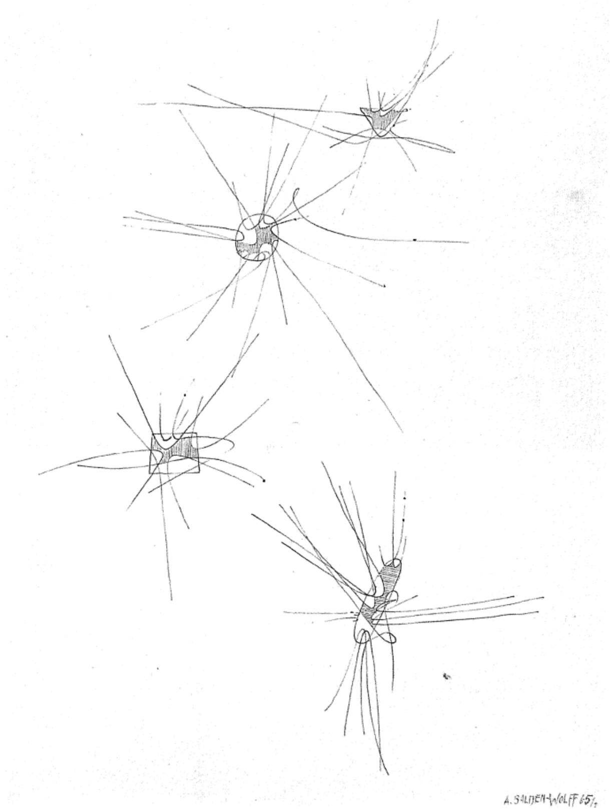
Perturbed quantum gravity bounces:

Approaching inhomogeneity in group field theory

Lisa Mickel



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Für meine Mama,
meine Oma,
und meinen Onkel.

'Das ist nicht leicht zu verstehen. Daß es für einen Menschen so wichtig sein kann, zu fragen und eine Antwort zu bekommen, daß es wichtiger ist als alles andere.

[...]

Man kann es nicht verstehen. Man muß nachgeben und sagen: Es ist so. Daß sie wissen müssen. Und zwar egal, was.'

- Peter Høeg in 'Der Plan von der Abschaffung des Dunkels'.

'It is not easy to understand. That it can be so important for a person to ask and receive an answer, that it is more important than anything else.

[...]

One cannot comprehend it. One has to give in and say: That is how it is. That they need to know. No matter what. '

Abstract

In this thesis we investigate cosmological perturbations around a quantum gravity bounce from a model-agnostic perspective as well as from the viewpoint of group field theory. Bounces provide a resolution to the Big Bang singularity and can be found in theories of quantum gravity, such as group field theory and loop quantum cosmology. They manifest in an effective Friedmann equation that is calculated directly from the quantum theory and encodes quantum corrections to high curvature regimes. Going beyond the dynamics of the homogeneous background spacetime to include also cosmological perturbations in a framework of quantum gravity is often a highly non-trivial task.

We investigate to what extent the separate universe framework can be used to extract information about the dynamics of gauge-invariant perturbations on super-horizon scales around a quantum gravity bounce. We find that for modified Friedmann equations similar to that of loop quantum cosmology, the conservation laws of general relativity continue to apply. More generally, however, these perturbations can become dynamical, as is the case in e.g. group field theory.

Independent from the cosmological setting, we make a more general proposal to reconstruct an effective metric from group field theory. Our proposition relies on symmetries that exist both in the classical as well as in the quantum framework. We promote the resulting Noether currents of the quantum theory to operators and identify their expectation values with their classical counterparts. Applying this procedure to the cosmological setting allows us to reconstruct expressions for effective metric perturbations from group field theory directly and opens up the possibility to investigate gauge-invariant quantities. While we can extract a flat Friedmann-Lemaître-Robertson-Walker metric at late times, the effective perturbations we recover for our choice of quantum state do no exhibit the dynamics of general relativity in the semiclassical regime.

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'Do what you want, and drink in the evening!'

- a dear friend motivating me through my PhD.

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I would also like to thank the rest of the CRAG research group, where a special mention goes to Elizabeth, for providing additional guidance as my advisor, and to Carsten, for the insightful discussions (and a presentation!) on cosmology. Not to be forgotten are the members of the G16 office crew who lightened up the often rainy days: Jonny, Abhinove, Siva, and George. There is a special group of people that made the time of my PhD a truly precious experience: Thank you to Ale for enduring me on a daily basis and being a pillar of steadfast support in times of academic or personal confusion. Thank you to Richard for the laughs, the sprinkles of chaos, and vibing on the same wavelength right from our first lunches in the common room to being my Escrima training partner for these years. And, thank you to Elsa for the ceaseless love and support that makes the joyful days jollier and turns even the gloomy ones into precious memories, for the wonderful evenings spent chatting away about the peculiarities of the world, and for instilling some of her unwavering confidence in me.

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Lastly, I am forever grateful to my family for accepting and supporting me on any path I may choose in life. *Danke, dass ich mich immer auf euch verlassen kann!*

Preface

This dissertation is presented for the degree of *Doctor of Philosophy* from the School of Mathematical and Physical Sciences at the University of Sheffield. The project has been supervised by Dr. Steffen Gielen.

The work presented in this thesis was carried out in collaboration with Dr. Steffen Gielen and is partly contained in the following two publications:

- Steffen Gielen and Lisa Mickel.
  'Gauge-invariant perturbations at a quantum gravity bounce.'
  Universe, vol. 9, p. 29, (2023). `arXiv:2211.04500`.

- Steffen Gielen and Lisa Mickel.
  'Reconstructing the metric in group field theory.'
  Class. Quant. Grav., vol. 41, p. 165002, (2024). `arXiv:2312.10016`.

These form the basis of chap. 5 and chap. 6, respectively, whereas a publication regarding the results of chap. 7 is still outstanding.

The thesis is divided in two parts. The first part introduces relevant background knowledge that the research results build on. It is structured as follows:

Chapter 1 gives an introduction to the core background concepts the rest of the thesis draws from. This includes a succinct overview of the concepts of general relativity, cosmology, and quantum theory. It establishes the need for a quantum theory of gravity and provides an overview of approaches.

Chapter 2 contains the details of group field theory, which is the theory of quantum gravity used in chap. 6 and chap. 7. It includes motivations for group field theory as well as the explicit construction used later in the thesis.

Chapter 3 provides an overview over concepts of cosmological perturbation theory, including the separate universe framework, which forms the basis of chap. 5. It also includes the perturbative analysis of the classical system the results of chap. 7 are compared to.

Chapter 4 combines previously discussed concepts to explain how bouncing cosmologies arise in group field theory and which approaches to including cosmological perturbations in the framework have been explored so far. This chapter also gives a short overview over bounces within the field of loop quantum cosmology.

The second part of the thesis is dedicated to research results and contains original work of the author. It consists of the following four chapters:

Chapter 5 explores how and whether the separate universe framework can be used to extract information about the evolution of long-wavelength perturbations around a quantum gravity induced bounce.

Chapter 6 contains a new proposal for group field theory operators that are constructed based on Noether currents of the theory. It argues that these operators can be used to extract an effective metric from group field theory.

Chapter 7 applies the proposal of chap. 6 to the cosmological setting. It compares the resulting dynamics for the background as well as for scalar perturbations to those of general relativity.

Chapter 8 concludes this thesis with a summary of our research results and points towards further research directions.

The appendices contain the following supplementary information:

Appendix A contains a more detailed definition of a manifold and its tangent vector space.

Appendix B gives explicit expressions for the gauge-invariant Einstein tensor for a general choice of lapse and shows how gauge-invariant forms of the energy-momentum tensor can be obtained.

Appendix C contains the calculations necessary for the analysis of scalar perturbations in a relational coordinate system, which is the topic of sec. 3.4.2 and does not form part of the standard literature. We compare the dynamics of group field theory perturbations in chap. 7 to these equations.

Appendix D is supplementary to the results of chap. 5 and was already included in the original publication [1].

Appendix E details the limit of the saddle-point approximation we use in chap. 7 to obtain analytical expressions for the group field theory metric. It forms part of the publication concerning the proposal of a group field theory metric [2].

Throughout this thesis we work in natural units

$$\hbar = c = 1 ,$$

where $\hbar$ denotes the reduced Planck constant and c the speed of light, leading to the following relation between units of mass, length, and time

$$[\text{mass}] = [\text{length}]^{-1} = [\text{time}]^{-1} .$$

We denote units in terms of length L. For instance, for the scale factor a, which has the unit of length, we write $[a] = L$.

We denote derivatives w.r.t. a time parameter in any coordinate system with a prime, i.e. $f' = \frac{\mathrm{d}f}{\mathrm{d}t}$, where t is an arbitrary choice of time coordinate.

The metric signature convention used throughout the thesis is $(-,+,+,+)$, also known as the 'East coast' convention. Greek letters are used to denote spacetime indices, e.g. $\mu = 0, 1, 2, 3$ and Latin letters are used for spatial indices, e.g. $i = 1, 2, 3$. A special set of indices is given by $A, B = 0, 1, 2, 3$, which denote field labels. We use $a, b = 1, 2, 3$ in some cases to denote spatial indices and in other cases to label 'spatial' fields. We do not use hats $\hat{\cdot}$ to denote operators in the quantum theory, the context should suffice to clarify whether a quantity should be understood as an operator.

Images for the title page of the thesis as well as title pages of the two parts are works by Annemarie Baden-Wolff and taken from "Mit Tusche Sticken - mit Fäden Zeichnen", *Kunstmuseum Dieselkraftwerk Cottbus*, (2015). All other image sources are contained in the respective captions.

Abbreviations

CMB Cosmic microwave background

EFE Einstein field equations

FLRW Friedmann-Lemaître-Robertson-Walker

GFT Group field theory

GR General relativity

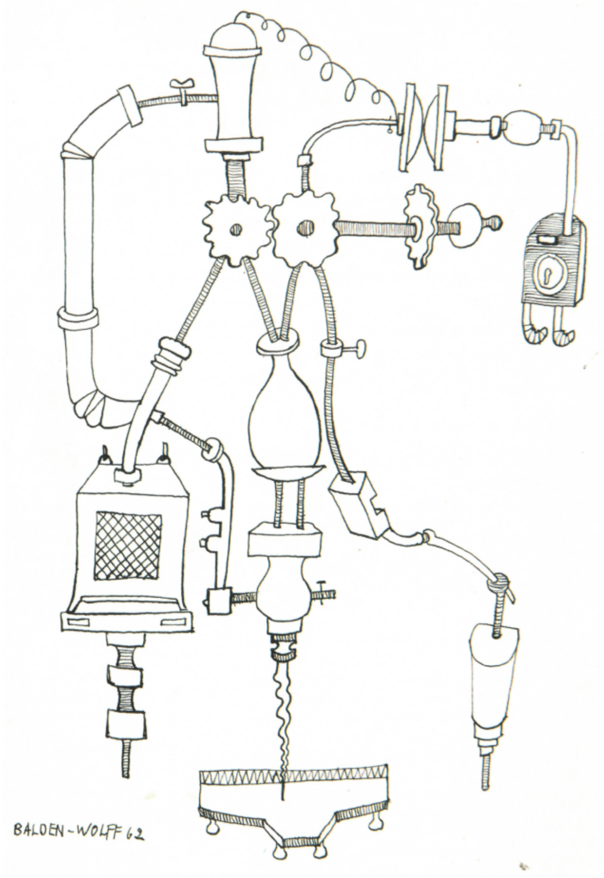
LQC Loop quantum cosmology

LQG Loop quantum gravity

Contents

Part I.

Foundations



Chapter 1.

Introduction: Gravity, the cosmos, and the quantum

'I wish to everyone a good trip in this strange but fascinating world.'

- Yvonne Choquet-Bruhat.

'Ich wünsche allen eine gute Reise in dieser wundersamen und zugleich faszinierenden Welt.'

Humanity's quest to understand the inner workings of the world has led to many discoveries and technological advancements. Some questions are arguably as old as humanity itself and, even though their answers might uncover practical wisdom during their pursuit, require no further justification to be answered other than the limitlessness of human curiosity. One of such questions is that of the beginning of our universe (a more concrete form of the naive 'Where do we come from?'). Clearly, such a question cannot be answered in a mere thesis, but we will focus on a small piece in the cosmic puzzle that will hopefully aid in bridging the gap between general relativity on the one, and the quantum world on the other side.

As gravity is a purely attractive force (to the best of our knowledge) it dominates at large scales despite its comparative weakness and therefore governs the evolution of the cosmos. The concept of what we now call gravity took on various forms throughout history, culminating in the formulation of Newtonian gravity in 1687 [3][1] In Newtonian gravity, masses attract each other, where the strength of the attraction weakens with

[1] See [4] for a translation.

increasing distance. This provided a satisfactory description of the gravitational force, until the measurement of Mercury's perihelion deviated from the Newtonian prediction [5][6, sec. 4.2]. Before this issue could be addressed directly, another crucial observation forced us to abandon the notion of absolute time and space: light travels at a constant speed, irrespective of the observer's motion [7]. This first led to the formulation of special relativity in 1905 [8][2], which excludes gravity from its domain of validity, and finally to the theory of general relativity in 1915 [10–12][3] - our current model for the force of gravity. The theory of general relativity could successfully resolve the discrepancy in the measurements of Mercury's perihelion [13] and accurately predicted the bending of light by massive objects, confirmed during a solar eclipse in 1919 [14].

At the heart of general relativity (GR) lie the Einstein equations, which relate the curvature of spacetime with the energy density of matter. Finding solutions to these equations is far from straightforward, still, rather remarkably, the first solution, namely the Schwarzschild solution [15, 16][4], was found shortly after Einstein's original publication. A solution that we still assume to describe our universe today, namely the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, was developed in the 1920s-1930s [19–23] and models the universe as homogeneous and isotropic, which should be accurate assumptions at large scales. General relativity has withstood many years of observational tests and we are yet to discover concrete experimental hints for its shortcomings. However, the theory predicts its own incompleteness through the presence of singularities that can e.g. be found at the centre of black holes, as well as at the beginning of our universe (Big Bang): curvature becomes infinite as one approaches a certain spacetime point, signalling the breakdown of the theory.

The field of cosmology has made tremendous progress in establishing the evolution of our universe. This relies on complementing the description of the homogeneous background by including small inhomogeneous perturbations, that ultimately give the seeds for structure formation. The cosmic microwave background (CMB) is our experimental window into the ancient universe. The CMB was emitted when electrons and protons combined to form atoms, making the universe transparent and allowing photons to travel uninhibited. Its release took place just 380 000 years after the Big Bang, where a common estimate for the age of the universe nowadays is around 13.8 billion years [24]. In addition to experimental evidence for isotropy in the early universe, the small fluctuations of the

[2]See [9] for a translation.
[3]See [9] for translations.
[4]For translations, see [17, 18].

CMB allow to extract a tremendous amount of information regarding the processes that took place before its release.

Simultaneously to the emergence of relativity forcing us to discard the notion of absoluteness, another revolutionising theory began to manifest itself: quantum mechanics. Quantum mechanics revealed that the world needs to be described as discrete and probabilistic at small scales. However, it does not incorporate the principle of relativity and therefore constitutes a remarkably accurate and useful theory at low energies only, similar to Newtonian gravity. Special relativity could successfully be incorporated into quantum mechanics in the form of quantum field theory (QFT) [25]. QFT describes three of the four fundamental forces of nature, namely the electromagnetic, the weak, and the strong force through the standard model of particle physics [26]. The standard model of particle physics is constantly tested at particle colliders such as the LHC and continues to be in alignment with measurements [27] despite its shortcomings [28].

The inclusion of gravity in the quantum description of the world remains elusive. This is not due to a lack of effort, in fact, there is a multitude of research activities occupied with the search for a theory of quantum gravity. One possible direction is to aim at including gravity in a quantum field theoretic description where the background remains fixed. Another is to follow the premise of general relativity that gravity is fundamentally different from the other forces and encoded in the structure of spacetime. This then amounts to finding a quantum description for the fabric of spacetime itself. Despite admirable mathematical and technical progress, a common struggle of proposals for a theory of quantum gravity is to make statements that are related to measurable reality. Perhaps, this is not surprising. Firstly, such formulations often exhibit a high degree of complexity, for instance, in the second avenue we mention above, a quantum theory has to include the geometric degrees of freedom of each spacetime point. Secondly, if one is to find signatures of quantum gravity, these should arise in extreme curvature regimes as corrections to GR, which are hardly accessible in laboratories on Earth.

The origin of the universe is then a particularly interesting testing ground for proposals of quantum gravity: in addition to high curvature extremes in the vicinity of the Big Bang, the symmetries of cosmological spacetimes allow to drastically restrict the amount of fundamental degrees of freedom that need to be included in the description of such a system. If the background cosmology can be accurately captured within a sector of a full quantum gravitational theory, one can attempt to incorporate also a characterisation of perturbations. A description of perturbations can help in connecting to cosmological observations, but is often difficult to include explicitly. The core concepts of cosmology are explained later in this chapter and the theory of cosmological perturbations within the framework of general relativity is the topic of chap. 3. We detail bouncing universes as they might appear in cosmological applications of quantum gravity theories, specifically in group field theory (GFT) and loop quantum cosmology (LQC), in chap. 4.

This brings us to the more concrete aim of this thesis, namely to explore avenues that reveal possible effects of quantum gravity on cosmological perturbations. The description of perturbations requires a notion of inhomogeneity in the quantum system. In approaches that are based on a quantum description of spacetime, localising perturbations is generally not straightforward and such a description needs to be established depending on the structures available in a given theory.

To circumvent this difficulty, we take a step towards finding a model independent description of modified perturbation equations in chap. 5, which encompasses the results of [1]. In this attempt, we focus on long-wavelength perturbations, such that we can work in the separate universe framework, where the details of inhomogeneities can approximately be neglected. We derive perturbation equations in the separate universe framework for a general form of a modified Friedmann equation. We then distinguish two cases of modifications: For the first, which is loop quantum cosmology-like, we can recover conservation laws of general relativity for gauge-invariant perturbations on super-horizon scales. The second case is more general and allows departures from the conservation laws of general relativity. To illustrate the possibility of dynamics of such perturbations we consider the example of a GFT bounce. We close with a comment on second order perturbation equations, and discuss how these do not provide further insights if one is limited to the strict separate universe picture. The results presented in chap. 5 draw on the separate universe framework introduced in chap. 3 and the concepts of quantum gravity bounces detailed in chap. 4.

In addition to the model agnostic approach, we investigate perturbations within the framework of GFT in detail. GFT is an approach to quantum gravity that takes seriously the background independence of general relativity and the quanta of the theory are to be interpreted as elementary spatial building blocks. Details of GFT are the topic of chap. 2. The application of GFT to the cosmological context leads to a bouncing universe and the theory has the potential for rich phenomenology, while retaining a clear connection to the fundamental theory.

The goal to extract cosmological perturbations from GFT will lead us to a more general proposal: reconstructing an entire metric from the quantum theory. The details of this construction are the topic of chap. 6 and have been published in [2]. We work in a relational coordinate system comprised of four massless scalar fields and make use of the fact that this setup introduces symmetries in the classical system as well as at the level of GFT. We propose an identification of the classically conserved currents with expectation values of novel GFT operators that are constructed from the translational symmetry of the scalar fields. We show that the classical conservation law holds also for the operators, irrespective of the choice of quantum state. The relational setup that we use for our construction is introduced in sec. 3.4.2 and has been employed in past studies of perturbation in GFT, as we discuss in chap. 4.

Having the potential to initiate serious progress in the field if proven to be physically useful, we consider the implications of this proposal in the cosmological context in chap. 7. This requires to make a choice of cosmological state, where we use Fock coherent states similar to previous GFT literature. We first study the dynamics of the homogeneous background mode and show that it can be consistently interpreted as a flat FLRW metric in the semiclassical regime and leads to a bounce. We then proceed to investigate the dynamics of inhomogeneous modes which are interpreted as cosmological perturbations. With the new operators, all perturbative quantities (including gauge-invariant ones) can be reconstructed from the effective metric, where we limit our study to scalar perturbations. We relate the dynamics of perturbations to the findings of the separate universe picture, to the classical scenario, and to previous findings in the context of GFT. We find that the dynamics we recover for perturbations from the quantum theory do not match GR at late times. In addition to the construction of chap. 6, this chapter uses concepts of GFT cosmology established in chap. 4 and compares to the classical perturbative analysis contained in sec. 3.4.2.

The remainder of this chapter elucidates some of the basics of the concepts mentioned above, whereas more specific details are explored in the rest of the first part of the thesis. We first provide a brief synopsis of the main components of GR and introduce the Einstein field equations (EFE) as well as the Hamiltonian formulation of GR in sec. 1.1. We then turn to the application of GR to cosmology based on the homogeneity and isotropy of the universe in sec. 1.2. We provide some details on the Big Bang singularity and, in order to provide a broader embedding of our later findings within the research field of cosmology, give an overview of the constituents and steps in the evolution of our universe. The core concepts of quantum mechanics and QFT are briefly described in sec. 1.3. Finally, in sec. 1.4 we turn to the topic of quantum gravity. We touch on three possible approaches to this problem, before explaining loop quantum gravity and spin foam models in more detail. We conclude in sec. 1.5.

1.1. General relativity

The theory of general relativity (GR) necessitates a paradigm shift of the very nature of gravity: in GR, gravity is no longer described as a force, but instead manifests as the curvature of spacetime. This curvature is caused by the presence of matter and in return determines its movement through spacetime. This relation is described by the Einstein equations. Here, 'matter' refers to anything that carries energy, and in particular, light is affected by gravity.

GR is formulated in the language of differential geometry and the two main principles of GR are the covariance and the equivalence principle. Roughly speaking, the equivalence principle states that gravity is a fictitious force. Indeed, any freely falling observer locally follows the laws of special relativity and gravity manifests itself only through the geometry of spacetime, which influences the observer's trajectory. Covariance can be summarised as the statement that 'all coordinate systems are equal', which refers to the fact that physical reality is invariant under the action of diffeomorphisms. General covariance leads to difficulties in defining observable quantities in GR.

We first give a short synopsis of the quantities necessary to describe spacetime and its properties in sec. 1.1.1. The Einstein field equations (EFE) give the relation between the geometry of spacetime and matter and are the topic of sec. 1.1.2. We also introduce the Hamiltonian formulation of GR in its original Arnowitt-Deser-Misner (ADM) formulation as well as in terms of Ashtekar-Barbero variables and briefly touch upon BF theories and the Plebanski formulation of GR. In sec. 1.1.3 we discuss one of the main peculiarities of GR: the occurrence of singularities. Furthermore, in sec. 1.1.4, we recount the issue of observables in GR and how this has been tackled through the notion of relational observables.

The content presented can be found in the many excellent introductory lectures and books such as [29–33]; we point to more specific literature where needed.

1.1.1. A manifold, a connection, and a metric

In more mathematical terms, spacetime is a 4-dimensional manifold, consisting of one time and three spatial dimensions, endowed with a Lorentzian metric.

We review the concept of a manifold and its structure and continue to introduce the metric as well as the connection, from which additional information about the manifold's properties, such as its curvature, can be extracted. The field of differential geometry is rich and wonderful and we refer the reader to the literature for specifics [29, 34, 35]. Our presentation relies on [29, 31].

Manifolds

Spacetime in general relativity is described by an $n = 4$ dimensional manifold. An $n-$dimensional manifold M is a topological space that can locally be described by a coordinate chart in $\mathbb{R}^n$.

Points p on a manifold can be characterised via coordinates in a specific coordinate system. We denote the coordinates of a point as x^μ with $\mu = 0, 1, 2, 3$. Furthermore, one can consider curves on a manifold, which are used to define the tangent vector space T_pM at each point. A vector $X \in T_pM$ can be decomposed as $X = X^\mu \frac{\partial}{\partial x^\mu}$, where $\frac{\partial}{\partial x^\mu}$ gives a basis of T_pM and X^μ denotes the components of the vector in this basis. Under a change of coordinate system $x^\mu \to \tilde{x}^\mu$ the vector X remains the same, but through a change of basis the components X^μ take on a different form in the new coordinate system. Explicitly, we have

$$X = X^\mu \frac{\partial}{\partial x^\mu} = \tilde{X}^\mu \frac{\partial}{\partial \tilde{x}^\mu}, \qquad \tilde{X}^\mu = X^\nu \frac{\partial \tilde{x}^\mu}{\partial x^\nu}. \tag{1.1}$$

We make the notion of a manifold and its tangent vector space more precise in app. A.

We can furthermore consider the dual of the tangent vector space T_p^*M, whose elements are 1-forms we denote as ω. Similar to vectors, the elements of the dual space are coordinate independent, but their components transform as follows:

$$\omega = \omega_\mu \mathrm{d}x^\mu = \tilde{\omega}_\mu \mathrm{d}\tilde{x}^\mu, \qquad \tilde{\omega}_\mu = \omega_\nu \frac{\partial x^\nu}{\partial \tilde{x}^\mu}\,, \tag{1.2}$$

where $\mathrm{d}x^\mu$ gives a basis of T_p^*M.

We can also define tensors, which are elements of the tensor product of an arbitrary number of tangent and dual vector spaces. The type (r,s) of a tensor is determined by the number r of vector spaces T_pM and the number s of dual vector spaces T_p^*M in the tensor product. For instance, a tensor T of type (2,1) is given by

$$T = T^{\mu\nu}{}_\lambda \frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu} \otimes \mathrm{d}x^\lambda\,, \qquad T \in T_pM \otimes T_pM \otimes T_p^*M\,. \tag{1.3}$$

The transformation properties of components of a tensor $T^{\alpha_1\ldots\alpha_n}{}_{\beta_1\ldots\beta_m}$ of type (n,m) are as follows

$$\tilde{T}^{\alpha_1\ldots\alpha_n}{}_{\beta_1\ldots\beta_m} = \frac{\partial \tilde{x}_1^\alpha}{\partial x^{\mu_1}} \cdots \frac{\partial \tilde{x}^{\alpha_n}}{\partial x^{\mu_n}} \frac{\partial x^{\nu_1}}{\partial \tilde{x}^{\beta_1}} \cdots \frac{\partial x^{\nu_m}}{\partial \tilde{x}^{\beta_m}}\, T^{\mu_1\ldots\mu_n}{}_{\nu_1\ldots\nu_m}. \tag{1.4}$$

According to the fundamental principle of covariance, coordinates are entirely unphysical and GR is invariant under coordinate transformations. As we saw, this does not imply that components of tensors do not change, however, tensorial equations are valid in any coordinate system.

Note that vectors and tensors live in their respective tangent vector spaces at a point p on M. As such, we cannot 'compare' tensors that belong to different T_pM and extra care needs to be taken when defining derivative operators. We introduce two notions of derivative, namely the Lie derivative and the covariant derivative, further below.

Metric

As already pointed out, the spacetime manifold needs to be equipped with a Lorentzian metric. This provides a notion of distances on a manifold which in turn allows one to define a causal structure. A metric $g_{\mu\nu}$ is a symmetric tensor of type (0,2), and can be denoted in the form of a 4×4 matrix or the so-called line element $\mathrm{d}s^2 := g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. It acts as a bilinear two-form on tangent vectors: $\langle X, Y \rangle = g_{\mu\nu}X^\mu Y^\nu$. If a metric has at least one eigenvalue with the value zero, we call it degenerate, otherwise non-degenerate. If all eigenvalues of a non-degenerate metric have the same sign, we call the metric Riemannian;

if one eigenvalue has a different sign, we call it Lorentzian. In this thesis we will use the convention that a 4-dimensional Lorentzian metric has one negative and three positive eigenvalues and set the metric signature to $(-,+,+,+)$.

Connection and derivatives

On a general manifold, there is no straightforward relation between vectors at different points p as each vector is an element of a different tangent vector space T_pM.[5] One then requires the notion of a connection to make sense of derivatives. General relativity uses the Levi-Civita connection, the components of which are called Christoffel symbols ${\Gamma^\lambda}_{\mu\nu}$ and are obtained from the metric as follows:

$$ {\Gamma^\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}\Big(\partial_\nu g_{\alpha\mu} + \partial_\mu g_{\alpha\nu} - \partial_\alpha g_{\mu\nu}\Big). \tag{1.5} $$

From the connection we can define the covariant derivative ∇_μ of a vector X or a dual vector ω, which fulfills the standard tensor transformation properties

$$ (\nabla_\mu X)^\nu = \partial_\mu X^\nu + {\Gamma^\nu}_{\mu\lambda} X^\lambda \,, \qquad (\nabla_\mu \omega)_\nu = \partial_\mu \omega_\nu - {\Gamma^\lambda}_{\mu\nu}\omega_\lambda \,. \tag{1.6} $$

This generalises to arbitrary tensors of type (k,l) as

$$ (\nabla_\mu T)^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_l} = \partial_\mu T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_l} + \sum_{i=1}^{k} {\Gamma^{\alpha_i}}_{\mu\lambda} T^{\alpha_1\ldots\lambda\ldots\alpha_k}{}_{\beta_1\ldots\beta_l} - \sum_{j=1}^{l} {\Gamma^{\lambda}}_{\mu\beta_j} T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\lambda\ldots\beta_l} \,, \tag{1.7} $$

where λ in the first sum is in the place of the i^{th} upper index, and λ in the second sum takes the place of the j^{th} lower index. The Levi-Civita connection is the unique metric compatible $\nabla_\lambda g_{\mu\nu} = 0$, torsion-free $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ connection.

Another notion of a derivative is that of the Lie derivative. The Lie derivative $\pounds$ of a (k,l) tensor T along a vector field v is given by

$$ \pounds_v T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_l} = v^\lambda \nabla_\lambda T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\beta_l} - \sum_{i=1}^{k} T^{\alpha_1\ldots\lambda\ldots\alpha_k}{}_{\beta_1\ldots\beta_l} \nabla_\lambda v^{\alpha_i} + \sum_{j=1}^{l} T^{\alpha_1\ldots\alpha_k}{}_{\beta_1\ldots\lambda\ldots\beta_l} \nabla_{\beta_j} v^\lambda \,. \tag{1.8} $$

[5] In special cases, e.g. for the Minkowski metric $\eta_{\mu\nu}$, the T_pM at each point can be identical, such that all vectors belong to the same vector space.

Curvature

The curvature of the spacetime manifold is encoded in the Riemann curvature tensor ${R^\lambda}_{\alpha\mu\nu}$, defined by

$$-{R^\lambda}_{\alpha\mu\nu}X^\alpha = \nabla_\mu\nabla_\nu X^\lambda - \nabla_\nu\nabla_\mu X^\lambda \,. \tag{1.9}$$

The components of the Riemann curvature tensor can explicitly be written as:

$${R^\lambda}_{\alpha\mu\nu} = \partial_\mu {\Gamma^\lambda}_{\alpha\nu} - \partial_\nu {\Gamma^\lambda}_{\mu\alpha} + {\Gamma^\lambda}_{\mu\sigma}{\Gamma^\sigma}_{\nu\alpha} - {\Gamma^\lambda}_{\nu\sigma}{\Gamma^\sigma}_{\mu\alpha}. \tag{1.10}$$

One also defines the Ricci tensor $R_{\mu\nu}$, the Ricci scalar R (also referred to as scalar curvature), and the Kretschmann scalar K as:

$$R_{\mu\nu} = {R^\alpha}_{\mu\alpha\nu}, \qquad R = g^{\mu\nu}R_{\mu\nu}, \qquad K = R^{\lambda\mu\alpha\nu}R_{\lambda\mu\alpha\nu}. \tag{1.11}$$

Distances between points and causality

The distance between two spacetime points or 'events' Q, P, corresponds to the length of the shortest curve connecting them. The length of a curve $x^\mu(\lambda)$ parametrised by λ between P and Q is given by the integral

$$\ell = \int_P^Q \sqrt{\pm g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}d\lambda \,, \tag{1.12}$$

where the dot denotes differentiation with respect to the curve parameter λ and the sign is chosen such that the term under the square root is positive. A curve that minimises (1.12) is called a geodesic and satisfies the following differential equation

$$\ddot{x}^\mu + {\Gamma^\mu}_{\sigma\nu}\dot{x}^\sigma\dot{x}^\nu = 0 \,, \tag{1.13}$$

if λ is an affine parameter. Inertial observers, also referred to as 'free' or 'test' particles, move through spacetime along geodesics.

The distance between two spacetime events contains information about their causal relation. There are three possibilities, depending on the the nature of the geodesic x^μ connecting them:

- Time-like: $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu < 0$.
  Two events are causally connected and any information that is exchanged travels below the speed of light.

- Light-like: $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$.
  Two events are causally connected and information connecting the two events travels at the speed of light.

- Space-like: $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu > 0$
  Two events are not causally connected. Only faster than light travel could lead to information exchange between them.

1.1.2. The relation of curvature and matter

The dynamics of general relativity are governed by the Einstein field equations (EFE):

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = \kappa T_{\mu\nu}, \tag{1.14}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the so-called Einstein tensor, $\Lambda \in \mathbb{R}$ denotes the cosmological constant, and $\kappa = 8\pi G$, where G is Newton's constant. The left-hand side encodes the geometry of the spacetime manifold and $T_{\mu\nu}$ encapsulates the energy-momentum tensor, whose explicit form is determined by the matter content of the spacetime under consideration. From the Bianchi identities it follows that $\nabla_\nu G^{\mu\nu} = 0$, such that $\nabla_\nu T^{\mu\nu} = 0$ follows from the Einstein equations, see e.g. [29, chap. 3]. Due to the symmetry of the Einstein tensor, which only has 10 independent components, and the Bianchi identity, which imposes four additional constraints, the EFE give six independent second order differential equations. Solving the EFE for the metric is an extremely complicated task and a limited number of exact solutions exist. Examples of such exact solutions are

- Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime [19–23]: This is the homogeneous and isotropic spacetime assumed to describe our universe. We will delve deeper into its specifics in sec. 1.2.1.

- Bianchi models [36, 37]: These are generalisations of the FLRW metric that relax the assumption of isotropy. While we have reason to believe that our universe is indeed isotropic, these solutions are still of interest in cosmological studies [38].

- Schwarzschild black hole [15, 16][6]: This is a static, spherically symmetric vacuum solution that describes the exterior of a black hole. While black holes we observe

[6] See [17, 18] for a translation.

are usually of the Kerr-type, the Schwarzschild metric can still be a good enough approximation to characterise certain observations of black holes [39].

- Reissner-Nordström black holes [40, 41]: These correspond to black holes that carry a charge. While they are to the best of current knowledge not realised in nature, they are interesting objects to study from a theoretical perspective.

- Kerr-Newman black holes [42, 43]: These describe rotating black holes (and would also allow for black holes with charge). As such, they form the most general class of black hole metrics and contain the former two. They give the most accurate description of astrophysical black holes (which are rotating but uncharged).

The EFE, like several breakthroughs in physics, came about as an excellent physically informed guess [12]. In hindsight, one can identify an action that encapsulates the geometric content, namely the Einstein-Hilbert action S_{EH}. The action of the full system, where n denotes the dimension of the spacetime manifold, is then given as a sum of the gravitational and matter part

$$S = \frac{1}{2\kappa} S_{\text{EH}} + S_{\text{matter}} \qquad \text{with} \qquad S_{\text{EH}} = \int \mathrm{d}^n x \sqrt{-g}(R - 2\Lambda) \tag{1.15}$$

and the EFE can be derived by varying w.r.t. the metric tensor $g_{\mu\nu}$. In particular, the energy-momentum tensor is defined as

$$T_{\mu\nu} := \frac{-2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g_{\mu\nu}} \,. \tag{1.16}$$

When considering paths to quantum gravity that aim for a direct quantisation of geometry, it has proven useful to consider alternative formulations of GR, which are more amenable to quantisation. In particular, loop quantum gravity (LQG) originates from a quantisation of the Hamiltonian Ashtekar-Barbero formulation of GR, and spin foam models derive from a quantisation of BF theories with additional constraints (the so-called Plebanski formulation). As we will motivate the construction of group field theory (GFT) from LQG and spin foam models in chap. 2, we give some insight into the Ashtekar-Barbero formulation, which is constructed from the Arnowitt-Deser-Misner (ADM) Hamiltonian formulation of GR, and BF theories in the following. Details of LQG and spin foam models are discussed in sec. 1.4.2.

Hamiltonian formulation: Arnowitt-Deser-Misner and Ashtekar-Barbero formalism

We first introduce the Arnowitt-Deser-Misner (ADM) Hamiltonian, which was the first Hamiltonian framework established for GR [44] and is a direct reformulation of the Einstein-Hilbert action. We will refrain from diving into technical details and focus on the major ideas; our explanations are inspired by [45, 46] and we refer the interested reader also to [29, app. E] for more details. Subsequently, we introduce Ashtekar-Barbero variables, which rely on introducing an internal reference frame.

The starting point of the Hamiltonian formalism is to carry out a 3+1 decomposition of the spacetime manifold M into 3-dimensional spatial hypersurfaces Σ of equal time, endowed with the spatial metric q_{ab}, $(a, b = 1, 2, 3)$, and a time direction. We thus have $M = \Sigma \times \mathbb{R}$. The time evolution can be further decomposed into the lapse N and shift vector N^a, which intuitively give the components of time translation orthogonal and parallel to the hypersurface, respectively. This 3+1 split is illustrated in fig. 1.1. Covariance is a fundamental principle of GR and such a split might seem counter-intuitive at first. One can however show that all foliations of spacetime are equivalent and lead to the same physical predictions, thus, covariance is simply hidden. The spacetime metric $g_{\mu\nu}$ decomposes as follows

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^a N_a & N_a \\ N_a & q_{ab} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1/N^2 & N^a/N^2 \\ N^a/N^2 & q^{ab} - N^a N^b/N^2 \end{pmatrix}. \tag{1.17}$$

The canonical variables in the ADM formalism are the spatial metric q_{ab} and its conjugate momentum $\pi^{ab} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{ab}}$, which satisfy the Poisson brackets $\{q_{ab}(x), \pi^{cd}(y)\} = \delta^c_{(a}\delta^d_{b)}\delta^{(3)}(x,y)$. The ADM Hamiltonian H is obtained from a Legendre transform and contains the so-called Hamiltonian and diffeomorphism constraint denoted $\mathcal{H}$ and $\mathcal{H}_a$, respectively

$$H = \mathcal{H}[N] + \mathcal{H}_a[N^a] = \int_\Sigma d^3x \left(N_a \mathcal{H}^a + N\mathcal{H} \right), \tag{1.18}$$

where the lapse and shift appear as Lagrange multipliers. GR is thus a fully constrained system determined by the first class constraints $\mathcal{H}$ and $\mathcal{H}^a$. Here we have focused only on the geometrical Hamiltonian; in general, the matter action is of course non-zero and gives additional terms in the Hamiltonian and diffeomorphism constraints.

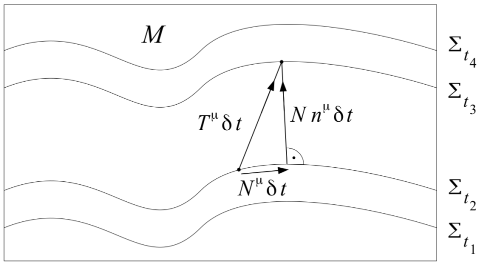


Figure 1.1.: In the Hamiltonian formulation of GR the spacetime manifold M is foliated into 3-dimensional hypersurfaces Σ. The time vector T^μ interpolating between the spatial hypersurfaces can be decomposed into a component normal to the Σ, denoted as the lapse N, and a component that lies in the surface called the shift vector N^a. Image taken from [46].

We now proceed to introduce the Ashtekar-Barbero formulation of GR [46–50]. The advantages of this formulation when looking for a quantum theory of gravity are twofold: firstly, by introducing a connection variable one achieves a Yang-Mills like formulation of gravity; secondly, the Hamiltonian constraint becomes a polynomial in the canonical variables, whereas in the ADM case it contains the highly non-linear 3-dimensional Ricci scalar. In the Ashtekar-Barbero formulation one introduces an internal flat reference frame at each point of the spatial hypersurfaces. Here 'flatness' refers to the fact that the internal space is described by a metric of the form δ_{ij} and internal indices can be trivially raised and lowered. The internal frame is referred to as the 'triad' frame (tetrad frames as internal coordinate systems for a 4-dimensional spacetime were introduced already by Einstein [51, 52]). The basis of the additional coordinate frame is given by triads e^a_i, where a is a spatial manifold index, whereas $i = 1, 2, 3$ denotes the internal index of the newly introduced frame. The components X^a of any tangent vector of a spatial hypersurface Σ can be decomposed over this triad basis as $X^a = e^a_k X^k$ and the inverse of the spatial metric is obtained from the triads through $q^{ab} = e^a_i e^{bi}$. The internal reference system needs to be invariant under rotations, i.e. it contains an SO(3), or equivalently SU(2), symmetry. The internal SU(2) symmetry is then encoded in the form of an additional, so-called Gauss-constraint, whose explicit form we omit (see e.g. [46]).

Ultimately, a desirable property of the Ashtekar-Barbero formulation is that one of the canonical variables is a connection. To define the connection variable, we need to consider some additional quantities first. Firstly, the connection in the internal frame spanned by the triads is the $su(2)$-algebra valued spin connection Γ^i_a, which is determined by

$$\nabla_a e^i_b := \partial_a e^i_b - \Gamma^c{}_{ab} e^i_c + \epsilon^{ijk} \Gamma^j_a e^k_b = 0\,, \tag{1.19}$$

where $\Gamma^c{}_{ab}$ are Christoffel symbols as defined in (1.5). Secondly, note that in the ADM formalism the extrinsic curvature K_{ab} of the spatial slices can be calculated from the spatial metric, the lapse and the shift as

$$K_{ab} = \frac{1}{2N}(q'_{ab} - 2\nabla_{(a}N_{b)}). \tag{1.20}$$

where the prime $'$ denotes derivation w.r.t. the time coordinate. Its decomposition in the triad frame reads

$$K_{ai}e^i_b = K_{ab}, \tag{1.21}$$

The extrinsic curvature 1-form K^i_a also takes values in the $su(2)$-algebra. One then introduces the Ashtekar-Barbero variables, namely the Ashtekar connection and the densitised triad, at last, which are, respectively, given by

$$A^i_a = \Gamma^i_a + \beta K^i_a, \qquad E^a_i = \sqrt{q}e^a_i\,, \tag{1.22}$$

where $\sqrt{q}$ is the determinant of the spatial metric and $\beta \in \mathbb{C}/0$ is the Barbero-Immirzi parameter. In classical GR, all values of β are equivalent and the different definitions of the Ashtekar connection are related by a simple canonical transformation (in the quantised theory, however, β carries physical significance). The Ashtekar connection has the properties of a connection due to its dependence on Γ^i_a, it is canonically conjugate to the densitised triad due to the appearance of K^i_a. Indeed, we have the following non-vanishing Poisson brackets

$$\{A^i_a(x), E^b_j(y)\} = \{\beta K^i_a(x), E^b_j(y)\} = \beta\,\kappa\,\delta^b_a\,\delta^i_j\,\delta^{(3)}(x,y)\,. \tag{1.23}$$

It is then possible to rewrite the ADM Hamiltonian in terms of the Ashtekar-Barbero variables, which at the classical level provides an equivalent description of GR. The goal of the LQG programme, as we will discuss in sec. 1.4.2, is to attain a description of quantum gravity by quantising the Ashtekar-Barbero formulation.

BF theories and the Plebanski formulation

Foreshadowing topics that will interest us in the discussion of quantum gravity approaches in sec. 1.4.2 we briefly sketch the main concepts of BF theories. These are topological field theories that are amenable to quantisation; the two fields appearing in the BF action are denoted as B and F and give the theory its name. Our presentation is based on [53–56].

BF theories can be defined for arbitrary spacetime dimension and are purely topological in any of them. They are hence not sufficient for a description of 4-dimensional Einstein gravity. Furthermore, by imposing additional constraints on the basic variables of the theory, 4-dimensional BF theories can be made equivalent to GR, which is known as the Plebanski formulation [53, 57, 58]. It is then no wonder that BF-theories and their extension have been of much interest to quantum gravity enthusiasts.

While BF theories alone are inadequate to describe 4-dimensional gravity due to their topological nature, they can reproduce GR in two and three dimensions (for specific choices in their construction). This is because GR itself is purely topological in these cases. What is meant by this statement is perhaps best illustrated by considering the Einstein-Hilbert action (1.15) with $\Lambda = 0$ in two dimensions, i.e. one considers a 2-dimensional manifold equipped with a metric with Lorentzian signature $(-,+)$. One finds that the two-dimensional integral can be evaluated explicitly and reduces to the Euler characteristic χ_E of the manifold in question, which is in turn determined by the genus h ('number of holes') of the manifold:

$$S_{\text{EH}} \propto \chi_E = 2 - 2h\,. \tag{1.24}$$

The Euler characteristic is a topological invariant and there are no local degrees of freedom in this theory; this is what we mean when we say gravity in two dimensions is purely topological. (For a classical treatment of 2-dimensional gravity, see [59, 60].)

BF theories can be defined on any $n-$dimensional manifold M $(n \geq 2)$. In addition to a manifold, one needs to specify a gauge group G, which can be any Lie group, and equip the group algebra with a bilinear form. The phase space variables of the theory are given by a connection[7] A and an $(n-2)$-form B, which takes values in the Lie algebra of G. The action of the theory, which inspires its name, reads

$$S_{BF} = \int_M \text{tr}(B \wedge F[A]) \quad \text{with} \quad F[A] := \text{d}A + A \wedge A\,, \tag{1.25}$$

where F encodes the curvature of the connection. The resulting field equations imply that

[7] One can formalise and generalise the notion of a connection we introduced earlier by defining connections on fibre bundles over manifolds. We will not concern ourselves with this further here and refer the reader to e.g. [34, chap. 10].

the connection is flat, hence, these are topological theories. To obtain e.g. a formulation of Lorentzian gravity in three dimensions, where the EFE simply state that the metric is flat, one would choose $G = SO(1,2)$; the Euclidean case it obtained for $G = SO(3)$.

Evidently, in four dimensions gravity has local degrees of freedom. To obtain Lorentzian GR from a BF theory one can choose $G = SO(1,3)$ and impose additional constraints on the theory

$$S = \int_M \mathrm{tr}(B \wedge F[A]) + \lambda_a \mathcal{C}^a \,, \tag{1.26}$$

where λ_a are Lagrange multipliers and $\mathcal{C}^a$ denote the so-called simplicity constraints, which impose additional conditions on the form of B. For further details, we refer the reader to [53–55].

1.1.3. Singularities

Singularities are a well-known problem of general relativity. The most intuitive notion of a singularity is in terms of a divergence of one of the curvature scalars, as is the case at the centre of black holes and at the origin of our universe ('Big Bang'). We will discuss the latter in more detail in sec. 1.2.2. It is important to distinguish between physical and coordinate singularities: components of e.g. the metric may diverge as a result of a poor choice of coordinate system, as e.g. the Schwarzschild metric in spherical coordinates at the black hole horizon, while curvature scalars remain finite.

Singularities are often seen as the regime where general relativity 'breaks down' as infinities in theoretical physics generally point to our ignorance (infinities are not physical). They then hint that a more complete theory of gravity, perhaps in the form of a quantum gravitational theory, is needed. We would like to point out that, even though generally understood to exist and posit a problem, defining the exact notion of a singularity is not straightforward and neither is the proof of its existence. Omitting mathematical details, we make the notion of a singularity slightly more precise, where our presentation is based on [29, sec. 9.1, 9.5].

Firstly, a singularity is, strictly speaking, not a point in spacetime, as the spacetime metric must be well-behaved on the entire spacetime manifold. We can therefore rather think about a singularity as a 'hole' or a boundary in spacetime.[8]. Secondly, the notion of diverging curvature scalars as signalling a singularity, as is readily illustrated for e.g.

[8] This intuitive notion is also imperfect, see examples in [29, sec. 9.1].

the Big Bang is not satisfactory in some cases: It is possible for the curvature scalars to be vanishing, while the singularity is noticeable only in the divergence of the Riemann curvature tensor [61].

A possible approach to a definition of a singularity is in terms of geodesic incompleteness: If any geodesic parametrised by an affine parameter is inextendable in at least one direction for a finite value of the same affine parameter, it is called incomplete. A spacetime is said to have a singularity if it contains at least one incomplete geodesic [62].

Singularity theorems provide us with the possibility to show that a spacetime is singular. They build on properties of the spacetime, such as conditions on curvature which can sometimes be rephrased in terms of energy conditions and causality as well as a choice of boundary or initial conditions. For a review of singularity theorems, see [63, 64]. For (original) literature regarding black holes please see [65, 66], whereas investigations of cosmological singularities can be found in [67] and references therein.

1.1.4. Relational observables

Coordinate independence, or covariance, is an integral principle of GR and naturally leads to the question which quantities are physically meaningful and can be observed by measurements. The answer to what is measurable in any theory is: observables. Observables are quantities that remain unchanged under symmetry transformations, as by definition such transformations do not impact the physical reality, but are merely a degeneracy in the system's description. The difficulty lies in identifying suitable observables of systems that have such a gauge dependence.

Gauge transformations are generated by the constraints $\mathcal{C}^a$ in a Hamiltonian formulation, and hence, so-called Dirac observables $\mathcal{O}$ are quantities that weakly Poisson commute with all constraints $\{\mathcal{C}^a, \mathcal{O}\} \approx 0$. As Dirac observables are invariant under gauge transformations they can be viewed as physical. The confusion of GR lies in the fact that the gauge group includes all diffeomorphisms, i.e. spatial as well as time translations. Hence, all Dirac observables are constants in space and time and therefore global quantities. Of course, this does not mean that we cannot use general relativity to make measurable predictions of a local nature (otherwise it would not be the successful theory it is today; gravitational waves do indeed propagate in space and time), but that these measurement outcomes depend explicitly on the coordinate system of the observer.

Formalising this apparent confusion and finding Dirac observables for GR has been studied enthusiastically and from these studies, the notion of relational observables emerged

[68–71]. Here, one distinguishes between partial and complete observables, which are extensively discussed in [72]. We consider this framework first from a more intuitive perspective, before outlining the idea more precisely with the notion of gauge orbits.

The core idea is that a partial observable is a measurable quantity (such as the reading of a clock), whereas complete observables are quantities that can be predicted by a theory. In particular, such a prediction will always be a relational statement: For example, one may predict that a homogeneous quantity Q will take a certain value when a clock reads a certain time $t = \tau$; or, forecast the value of an inhomogeneous quantity P at the top of the Eiffel tower $x = E$ at sunrise $t = S$. The values of $P(S, E)$ and $Q(\tau)$ for these specific events can be predicted, whereas, on their own P, Q, as well as t and x are partial observables. Complete observables, or relational observables, are then based on the fact that quantities can be unambiguously defined in relation to the value of another physical quantity. A simple example that we will use throughout this thesis is the use of a scalar field as a relational clock or coordinate. The intuitive notion is that if we have a function of spacetime $f(x^\mu)$ and a physical scalar quantity such as a scalar field $\chi(x^\mu)$ we can define a relational observable from $f(\chi(x^\mu))$, i.e. consider the value of f given a value of χ.

To discuss the construction of local Dirac observables (complete observables) within GR, we recall some structure of the Hamiltonian formulation of gauge theories. Firstly, for a constrained system not all phase space points are physical. Imposing the constraints $\mathcal{C}^a$ then reduces the phase space to the constraint surface $\mathcal{S}$, which contains all phase space points 'allowed' by the physical system. In a system with gauge freedom, not all points in $\mathcal{S}$ are physically distinguishable. Instead, the constraint surface contains gauge orbits $\alpha_\mathcal{C}$, where phase space points on a gauge orbit are related by gauge transformations generated by one of the constraints $\mathcal{C}$ of the system. All points on a gauge orbit are physically equivalent, and therefore, the entire gauge orbit is representative of a single 'physical scenario'. We can also consider the values of a phase space function f along gauge orbits, $\alpha_\mathcal{C}(f)$. To construct relational observables, we need to include an additional phase space function T, which takes on the role of a so-called 'clock' variable. A relational observable $F_{[f;T]}(\tau, x)$ that gives the value of a phase space function f with relation to T can be constructed in the following manner: For any point x on the constraint surface $\mathcal{S}$, find the point on the gauge orbit $\alpha_\mathcal{C}(T)$ that passes through x for which $\alpha_\mathcal{C}(T) = \tau$. At this phase space point, find the value of f. This then gives the value of the observable $F_{[f;T]}(\tau, x)$, which indeed assigns to each point on the constraint surface a value unambiguously. An important requirement for the clock field is that the phase space point where $\alpha_\mathcal{C}(T) = \tau$

is unique. If a system has M constraints, its gauge orbits are $M-$dimensional and one needs to include M functions that take on the role of T in the description above.

This procedure to define local Dirac observables is not always straightforward in practice and can be rather involved. Particularly suitable clocks are dust fields [73, 74], but also scalar fields have been of interest in the literature [75]. Relational coordinate systems based on matter reference frames are furthermore useful in quantum gravity, where they allow to evade the problem of time [76, 77], as well as providing a framework to construct local quantities [78–80]. Relational coordinate systems have also been used in manifestly gauge-invariant formulations of cosmological perturbation theory [81] and as a starting point for canonical quantisation [82, 83].

Finally, let us point out that the problem of gauge dependence of a theory can also be resolved by fixing a gauge. In e.g. perturbative systems it is generally possible to describe gauge-invariant quantities up to a given order in perturbation theory. This is the approach adopted in cosmological perturbation theory, were gauge-invariant quantities play a crucial role for observations (see sec. 3.2). Relational observables as described above have been constructed for a matter reference frame in the context of cosmological perturbation theory and were compared to results retrieved in 'standard' cosmological perturbation theory [81]. It was found that results agree, up to small corrections [84].

We will make use of a relational coordinate system spanned by four massless scalar fields in sec. 2.2.3 and sec. 3.4 as well as in chap. 6 and only work with quantities that are functions of these matter fields. We will not explicitly construct Dirac observables as detailed above, but instead gauge fix the coordinate system to that given by the physical matter fields.

1.2. Cosmology

This section describes how GR can be used to describe the evolution of our universe with a Friedmann-Lemaître-Robertson-Walker (FLRW) metric. We will begin with a general description of the universe's metric and matter types of interest in sec. 1.2.1, followed by a discussion of the initial Big Bang singularity (sec. 1.2.2). We give a brief overview of the evolution of our cosmos in sec. 1.2.3, including also a timeline of various processes that came about as the universe expanded and thereby cooled. We also introduce the basic concepts of inflation, which comprises the current paradigm for the evolution of the very early universe. Finally, we touch on the idea of cosmological perturbations, which we will revisit in chap. 3. A more thorough account of topics discussed in this section can be found in the standard literature, such as [29, 85–88].

1.2.1. General relativistic description of the universe

Within GR, our cosmos can be described by a specific solution to the EFE as given in (1.14): the Friedmann-Lemaître-Robertson-Walker (FLRW) metric [19–23]

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2\left(\frac{1}{1-Kr^2}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right), \tag{1.27}$$

where $N(t)$ is the lapse function corresponding to a choice of time coordinate, $a(t)$ is the scale factor of the universe, K gives the spatial curvature, and $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$ is the line element of the unit sphere. The scale factor $a(t)$ is the only degree of freedom of the FLRW metric. Regarding units, one can assign units of length to the scale factor and the lapse, or to the coordinates t and r. In the latter case, the curvature K has units $[K] = L^{-2}$, while in the former, K is dimensionless. Experimental evidence shows that the curvature contribution to our universe is very small, possibly even exactly zero [89], and it is widely assumed that our universe is indeed flat. However, we shall not withhold that this premise has been questioned [90].

The Einstein tensor (1.14) for the FLRW metric reads ($G^0{}_i = 0 = G^i{}_{i\neq j}$)

$$G^0{}_0 = -\frac{3(a')^2}{a^2N^2} - \frac{3}{a^2}K, \qquad G^i{}_i = -\frac{2a''}{aN^2} + \frac{2a'N'}{aN^3} - \frac{(a')^2}{a^2N^2} - \frac{K}{a^2}. \tag{1.28}$$

Here and throughout the prime ' denotes derivation w.r.t. the time coordinate in any lapse N. We leave the lapse function general, as we will use a specific, non-standard choice of time coordinate in later chapters. Popular lapse choices are *cosmic time*, $N = 1$, and *conformal time*, $N = a$. It is useful to introduce the Hubble rate

$$H = \frac{a'}{a}, \tag{1.29}$$

which characterises the evolution of the scale factor and thereby parametrises the rate of expansion of the universe.

We can use the FLRW metric for the study of cosmology under the assumption that our universe is homogeneous and isotropic on large scales. The isotropy assumption is based on observational evidence: From the viewpoint of planet Earth the universe looks the same in all directions on sufficiently large scales. ‘Sufficiently large scales’ can refer to e.g. approximately 100 Mpc,[9] where the distribution of galaxies has been found to be isotropic up to a few percent [86, 87, 91]. Together with the cosmological principle, which states that our position in the universe does not correspond to a particularly special one, homogeneity follows from isotropy at every spacetime point [92]. Without attempting to give an overview of observational evidence for the symmetries of our universe, we mention that the cosmic microwave background (CMB), which we will detail further below, (mostly) supports the isotropy assumption [93, 94]. Given homogeneity and isotropy on large scales one can still question whether the FLRW metric gives an accurate enough description of the universe: as GR is a non-linear theory, considering the evolution of averaged quantities might not sufficiently capture the true evolution. We discuss this so-called averaging problem further in sec. 3.1.2.

Having determined the form of a suitable metric for the universe from symmetry considerations, the question of the matter content remains, which, naturally, must be compatible with homogeneity and isotropy. This requirement is fulfilled by perfect fluids. A perfect fluid can be completely characterised by its energy density ρ and isotropic pressure P, which are related by an equation of state parameter w through $P = w\rho$. The energy-momentum tensor of a perfect fluid reads

$$T^{\mu}{}_{\nu} = (\rho + P)u^{\mu}u_{\nu} + P\delta^{\mu}_{\nu}\,. \tag{1.30}$$

where u^{μ} denotes the four velocity of the respective fluid, which satisfies $u_{\mu}u^{\mu} = -1$. In the fluid rest frame ($u_i = 0$) we have

$$T^{\mu}{}_{\nu} = \mathrm{diag}(-\rho, P, P, P)\,. \tag{1.31}$$

The continuity equation follows from $\nabla_{\mu}T^{\mu}_{0} = 0$ (see sec. 1.1.2):

$$\rho' = -3H(\rho + P)\,. \tag{1.32}$$

With a constant equation of state parameter, $w =$ const., the continuity equation can be integrated to give the following dependence of the energy density on the scale factor

[9] Approximately 3×10^{21} km.

(excluding the case $w = -1$),

$$\rho \propto a^{-3(1+w)}. \tag{1.33}$$

The equation of state parameter then fully characterises the evolution of a perfect fluid in an FLRW spacetime.

In addition to the equation of state parameter w, it is useful to define the adiabatic sound speed $c_s^2 = \frac{P'}{\rho'}$. Together with the continuity equation (1.32), one finds that

$$w' = -3H(w+1)(c_s^2 - w)\,. \tag{1.34}$$

For a constant equation of state parameter we have $w = c_s^2$.

Complementary to the description above, one often encounters universes filled with scalar fields in the literature, which can be interpreted as perfect fluids with a dynamic equation of state parameter and provide a convenient way to capture matter degrees of freedom. The energy-momentum tensor of a canonical scalar field χ is given by

$$T^{\mu}{}_{\nu} = \partial^{\mu}\chi\partial_{\nu}\chi - \left(\frac{1}{2}\partial^{\alpha}\chi\partial_{\alpha}\chi - U(\chi)\right)\delta^{\mu}_{\nu}\,, \tag{1.35}$$

where $U(\chi)$ denotes the scalar field potential. The energy density and pressure of a scalar field read

$$\rho = \frac{(\chi')^2}{2N^2} + U(\chi), \qquad P = \frac{(\chi')^2}{2N^2} - U(\chi). \tag{1.36}$$

The dynamics of the field are governed by the Klein–Gordon equation

$$\chi'' - \frac{N'}{N}\chi' + N^2\frac{\mathrm{d}U(\chi)}{\mathrm{d}\chi} + 3H\chi' = 0\,. \tag{1.37}$$

Depending on the choice of potential $U(\chi)$ a scalar field can mimic a perfect fluid with a constant equation of state parameter (in some cases such a behaviour is limited to certain time scales of interest). The cosmological considerations we make in the second part of the thesis will assume matter in the form of massless scalar fields ($U(\chi) = 0$ and hence $w = 1$). The reasons for restricting our considerations to such a type of matter only will become clear in our discussion of GFT in chap. 2 and the introduction to quantum gravity bounces in chap. 4.

The first and second Friedmann equation, respectively obtained from the purely time and spatial components of the EFE for an energy-momentum tensor as given in (1.31), read

$$3\left(\frac{a'^2}{a^2}+K\frac{N^2}{a^2}\right)-N^2\Lambda = N^2\kappa\rho\,, \tag{1.38}$$

$$2\frac{a'}{a}\frac{N'}{N}-\left(\frac{2a''}{a}+\frac{a'^2}{a^2}\right)-\frac{KN^2}{a^2}+N^2\Lambda = N^2\kappa P\,. \tag{1.39}$$

Solving the Friedmann equation in general for a universe that (like ours) consists of multiple matter components with a different equation of state parameter is not straightforward. However, as we saw in (1.33) different components have a different dependence on the scale factor and as the universe evolves there will typically be various periods in which a single component has a dominant contribution. For instance, the early universe is dominated by radiation, which has $\omega = \frac{1}{3}$, such that $\rho \propto a^{-4}$ and the curvature and cosmological constant contribution can be neglected.

1.2.2. The initial singularity: It all started with a bang

'Big Bang' singularity is the name that has been assigned to the initial moment of ignorance in which the universe began and that has long evaded a complete theoretical description. While, as we noted in sec. 1.1.3, finding a precise definition of a singularity is far from trivial, the singularity at the origin of an FLRW spacetime is undisputed.

The physicality of the Big Bang singularity was a topic of interest in the 1960s and it was found that its occurrence is a rather general feature of cosmological spacetimes. For instance, it could be shown that the singularity persists in homogeneous but anisotropic spacetimes [95]. The idea (or maybe even hope) that small perturbations away from exact homogeneity could render the universe non-singular in its origin proved to be futile, and it could be shown that the physical singularity arises even in such cases [96]. In [67] it was concluded that singularities occur for universes with $\Lambda \leq 0$ as long as the following conditions on the energy density and pressure components P_i hold:[10] $\rho + \sum_i P_i \geq 0$ and $\rho + P_i \geq 0$, where $i = 1, 2, 3$. As we mentioned, the early universe is dominated by radiation and Λ plays a negligible role, such that the above conditions are satisfied. For a

[10] The pressure components P_i correspond to the elements of the spatial diagonal of the energy-momentum tensor in the fluid rest frame (which corresponds to the frame in which the energy-momentum tensor is diagonal). These conditions are then more general than for a perfect fluid, where all P_i are equal.

more detailed description of the Big Bang singularity see e.g. [63, chap. 3].

We shall not delve into the mathematical richness of singularity theorems, but illustrate the occurrence of the initial singularity through the more intuitive notion of the divergence of the Ricci scalar. In an FLRW spacetime described by (1.27) the Ricci scalar (1.11) is given by

$$R = 6\left(\frac{K}{6} - \frac{1}{N^2}\frac{a'}{a}\frac{N'}{N} + \frac{1}{N^2}\left(\frac{a'^2}{a^2} + \frac{a''}{a}\right)\right) . \tag{1.40}$$

As we argued previously, in the presence of multiple fluids one can approximate the Friedmann equation (1.27) by assuming dominance of a single component in certain regimes. Furthermore, if $w > -\frac{1}{3}$, the energy density of a fluid (1.33) will dominate over the curvature contribution. If we additionally assume that the cosmological constant is small, one can solve the Friedmann equation (1.38) and e.g. in cosmic time $N = 1$ we find

$$a(t) \propto t^{\frac{2}{3(w+1)}} . \tag{1.41}$$

It follows that $a'/a \propto t^{-1}$ and $a''/a \propto t^{-2}$, leading to a divergence of the Ricci scalar (1.40) as $t \to 0$: we have found the Big Bang singularity.

1.2.3. A brief history of the universe

In the following, we briefly illustrate the evolution history of the cosmos as we understand it today. As this thesis is concerned with quantum gravity effects at the universe's origin, we state only the basics, aiming to simply provide the broader context of what follows after the extreme high curvature regime that will be of interest in later chapters.

It has long been manifest that we live in an expanding universe [97–100]; in fact, the expansion of the universe is accelerating [101–103]. We saw that within GR, we can describe the expanding universe with an FLRW metric and several perfect fluid components. The central question is then which matter types are contained in our universe and in which relative abundance. It is observationally straightforward to establish that the universe contains radiation and dust matter,[11] which have equation of state parameters $w_{\text{rad}} = \frac{1}{3}$

[11] We will use the word 'matter' to refer to any contribution to the r.h.s. of the Friedmann equation and

and $w_{\text{dust}} = 0$, respectively. Furthermore, there are possible contributions from the spatial curvature K, as well as the cosmological constant Λ.

As the universe proceeds to expand, different matter types dominate its energy density. As can be understood from (1.33) and the Friedmann equation (1.38), at early times, radiation dominates, followed by a period of dust matter domination. At later times, the curvature term will take over, unless we live in an exactly flat universe. Lastly, the cosmological constant will be all that is left to drive the expansion of the universe.

It is possible to observationally establish the relative abundance of different matter types in our universe today, which permits us to trace back the evolution history of these components. Radiation contributes less than $10^{-3}\,\%$ to the current universe's energy budget; dust matter makes up around $30\,\%$ of our universe, and the cosmological constant, also referred to as 'dark energy', gives the dominant contribution with around $69\,\%$. Spatial curvature can only contribute up to a maximum of $1\,\%$. For exact numbers and more details see [86, sec. 2.4].

This touches on some of the unsolved mysteries that lurk within our current cosmological model: Firstly, less than $5\,\%$ of the universe's total energy budget can be attributed to baryonic matter (protons and neutrons, neglecting the mass of electrons) that we observe in galaxies or other structures [24]. The primary contribution comes from a matter species unknown to us, fittingly dubbed 'dark matter': in order to describe observations it is necessary to assume a type of matter that interacts gravitationally, but only weakly through the other forces [104]. The true nature of the cosmological constant Λ is also an area of active research. For a discussion of different approaches to and viewpoints of the cosmological constant, see [105, 106].

The ΛCDM model is the currently prevalent standard model of cosmology. Two of its core ingredients are in the name: a non-zero cosmological constant Λ and cold dark matter (CDM), where the term 'cold' refers to the fact that it is non-relativistic. The ΛCDM model assumes that the universe is flat ($K = 0$ in (1.27)). It builds on a general relativistic description of gravity in the form of the FLRW metric and has six independent parameters [24]. Three of these parameters that are determined by the background evolution of the universe are related to the value of the Hubble constant, the dust matter density (CDM and baryonic), and the baryon density. Another parameter is the inte-

refer to the specific contribution with $w = 0$ as 'dust matter'. Note that in the literature, the latter often referred to simply as 'matter'. Dark matter is a form of dust ($w = 0$), whereas for baryonic matter this is only true after it decouples from photons and electrons just after the surface of last scattering (see below) [86].

grated optical depth τ to recombination, which gives the integrated scattering probability of photons and is thus a measure for the opacity of the universe [86, 89, 7.5]. The remaining two are related to perturbations of the homogeneous background and we will come back to these in sec. 1.2.4.

We proceed with a swift synopsis of the various processes that took place as the universe expanded. Their description emanates from the behaviour of matter at various energy scales, drawing on the standard model of particle physics, statistical physics, and atomic physics. From the viewpoint of GR, nothing but the energy density of the various perfect fluids that inhabit our universe are of importance. Still, the multitude of processes that unfolded as the universe expanded and cooled are not only extremely interesting, but their understanding also provides us with observational tests of the universe's evolution. Our description is based on [86].

Leaving out the Big Bang and a possible inflationary phase, which are discussed further below, the first event that followed in the early universe was baryogenesis, which is the term for the process that induced the observed disparity in the abundance of matter and anti-matter in the universe. Then, at a temperature of around 10^{15} K electroweak symmetry breaking took place and the gauge bosons of the weak force ($W^{\pm}$ and Z bosons) gained mass. At around 10^{12} K the universe had cooled down enough such that hadrons could form (QCD phase transition). Neutrinos decoupled at energies of around 10^{10} K and became free streaming relativistic particles. Finally, the universe cooled enough (10^{9} K) for the annihilation of positrons and electrons and the first elements were formed during nucleosynthesis. At this point, we exit the era of radiation domination and enter the era of dust matter domination. The next two processes are extremely important for our understanding of the history of the cosmos, as they culminated in the event at the origin of our earliest snapshot of the universe, namely the cosmic microwave background (CMB) (see also sec. 1.2.4): At 3400 K electrons and protons combined to form hydrogen atoms for the first time, a process called recombination. Whereas the universe was a dense 'soup' of charged particles until this point, the formation of neutral atoms meant photons could now stream freely. The surface of last scattering (2900 K) marks the moment after which the universe became completely transparent for photons and the CMB photons that we measure today were emitted. Subsequently, stars and galaxies began to form. The final event in our story is the moment in which the dust matter contribution became subdominant to the cosmological constant, which since drives the universe's expansion. Today, our universe has a temperature of around 2.7 K [24].

This picture that we have drawn does not come without its peculiarities, which are mainly the necessity of extremely fine tuned initial conditions. One of these is the fact that the curvature parameter in the very early universe must have been extremely small without vanishing entirely. This is known as the flatness problem. Another, known as the horizon problem, arises due to the fact that, despite the observed homogeneity of the CMB, the CMB photons could not have been in thermal equilibrium in the early universe as they would not be causally connected if the universe began in a period of radiation domination. Hence, their homogeneity must have been a (fine tuned) initial condition. In the following, we give a more detailed explanation of the horizon problem, also because horizons play an important role in the study of perturbations, as we will see in sec. 3.3. The flatness and the horizon problem can both be alleviated by an early phase of accelerated expansion, which provides the main motivation for models of inflation. We will discuss the basic premise of inflation at the end of this section.

The horizon problem

To understand the horizon problem we consider the causal structure of the FLRW space-time. The particle horizon[12] describes the furthest distance to particles whose light reaches us today. These particles may long have receded away from us, such that we can never observe them again, but the light they emitted in the past reaches us today.

From the comoving distance Δr that light travels between time t_1 and t_2 we can calculate the size of the (comoving) particle horizon r_{hor} at a given time

$$\Delta r(t_2, t_1) = \int_{t_1}^{t_2} \frac{\mathrm{d}t}{a(t)} \quad \Rightarrow \quad r_{\text{hor}}(t) = \int_0^t \frac{\mathrm{d}t}{a(t)} = \int_{a(0)}^{a(t)} \frac{\mathrm{d}a}{\dot{a}a} \,. \tag{1.42}$$

which assumes the universe 'began' at $t = 0$. Note that physical distances d are obtained from comoving ones through $d(t) = a(t)\,r(t)$. In a period where the universe's matter content is dominated by a perfect fluid, such that contributions from the curvature and cosmological constant can be neglected, one finds, using the Friedmann equation (1.38) in conformal time $H^2 = \frac{\kappa}{3}a^2\rho$ and (1.33) (thus assuming a constant equation of state

[12]Not to be confused with the event horizon, which is unrelated to the horizon problem. The event horizon is a horizon concerned with the future of an observer - all events outside the event horizon cannot be influenced by the observer and are therefore causally disconnected.

parameter), that the particle horizon is approximately given by the comoving Hubble horizon $(aH)^{-1}$. Therefore, the physical distance d_{hor} of the particle horizon can be approximated by the Hubble horizon R_H:

$$r_{\text{hor}} \sim \frac{1}{aH} \quad \Rightarrow \quad d_{\text{hor}} = a\, r_{\text{hor}} \sim \frac{1}{H} =: R_H \,. \tag{1.43}$$

The horizon problem concerns the observed high level of homogeneity of the CMB, which would be a natural consequence of thermal equilibrium in the early universe. However, if we consider the particle horizon at the surface of last scattering, where the CMB was emitted, we find that not all regions could have been in causal contact at this stage, assuming the universe began in an era of radiation domination. To make this more precise, if we compare the size of the surface of last scattering given by the distance CMB photons have travelled since their emission $\Delta r(t_{\text{today}}, t_{\text{CMB}})$, to the particle horizon of each CMB photon at time of emission, $r_{\text{hor}}(t_{\text{CMB}})$, we find that $r_{\text{hor}}(t_{\text{CMB}}) \ll \Delta r(t_{\text{today}}, t_{\text{CMB}})$. Different regions of the CMB photons are therefore not causally connected and without introducing a phase that predates the radiation domination era, their homogeneity cannot be explained through a thermalisation process that took place in the early universe. Instead, to agree with observation, homogeneity must be imposed in the form of initial conditions, which may be seen as a fine tuning problem. An era of rapid expansion in the early universe can resolve the horizon problem: The universe could initially be in thermal equilibrium, while still reaching its size today. Alternatively, in any bouncing universe, where the expanding phase we live in is preceded by an era of contraction, the horizon problem does not exist, as causal contact and thermalisation can take place in the contracting branch.

Early evolution: Inflation

Inflationary models lead to an extended phase of accelerated expansion $\frac{a''}{a} > 0$ in the early universe, which solves the horizon problem and furthermore provides a mechanism that leads to a naturally small value of the curvature density (thus resolving also the flatness problem; for details we refer the reader to [86, 4.1.2]). (The Big Bang singularity persists in simple inflationary models.) A phase of accelerated expansion can be achieved by assuming that the universe was dominated by a single scalar field with a potential that fulfills certain properties prior to the era of radiation domination. Such models are known as single field inflation, where the field of the model is referred to as the inflaton, which we will denote as ϕ.

Inflation has become a dominant paradigm today (although it is not perfect and many extensions are continuously studied and proposed). We outline the basics of single field inflation below; our presentation is based on [87, sec. 1.5.2]. At the core of single field inflation lies the inflaton ϕ with a matter action given by

$$S = -\frac{1}{2}\int \sqrt{-g}(\partial_\mu\phi\partial^\mu\phi + U(\phi)), \tag{1.44}$$

and energy density and pressure as given in (1.36). To attain a period of accelerated expansion we require $w < -\frac{1}{3}$. In slow-roll inflation, one assumes that the scalar field potential dominates over the kinetic term, such that $w \approx -1$, and furthermore that the kinetic term varies only slowly to ensure that the accelerated phase can last long enough. These requirements are captured by the slow-roll conditions, which in cosmic time $N = 1$ read

$$\frac{1}{2}(\phi')^2 \ll U(\phi) \qquad |\phi''| \ll 3H|\phi'|\,. \tag{1.45}$$

The latter follows from the Klein-Gordon equation (1.37), which governs the field dynamics. As inflation proceeds, the field gains kinetic energy and approaches the minimum of the potential. Inflation ends and the inflaton decays to standard model particles in a process known as 'reheating', initiating the radiation dominated era of the universe.

Various potentials $U(\phi)$ can satisfy the conditions (1.45) and a plethora of inflationary models exists. Inflation appears to agree well with observations, however, it is not without caveats and criticisms. For a review of successes and criticisms of inflation in light of recent data, please see [107].

With regards to the topic of this thesis, we point out that in some sense, one can view inflation as a coarse grained description for the happenings in the early universe, as we have no concrete knowledge of the prevalent degrees of freedom. Also, it is unclear whether the description of the universe's dynamics in classical terms is useful above the Planck scale, which is reached at around 10^{19} GeV, or 10^{32} K, where one expects quantum effects to become dominant. A satisfactory description of these regimes is expected only within a theory of quantum gravity, which, as we will discuss in chap. 4, may induce a bounce to replace the Big Bang singularity.

1.2.4. Beyond homogeneity

Needless to say, the universe is not perfectly homogeneous as any look around us or into the night sky reveals. In our universe today, matter has clumped to form structures, such as planets and stars on smaller, as well as galaxies on larger scales. The formation of structures can be modelled from the growth of initially very small inhomogeneities on top of a homogeneous 'matter soup'. These inhomogeneities are described by cosmological perturbation theory; the mathematical treatment of perturbations in the early universe will be discussed in detail in chap. 3.

As explained in sec. 1.2.3, the CMB can be seen as our window to the origin of our universe. The CMB has been mapped in three successive experiments [108, 109], the latest of which, the *Planck* satellite [24], reached unprecedented precision. Overall, the CMB is well described by an (almost) perfect black body spectrum with a temperature of $\sim 2.7\,\mathrm{K}$. The crucial information lies in the anisotropies, however. From these small anisotropies one can extract a power spectrum, which encodes correlations in the anisotropies on a given scale. Among other things, it can be used to test theories of the early universe and it is this property that makes it interesting in the context of quantum gravity.[13] It should be noted that effects on the power spectrum are highly degenerate - many modifications in the theory lead to the same effects. We can now name the final two ΛCDM parameters, A_s and n_s, which describe the amplitude and tilt of the CMB power spectrum, respectively [86, 89].

The focus of this thesis lies on quantum gravity effects in the very early universe, specifically, we will consider models that replace the Big Bang singularity with a bounce. Regarding the connection to cosmology, we may hope that such scenarios can make inflation superfluous, or consistently precede inflation. How exactly a quantum gravitational theory of the universe connects to cosmological measurements, can be brought in agreement with them, and leaves observational imprints is a highly model dependent, non-trivial task.

1.3. Quantum theory

Quantum mechanics fundamentally revolutionised the field of physics by revealing that the world at small scales follows a set of laws that is completely distinct from our every-

[13] The anisotropies of the CMB encode far more than that, but these processes are not only outside of the scope of this thesis, but also outside of the scope of knowledge of the author.

day experience. Before proceeding to discuss approaches to quantum gravity, we briefly mention the concepts that have led to an accurate and well established description of several quantum phenomena. We include the axioms of quantum mechanics in sec. 1.3.1 and outline the canonical quantisation as well as the path integral approach to quantum field theory in sec. 1.3.2.

1.3.1. Quantum mechanics

The field of quantum mechanics was established through a string of experimental evidence that culminated in an axiomatic formulation of the theory. The axioms of quantum mechanics establish the link between the mathematical formalism and the real world. They can be found in different forms and in different order in a multitude of places, such as [34]. Here we give a short overview over these axioms.

- States: A physical state $|\psi\rangle$ is described as a vector of a Hilbert space $\mathscr{H}$. Projection of a state on a specific basis gives the wavefunction ψ, e.g. $\langle x|\psi\rangle = \psi(x)$.
- Operators: To every physically relevant quantity A there is associated a hermitian operator $A = A^\dagger$.
- Commutators: Poisson brackets of phase space functions are replaced with commutators of operators $[A,B] = AB - BA$.
- Dynamics: Time evolution is unitary. In the Schrödinger picture, evolution of the wavefunction satisfies $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi = H\psi$, where H is the Hamiltonian operator of the system. In the Heisenberg picture, time evolution is obtained from the commutator of an operator with the Hamiltonian $\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}A = [A, H]$.
- Measurement (Born's rule/ Wave function collapse): We can only make probabilistic statements of measurement outcomes. The expected value, or expectation value, of an operator for a specific state is given by $\langle\psi| A |\psi\rangle$. Upon measurement, the system collapses into an eigenstate of the measured operator. Successive measurements within short time intervals will give the same result.

Quantum mechanics works extremely well for all non-relativistic processes, such as those found in atomic physics. In order to combine the quantum nature of small scales with special relativity, quantum mechanics had to be extended to quantum field theory, which is the subject of the next section.

1.3.2. Quantum field theory

Quantum mechanics works extremely well at low energy scales, or equivalently, for slow processes. With the advent of special relativity, however, it became clear that quantum mechanics is applicable only in such regimes and an extension of the theory, which respects Lorentz invariance and allows for particle creation and annihilation, is required. In regimes with high energy, the equivalence of energy and matter allows for particle creation and special relativistic effects can no longer be neglected. Quantum field theory (QFT) is the framework in which such effects are accurately captured. It provides the theoretical framework for the standard model of particle physics, which has so far stood the test of time and continues to be confirmed at particle accelerators [27]. The two main principles of QFT are Lorentz invariance and unitarity and the main objects are quantum fields living on a flat spacetime described by the Minkowski metric. These fields capture the matter content of our universe. Their quantisation can be achieved in a canonical or a path integral approach. Details of QFT can be found in e.g. [25, 110].

In canonical quantisation, one promotes the field of a classical field theory to an operator, where each field mode is quantised in the same fashion as a harmonic oscillator. For a scalar field ϕ this leads to the following form of the field operator ($x = (t, \vec{x})$)

$$\phi(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(a_p e^{\mathrm{i}(-\omega t + \vec{p}\vec{x})} + a_p^\dagger e^{\mathrm{i}(\omega t - \vec{p}\vec{x})} \right) , \tag{1.46}$$

where $\vec{p}$ denotes the three-momentum and ω the energy eigenvalue of each field mode. The interpretation of the above is that the field operator acting on the vacuum $\phi(x) |0\rangle$ creates a particle at position $\vec{x}$ and time t. The above form of $\phi(x)$ can be obtained by first carrying out a Fourier decomposition in space and noticing that the Klein-Gordon equation $(\partial_t^2 - \nabla^2 + m^2)\phi = 0$ imposes the dynamics of a harmonic oscillator on each field mode. The procedure is then to quantise each field mode as a single harmonic oscillator and introduce mode dependent ladder operators a_p and $a_p^\dagger$ that satisfy the following commutation relations

$$[a_p, a_q^\dagger] = \delta(\vec{p} - \vec{q}) \, . \tag{1.47}$$

The probability of a particle to propagate from x to y can be calculated from the 2-point correlation function or propagator $\langle 0| \phi(x)\phi(y) |0\rangle$. In an interacting theory, one can ask more interesting questions, such as the likelihood to transition between two states with given particle number and momenta. The transition amplitude between an initial state $|i\rangle$ at time t_i and a final state $|f\rangle$ at time t_f is denoted as $|\langle f, t_f | i, t_i\rangle|^2$. In particle colliders, QFT predictions are compared to experimental results based on scattering amplitudes. These are calculated from asymptotic transition amplitudes, which are encoded in the so-called S-matrix S

$$\langle f; \infty | i; -\infty\rangle = \langle f| S |i\rangle \ . \tag{1.48}$$

S-matrix elements can be related to $n-$point correlation functions through e.g. the LSZ reduction formula. These higher order correlation functions in interacting theories can be calculated from a perturbative expansion characterised by Feynman diagrams.

A particularly convenient technique for calculating $n-$point correlation functions arises in the path integral approach to QFT. Here, they can be calculated from derivatives of the generating functional $Z[J]$, also referred to as the partition function. For a scalar field ϕ we have

$$Z[J] = \int \mathcal{D}\phi \ e^{\mathrm{i}S[\phi]+\mathrm{i}J[\phi]} \tag{1.49}$$

where the action S contains a kinetic and an interacting term, J denotes the source term, and $\int \mathcal{D}\phi$ stands for the path integral.

1.4. (The trouble with) Quantum gravity

Having outlined the basics of GR as well as QFT, one may now ask whether it is possible to achieve a quantum description of gravity with QFT methods. The answer is not in the affirmative, at least not with the standard methods used by particle physics (otherwise this thesis would be obsolete). We outline the reasons for this below and refer the reader to [29, sec. 14.1] [56] for further details.

In an attempt to quantise GR as a field theory, the naive approach would be to promote the metric to an operator. This however, leads to two conundrums: Firstly, as we saw in the previous section, QFT is defined on a fixed Minkowski spacetime described by $g_{\mu\nu} = \eta_{\mu\nu}$. It is not clear how a formulation of QFT could be achieved where the metric, which describes the background, is at the same time a quantum field in the theory.[14]

[14] One can however investigate QFT on a curved background spacetime, which is indeed an active research

Secondly, the metric encodes causality of the spacetime, which is an integral part of the formulation of QFT, and a quantum metric would render the causal structure ambiguous.

Maybe the most straightforward attempt to circumvent these issues is found by considering a perturbation around the Minkowski background $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where the field theory is still formulated on a flat spacetime and only the perturbation $h_{\mu\nu}$ is treated as a quantum field. The gravitational force is then mediated by a spin-2 particle called the graviton, corresponding to the excitation of $h_{\mu\nu}$. Unfortunately, it turns out that this theory is not renormalizable [112, 113] and should thus be viewed as a low energy effective field theory [114]. (In our brief overview of QFT we did not touch on issues of renormalizability, which concerns itself with absorbing infinities that appear in the theory in a suitable manner. The interested reader is referred to [29, sec. 14.1], which contains a better overview of renormalisation than we could ever provide. A way out of non-renormalizability is the inclusion of a non-Gaussian fixed point of the renormalisation group flow, which is the premise of the asymptotic safety approach [115].)

With straightforward methods proving futile, one may of course ask whether it might not be possible that gravity simply needs to be described in a classical manner and no quantum analogue exists. However, the EFE (1.14) give a precise relation between matter and geometry, and we know very well that matter is quantum. Any 'typically' quantum behaviour of matter (such as the collapse of a wavefunction) then needs to be mimicked by the metric.[15] We have furthermore seen that general relativity predicts its own breakdown in the form of singularities in sec. 1.1.3, which are hopefully resolved in a quantum treatment. It is then not only unsatisfactory to have a quantum theory for all forces but one, but a quantum description of gravity appears to almost be a physical necessity. There is then an ongoing quest for a theory of quantum gravity to reveal how the quantum nature of reality manifests in gravity.

This quest is a difficult one and there exist a plethora of approaches, none of which, at this point are complete from a theoretical viewpoint or can make verifiable predictions. Different approaches have vastly different starting points and it is not at all clear how they connect to each other - where connections between some are more apparent than between others. Broadly speaking, approaches can be classified as background dependent and background independent. The former assumes an underlying Minkowski background and

field [111] that we touch on again in sec. 1.4.1.

[15]E.g., if matter is in a superposition of two locations, by the virtue of the EFE this would be reflected by the spacetime geometry. The collapse of the matter wave function would then induce an instantaneous change also of the metric.

constructs a quantum theory of gravity on top, where the apparent curvature of spacetime described by general relativity is emergent. Background independent approaches, on the other hand, aim to quantise gravity or spacetime directly.

We continue to give a rough overview of possible approaches in sec. 1.4.1. In this thesis, we are particularly interested in GFT, which is a background independent approach to quantum gravity and is related to the LQG and spin foams approach, such that we explain the motivations behind these approaches and describe them in some detail in sec. 1.4.2. GFT itself and its motivations will be the subject of chap. 2.

1.4.1. Approaches to quantum gravity

One can approach the problem of quantum gravity from different viewpoints:

- **Canonical quantisation**
  In the canonical quantisation approach, the goal is to find a suitable quantisation of the Hamiltonian formulation of GR. A major obstacle in this approach is the fact that the GR Hamiltonian H, which is responsible for time evolution, is a constraint (see sec. 1.1.2). Hence, for any wave function Ψ, which satisfies the Schrödinger equation $\mathrm{i}\partial_t \Psi = H\Psi$, we find $H\Psi = 0$ and thus no time evolution. This is known as the problem of time. LQG is a candidate theory of quantum gravity that is based on canonical quantisation and we will give further details on its formulation in sec. 1.4.2.

- **Path integral approach**
  In the path integral approach, one can attempt to define a path integral over all possible metrics, with a generating functional along the lines of $Z = \int e^{\mathrm{i}S[g_{\alpha\beta}]} \mathrm{d}\mu[g_{\alpha\beta}]$. Here, $\mathrm{d}\mu[g_{\alpha\beta}]$ denotes a measure on the space of all possible metrics, the definition of which is far from trivial. This approach can include transitions between metrics of a different topology. An example is the field of causal dynamical triangulations, which is based on a path integral over all causality preserving triangulations of spacetime [116, 117]. Spin foams are another path integral approach to quantum gravity and the topic of sec. 1.4.2.

- **String theory**
  In string theory, which is often hailed as the most promising approach to quantum gravity, the notion of point particles is replaced with one dimensional strings, which

can be either open or closed [118, 119]. The theory is otherwise formulated on a classical Minkowski background and we therefore refer to it as a 'background dependent approach'.[16] A great advantage of string theory is that it solves the problem of renormalizability. String theory requires multiple (at least 6) additional compact spatial dimensions. To be consistent with observations, the theory needs to reduce to the four spacetime dimensions we live in the low energy limit. In principle, the additional dimensions can have observable effects at higher energies, which could potentially be found in particle colliders. An integral part of string theory is then to construct the low energy limit in the form of effective field theories (EFTs) and the aim of the so-called Swampland programme is to determine which EFTs are incompatible with the world we observe [121]. Another possible ingredient is supersymmetry, which predicts a supersymmetric partner for each standard model particle; however, despite ongoing efforts, these have evaded detection at any of the particle colliders. Overall, string theory has a long history and has substantially developed over the years, in particular, it has lead to mathematical advances. Still, no observational evidence for string theory has been found.

Perhaps the most conservative approach that pushes the limits of our understanding within the bounds of the well-established theories is that of QFT on curved spacetimes, also known as the semi-classical approach [111, 122]. It sits at the boundaries of general relativity and quantum field theory and relates the two through the semiclassical EFE as a first approximation without introducing new physics. It is not a full theory of quantum gravity, but instead should give direct and robust hints of its effects. A major result within this framework was the (theoretical) discovery that a Schwarzschild black hole can emit so-called Hawking radiation [123, 124].

At the heart of the semi-classical approach lie the semiclassical EFE, where the geometrical, left-hand side is given by classical general relativity, whereas on the right-hand side the energy-momentum tensor is replaced by the expectation value of a quantum operator

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = \kappa \langle T_{\mu\nu} \rangle \ . \tag{1.50}$$

The first step to study quantum effects in any spacetime is then to quantise the energy-momentum tensor on a given curved background and establish its expectation value; the

[16] We do not wish to comment on whether and how string theory can be formulated in a background independent manner and leave such a discussion to the experts. For a discussion of the AdS/CFT correspondence and its consequences for background independence see e.g. [120].

second is to investigate the backreaction problem, namely, the impact of quantum fluctuations on the geometry. Though straightforward in philosophy, quantum field theory on curved spacetime is technically and computationally rather involved.

In the following section we will delve deeper into the world of LQG and spin foams, as these form the basis of GFT. We have omitted many fascinating approaches to quantum gravity, and we want to point the curious reader to [125], where a more complete and comprehensive overview can be found.

1.4.2. Loop quantum gravity and spin foams

Loop quantum gravity is a background independent approach to quantum gravity that applies the canonical quantisation programme to the Ashtekar-Barbero formulation of general relativity, which we described in sec. 1.1.2. Spin foam models aim for a path integral formulation of quantum gravity and include sums over weighted triangulations of spacetime. LQG and spin foams are fascinating approaches to quantum gravity in their own right, but we give particular attention to their conceptual ideas because they form the basis for the motivation of GFT in chap. 2.

Kinematical states of LQG: Spin networks

As we will explore in the next chapter, the elementary quanta of GFT can be interpreted in close correlation to spin network states, which form the elements of the kinematical Hilbert space of LQG. We therefore describe the construction of these states within LQG in some detail here. The introduction to LQG of this section is primarily based on [46], for further LQG literature we also refer the curious reader to [50, 126, 127].

The aim of the LQG programme is to achieve a canonical quantisation of general relativity and it relies on a quantisation of the Hamiltonian formulation in Ashtekar-Barbero variables (see sec. 1.1.2). Recall that the Hamiltonian formulation relies on a splitting of the spacetime manifold M into a spatial hypersurface Σ and a time direction, such that $M = \Sigma \times \mathbb{R}$. The basic canonical variables of the theory are the densitised triad E^a_i and the Ashtekar connection A^i_a. To circumvent the distributional nature of their Poisson brackets (1.23), these variables are smeared in a suitable manner. Specifically, the variables suitable for quantisation are obtained by smearing the Ashtekar connection along curves and the densitised triads along surfaces to obtain holonomies h_c^j and fluxes $E_n(S)$, respectively:

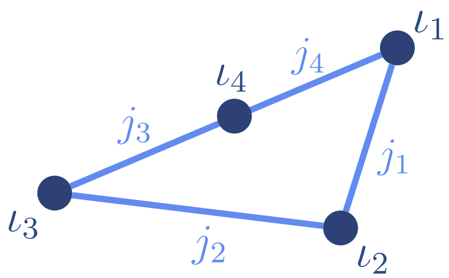
$$h_c^j = \mathcal{P}\exp\left(\int_c A^{(j)}\right) \quad \text{with} \quad A^{(j)} = A_a^i(c(t))\dot{c}^a(t)\tau_i^{(j)}\,, \qquad E_n(S) = \int_S E_i^a n^i \mathrm{d}S_a\,. \tag{1.51}$$

The holonomies are elements of SU(2), $h_c^j \in \mathrm{SU}(2)$, and are labelled by the curve c the Ashtekar connection is smeared over as well as a group representation label j. The terms that enter h_c^j are as follows: $c(t)$ gives a curve on a spatial hypersurface Σ parametrised by $t \in \mathbb{R}$ and the dot denotes the derivative w.r.t. the curve parameter. The $\tau_i^{(j)}$ are $su(2)$ generators in a specific representation. Finally, $\mathcal{P}$ denotes the path ordered exponential. The fluxes $E_n(S)$ are obtained by integrating over a two-dimensional surface S, where the n^i denote suitable test functions $n : S \to su(2)$. One can rewrite the Hamiltonian in terms of these variables and promote the holonomies and fluxes to operators. While straightforward in philosophy, this procedure contains several ambiguities related to operator ordering and a choice of regularisation procedure in practice.[17]

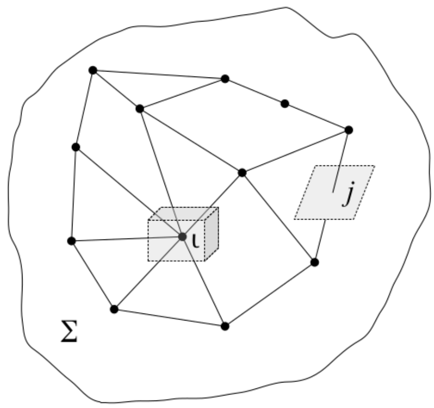
Kinematical states in LQG are commonly referred to as cylindrical functions Ψ. To be more precise, a cylindrical function is a functional of the Ashtekar connection A defined by a collection of curves on the spatial manifold, which we shall refer to as a 'graph' Γ, and a smooth function ψ that takes as input n holonomies $f : \mathrm{SU}(2)^n \to \mathbb{C}$. The tuple (Γ, ψ) defines a cylindrical function $\Psi_{\Gamma,\psi}(A) = \psi(h_{c_1}^{j_1} \dots h_{c_n}^{j_n})$ and the kinematical Hilbert space is obtained from the collection of all cylindrical functions obtained from any Γ or ψ. We can visualise cylindrical functions, which as we discussed are depend on holonomies, which have been attained by smearing the Ashtekar connection along a curve. We can draw this curve as a line, and add a representation or 'spin' label j to have a pictorial representation of a holonomy, and by extension, a cylindrical function.

Elements of the kinematical Hilbert space furthermore need to satisfy the SU(2) symmetry of the internal coordinate system spanned by the triads. Cylindrical functions that are invariant under SU(2) transformations are called *spin networks*. Spin networks form the basis of the discussions that follow below and we will encounter them again when discussing the motivations for GFT in sec. 2.1. It turns out that SU(2) transformations act on the ends of holonomies, and invariant states are obtained by contracting two or

[17] As we detail further in sec. 4.2 these ambiguities translate into different phenomenologies in the context of loop quantum cosmology (LQC).



(a) A spin network can be depicted as a graph with vertices labelled by spins j_i and vertices labelled by intertwiners ι_i.



(b) The edges of a spin network can be associated with quanta of area, whereas the vertices are associated with quanta of volume. This will become more intuitive when we consider triangulations of manifolds. Image taken from [46].

Figure 1.2.: Spin networks can be represented as graphs, where each edge corresponds to a holonomy embedded in a spatial hypersurface Σ and is labelled by a representation label j. The vertices depict gauge-invariant gluings of holonomies at their edge points and are labelled by intertwiners.

more holonomies with one another. Imposing invariance under SU(2) transformations thus requires that a spin network does not contain holonomies with open edges; instead, in the pictorial representation, they must be 'glued' to one another at their open ends. We can represent such a gluing by a vertex, thus obtaining a graph with vertices and edges. These vertices are then labelled by so called intertwiners ι, which form a basis of the subspace invariant under SU(2) transformations. In short, spin networks can be represented by graphs, whose edges represent holonomies and are decorated with SU(2) spin indices j_i, and whose vertices, which represent the gauge-invariant gluing of holonomies, are decorated with intertwiners ι; see fig. 1.2a. Spin networks are eigenfunctions of the holonomy operators and the flux operators act as derivatives. We would like to point out that through their smearing by functions on the spatial hypersurface Σ, spin networks are explicitly embedded in Σ.

Geometrical quantities of the spatial surfaces are encoded by the spin networks and can be obtained from the area and volume operator. These operators are obtained by first discretising the classical quantities in terms of fluxes, and subsequently promoting these to operators. A discretised version of the area of a surface S and the volume of a region

R in terms of fluxes are given by

$$A_S = \sum_i \sqrt{E^n(U_i)E_n(U_i)}\,, \qquad V_R = \sum_i V_{R_i}\,. \tag{1.52}$$

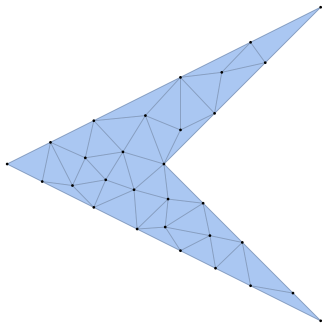
Here, the area of a surface S is obtained by partitioning S into a finite number of smaller surfaces U_i and summing over (the square root of) the fluxes of each of the U_i. The non-discretised area is recovered in the limit of infinitely small U_i. Similarly, the discretised volume of a region R is obtained by separating R into smaller regions R_i. The explicit expression involves a square root over the product of three fluxes. Operator analogues are obtained by promoting the above expressions to operators, which, due to the presence of a square root is not a straightforward procedure. One finds that quanta of area are associated with the edges of a spin network and quanta of volume to its vertices fig. 1.2b. Both give a discrete spectrum, where the eigenvalues are determined by the quantum numbers associated with the edges and vertices. For both operators, the spectrum is bounded from below and gives a minimum value for the area and volume on spatial hypersurfaces. The spectrum of the area operator is included in most reviews, such as [46], whereas for the spectrum of the volume operator we refer the reader to e.g. [128].

So far, we have constructed the kinematical Hilbert space of LQG. It remains to impose the constraints and to identify the states in the kinematical Hilbert space that satisfy the dynamics of the theory and are therefore physical. While the imposition of the diffeomorphism constraint has been achieved, solving the scalar constraint is an ongoing quest in LQG [129].

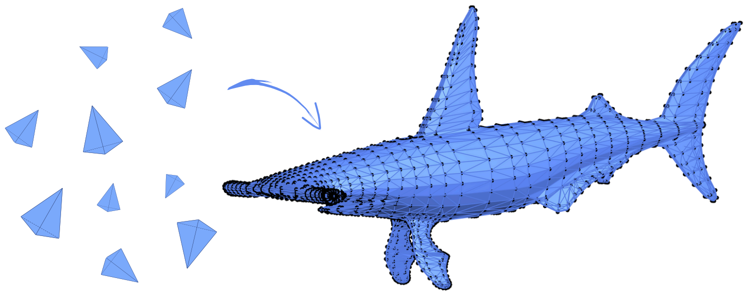
Loop quantum cosmology (LQC) applies LQG techniques to cosmological spacetimes and is a field with rich cosmological phenomenology. The main idea is to restrict to the cosmological setting before quantisation, and quantise only the cosmological sector, which significantly simplifies the procedure. We include further details on LQC in sec. 4.2.

Spin networks and triangulated manifolds

In the following, we discuss the concept of triangulated manifolds and how they can be encoded in spin networks. Spin foams, which we will discuss in the next section, are to be interpreted in terms of triangulations of manifolds and the connection to spin networks will become apparent shortly. The concept of a triangulation refers to the splitting of a manifold into discrete building blocks. (In more formal terms, such a triangulation is



(a) Triangulation of a shape with triangles.



(b) Triangulation of a hammerhead shark with tetrahedra.

Figure 1.3.: Illustration of triangulations in two and three dimensions.

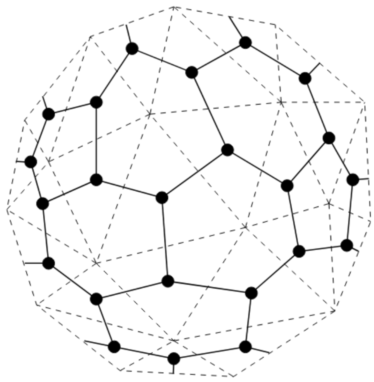
given by a simplicial complex, see e.g. [130, sec. 3].) We illustrate the idea of triangulated manifolds with the following examples (see also fig. 1.3):

- A 2-dimensional surface can be triangulated by multiple triangles, which are glued along their edges.

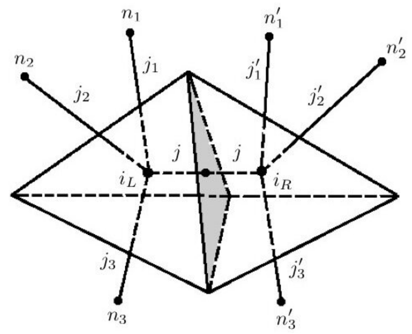
- A 3-dimensional manifold can be triangulated by tetrahedra, the higher dimensional analogue of triangles. The boundary of a tetrahedron can be obtained by gluing four triangles along their edges. If we were to consider 3-dimensional gravity, we could then triangulate a spatial hypersurface with triangles and build a three dimensional spacetime manifold by gluing triangles to obtain tetrahedra.

- A 4-dimensional manifold can be triangulated by the higher-dimensional analogue of tetrahedra (4-simplices), which are difficult if not impossible to be imagined by humans. The boundary of a 4-simplex is obtained by gluing five tetrahedra along their faces. 3-dimensional spatial hypersurfaces of a 4-dimensional manifold are triangulated by tetrahedra.

The aim of the following is to illustrate how the graph of a spin network can be related to a triangulation of a manifold. More specifically we introduce how triangulations of spatial surfaces can be related to 'abstract' spin network states. The term 'abstract' refers to the fact that, unlike in LQG, these spin networks are no longer embedded in a spatial slice Σ, but live on triangulations of Σ instead. From a more philosophical point of view, one might indeed hope that background independent approaches allow to replace the notion of a manifold by a more fundamental combinatorial structure. We focus on



(a) Three valent spin networks correspond to 2-dimensional triangulations. Image taken from [54].



(b) Four valent spin networks give triangulations of 3-dimensional manifolds with tetrahedra. Image taken from [131].

Figure 1.4.: Spin networks can be associated with triangulations of manifolds.

the more intuitive picture and omit technical details, since the primary purpose of the following is to facilitate a more intuitive understanding of GFT and its motivations later on. (E.g., we will be sloppy and use the term 'spin network' for any state that can be represented as a graph with edges and vertices labelled by group theoretic data.) More details can be found in [54, 56].

To relate the concept of a triangulation to spin network states, we first consider the triangulation of a surface encoded in a graph: For a graph with three-valent vertices (3 edges at each vertex), we can associate a triangle to each vertex and each of the edges corresponds to a side of the triangle. Then, a connected graph encodes glued triangles, which represent a triangulation as depicted in fig. 1.4a. For a triangulation of a 3-dimensional hypersurface we restrict ourselves to graphs that consist only of four-valent vertices. Such graphs can be related to a triangulation by assigning to each vertex a tetrahedron (the basic building block of 3-dimensional space) and to each edge of the spin network a face of the tetrahedron. Edges connecting two vertices then represent glued faces of two tetrahedra, see fig. 1.4b. When discretising a gravitational spacetime, one does not only consider a triangulation of the manifold, but the basic building blocks of the triangulation are endowed with additional geometrical information. A spin network is then not described solely by its graph, but carries spin and intertwiner labels on each edge and vertex, respectively. Recalling the area and volume operators in LQG, we can interpret these labels as encoding the geometric structure, such as the volume of the tetrahedra and the area of their faces.

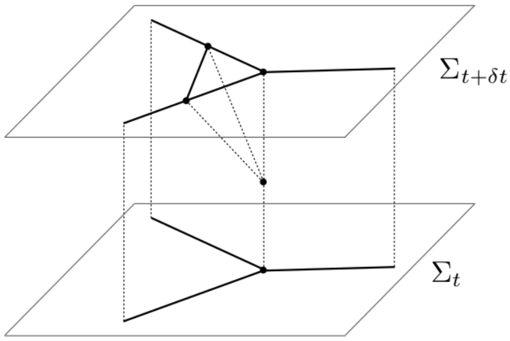


Figure 1.5.: A spin foam interpolates between boundary spin networks and can be understood as a triangulation of spacetime encoded with additional geometric information in the form of group theoretic information. Image taken from [46]

Spin foams

Spin foam models are a path integral approach to quantum gravity, where the idea is to integrate over all possible metrics between fixed spatial hypersurfaces. We restrict the presentation to the main conceptual ideas and refer the reader to [54, 56].

The main quantity of interest in spin foams are transition amplitudes between triangulated spatial slices. These are calculated from a weighted sum over all possible triangulations of spacetime that interpolate between the boundary states. The boundary states representing triangulated hypersurfaces are given by abstract spin network states, as discussed in the previous section, with the generalisation that the graphs can carry group representation data of more general groups and are not restricted to SU(2). While spin networks encode triangulations of *space*, spin foams give triangulations of *spacetime*, and their boundaries are given by spin networks. A spin foam is in essence a higher dimensional analogue of a spin network: Where a spin network is a graph consisting of edges and vertices decorated with group theoretic data, a spin foam is built from vertices, edges, and faces, endowed with the same information, see fig. 1.5. (In the language of simplicial geometry, the 'graph' of a spin foam is a 2-complex.) In relation to quantum gravity, the idea is that a spin foam is a triangulated manifold decorated with group representation data.

To calculate the transition amplitude between two given spin networks, one sums over all possible spin foams that could interpolate between them, where each spin foam is assigned a certain weight. This weight is referred to as the spin foam amplitude. Which spin foams are to be considered in this sum and their respective weight is what determines a specific spin foam model. The partition function of a spin foam model can be written as:

$$Z = \sum_{\Gamma} w(\Gamma) \sum_{j_f, \iota_e} \prod_{f} A_f(j_f) \prod_{e} A_e(j_f, \iota_e) \prod_{v} A_v(j_f, \iota_e) \,, \tag{1.53}$$

where the sum in Γ is over all 2-complexes weighted by $w(\Gamma)$ and the products contain amplitudes associated with faces A_f, edges A_e, and vertices A_v, respectively. These amplitudes depend on the spins associated with faces j_f and intertwiners associated with edges ι_e.

To obtain a spin foam model for a specific theory of quantum geometry, one discretises the action to obtain a prescription for the discretised partition function, e.g. $Z = \int \mathcal{D}A\,\mathcal{D}B\; e^{\mathrm{i}S_{BF}}$ and furthermore assigns a weight $w(\Gamma)$ to possible triangulations. For instance, the discretisation of the BF action in four dimensions with $G = SO(4)$ reduces the sum over 2-complexes in Z to a single element; it furthermore determines which representations j are to be summed over and gives a prescription for the vertex amplitude, see [56] for details.

We already mentioned BF theory as a topological theory. In four dimensions GR is evidently not topological and we need to impose additional constraints to obtain general relativity from the BF action. The implementation of these so-called simplicity constraints at the spin foam level is the final step to a theory of quantum gravity within the spin foam approach. Unfortunately, this is far from trivial and an ongoing research topic [132, 133]. We briefly mention two approaches for a spin foam model for 4-dimensional gravity:

- Barrett-Crane model [134, 135]: Starting from a discretised 4-dimensional BF theory, the Barrett-Crane model gives a prescription on how to impose the simplicity constraints at the quantum level. This can be achieved by imposing restrictions on the group representations that are associated with the triangulations of the boundary states. In particular, the sum in the spin foam amplitude is limited to a specific class of representations of the group. For the Lorentzian case, the quantum model is based on representations of $SL(2,\mathbb{C})$, which makes the study of such models more complicated than in the Euclidean case where $G = SO(4)$, due to non-compactness of the group.

- EPRL model [133, 136, 137]: The EPRL model similarly imposes the simplicity constraints on a 4-dimensional BF theory at the quantum level, where in the Lorentzian case the boundary networks are labelled by $SL(2,\mathbb{C})$ representation data. The model introduces a projection from these boundary states to spin networks labelled by SU(2) representation data, thus linking the boundary states of the EPRL model back to LQG.

1.5. Conclusion: Back to the basics

In this chapter, we gave a broad overview of concepts that are related to the research aims of this thesis. While some concepts might be well-known to a large number of physicists, others may appear rather basic only within a specialised field of research. Specific background knowledge that interpolates between cosmology on the one hand and quantum gravity on the other is found in the remainder of the first part of the thesis.

We began our journey with a quick survey of GR and its formulation in the language of differential geometry. The Einstein Field Equations (EFE) encode the dynamics of GR and can be derived from an action principle, where the geometrical part is given by the Einstein-Hilbert action. As an alternative to the Lagrangian formulation, there exists also a Hamiltonian approach to GR, where spacetime is decomposed into spatial hypersurfaces and a temporal direction. One finds that the Hamiltonian of GR is given by a sum of constraints, namely, the Hamiltonian or scalar constraint, which gives time translation, and the spatial diffeomorphism constraint, which generates translations within a spatial slice. In the original Hamiltonian formalism by Arnowitt-Deser-Misner the spatial metric gives one of the canonical phase space variables, whereas the Ashtekar-Barbero formulation makes use of the triad formalism and introduces an internal reference frame. As the canonical variables of the latter are more suitable for quantisation, this formulation forms the basis of LQG. One can furthermore obtain GR from the action of a BF theory with the correct choice of dimension and gauge group if one introduces additional constraints, which forms the basis for spin foam models.

Despite its success, GR predicts its own breakdown in the form of singularities, which manifest in the divergence of curvature scalars, but can more specifically be defined in terms of geodesic incompleteness. Due to general covariance the definition of observables requires some additional care in GR, where it is useful to make a distinction between partial and complete observables. The latter are defined in relational terms in the sense that any physical statements can only be made by relating the values of two physical quantities. Much of this thesis will be concerned with matter reference frames made from four scalar fields that allow to make statements in a relational coordinate system.

A homogeneous and isotropic solution that can be used to describe our universe is the FLRW metric. We included a curvature term as well as a cosmological constant in the discussion, but will only be concerned with the case of a flat FLRW spacetime with vanishing cosmological constant from here onwards. The universe's matter content can be described in the form of a perfect fluid. Another useful description is in the form of a scalar field. From the EFE one obtains the Friedmann equations, which capture the evolution of the scale factor and thereby the evolution of the cosmos. We keep the lapse function general throughout, as we will require a specific choice of time coordinate when considering quantum gravitational models. The Big Bang singularity naturally occurs in a cosmological spacetime: GR cannot describe the origin of the universe, a gap in our understanding that can hopefully and ideally be resolved by a theory of quantum gravity, which is indeed the case for GFT cosmology and LQG as we discuss in chap. 4. In the description of the universe's evolution and the particles contained within we have to deal with an intricate interplay of particle, nuclear and statistical physics. Many of these processes are experimentally constrained such that any alterations to the early universe we consider in the quantum gravitational context are assumed to affect only regimes in the extremely early universe.

The standard model of cosmology is not without troubles of its own. We focused on the horizon problem for two reasons: Firstly, to introduce a relation between the particle horizon, which plays the role of a scale that determines past causal relations, and the Hubble horizon. This is of importance for the study of large scale cosmological perturbations as we explain in sec. 3.3 and study further in chap. 5. Secondly, quantum gravity bounces straightforwardly provide a solution to the horizon problem.

The evolution of the universe as modelled in GR by means of the FLRW metric together with a period of inflation and different types of perfect fluids agrees well with various observations if we allow for small inhomogeneities. Importantly, fluctuations in the early universe are imprinted in the CMB. Any quantum gravity theory needs to be consistent with the CMB, therefore, in addition to the FLRW background, cosmological perturbations provide important guidance for constructing quantum gravitational models. Possible quantum gravity effects on large scale perturbations will be investigated in chap. 5 in a model independent manner, whereas chap. 7 concerns itself with perturbations within the GFT framework.

Aside from gravity, the other ingredient to quantum gravity is the 'quantum'. The requirement for a quantum theory of gravity is motivated not just from the occurrence of singularities, but also from the fact that we know matter exhibits a quantum nature which, due to the relation of matter and geometry, must carry over to the spacetime description. While there are various pathways to approaching the problem of quantum gravity, we focus on LQG and spin foams as a preliminary to the following chapter. LQG is based on the Ashtekar-Barbero formalism of GR and variables suitable for quantisation are obtained by smearing the classical phase space variables. One can identify a kinematical Hilbert space that satisfies the necessary gauge-invariance w.r.t. the internal triad frame. The gauge-invariant states are called spin network states and are explicitly embedded in a spatial hypersurface. Spin networks can be interpreted as graphs decorated with group theoretic data and they allow to recover geometric quantities of spacetime via area and volume operators. One can furthermore introduce a more abstract notion of spin network graphs, where all vertices have the same valency. These can be interpreted as triangulations of a spatial manifold. Based on the notion of triangulated spatial surfaces, spin foams are a path integral formulation for quantum gravity. The aim of spin foam models is to calculate transition amplitudes between triangulated manifolds as encoded in spin networks and a given spin foam model determines how possible interpolations between boundary spin networks should be weighted.

Having recalled the basics of the theories we know, and reiterated the need for a quantum gravity theory, as well as detailing some possible approaches, we are now in a position to introduce the quantum gravity theory we primarily focus on in this thesis: Group field theory.

We would like to end this chapter with a more philosophical note: approaches to quantum gravity are vast and at this moment in time, incomplete. The quantum gravitational problem can be approached from the mathematical and more rigorous perspective, hoping that a suitable theory can be found eventually if theoretical inconsistencies are taken seriously and resolved. Alternatively, one may hope for phenomenological guidance by applying perhaps theoretically incomplete theories to simpler scenarios, such as cosmology, in the hope to learn a) whether considerations made in the theory so far are compatible with previous knowledge and/ or b) which alterations to the fundamental theory lead to (more) suitable models for reality. While the main curiosity of this thesis is to explore the latter, we believe that ultimately progress is to be made by allowing both avenues of inquiry to inform one another.

Chapter 2.

Group field theory

'Das Dreieck ist das Maß aller Dinge.'

- Jessica Volz.

'Triangles are the measure of all things.'

One of the core aims of this thesis is to illuminate a new path for extracting cosmological phenomenology from group field theory (GFT), which is a background independent approach to quantum gravity. GFTs first appeared in 1992 in a 3-dimensional quantum gravity model introduced by Boulatov [138]. They have since been studied in the context of LQG and spin foam models [139–143] and have developed into their own research field. In particular, much progress has been made in the realm of homogeneous and isotropic cosmology. The application of GFT to the cosmological context leads to a resolution of the Big Bang singularity with a bounce [144, 145], which is a desirable feature of any quantum gravity theory. Further studies have revealed that GFT can introduce additional phenomenologically interesting features in the cosmological evolution [146–148] and the GFT setting has been extended to include cosmological perturbations, which constitutes a first step in bridging the gap to observations [149–152]. In GFT, the relation between the cosmological sector and the full theory is straightforward, and it thus appears as a promising candidate to connect quantum gravity to cosmological phenomena. We introduce the foundations and core concepts of the GFT approach in this section; the application of GFT to the cosmological sector is discussed in the context of a later chapter can be found in sec. 4.1. We first give a succinct overview of GFT and the specific formulation we use later on before we proceed to motivate and explain the theory in more detail, such that concepts that appear *ad-hoc* at first should become successively more clear. While we aim to give an overview that is sufficient to follow the research results of this thesis, the scope will inevitably be limited and for further intriguing details of GFT, the reader is referred to excellent reviews such as [130, 139, 153, 154] . For details regarding group theoretic ideas used below we refer the reader to [155].

GFT, in its essence, is a field theory of one or multiple fields on a group manifold, together with an action. A specific GFT is then obtained by specifying a dimension for the group manifold, a group, a kinetic and potential term for the GFT action, as well as additional constraints on the group field. GFT does not presuppose a spacetime manifold, instead, spacetime is dynamically emergent from a large number of GFT quanta, which should be understood as the building blocks of spacetime. The setup of group field theories has a wide range of flexibility and can accommodate a multitude of quantum gravity models, depending on their concrete implementation. Most of the literature is concerned with a scalar group field that obeys bosonic statistics and can be real or complex. We also consider only scalar fields; for considerations regarding fermionic fields see [156]. For a $d-$dimensional GFT, which should give a quantum theory for $d-$dimensional gravity, the domain of the group field φ contains $d-$copies of a group G and n additional real arguments (where $n = 0$ is possible):

$$\varphi : G^d \times \mathbb{R}^n \to \mathbb{R} \text{ or } \mathbb{C}\,. \tag{2.1}$$

We will denote the group field as $\varphi(g_I, \chi^A)$, where $I = 1 \dots d$, $A = 0 \dots n-1$, the g_I denote the group elements, and the χ^A the real valued functions.[1] The dynamics of the field is governed by an action

$$S[\varphi] = K[\varphi] + V[\varphi], \tag{2.2}$$

where $K[\varphi]$ and $V[\varphi]$ denote the kinetic and potential term, respectively. The choice of the group G, the dimension of the GFT domain d, the coupling of real parameters χ^A, the form of the kinetic and potential term, as well as additional requirements on the fields are all motivated by physical considerations. We stress that $\varphi(g_I, \chi^A)$ is not defined on a spacetime manifold, but on an abstract group manifold. One might rightfully ask about the origin of the interpretation of GFT as a theory of quantum gravity. The excitations

[1]We start the index A at zero, in accordance with spacetime indices; later we will interpret χ^0 as a relational clock.

in the quantum theory, which we refer to as GFT quanta or particles, are interpreted as spatial building blocks from which a macroscopic geometry that can be related to the classical spacetime structure of GR emerges in the limit of a large amount of quanta. The picture is then that these quanta create spatial hypersurfaces and their collective behaviour resembles that of a classic spacetime (in appropriate limits and for certain state choices). How this interpretation arises from the relation to spin networks, spin foams and triangulations of manifolds as discussed in sec. 1.4.2 and how this idea can be implemented in practice will become clear in the remainder of this chapter. Ultimately, GFT can be seen as a field theoretic formulation of a quantum gravity model that incorporates ideas from LQG and spin foam approaches. Independent avenues to investigate how spacetime could emerge from GFTs have progressively evolved, where the rigorous relation to the spacetime description within GR is in some cases not so clear and poses an open research question.

The rest of this chapter is structured as follows: The motivation behind interpreting GFT as an approach to quantum gravity is the subject of sec. 2.1, where we relate its quanta and formulation to spin networks and spin foam models. We detail the construction of GFT in sec. 2.2, where we motivate the main choices that give a concrete implementation of a GFT. We furthermore discuss how dynamics in GFT should be understood as a relational evolution with respect to a physical matter clock and examine possible forms of the GFT action and the restrictions imposed by a matter clock field. In sec. 2.2.3 we present quantisation approaches of GFT that have been established in the literature, with a particular focus on the Hamiltonian, or 'deparametrised' formalism, which we employ in the second part of this thesis. We dedicate sec. 2.2.3 to the concrete GFT with four massless scalar fields that was established in [78] and gives the construction we use for the research conducted in chap. 6 and 7. Finally, we comment on further research directions in sec. 2.3 and postpone the discussion of applications of GFT to cosmology to sec. 4.1. We conclude with an overview of the material presented in this chapter in sec. 2.4.

For the hasty reader, we disclose already at this point that the research in chap. 6 and 7 will be concerned with a real group field φ that is a function of four copies of SU(2), as well as four massless scalar fields χ^A:

$$\varphi(g_I, \chi^A) = \varphi(g_1,\, g_2,\, g_3,\, g_4; \chi^0,\, \chi^1\,, \chi^2, \chi^3), \qquad g_I \in \mathrm{SU}(2),\; \chi^A \in \mathbb{R}\,. \tag{2.3}$$

For $G = \mathrm{SU}(2)$ the group field can be decomposed into Peter-Weyl modes, which simplifies further calculations

$$\varphi(g_I, \chi^A) = \sum_J \varphi_J(\chi^A)\mathcal{D}_J(g_I)\,, \tag{2.4}$$

where $J = (\vec{j}, \vec{m}, \iota)$ is a multi-index consisting of spins $\vec{j} = (j_1, j_2, j_3, j_4)$ that label irreducible representations of SU(2), magnetic indices $\vec{m} = (m_1, m_2, m_3, m_4)$ with $m_i \in \{-j_i, -j_i+1, \ldots, j_i-1, j_i\}$, and intertwiner labels ι, which we already encountered in the description of spin networks in sec. 1.4.2. We make this decomposition precise in sec. 2.2 and the definition of $\mathcal{D}_J(g_I)$ is given in (2.11). The theory can then be formulated in terms of the modes $\varphi_J(\chi^A)$. We restrict our analysis to an action consisting only of a kinetic term, which takes on the following form

$$K[\varphi] = \int d^4\chi \sum_J \left(\frac{1}{2}\mathcal{K}_J^{(0)}\varphi_J^2 - \frac{1}{2}\mathcal{K}_J^{(2)} \sum_A (\partial_A \varphi_J)^2 \right)\,, \tag{2.5}$$

where $\partial_A = \frac{\partial}{\partial \chi^A}$ and $\mathcal{K}_J^{(0)}$ and $\mathcal{K}_J^{(2)}$ are mode dependent constants. The considerations that motivate this particular choice of GFT action are laid out below and its construction will specifically be the topic of sec. 2.2.3.

2.1. Motivations and relation to other approaches

The purpose of this section is to provide a heuristic interpretation of the GFT quanta and the motivations behind the choices made to construct a specific quantum gravity model. This should clarify in which sense GFTs are viewed as candidate theories for a background independent approach to quantum gravity. LQG, spin foams and GFT are actively developing fields of research and one might hope that GFT might provide insights also for the other approaches, and vice versa. We refer the reader to the description of LQG and manifold triangulations given in sec. 1.4.2; we will assume knowledge of the concepts introduced therein in what follows.

2.1.1. Interpretation of GFT quanta

This section discusses the interpretation of GFT quanta and how the dimension, gauge group and constraint on the group field can be motivated from (abstract) spin networks. The underlying idea is that the GFT quanta can be interpreted as open spin network vertices, where we recall that spin networks form the kinematical Hilbert space of LQG and abstract spin networks can be related to triangulations of manifolds (see sec. 1.4.2). More specifically, spin networks can be depicted as closed graphs decorated with group theoretic data on their vertices and edges. If one restricts to graphs with d edges at every vertex, each graph can be associated with the triangulation of a $d-1$ dimensional manifold. The group theoretic data encode geometric degrees of freedom, specifically, in LQG we recall that they are related to the holonomies, which encode the connection. The interpretation of GFT quanta as a spin network vertex of LQG then implies $G = \mathrm{SU}(2)$. As we saw, due to the SU(2) gauge-invariance of the internal reference system, spin networks contain no holonomies with open edges. Hence, in LQG, no open spin network vertices exist, as they would break the rotational symmetry. If GFT field excitations are to correspond to open spin network vertices, we then need to impose an additional condition on the GFT field to satisfy the SU(2) symmetry, which is commonly done by imposing right invariance of the group field under the action of the group, namely, for a 4-dimensional GFT,

$$\varphi(g_1 h, g_2 h, g_3 h, g_4 h) = \varphi(g_1, g_2, g_3, g_4) \quad \forall h \in \mathrm{SU}(2)\,. \tag{2.6}$$

In the Peter-Weyl decomposition, GFT quanta can be interpreted as open spin network nodes labelled by an intertwiner ι, where the edges are decorated with spin labels j as well as magnetic indices m. Recalling the relation between abstract spin networks and triangulations of manifolds, one can also interpret the GFT quanta as simplicial complexes. In order to obtain triangulations of spatial slices in 4-dimensional gravity, we can fix the dimension of the domain of the group field to $d = 4$ such that a GFT quantum corresponds to a tetrahedron. These tetrahedra are then the fundamental building blocks for triangulations of 3-dimensional spatial slices, consistent with the portrayal of GFT quanta as 'building blocks of space'.

We postpone the detailed discussion of quantisation approaches in GFT to sec. 2.2.3, but independently of the approach, the Hilbert space of GFT is given by a Fock space

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathscr{H}_S^{(n)}\,, \qquad \mathscr{H}_S^{(n)} = S \otimes^n \mathscr{H}^{(1)}\,, \tag{2.7}$$

where $\mathscr{H}^{(1)}$ is a single particle Hilbert space and $\mathscr{H}_S^{(n)}$ denotes the symmetrised tensor product of n single particle Hilbert spaces, as dictated by the bosonic statistics of the group field. In accordance with the interpretation that GFT quanta constitute the building blocks of a spatial manifold, the Fock vacuum $|0\rangle$ is to be understood as a 'no-space' state. Each GFT quanta is endowed with the same group theoretic data as a spin network vertex of the same valency and it was argued in [157] that the canonical formulation of GFT can be understood as a second quantisation of LQG. The interpretation of GFT quanta as spin network vertices allows us to import the area and volume operators, where we recall that quanta of area are associated with the edges and quanta of volume to the nodes of spin network states. Similarly, the GFT quanta then carry quanta of volume.

In the case where one considers a single group field with fixed dimension $d = 4$, the Fock space includes only spin networks with four-valent vertices, whereas in LQG a spin network can contain vertices with arbitrary valency. It is furthermore important to note that while the GFT Fock space elements carry the same group theoretic data as spin networks, they do not contain the same connectivity information and do not directly correspond to a graph. In general, one can construct a GFT state inspired by a spin network, but cannot uniquely reconstruct a spin network graph from a given GFT state, see [154] for a more detailed discussion. (We comment on considerations that have been made for connected states in GFT in sec. 2.3.)

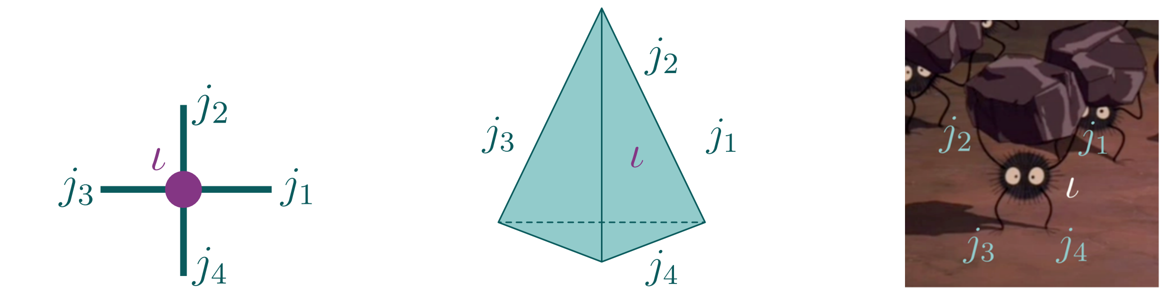
Another main conceptual difference between GFT excitations and spin network states is that, while the latter are embedded in a classical spatial surface of a spacetime and can be related to the triangulation of said surface, in GFT we are concerned with field excitations over a group manifold and spacetime emerges only in a many particle limit. In this sense, the vertices created by the GFT quanta are more abstract than the spin networks related to triangulations discussed in sec. 1.4.2. This is what we refer to when we say GFT is a theory *of*, not *on* spacetime: before spacetime emerges through the multitude of excitations, it is simply absent. The philosophy is then that in a GFT that poses as a suitable candidate for a theory of quantum gravity, fluctuations in (geometric) observables should be sufficiently suppressed for a multitude of GFT quanta, such that the emergent spacetime exhibits semiclassical properties and can be described using the methods of GR. This process is often compared to hydrodynamics: the parallel is drawn by viewing the GFT quanta as water molecules, the multitude of which leads to the emergence of a fluid, where the molecules (GFT quanta) are described by a different set of physical laws than the fluid (spacetime).

In light of these differences in the formulation of the GFT Fock space and the Hilbert space of (abstract) spin networks, we would like to reflect on the pictorial interpretation of GFT on a more personal note: While it is of course helpful to associate abstract concepts and elements of theories to visual representations that can allow to build an intuitive understanding, one should be cautious in taking these at face value. GFT is a rather abstract formulation of a rather abstract phenomenon (we have of yet no reliable quantum gravitational implications for reality, let alone from GFT, after all) and the relation to phenomena within our comprehension (i.e. classical spacetime physics) is to be found in a suitable semiclassical limit. Perhaps a more playful alternative to the standard interpretation as tetrahedra or spin network vertices that also emphasises that while well motivated from these constructions, the GFT quanta are more abstract, can be given by thinking of the quanta as four-legged spiders (or the little coal monsters 'Susuwatari' in Studio Ghibli's 'Spirited Away'): each of the four legs would then carry a spin j and a magnetic index m and the 'body' would be associated with an intertwiner label ι. The different possible pictorial interpretations of GFT quanta are summarised in fig. 2.1.

This concludes our discussion of the interpretation of GFT quanta and the GFT Fock space. The reader may use whichever interpretation they find most satisfactory. We discussed the motivations behind setting $G = \mathrm{SU}(2)$, $d = 4$, and imposing right invariance on the group field. Ultimately, GFT can be seen as a tool that can be utilised to simplify calculations in other approaches, or establish new avenues to tackling the problem of quantum gravity in a background independent manner. We proceed to describe how GFTs naturally occur in spin foams, where they provide a machinery to calculate spin foam amplitudes.

2.1.2. Spin foam amplitudes from GFT

We remind ourselves that the aim of spin foam models is to compute spin foam amplitudes, which represent the probability to transition between quantum geometries on spatial hypersurfaces (see sec. 1.4.2). More explicitly, they are obtained from a weighted sum over spin foams that interpolate between boundary spin network states. If we restrict to spin networks with a fixed number of edges, the boundary states can be interpreted as triangulations of the spatial manifolds and the weighted spin foam in each term of the sum corresponds to a triangulation of spacetime. For any given spin foam model, this amplitude can be calculated from a GFT, provided one makes adequate choices for the group, additional constraints, and action [141–143]. Specifically, for a suitable choice of GFT action $S[\varphi]$, a perturbative expansion of the partition function $Z = \int \mathcal{D}\varphi e^{-S[\varphi]}$



(a) GFT quanta as a four-valent spin network vertex. The intertwiner ι labels the vertex and the spins j_i are associated with the edges.

(b) GFT quanta as a tetrahedron. The intertwiner ι labels the body of the tetrahedron and the spins j_i are associated with the faces.

(c) GFT quanta as an abstract Susuwatari. The intertwiner ι labels the body the spins j_i are associated with the legs. Image from 'Spirited Away' by Studio Ghibli (2001).

Figure 2.1.: Pictorial representation of the different interpretations for GFT quanta suggested in the text (where the last one is a light-hearted reminder that all visual representations are incomplete). The magnetic indices m are not shown, but are associated with the ends of the open edges (and are therefore absent in LQG spin networks, which are closed graphs).

generates all possible Feynman diagrams (which correspond to spin foams and thereby triangulations) that interpolate between given boundary spin networks. GFT thus gives a systematic way to compute transition amplitudes.

To illustrate this further, we consider spin foam models for 3- and 4-dimensional BF theory. As mentioned already in the introduction to this chapter, the first occurrence of GFT was as a 3-dimensional quantum gravity model, dubbed the Boulatov model [138]. The Boulatov model was extended to four dimensions by Ooguri [158] to yield a quantum theory for 4-dimensional BF theory (which is still a purely topological theory). Both the Boulatov and the Ooguri model are respectively formulated as a 3- and 4-dimensional GFT with a bosonic scalar field, where the group for concreteness is assumed to be SU(2). The action is chosen such that it generates triangulations of spacetime and both models have a kinetic term that is local in the group elements. In the case of the Boulatov model, the interaction term couples four copies of the group field φ^4, which is pictorially to be understood as a prescription to glue four triangles to obtain a tetrahedron. The Ooguri model contains a quintic interaction, φ^5. In this case, the GFT quanta can be interpreted as tetrahedra, and the interaction term contains the prescription to glue five tetrahedra to a four-simplex. Ultimately, through several interaction vertices one then creates triangulations of spacetime. These triangulations are obtained in the form of an expansion of the generating functional as Feynman graphs, following the standard procedure of QFT. The Feynman graphs are to be interpreted as 4-dimensional simplicial complexes, which correspond to a triangulation of the 4-dimensional spacetime manifold, and their weight is determined by the details of the respective triangulation. In this manner, one creates a weighted sum over all triangulations allowed by the choice of action.

Recall that in GR, imposing the simplicity constraint on a 4-dimensional BF theory turns the topological theory into a description of Einstein gravity (known as the Plebanski formulation, see sec. 1.1.2). One can then hope to formulate a path integral for full quantum gravity by imposing the simplicity constraint on the quantised 4-dimensional BF theory, as is the aim of e.g. the EPRL model [133, 136, 137]. One can use insights from the EPRL model to restrict the form of kinetic as well as the interaction term of the GFT action (even though few explicit studies of interactions within GFT exist and their role in cosmological applications has so far mostly been neglected). Furthermore, the boundary states of the EPRL model are spin network states decorated with SU(2) variables, akin to LQG. In general, the relation between GFT and spin foam models can be used to infer desirable properties of the GFT action.

2.2. Constructing a GFT

Having explored the motivations behind the original formulation of GFTs, we now delve deeper into the details of their construction. We recall that a specific GFT is determined by the choice of gauge group, field type, dimension, additional constraints, and finally, the GFT action. We detail a GFT based on the aim to construct a quantum model for 4-dimensional gravity, using the relation to spin networks and spin foams discussed in the previous section. After re-capping the choices for the above-mentioned main ingredients, we introduce a mode decomposition of the group field. We proceed to explain how dynamics in GFT are achieved by introducing a matter field as a clock and should be understood as relational in the sense of relational observables as explained in sec. 1.1.4. The matter clock imposes restrictions on the GFT action and we consider a concrete example for the kinetic term, which will become relevant for the discussion of GFT cosmology and the results of the second part of this thesis. We also comment on possible interaction terms. Finally, we discuss quantisation techniques for GFT, where two different approaches to the quantum theory, which have been dubbed the ‘deparametrised’ and ‘algebraic’ approach by [148], have evolved. We focus on the deparametrised approach, which is the one relevant for the rest of this thesis, but provide a short introduction to the latter as well.

We choose a real bosonic group field φ in what follows; the construction with a complex field follows the same steps. While the results of chap. 6 and chap. 7 are not tied to a specific choice of gauge group, and in principle could be carried out for any compact group, the relation to LQG would be less clear. For concreteness we then choose $G = \mathrm{SU}(2)$ in the following. To obtain 4-dimensional gravity from GFT we include four copies of SU(2) in the field domain, i.e. [2]

$$\varphi : \mathrm{SU}(2)^4 \to \mathbb{R}\,, \qquad \varphi(g_I) := \varphi(g_1, g_2, g_3, g_4) \quad \text{with each } g_I \in \mathrm{SU}(2)\,. \tag{2.8}$$

Akin to LQG, it is useful to carry out a decomposition of the group field into modes labelled by SU(2) representation data. The Peter-Weyl theorem states that any (square integrable) function on a compact group G can be decomposed as a direct sum over matrix coefficients of irreducible unitary representations, which form an orthonormal basis of $L^2(G)$ [159]. Specifically, in the case of SU(2), such a basis is given by the Wigner D-matrices [160, chap. 4]. A Wigner D-matrix $\mathcal{D}^j(g)$, $g \in \mathrm{SU}(2)$, is a unitary square matrix labelled by an irreducible representation j of SU(2) and has dimension $2j+1$. For $g = \mathcal{R}(\alpha,\beta,\gamma) = e^{-i\alpha\tau_x}e^{-i\beta\tau_y}e^{-i\gamma\tau_z}$ and τ_i the generators of SU(2) in a specific representation j, the matrix elements of $\mathcal{D}^j(g)$ are explicitly given by

$$\mathcal{D}^j_{m,m'}(\alpha,\beta,\gamma) = \langle jm'|\, \mathcal{R}(\alpha,\beta,\gamma)\, |jm\rangle \ , \tag{2.9}$$

where $|jm\rangle$ denotes the basis vectors of the $2j+1$ dimensional Hilbert space associated with each spin, where $m \in \{-j, -j+1, \ldots, j-1, j\}$ and $\langle jm'|jm\rangle = \delta_{m'm}$. For details, such as explicit calculations of these matrix elements see [160, chap. 4], [161, sec. 8], [155]. The GFT field $\varphi(g_I) \in L^2(\mathrm{SU}(2)^4)$ can then be decomposed as a mode sum over products of four Wigner D-matrices

$$\varphi(g_I) = \sum_{\vec{j}\vec{m}\vec{n}\iota} \varphi_{\vec{j}\vec{m}\iota} \mathcal{I}_{\vec{j}\vec{n}\iota} \prod_{i=1}^{4} \sqrt{2j_i+1}\,\mathcal{D}^{j_i}_{m_i n_i}(g_i)\,, \tag{2.10}$$

where the indices in the sum mark different modes and are referred to as 'spins' $\vec{j} = (j_1, j_2, j_3, j_4)$, 'magnetic indices' $\vec{m} = (m_1, m_2, m_3, m_4)$ and $\vec{n} = (n_1, n_2, n_3, n_4)$, and 'in-

[2] We will introduce real valued arguments of the GFT field as contained in (2.3) in a consecutive manner, emphasising their motivations and implications for the construction.

tertwiner labels' ι. While the spins label irreducible representations of SU(2), we have $m_i, n_i \in \{-j_i, -j_i+1, \ldots, j_i-1, j_i\}$. The intertwiner $\mathcal{I}_{\vec{j}\vec{n}\iota}$ appears in the decomposition when we impose gauge-invariance on the group field (2.6) and use the properties of the Wigner matrices. The intertwiner label ι then again, just like in LQG, numbers the basis of the subspace invariant under SU(2) transformations. For more details see, e.g., [162]. To ease the notational load, we introduce the multi-index $J = (\vec{j}, \vec{m}, \iota)$ and the following shorthand

$$\mathcal{D}_J(g_I) := \sum_{\vec{n}} \mathcal{I}_{\vec{j}\vec{n}\iota} \prod_{i=1}^{4} \sqrt{2j_i+1}\mathcal{D}_{j_i m_i n_i}(g_I)\,, \qquad \Rightarrow \qquad \varphi(g_I) = \sum_J \varphi_J \mathcal{D}_J(g_I)\,. \tag{2.11}$$

The Peter-Weyl theorem holds for any compact group, and hence a similar decomposition can be carried out for other choices of compact G. For non-compact groups, e.g. SL(2,$\mathbb{C}$) which is of interest in the construction of the Lorentzian Barrett-Crane model [135], additional subtleties arise due to continuous representations. These additional technical difficulties seem manageable, however, and the construction of GFT models based on SL(2,$\mathbb{C}$) has been considered in e.g. [151, 152].

2.2.1. A massless scalar field as a matter clock

Dynamics in GFT are usually described relationally with respect to a massless scalar field clock, which we denote as χ^0 [144, 154]. As disused in sec. 1.1.4, relational observables are a useful tool in GR and its quantisation approaches, as they allow to evade the issue of being limited to global observables. For a relational observable, one considers the value of a quantity of interest w.r.t. another physical quantity, allowing for a coordinate independent statement. In the case of time evolution, a good clock variable is then typically given by quantity that evolves monotonically. Matter clocks make common appearances in cosmology, maybe most famously in the form of pressureless matter fields, known as dust [76, 77, 81] as introduced by the Brown-Kuchař model [73]. Clock fields are also frequently used in quantum cosmology, where they allow to evade the problem of time [163–165], as well as in LQG [82] and LQC [166, 167]. Similar to the GFT case, LQC employs a massless scalar field as a clock. There are other possibilities, and one can also consider evolution w.r.t. geometric quantities, such as the shear in Bianchi models [168], or, for an example in GFT, an anisotropy parameter as defined in [148].

So far we have focused on a group field whose domain is given solely by (multiple copies of) elements of the chosen gauge group. We already noted in the introduction of this chapter that it is possible to extend this domain to include real valued functions. The massless scalar field that is to serve as a relational clock is included in GFT by extending the domain of the group field [144]

$$\varphi : \mathrm{SU}(2)^4 \times \mathbb{R} \to \mathbb{R}\,, \qquad \varphi(g_I, \chi^0) = \sum_J \varphi_J(\chi^0)\mathcal{D}_J(g_I)\,, \tag{2.12}$$

and carries over as an argument of the Peter-Weyl modes $\varphi_J(\chi^0)$. When connecting to GR, GFT quanta with the same scalar field value are then interpreted as belonging to the same spatial slice, and evolution takes place w.r.t. the value of the scalar field. A massless scalar field constitutes a suitable matter clock due its monotonicity. Its action in a general relativistic spacetime reads

$$S_\chi = -\frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}\,g^{\mu\nu}\partial_\mu\chi^0\partial_\nu\chi^0\,. \tag{2.13}$$

Imposing that the shift $\chi^0 \to \chi^0 + \epsilon$ and reflection $\chi^0 \to -\chi^0$ symmetries of the classical scalar field action (2.13) are respected at the level of the GFT action reinforces the interpretation of χ^0 as a massless scalar field. As we will see below, these symmetries significantly restrict the permissible form of terms that can be included in $S[\varphi]$.

In this thesis we treat the scalar field classically, however, this does not imply that the scalar field cannot be included in the quantum system; see also the discussion in sec. 2.2.3. One may raise the question about the effects of potential non-monotonicity of a quantised scalar field. Quantum reference frames have been considered more broadly in the literature [169–172], as well as specifically in the context of the relational dynamics from a quantum field in GFT [173–175]. Following e.g. the results of [175], roughly speaking, relational dynamics of GFT operators can be obtained by projecting on a clock state. One should then view any observables of the quantum theory as conditioned on the value of the clock field, in the sense that $\langle\hat{\mathcal{O}}(\chi^0)\rangle$ gives 'the expectation value of $\hat{\mathcal{O}}$, given that the measurement of the scalar field reads χ^0'. The operators constructed in this manner recover the same dynamics as when treating the field classical throughout [175]. This viewpoint differs from scenarios where (the fluctuations of) a quantum field are considered on a classical background – in such cases, quantum fluctuations can be characterised in relation to a classical background variable.

The setup with a relational clock can be extended to include multiple massless scalar fields to establish a full relational coordinate system [79, 80, 83], which will be discussed in the context of GFT in sec. 2.2.3.

2.2.2. Action

The GFT action $S[\varphi]$ consists of a kinetic $K[\varphi]$ and an interaction term $V[\varphi]$, where the latter includes any terms that contain GFT fields at higher than quadratic order,

$$S[\varphi] = K[\varphi] + V[\varphi] = \int \mathrm{d}\chi^0 \mathcal{L}[\varphi] \,. \tag{2.14}$$

(We also introduced the GFT Lagrangian, where we will drop the explicit group field label in the following $\mathcal{L} := \mathcal{L}[\varphi]$.)

As previously mentioned, one can take inspiration from spin foam models to restrict the form of the GFT action [141, 144, 176]. In this picture, the Feynman graphs obtained from GFT are interpreted as spin foams, which correspond to triangulations of spacetime. Recall from sec. 1.4.2 that spin foam amplitudes are calculated from vertex and edge amplitudes of spin foams. While the kinetic term in the GFT action is related to the edge amplitude, an appropriate choice of the interaction term can reproduce the vertex amplitude of any spin foam model [142].

It is moreover customary to demand invariance of $S[\varphi]$ under the symmetries of the matter field coupled to the group field, which allows one to constrain the form of the action further [144, 145]. Such constraints are welcome as they provide physical motivations for e.g. a specific form of the kinetic term. In principle, it is straightforward to couple scalar fields with a non-vanishing potential to the group field in (2.12). However, in addition to possibly limiting the scalar field's suitability as a matter clock to finite regions in which it changes monotonically, general scalar field Lagrangians would violate the above-mentioned shift and reflection symmetries. Without imposing these symmetries on the GFT action it is less clear from which physical principles its form should be restricted.

We first consider more concrete forms of the kinetic term based on these symmetry arguments and comment on possible interaction terms further below.

Kinetic term

The general form of the kinetic term reads

$$\begin{aligned} K[\varphi] =& \frac{1}{2}\int \mathrm{d}^4 g\, \mathrm{d}^4 g'\, \mathrm{d}\chi^0 \mathrm{d}\tilde{\chi}^0 \varphi(g_I, \chi^0)\mathcal{K}(g_I, g'_I; \chi^0, \tilde{\chi}^0)\varphi(g'_I, \tilde{\chi}^0) \\ =& \frac{1}{2}\sum_J \int \mathrm{d}\chi^0\, \mathrm{d}\tilde{\chi}^0 \varphi_J(\chi^0)\mathcal{K}_J(\chi^0, \tilde{\chi}^0)\varphi_J(\tilde{\chi}^0)\,, \end{aligned} \tag{2.15}$$

where $\mathcal{K}(g_I, g'_I; \chi, \chi')$ denotes the kinetic kernel and $\mathcal{K}_J(\chi, \chi')$ its decomposition in Peter-Weyl modes. We have skipped a step in carrying out this decomposition and in the second line also imposed that $\mathcal{K}_{J,J'}(\chi^0, \tilde{\chi}^0) \propto \mathcal{K}_J(\chi^0, \tilde{\chi}^0)\delta_{J,J'}$. This assumes locality in the group elements for the kinetic kernel and can be motivated e.g. from the EPRL spin foam model [143, 144]. We note that in this case, different J modes evolve independently from one another, which would not be the case in more general models [154]. In its general form as reported above, $K[\varphi]$ is non-local in the scalar field, as the group fields appear with different values of the matter field. If we impose that the translational as well as reflection symmetries of the classical matter action (2.13) hold for $S[\varphi]$, $\mathcal{K}_J(\chi^0, \tilde{\chi}^0)$ can depend only on the squared difference of the scalar fields, i.e.

$$\mathcal{K}_J(\chi^0, \tilde{\chi}^0) = \mathcal{K}_J((\chi^0 - \tilde{\chi}^0)^2)\,. \tag{2.16}$$

For this form of $\mathcal{K}(\chi^0, \tilde{\chi}^0)_J$ we can then perform a Taylor expansion of (2.15) around the difference of the matter field values. Explicitly, we introduce $\tilde{\chi}^0 = \chi^0 + \epsilon$ and expand $\varphi(\chi^0 + \epsilon)$ around $\epsilon = 0$, which gives a kinetic term that is an infinite sum of even derivatives[3] w.r.t. the matter field

$$\begin{aligned} K[\varphi] = \frac{1}{2}\int \mathrm{d}\chi^0 \sum_J \sum_{n=0}^{\infty} \varphi_J(\chi)\mathcal{K}_J^{(2n)} \left(\frac{\partial}{\partial\chi^0}\right)^{2n} \varphi_J(\chi^0)\,, \\ \text{with} \quad \mathcal{K}_J^{(2n)} = \int \mathrm{d}\epsilon \frac{\epsilon^{2n}}{(2n)!}\mathcal{K}_J(\epsilon^2)\,. \end{aligned} \tag{2.17}$$

While this expansion in principle includes an infinite amount of terms one needs to introduce a truncation at finite order in practice. A common choice in the literature [144, 177]

[3]Terms with odd powers in derivatives w.r.t. the scalar field need to vanish due to reflection symmetry.

is to consider only the first two terms, $n = 0$ and $n = 1$, such that

$$K[\varphi] = \frac{1}{2}\sum_J \int \mathrm{d}\chi^0 \varphi_J(\chi^0) \left(\mathcal{K}_J^{(0)} + \mathcal{K}_J^{(2)} \left(\frac{\partial}{\partial\chi^0}\right)^2 \right) \varphi_J(\chi^0) \,. \tag{2.18}$$

Note that, while the non-local form of the kinetic term (2.16) is perhaps the most general after imposing symmetry requirements on the fields and leads to an expansion as given above, some studies in the literature [178] do not see a second order kinetic term as given in (2.18) as a truncation of a possibly infinite sum.

We include a concrete form of the kinetic term that is frequently considered in the literature and was introduced in [176]. Specifically, the mass term in the expansion (2.17) is set as $\mathcal{K}_J^{(0)} = \mu + \Delta_g$, where $\mu \in \mathbb{R}$ and $\Delta_g := \sum_i \Delta_{g_i}$ is the Laplace-Beltrami operator on $\mathrm{SU}(2)^4 \times \mathbb{R}^4$, acting on the group elements g_I. Including a Laplacian in the kinetic kernel is also considered in [179] and further motivated by studies of renormalisation [180–184]. The action of Δ_g can be understood by considering the Peter-Weyl decomposition of the fields, where Δ_{g_i} acts as the Casimir on the Wigner matrices $\Delta_{g_i}\mathcal{D}^{j_i}_{m_i,n_i}(g_i) = -j_i(j_i+1)\mathcal{D}^{j_i}_{m_i,n_i}(g_i)$. We furthermore set $\mathcal{K}_J^{(2)} = \tau$ with $\tau \in \mathbb{R}$. The constants τ and μ are independent of the $J-$mode, such that in this implementation, the kinetic term depends only on the spins $\vec{j}$ of the Peter-Weyl mode. To summarise, the terms appearing in (2.18) can be written as

$$\mathcal{K}_J^{(0)} = \mu - \sum_i j_i(j_i+1) \,, \qquad \mathcal{K}_J^{(2)} = \tau \,. \tag{2.19}$$

We note however that many results, including the ones in later chapters, are independent of these specific expressions and would hold also for a broader range of possible definitions. We will detail in sec. 2.2.3 how the relative signs of $\mathcal{K}_J^{(2)}$ and $\mathcal{K}_J^{(0)}$ determine the dynamics of the respective Peter-Weyl mode.

Interaction term

While we will neglect the interactions in the remainder of this thesis, we still want to comment on potential forms.

Recall that the interaction term can be specified to reproduce the vertex amplitude of a given spin foam model. For triangulations of a 4-dimensional spacetime, where a spin foam vertex represents a four-simplex, one would then expect a quintic interaction term φ^5, interpreted as the gluing of five tetrahedra [141, 158]. Furthermore, as the interaction is localised on a vertex, as is the clock field χ^0 in the construction we consider, one assumes that the GFT interaction term is local in χ^0. With these considerations, the GFT interaction takes on a form similar to [177]

$$V[\varphi] = \int \mathrm{d}\chi^0 \sum_{\vec{J}} \mathcal{V}^{\vec{J}} \prod_{i=1}^{5} \varphi_{J_i}(\chi^0)\,, \tag{2.20}$$

where $\vec{J} := J_1, J_2, J_3, J_4, J_5$. Specification of a particular spin foam model then induces combinatorial constraints between the J_i labels (see e.g. [144] for an interaction term of the EPRL model).

In cosmological applications of GFT, interactions are often neglected as a first approximation. As GFT cosmology progresses, one might examine GFT interaction terms from a purely phenomenological perspective [146, 185, 186], where we will discuss some results in sec. 4.1.

2.2.3. Canonical quantisation and dynamics

Two canonical quantisation approaches for GFT have evolved, whose relation (and ideally, equivalence) is still an open question. The original approach, titled as the 'algebraic approach' in [148], relies on Schwinger-Dyson equations derived from an action principle, whereas the alternative 'deparametrised' approach uses a GFT Hamiltonian and was first introduced in [177]. The algebraic approach is still the one most commonly found in the literature, even though increasing progress has been made also in the deparametrised approach [2, 78, 148, 185, 187].

Below, we proceed to detail the deparametrised approach, which the work presented in chap. 6-7 is based on. We give a short summary of the algebraic approach at the end of this section. For a more in depth illumination of the similarities and differences of these approaches, we refer the reader to [187], which contrasts the implementation of different classes of quantum states in the two approaches, and the introduction of [148].

Deparametrised approach

This approach was first introduced in [177] and is based on a GFT Hamiltonian that is constructed from the GFT action through a Legendre transform w.r.t. the real valued argument χ^0 of the group field. The scalar field is thereby assigned the role of an evolution parameter, in line with its interpretation as a clock field. The theory can be canonically quantised in a straightforward manner and operator dynamics can be obtained e.g. by solving the Heisenberg equations of motion. Below we summarise the procedure and refer the reader to the original paper [177] for further details.

Before we proceed, let us point out that there are different ways to incorporate relational coordinates in a quantum theory, see [169] for a detailed introduction. In the perhaps simplest approach, one singles out a clock as an external parameter prior to quantisation. This is the approach that was followed in the construction of the GFT Hamiltonian [177], and we will adopt this viewpoint also here. For a discussion of deparametrisation for many body systems in GFT at the classical and at the quantum level, see [188]. It is often advocated that in the deparametrised approach the clock should be interpreted as classical and does not form part of the quantum system. However, the investigations of [172, 173, 175] show that such a viewpoint might be a bit simplistic. Instead, one should interpret any observables of the resulting quantum theory as conditioned on the clock reading, where the clock forms part of the quantum system. An alternative approach is to carry out a clock neutral quantisation of the entire system and identify a suitable clock variable afterwards [174]. There has been recent progress in the study of quantum reference frames [172] and the results of [170, 171] suggest that consistent switching between different quantum clocks is possible (in cosmology). Furthermore, the relation between the two approaches of incorporating a clock field in GFT was recently investigated [175] and the authors find that for the cosmological case (which will be of interest in the second part of this thesis and which is the most studied application of GFT with a relational clock in the literature) the two procedures lead to equivalent effective dynamics.

To sketch the derivation of the Hamiltonian, let us start from a GFT action with a kinetic term as given in (2.18). As a first step, one obtains a form of the GFT Lagrangian that contains only first derivatives of the GFT field by integrating by parts and neglecting boundary terms, (in the following we drop the explicit χ^0 label $\varphi_J = \varphi_J(\chi^0)$ and use $\partial_0\varphi_J := \frac{\partial}{\partial\chi^0}\varphi_J$)

$$\mathcal{L} = \sum_J \left(\frac{1}{2}\mathcal{K}_J^{(0)}\varphi_J^2 - \frac{1}{2}\mathcal{K}_J^{(2)}(\partial_0\varphi_J)^2 \right) - V(\varphi)\,. \tag{2.21}$$

This form of the Lagrangian is amenable to canonical quantisation. If we had included higher order terms in the expansion of the kinetic term and thus encountered higher order derivatives in the Lagrangians, the quantisation would have to be carried out with more involved methods, see e.g. [189]. To proceed with the construction, one defines the momentum conjugate to a Peter-Weyl mode of the group field $\pi_J = \pi_J(\chi^0)$,

$$\pi_J := \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi_J)} = -\mathcal{K}_J^{(2)} \partial_0 \varphi_J \,. \tag{2.22}$$

A Hamiltonian $\mathcal{H}$ for GFT is then obtained from a Legendre transform of (2.21) w.r.t. the clock parameter

$$\mathcal{H} = \sum_J \pi_J \partial_0 \varphi_J - \mathcal{L} = \sum_J \frac{\mathcal{K}_J^{(2)}}{2} \left(-\frac{\pi_J^2}{|\mathcal{K}^{(2)}|^2} + m_J^2 \varphi_J^2 \right) + V(\varphi) \,, \tag{2.23}$$

where we introduced the parameter $m_J^2 := -\frac{\mathcal{K}_J^{(0)}}{\mathcal{K}_J^{(2)}}$. For quantisation, we promote the GFT field and its conjugate momentum to operators that satisfy the standard canonical commutation relations, which (in contrast to the algebraic approach) are defined on *equal time* slices, i.e. for operators with the same value of the scalar field:

$$[\varphi_J(\chi^0), \pi_{J'}(\chi^0)] = \mathrm{i}\, \delta_{J,J'} \,. \tag{2.24}$$

The dynamics are dictated by the Hamiltonian and working in the Heisenberg picture yields operators that satisfy the following equations of motion

$$\partial_0 \pi_J = -\mathrm{i}\, [\pi_J, \mathcal{H}] = -\mathcal{K}_J^{(2)} m_J^2 \, \varphi_J \,, \qquad \partial_0 \varphi_J = -\frac{\pi_J}{\mathcal{K}_J^{(2)}} \,. \tag{2.25}$$

Despite the suggestive notation, m_J^2 can take on negative values, and its sign is solely determined by the relative signs of the parameters of the kinetic kernel, namely $\mathcal{K}_J^{(0)}$ and $\mathcal{K}_J^{(2)}$. The sign of m_J^2 dictates whether the kinetic part of the Hamiltonian takes on the form of a harmonic oscillator ($m_J^2 < 0$, which is the case if $\mathrm{sgn}(\mathcal{K}_J^{(2)}) = \mathrm{sgn}(\mathcal{K}_J^{(0)})$), or alternatively, an inverted harmonic oscillator ($m_J^2 > 0$, i.e. $\mathrm{sgn}(\mathcal{K}_J^{(2)}) \neq \mathrm{sgn}(\mathcal{K}_J^{(0)})$). As we will see in more detail below, the inverted Harmonic oscillator leads to exponentially growing modes (2.44), which we will interpret as leading to a universe with unbounded expansion in sec. 4.1. For the particular forms of the coefficients in the expansion of the kinetic term given in (2.19), we have $m_J^2 = \frac{\sum_i j_i(j_i+1)-\mu}{\tau}$. The sign of m_J^2 then not only depends on the choice of the parameters τ and μ, but also on the spins of the respective $J-$mode. In particular we find that $m_J^2 > 0$ for a finite number of modes only if $\mu > 0$

and $\tau < 0$. In this case, $\sum_i j_i(j_i + 1) - \mu$ becomes positive for modes with large (enough) spins j_i such that these modes have $m_J^2 < 0$. The number of Peter-Weyl modes whose dynamics are determined by a Hamiltonian with a kinetic term that resembles an inverted harmonic oscillator and therefore grow exponentially for large χ^0 is then finite and limited to small j modes [190], which might be desirable from a phenomenological perspective. Conceptually, one might also expect an effective spacetime to emerge from a large number of Planck-sized quanta, where the volume carried by the GFT quanta scales with their spin labels if one uses the volume operator of LQG.

We describe the extension of this framework to include additional scalar fields in the next section, which will be the GFT formulation we use in the second part of the thesis.

Hamiltonian GFT with four massless scalar fields

In this section we introduce the GFT model with four massless scalar fields, which will serve as the basis for our research results presented in chap. 6 and chap. 7. The motivation to include three additional scalar fields stems from the desire to extend the relational idea of GFT dynamics, where any changes to the spacetime geometry happen w.r.t. the value of the matter clock, to an entire coordinate system. In this cases, one field serves as relational clock, whereas the others pose as spatial rods, i.e. they take on the roles of spatial coordinates. Together, the fields are assumed to span a suitable coordinate system, at least locally. Such an extension to the deparametrised approach [177] explained in the previous section to the case of a GFT with four scalar fields was proposed in [78]; similar constructions were considered in [79, 80, 151, 152].

Analogously to (2.12) we extend the domain of the group field to include four massless scalar fields, which we label by $A = 0, 1, 2, 3$, such that we have

$$\varphi : \mathrm{SU}(2)^4 \times \mathbb{R}^4 \to \mathbb{R}\,, \tag{2.26}$$

$$\varphi(g_I, \chi^A) := \varphi(g_1,\, g_2,\, g_3,\, g_4; \chi^0,\, \chi^1\,, \chi^2, \chi^3) \quad \text{with} \quad g_I \in \mathrm{SU}(2),\; \chi^A \in \mathbb{R}\,, \tag{2.27}$$

and the field modes of the Peter-Weyl decomposition now depend on the values of all four scalar fields

$$\varphi(g_I,\chi^A) = \sum_J \varphi_J(\chi^A)\, \mathcal{D}_J(g_I)\,. \tag{2.28}$$

We emphasise again that in GFT there a priori exists no spacetime manifold. Associating the matter fields to GFT quanta allows us to define hypersurfaces associated with quanta with the same field value and in the limit where a spacetime has emerged from a large number of GFT quanta, the relational coordinate system can be related to a specific choice of coordinate system on a spacetime manifold within GR. In GR, we can switch to the relational coordinate system spanned by the fields by demanding $\partial_\mu \chi^A \propto \delta^A_\mu$. The assumption is then that these four fields fulfill the non-degeneracy condition $\det\left(\partial_\mu \chi^A\right) \neq 0$, such that they locally span a Cartesian coordinate system. We discuss this kind of coordinate system within GR further in sec. 3.4 and sec. 6.1.

The classical matter action is given by a simple sum over the Lagrangians of the fields,

$$S_\chi = -\frac{1}{2}\int \mathrm{d}^4x \sum_A \sqrt{-g}\, g^{\mu\nu}\partial_\mu\chi^A\partial_\nu\chi^A \,. \tag{2.29}$$

In addition to the shift and reflection symmetry for each field separately, the above satisfies an additional rotational symmetry between the different fields $\chi^A \mapsto R^A{}_B\chi^B$ (where R^A_B denotes the rotation matrix) i.e. the matter fields are physically indistinguishable [78]. The action (2.29) is therefore invariant under the action of the Euclidean group $E(4)$ on the massless scalar fields.

The additional fields enter the GFT action as a natural extension of the single field case, where we again focus on the kinetic term:

$$K[\varphi] = \sum_J \int \mathrm{d}\chi^A\, \mathrm{d}\tilde{\chi}^A \varphi_J(\chi^A)\mathcal{K}_J(\chi^A, \tilde{\chi}^A)\varphi_J(\tilde{\chi}^A)\,. \tag{2.30}$$

We again assume that the symmetries of the classical action (i.e. invariance under shifts and reflections of each of the scalar fields separately, as well as rotations between the fields) carry over to GFT and use these symmetries to constrain the allowed terms in the GFT action. In particular, this imposes that $\mathcal{K}_J(\chi^A,\, \tilde{\chi}^A) = \mathcal{K}_J(\sum_A(\chi^A - \tilde{\chi}^A)^2)$ and the action can only depend on derivatives of the group field of even order. Similar to the considerations made in sec. 2.2.2 we can then carry out a Taylor expansion of $\varphi(\chi^A + \epsilon^A)$

around $\epsilon^A = 0$ [4]

$$S[\varphi] = \int \mathrm{d}^4\chi \left(\frac{1}{2} \sum_J \sum_{n=0}^{\infty} \mathcal{K}_J^{(2n)} \, \varphi_J(\chi^A) \Delta^n \varphi_J(\chi^A) - V(\varphi) \right) , \qquad (2.31)$$

where $\Delta = \sum_A \left(\frac{\partial}{\partial \chi^A} \right)^2$ is the Laplacian on $\mathbb{R}^4$ and the $\mathcal{K}_J^{(2n)}$ take the same form as in (2.17), with a multi-dimensional integral $\mathrm{d}\epsilon \to \mathrm{d}^4\epsilon$. This leads to the following Lagrangian (using integration by parts and neglecting boundary terms), where we have again restricted ourselves to the first two terms in the expansion

$$\mathcal{L} = \sum_J \left(\frac{1}{2} \mathcal{K}_J^{(0)} \varphi_J^2 - \frac{1}{2} \mathcal{K}_J^{(2)} \sum_A (\partial_A \varphi_J)^2 \right) - V(\varphi) . \qquad (2.32)$$

For the definition of the Hamiltonian one needs to single out a field that will be used as deparametrisation parameter, which is then to be interpreted as the clock field. As hinted by the suggestive notation, we appoint χ^0 to take on this role. The Hamiltonian associated with the action (2.32) reads ($b = 1, 2, 3$)

$$H = \int \mathrm{d}^3\chi \sum_J \frac{\mathcal{K}_J^{(2)}}{2} \left(- \frac{\pi_J^2}{|\mathcal{K}_J^{(2)}|^2} + m_J^2 \varphi_J^2 + \sum_b (\partial_b \varphi_J)^2 \right) + V(\varphi) . \qquad (2.33)$$

The symmetry requirements used to constrain the kinetic term can also be imposed on the interaction term. In our work we will neglect any GFT interactions and hence omit $V(\varphi)$ in what follows. With an increasing number of GFT quanta this assumption becomes increasingly invalid [185].

Before quantisation, we carry out Fourier decomposition of $\varphi_J(\chi^A)$ and $\pi_J(\chi^A)$ w.r.t. the spatial fields $\vec{\chi} = (\chi^1, \chi^2, \chi^3)$, namely

$$\varphi_J(\chi^A) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{\mathrm{i}\vec{k}\vec{\chi}} \varphi_{J,k}(\chi^0), \qquad \pi_J(\chi^A) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} e^{\mathrm{i}\vec{k}\vec{\chi}} \pi_{J,k}(\chi^0) . \qquad (2.34)$$

By applying a Fourier transform, we have assumed absolute integrability [5] of the GFT field and its conjugate momentum over spatial slices defined by the χ^a ($a = 1, 2, 3$), an

[4] In the expansion we use the same value of $\epsilon^A = \epsilon$ for all fields; the differences between the ϵ^A will have the same order of magnitude as some higher power of ϵ^A and can be neglected in a truncated expansion.

[5] Many discussions on Fourier transforms use square integrability as a condition instead. For a detailed discussion on Fourier transforms and the required notion of integrability, see e.g. [191].

assumption that, as discussed at length in [78], breaks the rotational symmetry between the fields. The reason for this is that general solutions to the GFT dynamics can grow unboundedly in the direction of the clock field (which is what permits the interpretation of the GFT cosmological sector as yielding an expanding universe). In such cases, a change of clock parameter would violate the integrability condition. In short, once we carry out a Fourier decomposition, we have fixed the clock parameter and the other fields are interpreted as spanning spatial hypersurfaces over which the GFT field is integrable.

In this decomposition (and for $V(\varphi)=0$) the Hamiltonian (2.33) takes on the form

$$\begin{aligned} H &= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \sum_J \frac{\mathcal{K}_J^{(2)}}{2} \left(-\frac{1}{|\mathcal{K}_J^{(2)}|^2} \pi_{J,-k}(\chi^0)\pi_{J,k}(\chi^0) + \omega_{J,k}^2\, \varphi_{J,-k}(\chi^0)\varphi_{J,k}(\chi^0) \right) \\ &=: \int \frac{\mathrm{d}^3k}{(2\pi)^3} \sum_J H_{J,k}\,, \end{aligned} \tag{2.35}$$

where we introduced $\omega_{J,k}^2 := m_J^2 + \vec{k}^2$. We conclude that, similar to the Peter-Weyl modes, the Fourier modes develop independently of one another. The equal time commutation relations (2.24) gain a dependence on the wavenumber associated with the spatial fields

$$[\varphi_{J,k}(\chi^0), \pi_{J',k'}(\chi^0)] = \mathrm{i}\,\delta_{JJ'}(2\pi)^3\delta(\vec{k}+\vec{k}')\,, \tag{2.36}$$

and the dynamics of each mode follow again from the Heisenberg equations of motion

$$\partial_0 \pi_{J,k} = -\mathrm{i}\,[\pi_{J,k}, H] = -\mathcal{K}_J^{(2)}\omega_{J,k}^2\varphi_{J,k}\,, \qquad \partial_0\varphi_{J,k} = -\frac{\pi_{J,k}}{\mathcal{K}^{(2)}}\,. \tag{2.37}$$

As proposed already in [177] and carried over for the case of four fields in [78], it is useful to introduce time-dependent creation and annihilation operators $A_{J,k}(\chi^0)$, $A_{J,k}^\dagger(\chi^0)$, which are defined from the GFT field operator and its momentum via

$$\begin{aligned} &\pi_{J,k}(\chi^0) = -\mathrm{i}\alpha_{J,k}(A_{J,k} - A_{J,-k}^\dagger)\,, \qquad \varphi_{J,k}(\chi^0) = \frac{1}{2\alpha_{J,k}}(A_{J,k} + A_{J,-k}^\dagger)\,, \\ &\text{with} \quad \alpha_{J,k} = \sqrt{\frac{|\omega_{J,k}||\mathcal{K}^{(2)}|}{2}}\,. \end{aligned} \tag{2.38}$$

From (2.25) it follows that they satisfy the equal-time commutation relations

$$[A_{J,k}(\chi^0), A_{J',k'}^\dagger(\chi^0)] = \delta_{JJ'}(2\pi)^3\delta(\vec{k}-\vec{k}')\,, \tag{2.39}$$

with all other commutators vanishing. As we work in the Heisenberg picture, we need to distinguish between time *dependent* operators denoted by $A_{J,k}(\chi^0)$, $A^\dagger_{J,k}(\chi^0)$ and time *independent* operators $a_{J,k}$, $a^\dagger_{J,k}$, defined by $a_{J,k} = A_{J,k}(0)$ and $a^\dagger_{J,k} = A^\dagger_{J,k}(0)$, which obey the same commutation relations

$$[a_{J,k}, a^\dagger_{J',k'}] = \delta_{JJ'}(2\pi)^3\delta(\vec{k} - \vec{k'})\,. \tag{2.40}$$

As we pointed out in the previous section and as explained thoroughly in [78, 177], the Hamiltonian can take on two different forms. In the case of a GFT with a single massless scalar field, this is determined solely by the sign of m_J^2, and hence through the signs of $\mathcal{K}_J^{(0)}$ and $\mathcal{K}_J^{(2)}$. For the Hamiltonian (2.35) we find that the type of the Hamiltonian now depends on the sign of $\omega^2_{J,k}$, which in general depends both on the sign of m_J^2 as well as the value of $\vec{k}^2$. In cases where $\omega^2_{J,k} < 0$ the mode Hamiltonian $H_{J,k}$ written in terms of the ladder operators[6] is that of a harmonic oscillator

$$H_{J,k} = -\,\mathrm{sgn}(\mathcal{K}_J^{(2)})\frac{|\omega_{J,k}|}{2}\left(a_{J,-k}a^\dagger_{J,-k} + a^\dagger_{J,k}a_{J,k}\right)\,. \tag{2.41}$$

In the alternative case, i.e. for modes with $\omega^2_{J,k} > 0$, we obtain a Hamiltonian that is of squeezing type[7]

$$H_{J,k} = \mathrm{sgn}(\mathcal{K}_J^{(2)})\frac{|\omega_{J,k}|}{2}\left(a_{J,k}a_{J,-k} + a^\dagger_{J,k}a^\dagger_{J,-k}\right)\,. \tag{2.42}$$

For both the oscillating as well as the squeezing case we can solve the Heisenberg equations of motion for $A_{J,k}$ and $A^\dagger_{J,k}$ explicitly (where we omit the explicit dependence on χ^0 from hereon). In the case of oscillating modes, i.e. for a Hamiltonian of the form (2.41) we find

$$A_{J,k} = a_{J,k}e^{\mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})|\omega_{J,k}|\chi^0}\,, \quad A^\dagger_{J,k} = a^\dagger_{J,k}e^{-\mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})|\omega_{J,k}|\chi^0}\,, \tag{2.43}$$

[6] At the level of the Hamiltonian it is irrelevant whether the expression is given in terms of time-dependent or time-independent operators.

[7] This form of the Hamiltonian is referred to as 'squeezed Hamiltonian' due its similarity with squeezed states that are commonly used in atomic optics. These are created from the Fock vacuum through $\exp\big(\zeta^*(a^\dagger)^2 + \zeta a^2\big)\,|0\rangle$, see e.g. [192].

whereas a squeezing Hamiltonian (2.42) results in exponentially growing expressions

$$\begin{aligned} A_{J,k} &= a_{J,k} \cosh\left(|\omega_{J,k}|\chi^0\right) - \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)}) a^\dagger_{J,-k} \sinh\left(|\omega_{J,k}|\chi^0\right) \,, \\ A^\dagger_{J,k} &= a^\dagger_{J,k} \cosh\left(|\omega_{J,k}|\chi^0\right) + \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)}) a_{J,-k} \sinh\left(|\omega_{J,k}|\chi^0\right) \,. \end{aligned} \tag{2.44}$$

From these solutions to the Heisenberg equations of motion within the free theory, we can obtain time-dependent expressions for any combinations of ladder operators, independent of the choice of state.

The Hilbert space of the theory is given by the Fock space obtained from the $a_{J,k}, a^\dagger_{J,k}$; the $a^\dagger_{J,k}$ create quanta of geometry, $a^\dagger_{J,k}\,|0\rangle = \left|J, \vec{k}\right\rangle$, interpreted as fundamental building blocks of space, and the $a_{J,k}$, annihilate the Fock vacuum $|0\rangle$. As already emphasized in [177], the Fock vacuum need not be an eigenstate of the Hamiltonian; indeed, the squeezing Hamiltonian (2.42) will excite the Fock vacuum and lead to an increasing number of quanta.

Foreshadowing already the discussion on cosmological spacetimes emerging from GFT, we mention that one typically uses Fock coherent states, as they have desirable semiclassical properties,

$$|\sigma\rangle = e^{-||\sigma||^2/2} \exp\left(\sum_J \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sigma_J(\vec{k}) a^\dagger_{J,k} \right) |0\rangle \,, \tag{2.45}$$

where $\sigma_J(\vec{k})$ is a (complex) function, to obtain an effective evolution of our universe. One then finds that the case of a squeezing Hamiltonian gives an expanding universe, whereas the oscillating Hamiltonian results in a scenario akin to a static cosmology. We will show this more explicitly in sec. 4.1.3.

Algebraic approach

Here we briefly summarise the algebraic approach to GFT, which is the original formulation of the theory and used in most of the literature.

While in the deparametrised approach we considered a real group field, in the algebraic approach one works with a complex field instead. We keep the rest of the construction the same as previously, but couple only a single massless scalar field for simplicity, where there is no hindrance to consider also multiple scalar fields (for studies with a relational coordinate system see e.g. [151, 152]). Explicitly, we have

$$\varphi : \mathrm{SU}(2)^4 \times \mathbb{R} \to \mathbb{C} \,. \tag{2.46}$$

To quantise the theory, the group field is promoted to an operator and one imposes the bosonic commutation relations

$$[\varphi_J(\chi^0), \varphi^\dagger_{J'}(\tilde{\chi}^0)] = \delta_{J,\,J'}\, \delta(\chi^0 - \tilde{\chi}^0) \,. \tag{2.47}$$

In contrast to the equal time commutation relations (2.36) used in the Hamiltonian approach, where operators are seen as evolving in the Heisenberg picture, the $\varphi_J(\chi^0)$ with unequal values of the clock field are treated as distinct operators in the algebraic approach.

The Hilbert space is given by a Fock space defined w.r.t. the field operators: The field creation operator $\varphi_J(\chi^0)^\dagger$ acts on the Fock vacuum $|0\rangle$, which is annihilated by $\varphi_J(\chi^0)\,|0\rangle = 0$, to create excitations of the GFT field

$$\varphi^\dagger_J(\chi^0)\,|0\rangle = \left|\vec{j},\vec{m},\iota;\chi^0\right\rangle \,. \tag{2.48}$$

The operators $\varphi^\dagger_J(\chi^0)$ can then be interpreted as creating open spin network vertices labelled with SU(2) representation data.

To extract the physical sector of the theory from the kinematical Hilbert space, one imposes dynamical relations. This can be achieved by demanding that physical states satisfy

$$\frac{\delta S}{\delta \varphi^\dagger}\,|\Psi\rangle = 0 \,, \tag{2.49}$$

which is the case for the Fock coherent state we include below (2.52), as was pointed out in [187].[8] The majority of the literature however focuses on the simplest Schwinger-Dyson equation [144] and requires that the equations of motion are satisfied at the level of an expectation value

$$\langle\Psi|\,\frac{\delta S}{\delta \varphi^\dagger}\,|\Psi\rangle = 0 \,. \tag{2.50}$$

[8]However, one does not impose the conjugate equation $\frac{\delta S}{\delta\varphi}\,|\Psi\rangle = 0$, as this does not commute with (2.49).

In which sense this procedure can be understood as assuming that the kinematical Hilbert space corresponds to the physical Hilbert space even before imposing dynamics is discussed in [187].

To give an example we consider the same action as in the previous section (no interaction term and a kinetic kernel truncated at second order), but for a complex field:

$$K[\varphi] = \frac{1}{2}\sum_J \int \mathrm{d}\chi^0 \varphi_J^\dagger(\chi^0) \left(\mathcal{K}_J^{(0)} + \mathcal{K}_J^{(2)} \left(\frac{\partial}{\partial \chi^0}\right)^2\right) \varphi_J(\chi^0) + \text{ h.c.}\,. \tag{2.51}$$

A popular state choice in the literature is that of a Fock coherent state

$$|\sigma\rangle = \exp\left(\sum_J \int \mathrm{d}\chi^0 \sigma_J(\chi^0) \varphi_J^\dagger(\chi^0)\right) |0\rangle \ , \tag{2.52}$$

which satisfies the property $\varphi_J(\chi^0)\,|\sigma\rangle = \sigma_J(\chi^0)\,|\sigma\rangle$ and results in the following equations of motion using (2.50)

$$(\mathcal{K}_J^{(0)} + \mathcal{K}_J^{(2)} \partial_0^2)\sigma_J(\chi^0) = 0\,, \tag{2.53}$$

which can be solved to obtain an explicit solution for $\sigma_J(\chi^0)$. Note that this is equivalent to the classical equations of motion for a GFT action of the form (2.51). Many of the phenomenological results in GFT cosmology, which will be the subject of sec. 4.1.5, are based on the algebraic approach.

At the level of mean field (coherent) states, one effectively solves the classical equations of motion in both cases, such that, at leading order, the results of both approaches agree. In the case of more general states as were investigated in [187] the relation between results in the different approaches is less clear.

2.3. Further directions

There are further research directions that go beyond the ideas presented in this chapter and are outside the context of GFT cosmology, which is discussed extensively in sec. 4.1. Despite their interesting nature, detailing these research avenues would be beyond the scope of this thesis. For the interested reader with a newly found enthusiasm for GFT, we would nonetheless like to point out some additional ideas:[9]

- **Coloured GFT models**
  These models are of interest from two perspectives:
  1. In general, GFT Feynman diagrams can include triangulations with topological singularities. In coloured GFT models one only creates triangulations of (pseudo-)manifolds [193]. Coloured GFTs are often considered in the study of renormalisation in GFT [180, 183].
  2. We discussed ambiguities in the graph structure belonging to a GFT state in sec. 2.1.1. Restricting to a coloured GFT reduces the ambiguity in the structure of multiparticle states with connected GFT quanta. This idea was explored in [194] to construct kinematical states with a given topology.
- **Black holes**
  The idea of constructing kinematical states with a given topology from coloured GFTs was applied to the black hole context in [195]. Here the authors construct a Schwarzschild black hole spacetime by gluing shells with the correct topology and propose to calculate the black hole entropy. The possibility of constructing a relational coordinate system for a black hole spacetime in GFT was considered in [80], where the authors focus on the shell volume of a Schwarzschild black hole.

2.4. Conclusion: The group field we build on

Group field theories are field theories on an abstract group manifold G. They were first introduced as a useful tool to compute spin foam amplitudes, where a specific GFT action can reproduce transition amplitudes for any spin foam model through a Feynman graph expansion, but have since evolved into their own research field. The specifics of a GFT that one hopes can be used to recover a quantum gravitational theory are inspired by LQG and spin foams. One can however also dare to go beyond the construction of these models and, for instance, explore new possible connections to classical space times, as we will propose in chap. 6. As we saw, the GFT framework is rather flexible and one can hope that by making progress within GFT one can also attain new insights into LQG or spin foam models.

[9] This selection is explicitly incomplete and based on a personal choice.

The quanta of GFT can be interpreted as spin network vertices (with their valency determined by the dimension of the GFT), or similarly, as building blocks of the triangulation of the spatial manifold (which is a tetrahedron in 4-dimensional gravity). GFT quanta, unlike LQG spin networks, are not embedded in a spatial hypersurface but live on an abstract group manifold. They themselves constitute the very building blocks of space and the interpretation is that spacetime emerges in the limit of a large amount of such quanta.

Just like the overwhelming majority of the literature, we use a scalar group field with bosonic commutation relations, additionally, we assume the field to be real-valued. The specifics of a GFT are determined by its action, dimension, choice of group as well as additional constraints. As we are interested in 4-dimensional gravity we choose $d = 4$. For a relation with LQG and the EPRL model, we choose $G = \mathrm{SU}(2)$ and impose right invariance under group action.

To extract dynamics from the theory one can couple a massless scalar field that serves as a relational clock, a strategy that is commonly employed in quantum cosmology. This construction can be extended to include three additional matter fields, which are interpreted as ‘rod’ fields and serve as spatial coordinates. One thereby obtains a 4-dimensional relational coordinate system that can be used to relate GFT quantities to a spacetime manifold. Together with the relation to EPRL models and imposing the classical symmetries of the scalar fields on the GFT action one can then restrict the form of the GFT kinetic term. Similar arguments can be applied to the interaction term, which will be of no concern for the results in this thesis.

We detailed the formulation of the quantum theory in the deparametrised approach to GFT. In this approach, a GFT Hamiltonian is constructed by deparametrising w.r.t. the clock field as an evolution parameter. In the case of multiple fields one needs to carry out a Fourier decomposition w.r.t. the spatial fields before quantisation and the rotational symmetry between the four matter fields is broken. One can extract operator dynamics from the Heisenberg equations of motion and finds that there are two types of Hamiltonian: one resembles a harmonic oscillator, the other corresponds to a squeezed Hamiltonian akin to squeezed states in quantum optics. Which one is realised depends on the fundamental parameters appearing in the kinetic term and has important consequences for the phenomenology of emerging spacetimes.

The crucial takeaway from this section, in addition to a broader understanding of the motivations behind studying GFTs, is the Hamiltonian theory for GFT with four massless scalar fields and the resulting dynamics for the creation and annihilation operators.

Chapter 3.

Cosmological perturbation theory

'[...] je mehr sich einer begrenzt, um so mehr ist er andererseits dem Unendlichen nahe [...]'

- Stefan Zweig in 'Die Schachnovelle'.

'[...] the more one limits oneself, the closer on is, on the other hand, to the infinite [...]'

While the universe is homogeneous and isotropic on large scales and its overall evolution can be well captured by a (flat) FLRW metric as introduced in sec. 1.2, these symmetries are evidently broken at smaller scales. Deviations from these large scale assumptions are modelled as small inhomogeneous perturbations of the FLRW metric and the matter fields. These perturbations play an important role for the evolution of the cosmos: small density fluctuations in the early universe ignited the aggregation of matter into larger structures that ultimately allowed for the formation of stars, galaxies, and other structures (see [196, chap. 1] and references therein.).

The field of cosmological perturbation theory started to emerge with [197] and it was soon realised that the covariant nature of GR introduces a gauge dependence and complicates the physical interpretation of inhomogeneous perturbations. This issue can be overcome by introducing gauge-invariant variables [198] that give physical quantities and ultimately allow to connect to measurements. The model of the early universe determines the behaviour of perturbations and thereby the initial conditions for all further evolution. The most direct observational access we have to this early era is the CMB, which carries imprints of several phases of the evolution of the early universe up to decoupling, see sec. 1.2.3. Ideally, gravitational waves, which can travel freely even before decoupling, will lift the veil of opacity and reveal information about the universe that predates the

CMB. First measurements of the stochastic GW background [199–202] have reinforced such hopes.

The early universe is a pertinent environment to look for possible quantum gravitational effects, since quantum corrections should become relevant in the vicinity of the Big Bang singularity and ideally provide us with a more complete understanding of such high curvature regimes. In addition to its effects on the initial singularity (ideally, in the form of singularity resolution) and consistency with the cosmological background evolution in the low curvature regime, a quantum gravitational theory can and should also be assessed by its effects on perturbations in the early universe. Consistency with the CMB can then be seen as an evaluation criterion in addition to theoretical consistency, making the study of cosmological perturbations within such theories an important tool. In an ideal world, the study of cosmological perturbations can furthermore point to possibly observable imprints of quantum gravity effects. The results presented in the second part of the thesis aim at relating quantum gravity to the study of cosmological perturbations.

This chapter first gives an overview over standard cosmological perturbation theory in sec. 3.1, where we discuss the perturbed metric. The issue of gauge-invariance, which is crucial for relating perturbation theory to observations, is the topic of sec. 3.2. We then introduce the separate universe framework as an approximation for the study of large-wavelength perturbations in sec. 3.3. Finally, we give the details for perturbation theory in a relational framework spanned by four massless scalar fields in sec. 3.4, which is the setup used in chap. 6 and chap. 7.

3.1. Perturbative principles

This section reviews the basics of cosmological perturbation theory, which is part of standard literature and can be found in e.g. [85–88, 203].

As discussed in sec. 1.2, we assume that the background spacetime is described by a flat ($K = 0$) FLRW metric (cf. (1.27)), which we can write in Cartesian coordinates as

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,, \quad \text{i.e.} \quad g_{\mu\nu} = \mathrm{diag}(-N(t)^2,\, a(t)^2,\, a(t)^2,\, a(t)^2)\,. \tag{3.1}$$

While the standard literature usually fixes the lapse and works with cosmic time $N = 1$ or conformal time $N = a$, we leave the lapse general in what follows. The motivation for this is that our choice of matter reference frame will impose a specific, non-standard form of N (see sec. 3.4.1, chap. 6 and 7). As we are interested in the early universe we neglect the contribution of the cosmological constant and set $\Lambda = 0$. The Friedmann equations (cf. (1.38) and (1.39)) in this case read

$$3\left(\frac{a'^2}{a^2}\right) = N^2\kappa\rho\,, \tag{3.2}$$

$$2\frac{a'}{a}\frac{N'}{N} - \left(\frac{2a''}{a} + \frac{a'^2}{a^2}\right) = N^2\kappa P\,. \tag{3.3}$$

The premise of cosmological perturbation theory is to perturb around the homogeneous background at linear order. To model small inhomogeneities on top of an FLRW universe, one makes a perturbative Ansatz and adds a small perturbation to each metric component $g_{\mu\nu}(t) \to g_{\mu\nu}(t) + \delta g_{\mu\nu}(t,\vec{x})$, where $g_{\mu\nu}(t)$ encompasses the homogeneous background metric, and $\delta g_{\mu\nu}(t,\vec{x})$ the perturbations. The absolute values of the perturbative components $|\delta g_{\mu\nu}(t,\vec{x})|$ are required to be much smaller than any non-zero component of $|g_{\mu\nu}|$. In principle, any homogeneous contribution to $\delta g_{\mu\nu}(t,\vec{x})$ could be reabsorbed in the definition of the background metric. In the following we will write $g_{\mu\nu} = g_{\mu\nu}(t)$ and demand that only perturbed quantities contain inhomogeneities. The general form of the perturbed FLRW metric then reads as follows

$$\begin{aligned}\mathrm{d}s^2 = &- N(t)^2\left(1 + 2\tilde{\Phi}(t,\vec{x})\right)\mathrm{d}t^2 + 2N(t)a(t)\,\left(\partial_i B(t,\vec{x}) - B_i^V(t,\vec{x})\right)\,\mathrm{d}t\,\mathrm{d}x^i \\ &+ a(t)^2\Big((1-2\psi(t,\vec{x}))\delta_{ij} + 2\partial_i\partial_j E(t,\vec{x}) \\ &\qquad - \left(\partial_i E_j^V(t,\vec{x}) + \partial_j E_i^V(t,\vec{x})\right) + 2E_{ij}^T(t,\vec{x})\Big)\mathrm{d}x^i\mathrm{d}x^j\,.\end{aligned} \tag{3.4}$$

Here, we have already included a division of the metric perturbations into scalar $\tilde{\Phi}$, ψ, B, E, vector B_i^V, E_i^V, and tensor E_{ij}^T parts. The vector components have vanishing divergence and the tensor component is transverse and traceless:

$$\delta^{ij}\partial_j B_i^V = 0, \quad \delta^{ij}\partial_j E_i^V = 0, \quad \delta^{ik}\partial_k E_{ij}^T = 0, \quad \delta^{ij}E_{ij}^T = 0\,. \tag{3.5}$$

Such a scalar-vector-tensor (SVT) decomposition [197] is commonly found in the literature and is based on the transformation properties of the respective components under a change of coordinate system. For details we refer the reader to e.g. [203]. One finds that the different components dynamically decouple at the level of linear perturbation theory, which significantly simplifies the analysis. The metric perturbations do not appear with raised indices in any of the expressions, e.g. the perturbed inverse metric component δg^{0i} is given by

$$\delta g^{0i} = \frac{1}{aN}\left(\partial_i B(t,\vec{x}) - B_i^V(t,\vec{x})\right) . \tag{3.6}$$

Vector modes are irrelevant in the absence of vector matter perturbations (which can appear e.g. in the form of anisotropic stress) and tensor modes correspond to gravitational waves. We will focus on scalar perturbations since these are the most relevant when connecting to CMB observations [24]. The perturbed metric for scalar perturbations reads

$$\begin{aligned} \mathrm{d}s^2 = &- N^2(t)\left(1+2\tilde{\Phi}(t,x^i)\right)\mathrm{d}t^2 + 2N(t)a(t)\,\partial_i B(t,x^i)\,\mathrm{d}t\,\mathrm{d}x^i \\ &+ a^2(t)\left[\left(1-2\psi(t,x^i)\right)\delta_{ij} + 2\partial_i\partial_j E(t,x^i)\right]\mathrm{d}x^i\,\mathrm{d}x^j . \end{aligned} \tag{3.7}$$

A similar procedure as carried out for the metric above can be used to perturb other quantities of the theory, such as the matter content. In the case of a scalar field χ, which is the only type of matter we will consider, we similarly perturb the field around its background value $\chi(t) \to \chi(t) + \delta\chi(t,\vec{x})$.

When introducing linear perturbations above, we assumed that these are small w.r.t. the background metric, a statement which might at first appear ambiguous in a covariant setting. At the background level, despite general covariance, there exists a preferred coordinate system due to the expansion of the universe, namely, the frame in which the metric takes the form given in (3.1). It is in this coordinate system in which the metric perturbations have to be small w.r.t. the background.[1] There is a remaining freedom, however, namely those coordinate changes that change $\delta g_{\mu\nu}$ only up to terms at the order of the perturbations themselves [203]. It is these *infinitesimal* coordinate transformations that induce the gauge freedom in cosmological perturbation theory, which we discuss in sec. 3.2.

Finally, we point out that it is often useful to carry out the analysis of perturbations in Fourier space, where the wavenumber k is related to the physical wavelength of perturbations as $\lambda \propto \frac{a}{k}$. Due to the linear dependence of all expressions on perturbation variables,

[1] The situation is slightly different in the study of gravitational waves, where no preferred coordinate system exists [204].

the Fourier decomposition is a straightforward step.[2] For scalar perturbations we have

$$\begin{aligned} \tilde\Phi(t,\vec{x}) &\to \tilde\Phi(t,\vec{k})\,, \qquad && \psi(t,\vec{x}) \to \psi(t,\vec{k})\,, \\ \partial_i B(t,\vec{x}) &\to \mathrm{i}\, k_i B(t,\vec{k})\,, \qquad && \partial_i\partial_j E(t,\vec{x}) \to -k_i k_j E(t,\vec{k})\,. \end{aligned} \tag{3.8}$$

3.1.1. Dynamics of perturbations

Dynamics of the perturbations are governed by the perturbed EFE (we neglect the cosmological constant, $\Lambda = 0$)

$$\delta G^{\mu}{}_{\nu} = \kappa \delta T^{\mu}{}_{\nu}\ , \tag{3.9}$$

which are calculated from the metric and matter content at linear order. We consider only scalar perturbations in the expressions below.

As we will later work with a non-standard form of the lapse we include the expressions for the perturbed Einstein tensor for a general lapse, which are not commonly found in the literature (and were therefore rederived for this thesis; no sum over i in $\delta G^{i}{}_{i}$):

$$\begin{aligned} \delta G^0{}_0 =& \frac{6}{N^2}\frac{a'}{a}\left(\frac{a'}{a}\tilde\Phi + \psi'\right) + \frac{2}{aN}\frac{a'}{a}\nabla^2 B - \frac{2}{a^2}\nabla^2\psi - \frac{2}{N^2}\frac{a'}{a}\nabla^2 E'\,, \\ \delta G^0{}_i =& -\frac{2}{N^2}\partial_i\left(\frac{a'}{a}\tilde\Phi + \psi'\right)\,, \\ \delta G^i{}_i =& \frac{2}{N^2}\left(\frac{(a')^2}{a^2} - 2\frac{a'}{a}\frac{N'}{N} + 2\frac{a''}{a}\right)\tilde\Phi + \frac{2}{N^2}\frac{a'}{a}\tilde\Phi' + \frac{2}{N^2}\left(3\frac{a'}{a} - \frac{N'}{N}\right)\psi' + \frac{2}{N^2}\psi'' \\ & + (\nabla^2 - \partial_i^2)\left(\frac{1}{aN}\left(2\frac{a'}{a}B + B'\right) + \frac{1}{a^2}(\tilde\Phi - \psi) + \frac{1}{N^2}\left(\frac{N'}{N} - 3\frac{a'}{a}\right)E' - \frac{1}{N^2}E''\right)\,, \\ \delta G^i{}_{\neq j} =& \partial_i\partial_j\left(-\frac{1}{aN}\left(2\frac{a'}{a}B + B'\right) - \frac{1}{a^2}(\tilde\Phi - \psi) + \frac{1}{N^2}\left(3\frac{a'}{a} - \frac{N'}{N}\right)E' + \frac{1}{N^2}E''\right)\,. \end{aligned} \tag{3.10}$$

One can similarly obtain a perturbed stress-energy tensor, where its explicit form is of course dependent on the specifics of the matter content. We first consider a description of matter perturbations starting from a perfect fluid, whose energy-momentum tensor is given by (1.30). We work in the fluid rest frame, where the spatial part of its velocity vanishes $u^i = 0$ and define the velocity perturbation as $\delta u^i =: \frac{N}{a}\mathsf{v}_i$.[3] Focusing on scalar

[2] Assuming that the perturbation variables are (absolutely) integrable over spatial slices.

[3] This is a definition; the unusual index structure arises because we have $\delta u_i = -aB_i + a\mathsf{v}_i$. The velocity perturbation v is not to be confused with the Mukhanov–Sasaki variable v introduced further below.

perturbations again, we have $\mathsf{v}_i \to \partial_i \mathsf{v}$. General matter perturbations away from the perfect fluid can contain also anisotropic stress, which we encode in $\Sigma_{ij} := \delta T^i_{\ j} - \frac{1}{3}\delta^i_k \delta T^k_{\ j}$. Just as for the metric perturbations one does not encounter expressions with raised indices for Σ_{ij}. Σ_{ij} is traceless and we can again decompose it into scalar, vector and tensor components, which fulfill the same requirements as given in (3.5). The scalar part of the anisotropic stress is given by $(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\Sigma$, where Σ denotes a scalar function. We then find for the perturbed energy-momentum tensor

$$\begin{aligned} \delta T^0_{\ 0} &= -\delta\rho, \qquad \delta T^i_{\ j} = \delta P \delta^i_j + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\Sigma\,, \\ \delta T^0_{\ i} &= \frac{a}{N}(\rho + P)(\partial_i B + \partial_i \mathsf{v})\,, \qquad \delta T^i_{\ 0} = -\frac{N}{a}(\rho+P)(\partial_i \mathsf{v})\,. \end{aligned} \tag{3.11}$$

One can furthermore separate the pressure perturbation into an adiabatic contribution proportional to the energy density and a non-adiabatic term, where we define the entropy perturbations δS and the sound speed c_s^2:

$$\delta P = c_s^2 \delta\rho + \tau \delta S \quad \text{with} \quad c_s^2 = \frac{P'}{\rho'}\,. \tag{3.12}$$

In later chapters we will be concerned only with scalar fields as matter content. For a single scalar field with energy-momentum tensor as given in (1.35) we obtain

$$\begin{aligned} \delta T^\mu_{\ \nu} =& \delta g^{\mu\alpha}\partial_\alpha\chi\partial_\nu\chi + g^{\mu\alpha}\partial_\alpha\delta\chi\partial_\nu\chi + g^{\mu\alpha}\partial_\alpha\chi\partial_\nu\delta\chi \\ &- \left(\frac{1}{2}\delta g^{\alpha\lambda}\partial_\lambda\chi\partial_\alpha\chi + g^{\alpha\lambda}\partial_\lambda\chi\partial_\alpha\delta\chi - \frac{\mathrm{d}U(\chi)}{\mathrm{d}\chi}\delta\chi\right)\delta^\mu_\nu\,. \end{aligned} \tag{3.13}$$

For a scalar field that is homogeneous at background level s.t. $\partial_\mu\chi \propto \delta^0_\mu$, the above simplifies considerably. In chap. 7 we use a relational coordinate system made from four massless scalar fields for which the perturbed energy-momentum tensor takes an unconventional form. This set up as well as the perturbative analysis of the corresponding energy-momentum tensor are explained in sec. 3.4.2.

The above expressions can be used to obtain and then solve the EFE. In particular, one finds the perturbed continuity equation

$$\delta\rho' = 3\psi'(\rho + P) - 3\mathcal{H}(\delta\rho + \delta P) - \frac{2\,k^2}{\kappa\, a^2}(\psi' + \mathcal{H}\psi)\,, \tag{3.14}$$

where k^2 denotes the wavenumber in a Fourier decomposition.

3.1.2. The averaging problem

The procedure we described above relies on the assumption that the description of a homogeneous background with small linear perturbations is a good approximation of the full, inhomogeneous system. Modelling the universe with an FLRW metric and describing the matter content as an averaged fluid averages over the 'lumpiness' we plainly observe. The EFE are expected to accurately describe the evolution of inhomogeneities at each point in the universe and due to their high degree of non-linearity, taking the average of the EFE is not the same as considering the EFE of averaged quantities, e.g. for the Einstein tensor $\langle G_{\mu\nu}[g_{\mu\nu}]\rangle \neq G_{\mu\nu}[\langle g_{\mu\nu}\rangle]$, where $\langle\cdot\rangle$ denotes a generic averaging procedure. The experimental success of cosmology gives good reason to trust the description of a perturbed FLRW universe as a first approximation. As cosmological data becomes increasingly accurate and tensions have begun to arise, it is however judicious to investigate the influence of non-linear effects. In a more complete analysis, one may consider corrections from the inhomogeneities to the equations of motion of averaged quantities [88, sec. 5.1.4][205–208].

3.2. A question of gauge

In this section we discuss the gauge freedom introduced by infinitesimal coordinate transformations of the coordinate system of inhomogeneous perturbations. We first show how perturbation variables transform under such a change of coordinates in sec. 3.2.1 and introduce gauge-invariant combinations of perturbation variables in sec. 3.2.2. Here we also comment on the physical degrees of freedom and briefly outline how gauge-invariant perturbations can be related to observations. In sec. 3.2.3 we give some examples of common gauge choices in the literature, paying special attention to the harmonic gauge which we will use in sec. 3.4. While most of these calculations are standard in the literature, their forms for an unspecified lapse are not. For convenience we therefore report them explicitly here.

3.2.1. Perturbations and covariance

General covariance is one of the main principles of general relativity. As explained in sec. 3.1, perturbative calculations in cosmology are carried out in a preferred background coordinate system dictated by the evolution of the universe. There remains a freedom in the choice of coordinate systems for the inhomogeneous perturbations: any coordinate system in which perturbations remain small is permitted.

In order to establish the gauge dependence of perturbative quantities δQ, we identify how they transform under such a change of coordinates $x^\mu \to \widetilde{x}^\mu$, which we characterise by a four-vector ξ^μ. Explicitly, perturbed quantities in different coordinate systems are related by the Lie derivative $\pounds$ as given in (1.8) (see e.g. [85])

$$x^\mu \to \widetilde{x}^\mu = x^\mu + \xi^\mu \quad \Rightarrow \quad \widetilde{\delta Q} = \delta Q + \pounds_\xi Q \,, \tag{3.15}$$

where Q denotes the background value of δQ.

A general coordinate transformation can mix scalar and vector components. For $\xi^\mu = (\xi^0, \xi^i)$ we can decompose the spatial part ξ^i into its scalar and vector components, $\xi^i =: \partial_i \xi + \xi^i_V$. In the case where we are interested in scalar perturbations only, as we will be in the following, one restricts to those coordinate transformations that preserve the scalar nature of perturbations. We thus have $\xi^i = \partial_i \xi$, where the unusual index structure arises from the decomposition we used above.

The Lie derivative of the metric tensor simplifies to $\pounds_\xi g_{\mu\nu} = \xi^\lambda g_{\mu\nu,\lambda} + g_{\mu\lambda}\partial_\nu \xi^\lambda + g_{\lambda\nu}\partial_\mu \xi^\lambda$ and for an FLRW background metric (3.1) we find

$$\pounds_\xi g_{00} = -\, 2N^2 \left(\frac{N'}{N}\xi^0 + \xi^{0'} \right) , \qquad \pounds_\xi g_{0i} = -N^2 \partial_i \xi^0 + a^2 \xi^{i'} \,, \tag{3.16}$$

$$\pounds_\xi g_{ii} = 2a^2 \left(\frac{a'}{a}\xi^0 + \partial_i^2 \xi \right) , \qquad \pounds_\xi g_{i\neq j} = a^2(\partial_i \xi^j + \partial_j \xi^i) = 2a^2 \partial_i \partial_j \xi \,. \tag{3.17}$$

From these we can deduce the transformation properties of the metric perturbation variables [4]

$$\tilde{\Phi} \to \tilde{\Phi} + \frac{N'}{N}\xi^0 + \xi^{0\prime} \qquad B \to B - \frac{N}{a}\xi^0 + \frac{a}{N}\xi' \qquad \psi \to \psi - \frac{a'}{a}\xi^0 \qquad E \to E + \xi\,. \tag{3.18}$$

For a scalar field χ, we have

$$\delta\chi \to \delta\chi + \chi'\xi^0\,. \tag{3.19}$$

3.2.2. Gauge-invariant variables

A natural consequence of the remaining coordinate freedom is that only quantities that are invariant under such infinitesimal coordinate transformations can be physical. We present some common choices of gauge-invariant perturbation variables in the following.

Gauge-invariant versions of the perturbations $\tilde{\Phi}$ and ψ are known as the Bardeen variables Φ_B and Ψ_B [198] and are given by

$$\Phi_B = \tilde{\Phi} + \frac{1}{N}\left(\left(B - \frac{a}{N}E'\right)a\right)', \qquad \Psi_B = \psi - \frac{a'}{N}\left(B - \frac{a}{N}E'\right). \tag{3.20}$$

In chap. 5 we will be concerned with the curvature perturbation on equal density hypersurfaces[5] ζ and the comoving curvature perturbation[6] $\mathcal{R}$:

$$-\zeta = \psi + \frac{H}{\rho'}\delta\rho\,, \qquad \mathcal{R} = \psi + \frac{H}{\chi'}\delta\chi\,. \tag{3.21}$$

As we detail in sec. 3.3.1, these two variables are equal on scales larger than the Hubble horizon and commonly used to calculate the primordial power spectrum. They can be related to the Bardeen variable Φ_B, which in turn can be related to the radiation density and thereby ultimately to the CMB (see e.g. [87, 88]).

Another frequently encountered gauge-invariant quantity is the Mukhanov-Sasaki vari-

[4] The additional terms would have opposite signs if we were working in a different metric signature convention, i.e. $(+,-,-,-)$ instead of $(-,+,+,+)$.

[5] Note that the use of the symbols ζ and $\mathcal{R}$ is not consistent across the literature. We use the same convention as e.g. [86, 88], but opposite of [87].

[6] We give the expression for a single scalar field, as we assume in chap. 5. In general, $\mathcal{R} = \psi + H\frac{a}{N}(B + \mathsf{v})$ [86].

able [85, 209], which for a single scalar field reads[7]

$$v = a\left(\delta\chi + \frac{\chi'}{H}\psi\right) . \tag{3.22}$$

The total perturbed action of the gravitational and matter system expanded at second order in perturbation variables can be written purely in terms of the Mukhanov-Sasaki variable (neglecting total derivatives) [85], namely

$$\delta S = \frac{1}{2}\int \mathrm{d}^4x \left((v')^2 - c_s^2\delta^{ij}\partial_i v\partial_j v + \frac{z''}{z}v^2\right) , \tag{3.23}$$

where $z = a\frac{\chi'}{H}$. The action thus takes the form of a scalar field with a dynamical mass term. In a Hamiltonian analysis, the Mukhanov-Sasaki variable is naturally obtained from a canonical transformation to the physically relevant scalar degrees of freedom as was first discussed in [210]. (We will discuss physical degrees of freedom in cosmological perturbation theory below.) When studying quantum perturbations, it is important to quantise the physically relevant sector; furthermore (3.23) has a form amenable for canonical quantisation.

It is possible to rewrite the entirety of the linearised EFE in a gauge-invariant manner and work only with gauge-invariant quantities. We report the form of the gauge-invariant Einstein tensor as well as that of the energy-momentum tensor in app. B.

Surveying the degrees of freedom

We depart on a short detour to discuss the physical degrees of freedom in cosmological perturbation theory. We have introduced four scalar metric perturbations and the freedom to change the coordinate system through ξ^0 and ξ. From the transformation properties (3.18) it is apparent that two of the metric variables can be set to zero by an appropriate choice of the perturbed coordinate system. Furthermore, if matter is given by a scalar field, one can similarly choose a coordinate system in which $\delta\chi = 0$, as seen in (3.19).

A discussion of the physical degrees of freedom is most easily carried out within a Hamiltonian description. In a Hamiltonian system the number of physical degrees of freedom P of a system with a $2n$ dimensional phase space is given by $P = n - M - \frac{S}{2}$, where M denotes the number of first, and S the number of second class constraints. We briefly recall that the constraints reduce the system to the physical phase space and that

[7] See [85] for the derivation in the case of a perfect fluid.

first class constraints are those whose Poisson brackets with all other constraints vanish (weakly, i.e. when the equations of motion are satisfied), see e.g. [211]. From the ADM-formulation of GR introduced in sec. 1.1.2 we find that the Hamiltonian of GR (1.18) is given by four first class constraints: the scalar or Hamiltonian constraint H and three diffeomorphism constraints H^a. The phase space is spanned by the six components of the spatial metric q_{ab} and their conjugate momenta. We thus have $n = 6\,, M = 4$, and $S = 0$, resulting in $P = 2$ physical degrees of freedom of the gravitational sector. These are exactly the 2 degrees of freedom we find for gravitational waves. Matter then adds additional degrees of freedom to the theory.

We can repeat the analysis above for an FLRW universe filled with a single scalar field. The Hamiltonian of a massless scalar field gives contributions to the scalar and diffeomorphism constraint as [210]

$$H_{\mathrm{matter}} = \frac{N}{2}\left(\frac{\pi_\chi^2}{\sqrt{q}} + \sqrt{q}q^{ab}\partial_a\chi\partial_b\chi + 2\sqrt{q}U(\chi)\right) + N^a\pi_\chi\partial_a\chi \tag{3.24}$$

and does not lead to additional constraints. Due to homogeneity the six degrees of freedom of the spatial metric reduce to a single one. Furthermore, the diffeomorphism constraint vanishes trivially and we have $M = 1$. The combined phase space with matter has $n = 2$, such that we have a single physical degree of freedom $P = 1$.

A Hamiltonian analysis of cosmological perturbation theory was first carried out in [210] and has since entered the literature [83, 212, 213]. Including perturbations of the spatial metric and the matter content enlarges the phase space of the theory, as these evolve separately from the background variables and are described by a perturbed Hamiltonian. The spatial metric perturbation can again be decomposed into scalar, vector and tensor parts, and the Hamiltonians describing the different perturbation types decouple. Explicitly, one finds the following: There are two vector components and two vector constraints; there are two tensor perturbations and no constraints; the scalar sector has two metric components and two constraints. The two geometric degrees of freedom of GR are contained in the tensor perturbations and the matter content introduces additional degrees of freedom. For instance, in a universe filled with a single scalar field, there is one scalar degree of freedom on addition to the tensor perturbations.

3.2.3. Gauge choices

Instead of working with gauge-invariant quantities, one can opt to fix the gauge instead, which can considerably simplify calculations. Common gauge choices are

- Longitudinal or Newtonian gauge $E = B = 0$:
  In this gauge the perturbed metric (3.7) becomes diagonal. Furthermore, in the absence of anisotropic matter ($\Sigma = 0$ in (3.11)), we have $\tilde{\Phi} = \psi$ from the Einstein equations (3.10). The Bardeen variables (3.20) also simplify considerably, such that in general scenarios this is arguably the simplest gauge for carrying out a perturbative analysis.

- Comoving gauge $\delta\chi = 0$ and $E = 0$ or $B = 0$:
  There are different implementations of this gauge. The idea is to be in the comoving frame of the fluid and set $\delta T^0_{\ i} = 0$. (For a scalar field we have $\delta T^0_{\ i} = \frac{\partial_0 \chi}{N^2}\partial_i \delta\chi$.) One requires an additional condition and can set either E or B to zero [86, 88]. In this gauge, the comoving curvature perturbation (3.21) takes on a simple form $\mathcal{R} = \psi$. We will use this gauge in chap. 5.

- Synchronous gauge $\tilde{\Phi} = 0$, $B = 0$:
  This is the gauge used in the original literature on cosmological perturbations [197]. It however does not completely fix the gauge and one retains unphysical gauge degrees of freedom in the formalism. Confusions related to this redundancy could be clarified by introducing a gauge-invariant treatment of perturbations [198].

Harmonic gauge

The cosmological perturbations we extract from GFT in chap. 7 are obtained within a coordinate system spanned by four massless scalar fields, which induces a harmonic gauge. For this reason, we discuss the harmonic gauge in some detail. The perturbative analysis of such a system within GR is the topic of sec. 3.4.2.

In harmonic gauge, the coordinates each satisfy the wave equation

$$\Box x^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = 0\,. \tag{3.25}$$

This gauge plays a crucial role in the history of GR, as it was used to prove that the EFE indeed give a well posed initial value problem [214] [29, sec. 10.2]. It has also been used in the study of perturbations through a non-singular bounce, as evolution equations remain well defined throughout [215, 216].

Generally, the harmonic gauge condition (3.25) introduces conditions also at the background level: For an FLRW metric (3.1) the relevant Christoffel symbols are $\Gamma^0_{ii} = \frac{aa'}{N^2}$, $\Gamma^0_{00} = \frac{N'}{N}$ and we find that (3.25) is satisfied if

$$3H - \frac{N'}{N} = 0\,. \tag{3.26}$$

This restricts admissible lapse choices to cases in which $N \propto a^3$.

Imposing (3.25) on the perturbative level ($\delta(g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}) = 0$) leads to the following harmonic gauge conditions, where we again restrict our discussion to scalar perturbations,

$$-2\frac{HB}{aN} - \frac{B'}{aN} + \frac{1}{a^2}(-\tilde{\Phi} + \psi + \nabla^2 E) = 0\,, \tag{3.27}$$

$$\frac{2\tilde{\Phi}}{N^2}\left(\frac{N'}{N} - 3H\right) - \frac{1}{aN}\nabla^2 B + \frac{1}{N^2}(-\tilde{\Phi} - 3\psi + \nabla^2 E)' = 0\,. \tag{3.28}$$

The first term of (3.28) vanishes due to (3.26). Note that (3.25) continues to be satisfied under a coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ (with $\xi^\mu = (\xi^0, \delta^{ij}\partial_j\xi)$ as introduced in sec. 3.2.1) as long as $\Box\xi^0 = 0 = \Box\xi$. Hence, the harmonic gauge contains a residual gauge freedom in the form of initial conditions for ξ^0 and ξ. This freedom can be used to initially set some perturbation variables to zero; whether such an additional gauge fixing is conserved under evolution depends on the physical scenario. Indeed, recalling how the metric perturbations transform under a change of coordinates (3.18), we see that (3.27) remains valid if $\frac{1}{N^2}\xi'' + \frac{k^2}{a^2}\xi = 0$, whereas (3.28) is satisfied as long as $\frac{1}{N^2}(\xi^0)'' + \frac{k^2}{a^2}\xi^0 = 0$.

3.3. Separate universe framework

The separate universe framework is built on the idea that perturbations larger than the Hubble horizon (1.43) appear homogeneous and gradient terms can be neglected in their perturbative analysis [217, 218]. In this regime, the universe can be modelled as independent Hubble sized regimes, or ‘patches’, over which perturbations can be seen as homogeneous. We will make this notion more precise below.

The separate universe framework can be useful not only to simplify calculations, but also to study beyond GR effects on perturbations [219, 220]. For quantum gravitational theories, where including a full treatment of perturbations is often far from straightforward, an analysis within the separate universe picture can already shed light on possible imprints [221, 222]. We will investigate the effect of quantum gravitationally modified Friedmann equations on large scale perturbations in the separate universe framework in chap. 5.

We begin this section by explaining the separate universe picture in more detail and discussing the super-horizon evolution of gauge-invariant perturbations in sec. 3.3.1. In sec. 3.3.2 we comment on the interplay between simplifications of the separate universe framework and the choice of gauge.

3.3.1. Homogeneous perturbations

The following introduction of the separate universe idea is based on [217, 219].

The Hubble horizon or Hubble radius $R_H := \frac{1}{H}$ is a term often used interchangeably with the particle horizon we defined in sec. 1.2.3. R_H gives the length scale at which spacetime events could have been in causal contact in the past. Perturbation modes whose wavelength λ exceeds the Hubble horizon are approximately homogeneous for an observer within, such that spatial gradient terms become inconsequential for their description. One can then carry out an expansion around small gradients where the condition $\lambda \gg R_H$ gives the expansion parameter $|\frac{k}{aH}| \ll 1$. Such an expansion is useful to study the evolution of perturbations on large scales and significantly simplifies their analysis. It naturally leads to the separate universe picture, where the universe is modelled by a multitude of patches approximately the size of the Hubble horizon. Different patches of the universe separated by R_H or further are causally disconnected and each evolve independently in accordance with the Friedmann equation. Long-wavelength perturbations, which is a term we reserve for perturbation modes that satisfy $\lambda \gg R_H$, can thus be treated as homogeneous within a Hubble patch. For scalar metric perturbations (3.7), this implies

$$\tilde{\Phi}(t, x^i) \to \tilde{\Phi}(t)\,, \quad \psi(t, x^i) \to \psi(t)\,, \quad \partial_i B \to 0\,, \quad \partial_i \partial_j E \to 0\,. \tag{3.29}$$

The evolution of a long wavelength perturbations can be obtained by considering the difference between its local parameter value in a patch and the average taken over all patches. Consequentially, the local values of quantities in a single patch can be described as the average plus a homogeneous perturbation. For instance, the local values of the lapse, scale factor, energy density, and pressure are respectively given by

$$N_{\rm loc} = N(1+\tilde{\Phi})\,, \quad a_{\rm loc} = a(1-\psi)\,, \tag{3.30}$$

$$\rho_{\rm loc} = \rho + \delta\rho\,, \quad P_{\rm loc} = P + \delta P\,. \tag{3.31}$$

This picture has two scales which are important: the first is the Hubble radius, which gives the scale beyond which spatial gradients are subdominant and can be neglected. The second is the scale of the overall background, i.e. the scale on which the averaging over disconnected patches takes place. We will use the terms 'super-horizon' scales and 'long wavelength' perturbation to refer to regimes in which spatial gradients can be neglected and a separate universe description is applicable.

In the separate universe picture, the dynamics of some gauge-invariant quantities simplify significantly. Consider first the curvature perturbation on equal density hypersurfaces, ζ (3.21). It was shown in [219] that ζ will be conserved on super-horizon scales for adiabatic perturbations as long as energy conservation in the form of $\nabla_\mu T^\mu_\nu = 0$ holds, independently of gravitational dynamics. Explicitly, the authors find that

$$\zeta' = -\frac{H}{\rho+P}\delta P_{\rm nad} - \frac{1}{3}\nabla^2(\sigma + \mathsf{v} + B)\,, \tag{3.32}$$

where $\delta P_{\rm nad}$ encodes the non-adiabatic contribution to the perturbations, which in the absence of anisotropies is given by $\tau\delta S$ in our convention, σ denotes the shear, v the perturbed 3-velocity of the fluid, and B the metric perturbation.

Furthermore, from the Einstein equations $\delta G^0{}_0 = \kappa\delta T^0{}_0$ and $\delta G^0{}_i = \kappa\delta T^0{}_i$, see (3.10), one can relate the expressions of ζ and $\mathcal{R}$ by establishing a relation between $\delta\rho$ and $\delta\chi$.[8] Specifically, one finds that they are related by a gradient term

$$\frac{\delta\rho}{\rho'} + \nabla^2 f = \frac{\delta\chi}{\chi'} \qquad \Rightarrow \qquad \mathcal{R} = -\zeta + H\nabla^2 f\,. \tag{3.33}$$

where $f = \frac{2}{3}\frac{1}{\kappa(\rho+P)}\left(-\frac{1}{aN}B + \frac{1}{a^2}\frac{1}{H}\psi + \frac{1}{N^2}E'\right)$ and will simplify further depending on the gauge choice. Thus, on super-horizon scales $-\zeta = \mathcal{R}$ are equally conserved. We will revisit this result in chap. 5 and find that in some quantum gravity models this need no

[8] Once again we focus on the case with a single scalar field, but the calculation and result also apply to more general matter content.

longer be the case.

3.3.2. Note on gauge choices

In the separate universe picture, the metric perturbations E and B can be neglected, as they enter the metric only as spatial gradients (3.29). Furthermore, the off-diagonal elements of the perturbed Einstein tensor (3.10) are trivially zero, as again they contain only gradient terms. In the separate universe framework with a single scalar field the system is then characterised by the perturbation variables $\psi, \tilde{\Phi}$, and $\delta\chi$. The gauge freedom of infinitesimal gauge transformations of the perturbed coordinate system furthermore reduces to the choice of ξ^0, which in accordance with (3.18) and (3.19) can be used to set one of the remaining variables to zero. For the counting of degrees of freedom we can proceed as follows: on the gravitational side we have reduced the perturbed metric phase space to a single variable, and the matter sector is unchanged. As already pointed out, the diffeomorphism constraints are trivially satisfied, such that one constraint, the Hamiltonian constraint, remains. We thus recover a single degree of freedom.

The Newtonian gauge (see sec. 3.2.3) is a popular gauge in the literature in which one fixes $E = B = 0$. Usually, from the off-diagonal spatial components of the EFE and in the absence of anisotropic stress this leads to

$$\partial_i \partial_j \psi - \partial_i \partial_j \tilde{\Phi} = 0 \qquad \Rightarrow \qquad \tilde{\Phi} = \psi \, . \tag{3.34}$$

The equation $\partial_i \partial_j \psi - \partial_i \partial_j \tilde{\Phi} = 0$ is however trivially satisfied for negligible spatial gradients. As we can see from these considerations, and as was further elaborated within the Hamiltonian framework for cosmological perturbations in [212], the Newtonian gauge does not lead to additional constraints for the perturbation variables in the separate universe framework. In [220], the authors then define a pseudo-longitudinal gauge. Here, the gauge is fixed in a limit where gradient terms contribute and the Newtonian gauge is meaningful. Even if the system evolves through a regime where spatial gradients are negligible, the relation $\tilde{\Phi} = \psi$ then remains valid through the previous gauge fixing. In [212] the authors redefine the Newtonian gauge to recover such a relation also within the Hamiltonian framework, by introducing relations similar to the diffeomorphism constraint. We work within the separate universe picture in chap. 5 and will revisit these considerations on suitable gauge choices.

3.4. Analysis in relational coordinates

In the following, we will carry out a perturbative analysis of a flat FLRW spacetime

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{3.35}$$

for two cases: In the first, we assume that the universe is filled with a single massless scalar field that takes on the role of a relational clock with respect to which the evolution takes place (sec. 3.4.1). In the second, the matter content is given by four massless scalar fields, which are assumed to span an entire relational coordinate system (sec. 3.4.2). We introduced the notion of relational observables in GR in sec. 1.1.4. The concept of including a massless scalar field as a clock in GFT was the topic of sec. 2.2.1 and we extended this case to four fields in sec. 2.2.3. The case of a single field suffices when one is interested only in the background evolution, as we discuss in chap. 4. The four field case becomes of interest when searching for a description of inhomogeneities and thereby cosmological perturbation theory in e.g. GFT, as this requires a complete matter reference frame. This will be the aim of chap. 6 and chap. 7. In GFT, the necessity of using four massless scalar fields instead of e.g. dust fields arises because such scalar fields are the only matter type that has been successfully included in the quantum theory at this point [144]. It then comes as no surprise that we are interested in the equivalent classical scenarios. As we will discuss further in chap. 4 and chap. 6, it is only in a relational coordinate system that it is meaningful to compare GFT quantities to those of GR.

The settings we consider deviate from standard cosmological analysis, where it is common to work in conformal time, $N = a$, and the Newtonian gauge, $E = B = 0$, in order to simplify the analysis. Instead, as we will see, the matter frame choice completely fixes the lapse as well as the gauge in the perturbed coordinate system. In the following, we include a background, as well as a perturbative analysis for both described cases of interest. In chap. 7 we will compare our findings within the GFT framework to the general relativistic evolution equations contained in what follows.

3.4.1. Relational clock

We first consider the case in which the matter content is given by a single scalar field χ^0. Demanding that the scalar field serves as a time parameter (relational clock) determines the form of the lapse function. For a massless scalar field with an action as given by (2.13) one obtains its conjugate momentum as $\pi_0 = \partial\mathcal{L}/\partial(\partial_0\chi^0) = a^3\partial_0\chi^0/N$. In a coordinate system in which χ^0 takes on the role of time $\partial_0\chi = 1$, the lapse is then fixed to

$$N = \frac{a^3}{\pi_0}\,. \tag{3.36}$$

As we are working within the relational clock frame, $'$ denotes derivatives with respect to χ^0 in what follows. The units of the lapse and scale factor are $[a] = [N] = L^2$ and $[\partial_\mu\chi^A] = L^0$.

The non-vanishing components of the energy momentum tensor (3.13) read ($i = 1, 2, 3$)

$$T^0_{\ 0} = -\frac{\pi_0^2}{2a^6}\,, \qquad T^i_{\ i} = \frac{\pi_0^2}{2a^6}\,, \tag{3.37}$$

thus corresponding to a perfect fluid with $w = 1$. The Friedmann equations (3.2) and (3.3) for the lapse choice (3.36) are

$$H^2 = \frac{\kappa}{6}\,, \qquad H' = 0\,. \tag{3.38}$$

These differ from standard expressions in the literature due to the unusual choice of N. For conformal time $N = a$, we would find $H^2 = \frac{\kappa}{6}\frac{\pi_0^2}{a^4}$ and $H' = -\frac{2}{3}\kappa\frac{\pi_0^2}{a^4}$ instead.

Perturbations

We proceed to carry out a perturbative analysis for the setting with a single relational clock field; we will return to these results for the interpretation of our results in chap. 7. We work with $N = \frac{a^3}{\pi_0}$ and in harmonic gauge (sec. 3.2.3). The harmonic gauge conditions for our choice of lapse read

$$\begin{aligned} -2\pi_0\frac{HB}{a^4} - \pi_0\frac{B'}{a^4} + \frac{1}{a^2}(-\tilde{\Phi} + \psi + \nabla^2 E) &= 0\,, \\ -\frac{\pi_0}{a^4}\nabla^2 B + \frac{\pi_0^2}{a^6}(-\tilde{\Phi} - 3\psi + \nabla^2 E)' &= 0\,, \end{aligned} \tag{3.39}$$

and can be combined to give

$$\pi_0^2(-\tilde{\Phi} - 3\psi + \nabla^2 E)'' = a^4\nabla^2(-\tilde{\Phi} + \psi + \nabla^2 E)\,. \tag{3.40}$$

The components of the perturbed Einstein tensor (3.10) for our lapse choice are explicitly given by

$$\delta G^0{}_0 = \frac{6\pi_0^2}{a^6} H \left(H\tilde{\Phi} + \psi' \right) + \frac{2\pi_0}{a^4} H \nabla^2 B - \frac{2}{a^2} \nabla^2 \psi - \frac{2\pi_0^2}{a^6} H \nabla^2 E' \,, \tag{3.41}$$

$$\delta G^0{}_i = -\frac{2\pi_0^2}{a^6} \partial_i \left(H\tilde{\Phi} + \psi' \right) \,, \tag{3.42}$$

$$\delta G^i{}_i = \frac{2\pi_0^2}{a^6} \left(2H' - 3H^2 \right) \tilde{\Phi} + \frac{2\pi_0^2}{a^6} H \tilde{\Phi}' + \frac{2\pi_0^2}{a^6} \psi'' \tag{3.43}$$

$$+ (\nabla^2 - \partial_i^2) \left(\frac{\pi_0}{a^4} (2HB + B') + \frac{1}{a^2} (\tilde{\Phi} - \psi) - \frac{\pi_0^2}{a^6} E'' \right) \,, \tag{3.44}$$

$$\delta G^i{}_{\neq j} = \partial_i \partial_j \left(-\frac{\pi_0}{a^4} (2HB + B') - \frac{1}{a^2} (\tilde{\Phi} - \psi) + \frac{\pi_0^2}{a^6} E'' \right) \,. \tag{3.45}$$

The energy momentum tensor (3.13), which, due to the role of χ^0 as a clock field, does not contain explicit matter perturbations, i.e. $\delta\chi^0 = 0$ (which amounts to an appropriate initial choice of ξ^0), is given by [9]

$$\delta T^0{}_0 = \frac{\tilde{\Phi}}{N^2} \,, \qquad \delta T^0{}_i = 0 \,, \qquad \delta T^i{}_0 = \frac{1}{aN} \partial_i B \,, \qquad \delta T^i{}_j = 0 \,, \qquad \delta T^i{}_i = -\frac{\tilde{\Phi}}{N^2} \,. \tag{3.46}$$

The perturbed EFE combined with the harmonic gauge conditions (3.39) give the following (for details of the derivation please see app. C.1)

$$E'' - \frac{a^4}{\pi_0^2} \nabla^2 E = 0 \,, \qquad \tilde{\Phi}'' - 4H\tilde{\Phi}' - \frac{a^4}{\pi_0^2} \nabla^2 \tilde{\Phi} = 0 \,, \qquad -\frac{a^4}{\pi_0^2} \nabla^2 \psi + \psi'' = 0 \,. \tag{3.47}$$

Furthermore, we can obtain B from (3.39) and the diffeomorphism constraint $H\tilde{\Phi} + \psi' = 0$ needs to be satisfied.

3.4.2. Massless scalar fields as coordinates

We now consider a scenario in which four massless scalar fields can be used to span an entire coordinate system. This idea is not new: For instance, in [75] the authors discuss scalar fields that serve as harmonic coordinates in the Lagrangian and Hamiltonian formulation of GR. In [83] the authors consider a system with four massless scalar fields and carry out a canonical quantisation with LQG techniques. These matter reference frames have also been employed to study perturbations within GFT [79, 80, 151, 152] (we will

[9] These are the same expressions as recovered in e.g. [215]. To see this, set $\pi_0 = 1$ above and $P = X$, $g = 0$ in the expressions in [215].

discuss these further in sec. 4.1.4). While some of the GFT models [151, 152] include a fifth matter field that is assumed to dominate the other four frame fields, thus effectively recovering the single field case discussed in the previous section, we will assume that the four reference fields are the only matter content in our investigations in chap. 6 and 7. While the perturbative analysis for this choice of matter frame is a straightforward application of cosmological perturbation theory within GR, we present the results here in some detail. They will be needed to interpret the results of chap. 7.

In a relational coordinate system spanned by four massless scalar fields we identify each of the scalar fields χ^A with a spacetime coordinate x^μ by demanding that surfaces of constant χ^A are also constant surfaces of said coordinate. In such a coordinate system we thus have $\partial_\mu \chi^A \propto \delta^A_\mu$, where $A = 0, 1, 2, 3$ denotes a *label* of the fields (and is not a spacetime index). We will use $a = 1, 2, 3$ to denote the spatial fields and 0 for the clock field. For such a relational coordinate system to be locally well-defined the fields have to satisfy the following non-degeneracy condition[10]

$$\det\big(\partial_\mu \chi^A\big) \neq 0\,. \tag{3.48}$$

The relational coordinate system naturally gives a special case of the harmonic gauge $\Box x^\mu = 0$ (see sec. 3.2.3) by virtue of the Klein-Gordon equation $\Box \chi^A = 0$ satisfied by each of the fields. While the harmonic gauge has a residual gauge freedom, fixing the relational coordinate system as we do here fixes the gauge completely. We already saw in the previous section how the lapse is fixed from the momentum of the clock field (3.36). More generally, using an ADM decomposition of the metric (1.17) we obtain the following relations between the canonical momenta of the scalar fields $\pi_A = \partial\mathcal{L}/\partial(\partial_0\chi^A)$ for the action as given in (2.29) and the lapse N and shift N^a vector ($g^{0i} = N^a/N^2$):

$$\pi_0 = \frac{\sqrt{|q|}}{|N|}\,, \qquad \pi_a = -N^a \frac{\sqrt{|q|}}{|N|} = -N^a \pi_0 \quad (a = 1, 2, 3)\,. \tag{3.49}$$

The lapse and the shift are then determined by the spatial metric q^{ab} together with the scalar field momenta. As N^a vanishes at the background level for (3.35), so do the conjugate momenta of the spatial fields. The Klein-Gordon equation for the clock field χ^0 is then equivalent to the statement $\partial_t \pi_0 = 0$.

[10] For a single scalar field χ^0 acting as a clock, the equivalent condition is $\partial_0 \chi^0 \neq 0$.

The setup we described is unusual from the viewpoint of standard cosmology, where all quantities are usually assumed to be purely homogeneous at the background level, see sec. 3.1. Purely homogeneous fields, however, would evidently not function as suitable coordinates. Importantly, only the gradients of the scalar fields enter the free scalar field action (2.29), therefore, as long as the gradients of the fields are homogeneous quantities, no inhomogeneities appear in the EFE at the background level. For compatibility with the FLRW background, the matter has to satisfy the requirement of isotropy. Thus, three fields that have homogeneous gradients and are related by rotations need to be included. Altogether, we then have one field of the form $\chi^0 = \chi^0(t)$ and three fields with $\chi^a = \chi^a(x^a)$, $a = 1, 2, 3$. As the gradients of the spatial fields are homogeneous, but non-vanishing, they contribute to the background dynamics, and in particular, the Friedmann equation, as we show below. Similar ideas have been discussed in the context of solid inflation, which considers a universe filled with three scalar fields with homogeneous gradients [223].

To establish the background dynamics, we recall the form of the energy momentum tensor $T^{\mu}{}_{\nu} = \sum_A T^{(A)\mu}{}_{\nu}$ (3.13) and the background Einstein tensor (1.28). From the space-time components (which correspond to the diffeomorphism constraint in a Hamiltonian analysis), $G^0{}_i = 0$, it follows that $\sum_A \partial_i\chi^A\partial_0\chi^A = 0$. Furthermore, from the off-diagonal space components $G^i{}_{\neq j} = 0$ we have $\sum_A \partial_i\chi^A\partial_j\chi^A = 0$. Additionally, since all diagonal components of the Einstein tensor $G^i{}_i$ are identical in a coordinate system where $\partial_\mu\chi^A = \lambda_A\,\delta^A_\mu$, with $\lambda_A \in \mathbb{R}$ (no sum over A), it follows that all spatial gradients λ_a must be equal. We set $\lambda_A = 1$ and assume $\partial_\mu\chi^A = \delta^A_\mu$ in the following, which evidently satisfies the above conditions imposed by the EFE. We then find the following expressions for the energy density and pressure

$$-T^0{}_0 = \rho = \frac{1}{2}\left(\frac{1}{N^2} + \frac{3}{a^2}\right) = \frac{(\pi_0)^2}{2a^6} + \frac{3}{2a^2}, \qquad T^i{}_i = P = \frac{1}{2}\left(\frac{1}{N^2} - \frac{1}{a^2}\right) = \frac{(\pi_0)^2}{2a^6} - \frac{1}{2a^2}. \tag{3.50}$$

The contribution of the spatial coordinate fields appears as an additional term $\propto a^{-2}$ that would be equivalent to negative spatial curvature (1.38) and which we refer to as gradient energy. For certain initial conditions where $\frac{(\pi_0)^2}{a^4} \gg 1$, the contribution of the spatial fields to the energy density can become negligibly small for a certain period of time, effectively recovering the standard cosmological background scenario with a single massless scalar field. This limit can be achieved for sufficiently early times, depending on the value of π_0, but at late times the gradient energy will dominate. In general, we have an equation of state parameter $w = P/\rho = \frac{1-a^4/\pi_0^2}{1+3a^4/\pi_0^2} \in (-\frac{1}{3}, 1)$ and similarly for the sound speed $c_s^2 = P'/\rho' = \frac{1-a^4/(3\pi_0^2)}{1+a^4/\pi_0^2} \in (-\frac{1}{3}, 1)$.

The resulting first (1.38) and second (1.39) Friedmann equations read

$$H^2 = \left(\frac{a'}{a}\right)^2 = \frac{\kappa}{6}\left(1 + 3\frac{a^4}{\pi_0^2}\right), \qquad \frac{a''}{a} = \frac{\kappa}{6}\left(1 + 9\frac{a^4}{\pi_0^2}\right), \tag{3.51}$$

where, again, the terms proportional to $\frac{a^4}{\pi_0^2}$ arise due to the spatial fields and would not appear in the case of a single (clock) scalar field, see (3.38). An alternative way of writing the second Friedmann equation is $H' = \kappa\frac{a^4}{\pi_0^2}$.

Perturbations

We give the results for the perturbative analysis for the gauge fixed system with four massless scalar fields. Perturbations within models of solid inflation, which include three massless scalar fields with homogeneous gradients have been investigated in the literature [223]. Similar equations of motion appear in the perturbation analysis in harmonic gauge and can be found in e.g. [215].

Our choice of coordinate system naturally limits us to the harmonic gauge and completely fixes the residual gauge freedom discussed in sec. 3.2.3. In particular, there are no perturbations in the scalar fields in the relational coordinate system where $\partial_\mu \chi^A = \delta^A_\mu$. The perturbed energy-momentum tensor (3.13) for four massless scalar fields in the relational coordinate system then reads [11]

$$\delta T^\mu{}_\nu = \sum_A \left(\delta g^{\mu\alpha}\delta^A_\alpha \delta^A_\nu - \frac{1}{2}\delta g^{\alpha\lambda}\delta^A_\lambda \delta^A_\alpha \delta^\mu_\nu\right), \tag{3.52}$$

and we therefore find

$$\begin{aligned}\delta T^0{}_0 &= \frac{\tilde\Phi}{N^2} - \frac{1}{a^2}(3\psi - \nabla^2 E), \qquad \delta T^0{}_i = \frac{1}{aN}\partial_i B = \delta T^i{}_0, \\ \delta T^i{}_j &= -\frac{2}{a^2}\partial_i\partial_j E, \qquad \delta T^i{}_i = -\frac{\tilde\Phi}{N^2} + \frac{1}{a^2}\left(-\psi + (\nabla^2 - 2\partial_i^2)E\right).\end{aligned} \tag{3.53}$$

With these, we can furthermore find explicit expressions for the gauge-invariant quantities

[11] We do not simplify further so the reader can follow the calculation with greater ease.

ζ and $\mathcal{R}$ (3.21). Note that, unlike in the single field case (3.46), $\delta T^0_{\ i} \neq 0$ due to the sum over spatial fields in (3.52). This also implies that $\mathcal{R} \neq \psi$ in the relational coordinate system, but instead, using $\mathcal{R} = \psi + H\frac{a}{N}(B + \mathsf{v})$ together with (3.11) and (3.53), we find

$$-\zeta = \psi + \frac{1}{3}\frac{\frac{\pi_0^2}{a^4}\tilde{\Phi} - (3\psi - \nabla^2 E)}{1 + \frac{\pi_0^2}{a^4}}\,, \qquad \mathcal{R} = \psi + \frac{\pi_0}{a^2}\frac{HB}{1 + \frac{\pi_0^2}{a^4}}\,, \tag{3.54}$$

where we used $\rho + P = \frac{\pi_0^2}{a^6} + \frac{1}{a^2}$ from (3.50). As for the background, the above reduces to the single field case in the limit $\frac{\pi_0^2}{a^4} \gg 1$. In particular, in this limit we find $-\zeta \to \psi + \frac{\tilde{\Phi}}{3}$ and $\mathcal{R} \to \psi$.

From the perturbed Einstein equations, where $\delta G^\mu_{\ \nu}$ is given by (3.45), and the harmonic gauge conditions (3.39) one can derive the following equations of motion for E and $\tilde{\Phi}$

$$E'' - \frac{a^4}{\pi_0^2}\nabla^2 E + 2\kappa\frac{a^4}{\pi_0^2}E = 0\,, \qquad 4H\tilde{\Phi}' - \tilde{\Phi}'' + \frac{a^4}{\pi_0^2}\nabla^2\tilde{\Phi} = 0\,, \tag{3.55}$$

and obtain ψ and B from

$$-2\nabla^2\psi + 3\kappa\psi = -3\kappa\tilde{\Phi} + 2\frac{\pi_0^2}{a^4}H\tilde{\Phi}' + \kappa\nabla^2 E\,, \qquad -\frac{2\pi_0}{a^2}\left(H\tilde{\Phi} + \psi'\right) = \kappa B\,. \tag{3.56}$$

In the above we made use of the background equations to replace $H' = \kappa\frac{a^4}{\pi_0^2}$. The details of the derivation are included in app. C.2.

3.5. Conclusion: A matter of perturbations

A theoretical description of perturbations is of pivotal importance for interpreting and understanding cosmological observations. Perturbations give insights into the evolution of our universe and can be used to evaluate the viability of any alterations to our understanding of the fundamental nature of gravity.

In this chapter we reviewed the basics of the well-established field of cosmological perturbation theory, which builds on the premise that deviations from a homogeneous isotropic background cosmology can be captured through linear inhomogeneous perturbations. After discussing the main concepts, we introduced the gauge freedom in the description of these perturbations as a natural consequence of the covariance of general relativity. Throughout the chapter, we mostly keep the lapse general and as a consequence include a more general version of perturbative equations than those frequently encountered in the literature. We introduced some gauge-invariant quantities that are widespread in the literature as they can be related to cosmological observations, in particular, the CMB. We also included some common gauge choices, with a particular focus on the harmonic gauge, which we will require for perturbative studies within GFT in the second part of the thesis.

Furthermore, we detailed the separate universe framework, in which perturbations whose wavelength exceeds the Hubble horizon are modelled as homogeneous quantities across distinct Hubble patches. We furthermore pointed out ambiguities in gauge choices, specifically, the Newtonian gauge, within this framework. Our results of chap. 5 are based on the separate universe approach.

Finally, we detailed two scenarios where massless scalar fields serve as relational coordinates, namely, the single field case where the scalar field is employed as a relational clock, as well as the four field case, where four such fields give an entire coordinate system. This is possible in the case of massless scalar fields, as compatibility with homogeneity requires only homogeneity of the field gradients, not the fields themselves. We analyse both cases at the background as well as the perturbative level. We find that in the latter case, the spatial fields contribute to the Friedmann equations through a term akin to negative spatial curvature. The dynamics of perturbative quantities will become important to interpret possible agreement and discrepancies of perturbative evolution equations recovered within GFT in chap. 7.

Chapter 4.

Quantum gravity bounces

'Ich bin ein Teil des Teils der anfangs alles war
Ein Teil der Finsternis, die sich das Licht gebar'

- Johann Wolfgang von Goethe in 'Faust' .

'Part of the Part am I, once All, in primal Night
Part of the Darkness which brought forth the Light'
(translated by Bayard Taylor)

In this chapter, we consider how background independent approaches to quantum gravity, specifically group field theory (GFT) and loop quantum cosmology (LQC), can lead to a resolution of the Big Bang singularity by replacing it with a bounce. Furthermore, we give a brief overview of ideas that have been developed to study cosmological perturbations within these approaches. Particularly in the case of GFT this avenue is considerably less established than the modifications to the cosmological background. While our main focus lies on GFT, we include a brief overview of LQC for two reasons: Firstly, to illustrate how similar ideas can manifest in different approaches, and secondly, because we will discuss an LQC-corrected Friedmann equation in chap. 5.

We would like to emphasise that the procedure of reconstructing bouncing cosmologies from GFT and LQC is entirely distinct from the route taken in classical approaches. One can realise the latter by modifying the gravitational sector e.g. through $f(R)$ models, where the main structures of general relativity remain intact and it is only the Einstein-Hilbert action that changes [224, 225]. Alternatively (and perhaps equivalently [226]) one can consider alterations to the universe's matter content that avoid the Big Bang singularity and give a bounce instead [224, 227]. Again, the structures of general relativity remain unaltered, one can trust and apply all the well-established tools of differential geometry, and relate such modified theories to cosmological observations in a conceptually straightforward (albeit possibly technically difficult) manner. Here, on the other hand, we are searching for a more fundamental theory of quantum gravity that reduces to general relativity in a suitable classical limit (in a similar spirit to how Newtonian gravity can be recovered from GR.) In such approaches the dynamics no longer follow from the EFE (1.14) (or their modified versions), instead, the underlying quantum theory dictates the evolution. Below we will demonstrate explicitly how, in the context of cosmology, the evolution of the scale factor is given directly by the quantum theory. We can use such a solution for the scale factor to *reconstruct* a Friedmann equation in hindsight and compare such an equation to GR. Ideally, we recover the Friedmann equation of GR in the classical limit, which in the case of cosmology is expected to be realised at late times. Modifications to the early universe dynamics, which we will see below introduce a bounce, are then to be interpreted as quantum effects from a more fundamental underlying theory.

This chapter is organised as follows: We first introduce the application of GFT to cosmological spacetimes in sec. 4.1, where we detail how GFT quantities can be related to GR and lead to a bouncing universe. Additionally, we discuss approaches to cosmological perturbations within the GFT framework that have been presented in the literature. In sec. 4.2 we give a brief overview of the LQC framework and sketch how a bounce is recovered in this approach. Finally, we make a more general observation about cosmological perturbations around a bounce in sec. 4.3, before concluding in sec. 4.4.

4.1. Group field theory cosmology: bounces and beyond

After having introduced GFT as an approach for quantum gravity in chap. 2, we now review its application to cosmology and the related phenomenological avenues that have been explored in the literature. It should come as no surprise that also in GFT the first phenomenological application was in the realm of cosmology. When introducing GFT we emphasised that a classical spacetime emerges in the limit of a multitude of quanta and suggested the analogy to hydrodynamics where a fluid emerges from the collective behaviour of water molecules. We will make this notion more precise below and show explicitly how an effective evolution of the universe can be reconstructed from GFT. With the considerations we detail below, one can obtain a sufficiently simple system, whose phenomenological implications can be studied. The aim of this section is to introduce the setup and results of GFT applied to cosmology. It is structured as follows: We start by explaining how a classical spacetime can be reconstructed from the GFT volume operator in sec. 4.1.1. In sec. 4.1.2 we discuss how a suitable state is chosen and how the cosmological sector is implemented in GFT. Neglecting interactions, we show explicitly that a bouncing universe can be obtained in the deparametrised framework and recover an effective Friedmann equation in sec. 4.1.3. Finally, we describe progress that has been made towards incorporating cosmological perturbations in GFT in sec. 4.1.4 and summarise some results that can be obtained by extending the framework in sec. 4.1.5.

4.1.1. Volume of the universe from GFT

Geometric quantities that can be interpreted as describing a semiclassical spacetime can be recovered from GFT by considering expectation values of suitable operators. More precisely, one is interested in the volume operator and its expectation value over suitable semiclassical states. (An alternative relation to classical quantities is introduced in chap. 6.) The volume operator V is given as the sum over the volumes of individual Peter-Weyl modes

$$V = \sum_J V_J = \sum_J v_J N_J \,, \tag{4.1}$$

where $N_J = A_J^\dagger A_J$ is the number operator of each mode and v_J denotes the volume eigenvalue of the respective mode, usually assumed to be that of an LQG spin network vertex labeled by J. To understand the reasoning behind this, we recall that the vertices of an LQG spin network carry quanta of volume and in a scenario where GFT quanta can be interpreted as spin network vertices, i.e. in the case where $G = \mathrm{SU}(2)$, it appears natural to assign to the quanta the same volume eigenvalues as their LQG counterparts.

The volume operator gives the spatial volume of the entire system and should therefore be interpreted as a global quantity. The equivalent quantity in a classical theory is the volume of a spatial hypersurface, obtained by integrating the determinant of the spatial metric. In the case of a flat FLRW universe (3.35) this leads to the identification

$$\frac{\langle V \rangle}{V_0} = a^3 \,, \tag{4.2}$$

where $\langle V \rangle$ is the expectation value of the volume operator and we have introduced a scale for the volume $V_0 \in \mathbb{R}^+$. The appearance of a regulator is not unexpected: the scale factor in cosmology does not allow to extract any information about the total volume of the universe, whereas the volume operator V assigns a concrete volume to a collection of quanta. A factor similar to V_0 makes a common appearance in the LQC literature [167, 228] and is known as the 'fiducial cell'. It is usually not discussed within the context of GFT, where one is usually interested in an effective Friedmann equation of the form $\langle V \rangle'^{\,2}/\langle V \rangle^2$, which is independent of V_0. We will set $V_0 = 1$ in the following. To obtain the so-called 'effective' scale factor from (4.2) the expectation value has to be taken over semiclassical states. If we recall that the spectrum of the LQG volume operator is bounded from below, it is intuitively clear that the effective scale factor $a = \langle V \rangle^{1/3}$ cannot vanish.[1] As we show explicitly in sec. 4.1.3 one can obtain a bouncing universe from GFT. Moreover, one finds an effective Friedmann equation that matches that of general relativity at late times.

4.1.2. Cosmological sector

The cosmological sector is a particularly advantageous testing ground for quantum gravity theories. This is due to its high degree of symmetry, which allows one to significantly restrict the number of degrees of freedom and retain manageable expressions.

The aim of this section is to demonstrate how such simplifications are implemented in the case of GFT. In this endeavour we proceed as follows: We first identify a suitable state for the cosmological scenario, where we include a historical motivation, and discuss the implementation of homogeneity and isotropy. An additional assumption that is commonly used in GFT cosmologies is to neglect interactions in the GFT action; we explain this further at the end of the section.

A question of state

The purpose of the following is to motivate the choice of state that is used to extract effective dynamics of the universe in the GFT literature. To cut straight to the chase, we reveal that Fock coherent states give the foundation for most results in GFT cosmology [154, 229] and we will use them to extract effective expressions for quantities of interest in later chapters. We already gave the expressions for Fock coherent states in the de-

[1] Save for some fine-tuned choices of initial conditions, which we do not discuss further.

parametrised approach (2.45) as well as the algebraic approach (2.52) in sec. 2.2.3. The motivation for using these states is simple: in order to extract effective dynamics from GFT and connect to classical spacetimes, we are required to take expectation values over *semiclassical* states. Only if the states we use exhibit sufficiently classical behaviour can the identification of expectation values with classical quantities (4.2) be justified. Fock coherent states are widespread in the literature, where they serve as excellent semiclassical states due to saturating the Heisenberg uncertainty bounds. The notion of semiclassicality in GFT cosmology is adjusted to requiring instead that the relative uncertainty of operator expectation values is small, e.g., in the case of the volume operator we require $(\langle V^2\rangle - \langle V\rangle^2)/\langle V\rangle^2 \ll 1$. That this is indeed the case for Fock coherent states in GFT was shown in [185] and we will give more details on this result toward the end of sec. 4.1.3.

We could end the discussion of state choices in GFT cosmology at this point, however, we proceed with a more historical introduction of their appearance. The key idea is that the cosmological scenario should be captured by a 'condensate' regime in GFT, similar to the description of Bose-Einstein-Condensates (BECs).

The idea that condensates can be used to describe an emergent classical spacetime from a more fundamental quantum theory predates the studies of GFT cosmology [230, 231]. Since the term 'condensates' is inspired by the study of BECs, let us briefly recap the fascinating nature of BECs: When a collective of bosonic atoms is cooled below its critical temperature (as has been realised in laboratories [232–234]), a large number of particles occupy the ground state of the system. Quantum mechanically, the multi-particle state of the atoms in the ground state can be described as the product of an identical single particle wavefunction, which is referred to as the 'condensate wavefunction' or 'macroscopic wavefunction' [235]. This description would be exact for an ideal Bose gas, where interactions between particles can be neglected, but even when weak interactions are included this description remains valid.

This idea is imported to the context of quantum gravity: In a so-called condensate phase, all fundamental geometric quanta are in the same state and can be described by the same single particle 'condensate' wavefunction, thus greatly reducing the degrees of freedom required to describe the quantum state. The condensate phase then naturally gives a coarse-graining procedure [2]: when using condensate states, the behaviour of geo-

[2] Coarse graining is a process in which many degrees of freedom can be distilled into few quantities that capture the essential properties of a system. This is most elegantly realised in statistical mechanics, where e.g. an ideal gas consisting of many molecules can be completely characterised by its temperature, pressure, volume and number of particles.

metric quanta is captured by a single particle wave function.

Condensate states were introduced for GFT in [176, 229], where the authors embed GFT quanta, understood as tetrahedra, in a spatial manifold. The geometry of a spatial hypersurface can then be probed by considering the geometric information of these tetrahedra. In order to be compatible with homogeneity, the geometric degrees of freedom of each tetrahedron must then be identical - and hence the wavefunction of each quantum the same.

Let us illustrate this concept more clearly by considering a general state with n GFT quanta, using the framework of the deparametrised approach (sec. 2.2.3), and considering only the $k = 0$ mode (or equally, a GFT with a single massless scalar field). We start with a general state $|\Psi\rangle$ given by a sum of Peter-Weyl modes, where each n-particle state in the sum below is associated with an n-particle wavefunction $\Psi_{J_1,\dots,J_n}$. In the case where all particles are described by the same single particle wavefunction we then have $\Psi_{J_1,\dots,J_n} \to \prod_i^n \Psi_{J_i}$ and get the following simplification for the state

$$|\Psi\rangle = \sum_{J_1,\dots,J_n} \Psi_{J_1,\dots,J_n} \prod_i^n a^\dagger_{J_i} |0\rangle \quad \to \quad |\Psi\rangle = \prod_i^n \sum_{J_i} \Psi_{J_i} a^\dagger_{J_i} |0\rangle \ . \tag{4.3}$$

(The $a^\dagger_J$ operators above correspond to the ladder operators of sec. 2.2.3 with $k = 0$.) In cosmology, we want to allow for varying particle numbers and therefore consider states that are a superposition of multi-particle states with different n. There are of course several ways to implement this, but perhaps the easiest (and most familiar) is to choose a coherent state

$$|\sigma\rangle = \mathcal{N}_\sigma \exp\left(\sum_J \sigma_J a^\dagger_J\right) |0\rangle \ , \tag{4.4}$$

where the single particle wave function is now encoded in σ_J. A state of the form (4.4) also ensures that the state fulfills the requirement of semiclassicality, which would not necessarily be the case if the exponential was replaced with a general function. There exist proposals in the literature to consider also dipole condensates, which incorporate two-particle interactions [154, 176].

This concludes the historical motivation for coherent states in GFT cosmology We would like to close with some comments:

- We started the discussion by drawing an analogy to BECs. We would like to point out that this analogy reaches only so far as that both systems are captured by a single particle state. In BECs, the condensate phase is reached through a macroscopic occupation of the system's ground state (this is the condensation process), which is not what happens in GFT cosmology. In GFT, one typically focuses on the free action and neglects interactions, where we saw in sec. 2.2.3 that this leads to two types of modes, namely oscillating and squeezed modes. The squeezed modes allow the interpretation of a GFT condensate as an expanding universe, as we will see explicitly below, but for such modes the concept of a ground state of the system is obsolete. In [186] the authors consider an interacting Hamiltonian that results in a Mexican hat potential. They find that in this case one can indeed recover a process akin to condensation, provided one uses a coherent state with certain initial conditions. It could be argued that a more suitable analogy would be coherent and squeezed states that appear in quantum optics [192, 236].

- As mentioned in the beginning of this section from a purely phenomenological perspective, the only requirement on the wavefunction is sufficient semiclassicality. While the concept of wavefunction homogeneity is intuitively clear and conceptually desirable, the relation to the scale factor is obtained from (the expectation value of) the total volume operator (4.2). The effective scale factor is thus only sensitive to the evolution of the collective volume of all quanta and oblivious to the distribution of geometric information among them. In essence, the Friedmann equation one obtains from GFT encapsulates the behaviour of the total volume, irrespective of the local geometrical properties such as homogeneity. Existing works on cosmological perturbations in the GFT framework (see sec. 4.1.4) are also based on coherent states, which are then no longer interpreted as spatially homogeneous despite having wave function homogeneity. There are indeed no clear indications based on general grounds that a simple coherent state should correspond to a purely homogeneous geometry (or another specific geometry for that matter). In order to make statements about the possible geometries of the spatial hypersurfaces one needs to consider additional operators. E.g., in [148] the authors define an anisotropy operator, and in chap. 6 we propose new operators that, as we further discuss in chap. 7, allow to explicitly interpret the effective cosmological dynamics arising from GFT as a flat FLRW metric.

Homogeneity and isotropy

As we already mentioned, the intuitive idea of homogeneity is reflected in the choice of a Fock coherent state. However, in the case where one considers a GFT with four massless scalar fields, three of which are interpreted as spatial rods, one can instead think of homogeneous quantities as being contained in the $k = 0$ mode. A homogeneous state should then be peaked around this value, which is the viewpoint we take in chap. 7.

Isotropy is imposed by demanding that all edges of a GFT quantum carry the same spin label j. In the picture where one interprets GFT quanta as a tetrahedron, this corresponds to demanding that each side has the same area,[3] see fig. 2.1. The effect of relaxing this assumption was studied in [148], the results of which we summarise in sec. 4.1.3.

Neglecting interactions

In chap. 2 we introduced the GFT action as a sum of a kinetic and interaction term (2.2). In many cosmological applications of GFT, the interaction term is neglected and one works with the kinetic term only [144, 154, 177]. The honest reason for this simplification is that it allows us to recover analytical solutions for the operator dynamics and thereby the effective evolution of the universe. In general, one expects the structure of interactions in GFT to be non-local (in the group elements) and rather complicated. The justification for why these results are of interest from a phenomenological point of view is that we are typically interested in the extremely early universe only, where there is few enough GFT quanta that interactions are subdominant.

Neglecting interactions then limits the time in which GFT cosmology can be seen as valid. At which point in cosmic time this breakdown occurs is however unclear and depends on the parameters of the theory. In general, the expectation of how GFT is situated in the broader context of cosmology is that it gives corrections only to the very early (pre-inflationary) universe. One may then assume that a general relativistic regime is reached before interactions become dominant and that GR accurately describes the remainder of the cosmological evolution. In that sense, the premise is that interactions are negligible for 'long enough', where the timescale is determined by the strength of the interactions. These arguments are rather on the philosophical side and should be clarified once the theory has developed further. A first step in that direction was taken in [185] and we will discuss these results further in sec. 4.1.3.

[3] This can also be seen from recalling that in spin networks the edges carry quanta of area, and the spectrum of the area operator is determined by the spin carried by the edges.

4.1.3. Bouncing universe

We proceed to show explicitly how a bouncing universe is obtained from GFT and how the resulting effective Friedmann equation can be compared to GR. The notion of the suitability of semiclassical states is made more concrete, and finally, we give some results that have been obtained for interacting GFTs.

We use the Hamiltonian construction, but emphasise that most cosmological GFT studies in the literature use the algebraic approach (we outlined both approaches in sec. 2.2.3). For the results presented in this section, the two approaches give very similar results (even though they give a slightly different form of the effective Friedmann equation [148, 185, 187]). The expressions we use in the following were explicitly derived for the case of four massless scalar fields in sec. 2.2.3. Here we work with a single massless scalar field only; the expressions for this case can be obtained from those in sec. 2.2.3 by setting $k = 0$ and we write e.g. $A_{J,0} =: A_J$.

Operator dynamics

We are now in the position to carry out explicit calculations within GFT cosmology and investigate the resulting effective evolution of the universe. This serves as the first test as to whether GFT allows us to recover a classical spacetime in the limit of large particle number. As we will consider a universe filled by a single massless scalar field, we do not expect to recover a realistic evolution of the universe in the sense that it could accurately describe our universe today (which is of course filled with a much greater variety of matter content) or in the era of radiation domination. However, if GFT can agree with GR for FLRW with a single massless scalar field, it will pass its first test. How and whether this can be embedded or extended into a more realistic description of cosmological evolution is an open question.

We briefly summarise the ingredients required for the calculation below: We work with a group field $\varphi : \mathrm{SU}(2)^4 \times \mathbb{R} \to \mathbb{R}$ that has been coupled to a single massless scalar field χ^0, restrict to the free GFT action ($V[\varphi] = 0$ and $K[\varphi]$ as given in (2.18)) and use the deparametrised framework (sec. 2.2.3). The field χ^0 plays the role of a matter clock, as explained in sec. 2.2.1. We work with a Fock coherent state $|\sigma\rangle$ [185]

$$|\sigma\rangle = \mathcal{N}\, e^{\sum_J (\sigma_J a_J^\dagger - \sigma_J^* a_J)}\, |0\rangle \,, \qquad a_J\, |\sigma\rangle = \sigma_J\, |\sigma\rangle \,, \quad \text{with} \quad \sigma_J = \mathcal{A}_J + \mathrm{i}\mathcal{B}_J \,, \tag{4.5}$$

where $\mathcal{A}_J, \mathcal{B}_J \in \mathbb{R}$ and $\mathcal{N}$ is a normalisation constant.

We are interested in the effective evolution of the volume operator, $\langle V \rangle$. Recall that the form of the Hamiltonian depends on the factors appearing in the expansion of the kinetic term (2.21), more specifically, their relative sign. These terms are mode dependent and we can therefore have different dynamics for different J modes. For completeness, we will consider both options.

We start with the case where the sign of the factors appearing in the kinetic term (2.21) is the same, $\mathrm{sgn}(\mathcal{K}^{(0)}) = \mathrm{sgn}(\mathcal{K}^{(2)})$, and therefore $m_J^2 < 0$, which leads to the Hamiltonian of a harmonic oscillator (cf. (2.41))

$$H_J = -\,\mathrm{sgn}(\mathcal{K}_J^{(2)})\frac{|m_J|}{2}\left(a_J a_J^\dagger + a_J^\dagger a_J\right) . \tag{4.6}$$

We will dub modes that follow such dynamics 'oscillating modes'. In the Heisenberg picture, the dynamics of the ladder operators is given by

$$A_J = a_J e^{\mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})|m_J|\chi^0} , \quad A_J^\dagger = a_J^\dagger e^{-\mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})|m_J|\chi^0} . \tag{4.7}$$

The number operator of each mode $N_J = A_J^\dagger A_J$ commutes with the Hamiltonian H_J for harmonic oscillator modes and is hence trivially a constant; taking the expectation value for the state given in (4.5) gives

$$\langle N_J \rangle = \langle A_J^\dagger A_J \rangle = |\sigma_J|^2 . \tag{4.8}$$

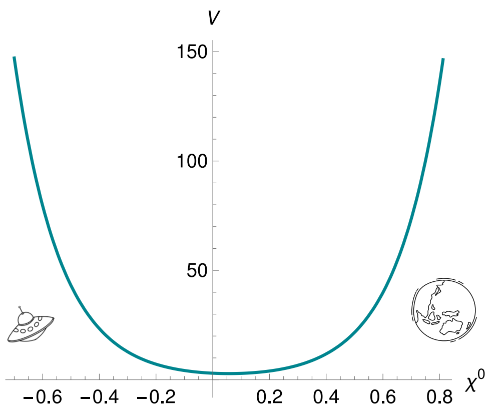
If we consider only a single J mode of this type in the volume operator (4.1), we would find no evolution in the effective scale factor $\langle V \rangle = \langle V_J \rangle = \text{const.} = a^3$. Thus, for a single oscillating mode we recover a static universe.

For the alternative case with $\mathrm{sgn}(\mathcal{K}^{(0)}) \neq \mathrm{sgn}(\mathcal{K}^{(2)})$, and therefore $m_J^2 > 0$, on the other hand, we find a squeezing Hamiltonian (cf. (2.42))

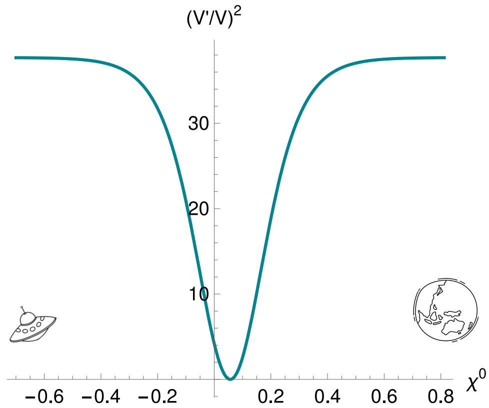
$$H_J = \mathrm{sgn}(\mathcal{K}_J^{(2)})\frac{|m_J|}{2}\left(a_J^2 + (a_J^\dagger)^2\right) , \tag{4.9}$$

and the operator dynamics read

$$\begin{aligned} A_J &= a_J \cosh\left(|m_J|\chi^0\right) - \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)}) a_J^\dagger \sinh\left(|m_J|\chi^0\right) , \\ A_J^\dagger &= a_J^\dagger \cosh\left(|m_J|\chi^0\right) + \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)}) a_J \sinh\left(|m_J|\chi^0\right) . \end{aligned} \tag{4.10}$$



(a) Expectation value of the volume operator.



(b) Effective Friedmann equation.

Figure 4.1.: The evolution of the universe's volume for a single squeezed mode w.r.t. the matter clock χ^0. The Big Bang singularity of GR is replaced by a bounce. In the pre-bounce regime the universe undergoes a period of contraction, whereas post-bounce the universe expands. This expanding branch is interpreted as the universe we live in today (marked by planet Earth in the plot). The volume expectation value $V = \langle V \rangle$ is taken over a Fock coherent state (4.5).

This gives the following dynamics of the number operator

$$\langle N_J \rangle = \langle A_J^\dagger A_J \rangle = \mathcal{C}_{1,J} e^{2m_J \chi^0} + \mathcal{C}_{2,J} e^{-2m_J \chi^0} - \frac{1}{2}\,, \tag{4.11}$$

where $\mathcal{C}_{1,J} := \frac{1}{2}(\mathcal{A}_J + \mathcal{B}_J)^2 + \frac{1}{4}$ and $\mathcal{C}_{2,J} := \frac{1}{2}(\mathcal{A}_J - \mathcal{B}_J)^2 + \frac{1}{4}$. If we again restrict to the contribution of a single mode in the volume operator and make the identification $\langle V \rangle = \langle V_J \rangle = a^3$, for a squeezing mode we find that the scale factor grows for $\chi^0 \to \infty$ (late times) and $\chi^0 \to -\infty$, and reaches a minimum at $\chi^0_{\text{min}} = \frac{1}{4m} \log\left(\frac{\mathcal{C}_{2,J}}{\mathcal{C}_{1,J}}\right)$. We can interpret this as a bouncing universe (fig. 4.1a): For $\chi^0 < \chi^0_{\text{min}}$ the universe is in a contracting phase, it bounces at $\chi^0 = \chi^0_{\text{min}}$ and enters an expanding phase thereafter. This expanding branch is then interpreted as the universe we live in. Indeed, as we detail further below, the late and early time limit gives exponential solutions that are consistent with GR.

As we saw, the volume expectation values of distinct Peter-Weyl modes, $V_J = v_J N_J$, evolve differently. For squeezed modes ($m_J^2 > 0$) the growth rate of $\langle N_J \rangle$ is determined by m_J^2 (4.11). If the values of m_J^2 are non-degenerate, the J mode with the maximum m_J^2 will dominate at late times, such that in these regimes the contributions of other modes to the total volume $\langle V \rangle = \sum_J \langle V_J \rangle$ (and thereby to the scale factor) can be neglected, as was first pointed out in [190]. The sum includes the modes with $m_J^2 < 0$, which quickly become subdominant away from the bounce.

There is a caveat: since in principle the total volume operator contains an infinite sum over J modes, one needs to ensure this sum is finite (at least for $\chi^0 < \infty$). The hope might be to achieve this with initial conditions in (4.5), however a squeezed Hamiltonian as we find for $m_J^2 > 0$ will excite also an initially empty mode with $\sigma_J = 0$. This issue can be avoided by limiting the number of squeezed modes to a finite amount and setting initial conditions for oscillating modes such that they give a finite contribution to the total volume. To illustrate how this could be realised in practice, we recall the explicit expressions we gave for $\mathcal{K}^{(0)}$ and $\mathcal{K}^{(2)}$ in (2.19), namely $\mathcal{K}_J^{(0)} = \mu - \sum_i j_i(j_i+1)$ and $\mathcal{K}_J^{(2)} = \tau$. As we already discussed at the end of sec. 2.2.3, in the case where μ is positive, $\mathcal{K}^{(0)} > 0$ for small spins j_i and negative for large spins.[4] To have a limited number of squeezed modes we then need to demand that $\tau < 0$, which fixes $\mathcal{K}^{(2)} < 0$.

We have then established the following picture for a bouncing universe from GFT: In the free theory coupled to a massless scalar field we find two types of contributions to the total volume operator. While some Peter-Weyl modes have an unchanging number of quanta and hence give a constant contribution to a, the number in other modes changes with time and leads to a bouncing universe. The number of dynamical modes can be restricted to a finite amount for specific choices of $\mathcal{K}^{(0)}$ and $\mathcal{K}^{(2)}$. We saw that a single squeezed mode dominates at late times, which is the reason why most of the cosmological studies in GFT focus on a single Peter-Weyl mode $J = J_0$ only. We adopt this assumption in all further discussions and in the results considered in chap. 7. We however would like to point out that the validity of GFT cosmology is judged based on two aspects: the occurrence of a bounce and the late time behaviour of the scale factor, neither of which would be influenced by the other modes. Their contribution would certainly change the evolution of the scale factor near the bounce, but in order to relate such effects to physical consequences for the universe today, one needs to incorporate the study of perturbations.

Effective Friedmann equation

To establish whether the late time behaviour of the effective scale factor recovered from GFT can be brought into agreement with GR at late times, we proceed to consider an ef-

[4] The exact cutoff is determined by the value of μ, and if we wanted to be exact, we would have to demand that $\mu > 3$ to have at least one J mode with $\mathcal{K}^{(0)} > 0$, as the minimum spin value is $j = \frac{1}{2}$.

fective Friedmann equation. From here on we will make use of the single mode assumption and set $J = J_0$. As we argued above and as first pointed out in [190], a single Peter-Weyl mode will dominate the condensate dynamics at late times, which is the regime in which one aims to connect with GR. In this case, the volume operator is directly proportional to the number operator $V = \sum_J V_J = V_{J_0} = v_0 N_{J_0}$, where v_0 is the volume eigenvalue of the J_0 mode. As we consider the contribution of a single mode only, we omit the J labels in what follows.

We first recall the background dynamics of an FLRW universe filled with a massless scalar field as described in sec. 3.4.1. As the GFT setup includes a single massless scalar field χ^0 as only matter content of the theory, the energy density is given by $\rho = \frac{(\chi^{0\,\prime})^2}{2N^2} = \frac{\pi_0^2}{2a^6}$ (where $N = \frac{a^3}{\pi_0}$ in this case denotes the lapse function, not to be confused with the number operator). Furthermore, by using the massless scalar field as a relational clock we have fixed $\chi^{0\,\prime} = 1$, such that the Friedmann equation in this frame reads

$$H^2 = \frac{\kappa}{3}\rho N^2 = \frac{\kappa}{6}\,, \tag{4.12}$$

i.e. we find a constant Hubble rate. Using $\langle V\rangle = a^3$, the Hubble rate $H = \frac{a'}{a}$ can be obtained from expectation values of the volume operator from

$$H = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}\chi^0}\chi^{0\,\prime} = \frac{1}{3}\frac{\mathrm{d}\langle V\rangle}{\mathrm{d}\chi^0}\frac{1}{\langle V\rangle}\chi^{0\,\prime} = \frac{1}{3}\frac{\mathrm{d}\langle V\rangle}{\mathrm{d}\chi^0}\frac{1}{\langle V\rangle} \quad \Rightarrow \quad H^2 = \frac{1}{9}\left(\frac{\mathrm{d}\langle V\rangle}{\mathrm{d}\chi^0}\frac{1}{\langle V\rangle}\right)^2 . \tag{4.13}$$

From the expectation value of the number operator of squeezed modes (4.11) we find

$$\left(\frac{d\langle V\rangle}{d\chi^0}\right)^2 \langle V\rangle^{-2} = 4m^2\left(1 + \frac{v_0}{\langle V\rangle} + \frac{\mathcal{Y}}{\langle V\rangle^2}\right) = 4m^2\left(1 + \frac{v_0}{a^3} + \frac{\mathcal{Y}}{a^6}\right)\,, \tag{4.14}$$

where $\mathcal{Y} = -v_0^2\left(\mathcal{A}^4 + \mathcal{B}^4 + \mathcal{A}^2 + \mathcal{B}^2 - 2\mathcal{A}^2\mathcal{B}^2\right) < 0$. The typical evolution of the above is depicted in fig. 4.1b. At late times, the r.h.s. of (4.14) goes to a constant, which agrees with the Friedmann equation of GR (4.12) for $m^2 = m^2_{J_0} = \frac{3}{8}\kappa$. We have thus found that we can match the late time behaviour of GR, where the value of $m^2_{J_0}$ (which we recall is fixed by the kinetic terms in the expansion of the GFT action) determines the value of the gravitational constant. The bounce occurs when $\langle V\rangle = \frac{1}{2}(-v_0 + \sqrt{v_0^2 + 4|\mathcal{Y}|})$. One of the phenomenologically desirable properties of quantum gravity theories is singularity resolution (this is one of the physical puzzles that sent us down the path to find a quantum theory of gravity after all!) and we indeed find that the Ricci scalar (1.40) at the bounce is finite:

$$R_{\text{bounce}} = \frac{1}{N^2}\frac{a''}{a} = \frac{\pi_0^2 m^2}{3\,\mathcal{C}_{1,\,J}\,\mathcal{C}_{2,\,J}}\,, \tag{4.15}$$

where we used $\frac{a''}{a}|_{\text{bounce}} = \frac{4}{3}m^2$ and $a_{\text{bounce}} = 2^{1/3}(\mathcal{C}_{1,\,J}\,\mathcal{C}_{2,\,J})^{1/6}$.

As we already discussed, including multiple modes in this discussion would not affect the late time behaviour. However, additional modes would contribute to the early evolution and could specifically change the value and behaviour of the Ricci scalar around the bounce.

Assessing semiclassicality

As described in sec. 4.1.2, Fock coherent states are widespread in the GFT literature and were originally motivated from the concept of a 'condensate phase' in GFT cosmology. However, ultimately, the essential criterion for suitable states is semiclassicality. This notion was studied in detail in [185], where the authors introduced a measure for semiclassicality and considered also more general classes of coherent states. We briefly summarise their findings in the following.

The authors make use of the fact that GFT cosmology as detailed above exhibits an $su(1{,}1)$ structure[5] and, in addition to the well-known Fock coherent states, consider also two types of $su(1{,}1)$ coherent states: Perelomov-Gilmore and Barut–Girardello coherent states. (For details on the definition of more general coherent states we refer the reader to [185, 237].) The measure for the 'quantumness' r of a state $|\Psi\rangle$ is calculated from the relative uncertainty of operators of interest, and e.g. for the volume operator V, reads

$$r(V, |\Psi\rangle) = \frac{\langle\Psi|\,V^2\,|\Psi\rangle - \langle\Psi|\,V\,|\Psi\rangle^2}{\langle\Psi|\,V\,|\Psi\rangle^2}\,. \tag{4.16}$$

Evidently, for a semi-classical state this quantity should be small. The asymptotic value of r at late times can be made arbitrarily small for Fock and Barut–Girardello coherent states, depending on initial conditions. The latter are however cumbersome for calculations due to the appearance of Bessel functions. Perelomov-Gilmore coherent states on the other hand do not 'classicalise' at late times and therefore do not qualify as semiclas-

[5] The algebra is formed by the following three operator combinations: $\frac{1}{4}(a^\dagger a + a a^\dagger)\,, \frac{1}{2}(a^\dagger)^2\,,$ and $\frac{1}{2}a^2$. For more details we refer the reader to [185, app. A].

sical states. Interestingly, the authors also find that for both, Fock and Barut–Girardello coherent states, the asymptotic values of the relative uncertainty are generally not symmetric in the far pre- and post-bounce regimes. Therefore, a sufficiently classical phase in the expanding branch of the universe does not imply a classical pre-bounce phase.

In [187] this analysis is substantially extended to a broader range of states. Specifically, the authors consider generalised Gaussian states that include displacement, squeezing, and thermal contributions.

A first look at interactions

As discussed in sec. 4.1.2, we will disregard interactions between GFT quanta in this thesis. We would however like to mention two studies of interactions in the deparametrised framework [185, 186]. Both of these consider a Hamiltonian of the form (and restrict to the single mode case)

$$H = -\frac{m}{2}\left(a^2 + (a^\dagger)^2\right) + \frac{\lambda}{4}\left(a + a^\dagger\right)^4 , \tag{4.17}$$

where [186] considers $\lambda > 0$ and [185] focuses on $\lambda < 0$. As it is no longer possible to derive analytical solutions for an interacting Hamiltonian of this form, any dynamics need to be solved numerically. The case with $\lambda > 0$ leads to a Mexican hat potential and a bounded Hamiltonian, as already mentioned in the discussion of condensate phases in GFT in sec. 4.1.2. This results in bound states that can be interpreted as static cosmologies. In the case with $\lambda < 0$, one recovers a dynamically growing particle number, permitting the interpretation as an expanding universe. The Friedmann equation at late times is no longer constant as in (4.14), but $(\langle V\rangle'/\langle V\rangle)^2$ increases with time. Furthermore, the mean field approximation as used in the algebraic approach to GFT, in which equations of motion are solved by replacing operators with their expectation values (resulting in the classical equations of motion), was shown to quickly break down once interactions become relevant.

In general, one can conclude that interactions can have a relevant impact on cosmological dynamics and one needs to be cautious with approximations that are viable for the free theory once interactions are no longer negligible.

4.1.4. Approaches to cosmological perturbations

Having introduced the ideas and machinery behind homogeneous GFT cosmology, we now turn to the considerations that have been made regarding perturbations. Recall that a single Peter-Weyl mode dominates the evolution of the GFT volume operator at late times, which is the region one matches to GR. While the presence of multiple modes would considerably change the dynamics at the bounce, evidence of such alterations could only be carried forward by perturbations. Furthermore, relating perturbations to conditions imposed by cosmological measurements can give stringent restrictions on permissible background dynamics. Including cosmological perturbations in a GFT description then provides an additional test of the theory's viability and specifics of its construction (in a cosmological context) as well as opening up the pathway to establishing whether quantum gravity effects could manifest in observations.

As we pointed out already, in the majority of the literature the volume operator is the only GFT operator used to extract information about the quantum geometry arising from GFT. Thus, perturbations arising in GFT are analysed as perturbations of the volume operator, which can be related to the spatial volume element of GR (4.2). In GR, the perturbation of the spatial volume element of a flat FLRW spacetime (without gauge fixing) reads

$$a^3(t,\vec{x}) \approx a^3(t)(1 - \nabla^2 E(t,\vec{x}) - 3\psi(t,\vec{x}))\,. \tag{4.18}$$

Any connection of GFT perturbations to cosmological perturbation theory is then facilitated by this quantity, and in particular, for small wavelength perturbations one cannot separate the combination of ψ and E, as would be necessary to study e.g. gauge-invariant quantities (sec. 3.2). This will be substantially different in the approach we introduce in chap. 6 and we show explicitly in chap. 7 how our new approach can be used to extract effective expressions of the scalar perturbation variables separately.

Furthermore, all research avenues detailed below make use of the algebraic approach, neglect interactions in the GFT action, and use a second order kinetic term (see sec. 2.2). They build on the notion of a coherent state as given in (2.52) with a mean field characterised by σ_J and the expectation value of the GFT volume operator of a single Peter-Weyl mode J over such a state reads $\langle V_J \rangle = |\sigma_J|^2$. We provide an overview of the results achieved in GFT perturbations so far and preview how our construction in later chapters will differ.

Ideas for including perturbations in GFT were already given with the introduction of condensate states in [176]. A concrete extension of the condensate state was proposed in [238] and is rather different to the research directions that were explored thereafter. In [238] the authors consider a single particle excitation on top of the condensate state, which is to be interpreted as a perturbation. As there are no correlations between the quanta, it is not possible to localise the perturbation over the condensate, resulting in a homogeneous perturbation. Still, the excitation changes the quantum system by increasing the available degrees of freedom. The interpretation of such a perturbation within 'conventional' cosmological perturbation theory is unclear.

In [149] the authors couple four massless scalar fields to the group field for the first time and establish the idea of a full relational coordinate system in GFT. These matter fields satisfy a shift and rotation symmetry separately as well as a rotational symmetry w.r.t each other. In this process, the kinetic term is amended to include also second order derivatives in the spatial fields (see (2.33) and more generally the description in sec. 2.2.3 for the Hamiltonian setting). With a relational coordinate system, it is possible to include inhomogeneities explicitly and this construction serves as the basis for further studies of the perturbative regime.

Two different suggestions to study volume perturbations $\delta V = \delta V_{J_0}$, where we again restrict our attention to a single mode condensate, have been put forward. In the first, the object of interest is the two-point function of the volume perturbations $\langle \delta V_k \delta V_{k'} \rangle$, which arise from local quantum fluctuations of the volume operator [80, 149, 239] [6]

$$\langle \delta V_k \delta V_{k'} \rangle = \langle V_k V_{k'} \rangle - \langle V_k \rangle \langle V_{k'} \rangle \, . \tag{4.19}$$

Here, k denotes the wavenumbers obtained from a Fourier decomposition of the spatial fields. The two-point function $\langle \delta V_k \delta V_{k'} \rangle$ is non-vanishing even for a purely homogeneous mean field $\sigma_{J_0} = \sigma_{J_0}(\chi^0)$, as is used to describe the homogeneous background cosmology (2.52). In such a scenario, perturbations are generated by quantum fluctuations in space itself and are not introduced as a perturbed quantity on top of a homogeneous background. Overall, the open question is how these quantities can be related to gauge-invariant cosmological observables. A step in this direction is taken in [239], where the authors compute the two-point function of the comoving curvature perturbation on equal density hyper-

[6] [239] uses a slightly different convention.

surfaces ζ (3.21) from fluctuations in GFT operators over a homogeneous background.[7] The authors recover a primordial power spectrum with spectral index $n_s = 4$, where the amplitude is determined by a single free parameter. While evidently disagreeing with observations ($n_s \approx 1$, [24]), this result matches the value obtained in a classical analogue system. Specifically, this result is reproduced in classical cosmology with a single scalar field (the contribution of the spatial fields is neglected) and an LQC modified background leading to a bounce. Alternatively, one can consider perturbations directly in the mean field σ_J

$$\sigma_J(\chi^0, \vec{\chi}) \approx \sigma_J(\chi^0) + \delta\sigma_J(\chi^0, \vec{\chi}) \,, \tag{4.20}$$

which translates into a perturbation of the volume operator[8] $V(\chi^0, \vec{\chi}) \approx V(\chi^0) + \delta V(\chi^0, \vec{\chi})$, $\delta V = \sum_J \delta V_J$. The authors of [222] construct a separate universe framework (see sec. 3.3) for GFT. The dynamics of the homogeneous volume perturbation over a single patch of the separate universe framework can be compared to the equivalent GR dynamics in the long wavelength limit. The authors find agreement between the two in the limit of large background volume.

This idea was extended further in [151], where explicitly inhomogeneous quantities localised by the relational fields are considered, allowing to investigate also sub-horizon dynamics. We note that, while the authors employ a reference frame consisting of four massless scalar fields, they additionally include a fifth (massless scalar) field, which is assumed to dominate the energy density of the universe. Using the strategy given in (4.20), the authors derive dynamics for the volume perturbation $\delta V(\chi^0, \vec{\chi})$, where the signature of the equation of motion for the volume perturbation depends on initial conditions. The authors find agreement of the dynamics of $\delta V(\chi^0, \vec{\chi})$ with GR at late times only in the separate universe ($k \to 0$) regime.

The discrepancy with could be mended for long wavelength perturbations in [152], where the results of [151] are extended to include two types of GFT fields: the quanta of one of the fields are to be interpreted as 'space-like' tetrahedra, the other as 'time-like' tetrahedra. This idea has its origin in considerations regarding causality in GFT. The main input for the phenomenology however is that the dynamics of the space-like field

[7] Recall that ζ is a gauge-invariant quantity that can be related to the CMB power spectrum in standard cosmology; sec. 3.2.

[8] To make this precise, we need to mention that usually the mean field is decomposed as $\sigma_J = \rho_J e^{i\theta_J}$, $\rho_J, \theta_J \in \mathbb{R}$, and the real and imaginary part are perturbed separately, $\rho_J \to \rho_J + \delta\rho_J$, $\theta_J \to \theta_J + \delta\theta_J$. One then finds $\delta V_J \propto \rho_J \delta\rho_J$.

depends only on the clock field, whereas the time-like field depends solely on the rod fields. The results also rely on a more involved state, namely, a coherent state with additional two-body excitations, which are interpreted as perturbations. Studying the dynamics of effective perturbations through the expectation value of the volume operator over such a state then gives enough freedom to reconstruct the corresponding dynamics of GR for long-wavelength perturbations. The authors define an 'almost' gauge-invariant quantity, which approximates $\mathcal{R}$ as defined in (3.21) in the long wavelength limit (where the k^2E contribution in (4.18) becomes negligible). Again, as one is limited to studying perturbations of the form (4.18), one cannot necessarily retrieve the dynamics of gauge-invariant perturbations in all regimes of interest from a cosmological perspective.

As detailed in sec. 3.4.2, in general the rod fields give a non-negligible contribution to the background dynamics, as well as to the equations of motion for perturbations. All approaches listed above either neglect the contribution of the spatial fields to the energy density based on physical considerations [222, 239], or include an additional field [151, 152] to effectively recover GR with a single massless scalar field on the classical side.

4.1.5. Further results

Having used the simplest possible GFT construction to obtain a first phenomenologically feasible result of a bouncing universe from GFT, one can now proceed to test the implications of certain aspects in the GFT construction on the evolution of the effective scale factor. Indeed, there is rich literature on GFT phenomenology that relaxes some of the assumptions made above. We summarise some of these avenues below, where we point out that we make no claim of completeness.

- The authors of [240] approach GFT interactions from a phenomenological perspective and ask whether including certain interaction terms can lead to a cyclic universe (i.e. a succession of bounces and re-collapses). They find that this can be achieved by including two interaction terms, namely a φ^5 and a φ^6 term. The results are obtained in the mean field regime of the algebraic approach, which, as we discussed above becomes increasingly inaccurate in the presence of interactions.

- In [148], the assumption that all the spin labels j associated with a single GFT quantum are the same is relaxed and anisotropic tetrahedra are included in the analysis. The authors consider several squeezed modes as contributing to the volume of the universe with anisotropic spin labels and find that there is no alteration to the late time Friedmann equation, as also here, a single mode dominates at late times. Furthermore, they define an anisotropy parameter and find that the universe isotropises with its expansions.

4.2. Bouncing universe in loop quantum cosmology

The application of LQG techniques to cosmology has led to the development of the field of loop quantum cosmology (LQC) [167, 241, 242]. The essential idea is to reduce the full phase space of gravity in the Ashtekar-Barbero formulation of GR (sec. 1.1.2) to the cosmological sector first, before carrying out the LQG quantisation programme as outlined in sec. 1.4.2. This drastically simplifies the system and the physical Hilbert space can be constructed explicitly, which is an open question in full LQG [127]. One can then study the evolution of physical states that have semiclassical properties, and in particular extract the universe's dynamics from the expectation value of the LQG volume operator, where the LQC description replaces the initial singularity with a bounce. The LQC programme started in the early 2000s [243–246] and pre-dates results in GFT cosmology by a considerable margin. LQC has since become a research field with rich phenomenology, where methods have been developed to study perturbations and compute e.g. CMB power spectra [247, 248]. However, the relation of LQC with the full theory of LQG remains rather elusive [242].

In the following we will outline the basic ideas of LQC, referring back to the LQG setup introduced in sec. 1.4.2. We then briefly explain how a so-called 'effective' description of LQC has been established: Effective models work in the classical framework and are based on a modified Hamiltonian that captures quantum corrections. It is this feature that allows one to study LQC effects with the classical machinery, thus making LQC particularly attractive from a phenomenological point of view.

The starting point of LQC is the description of a cosmological spacetime within the Ashtekar-Barbero formalism, more specifically one focuses on a (flat) FLRW spacetime with a massless scalar field χ^0. The Hamiltonian constraint is then given by the sum of the gravitational and matter part

$$\mathcal{H} = \mathcal{H}_{\text{geom}} + \mathcal{H}_{\text{matter}} \,, \tag{4.21}$$

and the diffeomorphism constraint vanishes trivially. The geometric phase space, generally characterised by the fluxes E^a_i and the Ashtekar connection A^i_a, reduces to a single pair of canonical variables, c and p, that are related to the scale factor and its time derivative as

$$E^a_i = a^2\delta^a_i =: pV_0^{-\frac{2}{3}}\delta^a_i\,, \quad A^i_a = \frac{\beta}{N}a'\delta^i_a =: cV_0^{-\frac{1}{3}}\delta^i_a\,, \qquad \{c{,}p\} = \frac{\kappa\beta}{3}\,, \tag{4.22}$$

where N is the lapse, V_0 is the fiducial cell in LQC, and β the Barbero-Immirzi parameter (see (1.22)). The fiducial cell is an arbitrary regularisation parameter and additional care needs to be taken to ensure that physical quantities are independent of V_0.

With (4.22), the Hamiltonian takes on a rather simple form

$$\mathcal{H}_{\text{geom}} = -\frac{3N}{\kappa\beta^2}\sqrt{|p|}c^2\,, \qquad \mathcal{H}_{\text{matter}} = N|p|^{-\frac{3}{2}}\frac{\pi_0^2}{2}\,, \tag{4.23}$$

where π_0 denotes the conjugate momentum of the scalar field χ^0. The Hamiltonian in (4.23) corresponds exactly to the description with GR and gives the same dynamics for the universe as would be obtained from the EFE, albeit written in a different set of variables. In particular, solving this system leads to the well known Big Bang singularity. The input from LQG enters by constructing operators corresponding to c and p from smearing, specifically, one constructs holonomies by integrating c along curves. This process contains the crucial ingredients of LQC: Firstly, one restricts oneself to three holonomies obtained from curves in the x, y and z direction. This is motivated by homogeneity: any holonomy of a different curve on the homogeneous spatial manifold should be indistinguishable. Still, this drastically reduces the spin networks contained in the kinematical Hilbert space. Secondly, integration along a curve introduces a regularisation parameter, usually denoted as μ, which should be understood as the length of the curve. Different choices of μ can lead to different phenomenology and the so-called $\bar{\mu}$-scheme, where the regulator is phase space dependent, has proven to be an attractive choice [249, 250]. This is because it sets an initial condition independent regime in which quantum gravity corrections occur, specifically, the upper curvature bound is independent of the fiducial cell V_0. Explicitly, the holonomies (1.51) of LQC are then constructed as follows (where the τ_i denote $su(2)$ generators and are related to the Pauli matrices σ_i as $\tau_i = -\frac{\mathrm{i}}{2}\sigma_i$):

$$h_{x,\,\mu} = \exp(-c\mu\tau_1)\,, \quad h_{y,\,\mu} = \exp(-c\mu\tau_2)\,, \quad h_{z,\,\mu} = \exp(-c\mu\tau_3)\,. \tag{4.24}$$

To obtain the quantum Hamiltonian, one first needs to find a form of the classical

Hamiltonian in terms of holonomies and fluxes, such that these can be promoted to operators. This procedure induces the departure from GR and is not free from ambiguities, as the symmetry reduction to the cosmological setting can be carried out at different stages of the construction. Indeed, different quantisation procedures have been established and depending on where this reduction is carried out, modifications to the general relativistic setting differ [242, 251, 252].

Upon quantisation, the scalar field momentum becomes a derivative operator $\mathcal{H}_{\text{matter}} \propto \partial^2_{\chi^0}$, such that physical states in LQC evolve w.r.t. the scalar field. Finally, one can obtain an effective evolution of the volume operator $\langle V \rangle$ by choosing a semiclassical state in the physical Hilbert space and computing its expectation value. Usually, one recovers a bouncing universe and the most commonly used form of the effective Friedmann equation reads [167]

$$H^2_{\text{LQC}} = \frac{\kappa}{6} N^2 \rho \left(1 - \frac{\rho}{\rho_c}\right) , \tag{4.25}$$

where $\rho = \frac{\pi_0^2}{a^6}$ and ρ_c is a fundamental parameter, independent of initial conditions.

It was noticed in [253] that the expectation value $\langle V \rangle$ follows a trajectory obtained from a modified Hamiltonian in the classical phase space, which has entered the literature as the so-called 'effective' Hamiltonian [167, 254, 255]. This effective Hamiltonian is obtained by 'polymerisation' of the classical Hamiltonian, where one makes the following replacements

$$c \to \frac{\sin(\mu c)}{\mu} , \qquad \mathcal{H}_{\text{geom}} \to \mathcal{H}_{\text{geom, effective}} = -\frac{3N}{\kappa\beta^2}\sqrt{p}\frac{\sin^2(\mu c)}{\mu^2} . \tag{4.26}$$

The procedure we have outlined above can be summarised as follows: First reduce to the phase space of the physical system in question and carry out a quantisation using LQG techniques at the level of the reduced system. This requires a choice of holonomies, regulators, and construction procedure for the Hamiltonian. To illustrate the significance of the latter, we note that one can e.g. recover an asymmetric bounce [252]. Due to the stark reduction of degrees of freedom by fixing the holonomies, which is based on (well-motivated) choices, the implications of LQC for LQG are unclear. For criticisms of LQC we refer the reader to [256].

There exist several proposals to incorporate cosmological perturbations in LQC and we refer the reader to [247] for a detailed analysis of these approaches and the current state of connecting to observations. In some cases one can obtain explicit equations of motion for cosmological perturbations that match GR in the classical regime. This can be achieved e.g. by quantising the background and perturbations separately [257], such that the perturbations evolve on an effectively classical background with LQC corrections; or by working in the effective framework entirely and ensuring that the algebra of the modified constraints is anomaly free [258]. Importantly, LQC alone cannot produce a scale invariant spectrum for the CMB and must therefore be combined with an additional early universe scenario, such as inflation, in order to match observations. When coupled with inflation, the form of the CMB power spectrum can be recovered accurately, with corrections appearing only for large scale multipoles, which are not strongly constrained; see e.g. [248] for implications for the CMB from different LQC quantisation schemes and for different treatments of perturbations.

4.3. Perturbations at a bounce

We have so far discussed two avenues to obtaining singularity resolution in the early universe from quantum gravity bounces. Here we comment more generally on the evolution of the Hubble horizon w.r.t. the scale factor and thereby the behaviour of perturbations around the bounce. We make use of these considerations in chap. 5, where we explore the behaviour of long-wavelength perturbations for modified gravitational dynamics around a bounce.

We recall that in GR, the dynamical evolution of perturbations, in particular, of gauge-invariant perturbations, is rather different depending on the size of their wavelength relative to the Hubble horizon. The notion of the Hubble horizon was detailed in sec. 1.2.3, and the behaviour of gauge-invariant perturbations was discussed in sec. 3.2. In the contracting branch of a bouncing universe, all perturbations are inside the Hubble horizon in the far pre-bounce regime. The Hubble horizon however decreases more rapidly than the scale factor, such that modes exit the horizon as one approaches the bounce. At the bounce itself, the Hubble horizon is infinite and all perturbations are inside the Hubble horizon, albeit for a short time only. Shortly after the bounce, all modes will again be inside the horizon and exit as the universe expands further. This scenario is illustrated in fig. 4.2.

The biggest 'quantum' effect of quantum gravity bounces is that they resolve the classical singularity. Hence, around the bounce regime, quantum gravity effects are the strongest and would have the most impact on the evolution of cosmological perturbations. As we just described, most perturbations will be in the super-horizon regime in the vicinity of the bounce. The super-horizon regime considerably simplifies the dynamics of perturbations in GR and can be studied with e.g. the separate universe framework (sec. 3.3).

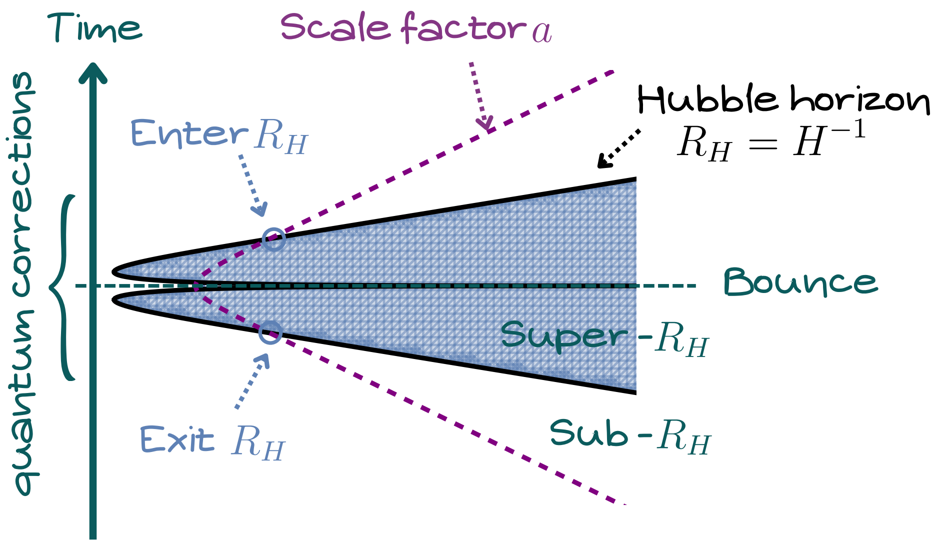


Figure 4.2.: Schematic diagram of the relative evolution of the Hubble horizon and scale factor in a bouncing universe. The dotted line indicates the evolution of the scale factor, which is representative of the evolution of the physical wavelength of perturbations $\lambda = \frac{a}{k}$. The blue shaded region marks the super-horizon regime and we marked the exit and entry points of the perturbations as the universe contracts and re-expands.

4.4. Conclusion: It all (re)started with a bounce

The aim of this chapter was to combine previously introduced concepts and show how GFT and LQG techniques can be used to extract the evolution of our universe (and its perturbations).

We saw that the application of GFT to cosmology allows us to recover a modified Friedmann equation that induces a bounce in the place of the Big Bang singularity, which is a desirable feature of any quantum gravity theory. The effective evolution of the scale factor is extracted from the expectation value of the volume operator and depends on the form of the GFT action as well as the choice of state. The majority of the literature focuses on a GFT action that consists of a second order kinetic term and neglects interactions. Regarding the choice of state, the most important criterion is that it is sufficiently semiclassical in the limit of large quanta, which in this case is equivalent to late times. Concretely, the most widespread state choice is that of a Fock coherent state, which has semiclassical properties, but was originally motivated by considerations regarding the implementation of homogeneity in GFT. Isotropy, on the other hand, is encoded at the level of the GFT quanta, where one restricts to modes that carry equal spin labels on their edges. We showed explicitly how the dynamics of the expectation value of the volume operator are obtained in the Hamiltonian framework of GFT and used the result to deduce an effective Friedmann equation. As the expectation value of the volume operator is bounded from below, the Big Bang singularity is absent. Instead, we find that the GFT effective dynamics result in a bouncing universe, where the expanding branch is pre-dated by a contracting one. From the evolution of the volume operator it is apparent that a single Peter-Weyl mode J_0 will dominate the evolution at late times, which motivates the often employed approximation to consider the dynamics of a single GFT mode only. At late times, the effective Friedmann equation reduces to that of general relativity if one fixes the constant m_{J_0} appearing in the kinetic term of the dominating mode accordingly. We stress that the GFT dynamics are obtained w.r.t a matter clock given by a massless scalar field, which fixes the choice of lapse. Hence, the Friedmann equation takes on a non-standard form.

Having established the considerations that lead to a bouncing universe in GFT, we elaborate how perturbations can be included in the framework. The main idea is that three additional massless scalar fields are included in addition to the clock field. These fields can be used to construct a relational coordinate system, such that perturbations can then be localised in space w.r.t. the three additional spatial fields. The effective dynamics of perturbation variables are again obtained in the form of expectation values over semiclassical states. These can be compared to the equations of motion within GR in the relational coordinate system, where a possible agreement between the two depends on the choice of GFT state, as well as the details of the GFT construction. Past studies are limited to considering the perturbed GR volume element, which is a combination of classical scalar metric perturbations. This restricts the ability to obtain effective expressions of gauge-invariant quantities. Our results on cosmological perturbations within GFT contained in chap. 7 differ from these constructions in various ways: firstly, we build on new operators introduced in chap. 6 that allow to retain all scalar metric perturbations separately; secondly, we work in a Hamiltonian framework; and we consider a finite width Gaussian coherent state that encodes both, the background dynamics as well as the perturbations.

So far there is no matter content in GFT that corresponds to the constituents of a realistic universe (see sec. 1.2). This can be rectified by limiting the validity of GFT cosmology to the vicinity of the universe's origin, such that it may resolve the Big Bang singularity and assuming that another mechanism (such an inflation) takes over to connect to the established description of the universe. Alternatively, one can hope that in the future, all types of matter can be included in GFT and lead to a realistic cosmology.

We also touched upon another quantum bounce scenario, namely that of LQC. LQC is obtained by applying techniques from LQG to the symmetry reduced sector of cosmology. This allows one to carry out the loop quantisation programme in full, the result is however dependent on certain choices made in the construction, which includes the restriction to certain holonomies as well as the procedure to construct the quantum Hamiltonian. Similar to GFT cosmology, the effective evolution of the scale factor is obtained from the expectation value of the volume operator over suitable semiclassical states and one can reconstruct an effective Friedmann equation. Due to the Hamiltonian structure of LQC and the fact that, unlike GFT, it is a direct quantisation of classical GR, it can be more easily related to cosmological perturbations. In connection with an inflationary phase, studies of LQC perturbations can be brought in agreement with the CMB power spectrum.

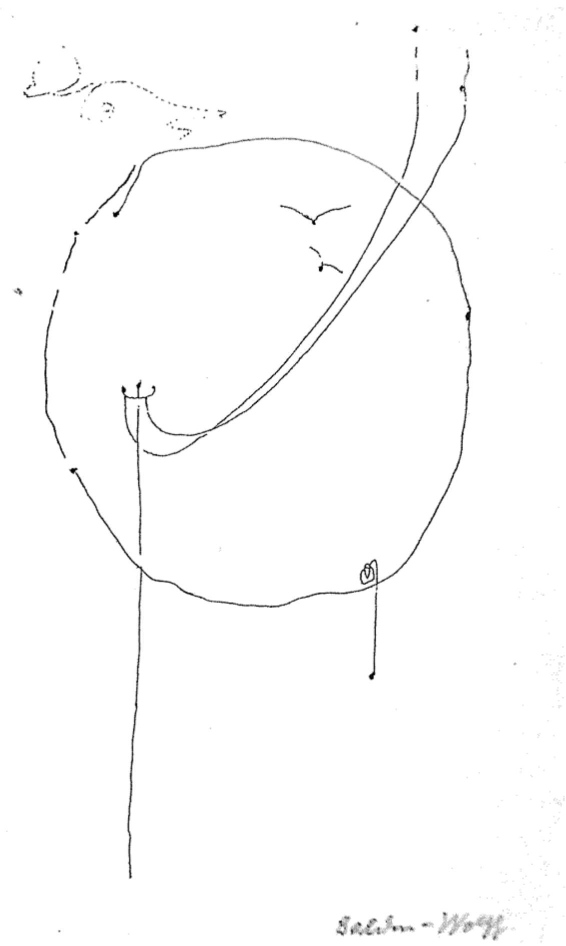
A downside of LQC is that its connection to the full theory of LQG is rather unclear. In the quest to establishing a full theory of quantum gravity, its potential to guide further developments could therefore be seen as obstructed. The GFT formalism on the other hand, has the advantage that the connections of the cosmological sector to the full theory are apparent. Momentarily, the cosmological sector of GFT relies on a number of simplifications and ingredients (no interactions, single mode, simple choice of state), however, if a connection to more physical scenarios, such as cosmological perturbations, can be made, it might be possible to gather further insights into the fundamental theory.

One can debate how exactly a full quantum theory of gravity within the GFT framework should behave. For instance, one could demand that such a theory should be defined by a specific form of the GFT action and the physical scenario is determined solely by the state choice. Alternatively, one could argue that different GFT actions capture different spacetimes. In the end, these are philosophical dreams, as momentarily, we are just beginning to attain an understanding of how GFT can be used to describe realistic cosmological spacetimes, which are arguable the simplest. If and how applications to other physical scenarios are possible and how they can be related only the future will show (some starting points for the study of black holes exist, as mentioned in sec. 2.3).

Finally, we ended this chapter on a more general remark about the behaviour of perturbations in a bouncing universe. Close to the bounce, most perturbations will be in the super-horizon regime, in which spatial gradients can be neglected in their description. In the next chapter we will make use of this fact in an attempt to study the behaviour of perturbations around a quantum gravity bounce in a model independent manner.

Part II.

Approaching inhomogeneity



Chapter 5.

Quantum gravity effects on super-horizon perturbations

'Nothing is static, nothing is final, everything is held provisionally.'

- Jocelyn Bell Burnell.

'Nichts ist statisch, nichts ist endgültig, alles ist provisorisch.'

This chapter contains the result of [1], which was published in *Universe* in December 2022.

We address a question that is simple in its nature: is it possible to make statements about dynamics of gauge-invariant cosmological perturbations from a modified Friedmann equation alone? In a context where one wishes to understand quantum gravitational implications for cosmological perturbations, this question arises naturally from the situation described in chap. 4: While a specific approach to quantum gravity may permit to derive corrections to the Friedmann equation, including inhomogeneous quantities can be a rather non-trivial task which additionally is highly model-dependent.

Therefore, in what follows, we use the Friedmann equation as our starting point and remain agnostic about its origin. As we begin with a homogeneous equation only, we cannot reconstruct any information about gradients of cosmological perturbations; instead we restrict our analysis to super-horizon perturbations and work in the separate universe picture as described in sec. 3.3. As explained in sec. 3.3, in the separate universe picture the universe is modelled as an ensemble of independent homogeneous patches and the difference between the value of relevant quantities in a local patch and the ensemble average is interpreted as a perturbation. Specifically, this results in homogeneous perturbations across each patch. This approximation is only applicable to perturbations on super-horizon scales, since these can be seen as approximately homogeneous inside the Hubble radius. The procedure is then simple: Perturbing the modified Friedmann equation at linear order leads to a first order equation of motion for long-wavelength perturbations. We additionally assume that the continuity equation remains unaffected by the changes that lead to alterations in the Friedmann equation, which is the case for LQC and GFT cosmology (see chap. 4).

If one considers modifications to the Friedmann equation that replace the Big Bang singularity with a bounce in a similar fashion to the quantum gravity bounces we described in chap. 4 (namely GFT and LQC), the evolution of the Hubble radius in relation to the scale factor can be described as follows (see also fig. 4.2):

- In the far past of the contracting branch, the horizon is arbitrarily large; it shrinks with the contraction of the universe.

- As the Hubble radius decreases faster than the scale factor, there exists a (wavenumber-dependent) point in the contracting phase where perturbations exit the Hubble horizon.

- At the bounce itself, the Hubble horizon is infinite, so all modes are inside the horizon, albeit for a very short time only.

- After the bounce, the majority of modes will again be outside the horizon and re-enter only after the universe has expanded further.

As quantum gravity effects are expected to affect regimes of very high curvature, their impact is strongest, and likely limited to, the region close to the bounce, where perturbations are super-horizon (again, save for the bounce itself). It is then consistent (at least as a first approximation) to model the dynamics of perturbations in the separate universe picture.

We focus on two gauge-invariant variables that are widespread in the cosmological literature, namely the comoving curvature perturbation $\mathcal{R}$ and the curvature perturbation on equal density hypersurfaces ζ as given in (3.21). All results for these variables rely on the assumption that the same notion of gauge-invariance remains unaltered, despite the modifications to the gravitational dynamics. The conservation of the curvature perturbation on equal density hypersurfaces ζ on super-horizon scales for adiabatic perturbations depends only on the continuity equation and therefore remains unaffected; for the comoving curvature perturbation $\mathcal{R}$ however, we find that its conservation can only be guaranteed for a certain class of modified Friedmann equations and modifications are possible in general.

This chapter is organised as follows: We first introduce a general form for a modified Friedmann equation that we will use throughout and derive the resulting perturbation equations in sec. 5.1. We then turn our attention to the dynamics of ζ and $\mathcal{R}$ in sec. 5.2. After re-deriving the conservation law for ζ, we show that the conservation law for $\mathcal{R}$ continues to hold on super-horizon scales if the modification to the Friedmann equation is a function of the energy density only. We then consider an example of the more general case, where a modification as it can appear in GFT introduces dynamics in $\mathcal{R}$ around the bounce and ζ is conserved only for adiabatic perturbations. Before concluding in sec. 5.4, we comment on the relation of our approach to second order equations as are common in the literature in sec. 5.3.

We denote the Hubble rate in any lapse by H and most of our results will be for general choices of lapse. We focus on conformal time $N = a$ in sec. 5.3 and denote the Hubble rate in conformal time as $\mathcal{H}$. Even though this is the same notation we use for the Hamiltonian constraint of general relativity (1.18), no confusion should arise, as the constraint plays no role in this chapter. For the most part, our results are for general types of matter; in places where we specifically consider the case of a single massless scalar field, we denote this field as χ.

We would like to point out that even though the overarching theme of this thesis is to explore quantum gravitational implications for cosmological perturbations from the perspective of GFT, the results presented in this chapter are *not* based on any details of said theory, but are model agnostic. The specifics of the GFT bounce only enter in the form of an example for a modified Friedmann equation in sec. 5.2.2.

5.1. Perturbation equations from a modified Friedmann equation

As a first step in our endeavour we introduce a general notation for modifications to the Friedmann equation that could encode non-GR effects from any origin. We proceed to derive first order perturbation equations from the modified Friedmann equation, which capture the dynamics of super-horizon perturbations.

Inspired by the form of modified Friedmann equations in LQC (4.25) and GFT (4.14), we introduce the following generic form of the modified Friedmann equation:

$$\frac{H^2}{N^2} = \frac{\kappa}{3}\rho\,\mathcal{F}\,, \tag{5.1}$$

where any possible modifications are contained in the model-dependent function $\mathcal{F}$. We recover the Friedmann equation of GR in the case of $\mathcal{F} = 1$, hence, for consistency with GR in the low curvature limit we require $\mathcal{F} \to 1$ at late times (i.e. far away from the bounce). For example, the concrete forms of $\mathcal{F}$ for the LQC (4.25) and GFT (4.14) Friedmann equations introduced in chap. 4, which both assume the matter content to be given by a single massless scalar field $\rho = \frac{\pi_\chi^2}{(2a^6)}$, read

$$\mathcal{F}_{\rm LQC} = 1 - \frac{\rho}{\rho_{\rm c}}\,, \tag{5.2}$$

$$\mathcal{F}_{\rm GFT} = 1 + \frac{v_0}{a^3} + \frac{\mathcal{Y}}{a^6}\,. \tag{5.3}$$

Importantly, ρ_c is a fundamental constant, and so is v_0, i.e. these quantities remain unperturbed, as they take on the same value in each patch. $\mathcal{Y}$ on the other hand is a constant of motion, that differs across patches in the separate universe picture. This is the reason $\mathcal{F}_{\rm GFT}$ falls in the second category of modified Friedmann equations, which we explore in sec. 5.2.2, whereas $\mathcal{F}_{\rm LQC}$ belongs to the family of modifications detailed in sec. 5.2.1.

From the effective Friedmann equation one can derive an equation of motion for the Hubble rate

$$\frac{H'}{H} = \frac{N'}{N} + \frac{1}{2}\left(\frac{\rho'}{\rho} + \frac{\mathcal{F}'}{\mathcal{F}}\right)\,. \tag{5.4}$$

To obtain perturbation equations, we perturb the modified Friedmann equation (5.1) at linear order, which results in

$$\begin{aligned} H\psi' =& - H^2\left(\tilde{\Phi} + \frac{\delta\rho}{2\rho} + \frac{\delta\mathcal{F}}{2\mathcal{F}}\right) \\ =& - \frac{\kappa}{3}N^2\left(\rho\mathcal{F}\,\tilde{\Phi} + \frac{1}{2}\left(\mathcal{F}\delta\rho + \rho\delta\mathcal{F}\right)\right)\,. \end{aligned} \tag{5.5}$$

As mentioned in the introduction of this chapter, we assume that the continuity equation (1.32) holds, leading to the following perturbed continuity equation (3.14) in each patch of the separate universe picture

$$\rho' + 3H(\rho + P) = 0 \qquad \Rightarrow \qquad \delta\rho' + 3H(\delta\rho + \delta P) - 3\psi'(\rho + P) = 0\,. \tag{5.6}$$

Taking the time derivative of (5.5), together with the above leads to the (perturbed) Raychaudhuri equation

$$\begin{aligned} -\psi'' =& \left(-\frac{N'}{N} - \frac{\mathcal{F}'}{2\mathcal{F}} + 3H\frac{\rho + P}{\rho}\right)\psi' + H\,\tilde{\Phi}' + \frac{H}{2}\left(\frac{\delta\mathcal{F}'}{\mathcal{F}} - \frac{\mathcal{F}'}{\mathcal{F}^2}\delta\mathcal{F}\right) \\ &+ \frac{\kappa}{2}N^2\mathcal{F}(\rho + P)\left(\frac{\delta\rho}{\rho} - \frac{\delta\rho + \delta P}{\rho + P}\right) \\ =& - \frac{N'}{N}\psi' + H\,\tilde{\Phi}' - \frac{\kappa}{2}N^2\mathcal{F}(P + \rho)\left(\frac{(\delta P + \delta\rho)}{P + \rho} + \frac{\delta\mathcal{F}}{\mathcal{F}} + 2\tilde{\Phi}\right) \\ &+ \frac{H}{2}\frac{\mathcal{F}'}{\mathcal{F}}\left(-\frac{\delta\mathcal{F}}{2\mathcal{F}} + \frac{\delta\rho}{2\rho} + \tilde{\Phi} + \frac{\delta\mathcal{F}'}{\mathcal{F}'}\right)\,. \end{aligned} \tag{5.7}$$

The different forms of the perturbation equation are obtained by making use of the Friedmann equation (5.1) and (5.5). The perturbation equations reduce to those of general relativity as reported in sec. 3.4.1 in the long-wavelength limit for $\mathcal{F} = 1$ (and hence $\mathcal{F}' = 0\,,\ \delta\mathcal{F} = 0\,,\ \delta\mathcal{F}' = 0$).

Recall that if the universe's matter content is assumed to be a single scalar field χ the energy density and pressure are given by (1.36)

$$\rho = \frac{\chi'^2}{2N^2} + U(\chi)\,, \qquad P = \frac{\chi'^2}{2N^2} - U(\chi) \tag{5.8}$$

and the perturbed energy density and pressure read

$$\delta\rho = \frac{\chi'^2}{N^2}\left(\frac{\delta\chi'}{\chi'} - \tilde{\Phi}\right) + \frac{\mathrm{d}U(\chi)}{\mathrm{d}\chi}\delta\chi\,, \qquad \delta P = \frac{\chi'^2}{N^2}\left(\frac{\delta\chi'}{\chi'} - \tilde{\Phi}\right) - \frac{\mathrm{d}U(\chi)}{\mathrm{d}\chi}\delta\chi\,. \tag{5.9}$$

As explained in sec. 3.2 it is possible to simplify the perturbative analysis by a suitable choice of gauge, where we have not specified a gauge choice so far. In the subsequent analysis we will work in the comoving gauge (see sec. 3.2.3), for the following reasons:

- It is a particularly convenient choice when studying the comoving curvature perturbation $\mathcal{R}$ (3.21), as in comoving gauge we have $\mathcal{R} = \psi$ (in the case where matter is given by a single scalar field).

- When working with relational settings such as GFT, where the scalar field takes the role of a physical clock [144, 177], the comoving gauge corresponds to the statement that at an instant of time all patches of the separate universe picture have the same clock value.

- We avoid subtleties related to the Newtonian gauge. As explained in sec. 3.3, the metric perturbations E and B do not appear in the strict separate universe limit and the Newtonian gauge would not lead to any simplifications of the equations of motion.

In the comoving gauge, the lapse perturbation is directly related to the perturbation of the energy density and pressure (5.9) for scalar matter, i.e.

$$\delta\rho = -\frac{\chi'^2}{N^2}\tilde{\Phi} = -(\rho + P)\tilde{\Phi} = \delta P\,. \tag{5.10}$$

We now proceed to analyse the dynamics of gauge-invariant perturbations, where we consider separately the class of modified Friedmann equations where $\mathcal{F}$ is a function of the energy density only.

5.2. Dynamics of gauge-invariant perturbations

With the perturbation equations derived in the previous section we can proceed to study the dynamics of gauge-invariant cosmological perturbations, where we focus on ζ and $\mathcal{R}$, which are introduced in sec. 3.2. For convenience, we repeat their explicit forms:

$$-\zeta = \psi + \frac{H}{\rho'}\delta\rho\,, \qquad \mathcal{R} = \psi + \frac{H}{\chi'}\delta\chi\,. \tag{5.11}$$

In comoving gauge one finds, using the definitions of $\mathcal{R}$ and ζ and the continuity equation (5.6),

$$-\zeta = \mathcal{R} + H\frac{\delta\rho}{\rho'} = \mathcal{R} + \frac{\tilde{\Phi}}{3}\,. \tag{5.12}$$

In GR, $\mathcal{R} = -\zeta$ on super-horizon scales. However, this equality depends on the diffeomorphism constraint and may therefore not hold for modified gravitational dynamics. Still, in scenarios where matter is given by a single scalar field and the scalar field potential dominates over its kinetic term, as is the case in e.g. slow roll inflation, one can approximate

$$\rho = \frac{\chi'^2}{2N^2} + U(\chi) \approx U(\chi) \quad \Rightarrow \; \rho' \approx \frac{\mathrm{d}U(\chi)}{\mathrm{d}\chi}\chi'\,, \quad \delta\rho \approx \frac{\mathrm{d}U(\chi)}{\mathrm{d}\chi}\delta\chi \tag{5.13}$$

and the equality of $-\zeta$ and $\mathcal{R}$ follows directly, independent of gravitational equations.

Before we proceed to analyse the dynamics of $\mathcal{R}$ as dictated by the modification to the Friedmann equation, we recall the well-known result (see e.g. [86, sec. 6.2.4]) that the conservation law for ζ on super-horizon scales depends only on the validity of the perturbed continuity equation (5.6) as long as perturbations are adiabatic, i.e. they satisfy $\delta P = \frac{P'}{\rho'}\delta\rho$. Explicitly, the equation of motion for ζ can be rewritten as

$$-\zeta' = \psi' + \left(\frac{H}{\rho'}\delta\rho\right)' = -\left(\frac{1}{3(\rho+P)}\right)'\delta\rho + \frac{H(\delta\rho+\delta P)}{\rho+P} = \frac{\rho'+P'}{3(\rho+P)^2}\delta\rho + \frac{H(\delta\rho+\delta P)}{\rho+P}\,, \tag{5.14}$$

making use of (5.6). Using the adiabaticity condition $\delta P = \frac{P'}{\rho'}\delta\rho$ together with (5.6) one then finds that $\zeta' = 0$ on super-horizon scales. We stress that details of the (modified) gravitational dynamics were not needed to derive this result. It is however valid in the long-wavelength limit only, since the perturbed continuity equation (5.6) generally contains gradient terms (see e.g. (3.14) for the perturbed continuity equation in general relativity and sec. 5.3.1 for further discussion on this matter).

5.2.1. Conservation laws for a special class of Friedmann equations

In this section we focus on a specific class of modified Friedmann equations, namely those where the modification depends on the perturbed energy density only. We show that the conservation law for $\mathcal{R}$ continues to hold on super-horizon scales in this case.

In the case where $\mathcal{F}$ is a function of the energy density ρ only, $\mathcal{F} = \mathcal{F}(\rho)$, its perturbation takes the form $\delta\mathcal{F} = \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\rho}\delta\rho$, which simplifies perturbation equations considerably. For convenience, we introduce the following quantities that we will make ample use of in the calculations carried out below

$$\mathcal{F}_\rho := \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}\rho}\,, \qquad \mathcal{F}_{\rho\rho} := \frac{\mathrm{d}^2\mathcal{F}}{\mathrm{d}\rho^2}\,, \qquad \mathcal{A} := \mathcal{F} + \mathcal{F}_\rho\,\rho\,. \tag{5.15}$$

From the above, we obtain the following relations (using (5.6))

$$\begin{aligned} &\delta\mathcal{F} = \mathcal{F}_\rho\delta\rho\,, \qquad \mathcal{F}' = -3H(\rho+P)\mathcal{F}_\rho\,, \qquad \mathcal{A}' = -3H(\rho+P)(2\mathcal{F}_\rho + \rho\mathcal{F}_{\rho\rho})\,,\\ &\frac{\delta\mathcal{F}'}{\mathcal{F}'} = \frac{\mathcal{F}_{\rho\rho}\delta\rho}{\mathcal{F}_\rho} + \frac{\delta\rho+\delta P}{\rho+P} - \frac{\psi'}{H}\,. \end{aligned} \tag{5.16}$$

With this, the second Friedmann equation (5.4) and the perturbed equations of motion (5.5) and (5.7) simplify for the $\mathcal{F}(\rho)$ class of modified Friedmann equations:

$$H' - \frac{N'}{N}H = -\frac{\kappa}{2}N^2(\rho+P)\mathcal{A}\,, \tag{5.17}$$

$$H\psi' = -H^2\tilde{\Phi} - \frac{\kappa}{6}N^2\mathcal{A}\,\delta\rho\,, \tag{5.18}$$

$$\begin{aligned} -\psi'' &= -\frac{\kappa}{2}N^2\Big(\mathcal{A}\,\delta P + 2(\rho+P)\mathcal{A}\,\tilde{\Phi} + (\mathcal{F} + \rho(\rho+P)\mathcal{F}_{\rho\rho} + (2P+3\rho)\mathcal{F}_\rho)\delta\rho\Big)\\ &\quad - \frac{N'}{N}\psi' + H\tilde{\Phi}'\\ &= -\kappa\chi'\mathcal{A}\,\delta\chi' - \Big(\mathcal{F}_\rho + \frac{\rho}{2}\mathcal{F}_{\rho\rho}\Big)\,\kappa\chi'^2\,\delta\rho - \frac{N'}{N}\psi' + H\tilde{\Phi}'\,. \end{aligned} \tag{5.19}$$

All expressions are general save for the last line, which holds only when the matter content is given by a single massless scalar field with $\rho + P = \frac{(\chi')^2}{N^2}$. The above equations reduce to those derived in [221] for LQC ($\mathcal{A} = 1 - 2\frac{\rho}{\rho_c}$ and $\mathcal{F}_\rho = -\frac{1}{\rho_c}$) in conformal time $N = a$ and a Newtonian(-like) gauge $\psi = \tilde{\Phi}$.

In general relativity, we have access to the diffeomorphism constraint given by the time-space components of the perturbed Einstein equations, $\delta G^0{}_i = \kappa \delta T^0{}_i$:

$$\partial_i \left(H\tilde{\Phi} + \psi' - \frac{\kappa}{2}\chi'\,\delta\chi \right) =: \partial_i D = 0\,. \tag{5.20}$$

In the strict separate universe limit, the above is trivially satisfied, as all spatial gradients need to vanish. If we however assume that GR holds in regimes where modifications are small (i.e. in the far pre- and post-bounce regime in the case where $\mathcal{F}$ introduces a bounce, as is the case for GFT and LQC), one can assume that $D = 0$ is satisfied. (This assumes that D is not just a function of time, i.e. that it is an inhomogeneous quantity, which is consistent with the idea that perturbations contain inhomogeneous terms, whereas homogeneous contributions can be absorbed into the background in standard cosmological perturbation theory; see sec. 3.1.) To establish conservation laws for ζ and $\mathcal{R}$ below, we will make use of the following: If the constraint holds in the general relativistic regime at $t = t_0$, i.e. $D(t_0) = 0$ and the differential equation

$$\mathcal{A}D' + W\mathcal{A}D - \mathcal{A}'D = 0 \tag{5.21}$$

holds *throughout the evolution*, i.e. also in the regions affected by modifications to the gravitational dynamics, $D = 0$ will remain true in the regime where high curvature corrections become relevant. In the case of bouncing cosmologies, this means that if general relativity holds in the contracting branch, $D = 0$ throughout the bounce.

One caveat is that D itself might receive $\mathcal{F}$-dependent modifications, as was found to be the case for LQC in [221]. As the diffeomorphism analogue cannot be derived from the modified Friedmann equation in the separate universe picture, the altered form of D needs to be assumed from an educated guess (and justified in hindsight). Inspired by the findings of [221], we propose that the alterations are contained in the term proportional to $\delta\chi$, i.e.

$$D = \psi' + \tilde{\Phi}H - \mathcal{A}\frac{\kappa}{2}\chi'\,\delta\chi\,. \tag{5.22}$$

We proceed to show that (5.21) is indeed satisfied for $W = 3H - \frac{N'}{N}$ also for the regions in which beyond GR effects become relevant. Here we conduct the calculation in comoving gauge[1], such that we have

$$D = \psi' + \tilde{\Phi} H \,. \tag{5.23}$$

From (5.18) and the expression for the perturbed energy density in comoving gauge (5.10) it follows that

$$\psi' = -H\tilde{\Phi} + \frac{\kappa}{6}\mathcal{A}\frac{\chi'^2}{H}\,\tilde{\Phi}\,, \tag{5.24}$$

such that D takes on the following form in our gauge choice

$$D = \frac{\kappa}{6}\frac{\chi'^2}{H}\mathcal{A}\tilde{\Phi}\,. \tag{5.25}$$

From the equation of motion for ψ'' (5.19) and the relation between $\delta\rho$ and $\tilde{\Phi}$ given in (5.10), we find that the derivative of D in comoving gauge (where $\delta\chi' = 0$) can be expressed as

$$D' = \frac{N'}{N}\psi' + H'\,\tilde{\Phi} + \kappa\chi'^2\left(\mathcal{F}_\rho + \frac{\rho}{2}\mathcal{F}_{\rho\rho}\right)\delta\rho = \frac{N'}{N}\psi' + H'\,\tilde{\Phi} - \kappa\frac{\chi'^4}{N^2}\left(\mathcal{F}_\rho + \frac{\rho}{2}\mathcal{F}_{\rho\rho}\right)\tilde{\Phi}\,. \tag{5.26}$$

Inserting this as well as (5.24) and the equation for H' (5.17), into (5.21), and then eliminating $\mathcal{A}'$ using (5.16) leads to

$$\mathcal{A}D' + W\mathcal{A}D - \mathcal{A}'D = \mathcal{A}\tilde{\Phi}\left(H' - \frac{N'}{N}H + \frac{\kappa\,\chi'^2}{6H}\left(\mathcal{A}\frac{N'}{N} - \mathcal{A}' + W\mathcal{A}\right) - \frac{\kappa\chi'^4}{N^2}\left(\mathcal{F}_\rho + \frac{\rho}{2}\mathcal{F}_{\rho\rho}\right)\right) \tag{5.27}$$

$$= \kappa\chi'^2\mathcal{A}\,\tilde{\Phi}\left(\frac{1}{6H}\left(\mathcal{A}\frac{N'}{N} - \mathcal{A}' + W\mathcal{A} - 3H\mathcal{A}\right) - \frac{\chi'^2}{N^2}\left(\mathcal{F}_\rho + \frac{\rho}{2}\mathcal{F}_{\rho\rho}\right)\right) \tag{5.28}$$

$$= \frac{\kappa\chi'^2\mathcal{A}^2\,\tilde{\Phi}}{6H}\left(\frac{N'}{N} + W - 3H\right)\,. \tag{5.29}$$

Hence, for the choice $W = 3H - \frac{N'}{N}$, as long as $D = 0$ initially, $\psi' + \tilde{\Phi}H = 0$ holds throughout the evolution.

[1] An explicit calculation for the longitudinal gauge is carried out in app. D for conformal time.

We can now use the result that an analogue to the diffeomorphism constraint holds for $\mathcal{F} = \mathcal{F}(\rho)$ to infer conservation laws for ζ and $\mathcal{R}$. As we already saw at the beginning of this section, ζ is conserved on super-horizon scales for adiabatic perturbations, irrespective of the modified gravitational dynamics (as long as the continuity equation remains unaltered). In GR, a single scalar field cannot induce non-adiabaticities, irrespective of its potential, which extends to the case in which modifications to the Friedmann equation are of the form $\mathcal{F} = \mathcal{F}(\rho)$ discussed in this section: From (5.9) it follows that in comoving gauge we have $\delta\rho = -(\rho + P)\tilde{\Phi} = \delta P$. As we have just shown, $D = \frac{\kappa}{6}\frac{\chi'^2}{H}\mathcal{A}\tilde{\Phi} = 0$, and hence $\frac{\delta P}{P'} = \frac{\delta\rho}{\rho'}$ is trivially satisfied. Therefore, ζ will always be conserved on super-horizon scales for this class of modifications when the matter content is given by a single scalar field. From the continuity equation (1.32), and again $\delta\rho = -(\rho + P)\tilde{\Phi}$ it furthermore follows that

$$-\zeta = \mathcal{R} + H\frac{\delta\rho}{\rho'} = \mathcal{R} + \frac{\tilde{\Phi}}{3} = \mathcal{R}\,, \tag{5.30}$$

where in the last step we used that $D = 0$ and hence $\tilde{\Phi} = 0$. The equality of ζ and $\mathcal{R}$ on super-horizon scales therefore remains valid if modifications to the Friedmann equation depend only on the energy density, and thereby the conservation law for $\mathcal{R}$ continues to hold as well.

While we carried out the calculations in a specific gauge and intermediate steps have gauge dependent expressions, we stress that the final result concerns gauge independent variables and therefore holds in any gauge.

5.2.2. General case: a GFT example

Having established a class of modified Friedmann equations for which the conservation laws of general relativity for the two gauge-invariant perturbations ζ and $\mathcal{R}$ continue to hold on super-horizon scales, we now turn to the more general case, $\mathcal{F} \neq \mathcal{F}(\rho)$. For such general modifications to the Friedmann equations it might not be possible to derive an analogue to the diffeomorphism constraint D and the statements made for the $\mathcal{F} = \mathcal{F}(\rho)$ case need no longer be true. Specifically, there is the possibility that a single scalar field with non-vanishing potential $U(\chi)$ introduces non-adiabaticities, which would invalidate the conservation law for ζ (for a massless field as appears in LQC or GFT this is of course not the case). Furthermore, $-\zeta$ and $\mathcal{R}$ need no longer be equal on super-horizon scales, such that for adiabatic perturbations the conservation law for $\mathcal{R}$ no longer follows from

$\zeta' = 0$, but its dynamics are instead governed through (5.5), which in comoving gauge reads

$$-\frac{\mathcal{R}'}{H} = \tilde{\Phi} + \frac{\delta\rho}{2\rho} + \frac{\delta\mathcal{F}}{2\mathcal{F}} = (1-w)\frac{\tilde{\Phi}}{2} + \frac{\delta\mathcal{F}}{2\mathcal{F}} \,. \tag{5.31}$$

We stress that the reverse is not applicable: for any specific form of $\mathcal{F} \neq \mathcal{F}(\rho)$, the general relativistic conservation laws *may* still hold; we can however not make any general statements and these needs to be confirmed on a case-by-case basis.

While the framework presented in this chapter is general, for the rest of this section we consider a concrete example where a modified Friedmann equation introduces dynamics in $\mathcal{R}$ around the bounce. In line with the overarching theme of this thesis, we focus on GFT cosmology and examine the dynamics of the comoving curvature perturbation $\mathcal{R}$ in a GFT toy model as established in [185], which leads to an effective Friedmann equation as specified by (5.3).

The standard cosmological GFT framework assumes that the only, or at least the dominant, matter contribution around the bounce is a single massless scalar field, that also serves as a relational matter clock.[2] For a single massless scalar field $U(\chi) = 0$ we can solve the Klein–Gordon equation (1.37) to obtain the following expressions for the energy density and its perturbation

$$\chi' = \frac{\pi_\chi N}{a^3} \quad \Rightarrow \quad \rho = \frac{\pi_\chi^2}{2a^6}\,, \qquad \delta\rho = 2\rho\left(\frac{\delta\pi_\chi}{\pi_\chi} + 3\psi\right)\,, \qquad \rho' = -3H\frac{\pi_\chi^2}{a^6} = -6H\rho\,, \tag{5.32}$$

where the scalar field momentum π_χ is a constant of motion. Furthermore, for a massless scalar field, the relation between the lapse perturbation and the energy density perturbation in comoving gauge as given in (5.10) reduces to $\frac{\delta\rho}{\rho} = -2\tilde{\Phi}$. One also obtains

$$\zeta = \frac{1}{3}\frac{\delta\pi_\chi}{\pi_\chi}\,, \tag{5.33}$$

so that the conservation of ζ follows directly from the fact that π_χ and its perturbation

[2] This differs from the construction we consider in the subsequent chapters, where the matter content is instead given by *four* massless scalar fields. This impacts also the classical analysis of the background and perturbations, which were the subject of sec. 3.4.1 and sec. 3.4.2.

$\delta\pi_\chi$ are constants of motion.

We first introduce the details of the GFT construction and how super-horizon perturbations can be obtained directly from the effective evolution of the GFT volume operator studied separately in each patch of the separate universe picture. The GFT framework is described in chap. 2, and details of GFT cosmology are given in sec. 4.1; we refer the reader to these sections for details on the GFT results used in this section. Here we will limit ourselves to recalling the form of the expectation value of the volume operator that leads to the modified Friedmann equation we use here. Subsequently, we turn to the perturbative dynamics that follow from the modified Friedmann equation as derived in sec. 5.1 and compare the two outcomes.

Long-wavelength perturbations in GFT

Before applying the perturbation equations we derived to the GFT modified Friedmann equation, we consider the dynamics of perturbations as they arise directly from the quantum theory in the separate universe picture. As is common in GFT cosmology, we restrict our considerations to the free theory and neglect any interactions between GFT quanta. In this regime, it is a widespread (and well motivated, see sec. 4.1) simplification to study a single field mode $J = J_0$ only. To obtain the dynamics of long-wavelength perturbations directly from a GFT cosmological model we follow the procedure demonstrated in sec. 4.1.3: we use the expectation value of the GFT volume operator to deduce the effective dynamics of the scale factor (4.2) and hence obtain an effective Friedmann equation (4.13). The evolution of the volume operator is determined by the details of the GFT. Specifically, for a cosmological model as we consider here, the squeezing Hamiltonian (4.9) determines the dynamics of creation and annihilation operators (4.10), which in turn determine the dynamics of the number operator (4.11), which in the single mode case is directly proportional to the volume operator $\langle V\rangle = v_0\langle N\rangle$. In order to relate the volume expectation value to the scale factor one needs to work with a sufficiently semiclassical state and we work again with a Fock coherent state as given in (4.5). This leads to the following expression for the volume expectation value

$$V(\chi) := \langle V\rangle = v_0 A e^{2m\chi} + v_0 B e^{-2m\chi} - \frac{v_0}{2}\,, \tag{5.34}$$

where $A := \mathcal{C}_{1,J_0}$ and $B := \mathcal{C}_{2,J_0}$ in comparison to (4.11).

Using the separate universe setup (see sec. 3.3.1) one can then define a perturbation of the scale factor as the difference between the local scale factor a_p in a patch p and the ensemble average a_{bg}, where dynamics of the perturbations are fixed from the operator dynamics and initial conditions. We can calculate the perturbation of the scale factor $\delta a_p = -a_{bg}\psi_p$ from the volume operator in each patch $V_p = (a_p)^3 = (a_{bg})^3(1-3\psi_p)$ and the ensemble average V_{bg}

$$\psi_p = \frac{1}{3}\left(1 - \frac{V_p}{V_{bg}}\right) , \quad \text{where } V_{bg} := \frac{1}{n}\sum_p V_p \,, \tag{5.35}$$

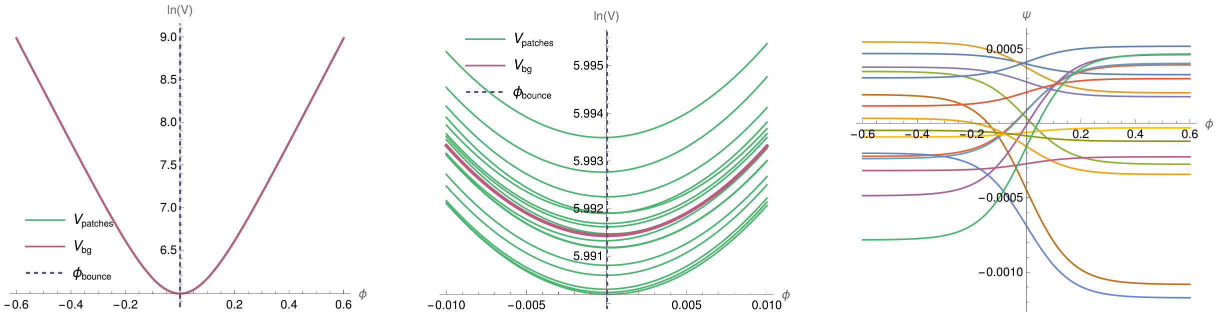
and n is the total number of patches. From the analytical solution of $V(\chi)$ (5.34) the evolution of ψ_p is readily calculated, as each patch has the same relational time parameter in comoving gauge (as $\delta\chi = 0$), making the comparison of the values of $V_p(\chi)$ straightforward (unlike in [222], where more general gauge choices were studied). Finally, we quote the asymptotic values of ψ_p resulting for $V(\chi)$ given in (5.34), which give the initial conditions for the comparison to the separate universe solution below. They read, in the far pre- and post-bounce regime, respectively,

$$\psi_{\text{pre, exact}} = \frac{1}{3}\left(1 - \frac{B_p}{B_{bg}}\right) = -\frac{1}{3}\frac{\delta B_p}{B_{bg}} \,, \qquad \psi_{\text{post, exact}} = \frac{1}{3}\left(1 - \frac{A_p}{A_{bg}}\right) = -\frac{1}{3}\frac{\delta A_p}{A_{bg}} \,. \tag{5.36}$$

To illustrate possible evolutions of ψ_p we construct an ensemble of $n = 16$ patches with perturbed initial conditions A_p and B_p that are determined through random fluctuations in the initial conditions $\mathrm{Re}(\sigma)$ and $\mathrm{Im}(\sigma)$ of the state (4.5) as generated from a white noise process. The resulting volume expectation value and resulting perturbation ψ_p in each patch are depicted in fig. 5.1. The most sizeable relative deviation in the patches occurs around the bounce of the ensemble average (given by the minimum of V_{bg}), where the minimum volume of each patch is reached at different χ values. Consequently, ψ_p fluctuates around the bounce, but asymptotically approaches a constant value in the far pre- and post-bounce regimes. Recalling that in comoving gauge we furthermore have $\psi = \mathcal{R}$, we conclude that for the GFT Friedmann equation (5.34), the comoving curvature perturbation $\mathcal{R}$ is not conserved around the bounce, and the classical conservation law holds in the early- and late-time regimes only.

Separate universe approach

Having established that the GFT modified Friedmann equation introduces dynamics in the comoving curvature perturbation $\mathcal{R}$ when considering dynamics as they arise directly



(a) Evolution of the volume in each patch V_p and the background (average) volume V_{bg}.

(b) The evolution of the volume of each patch V_p and V_{bg} close to the bounce. The minimum volume is reached at a different value of χ in each patch.

(c) Evolution of ψ_p (5.35) as determined by V_p and V_{bg} for all patches.

Figure 5.1.: The evolution of the expectation value of the volume operator for Fock coherent states $V(\chi)$ and the resulting dynamics for ψ in an example ensemble with $n = 16$ patches and different values of A_p, B_p in each patch, where $v_0 = 1$. The values of initial conditions in different patches (generated from a white noise process) are given in the table below. Note that the behaviour presented above is generic and the specific values are only reported for completeness.

A_p	200.03	200.103	200.391	199.99	200.119	200.876	199.984	200.244	199.948	200.298	200.046	200.93	200.432	200.361	199.946	199.915
B_p	199.981	199.934	200.053	200.395	200.035	200.146	200.405	200.317	200.554	200.29	200.192	200.382	200.242	200.453	200.733	200.079

from the quantum theory, we now turn to the evolution of $\mathcal{R}$ as it results from the modified perturbation equations established in sec. 5.1.

For this, we first solve the background dynamics arising from the modified GFT Friedmann equation (5.1), which then allows us to find a solution for ψ' from (5.5). We continue to work in comoving gauge, such that the dynamics of $\mathcal{R}$ are given by (5.31). We will see that the generalised perturbation equations accurately capture the evolution of $\mathcal{R}$ through the bounce at linear order in the perturbations.

The **background dynamics** are determined by the modified Friedmann equation, which for the GFT cosmological scenario we consider is given by

$$\mathcal{F}_{\text{GFT}} = 1 + \frac{v_0}{a^3} + \frac{\mathcal{Y}}{a^6}\,, \tag{5.37}$$

where the constant of motion $\mathcal{Y}$ is determined by the coefficients in (5.34) as (see also sec. 4.1.3)

$$\mathcal{Y} = \frac{v_0^2}{4} - 4\,v_0^2\,A\,B\ \ < 0\,. \tag{5.38}$$

For ease of comparison with the evolution of the effective volume as given in (5.34), we solve the Friedmann equation for $V = a^3$. For this, we rewrite $H = \frac{\mathrm{d}a}{\mathrm{d}\chi}\frac{1}{a}\chi' = \frac{1}{3}\frac{\mathrm{d}V}{\mathrm{d}\chi}\frac{1}{V}\chi'$ and make use of the fact that the matter content is given by a single massless scalar field with $\rho = \frac{\pi_\chi^2}{2V^2}$ and we are working in relational time $\chi' = \frac{\pi_\chi N}{V}$. This leads to the following form of the modified Friedmann equation

$$H^2 = \frac{1}{9}\left(\frac{\mathrm{d}V}{\mathrm{d}\chi}\frac{1}{V}\right)^2\left(\frac{\pi_\chi N}{V}\right)^2 = \frac{\kappa}{3}N^2\rho\mathcal{F} \qquad \Rightarrow \qquad \left(\frac{\mathrm{d}V}{\mathrm{d}\chi}\frac{1}{V}\right)^2 = \frac{3}{2}\kappa\mathcal{F}\,, \tag{5.39}$$

which has the solution

$$V(\chi) = \frac{\mathcal{C}}{4}e^{\sqrt{3\kappa/2}\,\chi} + \left(-\mathcal{Y} + \frac{v_0^2}{4}\right)\mathcal{C}^{-1}e^{-\sqrt{3\kappa/2}\,\chi} - \frac{v_0}{2}\,. \tag{5.40}$$

Here, $\mathcal{C}$ is an integration constant fixed by the initial condition for V and the value of $\mathcal{Y}$ is determined by $\mathcal{F}_{\mathrm{GFT}}$. If we compare to (5.34) we find that

$$A = \frac{\mathcal{C}}{4}\,, \qquad B = \left(-\mathcal{Y} + \frac{v_0^2}{4}\right)\mathcal{C}^{-1}\,. \tag{5.41}$$

The solution to the modified Friedmann equation $V(\chi)$ as given in (5.40) then agrees with the exact expression for V_{bg} obtained from (5.34) and (5.35).

Before proceeding with the analysis of the dynamics of perturbations, we comment on averaging effects that arise as part of the separate universe treatment. The first comment concerns the definition of $\mathcal{F}$ in the Friedmann equation, $\mathcal{F} = \mathcal{F}_{\mathrm{bg}}$: In the separate universe picture each patch follows the (modified) Friedmann equation $\left(\frac{\mathrm{d}V_p}{\mathrm{d}\chi}\frac{1}{V_p}\right)^2 = \frac{3}{2}\kappa\mathcal{F}_p$, and the background satisfies $\left(\frac{dV_{bg}}{d\chi}\frac{1}{V_{bg}}\right)^2 = \frac{3}{2}\kappa\mathcal{F}_{bg}$. For a definition of $\mathcal{F}_{bg}$ in analogy with that of the background volume V_{bg} we have

$$\mathcal{F}_{bg} = \frac{1}{N_p}\sum_p \mathcal{F}_p = 1 + v_0\sum_p \frac{1}{V_{bg}+\delta V_p} + \sum_p \frac{\mathcal{Y}_{bg}+\delta\mathcal{Y}_p}{(V_{bg}+\delta V_p)^2} \approx 1 + \frac{v_0}{V_{bg}} + \frac{\mathcal{Y}_{bg}}{V_{bg}^2}\,. \tag{5.42}$$

The approximation in the last step ensures that $\mathcal{F}_{bg}$ as given in (5.37) is defined solely from background quantities. It holds for small perturbations ($\frac{\delta V_p}{V_{bg}} \ll 1\,,\ \frac{\delta\mathcal{Y}_p}{\mathcal{Y}_{bg}} \ll 1$), i.e. in the regime where linear perturbation theory is applicable. Secondly, there is an ambiguity in the definition of $\mathcal{Y} = \mathcal{Y}_{bg}$: One could define it as the ensemble average $\mathcal{Y}_{bg} = \frac{1}{N_p}\sum_p \mathcal{Y}_p$ or as $\mathcal{Y}_{bg} = \frac{v_0^2}{4} - 4\,A_{bg}\,B_{bg}$, with $A_{bg} := \frac{1}{N_{\mathrm{patches}}}\sum_p A_p$ and $B_{bg} := \frac{1}{N_{\mathrm{patches}}}\sum_p B_p$ (which again agree at the linear perturbation level). These are inequivalent due to the non-linearity of $\mathcal{Y}$. The background volume V_{bg} as given in (5.35) is obtained by replacing A, B with their background values $A, B \to A_{bg},\ B_{bg}$ in (5.34), and hence we choose the latter

$$\mathcal{Y} := \frac{v_0^2}{4} - 4A_{bg}B_{bg}\,. \tag{5.43}$$

We note that the given alternative would not alter any of the qualitative statements made, but would introduce non-linear averaging effects in V_{bg} around the bounce. The perturbations are defined equivalently to all other perturbed quantities, $\delta\mathcal{Y} := \mathcal{Y}_p - \mathcal{Y}_{bg} = -4\,v_0^2\,(\delta A_p\,B_{bg} + \delta B_p\,A_{bg} + \delta A_p\,\delta B_p)$. This failure of averaged quantities to capture the true evolution is known as 'the averaging problem' in standard cosmology and is discussed in sec. 3.1.

We now turn to the **perturbative dynamics** and consider $\mathcal{R} = \psi$. For a GFT modified Friedmann equation specified by (5.37), the perturbed modification reads

$$\delta\mathcal{F} = 3\frac{v_0}{a^3}\psi + 6\frac{\mathcal{Y}}{a^6}\psi + \frac{\delta\mathcal{Y}}{a^6}\,. \tag{5.44}$$

Recalling that the matter content is given by a massless scalar field with $w = 1$ the dynamics of $\mathcal{R}$ are determined by $\mathcal{R}' = -H\frac{\delta\mathcal{F}}{2\mathcal{F}}$, see (5.31). To avoid division by zero at the bounce where $\mathcal{F} = 0$, we use the following form of the equation of motion, and rewrite in relational time similar to (5.39)

$$2H\psi' = -\frac{\kappa}{6}\chi'^2\delta\mathcal{F} \qquad \Rightarrow \qquad \frac{\mathrm{d}V}{\mathrm{d}\chi}\frac{1}{V}\frac{\mathrm{d}\psi}{\mathrm{d}\chi} = -\frac{\kappa}{4}\delta\mathcal{F}\,. \tag{5.45}$$

Note that this is independent of the explicit form of the lapse N, like the relational Friedmann equation (5.39). The above gives the following solution:

$$\psi = \frac{\mathcal{C}_\psi\left(\mathcal{C}^2 e^{\sqrt{6\kappa}\,\chi} - v_0^2 + 4\mathcal{Y}\right) + \frac{4}{3}\delta\mathcal{Y}}{\left(\mathcal{C}\,e^{\sqrt{3\kappa/2}\,\chi} - v_0\right)^2 - 4\mathcal{Y}}\,, \tag{5.46}$$

where we used the background solution (5.40) in (5.45) and $\mathcal{C}_\psi$ is an initial condition. The asymptotic values of the solution to the perturbation equations $\psi = \psi_{\mathrm{pert}}$ as given in (5.46) are used to fix initial conditions, where we find in the far pre- and post-bounce regime, respectively,

$$\psi_{\text{pre, pert}} = \frac{4\delta\mathcal{Y}}{3\,(v_0^2 - 4\mathcal{Y})} - \mathcal{C}_\psi\,, \qquad \psi_{\text{post, pert}} = \mathcal{C}_\psi\,. \tag{5.47}$$

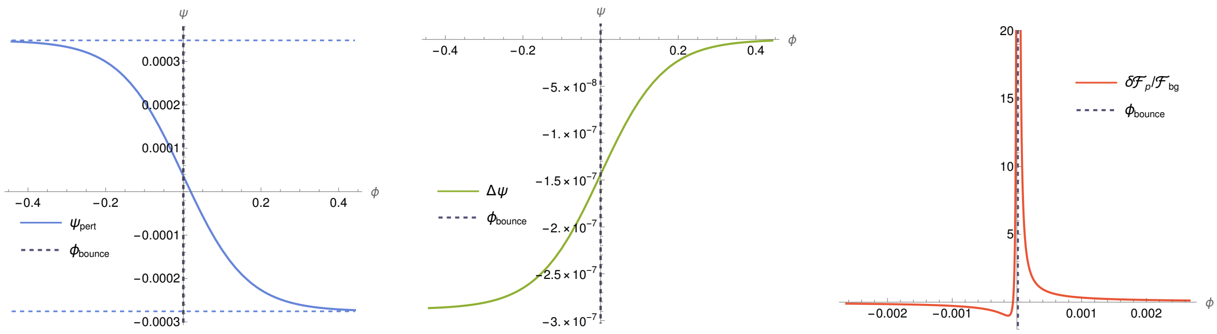
Setting the initial condition in the post-bounce regime from the exact values (5.36) obtained for a GFT ensemble as studied in sec. 5.2.2, we have $\mathcal{C}_\psi = \psi_{\text{post, exact}}$ and thus, inserting this into (5.47), we obtain

$$\psi_{\text{pre, pert}} = \frac{1}{3}\frac{A_p}{A_{bg}}\left(1 - \frac{B_p}{B_{bg}}\right) = -\frac{1}{3}\frac{\delta B_p}{B_{bg}}\left(1 + \frac{\delta A_p}{A_{bg}}\right) = \psi_{\text{pre, exact}} + \mathcal{O}(\epsilon^2)\,. \tag{5.48}$$

Thus, if perturbations are of order ϵ, the discrepancy to the exact solution is second order in perturbations and is negligible in linear perturbation theory; explicitly the mismatch is determined by the magnitude of $\delta A_p\,\delta B_p$.

In Fig. 5.2 we explicitly show the the difference between the exact evolution of $\mathcal{R}$ as obtained from (5.35) and the solution we obtain with the separate universe equations (5.46) for a patch of the exemplary ensemble studied above (see fig. 5.1). They are in excellent agreement as long as perturbations are small (which is a necessary prerequisite for the equations derived in sec. 5.1). The validity of perturbation theory is based on the relative smallness of perturbations, e.g. $\frac{\delta\mathcal{F}}{\mathcal{F}} \ll 1$, however, this condition breaks down at the bounce where $\mathcal{F} = 0$ but $\delta\mathcal{F} \neq 0$ (see fig. 5.2c). Nonetheless, for small initial values of ψ, the solution to the generalised perturbation equations (5.31) accurately traces the exact solution and thereby the non-trivial dynamics of $\mathcal{R}$ around the bounce. For large initial perturbations, this will no longer be the case, as is apparent from the comparison of asymptotic values given in (5.48).

We conclude with a final comment on a possible confusion that may arise when considering perturbations at early and late times, away from the bounce, where beyond-GR effects become negligible. We have established that the conservation of ζ on large scales is unchanged for adiabatic perturbations independent of the type of modification to the Friedmann equation. In the general relativistic regimes (far from the bounce) we expect $-\zeta = \mathcal{R}$, however, as we have shown the value of $\mathcal{R}$ can change through the bounce, leading to $-\zeta \neq \mathcal{R}$ in either the pre- or post-bounce regime. This apparent discrepancy arises from the non-conservation of D for general Friedmann equations where $\mathcal{F} \neq \mathcal{F}(\rho)$: while D is constant in each semi-classical regime, its value can change in the bounce phase, introducing a constant shift between the two quantities. (Recall that $D \propto \tilde{\Phi}$ (5.24) and



(a) Evolution of ψ_{pert} (5.46) in an exemplary patch. The horizontal dashed lines represent the asymptotic values of the solution (5.48).

(b) The difference between the exact solution (5.35) and the solution obtained from the perturbation equations (5.46), $\Delta\psi = \psi_{\text{exact}} - \psi_{\text{pert}}$.

(c) In the immediate vicinity of the bounce $\frac{\delta\mathcal{F}_p}{\mathcal{F}_{bg}}$ is large, indicating a breakdown of linear perturbation theory.

Figure 5.2.: ψ for a single patch of an ensemble with $n = 16$ (see also fig. 5.1) as given in (5.46) (ψ_{pert}) compared to the exact solution (5.35) (ψ_{exact}). The difference between the two solutions increases in the bounce region and asymptotically approaches a constant value, but remains small throughout. Initial conditions are set in the post-bounce regime at $\chi = 4$. The asymptotic values of ψ are given by (5.36) and (5.48). While the qualitative behaviour of the plots and the conclusions we draw in the main text are independent of the specific choice of initial conditions, we quote the numerical values of parameters in the solution of ψ_{pert} for reference: $A_{bg} = 200.226\,,\ B_{bg} = 200.262\,,\ \mathcal{Y} = -160390\,,\ \mathcal{C} = 800.903\,,\ \delta\mathcal{Y} = 35.0685\,,\ \mathcal{C}_\psi = 2.75576 \times 10^{-4}\,,\ v_0 = 1\,.$ The bounce time is $\chi_{\text{bounce}} = 1.47351 \times 10^{-5}$ and we set $\kappa = 8\pi$.

$-\zeta = \mathcal{R} + \frac{\tilde{\Phi}}{3}$ in comoving gauge (5.30)). The far pre- and post-bounce phases should therefore be seen as independent general relativistic regimes. Note that in the GFT example considered here, there exists a special case in which the pre- and post-bounce asymptotic value of $\mathcal{R}$ remains unchanged, despite ψ being dynamical around the bounce. This happens when the ratio $\frac{A_p}{B_p}$ is the same across all patches, such that all patches 'bounce' at the same value of χ.

5.3. Second order dynamics and the relation to sub-horizon dynamics

The results of the previous two sections can be summarised as follows: We derived an analogue to the diffeomorphism constraint for a specific class of modified Friedmann equations, where the modification depends on the energy density only. We showed that the relation $\mathcal{R} = -\zeta$ remains valid on super-horizon scales in this case, and that a single scalar field cannot induce non-adiabatic perturbations, leading to the conservation of ζ. We then turned to the more general case and considered a solution to the first order equation in ψ for a specific GFT model, which introduces dynamics in $\mathcal{R}$ around the bounce region and therefore leads to a violation of the conservation law. (For adiabatic perturbations, ζ remains conserved independent of the modification of the Friedmann equation, as long as the continuity equation holds.)

In the GFT example of sec. 5.2.2, the first order equation (5.5) in ψ could be solved directly and sufficed to determine the dynamics of $\mathcal{R}$. This was possible due to a specific choice of matter content that allowed to eliminate all perturbation variables but one. In more general cases, one can still combine perturbation equations to obtain second order equations in a single perturbation variable that can be solved directly. In this section we discuss two possible approaches to second order equations that can be found in the literature for the separate universe picture, namely, an equation for ψ (see e.g. [220]) as well as for the Mukhanov–Sasaki variable v as in [259]. In order to relate our results to some of the literature, we summarise the above-mentioned two second order approaches of main interest in standard cosmology and comment how they would apply to the more general types of cosmological dynamics we study in this chapter. Considering the second order equations for ψ in sec. 5.3.1 and for the Mukhanov–Sasaki variable v in sec. 5.3.2, we find that already in general relativity these two approaches lead to different dynamics for ζ in the strict separate universe limit. If one obtains its evolution from a second order equation in ψ , ζ remains constant, in agreement with our considerations in sec. 5.2. On the other hand, if one solves the long-wavelength limit of the Mukhanov–Sasaki equation, the solution for ζ has a constant and a dynamical part, where the latter is particularly important in the contracting branch. We comment how this can be understood as revealing the limits of the separate universe picture, which is fully agnostic about $k-$dependent dynamics. In sec. 5.3.1, we consider general $k-$dependent modifications to the continuity equation and first order equation of motion for ψ and derive the resulting $k-$dependence in the dynamical equations for ζ.

To simplify calculations as well as facilitate comparison with the literature, we work in conformal time ($N = a$ and we denote the Hubble parameter as $\frac{a'}{a} = \mathcal{H}$) and longitudinal gauge ($E = B = 0$ and $\psi = \tilde{\Phi}$, where we discussed the origin of the last relation in sec. 3.3.2). We work with adiabatic perturbations, but do not assume a constant equation of state parameter. In this gauge, the relevant linearised Einstein equations involving the perturbation variable ψ are (see, e.g., [85])

$$-k^2\psi - 3\mathcal{H}\left(\mathcal{H}\psi + \psi'\right) = \frac{\kappa}{2}a^2\delta\rho\,, \tag{5.49}$$

$$\psi\left(2\mathcal{H}' + \mathcal{H}^2\right) + 3\mathcal{H}\psi' + \psi'' = \frac{\kappa}{2}a^2\delta P\,. \tag{5.50}$$

5.3.1. Second order dynamics in ψ

To facilitate comparison with the literature, we first recap the general relativistic calculation, namely we solve the inhomogeneous second order GR equation for ψ and consider the resulting expression for ζ. Subsequently, we consider how general $k-$dependent terms could affect the dynamics of ζ.

In GR, for conformal time and longitudinal gauge, the combination of (5.49) and (5.50) yields the following second order equation of motion for the variable ψ

$$3\mathcal{H}^2\left(c_s^2 - w\right)\psi + 3\mathcal{H}(c_s^2+1)\psi' + \psi'' = -c_s^2k^2\psi\,, \tag{5.51}$$

assuming adiabatic perturbations, where $c_s^2 = \frac{\delta P}{\delta\rho} = \frac{P'}{\rho'}$ and we made use of the background equation $\mathcal{H}' = -\frac{1}{2}\mathcal{H}^2(1+3w)$. In the separate universe limit with $k \to 0$ the right hand side can be neglected and one obtains an equation equivalent to the combination of (5.5) and (5.7) in conformal time for $\mathcal{F} = 1$. Solutions will therefore agree with those found in the separate universe picture. An explicit solution to the separate universe dynamics reads $\psi(\eta) = \frac{\mathcal{H}}{a^2}\left(\frac{3}{2}C_1\int \mathrm{d}\eta\,(a^2\,(w+1)) + C_2\right)$ with C_1 and C_2 $k-$dependent constants (to confirm that this indeed satisfies the above, use the background equations $\mathcal{H}'$ and $\mathcal{H}''$ as well as the continuity equation (1.34); see, e.g., [220, 260]).

To find an explicit form of ζ from this solution, note that, using the $k-$dependent expression for the perturbed energy density (5.49) and continuity equation (5.6), ζ (5.11) can be rewritten as

$$-\zeta = \frac{2k^2}{9\mathcal{H}^2(w+1)}\psi + \frac{3w+5}{3(w+1)}\psi + \frac{2\psi'}{3\mathcal{H}(w+1)} \tag{5.52}$$

and it follows from (5.51) and the background equations for $\mathcal{H}'$ and w', that

$$-\zeta' = \frac{2\,k^2\left(\mathcal{H}\psi + \psi'\right)}{9\,\mathcal{H}^2(w+1)}\,, \tag{5.53}$$

resulting directly in $\zeta' = 0$ for long-wavelength perturbations $k \to 0$. Explicitly, with the solution of ψ one obtains

$$-\zeta(\eta) = C_1 - k^2 \frac{2C_2 + 3C_1 \int \mathrm{d}\eta \, (a^2(w+1))}{9a^2\mathcal{H}(w+1)} \tag{5.54}$$

which is constant in the separate universe limit as the dynamical part of the solution can be neglected.

Generalised inhomogeneous equations

We now investigate how $k-$dependent modifications affect the dynamics of ζ following the same procedure as above. In GR, we obtained a second order equation for ψ by combining the derivative of the first order equation with the perturbed continuity equation. If we want to consider possible effects of inhomogeneous terms, both equations are required to have inhomogeneous corrections in order to ensure a consistent GR limit. This is a manifestation of the fact that changes to the gravitational dynamics must result in changes to the matter dynamics. We make the following simple Ansatz on how inhomogeneities enter the above-mentioned equations:

$$\mathcal{H}\psi' = -\mathcal{H}^2 \left(\psi + \frac{\delta\rho}{2\rho} + \frac{\delta\mathcal{F}}{2\mathcal{F}} \right) + G_k \,, \tag{5.55}$$

$$\delta\rho' = 3\psi'(\rho + P) - 3\mathcal{H}(\delta\rho + \delta P) + Z_k \,, \tag{5.56}$$

where Z_k and G_k are general functions that reduce to $G_k \to -\frac{k^2}{3}\psi$ and $Z_k \to -\frac{2k^2}{\kappa a^2}(\psi' + \mathcal{H}\psi)$ in the classical limit. Furthermore, we impose that in the separate universe limit (small k) we have $Z_k \to 0$ and $G_k \to 0$; this is required to ensure consistency with previous results (i.e. the general perturbed equations of motion (5.5), (5.7) and the perturbed continuity equation (5.6)).

With these general corrections, we obtain the following modified second order perturbation equation

$$3\mathcal{F}\mathcal{H}^2(c_s^2 - w)\psi + \left(3\mathcal{F}\mathcal{H}(c_s^2+1) - \frac{\mathcal{F}'}{2} \right)\psi' + \mathcal{F}\psi'' + \left(3\mathcal{H}^2(c_s^2 - w) - \mathcal{H}\frac{\mathcal{F}'}{\mathcal{F}} \right)\frac{\delta\mathcal{F}}{2} + \mathcal{H}\frac{\delta\mathcal{F}'}{2}$$
$$= -\frac{\kappa a^2 \mathcal{F}^2 Z_k}{6\mathcal{H}} + G_k \left((3c_s^2+1)\mathcal{F} - \frac{\mathcal{F}'}{\mathcal{H}} \right) + \frac{G_k'}{\mathcal{H}}\mathcal{F} \,. \tag{5.57}$$

This is derived from the derivative of (5.55) by inserting (5.56), the continuity equation and replacing $\delta\rho$ as given by (5.55), as well as making use of the background equation $\mathcal{H}' = -\frac{1}{2}\left(\mathcal{H}^2(1+3w) - \mathcal{H}\frac{\mathcal{F}'}{\mathcal{F}}\right)$ and the modified Friedmann equation. A consistency check reveals that this reduces to (5.51) in the classical limit ($\mathcal{F} \to 1$, $\delta\mathcal{F} \to 0$, $\mathcal{F}' \to 0, \delta\mathcal{F}' \to 0$ and G_k, Z_k as given above). Note that (5.57) (and its long-wavelength limit) can depend on other perturbations variables than ψ through the explicit form of $\delta\mathcal{F}$ and therefore need not yield an explicit solution for ψ.

We again compute an expression for ζ by replacing $\delta\rho$ from (5.55) and inserting the continuity and Friedmann equation in the definition of ζ (5.11)

$$-\zeta = -\frac{2G_k}{3\mathcal{H}^2(w+1)} + \frac{(3w+5)\psi}{3(w+1)} + \frac{2\psi'}{3\mathcal{H}(w+1)} + \frac{\delta\mathcal{F}}{3\mathcal{F}(w+1)} . \tag{5.58}$$

Its derivative reads (replace $\delta\rho'$ from (5.56), $\delta\rho$ from (5.55) and use the background equations)

$$-\zeta' = -\frac{Z_k}{3(1+w)\rho} = -\frac{\kappa a^2 \mathcal{F} Z_k}{9\mathcal{H}^2(1+w)} , \tag{5.59}$$

which reduces to (5.53) in the classical limit and vanishes when spatial gradients can be neglected. Hence, we see that the dynamics of ζ are governed by the $k-$dependence of the continuity equation, as in GR. Furthermore, in agreement with the strict separate universe picture employed in previous sections, the above leads to a conservation of ζ for $Z_k \to 0$.

5.3.2. Limitations of the separate universe picture

As explained in sec. 3.1, the Mukhanov–Sasaki variable is a commonly used variable in cosmological perturbation theory, as its action takes a convenient form for quantisation. Specifically, it behaves like a scalar field in an expanding background, thus permitting quantisation as a canonical scalar field (see, e.g., [261]). Here we focus on the dynamics of ζ as obtained from the Mukhanov–Sasaki variable and comment on the related limits of the separate universe picture.

First, consider the form of the Mukhanov–Sasaki variable as introduced in sec. 3.2.2 for scalar matter content, $v = a(\delta\chi + \frac{\chi'}{\mathcal{H}}\psi) = z\mathcal{R}$, with $z = a\frac{\chi'}{\mathcal{H}}$ (in GR, it follows also that

$v = -z\zeta$ for adiabatic perturbations on super-horizon scales), and its equation of motion, the so-called Mukhanov–Sasaki equation, in Fourier space [85]

$$v'' + c_s^2 k^2 v - \frac{z''}{z} v = 0\,. \tag{5.60}$$

The above can be derived from the matter and gravity actions (rewritten in terms of v), or, in the separate universe picture, from algebraic manipulations of the homogenous perturbations equations, as was done in [221] for LQC. In either case, the derivation makes use of the constraint equation $D = H\tilde{\Phi} + \psi' - \frac{\kappa}{2}\chi'\delta\chi = 0$ (or its modified version (5.22)). As discussed in sec. 5.2.1 this comes from inhomogeneous terms in the Einstein equations and is therefore not available in the strict separate universe limit. As we showed in sec. 5.2.1, an alternative form can be derived for modified Friedmann equations of the form $\mathcal{F} = \mathcal{F}(\rho)$, but not for the general case as e.g. in sec. 5.2.2. It is then only for the former that one has the chance of recovering a long-wavelength analogue to (5.60).

Solving the long-wavelength limit ($k \to 0$) of (5.60), which is derived for GR without modifications, together with $v = -z\mathcal{R}$ leads to the following solution for $\mathcal{R}$ [85, 259]

$$\mathcal{R} = V + S \int \frac{d\eta}{z^2}\,, \tag{5.61}$$

where V and S are ($k-$dependent) constants and η denotes conformal time. In the super-horizon limit and in the case of adiabatic perturbations we have $\mathcal{R} = -\zeta$ and therefore the above solution gives the dynamics for ζ as well. Unlike (5.53), the dynamical part of the solution does not vanish for long-wavelength perturbations; instead, ζ is dynamical even on large scales. Note that this dynamical part is negligible in an expanding phase as e.g. in standard inflation, but can dominate in a contracting phase, as is pointed out in [221, 259]. Specifically, ζ' can increase as one approaches the bounce (i.e. $-\eta \to 0$) in cases where the dynamics take on the form $\zeta' \sim k^2(-\eta)^{-|p|}$ ($p \in \mathbb{R}$). In these cases, k has to remain sufficiently small ($k \ll \mathcal{H}$) for the separate universe picture to be applicable; however, at the bounce point where $\mathcal{H} = 0$ the separate universe limit cannot be consistently applied. The dynamics in (5.61) arise because the long-wavelength limit is imposed on the Mukhanov–Sasaki equation (5.60), instead of the dynamical equation for ζ (5.53). This amounts to neglecting $k-$dependent terms in (5.60), but not in (5.53); indeed the long-wavelength limit of (5.53) would impose $S = 0$, thus recovering a constant solution. A solution such as (5.61) is therefore obtained by neglecting $k-$dependent terms in the second order, but not in the first order equation, thus acknowledging that outside of the strict separate universe picture there are of course dynamics in ζ for small (but non-zero) k values. Hence, deriving the Mukhanov–Sasaki equation (and its possible modifications) requires knowledge of inhomogeneous dynamics, which are not available in the separate universe picture in general. It then becomes apparent that for a full treatment one would need to understand the finite theory: it seems necessary to verify any statements made about the dynamics of perturbations in the separate universe limit around the bounce region against the full dynamics including gradient terms. Furthermore, note that from $\mathcal{R}' = 0$ (which is the case in the separate universe limit for a range of modified Friedmann equations as we have shown in sec. 5.2.1) it immediately follows that the Mukhanov–Sasaki equation (5.60), which can be rewritten as $(z\mathcal{R}')' + z'\mathcal{R}' = 0$, is trivially satisfied, irrespective of the choice of z.

The LQC case differs from the general considerations we are concerned with here: In LQC there exists an effective Hamiltonian that allows one to study perturbations also outside the strict $k \to 0$ limit. Taking into account inhomogeneous contributions, one can then derive an analogue to the Mukhanov–Sasaki equation as was done in [258]. The authors find that the LQC modifications impact only the k^2-term, such that the Mukhanov–Sasaki equations remains unchanged for long-wavelength perturbations (while the background dynamics are LQC-corrected). Moreover, it is possible to algebraically derive a Mukhanov–Sasaki like equation making use of a modified diffeomorphism constraint, as was done in [221]. In the model-agnostic approach with only a modified Friedmann equation, it is uncertain whether an algebraic derivation of an equation like (5.60) is possible in absence of a diffeomorphism constraint. It then follows that the applicability of (5.60) to a scenario with a modified Friedmann equation is far from clear and a similar equation needs to be established from the full dynamics for a specific model in question to make further progress on this matter. We saw that, already in GR, the dynamics of the strict separate universe picture are not necessarily identical with the dynamics arising in the separate universe limit of equations of motion that are derived using non-homogeneous dynamics. In this sense, the applicability of the separate universe picture has its limits. One has to assume that the wavelength of perturbations is always large enough (with respect to the Hubble horizon) for it to hold, however, as one approaches the bounce, at which the Hubble horizon diverges, this assumption breaks down.

Concluding the considerations of the above, we see that second order equations as are used in standard cosmology do not provide additional insight into the evolution of gauge-invariant perturbation variables for general theories with a modified Friedmann equation when the full (model-dependent) dynamics are unknown. They may nonetheless be useful for specific theories where additional information is available.

5.4. Conclusion: To evolve or not to evolve

In this chapter we examined the effects of alterations to the Friedmann equation on long-wavelength scalar perturbations. Modifications to the Friedmann equation may arise from various alterations to gravity, including quantum gravitational theories, such as LQC or GFT cosmology.

Our approach is model agnostic, such that the results have a broad range of applicability. To this extent, we worked in the separate universe framework, where one considers the universe as a collection of homogeneous independent patches, and perturbations are understood as the deviations between a variable's value in the patch and its average over the ensemble of patches. They are thus homogeneous *in each patch* and all spatial gradients vanish. We make two assumptions that would need to be checked in scenarios where the underlying theory allows: Firstly, we posit an unaltered continuity equation; secondly, we import gauge-invariant variables from general relativity and therefore require that the modified gravitational dynamics do not affect the notion of gauge-invariance.

We focus on the two gauge-invariant variables ζ and $\mathcal{R}$, which are commonly found in the cosmological literature due to their relation to the CMB power spectrum, and due to the second assumption stated above retain their physical meaning. From the continuity equation it directly follows that ζ is conserved on super-horizon scales as long as perturbations are adiabatic, which is a statement that carries over from general relativity.

The equations of motion for perturbations in the separate universe picture are obtained by perturbing the modified Friedmann equation at linear order. We derive these equations in general, however, when considering the two classes of possible Friedmann equations in detail, we focus on scenarios where the matter content is given by a single massless scalar field, as can be found in LQC [3] and GFT cosmology.

We could extend results obtained for scalar perturbations in the separate universe regime in LQC [221] to a wider class of modified Friedmann equations for which one can

[3]At least for the quantum theory; in effective studies of the phenomenological implications of LQC one often assumes a scalar field potential that leads to an inflationary phase, see e.g. [254].

recover the conservation laws of general relativity for the gauge-invariant variables $\mathcal{R}$ and ζ. Specifically, we found that as long as the modification to the Friedmann equation is a function of the energy density only, $\mathcal{F} = \mathcal{F}(\rho)$, such that $\delta\mathcal{F} \propto \delta\rho$, one can demonstrate that an analogue of the diffeomorphism constraint holds, which is necessary to obtain the general relativistic conservation laws. With an analogue to the diffeomorphism constraint, which in GR follows from the time-space components of the Einstein equations, we showed that ζ is conserved for a single scalar field, irrespective of the scalar field potential. In other words, a single scalar field can only induce adiabatic perturbations. One furthermore recovers $\mathcal{R} = -\zeta$ on super-horizon scales.

More general modified Friedmann equations, on the other hand, can induce dynamics in the comoving curvature perturbation $\mathcal{R}$. An example for this class of models is a Friedmann equation obtained from GFT, which we study in further detail to illustrate the dynamics of $\mathcal{R}$ across the bounce. We furthermore show that the difference between the evolution obtained from the perturbed equations of motion to that retrieved from the expectation values of the GFT volume operator directly is second order in perturbation theory.

Lastly, we establish whether second order equations as are commonly used in the literature, can give further insights. Neither of the approaches leads to further model-independent insights about the evolution of perturbations. We discuss that already in general relativity, a solution to the second order equation in ψ leads to different dynamics for ζ than the Mukhanov–Sasaki equation in the separate universe framework: the former results in a conservation law, whereas the latter gives a dynamical ζ. We conclude that this discrepancy can be seen as a limitation of the separate universe framework. For a full treatment of perturbative dynamics around the bounce, knowledge of the full dynamics is necessary, and it indeed seems that any dynamics obtained in the separate universe picture would have to be verified against the inhomogeneous dynamics. Another way to put this is as follows: Separate universe dynamics for modified Friedmann equations show where dynamics might be introduced to otherwise constant quantities; one cannot necessarily assume that constants in the separate universe picture remain constants on large, but inhomogeneous scales, as can be the case for ζ in a contracting branch.

While our results are limited to the separate universe assumption, the dynamics in $\mathcal{R}$ on super-horizon scales demonstrate the possibility of crucial departures from standard cosmology for general modified Friedmann equations. On the other hand, our results show that this is of no concern for LQC-like scenarios. To overcome the limitations of the separate universe picture, details of the theory need to be known that allow to establish specific dynamics for a certain model. In the following chapter we make a proposition that allows us to extract inhomogeneous dynamics in GFT, and finally, in chap. 7 we investigate the resulting dynamics for cosmological perturbations. We relate those findings to the framework presented in this chapter in sec. 7.5.

As a final remark, we would like to comment on an ambiguity in the definition of ζ: Here we assumed an unaltered definition of ζ also in the non-general relativistic regime. It would instead be possible to take the viewpoint of [220] and interpret our modified Friedmann equation (5.1) as a modification of the energy density $\rho_{\text{eff}} = \rho\mathcal{F}$, which would carry over into the definition of curvature perturbation on uniform density hypersurfaces ζ_{eff}. This would result in a modified continuity equation $\rho'_{\text{eff}} = -3H(\rho + P)\mathcal{F} + \rho\mathcal{F}'$ and thus affect the dynamics of ζ. This highlights the degeneracy of the Friedmann equation, which does not permit conclusions on whether its alteration arises from the matter or gravitational sector; instead, this input must be given by the theory that induces the modification in the first place. In the case of LQC and GFT, it is indeed the gravitational sector that deviates from general relativity.

Chapter 6.

Introducing an effective metric in group field theory

'[...] fascination with symmetries and groups is one way of coping with frustrations of life's limitations [...] '

- Pierre de la Harpe in 'Topics in Geometric Group Theory'.

'[...] Faszination mit Symmetrien und Gruppen is ein Weg mit den Frustrationen und Limitationen des Lebens umzugehen [...] '

The results of this chapter are contained in [2] and were published in *Classical and Quantum Gravity* in July 2024.

In the previous chapter we explored a model independent approach to studying the impact of modified gravitational dynamics on cosmological perturbations. Due to the generality of the approach we were limited to studying super-horizon perturbations in the separate universe picture, as this does not require any information beyond the modified Friedmann equation. While it is possible to make some general statements, we saw that knowledge of sub-horizon dynamics is required to gain more complete insights into perturbative dynamics. As hinted at by the title of this thesis, our overarching goal is then to investigate cosmological perturbations within the framework of GFT cosmology. As detailed in sec. 4.1, there exist studies into perturbations in the GFT framework from the separate universe perspective, and recently research efforts have focused on studying inhomogeneities in the quantum framework using a relational coordinate system spanned by four massless scalar fields. We will adopt this four field setup, where one field serves as a clock whereas the others pose as 'spatial rods'. This allows us to introduce inhomogeneous quantities in GFT, thus generating a framework in which perturbations can be studied. The main idea and construction of GFT, including the relational setup with four scalar fields, is contained in chap. 2. The details of GFT cosmology as outlined in sec. 4.1 are not important for this chapter, save for illustrating the conventional GFT route to establishing an effective quantity that can be related to general relativity: The GFT operator of interest has been the volume operator imported from LQG and in cosmology, its expectation value over suitable semiclassical states[1] is identified with the scale factor of the universe, $\langle V \rangle \propto a^3$.

In this chapter, we will introduce a new proposal for obtaining general relativistic quantities from GFT. We define novel GFT operators using symmetries of the GFT action and show how these operators can be used to reconstruct an effective metric directly from the quantum theory. Specifically, these new operators arise from the same symmetries as classically conserved currents that are directly related to the metric components in the relational coordinate system we are forced to employ in GFT. The expectation values of said operators, that form the components of a GFT energy-momentum tensor, then relate to the components of an effective metric, which replaces the identification of the volume operator with the spatial volume element of previous GFT works. As the metric forms the central object of general relativity, the possibility of reconstructing an effective metric directly from the quantum theory opens up the pathway to studying quantum corrections of several quantities of interest and making contact with observations.

Here, we focus solely on the construction of these operators and the reasoning behind an identification with an effective metric. This construction is completely general and could potentially be applied to a variety of spacetimes. As a first application to evaluate the usefulness of our proposal, we study the effective GFT metric for a universe described by a perturbed flat FLRW metric in the subsequent chapter, where we extract dynamics of the scale factor and consider scalar perturbations. Exploring the consequences and validity of the proposal detailed in this chapter beyond cosmology is left for future work.

This chapter is organised as follows: We first establish the classical Noether currents that we find within GR for a spacetime with four massless scalar fields in sec. 6.1. We consider their form in the relational coordinate system spanned by these four massless scalar fields and notice that in this specific coordinate system, their components are directly

[1] What constitutes a 'suitable semiclassical state' is detailed in sec. 4.1.

related to the inverse metric. We then turn to GFT and use the symmetry of the GFT action under translations of the scalar fields to define a GFT energy–momentum tensor in sec. 6.2.1. We propose its relation to the classical Noether currents and establish the components of the energy momentum tensor in terms of GFT operators, where we discuss normal ordering and include their explicit dynamics. In sec. 6.2.2 we show explicitly that the conservation law for GFT energy momentum tensor remains valid after quantisation. We conclude in sec. 6.3.

6.1. Effective metric from conserved currents

In this section, we will not be interested in GFT (yet), but focus solely on classical conservation laws that follow from symmetries of the scalar field action. We will consider the Noether currents associated with the shift symmetry of massless scalar fields in GR and consider their form in a relational coordinate system such as the one utilised to study inhomogeneities in GFT. For this choice of coordinate system, the classically conserved currents are directly related to the inverse metric.

Recall that the standard action for a free massless scalar field χ on a curved background with Lorentzian metric $g_{\mu\nu}$ is given by

$$S_\chi = -\frac{1}{2}\int \mathrm{d}^4x \, \sqrt{-g}\, g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi \,. \tag{6.1}$$

This action is invariant under constant shifts in the field $\chi \mapsto \chi + \epsilon$ ($\epsilon \in \mathbb{R}$) and by virtue of Noether's theorem [262, 263] thereby implies the existence of a conserved current j^μ that satisfies a conservation law of the form

$$\partial_\mu j^\mu = 0\,, \quad j^\mu = -\sqrt{-g}\, g^{\mu\nu}\partial_\nu\chi \,. \tag{6.2}$$

Indeed, this conservation law gives the familiar Klein-Gordon equation $\Box\chi = 0$ and the canonical momentum π_χ of χ is given by the time component of this current, $\pi_\chi = -\sqrt{-g}\, g^{0\nu}\partial_\nu\chi = j^0$. For a *diagonal* metric we have $\pi_\chi = \sqrt{|q|}/|N|\partial_0\chi$, where $|q|$ denotes the determinant of the spatial metric and N the lapse. (We included the absolute values for the lapse to emphasise that the sign of the lapse function has no physical relevance and is a convention, in the following we will simply write $N = |N|$, as we implicitly did in earlier chapters.) It follows that $\mathrm{sgn}(\pi_\chi) = \mathrm{sgn}(\partial_0\chi)$. In a coordinate system, where $\partial_0\chi > 1$, the scalar field momentum, and equivalently j^0, is therefore always positive.

In the spirit of relational coordinate systems as were discussed in sec. 1.1.4 and sec. 3.4.2, we consider the case of four massless scalar fields χ^A, (with $A = 0, \ldots, 3$), each described by an action of the form (6.1), and use them to span a relational coordinate system. Each field is then identified with its respective coordinate x^A such that $\partial_\mu\chi^A = \delta^A_\mu$ and we impose the non-degeneracy condition (3.48), in order for the relational coordinate system to be locally well-defined. We also remind the reader of the relation between the scalar field momenta and the lapse and shift for such a choice of coordinates (3.49).

There then exists a conserved current for each of the relational fields, which, in the relational coordinate system, takes on the following form (cf. (6.2)):

$$(j^\mu)^A = -\sqrt{-g}\, g^{\mu A}\,. \tag{6.3}$$

In the relational coordinate system, gradients of the scalar fields are dimensionless, hence the metric components have units of length^4, $[g_{AB}] = L^4$, and for the conserved current we find $[(j^\mu)^A] = L^4$. We could introduce an arbitrary dimensionful parameter ξ when fixing the relational coordinate system, such that $\partial_\mu\chi^A = \xi\delta^A_\mu$, but as this resembles a convention, we employ the simplest choice $\xi = 1$. From (6.3), it is apparent that in the relational coordinate system, the components of each conserved current corresponding to one of the relational fields are directly related to the respective inverse metric components.

We can then define a symmetric matrix field $j^{AB} := \left(j^B\right)^A$ $(B = 0, \ldots, 3)$ and invert (6.3) to define the inverse metric components in terms of the conserved currents

$$j^{AB} = -\sqrt{-g}g^{AB} \qquad \Rightarrow \qquad g^{AB} = -\left(-\det\left(j^{AB}\right)\right)^{-1/2} j^{AB}\,. \tag{6.4}$$

In short, knowing the conserved currents of all four massless scalar fields allows us to reconstruct the (inverse) metric *in the relational coordinate system*. Even though one of the indices on the right hand side of (6.4) is a mere label whose positioning is determined by convention, as long as there is no change of coordinate system and with the above definition of j^{AB}, we can intuitively think of A, B as contravariant indices. Due to its direct relation to the metric, j^{AB} for the scalar field action given in (6.1) is symmetric in its two indices. (For more general scalar field actions, this might change and the construction would not work as proposed here; we discuss this further in sec. 7.6.)

The relational coordinate system defined by the condition $\partial_\mu\chi^A = \delta^A_\mu$ represents a local gauge-fixing within the diffeomorphism-invariant theory of general relativity. It can be seen as a specific case of the harmonic gauge condition $\Box x^\mu = 0$, which was already discussed in sec. 3.4. As also discussed in sec. 3.4, unlike the harmonic gauge condition, working in a relational coordinate system as we do here, which one might want to refer to as 'scalar field gauge', fixes the gauge completely (presuming (3.48)), such that there is no residual gauge freedom. It does not define which directions should be seen as time-like or space-like, whereas the GFT construction of the next section requires to single out a clock field.

At this point we include a small discussion of signs that appear in the definitions above. The minus sign inside the bracket in (6.4) originates from having a negative metric determinant (and thus $\det\left(j^{AB}\right) < 0$), i.e. working with a Lorentzian signature. Adopting the convention where the metric $g_{\mu\nu}$ has one negative and three positive eigenvalues (the 'East Coast' signature convention) as we do here, leads to the overall minus sign of the scalar field action (6.1), which propagates as an overall minus sign in (6.4). The alternative 'West Coast' signature convention would, in addition to flipping the signs of metric components, also introduce an additional minus sign in (6.3), leaving the signature of $(j^\mu)^A$ unchanged (namely, $(+---)$). When the components of j^{AB} are obtained from GFT operator expectation values, as we propose below, it is a priori unclear whether one can fix the metric signature. The results for and FLRW background metric described in sec. 7.3 illuminate this more clearly.

The important insight of this section is the established link between the conserved currents of the scalar fields that span the relational coordinate system and the inverse metric components, (6.3). It then follows that in a quantum theory where one can establish operator analogues to the components of j^{AB}, their expectation values over suitable semi-classical states can be used to reconstruct an effective metric through (6.4). Here one should clearly distinguish between the notion of an effective metric *emerging* from the *expectation values* of such operators and explicit *metric operators*. The effective metric is calculated from expectation values of operators corresponding to the conserved currents via (6.4) and it is not clear how an operator for g^{AB} could be defined directly from (6.4) due to the occurrence of the square root; questions about eigenvalues of the metric or similar are then undefined. We propose instead that for suitable states the expectation values of the conserved currents give information about the classical geometry and the metric arises in the semi-classical regime. Hence, we refer to the metric as a purely effective quantity. A prerequisite for this construction is that the symmetry of j^{AB} is respected by their operator analogues. In what follows we will explicitly construct such operators for the free GFT action in the form of a GFT energy-momentum tensor.

6.2. GFT energy-momentum tensor

In this section we define the GFT energy-momentum tensor and its corresponding operators. As detailed in sec. 2.2.1, dynamics can be introduced to GFT by coupling a single massless scalar field that serves as a relational clock. Similarly, one can couple additional scalar fields to serve as spatial rods to the group field, such that altogether the fields span a four-dimensional relational coordinate system. The symmetries of the scalar field action can then be used to constrain the kinetic and interaction term by imposing that the symmetries of the classical action are preserved in GFT. We use the Hamiltonian construction of GFT with multiple massless scalar fields introduced in [78] and detailed in sec. 2.2.3, where we have $\varphi : \mathrm{SU}(2)^4 \times \mathbb{R}^4 \to \mathbb{R}$. We work with a Peter-Weyl decomposition of φ, where our results depend only on the existence of such a decomposition and are independent of the details of SU(2). They could therefore be applied for alternative choices of a compact group. We assume a GFT action of the form (2.31)

$$S[\varphi] = \int d^4\chi \; \mathcal{L} \,, \qquad \mathcal{L} = \sum_J \left(\frac{1}{2}\mathcal{K}_J^{(0)}\varphi_J^2 - \frac{1}{2}\mathcal{K}_J^{(2)} \sum_A (\partial_A \varphi_J)^2 \right) , \tag{6.5}$$

where we have neglected interactions, i.e. $V(\varphi) = 0$. Each Peter-Weyl mode of the group field is a function of all four scalar fields $\varphi_J = \varphi_J(\chi^0, \chi^1, \chi^2, \chi^3)$ and the partial derivative is taken w.r.t. the scalar fields $\partial_A := \frac{\partial}{\partial \chi^A}$. (Note that to obtain the form of the Lagrangian (6.5) we performed an integration by parts and neglected the boundary terms.)

After defining the GFT energy-momentum tensor from the Lagrangian and carrying out its quantisation, we use the solutions to the Heisenberg equations of motion to obtain explicit dynamical expressions for the operators of interest and discuss an appropriate normal ordering procedure. We show explicitly that the conservation law of the GFT energy-momentum tensor carries over to the quantised theory. For further details of the Hamiltonian framework of GFT with four massless scalar fields, including the Fourier decomposition and solutions to the operator dynamics we use below, we refer the reader to sec. 2.2.3.

6.2.1. Definition

The classical scalar field action (6.1) can be used to restrict the form of the kinetic kernel by demanding that the GFT action exhibits the same symmetries. In particular, the GFT action is invariant under translations of any of the scalar fields $\chi^A \mapsto \chi^A + \epsilon^A$ for arbitrary constant ϵ^A. This symmetry leads to a conserved current ([25, 262, 263]), namely the GFT energy-momentum tensor

$$T^{AB} := -\frac{\partial \mathcal{L}}{\partial(\partial_A \varphi)} \partial_B \varphi + \delta^{AB} \mathcal{L} = \sum_J \left(\mathcal{K}_J^{(2)} \partial_A \varphi_J \, \partial_B \varphi_J \right) + \delta^{AB} \mathcal{L} \tag{6.6}$$

with the Lagrangian density given in (6.5). By construction, the energy-momentum tensor satisfies the conservation law $\partial_A T^{AB} = 0$. As T^{AB} and j^{AB} both represent the conserved current arising from the shift symmetry of the scalar fields, it appears to be a sensible proposition to identify these two quantities with one another. As we have seen in sec. 6.1, this directly implies that the GFT energy-momentum tensor is related to the inverse metric through (6.4) and we thus obtain the desired GFT quantity required to complete the construction of an effective metric by quantising T^{AB} (see below). We would like to stress again that we are not proposing that the novel operators we obtain upon quantising the T^{AB} should be interpreted as "metric operators" of any form; it is j^{AB} whose operator analogue we construct in this section. Indeed, from (6.4), we see that the inverse metric components are related to j^{AB} in a non-polynomial form, making the interpretation of g^{AB} as an operator unclear. Instead, it should be understood that an effective metric can be obtained from the expectation values of j^{AB} over semiclassical states, to be interpreted as a macroscopic effective geometry emerging in the semi-classical regime. This implies that quantum fluctuations are sufficiently small for this interpretation to be viable, see also sec. 4.1 for a discussion of suitable semiclassical states in GFT. The emergence of an effectively classical geometry is then similar to the reconstruction of an effective scale factor from the expectation value of the volume operator in GFT cosmology, $a^3 = \langle V \rangle$. A final note on the expression of T^{AB}: Due to the $E(4)$ symmetry of the GFT action the positioning of the A, B indices does not matter at this point; they are raised and lowered with the Kronecker delta δ_{AB}. This is of course not the case for j^{AB} related to the inverse metric as defined in the previous section, which needs to be kept in mind to enable a sensible identification of the two quantities.

The remainder of this section is dedicated to promoting the components of the GFT energy-momentum tensor T^{AB} to operators. The procedure goes as follows: We first write all components as defined by (6.6) in terms of the GFT field modes φ_J and their canonical momenta π_J and carry out a Fourier decomposition over the spatial fields. The resulting expressions can be quantised by promoting φ_J and π_J to operators, which are finally replaced with ladder operators $A_J, A_J^\dagger$ and a normal ordering is imposed. We show that the conservation law for T^{AB} holds also at operator level in sec. 6.2.2. To ease the notational load we restrict to a single J-mode and omit the $J-$label in the following. In GFT cosmology, one often limits oneself to the study of a single mode, see e.g. [190], still, phenomenologically interesting effects of considering multiple modes exist [147, 148, 240] (for details, please refer to sec. 4.1.5). Extending the results to the multimode scenario is straightforward, as we saw in sec. 2.2 that different J-modes evolve independently and the energy-momentum tensor is simply a sum over the various modes $T^{AB} = \sum_J T_J^{AB}$.

In a first step we insert the expression for the canonical momentum $\pi_J = -\mathcal{K}_J^{(2)} \partial_0 \varphi_J$ to obtain explicit expressions for the energy-momentum tensor, which depend on the clock as well as the spatial fields $T^{AB} = T^{AB}(\chi^0, \vec{\chi})$:

$$
\begin{aligned}
T^{00} &= \frac{\pi^2}{2\mathcal{K}^{(2)}} - \frac{\mathcal{K}^{(2)}}{2}\left(m^2\varphi^2 + \sum_b (\partial_b\varphi)^2\right) , \\
T^{0b} &= -\pi\partial_b\varphi , \qquad T^{a\neq b} = \mathcal{K}^{(2)}\partial_a\varphi\partial_b\varphi , \\
T^{aa} &= -\frac{\pi^2}{2\mathcal{K}^{(2)}} - \frac{\mathcal{K}^{(2)}}{2}\left(m^2\varphi^2 - (\partial_a\varphi)^2 + \sum_{b\neq a}(\partial_b\varphi)^2\right) \quad \text{(no sum over } a) .
\end{aligned}
\tag{6.7}
$$

As explained in sec. 2.2.3, before quantisation, we need to perform a Fourier transform of the energy-momentum tensor $T_k^{AB} = T_k^{AB}(\chi^0)$. From (6.7), this results in the following combination of convolutions, where $\varphi_k = \varphi_k(\chi^0)$ and $\pi_k = \pi_k(\chi^0)$ denote the Fourier transforms of $\varphi(\chi^0, \vec{\chi})$ and $\pi(\chi^0, \vec{\chi})$, respectively:

$$
\begin{aligned}
T_k^{00} &= \frac{1}{2}\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\Big[\frac{\pi_\gamma\pi_{k-\gamma}}{\mathcal{K}^{(2)}} - \mathcal{K}^{(2)}\left(m^2 - \vec{\gamma}\cdot(\vec{k}-\vec{\gamma})\right)\varphi_\gamma\varphi_{k-\gamma}\Big] , \\
T_k^{0b} &= -\mathrm{i}\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\gamma_b\pi_{k-\gamma}\varphi_\gamma , \qquad T_k^{a\neq b} = -\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\mathcal{K}^{(2)}\gamma_a(k_b-\gamma_b)\varphi_\gamma\varphi_{k-\gamma} , \\
T_k^{aa} &= \frac{1}{2}\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\Big[\mathcal{K}^{(2)}\left(-\gamma_a(k_a-\gamma_a) + \sum_{b\neq a}\gamma_b(k_b-\gamma_b) - m^2\right)\varphi_\gamma\varphi_{k-\gamma} - \frac{\pi_\gamma\pi_{k-\gamma}}{\mathcal{K}^{(2)}}\Big] .
\end{aligned}
\tag{6.8}
$$

We can now proceed to quantise the above expressions by promoting the GFT field and its momentum to operators that satisfy

$$[\varphi_{J,k}(\chi^0),\pi_{J',k'}(\chi^0)] = \mathrm{i}\,\delta_{JJ'}(2\pi)^3\delta(\vec{k}+\vec{k'})\,. \tag{6.9}$$

We will denote the operator equivalent to T^{AB} as $\mathcal{T}^{AB}$. As detailed in sec. 2.2.3, it is convenient to introduce (time dependent) ladder operators $A_{J,k}(\chi^0)$, $A^\dagger_{J,k}(\chi^0)$, that are related to the time independent operators via $a_{J,k} = A_{J,k}(0)$ and $a^\dagger_{J,k} = A^\dagger_{J,k}(0)$ and satisfy

$$[A_{J,k}(\chi^0), A^\dagger_{J',k'}(\chi^0)] = \delta_{JJ'}(2\pi)^3\delta(\vec{k}-\vec{k'})\,. \tag{6.10}$$

To avoid divergences we impose normal ordering on the level of time independent operators $a_{J,k}$, $a^\dagger_{J,k}$. Normal ordering in terms of the time dependent operators $A_{J,k}(\chi^0)$, $A^\dagger_{J,k}(\chi^0)$ would lead to the same normal ordering prescription as in the Schrödinger picture; however, as detailed in [78], the vacuum expectation value still diverges in this case and further renormalisation is necessary. Such a procedure is then equivalent to the normal ordering at the level of the $a_{J,k}$, $a^\dagger_{J,k}$ as we propose here. Note that the question of normal ordering arises only in the case of the $k=0$ mode, as operators for different $k-$modes commute. Using (2.43) and (2.44), one can now write the Fourier modes of the energy-momentum tensor in terms of time-dependent functions of ladder operators a_k and $a^\dagger_k$, and implement the normal ordering procedure, such that the expressions for the operators corresponding to the the energy-momentum tensor components read

$$
\begin{aligned}
:\mathcal{T}^{00}_k: &= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{4\sqrt{|\omega_\gamma||\omega_{k-\gamma}|}}\left[2\beta^+_{k,\gamma} : A^\dagger_{-\gamma}A_{k-\gamma} : +\beta^-_{k,\gamma}\left(: A^\dagger_{-\gamma}A^\dagger_{\gamma-k} : + : A_\gamma A_{k-\gamma} :\right)\right], \\
:\mathcal{T}^{0b}_k: &= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\frac{1}{2}\sqrt{\frac{|\omega_{k-\gamma}|}{|\omega_\gamma|}}\gamma_b\left(: A^\dagger_{\gamma-k}A_\gamma : - : A^\dagger_{-\gamma}A_{k-\gamma} : - : A_{k-\gamma}A_\gamma : + : A^\dagger_{\gamma-k}A^\dagger_{-\gamma} :\right), \\
:\mathcal{T}^{a\neq b}_k: &= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{2\sqrt{|\omega_\gamma||\omega_{k-\gamma}|}}\gamma_a\left(\gamma_b - k_b\right)\Big(: A^\dagger_{-\gamma}A_{k-\gamma} : + : A^\dagger_{\gamma-k}A_\gamma : \\
&\qquad + : A^\dagger_{-\gamma}A^\dagger_{\gamma-k} : + : A_\gamma A_{k-\gamma} : \Big), \\
:\mathcal{T}^{aa}_k: &= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{4\sqrt{|\omega_\gamma||\omega_{k-\gamma}|}}\Big[2(\beta^-_{k,\gamma} - 2\gamma_a(k_a-\gamma_a)) : A^\dagger_{-\gamma}A_{k-\gamma} : \\
&\qquad + (\beta^+_{k,\gamma} - 2\gamma_a(k_a-\gamma_a))\left(: A^\dagger_{-\gamma}A^\dagger_{\gamma-k} : + : A_\gamma A_{k-\gamma} :\right)\Big],
\end{aligned}
\tag{6.11}
$$

where we defined $\beta^{\pm}_{k,\gamma} = -m^2 + \vec{\gamma}\cdot(\vec{k}-\vec{\gamma}) \pm |\omega_\gamma||\omega_{k-\gamma}|$. Notice that only four independent combinations of ladder operators are needed to define all components of $\mathcal{T}^{AB}$. Recall that we identified two types of modes in sec. 2.2.3: oscillating (2.43) and squeezing modes (2.44). The former arise in the case where $\omega_k^2 = m^2 + k^2 < 0$ and for the latter we have $\omega_k^2 > 0$. If $m^2 < 0$ and for $\vec{\gamma}^2 < |m^2|$, A_γ and $A^\dagger_\gamma$ operators have the dynamics of oscillating modes (2.43); for all other cases they follow the dynamics of squeezing modes (2.44). Thus, all oscillating modes have $m^2 < 0$, but for large enough k values, all modes are of squeezing type. If we now consider the expressions appearing in (6.11), it is apparent that there are three different types of mode combinations that can appear, namely those that contain only oscillating or squeezed modes, or those that are a product of operators of that belong to different mode types. The mixed case will inevitably arise for modes with $m^2 < 0$, since the integral over $\vec{\gamma}$ will include modes with $\omega_k^2 < 0$ as well as those with $\omega_k^2 > 0$. In everything that follows we will provide the expressions for operator pairs of the same mode type, i.e. only oscillating or only squeezed modes. The extension to the mixed mode case is straightforward using (2.43) and (2.44). For purely oscillating modes

($\omega_\gamma^2 < 0$ and $\omega_{k-\gamma}^2 < 0$) we have (using (2.43))

$$\begin{aligned} :A^\dagger_{-\gamma}A_{k-\gamma}: &= a^\dagger_{-\gamma}a_{k-\gamma}\,, \qquad :A^\dagger_{-\gamma}A^\dagger_{\gamma-k}: = a^\dagger_{-\gamma}a^\dagger_{\gamma-k}e^{-\mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})(|\omega_{-\gamma}|+|\omega_{\gamma-k}|)\chi^0}\,, \\ :A^\dagger_{\gamma-k}A_{\gamma}: &= a^\dagger_{\gamma-k}a_{\gamma}\,, \qquad :A_{\gamma}A_{k-\gamma}: = a_\gamma a_{k-\gamma}e^{\mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})(|\omega_{\gamma}|+|\omega_{k-\gamma}|)\chi^0}\,. \end{aligned} \tag{6.12}$$

For purely squeezed modes ($\omega_\gamma^2 > 0$ and $\omega_{k-\gamma}^2 > 0$) we find (using (2.44))

$$\begin{aligned} :A^\dagger_{-\gamma}A_{k-\gamma}: =\;& a^\dagger_{\gamma-k}a_\gamma \sinh\left(\omega_{-\gamma}\chi^0\right)\sinh\left(\omega_{k-\gamma}\chi^0\right) + a^\dagger_{-\gamma}a_{k-\gamma}\cosh\left(\omega_{-\gamma}\chi^0\right)\cosh\left(\omega_{k-\gamma}\chi^0\right) \\ &+ \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})\Big(a_\gamma a_{k-\gamma}\sinh\left(\omega_{-\gamma}\chi^0\right)\cosh\left(\omega_{k-\gamma}\chi^0\right) \\ &\qquad - a^\dagger_{-\gamma}a^\dagger_{\gamma-k}\cosh\left(\omega_{-\gamma}\chi^0\right)\sinh\left(\omega_{k-\gamma}\chi^0\right)\Big) \\ =\;& A^\dagger_{\gamma-k}A_\gamma - \sinh\left(\omega_\gamma\chi^0\right)^2(2\pi)^3\delta(\vec{k})\,, \\ :A^\dagger_{\gamma-k}A_{\gamma}: =\;& a^\dagger_{-\gamma}a_{k-\gamma}\sinh\left(\omega_\gamma\chi^0\right)\sinh\left(\omega_{\gamma-k}\chi^0\right) + a^\dagger_{\gamma-k}a_\gamma\cosh\left(\omega_\gamma\chi^0\right)\cosh\left(\omega_{\gamma-k}\chi^0\right) \\ &+ \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})\Big(-a^\dagger_{\gamma-k}a^\dagger_{-\gamma}\sinh\left(\omega_\gamma\chi^0\right)\cosh\left(\omega_{\gamma-k}\chi^0\right) \\ &\qquad + a_{k-\gamma}a_\gamma\cosh\left(\omega_\gamma\chi^0\right)\sinh\left(\omega_{\gamma-k}\chi^0\right)\Big) \\ =\;& A^\dagger_{-\gamma}A_{k-\gamma} - \sinh\left(\omega_\gamma\chi^0\right)^2(2\pi)^3\delta(\vec{k})\,, \\ :A^\dagger_{-\gamma}A^\dagger_{\gamma-k}: =\;& a^\dagger_{-\gamma}a^\dagger_{\gamma-k}\cosh\left(\omega_{-\gamma}\chi^0\right)\cosh\left(\omega_{\gamma-k}\chi^0\right) - a_\gamma a_{k-\gamma}\sinh\left(\omega_{-\gamma}\chi^0\right)\sinh\left(\omega_{\gamma-k}\chi^0\right) \\ &+ \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})\Big(a^\dagger_{-\gamma}a_{k-\gamma}\cosh\left(\omega_{-\gamma}\chi^0\right)\sinh\left(\omega_{\gamma-k}\chi^0\right) \\ &\qquad + a^\dagger_{\gamma-k}a_\gamma\sinh\left(\omega_{-\gamma}\chi^0\right)\cosh\left(\omega_{\gamma-k}\chi^0\right)\Big) \\ =\;& A^\dagger_{-\gamma}A^\dagger_{\gamma-k} - \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})\sinh\left(\omega_\gamma\chi^0\right)\cosh\left(\omega_\gamma\chi^0\right)(2\pi)^3\delta(\vec{k})\,, \\ :A_\gamma A_{k-\gamma}: =\;& -a^\dagger_{-\gamma}a^\dagger_{\gamma-k}\sinh\left(\omega_\gamma\chi^0\right)\sinh\left(\omega_{k-\gamma}\chi^0\right) + a_\gamma a_{k-\gamma}\cosh\left(\omega_\gamma\chi^0\right)\cosh\left(\omega_{k-\gamma}\chi^0\right) \\ &- \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})\Big(a^\dagger_{-\gamma}a_{k-\gamma}\sinh\left(\omega_\gamma\chi^0\right)\cosh\left(\omega_{k-\gamma}\chi^0\right) \\ &\qquad + a^\dagger_{\gamma-k}a_\gamma\cosh\left(\omega_\gamma\right)\sinh\left(\omega_{k-\gamma}\chi^0\right)\Big) \\ =\;& A_\gamma A_{k-\gamma} + \mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})\sinh\left(\omega_\gamma\chi^0\right)\cosh\left(\omega_\gamma\chi^0\right)(2\pi)^3\delta(\vec{k})\,. \end{aligned} \tag{6.13}$$

The normal ordering procedure only affects $\mathcal{T}^{00}$ and $\mathcal{T}^{aa}$; the contributions arising from the re-ordering vanish for the other components. Explicitly, the relation between the normal-ordered operators and those before normal ordering depends on the type of mode through the sign of ω_γ^2 and is as follows

$$: \mathcal{T}_k^{00} : = \mathcal{T}_k^{00} - \delta(\vec{k}) \frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{4} \int \mathrm{d}^3\gamma \, |\omega_\gamma| (1 - \mathrm{sgn}(\omega_\gamma^2)) \,, \tag{6.14}$$

$$\begin{aligned} : \mathcal{T}_k^{aa} : = \mathcal{T}_k^{aa} &+ \delta(\vec{k}) \frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{4} \int \mathrm{d}^3\gamma \left(|\omega_\gamma| (1 + \mathrm{sgn}(\omega_\gamma^2)) - \frac{2\gamma_a^2}{|\omega_\gamma|} \right) \\ &+ \delta(\vec{k}) \, \mathrm{sgn}(\mathcal{K}^{(2)}) \int \mathrm{d}^3\gamma \; \Theta(\omega_\gamma^2) \left(|\omega_\gamma| - \frac{\gamma_a^2}{|\omega_\gamma|} \right) \sinh^2(\omega_\gamma \, \chi^0) \,. \end{aligned} \tag{6.15}$$

The additional terms are multiplied by delta distributions and vanish for $k \neq 0$. For $: \mathcal{T}_k^{00} :$ the integral is only relevant if $m^2 < 0$ and has non-vanishing contributions only for $|\vec{\gamma}| < |m|$, recalling that $\omega_\gamma^2 = m^2 + \gamma^2$. For $m^2 > 0$ it vanishes entirely. For $: \mathcal{T}_k^{aa} :$ the terms multiplied by the delta distribution are divergent. The last term includes a Heaviside function $\Theta(\omega_\gamma^2)$, and is relevant for squeezed modes only.

This concludes the construction of the operator expressions. We proceed to show that the normal ordered operators continue to satisfy the conservation law $\partial_A \mathcal{T}^{AB} = 0$. How the procedure to reconstruct an effective metric from the $\mathcal{T}^{AB}$ operators introduced in this chapter works in practice will be illustrated in the next chapter, where we consider a cosmological spacetime.

6.2.2. Conservation law

One would expect the conservation law for the GFT stress-energy tensor to hold also at the level of the operators, such that $\partial_0 \mathcal{T}_k^{0B} + \mathrm{i} \sum_a k_a \mathcal{T}_k^{aB} = 0$. Any violation could only arise from terms appearing due to operator re-ordering. In this section we discuss that the conservation law remains unaffected and holds for the normal ordered operators (6.11). This is most easily understood from the following: Any terms appearing from operator reordering of the expressions appearing in (6.8) would be proportional to $\delta(\vec{k})$ (see (6.9)), such that alterations to the conservation law would be of the form $\partial_0 \mathcal{T}_k^{0B} + \mathrm{i} \sum_a k_a \mathcal{T}_k^{aB} = \partial_0 \xi^{(0)} \delta(\vec{k}) + \mathrm{i} \sum_a \xi^{(a)} k_a \delta(\vec{k}) = 0$, where $\xi^{(A)}$ are independent of χ^0. For completeness we demonstrate the validity of the conservation law explicitly for the energy-momentum tensor given in terms of the φ_k, π_k operators, where we use the ordering given in (6.8), and show that this calculation is unaffected by the normal ordering we enforce in (6.11). For convenience, we quote the following property of convolutions, which we will use continuously below

$$\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3} f(\vec{\gamma})g(\vec{k}-\vec{\gamma}) = \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3} f(\vec{k}-\vec{\gamma})g(\vec{\gamma})\,. \tag{6.16}$$

The terms of the conservation law for the $B=0$ component, $\partial_0 \mathcal{T}_k^{00} + \mathrm{i}\sum_a k_a \mathcal{T}_k^{0a} = 0\,,$ are explicitly given by

$$\begin{aligned}
\partial_0 \mathcal{T}_k^{00} = &\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\frac{1}{2}\Bigg[\left(\frac{\partial_0\pi_\gamma}{\mathcal{K}^{(2)}} + \left(m^2 - \vec{\gamma}\cdot(\vec{k}-\vec{\gamma})\right)\varphi_\gamma\right)\pi_{k-\gamma} \\
&+ \pi_{k-\gamma}\left(\frac{\partial_0\pi_\gamma}{\mathcal{K}^{(2)}} + \left(m^2 - \vec{\gamma}\cdot(\vec{k}-\vec{\gamma})\right)\varphi_\gamma\right)\Bigg]\,, \\
\mathrm{i}\sum_a k_a \mathcal{T}_k^{0a} = &\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\frac{1}{2}\vec{k}\cdot\vec{\gamma}\left(\pi_{k-\gamma}\varphi_\gamma + \varphi_\gamma\pi_{k-\gamma} + [\pi_{k-\gamma},\varphi_\gamma]\right)\,,
\end{aligned} \tag{6.17}$$

where we have used (6.16) to rewrite the last line. Combining these terms, we obtain

$$\begin{aligned}
&\partial_0 \mathcal{T}_k^{00} + \mathrm{i}\sum_a k_a \mathcal{T}_k^{0a} \\
&= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\frac{1}{2}\left[\left(\frac{\partial_0\pi_\gamma}{\mathcal{K}^{(2)}} + \omega_\gamma^2\varphi_\gamma\right)\pi_{k-\gamma} + \pi_{k-\gamma}\left(\frac{\partial_0\pi_\gamma}{\mathcal{K}^{(2)}} + \omega_\gamma^2\varphi_\gamma\right) + \vec{k}\cdot\vec{\gamma}[\pi_{k-\gamma},\varphi_\gamma]\right] \\
&= 0\,.
\end{aligned} \tag{6.18}$$

In the final step we used the equation of motion for the operators (2.37) as well as the commutator (6.9), $[\pi_{k-\gamma},\varphi_\gamma] \propto \delta(\vec{k})$. The last term indeed vanishes as we have $\vec{k}\,\delta(\vec{k}) = 0$.

In the case of the spatial components of the conservation law, $\partial_0 \mathcal{T}_k^{0b} + \mathrm{i}\sum_a k_a \mathcal{T}_k^{ab} = 0$, we have

$$\begin{aligned}
\partial_0 \mathcal{T}_k^{0b} &= \mathrm{i}\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\gamma_b\left(\mathcal{K}^{(2)}\omega_{k-\gamma}^2\varphi_{k-\gamma}\varphi_\gamma + \frac{\pi_{k-\gamma}\pi_\gamma}{\mathcal{K}^{(2)}}\right)\,, \\
\mathrm{i}\sum_a k_a \mathcal{T}_k^{ab} &= \mathrm{i}\int \frac{\mathrm{d}^3\gamma}{(2\pi)^3}\left[\left(-(k_b-\gamma_b)\vec{k}\cdot\vec{\gamma} + \frac{k_b}{2}(\vec{k}-\vec{\gamma})\cdot\vec{\gamma} - \frac{m^2}{2}k_b\right)\mathcal{K}^{(2)}\varphi_\gamma\varphi_{k-\gamma} - \frac{k_b}{2}\frac{\pi_\gamma\pi_{k-\gamma}}{\mathcal{K}^{(2)}}\right]\,.
\end{aligned} \tag{6.19}$$

We see that the above contains no products of the form $\varphi{\cdot}\pi$ and therefore no commutators. Still, we demonstrate how the conservation law can be shown. Namely, use the property

(6.16) to add the expression with $\gamma \leftrightarrow k - \gamma$ to itself and divide by two, to obtain

$$\begin{aligned}\partial_0 \mathcal{T}_k^{0b} &= \frac{\mathrm{i}}{2} \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3} \left[\left(\gamma_b \omega_{k-\gamma}^2 + (k_b - \gamma_b)\omega_\gamma^2 \right) \mathcal{K}^{(2)} \varphi_{k-\gamma}\varphi_\gamma + k_b \frac{\pi_{k-\gamma}\pi_\gamma}{\mathcal{K}^{(2)}} \right] \\ &= \frac{\mathrm{i}}{2} \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3} \Big[\left(\gamma_b(\vec{k} - \vec{\gamma})^2 + (k_b - \gamma_b)\vec{\gamma}^2 + k_b m^2 \right) \mathcal{K}^{(2)} \varphi_{k-\gamma}\varphi_\gamma \\ &\qquad + k_b \frac{\pi_{k-\gamma}\pi_\gamma}{\mathcal{K}^{(2)}} \Big] .\end{aligned} \tag{6.20}$$

With this, we finally find

$$\partial_0 \mathcal{T}_k^{0b} + \mathrm{i} \sum_a k_a \mathcal{T}_k^{ab} = \frac{\mathrm{i}}{2} \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3} \left[\left(\gamma_b \vec{k}^2 - k_b(\vec{k} \cdot \vec{\gamma}) \right) \mathcal{K}^{(2)} \varphi_{k-\gamma}\varphi_\gamma \right] = 0 , \tag{6.21}$$

where the last equality follows from the fact that the integral can be transformed into minus itself, again from (6.16).

Finally, we consider the additional terms that occur when imposing normal ordering to obtain $:\mathcal{T}_k^{00}:$ and $:\mathcal{T}_k^{aa}:$, see (6.15). For $:\mathcal{T}_k^{00}:$ the additional term is time independent and therefore does not contribute to the conservation law; for $:\mathcal{T}_k^{aa}:$ we have a term proportional to $\delta(\vec{k})$, giving again a term of the form $\vec{k}\,\delta(\vec{k})$. Therefore, the conservation $\partial_0 :\mathcal{T}_k^{0B}: + \mathrm{i}\sum_a k_a :\mathcal{T}_k^{aB}: = 0$ holds on the level of the normal ordered operators, and is therefore *independent* of the choice of state. Recall that the conservation law for the classical current $\partial_\mu (j^\mu)^A$ corresponds to the Klein-Gordon equation for each field (sec. 6.1). The conservation law of $\mathcal{T}^{AB}$ together with the identification with j^{AB} then guarantees that the Klein-Gordon equation is exactly satisfied also at the level of the quantum theory.

The conservation laws $\partial_0 \mathcal{T}_0^{00} = \partial_0 \mathcal{T}_0^{0a} = 0$ deserve some additional attention, as they relate to the globally conserved charges discussed already in [78]. Specifically, the expectation value of $\mathcal{T}_0^{00}$ corresponds to the conserved momentum of the clock field χ^0 in a homogeneous cosmological setting, as we will discuss in the next chapter.

6.3. Conclusion: A metric from Noether currents

In this chapter we introduced a novel set of GFT operators that we propose can be used to reconstruct an effective metric directly from GFT. This proposition goes beyond the available GFT literature, which focuses predominantly on the volume operator (even though other proposals for operators based on geometric quantities exist, see e.g. [148]). It thus opens up a new route to investigate phenomenological implications of GFT that can hopefully give insight into the fundamental construction of the theory, as well as possibly observable quantum gravity effects.

Our construction relies on a relational framework, where four massless scalar fields are coupled to the group field. One field takes on the role of a relational clock and the remaining three serve as spatial rods that can be used to locally span a spatial coordinate system (if the condition of non-degeneracy holds). We work in the deparametrised approach and single out a clock field before quantisation. We carried out the quantisation of the GFT energy-momentum tensor in the Hamiltonian approach to GFT, where the dynamics for the simple free GFT action we use here can be solved independently of the choice of state.

The Hamiltonian in the deparametrised approach with a clock and three spatial fields was established previously [78], but in their work the authors focus on the globally conserved charges resulting from the conservation law for the GFT energy-momentum tensor. The construction of the local operators $\mathcal{T}^{AB}$ was formerly unknown to the literature. During quantisation, we imposed normal ordering at the level of time-independent ladder operators to cancel any vacuum divergences and we showed explicitly that the conservation law holds at the level of the $\mathcal{T}^{AB}$ and is unaffected by operator re-ordering. As pointed out in [78], depending on the sign of the fundamental parameters appearing in the GFT kinetic term, as well as the magnitude of the wave number k, one obtains different dynamics for the operators, which can be classified as oscillating and squeezed modes. In both cases, the dynamics for the ladder operators can be solved explicitly and determine the evolution of the $\mathcal{T}^{AB}$.

The GFT energy-momentum tensor arises from the conservation law associated with the translational symmetry of the four massless scalar fields, in accordance with Noether's theorem. This shift symmetry is therefore crucial for the definition of T^{AB} and while its definition is not restricted to the free action of GFT with a kinetic term truncated at second order, any additional terms, such as interactions would have to respect this symmetry for the procedure proposed in this chapter to remain valid. The crucial point is the identification of T^{AB}, a function on a group manifold, with the conserved currents that arise on the spacetime manifold due to the same shift symmetry for each of the scalar fields $(j^\mu)^A$. As the GFT construction relies on a relational coordinate system, this identification is valid only in a relational frame, where the scalar fields serve as coordinates. In this specific frame, the conserved currents $(j^\mu)^A$ can be used to identify a symmetric matrix field j^{AB} whose components are directly related to those of the inverse metric. It is in that sense that the metric can be effectively reconstructed from GFT. Here, 'effective' refers to the fact that, while we propose to interpret the $\mathcal{T}^{AB}$ as an operator equivalent to j^{AB} the reconstruction of the metric happens only after the expectation values of said operators have been taken over states that are sufficiently semiclassical. The conservation law of $\mathcal{T}^{AB}$ then directly implies that the Klein-Gordon equation (which is just $\partial_\mu j^\mu = 0$ in GR, for each of the fields) holds exactly, irrespective of the choice of state.

At this point, reconstructing an effective metric from GFT is merely a proposal, opening up the path to exciting GFT phenomenology that goes substantially beyond previous literature where one relies on the volume operator to extract geometric quantities. So far, the operators we propose are general and not tied to a specific general relativistic spacetime. When reconstructing a metric one would expect that, details of a physical scenario, such as the spacetime symmetries enter through the choice of state. The dynamics of the GFT energy-momentum tensor, however, are fixed by the choice of the GFT action in the deparametrised approach. Here, a mismatch with the dynamics of GR could lead to insights for the construction of the fundamental theory. Overall, there are plentiful directions in which our findings can be extended either in phenomenological applications, or in understanding the viability of the proposal further at a fundamental level.

In the next chapter, we conduct a first test of our proposition and apply it to quantum gravity's favourite example of homogeneous cosmology as given by a flat FLRW metric. For this, we shall make a simple state choice based the requirements of respecting the symmetry of the spacetime (spatial homogeneity and isotropy) and sufficient semiclassicality. We can already reveal that agreements as well as mismatches with general relativity arise, especially at the level of cosmological perturbations (this might in some sense might be expected from a first naive application of a proposal of this kind). Comments on further extensions are also left for the subsequent chapter.

As a final note we remark that while the construction of the $\mathcal{T}^{AB}$ operators is carried out in the deparametrised approach to GFT, an extension to the algebraic approach sec. 2.2.3 is in principle straightforward. The main difference is that one would have a complex GFT field, which would need to be reflected in the definition of the GFT energy-momentum tensor (6.6) by including an appropriate sum.

Chapter 7.

Group field theory metric for the universe

'Immer mit den einfachsten Beispielen anfangen.'

- David Hilbert.

'Begin with the simplest examples.'

The possibility to reconstruct an effective metric from GFT through the procedure we introduced in chap. 6 appears attractive, as it gives a direct connection between the quantum theory and general relativity. To test the validity of this bold and exciting proposal, we investigate its application to the cosmological setting in this chapter. The analysis of the FLRW metric reconstructed from GFT (parts of sec. 7.1, sec. 7.2 and 7.3) is contained in [2], which was published in *Classical and Quantum Gravity* in July 2024. The application to cosmological perturbations and possible extensions constitute unpublished results.

Overall, the FLRW metric appears to be an accurate description for the cosmos (see sec. 1.2), and in standard cosmology the universe is modelled as a (flat) FLRW spacetime with small inhomogeneous perturbations. The details of cosmological perturbation theory were given in sec. 3.1, including the analysis of the background cosmology as well as perturbative dynamics in a relational coordinate system spanned by four massless scalar fields, which deviates from standard systems that are commonly found in the literature. The application of our proposal for an effective metric from GFT to the cosmological scenario is implemented through an appropriate choice of semi-classical state. As explained in sec. 4.1, it has been shown that for certain (well-motivated) choices of the kinetic term in the GFT action, a single mode will dominate at late times, and thereby determine the semi-classical limit that can be related to general relativity [190]. We therefore restrict to the analysis of a single Peter–Weyl mode with $J = J_0$ as is common in cosmological GFT studies [151, 185]. As seen in sec. 6.2, considering multiple $J-$modes gives a sum of the operators in question and is straightforward in principle.

This chapter is structured as follows: We first establish the relation between expectation values of the GFT energy-momentum tensor components and the perturbed FLRW metric in sec. 7.1. For this, we calculate the components of j^{AB} for a perturbed FLRW metric and invert the resulting expressions to obtain background and perturbative quantities in terms of the $\langle : \mathcal{T}^{AB} : \rangle$. In sec. 7.2 we introduce our choice of state that reflects the required symmetries of the cosmological setting. We proceed to calculate the expectation values of the $\mathcal{T}^{AB}$ operators introduced in the previous chapter. From the homogeneous background mode, which is the subject of sec. 7.3, we establish an effective Friedmann equation in the case of squeezed modes and point out differences of our approach to previous GFT studies. We include also the results for oscillating modes. Inhomogeneous modes, which give the effective expressions for cosmological perturbations, are treated in sec. 7.4 for squeezed as well as oscillating modes. We compare the resulting dynamics of a suitable effective perturbative quantity, namely the scalar perturbation E, to classical equations of motion. In sec. 7.5 we compare the findings of the explicit perturbative analysis in GFT established in this chapter to the results of chap. 5, which utilised the separate universe picture. Before concluding in sec. 7.7 we make some comments on possible extensions of our setup in 7.6.

We drop the normal ordering symbol for the GFT energy-momentum tensor operators in this chapter, it should be understood that we always use the normal ordered version, i.e. $\langle \mathcal{T}^{AB} \rangle := \langle : \mathcal{T}^{AB} : \rangle$, where we recall that normal ordering only influences the $\vec{k} = 0$ mode. Otherwise we keep the notation of chap. 6.

7.1. Effective metric components

As is apparent from the title of this chapter, and in accordance with the overarching agreement that the holy grail of testing grounds for quantum gravity is cosmology, the first application of our proposed effective GFT metric is the perturbed FLRW metric. As explained in sec. 3.2, including perturbations introduces an additional local gauge freedom regarding the choice of the perturbed coordinate system, which is freedom completely fixed in the relational coordinate system.

In this section we calculate the components of the symmetric tensor j^{AB} defined in sec. 6.1 resulting from the classically conserved currents for a perturbed FLRW metric. These are defined only in the relational coordinate system spanned by four massless scalar fields, and this is the matter content of the universe we are thus forced to assume. Making the identification with the GFT operators $\langle \mathcal{T}^{AB} \rangle$ and inverting the expressions gives metric quantities in terms of operator expectation values. Up to this point, one need not assume any knowledge of the form of the GFT operators or the semiclassical state chosen for such an identification and the relations we present below are hence independent of the concrete analysis that follows in the rest of this chapter. Needless to say that the state choice as well as the GFT dynamics determine whether we can in fact interpret the resulting effective metric as perturbed FLRW; the expressions below simply transcend the state choice of sec. 7.2 and can be used for any alternative proposal. In the following we use i, j to denote general spatial indices and a, b for spatial indices in the relational coordinate system.

The general perturbed FLRW metric (as introduced in sec. 3.1) reads

$$\begin{aligned} \mathrm{d}s^2 = &- N(t)^2(1+2\tilde{\Phi}(t,\vec{x}))\mathrm{d}t^2 + 2N(t)a(t)\,\left(\partial_i B(t,\vec{x}) - B_i^V(t,\vec{x})\right)\,\mathrm{d}t\mathrm{d}x^i \\ &+ a(t)^2\Big((1-2\psi(t,\vec{x}))\delta_{ij} + 2\partial_i\partial_j E(t,\vec{x}) \\ &\qquad - \left(\partial_i E_j^V(t,\vec{x}) + \partial_j E_i^V(t,\vec{x})\right) + 2E_{ij}^T(t,\vec{x})\Big)\mathrm{d}x^i\mathrm{d}x^j, \end{aligned} \tag{7.1}$$

where N denotes the lapse function, a the scale factor, and we have carried out a decomposition of metric perturbations into scalar (ψ , $\tilde{\Phi}$, E, B), vector (B_i^V, E_i^V) and tensor (E_{ij}^T) components, the properties of which are given in (3.5).

For the metric (7.1) and the standard matter action for four massless scalar fields[1] (6.1), the classically conserved currents *in the relational coordinate system* take on the following

[1] We discuss k-essence models in sec. 7.6.2.

form

$$
\begin{aligned}
j^{00} &= \frac{a^3}{N}\left(1 + (-\tilde{\Phi} - 3\psi + \nabla^2 E)\right) , \qquad j^{0a} = a^2(B^V_a - \partial_a B) , \\
j^{a\neq b} &= aN\left(2\partial_a\partial_b E - \partial_a E^V_b - \partial_b E^V_a + 2E^T_{ab}\right) , \\
j^{aa} &= -\, aN\left(1 + \tilde{\Phi} - \psi + \nabla^2 E - 2\partial_a^2 E + 2\partial_a E^V_a - 2E^T_{aa}\right) \quad \text{(no sum over } a\text{)}.
\end{aligned}
\tag{7.2}
$$

We have left the lapse function N general, but it should be understood that the identification of the j^{AB} components with the GFT energy-momentum tensor is only possible in a relational coordinate system. From (3.49) we have $N = a^3/\pi_0$ and $N^a = -\pi_a/\pi_0$, where π_0 and π_a are the momenta of the clock and rod fields, respectively. In particular, in the case of scalar perturbations, we find $\partial_a B = -\pi_a/a^2$.

The conserved current (7.2) for a flat FLRW universe (i.e. taking into account homogeneous background quantities only) thus takes the form

$$
j^{AB} = \begin{pmatrix} \pi_0 & 0 \\ 0 & -\frac{a^4}{\pi_0}\delta^{ab} \end{pmatrix} , \tag{7.3}
$$

where we recall that $\pi_0 > 0$ in the relational coordinate system (see the discussion below (6.2)). Notice that the signs of the components are fixed by the Lorentzian signature of (7.1); in the case of a Euclidean signature, all entries would be positive.

Recall that the $\mathcal{T}^{AB}$ operators constructed in chap. 6 are defined in terms of Fourier modes of the spatial fields and we therefore relate them to the Fourier modes of j^{AB}. For any classically perturbed quantity we have $f(t,x) = \bar{f}(t) + \delta f(t,x)$, where the background quantity is given by $\bar{f}(t) = \int dx f(t,x) = f_{k=0}(t)$. Hence, the $k = 0$ mode determines the homogeneous part and the Fourier transform of the perturbation $\delta f(t,x)$ is given by $\delta f_k(t) = f_k(t) - f_{k=0}(t)\delta(k)$. The conjugate momentum of the clock field and scale factor are then determined by the $k = 0$ mode of the diagonal components of $\langle \mathcal{T}_0^{AB} \rangle$:

$$
\pi_0 = \langle \mathcal{T}_0^{00} \rangle , \qquad a^4 = -\langle \mathcal{T}_0^{00} \rangle \langle \mathcal{T}_0^{aa} \rangle . \tag{7.4}
$$

If the off-diagonal components of $\langle \mathcal{T}^{AB} \rangle$ vanish and all spatial diagonal components $\langle \mathcal{T}_0^{aa} \rangle$ are identical (which we will find to be the case for the state we consider below) the effective metric can consistently be interpreted as *flat* FLRW.

The non-zero k-modes correspond to perturbations and in general, the $\langle \mathcal{T}_k^{AB} \rangle$ include scalar, vector, and tensor modes (7.2). For our choice of state, which is the subject of the next section, we will see that the operator expectation values can consistently be interpreted as containing only scalar perturbations and we neglect vector and tensor perturbations in the expressions that follow. A more complete analysis that reveals which types of state choices can give rise also to vector and tensor perturbations is left for future work.

Considering only scalar perturbations, the identification $j_k^{AB} = \langle \mathcal{T}_k^{AB} \rangle$ leads to the following:

$$\begin{aligned} \langle \mathcal{T}^{00}_{k\neq 0} \rangle &= -\frac{a^3}{N}(\tilde{\Phi} + 3\psi + k^2 E), \qquad \langle \mathcal{T}^{0a}_{k\neq 0} \rangle = -\mathrm{i}a^2 k_a B, \\ \frac{1}{3}\,\mathrm{tr}\langle \mathcal{T}^{aa}_{k\neq 0} \rangle &= -aN\left(\tilde{\Phi} - \psi - \frac{k^2}{3}E\right), \qquad \langle \mathcal{T}^{a\neq b}_{k\neq 0} \rangle = -2aN k_a k_b E. \end{aligned} \tag{7.5}$$

Inverting the above gives expressions for effective perturbations ($k_a \neq 0, k_b \neq 0$):

$$\begin{aligned} \tilde{\Phi} &= -\frac{\langle \mathcal{T}^{00}_{k\neq 0} \rangle N}{4a^3} - \frac{\mathrm{tr}\langle \mathcal{T}^{aa}_{k\neq 0} \rangle}{4aN}, \qquad E = -\frac{1}{2aN}\frac{\langle \mathcal{T}^{a\neq b}_{k\neq 0} \rangle}{k_a k_b}, \\ \psi &= -\frac{\langle \mathcal{T}^{00}_{k\neq 0} \rangle N}{4a^3} + \frac{\mathrm{tr}\langle \mathcal{T}^{aa}_{k\neq 0} \rangle}{12aN} + \frac{k^2}{k_a k_b}\frac{1}{6aN}\langle \mathcal{T}^{a\neq b}_{k\neq 0} \rangle, \qquad B = \frac{\mathrm{i}}{a^2}\frac{\langle \mathcal{T}^{0a}_{k\neq 0} \rangle}{k_a}. \end{aligned} \tag{7.6}$$

With these identifications, it is possible to study also gauge-invariant perturbation variables as given in (3.20) and (3.54).

We make a final comment, risking repetitiveness: As these expressions are independent of the GFT details and a result of identifying the expectation values of the GFT energy-momentum tensor $\langle \mathcal{T}^{AB} \rangle$ with the conserved currents of a perturbed FLRW metric in a coordinate system spanned by four massless scalar fields, they can be used in alternative state proposals as well (our state choice detailed in the next section should be understood as a naive first guess). In some way, this illustrates the nature of the effective GFT metric proposal: In itself, for any (suitably semiclassical) state an effective metric can be reconstructed, however, the task is to interpret the resulting physical scenario. Here we take the approach that we compare the effective metric to a specific solution of GR, which reflects our *belief* (or rather, hope) that we have found a state that will recover said metric (in the semi-classical limit). How to interpret a general metric, without assuming a classical counterpart from the beginning is less clear.

7.2. Choice of state

To obtain explicit expressions for the operator expectation values, enabling us to concretely reconstruct an FLRW metric as well as its perturbations from the identifications (7.4) and (7.5), we have to make a choice of state. The goal is to select a state that satisfies the condition of semiclassicality, such that the expectation values $\langle \mathcal{T}^{AB} \rangle$ can indeed be related to an effective metric. Furthermore, it must incorporate properties of the cosmological spacetime, in this case are homogeneity and isotropy at the background level.

We discussed the semiclassicality of states in sec. 4.1, and recall that Fock coherent states satisfy the requirement of relatively small uncertainty in operator expectation values throughout the evolution [185] (see also [187] for a more in-depth analysis of a broader class of semiclassical GFT states). Even though the analysis in [185] focused on the expectation value of the volume operator, we assume that this property will carry over to the GFT energy-momentum tensor, since it contains similar operator combinations. We therefore work with a Fock coherent state $|\sigma\rangle$ which is an eigenstate of the (time-independent) annihilation operator $a_{J,k} |\sigma\rangle = \sigma_J(\vec{k}) |\sigma\rangle$:

$$|\sigma\rangle = e^{-||\sigma||^2/2} \exp\left(\sum_J \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sigma_J(\vec{k}) a^\dagger_{J,k} \right) |0\rangle \,, \tag{7.7}$$

where $|0\rangle$ is the GFT Fock vacuum and $||\sigma||^2 = \sum_J \int \frac{\mathrm{d}^3 k}{(2\pi)^3} |\sigma_J(\vec{k})|^2$. In an FLRW background there exist only homogeneous quantities, and especially all components of j^{AB} are homogeneous. To reflect this homogeneity in the quantum state, we choose a sharply peaked Gaussian for $\sigma(\vec{k})$,

$$\sigma_J(\vec{k}) = \delta_{J,J_0} \frac{\mathcal{A} + \mathrm{i}\mathcal{B}}{c_\sigma} \, e^{-\frac{(\vec{k}-\vec{k}_0)^2}{2s^2}} \,, \tag{7.8}$$

where $\mathcal{A}$, $\mathcal{B} \in \mathbb{R}$, s determines the peakedness of the state, and we set the homogeneous $k = 0$ mode as the initially dominantly excited Fourier mode, $\vec{k}_0 = 0$. The normalisation factor $c_\sigma = \left(\frac{s}{2\sqrt{\pi}} \right)^{3/2}$ is fixed for convenience regarding later calculations. The state reflects our restriction to a single Peter-Weyl mode; in the more general case of multiple modes, the initial conditions, namely $\mathcal{A}$, $\mathcal{B}$ and s, could be $J-$dependent. Whether and how this would have phenomenologically interesting consequences for perturbations is left for future work. We would like to point out the difference between this state choice and standard cosmological perturbation theory: While the Gaussian is strongly peaked on the background mode, it has a finite width, such that inhomogeneous modes will always be excited. A strictly homogeneous state is reached in the limit of $s \to 0$, corresponding to an infinitely peaked state, which would introduce divergences that are avoided for $0 < s \ll 1$. In standard cosmological approaches, one usually assumes a delta-peak for the background mode and fixes the initial spectrum for the perturbations by assuming the Bunch-Davies vacuum, see e.g. [85]. This is conceptually different to our proposed state, which does not allow to excite solely the homogeneous background. The $k \neq 0$ modes are then to be regarded as inhomogeneous perturbations over said background and are studied in sec. 7.4. Since we have chosen $\sigma(\vec{k})$ to be sharply peaked on the background mode $\vec{k} = 0$, the expectation value of the energy-momentum tensor (6.11) will be determined by (low $|\vec{k}|$) squeezed modes for $m^2 > 0$ and by (low $|\vec{k}|$) oscillating modes for $m^2 < 0$. Therefore, it appears sufficient to focus on explicit expressions for the $\mathcal{T}^{AB}$ that are either fully determined by squeezed or oscillating modes, as we did in chap. 6. We consider the background metric, governed by the $\vec{k} = 0$ mode, in sec. 7.3. Whether this choice of state is well suited to obtain cosmological perturbations that agree with the general relativistic treatment is a question we will revisit after concluding the perturbative analysis in sec. 7.4.

7.3. Background

We can now determine explicitly the effective metric resulting from our proposal made in chap. 6, having made a choice of state that for the reasons stated above reflects an FLRW spacetime with perturbations. Here, we consider only the background mode $\vec{k} = 0$ and its resulting effective metric, which we will see does indeed correspond to a flat FLRW spacetime for a specific range of initial conditions. The remaining $\vec{k} \neq 0$ modes, interpreted as perturbations to the homogeneous spacetime, are the subject of sec. 7.4.

The procedure to recover the effective metric is now as follows:

- Calculate the expectation values of $\mathcal{T}^{AB}_{k=0}$ in our choice of state (7.7). For this choice, the convolutions appearing in the operator expressions can be simplified with the saddle-point approximation to obtain explicit expressions for the components.

- Make the identification $\langle \mathcal{T}^{AB}_{k=0} \rangle = j^{AB}$ for an FLRW spacetime and thereby reconstruct the components of the inverse metric. This allows us to determine which initial conditions lead to a Lorentzian metric and one obtains the evolution of the scale factor a and thereby an effective Friedmann equation that can be compared to that of GR.

As discussed in chap. 6, the dynamics of $\mathcal{T}^{AB}$ depend on the type of modes we are considering - squeezed or oscillating. As we are interested in recovering an expanding universe, our focus lies on squeezed modes $m^2_{J_0} = m^2 > 0$, which have a growing number of quanta over time. We report the contribution to an effective metric from oscillating modes $m^2 < 0$ at the end of this section, on the one hand for completeness, on the other to assess possible contributions in a scenario in which multiple $J-$modes are excited. We will not be concerned with the special case $m^2 = 0$ which requires a separate analysis as carried out in [186], where the authors show that this scenario contains fine-tuning instabilities.

For squeezed modes, $m^2 > 0$, the expectation values of the normal ordered components of the GFT energy-momentum tensor (6.11) for the state defined in (7.7) and (7.8) are given by

$$\begin{aligned}
\langle \mathcal{T}_0^{00} \rangle &= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3} \mathrm{sgn}(\mathcal{K}^{(2)}) |\omega_\gamma| (\mathcal{B}^2 - \mathcal{A}^2) \frac{e^{-\gamma^2/s^2}}{c_\sigma^2} \approx \mathrm{sgn}(\mathcal{K}^{(2)}) |m| (\mathcal{B}^2 - \mathcal{A}^2)\,, \\
\langle \mathcal{T}_0^{0b} \rangle &= 0\,, \qquad \left\langle \mathcal{T}_0^{a\neq b} \right\rangle = 0\,, \\
\langle \mathcal{T}_0^{aa} \rangle &= \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3} \mathrm{sgn}(\mathcal{K}^{(2)}) \frac{e^{-\gamma^2/s^2}}{c_\sigma^2} \Bigg(\left(-|\omega_\gamma| + \frac{\gamma_a^2}{|\omega_\gamma|} \right) \Big((\mathcal{A}^2 + \mathcal{B}^2) \cosh\big(2\omega_\gamma \chi^0\big) \\
&\qquad - 2\,\mathrm{sgn}(\mathcal{K}^{(2)}) \mathcal{A}\mathcal{B} \sinh\big(2\omega_\gamma \chi^0\big) \Big) + \frac{\gamma_a^2}{|\omega_\gamma|} (\mathcal{A}^2 - \mathcal{B}^2) \Bigg) \\
&\approx -\,\mathrm{sgn}(\mathcal{K}^{(2)}) |m| \left((\mathcal{A}^2 + \mathcal{B}^2) \cosh\big(2|m|\chi^0\big) - 2\,\mathrm{sgn}(\mathcal{K}^{(2)}) \mathcal{A}\mathcal{B} \sinh\big(2|m|\chi^0\big) \right)\,.
\end{aligned} \tag{7.9}$$

To carry out the integrals explicitly for $\langle \mathcal{T}_0^{00} \rangle$ and $\langle \mathcal{T}_0^{aa} \rangle$, we made use of the saddle-point approximation

$$\int \mathrm{d}^3x\, e^{-\frac{(\vec{x}-\vec{\mu})^2}{s^2}} g(\vec{x}) \approx g(\vec{\mu}) \int \mathrm{d}^3x\, e^{-\frac{(\vec{x}-\vec{\mu})^2}{s^2}} = g(\vec{\mu}) (\sqrt{\pi} s)^3, \tag{7.10}$$

which holds for sharply peaked Gaussians such that $g(\vec{x})$ can be considered approximately constant in the region $|\vec{x} - \vec{\mu}| \leq s$ and is applicable for our state choice due to $\sigma(\vec{k})$ being highly peaked (7.8). Indeed, employing the saddle-point approximation at the background level is equivalent to considering the limit in which $\sigma(\vec{k} = 0)$ is given exactly by a delta-peak. (This approximation will become inaccurate for $\langle\mathcal{T}_0^{aa}\rangle$ at late times. We investigate the limits of this approximation in more detail in app. E, demonstrating that the saddle-point approximation is justified for the times we are interested in in the case of a sufficiently sharply peaked state.) Importantly, the off-diagonal components, i.e. $\langle\mathcal{T}^{0a}\rangle$ and $\langle\mathcal{T}^{a\neq b}\rangle$, vanish exactly due to the antisymmetry of the integrals, giving a spatially flat metric. For the $\langle\mathcal{T}_0^{0b}\rangle$ components this can be interpreted as the vanishing of the canonical momenta conjugate to the spatial fields χ^b, since $j^{0b} \propto \pi^b$. Furthermore, $\langle\mathcal{T}_0^{00}\rangle$ is exactly constant in time, which is an important consistency check, since it shows that π_0 is exactly conserved at the background level for our choice of state.[2] We know that the energy-momentum tensor satisfies the conservation law $\partial_0\mathcal{T}^{0B} + \mathrm{i}\sum_a \mathcal{T}^{aB} = 0$ independent of the state, and thereby that the Klein-Gordon equation must be satisfied. The conservation of π_0 is exactly the Klein-Gordon equation for the classical FLRW model.

Before discussing the effective metric we recover further, we comment on the importance and ambiguities of the signs appearing in the above expressions. Through the identification (6.4) the signs of the components of the conserved current are directly related to the metric signature: all entries of the conserved current will either have the same sign (Euclidean case) or the spatial diagonal will have opposite sign of the j^{00} entry (Lorentzian case). Given that $\mathrm{sgn}(\langle\mathcal{T}_0^{aa}\rangle) = -\mathrm{sgn}(\mathcal{K}^{(2)})$ and $\mathrm{sgn}(\langle\mathcal{T}_0^{00}\rangle) = \mathrm{sgn}(\mathcal{K}^{(2)})\mathrm{sgn}(\mathcal{B}^2 - \mathcal{A}^2)$ (from (7.9)), and a Lorentzian (Euclidean) effective metric is obtained when these expectation values have the opposite (same) sign, the initial conditions $\mathcal{A}$, $\mathcal{B}$ determine the signature of the effective metric we reconstruct. The signature of the effective metric has so far not been fixed and we can deduce that the Lorentzian case is found for $\mathcal{B}^2 > \mathcal{A}^2$, whereas $\mathcal{B}^2 < \mathcal{A}^2$ results in a Euclidean metric.[3] Let us now turn to the factor of $\mathrm{sgn}(\mathcal{K}^{(2)})$. The sign of the j^{00} component is determined by the overall sign of the matter Lagrangian in (6.1), which is usually chosen such that π_0 is positive (as we did in our case). This means that if we had decided to work with a different metric signature convention, i.e. 'East Coast' $(+,-,-,-)$ instead of 'West Coast' $(-,+,+,+)$, we would have chosen a different sign for the Lagrangian in (6.1), and ended up with the same components for j^{AB} as given in (7.3). The sign in the identification $j^{AB} = \langle\mathcal{T}^{AB}\rangle$ is also pure convention and we could equally have chosen $j^{AB} = -\langle\mathcal{T}^{AB}\rangle$, since the sign of T^{AB} is arbitrary already in its definition (6.6). Restricting to the Lorentzian case with $\mathcal{B}^2 > \mathcal{A}^2$, the requirement $j^{00} > 0$

[2] In fact, it is possible to solve the integral for $\langle\mathcal{T}_0^{00}\rangle$ explicitly to give a Tricomi confluent hypergeometric function. As the explicit form of $\langle\mathcal{T}_0^{00}\rangle$ would not aid in further illuminating our results, we omit it.

[3] The special case of $\mathcal{B}^2 = \mathcal{A}^2$ corresponds to vanishing momentum of the clock field and is therefore excluded.

then gives the following prescription[4]: For $\text{sgn}(\mathcal{K}^{(2)}) = 1$, identify $j^{AB} = \langle \mathcal{T}^{AB} \rangle$, whereas for $\text{sgn}(\mathcal{K}^{(2)}) = -1$, set $j^{AB} = -\langle \mathcal{T}^{AB} \rangle$. (The resulting expressions are independent of this choice, but for concreteness we can choose the former, i.e. $\text{sgn}(\mathcal{K}^{(2)}) = 1$, in what follows. We will leave $\text{sgn}(\mathcal{K}^{(2)})$ general in any expressions of expectation values and insert its value only when identifications with classical quantities are made.)

Here we are interested in the Lorentzian case and therefore restrict to initial conditions with $\mathcal{B}^2 > \mathcal{A}^2$. Comparison with the conserved current for the FLRW case as given in (7.4) then gives the following identifications for the momentum of the clock field and the scale factor in the case of squeezed modes

$$\begin{aligned}
\pi_0 =& \left\langle \mathcal{T}_0^{00} \right\rangle = |m|(\mathcal{B}^2 - \mathcal{A}^2), \\
a^4 =& -\pi_0 \left\langle \mathcal{T}_0^{aa} \right\rangle = m^2(\mathcal{B}^2 - \mathcal{A}^2)\left((\mathcal{A}^2 + \mathcal{B}^2)\cosh\big(2|m|\chi^0\big) - 2\mathcal{A}\mathcal{B}\sinh\big(2|m|\chi^0\big)\right) \\
=& \frac{m^2}{2}(\mathcal{B}^2 - \mathcal{A}^2)\left((\mathcal{A} - \mathcal{B})^2 e^{2|m|\chi^0} + (\mathcal{A} + \mathcal{B})^2 e^{-2|m|\chi^0}\right) .
\end{aligned} \tag{7.11}$$

If we were to consider a Euclidean signature instead, we would have to restrict to $\mathcal{A}^2 > \mathcal{B}^2$, as all entries in (7.3) would then have the same sign. The above identifications would then be given by $\pi_0 = |m||\mathcal{B}^2 - \mathcal{A}^2|$, whereas $\frac{a^4}{\pi_0} = -\left\langle \mathcal{T}_0^{aa} \right\rangle$ is unaltered.

From the identification (7.11) we obtain the following effective Friedmann equation

$$\begin{aligned}
H^2 = \left(\frac{a'}{a}\right)^2 = \frac{1}{4}m^2\left(1 - \frac{4(\mathcal{A}^2 - \mathcal{B}^2)^2}{((\mathcal{A} - \mathcal{B})^2 e^{2m\chi^0} + (\mathcal{A} + \mathcal{B})^2 e^{-2m\chi^0})^2}\right) =& \frac{1}{4}m^2\left(1 - \frac{\pi_0^4}{a^8}\right) \\
\underset{\text{late times}}{\longrightarrow}& \frac{1}{4}m^2 .
\end{aligned} \tag{7.12}$$

In addition to a constant Hubble rate at late times, the effective metric gives a bouncing universe, with the bounce occurring at $a^4 = \pi_0^2$, or equivalently, $\langle \mathcal{T}_0^{aa} \rangle^2 = \langle \mathcal{T}_0^{00} \rangle^2$.

Recovering a constant Hubble rate in the late time limit is in agreement with all the Friedmann equations previously obtained for GFT models with a single clock field (see, e.g., [185] and sec. 4.1) as well as with the general relativistic Friedmann equation for a single massless scalar field. We find a mismatch of the late time effective Friedmann equation with general relativity with four massless scalar fields, where the gradients of the spatial fields contribute. To see this more explicitly, recall that the general relativistic

[4]The opposite is true for the Euclidean case $\mathcal{A}^2 > \mathcal{B}^2$: $\text{sgn}(\mathcal{K}^{(2)}) = 1 \to j^{AB} = -\langle \mathcal{T}^{AB} \rangle$ and $\text{sgn}(\mathcal{K}^{(2)}) = -1 \to j^{AB} = \langle \mathcal{T}^{AB} \rangle$.

Friedmann equations for the case of a single and four massless scalar fields, respectively (see sec. 3.4), are given by

$$H^2_{\text{single field}} = \frac{\kappa}{6}\,, \qquad H^2_{\text{four fields}} = \frac{\kappa}{6}\left(1 + 3\frac{a^4}{\pi_0^2}\right)\,. \tag{7.13}$$

In the four field case the contribution of the spatial fields dominates at late times, rendering a match with the effective Friedmann equation (7.12) we recovered unfeasible. In fact, as the bounce happens at $\frac{a^4}{\pi_0^2} = 1$ there is no (early time) regime in which their contribution can be neglected with respect to that of the clock field while being away from the GFT bounce. Instead, we recover the late time behaviour of general relativity with a single scalar field if we fix $m^2 = \frac{2}{3}\kappa$.[5] Singularity resolution through bounce is a common feature of GFT cosmology (sec. 4.1). The explicit form of the Friedmann equation we find here however differs due to recovering an effective GR regime through the GFT energy-momentum tensor instead of the volume operator V, which is used in previous works. Recall that for the single J mode case, the volume operator is proportional to the number operator $N = A^\dagger A$. Explicitly, the dynamical solution for N in the case of squeezed modes is found to be $\langle N\rangle = (\mathcal{A}^2 + \mathcal{B}^2)\cosh(2|m|\chi^0) - 2\mathcal{A}\mathcal{B}\sinh(2|m|\chi^0)$, which we recognize from (7.9) and find $\langle \mathcal{T}_0^{aa}\rangle = -\text{sgn}(\mathcal{K}^{(2)})|m|\,\langle N\rangle$. We emphasise that this relation between the expectation value of the number operator and the spatial diagonal components of the GFT energy-momentum tensor depends on the choice of state and the implementation of the saddle-point approximation, and therefore does not hold in general.

We can contrast our result to those previously reported in the GFT literature in more detail. For this, recall the calculation in sec. 4.1.3 to obtain an effective Friedmann equation in the deparametrised approach to GFT with a single clock field. The effective expression for the number operator for squeezed modes (4.11) differs to the one above only by an additional constant term. This term is absent in the scenario considered here due to the normal ordering procedure we impose, as was already pointed out in [78]. We consider a similar state to the one used in the calculation outlined in 4.1.3 and found that the contributions from the spatial fields are irrelevant for the $\vec{k} = 0$ mode. It is then rather expected that we recover a similar evolution of the number operator as previously. From the relation of $\langle \mathcal{T}_0^{aa}\rangle$ and $\langle N\rangle$ that follows for our state choice as explained above,

[5] We discuss a possible alleviation of this discrepancy by considering k-essence models in sec. 7.6.2.

we find that the Hubble rate is related to operator expectation values through (cf. (4.13))

$$H^2 = \frac{1}{16}\left(\frac{\langle \mathcal{T}_0^{aa}\rangle'}{\langle \mathcal{T}_0^{aa}\rangle}\right)^2 = \frac{1}{16}\left(\frac{\langle N\rangle'}{\langle N\rangle}\right)^2 . \tag{7.14}$$

The difference in the form of the effective Friedmann equation (7.12) in comparison to previous results, such as (4.14), then originates from the different identification of classical quantities with operator expectation values in GFT. In our new proposal, we assign a different power of the scale factor a to the same dynamical expression, leading to $\langle N\rangle \propto a^4$ instead of $\langle N\rangle \propto a^3$ as previously considered. Thus, our proposal suggests that the number of excited GFT quanta is not directly proportional to the increase in classical spatial volume, but grows more rapidly. This is in contrast to the usual physical picture of Planck-scale quanta carrying fixed amounts of volume, where the combined volume of all quanta amounts to the spatial volume in a suitable classical limit. While conceptually interesting, the different identification of the scale factor does not alter the late time Friedmann equation, indeed, one can identify the scale factor with any power of the dynamical expression found in the Friedmann equation to obtain a constant Hubble rate at late times. This would simply lead to a different numerical value of $|m|$ as fixed in the late time limit.

The effective Friedmann equation introduces a bounce at $a^4 = \pi_0^2$, which resolves the singularity found in a general relativistic FLRW spacetime. We saw in sec. 1.2.2 that the Big Bang singularity can be seen in the Ricci scalar, which diverges as $a \to 0$ in a classical FLRW setting, resulting in the singularity quantum gravity might be expected to solve. For our choice of lapse the Ricci scalar reads $R = \frac{6\pi_0^2}{a^6}(-2H^2 + \frac{a''}{a})$ and for the effective Friedmann equation in (7.12) we find that the Ricci scalar (1.40) at the bounce ($a' = 0$ and $\pi_0 = a^2$) is indeed finite. It is explicitly given by

$$R_{\text{bounce}} = 6\frac{m^2}{\pi_0} , \tag{7.15}$$

where we used $R_{\text{bounce}} = 6\frac{a''}{a^3}$ and $a''_{\text{bounce}} = m^2 a_{\text{bounce}}$ from (7.11). Its value is determined entirely by the value of π_0 and thus by the initial conditions $\mathcal{A}$, $\mathcal{B}$ appearing in the choice of coherent state (7.7) through (7.11), if m is fixed by requiring consistency with the late time Friedmann equation of general relativity with a single massless scalar field.

Finally, we consider the expectation values for oscillating modes $m^2 < 0$. Here σ needs to be especially peaked, so that contributions from squeezing modes can be neglected in the integral and only the region near $\vec{\gamma} = 0$ (which consists entirely of oscillating modes) contributes; see also the discussion in chap. 6. If squeezed contributions can be neglected, such that the dynamics of the time dependent ladder operators are given by (6.12), and we again make use of the saddle-point approximation (7.10) we obtain from (6.11)

$$\begin{aligned}
\langle \mathcal{T}_0^{00} \rangle &\approx \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3} \mathrm{sgn}(\mathcal{K}^{(2)}) \frac{e^{-\gamma^2/s^2}}{c_\sigma^2} |\omega_\gamma| (\mathcal{A}^2 + \mathcal{B}^2) \approx \mathrm{sgn}(\mathcal{K}^{(2)}) |m| (\mathcal{A}^2 + \mathcal{B}^2)\,, \\
\langle \mathcal{T}_0^{0b} \rangle &= 0\,, \qquad \left\langle \mathcal{T}_0^{a\neq b} \right\rangle = 0\,, \\
\langle \mathcal{T}_0^{aa} \rangle &\approx \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3} \frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{|\omega_\gamma|} \frac{e^{-\gamma^2/s^2}}{c_\sigma^2} \\
&\qquad \times \left(\gamma_a^2 (\mathcal{A}^2 + \mathcal{B}^2) + (|\omega_\gamma|^2 + \gamma_a^2) \left((\mathcal{A}^2 - \mathcal{B}^2) \cos\!\big(2|\omega_\gamma|\chi^0\big) + 2\mathcal{A}\mathcal{B} \sin\!\big(2|\omega_\gamma|\chi^0\big) \right) \right) \\
&\approx |m| \left(\mathrm{sgn}(\mathcal{K}^{(2)}) (\mathcal{A}^2 - \mathcal{B}^2) \cos\!\big(2|m|\chi^0\big) + 2\mathcal{A}\mathcal{B} \sin\!\big(2|m|\chi^0\big) \right) .
\end{aligned} \tag{7.16}$$

Through the identification (7.4) this results in

$$\begin{aligned}
\pi_0 &= |m| (\mathcal{A}^2 + \mathcal{B}^2)\,, \\
a^4 &= -\,|m|^2 (\mathcal{A}^2 + \mathcal{B}^2) \left((\mathcal{A}^2 - \mathcal{B}^2) \cos\!\big(2m\chi^0\big) + 2\mathcal{A}\mathcal{B} \sin\!\big(2m\chi^0\big) \right) .
\end{aligned} \tag{7.17}$$

The sign of π_0 is independent of the initial conditions, and the sign convention for the identification $j^{AB} = \pm\langle \mathcal{T}^{AB} \rangle$ should be adjusted depending on the sign of $\mathcal{K}^{(2)}$. On the other hand, the sign of a^4 is not fixed and fluctuates throughout the evolution, thus, an oscillating mode on its own cannot be reconciled with a viable cosmological interpretation[6], such that they can only appear in conjunction with squeezed modes. In this case, their presence would modulate the evolution of the scale factor found for squeezed modes (7.11), an effect whose relative importance would diminish as the universe evolves. Phenomenologically, the effects of oscillating modes are particularly interesting in the early universe, close to the bounce. Outside the cosmological context, oscillating modes could be interpreted as resulting in an effective spacetime with an alternating metric signature, but this is to be clarified further. We emphasise that the result for oscillating modes significantly differs to that found for the case where the connection to an effective classical theory is made through the volume and thereby the number operator $a^3 = \langle V \rangle \sim \langle N \rangle$ (in the case of a single mode). I these cases the scale factor remains constant and is always

[6] Unless one restricts to a very short evolution time in which $a^4 > 0$, the motivation for which would be unclear.

positive for oscillating modes, as it follows the behaviour of $\langle N \rangle$.

This concludes the discussion of the cosmological background metric, which we reconstructed from the $\vec{k} = 0$ mode of the GFT energy-momentum tensor for a suitable state. For a squeezed Peter-Weyl mode, we recover an effective expression for the scale factor that leads to an effective Friedmann equation with a bounce. The interpretation of recovering a flat FLRW spacetime is substantially strengthened in our approach, since we can explicitly consider metric components.

In the following we will extend the analysis also to inhomogeneous modes and compare the results to cosmological perturbation theory.

7.4. Cosmological perturbations

We now focus on the $\vec{k} \neq 0$ components of the $\mathcal{T}^{AB}$ components for the state introduced in sec. 7.2 that was used to determine the effective background metric in the previous section. Recall that for this naive choice of state, even though it is highly peaked on the homogeneous background mode, inhomogeneous modes will always be excited as well. In the following we examine the dynamics that arise for cosmological perturbations if we identify these inhomogeneous modes with inhomogeneities of the perturbed FLRW metric (7.1). Maybe unsurprisingly, we find a strong mismatch with the general relativistic dynamics. Still, the following can be seen as a guidance to construct perturbative quantities and may give hints which adjustments could lead to an agreement with general relativity at late times.

Recall that all components of the GFT energy-momentum tensor (6.11) depend on the same operator combinations. In particular, each term is a product of the time-dependent ladder operators A_k and $A_k^\dagger$. From the state choice (7.7) with (7.8) and the linear dependence of A_k , $A_k^\dagger$ on the time-independent creation and annihilation operators (see (2.43) and (2.44)) we find that each of the terms in the expectation values for $\langle \mathcal{T}_k^{AB} \rangle$ will be proportional to $e^{-\frac{\gamma^2}{2s^2}} e^{-\frac{(k-\gamma)^2}{2s^2}}$. Similar to the background dynamics, we can then employ the saddle-point approximation to obtain explicit dynamics for the $\langle \mathcal{T}^{AB} \rangle$ components. For this, we rewrite the exponentials appearing in the integrals as

$$e^{-\frac{k^2-2k\gamma+2\gamma^2}{2s^2}} = e^{-\frac{1}{s^2}(\gamma-\frac{k}{2})^2} e^{-\frac{k^2}{4s^2}}, \tag{7.18}$$

and find a Gaussian peaked on $\vec{\gamma} = \frac{\vec{k}}{2}$, such that we have $\vec{\gamma} \to \frac{\vec{k}}{2}$ after applying the saddle-point approximation (7.10). This approximation, which requires $s \ll 1$, will not hold for all times or for large (enough) values of k. Note furthermore that for our choice of $\sigma(\vec{k})$ (7.8) we have $A_k |\sigma\rangle = A_{-k} |\sigma\rangle$ (and similarly for $A_k^\dagger$) for oscillating as well as squeezed modes, due to $\omega_k = \omega_{-k}$. Expressions for the operator expectation values (6.11) then simplify to

$$\begin{aligned}
\langle \mathcal{T}_k^{00} \rangle &\approx \frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{4|\omega_{k/2}|} c_\sigma^2 \left[k^2 \langle A_{k/2}^\dagger A_{k/2} \rangle - 2m^2 \left(\langle A_{k/2}^{\dagger\,2} \rangle + \langle A_{k/2}^{2} \rangle \right) \right], \\
\langle \mathcal{T}_k^{0b} \rangle &\approx \frac{k_b}{4} c_\sigma^2 \left[\langle A_{k/2}^{\dagger\,2} \rangle - \langle A_{k/2}^{2} \rangle \right], \\
\langle \mathcal{T}_k^{a\neq b} \rangle &\approx - \frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{|\omega_{k/2}|} \frac{k_a k_b}{8} c_\sigma^2 \left[2 \langle A_{k/2}^\dagger A_{k/2} \rangle + \langle A_{k/2}^{\dagger\,2} \rangle + \langle A_{k/2}^{2} \rangle \right], \\
\langle \mathcal{T}_k^{aa} \rangle &\approx \frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{4|\omega_{k/2}|} c_\sigma^2 \left[-(4m^2 + k_a^2) \langle A_{k/2}^\dagger A_{k/2} \rangle + \frac{1}{2}(k^2 - k_a^2) \left(\langle A_{k/2}^{\dagger\,2} \rangle + \langle A_{k/2}^{2} \rangle \right) \right],
\end{aligned} \tag{7.19}$$

where the factor c_σ^2 enters from the integral over the exponential in the saddle-point approximation (7.10) and is cancelled by our choice of state (7.8). We use equality signs in the expressions that follow; it should be understood that statements below rely on the applicability and sufficient accuracy of the saddle-point approximation.

As detailed in the previous section, recovering an FLRW background metric with a single Peter–Weyl mode is only possible in the case of a squeezed mode. Since there is no split between background and perturbations in the simple state choice we employ here, perturbations are then also of squeezing type in the single mode case, $J = J_0$. In the more general case, where a minimum of two $J-$modes are excited, one of them can be of the oscillating type, as this will not alter the background dynamics at late times. For completeness we then also consider the perturbations arising from oscillating modes.

Cosmological perturbation theory in a relational coordinate system spanned by four massless scalar fields was the topic of sec. 3.4.2. Our choice of relational coordinate system is a specific case of the harmonic gauge determined by $\Box x^\mu = 0$ or equivalently $\delta(g^{\mu\nu}\Gamma^\lambda_{\mu\nu}) = 0$, which leads to the relations between perturbation variables given in (3.39). As noted already in chap. 6, the Klein-Gordon equation is always satisfied in our setup and follows from the conservation laws for $\mathcal{T}^{AB}$, which hold at operator level. Hence, the harmonic gauge condition is fulfilled exactly for any choice of state, and since it determines the choice of coordinate frame, satisfying the harmonic gauge condition at the quantum level is an important consistency check.

From the relation of perturbation variables to operator expectation values as given in (7.6) we can establish equations of motion for effective perturbations arising from the GFT effective metric, in terms of the dynamics of operator expectation values, independent of the explicit state choice. From the identifications in (7.6) we obtain

$$\begin{aligned}
B'' + 4HB' + 2\left(H' + 2H^2\right)B &= \mathrm{i}\,\frac{\langle \mathcal{T}^{0a}\rangle''}{k_a a^2}\,,\\
E'' + 8HE' + 4\left(H' + 4H^2\right)E &= -\,\frac{\pi_0}{2k_a k_b a^4}\langle \mathcal{T}^{a\neq b}\rangle''\,,\\
\tilde{\Phi}'' + 8H\tilde{\Phi}' + 4\left(H' + 4H^2\right)\tilde{\Phi} &= -\left(H' + 4H^2\right)\frac{\langle \mathcal{T}^{00}\rangle}{\pi_0} - \frac{2H\langle \mathcal{T}^{00}\rangle'}{\pi_0} - \frac{\langle \mathcal{T}^{00}\rangle''}{4\pi_0} - \frac{\pi_0\,\mathrm{tr}\langle \mathcal{T}^{aa}\rangle''}{4a^4}\,,\\
\psi'' + 8H\psi' + 4\left(H' + 4H^2\right)\psi &= -\left(H' + 4H^2\right)\frac{\langle \mathcal{T}^{00}\rangle}{\pi_0} - 2H\frac{\langle \mathcal{T}^{00}\rangle'}{\pi_0} - \frac{\langle \mathcal{T}^{00}\rangle''}{4\pi_0}\\
&\quad + \frac{k^2\pi_0\langle \mathcal{T}^{a\neq b}\rangle''}{6k_a k_b a^4} + \frac{\pi_0\,\mathrm{tr}\langle \mathcal{T}^{aa}\rangle''}{12a^4}\,.
\end{aligned} \tag{7.20}$$

We proceed to analyse squeezed and oscillating modes separately, due to their differing late time limits.

The perturbative analysis for a flat FLRW spacetime filled with four massless scalar fields was completed in sec. 3.4.2. Due to its comparative simplicity, we will focus on comparing the dynamics of the scalar perturbation E as obtained from the quantum theory to those of GR. As the effective Friedmann equation derived in (7.12) has the late time limit of general relativity with a single scalar field devoid of the spatial field contribution, we compare the effective GFT perturbation equations to the single field case as well. The classical equation of motion for E in a scenario with four fields as well as a single massless scalar field in GR read, respectively:

$$\text{four fields:}\quad \frac{\pi_0^2}{a^4}E'' + k^2 E = -2\kappa E, \qquad\qquad \text{single field:}\quad \frac{\pi_0^2}{a^4}E'' + k^2 E = 0\,. \tag{7.21}$$

In principle, one could carry out a comparative analysis for all scalar perturbation variables, however, as we will find a considerable mismatch between effective GFT dynamics and general relativity, focusing on E should suffice at this stage. The full analysis would become relevant once agreement with general relativity has been established in the late time regime.

7.4.1. Squeezed modes

The inhomogeneous squeezed modes, which we recall have $\omega_k^2 > 0$, have similar dynamics to the background mode with additional $k-$dependent terms. In particular, all components of $\langle \mathcal{T}_k^{AB} \rangle$ grow exponentially. To obtain explicitly their dynamics from (7.19) it is useful to define the following expressions (we assume $\omega_{k/2} > 0$)

$$\begin{aligned}\langle A_{k/2}^\dagger A_{k/2} \rangle &= \frac{1}{2} e^{-\frac{k^2}{4s^2}} \left((\mathcal{A} - \mathrm{sgn}(\mathcal{K}^{(2)})\mathcal{B})^2 e^{2\omega_{k/2}\chi^0} + (\mathcal{A} + \mathrm{sgn}(\mathcal{K}^{(2)})\mathcal{B})^2 e^{-2\omega_{k/2}\chi^0} \right) \\ &=: \mathfrak{n}_k(\chi^0)\,, \\ \langle A_{k/2}^{\dagger\,2} \rangle + \langle A_{k/2}^2 \rangle &= 2\, e^{-\frac{k^2}{4s^2}} (\mathcal{A}^2 - \mathcal{B}^2) =: \mathfrak{c}_k\,,\end{aligned} \tag{7.22}$$

in terms of which the expectation values for the GFT energy-momentum tensor (7.19) read

$$\begin{aligned}\langle \mathcal{T}_k^{00} \rangle =& \frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{2\omega_{k/2}} \left(\frac{k^2}{2} \mathfrak{n}_k(\chi^0) - m^2 \mathfrak{c}_k \right)\,, \qquad \langle \mathcal{T}_k^{a\neq b} \rangle = -\frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{8\omega_{k/2}} k_a k_b \left(2\mathfrak{n}_k(\chi^0) + \mathfrak{c}_k \right)\,, \\ \langle \mathcal{T}_k^{aa} \rangle =& \frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{2\omega_{k/2}} \left[-\left(m^2 + \frac{k_a^2}{4} \right) 2\mathfrak{n}_k(\chi^0) + \left(\frac{k^2 - k_a^2}{4} \right) \mathfrak{c}_k \right]\,, \\ \langle \mathcal{T}_k^{0b} \rangle =& \frac{\mathrm{i}\,\mathrm{sgn}(\mathcal{K}^{(2)})}{8\omega_{k/2}} k_b \mathfrak{n}_k'(\chi^0)\,.\end{aligned} \tag{7.23}$$

If then follows that $\frac{1}{3}\,\mathrm{tr}\langle \mathcal{T}_k^{aa} \rangle = \frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{2\omega_{k/2}} \Big[-\Big(m^2 + \frac{k^2}{12} \Big) 2\mathfrak{n}_k(\chi^0) + \frac{k^2}{6}\mathfrak{c}_k \Big]$, which will be a useful expression in the following analysis. For our choice of state, $\mathfrak{n}_k(\chi^0)$ corresponds to the expectation value of the number operator $N_k = \int \frac{\mathrm{d}^3\gamma}{(2\pi)^3} : A_{k-\gamma}^\dagger A_\gamma :$ (not to be confused with the lapse function), i.e. $\mathfrak{n}_k(\chi^0) = \langle N_k \rangle$. This relation is valid as long as $\sigma(\vec{k})$ (7.8) is symmetric in k and we are within the range of validity of the saddle-point approximation. In particular, the exact form of $\sigma(\vec{k})$ is irrelevant, as long as it is sufficiently peaked on the $\vec{k} = 0$ mode.

To analyse the dynamics of the energy-momentum tensor components we first note that

$\mathfrak{n}_k(\chi^0)$ satisfies the following equation of motion

$$\mathfrak{n}_k(\chi^0)'' = 4\omega_{k/2}^2 \mathfrak{n}_k(\chi^0)\,. \tag{7.24}$$

As $\mathfrak{n}_k(\chi^0)$ fully governs the dynamics of the squeezed energy-momentum tensor, the $\langle \mathcal{T}_k^{AB} \rangle$ satisfy similar dynamics, namely

$$\begin{aligned}
\langle \mathcal{T}_k^{00} \rangle'' &= 4\omega_{k/2}^2 \langle \mathcal{T}_k^{00} \rangle + \mathrm{sgn}(\mathcal{K}^{(2)}) 2\omega_{k/2} m^2 \mathfrak{c}_k\,,\\
\langle \mathcal{T}_k^{a\neq b} \rangle'' &= 4\omega_{k/2}^2 \langle \mathcal{T}_k^{a\neq b} \rangle + \mathrm{sgn}(\mathcal{K}^{(2)}) \frac{k_a k_b}{2} \omega_{k/2} \mathfrak{c}_k\,,\\
\langle \mathcal{T}_k^{aa} \rangle'' &= 4\omega_{k/2}^2 \langle \mathcal{T}_k^{aa} \rangle - \mathrm{sgn}(\mathcal{K}^{(2)}) \omega_{k/2} \left(\frac{k^2 - k_a^2}{2} \right) \mathfrak{c}_k\,,\\
\langle \mathcal{T}_k^{0b} \rangle'' &= 4\omega_{k/2}^2 \langle \mathcal{T}_k^{0b} \rangle\,,
\end{aligned} \tag{7.25}$$

from which it follows that $\frac{1}{3}\,\mathrm{tr}\langle \mathcal{T}_k^{aa} \rangle'' = \frac{4}{3}\omega_{k/2}^2\,\mathrm{tr}\langle \mathcal{T}_k^{aa} \rangle - \mathrm{sgn}(\mathcal{K}^{(2)})\omega_{k/2}\frac{k^2}{3}\mathfrak{c}_k$. They also satisfy ($k_a \neq 0$, $k_b \neq 0$)

$$\langle \mathcal{T}_k^{00} \rangle' = -2\,\mathrm{i}\,\frac{k^2}{k_a}\langle \mathcal{T}_k^{0a} \rangle, \qquad \langle \mathcal{T}_k^{a\neq b} \rangle' = 2\,\mathrm{i}\,k_a \langle \mathcal{T}_k^{0b} \rangle, \qquad \langle \mathcal{T}_k^{aa} \rangle' = 2\,\mathrm{i}\,\left(4m^2 + k_a^2\right)\frac{\langle \mathcal{T}_k^{0b} \rangle}{k_b}\,, \tag{7.26}$$

where the index b on the right-hand side of the last expression can refer to any spacetime component of the energy-momentum tensor. Note in particular that due to the exponential growth of $\mathfrak{n}_k(\chi^0)$, the constant terms in the expressions can be neglected at late times, leading to closed second order equations for the $\langle \mathcal{T}_k^{AB} \rangle$ that are exactly those of the number operator.

The comparison of (7.2) and (7.23) allows a naive identification regarding the nature of the perturbations focusing on the matching of factors of k, in particular, the $\langle \mathcal{T}^{AB} \rangle$ resulting from our state choice are consistent with purely scalar perturbations. We first note that the overall factor of k_b in $\langle \mathcal{T}^{0b} \rangle$ is consistent with vanishing vector modes $B_a^V = 0$. Similarly, from $\langle \mathcal{T}^{a\neq b} \rangle$ we find $E_{a\neq b}^T = 0$ and $\partial_a E_b^V + \partial_b E_a^V = 0$ ($a \neq b$); we also get $\partial_a E_a^V = 0$ from $\langle \mathcal{T}^{aa} \rangle$. Finally, we conclude that $E_{aa}^T = 0$ by noticing that the k_a^2 terms in $\langle \mathcal{T}^{aa} \rangle$ give exactly the $k_a^2 E$ term in j^{aa}, using the identification $T^{a\neq b} = j^{a\neq b}$. The possibility of obtaining vector and tensor perturbations from the effective GFT metric we construct here is should be clarified in future studies; in what follows we focus solely on scalar perturbations.

We can then use the above results and the relations found in (7.6) to write down explicit expressions for the scalar metric perturbations arising from squeezed modes[7]

$$
\begin{aligned}
E =& \frac{1}{16\omega_{k/2}} \frac{\pi_0}{a^4} (\mathfrak{c}_k + 2\mathfrak{n}_k(\chi^0)) \,, \\
B =& \frac{1}{8\omega_{k/2}} \frac{1}{a^2} \mathfrak{n}_k(\chi^0)' \,, \\
\psi =& \frac{1}{16\omega_{k/2}} \left(2\frac{m^2}{\pi_0} \mathfrak{c}_k - \mathfrak{n}_k(\chi^0) \left(k^2 \left(\frac{\pi_0}{a^4} + \frac{1}{\pi_0} \right) + \frac{4m^2\pi_0}{a^4} \right) \right) \,, \\
\tilde{\Phi} =& - \frac{1}{16\omega_{k/2}} \left(\mathfrak{c}_k \left(\frac{k^2\pi_0}{a^4} - \frac{2m^2}{\pi_0} \right) + \mathfrak{n}_k(\chi^0) \left(k^2 \left(-\frac{\pi_0}{a^4} + \frac{1}{\pi_0} \right) - \frac{12m^2\pi_0}{a^4} \right) \right) \,.
\end{aligned}
\tag{7.27}
$$

From the effective expressions above we can make some basic observations regarding the behaviour of perturbations arising from squeezed modes:

- The magnitude of perturbations at the bounce, where we have $a^4 = \pi_0^2$, is determined by their wavenumber. Recalling that the amplitudes of $\mathfrak{n}_k(\chi^0)$ and $\mathfrak{c}_k$ as defined in (7.22) scale as $e^{-\frac{k^2}{4s^2}}$, we conclude that there is a minimum value of k for which perturbations can be assumed to be small at the bounce. This value is determined by the value of s, which regulates the peakedness of the state (7.7), and can therefore be made arbitrarily small. Nevertheless, for our state choice there are always small, but non-zero $k-$modes that are of the same order as the background mode. This differs from standard cosmological perturbation theory, where all perturbations are assumed to be small w.r.t. the background, and is a finite width effect of the state we are considering; the situation of standard cosmology corresponds to the case of $s \to 0$. An avenue to potentially reconcile our state choice with conventional cosmology would be to include a range of $k-$modes, with a cut-off scale determined by s, in the GFT background. One would then have to establish how such a definition of the GFT background could be translated appropriately to the classical context.

- As the universe expands, $\mathfrak{n}_k(\chi^0)$ increases and hence the perturbations grow in time. In particular, $\mathfrak{n}_k(\chi^0)/a^4$ grows (recall that at late enough times $a^4 \propto e^{2m\chi^0}$ and $\mathfrak{n}_k(\chi^0) \propto e^{2\omega_{k/2}\chi^0}$), such that all perturbations increase and are smallest/ take their minimum value at the bounce. This is undesirable from the perturbative point of view, but can be reconciled by recalling that the free GFT theory as well as the saddle-point approximation are applicable for a finite time only and furthermore,

[7] Recall the discussion above (7.11), where we concluded that the sign of the effective expressions is independent of $\mathrm{sgn}(\mathcal{K}^{(2)})$, but for concreteness we can again choose $\mathrm{sgn}(\mathcal{K}^{(2)}) = 1$ and $j^{AB} = \langle \mathcal{T}^{AB} \rangle$.

the perturbations are exponentially suppressed in k, i.e. the faster they grow, the smaller their initial amplitude. At late times, the term proportional to $\frac{k^2}{\pi_0}\mathfrak{n}_k(\chi^0)$ will be dominant in the expressions for $\tilde{\Phi}$ and ψ.[8] However, an approximation of the form $\tilde{\Phi} \approx \psi \approx \frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{16\omega_{k/2}}\frac{k^2}{\pi_0}\mathfrak{n}_k(\chi^0)$ would be invalid, as it violates (3.40), which is derived directly from the harmonic gauge conditions. The harmonic gauge conditions are equivalent to the conservation law $\partial_0\mathcal{T}^{0B} + \mathrm{i}\sum_a k_a\mathcal{T}^{aB} = 0$, which we showed holds exactly at operator level in sec. 6.2.2.

- Using the above one can obtain explicit expressions for gauge-invariant perturbation variables using (3.20) and (3.54). Recall that in the limit $\frac{\pi_0^2}{a^4} \gg 1$ we have $\psi \to \mathcal{R}$.

- In the $k \to 0$ limit, ψ and $\tilde{\Phi}$ tend towards constants. The same applies to E, however in the strict separate universe limit, E and B do not appear as they enter the description only as terms proportional to the wavenumber (or, equivalently spatial gradients, see e.g. (7.2)). The role of E and B in the separate universe limit was also discussed in sec. 3.3; a comparison of the separate universe limit of (7.6) to the results of chap. 5 will be the topic of sec. 7.5.

We proceed to analyse the concrete form of the equation of motion for the perturbation variable E arising for squeezed GFT modes and compare to its classical counterpart. The effective dynamics of the variable E can be written as (using (7.20) and (7.25))

$$E'' + 8HE' + 4(H' + 4H^2 - \omega_{k/2}^2)E + \frac{\omega_{k/2}}{4}\frac{\pi_0}{a^4}\mathfrak{c}_k = 0\,. \tag{7.28}$$

In the late time limit we can neglect the $\mathfrak{c}_k-$term as it falls off as a^{-4}, and approximate $H' \sim 0$ and $H^2 \sim \frac{m^2}{4}$ (see sec. 7.3). If we also insert $\omega_{k/2}^2 = \frac{k^2}{4} + m^2$, we find

$$E'' + 8HE' - k^2E \sim 0\,. \tag{7.29}$$

This can be simplified further by considering the expression for E': At late times, we can assume that $E \sim \frac{\pi_0}{8\omega_{k/2}a^4}\mathfrak{n}_k(\chi^0)$, again neglecting the $\mathfrak{c}_k-$term and make use of $\mathfrak{n}_k(\chi^0)' \sim 2\omega_{k/2}\mathfrak{n}_k(\chi^0)$ (see (7.22)), leading to

$$E' \sim -4HE + 2\omega_{k/2}E\,. \tag{7.30}$$

[8] Here we have assumed that the saddle-point approximation is still applicable in this regime.

For small wavenumbers $\frac{k^2}{4} \ll m^2$ we furthermore have $\omega_{k/2} \sim 2H$, such that $E' \sim 0$ and the equation of motion for E simplifies to

$$E'' - k^2 E = 0\,. \tag{7.31}$$

Comparing to (7.21) we find that the effective equation (7.31) has a Euclidean signature instead of the Lorentzian one of GR and is missing a factor of $\frac{\pi_0^2}{a^4}$. It furthermore resembles the general relativistic single field case instead of that of four massless scalar fields, which is similar to what we found for the effective Friedmann equation in sec. 7.3.

In previous works the signature of perturbations was found to be dependent on initial conditions, where both the Lorentzian as well as the Euclidean case could be recovered [151]. It is then evident that alterations to the setup we present here will be necessary to recover agreement with Lorentzian general relativity, the dynamical equations we included in (7.20) are general and can give guidance also for future work that might e.g. change the construction of the model or look at different states.

In the GR perturbation equations, the factor π_0^2/a^4 more generally reads a^2/N^2, i.e. depends on the ratio of the scale factor and the lapse, and would hence be absent in the case of conformal time $N \sim a$. The lapse is however determined by our choice of coordinate system and the expression of the conjugate momentum of the clock field, such that one would have to consider alternative matter actions to obtain a different form of N. For this reason, one might want to consider k-essence models that include a more general function of the kinetic term in the Lagrangian for the four massless scalar fields. The challenge is then to obtain a model in which $N \sim a$ and $H^2 \sim$ const. at late times; we discuss an extension of our setup to k-essence models in sec. 7.6.2.

A discussion similar to the one we included for E above could be carried out for the other three scalar perturbation variables. As these will generally suffer from similar deviations, we leave this analysis for future work, once the discrepancies to GR have been better understood.

This concludes the analysis of squeezed modes. In the following we repeat the analysis of perturbative dynamics for the case of oscillating modes.

7.4.2. Oscillating modes

We follow the equivalent procedure for oscillating modes, which occur when $\omega_{k/2}^2 < 0$. Recall that for the action used for our construction (6.5) and the definition of $\omega_k^2 = m^2 + k^2$

this will only hold in the case where $m^2 < 0$ and for sufficiently small wavenumbers. In particular, as the saddle-point approximation reduces all frequencies to $\omega_{k/2}$, we only find oscillating modes for $k^2 < 4m^2$. For the operator expectation values appearing in (7.19) we obtain

$$\begin{aligned}
\langle A^\dagger_{k/2} A_{k/2}\rangle &= e^{-\frac{k^2}{4s^2}}(\mathcal{A}^2+\mathcal{B}^2) =: \mathfrak{d}_k\,,\\
\langle {A^\dagger_{k/2}}^2\rangle + \langle {A_{k/2}}^2\rangle &= e^{-\frac{k^2}{4s^2}}\left((\mathcal{A}-\mathrm{i}\mathcal{B})^2 e^{-2\mathrm{i}\mathrm{sgn}(\mathcal{K}^{(2)})|\omega_{k/2}|\chi^0} + (\mathcal{A}+\mathrm{i}\mathcal{B})^2 e^{2\mathrm{i}\mathrm{sgn}(\mathcal{K}^{(2)})|\omega_{k/2}|\chi^0}\right)\\
&= e^{-\frac{k^2}{4s^2}} 2\left((\mathcal{A}^2-\mathcal{B}^2)\cos\big(2|\omega_{k/2}|\big) + 2\mathrm{sgn}(\mathcal{K}^{(2)})\mathcal{A}\mathcal{B}\sin\big(2|\omega_{k/2}|\big)\right)\\
&=: \mathfrak{f}_k(\chi^0)\,,
\end{aligned} \tag{7.32}$$

which leads to

$$\begin{aligned}
\langle \mathcal{T}^{00}_k\rangle &= \frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{4|\omega_{k/2}|}\left[k^2\mathfrak{d}_k - 2m^2\mathfrak{f}_k(\chi^0)\right],\\
\langle \mathcal{T}^{0b}_k\rangle &= \frac{\mathrm{i}k_b}{2}e^{-\frac{k^2}{4s^2}}\left[(\mathcal{A}^2-\mathcal{B}^2)\sin\big(2|\omega_{k/2}|\chi^0\big) - 2\mathcal{A}\mathcal{B}\cos\big(2|\omega_{k/2}|\chi^0\big)\right],\\
&= -\frac{\mathrm{i}k_b}{8|\omega_{k/2}|}\mathfrak{f}_k(\chi^0)'\\
\langle \mathcal{T}^{a\neq b}_k\rangle &= -\frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{8|\omega_{k/2}|}k_a k_b\left[2\mathfrak{d}_k + \mathfrak{f}_k(\chi^0)\right],\\
\langle \mathcal{T}^{aa}_k\rangle &= \frac{\mathrm{sgn}(\mathcal{K}^{(2)})}{4|\omega_{k/2}|}\left[-(4m^2+k_a^2)\mathfrak{d}_k + \frac{1}{2}(k^2-k_a^2)\mathfrak{f}_k(\chi^0)\right].
\end{aligned} \tag{7.33}$$

The dynamics of oscillating modes are governed by $\mathfrak{f}_k(\chi^0)$, which satisfies

$$\mathfrak{f}_k(\chi^0)'' = -4|\omega_{k/2}|^2\mathfrak{f}_k(\chi^0)\,. \tag{7.34}$$

This leads to the following equations of motion for the GFT energy-momentum tensor

$$\begin{aligned}
\langle \mathcal{T}^{00}\rangle'' &= -4\omega_{k/2}^2\langle \mathcal{T}^{00}\rangle + \mathrm{sgn}(\mathcal{K}^{(2)})|\omega_{k/2}|k^2\mathfrak{d}_k,\\
\langle \mathcal{T}^{a\neq b}\rangle'' &= -4\omega_{k/2}^2\langle \mathcal{T}^{a\neq b}\rangle - \mathrm{sgn}(\mathcal{K}^{(2)})|\omega_{k/2}|k_a k_b\mathfrak{d}_k,\\
\langle \mathcal{T}^{aa}\rangle'' &= -4\omega_{k/2}^2\langle \mathcal{T}^{aa}\rangle - \mathrm{sgn}(\mathcal{K}^{(2)})|\omega_{k/2}|\left(4m^2+k_a^2\right)\mathfrak{d}_k\,,\\
\langle \mathcal{T}^{0b}\rangle'' &= -4\omega_{k/2}^2\langle \mathcal{T}^{0b}\rangle\,.
\end{aligned} \tag{7.35}$$

Note that this mimics the dynamical equations of squeezed modes, with an opposite sign, which hints at the possibility to recover a Euclidean signature in the perturbation equations.

The expressions for the perturbation variables are those of the squeezing case (7.27), with $\mathfrak{c}_k \to \mathfrak{f}_k(\chi^0)$ and $\mathfrak{n}_k(\chi^0) \to \mathfrak{d}_k$, which is clear from comparing (7.22) and (7.32); explicitly we have ($\mathrm{sgn}(\mathcal{K}^{(2)}) = 1$)

$$\begin{aligned}
E =& \frac{\pi_0}{16|\omega_{k/2}|a^4}\left(\mathfrak{f}_k(\chi^0) + 2\mathfrak{d}_k\right) , \\
B =& \frac{1}{8|\omega_{k/2}|a^2}\mathfrak{f}_k(\chi^0)' , \\
\psi =& -\frac{1}{16|\omega_{k/2}|}\left(-2\frac{m^2}{\pi_0}\mathfrak{f}_k(\chi^0) + \mathfrak{d}_k\left(k^2\left(\frac{\pi_0}{a^4} + \frac{1}{\pi_0}\right) + 4m^2\frac{\pi_0}{a^4}\right)\right) , \\
\tilde{\Phi} =& -\frac{1}{16|\omega_{k/2}|}\left(\mathfrak{f}_k(\chi^0)\left(k^2\frac{\pi_0}{a^4} - \frac{2m^2}{\pi_0}\right) + \mathfrak{d}_k\left(k^2\left(-\frac{\pi_0}{a^4} + \frac{1}{\pi_0}\right) - 12m^2\frac{\pi_0}{a^4}\right)\right) .
\end{aligned} \tag{7.36}$$

Importantly, there are no growing terms in the perturbations (they are called oscillating modes after all!), such that terms proportional to $\mathfrak{d}_k$ cannot be neglected at late times. The only applicable late time limit is that the amplitude of terms proportional to a^{-4} decreases. In particular, this implies that E and B decay whereas ψ and $\tilde{\Phi}$ oscillate around a set value.

Using (7.20) together with (7.35) we find for oscillating modes

$$E'' + 8HE' + 4E(4H^2 + H') = -4\omega_{k/2}^2 E + \frac{|\omega_{k/2}|\pi_0^2}{2a^4}\mathfrak{d}_k . \tag{7.37}$$

The late time limit is different to the squeezed case, in particular, the last term proportional to $\frac{\mathfrak{d}_k}{a^4}$ is of the same order as E and cannot be disregarded. At late times we have $4H^2 \sim m^2$ and $H' \sim 0$ leading to (using $\omega_{k/2}^2 = \frac{k^2}{4} + m^2$)

$$E'' + 8HE' + 32H^2E = -k^2E + \frac{|\omega_{k/2}|\pi_0^2}{2a^4}\mathfrak{d}_k . \tag{7.38}$$

While we recover a Lorentzian signature, the discrepancy of the a^4/π_0^2 factor remains, furthermore, unlike in the squeezed case, the H^2E term does not cancel with the m^2-dependent part of the $\omega_{k/2}^2$-term. Lastly, as no terms can be neglected at late times, E' cannot be simplified and we are left with additional terms that are absent in GR.

This concludes the analysis of scalar perturbations within our proposal to extract an effective metric from GFT for a first naive state choice. We proceed by comparing these results to those obtained from the separate universe methods we used in chap. 5.

7.5. Separate universe considerations

In chap. 5 we derived dynamics for super-horizon perturbations from a modified Friedmann equation making use of the separate universe picture, which allows us to remain agnostic about finite k dynamics. As discussed in sec. 5.3.2 the separate universe picture has its limits and to fully comprehend the implications of modified gravitational dynamics for cosmological perturbations in any given model, information about inhomogeneous dynamics is required. In what follows, we compare the evolution of ψ obtained through the methods of chap. 5 to the explicit solutions given in sec. 7.4.1.

We briefly summarise the main idea of chap. 5: For perturbations with sufficiently large wavelengths, gradients can be neglected and perturbations can be modelled in the separate universe picture, where the universe is modelled as a collection of independent, homogeneous patches. Perturbations in a single patch are then given by the difference between the value in the patch and the ensemble average and perturbation equations can be obtained by perturbing the background equations.

As discussed in sec. 7.3, the effective Friedmann equation (7.12) we recover from an effective GFT metric has the late time limit of general relativity with a single massless scalar field. The method of chap. 5 pre-supposes that the modified Friedmann equation agrees with general relativity at late times, and therefore we neglect rod field contributions to the energy density, such that $\rho = \frac{\pi_0^2}{2a^6}$.

Using the results of chap. 5, for the general Ansatz that the Friedmann equation is given by $H^2 = \frac{\kappa}{3}N^2\rho\mathcal{F}$, which in the case of a single massless scalar field that serves as a clock reduces to $H^2 = \frac{\kappa}{6}\mathcal{F}$, we have the following equation of motion for ψ (5.31):

$$-\psi' = H\frac{\delta\mathcal{F}}{2\mathcal{F}}\,. \tag{7.39}$$

From (7.12) we have $\mathcal{F} = 1 - \frac{\pi_0^4}{a^8}$ and $\delta\mathcal{F} = -4\frac{\pi_0^4}{a^8}\left(\frac{\delta\pi_0}{\pi_0} + 2\psi\right)$, with $\delta a = -a\psi$. Note that this differs from the effective Friedmann equation that was used as an example in sec. 5.2.2 due to the normal ordering procedure employed for the $\mathcal{T}^{AB}$ operators and the different identification of the scale factor, which here is obtained from the GFT energy-momentum tensor. Note also that we can write the right-hand side of the effective Friedmann equation (7.12) as a function of the energy density, namely $\mathcal{F} = 1 - (2\,\pi_0\,\rho)^{4/3}$. We reiterate that, as the clock field momentum π_0 does not constitute a fundamental parameter but is determined by initial conditions, the GFT effective Friedman equation belongs to the more general class of modified Friedmann equations with $\mathcal{F} \neq \mathcal{F}(\rho)$.

The equation of motion for ψ (7.39) can be solved, using the effective solution for a (7.11), namely we obtain

$$\psi = -\frac{\delta\pi_0}{2\pi_0} + \left(1 - \frac{\pi_0^4}{a^8}\right)^{\frac{1}{2}} c_\psi \,, \tag{7.40}$$

where $c_\psi \in \mathbb{R}$. At the bounce, where $a^4 = \pi_0^2$ we have $\psi = -\frac{\delta\pi_0}{2\pi_0}$, whereas in the far pre- and post bounce regime, where π_0^2/a^4 small, we have $\psi = c_\psi - \frac{\delta\pi_0}{2\pi_0}$. The effective Friedmann equation (7.12) hence leads to the special case where the evolution of ψ around the bounce is symmetric, which is rather different from the exemplary evolutions displayed in fig. 5.1c. The second term in (7.40) introduces dynamics of ψ in the vicinity of the bounce.

In the limit $k^2 \ll m^2$, the expression for ψ as given in (7.27) reads

$$\psi \sim -e^{-\frac{k^2}{4s^2}} \left(\frac{1}{2} + \frac{k^2}{m}\left(1 + \frac{a^4}{\pi_0^2}\right)\right) . \tag{7.41}$$

Note that for the effective GFT expressions, ψ in the $k \to 0$ limit is exactly constant throughout the evolution, whereas any $k \neq 0$ mode (which is strictly speaking required for a perturbation) would exponentially grow at late times. This instability is related to the Euclidean signature that we find for perturbations and we conclude that for exponentially growing solutions the separate universe limit is not applicable beyond the $k = 0$ mode. For the $k = 0$ mode agreement with the separate universe analysis (7.40) is then found only at late times.

To summarise, in the separate universe picture, we find that ψ is dynamical around the bounce point, which does not agree with the small k solution of ψ as found in the relational GFT picture; instead, for non-zero k, ψ is minimal at the bounce and continues to increase outside the bounce region . This mismatch is unsurprising, as the separate universe analysis demands that general relativity is recovered at late times, which is however not the case for the explicit perturbative expressions we found in our setup.

It would not be meaningful to carry out a separate universe analysis for oscillating modes: Their background dynamics to not lead to a general relativistic Friedmann equation at late times, and in the case were they 'live' on a background determined by squeezed modes, their dynamics are independent from the modified Friedmann equation that dictates the evolution of the background, thus making the separate universe procedure inapplicable.

7.6. Possible extensions

In our analysis of the effective GFT metric recovered for the cosmological setting, we encountered two main discrepancies:

1. The effective Friedmann equation disagrees with that of general relativity with four massless scalar fields at late times. Instead, consistent with previous literature, the Hubble rate (in the coordinate system where χ^0 serves as a clock) becomes approximately constant and resembles the case of a single scalar field as the universe expands.

2. Effective dynamics of perturbations do not agree with those of GR. In particular, we find a Euclidean signature for effective perturbations in the case of squeezed modes and a factor of a^4/π_0^2 is missing from the equations of motion.

It is then apparent that in order to recover a suitable semiclassical regime for cosmology, alterations have to be introduced to the setup described above. In this section we consider two routes to such alterations that focus on the manner in which the scalar fields are included in the theory and restrictions imposed by the setup. In particular, we consider including the clock and spatial scalar fields differently in the GFT action. On the classical side, we consider alternative scalar field actions in the form of k-essence models. We find that both cases are restricted by symmetry requirements on the form of the GFT energy-momentum tensor and the conserved classical currents. We present both considerations separately, whether and in which way k-essence models could be related to more general GFT actions is left for future work.

7.6.1. Extensions of the GFT action

In the construction of the GFT action with four massless scalar fields, one imposes the symmetries that are fulfilled by the classical scalar field action in GR, namely, shifts, rotations and reflections (sec. 2.2.3). The Laplacian on $\mathbb{R}^4$ in the GFT action (2.31) fulfills the above-mentioned symmetries. In particular, the derivatives w.r.t. the scalar fields enter with the same pre-factor to preserve the rotational symmetry. As the $E(4)$ symmetry is broken upon singling out a clock field for quantisation in the deparametrised approach to GFT, one might want to impose an $E(3)$ symmetry between the spatial fields only and allow for a different factor in front of the derivatives w.r.t the clock field in the GFT action, as was considered already in [149]. This more general Laplacian would then read $\Delta = \left(\frac{\partial}{\partial\chi^0}\right)^2 + c_a \sum_a \left(\frac{\partial}{\partial\chi^a}\right)^2$, where $c_a \in \mathbb{R}$, and lead to the free action (truncated at second order):[9]

$$S = \int \mathrm{d}^4\chi \mathcal{L}, \qquad \mathcal{L} = \sum_J \left(\frac{1}{2}\mathcal{K}_J^{(0)}\varphi_J^2 - \frac{1}{2}\mathcal{K}_J^{(2)} \left((\partial_0\varphi_J)^2 + c_a \sum_a (\partial_a\varphi_J)^2 \right) \right) . \tag{7.42}$$

The action (6.5) used in our analysis so far is recovered for $c_a = 1$.

Following the process of sec. 2.2.3 to construct a Hamiltonian from the GFT action, it is apparent that the additional factor in front of the spatial gradient term enters the definition of ω_k, namely $\omega_k^2 = m^2 + k^2 \;\rightarrow\; \omega_k^2 = m^2 + c_a k^2$ (see (2.32), (2.33), and (2.35)). Recall that we obtain a squeezing type Hamiltonian in the cases where ω_k^2 and the term containing time derivatives (or, equivalently, the canonical momenta) of the GFT field have a different sign in (2.35), which is the case for $\omega_k^2 > 0$. The opposite scenario ($\omega_k^2 < 0$) results in an oscillating Hamiltonian instead. If we consider the case with $c_a = -1$, we find that all modes with $m^2 < 0$ are oscillating modes. For $m^2 > 0$, we find squeezed modes only in the cases where $k^2 < m^2$; all other modes are of oscillating type. (For $c_a = 1$, which we considered so far, a change in mode type occurs in the case of $m^2 < 0$, where all modes with $k^2 < m^2$ are oscillating modes, whereas for large $k-$values all modes are of squeezing type.) This might be phenomenologically desirable as it limits the number of squeezed (and therefore, exponentially growing) modes.

Despite this enticing feature, the symmetry requirements on the GFT energy-momentum tensor (6.6) exclude such an alteration to the GFT action. In particular, in absence of the $E(4)$ symmetry the GFT energy-momentum tensor T^{AB} is no longer symmetric, as

[9] We could have more generally also introduced an arbitrary parameter in front of the $\left(\frac{\partial}{\partial\chi^0}\right)^2$ term, but this could be reabsorbed into $\mathcal{K}^{(2)}$ below, so we omit it.

$T^{AB} = T^{BA}$ requires that

$$\frac{\partial \mathcal{L}}{\partial(\partial_A \varphi)} \partial_B \varphi = \frac{\partial \mathcal{L}}{\partial(\partial_B \varphi)} \partial_A \varphi \,, \tag{7.43}$$

for all A,B, i.e. the Lagrangian must be invariant under a re-labelling of the scalar fields. In the Hamiltonian framework, this symmetry is broken by making a choice of clock field; but a priori this deparametrisation can be carried out with respect to any of the fields. Introducing the clock field on a different footing in the Lagrangian would break this symmetry and therefore negate the identification of T^{AB} with j^{AB} in the relational coordinate system (assuming the standard scalar field action (6.1)).

To uphold the premise of our proposal to identify the expectation values of the GFT energy-momentum tensor with the respective classical currents, any generalisation of the GFT action, must respect this symmetry. We can thus restrict the form of the action to containing only symmetric combinations of second order derivatives of the group field with respect to the χ^A. If one wanted to extend the GFT action to include e.g. higher order derivatives w.r.t. the scalar fields, such higher order terms must appear equally for all four scalar fields. Such a modification will then inevitable affect also the background dynamics and one cannot include additional terms solely for the spatial fields (which might have been desirable from a purely phenomenological perspective).

7.6.2. K-essence models

We now consider changes not on the side of GFT, but in the classical theory that we would like to find agreement with. A priori, our construction within GFT assumes only that the classical action of the scalar fields contains the same symmetries as the GFT action and makes no assumption about its specific form.[10] In the following, we first introduce a more general form of the classical scalar field action compatible with the GFT construction and continue to assess how this would affect the effective dynamics of the scale factor as recovered in sec. 7.3. Throughout this section, we assume that the GFT action is of the 'standard' form used throughout the thesis (6.5) and hence that the GFT energy-momentum tensor is symmetric.

When positing four massless scalar fields to be used as a relational coordinate system,

[10] One also needs to assume that the fields are 'good' coordinates. For instance, if the clock field does not evolve monotonically overall, one needs to restrict to a time span in which it does.

as is done in our work explained above [2], as well as in previous GFT works [78, 151, 152] it seems natural to choose the standard Lagrangian for all four massless scalar fields on the classical side ($A = 0,1,2,3$)

$$\mathcal{L}_\chi = -\frac{1}{2}\sqrt{-g}\sum_A g^{\mu\nu}\partial_\mu\chi^A\partial_\nu\chi^A\,. \tag{7.44}$$

As found in sec. 7.3, our proposal to reconstruct an effective metric from GFT gives a mismatch already at the background level with the Friedmann equation for GR with four massless scalar fields, since the effective Friedmann equation resembles GR with a single massless scalar field instead. Furthermore, the dynamics of perturbations studied in sec. 7.4 reveal a similarity with general relativistic perturbation equations in conformal time. The proposed setup requires only that the classical and quantum system exhibit a shift symmetry in the Lagrangian and in the following we extend the classical analysis to a more general Lagrangian that contains functions of the kinetic terms of the scalar fields and thereby respects the shift symmetry. In particular, we consider a Lagrangian of the form

$$\mathcal{L} = \sqrt{-g}P(X_0, X_a), \quad \text{with} \quad X_0 = -\frac{1}{2}g^{\mu\nu}\partial_\mu\chi^0\partial_\nu\chi^0,\ X_a = -\frac{1}{2}g^{\mu\nu}\partial_\mu\chi^a\partial_\nu\chi^a\,, \tag{7.45}$$

where P denotes a general function. For a flat FLRW spacetime and in a relational coordinate system with $\partial_\mu\chi^A = \delta^A_\mu$ we have $X_0 = \frac{1}{2N^2}$ and $X_a = -\frac{1}{2a^2}$. Models of this type are referred to as k-essence models, see e.g. [264, 265]. For a Lagrangian as given in (7.45), the energy-momentum tensor is given by [11]

$$^{(\chi)}T^\mu{}_\nu = \delta^\mu_\nu\,P + \sum_A \frac{\partial P}{\partial X_A}g^{\mu\alpha}\partial_\alpha\chi^A\partial_\nu\chi^A = \delta^\mu_\nu\,P + \sum_A \frac{\partial P}{\partial X_A}g^{\mu A}\delta^A_\nu\,. \tag{7.46}$$

and the classically conserved currents read

$$(j^\mu)^A = -\sqrt{-g}\left(\frac{\partial P}{\partial X_0}g^{\mu 0}\delta^A_0 + \sum_a \frac{\partial P}{\partial X_a}g^{\mu a}\delta^A_a\right)\,, \tag{7.47}$$

where we imposed relational coordinates $\partial_\mu\chi^A = \delta^A_\mu$. If we consider the time-space com-

[11] We do not carry out the sum explicitly in the last step to avoid a confusing index structure.

ponents of the currents, we find

$$(j^0)^a = -\sqrt{-g}g^{0a}\frac{\partial P}{\partial X_a}, \qquad (j^a)^0 = -\sqrt{-g}g^{0a}\frac{\partial P}{\partial X_0} \tag{7.48}$$

and therefore, in order to relate to a symmetric energy-momentum tensor on the GFT side, we must demand $j^{0a} = j^{a0}$. This imposes $\frac{\partial P}{\partial X^0} = \frac{\partial P}{\partial X^a}$, i.e. the Lagrangian has to include all fields in the same manner,

$$\begin{aligned}\mathcal{L} =& \sqrt{-g}P(X), \quad \text{with} \quad X := -\frac{1}{2}\sum_A g^{\mu\nu}\partial_\mu\chi^A\partial_\nu\chi^A = \frac{1}{2N^2} - \frac{1}{2a^2} \\ (j^\mu)^A =& -\sqrt{-g}g^{\mu A}\frac{\partial P}{\partial X},\end{aligned} \tag{7.49}$$

where we assumed a flat FLRW background and a relational coordinate system for the explicit form of X. (Note that the symmetry requirement strictly speaking only applies when considering non-diagonal metrics, as in the diagonal case we trivially have $j^{0a} = 0 = j^{a0}$; however, one might strive for a general construction that can hold for various metrics, including their perturbed variants.) The clock field momentum is given by the following, where in the last step we restricted to the flat FLRW case and assumed a Lagrangian of the form (7.49)

$$\pi_0 = -\sqrt{-g}\frac{\partial P}{\partial X_0}g^{00} = \frac{a^3}{N}\frac{\partial P}{\partial X}, \tag{7.50}$$

which upon fixing P gives an equation that can be solved for the lapse N. Recall that in a relational coordinate system with the standard matter Lagrangian (7.44), the lapse is fixed to be $N = a^3/\pi_0$. Importantly, as π_0 corresponds to the j^{00} component it is conserved also in this more general case, as long as $g_{\mu\nu}$ is diagonal (which is the case for flat FLRW at the background level).

For (7.49) and the perturbed FLRW metric given in (7.1) the classically conserved currents in the relational coordinate system take on the following form

$$\begin{aligned}
j^{00} =&\frac{a^3}{N}\left(\frac{\partial P}{\partial X}\Big(1+(-\tilde{\Phi}-3\psi+\nabla^2 E)\Big)+\delta\frac{\partial P}{\partial X}\right) ,\\
j^{0a} =&-a^2\left(\frac{\partial P}{\partial X}(-B_a^V+\partial_a B)\right) ,\\
j^{a\neq b} =&-aN\left(\frac{\partial P}{\partial X}\left(-2\partial_a\partial_b E+\partial_a E_b^V+\partial_b E_a^V-2E_{ab}^T\right)\right) ,\\
j^{aa} =&-aN\left(\frac{\partial P}{\partial X}\Big(1+\tilde{\Phi}-\psi+\nabla^2 E-2\partial_a^2 E+2\partial_a E_a^V-2E_{aa}^T\Big)+\delta\frac{\partial P}{\partial X}\right) .
\end{aligned} \tag{7.51}$$

In the following we consider the background dynamics for k-essence models of the form (7.49), where we limit our attention to squeezed modes. We focus on two aspects: Firstly, the impact of a changed matter action on the effective dynamics for the background mode; secondly, whether the change in matter Lagrangian could lead to conformal time and thereby aid in the mismatch of the perturbation equations.

At the background level, the identification $j^{AB}=\langle\mathcal{T}^{AB}\rangle$ leads to the following expressions (the expectation values $\langle\mathcal{T}^{AB}\rangle$ are the same as in (7.9), as we assume that all GFT related aspects remain unchanged; we also use $\mathrm{sgn}(\mathcal{K}^{(2)})=1$)

$$\begin{aligned}
\langle\mathcal{T}_0^{00}\rangle =&|m|(\mathcal{B}^2-\mathcal{A}^2)=\frac{a^3}{N}\frac{\partial P}{\partial X} ,\\
\langle\mathcal{T}_0^{aa}\rangle =&-|m|\left((\mathcal{A}^2+\mathcal{B}^2)\cosh\big(2|m|\chi^0\big)-2\,\mathcal{A}\mathcal{B}\sinh\big(2|m|\chi^0\big)\right)\\
=&-aN\frac{\partial P}{\partial X} ,\\
\langle\mathcal{T}_0^{0a}\rangle =&0, \qquad \langle\mathcal{T}_{k=0}^{a\neq b}\rangle=0 .
\end{aligned} \tag{7.52}$$

For $P=X$ we recover the expressions reported in sec. 7.3 and $N=a^3/\pi_0$ from (7.50). The identification with the operator expectation values (7.52) reveals that $\langle\mathcal{T}_0^{00}\rangle=\frac{a^3}{N}\frac{\partial P}{\partial X}\overset{!}{=}$ const. For an expanding universe, the value of the scale factor should be increasing and as we have $\langle\mathcal{T}_0^{aa}\rangle\propto e^{2m\chi^0}$ at late times, it seems justified to demand that $-\langle\mathcal{T}_0^{aa}\rangle=aN\frac{\partial P}{\partial X}\overset{!}{\propto}a^n$ (approximately) with $n>0$.[12] Furthermore, combining the expressions above we obtain $\langle\mathcal{T}_0^{aa}\rangle/\langle\mathcal{T}_0^{00}\rangle=N^2/a^2$ and thus $N^2/a^2\propto a^n$, which excludes the case of conformal time ($N=a$).

[12] Note that the exact value of n does not matter and will give a similar Friedmann equation with a different value of m, which is fixed by the late time limit.

From (7.46) we can obtain the energy density and the Friedmann equation:

$$-\rho = {}^{(x)}T^0_{\ 0} = P + \frac{\partial P}{\partial X} g^{00}, \qquad H^2 \propto \rho N^2 = -N^2 P + \frac{\partial P}{\partial X}\,. \tag{7.53}$$

We can use these expressions for some general considerations on whether it would be possible to recover a constant Hubble rate at late times, as we found in the GFT case (see sec. 7.3). First we make the assumption that the lapse can be approximated as $N \propto a^q$, at least in some regime. (In general, this need not be the case as the solution to (7.50) can be more involved.) From the condition that $N^2/a^2 \propto a^n \, n > 0$ we discussed above, it follows that $q > 1$. If we furthermore consider the late time regime in which $X \propto a^{-2}$ (7.49) we find that we obtain a constant Hubble rate at late times for $P \propto X^q$. Specifically, we have

$$N^2 P \to a^{2q} a^{-2q} \qquad \frac{\partial P}{\partial X} \to a^{-2q+2}\,, \tag{7.54}$$

where the former term dominates the late time regime. As already mentioned, solving (7.50) for $P \neq X$ is rather complicated however, and in general does not lead to a simple analytical relation of the desired form $N \propto a^q$. If we focus on the case with a single scalar field instead, where we have $X = \frac{1}{2N^2}$, and assume $P \propto X^q$, a constant Friedmann equation is obtained only for $P = X$, as $H^2 \propto N^{-2(q-1)}$ in such a scenario. This then recovers the simplest case that was studied previously. Including the function P in the classical setup therefore does not seem to aid the discrepancies in the late time Friedmann equation that were found in sec. 7.3, at least not in a straightforward manner.

As a side note, we would like to point out that if we were only concerned with the background dynamics and would not have to take the symmetry requirement that restricts the general Lagrangian (7.45) to be of the form given in (7.49) into account, we could obtain a constant Hubble rate within k-essence models. In such cases we would have the freedom to choose e.g. $P = (X_0)^u + \sum_a (X_a)^v$, with $u, v \in \mathbb{R}$, such that the conserved currents at the background level read

$$j^{00} = \pi_0 = u(X_0)^{u-1}\frac{a^3}{N} = \frac{u}{2^{u-1}}\frac{a^3}{N^{2\,u-1}}, \qquad j^{aa} = -v(X_a)^{v-1} a N = (-1)^v \frac{v}{2^{v-1}}\frac{N}{a^{2\,v-3}}\,. \tag{7.55}$$

We assume again that $N \propto a^q$ and use the expressions of the operator expectation values (7.52) to restrict the values of u and v. Firstly, demanding a constant clock momentum $\pi_0 = j^{00} = \text{const.}$ fixes $q = 3/(2u-1)$. Secondly, we saw that j^{aa} grows exponentially at late times, such that we recover an expanding universe if $j^{aa} \propto a^n$, with $n > 1$. Together with the above, this leads to $q - 2v + 3 > 0$. The energy density for this type of model reads

$$-\rho = (1-2u)\frac{1}{2^u N^{2u}} + (1-2v)(-1)^v \frac{3}{2^v a^{2v}}\,. \tag{7.56}$$

The Friedmann equation $H^2 \propto N^2\rho$ then has two terms: the first scales as $N^{-2u+2} \sim a^{(-2u+2)q}$ and the second as a^{-2v+2q}. One then needs to find suitable combinations of u and v that give a constant π_0, an expanding universe, and a constant Hubble rate at late times. An example is $v = 2$ and $u = \frac{5}{4}$, giving $N \propto a^2$. The discrepancy between the GFT effective Friedmann equation and the GR Friedmann equation at late times for four massless scalar fields would be resolved in this type of model.

In summary, we considered a more general form of the classical matter Lagrangian that retains the shift symmetry of the scalar fields. We saw how for a non-diagonal metric, the form of the matter Lagrangian is restricted by the symmetry of the GFT energy-momentum tensor, assuming the GFT construction remains unaltered. By considering the evolution of the expectation values of the GFT energy-momentum tensor, which determine the scale factor and the value of the clock momentum, we could impose restrictions on models that give desirable phenomenology, for instance we could conclude that $N^2/a^2 \neq$ const. In the case where one is interested in a flat FLRW metric at the background level only, where off-diagonal components of the classical currents trivially vanish, we showed that some k-essence models allow to recover a constant Friedmann equation at late times even when the matter content is given by four massless scalar fields, which agrees with the effective Friedmann equation we found from GFT in sec. 7.3. As such models cannot be extended to the perturbative analysis however, the impact of this result should be seen as limited.

7.7. Conclusion: Insights from cosmology

The objective of this chapter was to reconstruct an effective metric from GFT for a universe described by a perturbed flat FLRW metric using the process proposed in chap. 6. The interest in the outcome of this investigation is twofold: On the one hand, we wish to establish the usefulness of the suggested effective GFT metric in connection to GR; on the other, we are interested in any phenomenological results that may be found, particularly for cosmological perturbations.

After re-capping the classical scenario for a perturbed FLRW metric we gave explicit expression for the general relativistic Noether currents and focused on the relation between background and perturbative variables to the expectation values of the GFT energy-momentum tensor.

While our proposal is very general, in the sense that an effective metric can be associated with any state that is sufficiently semiclassical, such that operator expectation values can be considered as effective quantities, the particular choice of state governs the specific form of such a metric and its symmetries. For our analysis, we chose a Fock coherent state highly peaked on the homogeneous $\vec{k} = 0$ mode. Fock coherent states are commonly used in the GFT literature as they satisfy the requirement of semiclassicality; peaking around the background mode reflects the spacetime symmetries that should be captured by the effective metric.

We examined the effective background metric resulting from the $k = 0$ modes of the GFT energy-momentum tensor. We found that the resulting canonically conjugate momentum of the clock field is conserved in time, as dictated by the Klein–Gordon equation, and for squeezed modes the effective scale factor generates a bouncing universe, thus resolving the singularity. Remarkably, only half of the possible space of initial conditions lead to an effective Lorentzian metric, while the other half results in a metric with Euclidean signature. This finding is consistent with the fact that the fundamental assumptions underlying our GFT model (specifically, the choice of a compact gauge group and the omission of interactions) would be compatible with both Lorentzian and Euclidean models of quantum gravity. Furthermore, the signature of the metric cannot be deduced from a Friedmann equation alone, which can take a similar form in both cases. At the level of perturbations, the spacetime signature becomes apparent from the equations of motion and in this context, the studies by [151, 152] similarly find that the effective signature is determined by the initial conditions rather than e.g. the choice of a compact or non-compact gauge group.

Our expression for the scale factor results in an effective Friedmann equation that agrees with general relativity coupled to a single scalar field at late times. We find a relation between the number operator and the effective scale factor that differs from the literature, namely $\langle N \rangle \propto a^4$ instead of $\langle N \rangle \propto a^3$. Squeezed modes give a bouncing universe similar to previous studies. For the contributions arising from oscillating modes, we find that a^4 can take negative values, therefore, it seems that such modes can only appear in conjunction with one or more squeezed modes. While the effective Friedmann equation we recover mirrors the standard GFT result for homogeneous cosmology, there are discrepancies with the classical Friedmann equation: given that the spatial coordinate fields have non-vanishing gradient energy, they would be expected to contribute additional terms in the energy density that can only be neglected when $\frac{\pi_0^2}{a^4} \gg 1$. Such terms are not found in the effective GFT dynamics, so that at best one might expect a matching between GFT and classical cosmology for early times, where these terms do not yet dominate. However, we found that the bounce occurs at $\frac{\pi_0^2}{a^4} = 1$, meaning there is no such early-time regime in the GFT setting we consider here.

In the relational coordinate system we employed in the construction in chap. 6 and used in this section, the value of the Ricci scalar at the bounce is determined by initial conditions, namely the value of π_0, and unrelated to quantities like the ratio of the energy density to the Planck density. This seems to imply that the bounce could occur at low curvatures (somewhat reminiscent of what happens in the so-called μ_0-scheme of loop quantum cosmology [254]). Extending our setup to allow the spatial and clock field to have unequal gradients in the relational coordinate system might allow for a bounce scale that is independent of initial conditions.

There are various routes one might consider to remedy the tension between the two Friedmann equations. If one focuses solely on the Friedmann equation (instead of the full effective metric), one might be inclined to introduce a curvature term in the GR expressions, which would cancel the additional contribution from the spatial matter fields, thus leading to a consistent interpretation of the GFT scenario if the flatness assumption is dropped. However, such an interpretation is only viable if one is limited to the investigation of an effective Friedmann equation, whereas in our setup we have access to all metric components, and since $\left\langle \mathcal{T}_0^{ab} \right\rangle \propto \delta^{ab}$, our choice of state clearly corresponds to a flat metric. More generally, the mismatch seems to arise from a general difficulty to include spatial gradients of the scalar fields in the GFT construction. As we have seen, spatial homogeneity implies that the mean field should be peaked around $\vec{k} = 0$; indeed we have neglected all finite $\vec{k}$ contributions in our saddle-point approximation. These assumptions then also entail that the canonical momenta (or, in other words, the kinetic energy) of the spatial coordinate fields must vanish. It would not be possible to introduce non-vanishing canonical momenta without also departing from homogeneity, as discussed from a slightly different perspective in [78]. In our classical setup we had to assume that while the spatial coordinate fields are not homogeneous in space (they would not be good coordinates otherwise), their energy-momentum tensor is. It may be that in our GFT scenario the imposition of spatial homogeneity by peaking on $\vec{k} = 0$ actually imposes a stronger condition of homogeneity on the matter fields themselves, which would be incompatible with the presence of gradient energy, and imply an inconsistency in our starting point of assuming that the matter fields are good spatial coordinates.

Our results here are consistent with previous literature, in which the spatial coordinate fields are either simply assumed to be negligible at background level [151, 152], or where effective Friedmann dynamics only show contributions from additional matter fields if one chooses a state peaked around $\vec{k}_0 \neq 0$ [78]. Introducing gradient energy into the effective Friedmann equation might require entirely different types of states, for instance states built from multiple Peter–Weyl modes. It might also require including the effects of GFT interactions, which we neglected here; as stated in sec. 4.1.2, this assumption usually applies to the early universe since interactions generally dominate at late times. If the late-time limit of a suitably defined interacting GFT matches with the expression for general relativity with four scalar fields, this could be seen as a phenomenological constraint on the allowed types of GFT interactions. A possible alternative to the model we studied would be to introduce an additional scalar matter field as in [151, 152]. This fifth field is not interpreted as a relational coordinate and has its own independent initial conditions; if this field dominates over the coordinate fields at some initial time, such a scenario might yield an intermediate regime where the effective Friedmann equation matches that of general relativity, before spatial gradient terms would be expected to dominate. Whether this could be realised in our new approach needs to be studied in more detail.

In succession to the study of effective background dynamics, we turned our focus to the perturbations arising from an effective GFT metric. With our new proposal, we were able to reconstruct expressions for all scalar metric perturbations explicitly for the first time in the GFT literature. We established general equations of motion for scalar perturbation variables in terms of the effective operator dynamics, which are independent of a specific state choice. These can hence be used for any state proposal that goes beyond the example we consider here, or for more general models with alternative operator dynamics that can occur e.g. through modification to the GFT action. Continuing with a concrete example, we used the same state as for the effective background metric, as non-zero

$k-$modes, which we interpret as perturbations, will always be excited in addition to the background mode. We again applied the saddle-point approximation, which restricts the validity of our results to perturbations with sufficiently small wavenumbers. For our state choice small wavelength perturbations are initially exponentially suppressed and neglecting interactions limits the validity of our analysis to a finite time. We considered the case of oscillating and squeezed modes separately, where for both mode types the dynamics of perturbations are naturally very similar to those we found for the background. Our choice of state leads to expressions for the GFT energy-momentum tensor components that are compatible with the interpretation of recovering only scalar perturbations, even though in principle the $\langle \mathcal{T}^{AB} \rangle$ contain all perturbation types. The effective scalar perturbations we found for squeezed modes grow away from the bounce, excluding a consistent interpretation as small alterations to a homogeneous background after a certain point in the evolution. Comparing the equation of motion for the perturbation variable E to those obtained in GR for the case of a single, as well as four massless scalar fields, revealed the following discrepancies to the effective dynamics: Firstly, the dynamics of the effective perturbation have a Euclidean signature instead of a Lorentzian one. Secondly, they resemble the general relativistic dynamics one might expect in the case of conformal time, whereas in the relational coordinate system the GR expressions contain a relative factor a^4/π_0^2. Finally, the late time limit of the effective dynamics for E resemble (save for the aforementioned discrepancies) the single field case of GR, similar to what we found for the effective Friedmann equation.

A comparison of the effective dynamics for ψ to those that would result from the separate universe picture as studied in chap. 5 revealed that an agreement can only be found for the $k = 0$ mode. This is maybe unsurprising, as the effective perturbations exhibit exponential growth at late times and as such disagree with GR - an agreement with GR in the classical limit is however a prerequisite for the applicability of the separate universe procedure. To recover a bouncing universe at the background level, we saw that at least one squeezed mode needs to be excited - for our state choice we then inevitably encounter perturbations that are of squeezing type.

Oscillating modes on the other hand, remain finite in amplitude throughout the evolution of the universe. While recovering a Lorentzian signature in the dynamical equations for the effective perturbation E, we again encountered the same discrepancy regarding a dynamical factor of a^4/π_0^2, moreover, additional terms which cancelled in the squeezing case and are not present in the GR dynamics arise.

To obtain effective dynamics that match GR alterations to the setup presented are necessary. These can go in several directions. We considered extensions to the construction proposed here in the direction of altering the GFT action or the action of the scalar fields in the GR setting. (These are studied separately, assuming in each case that the other remains unchanged.) Both cases lead to interesting effects and are limited by symmetry requirements on $\mathcal{T}^{AB}$ and j^{AB}, which are crucial to allow a consistent identification of these two quantities with one another. Adjusting how the derivatives w.r.t. the clock and spatial fields enter the GFT action has the potential to alter which values of ω_k result in oscillating or squeezed modes. Interestingly, it is possible to introduce a maximum wavenumber for squeezed modes, such that all modes with larger k will be of oscillating type. In the case of k-essence models we assessed whether it is possible to find a form of the matter Lagrangian that gives conformal time and a constant general relativistic Friedmann equation at late times. The desired result can only be achieved if one is concerned solely with the background metric (which, clearly, we are not, as we explicitly want to study also the perturbations, the access to which makes the proposal of a GFT metric so appealing). In this case, the clock and rod fields can be included in the action in such a way that one recovers a constant Friedmann equation in GR with four massless scalar fields, thus matching the result of the GFT case.

There is a plethora of possibilities to extend the results of this chapter in future studies, some of which we mention below

- Alternative state choices. The state we considered here is characterised by a single $k-$dependent function that determines both the background as well as the perturbations. One might want to consider states that more closely resemble the scenario of standard cosmology, e.g., one could consider a state in which $\sigma(\vec{k})$ is given by a delta-peak for the background mode plus a small $k-$dependent contribution for the perturbations, which can exhibit an entirely different spectrum. As discussed in sec. 4.1.4, such state choices have been considered in past studies of perturbations in GFT [151, 152, 222].

- Changes to the operator dynamics. The dynamics of perturbations are directly related to the dynamics of the components of the GFT energy-momentum tensor. The dynamics of the $\mathcal{T}^{AB}$ are determined by the GFT action and while the state plays the important role of imposing homogeneity, the resulting physical system is also dictated by the GFT dynamics. Including higher order kinetic terms or interactions in the GFT action might then lead to effective dynamics that are closer to those of GR at late times.

- Including additional group fields. In [152], the authors consider a GFT model which includes two types of GFT field, a so-called ‘space-like’ and a ‘time-like’ one, which enter the GFT action in a different manner. Their interplay allows one to recover dynamical equations for the perturbations of the spatial volume element that agree with GR in a certain limit. Similar extensions would change the explicit form of the GFT energy-momentum tensor and symmetry requirements would likely impose certain conditions (and possibly limitations) on such a construction.

In conclusion, the effective metric we reconstruct here can consistently be interpreted as a flat FLRW metric at the background level, which is a most welcome result. The effective Friedmann equation differs from previous GFT literature, as the scale factor is no longer recovered from a relation to the number operator (via the volume operator). Furthermore, the effective Friedmann equation does not match that of GR in the case of four massless scalar fields, but rather recovers the single field case. The exceptional advantage of having access to a reconstructed metric lies in the fact that we can explicitly reconstruct any combination of perturbative quantities, in particular, we can construct effective gauge-invariant perturbations. This is of particular interest as gauge-invariant quantities are those that can be related to observations. Access to such quantities goes beyond previous GFT literature, which was limited to the study of perturbations of the volume operator. For the construction used here we find a mismatch between the dynamics of effective perturbations and the evolution obtained in GR at late times for oscillating as well as squeezed modes, which needs to be reconciled if one wishes to proceed with a phenomenological analysis of quantum gravity effects on perturbations. Similarity to previous GFT results for perturbations in a relational framework, which displayed similar discrepancies at first [151] but could be resolved in later work [152], give hope that a reconciliation is feasible. Indeed, as we have seen, possibilities to extend the results presented here are plentiful.

Finally, we emphasise that the setup from chap. 6 is general and not limited to the cosmological framework. The usefulness of the proposal to reconstruct an effective GFT metric should be established by investigating its application also outside of the context of homogeneous and isotropic cosmology. Here, anisotropic Bianchi models might be best suited and black hole spacetimes would be of particular phenomenological interest. If proven suitable to obtain a variety of spacetimes, the effective GFT metric could pave the way for a variety of fruitful future research directions.

Chapter 8.

Conclusion and outlook

As all things, this thesis must find its end. We summarise our results and give modest suggestions for further research directions.

8.1. Conclusion: Quantum gravity and the universe

'Sometimes the only scientific answer we can give is "We don't know".'

- Sabine Hossenfelder.

'Manchmal lautet die einzig wissenschaftliche Antwort auf eine Frage "Das wissen wir nicht".'

Our quest for describing perturbations within a quantum theory of gravity sent us on a journey that touched upon various topics and concepts of physics.

In chap. 1 we reviewed the main concepts of general relativity and discussed alternative formulations in the form of Ashtekar-Barbero variables and the Plebanski formulation. We also touched upon the difficulty of finding observables in GR, which can be mended by focusing on relational quantities instead. We detailed the application of GR to cosmology, where our universe is described by a flat FLRW metric, where we saw that for standard matter types, one recovers the Big Bang singularity at the origin of the universe. After briefly reviewing the main concepts of quantum mechanics and quantum field theory, we motivated the need for quantum gravity from the basic premise that the quantum nature of matter is well-established and the Einstein equations posit an equivalence between geometry and matter. We focused on two background independent approaches to quantum gravity, namely loop quantum gravity, which is based on the Ashtekar-Barbero formulation of GR, and spin foams, which rely on a path integral formulation. In both frameworks, spin networks play a crucial role: they give the kinematical Hilbert space of LQG and the boundary states for the path integral in spin foams. Spin networks can be depicted as graphs which in turn can be interpreted as triangulations of spatial slices.

In chap. 2 we introduced the GFT framework and gave a detailed account of the specific formulation of GFT we use in following chapters. We saw how GFT can be motivated from loop quantum gravity and spin foams approaches, leading to an interpretation of GFT quanta in terms of spin network vertices. However at this point, GFT has become an approach of its own, especially with regards to cosmological applications. As GFT is formulated on an abstract group manifold, there is initially no notion of spacetime, instead, spacetime is emergent. The classical limit in which the theory should recover GR is reached for a large number of GFT quanta, which are to be understood as the 'building blocks of space'. Dynamics are introduced to the theory by coupling a massless scalar field that serves as a relational clock to the group field. We detailed the specific form of the free GFT action used throughout this thesis before describing the two quantisation schemes commonly found in the literature, namely the deparametrised and the algebraic approach. The former is based on a GFT Hamiltonian and is the approach we use throughout. We gave details of the Hamiltonian framework with four massless scalar fields that are used to span a local relational coordinate system as an extension to the relational time coordinate.

In chap. 3 we provided more details on the well-established field of cosmological perturbation theory, where one includes small deviations from exact homogeneity in the description of the universe. Due to the ambiguity of the coordinate system these perturbations live in, this induces an additional gauge ambiguity, and only gauge-invariant quantities can be related to observations. We introduced the separate universe framework as a useful description for large-scale perturbations, which forms the basis of our results in chap. 5. Finally, we carried out a perturbative analysis in a universe filled with one as well as four massless scalar fields. Background independent approaches can be supplemented with matter reference frames to allow to make statements about spacetime events. In particular, a single massless scalar field can serve as a clock and four such fields can span a local Cartesian coordinate system. As such a system is rather uncommon from a cosmologist's point of view, we included an analysis of the background cosmology and perturbations in such a frame. The relational coordinate system leads to a specific lapse choice and imposes harmonic gauge conditions. The results of later chapters are compared to these perturbation equations.

In chap. 4 we focused on the application of group field theory to cosmology. In past literature, an effective evolution of the universe could be reconstructed from the expectation value of the volume operator, which is assumed to be directly related to the scale factor. We gave a historical introduction behind the choice of a Fock coherent state as a semiclassical state that is used to extract the effective evolution of the universe. We demonstrated explicitly how an effective Friedmann equation is obtained from GFT, making use of the deparametrised framework established in chap. 2. The GFT Friedmann equation introduces a bounce that resolves the Big Bang singularity and agrees with general relativity at late times. We provided an overview over past results regarding cosmological perturbations in GFT. Additionally, we briefly summarised how a bouncing universe can be obtained within LQC, which applies LQG techniques to quantise the symmetry reduced setting of cosmology. Finally, we made the observation that in the vicinity of such quantum bounces, many perturbations are outside the Hubble horizon and can therefore be described by super-horizon dynamics.

We began the second part of the thesis by investigating the behaviour of perturbations around a quantum gravity bounce in chap. 5. Ideally, one would not have to find a way to include perturbations in each quantum gravity proposal separately. We explored a more general, model-agnostic approach by focusing on long-wavelength perturbations near a quantum gravity bounce. We found that one can make limited statements about whether conservation laws of GR that hold for gauge-invariant perturbation on super-horizon scales carry over to a specific theory by considering only on a modified Friedmann equation. We first established perturbation equations for a general modified form of the Friedmann equation and continued to analyse the dynamics of gauge-invariant perturbation variables. We found that one can separate modified Friedmann equation into two classes. In the case where alterations depend only on the energy density, as e.g. in LQC, the conservation laws of GR for super-horizon perturbations remain valid. This statement does not carry over to the more general case and as an example for such a scenario we illustrated how the comoving curvature perturbation $\mathcal{R}$ becomes dynamical around a GFT bounce. The curvature perturbation on equal density hypersurfaces ζ on the other hand is always conserved for adiabatic perturbations. While one cannot make more detailed statements, as no analogue to e.g. the Mukhanov–Sasaki equation can be derived for general theories, this can still hint at crucial departures from general relativity. Our analysis assumed that modifications arise in the gravitational sector, and the continuity equation remains unaltered.

Having seen that information about inhomogeneities is required to make further progress in the study of cosmological perturbations within quantum gravity, we set out to study perturbations in GFT. Similar to previous approaches in the literature we make use of a relational coordinate system spanned by four massless scalar fields to localise perturbations. We made a more general new proposal on how to extract classical spacetime physics from GFT in chap. 6: The matter reference frame introduces conserved Noether currents in the classical setting as well as in the quantum theory and we propose that these currents should be directly related. Perhaps this can be seen as taking the idea that 'physics is in the symmetries' to the extreme. We first introduced the classical currents and showed how they are explicitly related to the inverse metric in the relational coordinate system. We then proceeded to construct the Noether currents on the GFT side in the form of an energy-momentum tensor. We gave its definition in the deparametrised GFT framework, imposed a normal ordering procedure and showed that the conservation law continues to hold at operator level, independently of the choice of state. As the expectation values of the GFT energy-momentum tensor are directly related to the classical currents, they allow to reconstruct an effective metric from GFT for suitable semiclassical states.

In chap. 7 we applied our proposal to the cosmological setting. For this, we first established the form of the classical currents for a perturbed flat FLRW metric. In order to extract effective expressions for these currents from the quantum theory, we used a coherent state characterised by a highly peaked Gaussian. The $k = 0$ mode corresponds to the homogeneous background mode, whereas we interpret all other modes as perturbations. At the background level, the effective metric we recovered is exactly diagonal, strengthening the interpretation of GFT cosmology as giving a flat FLRW spacetime. We obtained an effective Friedmann equation that has a different dependence on the scale factor in comparison to previous GFT results, while it induces a bounce and recovers the same late time behaviour as previous GFT studies. However, on the GR side the spatial fields give a non-negligible contribution to the Friedmann equation, such that a mismatch arises. We furthermore investigated perturbations, where the concept of an effective metric allowed us to obtain effective expressions for all scalar metric perturbations explicitly. In principle, this would allow us to investigate also gauge-invariant quantities, which was not possible in past GFT studies. However, we found that the dynamics of perturbations as calculated in the GFT framework differ from GR in substantial ways, such that changes to the framework are required before such an investigation becomes worthwhile. We considered two naive alterations to the framework we used. The first was concerned with including the spatial fields differently to the clock field in the GFT action, which is however ruled out by symmetry demands for the GFT energy-momentum tensor. We also considered k-essence models, which are similarly restricted by symmetry requirements.

In summary, starting from the application of quantum gravity to cosmology we aimed to take a step towards establishing cosmological perturbations within such theories in order to obtain an additional guiding principle in the ongoing search for quantum gravity.

8.2. Outlook: The search must go on

'Man merkt nie, was schon getan wurde, man sieht immer nur, was noch zu tun bleibt.'

- Marie Curie.

'One never notices what has been done, one only sees what remains to be done.'

Many studies of quantum gravitational theories aim to establish a first relation to GR through the study of cosmological models. To achieve such a feat in a model independent manner requires more advanced methods than the separate universe picture we used in chap. 5. Further investigations in such directions would certainly be helpful, but their starting point might be less clear.

Regarding GFT specifically, the perhaps rather optimistic proposal of reconstructing an effective metric from the quantum theory as introduced in chap. 6 opens up the pathway to a plethora of research directions and seemingly the hardest task will be to determine which are worth pursuing. We have included here only the first naive application to a physical scenario that is more complex than the flat FLRW background and encountered rather fundamental deviations from the general relativistic system we hoped to capture. There are several possibilities to alleviate these discrepancies, some of which we already detailed in sec. 7.7. Considering results that have been achieved in previous studies of GFT perturbations, it might well be possible that an agreement with GR can be found. In general, when considering more involved models it would be interesting to establish general structures within the setup that we proposed. Specifically, one can try to address questions such as: Does the concrete form of the GFT Noether currents as encoded in the GFT energy-momentum tensor restrict the form of effective metrics? How do the properties of a GFT state translate to properties of the reconstructed spacetime? Any such further extensions could help to establish whether the notion of an effective metric is a useful one.

In the end, it may be that GFT is another attempt at quantum gravity that like its predecessors will evade the realm of verifiable claims, or prove sufficiently malleable to account for any mismatches. One must then honestly acknowledge that the quest for quantum gravity is a difficult one and one cannot hope to build a theory in a day (or decade), but ultimately is limited to chiselling away at the wall of the unknown one bit at a time. Hopefully, the combined efforts will reveal a more coherent picture one day.

Appendices

Appendix A.

Definition of a manifold and its tangent vector space

We include the definition of a manifold and detail how vectors are defined on a manifold, where our description is based on [34]. This appendix is intended to complement the introduction to general relativity we gave in sec. 1.1.

An $n-$dimensional manifold M is a topological space that can locally mapped to $\mathbb{R}^n$. To make this notion precise, consider a family of open sets $\{U_i\}$, whose union covers the set M, i.e. $\underset{i}{\cup} U_i = M$. To each U_i we can assign a homeomorphism $\varphi_i : U_i \to U_i' \in \mathbb{R}^n$, i.e. an injective map with an open image in $\mathbb{R}^n$. For a point $p \in M$ we have $\varphi(p) = \{x^1(p), \ldots, x^n(p)\}$, where we refer to $\{x^\mu(p)\}$ as the coordinate of p. In order to locate points p on the ma- nifold we need to introduce a coordinate system, which is exactly the role of the maps φ_i. The pair (U_i, φ_i) is referred to as a chart, and the set of all charts for a manifold that have a smooth transition between coordinate system $\{(U_i, \varphi_i)\}$ is called an atlas. Concretely, the requirement of a smooth transition states that for any non-empty intersection between coordinate neighbourhoods $U_i \cap U_j \neq \emptyset$ the map $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$ is (infinitely) differentiable and surjective. We can now rephrase the definition of a manifold: An $n-$dimensional manifold is a set M with a maximal atlas (an atlas that contains all allowed charts).[1]

Having local charts, we can describe points on a manifold in their assigned coordinate system. We can also consider curves $c(t)$ on a manifold $c : (a, b) \to M$, with (a, b) an open interval in $\mathbb{R}$. Let $f : M \to \mathbb{R}$ be a function whose gradient we wish to consider along the

[1] Some care needs to be taken in the case of manifolds with boundaries, which we do not describe further here.

curve $c(t)$ at a given point. For simplicity, we choose this point to correspond to $t = 0$ and demand that $t \in (a, b)$. The gradient of f is given by

$$\frac{\mathrm{d}f(c(t))}{\mathrm{d}t}\Big|_{t=0} = \frac{\partial\left(f \circ \varphi^{-1}(x)\right)}{\partial x^\mu}\frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}\Big|_{t=0} =: \left(X^\mu \frac{\partial}{\partial x^\mu}\right) f =: X[f] \tag{A.1}$$

i.e. we have defined a differential operator X that gives us the derivative of a function on the manifold along a curve. As we can see from the definition above, this differential operator is determined by the curve and its derivative at the given point, $X^\mu = \frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}|_{t=0}$, and hence all curves with the same value and first derivative at $t = 0$ define the same differential operator. X is then uniquely defined from this equivalence class of curves. The differential operators that arise from all equivalence classes of curves at a given point on M form a vector space, called the tangent vector space of M at point p, denoted T_pM. A basis for this vector space is given by $\frac{\partial}{\partial x^\mu}$ and X^μ gives the components of a vector. As we encountered in the definition of the manifold, charts can overlap and we can have different coordinate systems for the same region of a manifold. In this case, the vector X remains the same, but clearly, since the basis takes a different form in the new coordinate system, the components X^μ transform under such a change. Explicitly, we have

$$X = X^\mu \frac{\partial}{\partial x^\mu} = \tilde{X}^\mu \frac{\partial}{\partial \tilde{x}^\mu}, \qquad \tilde{X}^\mu = X^\nu \frac{\partial \tilde{x}^\mu}{\partial x^\nu}. \tag{A.2}$$

Starting from the notion of a tangent vector space at each point on the manifold, we can make use of the properties of vector spaces to define further quantities. For instance, one can consider dual vector spaces and define tensors, for which we refer the reader back to the main text in sec. 1.1.

Appendix B.

Gauge-invariant Einstein equations

We reported the expressions for the perturbed Einstein tensor and energy momentum tensor for a general lapse choice in sec. 3.1.1. In the situation we consider in the main text, the gauge is explicitly fixed by a choice of relational coordinate system. For completeness, we report gauge-invariant expressions for the Einstein tensor for a general lapse in the following and show how a gauge-invariant energy momentum tensor can be obtained.

Recall that the transformation of a quantity under a change under the perturbed coordinate system (3.15) is determined by the Lie derivative (1.8) and we have

$$x^\mu \to \widetilde{x}^\mu = x^\mu + \xi^\mu \quad \Rightarrow \quad \widetilde{\delta Q} = \delta Q + \pounds_\xi Q\,, \tag{B.1}$$

with $\xi^\mu = (\xi^0, \xi^i)$.

Gauge-invariant Einstein tensor

The Lie derivatives of the components of the Einstein tensor $G^\mu{}_\nu$ for a flat FLRW metric read

$$\pounds_\xi G^0{}_0 = \frac{6}{N^2}\left(\frac{(a')^2 N'}{a^2 N} + \frac{(a')^3}{a^3} - \frac{6a'a''}{a^2}\right)\xi^0\,, \tag{B.2}$$

$$\pounds_\xi G^0{}_i = -\frac{2}{a^2 N^3}\left(aa'N' + N\left((a')^2 - aa''\right)\right)\partial_i \xi^0\,, \tag{B.3}$$

$$\pounds_\xi G^i{}_i = \frac{2}{a^3 N^4}\left(-3a^2 a'(N')^2 + N^2\left((a')^3 - a^2 a'''\right) + a^2 N\left(3a''N' + a'N''\right)\right)\xi^0\,, \tag{B.4}$$

$$\pounds_\xi G^i{}_{i\neq j} = 0\,. \tag{B.5}$$

We observe that the following combination of perturbations transforms conveniently under a change of perturbed coordinate system, which follows directly from (3.18):

$$\frac{a}{N}\left(B - \frac{a}{N}E'\right) =: f_{B,E} \to f_{B,E} - \xi^0 \,. \tag{B.6}$$

The components of the perturbed Einstein tensor (3.10) can then be made gauge-invariant through the following combinations ($\delta G^i{}_{\neq j}$ is unaffected by a change of perturbed coordinate system; no sum over i below):

$$\begin{aligned} \delta G^0{}_0{}^{(\mathrm{gi})} &= \delta G^0{}_0 + G^0{}_0{}' f_{B,E}\,, \qquad \delta G^i{}_i{}^{(\mathrm{gi})} = \delta G^i{}_i + G^i{}_i{}' f_{B,E}\,, \\ \delta G^0{}_i{}^{(\mathrm{gi})} &= \delta G^0{}_i + (G^0{}_0 - \frac{1}{3}G^k{}_k)\partial_i f_{B,E}. \end{aligned} \tag{B.7}$$

If we insert the Bardeen potentials (3.20), we finally obtain manifestly gauge-invariant expressions for the components of the perturbed Einstein tensor

$$\begin{aligned} \delta G^0{}_0{}^{(\mathrm{gi})} &= \frac{2}{a^2N^2}\left(3aa'\Psi_B' + 3(a')^2\Phi_B - \left(N^2\nabla^2\Psi_B\right)\right)\,, \\ \delta G^0{}_i{}^{(\mathrm{gi})} &= -\frac{2}{N^2}\partial_i\left(\frac{a'}{a}\Phi_B + \Psi_B'\right)\,, \\ \delta G^i{}_i{}^{(\mathrm{gi})} &= \frac{1}{a^2N^3}\Big[-2aN'\left(2a'\Phi_B + a\Psi_B'\right) + 2N\Big(a\left(a'\left(3\Psi_B' + \Phi_B'\right) + a\Psi_B''\right)\,, \\ &\qquad + \left(2aa'' + (a')^2\right)\Phi_B\Big) + N^3(\nabla^2 - \partial_i^2)\left(-\Psi_B + \Phi_B\right)\Big] \\ \delta G^i{}_{i\neq j}{}^{(\mathrm{gi})} &= \frac{1}{a^2}\partial_i\partial_j(\Psi_B - \Phi_B)\,. \end{aligned} \tag{B.8}$$

Gauge-invariant stress energy tensor

The Lie derivatives of the background stress energy tensor are

$$\pounds_\xi T^0{}_0 = T^0{}_0{}'\xi^0, \qquad \pounds_\xi T^i{}_i = T^i{}_i{}'\xi^0, \qquad \pounds_\xi T^0{}_i = (T^0{}_0 - \frac{1}{3}T^k{}_k)\partial_i\xi^0, \qquad \pounds_\xi T^i{}_{\neq j} = 0\,, \tag{B.9}$$

such that one can obtain gauge-invariant expressions in equivalent fashion to the perturbed Einstein tensor (B.7). One can then combine the gauge-invariant expressions to obtain an equation of motion e.g. for the Bardeen variable Φ_B.

Appendix C.

Perturbations in relational coordinate systems

We explicitly include calculations to derive perturbation equations for a universe filled with a single and four massless scalar fields, discussed in sec. 3.4.1 and 3.4.2, respectively.

Choosing one field as a clock, the lapse is fixed to $N = a^3/\pi_0$. We work in harmonic gauge (see sec. 3.2.3) and for convenience we include again the harmonic gauge conditions (3.39)

$$\begin{aligned} -\frac{\pi_0}{a^4}(HB + B') + \frac{1}{a^2}(-\tilde{\Phi} + \psi + \nabla^2 E) &= 0\,, \\ -\frac{\pi_0}{a^4}\nabla^2 B + \frac{\pi_0^2}{a^6}(-\tilde{\Phi} - 3\psi + \nabla^2 E)' &= 0\,, \end{aligned} \tag{C.1}$$

and their combination (3.40)

$$\pi_0^2(-\tilde{\Phi} - 3\psi + \nabla^2 E)'' = a^4\nabla^2(-\tilde{\Phi} + \psi + \nabla^2 E)\,. \tag{C.2}$$

C.1. A single massless scalar field

We show explicitly the steps to derive the perturbative dynamics for a universe filled with a single massless scalar field that serves as a clock, working in harmonic gauge. This system was discussed in sec. 3.4.1. This analysis is similar to the discussion of the single field case in [151, 215].

The space-time components of the Einstein equations $\delta G^0{}_i = \kappa\delta T^0{}_i = 0$ give the constraint

$$H\tilde{\Phi} + \psi' = 0\,. \tag{C.3}$$

The spatial-diagonal $\delta G^i{}_j = \kappa\delta T^i{}_j = 0$ components give

$$-\frac{\pi_0}{a^4}(2HB + B') - \frac{1}{a^2}(\tilde{\Phi} - \psi) + \frac{\pi_0^2}{a^6}E'' = 0\,, \tag{C.4}$$

which together with the first harmonic gauge condition (C.1) leads to an equation of motion for E

$$E'' - \frac{a^4}{\pi_0^2}\nabla^2 E = 0\,. \tag{C.5}$$

Inserting this into (C.2) leads to

$$\pi_0^2(-\tilde{\Phi} - 3\psi)'' = a^4\nabla^2(-\tilde{\Phi} + \psi)\,. \tag{C.6}$$

Finally, (C.4) simplifies the expressions for $\delta G^0{}_0 = \kappa\delta T^0{}_0$ and $\delta G^i{}_i = \kappa\delta T^i{}_i$, leading respectively to

$$3H(H\tilde{\Phi} + \psi') + \frac{a^2}{\pi_0}\nabla^2 B - \frac{a^4}{\pi_0}\nabla^2\psi - H\nabla^2 E' = \frac{\kappa}{2}\tilde{\Phi}\,, \tag{C.7}$$

$$(2H' - 3H^2)\tilde{\Phi} + H\tilde{\Phi}' + \psi'' = -\frac{\kappa}{2}\tilde{\Phi}\,. \tag{C.8}$$

These simplify using the background equations as given in (3.38), namely $3H^2\tilde{\Phi} = \frac{\kappa}{2}\tilde{\Phi}$. Inserting $\nabla^2 B = -\frac{\pi_0}{a^2}(\tilde{\Phi}' + 3\psi' - \nabla^2 E)$ from (C.1) into (C.7) we get

$$-\frac{a^4}{\pi_0^2}\nabla^2\psi - H\tilde{\Phi}' = -\frac{a^4}{\pi_0^2}\nabla^2\psi + \psi'' = 0\,, \tag{C.9}$$

where we also made use of (C.3) and thus obtain an equation of motion for ψ, which is equivalent to that of E. From (C.8) an equation of motion for $\tilde{\Phi}$ can be derived using (C.6) to replace $\nabla^2\psi$

$$\tilde{\Phi}'' - 4H\tilde{\Phi}' - \frac{a^4}{\pi_0^2}\nabla^2\tilde{\Phi} = 0\,. \tag{C.10}$$

Using the background solution $a = a_0 e^{H\chi}$, where a_0 denotes a constant with $[a_0] = L$, explicit solutions for the perturbation variables can be found in the form of Bessel functions:

$$\begin{aligned} E =& b_1 J_0\left(\frac{e^{2Ht}k}{2H\pi_0}\right) + b_2 Y_0\left(\frac{e^{2Ht}k}{2H\pi_0}\right), \qquad \psi = c_1 J_0\left(\frac{e^{2Ht}k}{2H\pi_0}\right) + c_2 Y_0\left(\frac{e^{2Ht}k}{2H\pi_0}\right), \\ \tilde{\Phi} =& d_1 e^{2Ht} J_1\left(\frac{e^{2Ht}k}{2H\pi_0}\right) + d_2 e^{2Ht} Y_1\left(\frac{e^{2Ht}k}{2H\pi_0}\right). \end{aligned} \tag{C.11}$$

An explicit form of B then follows from e.g. the second gauge condition in (3.39). The diffeomorphism constraint (C.3) and the harmonic gauge conditions (C.1) impose additional constraints on the initial conditions of the perturbation variables.

C.2. Four massless scalar fields

In sec. 3.4.2 we reported the perturbation equations for the case of a universe filled with four massless scalar fields χ^A that also serve as coordinates $\partial_\mu \chi^A = \delta^A_\mu$. In the following we include the calculations carried out to obtain those results.

The perturbed Einstein tensor can be found in (3.45) and the perturbed energy momentum tensor for the scenario with four massless scalar fields is given in (3.53). The constraint equation $\delta G^0{}_i = \kappa T^0_{\ i}$ gives

$$-\frac{2\pi_0}{a^2}\left(H\tilde{\Phi} + \psi'\right) = \kappa B\,. \tag{C.12}$$

From $\delta G^i{}_j = \kappa \delta T^i_{\ j}$ it follows that

$$-\frac{\pi_0}{a^2}(2HB + B') - (\tilde{\Phi} - \psi) + \frac{\pi_0^2}{a^4}E'' = -2\kappa E\,, \tag{C.13}$$

which together with the first gauge condition (C.1) results in the following equation of motion for E

$$E'' - \frac{a^4}{\pi_0^2}\nabla^2 E + 2\frac{a^4}{\pi_0^2}\kappa E = 0\,. \tag{C.14}$$

The other two Einstein equations give the following relations

$$\begin{aligned}\frac{1}{3}\operatorname{tr}\delta G^i{}_i =&\frac{2\pi_0^2}{a^6}\left(2H'-3H^2\right)\tilde{\Phi}+\frac{2\pi_0^2}{a^6}H\tilde{\Phi}'+\frac{2\pi_0^2}{a^6}\psi''+\frac{4}{3}\frac{\kappa}{a^2}\nabla^2E\\ =&\kappa\left(-\frac{\pi_0^2}{a^6}\tilde{\Phi}-\frac{\psi}{a^2}+\frac{1}{3}\nabla^2\frac{E}{a^2}\right),\\ \delta G^0{}_0 =&\frac{6\pi_0^2}{a^6}H^2\tilde{\Phi}-\frac{2\pi_0^2}{a^6}H\tilde{\Phi}'-\frac{2}{a^2}\nabla^2\psi=\kappa\left(\frac{\pi_0^2}{a^6}\tilde{\Phi}-\frac{1}{a^2}(3\psi-\nabla^2E)\right),\end{aligned} \tag{C.15}$$

where we inserted (C.14) and the expression for $\nabla^2 B$ from (C.1). We can combine $-\delta G^0{}_0+\operatorname{tr}\delta G^i{}_i$ to obtain

$$\frac{6\pi_0^2}{a^6}\left(2H'-4H^2\right)\tilde{\Phi}+\frac{8\pi_0^2}{a^6}H\tilde{\Phi}'+\frac{6\pi_0^2}{a^6}\psi''+4\frac{\kappa}{a^2}\nabla^2E+\frac{2}{a^2}\nabla^2\psi=-\kappa\frac{\pi_0^2}{a^6}4\tilde{\Phi}, \tag{C.16}$$

which results in an equation for $\tilde{\Phi}$, if we use (C.2) to replace $\frac{3\pi_0^2}{a^4}\psi''+\nabla^2\psi=-\frac{\pi_0^2}{a^4}\tilde{\Phi}''+\frac{\pi_0^2}{a^4}\nabla^2E''+\nabla^2\tilde{\Phi}-\nabla^4E$ and (C.14):

$$\frac{6\pi_0^2}{a^6}(H'-2H^2)\tilde{\Phi}+\frac{4\pi_0^2}{a^6}H\tilde{\Phi}'-\frac{\pi_0^2}{a^6}\tilde{\Phi}''+\frac{\nabla^2}{a^2}\tilde{\Phi}=-2\kappa\frac{\pi_0^2}{a^6}\tilde{\Phi}. \tag{C.17}$$

Using the background equations of motion (3.51) we have $H'=\kappa\frac{a^4}{\pi_0^2}$, such that the above simplifies to

$$4H\tilde{\Phi}'-\tilde{\Phi}''+\frac{a^4}{\pi_0^2}\nabla^2\tilde{\Phi}=0 \tag{C.18}$$

and we retain the same equation of motion as in the single field case.

The combination $\delta G^0{}_0+\frac{1}{3}\operatorname{tr}\delta G^i{}_i$ on the other hand leads to

$$\frac{4\kappa}{a^2}\psi+\frac{2\pi_0^2}{a^6}\psi''-\frac{2}{a^2}\nabla^2\psi=-\kappa\frac{4}{a^2}\tilde{\Phi}, \tag{C.19}$$

where we inserted the expression for H'.

We can obtain an expression for ψ as a function of $\tilde{\Phi}$ and E by using the equations of motion for E (C.14) and $\tilde{\Phi}$ (C.18) to simplify (C.2) and finally inserting (C.19)

$$\frac{3\pi_0^2}{a^4}\psi''+\nabla^2\psi=-2\kappa\nabla^2E-4\frac{\pi_0^2}{a^4}H\tilde{\Phi}' \qquad\Rightarrow\qquad 3\kappa\tilde{\Phi}-2\frac{\pi_0^2}{a^4}H\tilde{\Phi}'-\kappa\nabla^2E=2\nabla^2\psi-3\kappa\psi. \tag{C.20}$$

Solving the equations for E and $\tilde{\Phi}$ then allow us to recover solutions for ψ from the above and B from (C.1).

Note that due to the altered background dynamics in the case of four scalar fields (3.51) finding explicit solutions for the perturbations is more involved than for the single field case. One could e.g. consider a regime in which the background dynamics are dominated by the single field such that $a \approx a_0 e^{H\chi}$ to circumvent this issue, however, we find that no such regime exists for the GFT bounce we recover in sec. 7.3.

Appendix D.

Constraint equation in Newtonian gauge

This appendix forms part of [1] and complements the results of chap. 5. The calculation below is included for completeness and does not impact the results of the main text.

In sec. 5.2.1 we showed that an analogue to the diffeomorphism constraint holds for mo- dified Friedmann equations of the general form $H^2 = \frac{\kappa}{3}N^2\rho\mathcal{F}$ in the cases where the modification depends on the energy density only, i.e. $\mathcal{F} = \mathcal{F}(\rho)$. This calculation was carried out for a general lapse function in the comoving gauge. In the following, we carry out a similar calculation in the longitudinal gauge $\psi = \tilde{\Phi}$ and for conformal time $N = a$, which is a direct generalisation of the results of [221]. To emphasise that we work in conformal time we denote the Hubble parameter as $\mathcal{H}$.

Inspired by the results in [221], we assume the following form of the constraint:

$$D := \psi' + \mathcal{H}\psi - \mathcal{A}\frac{\kappa}{2}\chi'\delta\chi\,. \tag{D.1}$$

We again show that an equation of the form (5.21)

$$\mathcal{A}D' + W\mathcal{A}D - \mathcal{A}'D = 0 \tag{D.2}$$

holds, and hence, if initial conditions are set in a regime where $D = 0$, it follows that $D' = 0$ and the constraint equation holds throughout the evolution. We use the result from chap. 5, where we showed that (D.2) is satisfied for $W = 3H - N'/N = 2\mathcal{H}$ in conformal time, so that the above reduces to

$$\mathcal{A}D' + 2\mathcal{H}\mathcal{A}D - \mathcal{A}'D = 0\,. \tag{D.3}$$

To show that this holds, we use the generalised equations for the $\mathcal{F}(\rho)$ case as given in (5.17)-(5.19) and first rewrite D' as

$$\begin{aligned}
D' =& \psi'' + \mathcal{H}'\psi + \mathcal{H}\psi' - \frac{\kappa}{2}\left(\mathcal{A}'\chi'\delta\chi + \mathcal{A}\chi''\delta\chi + \mathcal{A}\chi'\delta\chi'\right) \\
=& \psi'' + \mathcal{H}'\psi - \mathcal{H}^2\psi - \frac{\kappa\mathcal{H}}{6\mathcal{H}}a^2\mathcal{A}\,\delta\rho - \frac{\kappa}{2}\left(\mathcal{A}'\chi'\delta\chi - a^2\frac{\mathrm{d}U}{\mathrm{d}\chi}\mathcal{A}\delta\chi - 2\mathcal{H}\chi'\mathcal{A}\delta\chi + \mathcal{A}\chi'\delta\chi'\right) \\
=& \psi'' - \frac{\kappa}{6}a^2\mathcal{A}\,\delta\rho + \frac{\kappa}{2}\left(-\mathcal{A}'\chi'\delta\chi + a^2\mathcal{A}\,\delta\rho - 2\chi'\mathcal{A}\,\delta\chi' + 2\mathcal{H}\chi'\mathcal{A}\,\delta\chi\right)\,.
\end{aligned} \tag{D.4}$$

In the first step we inserted the Klein-Gordon equation $\chi'' = -a^2\frac{\mathrm{d}\tilde{V}}{\mathrm{d}\chi} - 2\mathcal{H}\chi'$ and ψ' from (5.18) and in the second made use of the background identity $\mathcal{H}' - \mathcal{H}^2 = -\frac{\kappa}{2}\chi'^2\mathcal{A}$ and used the expression for $\frac{\mathrm{d}U(\chi)}{\mathrm{d}\chi}\delta\chi$ obtained from (5.9). If we then insert ψ'' from (5.19), the expression for D' can be written as

$$D' = \kappa\delta\rho\left((\mathcal{F}_\rho + \frac{\rho}{2}\mathcal{F}_{\rho\rho})\chi'^2 + \frac{1}{3}a^2\mathcal{A}\right) + \frac{\kappa}{2}\delta\chi\left(-\mathcal{A}'\chi' + 2\mathcal{H}\chi'\mathcal{A}\right)\,. \tag{D.5}$$

Furthermore, making again use of (5.18), we can write $D = -\frac{\kappa}{6\mathcal{H}}a^2\mathcal{A}\delta\rho - \mathcal{A}\frac{\kappa}{2}\chi'\delta\chi$, such that $2\mathcal{H}\mathcal{A}D - \mathcal{A}'D$ reads

$$2\mathcal{H}\mathcal{A}D - \mathcal{A}'D = \left(-\mathcal{A}^2\frac{\kappa}{3}a^2 + \mathcal{A}'\frac{\kappa}{6\mathcal{H}}a^2\mathcal{A}\right)\delta\rho + \left(-\mathcal{H}\mathcal{A}^2\kappa\chi' + \mathcal{A}'\mathcal{A}\frac{\kappa}{2}\chi'\right)\delta\chi\,. \tag{D.6}$$

Combining the above expressions, we finally obtain

$$\mathcal{A}D' + 2\mathcal{H}\mathcal{A}D - \mathcal{A}'D = \kappa\left(\mathcal{F}_\rho + \frac{\rho}{2}\mathcal{F}_{\rho\rho}\right)\chi'^2\mathcal{A}\,\delta\rho + \mathcal{A}'\frac{\kappa}{6\mathcal{H}}a^2\mathcal{A}\,\delta\rho = 0\,, \tag{D.7}$$

where in the last step we inserted $\mathcal{A}'$ as given in (5.16).

We have thus shown explicitly that a constraint equation holds also in longitudinal gauge in the case where $\mathcal{F} = \mathcal{F}(\rho)$. Note that in this gauge the conservation of $\mathcal{R}$ is not immediately apparent from the constraint. While it can be shown explicitly, it straightforwardly follows from $\mathcal{R}' = 0$ in comoving gauge, making use of the fact that $\mathcal{R}$ is a gauge-invariant quantity.

Appendix E.

Next to leading order saddle-point approximation

In order to obtain analytical expressions for the expectation values of the GFT energy-momentum tensor in chap. 7 we made use of the saddle-point approximation (7.10) to approximate the convolutions that appear in the general expressions (6.11).

The saddle-point approximation relies on the fact that the integrand is dominated by the contribution at the maximum of the exponential. However, the functions that appear in the expressions for the energy momentum tensor generally evolve in time, such that the validity of the saddle-point approximation is limited.

Here we consider the next to leading order contribution for the background mode with $k = 0$ only. In general, the saddle-point approximation will become invalid for large values of the wave number as well. The procedure to find such limits are the same as that given below, but we leave a concrete analysis for future work. We note that the analysis for the saddle-point approximation given below was part of the results published in [2].

Expressions contained in chap. 7 consider the leading-order term in the saddle-point approximation only, effectively recovering the scenario of an infinitely peaked Gaussian, where only the $\vec{k} = 0$ mode contributes to the effective dynamics. However, as pointed out previously, in our construction this limit cannot be realised exactly and inhomogeneous modes will inevitably contribute to what is usually understood as background dynamics. In what follows we therefore include the next-to-leading-order term in the saddle-point approximation and consider its implications for the effective dynamics. We also establish the latest time at which the saddle-point approximation can be valid.

Including the next-to-leading-order term in (7.10), the saddle-point approximation reads

$$\int_a^b \mathrm{d}^3x\, g(\vec{x}) e^{-\lambda(\vec{x}-\vec{\mu})^2} \approx \sqrt{\frac{\pi}{\lambda}}^3 \left(g(\vec{\mu}) + \frac{1}{4\lambda}\Delta g(\vec{x})|_{\vec{x}=\vec{\mu}} \right) , \tag{E.1}$$

where $\lambda = \frac{1}{s^2} > 0$ and $g(\vec{x})$ is integrable, i.e., $\int_a^b |g(\vec{x})| \mathrm{d}^3x < \infty$.[1] The saddle-point approximation becomes increasingly accurate for $\lambda \to \infty$. In our case, we have $\vec{\mu} = 0$. For a derivation of the saddle-point approximation and its higher-order terms, see [266].

We apply the above to the components of the GFT energy-momentum tensor $\langle \sigma | \mathcal{T}^{AB} | \sigma \rangle$, with σ specified in (7.7) and the $\mathcal{T}^{AB}$ operators given in (6.11). Recalling that the off-diagonal components $\langle \mathcal{T}^{0b} \rangle$ and $\langle \mathcal{T}^{a\neq b} \rangle$ are exactly zero, independent of the saddle-point approximation, we need to consider only $\langle \mathcal{T}_0^{00} \rangle$ and $\langle \mathcal{T}_0^{aa} \rangle$.

For $\langle \mathcal{T}^{00} \rangle$, we have $g(\vec{\gamma}) = |\omega_\gamma| = \sqrt{\vec{\gamma}^2 + m^2}$ and thus $\Delta g(\vec{\gamma})|_{\vec{\gamma}=0} = \frac{3}{|m|}$. If we include the next-to-leading-order correction to the expression in (7.9), we obtain a small constant shift in the value of $|\pi_0| = \langle \mathcal{T}_0^{00} \rangle$, namely

$$\left\langle \mathcal{T}_0^{00} \right\rangle_{\mathrm{NLO}} \approx \mathrm{sgn}(\mathcal{K}^{(2)}) \left(|m| + \frac{3}{4|m|\lambda} \right) (\mathcal{B}^2 - \mathcal{A}^2) \,. \tag{E.2}$$

For the $\langle \mathcal{T}^{aa} \rangle$ component, on the other hand, we have more complex expressions that include also a time dependence (see (7.9)):

$$\begin{aligned} g(\vec{\gamma}) =& \left(-|\omega_\gamma| + \frac{\gamma_a^2}{|\omega_\gamma|} \right) \left((\mathcal{A}^2 + \mathcal{B}^2) \cosh\bigl(2|\omega_\gamma|\chi^0\bigr) - 2\,\mathrm{sgn}(\mathcal{K}^{(2)}) \mathcal{A}\mathcal{B} \sinh\bigl(2|\omega_\gamma|\chi^0\bigr) \right) \\ &+ \frac{\gamma_a^2}{|\omega_\gamma|} (\mathcal{A}^2 - \mathcal{B}^2) \,, \\ \Delta g(\vec{\gamma})|_{\vec{\gamma}=0} =& -\frac{1}{|m|} \left(\bigl(\mathcal{A}^2 + \mathcal{B}^2\bigr) \cosh\bigl(2|m|\chi^0\bigr) - 2\mathcal{A}\mathcal{B}\mathrm{sgn}(\mathcal{K}^{(2)}) \sinh\bigl(2|m|\chi^0\bigr) \right) \\ &- 6\chi^0 \left(\bigl(\mathcal{A}^2 + \mathcal{B}^2\bigr) \sinh\bigl(2|m|\chi^0\bigr) - 2\mathcal{A}\mathcal{B}\mathrm{sgn}(\mathcal{K}^{(2)}) \cosh\bigl(2|m|\chi^0\bigr) \right) \\ &+ \frac{2}{|m|} \left(\mathcal{A}^2 - \mathcal{B}^2 \right) \,. \end{aligned} \tag{E.3}$$

As λ is large, but finite, $g(\vec{\gamma})$ as given above will dominate the integral (E.1) for late enough times, leading to an inapplicability of the saddle-point approximation. To establish the

[1] Note that strictly speaking the integrals we consider are over the range $(-\infty, \infty)$, with $g(\vec{x})$ increasing monotonically with $\vec{x}$. We then need to limit integration to a finite range, which is tantamount to excluding modes with very large wave numbers.

maximum value of χ^0 for which the saddle-point approximation is viable, we consider the late-time limit where $g(\vec{\gamma}) \propto e^{2\omega_\gamma \chi^0}$. The integrand then behaves as $e^{-\lambda\vec{\gamma}^2} g(\vec{\gamma}) \propto e^{-\lambda\vec{\gamma}^2 + 2\omega_\gamma \chi^0}$. The saddle-point approximation is applicable at a maximum of $-\lambda\vec{\gamma}^2 + 2\omega_\gamma\chi^0$, which occurs at $\vec{\gamma} = 0$ for $\chi^0 < |m|\lambda$. Additionally, the condition $g(0) > \frac{1}{4\lambda}\Delta g(\vec{\gamma})|_{\vec{\gamma}=0}$ needs to be satisfied to ensure that $g(0)$ remains the leading-order term throughout the evolution. In the late-time limit, (E.3) translates to

$$\begin{aligned} g(0) &\approx -\frac{|m|}{2}\left(\mathcal{A} - \operatorname{sgn}(\mathcal{K}^{(2)})\mathcal{B}\right)^2 e^{2|m|\chi^0} , \\ \Delta g(\vec{\gamma})|_{\vec{\gamma}=0} &\approx -\left(\mathcal{A} - \operatorname{sgn}(\mathcal{K}^{(2)})\mathcal{B}\right)^2 e^{2|m|\chi^0}\left(\frac{1}{2|m|} + 3\chi^0\right) , \\ \Rightarrow \quad g(0) &> \frac{1}{4\lambda}\Delta g(\vec{\gamma})|_{\vec{\gamma}=0} \qquad \Leftrightarrow \qquad \chi^0 < \frac{2}{3}\lambda|m| - \frac{1}{6|m|} , \end{aligned} \tag{E.4}$$

which gives a similar, but more stringent constraint on the latest time the saddle-point approximation can be considered as valid.

Restricting to this regime, the expression for $\langle \mathcal{T}_0^{aa} \rangle$ then reads

$$\begin{aligned} \langle \mathcal{T}_0^{aa} \rangle_{\text{NLO}} \approx \operatorname{sgn}(\mathcal{K}^{(2)}) \Big[&\frac{\mathcal{A}^2 - \mathcal{B}^2}{2\lambda|m|} + \left(|m| + \frac{1}{4\lambda|m|}\right)\Big(2\mathcal{A}\mathcal{B}\operatorname{sgn}(\mathcal{K}^{(2)})\sinh\big(2|m|\chi^0\big) \\ &- \left(\mathcal{A}^2 + \mathcal{B}^2\right)\cosh\big(2|m|\chi^0\big)\Big) + \frac{3\chi^0}{2\lambda}\Big(2\mathcal{A}\mathcal{B}\operatorname{sgn}(\mathcal{K}^{(2)})\cosh\big(2|m|\chi^0\big) \\ &- \left(\mathcal{A}^2 + \mathcal{B}^2\right)\sinh\big(2|m|\chi^0\big)\Big)\Big] . \end{aligned} \tag{E.5}$$

If we recall that the above is related to the scale factor as $a^4 = -|\pi_0|\,\langle \mathcal{T}_0^{aa} \rangle$, it is apparent that the additional terms will influence the form of the effective Friedmann equation, as we have

$$\frac{a'}{a} = \frac{1}{4}\frac{\langle \mathcal{T}_0^{aa} \rangle'_{\text{NLO}}}{\langle \mathcal{T}_0^{aa} \rangle_{\text{NLO}}} = \frac{1}{4}\frac{g'(0) + \frac{1}{4\lambda}\Delta g'(\vec{\gamma})|_{\vec{\gamma}=0}}{g(0) + \frac{1}{4\lambda}\Delta g(\vec{\gamma})|_{\vec{\gamma}=0}} . \tag{E.6}$$

We first focus our attention on the implications of the next-to-leading-order terms in the late-time limit, in which the expectation value (E.5) and its logarithmic derivative

(E.6) simplify to

$$\langle \mathcal{T}_0^{aa} \rangle_{\mathrm{NLO}} \approx -\mathrm{sgn}(\mathcal{K}^{(2)}) e^{2|m|\chi^0} \left(\mathcal{A} - \mathrm{sgn}(\mathcal{K}^{(2)})\mathcal{B}\right)^2 \left(\frac{|m|}{2} + \frac{3\chi^0}{4\lambda} + \frac{1}{8\lambda|m|}\right) , \tag{E.7}$$

$$\left(\frac{\langle \mathcal{T}_0^{aa} \rangle'}{\langle \mathcal{T}_0^{aa} \rangle}\right)^2_{\mathrm{NLO}} \approx \frac{16m^2 \left(2\lambda m^2 + 3|m|\chi^0 + 2\right)^2}{\left(4\lambda m^2 + 6|m|\chi^0 + 1\right)^2} , \tag{E.8}$$

where we neglected the subdominant constant term in (E.5).

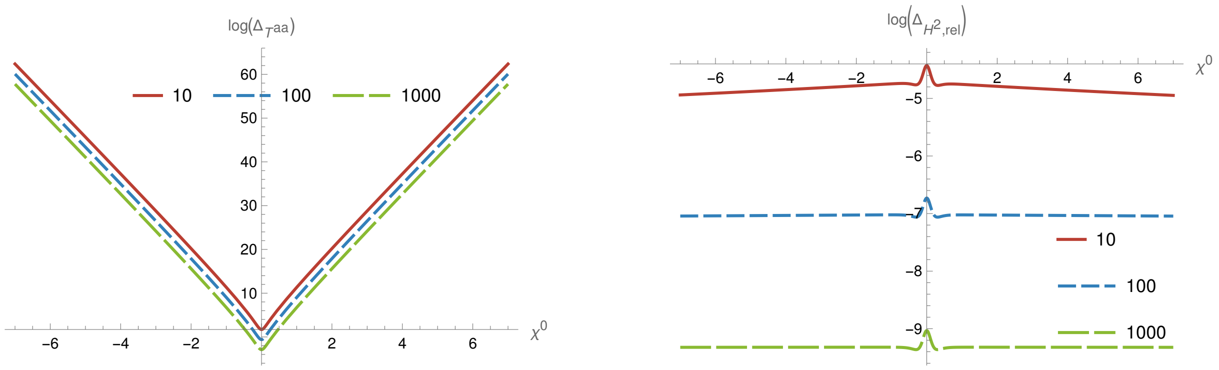
For large values of χ^0 we have $\left(\langle \mathcal{T}_0^{aa} \rangle' / \langle \mathcal{T}_0^{aa} \rangle\right)^2_{\mathrm{NLO}} \to 4m^2$ such that $H^2_{\mathrm{NLO}} \to \frac{m^2}{4}$, as we found for the leading-order saddle-point approximation in chap. 7. Note however that at the end of the period of validity for the saddle-point approximation, one has $H^2_{\mathrm{NLO}} = \frac{1}{16}(2m + \frac{3}{4\lambda m})^2$, which is slightly larger than for the leading-order contribution only.

To establish the effect of the next-to-leading-order term on the bounce behaviour we consider the absolute difference between $\langle \mathcal{T}_0^{aa} \rangle_{\mathrm{NLO}}$ and $\langle \mathcal{T}_0^{aa} \rangle_{\mathrm{LO}}$ as well as the relative difference of the leading-order and next-to-leading-order Hubble rate, defined as

$$\Delta_{H^2,\,\mathrm{rel}} := \left(\left(\frac{\langle \mathcal{T}_0^{aa} \rangle'}{\langle \mathcal{T}_0^{aa} \rangle}\right)^2_{\mathrm{NLO}} - \left(\frac{\langle \mathcal{T}_0^{aa} \rangle'}{\langle \mathcal{T}_0^{aa} \rangle}\right)^2_{\mathrm{LO}}\right) \left(\frac{\langle \mathcal{T}_0^{aa} \rangle'}{\langle \mathcal{T}_0^{aa} \rangle}\right)^{-2}_{\mathrm{LO}} , \tag{E.9}$$

where we correct for different bounce times, such that both bounces occur at $\chi^0 = 0$.[2] Both are depicted for different values of λ in fig. E.1. The qualitative bounce behaviour remains unchanged by the contribution of inhomogeneous modes. The relative error peaks around the bounce, where H^2 is minimal, and decreases with increasing χ^0, as can also be seen from (E.6). Overall, next-to-leading-order contributions have minimal impact on the dynamics, demonstrating that they can be neglected as is done in chap. 7.

[2]For the leading-order term only, the bounce happens at $\chi^0 = \frac{1}{2m} \ln\left(\frac{|\mathcal{A}+\mathrm{sgn}(\mathcal{K}^{(2)})\mathcal{B}|}{|\mathcal{A}+\mathrm{sgn}(\mathcal{K}^{(2)})\mathcal{B}|}\right)$.



(a) $\ln(|\langle \mathcal{T}_0^{aa} \rangle_{\mathrm{NLO}}| - |\langle \mathcal{T}_0^{aa} \rangle_{\mathrm{LO}}|)$. As $\mathcal{B} > \mathcal{A}$, it is clear from (E.5) that the next-to-leading-order expression is greater than the leading-order expression alone.

(b) Relative difference of H^2, as given in (E.9). The relative difference is largest around the bounce. Taking the limit $\chi^0 \to 0$ gives a finite value, as depicted above (e.g., $\Delta_{H^2,\,\mathrm{rel}} \approx 0.002$ for $\lambda = 10$).

Figure E.1.: Comparison of dynamics arising from including also the next-to-leading-order term, $\langle \mathcal{T}_0^{aa} \rangle_{\mathrm{NLO}}$, to those with the leading-order only, $\langle \mathcal{T}_0^{aa} \rangle_{\mathrm{LO}}$, for different values of λ, namely $\lambda = 10$ (red, full) , $\lambda = 100$ (blue, small dashed), $\lambda = 1000$ (green, large dashed). The other parameter values are $|m| = 4\sqrt{\pi/3}$, $\mathcal{A} = 10$, $\mathcal{B} = 20$, $\mathrm{sgn}(\mathcal{K}^{(2)}) = 1$. The times were adjusted such that the bounces coincide and occur at $\chi^0 = 0$.

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