introduction to [TFMM23] as a set of '$\mathbb{Z}_2$-valued field configurations' on the space $[0,1]^{d-1}$, cf. e.g. [Kog79, Mar91]) and evolutions of these are particle trajectories - composed by stacking to make longer particle trajectories. Taking $d - 1 = 2$ and finite point-set subsets this yields 'braids' - except that so far with a non-associative composition. Torzewska et al rapidly pass to equivalence classes of evolutions under equivalences that are natural from a low-dimensional geometric topology perspective. This allows them to make contact with and generalise classical constructions such as braid groups and loop braid groups. At this point the interest in [TFMM23] comes from the kind of particles considered. But *for us* it is an interesting question why to use equivalence essentially under the geometric-topological notion of 'ambient isotopy' as they do. Mathematically the reason is that it leads to beautiful groups and groupoids. But from a Physics perspective one can observe (even as a mathematician knowing nothing about any relevant experiments!) that the prevailing choice should be for organisational power in Physics, not simple mathematical beauty (although cf. e.g. [LM77, Bai80] and references therein). So with this in mind, we are motivated to consider much more general notions of 'particle evolution' and in particular other notions of equivalence. In practice both departures lead quickly to unrealistically hard thesis problems, so the constructions we attempt are relatively very modest. But at least this context explains our ambit, and starts to set up the more general class of problems, which is our aim.

Let us now discuss, in overview, our main results.

We built the magma $\mathfrak{M}$, then we would like to 'understand' this magma, by studying magma maps to and from $\mathfrak{M}$ — so for example to study submagmas (Chapter 4); and to do *Representation theory* of this magma (Chapters 5-6). We have first a general question: What does the representation theory of magmas look like? Representations are structure preserving maps, but representation *theory* is about organisation - compared to the representation theory of groups, with the notion of irreducible representations for example. So, in this case, we study quotient magmas, by defining a congruence relation. In particular we can aim for associative quotients (to move us closer to classical group representation theory).

In one approach, looking for congruences, we construct equivalences on the underlying set $\mathcal{P}([0,1]^2)$ up to certain transformations in the y-direction. We start with transformations, which are *necessary* for associativity, and which, coincidentally, lie in the Thompson group F (for a review of group F see e.g. [CFP94]), but do not generate the whole of F. To develop this into a congruence we first put a new composition rule (in (5.6)) on the underlying set of the Thompson group F (then we consider, is this associative? - It is not! but we can take the closure of the subset of F that we have so far under this composition, and then take the group closure). (See e.g Section 5.2.3).

In §5.3-5.4 we take a different approach. We construct another equivalence relation on our underlying set of the magma $\mathfrak{M}$ using paths in elements (see definition 5.4.9). We call it R_α in 5.4.2, and this relation reaches the congruence (see 5.4.3). Finally, we arrive at the magma quotient by R_α (see definition 5.4.21) that leads us to get the *Monoid* which is $\mathsf{M}(\alpha, \alpha)/R_\alpha$ (see theorem 5.4.22). We return to discuss interesting properties of this monoid in Chapter 6.

In Chapter 6, we aim to build a map from our magma $\mathfrak{M}$ to a *Partition Algebra*, specifically a *Partition Monoid* (see for example [HR05]). There is a pictorial representation of the defining basis of the partition algebra. This pictorial representation can be seen as elements of our magma. Suppose for example we indicate a partition of six vertices. It is also an element of the passport photograph magma, what we mean by this is we allow that there is a closing box, which is the unit square $[0, 1]^2$. It is not yet clear where the vertices would go. But if we decide where the vertices would go, then we would definitely say that this element of the passport photograph magma can be used to determine a partition, according to the connected components (see figure 1.2). This is practically complicated because

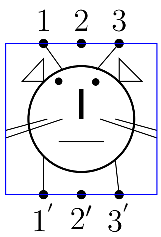


Figure 1.2: An example of a partition of six vertices

that passport photograph could be any 'picture', who knows what this is as precise element in $[0, 1]^2$. We may need to know a lot of information before we can figure out whether there are paths or not. When is there a path between two points in an element of the passport photograph magma (in $[0, 1]^2$)? In Chapter 6 we discuss these points before coming to the mathematical construction. Again this is related to the paper of Torzewska in [TFMM23]. Furthermore, we may say if there are paths, and they have tangents, and paths cross, and the tangents are different at the crossing point then we may try to refine the path-connected rule. Then we don't follow the kink path. We only follow the non-kink - so we don't change the tangent at the point. Recall, in our original setup, we are trying to construct congruences as we can in principle make lots of choices about what the map would be from the whole of ($\mathcal{P}([0,1]^2)$) down to the quotient. So that is the playground of Chapter 6. We just tried to construct maps from the magma to a nicer (in the sense of known) algebraic structure, which is the partition algebra, because it already has a pictorial realisation similar to the one indicated above.

1.1.1 Outline of the Thesis

We give a brief overview of this thesis:

In Chapter 2: We recall the basic notation and definition. We review some definitions such as relation, partition, topological space,... also, we look at the Jordan curve theorem, Magma definition and Category.

In Chapter 3: We define a 'Passport Photograph Magma' which is Magma $\mathfrak{M}$, where the underlying set is $\mathcal{P}([0,1]^2)$ with the binary operation a stack/shrink composition.

In Chapter 4: We describe submagmas in two different ways. The first one is generating this magma by a certain subset of $\mathcal{P}([0,1] \times [0,1])$. Then we look at braid-like submagma.

In Chapter 5: We consider several kinds of equivalence relation on $\mathcal{P}([0,1]^2$ that may become congruences in our magma, or in certain submagmas. We look at relations and congruences generally, but we are particularly interested in associative quotients.

In Chapter 6: We use our magma to represent the partition then we use the partition to describe an equivalence relation on our magma.

Chapter 2

Basic notation and definitions

In this chapter, we introduce some notations and constructions that will be useful later.

2.1 Basic definitions

Definition 2.1.1. *Given a set S, the power set of S is the set of subsets, denoted by $\mathcal{P}(S)$.*

Example 2.1.2. *If $S = \{1,2\}$ then $\mathcal{P}(\{1,2\}) = \{S, \emptyset, \{1\}, \{2\}\}$.*

(**2.1.3**) Fix a set S. We observe that the usual union operation on sets defines a binary operation on $\mathcal{P}(S)$ by $(a,b) \mapsto a \cup b$.

(**2.1.4**) Notation (see e.g. [Gre80, Mar21]): Let $n \in \mathbb{N}$. We use the notation $\underline{n}$ to denote the set with n elements

$$\underline{n} := \{1, 2, 3, ..., n\}. \tag{2.1}$$

and similarly, $\underline{n}' := \{1', 2', ..., n'\}$, and so on.

2.2 Equivalence relation and Partition of set

Definition 2.2.1. *Let A and B be sets. A relation R from set A to set B is a set such that $R \subset A \times B$.*

Definition 2.2.2. *An* equivalence relation *on set A is a relation on A to A that is reflexive, symmetric and transitive.*

Definition 2.2.3. *For A a set, let $\mathbb{E}\mathrm{qu}(A)$ be the set of equivalence relations on A.*

Example 2.2.4. *Consider $\underline{2} = \{1, 2\}$. Then the set $\mathbb{E}\mathrm{qu}(\underline{2})$ consists of $\{(1, 1), (2, 2)\}$ and $\{(1, 1), (2, 2), (1, 2), (2, 1)\}$.*

(**2.2.5**) Given a relation ρ then we may write $a\,\rho\,b$ to mean that $(a, b) \in \rho$.

Definition 2.2.6. *(e.g [Mar11, How72]) Let ρ be an equivalence relation on A and $x \in A$, Then an* equivalence class of x *defined as*

$$[x] = \{a \in A \mid \quad a\rho x\}.$$

Definition 2.2.7. *For $\rho \subseteq A \times A$ a relation on set A to A we define $\overline{\rho}$ for the smallest equivalence relation containing ρ.*

We will use this in section 5.1.2.4. We talk about how to construct $\overline{\rho}$ in section 5.1.2.5.

Definition 2.2.8. *For A a set let $\mathfrak{Par}_A$ be the set of partitions of A. A partition of a set A is a set of non-intersecting subsets whose union is A.*

Example 2.2.9. *Consider $\underline{2} = \{1, 2\}$. Then the set $\mathfrak{Par}_{\underline{2}}$ consists of $\{\{1\}, \{2\}\}$ and $\{\{1, 2\}\}$.*

Note that: In the partition of A, each subset is referred to as a 'part' or 'an element',
such as $\{1\}$ is a part of the partition$\{\{1\}, \{2\}\}$ in the above example.

(**2.2.10**) Note that there is a natural bijection between a set of equivalence relation on $\underline{n}$ and a set of partition on $\underline{n}$.

$$\mathbb{E}qu_{\underline{n}} \rightleftarrows \mathfrak{Par}_{\underline{n}}$$

(For the proof see e.g. [PMF, Lecture 17]).

2.3 Some basic topology

Definition 2.3.1. *(see e.g. [Mar21, Arm13]) A topological space is a set X with a collection τ of subsets of X (called 'open' sets) such that: any union of open sets is open; any finite intersection of open sets is open; and the empty set $\emptyset$ and whole set X are open.*

Note that: A set X with the collection τ (the topology on it) is called a topological space (X, τ).

Example 2.3.2. *1. For any set X the collection $\tau_1 = \{X, \emptyset\}$ is a topology on X called the trivial topology or indiscrete topology.*
2. The power set $\tau_2 = \mathcal{P}(X)$ is topological space on X, called discrete topology.
3. The collections $\tau_3 = \{\{1,2,3\}, \emptyset, \{1\}\}$ and $\tau_4 = \{\{1,2,3\}, \emptyset, \{1\}, \{1,3\}\}$ are topologies on the set $\{1,2,3\}$.

(**2.3.3**) We assume familiarity with the metric topology on a metric space. In particular if we say $\mathbb{R}$ is a topological space, we will mean with the Euclidean metric topology, unless we say otherwise.
Similarly, if we say $\mathbb{R}$ is a topological space with the 'standard' topology, we will mean with the Euclidean metric topology.

That is, a set $V \subset \mathbb{R}$ is open if, for every $v \in V$, there exists an $\epsilon > 0$ such that $(v - \epsilon, v + \epsilon) \subset V$.

(**2.3.4**) Similarly, $\mathbb{R}^2$ and other such sets with a natural metric space structure will be understood to be topological spaces with the metric topology.

Definition 2.3.5. *(see e.g. [Mar21, Arm13]) Let X and Y be two topological spaces. A function $f : X \to Y$ is* continuous *if the preimage (or inverse image) of each open set of Y is an open set in X.*

Example 2.3.6. *Let $X = \{1,2,3\}$ and $Y = \{a,b,c\}$. Then $\tau_X = \{X, \emptyset, \{1\}, \{1,2\}\}$ and $\tau_Y = \{Y, \emptyset, \{a\}, \{a,b\}\}$ are topologies on X and Y respectively.*
1. Let $f : X \to Y$ given by $f(1) = a, f(2) = b$ and $f(3) = c$.
Then for any open set $U \subset Y$, $f^{-1}(U)$ is open in X. So in this example, the preimage of each open set in Y is an open set in X.

Therefore,f is the continuous map.

2. Let $g : X \to Y$ given by $g(1) = a, g(2) = c$ and $g(3) = b$. Then g is not continuous since

$$g^{-1}(\{a, b\}) = \{1, 3\} \notin \tau_X$$

is not open set in X.

Example 2.3.7. *Let $X = [0, 1] \subset \mathbb{R}$ with the subspace topology, and consider the map $id : [0, 1] \to [0, 1]$, given by $id(x) = x$.*
Consider any open set U in $[0, 1]$, since $id(x) = x$, the preimage $id^{-1}(U)$ is exactly U. Because U is open in the topology of $[0, 1]$, $id^{-1}(U)$ is open. Therefore, the map id is continuous.

Lemma 2.3.8. *Let $X = \mathbb{R}$ with the Euclidean metric topology. Let $a, b \in \mathbb{R}, a \neq 0$ and consider the map $f : \mathbb{R} \to \mathbb{R}$, given by $f(x) = ax + b$. Then f is continuous.*

Proof. Let $U \subset \mathbb{R}$ is any open set, then for every $u \in U$ there is $\epsilon > 0$ such that $(u - \epsilon, u + \epsilon) \subset U$. We want to prove the preimage $f^{-1}(U) \subset \mathbb{R}$ is open.
$f^{-1}(U) = (f^{-1}(u - \epsilon), f^{-1}(u + \epsilon)) = (\frac{u-b-\epsilon}{a}, \frac{u-b+\epsilon}{a})$ is open in $\mathbb{R}$. Since U is open in the topology of $\mathbb{R}$, $f^{-1}(U)$ is open on $\mathbb{R}$. Therefore, the map f is continuous. □

Note that: proofs of the following propositions or lemmas are in [Mar21], unless we write the proof.

Definition 2.3.9. *(see e.g. [Mar21, Arm13]) Let X be a topological space with topology τ and let L is a subset of X ($L \subset X$). The* subspace topology *on L is the set*

$$\tau_L = \{L \cap U | U \in \tau\}.$$

(**2.3.10**) Note the previous definition (2.3.9) effectively claims that τ_L is a topology □.

Note that: the topological space (L, τ_L) above is called a subspace of (X, τ).

Example 2.3.11. *1. Let* $(X = \{1,2,3\}, \tau_4)$ *as in example (2.3.2) and* $L \subset X$ *where* $L = \{1,2\}$. *Then* $\tau_L = \{L, \phi, \{1\}\}$.

2. Let $(\mathbb{R}, \tau)$ *be the set of real numbers with* τ *being the Euclidean metric topology on* $\mathbb{R}$ *as in (2.3.3). Let* $L = [0,1]$, *hence a subset of* $\mathbb{R}$. *Then* (L, τ_L) *is the subspace topology on* $[0,1]$ *induced from* $\mathbb{R}$.

Note that any sets as $[0,y)$ *and* $(x,1]$ *are not open in* $\mathbb{R}$. *But they are open in the subspace topology of* $[0,1]$. *Since there exists an open set such as*$(-x,y)$ *and* $(x,y+1)$ *in* $\mathbb{R}$. *Then we have* $(-x,y) \cap [0,1] = [0,y)$ *when* $y < 1$ *and* $(x,y+1) \cap [0,1] = (x,1]$ *when* $x > 0$. *So both intervals are open on* $[0,1]$.

Proposition 2.3.12. *(See e.g. [Mar21, Proposition 7.32, p(73)]) Let* X, Y *and* Z *be topological spaces and let* $f : X \to Y$ *and* $g : Y \to Z$ *be continuous functions. Then* $g \circ f : X \to Z$ *is continuous.* □

Proposition 2.3.13. *(See e.g. [Mar21, Proposition 7.33, p(74)]) Suppose that* A *is a subspace of a topological space* X. *Then*

(a) the inclusion map $\iota_A : A \to X, x \mapsto x$ *is continuous.*

(b) if Y *is any topological space and* $f : X \to Y$ *is continuous, the restriction* $f|_A : A \to Y$ *of* f *to* A *is continuous.*

(c) if Z *is any topological space and* $f : Z \to X$ *satisfies* $f(Z) \subset A$, *then* $f : Z \to X$ *is continuous* $\Leftrightarrow$ $f : Z \to A$ *is continuous.* □

Proposition 2.3.14. *(See e.g. [Mar21, Proposition 7.34, p74]) Let* X *and* Y *be two topological spaces. Then*

(a) the projection maps $\pi_X : X \times Y \to X$, $(x,y) \mapsto x$ *and* $\pi_Y : X \times Y \to Y$, $(x,y) \mapsto y$ *are continuous.*

(b) if Z *is any topological space then* $f : Z \to X \times Y$ *is continuous if and only if* $\pi_X \circ f$ *and* $\pi_Y \circ f$ *are both continuous.* □

Example 2.3.15. *Consider* $X = [0,1]^2$ *be a subset of* $\mathbb{R}^2$ *with the subspace topology. Then let* $f : [0,1] \to [0,1]^2$, *defined as* $f(x) = (x,0)$. *(Note care is needed with the notation here:* (x,y) *means a vector or an open interval, depending on context.)*

For a single point such as $(u,0) \in \mathbb{R}^2$ *the preimage is* $f^{-1}((u,0)) = \{x \in [0,1] : f(x) = (x,0) = (u,0)\} = \{u\}$. *Now we look for any point such as* (u,v) *where* $u, v \in (0,1)$. *Then the preimage is* $f^{-1}((u,v)) = \{x \in [0,1] : f(x) = (x,0) = (u,v)\} = \emptyset$.

Let $U \subset [0,1]^2$ *be an open set in* $[0,1]^2$. *We want to show that* $f^{-1}(U)$ *is an open set in* $[0,1]$. *Before we show* $f^{-1}(U)$ *is open, I will prove the projection map is continuous. Let*

$$\pi_1 : [0,1] \times [0,1] \to [0,1],$$
$$(x,y) \mapsto \pi_1(x,y) = x$$

$$\pi_2 : [0,1] \times [0,1] \to [0,1],$$
$$(x,y) \mapsto \pi_2(x,y) = y$$

Suppose V *is an open set of* $[0,1]$. *Then* $\pi_1^{-1}(V) = V \times [0,1]$ *this is an open set in* $[0,1] \times [0,1]$. *Since* $V \subset [0,1]$ *and* $[0,1] \subset [0,1]$ *are open. These for all* $(x,y) \in V \times [0,1]$. *Then we have* $(x,0) \in V \times [0,1] \subset V \times [0,1]$. *Thus* π_1 *is continuous.*

Suppose U *is an open set of* $[0,1]$. *Then* $\pi_1^{-1}(U) = [0,1] \times U$ *this is an open set in* $[0,1] \times [0,1]$. *Since* $U \subset [0,1]$ *and* $[0,1] \subset [0,1]$ *are open. These for all* $(x,y) \in [0,1] \times U$. *Then we have* $(x,0) \in [0,1] \times U \subset [0,1] \times U$. *Thus* π_2 *is continuous.*

Now let us assume $\pi_1 \circ f$ *and* $\pi_2 \circ f$ *are continuous by 2.3.14 , and Let* U *be open subset* $[0,1] \times [0,1]$. *We can write* $U = W \times V$ *where* $W, V \subset [0,1]$ *then*

$$\begin{aligned} f^{-1}(U) &= f^{-1}(W \times V) \\ &= \{x \in [0,1] \mid f(x) \in W \times V\} \\ &= \{x \in [0,1] \mid \pi_1 \circ f(x) \in W \quad \text{and} \quad \pi_2 \circ f(x) \in V\} \\ &= (\pi_1 \circ f)^{-1}(W) \cap (\pi_2 \circ f)^{-1}(V) \end{aligned}$$

Since $\pi_x \circ f$ *and* $\pi_y \circ f$ *are continuous by proposition 2.3.14 this set is the intersection of two open sets and hence it is open. Thus* $f^{-1}(U)$ *is a union of open sets, hence open on* $[0,1]$. *Therefore* f *is continuous.*

Lemma 2.3.16. *(See e.g [Mar21, glue lemma, p75]) Let* X *and* Y *be two topological spaces and let* A *and* B *be subsets of* X *such that* $X = A \cup B$. *Let* $g : A \to Y$ *and*

$h : B \to Y$ be two continuous functions such that $g(x) = h(x)$ for all $x \in A \cap B$. Define $f : X \to Y$ by

$$f(x) = \begin{cases} g(x) & \text{if} \quad x \in A \\ h(x) & \text{if} \quad x \in B \end{cases}$$

Then

(a) if A and B are both open subsets of X then f is continuous; and

(b) if A and B are both closed subsets of X then f is continuous. □

Definition 2.3.17. *(See e.g. [Mar21, Definition 9.1, p87]). Let X and Y be two topological spaces and let $f : X \to Y$. Then f is called a* homeomorphism *if it is bijective and both f and f^{-1} are continuous.*
In this case, X is homeomorphic to Y, denoted by $X \cong Y$.

We will use homeomorphisms for example in section 5.2.1.

Example 2.3.18. *Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + 1/2$. The inverse is $f^{-1}(x) = x - 1/2$.*

(**2.3.19**) Notation. Since a group is a set with a closed binary operation (obeying some axioms) then we may write a group just as G, or write $G = (G, \circ)$, say, meaning that the set is G and the operation is $\circ$.

(**2.3.20**) Recall that in 2.3.17 we defined homeomorphism between topological spaces. Given topological spaces A and B, we write $Homeo(A, B)$ for the set of homeomorphisms between them. We note that $Homeo(A, A)$ is a group under function composition (See e.g. [Wen15, Definition 1.2.8, p13]).

(**2.3.21**) (See e.g. [Moi13, Ch.1].) Let A be a topological space. Two points $a, b \in A$ are *path-connected* if there is a *path from a to b in A*, i.e. a continuous map $f : [0, 1] \to A$ with $f(0) = a$ and $f(1) = b$.

Example 2.3.22. *Let $(X = \{1, 2, 3\}, \tau_3)$ be the topological space as in example (2.3.2).*
Then the function $f : [0, 1] \to X$ defined as $f(t) = 1 \ \ \forall \ \ t \in [0, 1]$. It gives a path from 1 to 1. Since the open sets in X are X,$\emptyset$ or $\{1\}$. Then the preimages are $[0, 1]$, $\emptyset$ and $[0, 1]$.These are all open in $[0, 1]$.

Example 2.3.23. *1. The id map in example 2.3.7 has* $id(0) = 0$ *and* $id(1) = 1$*, so it gives a path from 0 to 1 in* $[0, 1]$*.*
2. The f *map in example 2.3.15 has* $f(0) = (0, 0)$ *and* $f(1) = (1, 0)$*, so it gives a path from (0,0) to (1,0) in* $[0, 1]^2$*.*

(**2.3.24**) A space A is *path-connected* if for every $a, b \in A$ there is a path from a to b.

Example 2.3.25. *1. Let* $X = \mathbb{R}$ *with the usual Euclidean metric topology. Then for any* $x, y \in \mathbb{R}$ *we can define a function* $\eta : [0, 1] \to \mathbb{R}$ *by*

$$\eta(t) = x + t(y - x).$$

— Note that this formula gives a well-defined function to the given codomain (and cf. the following example).
This η *is a continuous map by (2.3.8). We have* $\eta(0) = x$ *and* $\eta(1) = y$*.*
So this is a path from x *to* y*.*
Thus $\mathbb{R}$ *is path-connected.*
2.Let $X = [0, 1] \cup [2, 3] \subset \mathbb{R}$ *with the subset topology. This is not path-connected. There is no continuous map* $\eta : [0, 1] \to X$ *with* $\eta(0) = 1/2$ *and* $\eta(1) = 2.5$*.*

Lemma 2.3.26. *Let X be topological space and q, p and $r \in X$. Suppose there are paths $\gamma, \sigma : [0,1] \to X$ with $\gamma(0) = q, \gamma(1) = \sigma(0) = r$ and $\sigma(1) = p$. Then there is a path in X from q to p.*

Proof. Let $\gamma : [0,1] \to X$ is path from q to r. Thus $\gamma(0) = q, \gamma(1) = r$. And let $\sigma : [0,1] \to X$ is path from r to p. Thus $\sigma(0) = r, \sigma(1) = p$. Since $\gamma(1) = \sigma(0) = r$, we have a well-defined function $h : [0,1] \to X$ given by

$$h(t) = \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}] \\ \sigma(2t-1), & t \in [\frac{1}{2}, 1] \end{cases}$$

For $t = 1/2$ we have $\gamma(1) = \sigma(0) = r$. To see $h(t)$ is a continuous function we proceed as follows. The restrictions to the closed subsets $[0, 1/2], [1/2, 1]$ are continuous because $t \mapsto 2t$ and $t \mapsto 2t - 1$ are continuous by 2.3.8. And the composition between two continuous functions is a continuous function by 2.3.12.Then we have $[0,1] = [0,1/2] \cup [1/2,1]$. So by the glueing Lemma 2.3.16 h is continuous. Since $h(0) = \gamma(0) = q$ and $h(1) = \sigma(1) = p$. Therefore it is a path from q to p. □

Lemma 2.3.27. *Recall from 2.3.9 that we consider $[0,1] \times [0,2]$ as a topological space with the Euclidean metric topology. Suppose $\gamma : [0,1] \to [0,1] \times [0,2]$ is a path from $\gamma(0)$ to $\gamma(1)$ in $[0,1] \times [0,2]$ (i.e. a path as defined in 2.3.21). Then there is a path from the image of $\gamma(0)$ to the image of $\gamma(1)$ in $Shrink_2(\gamma([0,1]))$ as defined as a set in 3.1.17, and with the subspace topology.*

Proof. Let $\gamma : [0,1] \to [0,1] \times [0,2]$ be a path, and write it as $\gamma(t) = (\gamma_x(t), \gamma_y(t))$. Since γ is a path, it is continuous. Thus γ_x is continuous by Proposition(2.3.14 (if part)). Also, γ_y is continuous by the same argument.
Define function $\tilde{\gamma} : [0,1] \to [0,1] \times [0,1]$ given by $\tilde{\gamma}(t) = (\gamma_x(t), \frac{\gamma_y(t)}{2})$. This function is continuous by the same Proposition (2.3.14 (if part)). noting that $f : x \mapsto x/2$ is continuous (by 2.3.8)so the y-component is continuous by proposition 2.3.12 Thus $\tilde{\gamma}$ is path from $\tilde{\gamma}(0)$ to $\tilde{\gamma}(1)$ in $[0,1] \times [0,1]$.
In particular $\tilde{\gamma}(0) = (\gamma_x(0), \frac{\gamma_y(0)}{2})$ and $\tilde{\gamma}(1) = (\gamma_x(1), \frac{\gamma_y(1)}{2})$.
Altogether $Shrink_2(\{\gamma(t)|t \in [0,1]\}) = \{(\gamma_x(t), \frac{\gamma_y(t)}{2})|t \in [0,1]\} = \{\tilde{\gamma}(t)|t \in [0,1]\}$ so the path $\tilde{\gamma}$ in $[0,1]^2$ is also a path in $Shrink_2(\gamma([0,1]))$. □

2.4 Definitions for spaces of the form $[0,1]^d$

In this section, we will start by providing some preliminary definitions which are piecewise-linear function and the Jordain curve theorem and friends.

2.4.1 Piecewise-linear functions

Our main object of study is the set $\mathcal{P}([0,1]^2)$. Many specific elements of this set, i.e. many subsets of $[0,1]^2$, are undefinable (in the sense that there are elements for which I cannot give you enough information to determine the element - compare for example [Gow]). Here we recall some tools for giving some special specific elements.

A relatively simple subset would be a curve drawn in this box $[0,1]^2$. And a relatively simple curve would be a straight line. So let us start with some tools built using straight lines.

(**2.4.1**) Another relatively simple curve would be the graph of a suitable function, that is a function with domain $[0,1]$ and the same codomain.

(**2.4.2**) (see e.g [Abr15]) The general form of a linear function $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = mx + c$ where $m \in \mathbb{R}$ represents the slope of the linear function, and c is the constant value (y-intercept) .

Definition 2.4.3. *A* linear function in d-dimensions *is a function*

$$\begin{aligned} f \quad &: [a,b] \quad \to \mathbb{R}^d \\ &\quad x \quad\; \mapsto (f_1(x), f_2(x), ..., f_d(x)) \end{aligned}$$

where $f_i : [a,b] \to \mathbb{R}$ are linear functions as in (2.4.2).

Definition 2.4.4. *A piecewise linear function on interval $[\alpha,\beta]$ is a continuous map from $[\alpha,\beta]$ to $\mathbb{R}^d$ given by*

$$f(x) = \begin{cases} h_1(x) & \text{if } x \in [\alpha = \alpha_1, \alpha_2] \\ h_2(x) & \text{if } x \in [\alpha_2, \alpha_3] \\ \vdots & \text{if } \quad \vdots \\ h_i(x) & \text{if } x \in [\alpha_i, \alpha_{i+1}] \\ \vdots & \text{if } \quad \vdots \\ h_j(x) & \text{if } x \in [\alpha_j, \alpha_{j+1} = \beta] \end{cases}$$

Where $\alpha = \alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_{j+1} = \beta$ are breakpoints and each $h_i = (h_{i1}, h_{i2}, \ldots h_{id})$ is linear function from $[\alpha,\beta]$to $\mathbb{R}$ as in 2.4.3. Note that $h_i(\alpha_{i+1}) = h_{i+1}(\alpha_{i+1})$ for $1 \leqslant i \leqslant j$.

Example 2.4.5. *Suppose $g : [0,3] \to \mathbb{R}$, given by*

$$g(x) = \begin{cases} 3x+4, & \text{if} \quad 0 \leqslant x \leqslant 1 \\ 5x+2, & \text{if} \quad 1 \leqslant x \leqslant 3. \end{cases}$$

In this example $a_2 = 1$. Then $f_1(x) = 3x+4$ and $f_2(x) = 5x+2$ both are linear. We have $f_1(1) = f_2(1) = 7$.

Example 2.4.6. *Consider the function $f : [0,1] \to [0.1]$ given by $f(x) = x^2$ that satisfies the domain and range conditions. This is NOT piecewise-linear, since there is no suitable choice of breakpoints.*

(**2.4.7**) Note that if we have two points α, β in $\mathbb{R}^d$ then there is a straight line between α and β. In this case, we can write $[\alpha,\beta]$ for the straight line, regarded as a subset of $\mathbb{R}^d$.

Example: For $\alpha = (1,2,3)$ and $\beta = (4,5,6)$ then

$$[\alpha,\beta] = \{\alpha + t(\beta - \alpha) | 0 \leq t \leq 1\}$$

(**2.4.8**) Note that the set visited by a PL function f can be made by union of intervals like $[\alpha,\beta]$, specifically $[\alpha_i,\alpha_{i+1}]$ where each α_i is the evaluated breakpoint $f(a_i)$.

Example 2.4.9. *Consider* $h:[0,1]\to[5,9]$ *with a breakpoint at* $a_2 = 1/2$ *given by*

$$h(x) = \begin{cases} 6x+5 \ , & 0 \leqslant x \leqslant \frac{1}{2} \\ 2x+7 \ , & \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

The evaluation breakpoint is $h(\frac{1}{2}) = 8$ *. Thus the piecewise linear function* $h(x)$ *is made by* $[5,8]\cup[8,9]$.

(**2.4.10**) Note that the set visited by PL function f does not determine f. To determine the function f from the $f(a_i)=\alpha_i$ data, we need also to keep the a_i values.

Example 2.4.11. *From example (2.4.9)* $\alpha_2 = 8$, $h(a_2) = h(\frac{1}{2}) = 8$.

(**2.4.12**) **Notation**: Let $a<b\in\mathbb{R}$. We write $PL(a,b)$ for the set of all PL functions of the form $f:[a,b]\to[a,b]$.

We will use this for example in section 4.3.1.

2.4.2 The Jordan curve theorem and friends

We will use some of the techniques in this section for example in §5.4.3. (Mainly this section is exploring some of the properties of the real line (and the real plane and its subset $[0,1]^2$) that are sometimes taken for granted, but which we will need later to treat carefully.)

Here we assume some basic notions of topology (see also section 2.3 or for example [Mar21]).

Euclidean metric topology has much in common with real analysis, and in particular, the Jordan curve theorem (JCT) below is related to, for example, the Intermediate value theorem (IVT: if $f:[a,b]\to\mathbb{R}$ is continuous then $f([a,b])$ is also a closed interval). Our construction of the PP magma does not require any geometric or topological structure on $\mathcal{P}([0,1]^2)$, but some of the equivalence relations that we will put on it do require such extra structures.

There is not in general a unique way to understand geometric properties of the plane. And 'intuitive' methods seem to correlate with hard proofs. So there are some choices to be made in how to do proofs. Here, I try to give 'enough' exposition for what I need to do later.

(**2.4.13**) Recall that an open subset of $\mathbb{R}^2$ (as a topological space with the metric topology) is connected if and only if it is path-connected (as defined in 2.3.24).

The Jordan curve theorem says that if J is a Jordan curve in $\mathbb{R}^2$ (a topological circle, such as a simple polygon) Then $\mathbb{R}^2 \setminus J$ is disconnected, with two connected components. Let us call these components E and I, exterior to and interior to J, respectively.

Note that $\mathbb{R}^2 = E \sqcup I \sqcup J$ and $\mathbb{R}^2 \setminus J = E \sqcup I$.

This theorem also says that for a point in E and a point in I then every path between them in $\mathbb{R}^2$ intersects J. For suppose there is a path that does not intersect J. Such a path is also a path in $E \sqcup I$, so then $\mathbb{R}^2 \setminus J$ is connected — a contradiction.

Example 2.4.14. *Consider J to be the rectangular boundary of the rectangular region $[-10, 10] \times [-1, 1/2]$. Then for $a \in [0, 1] \times (1/2, 1]$ and $b \in [0, 1] \times [0, 1/2]$ every path between a and b intersects J.*

(**2.4.15**) Observe that if A is a simple connected region of $\mathbb{R}^2$ (such as a disk or rectangle, or in particular such as $[0,1]^2$) and J lies in A then $A \setminus J$ is again disconnected; and every path in A between the two components intersects J.

(**2.4.16**) Fix a Jordan curve J in $\mathbb{R}^2$, and hence (by the Theorem) a partition of $\mathbb{R}^2 \setminus J$ into E and I. Let R be a subset of $\mathbb{R}^2$. Observe that $R \setminus J = (R \cap E) \sqcup (R \cap I)$. If neither part is empty then every path between them (i.e. path between a point in $(R \cap E)$ and a point in $(R \cap I)$) in R intersects $R \cap J$. (For if there is a path not intersecting J then this is also a path in $\mathbb{R}^2$ not intersecting J, contradicting the Theorem.)

(**2.4.17**) In our case, we take the $[0,1]^2$, not the $\mathbb{R}^2$ her we can see the J curve is a line inside $[0,1]^2$ not a circle. So we can see the topology because $[0,1]^2$ lives inside $\mathbb{R}^2$. However, we can have a close curve or circle which comes around half of our space to make the J line closed in the ends as in Figure 2.1. But of course, we allow paths to be inside $[0,1]^2$ from top to bottom.

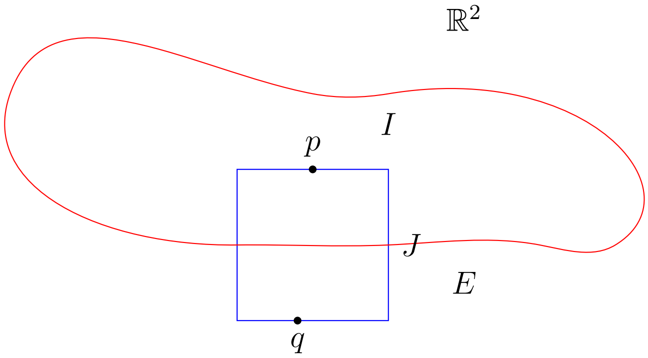


Figure 2.1: JCT

2.4.2.1 The sheep, the ring-fence and the Jordan Curve Theorem

This Theorem can help us to formalise the connections between pictures and mathematical geometry, so we recall it briefly here. One way of saying the Theorem is that if there is a field with a sheep-proof ring-fence inside it, and a water-hole (oasis?) at the edge of the field, then if a sheep is inside the fence it cannot get to the water - as in Figure 2.2

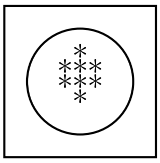


Figure 2.2: Field with a sheep-proof ring-fence

More mathematically we can say this as follows.

Theorem 2.4.18. *(See. [Moi13, The Jordan curve theorem, p(31)]),*
Let J be a topological 1−sphere in $\mathbb{R}^2$. Then $\mathbb{R}^2 - J$ is the union of two disjoint connected sets I and E, such that $J = FrI = FrE$. □

Instead of the JCT we could use the IVT:

(**2.4.19**) Suppose we have a point p on the top edge of $[0,1]^2$ and a point q on the bottom edge. Now suppose we have a continuous path $\sigma : [0,1] \to [0,1]^2$ from p to q in $[0,1]^2$. Then by the IVT there is a value s for which $\sigma(s) = (x, 1/2)$ for some x.

2.5 Magma preliminaries

Definition 2.5.1. *A magma is a pair consisting of a set with a closed binary operation on that set.*
(See for example [Bou89], where this is perhaps first called 'magma' in print; and also for example [Kan02, Sch99, Mar08]).

We write $(A, *)$ for set A with operation $* : A \times A \to A$. We may give a name, such as $M = (A, *)$, for this pair.

In many texts, the term *groupoid* is used instead of magma. But the term groupoid is also used (in other places) for a special kind of category — so these two uses of 'groupoid' are not compatible. For this reason, we will use 'magma'. We will do this even when referring to texts that call magmas groupoids.

Example 2.5.2. *Every group gives a magma.*

(**2.5.3**) Given a magma $(A, *)$, then every subset B of A that is closed under $*$ gives a magma, $(B, *)$ (where the domain and codomain of $*$ are understood as restricted in the obvious way to B).

(**2.5.4**) A magma on a finite set A may be given by a multiplication table. For example

$$\begin{array}{c|cc} * & a & b \\ \hline a & a & b \\ b & a & a \end{array} \tag{2.2}$$

Example 2.5.5. *A magma is given by $M = (\mathbb{Z}, -)$, where $\mathbb{Z}$ is the set of integers and $-$ is the subtract operation.*

Remark 2.5.6. *A magma does not necessarily need to be associative.*

Example 2.5.7. *The magma in example 2.5.5 is not associative because*

$$(3 - 2) - 6 \neq 3 - (2 - 6).$$

2.5.1 An extended example

The next few paragraphs contribute to an extended example of a magma, which we will not use later as such, but which will help to establish some terms.

(**2.5.8**) Here we write $\mathfrak{S}$ for the class of all sets.

Given a property p, we can define a subclass S_p of $\mathfrak{S}$ by $T \in S_p$ if each element t of T has the property p. Depending on the property p, S_p may be a full class (i.e. not a set, in the Russell's Paradox sense), or it may be a set. For example, if p for some element t is: t is a set and t has order 4 (which can be either true or false), then S_p is a class that is too big to be a set. But (for a different example) if property p is: t is a subset of some fixed set U containing some fixed subset $V \subset U$, then S_p is a set.

(**2.5.9**) The intersection of two sets is a set by definition, so if we have a subset S_p of the class $\mathfrak{S}$ as in next example then S_p is closed under intersection, and hence has a *smallest* element - a unique element contained in every other element.

Example 2.5.10. *Consider a set S of sets. This is a subset of the class of all sets $\mathfrak{S}$. On the class of all sets, a cartesian product is defined, and the cartesian product of any two sets is a set. So the 'set' of all sets would be a magma, except that it is not a set, by Russell's paradox! [How72, ID95] However, from this, we see that the cartesian product of two sets is a set. So we can think about a magma constructed as follows. We start with a set T of sets, such as $T = \{A\}$, where A is some non-empty set. We have*

$$\times : T \times T \to \mathfrak{S}$$

but this does not close on T.

Then we ask, formally, what is the smallest magma containing T. Let us call it S_T. As well as A, this set must contain $A \times A$, because $A \neq A \times A$, and we require closure. We also need to include in S_T the products $A\times(A\times A)$ and $(A\times A)\times A$. Note that these are not the same (see also below). And we need to contain the products $(A \times (A \times A)) \times A$, $(A \times A) \times (A \times A)$, $A \times (A \times (A \times A))$, $A \times ((A \times A) \times A)$,

and $((A \times A) \times A) \times A$. Also, these are not the same. Next, we need to involve the products $((A \times A) \times A) \times (A \times A)$, $(A \times A) \times (A \times (A \times A))$, ...and the rest of the 12 ways, because these are not the same. And so on.

(These different ways of inserting n pairs of brackets in word with $n+1$ letters comes from Catalan numbers *(see [Pak, Sta13]) which is given by $C_n = \frac{1}{n+1}\binom{2n}{n}$.)*
As a result, we can continue this process to create an infinite magma. By containing all of these sets we get closed magma under the cartesian product.

Now we can ask if such a magma $S_{\{A\}}$ is associative.
So, we need to check if $(A \times A) \times A = A \times (A \times A)$.
Let $A = \{a\} \in T$.
$L.H.S = (A \times A) \times A = \{((a,a),a)\}$.

However, $R.H.S = A \times (A \times A) = \{(a,(a,a))\}$.
Where, $(a,a) \in A \times A$.
Hence, $(a,a) \neq a$.
Therefore $(A \times A) \times A \neq A \times (A \times A)$.
We have shown that the magma is not associative.

2.5.2 Magma homomorphisms and representations

(**2.5.11**) A magma homomorphism $f : S \to T$ is a set map which commutes with the binary operation. That is,

$$f(s \ *_S \ s') \ = \ f(s) \ *_T \ f(s')$$

where $*_S$ is the composition in S, and so on.

Example 2.5.12. *Consider*

$$f : M \to \{1\}$$
$$m \mapsto 1$$

Then

$$f(m \ *_M \ m') \ = 1 \ = \ 1 \ *_{\{1\}} 1 \ = f(m) \ *_{\{1\}} \ f(m').$$

(**2.5.13**) The category of magmas and magma homomorphisms is sometimes called **Mag** (see for example 2.6.8).

(**2.5.14**) A *representation* of a magma is, in a general sense, the same thing as a

magma homomorphism. However, for a representation we sometimes restrict the target T to be some magma that is relatively well understood, so that S has an image that may be better understood than S itself, but which is a 'model' of S. For example, we may take T to be a monoid. This is analogous to representation theory of groups, where we choose the target to be a full matrix group.
We will give examples later.

For group representations, non-injectivity (non-faithfulness) can be characterised in terms of a kernel - the set of elements that are mapped to the identity. But for a magma we do not generally have an identity element. One alternative approach is to talk about a representation $f : S \to T$ inducing a congruence on S: we say $a \sim b$ if $f(a) = f(b)$.

Every set map $\psi : S \to T$ induces an equivalence relation on S in this way. But not every equivalence relation is a congruence. We talk about relations generally in §2.2, 5.1.26, 5.4.2.

Again we will give examples later, when we try to study representations of the magmas that we introduce in 3.1.

2.6 Categories and Functors

In this section, we introduce Category theory, see e.g [Lei14, Lei16, ML13]. This section contains a category 2.6.1 and functor 2.6.2.

2.6.1 Categories

Definition 2.6.1. *A (locally small) Category may be defined as a quadruple*

$$C = (A, B, \circ, id)$$

where

- *A is a collection such as a set;*
- *B is a collection of pairwise disjoint sets: a set $C(x, y)$ for each pair (x, y) in $A \times A$;*
- *$\circ$ is a composition for every triple (x,y,z) in A which means we have a binary operation $\circ : C(x, y) \times C(y, z) \to C(x, z)$;*
- *id is a map from A to B taking $a \in A$ to an element id_a of $C(a, a)$.*
  This quadruple verifies:
  - *Associativity axioms:*
    for each quadruple (x, y, z, t) of elements in A, and for each $b_1 \in C(x, y)$, $b_2 \in C(y, z)$ and $b_3 \in C(z, t)$ we have

    $$(b_1 \circ b_2) \circ b_3 = b_1 \circ (b_2 \circ b_3)$$

  - *Unity axioms: for $C(a_1, a_2) \times C(a_2, a_2) \to C(a_1, a_2)$ this require $(b, id_{a_2}) \mapsto b = b \circ id_{a_2}$ and $C(a_1, a_1) \times C(a_1, a_2) \to C(a_1, a_2)$ this also, require $(id_{a_1}, b) \mapsto b = id_{a_1} \circ b$.*

Example 2.6.2. *Consider a quadruple*

$$M_{\mathbb{R}} = (\mathbb{N}, M_{\mathbb{R}}(n,m), \circ, id)$$

where

- $\mathbb{N}$ *is the set of all natural numbers,*
- $\mathbb{R}$ *is the commutative ring of real numbers,*
- $M_{\mathbb{R}}(n,m)$ *is the set of* $n \times m$ *real matrices,*
  a composition $\circ : M_{\mathbb{R}}(n,k) \times M_{\mathbb{R}}(k,m) \to M_{\mathbb{R}}(n,m)$ *is given by* $(A,B) \mapsto AB$ *meaning matrix multiplication where* $A = \begin{pmatrix} A_{11} & A_{12} & \dots \\ A_{21} & A_{22} & \dots \\ \vdots & & \ddots \end{pmatrix}$, $B = \begin{pmatrix} B_{11} & B_{12} & \dots \\ B_{21} & B_{22} & \dots \\ \vdots & & \ddots \end{pmatrix}$ *such that* $(AB)_{ij} = \sum_{p=1}^{k} A_{ip}B_{pj}$, *and*
- $$id_k = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{pmatrix}$$

Proposition 2.6.3. $M_{\mathbb{R}}$ *is a category.*

Proof. Notice that our quadruple has the right kind of components to be a category. So, to show M_R is a category, we need it to satisfy the axioms.

- Associativity axiom:
  Let $(n,k,m,t) \in \mathbb{N}$ and $A_1 \in M_R(n,k), A_2 \in M_R(k,m), A_3 \in M_R(m,t)$ we need to proof $(A_1 \circ A_2) \circ A_3 = A_1 \circ (A_2 \circ A_3)$

$$\begin{aligned} L.H.S &= (A_1 \circ A_2) \circ A_3 \\ &= (A_1A_2) \circ A_3 \\ &= (A_1A_2)A_3 \\ &= A_1(A_2A_3) \\ &= A_1 \circ (A_2A_3) \\ &= A_1 \circ (A_2 \circ A_3) = R.H.S \end{aligned}$$

- Unity axiom:

  Let $M_R(n,k) \times M_R(k,k) \to M_R(n,k)$, we need to show $(A_1, id_k) \mapsto A_1$, so $A_1 \circ id_k = A_1$ and $M_R(k,k) \times M_R(k,n) \to M_R(k,n)$, we need to show $(id_k, A_2) \mapsto A_2$, so $id_k \circ A_2 = A_2$

  Thus M_R is a category. □

We see that the construction of our example had two stages. Firstly we gave a quadruple with components 'like a category'; and then we checked the axioms. This is a useful scheme in general. So let us make it systematic, as follows.

Definition 2.6.4. *A precategory maybe defined as a quadruple*

$$C = (A, B, \circ, id)$$

where

- *A is a collection such as a set;*
- *B is a collection of pairwise disjoint sets, a set $C(x,y)$ for each pair (x,y) in $A \times A$;*
- *$\circ$ is a composition for every triple (x,y,z) in A^3 which means we have a binary operation $\circ : C(x,y) \times C(y,z) \to C(x,z)$;*
- *id is a map from A to B taking $a \in A$ to an element of $C(a,a)$.*

Definition 2.6.5. *The precategory of sets is the quadruple of*

$$\boldsymbol{Set} = (A, B, \circ, id)$$

where

- *A is the collection of all sets,*
- *B is a collection of disjoint sets $\hom(X,Y)$ for each pair $(X,Y) \in A \times A$ given by the set of mappings/functions from X to Y $(X \to Y)$,*
- *$\circ$ is a composition of morphism for every three objects X, Y, Z of the form $\circ : \hom(X,Y) \times \hom(Y,Z) \to \hom(X,Z)$*

such that for $f : X \to Y$ *and* $g : Y \to Z$ *we have* $h : X \to Z$ *given by*

$$h(x) = g(f(x))$$

- *id is a map from* A *to* B *taking* $X \in A$ *to the element* id_X *(the identity map) of* $\hom(X, X)$.

Proposition 2.6.6. *The quadruple **Set** is a category.*

Proof. To show **Set** is a category, we need to satisfy the axioms of a category.

- Associativity axiom:
  Let $(X, Y, Z, T) \in A$ and $f_1 \in \hom(X, Y)$, $f_2 \in \hom(Y, Z)$
  and $f_3 \in \hom(Z, T)$.
  We need to proof $(f_1 \circ f_2) \circ f_3 = f_1 \circ (f_2 \circ f_3)$.

$$\begin{aligned} L.H.S &= (f_1 \circ f_2) \circ f_3 \\ &= (f_1 f_2) \circ f_3 \\ &= (f_1 f_2) f_3 \\ &= f_1 (f_2 f_3) \\ &= f_1 \circ (f_2 f_3) \\ &= f_1 \circ (f_2 \circ f_3) = R.H.S \end{aligned}$$

- Unity axiom:
  Let $\hom(X, Y) \times \hom(Y, Y) \to \hom(X, Y)$, we need to show
  $(f, id_Y) \mapsto f$,so $f \circ id_Y = f$ and
  $\hom(X, X) \times \hom(X, Y) \to \hom(X, Y)$
  we need to show $(id_X, f) \mapsto f$,so $id_X \circ f = f$.
  Thus **Set** is a category.

□

Example 2.6.7. *The quadruple* $\boldsymbol{Grp}$ *is*

$$\boldsymbol{Grp} = (\boldsymbol{Grp}_0, \boldsymbol{Grp}(-,-), \circ, id)$$

where

- $\boldsymbol{Grp}_0$ *means the class of all groups;*
- $\boldsymbol{Grp}(-,-)$ *means for every pair of groups the set of group homomorphisms;*
- $\circ$ *is the same as in* $\boldsymbol{Set}$*;*
- id *is the same as in* $\boldsymbol{Set}$*.*

We can check that the quadruple $\boldsymbol{Grp}$ *is a category.*

Note that for groups G, G' *we have underlying sets. We can also call them* G, G'*. Then we can compare* $\boldsymbol{Grp}(G, G')$ *and* $\boldsymbol{Set}(G, G')$*.*

For example consider $G = G' = Z_2$ *— the group of order 2:* $Z_2 = (\{1, -1\}, \times)$*. Then* $\boldsymbol{Grp}(Z_2, Z_2)$ *is a subset of* $\boldsymbol{Set}(Z_2, Z_2)$*. In particular* $f \in \boldsymbol{Set}(Z_2, Z_2)$ *given by* $f(1) = f(-1) = -1$ *is NOT a group homomorphism.*

Moreover, $g \in \boldsymbol{Set}(Z_2, Z_2)$ *given by* $g(1) = g(-1) = 1$ *is a group homomorphism.*

Example 2.6.8. *The quadruple* $\boldsymbol{Mag}$ *is*

$$\boldsymbol{Mag} = (\boldsymbol{Mag}_0, \boldsymbol{Mag}(-,-), \circ, id)$$

where

- $\boldsymbol{Mag}_0$ *means the class of all magmas;*
- $\boldsymbol{Mag}(-,-)$ *means for every pair of magmas the set of magma homomorphisms;*
- $\circ$ *is the same as in* $\boldsymbol{Set}$*;*
- id *is the same as in* $\boldsymbol{Set}$*.*

We can check that the quadruple $\boldsymbol{Mag}$ *is a category.*

Example 2.6.9. *Consider categories given by*

$$C = (A, B, \circ, id, ..., axioms)$$

and

$$C' = (A', B', \circ', id', ..., axioms)$$

. Then there is a product category defined as the following:

$$C \times C' = (A \times A', B \times B', \circ \times \circ', id \times id', ..., axioms)$$

where

- $A \times A'$ *means a collection of all pairs of objects. It is given by pairs* (a, a'), *where* $a \in A$ *and* $a' \in A'$.
- $B \times B'$ *is a class of all maps which are pairs* $(f, g) : (a, a') \to (b, b')$, *where* $f : a \to b \in C$ *and* $g : a' \to b' \in C'$.
- $\circ \times \circ'$ *is a composition of morphism for every morphism of the form* $\circ \times \circ' : (a, a') \to (c, c')$, $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$,*where* $f : a \to b$*and*$f' : b \to c$*in*C , $g : a' \to b'$*and*$g' : b' \to c'$*in*C'. $f' \circ f : a \to c$*and*$g' \circ g : a' \to c'$. *More explicitly,*

$$(a, a') \xrightarrow{(f,g)} (b,\ b') \xrightarrow{(f',g')} (c,\ c')$$
$$(f' \circ f\ ,\ g' \circ g)$$

- $id \times id'$ *are pairwise identities* $1_{(a,a')} : (a, a') \to (a, a')$, $1_{(a,a')} = (1_a, 1_{a'})$ *where* $1_a : a \to a$ *and* $1_{a'} : a' \to a'$.

To show $C \times C'$ *is a category, we need to satisfy the axioms of the category which are Associativity and unity axioms.*

- *Associativity axiom:*
  For each $((a, a'), (b, b'), (c, c'), (d, d'))$ *of pairs in* $A \times A'$ *and for each*

$$(f, g) : (a, a') \to (b, b'),$$

$$(f^{'}, g^{'}) : (b, b^{'}) \to (c, c^{'}),$$

$$(f^{''}, g^{''}) : (c, c^{'}) \to (d, d^{'}).$$

We need to prove $((f^{''}, g^{''}) \circ (f^{'}, g^{'})) \circ (f, g) = (f^{''}, g^{''}) \circ ((f^{'}, g^{'}) \circ (f, g))$.

$$\begin{aligned}
L.H.S &= ((f^{''}, g^{''}) \circ (f^{'}, g^{'})) \circ (f, g) \\
&= (f^{''} \circ f^{'}, g^{''} \circ g^{'}) \circ (f, g) \\
&= ((f^{''} \circ f^{'}) \circ f, (g^{''} \circ g^{'}) \circ g) \\
&= (((f^{''}(f^{'}))(f)), (g^{''}(g^{'}))(g)) \\
&= (f^{''}((f^{'})(f)), g^{''}((g^{'})(g))) \\
&= (f^{''}(f^{'} \circ f), g^{''}(g^{'} \circ g)) \\
&= (f^{''} \circ (f^{'} \circ f), g^{''} \circ (g^{'} \circ g)) \\
&= (f^{''}, g^{''}) \circ ((f^{'} \circ f), (g^{'} \circ g)) \\
&= (f^{''}, g^{''}) \circ ((f^{'}, g^{'}) \circ (f, g)) = R.H.S
\end{aligned}$$

Thus the associative axiom is hold.

- *Unity axiom:*

  Let $(f, g) : (a, a^{'}) \to (b, b^{'})$ *and* $(1_b, 1_{b'}) : (b, b^{'}) \to (b, b^{'})$ *we want to show* $(1_b, 1_{b'}) \circ (f, g) \mapsto (f, g)$, *so* $(1_b, 1_{b'}) \circ (f, g) = (1_b \circ f, 1_{b'} \circ g) = (f, g)$.

  Also,

  $(1_a, 1_{a'}) : (a, a^{'}) \to (a, a^{'})$ *and* $(f, g) : (a, a^{'}) \to (b, b^{'})$ *we want to show* $(f, g) \circ (1_a, 1_{a'}) \mapsto (f, g)$, *so* $(f, g) \circ (1_a, 1_{a'}) = (f \circ 1_a, g \circ 1_{a'}) = (f, g)$.

  Thus the unity axiom is hold.

  So, $C \times C^{'}$ *is the category.*

2.6.2 Functor

A functor is a map between two categories (see for example [Lei14]).

Definition 2.6.10. *Let* $\mathfrak{A} = (ob(\mathfrak{A}), hom_{\mathfrak{A}}(-,-), \circ_{\mathfrak{A}}, id)$ *and* $\mathfrak{B} = (ob(\mathfrak{B}), hom_{\mathfrak{B}}(-,-), \circ_{\mathfrak{B}}, Id)$ *be categories. A functor* $F : \mathfrak{A} \to \mathfrak{B}$ *consists of*

- *A function*

$$\begin{aligned} F_0 : ob(\mathfrak{A}) &\longrightarrow ob(\mathfrak{B}) \\ A &\longmapsto F_0(A) \end{aligned}$$

- *For each* $A_1, A_2 \in \mathfrak{A}$*, a function*

$$F_1 : hom_{\mathfrak{A}}(A_1, A_2) \to hom_{\mathfrak{B}}(F_0(A_1), F_0(A_2))$$

$$f \longmapsto F_1(f)$$

  satisfying the following axioms:

  - $F_1(f' \circ f) = F_1(f') \circ F_1(f)$
    whenever $A_1 \xrightarrow{f} A_2 \xrightarrow{f'} A_3$ *in* $\mathfrak{A}$.
  - $F_1(id_A) = Id_{F_0(A)}$ *whenever* $A \in \mathfrak{A}$

Just like we introduced precategory to help with construction of categories, so we can introduce 'prefunctor' to help construct functors. A prefunctor will be a pair: a map between object classes; and a collection of maps between morphism sets — this is just as in a functor, except that we do not (yet) check the axioms.

Example 2.6.11. *Consider a 'map'* $F : \boldsymbol{Grp} \to \boldsymbol{Set}$ *between*
Grp *is a category of all groups and*
Set *is a category of all sets. A function*

$$F_0 : ob(\boldsymbol{Grp}) \to ob(\boldsymbol{Set})$$

$$G \mapsto F_0(G)$$

Where $F_0(G)$ *is the underlying set and* *for each $G_1, G_2 \in$ **Grp**, a function*

$$F_1 : hom_{\boldsymbol{Grp}}(G_1, G_2) \rightarrow hom_{\boldsymbol{Set}}(F_0(G_1), F_0(G_2))$$

$$f \mapsto F_1(f)$$

Proposition 2.6.12. *The map F : **Grp** $\rightarrow$ **Set** is a functor.*

Proof. To show F is a functor, we need to satisfy the axioms of functor.

- Let $f_1 : G_1 \rightarrow G_2$ and $f_2 : G_2 \rightarrow G_3$, where $G_1, G_2, G_3 \in$ **Grp** ,
  f_1, f_2 are functions of a group homomorphism and $F_1(f_1), F_1(f_2)$ are the underlying set.
  We need to Proof $F_1(f_2 \circ f_1) = F_1(f_2) \circ F_1(f_1)$.
  $L.H.S = F_1(f_2 \circ f_2) = F_1(f_2(f_1))$,
  however, $R.H.S = F_1(f_2) \circ F_1(f_1) = F_1(f_2)(F_1(f_1)) = F_1(f_2(f_1))$.
  Therefor, $F_1(f_2 \circ f_1) = F_1(f_2) \circ F_1(f_1)$

- Let $G \in$ **Grp** then $F_1(id_G) = Id_{F_0(G)}$.
  Therefor F : **Grp** $\rightarrow$ **Set** is the functor.

□

(**2.6.13**) Note that (see e.g [Che02, DF04]) this functor F : **Grp** $\rightarrow$ **Set** called forgetful functor that means forget a group structure .

Example 2.6.14. *Let U : **Mag** $\rightarrow$ **Set** is a map between*
***Mag** means a category of all magmas and*
***Set** means a category of all sets. A function*

$$U_0 : ob(\boldsymbol{Mag}) \rightarrow ob(\boldsymbol{Set})$$

$$M \mapsto U_0(M)$$

Where $U_0(M)$ is the underlying set and
*for each $M_1, M_2 \in$ **Mag**, a function*

$$U_1 : hom_{\boldsymbol{Mag}}(M_1, M_2) \to hom_{\boldsymbol{Set}}(U_0(M_1), U_0(M_2))$$

$$u \mapsto U_1(u)$$

Proposition 2.6.15. *The map U : **Mag** $\to$ **Set** is a functor.*

Proof. To show U is a functor, we need to satisfy the axioms of functor.

- Let $u_1 : M_1 \to M_2$ and $u_2 : M_2 \to M_3$, where $M_1, M_2, M_3 \in$ **Mag** , u_1, u_2 are magmas homomorphism and $U_1(u_1), U_1(u_2)$ are the underlying set.

  We need to Proof $U_1(u_1 \circ u_2) = U_1(u_1) \circ U_1(u_2)$.
  $L.H.S = U_1(u_1 \circ u_2) = U_1(u_1(u_2))$,
  however, $R.H.S = U_1(u_1) \circ U_1(u_2) = U_1(u_1)(U_1(u_2)) = U_1(u_1(u_2))$.
  Therefor, $U(u_1 \circ u_2) = U(u_1) \circ U(u_2)$

- Let $M \in$ **Mag** then $U_1(id_M) = Id_{U_0(M)}$.
  Therefor U : **Mag** $\to$ **Set** is the functor.

□

Note that the functor U : **Mag** $\to$ **Set** is also a forgetful functor because it is forgetting the magma structure.

Chapter 3

Constructing a 'passport photo' magma

In this chapter, we put an algebra structure on a fairly general idea, that is, bringing two pictures together to make a new picture.

So the underlying idea here is a 'demand side' one: that humans have ways of doing this operation of bring-together of pictures that extend the possible extraction of 'meaning' from the pictures separately. (Obviously, the extraction of meaning itself is a separate hard problem. We will not address this problem directly here. One hope is that progress can arise in an analogous way to that arising via the study of the PageRank algorithm - see e.g. [Gle15, Mar11] - on the 'importance' of web pages.)

For example, there may be a natural way of bringing two pictures together - composing them - if they were created by cutting a single picture into two pieces in the first place. But specifically because of the demand-side perspective, we want to focus on *construction* and the choices made in the construction of our algebraic structure. In construction, we may start with a binary operation on a set - where the construction of the binary operation leads, rather than the wish for it to satisfy any specific axioms (a 'supply side' driver). For this reason, we start with algebraic structures having few (or no) axioms *ab initio*. Thus magmas.

We can then use magmas to frame choices for congruences which lead to 'associative-isation' (congruences that make associative) and other target axioms.

A congruence on an algebraic structure starts with a relation (for example 'identifying' some elements that must be equal if some axiom is to be imposed). The initial relation, then, can be characterised by some set of identifications. Thus it is immediately reflexive and symmetric. But for an initial relation characterised in this way to become a congruence, it may have to be larger than this initial set - the initial set may not manifest transitivity, for example. Part of our project is to investigate congruences 'seeded' by various initial identifications. Around this, there are both general issues and issues associated to our specific case, as we address below.

In §3.1 we give our basic definition. In §3.2 we give generalizations.

3.1 Passport photo magma

In this section, we construct a magma by stacking and shrinking 'passport photographs'. We will explain exactly what all these terms mean. We will also explain why we have chosen 'passport photo' for our magma elements, and discuss why 'stack-shrink' for composition.

3.1.1 Aside on other motivating ideas behind our setup

It is fair to say that we do not have a single coherent motivating aim guiding our construction choices here. Apart from the notably vague 'bringing pictures together' mentioned above, the closest to a coherent aim is perhaps to understand the possible emergence of topological phenomena in certain physical materials. But our grasp of any such physics is essentially zero, so at heart, ours is a pure-mathematical exercise.

With that caveat, we can 'mathematicise' the physical ideas as follows.

The square lattice Ising model is a 'two-dimensional' model of a system like a ferromagnet arising in statistical mechanics (see e.g. [Bax82]). A ferromagnet is a piece of solid metal material sitting in physical space, made of metal molecules with many physically important properties, among which is that each is a tiny magnet, with a magnetic-field orientation. So to describe the configuration of the material we should give the orientation of each molecule. The molecules sit in a lattice pattern. So their positions can be given (in the two-dimensional (2d) case) by pairs of coordinates. The coordinates can be taken to be integers by choosing units where the molecule spacing is 1. In real terms, the whole piece might lie in a 1cm-by-1cm square in the (2d) laboratory, and there might be $N = 10^{10}$ molecules in each direction. Recall that we write $\underline{n} = \{1, 2, ..., n\}$, so the set of coordinates of all molecules can be given by $\underline{N} \times \underline{N}$. Then if S is the set of possible orientations of a single molecule, the set ς of configurations of the whole magnet is

$$\varsigma \quad = \quad \hom(\underline{N}^2, S)$$

- where $\hom(A, B)$ means the set of all maps from set A to B. In our case, we choose $|S| = 2$ - for example $S = \{1, 2\}$ or $S = \{0, 1\}$.

Now we have the standard bijection

$$\iota : \hom(R, \{0, 1\}) \to \mathcal{P}(R)$$

where $\iota(f)$ is given by $r \in \iota(f)$ if $f(r) = 1$. Therefore our configuration set is given by

$$\varsigma \cong \mathcal{P}(\underline{N}^2)$$

The set $\underline{N}^2$ is of course finite but very large. It is a useful approximation to it to consider the interval $[0, 1]^2$ inside which it sits (the underlying physical space, instead of just the molecule points in that space). So this brings us to consider $\mathcal{P}([0, 1]^2)$.

... And in this setting, joining two such intervals corresponds (loosely) to bringing two sets of molecules together ...

3.1.2 The power set $\mathcal{P}([0, 1]^2)$

We defined the power set function in 2.1.1. The underlying set of the magma we will construct is $\mathcal{P}([0, 1]^2)$. That is, the elements are subsets of the unit square $[0, 1]^2$.

This is a well-defined set. But as we have noted in 4.2 most of its elements cannot be described explicitly (in the same sense as this is true for the real line or the unit interval — see for example [Gow].

The set $\mathcal{P}([0, 1]^2)$ is not hard to manipulate as an abstract set - describing functions to and from this set for example. But it will also be useful to give some specific elements, and work with some individual elements. Just as for the real line, there are many elements, and collections of elements, that can be given explicitly. Next we discuss this.

3.1.3 Representations of elements of $\mathcal{P}([0,1] \times [0,h])$

We say that a *monochrome picture* in a region $R \subset \mathbb{R}^2$ is an assignment of a colour, either black or white, to each point in R. That is, it is a function $p : R \to \{black, white\}$. (There are many potential mathematical subtleties to this, but we will pass over most of them for now. We address some of them in this Section.)

Definition 3.1.1. *For $h \in \mathbb{R}$ (usually $h \geq 0$) we define*

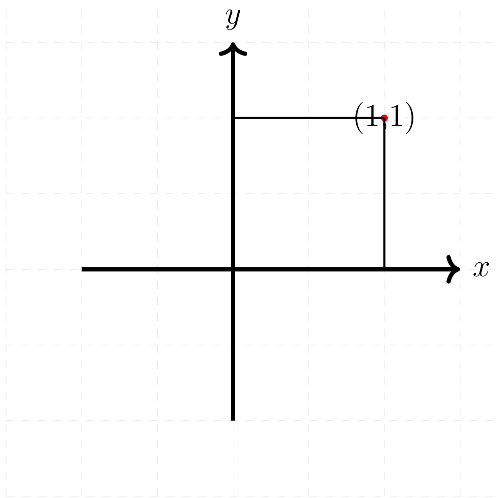
$$\mathfrak{S}_h := \mathcal{P}([0,1] \times [0,h]).$$

(**3.1.2**) Because $\mathfrak{S}_h$ includes all subsets of $[0,1] \times [0,h]$ it includes all monochrome 'pictures' in a frame of this size, by assigning black at point x if x is included, and white otherwise. Hence we think of 'passport photographs' and other images of such size.

(**3.1.3**) Next we consider how drawings of such functions $p : R \to \{black, white\}$ can be achieved in practice - specifically in the context of this document. And hence how some such functions can be communicated by drawings.
(Note in particular that we are using the word 'picture' in a mathematical sense, but using the word 'drawing' in the dictionary sense. It will be a main point of this Section to investigate the connection between these ideas.)

Note that: In some drawings here, a blue frame is used to indicate a framework for $R = [0,1]^2$ when the frame is not part of the picture (i.e. white not black). See for example 4.2.10

(**3.1.4**) Next, we draw some examples. Here we use a scale for pictures on our page which is given by:



.

Just as most individual real numbers are undefinable, so it is for elements of $\mathfrak{S}_1$. So the examples we can give explicitly are limited. But just as, in practice, the undefinability problem is easily circumvented (in various senses - see for example [Moi63, Moi13]), so it will be (up to a point) for us.

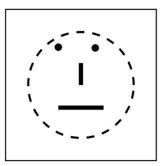
(**3.1.5**) We write the 'subset' $[0,1]^2$ of $[0,1]^2$ explicitly by

$$[0,1]^2 \;=\; \{(x,y) : x \in [0,1], y \in [0,1]\}$$

(**3.1.6**) The picture for the element $[0,1]^2 \in \mathcal{P}([0,1]^2)$ has the drawing

Example 3.1.7. *Consider a picture given by a drawing*

$$P_A \;=$$



$$\in \;\; \mathfrak{S}_1 \tag{3.1}$$

We have drawn this picture in an 'equation' $P_A = \text{something}$ above. By doing this, the writer is trying to convey a specific element of $\mathfrak{S}_1$ to the reader (as well as to give it the convenient and portable name P_A). It is an interesting question:

What element does the reader 'see'? Most of the time, for us, it will be fine if the reader sees approximately the same element (in the sense, say, of [Moi63, Moi13] - with a tolerance corresponding to the natural quantisation in the medium of this LaTex document!), but sometimes later we will need to be more careful.

3.1.3.1 Simple tools for building elements of $\mathcal{P}([0,1]^2)$

Before we describe any pictures we will define some operations which help to write a mathematical form of creating any pictures.

(**3.1.8**) The simple *union operation* $\cup$ (see e.g. [Cam99, uno]) (from 2.1.3) may be used to combine or merge more than one picture to create a composite picture within the same space.

(**3.1.9**) The *difference* (see eg[Cam99, uno]) $A \setminus B$ is used to define the elements in set A (first picture) not in set B (another picture).

(**3.1.10**) Note that the coordinate-flip map $\tau : (x,y) \mapsto (y,x)$ takes a point in $[0,1]^2$ to another such point, and hence also takes an element of $\mathcal{P}([0,1]^2)$ to another element.
(For examples of this see 4.2.)

3.1.3.2 Back to the pictures

Here we describe the picture P_A in (3.1) with mathematics and pictures as follows:

- First, consider the 'mathematical frame' subset

$$S_F = \{(x,0) : x \in [0,1]\} \bigcup \{(x,1) : x \in [0,1]\} \bigcup \{(0,x) : x \in [0,1]\} \bigcup \{(1,x) : x \in [0,1]\}$$

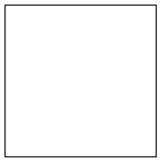
We can ask: what is the picture for this set? It is a set of mathematical lines, but these lines are very thin so, strictly speaking, they are too thin to see. There are a few different ways we can address this problem.

Let $e = 0.001$. We will use this to make lines into thick lines, with thickness e.

Then

$$\begin{aligned} S_e &= \{(x,y) : x \in [0,1], y \in [0,e]\} \cup \{(x,y) : x \in [0,1], y \in [1-e,1]\} \\ &\cup \{(x,y) : x \in [0,e], y \in [0,1]\} \cup \{(x,y) : x \in [1-e,1], y \in [0,1]\}, \end{aligned}$$

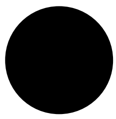
has the picture:



- We represent the face in P_A (3.1) by using circles. To start with, consider

$$\{(x-\frac{1}{2}, y-\frac{1}{2}) : x^2+y^2 < \frac{1}{16}\},$$

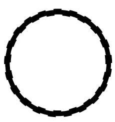
This has a picture:



For annulus:

$$\{(x-\frac{1}{2}, y-\frac{1}{2}) : \frac{1}{16+e} < x^2+y^2 < \frac{1}{16}\},$$

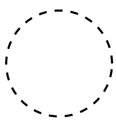
the picture is:



The dashed version of a face is the set :

$$S_S = \left\{(r\cos\theta, r\sin\theta) + (\frac{1}{2}, \frac{1}{2}) \ : \ \frac{1}{4+e} < r < \frac{1}{4-e}, \quad \theta \in [\frac{n}{9}\pi, \frac{2n+1}{18}\pi], \quad n = 0, ..., 17\right\}.$$

The picture is:



- Then the sets of left and right eyes are

$$S_L = \{(x - \frac{1}{4}, y - \frac{3}{4}) : x^2 + y^2 \leqslant 1/10000\},$$

$$S_R = \{(x - \frac{3}{4}, y - \frac{3}{4}) : x^2 + y^2 \leqslant 1/10000\}$$

respectively.

The picture is:

$S_L \cup S_R =$

- However, the set of nose is

$$S_{Nose} = \{(x, y) : \ \frac{1}{2} \leqslant x \leqslant \frac{1}{2} + 2e \ , \ \frac{1}{2} \leqslant y \leqslant \frac{5}{8}\}.$$

The picture is :

- Then, the set of the mouth in P_A is

$$S_M = \{(x, y) : \ \frac{2}{5} \leqslant x \leqslant \frac{13}{20} \ , \ \frac{2}{5} \leqslant y \leqslant \frac{2}{5} + 2e\}.$$

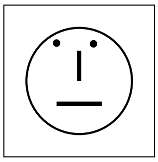
The picture is :

Finally, the mathematical form of the whole picture P_A is

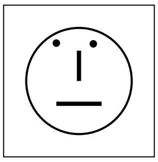
$$P_A = S_e \cup S_S \cup S_L \cup S_R \cup S_{Nose} \cup S_M.$$

How can we decide if two elements of $\mathfrak{S}_1$ are equal or not? If their pictures here do not 'look the same' then they are unequal. But in all other cases, we may have to be careful. We will return to this point later.

Example 3.1.11. *Consider a picture*

P_B =  $\in \mathfrak{S}_1.$

Then

$P_B =$ 

$$= S_e \cup \{(x - \frac{1}{2}, y - \frac{1}{2}) : \frac{1}{16+e} \leqslant x^2 + y^2 \leqslant \frac{1}{16}\} \cup S_L \cup S_R \cup S_{Noes} \cup S_M.$$

3.1.4 Binary operation ⊠ on $\mathcal{P}([0,1]^2)$

In this section we define a binary operation. The key points will be that it is well-defined and closed. These properties will probably be 'inutitively clear', but the construction is crucial for us, so we will proceed carefully.

In our construction there are various 'Definitions'. In some cases it is (perhaps implicitly) an assertion that these constructions-by-definition are well-defined. In other words some of our definitions are Proposition-Definitions. In these cases we will make sure to give whatever proof is needed *before* the Definition.
(For example it is formally possible to claim to 'define' a function $f : [0,1] \to [0,1]$ by $x \mapsto x + 1$, but of course this is not well-defined because the codomain does not contain the image.)

Recall from Defn.2.1.1 that for a set S then $\mathcal{P}(X)$ means the power set of X — the set of all subsets.

(**3.1.12**) **Convention**: Given a function $f : A \to B$, we understand

$$f : \mathcal{P}(A) \quad \to \quad \mathcal{P}(B)$$

by

$$S \subset A \quad \mapsto \quad f(S) = \{f(s) | s \in S\}$$

(**3.1.13**) Observe that for $(x, y) \in \mathbb{R}^2$ and $h \in \mathbb{R}$ we have $(x, y+h) \in \mathbb{R}^2$, because $\mathbb{R}$ is closed under addition. Therefore the following functions are well-defined.

Definition 3.1.14. *For $h \in \mathbb{R}$ define*

$$\mathsf{shift}_h : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$$

$$(x, y) \mapsto (x, y+h)$$

In the following, we define a *stack operation* between two pictures by glueing one picture above another (here 'above' means higher up the page).

Observe that for $x \in [0, h]$ and $y \in [0, h']$ then $x + y \in [0, h + h']$. Therefore the following functions are well-defined.

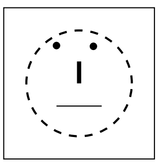
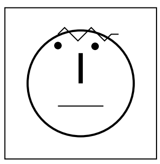
Definition 3.1.15. *For $h, h' \in \mathbb{R}$:*

$$\begin{array}{rcccl} \mathsf{stack}: & \mathcal{P}([0,1] \times [0,h]) \times \mathcal{P}([0,1] \times [0,h']) & \to & \mathcal{P}([0,1] \times [0,h+h']) \\ & (a, b) & \mapsto & \mathsf{shift}_h(a) \cup b. \end{array}$$

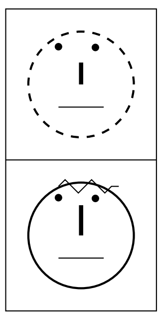
and $\overline{\mathsf{stack}}$ with the same domain and codomain, given by

$$(a, b) \mapsto a \cup \mathsf{shift}_{h'}(b).$$

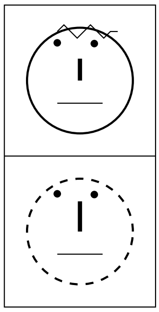
Example 3.1.16. *Consider*

$a =$  , $b =$  $\in \mathcal{P}([0,1]^2)$

Then,

$\mathsf{stack}(a, b) = \mathsf{shift}_1(a) \cup b =$ 

and

$$\overline{\mathsf{stack}}(a,b) = \mathsf{stack}(b,a) = a \cup \ \mathsf{shift}_1(b) =$$  .

Now we define a *shrink function* (or scale function) to squeeze (or rescale) the size of a passport photo.

Definition 3.1.17. *For $h > 0$,*

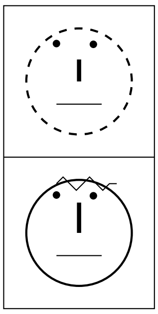
$$\mathsf{shrink}_h : \mathcal{P}([0,1] \times [0,h]) \quad \to \quad \mathcal{P}([0,1] \times [0,1])$$

$$a \ \mapsto \mathsf{shrink}_h(a)$$

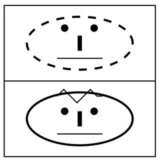
where

$$\mathsf{shrink}_h(a) := \{(x, \frac{y}{h}) | (x,y) \in a\}.$$

Example 3.1.18. *Consider*

$$A =$$  $$\in \mathcal{P}([0,1] \times [0,2]).$$

Then,

$$\mathsf{shrink}_2(A) =$$  .

Definition 3.1.19. *The binary operation $\boxtimes$ on $\mathcal{P}([0,1]^2)$ (which is a 'stack and shrink' of the photos from $\mathfrak{S}_1$) is defined by commutativity of the following diagram:*

$$\begin{array}{ccc} \mathcal{P}([0,1]^2) \times \mathcal{P}([0,1]^2) & \xrightarrow{\quad\boxtimes\quad} & \mathcal{P}([0,1]^2) \\ & {\scriptstyle \mathsf{stack}} \searrow \qquad \nearrow {\scriptstyle \mathsf{shrink}_2} & \\ & \mathcal{P}([0,1] \times [0,2]) & \end{array} .$$

Thus, we have the following proposition.

Proposition 3.1.20. *The pair*

$$\mathfrak{M} := (\mathcal{P}([0,1]^2), \boxtimes)$$

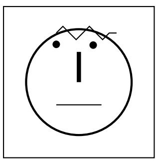
is a magma.

Proof. The operation is closed by construction as in Def.3.1.19. □

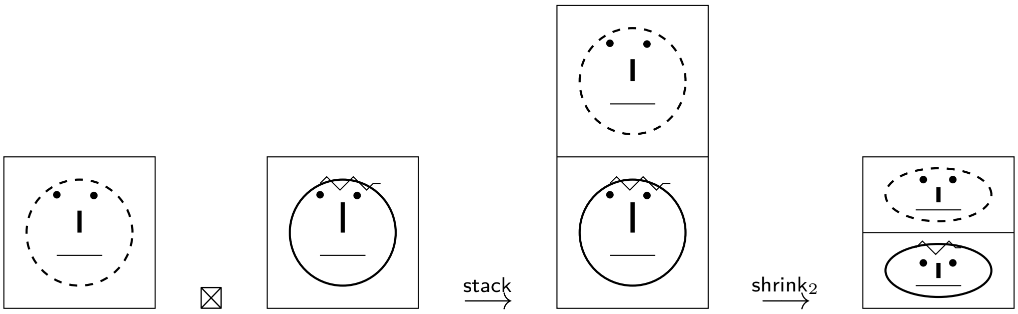
(**3.1.21**) Note that both stack's in 3.1.15 give magmas in this way, but these are different.

In the following we use the stack with $\mathsf{shift}_h(a)$.

Example 3.1.22. *Consider*

 ,  $\in \mathcal{P}([0,1]^2)$

Then,



. (3.2)

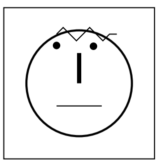
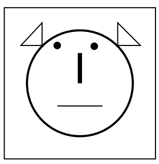
In this example the power set of interval square $\mathcal{P}([0,1]^2)$ is closed under the binary operation $\boxtimes$.

(**3.1.23**) Note that the stack function takes a union of two pictures which is 'mainly' a disjoint union, but which can have overlap where the two square regions meet. This fact does not look significant when we look at pictures by eye. But we will see later that it becomes significant when we consider submagmas, which are of course defined mathematically and not 'by eye' (see for example 4.1). For example, when two pictures are joining, then if we have two strings from top to bottom in each, the number of intersection points with the horizontal mid-point line could be four after composition. The result will no longer be two 'strings' in general, and will not satisfy certain more general notions of 'stringy-ness' either. We discuss this in Chapter 4.

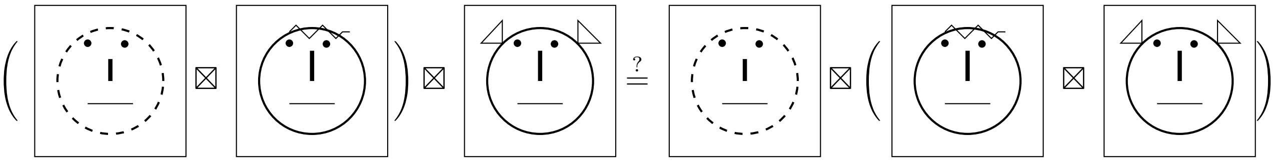
3.1.5 Some computations in the magma $\mathfrak{M} = (\mathcal{P}([0,1^2]), \boxtimes)$

In this part, we examine the composition operation $\boxtimes$ on $\mathfrak{S}_1$ to see if it is associative - to see if our magma is a semigroup. (We will show that it is not.)

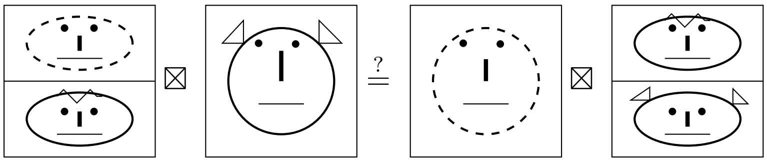
Consider

 ,  and  $\in \mathfrak{S}_1$.

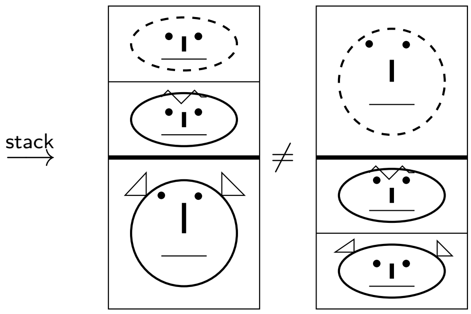
We want to check for an associative law: Why? To get a monoid from this magma that requires an associative algebraic structure. Then we can use representation theory (which has an associative target.)



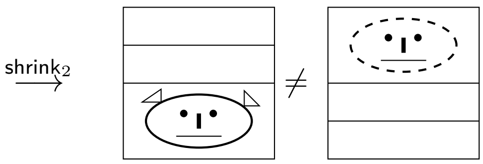
Here we do both sides together. First, we start by *stack* then shrink pictures in brackets, we get



Then, we apply a stack on both sides, we get



Finally, we apply shrink both sides , we get



(3.3)

Therefore $\mathfrak{S}_1$ is not associative with $\boxtimes$.

In chapter 5 we will discuss several relations on $\mathcal{P}([0,1]^2)$ that might be or might not be congruences on our magma (in the sense defined in section 5.1), aiming to make an associative quotient 5.4.1.

3.2 Generalisations: magma on $\mathcal{P}([0,1]^d \times [0,1])$

This section will construct magmas generalising the passport photograph magma (PP Magma). One aim is to construct a magma that will contain a submagma of 'braids' or braid-like elements. For this, we consider varying the underlying set from $\mathcal{P}([0,1] \times [0,1])$ to $\mathcal{P}([0,1]^d \times [0,1])$, where d is some fixed 'dimension'. Thus the PP Magma is $d = 1$; while the sculpture magma will be $d = 2$. It is also interesting to consider $d = 0$.

Next, we show how to define the composition for general d, generalising $\boxtimes$ from 3.1.19.

We do the generalisation by upgrading the underlying set $\mathfrak{S}_1$ to

$$\mathfrak{S}_1^d \;:=\; \mathcal{P}([0,1]^d \times [0,1]) \;=\; \mathcal{P}([0,1]^{d+1})$$

and adjusting the composition accordingly.

3.2.1 Representations of elements of $\mathcal{P}([0,1]^3)$

Here we want to draw $[0,1]^3$ but there are at least two problems here. Firstly, a new problem - how do we draw a 3D object in 2D? Then we have the other problem - more similar to the problem in ordinary PP Magma - of how to draw "all black" for the object $[0,1]^3$.

For the first problem, we will 'project' 3d onto 2d. We can do this in various ways, but roughly we project so that $(1,1,1)$ is close to $(0,0,0)$.

For the second problem we may draw just a framework - the same idea as for 2d above. Altogether then, $[0,1]^3$ is represented by figure 3.1.

3.2.2 Binary operation $\boxtimes$ on $\mathcal{P}([0,1]^{d+1})$

In this part, we define a stack-operation, shrink-function then the binary operation.

Definition 3.2.1. *Fix $d \in \mathbb{N}_0$. The stack binary operation* stack *given by*

$$\begin{aligned} \mathsf{stack} : \mathcal{P}([0,1]^{d+1}) \times \mathcal{P}([0,1]^{d+1}) &\to \mathcal{P}([0,1]^d \times [0,2]) \\ (a,b) \qquad\qquad &\mapsto \qquad \mathsf{shift}_1(a) \cup b \end{aligned}$$

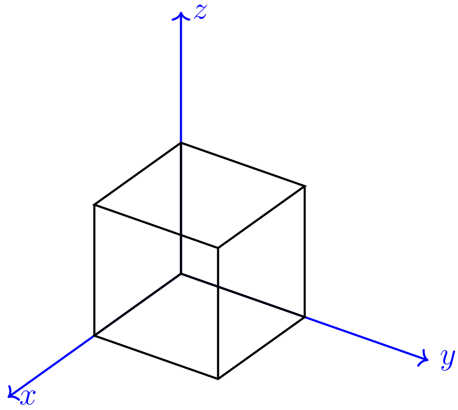


Figure 3.1: Picture of containing box for sculpture magma

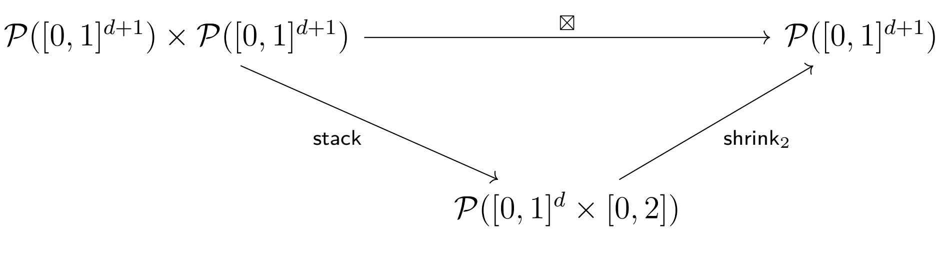
and $\overline{\mathsf{stack}}$ *with the same domain and codomain, given by*

$$(a, b) \quad \mapsto \quad a \cup \mathsf{shift}_1(b).$$

Definition 3.2.2. *Fix $d \in \mathbb{N}_0$. Define a function of* shrink_2 *by*

$$\begin{aligned} \mathsf{shrink}_2 \ : \ \mathcal{P}([0,1]^d \times [0,2]) &\rightarrow \mathcal{P}([0,1]^{d+1}) \\ a &\mapsto \mathsf{shrink}_2(a) = \{(x_1, x_2, ..., x_d, \frac{z}{2}) | (x_1, x_2, ..., x_d, z) \in a\}. \end{aligned}$$

Definition 3.2.3. *The binary operation $\boxtimes$ on $\mathcal{P}([0,1]^d \times [0,1])$ (which is a 'stack and shrink' of the photos from $\mathfrak{S}_1^d$) is defined by the commutativity of the following diagram:*



Proposition 3.2.4. *Fix $d \in \mathbb{N}_0$, and let $\boxtimes$ be defined as in Defn.3.2.3. The Pair*

$$\mathfrak{M}_d = (\mathcal{P}([0,1]^{d+1}), \boxtimes).$$

is a magma.

Proof. The operation is closed by construction as in Def.3.2.3 $\square$

Example 3.2.5. *1. Consider* $d = 1$*. Then* $\mathfrak{M}_1 = (\mathcal{P}([0,1]^2), \boxtimes)$ *is the magma in 3.1.20.*

2. Consider $d = 2$*. Then*

- *The stack binary operation is*

$$\begin{array}{rcl} \mathsf{stack} : \mathcal{P}([0,1]^3) \times \mathcal{P}([0,1]^3) & \to & \mathcal{P}([0,1]^2 \times [0,2]) \\ (a,b) & \mapsto & \mathsf{shift}_1(a) \cup b \end{array}$$

where

$$\mathsf{shift}_1(a) = \{(x_1, x_2, z+1) \mid (x_1, x_2, z) \in a\}$$

(See figure 3.2a).

- *The shrink function is*

$$\begin{array}{rcl} \mathsf{shrink}_2 \ : \ \mathcal{P}([0,1]^2 \times [0,2]) & \to & \mathcal{P}([0,1]^3) \\ a & \mapsto & \{(x_1, x_2, \frac{z}{2}) | (x_1, x_2, z) \in a\} \end{array}$$

(See figure 3.2b).

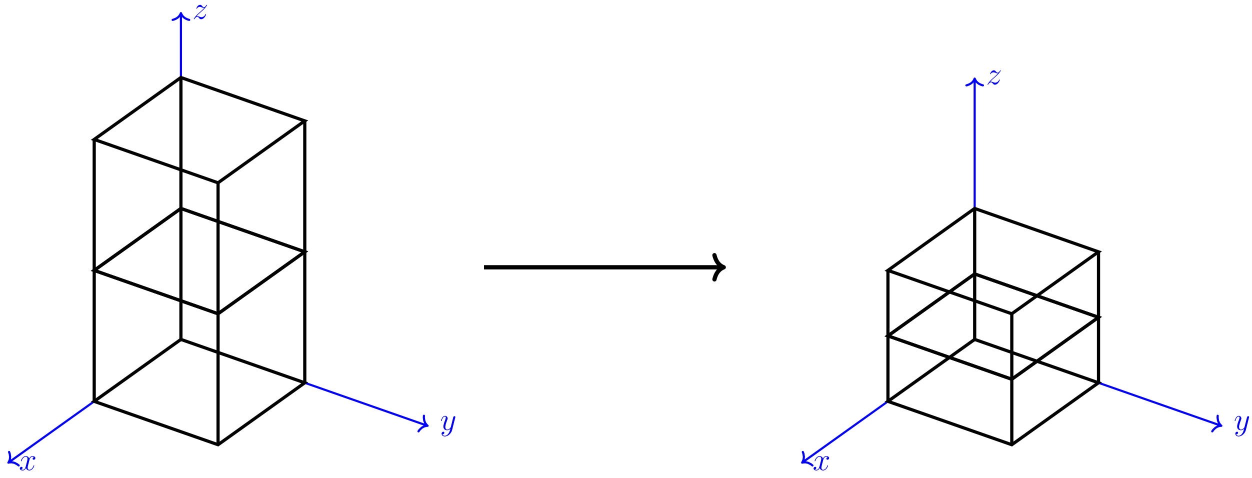
- *Followed by the binary operation on* $\mathcal{P}([0,1]^2 \times [0,1])$ *that is given by the commutativity*

$$\begin{array}{ccc} \mathcal{P}([0,1]^3) \times \mathcal{P}([0,1]^3) & \xrightarrow{\quad\boxtimes\quad} & \mathcal{P}([0,1]^3) \\ & {\scriptstyle\mathsf{stack}} \searrow \quad \nearrow {\scriptstyle\mathsf{shrink}_2} & \\ & \mathcal{P}([0,1]^2 \times [0,2]) & \end{array} .$$

Thus we get the magma $\mathfrak{M}_2 = (\mathcal{P}([0,1]^3), \boxtimes)$*.*

(**3.2.6**) Note that: in the example above (see 2), we refer to $\mathfrak{M}_2$ as the 'sculpture magma'.

(**3.2.7**) The stack and shrink representation of the sculpture magma is given in the following figure (3.2).



(a) Stack sculpture (b) Shrink sculpture

Figure 3.2: Stack and Shrink sculpture- frame only

Chapter 4

Submagmas of passport photograph magma

In this chapter, we address some submagmas of our magma $\mathfrak{M}$ defined in 3.1.20.

One 'direction of travel' is to look for submagmas that are somehow like the braid groups (but without the heavy layers of equivalence used in actual braid groups - see later). But before this we will look at possible submagmas more generally.

Given a subset of $\mathcal{P}([0,1]^2)$ we can ask if it is closed under $\boxtimes$, and thus search for submagmas this way. If the subset is not closed, we can ask what is the smallest submagma that contains our subset: the submagma 'generated' by the subset. For subset S this is sometimes written $\langle S \rangle$ or $\langle S \rangle_{\mathfrak{M}}$.

(**4.0.1**) For a simple example, let $\mathcal{P}^f(S)$ denote the subset of $\mathcal{P}(S)$ of finite subsets. Consider the subset $\mathcal{P}^f([0,1]^2)$ of our magma. Since the magma product is a kind of (almost) disjoint union, the subset $\mathcal{P}^f([0,1]^2)$ is closed under the product, and hence gives a submagma.

(**4.0.2**) For another example consider $(\mathbb{Q} \times \mathbb{Q}) \cap [0,1]^2 \subset [0,1]^2$. Note that this is an infinite but countable subset. The corresponding subset $\mathcal{P}((\mathbb{Q} \times \mathbb{Q}) \cap [0,1]^2)$ of the magma is not countable, but note that it is closed under the product. So this gives another submagma.

(**4.0.3**) Indeed for any uncountable subset S of $[0,1]^2$ it is interesting to consider the countable (but dense) subset $\mathbb{Q}^2 \cap S$. However we will mainly leave this aspect for later investigation.

This chapter contains three sections. Section 4.1 considers a submagma generated by $\mathcal{P}([0,1]\times[0,1))$. Then section 4.2 concerns the "$\#_t$ function" — another device for introducing relatively tame subsets. Finally, the goal of section 4.3 is to study braid-like submagmas.

4.1 Submagma of $\mathfrak{M}$ generated by $\mathcal{P}([0,1]\times[0,1))$

In this section, we define a new submagma called $\mathfrak{M}_S$ from the magma $\mathfrak{M}$ in Def 3.1.20.This example of a submagma is relatively straightforward, but it will be useful in 4.2.5 later.

(**4.1.1**) Note that if S is a set and S' is a subset of set S then $\mathcal{P}(S')$ is a subset of $\mathcal{P}(S)$.

For example $\mathcal{P}([0,1]\times[0,1))\subset\mathcal{P}([0,1]\times[0,1])$.

(**4.1.2**) Let D, C be sets. Let D' be a subset of D. Note that a function $f : D \to C$ (here D stands for Domain; C for Co-domain) gives a function from D' to C by restriction of domain. We will also write f as the name for this function, where no ambiguity arises. Thus we have immediately $f : D' \to C$ as well as for the original domain.

(**4.1.3**) For example all our Stack and Shrink functions from 3.1.19 restrict to functions on subsets of their domains.

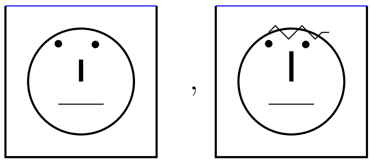
(**4.1.4**) Recall from 2.1.1 that $\mathfrak{S}_{1*} := \mathcal{P}([0,1]\times[0,1))$.

Proposition 4.1.5. *The subset $\mathcal{P}([0,1]\times[0,1))$ of $\mathcal{P}([0,1]\times[0,1])$ is closed under $\boxtimes$. Therefore $\mathfrak{M}_S = (\mathcal{P}([0,1]\times[0,1)),\ \boxtimes)$ is a submagma of $\mathfrak{M}$.*

Proof. We require to show that $a, b \in \mathfrak{S}_{1*} \implies a \boxtimes b \in \mathfrak{S}_{1*}$. Observe from 3.1.19 that in general the 'top' of $a \boxtimes b$, i.e. $a \boxtimes b\ \cap\ [0,1]\times\{1\}$, agrees with the top of a. For $a \in \mathfrak{S}_1$ we have $a \in \mathfrak{S}_{1*}$ if and only if this top is empty. □

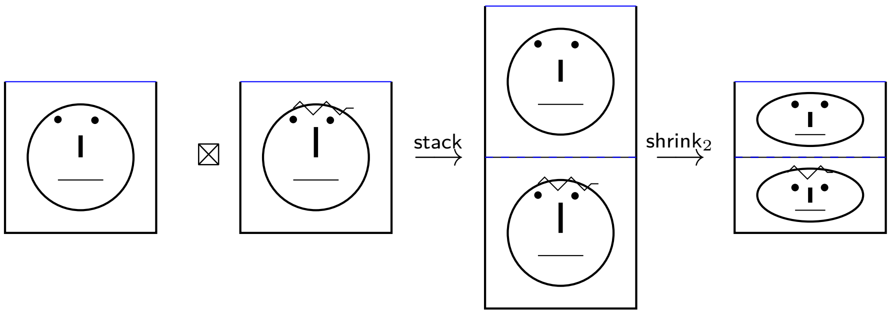
We introduce the following example to indicate the stack-shrink operation in this case.

Example 4.1.6. *Consider*



$\in \mathfrak{S}_{1*}$.

Then



.

From 4.1.6, we can see schematically that the 'semi-open' passport photo submagma is closed under the binary operation ⊠. *Notice that the blue dashed line represents the open end of the second element after the Stack operation; and then after Shrink.*

We have illustrated that the power set of $[0,1] \times [0,1)$, *(the set* $\mathcal{P}([0,1] \times [0,1))$*) is closed under the binary operation* ⊠.

4.2 The $\#_t$ function for subsets

In this section we return to the problem of characterising specific kinds of elements of our magmas, so elements of the set $\mathcal{P}([0,1]^{d+1})$ (in particular with $d = 1$). Sets of such elements are then candidates for submagmas. Here we have in mind elements built from components that are "low-dimensional", such as lines....

Just as there are many undefinable elements of $\mathcal{P}([0,1]^2)$, so there are many undefinable functions $f : [0,1] \to [0,1]$. However, we have many tools for giving some such functions (expressions such as $f(x) = x^2$ for example). So it is useful to be able to build elements of $\mathcal{P}([0,1]^2)$ from functions. On way to do this is as follows.

(**4.2.1**) Given a function $f : [0,1] \to [0,1]$ we may define an element

$$f^* \ := \ \{(x, f(x)) \mid x \in [0,1]\}$$

in $\mathcal{P}([0,1]^2)$. See also 4.3.1.

A *graph* of a function $f : [0,1] \to [0,1]$ is a drawing of the element f^* (possibly with the blue frame to give coordinate axes).

(**4.2.2**) For A, B sets, let us write $\hom(A, B)$ for the set of all functions from A to B.

Example: We are interested in $f \in \hom([0,1],[0,1])$.

We can define a subset of $\mathcal{P}([0,1]^2)$ by

$$\mathfrak{D}_1 = \{f^* \mid f \in \hom([0,1],[0,1])\}$$

There is also the subset given by applying the flip map from 3.1.10:

$$\mathfrak{D}_1^* \ := \ \tau(\mathfrak{D}_1).$$

(**4.2.3**) We can now ask if subset $\mathfrak{D}_1$ is closed under $\boxtimes$.

It will be clear that the answer is no.

It will also be clear that the submagma generated by $\mathfrak{D}_1$ is a proper submagma.

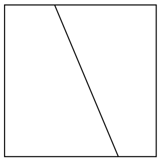
This leads us to consider an alternative characterisation of $\mathfrak{D}_1$.

(**4.2.4**) If S is an infinite set then we may write $|S|=\infty$ for the order. Thus the set of possible orders of a set is $\mathbb{N}_0 \cup \{\infty\}$.

Definition 4.2.5. *Consider an element $p \in \mathcal{P}([0,1]^2)$ and define the number of intersection points with any horizontal line $y = t$ as*

$$\#_t(p) \;=\; |\{(x,y) \in p : y = t\}| \tag{4.1}$$

Example 4.2.6. *Consider*

$$p_1 =$$

$$= \{(x,y) : x = \frac{2-y}{3}\}.$$

Then

$$\#_t(p_1) \;=\; |\{(\frac{2-y}{3}, y) \in p_1 : y = t\}|$$

We consider several values of t as the following:

$$\#_0(p_1) \;=\; |\{(\frac{2-y}{3}, y) \in p_1 : y = 0\}| = |\{(2/3, 0)\}| = 1.$$

and

$$\#_{\frac{1}{2}}(p_1) \;=\; |\{(\frac{2-y}{3}, y) \in p_1 : y = \frac{1}{2}\}| = |\{(\frac{1}{2}, \frac{1}{2})\}| = 1.$$

Finally,

$$\#_1(p_1) \;=\; |\{(\frac{2-y}{3}, y) \in p_1 : y = 1\}| = |\{(\frac{1}{3}, 1)\}| = 1.$$

(**4.2.7**) Now we can define subsets $\mathfrak{E}_{m,n}$ of $\mathcal{P}([0,1]^2)$ for $m, n \in \mathbb{N}$ by

$$\mathfrak{E}_{m,n} \;:=\; \{p \in \mathcal{P}([0,1]^2) \mid \; m \leq \#_t(p) \leq n \;\; \forall t \in [0,1]\}$$

and similarly

$$\mathfrak{E}^{m,n} \;:=\; \tau(\mathfrak{E}_{m,n})$$

- this bounds the number of intersections with any *vertical* line.

For example $\mathfrak{E}_{0,0} = \{\emptyset\}$.

(**4.2.8**) Note that an element of $\mathcal{P}([0,1]^2)$ is also a relation on $[0,1]$. A function

$f : A \to B$ is a subset of $A \times B$ with the property that every $a \in A$ appears exactly once as the left-hand side of a pair $(x, y) \in \rho$. In this sense, $f = f^*$. But we will avoid this identification in our notation.

Note from this that we have

$$\mathfrak{D}_1 = \mathfrak{E}^{1,1}$$

Note that for most elements $a, b \in \mathfrak{D}_1 = \mathfrak{E}^{1,1}$ we have $a \boxtimes b \in \mathfrak{E}^{2,2}$, which verifies that $\mathfrak{D}_1$ is not closed.

The corresponding question for $\mathfrak{D}_1^*$ is more interesting. (Because $\boxtimes$ treats the x and y directions differently.) If we consider $a, b \in \mathfrak{D}_1^*$ then $\#_t(a \boxtimes b) = 1$ for almost all t. Only $t = 1/2$ may have value 2. Here then, we are close to a submagma. There are two quick ways to make submagmas here. One is to modify $\mathfrak{D}_1^*$ slightly to remove the point with $y = 1$ in each element, so that $\#_1(p) = 0$. Let us call the corresponding subset $\mathfrak{D}_{10}^*$. This is similar to considering $\mathcal{P}([0, 1] \times [0, 1))$ as in 4.1.

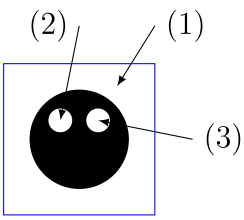
We have that the subset $\mathfrak{D}_{10}^*$ is closed under $\boxtimes$.

(**4.2.9**) Another way to proceed is to consider a subset of elements of $\mathfrak{D}_1^*$ where the subset of the picture restricted to $y = 0$ and $y = 1$ is fixed. Let us consider the subset of $\mathfrak{D}_1^*$ where the points $(1/2, 0)$ and $(1/2, 1)$ are present. Let us call this subset $\mathfrak{D}_{0.5}^*$.
(We will return to this kind of construction in 5.4.1.)

We have that the subset $\mathfrak{D}_{0.5}^*$ is closed under $\boxtimes$.

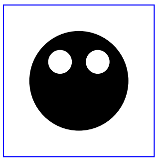
4.2.1 An extended example for $\#_t$

(**4.2.10**) In the following there is an example to understand calculating how many intersection points with any horizontal lines that go through any element in $\mathcal{P}([0, 1]^2)$.That is, we calculate $\#_t(p)$ as in 4.1 where $p \in \mathcal{P}([0, 1]^2)$.
Consider

$p =$ 

Here the black circle (label (1)) represents $f_1^* = \{(x-\frac{1}{2}, y-\frac{1}{2}) : x^2+y^2 \leqslant \frac{1}{16}\}$. Then the first small white circle (label (2)) represents $f_2^* = \{(x-\frac{3}{8}, y-\frac{5}{8}) : x^2+y^2 \leqslant 1/100\}$.Finally, the second white circle (label (3)) represents $f_3^* = \{(x-\frac{5}{8}, y-\frac{5}{8}) : x^2+y^2 \leqslant 1/100\}$.

Then,

$p =$  $= f_1^* \setminus (f_2^* \cup f_3^*)$

Now we look at different cases of a horizontal line $y = t$.

1. When a horizontal line $y = t = 0$ cut through p the calculation of $\#_t(p)$ is

$$\begin{aligned}\#_0(p) = &|\{(x-\frac{1}{2}, 0-\frac{1}{2}) : x^2+y^2 \leqslant \frac{1}{16}\} \setminus (\{(x-\frac{3}{8}, 0-\frac{5}{8}) : x^2+y^2 \leqslant 1/100\} \cup \\ &\{(x-\frac{5}{8}, 0-\frac{5}{8}) : x^2+y^2 \leqslant 1/100\})| \\ = &|\{(x-\frac{1}{2}, \frac{-1}{2}) : x^2+y^2 \leqslant \frac{1}{16}\} \setminus (\{(x-\frac{3}{8}, \frac{-5}{8}) : x^2+y^2 \leqslant 1/100\} \cup \\ &\{(x-\frac{5}{8}, \frac{-5}{8}) : x^2+y^2 \leqslant 1/100\})| \\ = &|\{(x-\frac{1}{2})^2 + (\frac{-1}{2})^2 \leqslant \frac{1}{16}\} \setminus (\{(x-\frac{3}{8})^2 + (\frac{-5}{8})^2 \leqslant 1/100\} \cup \\ &\{(x-\frac{5}{8})^2 + (\frac{-5}{8})^2 \leqslant 1/100\})| \\ = &|\{(x-\frac{1}{2})^2 \leqslant \frac{-3}{16}\} \setminus (\{(x-\frac{3}{8})^2 \leqslant \frac{-605}{100}\} \cup \{(x-\frac{5}{8})^2 \leqslant \frac{-605}{100}\})| \\ = &0\end{aligned}$$

there is no square number less than or equal to a negative number which mean there is no intersection point between p and the horizontal line $t = 0$ as seen in the graph 4.2 below.

2. When a horizontal line $y = t = 0.1$ cut through p the calculation of $\#_t(p)$ is

$$\begin{aligned}\#_{0.1}(p) = &|\{(x-\frac{1}{2}, 0.1-\frac{1}{2}) : x^2+y^2 \leqslant \frac{1}{16}\} \setminus \{(x-\frac{3}{8}, 0.1-\frac{5}{8}) : x^2+y^2 \leqslant 1/100\} \\ &\cup \{(x-\frac{5}{8}, 0.1-\frac{5}{8}) : x^2+y^2 \leqslant 1/100\}| \\ = &0\end{aligned}$$

this means there is no intersection point between p and the horizontal line

$t = 0.1$ as seen in the graph 4.2 below.

3. When a horizontal line $t = \frac{1}{4}$ cut throw p the calculation of $\#_t(p)$ is

$$\begin{aligned}\#_{\frac{1}{4}}(p) &= |\{(x-\frac{1}{2}, \frac{1}{4}-\frac{1}{2}) : x^2+y^2 \leqslant \frac{1}{16}\} \setminus \{(x-\frac{3}{8}, \frac{1}{4}-\frac{5}{8} : x^2+y^2 \leqslant 1/100\} \\ &\qquad \cup \{(x-\frac{5}{8}, \frac{1}{4}-\frac{5}{8}) : x^2+y^2 \leqslant 1/100\}| \\ &= |\{(\frac{1}{2}, \frac{1}{4})\}| \\ &= 1\end{aligned}$$

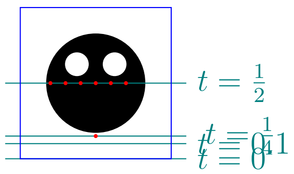
   this mean there is a one intersection point between p and the horizontal line $t = \frac{1}{4}$ as seen in the graph 4.2.

4. When a horizontal line $t = \frac{1}{2}$ cut through p the calculation of $\#_t(p)$ is

$$\begin{aligned}\#_{\frac{1}{2}}(p) &= |\{(x-\frac{1}{2}, \frac{1}{2}-\frac{1}{2}) : x^2+y^2 \leqslant \frac{1}{16}\} \setminus \{(x-\frac{3}{8}, \frac{1}{2}-\frac{5}{8}) : x^2+y^2 \leqslant 1/100\} \\ &\qquad \cup \{(x-\frac{5}{8}, \frac{1}{2}-\frac{5}{8}) : x^2+y^2 \leqslant 1/100\}| \\ &= \infty\end{aligned}$$

   this means there are many intersection point between p and the horizontal line $t = \frac{1}{2}$ as seen in the graph 4.2.

In graph 4.2, We draw a picture with horizontal lines indicated in teal and intersection points in red.



(4.2)

4.3 'Braid-like' Submagma

In this section, we aim to construct a braid-like substructure in the passport photograph magma $\mathfrak{M}$ (as defined in 3.1.20 above). The first question here is: what is braid-like? A lot of geometric topology is used in the definitions of braid groups — see for example [KT08, Kas12].

Can we find a sub*set* of the passport photograph magma with elements that we may call braid-like? In general a braid contains many strands of hair in a certain configuration. We can start with a single strand, or 'string'. That means we may look at a string as a 'braid'. So, roughly speaking, a mathematical string is a line or curve or path that joins two points in $\mathbb{R}^3$. However, in our construction, we need to look at strings to be elements in our magma $\mathfrak{M}$.

We will look for string that start at the top and ends at the bottom of $[0, 1]^2$. Now we describe what function we will use in our construction. (In general, hair strands have several types including straight, wavy and curly. For all of these types of hair, people just care about how they style it. Except mathematicians, who are looking for mathematics seeds that lead them to connect life with math.) So this is a reason for us to choose piecewise linear functions 2.4.12 for making strings inside our magma.

As an example from real life is the process of braiding hair (changing the hair configuration from unbraided to braided), the hair moves to the right or left and also pushes up.

We can look at this case mathematically by using our composition: combining hair configurations such as weaves is like stacking; and pushing the weaves up towards the scalp is like shrinking.

This section is about finding the subset of elements which are like braids - then we will do some math on it in our magma setting.

So here we are interested in the subset that has at least one string (line) going from top to bottom.

4.3.1 Submagma: Braid-like elements in two dimensions

Here we make some experiments with various forms of string-like or similarly special elements in $\mathcal{P}([0,1]^{d+1})$ with $d = 1$.

(**4.3.1**) Consider a function $f \in PL(0,1)$ as in 2.4.12. Observe that the set $\{(f(y), y) : y \in [0,1]\}$ is a subset of $[0,1]^2$.

Definition 4.3.2. *The set L_1 is defined as*

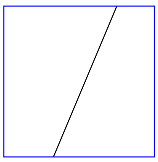
$$L_1 \;=\; \{\{(f(y), y) : y \in [0,1]\} \mid f \in PL(0,1)\} \tag{4.3}$$

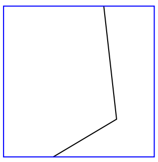
(**4.3.3**) Observe that L_1 is a subset of $\mathcal{P}([0,1]^2)$. We will prove this in 4.3.8 below.

(**4.3.4**) Notation: If we write f_* for an element of L_1, we mean

$$f_* = \{(f(y), y) : y \in [0,1]\} \quad \text{where} \quad f \in PL(0,1) \quad (2.4.12).$$

Example 4.3.5. *Using our pictures as in 3.1.3 consider*

$f_* =$  $= \{(\frac{y+1}{3}, y) : y \in [0,1]\} \in L_1.$

$g_* =$  $= \{(g(y), y) : y \in [0,1]\} \in L_1.$

where

$$g(y) = \begin{cases} \frac{5y+1}{3}, & y \in [0, \frac{1}{4}] \\ \frac{7-y}{9}, & y \in [\frac{1}{4}, 1]. \end{cases}$$

Proposition 4.3.6. *If $f_* \in L_1$ as in (4.3) we have $\#_t(f_*) = 1, \quad \forall t \in [0,1]$.*

Proof. Let $h_* \in L_1 \subset \mathcal{P}([0,1]^2)$, and let $t \in [0,1]$. So,

$$\begin{aligned}\#_t(h_*) &= |\{(x,y) \in \{(h(u),u) : u \in [0,1]\} \ : \ y = t\}| \\ &= |\{(h(t),t)\}| \\ &= 1 \quad \forall \quad t\end{aligned}$$

□

(**4.3.7**) Now let us ask if L_1 gives a submagma of the (PPM) under the composition. To be a submagma we need to see

(1) if $L_1 \subset \mathcal{P}([0,1]^2)$; then need to check

(2) if L_1 is closed under the composition of binary operation $\boxtimes$.

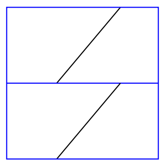
(**4.3.8**) First, we want to prove $L_1 \subset \mathcal{P}([0,1]^2)$, that means we need to show that every element of L_1 is also an element in $\mathcal{P}([0,1]^2)$. Suppose $h_* \in L_1$ this implies $h_* \subset f_*$. Then $h_* \subset [0,1]^2$. Thus $h_* \in \mathcal{P}([0,1]^2)$, Therefore $L_1 \subset \mathcal{P}([0,1]^2)$.

□

(**4.3.9**) Second, we use the next example to see if L_1 is closed under the composition.

Example 4.3.10. *Consider $f_* \in L_1$ as in 4.3.5. Then*

$h_* = f_* \boxtimes f_* =$



$$= \{(\frac{2y+1}{3}, y) : y \in [0, \frac{1}{2}]\} \cup \{(\frac{2y}{3}, y) : y \in [\frac{1}{2}, 1]\}.$$

Now I will show how many intersection points with the horizontal line $y = t = \frac{1}{2}$. So, I will use 4.3.6 to calculate how many intersection points

$$\begin{aligned}\#_{t=\frac{1}{2}}(h_*) &= |\{(\frac{2y+1}{3}, y) : y = \frac{1}{2}\} \cup \{(\frac{2y}{3}, y) : y = \frac{1}{2}\}| \\ &= |\{(\frac{2}{3}, \frac{1}{2})\} \cup \{(\frac{1}{3}, \frac{1}{2})\}| \\ &= |\{(\frac{1}{3}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{2})\}| \\ &= 2\end{aligned}$$

There are two intersection points. By 4.3.6 this means $h_* \notin L_1$. This implies L_1 is not closed under $\boxtimes$.

So, here we proved L_1 is not closed under $\boxtimes$. Thus L_1 is not submagma.

(**4.3.11**) There are several ways we could have shown this non-closure. We will look at some others later (consider the gluing lemma in 2.3.16 for example). But the $\#_t$ machinery will also be useful later.

Definition 4.3.12. *The set L_2 is defined as*

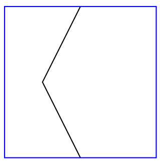
$$L_2 = \{\{(f(y), y) : y \in [0,1]\} \mid \ f \in PL(0,1) \ \ , \ \ f(0) = f(1) = 1/2\}. \tag{4.4}$$

where the set $PL(0,1)$ is as in 2.4.12

(**4.3.13**) Note from the definitions that $L_2 \subset L_1$.

(**4.3.14**) Notation: Given a function γ in $PL(0,1)$ with $\gamma(0) = \gamma(1) = \frac{1}{2}$, then if we write (γ) for an element of L_2, we mean $(\gamma) = \{(\gamma(y), y) : y \in [0,1]\}$.

Example 4.3.15. *Consider*

$(f) =$  $= \{(f(y), y) : y \in [0,1]\}$

$\in L_2$

where

$$f(y) = \begin{cases} \frac{1}{2} - \frac{y}{2}, & y \in [0, \frac{1}{2}] \\ \frac{y}{2}, & y \in [\frac{1}{2}, 1] \end{cases}$$

Lemma 4.3.16. *The set L_2 as in 4.3.12 is subset of $\mathcal{P}([0,1]^2)$.*

Proof. To prove $L_2 \subset \mathcal{P}([0,1]^2)$, we need to show every element of L_2 is an element in $\mathcal{P}([0,1]^2)$. But $L_2 \subset L_1$ so we are done. □

Next, we want to prove that $(L_2, \boxtimes)$ is a submagma. First, we will give a Lemma that we will use in the proof.

The following is a variation on the glue lemma, which can be found for example in [Mar21, §7.4]. The glue lemma considers two topological spaces, X and Y, say. It then considers subsets A, B of X so that $A \cup B = X$. On these subsets it considers functions $f : A \to Y$ and $g : B \to Y$, having the property that $f(x) = g(x)$ for all $x \in A \cap B$. So far here it is possible that A and B are disjoint. But in any case we can make a new function $h : X \to Y$ by setting $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ otherwise. The lemma then considers conditions under which h is continuous on X, if given that f and g are continuous on their domains.

It is helpful (and most relevant to us) to think of X above as some subset of $\mathbb{R}$, with the usual topology. For example if X is a closed interval and A and B are also closed intervals then the union property implies that they intersect. This is the usual 'intuitive' setting for continuity. This is the setting that we will consider, and we will go further and suppose that f and g are piecewise linear.

Definition 4.3.17. *Let A, B, Y be sets. Consider functions $f : A \to Y$ and $g : B \to Y$. If*
(I) $f(x) = g(x)$ for $x \in A \cap B$
then we define a 'union' function of f and g,

$$f \cup g : A \cup B \to Y \qquad \textit{given by}$$
$$x \quad \mapsto \begin{cases} f(x) & \textit{if} \quad x \in A \\ g(x) & \textit{if} \quad x \in B \setminus A. \end{cases}$$

Lemma 4.3.18. *(Piecewise linear glueing lemma) Let $\alpha < \delta \in \mathbb{R}$ and let $d \in \mathbb{N}$. Let $f : [\alpha, \beta] \to \mathbb{R}^d$ and $g : [\gamma, \delta] \to \mathbb{R}^d$ be piecewise linear functions, as defined in 2.4.3 Then the union $f \cup g$ exists and is a piecewise linear function if $\beta \geq \gamma$ and the functions agree on the intersection of domains.*

Proof. The intersection condition (I) is satisfied so $f \cup g$ exists. In particular here $[\alpha, \beta] \cup [\gamma, \delta] = [\alpha, \delta]$. We want to show that $f \cup g$ is a piecewise linear function on $[\alpha, \delta]$.

Let us define a new function

$$h := f \cup g : [\alpha, \delta] \to \mathbb{R}^d.$$

Since f is PL it takes the form $f : [\alpha, \beta] \to \mathbb{R}^d$

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in [\alpha = \alpha_1, \alpha_2] \\ f_2(x) & \text{if } x \in [\alpha_2, \alpha_3] \\ \vdots & \text{if } \quad \vdots \\ f_i(x) & \text{if } x \in [\alpha_i, \alpha_{i+1}] \\ \vdots & \text{if } \quad \vdots \\ f_j(x) & \text{if } x \in [\alpha_j, \alpha_{j+1} = \beta] \end{cases}.$$

Similarly, since g is PL it takes the form $g : [\gamma, \delta] \to \mathbb{R}^d$

$$g(x) = \begin{cases} g_1(x) & \text{if } x \in [\gamma = \gamma_1, \gamma_2] \\ g_2(x) & \text{if } x \in [\gamma_2, \gamma_3] \\ \vdots & \text{if } \quad \vdots \\ g_i(x) & \text{if } x \in [\gamma_i, \gamma_{i+1}] \\ \vdots & \text{if } \quad \vdots \\ g_k(x) & \text{if } x \in [\gamma_k, \gamma_{k+1} = \delta] \end{cases}.$$

Since $\beta \in [\gamma, \delta]$ there is an l so that $\beta \in [\gamma_l, \gamma_{l+1}]$. Here we add a breakpoint β, so we break g_l into two linear pieces with domains $[\gamma_l, \beta]$ and $[\beta, \gamma_{l+1}]$. Now we have

$$h(x) = \begin{cases} f_1(x) & \text{if } x \in [\alpha = \alpha_1, \alpha_2] \\ f_2(x) & \text{if } x \in [\alpha_2, \alpha_3] \\ \vdots & \text{if } \quad \vdots \\ f_j(x) & \text{if } x \in [\alpha_i, \alpha_{i+1} = \beta] \\ g_l(x) & \text{if } x \in [\beta, \gamma_{l+1}] \\ \vdots & \text{if } \quad \vdots \\ g_k(x) & \text{if } x \in [\gamma_k, \gamma_{k+1} = \delta] \end{cases}$$

Thus h is a Piecewise linear function. □

4.3.1.1 Submagma proof

Proposition 4.3.19. *Consider $L_2 \subset \mathcal{P}([0,1]^2)$, from (4.3.12). The set L_2 gives a submagma of $(\mathcal{P}([0,1]^2), \boxtimes)$.*

Proof. We want to check if L_2 is closed under the binary operation $\boxtimes$.
Let us take two elements (h) (notation as in 4.3.14) and (g) in L_2. We want to show that $(h) \boxtimes (g)$ is in $L_2 \subset \mathcal{P}([0,1]^2)$. We compute:

$$
\begin{aligned}
(h) \boxtimes (g) &= \mathsf{shrink}_2(\mathsf{stack}((h),(g))) \qquad \text{by definition 3.1.19 (this uses 3.1.15, 3.1.17)}\\
&= \mathsf{shrink}_2((h) \cup shift_1((g)))\\
&= \mathsf{shrink}_2(\{(h(y),y) : y \in [0,1]\} \cup \{(g(y), y+1) : y \in [0,1]\})\\
&= \{(h(y), \frac{y}{2}) : y \in [0,1]\} \cup \{(g(y), \frac{y+1}{2}) : y \in [0,1]\}
\end{aligned}
$$

In the first part of the last result we will re-write $\frac{y}{2} = x$, which implies $y = 2x$. Then $(h(y), \frac{y}{2})$ change to be $(h(2x), x)$ when $x \in [0, \frac{1}{2}]$.
In the second part

$$
\begin{aligned}
&\text{we will re-write} \quad \frac{y+1}{2} = x, \quad \Longrightarrow \quad y = 2x - 1.\\
&\text{then} \quad (g(y), \frac{y+1}{2}) \quad \text{change to be} \quad (g(2x-1), x) \quad \text{when} \quad x \in [\frac{1}{2}, 1].
\end{aligned}
$$

We have:

$$
(h) \boxtimes (g) = \{(h(2x), x) : x \in [0, \frac{1}{2}]\} \cup \{(g(2x-1), x) : x \in [\frac{1}{2}, 1]\}. \qquad (4.5)
$$

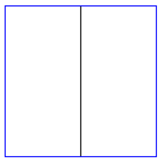
Note that this element of $\mathcal{P}([0,1]^2)$ is the union of a piece which is a function on $[0, 1/2]$ and a function on $[1/2, 1]$.
Since h and g are $PL(0,1)$ they are linear in each segment. Therefore, h and g are continuous on $[0,1]$.

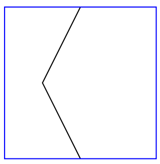
Next, we want to see if $h(2x)$ and $g(2x-1)$ are continuous or not. Note that $\gamma : x \mapsto 2x$ is continuous. Since it is a linear on $[0, \frac{1}{2}]$. Then $h(\gamma(x))$ is continuous. Since its composition of two continuous functions 2.3.12. Similarly, $g(2x-1)$ is continuous. Thus we observe $[0,1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$.
So, in this case, we can use the Glueing Lemma 2.3.16 to decide if the union in (4.5)

agrees with a continuous function on $[0,1]$. The Gluing Lemma says this union of continuous functions is continuous if they agree on their intersection (at $x = 1/2$). We have a slight strengthening of this glueing lemma, to the Piecewise linear glueing lemma 4.3.18).

Note that the axioms of this Lemma 4.3.18 are satisfied by the union given by our $(h) \boxtimes (g)$. So by this Lemma $(h) \boxtimes (g)$ agrees with a piecewise linear function on $[0,1]$. Note also that this function evaluates to 1/2 at the beginning and the end. So we are done. $\square$

Example 4.3.20. *Consider* (g) *and* $(f) \in L_2$ *given by*

$(g) =$  $= \{(1/2, y) : y \in [0,1]\}$

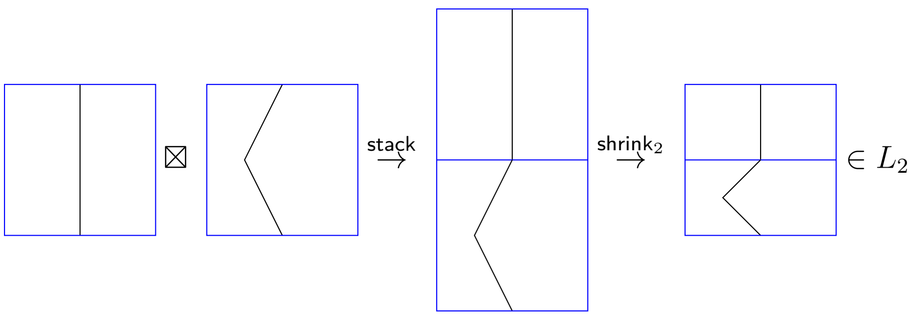
$(f) =$  $= \{(f(y), y) : y \in [0,1]\}$

$\in L_2$

where

$$f(y) = \begin{cases} \frac{1}{2} - \frac{y}{2}, & y \in [0, \frac{1}{2}] \\ \frac{y}{2}, & y \in [\frac{1}{2}, 1] \end{cases}$$

Then



$$(g) \boxtimes (f) = \begin{cases} \frac{1}{2} - y & \textit{if } y \in [0, 1/4] \\ y & \textit{if } y \in [1/4, 1/2] \\ 1/2 & \textit{if } y \in [1/2, 1]. \end{cases}$$

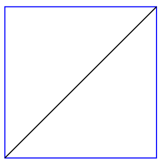
Definition 4.3.21. *The set L_3 is defined as*

$$L_3 \ := \ \{\{(f(y), y) : y \in [0,1]\} \mid f \quad \textit{any function from [0,1] to itself}\} \qquad (4.6)$$

(**4.3.22**) Notation: If we write f' for an element of L_3, we mean

$$f' = \{(f(y), y) : y \in [0,1]\} \quad \text{where} \quad f \text{ any function } (f : [0,1] \to [0,1]).$$

Example 4.3.23. *Using our picture as in 3.1.3 consider $f' \in L_3$ given by*

$f' =$  $= \{(f(y), y) : y \in [0,1]\}, \textit{where} f(y) = y.$

(**4.3.24**) Let us ask is L_3 give submagma of $\mathfrak{M}$ under $\boxtimes$. To be submagma we would need to check the following:

1. Is $L_3 \subset \mathcal{P}([0,1]^2)$? (next lemma answers.)
2. Is L_3 closed under the composition $\boxtimes$?

Lemma 4.3.25. *The set L_3 as in 4.3.21 is subset of $\mathcal{P}([0,1]^2$.*

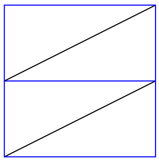
Proof. We want proof $L_3 \subset \mathcal{P}([0,1]^2)$. Suppose $h' \in L_3$ this implies $h' \subset f'$. Then $h' \subset [0,1]^2$. Thus $h' \in \mathcal{P}([0,1]^2)$, Therefore $L_3 \subset \mathcal{P}([0,1]^2)$.

□

(**4.3.26**) Now we check if L_3 is closed under composition $\boxtimes$. So the following example is used to verify it is not.

Example 4.3.27. *Consider $f' \in L_3$ as 4.3.23. Then*

$$f' \boxtimes f' =$$



$$= \{(2y, y) : y \in [0, \frac{1}{2}]\} \cup \{(2y-1, y) : y \in [\frac{1}{2}, 1]\}$$

Now I will show how many intersection points with the horizontal line $y = t = \frac{1}{2}$. So, I will use 4.3.6 to calculate how many intersection points

$$\begin{aligned}
\#_{t=\frac{1}{2}}(f' \boxtimes f') &= |\{(2y, y) : y = \frac{1}{2}\} \cup \{(2y-1, y) : y = \frac{1}{2}\}| \\
&= |\{(1, \frac{1}{2})\} \cup \{(0, \frac{1}{2})\}| \\
&= |\{(1, \frac{1}{2}), (0, \frac{1}{2})\}| \\
&= 2
\end{aligned}$$

There are two intersection points. By 4.3.6 this means $f' \notin L_3$. This implies L_3 is not closed under $\boxtimes$. So, here we proved L_3 is not closed under $\boxtimes$. Thus L_3 is not submagma.

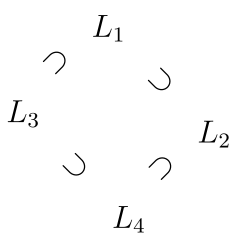
Definition 4.3.28. *The set L_4 defined as*

$$L_4 := \{\{(f(y), y) : y \in [0, 1]\} \mid f(0) = f(1) = 1/2\}. \tag{4.7}$$

Lemma 4.3.29. *The set L_4 as in 4.3.28 is subset of $\mathcal{P}([0, 1]^2)$.*

Proof. To prove $L_4 \subset \mathcal{P}([0, 1]^2)$, we need to show every element of L_4 is an element in $\mathcal{P}([0, 1]^2)$. But $L_4 \subset L_3$ so we are done. □

(**4.3.30**) Note that



4.3.2 Generalisations

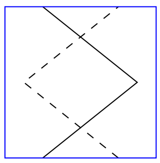
(**4.3.31**) There are many other subsets of $[0,1]^2$ of these kinds that we could try. We will leave most of these for future work. But we note that L_2 is, in a sense, a subset of 'single strings'. We can make a generalisation to more strings. In particular if we consider a subset of $\mathfrak{E}_{2,2}$ made from PL functions, just as L_1 is for $\mathfrak{E}_{1,1}$ then we will get elements of our magma that look like braids with n strings.

(**4.3.32**) We may define subsets of $[0,1]^2$ with 2-strings as

$$B_1 = \{ \ (f) \cup (g) \mid f, g \in PL(0,1), \ f(0) = f(1) = 1/3, \ g(0) = g(1) = 2/3 \ \}$$

But B_1 is not in $\mathfrak{E}_{2,2}$ as shown in the next example.

Example 4.3.33. *Consider $b_1 \in B_1$ given by*

$b_1 =$ 

where

$$f(y) = \begin{cases} \frac{5}{6}y + \frac{1}{3} & if \quad 0 \leq y \leq \frac{1}{2} \\ \frac{7}{6} - \frac{5}{6}y & if \quad \frac{1}{2} \leq y \leq 1 \end{cases}$$

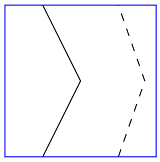
and

$$g(y) = \begin{cases} \frac{2}{3} - \frac{5}{6}y & if \quad 0 \leq y \leq \frac{1}{2} \\ \frac{5}{6}y - \frac{1}{6} & if \quad \frac{1}{2} \leq y \leq 1 \end{cases}$$

(**4.3.34**) The set B_2 define as

$$B_2 = \{ \ (f) \cup (g) \mid f, g \in PL(0,1), \ f(0) = f(1) = 1/3, \ g(0) = g(1) = 2/3 \ , g(y) > f(y), y \in [0,1]\}$$

Example 4.3.35. *Consider $b_2 \in B_2$ given by*

$b_2 \quad =$ 

where

$$f(y) = \begin{cases} \frac{y+1}{3} & if \quad 0 \leq y \leq \frac{1}{2} \\ \frac{2-y}{3} & if \quad \frac{1}{2} \leq y \leq 1 \end{cases}$$

and

$$g(y) = \begin{cases} \frac{4}{15}y + \frac{2}{3} & if \quad 0 \leq y \leq \frac{1}{2} \\ \frac{14}{15} - \frac{4}{15}y & if \quad \frac{1}{2} \leq y \leq 1 \end{cases}$$

Here B_2 is a subset of $\mathfrak{E}_{2,2}$ and closed under $\boxtimes$.

(**4.3.36**) Note that B_2 is closed under $\boxtimes$. Therefore, it is a submagma.

(**4.3.37**) If we use the same idea we can make n-string subsets of $\mathfrak{E}_{n,n}$.

Note that when d=1 the strings can not be touch or intersect, so they cannot twist around each other, but in d=2 strings can twist around each other.

We can also consider generalisations to higher d, and in particular to $d = 2$. Again we will leave these generalisations for future work.

Chapter 5

Quotients on passport photograph magma

This chapter will consider equivalence relations on the set $\mathcal{P}([0,1]^2)$ of passport photographs. And hence congruences, and hence quotients, on the magma. Broadly we are aiming for associative quotients, as a first step in doing representation theory.

We discuss magma congruences in general in section 5.1.

In section 5.1 (containing 5.1.18) we look at minimal relations that would be enough to make an associative quotient.

A quotient using the Thompson group F is in section 5.2. The idea here is that F is a subgroup of the group $Homeo([0,1],[0,1])$, and its action contains the action in 5.1.18.

The quotient under a relation R_α (which arranges elements into classes indexed by set partitions) is in 5.4.1 .

5.1 Generalities

5.1.1 Congruence

In this section we define magma congruence.

(**5.1.1**) Let $M = (M, *)$ be a magma and ρ an equivalence relation on M, with the class of $m \in M$ written $[m]$. We can try to define a magma on the set of equivalence classes of M. The product is given formally by

$$[m] \circ [m'] = [m * m'], \tag{5.1}$$

but this does not give a well-defined product in general. Suppose $l \in [m]$. Then $[l] = [m]$ so we require $[l] \circ [m'] = [m] \circ [m'] = [m * m'] = [l * m']$ for all $m' \in M$. In other words, we require that if $l \rho m$ then $(l * m')\ \rho\ (m * m')$ for all m'. (And similarly for some other identities.)

Definition 5.1.2. *Let a, b, a' and $b' \in \mathcal{P}([0,1]^2)$. Then a magma congruence on magma $\mathfrak{M}$ is an equivalence relation such that:*

$$a \sim a' \text{ and } b \sim b' \text{ implies } a \boxtimes b \sim a' \boxtimes b'.$$

(**5.1.3**) Given a magma congruence then we can define a quotient magma using (5.1).

(**5.1.4**) Now suppose that a group G acts on a set S. We can write gs for the result of acting with $g \in G$ on element $s \in S$. We have in mind an action so that for $g, g' \in G$ then $(gg')s = g(g's)$ (sometimes called a left-action of G on S) and $1s = s$, where 1 is the identity element in G.

This action of a group G on S gives a relation on S by $s \sim gs$ for all g.

(**5.1.5**) Note that this G-action relation obeys
reflexive (since $s \sim 1s = s$);
symmetric (since $s \sim gs$ and $gs \sim g^{-1}(gs) = 1s = s$;
transitive (since $s \sim gs$ and $gs \sim g'(gs)$ implies $s \sim g'(gs)$ since $g'(gs) = (g'g)s$).
So this $\sim$ is an equivalence relation.

That is, the classes of this relation are the sets $[s] = \{gs | g \in G\}$.

(**5.1.6**) Now suppose that we have a group action of a group G on the underlying set of a magma $M = (M, *)$. Since the relation we get from this action is an equivalence relation, we can ask if it is a congruence. That is we can ask if

$$[a * b] = [a' * b'] \text{ whenever } a' \in [a],\ b' \in [b]$$

In this case we know that $a' = ga$ for some $g \in G$ and $b' = g'b$ for some $g' \in G$. So we are asking if $a * b = g''(ga * g'b)$ for some $g'' \in G$.

(**5.1.7**) For example, for our magma (with some group action on it), we are asking if for every g, g' there is a g'' so that $a \boxtimes b = g''(ga \boxtimes g'b)$. Later we will look at examples of groups acting on our magma, and consider this equality requirement in these cases explicitly.

The idea is that $\boxtimes$ is stack-shrink, while $a \mapsto ga$ may also correspond to a change in the y direction. So for some groups G the action will be such that there is an element which we can write as $g \boxtimes g'$, which acts on $a \boxtimes b$ to take it to $ga \boxtimes g'b$.

One quick example of this is if $G = Homeo([0, 1], [0, 1])$. The action of g can be 'squeezed' into the interval $[1/2, 1]$ by rescale and shift. And the action of g' into $[0, 1/2]$ similarly. Both new actions, and the composite, are again homeomorphisms. So we get a congruence this way.

(**5.1.8**) The above example is close to a common equivalence that is used, sometimes called isotopy (see for example [Kas12]). In this section we will explore away from this common example, both by considering relations with smaller classes, and considering relations with bigger classes.

5.1.2 Some relations on the set $\mathfrak{S}_1$ of 'passport photos' (aiming for an associative magma)

Here we consider (equivalence) relations that might work as, or become, congruences on our magma. (We define magma congruences in (5.1.2)).

We can ask what is a relation we can describe such that the two sides of our associativity test as in (3.3) which are different, are related - for all compositions.

There are very 'heavy' relations that relate many things together. (For example the relation that has only one equivalence class. But in general there are many others.)

There are minimal relations that exactly relate as we require but do no more (and perhaps these are not equivalences??).

And then there might be relations in between ...which are somehow more or less 'nice'.

5.1.2.1 Experimental relation 1

Here we start to consider some explicit examples of relations on $\mathfrak{S}_1$.

Definition 5.1.9. *For $P_1, P_2 \in \mathfrak{S}_1$ we have a relation $\rightsquigarrow$ on $\mathfrak{S}_1$ given by $P_1 \rightsquigarrow P_2$ if P_2 obtained from P_1 by applying a function $f : [0,1] \to [0,1]$ to the y values of elements $(x, y) \in P_1$. That is, if*

$$P_2 \ = \{(x, f(y)) \mid (x, y) \in P_1\} \tag{5.2}$$

for some such function f.

An important question about our relations is if they are equivalence relations. We require the relations we use to make congruences to be equivalence relations, as noted in 5.1.1.

(**5.1.10**) Note that this relation $\rightsquigarrow$ is reflexive, because we can use the identity function for f as in the definition.

But this relation is not symmetric! For proof, we give an example. Consider $P = \{(0, 1/3), (0, 2/3)\}$ and consider the function $f(x) = 1/2$. From this we have $P \rightsquigarrow P' = \{(0, 1/2)\}$, because we have

$$\{(x, f(y))|(x, y) \in \{(0, 1/3), (0, 2/3)\}\} = \{(0, 1/2), (0, 1/2)\} = \{(0, 1/2)\}$$

But the photos P'' having the relation $P' \rightsquigarrow P''$ are all photos with one point because a function takes one point to one point. Therefore $P' \not\rightsquigarrow P$, because P has two points. Thus $\rightsquigarrow$ is not symmetric.

(**5.1.11**) We could 'fix' this particular issue for our example by taking the equivalence relation closure of $\rightsquigarrow$ (as defined in 5.1.2.5). But instead, let us move on to a more interesting example.

5.1.2.2 Experimental relation 2

Our previous example in 5.1.2.1 was not an equivalence relation. The problem was related to the kind of functions f that we allowed in the construction. We can try to fix this by restricting the kind of functions that we allow.

Definition 5.1.12. *We define a relation* $\overset{h}{\rightsquigarrow}$ *on* $\mathfrak{S}_1$ *as follows. For* $P_1, P_2 \in \mathfrak{S}_1$ *we have* $P_1 \overset{h}{\rightsquigarrow} P_2$ *if* P_2 *obtained from* P_1 *by applying a continuous bijective function* $f : [0,1] \to [0,1]$ *to the* y *values of elements* $(x, y) \in P_1$*. That is, if*

$$P_2 = \{(x, f(y)) \mid (x, y) \in P_1\} \tag{5.3}$$

for some such a continuous bijective function f*.*

Lemma 5.1.13. *This relation* $\overset{h}{\rightsquigarrow}$ *on* $\mathfrak{S}_1$ *from Def.5.1.12 is an equivalence relation.*

Proof. We need to check if $\overset{h}{\rightsquigarrow}$ is reflexive symmetric and transitive.

1. *Reflexive:*
   We want to show that $(P, P) \in \overset{h}{\rightsquigarrow}$ for all $P \in \mathfrak{S}_1$.
   Do we have a continuous bijective f such that $P = \{(x, f(y)) | (x, x) \in P\}$?
   Yes, because we can note that the identity function

   $$id : [0,1] \to [0,1]$$

   is continuous and bijective, so we can take $f(y) = y$.
   Therefore $\overset{h}{\rightsquigarrow}$ is reflexive.

2. *Symmetric:*
   Suppose $(P_1, P_2) \in \overset{h}{\rightsquigarrow}$ then is it true that $(P_2, P_1) \in \overset{h}{\rightsquigarrow}$?
   By our assumption, we know that there is continuous bijective f such that

   $$P_2 = \{(x, f(y)) | (x, y) \in P_1\}. \tag{5.4}$$

Can we show that there is a continuous bijective function — we can call it g — such that $P_1 = \{(x, g(y))|(x, y) \in P_2\}$?

Because f is bijective it has an inverse. That is, we have a function $g : [0, 1] \to [0, 1]$ such that $f \circ g = id$. Here, to stop the proof from becoming too long, we will assume that this g is continuous. If it is, then we can use it to make two pictures related by $\overset{h}{\rightsquigarrow}$. Let us see what happens when we apply g to P_2:

$$\{(x, g(y))|(x, y) \in P_2\} = \{(x, g(f(y)))|(x, f(y)) \text{ for } (x, y) \in P_1\}$$

$$= \{(x, g(f(y)))|(x, y) \in P_1\} = \{(x, y)|(x, y) \in P_1\} = P_1$$

where we used (5.4) to rewrite P_2. So $P_2 \overset{h}{\rightsquigarrow} P_1$.

3. *Transitive:*

Suppose (P_1, P_2) and $(P_2, P_3) \in \overset{h}{\rightsquigarrow}$, then it is true that $(P_1, P_3) \in \overset{h}{\rightsquigarrow}$?

By our assumption, we know that there are continuous bijective f, g such that $P_2 = \{(x, f(y))|(x, y) \in P_1\}$ and $P_3 = \{(x, g(y))|(x, y) \in P_2\}$. Substituting for P_2 we obtain

$$P_3 = \{(x, g(f(y))) \mid (x, f(y)) \text{ for } (x, y) \in P_1\}.$$

Here $g \circ f : [0, 1] \to [0, 1]$ is also a continuous bijective, because the composition of bijections is bijection; and the composition of continuous functions is continuous. So, $P_1 \overset{h}{\rightsquigarrow} P_3$. Therefore $\overset{h}{\rightsquigarrow}$ is transitive.

Thus $\rightsquigarrow$ is an equivalence relation.

□

(**5.1.14**) Since $\overset{h}{\rightsquigarrow}$ is an equivalence relation we can move on to the two other questions:

does it make the two bracketings in the associativity test lie in the same class?

is it a congruence?

(**5.1.15**) The inequality from 3.3 illustrates that the operation $\boxtimes$ is not associative. Here we call $P_1 = (Q_1 \boxtimes Q_2) \boxtimes Q_3$ for the left side of this associativity and $P_2 = Q_1 \boxtimes (Q_2 \boxtimes Q_3)$ for the right hand side. What we get is $P_1 \neq P_2$.

We claim the equivalence class of P_1 under relation ($\overset{h}{\leadsto}$) will be equal to the equivalence class of P_2. The class of P_1 is denoted by $[P_1]$. So then we claim $[P_1] = [P_2]$. That means we claim $f(P_1) = \{(x, f(y))|(x, y) \in P_1)\} = P_2$ for some bijective function f for every case of P_1 and P_2.

Specifically, we claim that in fact the single function f_m given in (5.1.18) below satisfied this identity in all cases; and is a continuous bijection.

5.1.2.3 The function f_m

In this section we define a key function in $Homeo([0,1],[0,1])$ and show its properties.

(**5.1.16**) Note that if we have a set S and a relation ρ on S then the equivalence closure of ρ (see e.g. 5.1.2.5) sorts S into classes. Let us call these the 'classes of ρ' (strictly they are the classes of the closure). Note that if we have a relation $\rho' \supset \rho$ as a subset of $S \times S$ then the classes of ρ' are unions of classes of ρ.

In particular, if $[s] = [t]$ in ρ (again we mean in the closure) then $[s] = [t]$ in $\rho' \supset \rho$.

(**5.1.17**) It follows that if a particular relation ρ (a set of relational pairs (s, t)) leads to an identity of classes, then a larger relation (a larger set of relational pairs - indeed any set containing the elements of ρ) gives the same identity.

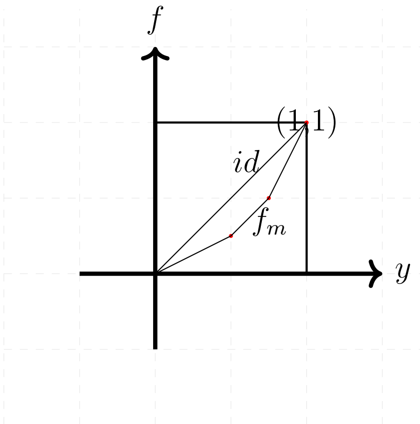
For example, if some relation yields the associative identities on classes in a magma, then a bigger relation will also give these identities.

In our case if we have a single function in our set acting on $\mathfrak{S}_1$ that leads to the identities on classes, then our relation will give these identities.
(Of course this does not mean that we have a congruence. But if we do have a congruence, it will be with a semigroup or monoid as quotient.)

(**5.1.18**) The function f_m is:

$$f_m(x) = \begin{cases} \frac{x}{2} & , \quad 0 \leqslant x \leqslant \frac{1}{2} \\ x - \frac{1}{4} & , \quad \frac{1}{2} < x \leqslant \frac{3}{4} \\ 2x - 1 & , \quad \frac{3}{4} < x \leqslant 1 \end{cases} \tag{5.5}$$

The graph of this function f_m is:



To see that f_m is a bijection consider the function:

$$g_m(x) = \begin{cases} 2x & , \quad 0 \leqslant x \leqslant \dfrac{1}{4} \\ x + \dfrac{1}{4} & , \quad \dfrac{1}{4} < x \leqslant \dfrac{1}{2} \\ \dfrac{x+1}{2} & , \quad \dfrac{1}{2} < x \leqslant 1 \end{cases}$$

We can compute $g_m \circ f_m$ by explicit computation, and thus confirm that $g_m \circ f_m = id_{[0,1]}$. Thus we have shown the following.

Proposition 5.1.19. *The function $f_m : [0,1] \to [0,1]$ is a bijection; and its inverse is ${f_m}^{-1} = g_m$ given as above.* □

Proposition 5.1.20. *Let $Q_1, Q_2, Q_3 \in \mathfrak{S}_1$, and recall product $\boxtimes$ from Def.3.1.19. Let $P_1 = (Q_1 \boxtimes Q_2) \boxtimes Q_3$ and $P_2 = Q_1 \boxtimes (Q_2 \boxtimes Q_3)$. (Remark: recall that $\boxtimes$ is not associative; so in general $P_1 \neq P_2$.) Then the equivalence class of P_1 under relation ($\overset{h}{\rightsquigarrow}$, as in 5.1.12) is equal to the equivalence class of P_2.*

Here the class of P_1 is denoted by $[P_1]$. So our statement becomes $[P_1] = [P_2]$. That means $f(P_1) = \{(x, f(y)) | (x, y) \in P_1)\} = P_2$ - for some continuous bijective function f.

In particular:

$$f_m(P_1) = P_2$$

Proof. Firstly, we can see from the graph above that this f_m is a bijection as required. Then, for the identity, notice, to begin with, that in P_1 the factor of this picture coming from Q_3 is living in the range of y values $y = 0$ to $y = 1/2$. After applying f_m, this range is changed to be from $y = 0$ to $y = 1/4$. Similarly, we can see that f_m changes Q_2 and Q_1 just as we need to change P_1 to P_2. More explicitly, looking at the second factor of this picture which is Q_2. It is living in the range of y values $y = 1/2$ to $y = 3/4$. After we apply f_m, this range is changed to be from $y = 1/4$ to $y = 1/2$. Finally, the last factor in P_1 is Q_1 which is living in the range of y values $y = 3/4$ to $y = 1$. After we apply f_m, this range is changes to be from $y = 1/2$ to $y = 1$. □

5.1.2.4 Minimal relation for formal associativity

In Prop.5.1.20 we proved that the presence of f_m in a subset of the $Homeo([0,1],[0,1])$ action on $\mathfrak{S}_1$ is sufficient to have the formal associative identities on classes. ('Formal' because this does not yet consider congruence.)
Next we consider the necessary (as opposed to sufficient) conditions for such a relation.

In the inequality from 3.3 we call

$$A_1 = (a \boxtimes b) \boxtimes c$$

for the left side of this inequality and $A_2 = a \boxtimes (b \boxtimes c)$ for the right-hand side. What we get is $A_1 \neq A_2$.

(**5.1.21**) Now we look for a relation on $\mathcal{P}([0,1]^2)$ that will identify the classes of A_1 and A_2. There are many possibilities - and some of these will be congruences on our magma. Minimally we can take the equivalence-relation closure of the relation obtained by identifying for all triples as above. This is easy to say, but not easy to implement in a controlled way (and it is possibly a lot smaller than the kinds of equivalences that might appear 'natural' in this setting - see later). Or, we could implement a more conceptual equivalence and see if it is a congruence and achieves the required identification. This is what we do in 5.1.26. (And this is where we will pass closer to consideration of the Thompson group.)

(**5.1.22**) Next we define a relation ρ_a that we will use to construct an equivalence relation such that the equivalence classes of A_1 and A_2 above are equal ($[A_1] = [A_2]$)

for all a, b, c. So here we will describe a minimal/necessary relation in order to get (after that) the smallest equivalence relation containing this relation.

We start with our underlying set $\mathcal{P}([0,1]^2)$. For A_1, A_2 as above we have $A_1 \neq A_2$. We want to force $[A_1] = [A_2]$ in all cases by defining the following:

Let $a, b, c \in \mathcal{P}([0,1]^2)$ and then define a recipe for making A_1 as

$$\begin{aligned} f_1 : \quad \mathcal{P}([0,1]^2)^{\times 3} &\longrightarrow \quad \mathcal{P}([0,1]^2) \\ (a, b, c) &\longmapsto \quad f_1(a, b, c) = (a \boxtimes b) \boxtimes c. \end{aligned}$$

and define making A_2 as

$$\begin{aligned} f_2 : \quad \mathcal{P}([0,1]^2)^{\times 3} &\longrightarrow \quad \mathcal{P}([0,1]^2) \\ (a, b, c) &\longmapsto \quad f_2(a, b, c) = a \boxtimes (b \boxtimes c). \end{aligned}$$

(**5.1.23**) *The relation* ρ_a on $\mathcal{P}([0,1]^2)$ can now be defined as

$$\rho_a \; = \; \{(x, y) \mid x = f_1(a, b, c) \quad \text{and} \quad y = f_2(a, b, c) \quad \text{for some} \quad a, b, c \in \mathcal{P}([0,1]^2)\}.$$

(**5.1.24**) Now we can ask: What is the smallest equivalence relation containing ρ_a (in the sense of (2.2.7))?

5.1.2.5 Closure of relation to Equivalence-relation

(**5.1.25**) For any relation ρ on a set X to itself, we can ask how to construct the smallest equivalence relation containing ρ. (Then we can apply this, for example, to relations such as ρ_a on $\mathcal{P}([0,1]^2)$, as in 5.1.23.)

Let's take any relation ρ on any set X defined as

$$\rho \subset \{(x, y) \in X \times X \mid x, y \in X\}.$$

Any equivalence relation containing ρ is reflexive, so must contain a reflexive closure. So first define

$$\overline{\rho}^r = \rho \cup \{(x, x) \in X \times X \;\mid x \in X\}$$

because we require equivalence relation closure. We also need to include in $\overline{\rho}$ the symmetric closure

$$\overline{\rho}^s \;=\; \rho \cup \{(y,x)|(x,y) \in \rho\}.$$

Next, we need to contain the transitivity closure. We first repeat the transformation

$$\rho \rightsquigarrow \rho \cup \{(x,z) \mid (x,y) \in \rho \quad and \quad (y,z) \in \rho\}.$$

Then we define $\overline{\rho}^t$ as the output of repeating this transformation until there is no more change.

Finally define

$$\overline{\rho}^e \;:=\; \overline{\overline{\overline{\rho}^r}^s}^t.$$

We will next show that the equivalence closure of the relation ρ on the set X is given by

$$\overline{\rho} = \overline{\rho}^e \;= \overline{\overline{\overline{\rho}^r}^s}^t.$$

(**5.1.26**) Therefore, in our case the equivalence closure ($\overline{\rho}_a$) of the original relation ρ_a on the set $\mathcal{P}([0,1]^2)$ is

$$\overline{\rho}_a = \overline{\overline{\overline{\rho}_a^r}^s}^t.$$

Proposition 5.1.27. *For any relation ρ on a set X to itself as above, the relation $\overline{\rho}^e$ is equivalence relation. And hence $\overline{\rho} = \overline{\rho}^e$.*

Proof.

1. *Reflexive*: by the reflexive closure, for all $x \in X$ this implies all pairs of the form (x,x) in $\overline{\rho}^r$.

   Also, we note that the symmetric and transitive closure only add pairs to this relation. Therefore, $\overline{\rho}^e$ is reflexive.

2. *Symmetric (first step)*: for $x, y \in X$ then by the s-closure for each (x,y) in $\overline{\rho}^r$, we must add (y,x). So $\overline{\overline{\rho}^r}^s$ is symmetric. Note that we need to return to check the symmetry of $\overline{\overline{\overline{\rho}^r}^s}^t$ after we do transitive closure.

3. *Transitive*: transitivity follows since we take transitive closure as last step in the construction. *Symmetric (final step)*: for x, y and $z \in X$ then for each (x, y) and (y, z) in $\overline{\overline{\rho}^r}^s$ we add (x, z) by the t-closure. We also have (y, x) and (z, y) in $\overline{\overline{\rho}^r}^s$ by symmetric closure. So by transitive closure, we add $(z, x) \in \overline{\rho}^e$. Then we have (z, x) and (x, z), by transitive closure. Thus $\overline{\rho}^e$ is symmetric.

Hence $\overline{\rho}^e$ is reflexive, symmetric and transitive. Therefore, $\overline{\rho}^e$ is an equivalence relation. It is the smallest equivalence relation that is containing ρ because, in each step in 5.1.25, we only add the minimum number of pairs that are necessary to satisfy the equivalence relation conditions. □

(**5.1.28**) Recall from 5.1.23 that for each pair A_1, A_2 which are images of the same triple of elements by the different orders of their composition, we have $A_1 \rho_a A_2$. We claim the equivalence class of A_1 under the relation $\overline{\rho}_a$ above will be equal to the equivalence class of A_2. The class of A_1 is denoted by $[A_1]$. So then we claim $[A_1] = [A_2]$.

(**5.1.29**) Now we want to prove our claim $[A_1] = [A_2]$. Since the closure only adds elements to the relation and does not remove any, A_1 and A_2 are related by the closure relation 5.1.26. Therefore, they are in the same class. □

5.2 Quotient of $\mathfrak{M}$ from Thompson Group F

In this section, we introduce the generalities of an action of a homeomorphism group 5.2.1. Then we recall the definition of Thompson Group F in §5.2.2. Then we introduce properties of the Thompson group action on $\mathcal{P}([0,1]^2)$ in §5.2.3. Finally, We address the subgroup generated by f_m in §5.2.4.

5.2.1 Generalities

(**5.2.1**) Recall 2.3.17. In particular the set $Homeo([0,1],[0,1])$ is a group. There is an action

$$* : Homeo([0,1],[0,1]) \times \mathcal{P}([0,1]^2) \to \mathcal{P}([0,1]^2)$$

of this group on $\mathcal{P}([0,1]^2)$ given by

$$f * p = \{(x, f(y)) \mid (x,y) \in p\}$$

(there are other actions, but this is the main one for us).

(**5.2.2**) For any group action on $\mathcal{P}([0,1]^2)$ we can partition $\mathcal{P}([0,1]^2)$ into orbits under the group action. This corresponds to an equivalence relation in the usual way.

In general an equivalence relation will not be a congruence on our magma. (And if it is a congruence, the quotient may not be associative.) (For example quotienting by the trivial group action obviously yields a non-associative congruence.) But we can investigate which equivalences and actions may lead to congruences.

(**5.2.3**) The action of $Homeo([0,1],[0,1])$ above yields a congruence with an associative quotient.

Here we are interested in finding subgroups which also yield an associative quotient.

A particularly interesting subgroup to consider is Thompson's F group, which we recall in §5.2.2. We will also consider even smaller/simpler subgroups that might yield associative quotients. (See e.g. §5.1.2, where we also consider the requirements for such a group.)

5.2.2 Thompson's group F

In this section, we provide one of three Thompson's groups called F.

In 5.2.3 we will use this group to make a quotient of our magma $\mathfrak{M}$.

We define dyadic rational numbers first then Thompson's group F.

Definition 5.2.4. *(see e.g.[Sti18, Bel07]) A dyadic rational number is a rational number whose denominator is a power of two. It is of the form $\frac{a}{2^b}$ where $a \in \mathbb{Z}$ and $b \in \mathbb{N}$.*

Example 5.2.5. *1. Consider $a = 1 \in \mathbb{Z}$ and $b = 1 \in \mathbb{N}$, the dyadic rational is $\frac{1}{2^1} = \frac{1}{2}$.*

2. Consider $a = -5 \in \mathbb{Z}$ and $b = 3 \in \mathbb{N}$, the dyadic rational is $\frac{-5}{2^3} = \frac{-5}{8}$.

Definition 5.2.6. *(see e.g. [Pen17, BM14, Bel07, CFP94, Chm18, Lev19])*
*A group $F = (F, \circ)$ is called **Thompson's group** F if it has the following properties:*

- *the underlying set F is the set of all piecewise-linear homeomorphisms from the close unit interval $[0, 1]$ to itself with finitely many breakpoints satisfying the following conditions:*

  1. *Every slope is a power of two, and*
  2. *Every breakpoint has dyadic rational coordinates,*

- *$\circ$ is representing the composition of functions (thus it is required to be closed in F - see the references above).*

Example 5.2.7. *Let $f : [0, 1] \to [0, 1]$ given by* $f(x) = \begin{cases} \frac{x}{2}, & \text{if} \quad 0 \leqslant x \leqslant \frac{1}{2} \\ x - \frac{1}{4}, & \text{if} \quad \frac{1}{2} \leqslant x \leqslant \frac{3}{4} \\ 2x - 1, & \text{if} \quad \frac{3}{4} \leqslant x \leqslant 1 \end{cases}$

Then by the definition 5.2.6 f is an element of F.

Proposition 5.2.8. *Thompson group F is closed under composition.*

For the proof (see. e.g [CFP94]).

(**5.2.9**) Note that Thompson group F generated by two functions A and B, given in next example 5.2.10.

Example 5.2.10. *Consider $A, B \in F$. Then $A, B : [0,1] \to [0,1]$ given by*

$$A(x) = \begin{cases} \frac{x}{2}, & \text{if} \quad 0 \leqslant x \leqslant \frac{1}{2} \\ x - \frac{1}{4}, & \text{if} \quad \frac{1}{2} \leqslant x \leqslant \frac{3}{4} \\ 2x - 1, & \text{if} \quad \frac{3}{4} \leqslant x \leqslant 1 \end{cases} \quad \text{and } B(x) = \begin{cases} x, & \text{if} \quad 0 \leqslant x \leqslant \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4}, & \text{if} \quad \frac{1}{2} \leqslant x \leqslant \frac{3}{4} \\ x - \frac{1}{8}, & \text{if} \quad \frac{3}{4} \leqslant x \leqslant \frac{7}{8} \\ 2x - 1, & \text{if} \quad \frac{7}{8} \leqslant x \leqslant 1 \end{cases}$$

Calculating $A \circ B(x)$ as follows:

Let $\{0 = x_0 \leqslant 1/2 \leqslant 3/4 \leqslant x_n = 1\}$ and $\{0 = x_0 \leqslant 1/2 \leqslant 3/4 \leqslant 7/8 \leqslant x_n = 1\}$ be breakpoints of A and B respectively.

Then since $B(0) = 0$, $A(B(0)) = A(0) = 0$, then $A(B(x)) = x/2$ for $0 \leqslant x \leqslant \frac{3}{4}$ where $1/2$ is a power of 2.

Likewise, since $A(B(x))$ is a dyadic rational number, and $A(B(x)) = x - 3/8$ for $\frac{3}{4} \leqslant x \leqslant \frac{7}{8}$, where $1 = 2^0$ is a power of 2 and $\frac{3}{8}$ is a dyadic rational number.

It follows that $A(B(x)) = 4x - 3$ for $\frac{3}{8} \leqslant x \leqslant 1$, where 4 is a power of 2 and $3 = \frac{3}{2^0}$ is a dyadic rational number.

$$\text{Thus } A \circ B(x) = \begin{cases} \frac{x}{2}, & \text{if} \quad 0 \leqslant x \leqslant \frac{3}{4} \\ x - \frac{3}{8}, & \text{if} \quad \frac{3}{4} \leqslant x \leqslant \frac{7}{8} \\ 4x - 3, & \text{if} \quad \frac{7}{8} \leqslant x \leqslant 1 \end{cases} \quad \text{is closed under composition.}$$

5.2.3 Properties of the Thompson group action on $\mathcal{P}([0,1]^2)$

Note that $f_m \in F$. It follows that the equivalence relation given by the action of F gives the associative identities on classes. It remains to check that this relation is a congruence.

(**5.2.11**) Given a set M and a set $S \subset \hom(M, M)$ we can define a relation $\overset{S}{\rightsquigarrow}$ on M by $m \overset{S}{\rightsquigarrow} m'$ if $m' = f(m)$ for some $f \in S$.

Observe that if $f, f' \in S$ but $f \circ f' \notin S$ (S is not closed under composition in the monoid $\hom(M, M)$) then $m \overset{S}{\rightsquigarrow} f(m) \overset{S}{\rightsquigarrow} f'(f(m)$ but we do not necessarily have $m \overset{S}{\rightsquigarrow} f'(f(m))$. That is, $\overset{S}{\rightsquigarrow}$ is not necessarily transitive. On the other hand, if S is closed (it is a sub-semigroup of $\hom(M, M)$) then we have transitivity.

Similarly if S includes the identity then we have reflexivity of $\overset{S}{\rightsquigarrow}$. And if S contains inverses then we have symmetricity.

In other words, given $S \subset \hom(M, M)$, and extending this to the subgroup $\langle S \rangle$ that it generates, then the relation determined by $\langle S \rangle$ is an equivalence relation. (Thus it defines classes, which we can then test for congruence if M is in fact a magma.)

(**5.2.12**) We have that any function $f : [0, 1] \to [0, 1]$ induces a function $f : \mathcal{P}([0, 1]^{d+1}) \to \mathcal{P}([0, 1]^{d+1})$ by

$$f(p) = \{(\underline{x}, f(y)) \mid (\underline{x}, y) \in p\}.$$

(**5.2.13**) Consider a subset $S \subset \hom([0, 1], [0, 1])$. We can define a relation $\overset{S}{\rightsquigarrow}$ on $\mathcal{P}([0, 1]^{d+1})$ as above. And we see that using the subgroup $\langle S \rangle$ will mean that this is an equivalence relation.

(**5.2.14**) Note that this is not enough to ensure that the relation is a congruence.
When is a relation of this (a, fa) form a congruence?
For a relation $\sim$ to be a magma congruence on a magma with composition $\boxtimes$, say, we require firstly that it is an equivalence - so here we write $[a]$ for the class of a. Then we require that $a' \sim a$ (i.e. $a' \in [a]$) and $b' \sim b$ implies $a \boxtimes b \sim a' \boxtimes b'$.
Let us check for the (a, fa) kind of relation.
$a \sim a'$ implies $a' = fa$ (for some $f \in S$)
so we must ask if
$a \boxtimes b \overset{?}{=} f''(fa \boxtimes f'b)$ for some $f'' \in S$, for all $f, f' \in S$.

To construct a candidate for f'' consider the following.

(**5.2.15**) Elements $f : \mathbb{I} \to \mathbb{I}$ of $Homeo(\mathbb{I}, \mathbb{I})$ compose by function composition. But elements in $Homeo_0(\mathbb{I}, \mathbb{I})$, the connected component of the identity, also compose by a form of stack-shrink:

$$\boxplus : Homeo_0(\mathbb{I}, \mathbb{I}) \times Homeo_0(\mathbb{I}, \mathbb{I}) \to Homeo_0(\mathbb{I}, \mathbb{I})$$

$$(f \boxplus g)(x) = \begin{cases} \frac{g(2x)}{2} & x \in [0,1/2] \\ \frac{1+f(2x-1)}{2} & x \in [1/2,1] \end{cases} \tag{5.6}$$

(note that this puts the first factor 'over' the second, as is our convention; and that being in the identity-component ensures closure/well-definedness). This $(Homeo_0(\mathbb{I},\mathbb{I}),\boxplus)$ is not associative. (But useful.)

There are a few interesting variations on this construction, but we will keep with this one here.

(**5.2.16**) Now we return to the congruence question for our particular $\overset{S}{\rightsquigarrow}$ kind of relation, where $S = \langle S \rangle$ for some subset/group of $Homeo_0(\mathbb{I},\mathbb{I})$. (The idea is that this S contains at least the piecewise-linear stack-shrink "associativising" function here denoted f_m - which takes $[1/2,3/4]$ to $[1/4,1/2]$, and of course the inverse and indeed the subgroup it generates.)

Proposition 5.2.17. *We have* $(f \boxplus g)a \boxtimes b = fa \boxtimes gb$ *for all* $f,g \in Homeo_0(\mathbb{I},\mathbb{I})$ *and* $a,b \in \mathcal{P}([0,1]^2)$.

Proof. Firstly note that for any f,g we have $fa = \{(x,f(y) \mid (x,y) \in a\}$ and

$$g(fa) \;=\; \{(s,g(t)) \mid (s,t) \in \{(x,f(y)) \mid (x,y) \in a\}\} \;=\; \{(x,g(f(y))) \mid (x,y) \in a\}$$

$$= \{(x,(g \circ f)(y)) \mid (x,y) \in a\} \;=\; (g \circ f)a$$

so the action is a group action and so $f^{-1}(fa) = a$. (So our identity implies that the congruence condition holds whenever S is closed under $\boxplus$.)

Next we have that

$$a \boxtimes b = \left\{(x,\frac{y+1}{2}) \mid (x,y) \in a\right\} \cup \left\{(x,\frac{y}{2}) \mid (x,y) \in b\right\}$$

while $f \boxplus g$ is as in (5.6). Observe that the first set in this expression for $a \boxtimes b$ has x values in $[1/2,1]$ so in $(f \boxplus g)a \boxtimes b$ it becomes

$$\left\{(x,\frac{1+f(2\frac{y+1}{2}-1)}{2}) \mid (x,y) \in a\right\} = \left\{(x,\frac{1+f(y)}{2}) \mid (x,y) \in a\right\}$$

Observe that this is the same as the image of fa in $fa \boxtimes gb$. The second set in $a \boxtimes b$

has x values in $[0, 1/2]$, so here the y coordinates become $\frac{g(2\frac{y}{2})}{2}$. Since the image of gb in $fa \boxtimes gb$ gives y-coordinates $g(y)/2$, we have the required identity. $\square$

(**5.2.18**) We have shown that a sufficient condition for congruence is closure of the acting subset - which so far must be a group $\langle S \rangle$ containing the associativising map f_m - under $\boxplus$.

So it now becomes a very interesting question to determine this group. (The $\boxplus$ action is not a group action, but we must also close to a group.) Note that f_m is in Thompson's group F, and $\boxplus$ also closes in F, so our group will be some subgroup of F. In this thesis we study the group generated by f_m. But it is easy to see that the smallest group containing this group and closed under $\boxplus$ is bigger. We propose to continue the study of this group later (just for reasons of time).

5.2.4 The subgroup generated by f_m

Here we consider the subgroup of F generated by f_m. We know that f_m is sufficient for the associative identities on classes. So we can ask what is the smallest subgroup that we need that contains f_m.

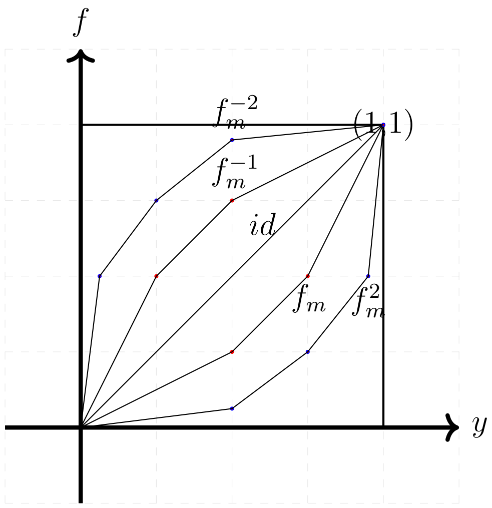


Figure 5.1: Plot of the functions $f_m^l, l \in \mathbb{Z}$

In figure 5.1 we try to take a step towards constructing equivalence classes of $\mathfrak{S}_1$ under a relation generated by f_m. As noted above, we should take the group generated by f_m. First, we need to show if the relation on $\mathfrak{S}_1$ is an equivalence relation: it is reflexive since the group contain the identity function 'id'. It is symmetric since the group is closed under inverses function which is ${f_m}^{-1}$ and so on. It is transitive since the group is closed under multiplication '${f_m}^2$' and so on.

(**5.2.19**) We already computed the inverse of f_m in 5.1.19.

Proposition 5.2.20. *A composition of the f_m 5.1.18 to itself is given by*

$$(f_m \circ f_m)(x) = \begin{cases} \dfrac{x}{4} & , \quad 0 \leqslant x \leqslant \dfrac{1}{2} \\ \dfrac{x}{2} - \dfrac{1}{8} & , \quad \dfrac{1}{2} < x \leqslant \dfrac{3}{4} \\ 2x - \dfrac{5}{4} & , \quad \dfrac{3}{4} < x \leqslant \dfrac{7}{8} \\ 4x - 3 & , \quad \dfrac{7}{8} < x \leqslant 1 \end{cases}$$

Proof. First, we consider the domain $0 \leqslant x \leqslant 1/2$, so we observe $f_m(x) = x/2$ in this domain, but the image is $f_m([0, 1/2]) = [0, 1/4]$, It is subset of $[0, 1/2]$, so $f_m(f_m(x)) = f_m(x/2) = x/4$ where $0 \leqslant x \leqslant 1/2$.
Now we consider the second domain which is $1/2 < x \leqslant 3/4$. So, we observe $f_m(x) = x - 1/4$ in this domain, but the image is $f_m((1/2, 3/4]) = (1/4, 1/2]$ and this range or 'output' is subset of $[0, 1/2]$. So, when we apply f_m again we use $f_m(x) = x/2$, so, for $1/2 < x \leqslant 3/4$, $f_m(x - 1/4) = x/2 - 1/8$.
Finally, we consider the last domain which is $3/4 < x \leqslant 1$. So, we observe $f_m(x) = 2x - 1$, but the image is $f_m((3/4, 1]) = (1/2, 1] = (1/2, 3/4] \cup (3/4, 1]$ that means this range spilt at $x = 3/4$. So the first range from $1/2$ to $3/4$ when we apply f_m again we use $f_m(x) = x - 1/4$, so here $f_m(2x - 1) = 2x - 5/4$ where $3/4 < x \leqslant 7/8$. The second range from $3/4$ to 1. We use $f_m(x) = 2x - 1$, so $f_m(2x - 1) = 4x - 3$ where $7/8 < x \leqslant 1$.
Thus

$$f_m \circ f_m(x) = \begin{cases} \dfrac{x}{4} & , \quad 0 \leqslant x \leqslant \dfrac{1}{2} \\ \dfrac{x}{2} - \dfrac{1}{8} & , \quad \dfrac{1}{2} < x \leqslant \dfrac{3}{4} \\ 2x - \dfrac{5}{4} & , \quad \dfrac{3}{4} < x \leqslant \dfrac{7}{8} \\ 4x - 3 & , \quad \dfrac{7}{8} < x \leqslant 1 \end{cases} = g(x)$$

□

(**5.2.21**) Next we compute $f_m \circ {f_m}^2(x)$.

First, we consider the domain $0 \leqslant x \leqslant 1/2$, so we observe ${f_m}^2(x) = x/4$ in this domain, but the image is ${f_m}^2([0, 1/2]) = [0, 1/8]$, It is a subset of $[0, 1/2]$, so $f_m({f_m}^2(x)) = f_m(x/4) = x/8$ where $0 \leqslant x \leqslant 1/2$.

Now we consider the second domain which is $1/2 < x \leqslant 3/4$. So, we observe ${f_m}^2(x) = x/2 - 1/8$ in this domain, but the image is ${f_m}^2((1/2, 3/4]) = (1/8, 1/4]$ and this range or 'output' is subset of $[0, 1/2]$. So, when we apply f_m again we use $f_m(x) = x/2$, so, for $1/2 < x \leqslant 3/4$, $f_m(x/2 - 1/8) = x/4 - 1/16$.

Next, we consider the third domain which is $3/4 < x \leqslant 7/8$. So, we observe ${f_m}^2(x) = 2x - 5/4$ in this domain, but the image is ${f_m}^2((3/4, 7/8]) = (1/4, 1/2]$, this range is subset of $[0, 1/2]$. So, when we apply f_m again we use $f_m(x) = x/2$, so, for $3/4 < x \leqslant 7/8$, $f_m(2x - 5/4) = x - 5/8$.

Finally, we consider the last domain which is $7/8 < x \leqslant 1$. So, we observe ${f_m}^2(x) = 4x - 3$, but the image is $f_m((7/8, 1]) = (1/2, 1] = (1/2, 3/4] \cup (3/4, 1]$ that means this range spilt at $x = 3/4$. So the first range from $1/2$ to $3/4$ when we apply f_m again we use $f_m(x) = x - 1/4$, so here $f_m(4x - 3) = 4x - 13/4$ where $7/8 < x \leqslant 15/16$. The second range from $3/4$ to 1. We use $f_m(x) = 2x - 1$, so $f_m(4x - 3) = 8x - 7$ where $15/16 < x \leqslant 1$.

Thus

$$f_m \circ {f_m}^2(x) = \begin{cases} \dfrac{x}{8} & , \quad 0 \leqslant x \leqslant \dfrac{1}{2} \\ \dfrac{x}{4} - \dfrac{1}{16} & , \quad \dfrac{1}{2} < x \leqslant \dfrac{3}{4} \\ x - \dfrac{5}{8} & , \quad \dfrac{3}{4} < x \leqslant \dfrac{7}{8} \\ 4x - \dfrac{13}{4} & , \quad \dfrac{7}{8} < x \leqslant \dfrac{15}{16} \\ 8x - 7 & , \quad \dfrac{15}{16} < x \leqslant 1 \end{cases} = {f_m}^3(x)$$

(**5.2.22**) The figure 5.2 below provides various iterated compositions of f_m with itself and $g_m = {f_m}^{-1}$ - the inverse of f_m (using the Maple program).

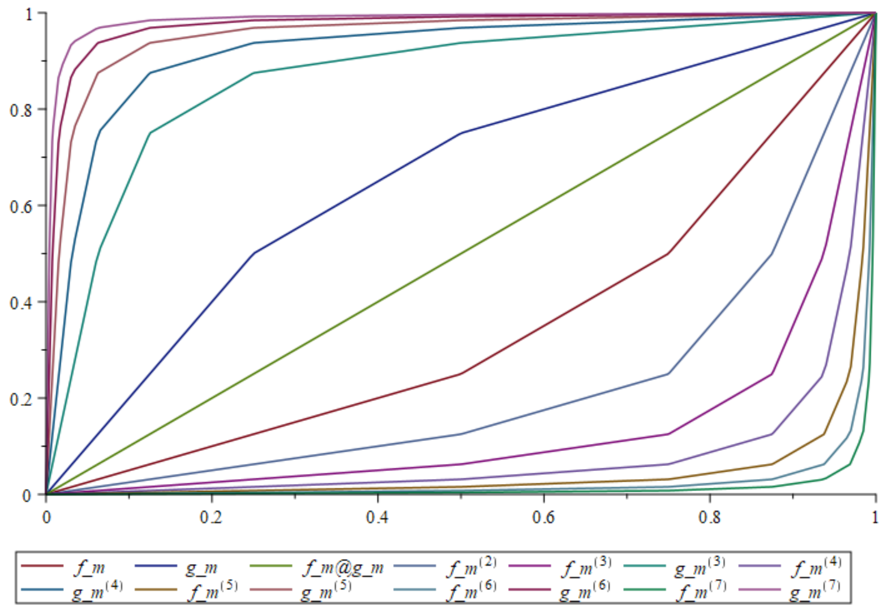


Figure 5.2: Plot of iterated compositions of f_m and g_m

(**5.2.23**) It remains to consider if the group generated by f_m gives a congruence. We will do this in future work.

5.3 Quotient by a path-relation R on the underlying set of $\mathfrak{M}$

In this section and section 5.4, we construct some equivalence relations on the underlying set of our magma using paths. We call the new relations R^p_n on $\mathcal{P}([0,1]^2)$ and and R_α on certain subsets. We will define the relation R^p_n on $\mathcal{P}([0,1]^2)$ in section 5.3.1 to see if it is congruence, but it fails. Then we introduce R_α in 5.4.2, which is successful.

5.3.1 The relation R^p_n (for a fixed $n \in \mathbb{N}$)

(**5.3.1**) Fix $n \in \mathbb{N}$. Recall from (2.1) that $\underline{n} = \{1, 2, ..., n\}$. Consider the finite sets

$$P = \{(\frac{i}{1+n}, 1) : i \in \underline{n}\} \tag{5.7}$$

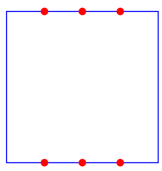
and

$$P' = \{(\frac{i'}{1+n}, 0) : i' \in \underline{n}\} \tag{5.8}$$

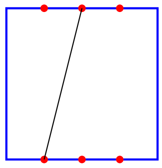
(**5.3.2**) Fix $n \in \mathbb{N}$, so that P and P' are fixed. Let $a, b \in \mathcal{P}([0,1]^2)$. The *relation* R^p_n on $\mathcal{P}([0,1]^2)$ is defined by:
$a \; R^p_n \; b$ if for each pair of points $x, y \in P \cup P'$ these points are path-connected in a if and only if they are path-connected in b.

(**5.3.3**) Note that, we use $\dashrightarrow$ to indicate passing to a "representative" picture (we will include examples to show the limitations of such pictures).

Example 5.3.4. *Consider $n = 3$. We can try to draw a picture with P and P' in red (that represents discrete points in the top and bottom $[0,1]^2$), the frame in blue and the elements of the magma in black.*

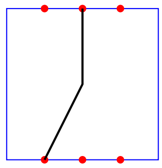
$a = \emptyset \dashrightarrow$  ,

$$b = \{(\frac{x+1}{4}, x) : x \in [0,1]\} \dashrightarrow \tag{5.9}$$



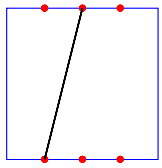
,

$$c = \{(\frac{2x+1}{4}, x) : x \in [0, \frac{1}{2}]\} \cup \{(\frac{1}{2}, x) : x \in [\frac{1}{2}, 1]\} \dashrightarrow$$



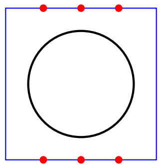
,

$$d = \{(\frac{x+1}{4}, x) : x \in [0,1)\} \dashrightarrow$$



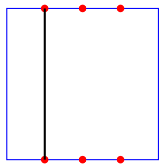
,

$$h = \phi \dashrightarrow$$



,

$$\textit{and} \quad k = \{(\frac{1}{4}, x) : x \in [0,1]\} \dashrightarrow$$



$\in \mathcal{P}([0,1]^2)$.

Then the relation between these elements in magma is as follows:

- *There is no relation between a and b, ($a \not R^p_3\, b$) because no points of $P \cup P'$ are path-connected in a, but two points are path-connected in b.*
- *There is no relation between b and h , ($b \not R^p_3\, h$) because two points are path-connected in b but no points of $P \cup P'$ are path-connected in h.*
- *There is no relation between b and d, ($b \not R^p_3\, d$) because, both elements look equal in the picture, but one point is missing mathematically at d.*
- *There is a relation between b and c, ($bR^p_3 c$) .*
- *There is a relation between b and k, ($bR^p_3 k$).*
- *There is a relation between a and d, ($aR^p_3 d$) because there are no points of $P \cup P'$ are path-connected in both a and d.*

(**5.3.5**) Any function $f : S \to T$ defines a relation $\sim^f$ on S by $a \sim^f b$ if $f(a) = f(b)$.

Example: let S be all the people, and f is height. Then person $a \sim^f b$ if they have the same height, i.e. if $f(a) = f(b)$.

Note that every such relation $\sim^f$ is an equivalence relation:

reflexive: because $f(a) = f(a)$;

symmetric: because $f(a) = f(b)$ implies $f(b) = f(a)$;

transitive: because $f(a) = f(b)$ and $f(b) = f(c)$ implies $f(a) = f(c)$.

(**5.3.6**) Fix n, then we define a map

$$H_n : \mathcal{P}([0,1]^2) \to \mathfrak{Par}_{P\cup P'}$$

such that i, j in same part in $H_n(a)$ if they are path-connected in a.

Example 5.3.7. *Consider magma element $a = \emptyset$ and $b = \{(\frac{x+1}{4}, x) : x \in [0,1]\}$ from (5.3.4). Then*

$$H_3(a) = \{\{(1/4,1)\},\{(1/2,1)\},\{(3/4,1)\},\{(1/4,0)\},\{(1/2,0)\},\{(3/4,0)\}\}.$$

$$H_3(b) = \{\{(1/4,1)\},\{(1/2,1),(1/4,0)\},\{(3/4,1)\},\{(1/2,0)\},\{(3/4,0)\}\}.$$

(**5.3.8**) Thus for each n we have an equivalence relation on $\mathcal{P}([0,1]^2)$ given by $\sim^{H_n}$.

Fix n. We claim $\sim^{H_n}$ is the same relation as R^p_n .

So we conclude that R^p_n is an equivalence relation.

Why is R^p_n same as $\sim^{H_n}$?

Firstly if $a\ R^p_n\ b$ then i, j path-connected in a if and only if i, j path-connected in b for all i, j. So aR^p_nb implies $H_n(a) = H_n(b)$. But also $H_n(a) = H_n(b)$ says i, j path-connected in a if and only if path-connected in b, so this implies $a\ R^p_n\ b$.

(**5.3.9**) So, by (5.3.5) we have that $\sim^{H_n}$ is an equivalence relation; and then by (5.3.8) aR^p_nb is an equivalence relation.

(**5.3.10**) Fix n. Note that if we write $[a]$ for the aR^p_nb-class of a, then $[a]$ is also the set of magma elements that have the same image under H_n.

In other words $b \in [a]$ if and only if $H_n(b) = H_n(a)$.

(**5.3.11**) We will determine if for each fixed value of n then aR^p_nb is a congruence on the PP magma.

Fix n. For the relation aR^p_nb to be a congruence under $\boxtimes$ defined in 3.1.19 we would need to show the following:

Claim (false): For $a_* \in [a]$ (meaning a_* in the same R^p_n class as a), and $b_* \in [b]$, then $[a_* \boxtimes b_*] = [a \boxtimes b]$.

For convenience let us say that the points of P at the top of $[0,1]^2$ are just given the names $1, 2, ..., n$. And the points of P' are given the names $1', 2', ..., n'$.

(**5.3.12**) Unfortunately we can see that this Claim is false for $n = 1$, as follows. (Similar examples then show it is false for all $n > 0$.)

First consider $a = \{(y/2, y) \mid y \in [0,1]\} \in \mathcal{P}([0,1]^2)$. This has $H_1(a) = \{\{1\}, \{1'\}\}$.

Next put $a_i = \{(x, i) \mid x \in [0,1]\}$, for $i = 0, 1$; and $b = \{(1/2, y) \mid y \in [0,1]\}$. Now let $c = b \cup a_0 \cup a_1$. We have $H_1(b) = H_1(c) = \{\{1, 1'\}\}$. Thus $b \in [c]$.

We have $H_1(a \boxtimes b) = \{\{1\}, \{1'\}\}$. But $H_1(a \boxtimes c) = \{\{1, 1'\}\}$. Thus $[a \boxtimes b] \neq [a \boxtimes c]$, although of course $a \in [a]$, and $b \in [c]$ (so for congruence we require equality).

5.4 An alternative approach

An alternative approach is to make the PP magma into a certain magmoid (a *magmoid* is a magma with partial composition — see for example [TFMM23]) and then try the same equivalence.

5.4.1 A passport-photograph magmoid

(**5.4.1**) We can make the PP magma into a magmoid by defining subsets $\mathsf{M}(\alpha, \beta)$ with $\alpha, \beta \in \mathcal{P}([0,1])$ as follows.

First, define $\pi_i : \mathcal{P}([0,1]^2) \to \mathcal{P}([0,1])$ by

$$\pi_i(a) = \{x | (x, i) \in a\}$$

Then for $\alpha, \beta \in \mathcal{P}([0,1])$ define

$$\mathsf{M}(\alpha, \beta) = \{a \in \mathcal{P}([0,1]^2) | \pi_0(a) = \alpha;\ \pi_1(a) = \beta\}$$

Proposition 5.4.2. *For all $\alpha, \beta, \gamma \in \mathcal{P}([0,1])$ the PP magma composition restricts to a composition*

$$\boxtimes : \mathsf{M}(\alpha, \beta) \times \mathsf{M}(\beta, \gamma) \to \mathsf{M}(\alpha, \gamma)$$

Proof. Let $\alpha, \beta, \gamma \in \mathcal{P}([0,1])$.
We must apply magma composition 3.1.19 to each $(a, b) \in \mathsf{M}(\alpha, \beta) \times \mathsf{M}(\beta, \gamma)$. We require to show:

$$\begin{aligned}\{a \quad \boxtimes \quad b \mid \quad & a \in \mathsf{M}(\alpha, \beta),\ b \in \mathsf{M}(\beta, \gamma)\} \\ \subseteq \quad & \{c \in \mathcal{P}([0,1]^2) \ \mid\ \pi_0(c) = \alpha;\ \pi_1(c) = \gamma\} \quad = \quad \mathsf{M}(\alpha, \gamma)\end{aligned}$$

The inclusion is true because, by 3.1.19, the part of a with $y = 0$ becomes the part of $a \boxtimes b$ with $y = 0$; and the part of b with $y = 1$ becomes the part of $a \boxtimes b$ with $y = 1$. □

Note that the inclusion above is proper because the part of a with $y = 1$ becomes the part of $a \boxtimes b$ with $y = 1/2$, so this part is always given by β in $a \boxtimes b$, whereas there are elements f of $\mathsf{M}(\alpha, \gamma)$ where $\pi_{1/2}(f) \neq \beta$.

(**5.4.3**) In particular, we have a submagma $\mathsf{M}(\alpha, \alpha)$ for each α.

5.4.2 Equivalence relations

(**5.4.4**) Now let us repeat the congruence test (on a path-based relation) on this submagma. But this time, instead of using n and $P \cup P'$, we consider paths between elements determined by α.

(**5.4.5**) For $\alpha \in \mathcal{P}([0,1])$ we understand

$$\alpha \sqcup \alpha \quad := \quad \{(x,0),(x,1) \mid x \in \alpha\} \quad \subset \mathcal{P}([0,1]^2)$$

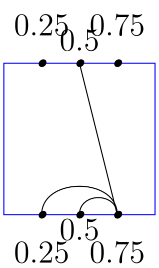
(**5.4.6**) Define

$$H_\alpha : \mathcal{P}([0,1]^2) \;\to\; \mathfrak{Par}_{\alpha \sqcup \alpha}$$

by $p \in [q]$ in $H_\alpha(a)$ (so note that $p, q \in \alpha \sqcup \alpha$) if there is a path between p and q in a. (Recall the notation $\mathfrak{Par}_S$ is defined in 2.2.8.)

Example 5.4.7. • *Let $\alpha = \{1/2\}$ and consider b from (5.9) (note that the red dots in the picture are not used here). We have $H_{\{1/2\}}(b) = \{\{(1/2,0)\},\{(1/2,1)\}\}$. Note that $b \notin M(\alpha, \alpha)$ here.*

• *Now let $\alpha = \{0.25, 0.5, 0.75\}$ and consider the following, using representative pictures as in (5.3.4):*

I. $a \dashrightarrow$ 

.

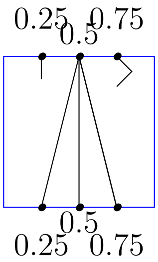
Here, and afterwards, we have a 'no-tricks' convention for pictures, so that if it looks like a continuous line it is *a continuous line, unless we say otherwise. And on the other hand, a small (but not infinitesimal) dot represents a single point, unless we say otherwise.*

In this example, we give the formula of a which is

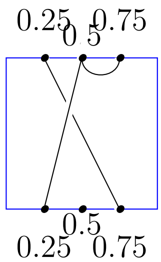
$$a = \{\{(\frac{3-x}{4}, x)|\ x \in [0,1]\} \ \cup \ \{(x-\frac{1}{2}, y+\frac{1}{4})|\ x^2+y^2 = \frac{2}{16}\ ,\ y \geq \frac{-1}{4}\} \\ \cup \ \{(x-\frac{5}{8}, y+\frac{1}{4})|x^2+y^2 = \frac{5}{64} \ \ ,y \geq \frac{-1}{4}\}\} \ \cup \ \{(0.25,1)\} \ \cup \ \{(0.75,1)\}.$$

However, For the rest of the examples, we will use only pictures.

We have $H_{\{0.25,0.5,0.75\}}(a) = \{\{(0.25,1)\}, \{(0.5,1),(0.25,0),(0.5,0),(0.75,0)\}, \{(0.75,1)\}\}$.

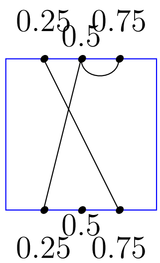
$II.$ $\qquad k \dashrightarrow$  .

We have $H_{\{0.25,0.5,0.75\}}(k) = \{\{(0.25,1)\}, \{(0.5,1),(0.25,0),(0.5,0),(0.75,0)\}, \{(0.75,1)\}\}$.

$III.$ $\qquad g \dashrightarrow$  .

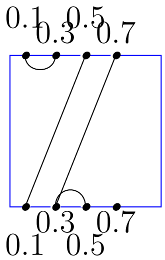
Note that: *the 'line' from* $(0.25,1)$ *to* $(0.75,0)$ *has a gap, so this line is not continuous. Then we have*

$H_{\{0.25,0.5,0.75\}}(g) = \{\{(0.25,1)\}, \{(0.75,0)\}, \{(0.5,1),(0.75,1),(0.25,0)\}, \{(0.5,0)\}\}$.

$IV.$ $\qquad h \dashrightarrow$  .

We have $H_{\{0.25,0.5,0.75\}}(h) = \{\{(0.25,1),(0.75,0)\}, \{(0.5,1),(0.75,1),(0.25,0)\}, \{(0.5,0)\}\}$.

- *Let* $\alpha = \{0.1, 0, 3, 0.5, 0.7\}$ *and consider*

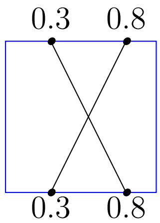
$c \dashrightarrow$ 

.

We have

$H_{\{0.1,0.3,0.5,0.7\}}(c) = \{\{(0.1,1),(0,3,1)\},\{(0.5,1),(0.1,0)\},\{(0.7,1),(0.3,0),(0.5,0)\},\{(0.7,0)\}\}.$

- *Let* $\alpha = \{0.3, 0.8\}$ *and consider*

$d \dashrightarrow$ 

.

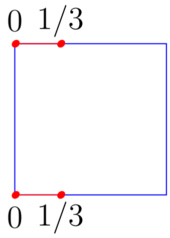
We have $H_{\{0.3,0.8\}}(d) = \{\{(0.3,1),(0.8,0)\},\{(0.8,1),(0.3,0)\}\}.$

(**5.4.8**) The investigation of possible congruence is much harder in principle here, since α can be uncountably infinite.

Definition 5.4.9. *Fix* $\alpha \in \mathcal{P}([0,1])$. *Let* $a, b \in \mathsf{M}(\alpha,\alpha)$. The relation R_α *on* $M(\alpha,\alpha)$ *is defined by* $a \; R_\alpha \; b$ *if for each pair of points* $p, q \in \alpha \sqcup \alpha$ *these are path-connected in* a *if and only if path-connected in* b.

Example 5.4.10. *Let* $\alpha = [0, 1/3]$ *and consider*

$$u = \{(x,0) : x \in [0,1/3]\} \cup \{(x,1) : x \in [0,1/3]\} \dashrightarrow$$



Observe that $u \in \mathsf{M}(\alpha,\alpha)$.

This example is hard to address because there are infinitely many points between 0 and 1/3 that of the top and the bottom but all of these points are connected in u

because the whole line is in u and there is a path in u from 0 to 1/3 (α is infinite).

As a first observation, we note that we can focus on the connected components of α, since for each of them its elements injected in the bottom line (and separately the top line) are immediately in the same part. This is simplest when α is in fact finite.

Example 5.4.11. *Let $\alpha = \{0.25, 0.5, 0.75\}$ and consider a and k from (5.4.7). We have $a\ R_\alpha\ k$.*

(**5.4.12**) For each α we have an equivalence relation on $\mathsf{M}(\alpha, \alpha)$ given by $\sim^{H_\alpha}$ as defined using 5.3.5.

Proposition 5.4.13. Fix α. Then $\sim^{H_\alpha}$ is the same relation as R_α .

Proof. If $a\ R_\alpha\ b$ then $p, q \in \alpha \sqcup \alpha$ path-connected in a if and only if p, q path-connected in b, for all p, q. So $a\ R_\alpha\ b$ implies $H_\alpha(a) = H_\alpha(b)$. But also $H_\alpha(a) = H_\alpha(b)$ says p, q path-connected in a if and only if path-connected in b, so this implies $a\ R_\alpha\ b$. □

(**5.4.14**) By (5.3.5) we have that $\sim^{H_\alpha}$ is an equivalence relation; and so then by (5.4.13) the relation R_α is an equivalence relation.

Fix α. Note that if we write $[a]$ for the R_α-class of a, then $[a]$ is the subset of magma elements that have the same image under H_α. In other words:
For $a \in \mathcal{P}([0, 1]^2)$, $a' \in [a]$ if and only if $H_\alpha(a') = H_\alpha(a)$.

5.4.3 Congruences

(**5.4.15**) In (2.3.21) we defined paths as parameterised by the interval $[0, 1]$. But note that for any proper interval $[u, v]$ a function $f : [u, v] \to \mathbb{R}^2$ can be re-parameterised to a path, for example by linearly rescaling the interval. If we refer to an f as above as a path we will mean the appropriate rescaling.

With this in mind we may call an unrescaled function such as f above a *pre-path*. (Cf. for example [CF63].)

(**5.4.16**) We say $\alpha \in \mathcal{P}([0, 1])$ is *finitary* if it has finitely many connected components.

Proposition 5.4.17. For each finitary α our equivalence R_α (as defined in 5.4.9) is a congruence on the magma $\mathsf{M}(\alpha,\alpha)$.

(Note – We will restrict to cases where α has finitely many connected components. Indeed let us start with the cases where α is finite.)

Proof (for finite case): Fix α. Consider $a, b \in \mathsf{M}(\alpha,\alpha)$; and consider $a' \in [a]$ and $b' \in [b]$. We need to investigate the paths between the points of $\alpha \sqcup \alpha$ in the magma product $a \boxtimes b$; and compare with $a' \boxtimes b'$.

Suppose $p, q \in \alpha \sqcup \alpha$ such that are path-connected in $a \boxtimes b$. We can divide into three kinds of cases. Case 1 is when p, q are on different edges (say p on the top edge); Case 2 is when p, q are both on the top edge; Case 3 is when both are on the bottom edge.

Let us first consider **Case** 1.

By assumption, there is a path that starts at q and ends at p on $a \boxtimes b$. Let J' be the $y = 1/2$ line of $a \boxtimes b$. Now let

$$\sigma : [0,1] \to a \boxtimes b \tag{5.10}$$

be such a path from q to p. We may assume that σ is non-self-crossing (injective), since given a self-crossing path the ‘loop’ between two crossing points can be removed to yield a ‘shorter’ path. Consider

$$C := \{s \in [0,1] \mid \sigma(s) \in J'\}$$

Observe that by assumption this set is finite, since it corresponds to a subset of α, which is finite here. By the Jordan curve Theorem (as in §2.4.2) it is also non-empty — our J' can be taken to be the part of a Jordan curve in $\mathbb{R}^2$ that intersects $[0,1]^2$ (it does not matter if the closure of J' to make J is taken either above or below - there are no paths outside of $[0,1]^2$ anyway). We can write $C = \{s_1, s_2, ..., s_l\}$ for some l — thus $\sigma(s_1) = (x_1, 1/2)$ (say) is the first point at which the path touches J' and so on.

It follows that there is a path from q to $(x_1, 1)$ in b. Note that this means there is a path between the same points in $b' \in [b]$. Of course, there may be many such paths, but for use later we pick one and call it σ'_1. Note that this also yields a 'shorter' path in $a' \boxtimes b'$ (for any $a' \in [a]$, or indeed any a').

Continuing along the path σ from $\sigma(s_1)$ to $\sigma(s_2)$ gives a path from $(x_1, 1/2)$ to $(x_2, 1/2)$. Since by construction, the interior of this path does not touch J' then by JCT (or IVT) it lies entirely either in a or b. Hence σ from $\sigma(s_1)$ to $\sigma(s_2)$ gives a path in either a or b.

Note that this implies a corresponding path, σ'_2 say, in either a' or b' respectively. And hence another one in $a' \boxtimes b'$ which composes with the image of σ'_1.

Now continuing along the path σ from $\sigma(s_i)$ to $\sigma(s_{i+1})$, with $i = 2, 3, ..., l-1$ (we will also understand s_{l+1} to be $s_{l+1} = 1$, and $s_0 = 0$, not in C) gives a path from $(x_i, 1/2)$ to $(x_{i+1}, 1/2)$. Since by construction, the interior of this path does not touch J' then by JCT (or IVT also as in §2.4.2) it lies entirely either in a or b. Hence σ from $\sigma(s_i)$ to $\sigma(s_{i+1})$ gives a path in either a or b. Indeed note that there is a last path, from $s = s_l$, that is in a.

Note that this implies a corresponding sequence of paths, σ'_i say, in either a' or b' respectively. And hence another sequence in $a' \boxtimes b'$ which composes in sequence. Note that this latter composite path in $a' \boxtimes b'$ starts at q and ends at p.

We have shown that given a path from q to p in $a \boxtimes b$ there is a path in $a' \boxtimes b'$. Since $a \in [a']$ and $b \in [b']$ a path in $a' \boxtimes b'$ yields a path in $a \boxtimes b$ by essentially the same argument. That is, the implication goes both ways. See Figures 5.3 and 5.4

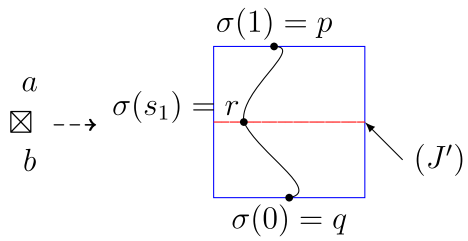


Figure 5.3: Case 1: Simple path where p, q are on different edges

(a) $a \boxtimes b$ (b) $a' \boxtimes b'$

Figure 5.4: Case 1: p, q are on different edges

Cases 2 and 3 are similar. In these cases, C could be empty. But the argument still works in essentially the same way.

Let us consider **case** 2.

By assumption, there is a path that starts at q and ends at p on $a \boxtimes b$. Then we use the previous setup except for how a path moves. So, a path start from q to $(x_1, 0)$ in a. Note that this means there is a path between the same points in $a' \in [a]$. Of course, there may be any such paths, but for us let us pick one and call it σ'_1. Note that this also yields a 'shorter' path in $a' \boxtimes b'$ (for any $b' \in [b]$, or indeed any b').

Continuing along the path σ from $\sigma(s_1)$ to $\sigma(s_2)$ gives a path from $(x_1, 1/2)$ to $(x_2, 1/2)$. Since by construction, the interior of this path does not touch J' then by JCT (or IVT) it lies entirely either in a or b. Hence σ from $\sigma(s_1)$ to $\sigma(s_2)$ gives a path in either a or b.

Note that this implies a corresponding path, σ'_2 say, in either a' or b' respectively. And hence another one in $a' \boxtimes b'$ which composes with the image of σ'_1.

Now continuing along the path σ from $\sigma(s_i)$ to $\sigma(s_{i+1})$, with $i = 2, 3, ..., l-1$ (we will also understand s_{l+1} to be $s_{l+1} = 1$, and $s_0 = 1$, not in C) gives a path from $(x_i, 1/2)$ to $(x_{i+1}, 1/2)$. Since by construction, the interior of this path does not touch J' then by JCT (or IVT also as in §2.4.2) it lies entirely either in a or b. Hence σ from $\sigma(s_i)$ to $\sigma(s_{i+1})$ gives a path in either a or b. Indeed note that there is a last path, from $s = s_l$, that is in a. Note that this implies a corresponding sequence of paths, σ'_i say, in either a' or b' respectively. And hence another sequence in $a' \boxtimes b'$ which composes in sequence. Note that this latter composite path in $a' \boxtimes b'$ starts at q and ends at p.

We have shown that given a path from q to p in $a \boxtimes b$ there is a path in $a' \boxtimes b'$. Since $a \in [a']$ and $b \in [b']$ a path in $a' \boxtimes b'$ yields a path in $a \boxtimes b$ by essentially the same argument. That is, the implication goes both ways. (see Figure 5.5)

(a) $a \boxtimes b$ (b) $a' \boxtimes b'$

Figure 5.5: Case 2: p, q are on the top edge

Since we have shown the implication for any pair of points p, q, we have $a' \boxtimes b' \in [a \boxtimes b]$ as required. □

(**5.4.18**) Note that we have repeatedly used the finiteness of α above. It is an interesting question if this condition can be relaxed. We will leave this question for a later work.

Lemma 5.4.19. *Let $a, b \in \mathcal{P}([0,1]^2)$. Consider q_x, p_x and $r_x \in [0,1]$. If σ is a path from $(r_x, 0)$ to $(p_x, 1)$ in a and γ is a path from $(q_x, 0)$ to $(r_x, 1)$ in b then there is a path from q to p in $a \boxtimes b$.*

Proof. Let us consider two continuous map $\sigma, \gamma : [0,1] \to [0,1]^2$ to be path from $(q_x, 0)$ to $(r_x, 1)$ and from $(r_x, 0)$ to $(p_x, 1)$. Then we want to show there is a path from q to p in $a \boxtimes b$. In order to get $a \boxtimes b$, we first stak (a, b) and then shrink it. By lemma 2.3.26 we have a path in $[0,1] \times [0,2]$ called h from $(q_x, 0)$ to $(p_x, 2)$ in $\mathsf{stack}(a, b)$ because the image of $(r_x, 1)$ in b is $(r_x, 1)$ in $\mathsf{stack}(a, b)$ and the image of $(r_x, 0)$ in a is $(r_x, 1)$ in $\mathsf{stack}(a, b)$. Then we apply shrink_2 by lemma 2.3.27. We get a path called g from $g(0)$ to $g(1)$ in $[0,1] \times [0,1]$. This is also a path in $\mathsf{shrink}_2(h([0,1]))$. So this is path from $(q_x, 0)$ to $(p_x, 1)$ in $a \boxtimes b$. □

(**5.4.20**) We will show in Sec.6 that in each case where the object α has order n then the quotient is isomorphic to the corresponding partition monoid of rank n.

Definition 5.4.21. *The magma* $\mathsf{M}(\alpha,\alpha)$ *quotient by* R_α *defined as*

$$\mathsf{M}(\alpha,\alpha)/R_\alpha = \{[a] : a \in \mathsf{M}(\alpha,\alpha)\}.$$

with an operation $\otimes : \mathsf{M}(\alpha,\alpha)/R_\alpha \times \mathsf{M}(\alpha,\alpha)/R_\alpha \to \mathsf{M}(\alpha,\alpha)/R_\alpha$ *given by*

$$[a] \otimes [b] = [a \boxtimes b]$$

Theorem 5.4.22. *The magma* $\mathsf{M}(\alpha,\alpha)$ *quotient by* R_α *is a monoid. The identity element is given by* $[\iota]$ *where* $\iota = \{(x,y) \mid x \in \alpha;\ y \in [0,1]\}$.

Proof. We need to show $\mathsf{M}(\alpha,\alpha)/R_\alpha$ is associative and it has an identity. Let $[a],[b]\text{and}[c] \in \mathsf{M}(\alpha,\alpha)/R_\alpha$.
We want to prove
I. Associative:

$$([a] \otimes [b]) \otimes [c] = [a] \otimes ([b] \otimes [c])$$

$$L.H.S = ([a] \otimes [b]) \otimes [c] = ([a \boxtimes b]) \otimes [c] = [(a \boxtimes b) \boxtimes c]$$

$$R.H.S = [a] \otimes ([b] \otimes [c]) = [a] \otimes ([b \boxtimes c]) = [a \boxtimes (b \boxtimes c)].$$

II. Identity: Let $[id],[a] \in \mathsf{M}(\alpha,\alpha)/R_\alpha$. Then

$$[id] \otimes [a] = [id \boxtimes a] = [a]$$

$$[a] \otimes [id] = [a \boxtimes id] = [a]$$

For I we can prove by noting that the drawing of the representative element on the LHS is the same as the drawing of the element on the RHS, except that the y direction is transformed ('stretched') by our f_m function from 5.1.18. But note that this rescaling does not affect the existence of paths from top to bottom.

And for II we can prove by noting that composition of any picture a with ι simply extends each path with $x \in \alpha$ from the image of the boundary of a to the boundary of the composite picture. □

(**5.4.23**) In 6.2 we will show that this monoid is isomorphic to a submonoid of the partition monoid.

Many interesting properties follow on from this (for example in representation theory), but we will leave these for a later work.

Chapter 6

Partitions from Passport photographs

In this section, we will recall the partition monoid, as defined in [Mar94]. We are going to use it, indirectly, to construct some equivalences on our magma - Which we hope will be congruences.

Note that we are going to use passport photos to represent partitions. But then afterwards we are going to use partitions to describe an equivalence on passport photos! So we will proceed carefully.

In section 6.1 we discuss how to related graphs and set partitions; and relate pictures and graphs, and hence to relate partitions and pictures. This allows us to give a 'graphical' construction of the partition monoid (i.e. a construction using elements of $\mathcal{P}([0,1]^2)$).

In section 6.2 we discuss how to relate to the magma $\mathfrak{M}$.

Section 6.1 is quite long, so let us give here a brief overview. We begin by experimenting with various concrete maps from the set of graphs on a given vertex set to $\mathcal{P}([0,1]^2)$. One of our maps $\mathcal{D}$ is shown, in 6.1.42, to be injective — meaning that we can recover the graph from the picture. Of course not every graph can be embedded in $[0,1]^2$, so no such $\mathcal{D}$ can be entirely straightforward. (For technical reasons our maps are maps to $\mathcal{P}([0,1]^2) \times \mathcal{P}([0,1]^2)$ rather than just $\mathcal{P}([0,1]^2)$, but the first component is relatively 'tame' and we will largely suppress it in this overview.) We introduce in 6.1.50 a subset C_2 of $\mathcal{P}([0,1]^2) \times \mathcal{P}([0,1]^2)$; and on this give a kind of inverse to $\mathcal{D}$ — a map $\mathcal{F}_2$ from ('good') pictures to graphs. Next we discuss maps between the set of relations on a given set and the set of graphs on that set - hence linking partitions to graphs. This allows us, in §6.1.3, to give the desired careful graphical realisation of the partition monoids.

6.1 Graphs and picture representations of set partitions

Let $n \in \mathbb{N}$, we define $\underline{n} = \{1, 2, ..., n\}$ and $\underline{n}' = \{1', 2', ..., n'\}$.
From 2.2.8 we see that $\mathfrak{Par}_{\underline{n} \cup \underline{n}'}$ is the set of all set partitions on $\underline{n} \cup \underline{n}'$.

Shortly we will show ways to represent a partition by a graph. First, we discuss graphs and *their* representations. (The construction is essentially standard, as for example in [KMY19], but we will need to be considerably more explicit and specific here.)

6.1.1 Graphs

(**6.1.1**) Recall that a *graph* (see e.g. [But00, Wil79, MCU]) maybe defined as a pair $G = (V, E)$, where V is a set (called the vertex set) and E is a set of unordered pairs of elements of V.

(**6.1.2**) For example, if the set $V = \{1, 2, 3, 1', 2', 3'\}$ and E might be $\{\{1, 2\}, \{1, 1'\}, \{3, 2'\}, \{1', 3'\}\}$. Together V and E are a graph G.

(**6.1.3**) *Remark.* There are several different definitions using the term 'graph' that are commonly used. All of them have sets of vertices and edges, but the precise notion of an edge varies from definition to definition. For example, an edge may be *directed* - corresponding to an *ordered* pair of vertices. Our definition above is convenient for our purpose.

(**6.1.4**) Fixing a vertex set V, we write $\mathbf{\Gamma}_V$ for the set of all graphs on this vertex set.

(**6.1.5**) For example

$$\mathbf{\Gamma}_{V=\{1,1'\}} \quad = \quad \{(V, \emptyset),\ (V, \{\{1, 1'\}\})\}$$

(**6.1.6**) A graph (V, E) is a *complete* graph if E contains every pair of vertices.

Example 6.1.7. *1. Let* $V = \{1, 1'\}$ *as above. Then* $(V, \{\{1, 1'\}\})$ *is complete graph.*

2. Let $V = \{1, 1', 1''\}$ *and* $E = \{\{1, 1'\}, \{1, 1''\}, \{1', 1''\}\}$*. Then* (V, E) *is a complete graph.*

3. Let V *as above, but* $E = \{\{1, 1'\}, \{1', 1''\}\}$*. Then* (V, E) *is not complete. Since* $\{1, 1''\} \notin E$

(**6.1.8**) Given a graph $G = (V, E)$, a *connected component* is a subset S of V such that for each $v, w \in V$ there is a chain of edges $\{v, v_1\}, \{v_1, v_2\}, ..., \{v_k, w\} \in E$; and that there is no chain from $v \in S$ to any vertex not in S.

Example 6.1.9. *Let* $V = \{1, 2, 3, 4, 1', 2', 3', 4'\}$ *and*
$E = \{\{1, 2\}, \{2, 3'\}, \{4.3'\}, \{1', 3\}, \{3, 2'\}, \{4, 4'\}\}$.
We have $\{1, 2\}, \{2, 3'\}, \{4.3'\}, \{4, 4'\} \in E$ *and* $\{1', 3\}, \{3, 2'\} \in E$ *are two chin of edges. Then* $S_1 = \{1, 2, 4, 3', 4'\}$ *and* $S_2 = \{3, 1', 2'\}$ *are subset of* V*. Therefore,* S_1, S_2 *are two connected components.*

(**6.1.10**) We write $\mathbf{\Gamma}$ for the collection of 'all' finite graphs. For us, this can be made into a set $\mathbf{\Gamma}^S$ by prescribing a set S from which vertices can be taken (although we do not fix this set here). Compare this with our notation $\mathbf{\Gamma}_V$ for the subset of graphs exactly on the vertex set V.

6.1.2 Picture representations of graphs

In this section, we will use some basic topology and geometry. They are not essential for our construction, but they are convenient. Some of our references are [Mar21], [Men90], [Arm13], [Moi13], [Bre93] and [RS05]. We will not add specific references in the text.

(**6.1.11**) We may represent a graph (such as the one in (6.1.2) above) as a diagram, such as the diagram in the figure 6.1.

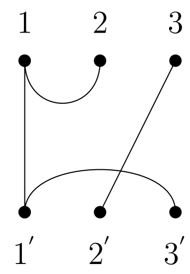


Figure 6.1: graph G

(**6.1.12**) To describe such diagrams mathematically, we need to make some explanations.

Firstly we consider a diagram as a subset of $[0,1]^2$. Of course this square is then embedded in the plane of the page (by choosing a coordinatisation of the page). Then for a graph $G = (V, E)$:

- each element $v \in V$ is denoted by a point $\mathcal{D}(v) \in [0,1]^2$. (We enlarge v to a small disk for visibility.)
- each element $\{v, w\} \in E$ is indicated by a line drawn between $\mathcal{D}(v)$ and $\mathcal{D}(w)$.

Remark. Instead of using $[0,1]^2$ we could use another space where lines can be embedded. We could use $[0,1]^3$ say. But a 2d space is convenient for inclusion on the page. We could use the disk $\mathbb{D} = \{(x,y) \mid x^2 + y^2 \leq 1\}$. But squares can be stacked, as we will use later for composition.

(**6.1.13**) The requirement for such a picture is that we can determine the graph just from the picture.

A potential problem with this is that two drawn lines may meet and then it may be unclear which path is the image of an edge. Indeed some graphs cannot be drawn on the plane without lines meeting.

There are various ways around this problem. Here we will arrange for lines only to meet 'transversally'.

(**6.1.14**) **Notation**. (See e.g. [KM20].) A pair of curves intersect *transversally* if both have tangents at the intersection point; and their tangents are not colinear.

Example 6.1.15. *Consider two lines* $y = 0$ *and* $y = x$*. The intersection point is* 0 *but a tangent for* $y = 0$ *is the line itself and the tangent of* $y = x$ *is the line itself. Therefore their tangent are not colinear. Then they intersect transversally.*

Lemma 6.1.16. *If two straight open lines in* $\mathbb{R}^d$ *intersect at a single point then they intersect transversely.*

(**6.1.17**) Before proof the previous lemma.Recall that $\mathbb{R}^d$ defined in 2.4.7.

Proof. Let $\alpha, \beta \in \mathbb{R}^d$ and $\mathbf{v} = \beta - \alpha \in \mathbb{R}^d$ be vector. Then the infinite line through α in the direction $\mathbf{v}$ is the set $L = \{\mathbf{v}t + \alpha \ \ |t \in \mathbb{R}\} = \{(v_1 t + \alpha_1, v_2 t + \alpha_2, ..., v_d t + \alpha_d)|t \in \mathbb{R}\}$. And our open line is the same but replacing $\mathbb{R}$ by some open interval (p, q) say.
So, the function $f(t)$ that represent L as $f(t) = (v_1 t_1 + \alpha_1, v_2 t_2 + \alpha_2, ..., v_d t_d + \alpha_d)$ and the gradient $\nabla f(t)$ (See e.g. [Mac86, p.170]), is $\nabla f(t) = (\frac{\partial}{\partial t_1}(v_1 t_1 + \alpha_1), \frac{\partial}{\partial t_2}(v_2 t_2 + \alpha_2), ..., \frac{\partial}{\partial t_d}(v_d t_d + \alpha_d)) = (v_1, v_2, ..., v_d) = \mathbf{v}$. Note that this is a constant vector independent of t.

Let $M = \{\mathbf{w}t + b \mid t \in \mathbb{R}\}$ for $\mathbf{w}, b \in \mathbb{R}^d$ be another infinite line in $\mathbb{R}^d$. So, the gradient of M is $\mathbf{w}$.

Now we show that if $\mathbf{v} = \mathbf{w}$ then either L and M never intersect or $L = M$.
Suppose $\mathbf{v} = \mathbf{w}$.
If $\alpha = b$ then $L = M$.
If $\alpha \neq b$, suppose L and M intersect at some points. Therefore there are some $t, t' \in \mathbb{R}$ such that $\mathbf{v}t + \alpha = \mathbf{v}t' + b$. So $b = \mathbf{v}(t - t') + \alpha = \mathbf{v}r + \alpha$ for some $r \in \mathbb{R}$. So, $M = \{\mathbf{v}t + \mathbf{v}r + \alpha \mid t \in \mathbb{R}\} = \{\mathbf{v}t + \alpha \mid t \in \mathbb{R}\} = L$. It follows that our open lines intersect like $(p, q) \cap (p', q')$. That is, they either do not intersect or they intersect in more than one point.

□

(**6.1.18**) We now consider some constructions of pictures of graphs, looking for ones that guarantee the transversal property for all lines.

Giving a construction means giving an explicit function

$$\mathcal{D} : \mathbf{\Gamma}_V \to \mathcal{P}([0, 1]^2)$$

for each vertex set of the type we have described. We can write $\mathcal{D}_i$ for some i to distinguish the various functions we try before fixing $\mathcal{D}$.

(**6.1.19**) Notation. For $A, B \in \mathbb{R}^2$,

$$[A, B] \;:=\; \{A + t(B - A) : 0 \leqslant t \leqslant 1\}$$

Note this gives a continuous path between A and B.

(**6.1.20**) Now we need to explain exactly how we will make a picture $\mathcal{D}(G) \subset [0,1]^2$ of a graph G with vertices $\underline{n} \cup \underline{n}'$ as above.

Consider the unit square frame $[0,1]^2$. Let vertex subset $\underline{n}$ be embedded on the top edge (ordered from left to right). Let $\underline{n}'$ vertices be placed on the bottom row of this square.

Consider a graph $G = (V, E)$ with $V = \underline{n} \cup \underline{n}'$. For each edge $e = \{v, w\}$, we define a path connecting vertices such that there exists a path or arc between v and w. Specifically we can try:

- an edge between two vertices i, j' (that is, on top and bottom) is represented by a direct connection $[A, B]$ where $A = \mathcal{D}(i)$ and $B = \mathcal{D}(j')$.
- a *curved* arc - such as a circle segment avoiding the frame of $[0,1]^2$ - represents a connection along the same row (same edge, either top or bottom).

(For now, we will not specify the centre of the circle used to construct such an arc.)

(**6.1.21**) Specifically we may represent the top vertex $i \in \underline{n}$ (recall $V = \underline{n} \cup \underline{n}'$) by a coordinate pair $(x_i, 1)$ where

$$x_i = \frac{i}{n+1} \in [0,1]$$

and the bottom vertex i' by $(x_{i'}, 0)$.
We aim next to describe the edge-path between vertices.

(**6.1.22**) We will use the interval $[A, B]$ in (6.1.19) to denote the path between $A = (x_i, 1)$ and $B = (x_j, 0)$.

(**6.1.23**) Perhaps the construction below works for some choice of curved arcs. This is not obviously easy here, so we consider some further "less geometrically complex" constructions. In particular we try piecewise linear constructions.

The reader may jump to 6.1.32 to pass over our reporting of some interesting experiments that failed.

(**6.1.24**) For example we can use a piecewise linear function (see (2.4.4)) to form an arc in the top edge that goes from $(x_i, 1)$ to $(x_{i+l}, 1)$ with evaluated breakpoint $(\frac{x_i+x_{i+l}}{2}, t)$ (at the breakpoint $a_2 = 1/2$, say), where we choose $t \neq \frac{1}{2}$ (we will fix the choice of t later).
Note that: the choice of t determines how higher or low the arc will be.

Example 6.1.25. *Consider* $(x_i, 1) = (1/5, 1)$, $(x_{i+l}, 1) = (2/5, 1)$ *and* $t = 7/10$ *then the evaluate breakpoint is* $(\frac{x_i+x_{i+l}}{2}, t) = (3/10, 7/10)$. *Then*

$$f(x) = \begin{cases} -3x + \frac{8}{5}, & \text{if} \quad \frac{1}{5} \leqslant x \leqslant \frac{3}{10} \\ 3x - \frac{1}{5}, & \text{if} \quad \frac{3}{10} < x \leqslant \frac{2}{5} \end{cases}$$

We represent this example by the figure 6.2 below.

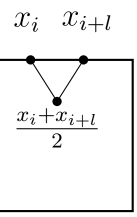


Figure 6.2: piecewise linear in 2-part to form the top arc

(**6.1.26**) Note that the above construction works for some choice of t. It does not has the transversal property we are looking for (6.1.14).

(**6.1.27**) To define the trial map $\mathcal{D}_2$ on a graph G explicitly, we use piecewise linear curves 2.4.4 in 2-part as:
• For an arc in the top edge from $(x_i, 1)$ to $(x_{i+l}, 1)$ we take the evaluated breakpoint at $(\frac{x_i+x_{i+l}}{2}, \frac{3}{4} - \frac{l}{10n})$.
• For an arc in the bottom edge from $(x_i, 0)$ to $(x_{i+l}, 0)$ we take the evaluated breakpoint at $(\frac{x_i+x_{i+l}}{2}, \frac{1}{4} + \frac{l}{10n})$.

Example 6.1.28. *Consider* $\underline{n} = \{1, 2, 3, 4\}$, $i = 1$, $l = 2$. *Then* $(x_i, 1) = (\frac{1}{5}, 1)$ *and* $(x_{i+l}, 1) = (\frac{3}{5}, 1)$. *The evaluate breakpoint is* $(\frac{4}{10}, \frac{7}{10})$.
The piecewise linear curve is

$$f(x) = \begin{cases} -\frac{3}{2}x + \frac{13}{10}, & \text{if} \quad \frac{1}{5} \leqslant x \leqslant \frac{4}{10} \\ \frac{3}{2}x + \frac{1}{10}, & \text{if} \quad \frac{4}{10} < x \leqslant \frac{3}{5}. \end{cases}$$

Note that this $\mathcal{D}_2$ (6.1.27) does not work because the images do not have the transversal property we are looking for (6.1.14)— see next example.

Example 6.1.29. *This example in fig.6.3 shows the possible failure of the transversal property (6.1.14) at the intersection point h.*
when $i < i' < i + l < i' + l'$.

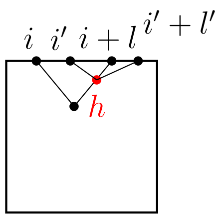


Figure 6.3: $i < i' < i + l < i' + l'$

(**6.1.30**) Next we try piecewise linear curves (see 2.4.4) in 3-part to define $\mathcal{D}_3(G)$ as follow:

- For an arc in the top edge from $(x_i, 1)$ to $(x_{i+l}, 1)$ with two evaluating breakpoints $(x_i, 1 - \frac{x_i}{1000n})$ and $(x_{i+l}, 1 - \frac{x_i}{1000n})$.
- For an arc in the bottom edge from $(x_i, 0)$ to $(x_{i+l}, 0)$. with evaluating breakpoints
$(x_i, \frac{x_i}{1000n})$ and $(x_{i+l}, \frac{x_i}{1000n})$.

Note that, this method does not work for the same reason above — see the example below.

Example 6.1.31. *In figure (6.4), we show for $\mathcal{D}_3$ the failure of the transversal property (6.1.14) at the intersection point h, when $i+l=i'$.*

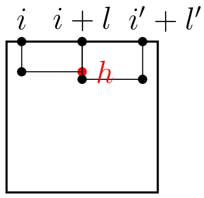


Figure 6.4: $i+l=i'$

(**6.1.32**) From now we use different piecewise linear curves (see 2.4.4) in 3-part to define $\mathcal{D}_4(G)$.

- For an arc in the top edge from $(x_i,1)$ to $(x_{i+l},1)$ with two evaluating breakpoints $(x_i+\frac{x_i}{1000n(2n+1)}, 1-\frac{x_i}{1000n})$ and $(x_{i+l}-\frac{x_i}{1000n(2n+1)}, 1-\frac{x_i}{1000n})$.
- For an arc in the bottom edge from $(x_i,0)$ to $(x_{i+l},0)$ with evaluating breakpoints $(x_i+\frac{x_i}{1000n(2n+1)}, \frac{x_i}{1000n})$ and $(x_{i+l}-\frac{x_i}{1000n(2n+1)}, \frac{x_i}{1000n})$.
- For a path from top to bottom use (6.1.22).

Here several choices have been made. For example, where the number 1000 is used this could be replaced by any sufficiently large number. However, this also follows the transversal property (6.1.14).

(**6.1.33**) Here we have various instances that are not done to scale, but are qualitatively correct:

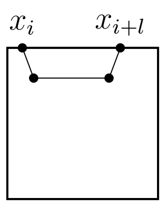


Figure 6.5: piecewise linear in 3-part to form a top arc.

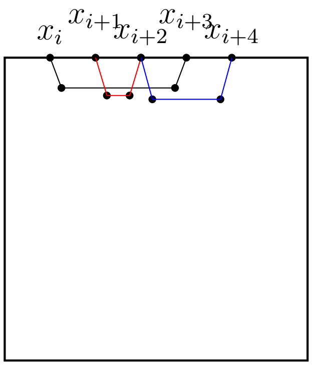


Figure 6.6: piecewise linear curves form 3-top arcs each arc in 3-part.

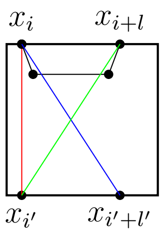


Figure 6.7: piecewise linear form a top arc in 3-part and 3-path from top to bottom.

Proposition 6.1.34. For any graph G the paths in $\mathcal{D}_4(G)$ intersect (excluding endpoints) at most pairwise transversally.

Proof. The cases to consider are:

1. two paths from top to top;
2. two paths from top to bottom;
3. one path from top to top with one path from top to bottom;
4. one path from top to top with one path from bottom to bottom.

Other cases can then be argued the same as these by symmetry.

$\star$ The case 1. from top to top: Suppose we have a curve between vertices i and $i + l$ on the top, with $l > 0$, and between i' and $i' + l'$ with $l' > 0$.

We will separate this case into a series of Lemmas, that starts at Lemma 6.1.35 and ends at Lemma 6.1.39. The proofs of all cases are completed by 6.1.40.

Lemma 6.1.35. *For any two curves on the top edge within $\mathcal{D}_4(G)$, if $i + l < i'$, then these curves do not touch.*

Proof. If $i+l<i'$ then the situation is as represented schematically by Figure (6.8)

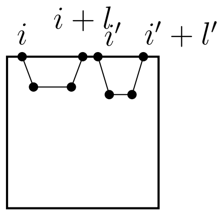


Figure 6.8: $i+l<i'$

In order to use the picture as part of a proof (i.e. to help organise a proof) we note that the 'drawn plane', as in the figure, can be a reliable organisational model of some aspects of plane geometry, at a schematic level. For example, we know that the range of x values of a curve is an interval, bounded by an upper and lower point. If the range of x values of one curve does not intersect the range for another curve 'by eye' in a figure in which the order (if not the precise value) of such points on the real line is manifestly respected, then it is safe to assume that they will not meet when the 'by eye' is replaced by a rigourous computation.

Thus we are guided to conclude here that the ranges of x values do not intersect, and hence that the curves do not touch. □

Lemma 6.1.36. *For any two curves on the top edge within $\mathcal{D}_4(G)$, if* $i<i'<i+l<i'+l'$ *, then these curves touch, but transversaly.*

Proof. If $i<i'<i+l<i'+l'$ then we claim that the curves must touch for $\mathcal{D}_4(G)$ (indeed for any $\mathcal{D}_i$ that follows the rules of (6.1.12), (6.1.14), (6.1.18)). This follows for example by the Jordan Curve Theorem (2.4.18), see e.g. [Moi13, p31].

For our specific $\mathcal{D}_4$ these curves have 2-breakpoints. For curve(1) the breakpoints between i and $i+l$ are $(i+\frac{i}{1000n(2n+1)}, 1-\frac{i}{1000n})$ and $((i+l)-\frac{x_i}{1000n(2n+1)}, 1-\frac{i}{1000n})$ respectively. Similarly, for curve(2) the breakpoints between i' and $i'+l'$ are $(i'+\frac{i'}{1000n(2n+1)}, 1-\frac{i'}{1000n})$ and $((i'+l')-\frac{i'}{1000n(2n+1)}, 1-\frac{i'}{1000n})$ respectively.
In more detail,

to prove these curves have their tangent. First, we look at curve (1) at its intersection point h. This curve has a horizontal line segment with a slope of zero. So at this point, the tangent is the line itself.

However, curve (2) has a line segment with a slope of $(-m)$. So at this point, the tangent of this curve is the line itself.

Thus, each curve has its own tangent at the intersection point$h$. Therefore, these tangents are distinct as in Figure (6.9).

This implies that the curves intersect transversely as in (6.1.14). $\square$

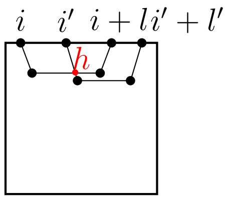


Figure 6.9: $i < i' < i + l < i' + l'$

Lemma 6.1.37. *For any two curves on the top edge within $\mathcal{D}_4(G)$, if $i + l = i'$, then the open curves do not touch.*

Proof. If $i + l = i'$ from our assumption in(6.1.34), we exclude the endpoint of these curves. Then the open curves do not touch. — see Figure(6.10). $\square$

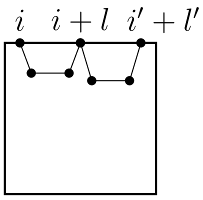


Figure 6.10: $i + l = i'$

Lemma 6.1.38. *For any two curves on the top edge within $\mathcal{D}_4(G)$, if $i < i' < i' + l' < i + l$, then these curves must touch, but transversaly.*

Proof. If $i < i' < i' + l' < i + l$ then the curves must touch see Figure 6.11. $\square$

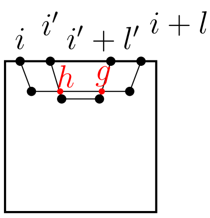


Figure 6.11: $i < i' < i' + l' < i + l$

Lemma 6.1.39. *For any two curves on the top edge within $\mathcal{D}_4(G)$, if $i+l=i'+l'$, then these curves must touch, but transversaly.*

Proof. If $i+l=i'+l'$ from our assumption, we exclude the endpoint so these curves do not touch.
However, if they do intersect at a point, each curve must have its tangent and these tangents must be distinct. Hence, this follow (6.1.14), — see Figure (6.12). □

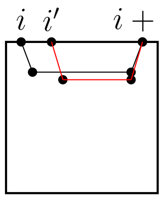


Figure 6.12: $i+l=i'+l'$

⋆ The case 2. Of paths from top to bottom is clear.
In this case, there are two situations:
I. Suppose we have two curves: the first curve from vertices i to j' (from top to bottom) and the second curve from $i+l$ to $j'+l'$ with $l, l' > 0$ (from top to bottom), these curves follow (6.1.22).

- If $[i, j'], [i+l, j'+l']$ with $l = l'$, these curves are connected eye-by-eye, which means they are parallel curves. Moreover, both curves are vertical lines with undefined slopes. So the tangents to these curves are the lines themselves. Therefore these curves do not cross — see Figure (6.13).

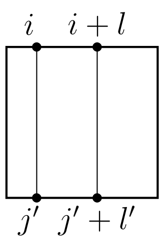


Figure 6.13: $[i, j'], [i+l, j'+l']$

II. Suppose we have a curve from vertices i to $j'+l'$ with $l' > 0$ (from top to bottom) and a curve from $i+l$ to j' with $l > 0$ (from top to bottom).

- If $\{\{i, j' + l'\}\}, \{\{i + l, j'\}\}$ then the curves must touch.

However, as both curves are line segments with distinct slopes (m, n respectively), their tangents at point h different. Thus, each curve has its tangent at the intersection point$h$. Therefore, these tangents are distinct.

This implies that the curves intersect transversely as following (6.1.14) and lemma (6.1.16)— see Figure (6.14).

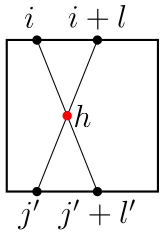


Figure 6.14: $\{\{i, j' + l'\}\}, \{\{i + l, j'\}\}$

⋆ The case 3. One path from top to top with one path from top to bottom.

Suppose we have a curve between vertices i and $i + l$ on the top edge, where $l > 0$, and another curve from top vertex i' to bottom vertex j follows (6.1.22).

- If $i + l < i'$ then the curves do not touch — see Figure (6.15)

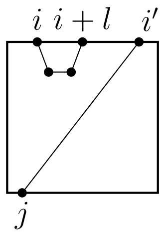


Figure 6.15: $i + l < i'$

- If $i + l = i'$ then the curves touch. However, from our assumption in 6.1.34, we exclude the endpoint of these curves – see Figure 6.16.

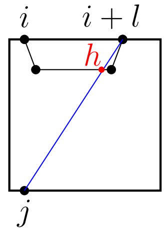


Figure 6.16: $i + l = i'$

(**6.1.40**) $\star$ The case 4. One path from top to top with one path from bottom to bottom.

Suppose we have a curve between vertices i and $i + l$ on the top, where $l > 0$, and between i' and $i' + l'$ on the bottom where $l' > 0$,, each with two breakpoints (as in 6.1.32).

- If $i < i + l$ on the top edge and $i' < i' + l'$ on the bottom edge, then the curves do not intersect. That is because the y-coordainate of the top breakpoints is $1 - \frac{i}{1000n}$ for the upper curve and the y-coordainate of the bottom breakpoints is $\frac{i'}{1000n}$ for the lower curve, as shown in (6.1.32) — see Figure (6.17).

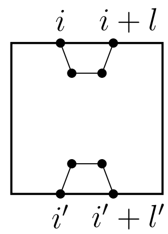


Figure 6.17: $i < i + l$ and $i' < i' + l'$

All cases for Proposition 6.1.34 are now covered. □

(**6.1.41**) **An alternative way to construct arcs for edges.** If rather than using $[0, 1]^2$ as our underlying space, with vertices on the top and bottom edge, we instead use the unit disk with vertices spaced around the boundary, then we can use the lines $[A, B]$ for every edge. Here it is clear that every pair of lines meets at most once, and then transversally in the interior of the disk.

We should not simply leave pictures on the disk, because stack-shrink does not work well with disks. But there exists a conformal (angle preserving and hence transversality preserving) map from the disk to the square. See for example: [Coo]

So we can make a picture on the disk, then transform to the square. This defines pictures $\mathcal{D}'(G)$ that are automatically transversal. In general, some vertices would end up on the vertical sides of the square, but this is easily avoided by clustering vertices near the top and bottom of the disk. (The conformal map is highly non-trivial, and not very nice to work with. But for us it is enough to have a construction that is specific and transversal.)

In fact there are many transversality preserving maps (including much simpler ones - e.g. proportional projection). So there is no shortage of alternative $\mathcal{D}$ functions. We specify one - an essentially arbitrary one - because we will often need concrete constructions later.

Proposition 6.1.42. The map $\mathcal{D} = \mathcal{D}_4$ is injective.

Proof. By construction, there exists a neighbourhood of each vertex point containing only the images of edges involving this vertex. By transversality, it is possible to follow each path from one vertex to the other without ambiguity. Thus $\mathcal{D}$ is invertible on its image. Suppose we have two graphs G_1 and G_2 in $\mathbf{\Gamma}_V$, then $\mathcal{D}(G_1) = \mathcal{D}(G_2) \implies G_1 = G_2$ based on the construction of the neighbourhood. Therefore, the map $\mathcal{D}$ is injective. □

(**6.1.43**) Note that we can formally apply stack-shrink to images, since they are in $\mathcal{P}([0,1]^2)$. Note however that the image of $\mathcal{D}_4$ is not closed under stack-shrink.

(**6.1.44**) Note that we can formally apply stack-shrink to images, since they are in $\mathcal{P}([0,1]^2)$. Note however that the image of $\mathcal{D}_4$ is not closed under stack-shrink.

(**6.1.45**) Now we need a kind of inverse to $\mathcal{D} = \mathcal{D}_4$. Note that $\mathcal{D}$ has

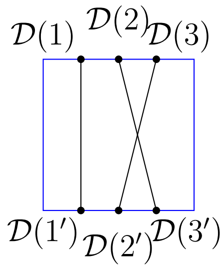
$$\mathcal{D} : \mathbf{\Gamma}_{\underline{n}\cup\underline{n}'} \to \mathcal{P}([0,1]^2)$$

The pseudo-inverse takes certain subsets of $[0,1]^2$ as input, and gives a graph as output. There are several possible choices for this function. We will consider a few of them. We will need some preparations.

(**6.1.46**) Function $\mathcal{F}_1$: Fix n. For each picture, we obtain a graph in $\mathbf{\Gamma}_{\underline{n}\cup\underline{n}'}$ as follows. For $X \in \mathcal{P}([0,1]^2)$ we have an edge in $\mathcal{F}_1(X)$ from v to w if there is a path in X from $\mathcal{D}(v)$ to $\mathcal{D}(w)$.

Remarks. Observe that every image is a union of complete graphs. Thus $\mathcal{F}_1$ is not a proper inverse on the image of $\mathcal{D}$. But note also that it is not even an inverse on the image of the subset of graphs that are disjoint unions of complete graphs. (These issues will no be a problem when we come to consider partitions.)

Example 6.1.47. *Consider $n = 3$ and*

$X =$  $\in \mathcal{P}([0,1]^2)$

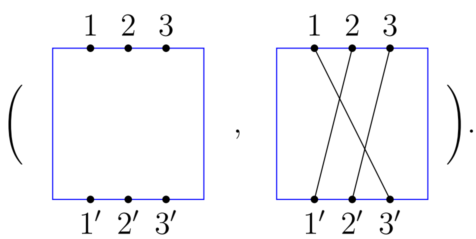
Then the edge set of graph $\mathcal{F}_1(X)$ is obtained from $\{\{\mathcal{D}(1),\mathcal{D}(1')\},\{\mathcal{D}(2),\mathcal{D}(3')\},\{\mathcal{D}(3),\mathcal{D}(2')\}\}$. That is, $\{\{1,1'\},\{2,3'\},\{3,2'\}\}$.

(**6.1.48**) For $n \in \mathbb{N}$, note that we can make a function

$$\mathcal{D}_{i2} : \mathbf{\Gamma}_{\underline{n}\cup\underline{n}'} \to \mathcal{P}([0,1]^2) \times \mathcal{P}([0,1]^2)$$

by adding a first subset recording only the $\mathcal{D}$-images of vertices. That is $\mathcal{D}_{i2}(G) = (a, b)$ where a is the set of vertex images and $b = \mathcal{D}(G)$ as considered before.

Example 6.1.49. *Let a set of vertex given by $V = \{1,2,3,1',2',3'\}$ and edges $E = \{\{1,3'\},\{2,1'\},\{3,2'\}\}$. Then graph $G = (V, E)$ gives*

$\mathcal{D}_{i2}(G) =$ 

(**6.1.50**) To define our next function, $\mathcal{F}_2$, we will need a subset of $\mathcal{P}([0,1]^2) \times \mathcal{P}([0,1]^2)$:

Consider a subset C_2 of $\mathcal{P}([0,1]^2) \times \mathcal{P}([0,1]^2)$ as follows.

For $(a,b) \in C_2$, the first subset a is finite, with $a \subseteq b$.

The subset b of $[0,1]^2$:

• has only finitely many points on the frame $\partial[0,1]^2$ (just on the top and bottom row), all belonging also to subset a.

• consist of the images of only finitely many smooth (or piecewise linear) embeddings of $[0,1]$, with these intersecting at most transversally or at their endpoints, which must be elements of a; and only intersecting a at endpoints.

We write $(b) \subseteq b$ for the union of images of $(0,1)$.

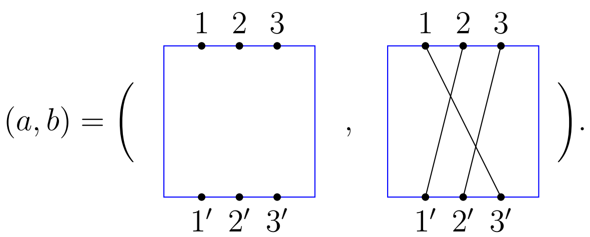
See 6.2.6 and 6.2.7 for some examples.

(**6.1.51**) The map $\mathcal{F}_2 : C_2 \to \mathbf{\Gamma}$ is then given as follows. Consider subsets $a, b \subset [0,1]^2$. Suppose b has n points on the top row and m points on the bottom row. This will give a graph with

$$V = \underline{n} \cup \underline{m}' \cup (a \setminus \partial[0,1]^2)$$

We label the top row points $1, 2, ..., n$ left to right; and similarly for $1, 2, .., m$ on the bottom. Note that the remaining points in a are interior, and left un-relabelled. We then have an edge $\{v, w\}$ whenever there is a continuous path in b from v to w not touching any other element of a, and this path only intersects other paths transversally or at an endpoint.

Example 6.1.52. *Let* $(a,b) \in C_2$ *given by*



Then $\mathcal{F}_2((a,b)) = (\{1,2,3,1',2',3'\} \ , \ \{\{1,3'\},\{2,1'\},\{3,2'\}\})$.

(**6.1.53**) **Claim**. *For any n the image of function $\mathcal{D}_{i2}$ is contained in C_2 as above.*

Proof. Note that each image $(a, b) \in \mathcal{P}([0,1]^2)^2$ has a finite (in the obvious sense) - in fact it has just the $2n$ vertices on the boundary. The interior contains only piecewise linear non-self-intersecting parts that intersect at most transversaly; and terminate at points of a. □

(**6.1.54**) **Claim**. *The set C_2 is closed under stack-shrink.* (This is in the sense that for (a, b) and (a', b') in C_2 we have that the product (c, d) is given by $d = b \boxtimes b'$ and c is given by $a \cup a'$, which contains the top vertices of a; the bottom vertices of a'; the interior vertices of both; and the not-necessarily-disjoint union of the bottom of a with the top of a'.)

Proof. Consider the product (c, d) as above, and consider in particular $d = b \boxtimes b'$. The two parts are disjoint except for the common boundary, so the linear and transversal properties will be upheld everywhere except possibly at the 'interior boundary' - the middle line. At this middle line paths from above or below either simply terminate at a vertex (now an interior vertex, coming from above or below), or, if vertices from above and below coincide, then paths may arrive from above and below. But they do so at the vertex, so transversality does not need to be checked here. □

(**6.1.55**) There is an example in 6.1.72 below.

(**6.1.56**) Since we have given a closed composition on C_2 - which composition we can also call $\boxtimes$, we have a magma $(C_2, \boxtimes)$.

Note that we can define a partition of C_2, essentially by $C_2(\alpha, \beta) = C_2 \cap \mathsf{M}(\alpha, \beta)$ (more formally this can be given in terms of the vertex sets of elements of C_2). Observe that we have

$$\boxtimes : C_2(\alpha, \beta) \times C_2(\beta, \gamma) \to C_2(\alpha, \gamma)$$

given by $m \times m' \mapsto m \boxtimes m'$.

Later we will discuss relations on C_2 that give congruences in this setting. These will be relations derived from $\mathcal{F}_2$, but using also maps to sets of partitions as we review next.

6.1.2.1 Graph representations of partitions

Recall from (2.2.10) that there is a bijection between the set $\mathbb{E}qu(S)$ of equivalence relations on a set S and the set $\mathfrak{Par}_S$ of partitions of S. Using this we can use relations to work with partitions. We can also connect graphs and relations. So we can connect graphs and partitions this way. We consider these connections next.

(**6.1.57**) Recall from (2.2.1) that a *relation* on a set S (to itself) is a subset of $S \times S$.

Given a relation $\rho \in \mathcal{P}(S \times S)$ we can define a graph $F_{rg}(\rho) = (V, E)$ in $\mathbf{\Gamma}_S$ by $\{a, b\} \in E$ if $(a, b) \in \rho$. Thus we have

$$F_{rg} : \mathcal{P}(S \times S) \to \mathbf{\Gamma}_S$$

Given a graph $G = (V, E)$ in $\mathbf{\Gamma}_S$ (so $V = S$) we can define a relation $\rho = F_{gr}(G)$ in $\mathcal{P}(S \times S)$ by $(a, b) \in \rho$ if $\{a, b\} \in E$. Thus we have

$$F_{gr} : \mathbf{\Gamma}_S \to \mathcal{P}(S \times S)$$

Let S be some set and $G \in \mathbf{\Gamma}_S$. Note that relation $F_{gr}(G)$ is always symmetric and never reflexive.

(**6.1.58**) A partition a belonging to $\mathfrak{Par}_{\underline{n} \cup \underline{n}'}$ may be represent by a graph $G_a = (V, E)$ as in (6.1.1), where $V = \underline{n} \cup \underline{n}'$ and E is the set of pairs such that $\{v, w\} \in E$ if v, w in the same part of a.

Example 6.1.59. *Consider* $a = \{\{1, 1'\}\} \in \mathfrak{Par}_{\underline{n} \cup \underline{n}'}$ *with* $n = 1$. *Here* G_a *has* $V = \{1, 1'\}$. *And* $E = \{\{1, 1'\}\}$.

Example 6.1.60. *Consider* $a = \{\{1, 1', 2'\}, \{2\}\} \in \mathfrak{Par}_{\underline{n} \cup \underline{n}'}$ *with* $n = 2$. *Here* $V = \{1, 1', 2, 2'\}$. *And* $E = \{\{1, 1'\}, \{1, 2'\}, \{1', 2'\}\}$.

(**6.1.61**) Remark. Note that G_a is always a union of complete graphs. So for $a \in \mathfrak{Par}_V$ this

$$G_{-} : \mathfrak{Par}_V \to \mathbf{\Gamma}_V$$

is not surjective on $\mathbf{\Gamma}_V$.

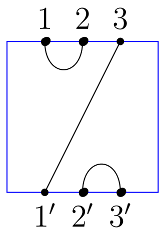
(**6.1.62**) Next we define a map

$$\mathbf{fl} : \mathbf{\Gamma}_V \to \mathfrak{Par}_V$$

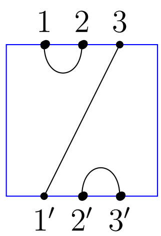
from graphs in $\mathbf{\Gamma}_V$ to partitions of V. (This map will be a kind of inverse to G_a.)

Given a graph $G \in \mathbf{\Gamma}_V$ then partition $\mathbf{fl}(G)$ is the partition of V according to the connected components in G.

Example 6.1.63. *Here and below we will, for convenience, use pictures to represent graphs (following enough of our rules to avoid ambiguity). Let graph $G \in \mathbf{\Gamma}_{\{1,2,3,1',2',3'\}}$ have a picture given by*

$G \mapsto$ 

Then

$\mathbf{fl}(G) = \mathbf{fl}\Big($  $\Big) = \{\{1,2\},\{3,1'\},\{2',3'\}\}.$

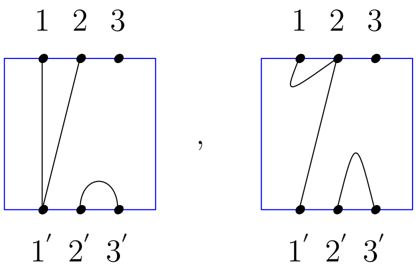
(**6.1.64**) For $W \subset V$ we also write $a|_W$ for the restriction of $a \in \mathfrak{Par}_V$ to $\mathfrak{Par}_W$ by deleting the elements not in W.

Example 6.1.65. *Consider a set of vertices given by $V = \{1,2,3,1',2',3'\}$ and the element of partition $\mathfrak{Par}_V$ given by $a = \{\{1,2\},\{1',2'\},\{3,3'\}\}$.*
Suppose $W = \{1,2,3,2'\}$ be subset of V. Then we have $a|_W = \{\{1,2\},\{3\},\{2'\}\}$ is an element of $\mathfrak{Par}_W$.

(**6.1.66**) We notice that the graph G and hence diagram $\mathcal{D}(G)$ is *not unique* for giving a set partition. That means we may find hundreds of diagrams representing the same partition. So we can say two diagrams are 'equivalent' if they have the same partition.

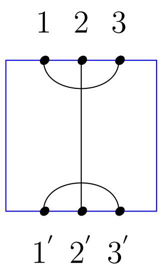
We also have to consider the resolution of the issues arising from a new example (which is in 6.1.68). This nice example illustrates the possible role of transversality in this setting.

Example 6.1.67. *Consider two graphs with* $V = \underline{3} \cup \underline{3}'$*:*



these graphs have the same partition which is $\{\{1, 2, 1'\}, \{3\}, \{2', 3'\}\} \in \mathfrak{P}_{\underline{3} \cup \underline{3}'}$.

Example 6.1.68. *Consider*



So the graph is hopefully clear, and the partition of this graph is $\mathfrak{P}_{\underline{3} \cup \underline{3}'} = \{\{1, 3\}, \{2, 2'\}, \{1', 3'\}\}$.

(**6.1.69**) Now we combine the G_a-map and the $\mathcal{D}$-map to make diagrams for partitions. For instance, consider the element $a \in \mathfrak{P}_{\underline{5} \cup \underline{5}'}$ given by

$$a = \{\{1, 1'\}, \{2, 3, 2'\}, \{4, 5, 5'\}, \{3', 4'\}\}.$$

Then the $\mathcal{D}_4(G_a)$ as shown in the figure 6.18 (here, as before, we do not take care to draw $\mathcal{D}_4$ with linear segments - just in order to proceed quickly).

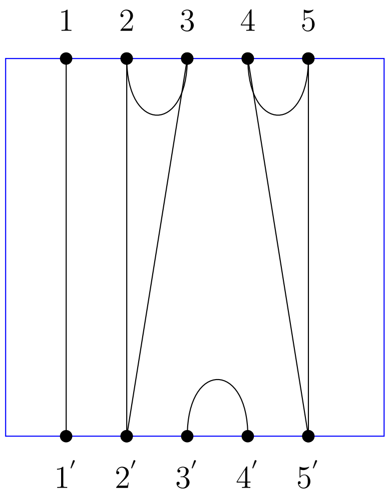


Figure 6.18: Sketch of $\mathcal{D}(G_a)$ for a partition $a \in \mathfrak{Par}_{\underline{5}\cup\underline{5}'}$

6.1.3 The partition monoid

In this section, we introduce the partition monoid in our construction (see also e.g. [Ban13, FL11, EG21]).

(**6.1.70**) For $n \in \mathbb{N}$ the *partition monoid* is given by the triple of

$$\mathfrak{PMo}_n \ = (\mathfrak{Par}_{\underline{n}\cup\underline{n}'}, \circ, \{\{1,1'\},\{2,2'\},...,\{n,n'\}\})$$

where

- $\mathfrak{Par}_{\underline{n}\cup\underline{n}'}$is the set of all set partitions on $\underline{n}\cup\underline{n}'$. So, in our construction, we will represent an element of $\mathfrak{Par}_{\underline{n}\cup\underline{n}'}$ ($a \in \mathfrak{Par}_{\underline{n}\cup\underline{n}'}$ by $\mathcal{D}(G_a)$).

- $\circ$ is a composition of the form

$$\circ : \mathfrak{Par}_{\underline{n}\cup\underline{n}'} \times \mathfrak{Par}_{\underline{n}\cup\underline{n}'} \to \mathfrak{Par}_{\underline{n}\cup\underline{n}'}.$$

  It is given as follows:

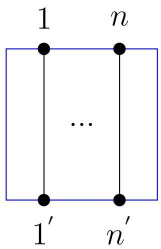
  To describe a composition of these elements. Let $a, b \in \mathfrak{Par}_{\underline{n}\cup\underline{n}'}$. Note here a, b represented by $\mathcal{D}(G_a), \mathcal{D}(G_b)$. Then we stack $\mathcal{D}(G_a)$ above $\mathcal{D}(G_b)$ by using

3.1.15 so that the lower vertices of $\mathcal{D}(G_a)$ are identified with the upper vertices of $\mathcal{D}(G_b)$. This result a new diagram consists of three parts. Those are a top row, a bottom row and the part where the boundary is in composition (we call it the middle row).

Then we apply the Shrink function 3.1.17 to this diagram which gives us altogether the composition between $\mathcal{D}(G_a)$ and $\mathcal{D}(G_b)$ of $\mathcal{D}(G_a) \boxtimes \mathcal{D}(G_b)$. Observe that this lies in C_2 by 6.1.53 and 6.1.54. We can then apply $\mathcal{F}_2$ from 6.1.51 to get the output graph. We apply **fl** from 6.1.62 to get a partition, and then restrict as in 6.1.64 to get the partition we need.
We will give examples shortly.

(**6.1.71**) It will be clear that $\circ$ is closed. But it may not be clear yet that this $\circ$ is associative. But this can be verified similar to earlier checks.

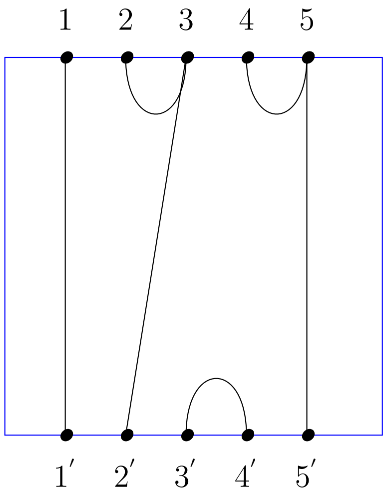
- The element $\{\{1,1'\},\{2,2'\},...,\{n,n'\}\}$ is given by diagram as



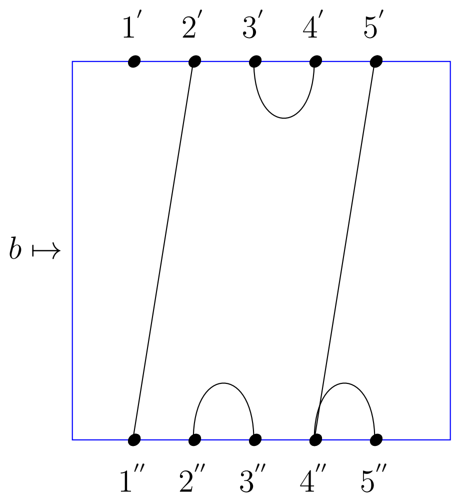
.

It will be clear from this that it is the identity of $\circ$.

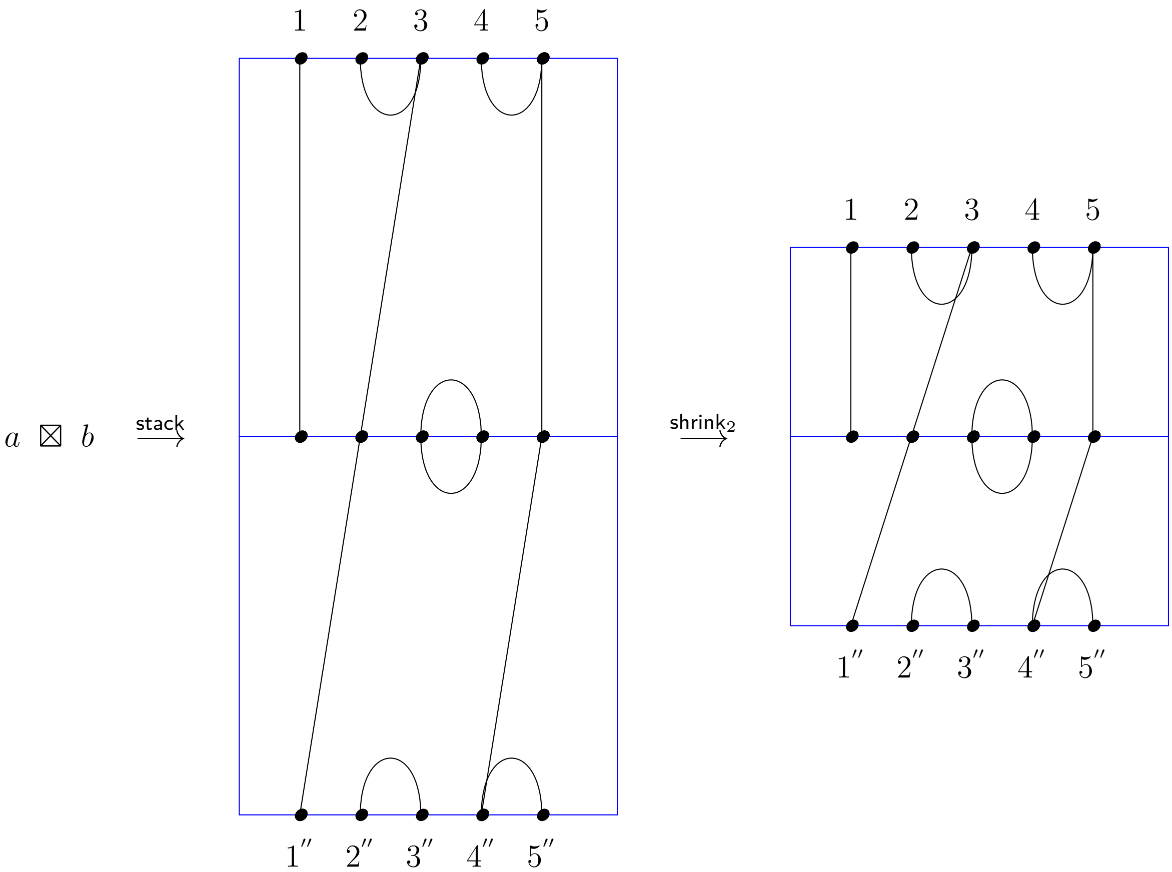
Example 6.1.72. *Consider* $a \in \mathfrak{Par}_{\underline{5} \cup \underline{5}'}$ *and* $b \in \mathfrak{Par}_{\underline{5}' \cup \underline{5}''}$. *Where,*

$a \mapsto$ 

,



Then the composition of the two partition diagram a and b



Therefore, the composition of the partitions is given by

$$a \boxtimes b \rightsquigarrow \{\{1\},\{2,3,1''\},\{4,5,4'',5''\},\{2'',3''\}\} \in \mathfrak{Par}_{\underline{5}\cup\underline{5}''}.$$

6.2 Relations on $\mathcal{P}([0,1]^2)$, derived from above 6.1.70

(**6.2.1**) Since we have maps $\mathcal{F}_2 : C_2 \to \Gamma$ (from 6.1.50) and $\mathbf{fl} : \Gamma \to \mathfrak{Par}$ (from 6.1.62 - by $\mathfrak{Par}$ we mean all the partitions taken together) we can define the composite map $\mathbf{fl} \circ \mathcal{F}_2$. We can define the corresponding equivalence relation $\sim^{\mathbf{fl}\circ\mathcal{F}_2}$ on C_2.

It follows essentially directly from 6.1.70 that this is a congruence on each of the magmas $(C_2(\alpha,\alpha), \boxtimes)$ as in 6.1.56. The quotient is thus given by the following.

Proposition 6.2.2. *The quotient magma* $(C_2(\alpha,\alpha)/\sim^{\mathbf{fl}\circ\mathcal{F}_2}, \boxtimes)$ *is isomorphic to the monoid* $\mathfrak{PMo}_{|\alpha|}$. □

(**6.2.3**) Indeed it is possible to lift this quotient so that the C_2 magmoid becomes the basic partition category (the partition category containing partition monoids rather than partition algebras). But we will not give details here.

(**6.2.4**) A more interesting question is the relationship between the monoid $(C_2(\alpha,\alpha)/\sim^{\mathbf{fl}\circ\mathcal{F}_2}, \boxtimes)$ above and the monoid $\mathsf{M}(\alpha,\alpha)/R_\alpha$ from 5.4.21.

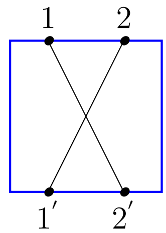
Proposition 6.2.5. In fact we have a monoid injection

$$\mathsf{M}(\alpha,\alpha)/R_\alpha \hookrightarrow (C_2(\alpha,\alpha)/\sim^{\mathbf{fl}\circ\mathcal{F}_2}, \boxtimes)$$

Proof. Observe that the partitions in the image of the map used to define R_α are precisely the partitions that can be drawn without any crossing paths. These are a subset of the partitions in the image of the map $\mathbf{fl} \circ \mathcal{F}_2$, and in fact the latter image contains all the partitions. □

In the next examples, we figure out how the above monoids look when graph classes belong to $C_2(\alpha,\alpha)/\sim^{\mathbf{fl}\circ\mathcal{F}_2}$ (as in 6.2.2) and $\mathsf{M}(\alpha,\alpha)/R_\alpha$ (as in 5.4.21). In particular, we explain why $\{\{1,2'\},\{2,1'\}\}$ cannot arise in $\mathsf{M}(\alpha,\alpha)/R_\alpha$.

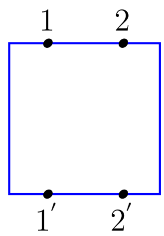
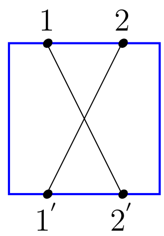
Example 6.2.6. *Consider the picture given by*

$a \dashrightarrow$  .

-recall the notation $\dashrightarrow$ *as explained in (5.3.3).*

Then, the different 'interpretations' that we have of this picture yield two different partitions as follows:

1. *When* $a \in C_2(\alpha, \alpha)$, *then*

   $a = (a_1, a_2) = \Big($  ,  $\Big)$.

   If $[a]_{\sim^{\mathbf{fl}\circ\mathcal{F}_2}} \in C_2(\alpha,\alpha)/\sim^{\mathbf{fl}\circ\mathcal{F}_2}$, *the element of* $\mathfrak{Par}$ *is* $\{\{1,2'\},\{2,1'\}\}$.

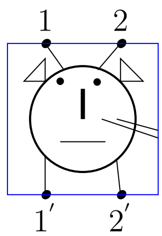
2. *Whereas if* $[a_2]_{R_\alpha} \in \mathsf{M}(\alpha,\alpha)/R_\alpha$, *then the element of* $\mathfrak{Par}$ *is* $\{\{1,2,1',2'\}\}$.

So far we have shown that $\{\{1,2'\},\{2,1'\}\}$ *does not arise from this picture* a *when we work in* $\mathsf{M}(\alpha,\alpha)/R_\alpha$. *To show that* $\{\{1,2'\},\{2,1'\}\}$ *cannot arise in* $\mathsf{M}(\alpha,\alpha)/R_\alpha$ *by* any *picture, we proceed as follows. First note that there must be a path from 1 to* $2'$. *But then every line from* 2 *to* $1'$ *must touch it by JCT 2.4.18. In detail, we are trying to make a partition with* 1 *in the same path as* $2'$, *there must be a path from* 1 *to* $2'$. *Then we try to make a path from* 2 *to* $1'$. *We do not say what path we make from* 1 *to* $2'$. *It can be any. But by the Jordan curve theorem, if there is a path from* 1 *to* $2'$, *then every path from* 2 *to* $1'$ *intersects that path. Now our relation says all those pictures, all of them are pictures of the* $\{\{1,2,1',2'\}\}$ *all together, partition. Because the two bits crossed. And so now there is a path from everything to everything. If we think of the line from* 1 *to* $2'$ *as being the fence and* *the sheep at 2 making the other, as it is trying to get to the water or grass in the place* $1'$*, then the Jordan curve theorem says it cannot, because the fence is there. It has to go through the fence.*

Recall from 6.1.50 that in $C_2(\alpha,\alpha)/\sim^{\mathbf{fl}\circ\mathcal{F}_2}$, we just allow a picture which has lines but in $\mathsf{M}(\alpha,\alpha)/R_\alpha$ can be any picture. In the next example we have a different picture. We see how the partition monoid image element looks like.

Example 6.2.7. *Consider the picture given by*

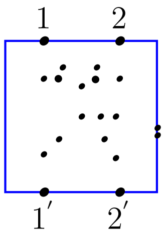
b - - →



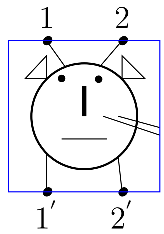
1. *When we look at this picture 'as' an element in* $C_2(\alpha,\alpha)$*, the first question is if we can interpret it as such an element. We need to figure out all the vertices and then the path or line between these vertices. So, here we have boundary vertices which are* $\{1,2,1',2'\}$ *and some other vertices inside, such as eyes. Note that if we have a 'trivalent point' (in the obvious sense), or a 'univalent' point, or indeed any odd-valent point, then this must be a vertex. So there are 4 vertices coming from trivalent points; and several from univalent points (the outer ends of the 'whiskers' are a challenge to interpret! - by the* C_2 *frame rule there can be no points on the vertical edges, so either this picture is NOT in* C_2 *or else the 'whiskers' end close to but not on the frame — so then there are two more interior vertices coming from these ends — note that it makes no difference to the final partition exactly where these vertices are). Note also that lines cannot touch other lines unless they have tangents, so we need a vertex for each of the ears, where they are touching the head.*

Then

$$b=(b_1,b_2)=\Bigg($$



,



$$\Bigg).$$

If $[b]_{\sim^{\mathbf{fl}\circ\mathcal{F}_2}} \in C_2(\alpha,\alpha)/\sim^{\mathbf{fl}\circ\mathcal{F}_2}$, *then the element of* $\mathfrak{Par}$ *is* $\{\{1,2,1',2'\}\}$.

2. *If* $[b_2]_{R_\alpha} \in \mathsf{M}(\alpha,\alpha)/R_\alpha$, *then the element of* $\mathfrak{Par}$ *is* $\{\{1,2,1',2'\}\}$.

Appendix A

Appendix: Background Notes

A.1 Basic definitions

(**A.1.1**) Note that a function $f : S \to T$ is a bijection whenever it is a injection and surjection.

Injection: if $\forall s_1, s_2 \in S, f(s_1) = f(s_2) \implies s_1 = s_2$.

Surjection: if $\forall \ t \in T, \exists \ s \in S$ such that $f(s) = t$.

(**A.1.2**) If $f : S \to T$ and $g : T \to W$ are function then the composition function is defined by $(g \circ f)(s) = g(f(s)) \forall \ s \in S$.

If f and g are injection, then $g \circ f : S \to W$ is injection .

If f and g are surjection, then $g \circ g$ is surjection.

Then $g \circ f$ is a bijection.

(**A.1.3**) A function $f : S \to T$ has an inverse function $g : T \to S$ if $\forall s \in S$ and $t \in T$

$$g \circ f(s) = 1_s \text{ and } f \circ g(t) = 1_t.$$

(**A.1.4**) A function has an inverse if and only if it is a bijection.

Proof. $\Rightarrow$ Suppose $f : S \to T$ has an inverse $f^{-1} : T \to S$ we will show f is bijection. First, we show f is a surjection. Suppose $t \in T$, Let $s = f^{-1}(t)$. Then $f(s) = f(f^{-1}(t)) = f \circ f^{-1}(t) = 1_t(t) = t$. So f is surjection.

Now show f is injective. Let $s_1, s_2 \in S$, $f(s_1) = f(s_2)$. We will show $s_1 = s_2$. Let

$t = f(s_1)$ and $s = f^{-1}(t)$. Then

$$s_2 = 1_S(s_2) = f^{-1} \circ f(s_2) = f^{-1}(f(s_2)) = f^{-1}(f(s_1)) = f^{-1}(t) = s.$$

at the same time

$$s_1 = 1_S(s_1) = f^{-1} \circ f(s_1) = f^{-1}(f(s_1)) = f^{-1}(t) = s.$$

Therefore $s_1 = s_2$, so f is bijection,

$\Leftarrow$ Suppose $f : S \to T$ is a bijection. We want to show there is an inverse. Define $g : T \to S$. Let $t \in T$, Since f is injective , there exist $s \in S$ such that $f(s) = t$. Let $f^{-1}(t) = s$. Since f is injective, this s is unique, so $f^{-1}$is well-defined. Now we must check that f^{-1} is the inverse of f. First, we will show that $f^{-1} \circ f = 1_S$. Let $s \in S$. Let $t = f(s)$. Then, by A.1.3, $f^{-1}(t) = s$. Then $f^{-1} \circ f(s) = f^{-1}(f(s)) = f^{-1}(t) = s$. Now we show that $f \circ f^{-1} = 1_T$. Let $t \in T$. Let $s = f^{-1}(t)$. Then, by A.1.3 , $f(s) = t$. Then $f \circ f^{-1}(t) = f(f^{-1}(t)) = f(s) = t$.

A.2 Homeomorphism group

Let A be topological space. Then a set of homeomorphisms from A to A given by $Homeo(A, A)$ is a group under function composition.

Proof. Let $f, g : A \to A \in Homeo(A, A)$ we want to show $Homeo(A, A)$ is group under composition.

1. closed under composition: Let $f : A \to A$ and $g : A \to A$. Since f, g are bijection by definition 2.3.17 then $f \circ g$ is a bijection. However, Since f, g are bijection Then f^{-1}, g^{-1} are bijection. So $f^{-1} \circ g^{-1}$ is a bijection. Also, since f, g are continuous by the definition 2.3.17 then $f \circ g$ is continuous by2.3.12. Since f, g since f^{-1}, g^{-1} are continuous by the definition 2.3.17 then $f^{-1} \circ g^{-1}$ is continuous by2.3.12. Therefore, $f \circ f$ is closed.

2, The identity: Let $id_A : A \to A$ be identity. Consider $f : A \to A$. Then $id_A \circ f = f$ and $f \circ id_A = f$.

3. Inverse: Let f be homeomorphism means it has continuous bijective and f^{-1}

is continuous. Then every bijection has the inverse. So, $f \in Homeo$, then $f^{-1} \in Homeo$.

4. Associativity: Let $f, g and h : A \to A \in Homeo$. We want to check $f \circ (g \circ h) = (f \circ g) \circ h$. Then $f \circ (g \circ h) = f \circ (g(h)) = f(g(h)) = (f(g))(h) = (f \circ g)(h) = (f \circ g) \circ h$.

A.3 Thompson Group F

The diagram of composing Thompson Group F.

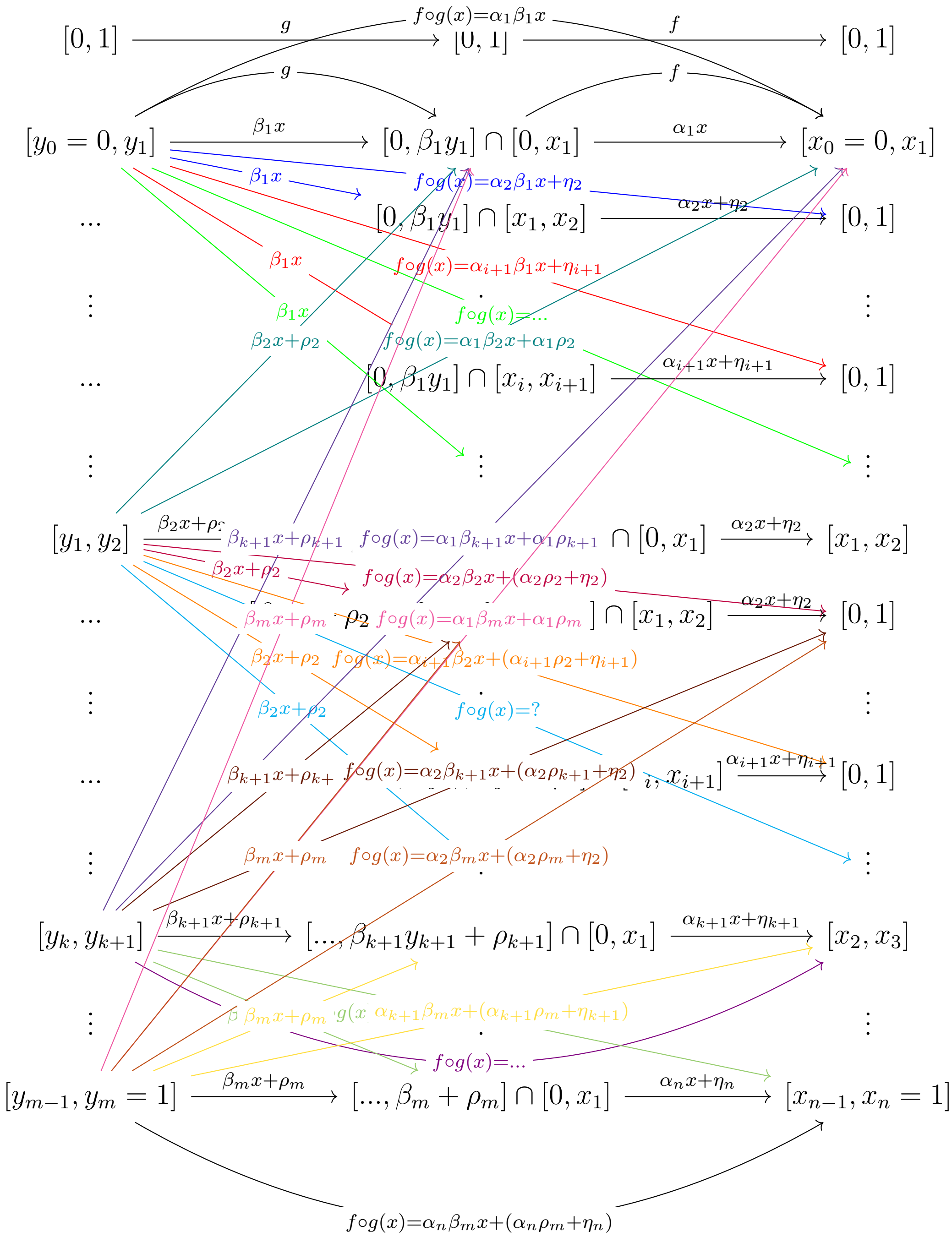


Figure A.1: The composition of Thompson Group F, $(f \circ g)$

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