Instantons on Asymptotically Conical $Spin(7)$-Manifolds

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Statement of Contributions

I confirm that the work submitted is my own and that appropriate credit has been given where reference has been made to the work of others.
Chapters 1 to 5 are based on the paper [28]. Chapter 6 is based on the preprint [29].

This copy has been supplied on the understanding that it is copyright material and that no quotation from the thesis may be published without proper acknowledgement.

To my parents and all my friends.

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Abstract

We develop the deformation theory of instantons on asymptotically conical $Spin(7)$-manifolds where the instanton is asymptotic to a fixed nearly G_2-instanton at infinity. By relating the deformation complex with spinors, we identify the space of infinitesimal deformations with the kernel of the twisted negative Dirac operator on the asymptotically conical $Spin(7)$-manifold.

We apply this theory to describe the deformations of the Fairlie–Nuyts–Fubini–Nicolai (FNFN) $Spin(7)$-instantons on $\mathbb{R}^8$, where $\mathbb{R}^8$ is considered to be an asymptotically conical $Spin(7)$-manifold asymptotic to the cone over S^7. We calculate the virtual dimension of the moduli space using the Atiyah–Patodi–Singer index theorem and the spectrum of the twisted Dirac operator.

We then apply the deformation theory to compute the deformations of Clarke–Oliveira's instanton on the Bryant–Salamon $Spin(7)$-Manifold. The Bryant–Salamon $Spin(7)$-Manifold $\not{S}^-(S^4)$ is an asymptotically conical manifold where the link is the squashed sphere $\frac{Sp(2)\times Sp(1)}{Sp(1)\times Sp(1)}$.

Finally, we show that with gauge groups $U(1)$ and $SU(2)$, no irreducible $Sp(2)\times U(1)$-invariant asymptotically conical instantons on $\mathbb{R}^8$ exist. Using this result, we prove that any asymptotically conical $U(1)$- or $SU(2)$-instanton on $\mathbb{R}^8$ asymptotic to the flat connection on S^7 satisfying certain conditions is obstructed.

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Chapter 1

Introduction

Instantons on 4-manifolds are connections whose curvatures are anti-self-dual. Instantons solve the Yang–Mills equation and hence have always been of interest to physicists in the contexts of quantum field theory, string theory, M-theory, supergravity etc. Instantons in dimensions higher than 4 were also studied by many physicists, Corrigan–Devchand–Fairlie–Nuyts [17], Fairlie–Nuyts [25], Fubini–Nicolai [27], before Donaldson–Thomas [19] and Donaldson–Segal [21] explained their importance and scope to a mathematical audience. Analogous to the 4-dimensional case, their prediction of the possibility to construct invariants from the moduli space has been one of the main sources of motivation behind the research on higher dimensional gauge theory for mathematicians.

The $Spin(7)$-instantons are instantons on 8-dimensional manifolds with $Spin(7)$-structures. The $Spin(7)$-instanton equation appeared in various places in the physics literature; Fairlie–Nuyts [25] and Fubini–Nicolai [27] have discussed $Spin(7)$-instantons on $\mathbb{R}^8$. Donaldson–Thomas [19] and Carrion [13] have discussed $Spin(7)$-instantons more generally, and around the same time, in 1998, Lewis also discussed $Spin(7)$-instantons in his PhD thesis [43]. In recent years, $Spin(7)$-instantons have been studied by Sá Earp [52], Tanaka [56], Walpuski [58], Lotay–Madsen [45] and many others.

1.1 Motivation for the Thesis

In this thesis, we develop the deformation theory of instantons on a particular type of non-compact $Spin(7)$-manifolds known as asymptotically conical $Spin(7)$-manifolds. These manifolds are complete $Spin(7)$-manifolds asymptotic to the cone over compact nearly G_2-manifolds.

The instantons on these manifolds also exhibit the asymptotically conical behaviour. Assuming that the instanton is unobstructed, we prove that the moduli space of these instantons is a manifold, and describe a way to calculate the (virtual) dimension. In the second part of the paper, we apply the deformation theory to certain instantons on $\mathbb{R}^8$, first constructed by Fairlie–Nuyts [25] and Fubini–Nicolai [27] independently, in the context of supergravity. The space $\mathbb{R}^8$ is indeed an asymptotically conical $Spin(7)$-manifold, and hence it is appropriate to study the deformations of these instantons using our theory. The main result for this part is the calculation of the virtual dimension of the moduli space of these instantons.

The study of asymptotically conical $Spin(7)$-manifolds goes back to 1989, when Bryant–Salamon [12] gave an example of a complete non-compact $Spin(7)$-manifold, namely, the negative spinor bundle over the 4-sphere. In 2014, Clarke [15] constructed a $Spin(7)$-instanton on this Bryant–Salamon $Spin(7)$-manifold. The manifolds $\mathbb{R}^8$, the 8-dimensional Euclidean space, and $\displaystyle{\not}S^-(S^4)$, the negative spinor bundle over the 4-sphere, are both examples of asymptotically conical $Spin(7)$-manifolds. $\mathbb{R}^8$ is asymptotic to the cone over S^7 with standard metric and $\displaystyle{\not}S^-(S^4)$ is asymptotic to the cone over S^7 with a squashed metric, where both S^7 with standard metric and S^7 with squashed metric are examples of nearly G_2-manifolds.

Asymptotically conical manifolds have been studied by many authors, e.g., asymptotically conical G_2 manifolds by Karigiannis–Lotay [36] and recently, asymptotically conical $Spin(7)$ manifolds were studied by Lehmann [42]. The analytic frameworks for studying asymptotically conical manifolds, namely, the weighted Sobolev theory and theory of asymptotically conical Fredholm and elliptic operators, have been developed by Lockhart–McOwen [44] and Marshall [47].

Our work on deformation theory in dimension 8 has been partially motivated by similar work in dimension 7, namely the deformation theory of asymptotically conical G_2-instantons, developed by Driscoll [22], utilising the works of Harland–Ivanova–Lechtenfeld–Popov [31], Charbonneau–Harland [14]. Asymptotically conical G_2-manifolds are asymptotic to the cone over nearly Kähler manifolds. Instantons on asymptotically conical G_2-manifolds have also been studied by many authors, Clarke [15], Oliveira [50], Lotay–Oliveira [46] and many others.

1.2 Outline of the Thesis

Here is a brief outline of this thesis.

After we discuss the basic notations and definitions, and fix conventions related to asymptotically conical $Spin(7)$-instantons in Chapter 2, we develop the deformation theory of asymptotically conical $Spin(7)$-instantons in Chapter 3. In the first part, we discuss the analytical framework to study instantons of asymptotically conical $Spin(7)$-manifolds. We use Lockhart–McOwen theory, and the relation between the Dirac operator on the cone and the Dirac operator on the link to show that the Dirac operator on the asymptotically conical manifold is Fredholm only when the rate of decay is not a critical weight, and the critical weights are precisely the rates that differ from the eigenvalues of the Dirac operator on the link by a fixed constant.

In the second part of Chapter 3, using the analytical framework and implicit function theorem, we prove that if the rate of decay is not a critical weight, the moduli space of asymptotically conical $Spin(7)$-instantons is a smooth manifold, given that the deformations are unobstructed; moreover the dimension of the moduli space is precisely the index of the Dirac operator on the asymptotically conical manifold.

In Chapters 4 and 5 we carry out an in-depth study of Fairlie–Nuyts–Fubini–Nicolai (FNFN) $Spin(7)$-instanton on $\mathbb{R}^8$ and its deformation theory. We apply the deformation theory developed in Chapter 3 by considering $\mathbb{R}^8$ to be the asymptotically conical $Spin(7)$-manifold asymptotic to the nearly G_2-manifold S^7.

In order to study the moduli space, we need to identify the critical weights, and hence need to calculate the eigenvalues of the Dirac operator on the link S^7 in a certain range determined by the fastest rate of convergence of FNFN-instanton. In Chapter 4 we use various techniques in representation theory and harmonic analysis, namely, the Frobenius reciprocity to decompose the space of L^2-sections of the spinor bundle into direct sums of finite dimensional Hilbert spaces indexed by $Spin(7)$-representations. Moreover, we express the Dirac operator as a sum of Casimir operators. We also calculate an eigenvalue bound which yields only six representations of $Spin(7)$ for which the eigenvalues of the Dirac operator could be in the prescribed range. Then we explicitly calculate the eigenvalues of the Dirac operator for these representations, and identify the critical rates.

In the first part of Chapter 5, we reconstruct the FNFN $Spin(7)$-instanton using algebraic techniques, by identifying S^7 with the homogeneous space $Spin(7)/G_2$. In the second part we use the Atiyah–Patodi–Singer theorem and the critical rates calculated in Chapter 4 to calculate the virtual dimension of the moduli space of the FNFN instanton. It turns out that the virtual dimensions of the moduli space are determined by precisely two known deformations of FNFN-instanton, namely dilation and translations.

Chapter 6 is devoted to computing the deformations of Clarke–Oliveira's Instanton on the Bryant–Salamon $Spin(7)$-Manifold. The Bryant–Salamon $Spin(7)$-Manifold is the negative spinor bundle of S^4 which is an asymptotically conical manifold where the link is the squashed 7-sphere $\frac{Sp(2)\times Sp(1)}{Sp(1)\times Sp(1)}$. We use the deformation theory of asymptotically conical $Spin(7)$-instantons developed in chapter 3 to calculate the deformations of Clarke–Oliveira's Instanton and calculate the virtual dimension of the moduli space.

In the final chapter, chapter 7, we prove the non-existence of irreducible $U(1)$ and $SU(2)$ asymptotically conical $Sp(2)\times U(1)$-invariant instantons on $\mathbb{R}^8$. Moreover, we prove that any asymptotically conical $U(1)$- or $SU(2)$-instantons on $\mathbb{R}^8$ asymptotic to the flat connection on S^7 satisfying certain condition are obstructed.

Chapter 2

Preliminaries

In this chapter we briefly discuss the preliminaries for studying asymptotically conical $Spin(7)$-instantons and fix notations and conversions. We review the notions of nearly G_2-manifolds, (asymptotically conical) $Spin(7)$-manifolds and (asymptotically conical) $Spin(7)$-instantons. We also briefly discuss Lockhart–McOwen analysis on asymptotically conical manifolds.

2.1 Nearly G_2-Manifolds

Definition 2.1.1. Let Σ be a Riemannian 7-dimensional manifold. A 3-form $\phi \in \Omega^3(\Sigma)$ is called a *G_2-structure on* Σ if in local orthonormal frame $e^1, \dots, e^7$, ϕ can be written as

$$\phi = e^{123} - e^{145} - e^{167} - e^{246} + e^{257} - e^{347} - e^{356}, \tag{2.1}$$

where $e^{ijk} := e^i \wedge e^j \wedge e^k$.

For more details on the group G_2, see Appendix B and for G_2-structures, see [9], [10], [11], [33].

Theorem 2.1.2 ([53]). *There are orthogonal decompositions*

$$\Omega^2(\Sigma) = \Omega^2_7 \oplus \Omega^2_{14},$$
$$\Omega^3(\Sigma) = \Omega^3_1 \oplus \Omega^3_7 \oplus \Omega^3_{27},$$

where Ω^k_d is a G_2-invariant subspace of Ω^k with point-wise dimension d and

$$\Omega^2_7 = \{u \lrcorner \phi : u \in \Omega^1(\Sigma)\} = \{w \in \Omega^2(\Sigma) : *(\phi \wedge w) = 2w\} \tag{2.2}$$

$$\Omega^2_{14} = \{w \in \Omega^2(\Sigma) : \psi \wedge w = 0\} = \{w \in \Omega^2(\Sigma) : *(\phi \wedge w) = -w\} \tag{2.3}$$

$$\Omega^3_1 = \{f\phi : f \in \Omega^0(\Sigma)\} \tag{2.4}$$
$$\Omega^3_7 = \{u \lrcorner \psi : u \in \Omega^1(\Sigma)\} \tag{2.5}$$
$$\Omega^3_{27} = \{w \in \Omega^3(\Sigma) : \phi \wedge w = 0, \psi \wedge w = 0\}. \tag{2.6}$$

for $\psi = *\phi$.

A Riemannian 7-manifold possesses a G_2-structure if and only if it is a spin manifold [40]. Hence, we now discuss the spinor bundle.

For our purpose, we start by fixing a representation of the Clifford algebra $Cl(7)$ in which the volume form Γ_7 acts as $-\,\mathrm{Id}$.

Let $\$(\Sigma)$ be the spinor bundle over a 7-manifold X with G_2-structure. Let $\xi \in \Gamma(\$(\Sigma))$ be a unit spinor such that $\omega \cdot \xi = 0$ for all $\omega \in \Omega^2_{14}$, where $\cdot$ denotes Clifford multiplication. The existence of ξ follows from the fact that $G_2 \subset SO(7)$ fixes a vector in the spinor representation, and the uniqueness (up to sign) follows from G_2 fixing everything in the trivial representation. Then we have an isomorphism given by

$$\begin{aligned} s : \Lambda^0(T^*\Sigma) \oplus \Lambda^1(T^*\Sigma) &\to \$(\Sigma) \\ (f, v) &\mapsto (f - v)\cdot \xi. \end{aligned} \tag{2.7}$$

Lemma 2.1.3. *The 3-form ϕ and 4-form $\psi = *\phi$ act on the subspaces Λ^0 and Λ^1 of $\$(\Sigma)$ with eigenvalues*

	Λ^0	Λ^1
ϕ	7	-1
ψ	-7	1

Proof. Since Λ^0 and Λ^1 are irreducible representations of G_2 and ϕ is G_2-invariant, by Schur's lemma ϕ preserves the decomposition. Furthermore, ϕ must act on each space as a constant and this action is traceless.

First let us take a look at Λ^0. We have $\psi = *\phi = \phi \cdot \mathrm{vol}_7$. Then $\phi \cdot \psi = 7\,\mathrm{vol}_7 - 6\phi$. Now, let $\phi \cdot \xi = \lambda\xi$. Then, using $\mathrm{vol}_7 \cdot \xi = -\xi$, we have

$$\phi \cdot \psi \cdot \xi = \phi^2\,\mathrm{vol}_7 \cdot \xi = -\phi^2\xi = -\phi\cdot(\phi\cdot\xi) = -\phi\cdot\lambda\xi = -\lambda^2\xi.$$

Also,

$$\phi\cdot\psi\cdot\xi = (7\,\mathrm{vol}_7 - 6\phi)\cdot\xi = (-7 - 6\lambda)\xi.$$

Hence,

$$\lambda^2 - 6\lambda - 7 = 0 \Rightarrow \lambda = 7 \text{ or } \lambda = -1.$$

Now, eigenvalues of ϕ acting on Λ^1 satisfies the same equation. Since ϕ is trace-less, we must have that ϕ acts on Λ^0 as 7 and on Λ^1 as -1. $\square$

Lemma 2.1.4. *Let $\omega \in \Omega^2(\Sigma)$. Then*

$$\omega \cdot \xi = -(\omega \lrcorner \phi) \cdot \xi.$$

Proof. Since $\pi_{14}(\omega) \cdot \xi = 0$, we have $\omega \cdot \xi = \pi_7(\omega) \cdot \xi$. Now, $\pi_7(\omega) = v \lrcorner \phi$ for some $v \in \Omega^1(\Sigma)$. Hence,

$$\omega \cdot \xi = (v \lrcorner \phi) \cdot \xi = -\frac{1}{2}(\phi \cdot v + v \cdot \phi) \cdot \xi = -\frac{1}{2}(-1v + 7v) \cdot \xi = -3v \cdot \xi.$$

Moreover,

$$\omega \lrcorner \phi = \pi_7(\omega) \lrcorner \phi = (v \lrcorner \phi) \lrcorner \phi = 3v.$$

Thus we have the result. $\square$

Corollary 2.1.5. *Let $f \in \Omega^0(\Sigma), v, u \in \Omega^1(\Sigma)$. Then Clifford multiplication of (f, v) by u is given by*

$$u \cdot (f, v) = (\langle u, v \rangle, -fu - (u \wedge v) \lrcorner \phi). \tag{2.8}$$

Proof.

$$\begin{aligned} s(u \cdot (f, v)) &= u \cdot ((f - v) \cdot \xi) \\ &= (fu - u \cdot v) \cdot \xi \\ &= (fu - u \wedge v + \langle u, v \rangle) \cdot \xi \\ &= [fu + \langle u, v \rangle] \cdot \xi - (u \wedge v) \cdot \xi \\ &= [fu + \langle u, v \rangle] \cdot \xi + ((u \wedge v) \lrcorner \phi) \cdot \xi \quad \text{by Lemma 2.1.4} \\ &= fu + \langle u, v \rangle + ((u \wedge v) \lrcorner \phi) \cdot \xi \\ &= s(\langle u, v \rangle, -fu - (u \wedge v) \lrcorner \phi). \end{aligned}$$

Hence the result follows from s being isomorphism. $\square$

Definition 2.1.6. Let Σ be a 7-dimensional Riemannian manifold and $\phi \in \Omega^3(\Sigma)$. Then ϕ is called a *nearly (parallel) G_2-structure* on Σ if it satisfies

$$d\phi = \tau_0 \psi, \tag{2.9}$$

where $\psi = *\phi$ and $\tau_0 \in \mathbb{R} \setminus \{0\}$. In this case, (Σ, ϕ) is called a *nearly G_2-manifold*.

Clearly ϕ is not closed, but is co-closed. For more on nearly G_2-structures, see [1], [26].

Definition 2.1.7. Let $\not{S}(\Sigma)$ be the spinor bundle on Σ. A real spinor $\xi \in \Gamma(\not{S}(\Sigma))$ is called a *Killing spinor* if there exists $\delta \in \mathbb{R} \setminus \{0\}$ such that for all $X \in \Gamma(T\Sigma)$, ξ satisfies the *Killing equation* given by

$$\nabla_X \xi = \delta X \cdot \xi. \tag{2.10}$$

The scalar δ is called the *Killing constant* for the Killing spinor ξ.

We note that a unit spinor ξ on a nearly G_2-manifold satisfying $\omega \cdot \xi = 0$ for all $\omega \in \Omega^2_{14}$ is a Killing spinor. Conversely, any Riemannian 7-manifold admitting a Killing spinor is a nearly G_2-manifold. In fact, there is a one to one correspondence between nearly G_2-structures and real Killing spinors on Σ [7].

$$\left\{\begin{matrix}\text{Nearly } G_2\text{-structure } \phi \\ \text{satisfying } d\phi = \tau_0 \psi\end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix}\xi \in \not{S}(\Sigma) \text{ such that} \\ \nabla_X \xi = \frac{\tau_0}{8} X \cdot \xi\end{matrix}\right\}$$

where we have used the fact that the Killing constant δ can be written in term of τ_0 as $\delta = \frac{\tau_0}{8}$. If g is the metric induced by the nearly G_2 structure ϕ, then the Ricci curvature is given by $\mathrm{Ric}_g = \frac{3}{8}\tau_0^2 g$, and hence every nearly G_2 manifold is Einstein. The scalar curvature is $\mathrm{Scal}_g = \frac{21}{8}\tau_0^2$.

We note that we can always re-scale τ_0. If we take $\tau_0 = 4$, then we have $d\phi = 4\psi$. The reason for this particular choice is that the unit 7-sphere S^7 has scalar curvature 42, and so, $\mathrm{Scal}_g = \frac{21}{8}\tau_0^2 = 42$, which implies $\tau_0 = 4$ (whereas taking $\tau_0 = -4$ would just change the orientation of the manifold). Hence we have

$$\nabla_X \xi = \frac{1}{2} X \cdot \xi. \tag{2.11}$$

For a nearly G_2-manifold (Σ, ϕ), we can define a 1-parameter family of affine connections on $T\Sigma$. Let $t \in \mathbb{R}$. Then ∇^t is a 1-parameter family of connections on $T\Sigma$ defined by

$$g(\nabla^t_X Y, Z) = g(\nabla_X Y, Z) + \frac{t}{3}\phi(X, Y, Z) \tag{2.12}$$

for $X, Y, Z \in \Gamma(T\Sigma)$.

Let T^t be the torsion $(1,2)$-tensor of the affine connection ∇^t. Then

$$g(X, T^t(Y, Z)) = g(X, \nabla^t_Y Z) - g(X, \nabla^t_Z Y) - g(X, [Y, Z]) = \frac{2t}{3}\phi(X, Y, Z) \tag{2.13}$$

using the fact that Levi–Civita connection is torsion-free. Hence the torsion tensor T^t is

$$T^t(X, Y) = \frac{2t}{3}\phi(X, Y, \cdot) \tag{2.14}$$

which is totally skew-symmetric, being proportional to ϕ.

Now, ∇^t lifts to the spinor bundle $\not{S}(\Sigma)$ given by

$$\nabla^t_X \eta = \nabla_X \eta + \frac{t}{6}(X \lrcorner \phi) \cdot \eta \tag{2.15}$$

where $\eta \in \Gamma(\not{S}(\Sigma))$ and $X \in \Gamma(T\Sigma)$. Then, using the eigenvalues (2.1.3), we find

$$\nabla^t_X \xi = -\frac{t-1}{2} X \cdot \xi. \tag{2.16}$$

Therefore, for $t = 1$, the Killing spinor ξ is parallel with respect to the connection ∇^1. Then the connection ∇^1 has holonomy group contained in G_2 with totally skew-symmetric torsion. This connection ∇^1 on the nearly G_2-manifold Σ is known as the *canonical connection*.

Remark 2.1.8. We note that there is a notion of canonical connection in the context of homogeneous spaces as well (see Appendix A). Following the work of [54], for the homogeneous nearly G_2-manifolds we consider, these two notions of canonical connection coincide.

Proposition 2.1.9. *[54] The Ricci tensor of the connection ∇^t is given by*

$$\mathrm{Ric}^t = \left(6 - \frac{2t^2}{3}\right) g.$$

As a corollary, we have the scalar curvature of the canonical connection to be $\frac{112}{3}$.

2.2 $Spin(7)$-Manifolds and $Spin(7)$-Instantons

Definition 2.2.1. Let X be an 8-dimensional Riemannian manifold equipped with a 4-form $\Phi \in \Omega^4(X)$ such that in local orthonormal basis $e^0, e^1, \ldots, e^7$, we have $\Phi = e^0 \wedge \phi + \psi$ where ϕ is as in (2.1) and $*(e^0 \wedge \phi) = \psi$. Then Φ is said to be a *$Spin(7)$-structure* on X and (X, Φ) is said to be an *almost $Spin(7)$-manifold*.

If Φ is torsion-free, i.e., if $\nabla\Phi = 0$ where ∇ is the Levi–Civita connection, or equivalently, if $d\Phi = 0$, then (X, Φ) is called a *$Spin(7)$-manifold*.

For more details on the group $Spin(7)$, see Appendix B and for $Spin(7)$-manifolds, see [9], [33].

Theorem 2.2.2 ([53]). *There are orthogonal decompositions*

$$\Omega^2(X) = \Omega^2_7 \oplus \Omega^2_{21}$$
$$\Omega^3(X) = \Omega^3_8 \oplus \Omega^3_{48}$$

$$\Omega^4(X) = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27} \oplus \Omega^4_{35}$$

where Ω^k_d is a $Spin(7)$-invariant subspace of Ω^k with point-wise dimension d and

$$\Omega^2_7 = \{w \in \Omega^2(X) : *(\Phi \wedge w) = 3w\} \tag{2.17}$$
$$\Omega^2_{21} = \{w \in \Omega^2(X) : *(\Phi \wedge w) = -w\} \tag{2.18}$$
$$\Omega^3_8 = \{u \lrcorner \Phi : u \in \Omega^1(X)\} \tag{2.19}$$
$$\Omega^3_{48} = \{w \in \Omega^3(X) : \Phi \wedge w = 0\} \tag{2.20}$$
$$\Omega^4_1 = \{f\Phi : f \in \Omega^0(X)\} \tag{2.21}$$
$$\Omega^4_7 = \{\mathcal{L}_\xi \Phi : \xi \in \mathfrak{so}(8)\} \tag{2.22}$$
$$\Omega^4_{27} = \{w \in \Omega^4(X) : *w = w, w \wedge \Phi = 0, w \wedge \mathcal{L}_\xi \Phi = 0 \textit{ for all } \xi \in \mathfrak{so}(8)\} \tag{2.23}$$
$$\Omega^4_{35} = \{w \in \Omega^4(X) : *w = -w\} \tag{2.24}$$

where $\mathcal{L}_\xi$ is the Lie derivative with respect to ξ.

Proposition 2.2.3 ([32]). *If Φ is a $Spin(7)$-structure on a manifold X, then X is a Spin manifold. Moreover, if Φ is torsion-free, then X admits a non-trivial parallel spinor.*

The canonical spin structure can be identified in the following way.

$$\cancel{S}^+ \cong \Lambda^0 \oplus \Lambda^2_7 \quad \text{and} \quad \cancel{S}^- \cong \Lambda^1_8.$$

Let X be a $Spin(7)$ manifold and P be a principal G-bundle on X for a compact group G. Let $\mathfrak{g}_P$ be the adjoint vector bundle. Then we have

$$\Omega^2(\mathfrak{g}_P) = \Omega^2_7(\mathfrak{g}_P) \oplus \Omega^2_{21}(\mathfrak{g}_P)$$

Definition 2.2.4. Let $\pi^2_7 : \Omega^2(\mathfrak{g}_P) \to \Omega^2_7(\mathfrak{g}_P)$ be the projection. Then a connection A on P is said to be a *$Spin(7)$-instanton* if $\pi^2_7(F_A) = 0$ where F_A is the curvature of the connection A. In this case $F_A \in \Omega^2_{21}(\mathfrak{g}_P)$.

Equivalently, A is a $Spin(7)$ instanton if it satisfies

$$*(\Phi \wedge F_A) = -F_A. \tag{2.25}$$

This follows from the fact that the operator on Λ^2 defined by $\omega \mapsto *(\Phi \wedge \omega)$ has eigenvalues -1 and 3 with eigenspaces Λ^2_{21} and Λ^2_7 respectively.

Moreover, A is an instanton if and only if F_A annihilates the parallel spinor, i.e., for parallel spinor ξ, we have $F_A \cdot \xi = 0$, where $\cdot$ denotes Clifford multiplication.

2.3 Asymptotically Conical $Spin(7)$-Manifolds

Let (Σ, g_Σ) be a Riemannian 7-manifold with a nearly G_2-structure ϕ satisfying $d\phi = 4\psi$ where $\psi = *\phi$. A *$Spin(7)$-cone* on Σ is $C(\Sigma) := (0,\infty) \times \Sigma$ together with a $Spin(7)$-structure $(C(\Sigma), \Phi_C)$ defined by

$$\Phi_C := r^3 dr \wedge \phi + r^4 \psi \tag{2.26}$$

where $r \in (0,\infty)$ is the coordinate. Σ is called the *link* of the cone. The metric g_C compatible with Φ_C is given by

$$g_C = dr^2 + r^2 g_\Sigma. \tag{2.27}$$

We note that condition $d\phi = 4\psi$ implies the torsion free condition $d\Phi_C = 0$, which implies that $(C(\Sigma), g_C, \Phi)$ is a $Spin(7)$-manifold.

Remark 2.3.1. We note that a $Spin(7)$-cone is not complete. Hence, we consider complete $Spin(7)$-manifolds whose geometry is asymptotic to the given (incomplete) G_2-cone.

Definition 2.3.2. Let (X, g, Φ) be a $Spin(7)$-manifold. X is called an *asymptotically conical (AC) $Spin(7)$-manifold with rate $\nu < 0$* if there exists a compact subset $K \subset X$, a compact connected nearly G_2 manifold Σ, and a constant $R > 1$ together with a diffeomorphism

$$h : (R,\infty) \times \Sigma \to X \setminus K \tag{2.28}$$

such that

$$\left|\nabla_C^j (h^*(\Phi|_{X\setminus K}) - \Phi_C)\right|(r,p) = O(r^{\nu-j}) \quad \text{as } r \to \infty \tag{2.29}$$

for each $p \in \Sigma$, $j \in \mathbb{Z}_{\geq 0}$, $r \in (R,\infty)$; where ∇_C is the Levi–Civita connection for the cone metric g_C on $C(\Sigma)$, and the norm is induced by the metric g_C.

$X \setminus K$ is called the *end* of X and Σ the *asymptotic link* of X.

Remark 2.3.3. For simplicity, we'll drop the points (r,p) while writing the norm, and will understand it from the context.

Remark 2.3.4. It can be proved that (see [36]) the metric g satisfies the same asymptotic condition

$$\left|\nabla_C^j (h^*(g|_{X\setminus K}) - g_C)\right| = O(r^{\nu-j}) \quad \text{as } r \to \infty.$$

Examples 2.3.5.

- $(\mathbb{R}^8, \Phi_0)$: Since $C(S^7) = \mathbb{R}^8 \setminus \{0\}$, $(\mathbb{R}^8, \Phi_0)$ is an AC manifold with any rate $\nu < 0$.

- Bryant–Salamon $Spin(7)$-manifold $\not{S}^-(S^4)$: This a rank 4 bundle, hence the total space is a manifold of dimension 8. This is an AC $Spin(7)$-manifold asymptotic to the cone over $(S^7$ with squashed metric with rate $\nu = -10/3$ (see [12]).
- Some more examples of AC $Spin(7)$-metrics can be found in the recent work of Lehmann [41].

2.4 Lockhart–McOwen Analysis on AC $Spin(7)$-manifold

Now we review Lockhart–McOwen analysis applied to AC $Spin(7)$ manifolds.

Let X be an AC $Spin(7)$ manifold. In order to define "weighted Banach spaces" on X, we first define a notion of radius function.

Definition 2.4.1. A *radius function* is a map $\varrho : X \to \mathbb{R}$ defined by

$$\varrho(x) := \begin{cases} 1 & \text{if } x \in \text{the compact subset } K \subset X \\ r & \text{if } x = h(r,p) \text{ for some } r \in (2R, \infty),\ p \in \Sigma \\ \widetilde{r} & \text{if } x \in h((R, 2R) \times \Sigma) \end{cases} \tag{2.30}$$

where $h : (R, \infty) \times \Sigma \to X \setminus K$ is the diffeomorphism, and $\widetilde{r}$ is a smooth interpolation between its definition at infinity and its definition on K, in a decreasing manner.

Let $\pi : E \to X$ be a vector bundle over X with a fibre-wise metric and a connection ∇ compatible with the metric.

Definition 2.4.2. Let $p \geq 1, k \in \mathbb{Z}_{\geq 0}, \nu \in \mathbb{R}$ and $C_c^\infty(E)$ be the space of compactly supported smooth sections of E. We define the *conically damped* or *weighted Sobolev space* $W_\nu^{k,p}(E)$ of sections of E over X of weight ν as follows:
For $\xi \in C_c^\infty(E)$, we define the *weighted Sobolev norm* $\|\cdot\|_{W_\nu^{k,p}(E)}$ as

$$\|\xi\|_{W_\nu^{k,p}(E)} = \left(\sum_{j=0}^{k} \int_X \left| \varrho^{-\nu+j} \nabla^j \xi \right|^p \varrho^{-8}\, \mathrm{dvol} \right)^{1/p} \tag{2.31}$$

which is clearly finite and indeed a norm. Then the weighted Sobolev space $W_\nu^{k,p}(E)$ is the completion of $C_c^\infty(E)$ with respect to the norm $\|\cdot\|_{W_\nu^{k,p}(E)}$.

Remark 2.4.3. [42]

- We note that $W_{-4}^{0,2}(E) = L^2(E)$.

- We have $\varrho^{\nu} W^{0,2}_{\mu}(E) = W^{0,2}_{\mu+\nu}(E)$. In particular, $W^{0,2}_{\nu}(E) = \varrho^{4+\nu} L^2(E)$.

Definition 2.4.4. Let $k \in \mathbb{Z}_{\geq 0}$ and $\nu \in \mathbb{R}$. Then for $\xi \in C^{\infty}_c(E)$, we define the *weighted C^k norm* $\| \cdot \|_{C^k_\nu(E)}$ as

$$\|\xi\|_{C^k_\nu(E)} = \sum_{j=0}^{k} \|\varrho^{-\nu+j} \nabla^j \xi\|_{C^0} \tag{2.32}$$

which is well defined and a norm. Then the *weighted C^k space* $C^k_\nu(E)$ is the closure of $C^{\infty}_c(E)$ with respect to this norm. We also define $C^{\infty}_\nu(E) := \bigcap\limits_{k\geq 0} C^k_\nu(E)$.

Theorem 2.4.5 (*Weighted Sobolev Embedding Theorem*)**.** *[47]*

1. *Let $k, l \geq 0$. If $k - \frac{8}{p} \geq l$, then*
$$W^{k,p}_\nu(E) \hookrightarrow C^l_\nu(E)$$
is a continuous embedding.

2. *Let $k \geq l \geq 0$, $p \leq q$ and $\mu \leq \nu$. If $k - \frac{8}{p} \geq l - \frac{8}{q}$, then*
$$W^{k,p}_\mu(E) \hookrightarrow W^{l,q}_\nu(E)$$
is a continuous embedding.

In order to ensure that we work with continuous sections, we shall always assume $k \geq 4$. This follows from the first part of the weighted Sobolev embedding theorem by putting $l = 0$ and $p = 2$.

Theorem 2.4.6 (*Weighted Sobolev Multiplication Theorem*)**.** *[22] Let $\xi \in W^{k,2}_\mu(E), \eta \in W^{l,2}_\nu(F)$. If $l \geq k > \frac{8}{2} = 4$, then the multiplication*

$$W^{k,2}_\mu(E) \times W^{l,2}_\nu(F) \to W^{k,2}_{\mu+\nu}(E \otimes F)$$

is bounded. In other words, there is a constant $C > 0$ such that

$$\|\xi \otimes \eta\|_{W^{k,2}_{\mu+\nu}(E\otimes F)} \leq C \|\xi\|_{W^{k,2}_\mu(E)} \|\eta\|_{W^{l,2}_\nu(F)}.$$

Proposition 2.4.7. *[42] Let $\xi \in W^{0,2}_\mu(E), \eta \in W^{0,2}_\nu(E)$. If $\mu + \nu < 8$, then*

$$\langle \xi, \eta \rangle_{L^2} = \int_M \langle \xi, \eta \rangle \,\mathrm{dvol}$$

is finite and satisfies,

$$\langle \xi, \eta \rangle_{L^2} \leq \|\xi\|_{W^{0,2}_\mu(E)} \|\eta\|_{W^{0,2}_\nu(E)}.$$

From Proposition 2.4.7, we have the pairing

$$\langle \cdot\,, \cdot \rangle_{L^2} : W^{0,2}_{\nu}(E) \times W^{0,2}_{-8-\nu}(E) \to \mathbb{R}.$$

This defines the isomorphism [42]

$$\left(W^{0,2}_{\nu}(E)\right)^* \cong W^{0,2}_{-8-\nu}(E). \tag{2.33}$$

Now, let $P \to X$ be a principal G bundle. Consider the associated vector bundle $E := T \otimes \mathfrak{g}_P$, where T is either $\Lambda^k T^*X$ of k-forms, or $\Lambda^* T^* X = \bigoplus_{k=0}^{8} \Lambda^k T^* X$, or the spinor bundle $\not{S}$ over X. If A is a connection on $\mathfrak{g}_P$, then E inherits a metric from T and a connection from the Levi–Civita connection on T and the connection on $\mathfrak{g}_P$. If $\xi \in C^\infty_c(T \otimes \mathfrak{g}_P) \subset \Gamma(T \otimes \mathfrak{g}_P)$, then the weighted Sobolev norm is given by Equation 2.31 where $\nabla = \nabla^{LC} \otimes 1_{\mathfrak{g}_P} + 1_T \otimes \nabla^A$, ∇^A being the connection on $\mathfrak{g}_P$.

Before moving forward let us fix few notations:

$$\begin{aligned}
\Omega^{m,k}_{\nu}(X) &:= W^{2,k}_{\nu}(\Lambda^m T^* X), \\
\Omega^{m,k}_{\nu}(\mathfrak{g}_P) &:= W^{2,k}_{\nu}(\Lambda^m T^* X \otimes \mathfrak{g}_P), \\
\Omega^{*,k}_{\nu}(X) &:= W^{2,k}_{\nu}(\Lambda^* T^* X), \\
\Omega^{*,k}_{\nu}(\mathfrak{g}_P) &:= W^{2,k}_{\nu}(\Lambda^* T^* X \otimes \mathfrak{g}_P).
\end{aligned}$$

2.5 Asymptotically Conical $Spin(7)$-Instantons and Moduli Space

Definition 2.5.1. Let X be an AC $Spin(7)$-manifold asymptotic to the cone $C(\Sigma)$. Let $P \to X$ be a principal G-bundle over X. Then an *asymptotically framed bundle* is the bundle P together with a choice of a principal bundle $Q \to \Sigma$ and an isomorphism

$$h^* P \cong \pi^* Q$$

where $\pi : C(\Sigma) \to \Sigma$ is the natural projection.

We note that such framing always exists [46]. So we fix a framing Q.

Definition 2.5.2. Let X be an AC $Spin(7)$-manifold asymptotic to the cone $C(\Sigma)$. Let $P \to X$ be an asymptotically framed bundle. A connection A on P is called an *asymptotically conical connection* with rate ν if there exists a connection A_Σ on $Q \to \Sigma$ such that

$$\left|\nabla^j_C(h^*(A) - \pi^*(A_\Sigma))\right| = O(r^{\nu-1-j}) \quad \text{as } r \to \infty \tag{2.34}$$

for each $p \in \Sigma$, $j \in \mathbb{Z}_{\geq 0}$, $\nu < 0$. The norm is induced by the cone metric and the metric on $\mathfrak{g}$.

A is called *asymptotic* to A_Σ and $\nu_0 := \inf\{\nu : A \text{ is AC with rate } \nu\}$ is called the *fastest rate of convergence of* A.

Remark 2.5.3.

- We have defined the rate of convergence in term of conical metric and the coordinate r on the cone. However, we could also have chosen in terms of the AC metric and the radius function ϱ. But in both cases the rate of convergence would be the same.
- The -1 in the term $O(r^{\nu-1-j})$ comes from the fact that a 1-from α on Σ satisfies $|\pi^*\alpha| = O(r^{-1})$.

Let $\mathcal{A}_P$ be the space of AC connections on P. Fix a reference connection $A \in \mathcal{A}_P$. Then, we can identify the spaces $\mathcal{A}_P$ and $\Omega^1(\mathfrak{g}_P)$ by $A' = A + \alpha$, for any other connection A' and $\alpha \in \Omega^1(\mathfrak{g}_P)$. Denote the space of $W^{k,2}_{\nu-1}$-connections by

$$\mathcal{A}_{k,\nu-1} := \{A + \alpha : \alpha \in \Omega^{1,k}_{\nu-1}(\mathfrak{g}_P)\} \tag{2.35}$$

and define

$$\mathcal{A}_{\nu-1} := \bigcap_{k=1}^{\infty} \mathcal{A}_{k,\nu-1} \tag{2.36}$$

which is the space of $C^\infty_{\nu-1}$-connections.

Now, a gauge transform is $\varphi \in \mathrm{Aut}(P)$ and acts on a connection A by $\varphi \cdot A = \varphi A \varphi^{-1} - d\varphi\varphi^{-1}$. Let $G \to GL(V)$ be a faithful representation of G, and consider the associated vector bundle $E := P \times_G V$. Moreover, consider the endomorphism bundle $\mathrm{End}(E)$ whose fibre at $x \in X$ is the vector space $\mathrm{End}(E_x) = \{\text{linear maps } E_x \to E_x\}$. Note that there is a natural embedding $\mathrm{Aut}(E_x) \to \mathrm{End}(E_x)$ and a canonical subgroup G_x of $\mathrm{Aut}(E_x)$ which is isomorphic to G (but not canonically isomorphic to G).

Then we define the *weighted gauge group* by (see [49])

$$\mathcal{G}_{k+1,\nu} := \{\varphi \in C^0(\mathrm{End}(E)) : \|\varphi - I\|_{k+1,\nu} < \infty, \varphi \in G\}. \tag{2.37}$$

We also define $\mathcal{G}_\nu := \bigcap\limits_{l=1}^{\infty} \mathcal{G}_{l,\nu}$.

Lemma 2.5.4. *[18] The point-wise exponential map defines charts for which $\mathcal{G}_{k+1,\nu}$ is a Hilbert Lie group with Lie algebra modelled on $\Omega^{0,k+1}_\nu(\mathfrak{g}_P)$ for $k \geq 3$. The group $\mathcal{G}_{k+1,\nu}$ acts on $\mathcal{A}_{k,\nu-1}$ smoothly via gauge transformations, for $k \geq 4$.*

Definition 2.5.5. Let X be an AC $Spin(7)$-manifold asymptotic to $C(\Sigma)$. Let $P \to X$ be a principal G-bundle asymptotically framed by $Q \to \Sigma$. Let A_Σ be an instanton on the nearly G_2 manifold Σ. Then the *moduli space of $Spin(7)$-instantons asymptotic to A_Σ with rate ν* is given by

$$\mathcal{M}(A_\Sigma, \nu) := \{Spin(7) \text{ instanton } A \text{ on } P \text{ satisfying (2.34) asymptotic to } A_\Sigma\}/\mathcal{G}_\nu. \tag{2.38}$$

Chapter 3

Deformation Theory of Asymptotically Conical $Spin(7)$-Instantons

In this chapter we describe the deformation theory of asymptotically conical $Spin(7)$-instantons. In the first part we discuss the necessary analytic framework, following the works of Lockhart–McOwen [44], Marshall [47], Karigiannis–Lotay [36] and Driscoll [23]. In the second part we develop the general theory, where we closely follow Donaldson [20] and Driscoll [23].

3.1 Fredholm and Elliptic Asymptotically Conical Operators

We begin this section by defining the operators that will be important in developing the deformation theory.

1. Let Σ be a nearly G_2-manifold and $Q \to \Sigma$ be a principal G-bundle. Let A_Σ be a connection on Q. Consider the bundle $\not{S}(\Sigma) \otimes \mathfrak{g}_Q$ where $\not{S}(\Sigma)$ is the spinor bundle on Σ and $\mathfrak{g}_Q = Q \times_{\mathrm{Ad}} \mathfrak{g}$. Then we have a twisted Dirac operator

$$\not{\mathfrak{D}}_{A_\Sigma} : \Gamma(\not{S}(\Sigma) \otimes \mathfrak{g}_Q) \to \Gamma(\not{S}(\Sigma) \otimes \mathfrak{g}_Q). \tag{3.1}$$

2. Let X be an AC $Spin(7)$-manifold with link Σ. Let $P \to X$ be an asymptotically framed bundle. Let $A \in \mathcal{A}_P$ be an AC connection asymptotic to A_Σ.

   Consider the bundle $\not{S}(X) \otimes \mathfrak{g}_P$ where $\not{S}(X)$ is the spinor bundle over X and $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$. Then we have a Dirac operator

$$\not{\mathfrak{D}}_A : \Gamma(\not{S}(X) \otimes \mathfrak{g}_P) \to \Gamma(\not{S}(X) \otimes \mathfrak{g}_P). \tag{3.2}$$

3. Let $C(\Sigma) = (0,\infty) \times \Sigma$ be a $Spin(7)$-cone over Σ and $\pi^*Q \to C(\Sigma)$ be a principal bundle over $C(\Sigma)$. Let $A_C = \pi^* A_\Sigma$.

   Now consider the bundle $\not{S}(C(\Sigma)) \times \mathfrak{g}_{\pi^*Q}$, where $\not{S}(C(\Sigma))$ is the spinor bundle on $C(\Sigma)$ and $\mathfrak{g}_{\pi^*Q} = \pi^*Q \times_{\mathrm{Ad}} \mathfrak{g}$. Then we have a Dirac operator

$$\not{\mathfrak{D}}_{A_C} : \Gamma(\not{S}(C(\Sigma)) \otimes \mathfrak{g}_{\pi^*Q}) \to \Gamma(\not{S}(C(\Sigma)) \otimes \mathfrak{g}_{\pi^*Q}). \tag{3.3}$$

   The objective behind introducing this Dirac operator $\not{\mathfrak{D}}_{A_C}$ is to study the Fredholm properties of the Dirac operator $\not{\mathfrak{D}}_A$ using Lockhart–McOwen theory.

Then we have the following sets of operators:

Base Manifold	Operator		Bundle
X	K	∇^A	$\Lambda^0(X) \otimes \mathfrak{g}_P$
		$\not{\mathfrak{D}}_A$	$\not{S}(X) \otimes \mathfrak{g}_P$
		$d_A^* d_A$	$\mathfrak{g}_P$
Σ	K_Σ	∇^{A_Σ}	$\Lambda^0(\Sigma) \otimes \mathfrak{g}_Q$
		$\not{\mathfrak{D}}_{A_\Sigma}$	$\not{S}(\Sigma) \otimes \mathfrak{g}_Q$
		$d_{A_\Sigma}^* d_{A_\Sigma}$	$\mathfrak{g}_Q$
$C(\Sigma)$	K_C	∇^{A_C}	$\Lambda^0(C(\Sigma)) \otimes \mathfrak{g}_{\pi^*Q}$
		$\not{\mathfrak{D}}_{A_C}$	$\not{S}(C(\Sigma)) \otimes \mathfrak{g}_{\pi^*Q}$
		$d_{A_C}^* d_{A_C}$	$\mathfrak{g}_{\pi^*Q}$

Definition 3.1.1. Let $E = \not{S}(C(\Sigma)) \otimes \mathfrak{g}_{\pi^*Q}$ or $\mathfrak{g}_{\pi^*Q}$ be a bundle over $C(\Sigma)$. Then a section σ of the bundle E is called *homogeneous of degree λ* if

$$\sigma = r^\lambda \eta_\Sigma$$

where η_Σ is a section of $\not{S}(\Sigma) \otimes \mathfrak{g}_Q$ or $\mathfrak{g}_Q$ respectively, lifted to the cone.

We note that here we use the identification $cl_C(u) = r cl_\Sigma(u)$ for all $u \in TC(\Sigma)$ where cl_Σ is the Clifford action of $T\Sigma$ and cl_C is the Clifford action of $TC(\Sigma)$.

Now, let K_C be either the Dirac operator $\not{\mathfrak{D}}_{A_C}$ or the coupled Laplace operator $d_{A_C}^* d_{A_C}$. Then we consider the set of critical weights $\mathscr{D}(K_C)$ given by

$$\mathscr{D}(K_C) = \{\lambda \in \mathbb{R} : \text{there exists non-zero } \sigma \in \Gamma(E) \text{ of homogeneous order } \lambda \text{ such that } K_C(\sigma) = 0\}. \tag{3.4}$$

Theorem 3.1.2. *[47] The extended map*

$$K : W^{k+l,p}_{\nu}(E) \to W^{k,p}_{\nu-\gamma}(F)$$

(where $K = \not{\mathfrak{D}}_A$ or $d_A^* d_A$, $E = F = \not{\$}(X) \otimes \mathfrak{g}_P$ or $\mathfrak{g}_P$ and γ is 1 or 2 respectively) is Fredholm if $\nu \in \mathbb{R} \setminus \mathscr{D}(K_C)$. Moreover, for $\nu < \nu'$, if $[\nu, \nu'] \cap \mathscr{D}(K_C) = \varnothing$, then the kernel $\ker K$ *is independent of the weight in the range $[\nu, \nu']$.*

Hence, we focus our attention on finding the set of critical weights for the operators $\not{\mathfrak{D}}_{A_C}$ and $d^*_{A_C} d_{A_C}$.

The set of critical weights for the Laplace operator $d^*_{A_C} d_{A_C}$

We want to find the set of critical weights for the Laplace operator $d^*_{A_C} d_{A_C}$, i.e., the set $\mathscr{D}(d^*_{A_C} d_{A_C})$. This set corresponds to a subset of the kernel of the operator containing elements of homogeneous order λ. Thus, if $\lambda \in \mathscr{D}(d^*_{A_C} d_{A_C})$, then there exists $\sigma \in \ker(d^*_{A_C} d_{A_C})$ such that $\sigma = r^{\lambda}\eta$ for $\eta \in \Omega^0(\mathfrak{g}_P)$. An easy calculation yields,

Lemma 3.1.3. *Let $\sigma = r^{\lambda}\eta$ for $\eta \in \Omega^0(\mathfrak{g}_P)$. Then,*

$$d^*_{A_C} d_{A_C} \sigma = r^{\lambda-2}(d^*_{A_\Sigma} d_{A_\Sigma}\eta - \lambda(\lambda+6)\eta).$$

Thus, ξ is in the kernel if and only if $\lambda(\lambda+6)$ is an eigenvalue of $d^*_{A_\Sigma} d_{A_\Sigma}$. But since the coupled Laplace operator is positive, $(-6, 0) \cap \mathscr{D}(d^*_{A_C} d_{A_C}) = \varnothing$. Hence, we have the following proposition.

Proposition 3.1.4. *Let A be an AC connection over an AC $Spin(7)$-manifold X. If $\nu \in (-6, 0)$, then the coupled Laplace operator*

$$d^*_A d_A : \Omega^{0,k+2}_{\nu}(\mathfrak{g}_P) \to \Omega^{0,k}_{\nu-2}(\mathfrak{g}_P)$$

is Fredholm.

The set of critical weights for the Dirac operator $\not{\mathfrak{D}}_{A_C}$

Now, we want to find the set of critical weights for the Dirac operator $\not{\mathfrak{D}}_{A_C}$, i.e., the set $\mathscr{D}(\not{\mathfrak{D}}_{A_C})$. This set corresponds to a subset of the kernel of the operator containing elements of homogeneous order λ. Thus, if $\lambda \in \mathscr{D}(\not{\mathfrak{D}}_{A_C})$, then there exists $\sigma \in \ker \not{\mathfrak{D}}_{A_C}$ such that $\sigma = r^{\lambda}\eta_{\Sigma}$ where η_{Σ} is a spinor on Σ.

Recall that the volume element Γ_8 acting on $\not{S}(C)$ satisfies $\Gamma_8^2 = 1$ and this gives an eigenspace decomposition $\not{S}(C) = \not{S}^+(C) \oplus \not{S}^-(C)$ corresponding to $+1$ and -1 eigenvalues respectively. This induces a decomposition of the Dirac operator $\mathcal{D}_C$ of the cone metric:

$$\mathcal{D}_C^{\pm} : \Gamma(\not{S}^{\pm}(C)) \to \Gamma(\not{S}^{\mp}(C)).$$

Proposition 3.1.5. *For the Dirac operators $\mathcal{D}_\Sigma : \Gamma(\not{S}(\Sigma)) \to \Gamma(\not{S}(\Sigma))$ and $\mathcal{D}_C^- : \Gamma(\not{S}^-(C)) \to \Gamma(\not{S}^+(C))$, we have,*

$$\mathcal{D}_C^- \eta = dr \cdot \left(\frac{\partial}{\partial r} + \frac{1}{r} \left(\frac{7}{2} - \mathcal{D}_\Sigma \right) \right) \eta.$$

Proof. We fix the convention where the indices i, j, k run from 1 to 7 and μ, ν run from 0 to 7. Let e_i be a local orthonormal frame of $T\Sigma$ and using the metric g_Σ, the dual e^i be that of $T^*\Sigma$. Then $E^0 := dr$ and $E^i := re^i$ for $i = 1, \ldots, 7$ form a local orthonormal frame for $T^*C(\Sigma)$.

Moreover, let ∂_i be the differentiation with respect to e_i and D_μ be the differentiation with respect to E_μ (vector field dual of E^μ using the metric g_C).

Now, let ω^i_j be the connection 1-form of the Levi–Civita connection on $T^*\Sigma$. Hence, $\nabla^\Sigma e^i = -\omega^i_j e^j$. Then

$$\omega^i_j = -\omega^j_i \tag{3.5}$$

and

$$de^i + \omega^i_j \wedge e^j = 0. \tag{3.6}$$

Moreover, let $\Omega^\mu_\nu = -\Omega^\nu_\mu$ be the Levi–Civita 1-form on $C(\Sigma)$. Now, $dE^0 = d^2 r = 0$. So, $dE^0 + \Omega^0_i \wedge e^i = 0 \Rightarrow \Omega^0_i \wedge e^i = 0$. For $i \geq 0$,

$$dE^i = -\Omega^i_\mu \wedge E^\mu = -\Omega^i_0 \wedge dr - \Omega^i_j \wedge re^j. \tag{3.7}$$

Also,

$$dE^i = dr \wedge e^i + rde^i = dr \wedge e^i - r\omega^i_j \wedge e^j. \tag{3.8}$$

Comparing (3.7) and (3.8), we get

$$\Omega^i_0 = e^i, \quad \Omega^i_j = \omega^i_j$$

Now let $\Gamma^\mu_{\sigma\nu}$ be the Christoffel symbols of the Levi–Civita connection on $C(\Sigma)$ and γ^i_{kj} be that of on Σ. Then,

$$\nabla^\Sigma_{e_k} e_j = \gamma^i_{kj} e_i.$$

But,

$$\nabla^\Sigma_{e_k} e_j = \omega^i_j(e_k) e_i.$$

Hence,
$$\gamma^i_{kj} = \omega^i_j(e_k) \Rightarrow e^k\gamma^i_{kj} = \omega^i_j.$$
Similarly, $E^\sigma\Gamma^\mu_{\sigma\nu} = \Omega^\mu_\nu$.

Now, $E^\sigma\Gamma^\mu_{\sigma\nu} = \Omega^\mu_\nu \Rightarrow E_0\Omega^\mu_\nu = \Gamma^\mu_{0\nu}$. But Ω^μ_ν is either $\Omega^0_i = e^i$ or $\Omega^i_0 = -e^i$ or $\Omega^i_j = \omega^i_j$ and hence $E_0\Omega^\mu_\nu = 0$, which implies
$$\Gamma^\mu_{0\nu} = 0$$
and
$$E^k\Gamma^i_{k0} = \Omega^i_0 = e^i \Rightarrow re^k\Gamma^i_{k0} = e^i \Rightarrow \Gamma^i_{k0} = \frac{1}{r}e_k e^i = \frac{1}{r}\delta^i_k.$$
Now, we consider the natural embedding of $Cl(7)$ into $Cl^0(8)$ by $e^i \mapsto E^0E^i$. Then the action of Dirac operator $\mathcal{D}_\Sigma$ on $\eta \in \Gamma(\$(\Sigma))$ is given by
$$\mathcal{D}_\Sigma\eta = E^0E^i\left(\partial_i\eta + \frac{1}{4}\gamma^k_{ij}E^0E^jE^0E^k\eta\right). \tag{3.9}$$
Now, $\Gamma_7 \cdot \eta = -\eta$ but since $\Gamma_7 = E^0E^1E^0E^2\cdots E^0E^7 = E^0E^1\cdots E^7 = \Gamma_8$, this implies $\Gamma_8 \cdot \eta = -\eta$. Thus, $\eta \in \Gamma(\$^-(C))$. The action of negative Dirac operator $\mathcal{D}^-_C$ on $C(\Sigma)$ is given by
$$\begin{aligned}
\mathcal{D}^-_C\eta &= E^0D_0\eta + E^i\left(D_i\eta + \frac{1}{4}\Gamma^\nu_{i\mu}E^\mu E^\nu\eta\right)\\
&= E^0\frac{\partial\eta}{\partial r} + E^i\left(D_i\eta + \frac{1}{4}(\Gamma^j_{i0}E^0E^j\eta^- + \Gamma^0_{ij}E^jE^0\eta + \Gamma^k_{ij}E^jE^k\eta)\right)\\
&= E^0\frac{\partial\eta}{\partial r} + E^i\left(\frac{1}{r}\partial_i\eta + \frac{1}{4r}(\delta_{ij}E^0E^j\eta - \delta_{ij}E^jE^0\eta + \gamma^k_{ij}E^jE^k\eta)\right)\\
&= E^0\frac{\partial\eta}{\partial r} - \frac{1}{2r}E^iE^iE^0\eta + \frac{1}{r}E^i\left(\partial_i\eta + \frac{1}{4}\gamma^k_{ij}E^jE^k\eta\right)\\
&= E^0\frac{\partial\eta}{\partial r} + \frac{7}{2r}E^0\eta + \frac{1}{r}E^i\left(\partial_i\eta + \frac{1}{4}\gamma^k_{ij}E^0E^jE^0E^k\eta\right).
\end{aligned} \tag{3.10}$$
The result follows from (3.9) and (3.10). ∎

Corollary 3.1.6. *Consider the following two twisted Dirac operators: $\not{\mathfrak{D}}_{A_\Sigma}$ and*
$$\not{\mathfrak{D}}^-_{A_C} : \Gamma(\$^-(C(\Sigma)) \otimes \mathfrak{g}_{\pi^*Q}) \to \Gamma(\$^+(C(\Sigma)) \otimes \mathfrak{g}_{\pi^*Q}).$$
Then,
$$\not{\mathfrak{D}}^-_{A_C} = dr \cdot \left(\frac{\partial}{\partial r} + \frac{1}{r}\left(\frac{7}{2} - \not{\mathfrak{D}}_{A_\Sigma}\right)\right). \tag{3.11}$$
Proof. We note that
$$\not{\mathfrak{D}}_{A_\Sigma} = cl_7 \circ (\nabla \otimes 1 + 1 \otimes \nabla^{A_\Sigma}) = \mathcal{D}_\Sigma \otimes 1 + cl_7 \circ (1 \otimes \nabla^{A_\Sigma})$$

and

$$\mathfrak{D}^-_{A_C} = cl_8 \circ (\nabla \otimes 1 + 1 \otimes \nabla^{A_C}) = \mathcal{D}^-_C \otimes 1 + cl_8 \circ (1 \otimes \nabla^{A_C}).$$

Hence we just focus on $cl_7 \circ (1 \otimes \nabla^{A_\Sigma})$ and $cl_8 \circ (1 \otimes \nabla^{A_C})$.

Let $\{g_b\}$ be a local frame of $\mathfrak{g}_Q$, $\widetilde{\omega}^c_b = e^i \widetilde{\gamma}^c_{ib}$ be the connection 1-form of ∇^{A_Σ} and $\widetilde{\Omega}^c_b = E^i \widetilde{\Gamma}^c_{ib}$ be that of ∇^{A_C}. Then $\pi^* \widetilde{\omega} = \widetilde{\Omega}$. Moreover, let $\{u_a\}$ be a local frame for the spin bundle. Then

$$\mathfrak{D}_{A_\Sigma}(f_{ab}u_a \otimes g_b) = \mathcal{D}_\Sigma(f_{ab}u_a) \otimes g_b + E^0 E^i f_{ab} u_a \otimes \widetilde{\gamma}^c_{ib} g_c. \tag{3.12}$$

Now,

$$\begin{aligned}
\mathfrak{D}^-_{A_C}(f_{ab}u_a \otimes g_b) &= \mathcal{D}^-_C(f_{ab}u_a) \otimes g_b + E^i f_{ab} u_a \otimes \widetilde{\Gamma}^c_{ib} g_c \\
&= E^0 \left(\frac{\partial}{\partial r} f_{ab} u_a + \frac{1}{r} \left(\frac{7}{2} - \mathcal{D}_\Sigma(f_{ab}u_a) \right) \right) \otimes g_b + \frac{1}{r} E^i f_{ab} u_a \otimes \widetilde{\gamma}^c_{ib} g_c \\
&= E^0 \left(\frac{\partial}{\partial r} + \frac{1}{r} \left(\frac{7}{2} - \mathcal{D}_\Sigma \right) \right) (f_{ab}u_a \otimes g_b) + \frac{1}{r} (E^i f_{ab} u_a) \otimes \widetilde{\gamma}^c_{ib} g_c \\
&= E^0 \left(\frac{\partial}{\partial r} + \frac{1}{r} \left(\frac{7}{2} - \mathcal{D}_\Sigma \right) \right) (f_{ab}u_a \otimes g_b) - \frac{1}{r} (E^0 E^0 E^i f_{ab} u_a) \otimes \widetilde{\gamma}^c_{ib} g_c \\
&= E^0 \left(\frac{\partial}{\partial r} + \frac{1}{r} \left(\frac{7}{2} - \mathfrak{D}_{A_\Sigma} \right) \right) (f_{ab}u_a \otimes g_b) \quad \text{(using 3.12)}.
\end{aligned}$$

□

Finally, we have the description of the set critical weights of the twisted Dirac operator.

Proposition 3.1.7. *Consider the Dirac operator*

$$\mathfrak{D}^-_A : W^{k+1,2}_{\nu-1}(\mathbb{S}^-(X) \otimes \mathfrak{g}_P) \to W^{k,2}_{\nu-2}(\mathbb{S}^+(X) \otimes \mathfrak{g}_P). \tag{3.13}$$

Then the set of critical weights is given by

$$\mathscr{D}(\mathfrak{D}^-_A) = \left\{ \nu \in \mathbb{R} : \nu + \frac{5}{2} \in \operatorname{Spec} \mathfrak{D}_{A_\Sigma} \right\}. \tag{3.14}$$

Thus, this Dirac operator $\mathfrak{D}^-_A$ is Fredholm if $\nu + \frac{5}{2} \in \mathbb{R} \setminus \operatorname{Spec} \mathfrak{D}_{A_\Sigma}$.

Proof. Consider the section σ of homogeneous degree $\nu - 1$. i.e., $\sigma = e^{(\nu-1)t}\eta$, where $e^t = r$. Then $\nu \in \mathscr{D}(\mathfrak{D}^-_A)$ if σ is also in the kernel of $\mathfrak{D}^-_A$. Now,

$$\begin{aligned}
0 = \mathfrak{D}^-_A \sigma &= E^0 e^{-t} \left(\frac{\partial}{\partial t} + \frac{7}{2} - \mathfrak{D}_{A_\Sigma} \right) e^{(\nu-1)t} \eta \\
&= E^0 e^{-t} e^{(\nu-1)t} e^{-(\nu-1)t} \left(\frac{\partial}{\partial t} + \frac{7}{2} - \mathfrak{D}_{A_\Sigma} \right) e^{(\nu-1)t} \eta \\
&= E^0 e^{-t} e^{(\nu-1)t} \left(\frac{\partial}{\partial t} + \frac{7}{2} + (\nu - 1) - \mathfrak{D}_{A_\Sigma} \right) \eta.
\end{aligned}$$

□

Index of the Dirac operator $\mathfrak{D}^-_{A_C}$

Definition 3.1.8. For $\lambda \in \mathbb{R}$, define the space

$$\mathcal{K}(\lambda)^-_C := \left\{ \sigma \in \ker \mathfrak{D}^-_{A_C} : \sigma(r,p) = r^{\lambda-1} P(r,p) \right\},$$

where

$$P(r,p) = \sum_{j=0}^{m} (\log r)^j \eta_j(\sigma)$$

and each $\eta_j \in \Gamma(\$(\Sigma) \otimes \mathfrak{g}_Q)$.

The following proposition is a consequence of the fact that the Dirac operator $\mathfrak{D}_{A_\Sigma}$ is self-adjoint.

Proposition 3.1.9. *If*

$$\mathfrak{D}^-_{A_C} \left(r^{\lambda-1} \sum_{j=0}^{m} (\log r)^j \eta_j(\sigma) \right) = 0, \tag{3.15}$$

with $\eta_m \neq 0$, then $m = 0$. Hence elements of $\mathcal{K}(\lambda)^-_C$ have no log terms.

Proof. Expanding the expression (3.15) using (3.11), we have

$$\begin{aligned} dr \cdot \Bigg[\Bigg((\lambda-1) r^{\lambda-2} \sum_{j=0}^{m} (\log r)^j \eta_j + r^{\lambda-1} \sum_{j=0}^{m} j (\log r)^{j-1} \frac{1}{r} \eta_j \Bigg) \\ + \frac{7}{2r} r^{\lambda-1} \sum_{j=0}^{m} (\log r)^j \eta_j - \frac{1}{r} r^{\lambda-1} \sum_{j=0}^{m} (\log r)^j \mathfrak{D}_{A_\Sigma} \eta_j \Bigg] = 0. \end{aligned}$$

Considering this as a polynomial in $\log r$ and comparing coefficients of $(\log r)^m$ and $(\log r)^{m-1}$ respectively, we get

$$\begin{aligned} & - r^{\lambda-2} \mathfrak{D}_{A_\Sigma} \eta_m + \frac{7}{2} r^{\lambda-2} \eta_m + (\lambda-1) r^{\lambda-2} \eta_m = 0 \\ \Rightarrow\ & \mathfrak{D}_{A_\Sigma} \eta_m - \frac{7}{2} \eta_m - (\lambda-1) \eta_m = 0 \\ \Rightarrow\ & \mathfrak{D}_{A_\Sigma} \eta_m = \left(\lambda + \frac{5}{2} \right) \eta_m, \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} & - r^{\lambda-2} \mathfrak{D}_{A_\Sigma} \eta_{m-1} + \frac{7}{2} r^{\lambda-2} \eta_{m-1} + (\lambda-1) r^{\lambda-2} \eta_{m-1} + r^{\lambda-2} m \eta_m = 0 \\ \Rightarrow\ & \mathfrak{D}_{A_\Sigma} \eta_{m-1} - \frac{7}{2} \eta_{m-1} - (\lambda-1) \eta_{m-1} - m \eta_m = 0 \end{aligned}$$

$$\Rightarrow m\eta_m = \not{\mathfrak{D}}_{A_\Sigma}\eta_{m-1} - \left(\lambda + \frac{5}{2}\right)\eta_{m-1}.$$

Then using the self-adjoint property of $\not{\mathfrak{D}}_{A_\Sigma}$, we get

$$\begin{aligned} m\langle \eta_m, \eta_m\rangle_{L^2(\Sigma)} &= \langle \not{\mathfrak{D}}_{A_\Sigma}\eta_{m-1}, \eta_m\rangle_{L^2(\Sigma)} - \langle (\lambda + 5/2)\eta_{m-1}, \eta_m\rangle_{L^2(\Sigma)} \\ &= \langle \eta_{m-1}, \not{\mathfrak{D}}_{A_\Sigma}\eta_m\rangle_{L^2(\Sigma)} - \langle (\lambda + 5/2)\eta_{m-1}, \eta_m\rangle_{L^2(\Sigma)} \\ &= \langle \eta_{m-1}, (\lambda + 5/2)\eta_m\rangle_{L^2(\Sigma)} - (\lambda + 5/2)\langle \eta_{m-1}, \eta_m\rangle_{L^2(\Sigma)} = 0 \quad \text{(using 3.16)}. \end{aligned}$$

Since $\eta_m \neq 0$, we have $m = 0$. □

Now, consider the Dirac operator (3.13) and denote its index by $\operatorname{Index}_\nu \not{\mathfrak{D}}_A^-$. Then, we have the following theorem.

Theorem 3.1.10. *[47] If $\nu, \nu' \in \mathbb{R} \setminus \mathscr{D}(\not{\mathfrak{D}}_A^-)$ such that $\nu \leq \nu'$, then*

$$\operatorname{Index}_{\nu'} \not{\mathfrak{D}}_A^- - \operatorname{Index}_\nu \not{\mathfrak{D}}_A^- = \sum\{\dim \mathcal{K}(\lambda)_C^- : \lambda \in (\nu, \nu') \cap \mathscr{D}(\not{\mathfrak{D}}_A^-)\}.$$

From the Proposition 3.1.9, we conclude that $\mathcal{K}(\lambda)_C^-$ is precisely the $\left(\lambda + \frac{5}{2}\right)$ eigenspace of the operator $\not{\mathfrak{D}}_{A_\Sigma}$. Summarising, we have the following theorem.

Theorem 3.1.11. *The Dirac operator*

$$\not{\mathfrak{D}}_A^- : W^{k+1,2}_{\nu-1}(\not{S}^-(X) \otimes \mathfrak{g}_P) \to W^{k,2}_{\nu-2}(\not{S}^+(X) \otimes \mathfrak{g}_P)$$

is Fredholm if ν is not a critical weight, i.e., $\nu + \frac{5}{2} \in \mathbb{R} \setminus \operatorname{Spec} \not{\mathfrak{D}}_{A_\Sigma}$. Moreover, for two non-critical weights ν, ν' with $\nu \leq \nu'$, the jump in the index is given by

$$\operatorname{Index}_{\nu'} \not{\mathfrak{D}}_A^- - \operatorname{Index}_\nu \not{\mathfrak{D}}_A^- = \sum_{\nu < \lambda < \nu'} \dim\ker\left(\not{\mathfrak{D}}_{A_\Sigma} - \lambda - \frac{5}{2}\right).$$

3.2 Deformations of Asymptotically Conical $Spin(7)$-Instantons

Let A be an asymptotically conical reference connection that also satisfies the $Spin(7)$-instanton equation. Then, we have $\pi_7(F_A) = 0$. Now, we can write any other connection in some open neighbourhood of A as $A' = A + \alpha$ for $\alpha \in \Omega^1(\mathfrak{g}_P)$. Then,

$$F_{A'} - F_A = d_A\alpha + \frac{1}{2}[\alpha, \alpha].$$

Hence the connection A' is a $Spin(7)$-instanton if and only if $\pi_7(F_{A+\alpha}) = 0$, i.e.,

$$\pi_7\left(d_A\alpha + \frac{1}{2}[\alpha, \alpha]\right) = 0.$$

We also have the gauge fixing condition $d_A^*\alpha = 0$ (which will be described in details later in page 27). We consider the non-linear operator

$$\begin{aligned} \not{\mathfrak{D}}_A^{\mathrm{NL}} : \Gamma(\Lambda^1 \otimes \mathfrak{g}_P) &\to \Gamma((\Lambda^0 \oplus \Lambda^2_7) \otimes \mathfrak{g}_P) \\ \alpha &\mapsto \left(d_A^*\alpha, \pi_7 \left(d_A\alpha + \frac{1}{2}[\alpha, \alpha] \right) \right) \end{aligned} \tag{3.17}$$

Hence, the local moduli space of $Spin(7)$-instanton can be expressed as the zero set of $\not{\mathfrak{D}}_A^{\mathrm{NL}}$, i.e., $\left(\not{\mathfrak{D}}_A^{\mathrm{NL}}\right)^{-1}(0)$.

Now, from the identifications of the positive and negative spinor bundles given by $\not{S}^+ = \Lambda^0 \oplus \Lambda^2_7$ and $\not{S}^- = \Lambda^1$ we can prove that the linearisation of the non-linear operator $\not{\mathfrak{D}}_A^{\mathrm{NL}}$ is precisely the twisted linear Dirac operator $\not{\mathfrak{D}}_A^-$,

$$\begin{aligned} \not{\mathfrak{D}}_A^- : \Gamma(\Lambda^1 \otimes \mathfrak{g}_P) &\to \Gamma((\Lambda^0 \oplus \Lambda^2_7) \otimes \mathfrak{g}_P) \\ \alpha &\mapsto (d_A^*\alpha, \pi_7(d_A\alpha)). \end{aligned} \tag{3.18}$$

In order to calculate the zero set of the non-linear operator, we calculate the kernel of the linearised Dirac operator, using the analytic techniques discussed in the previous subsections.

First we want to investigate the moduli space of AC $Spin(7)$-connections. We start with the following lemma.

Lemma 3.2.1. *Let $\alpha \in \Omega_\mu^{m-1,k}(\mathfrak{g}_P)$ and $\beta \in \Omega_\nu^{m,l}(\mathfrak{g}_P)$. If $k, l \geq 4$ and $\mu + \nu < -7$, then*

$$\langle d_A\alpha, \beta\rangle_{L^2} = \langle \alpha, d_A^*\beta\rangle_{L^2}.$$

Proof. Let us consider the manifold with boundary

$$X_{\leq R} := \{x \in X : \varrho(x) \leq R\}.$$

Then the boundary is given by $\partial(X_{\leq R}) = \{R\} \times \Sigma$. We note that

$$d(\alpha \wedge *\beta) = d_A\alpha \wedge *\beta - \alpha \wedge *d_A^*\beta$$

Now we apply Stoke's theorem

$$\int_{X_{\leq R}} \langle d_A\alpha, \beta\rangle \,\mathrm{dvol}_X - \int_{X_{\leq R}} \langle \alpha, d_A^*\beta\rangle \,\mathrm{dvol}_X = \int_{X_{\leq R}} d(\alpha \wedge *\beta) = \int_{\{R\}\times\Sigma} (\alpha \wedge *\beta).$$

Now, by the Sobolev embedding theorem, $|\alpha| \leq CR^\mu$ and $|\beta| \leq CR^\nu$ on the end. Hence, we have

$$\left| \int_{\{R\}\times\Sigma} (\alpha \wedge *\beta) \right| \leq \int_{\{R\}\times\Sigma} |\alpha \wedge *\beta| \,\mathrm{dvol}_\Sigma \leq CR^{\mu+\nu+7}$$

which goes to zero as $R \to \infty$, since $\mu + \nu < -7$. □

As an immediate consequence, we have,

Corollary 3.2.2. *Let $f \in \Omega^{0,k+2}_{\nu}(\mathfrak{g}_P)$ for $\nu < 0$ and $d_A^* d_A f = 0$. Then $d_A f = 0$.*

Proof. Since there are no critical weights in $(-6, 0)$, then if $d_A^* d_A f = 0$ and $\nu < 0$, we have $f \in \Omega^{0,k+2}_{\mu}(\mathfrak{g}_P)$ for some $\mu < -3$ for any k. Then $d_A f \in \Omega^{1,k+1}_{\mu-1}(\mathfrak{g}_P)$ and

$$\|d_A f\|_{L^2} = \langle d_A^* d_A f, f \rangle_{L^2} = 0$$

by integration by parts. □

The proof of the following lemma follows from the maximum principle.

Lemma 3.2.3 ([47]). *Let (X, g) be an asymptotically conical Riemannian manifold. Let $f \in \Omega^{0,k+2}_{\nu}(X)$ for $\nu < 0$ and $(d^* d) f = 0$. Then $f = 0$.*

Then we obtain the gauged version of Lemma 3.2.3 as follows.

Corollary 3.2.4. *Let $f \in \Omega^{0,k+2}_{\nu}(\mathfrak{g}_P)$ for $\nu < 0$ and $d_A^* d_A f = 0$. Then $f = 0$.*

Proof. Since $f \in \ker(d_A^* d_A)_{\nu}$, we have $d_A f = 0$. Then,

$$d^* d |f|^2 = 2 d^* \langle d_A f, f \rangle = 0.$$

Thus $|f|^2$ is a harmonic function and hence by Lemma 3.2.3, is zero. Thus, $f = 0$. □

The following lemma can easily be proved using inverse mapping theorem.

Lemma 3.2.5. *If $\nu \in (-6, 0)$, then the coupled Laplace operator*

$$d_A^* d_A : \Omega^{0,k+2}_{\nu}(\mathfrak{g}_P) \to \Omega^{0,k}_{\nu-2}(\mathfrak{g}_P)$$

is an isomorphism of topological vector spaces.

Moreover, we have the following simple result from the theory of Banach spaces.

Lemma 3.2.6. *Let X, Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. Then* $\ker T$ *is closed and a closed subspace $X_0 \subset X$ is a complement of* $\ker T$ *if and only if $T|_{X_0}$ is injective and $T(X) = T(X_0)$.*

First let us show that the image of d_A is closed.

Lemma 3.2.7. *For $\nu \in (-6, 0)$,*

$$\operatorname{Im}(d_A : \Omega^{0,k+2}_{\nu}(\mathfrak{g}_P) \to \Omega^{1,k+1}_{\nu-1}(\mathfrak{g}_P))$$

is a closed subspace of $\Omega^{1,k+1}_{\nu-1}(\mathfrak{g}_P)$.

Proof. Let $\{d_A f_n\}_{n=1}^{\infty}$ be a sequence in $d_A(\Omega_{\nu}^{0,k+2}(\mathfrak{g}_P))$. Let $\alpha \in \Omega_{\nu-1}^{1,k+1}(\mathfrak{g}_P)$ such that

$$\lim_{n\to\infty} \|d_A f_n - \alpha\|_{k+1,\nu-1} = 0$$

in $\Omega_{\nu-1}^{1,k+1}(\mathfrak{g}_P)$. Applying the bounded operator d_A^*, we get

$$\lim_{n\to\infty} \|d_A^* d_A f_n - d_A^* \alpha\|_{k,\nu-2} = 0$$

in $\Omega_{\nu-2}^{0,k}(\mathfrak{g}_P)$. Since $d_A^* d_A$ admits a bounded inverse, we can define $f := (d_A^* d_A)^{-1} d_A^* \alpha$ and then,

$$\lim_{n\to\infty} \|f_n - f\|_{k+2,\nu} = 0$$

in $\Omega_{\nu}^{0,k+2}(\mathfrak{g}_P)$. Now, applying the bounded operator d_A, we get

$$\lim_{n\to\infty} \|d_A f_n - d_A f\|_{k+1,\nu-1} = 0.$$

Hence, by uniqueness of limits, we get $\alpha = d_A f$, and hence $\operatorname{Im} d_A$ is closed. □

Proposition 3.2.8. *If $\nu \in (-6, 0)$, then we have the decomposition*

$$\Omega_{\nu-1}^{1,k+1}(\mathfrak{g}_P) = \ker d_A^* \oplus \operatorname{Im} d_A$$

where $d_A : \Omega_{\nu}^{0,k+2}(\mathfrak{g}_P) \to \Omega_{\nu-1}^{1,k+1}(\mathfrak{g}_P)$ and $d_A^* : \Omega_{\nu-1}^{1,k+1}(\mathfrak{g}_P) \to \Omega_{\nu-2}^{0,k}(\mathfrak{g}_P)$.

Proof. Let us consider the operator $d_A^* : \Omega_{\nu-1}^{1,k+1}(\mathfrak{g}_P) \to \Omega_{\nu-2}^{0,k}(\mathfrak{g}_P)$. This is a bounded operator. Hence the kernel is a closed subspace. We want to show that $X_0 := \operatorname{Im} d_A$ satisfies the conditions of Lemma 3.2.6. From Lemma 3.2.7 we note that $\operatorname{Im} d_A$ is closed. Then for $T := d_A^*$ and $X := \Omega_{\nu-1}^{1,k+1}(\mathfrak{g}_P)$ we have the result.

First we show that d_A^* restricted to $\operatorname{Im} d_A$ is injective. Let, for $f, g \in \Omega_{\nu-2}^{0,k}(\mathfrak{g}_P)$, $d_A^* d_A f = d_A^* d_A g$. Then $f - g$ is harmonic function and hence zero, which implies $d_A f - d_A g = 0$, which establishes the injectivity.

Now we need to show that $d_A^*(\Omega_{\nu-1}^{1,k+1}(\mathfrak{g}_P)) = d_A^*(\operatorname{Im} d_A) = d_A^* d_A(\Omega_{\nu}^{0,k+2}(\mathfrak{g}_P))$. This follows from the topological isomorphism in Lemma 3.2.5. □

Let us consider the moduli space of connections $\mathcal{B}_{k+1,\nu} = \mathcal{A}_{k+1,\nu-1}/\mathcal{G}_{k+2,\nu}$. Then the infinitesimal action of the gauge group $\mathcal{G}_{k+2,\nu}$ is given by

$$-d_A : \Omega_{\nu}^{0,k+2}(\mathfrak{g}_P) \to \Omega_{\nu-1}^{1,k+1}(\mathfrak{g}_P).$$

Thus we can view the Proposition 3.2.8 as a "*slice theorem*" which gives us the complement of the action of the gauge group.

Lemma 3.2.9. *The action of the gauge group $\mathcal{G}_{k+2,\nu}$ on the space of connections $\mathcal{A}_{k+1,\nu-1}$ is free.*

Proof. Let us consider the stabilizer group

$$\Gamma_{A,\nu} = \{\varphi \in \mathcal{G}_{k+2,\nu} : \varphi \cdot A = A\}.$$

We consider gauge transformations as sections of $\mathrm{End}(V)$. Now, the connection A has holonomy contained in G and hence it preserves the inner product on V as well as on $\mathrm{End}(V)$. Since, $\varphi \in \Gamma_{A,\nu}$ is a gauge transformation, by definition 2.37, we have $\|\varphi - I\| \in \Omega^{0,k+2}_{\nu}(\mathfrak{g}_P)$ and hence,

$$d^* d\|\varphi - I\|^2 = 2d^*\langle d_A(\varphi - I), \varphi - I\rangle = 0 \Rightarrow \varphi - I = 0,$$

since, $\varphi \cdot A = A$ implies $d_A\varphi = 0$. Hence, we have $\Gamma_{A,\nu} = \{I\}$. □

We note that unlike the case where the manifold X is compact, when X is AC, reducible connections do not produce singularities in the space of connections modulo gauge.

Let us define the set

$$T_{A,\nu,\epsilon} := \left\{\alpha \in \Omega^{1,k+1}_{\nu-1}(\mathfrak{g}_P) : d_A^*\alpha = 0, \|\alpha\|_{W^{k+1,2}_{\nu-1}} < \epsilon\right\}.$$

Then $T_{A,\nu,\epsilon} \subset \ker d_A^*$ models a local neighbourhood of the moduli space $\mathcal{B}_{k+1,\nu}$. We note that studying the moduli space using the local model $T_{A,\nu,\epsilon}$ is basically same as solving the Coulomb gauge fixing condition $d_A^*\alpha = 0$. This condition is local: locally, near A it selects a unique gauge equivalent class. The following lemma provides a sufficient condition for solving the gauge fixing condition. It is the weighted version of Proposition 2.3.4 of [20].

Lemma 3.2.10. *[23] If $\nu \in (-6, 0)$, then there is a constant $c(A) > 0$ such that if $A' \in \mathcal{A}_{\nu-1}$ and $A' = A + \alpha$ satisfies*

$$\|\alpha\|_{W^{4,2}_{\nu-1}} < c(A)$$

then there is a gauge transformation $\varphi \in \mathcal{G}_\nu$ such that $\varphi(A')$ is in Coulomb gauge relative to A.

Proposition 3.2.11. *If $\nu \in (-6, 0)$, then the moduli space $\mathcal{B}_{k+1,\nu}$ is a smooth manifold and the sets $T_{A,\nu,\epsilon}$ provide charts near $[A] \in \mathcal{B}_{k+1,\nu} = \mathcal{A}_{k+1,\nu-1}/\mathcal{G}_{k+2,\nu}$.*

Proof. The smoothness follows from Lemma 3.2.9 and the surjectivity follows from Proposition 3.2.10. The homeomorphism between $T_{A,\nu,\epsilon}$ and a neighbourhood of $[A] \in \mathcal{B}_{k+1,\nu}$ follows from a weighted version of Proposition 4.2.9 of [20] and the fact that $\Gamma_{A,\nu} = \{I\}$. □

Now, we turn out focus to the main objective of this section: the moduli space of AC $Spin(7)$-instantons.

Let us define the spaces,

$$\mathcal{M}(A_\Sigma,\nu)_{k+1} := \{A \in \mathcal{A}_{k+1,\nu-1} : A \text{ is a } Spin(7)\text{-instanton on } P\}/\mathcal{G}_{k+2,\nu}.$$

The proof of the following proposition is a weighted version of Proposition 4.2.16 of [20] and very similar to the proof for the 7-dimensional case given by Driscoll [23].

Proposition 3.2.12. *If $k \geq 4$ and $\nu \in (-6,0)$, then the natural inclusion given by $\mathcal{M}(A_\Sigma,\nu)_{k+1} \hookrightarrow \mathcal{M}(A_\Sigma,\nu)_k$ is a homeomorphism.*

Hence by the same elliptic regularity arguments that show Proposition 3.2.12 and weighted Sobolev embedding theorem, we see that $\mathcal{M}(A_\Sigma,\nu)_k$ consists of smooth connections. We obtain the following important corollary.

Corollary 3.2.13. *If $\nu \in (-6,0)$, then the zero set of the non-linear twisted Dirac operator (defined in 3.17) given by*

$$\mathfrak{D}_A^{NL} : W^{k+1,2}_{\nu-1}(\$^-(X)\otimes\mathfrak{g}_P) \to W^{k,2}_{\nu-2}(\$^+(X)\otimes\mathfrak{g}_P)$$

is independent of $k \geq 4$. Moreover, a neighbourhood of $[A] \in \mathcal{M}(A_\Sigma,\nu)$ is homeomorphic to 0 in $\left(\mathfrak{D}_A^{NL}\right)^{-1}(0)$.

Finally, we have all the tools necessary to define the deformation and obstruction spaces, state and prove the main theorem of this section.

Definition 3.2.14. For $\nu < 0$ the *space of infinitesimal deformations* is defined to be the kernel of the Dirac operator. That is,

$$\mathcal{I}(A,\nu) := \left\{\alpha \in \Omega^{1,k+1}_{\nu-1}(\mathfrak{g}_P) : \mathfrak{D}_A^-\alpha = 0\right\}. \tag{3.19}$$

The *obstruction space* $\mathcal{O}(A,\nu)$ is defined to be the cokernel of the Dirac operator. That is $\mathcal{O}(A,\nu) = (\Omega^{0,k}_{\nu-2}(\mathfrak{g}_P)\oplus\Omega^{2,k}_{7,\nu-2}(\mathfrak{g}_P))/\mathfrak{D}_A^-\left(\Omega^{1,k+1}_{\nu-1}(\mathfrak{g}_P)\right)$.

We can identify $\mathcal{O}(A,\nu)$ to be a finite-dimensional subspace of $\Omega^{0,k}_{\nu-2}(\mathfrak{g}_P)\oplus\Omega^{2,k}_{7,\nu-2}(\mathfrak{g}_P)$ such that,

$$\Omega^{0,k}_{\nu-2}(\mathfrak{g}_P)\oplus\Omega^{2,k}_{7,\nu-2}(\mathfrak{g}_P) = \mathfrak{D}_A^-\left(\Omega^{1,k+1}_{\nu-1}(\mathfrak{g}_P)\right)\oplus\mathcal{O}(A,\nu). \tag{3.20}$$

We have the main theorem:

Theorem 3.2.15. *Let A be an AC $Spin(7)$-instanton asymptotic to a nearly G_2 instanton A_Σ. Moreover, let $\nu \in (\mathbb{R} \setminus \mathscr{D}(\not{\mathfrak{D}}_A^-)) \cap (-6, 0)$. Then there exists an open neighbourhood $\mathcal{U}(A,\nu)$ of 0 in $\mathcal{I}(A,\nu)$, and a smooth map $\kappa : \mathcal{U}(A,\nu) \to \mathcal{O}(A,\nu)$, with $\kappa(0) = 0$, such that an open neighbourhood of $0 \in \kappa^{-1}(0)$ is homeomorphic to a neighbourhood of A in $\mathcal{M}(A_\Sigma,\nu)$. Hence, the virtual dimension of the moduli space is given by $\mathrm{Index}(\not{\mathfrak{D}}_A^-) = \dim \mathcal{I}(A,\nu) - \dim \mathcal{O}(A,\nu)$. Moreover, $\mathcal{M}(A_\Sigma,\nu)$ is a smooth manifold if $\mathcal{O}(A,\nu) = \{0\}$.*

Proof. Let $\mathbf{X} = \Omega^{1,k+1}_{\nu-1}(\mathfrak{g}_P) \times \mathcal{O}(A,\nu)$ and $\mathbf{Y} = \Omega^{0,k}_{\nu-2}(\mathfrak{g}_P) \oplus \Omega^{2,k}_{\nu-2}(\mathfrak{g}_P)$. We define a Banach space morphism as

$$\begin{aligned} \mathbf{F} : \mathbf{X} &\to \mathbf{Y} \\ (\alpha,\beta) &\mapsto \not{\mathfrak{D}}_A^{\mathrm{NL}}\alpha + \beta. \end{aligned}$$

where $\not{\mathfrak{D}}_A^{\mathrm{NL}}$ is the nonlinear twisted Dirac operator (3.17). Then, $\mathbf{F}(0,0) = 0$, and the differential at $(0,0)$ is given by

$$\begin{aligned} d\mathbf{F}|_{(0,0)} : \mathbf{X} &\to \mathbf{Y} \\ (\alpha,\beta) &\mapsto \not{\mathfrak{D}}_A^-\alpha + \beta. \end{aligned}$$

Then $d\mathbf{F}|_{(0,0)}$ is surjective and $d\mathbf{F}|_{(0,0)} = 0$ if and only if $(\not{\mathfrak{D}}_A\alpha, \beta) = (0,0)$. Hence $\ker d\mathbf{F}|_{(0,0)} =: \mathbf{K} = \mathcal{I}(A,\nu) \times \{0\}$ is finite dimensional, and we have a decomposition of $\mathbf{X}$ as $\mathbf{X} = \mathbf{K} \oplus \mathbf{Z}$, where $\mathbf{Z} \subset \mathbf{X}$ is a closed subspace. Moreover, we can write $\mathbf{Z} = \mathcal{Z} \times \mathcal{O}(A,\nu)$ for a closed subset $\mathcal{Z} \subset \Omega^{1,k+1}_{\nu-1}(\mathfrak{g}_P)$. By implicit function theorem, we choose the open subsets $\mathcal{U} \subset \mathcal{I}(A,\nu)$, $\mathcal{V}_1 \subset \mathcal{Z}$ and $\mathcal{V}_2 \subset \mathcal{O}(A,\nu)$, and smooth maps $\mathcal{F}_i : \mathcal{U} \to \mathcal{V}_i$ for $i = 1,2$, such that

$$\mathbf{F}^{-1}(0) \cap ((\mathcal{U} \times \mathcal{V}_1) \times \mathcal{V}_2) = \{((\alpha, \mathcal{F}_1(\alpha)), \mathcal{F}_2(\alpha)) : \alpha \in \mathcal{U}\}$$

in $\mathbf{X} = (\mathcal{I}(A,\nu) \oplus \mathcal{Z}) \times \mathcal{O}(A,\nu)$. Hence the kernel of $\mathbf{F}$ near $(0,0)$ is diffeomorphic to an open subset of $\mathcal{I}(A,\nu)$ containing 0.

Now, we define $\mathcal{U}(A,\nu) := \mathcal{U}$ and the map

$$\begin{aligned} \kappa : \mathcal{U}(A,\nu) &\to \mathcal{O}(A,\nu) \\ \alpha &\mapsto \mathcal{F}_2(\alpha). \end{aligned}$$

Then we have a homeomorphism from an open neighbourhood of 0 in $\kappa^{-1}(0)$ to an open neighbourhood of 0 in $\left(\not{\mathfrak{D}}_A^{\mathrm{NL}}\right)^{-1}(0)$ given by $\alpha \mapsto (\alpha, \mathcal{F}_1(\alpha))$. Now, corollary 3.2.13 tells us that a neighbourhood of $[A] \in \mathcal{M}(A,\nu)$ is homeomorphic to a neighbourhood of 0 in $\left(\not{\mathfrak{D}}_A^{\mathrm{NL}}\right)^{-1}(0)$. Hence the theorem. $\square$

Chapter 4

Eigenvalues of the Twisted Dirac Operator on S^7

In order to study the deformations of FNFN instantons, we need to calculate the spectrum of the twisted Dirac operator on the link S^7. We use representation theory, Frobenius reciprocity, and Casimir operators, to write the Dirac operators as a sum of Casimir operators. Then the problem of finding the spectrum of the Dirac operator reduces to finding the eigenvalues of the Casimir operators. This method relies on S^7 being a homogeneous manifold and is developed based on the works of [55], [5], [6], [22].

4.1 Dirac operators on Homogeneous Nearly G_2-Manifolds

Let $\Sigma = G/H$ be a reductive homogeneous nearly G_2-manifold. We consider the principal H-bundle $G \to \Sigma$. Let $\rho_V : H \to \mathrm{Aut}(V)$ be a representation of H. Then we have the associated vector bundle $E := G \times_\rho V \to \Sigma$ and the space of smooth sections $\Gamma(E)$ can be identified with the space of H-equivariant smooth function $G \to V$, i.e. the space $C^\infty(G, V)^H$.

Now, the following left action of G on the space $L^2(G, V)^H$ gives a representation ρ_L, called the *left regular representation* defined by

$$(\rho_L(h)\eta)(g) = \eta(h^{-1}g) \tag{4.1}$$

for $\eta \in L^2(G, V)^H$, and $g, h \in G$.

The right action of G on the space $L^2(G, V)$ gives a representation ρ_R, called the *right* *regular representation* defined by

$$(\rho_R(h)\eta)(g) = \eta(gh). \tag{4.2}$$

We note that $\rho_R(h)\rho_R(k) = \rho_R(hk)$, that is, it is a left action. However, "right" in the name reflects that it is defined using the right action of G on itself.

Then from H-equivariance,

$$L^2(G,V)^H = \{\eta \in L^2(G,V) : \rho_R(h)\eta = \rho_V(h)^{-1}\eta \text{ for all } h \in H\}.$$

If we use the same notations for Lie algebra representations, then,

$$L^2(G,V)^H = \{\eta \in L^2(G,V) : \rho_R(X)\eta = -\rho_V(X)\eta \text{ for all } X \in \mathfrak{h}\}.$$

Let $\widehat{G}$ be the set of equivalence classes of irreducible representations of G and for $\gamma \in \widehat{G}$ we have a representative (V_γ, ρ_γ). Then *Frobenius reciprocity* implies the decomposition

$$L^2(E) \cong L^2(G,V)^H \cong \bigoplus_{\gamma\in\widehat{G}} \mathrm{Hom}(V_\gamma, V)^H \otimes V_\gamma. \tag{4.3}$$

Now, since G/H is reductive, we have an orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ induced by the Killing form K on G, defined by

$$K(X,Y) = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}(X)\,\mathrm{ad}(Y)). \tag{4.4}$$

Let us assume that for some constant c the metric given by

$$g(X,Y) = -c^2 K(X,Y) \tag{4.5}$$

is a nearly G_2-metric. Let $\{I_A\}$ be an orthonormal basis for $\mathfrak{g}$, $\{I_a : a = 1,\ldots,\dim(G/H) = 7\}$ is a orthonormal basis for $\mathfrak{m}$ and $\{I_i : i = 8,\ldots,\dim(G)\}$ is a orthonormal basis of $\mathfrak{h}$.

We note that in this framework G-invariant tensors on the tangent bundle $T(G/H)$ correspond to H-invariant tensors on $\mathfrak{m}$ [38].

Now, we consider the complex spinor bundle $\not{S}(\Sigma) = G \times_{\rho,H} \Delta$ where Δ is the spinor space (that is, an 8-dimensional representation of $Cl(7)$). From the splitting $\not{S}_{\mathbb{C}}(\Sigma) \cong \Lambda^0_{\mathbb{C}} \oplus \Lambda^1_{\mathbb{C}}$, we have $\Delta \cong \mathbb{C} \oplus \mathfrak{m}^*_{\mathbb{C}}$. We now twist the spinor bundle by the associated bundle $E = G \times_{\rho_V,H} V$ for a representation V of H. Then

$$\not{S}_{\mathbb{C}}(\Sigma) \otimes E = G \times_{(\rho_\Delta \otimes \rho_V, H)} (\Delta \otimes V).$$

The canonical connection $\nabla^{1,A_\Sigma} : L^2(G, \Delta \otimes V)^H \to L^2(G, \mathfrak{m}^* \otimes \Delta \otimes V)^H$ can be written as

$$\nabla^{1,A_\Sigma}\eta = e^a \otimes \rho_R(I_a)\eta,$$

where ∇^1 is the canonical connection for the spinor bundle and A_Σ is the canonical connection of E, e^a is the basis of $\mathfrak{m}^*$ dual to I_a and $\eta \in L^2(G, \Delta \otimes V)^H$. Then the Dirac operator $\mathfrak{D}^1_{A_\Sigma}$ is given by

$$\mathfrak{D}^1_{A_\Sigma} = I_a \cdot \rho_R(I_a). \tag{4.6}$$

Then, from (4.6), for the G_2 3-form ϕ, we have a family of Dirac operators

$$\mathfrak{D}^t_{A_\Sigma} = \mathfrak{D}^1_{A_\Sigma} + \frac{(t-1)}{2}\phi \tag{4.7}$$

where for $t = 0$, we have $\mathfrak{D}^0_{A_\Sigma} = \mathfrak{D}_{A_\Sigma}$ (defined in (3.1)). Now,

$$L^2(\$_{\mathbb{C}}(\Sigma) \otimes E) \cong L^2(G, \Delta \otimes V)^H \cong \bigoplus_{\gamma \in \widehat{G}} \mathrm{Hom}(V_\gamma, \Delta \otimes V)^H \otimes V_\gamma \tag{4.8}$$

where the action of G on V_γ of the right hand side of the expression corresponds to the action of ρ_L on $L^2(\$_{\mathbb{C}}(\Sigma) \otimes E)$. We note that the Dirac operator commutes with the left action of G and hence it respects the decomposition (4.8). Then, by Schur's lemma, for every $t \in \mathbb{R}$, the Dirac operator $\mathfrak{D}^t_{A_\Sigma}$, restricted to $\mathrm{Hom}(V_\gamma, \Delta \otimes V)^H \otimes V_\gamma$ is given by

$$\mathfrak{D}^t_{A_\Sigma}|_{\mathrm{Hom}(V_\gamma, \Delta \otimes V)^H \otimes V_\gamma} = \left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma \otimes \mathrm{Id} \tag{4.9}$$

where $\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma : \mathrm{Hom}(V_\gamma, \Delta \otimes V)^H \to \mathrm{Hom}(V_\gamma, \Delta \otimes V)^H$ is the Dirac operator [22]

$$\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma \eta = -I_a \cdot (\eta \circ \rho_{V_\gamma}(I_a)) + \frac{t-1}{2}\phi \cdot \eta. \tag{4.10}$$

If $\{I_A\}$ is an orthonormal basis of $\mathfrak{g}$, then the *Casimir Operator* $\mathrm{Cas}_{\mathfrak{g}} \in \mathrm{Sym}^2(\mathfrak{g})$ is the inverse of the metric on $\mathfrak{g}$, defined by

$$\mathrm{Cas}_{\mathfrak{g}} = \sum_{A=1}^{\dim G} I_A \otimes I_A. \tag{4.11}$$

If (ρ, V) is any representation of $\mathfrak{g}$, then

$$\rho(\mathrm{Cas}_{\mathfrak{g}}) = \sum_{A=1}^{\dim G} \rho(I_A)^2$$

Similarly we define,

$$\mathrm{Cas}_{\mathfrak{h}} = \sum_{A=8}^{\dim G} I_A \otimes I_A. \tag{4.12}$$

Using Lichnerowicz formula, we can write the square of the Dirac operator as sum of Casimir operators.

Proposition 4.1.1. *[54] Let V be a representation of H, E be the associated vector bundle $G \times_H V \to G/H$, and A_Σ be the canonical connection of E. Then,*

$$\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2 \eta = (-\rho_L(\mathrm{Cas}_{\mathfrak{g}}) + \rho_V(\mathrm{Cas}_{\mathfrak{h}}) + 49/9)\eta \tag{4.13}$$

for $\eta \in \Gamma(\$_{\mathbb{C}}(\Sigma) \otimes E)$.

The expression for $\left(\mathfrak{D}^{t}_{A_\Sigma}\right)^2 \eta$ significantly simplifies as above only for $t = 1/3$.

Restricting the operator $\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2$ to $\mathrm{Hom}(V_\gamma, \Delta \otimes V)^H \otimes V_\gamma$, we get

$$\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2 |_{\mathrm{Hom}(V_\gamma, \Delta\otimes V)^H \otimes V_\gamma} = \left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2_\gamma \otimes \mathrm{Id}\,. \tag{4.14}$$

The self-adjointness of this operator implies that it is diagonalisable with real eigenvalues. Frobenius reciprocity and Proposition 4.1.1 implies

$$\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2_\gamma = -\rho_{V_\gamma}(\mathrm{Cas}_{\mathfrak{g}}) + \rho_V(\mathrm{Cas}_{\mathfrak{h}}) + 49/9. \tag{4.15}$$

Eigenvalue Bounds

We have the nearly G_2-manifold G/H which is a reductive homogeneous space. Now, the Casimir operators commute with the group action, and hence on irreducible representation, they act as a multiple of the identity. That is,

$$\rho_\gamma(\mathrm{Cas}_{\mathfrak{g}}) = c^{\mathfrak{g}}_\gamma\, \mathrm{Id},$$
$$\rho_\gamma(\mathrm{Cas}_{\mathfrak{h}}) = c^{\mathfrak{h}}_\gamma\, \mathrm{Id},$$

where $c^{\mathfrak{g}}_\gamma$ and $c^{\mathfrak{h}}_\gamma$ are real numbers, called *Casimir eigenvalues*. Now, let V_γ be an irreducible representation of G. Then we have the decomposition of V_γ as

$$V_\gamma = \bigoplus_{\sigma \in I} W^\gamma_\sigma,$$

where W_σ are irreducible representations of H and I is a finite sequence in $\widehat{H}$ (the set of equivalence classes of irreducible representations of H) which may have repeated entries. Similarly, for finite sequences J, K in $\widehat{H}$, we have the decomposition

$$\Delta = \bigoplus_{\alpha \in J} W_\alpha, \quad V = \bigoplus_{\beta \in K} W_\beta.$$

Let us assume that in the decomposition of $W_\alpha \otimes W_\beta$ into irreducible representations, W_σ^γ occurs with multiplicity 1. Then we consider the composition map

$$q^\sigma_{\alpha\beta} : V_\gamma \to W^\gamma_\sigma \to W_\alpha \otimes W_\beta \hookrightarrow \Delta \otimes V$$

where the first map is the projection map and the third one is equivariant embedding. Since the decompositions of V_γ, Δ and V into irreducible representations of H are orthogonal, $\{q^\sigma_{\alpha\beta}\}$ is an orthogonal basis of $\mathrm{Hom}(V_\gamma, \Delta \otimes V)^H$. Hence, $q^\sigma_{\alpha\beta}$ are eigenvectors of (4.15) and

$$\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2_\gamma (q^\sigma_{\alpha\beta}) = \left(-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\beta + 49/9\right) q^\sigma_{\alpha\beta}. \tag{4.16}$$

Then $\{q^\sigma_{\alpha\beta}\}$ diagonalizes the twisted Dirac operator $\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2_\gamma$. The eigenvalues are given by $-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\beta + 49/9$ with multiplicities $\dim \mathrm{Hom}(V_\gamma, \Delta \otimes W_\beta)^H$. Hence the eigenvalues of $\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2_\gamma$ are $\sqrt{-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\beta + 49/9}$ and $-\sqrt{-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\beta + 49/9}$.

Now, we want to find an eigenvalue bound for the operator $\left(\mathfrak{D}^{0}_{A_\Sigma}\right)_\gamma$. First, we have the following lemma.

Lemma 4.1.2. *Let A and B be $n \times n$ Hermitian matrices with eigenvalues $\{\lambda^A_1, \dots, \lambda^A_n\}$ and $\{\lambda^B_1, \dots, \lambda^B_n\}$ respectively. If $\{\lambda^{A+B}_1, \dots, \lambda^{A+B}_n\}$ are eigenvalues of $A + B$, then*

$$\min_i \left\{\left|\lambda^{A+B}_i\right|\right\} \geq \min_i \left\{\left|\lambda^A_i\right|\right\} - \max_i \left\{\left|\lambda^B_i\right|\right\}.$$

Proof. Since $\min_i \{|\lambda^A_i|\} = \min_{\|v\|=1} \langle Av, v\rangle$ and $\max_i \{|\lambda^B_i|\} = \max_{\|v\|=1} \langle Bv, v\rangle$, we have

$$\begin{aligned} |\langle (A+B)v, v\rangle| &= |\langle Av, v\rangle + \langle Bv, v\rangle| \\ &\geq |\langle Av, v\rangle| - |\langle Bv, v\rangle| \\ &\geq \min_i \left\{\left|\lambda^A_i\right|\right\} - \max_i \left\{\left|\lambda^B_i\right|\right\} \end{aligned}$$

for all $v \in \mathbb{C}^n$ with $\|v\| = 1$. Hence $\min_i \{|\lambda^A_i|\} - \max_i \{|\lambda^B_i|\}$ is a lower bound on $\left\{\left|\lambda^{A+B}_i\right|\right\}^n_{i=1}$. □

Theorem 4.1.3. *Let V_γ be an irreducible representation of G. If*

$$L_\gamma := \sqrt{\min_\beta\left\{-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\beta + 49/9\right\}} - \frac{7}{6} > 0$$

for $\beta \in \widehat{H}$, then L_γ is a lower bound on the smallest positive eigenvalues of $\left(\mathfrak{D}^0_{A_\Sigma}\right)_\gamma$.

Proof. We have $\left(\mathfrak{D}^0_{A_\Sigma}\right)_\gamma = \left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)_\gamma - \frac{1}{6}\phi$. Now, ϕ acts on Λ^0 and Λ^1 with eigenvalues 7 and -1 respectively. Hence $\max\{|7|, |-1|\} = 7$. Now, if $\lambda_1^2, \dots, \lambda_n^2$ are eigenvalues for $\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2_\gamma$, then $\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)_\gamma$ has eigenvalues $\pm\lambda_1, \dots, \pm\lambda_n$. Now, the smallest positive eigenvalue of $\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)_\gamma$ is given by $\sqrt{\min_\beta\left\{-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\beta + 49/9\right\}}$. Hence, by lemma 4.1.2,

$$\sqrt{\min_\beta\left\{-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\beta + 49/9\right\}} - \frac{7}{6}$$

is a lower bound on the smallest positive eigenvalues of $\left(\mathfrak{D}^0_{A_\Sigma}\right)_\gamma$. □

4.2 The Twisted Dirac Operator on S^7

We identify S^7 with the homogeneous space $Spin(7)/G_2$. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{g}_2 \subset \mathfrak{spin}(7)$ with respect to the Killing form (4.4) on the Lie algebra $\mathfrak{spin}(7)$. Clearly, $[\mathfrak{g}_2, \mathfrak{m}] \subset \mathfrak{m}$ and hence the homogeneous space $Spin(7)/G_2$ is reductive. Consider the Maurer–Cartan form θ on $Spin(7)$ and the splitting $\theta = \theta_{\mathfrak{g}_2} \oplus \theta_{\mathfrak{m}}$ induced by the decomposition $\mathfrak{spin}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}$. Then $\theta_{\mathfrak{g}_2} =: A_\Sigma$ is canonical connection on the bundle $G_2 \to Spin(7) \to S^7$ whose curvature is given by

$$F(X,Y) = -[X,Y]_{\mathfrak{g}_2} \tag{4.17}$$

for $X, Y \in \mathfrak{m}$. This is a G_2-invariant element in $\Lambda^2\mathfrak{m}^* \otimes \mathfrak{g}_2$. The torsion is given by

$$T(X,Y) = -[X,Y]_{\mathfrak{m}}. \tag{4.18}$$

In (2.14) putting $t = 1$ for canonical connection and denoting $T^1(X,Y)$ by $T(X,Y)$, we have

$$T(X,Y) = \frac{2}{3}\phi(X,Y,\cdot). \tag{4.19}$$

The nearly G_2 metric, normalised such that the scalar curvature of the canonical connection is $\frac{112}{3}$, can be written as (4.5) where K is the Killing form (4.4) and c is a constant to be determined. Let $\{I_A : A = 1,\dots,21\}$ be an orthonormal basis for $\mathfrak{spin}(7)$, $\{I_a : a = 1,\dots,7\}$ be a basis for $\mathfrak{m}$ and $\{I_i : i = 8,\dots,21\}$ be a basis for $\mathfrak{g}_2$. Let f_{AB}^C be the structure constants defined by

$$[I_A, I_B] = f_{AB}^C I_C.$$

We lower the indices as $f_{ABC} := f_{AB}^D \delta_{DC}$. Then,

$$[I_a, I_b] = f_{ab}^c I_c + f_{ab}^i I_i.$$

Then for $I_a, I_b \in \mathfrak{m}$, (4.18) and (4.19) imply

$$T_{ab}^c = -f_{ab}^c = \frac{2}{3}\phi_{abc}. \tag{4.20}$$

A simple calculation involving the relations shows that, for $c^2 = \frac{3}{40}$ we have

$$g(X,Y) = -c^2 K(X,Y) = -\frac{3}{40}\operatorname{Tr}_{\mathfrak{spin}(7)}(\operatorname{ad}(X)\operatorname{ad}(Y)). \tag{4.21}$$

is a nearly G_2-metric, normalised so that the scalar curvature of the canonical connection is $\frac{112}{3}$.

Let us consider $Cl(7)$, the Clifford algebra over $\mathbb{R}^7$. Let Δ be a 8-dimensional representation of $Cl(7)$ and $\rho_\Delta : \mathfrak{spin}(7) \to \operatorname{End}(\Delta)$ be the restriction to $\mathfrak{spin}(7)$. From (4.8), recall the identification $\Gamma(\$_{\mathbb{C}}(\Sigma) \otimes (\mathfrak{g}_P)_{\mathbb{C}}) \cong L^2(Spin(7), \Delta \otimes V)^{G_2}$, where $V \cong \mathfrak{spin}(7)_{\mathbb{C}}$. Consider the operator

$$\begin{aligned} \mathfrak{D}^\rho_{A_\Sigma} &: L^2(Spin(7), \Delta \otimes V)^{G_2} \to L^2(Spin(7), \Delta \otimes V)^{G_2} \\ \mathfrak{D}^\rho_{A_\Sigma} &= \rho_\Delta(I_a)\rho_R(I_a). \end{aligned} \tag{4.22}$$

We want to compare the operator (4.22) with the Dirac operator (4.6). That is, compare $\rho_\Delta(I_a)$ with the Clifford multiplication by I_a.

Let $\{e^a : a = 1,\dots,7\}$ be a orthonormal basis of $\mathfrak{m}^*$ dual to I_a of $\mathfrak{m}$. Now, we have the decomposition $\mathfrak{spin}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}$. We identify $\mathfrak{m}$ with $\mathbb{R}^7$ via an isomorphism $F : \mathfrak{m} \to \mathbb{R}^7$ as representations of $\mathfrak{g}_2$. Then we note that F is unique by Schur's lemma and imposing the condition that F commutes with the G_2-structures, i.e., F maps the G_2-invariant 3-form of $\mathfrak{m}$ to G_2-invariant 3-form of $\mathbb{R}^7$. Then F induces an isomorphism $Cl(\mathfrak{m}) \cong Cl(7)$. Then from (2.7) we have an isomorphism $\Delta \cong \mathbb{C} \oplus \mathfrak{m}^*_{\mathbb{C}}$. The basis e^a for $\mathfrak{m}^*$ gives orthonormal vectors e^a in Δ. We choose $e^0 \in \Delta$ so that $\{e^0, e^1, \dots, e^7\}$ is an orthonormal basis for Δ and the $Spin(7)$-invariant 4-form Φ is given by the usual formula $e^0 \wedge \phi + *\phi$. We identify $(f, v) \in \mathbb{C} \oplus \mathfrak{m}^*_{\mathbb{C}}$ with $fe^0 + v \in \Lambda^1(\Delta)^*$.

Lemma 4.2.1. *The action of $\rho_\Delta(I_a)$ on Δ is given by*

$$\rho_\Delta(I_a)(f,v) = \left(-\langle e^a, v\rangle, fe^a - \frac{1}{3}((e^a \wedge v) \lrcorner \phi)\right) \tag{4.23}$$

Proof. Consider the isomorphism $\mathfrak{so}(8) \cong \Lambda^2(\Delta)^*$ given by

$$\beta(w)(u) = -u \lrcorner w \quad \text{for } u \in \Lambda^1(\Delta)^* \text{ and } w \in \Lambda^2(\Delta)^*, \tag{4.24}$$

where $\lrcorner$ is the contraction from inner product. We consider the embedding $\mathfrak{spin}(7) \hookrightarrow \mathfrak{so}(8) \cong \Lambda^2(\Delta)^*$. Now, $\Lambda^2(\Delta)^* = \Lambda^2_7 \oplus \Lambda^2_{21}$ where $\Lambda^2_7, \Lambda^2_{21}$ are irreducible representations of $Spin(7)$. Now the image of the embedding $\mathfrak{spin}(7) \hookrightarrow \Lambda^2(\Delta)^*$ is the space Λ^2_{21} of 2-forms α satisfying $\alpha \lrcorner \Phi = -\alpha$. Then $\Lambda^2(\Delta)^* \cong \Lambda^2\mathfrak{m}^* \oplus (e^0 \wedge \mathfrak{m}^*)$. Moreover, $\Lambda^2\mathfrak{m}^* = \Lambda^2_7 \oplus \Lambda^2_{14}$ where $\Lambda^2_{14} \cong \mathfrak{g}_2$. Then,

$$\mathfrak{m} \cong \Lambda^2_{21}(\Delta)^* \cap (\Lambda^2_7\mathfrak{m}^* \oplus (e^0 \wedge \mathfrak{m}^*)), \tag{4.25}$$

i.e., the image of $\mathfrak{m}$ is isomorphic to $\left\{\alpha \in \Lambda^2_7\mathfrak{m}^* \oplus (e^0 \wedge \mathfrak{m}^*) : \alpha \lrcorner \Phi = -\alpha\right\}$. Now, for $a = 1, \ldots, 7$,

$$\left(-e^0 \wedge e^a + \frac{1}{3}e^a \lrcorner \phi\right) \lrcorner \Phi = -\left(-e^0 \wedge e^a + \frac{1}{3}e^a \lrcorner \phi\right).$$

Hence, we have

$$\mathfrak{m} \cong \operatorname{Span}\left\{\widetilde{I}_a := -e^0 \wedge e^a + \frac{1}{3}e^a \lrcorner \phi : a = 1, \ldots, 7\right\}. \tag{4.26}$$

We calculate

$$\begin{aligned}
\beta(\widetilde{I}_a)(f,v) &= \beta\left(e^a \wedge e^0 + \frac{1}{3}e^a \lrcorner \phi\right)(f,v) \\
&= -(fe^0 + v) \lrcorner \left(e^a \wedge e^0 + \frac{1}{3}e^a \lrcorner \phi\right) \\
&= fe^a - \langle e^a, v\rangle e^0 - \frac{1}{3}((e^a \wedge v) \lrcorner \phi) \\
&= \left(-\langle e^a, v\rangle, fe^a - \frac{1}{3}((e^a \wedge v) \lrcorner \phi)\right).
\end{aligned} \tag{4.27}$$

We find that $\widetilde{I}_a$ is an orthonormal basis for the right hand side of (4.25). We prove that I_a maps exactly to $\widetilde{I}_a$ under the isomorphism. By Schur's lemma, we have $\rho_\Delta(I_a) = c\beta(\widetilde{I}_a)$ for some constant c. Now, an explicit calculation shows that $[\beta(\widetilde{I}_a), \beta(\widetilde{I}_b)]_\mathfrak{m} = -\frac{2}{3}\phi_{abc}\beta(\widetilde{I}_c)$. But from (4.19) we also have $[\rho_\Delta(I_a), \rho_\Delta(I_b)]_\mathfrak{m} = -\frac{2}{3}\phi_{abc}\rho_\Delta(I_c)$. Hence $c = 1$ and $\rho_\Delta(I_a) = \beta(\widetilde{I}_a)$. □

Now, the formula (4.23) does not agree with the formula for Clifford multiplication by I_a, which from (2.8), is given by

$$I_a \cdot (f,v) = (\langle e^a, v\rangle, -fe^a - (e^a \wedge v) \lrcorner \phi). \tag{4.28}$$

To fix this, we consider another representation of $\mathfrak{spin}(7)$ defined by

$$\widetilde{\rho}_\Delta(X) := M^{-1} \cdot \rho_\Delta(X) \cdot M$$

where $M := \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{Id}_{\mathfrak{m}} \end{pmatrix} = M^{-1}$. Then we calculate

$$\widetilde{\rho}_\Delta(I_a)(f, v) = \left(\langle e^a, v \rangle, -fe^a - \frac{1}{3}((e^a \wedge v) \lrcorner \phi) \right). \tag{4.29}$$

Thus from (4.28), (4.23) and (4.29), the Clifford multiplication of a spinor η by I_a can be rewritten as

$$I_a \cdot \eta = (\rho_\Delta(I_a) + 2\widetilde{\rho}_\Delta(I_a))\, \eta. \tag{4.30}$$

Now, consider the twisted Dirac operator

$$\mathfrak{D}^1_{A_\Sigma} : \Gamma(\$_{\mathbb{C}}(\Sigma) \otimes (\mathfrak{g}_P)_{\mathbb{C}}) \to \Gamma(\$_{\mathbb{C}}(\Sigma) \otimes (\mathfrak{g}_P)_{\mathbb{C}}).$$

Consider the operators $\mathfrak{D}^\rho_{A_\Sigma}$ given by (4.22) and

$$\begin{aligned} &\widetilde{\mathfrak{D}}^\rho_{A_\Sigma} : L^2(Spin(7), \Delta \otimes V)^{G_2} \to L^2(Spin(7), \Delta \otimes V)^{G_2} \\ &\widetilde{\mathfrak{D}}^\rho_{A_\Sigma} = \widetilde{\rho}_\Delta(I_a)\rho_R(I_a). \end{aligned} \tag{4.31}$$

Note that the operators $\mathfrak{D}^\rho_{A_\Sigma}$ and $\widetilde{\mathfrak{D}}^\rho_{A_\Sigma}$ commute with the left action of $Spin(7)$ and hence respect the decomposition (4.8). From (4.30), we have

$$\begin{aligned} \left(\mathfrak{D}^\rho_{A_\Sigma} + 2\widetilde{\mathfrak{D}}^\rho_{A_\Sigma} \right) &= (\rho_\Delta(I_a)\rho_R(I_a) + 2\widetilde{\rho}_\Delta(I_a)\rho_R(I_a)) \\ &= (\rho_\Delta(I_a) + 2\widetilde{\rho}_\Delta(I_a))\, \rho_R(I_a) \\ &= I_a \cdot \rho_R(I_a) = \mathfrak{D}^1_{A_\Sigma}. \end{aligned} \tag{4.32}$$

Hence from (4.7) and (4.32), we have

$$\mathfrak{D}^t_{A_\Sigma} = \mathfrak{D}^1_{A_\Sigma} + \frac{t-1}{2}\phi = \left(\mathfrak{D}^\rho_{A_\Sigma} + 2\widetilde{\mathfrak{D}}^\rho_{A_\Sigma} \right) + \frac{t-1}{2}\phi. \tag{4.33}$$

In terms of Casimir operators, we can write

$$\mathfrak{D}^\rho_{A_\Sigma} = \frac{1}{2}\left(\rho_{\Delta\otimes R}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_\Delta(\mathrm{Cas}_{\mathfrak{m}}) - \rho_R(\mathrm{Cas}_{\mathfrak{m}})\right)$$

and

$$\widetilde{\mathfrak{D}}^\rho_{A_\Sigma} = \frac{1}{2}\left(\widetilde{\rho}_{\Delta\otimes R}(\mathrm{Cas}_{\mathfrak{m}}) - \widetilde{\rho}_\Delta(\mathrm{Cas}_{\mathfrak{m}}) - \rho_R(\mathrm{Cas}_{\mathfrak{m}})\right)$$

where $\rho(\mathrm{Cas}_{\mathfrak{m}}) := \rho(\mathrm{Cas}_{\mathfrak{spin}(7)}) - \rho(\mathrm{Cas}_{\mathfrak{g}_2}) = \rho(I_a)\rho(I_a)$. Then restricting $\mathfrak{D}^{\rho}_{A_\Sigma}$ and $\widetilde{\mathfrak{D}}^{\rho}_{A_\Sigma}$ to $\mathrm{Hom}(V_\gamma, \Delta \otimes \mathfrak{spin}(7)_{\mathbb{C}})^{G_2}$, we get

$$\left(\mathfrak{D}^{\rho}_{A_\Sigma}\right)_\gamma = \frac{1}{2}\left(\rho_{\Delta\otimes V_\gamma^*}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_\Delta(\mathrm{Cas}_{\mathfrak{m}}) - \rho_{V_\gamma^*}(\mathrm{Cas}_{\mathfrak{m}})\right), \tag{4.34}$$

$$\left(\widetilde{\mathfrak{D}}^{\rho}_{A_\Sigma}\right)_\gamma = M_\gamma^{-1}\left(\mathfrak{D}^{\rho}_{A_\Sigma}\right)_\gamma M_\gamma, \tag{4.35}$$

where we define M_γ in the following way. Recalling $M = \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{Id}_{\mathfrak{m}} \end{pmatrix} : \Delta \to \Delta$, for $\xi \in \mathrm{Hom}(V_\gamma \otimes \Delta, \mathfrak{spin}(7)_{\mathbb{C}})^{G_2}$, we define M_γ by

$$M_\gamma \xi(v \otimes \delta) := \xi(v \otimes M\delta). \tag{4.36}$$

Then from (4.33) we have

$$\left(\mathfrak{D}^{t}_{A_\Sigma}\right)_\gamma = \left(\mathfrak{D}^{\rho}_{A_\Sigma}\right)_\gamma + 2\left(\widetilde{\mathfrak{D}}^{\rho}_{A_\Sigma}\right)_\gamma + \frac{t-1}{2}\phi. \tag{4.37}$$

Now, we want to calculate the eigenvalues of the Casimir operators appearing in the expression of the Dirac operator. Let $V_{(a,b,c)}$ be an irreducible representation of $\mathfrak{spin}(7)$ with highest weight (a,b,c) and $V_{(a,b)}$ be an irreducible representation of $\mathfrak{g}_2$ with highest weight (a,b). The Casimir operators can be written as

$$\rho_{(a,b,c)}\left(\mathrm{Cas}_{\mathfrak{spin}(7)}\right) = c^{\mathfrak{spin}(7)}_{(a,b,c)}\,\mathrm{Id},$$
$$\rho_{(a,b)}(\mathrm{Cas}_{\mathfrak{g}_2}) = c^{\mathfrak{g}_2}_{(a,b)}\,\mathrm{Id}\,.$$

The Casimir eigenvalues are given by,

$$c^{\mathfrak{g}_2}_{(a,b)} = -\frac{8}{9}(a^2 + 3b^2 + 3ab + 5a + 9b), \tag{4.38}$$

$$c^{\mathfrak{spin}(7)}_{(a,b,c)} = -\frac{1}{3}(4a^2 + 8b^2 + 3c^2 + 8ab + 8bc + 4ca + 20a + 32b + 18c). \tag{4.39}$$

These expressions differ from that of [54], because we use a different normalisation of the Casimir operator and an opposite convention for the order of a, b, c.

4.3 Eigenvalues of the Twisted Dirac Operator

For an FNFN $Spin(7)$-instanton (see section 5.1), the fastest rate of convergence is -2. Hence we consider the family of moduli spaces $\mathcal{M}(A_\Sigma, \nu)$ for $\nu \in (-2, 0)$. We want to find the critical weights in $(-2,0)$, i.e., $\nu \in (-2,0)$ such that $\nu + \frac{5}{2} \in \operatorname{Spec} \mathfrak{D}_{A_\Sigma}$. Hence, we are interested in finding all the eigenvalues of the twisted Dirac operator in the interval $\left[\frac{1}{2}, \frac{5}{2}\right]$.

Since $\mathfrak{spin}(7)_{\mathbb{C}} = V_{(0,1,0)} = V_{(1,0)} \oplus V_{(0,1)}$, we have

$$\operatorname{Hom}\left(V_\gamma, \Delta \otimes \mathfrak{spin}(7)_{\mathbb{C}}\right)^{G_2} = \operatorname{Hom}\left(V_\gamma, \Delta \otimes V_{(1,0)}\right)^{G_2} \oplus \operatorname{Hom}\left(V_\gamma, \Delta \otimes V_{(0,1)}\right)^{G_2}.$$

Then, since $c^{\mathfrak{g}_2}_{(1,0)} = -48/9$ and $c^{\mathfrak{g}_2}_{(0,1)} = -96/9$, we have

$$\begin{aligned} \left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2_\gamma \Big|_{\operatorname{Hom}\left(V_\gamma, \Delta\otimes V_{(1,0)}\right)^{G_2}} &= -c^{\mathfrak{spin}(7)}_\gamma + c^{\mathfrak{g}_2}_{(1,0)} + 49/9 = -c^{\mathfrak{spin}(7)}_\gamma + 1/9, \\ \left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2_\gamma \Big|_{\operatorname{Hom}\left(V_\gamma, \Delta\otimes V_{(0,1)}\right)^{G_2}} &= -c^{\mathfrak{spin}(7)}_\gamma + c^{\mathfrak{g}_2}_{(0,1)} + 49/9 = -c^{\mathfrak{spin}(7)}_\gamma - 47/9. \end{aligned} \tag{4.40}$$

Hence the eigenvalues and their multiplicities of $\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2_\gamma$ are

Eigenvalues	Multiplicities
$-c^{\mathfrak{spin}(7)}_\gamma + \frac{1}{9}$	$\dim \operatorname{Hom}\left(V_\gamma, \Delta \otimes V_{(1,0)}\right)^{G_2}$
$-c^{\mathfrak{spin}(7)}_\gamma - \frac{47}{9}$	$\dim \operatorname{Hom}\left(V_\gamma, \Delta \otimes V_{(0,1)}\right)^{G_2}$

Hence, we can restate Theorem 4.1.3 as: for $V_\gamma = V_{(a,b,c)}$ an irreducible representation of $Spin(7)$, if

$$L_\gamma := L_{(a,b,c)} := \sqrt{-c^{\mathfrak{spin}(7)}_{(a,b,c)} - 47/9} - \frac{7}{6} > 0, \tag{4.41}$$

then L_γ is a lower bound on the smallest positive eigenvalues of $\left(\mathfrak{D}^0_{A_\Sigma}\right)_\gamma$.

This yields the following important corollary, which follows from (4.41) and (4.39).

Corollary 4.3.1. *Consider the irreducible representations of $Spin(7)$ given by*

$$V_{(0,0,0)},\ V_{(1,0,0)},\ V_{(0,0,1)},\ V_{(0,1,0)},\ V_{(2,0,0)},\ V_{(1,0,1)}.$$

If V_γ is not one of these irreducible representations, then the operator

$$\left(\mathfrak{D}^0_{A_\Sigma}\right)_\gamma : \operatorname{Hom}\left(V_\gamma, \Delta \otimes \mathfrak{spin}(7)_{\mathbb{C}}\right)^{G_2} \to \operatorname{Hom}\left(V_\gamma, \Delta \otimes \mathfrak{spin}(7)_{\mathbb{C}}\right)^{G_2}$$

has no eigenvalues in the interval $\left(-\frac{5}{2}, \frac{5}{2}\right)$.

4.4 Calculation of Eigenvalues of the Twisted Dirac Operator

In this section we explicitly calculate the eigenvalues of the twisted Dirac operator corresponding to the representations mentioned in Corollary 4.3.1. Let us describe the outline of the method.

Let V_γ be an irreducible representation of $Spin(7)$. We want to find the matrix of the operator given in (4.37).

We note that, by G_2-equivariance, on $\mathrm{Hom}(V_{(0,0,1)} \otimes V_\gamma, \mathfrak{spin}(7)_{\mathbb{C}})^{G_2}$, we have

$$\rho_{\Delta\otimes V_\gamma}(\mathrm{Cas}_{\mathfrak{g}_2}) = \rho_{\mathfrak{spin}(7)_{\mathbb{C}}}(\mathrm{Cas}_{\mathfrak{g}_2}). \tag{4.42}$$

Hence, from (4.34), (4.42) and the isomorphism $V_\gamma^* \cong V_\gamma$, we can rewrite the operator $\left(\mathfrak{D}^\rho_{A_\Sigma}\right)_\gamma$ as

$$\left(\mathfrak{D}^\rho_{A_\Sigma}\right)_\gamma = \frac{1}{2}\left(\rho_{\Delta\otimes V_\gamma}(\mathrm{Cas}_{\mathfrak{spin}(7)}) - \rho_{\mathfrak{spin}(7)_{\mathbb{C}}}(\mathrm{Cas}_{\mathfrak{g}_2}) - \rho_\Delta(\mathrm{Cas}_{\mathfrak{m}}) - \rho_{V_\gamma}(\mathrm{Cas}_{\mathfrak{m}})\right). \tag{4.43}$$

- First, we want to find a basis of $\mathrm{Hom}(V_{(0,0,1)} \otimes V_\gamma, \mathfrak{spin}(7)_{\mathbb{C}})^{G_2}$ that diagonalizes $\rho_{\mathfrak{spin}(7)_{\mathbb{C}}}(\mathrm{Cas}_{\mathfrak{g}_2}) + \rho_\Delta(\mathrm{Cas}_{\mathfrak{m}}) + \rho_{V_\gamma}(\mathrm{Cas}_{\mathfrak{m}})$. We construct the basis by non-zero G_2-equivariant maps
$$q^{(i,j)(k,l)}_{(m,n)} : V_{(0,0,1)} \otimes V_\gamma \to V_{(i,j)} \otimes V_{(k,l)} \to V_{(m,n)} \to \mathfrak{spin}(7)_{\mathbb{C}}, \tag{4.44}$$
where $V_{(m,n)}$ is either $V_{(1,0)}$ or $V_{(0,1)}$. We identify $\mathfrak{spin}(7)_{\mathbb{C}}$ with $\Lambda^2(\mathbb{C}^7)$, and the models for the representations $V_{(i,j)}$ and $V_{(i,j,k)}$ are described in Appendix B.3.5. We use the identities in Appendix B.3.4 and the projection formulae from Appendix B.3.5 to write down explicit expressions of these maps.

- Then, we want to find a basis that diagonalizes $\rho_{\Delta\otimes V_\gamma}(\mathrm{Cas}_{\mathfrak{spin}(7)})$. We consider the maps
$$p^{(i,j,k)}_{(m,n)} : V_{(0,0,1)} \otimes V_\gamma \to V_{(i,j,k)} \to V_{(m,n)} \to \mathfrak{spin}(7)_{\mathbb{C}}. \tag{4.45}$$
Then, $p^{(i,j,k)}_{(m,n)}$ are eigenvectors of $\rho_{\Delta\otimes V_\gamma}(\mathrm{Cas}_{\mathfrak{spin}(7)})$ with eigenvalues $c^{\mathfrak{spin}(7)}_{(i,j,k)}$.

- From the explicit expressions of q-basis and p-basis elements, we write $p^{(i,j,k)}_{(m,n)}$ in terms of $q^{(i,j)(k,l)}_{(m,n)}$ and the change of basis matrix.

- Now, $q^{(i,j)(k,l)}_{(m,n)}$ are eigenvectors of $-\rho_{\mathfrak{spin}(7)_{\mathbb{C}}}(\mathrm{Cas}_{\mathfrak{g}_2}) - \rho_\Delta(\mathrm{Cas}_{\mathfrak{m}}) - \rho_{V_\gamma}(\mathrm{Cas}_{\mathfrak{m}})$ with eigenvalues $c^{\mathfrak{g}_2}_{(i,j)} + c^{\mathfrak{g}_2}_{(k,l)} - c^{\mathfrak{g}_2}_{(m,n)} - c^{\mathfrak{spin}(7)}_{(0,0,1)} - c^{\mathfrak{spin}(7)}_\gamma$ and $p^{(i,j,k)}_{(m,n)}$ are eigenvectors of $\rho_{\Delta\otimes V_\gamma}(\mathrm{Cas}_{\mathfrak{spin}(7)})$ with eigenvalues $c^{\mathfrak{spin}(7)}_{(i,j,k)}$. Then using the change of basis matrix, we write down the matrix of $\left(\mathfrak{D}^\rho_{A_\Sigma}\right)_\gamma$ in the q-basis (4.43).

- Next, we calculate the matrix of M_γ in the q-basis. From (4.36), we see that it is a diagonal matrix with entries either 1 or -1, since q_i factors through either $V_{(0,0)} \cong \Lambda^0 \subset \Delta$, or $V_{(1,0)} \cong \Lambda^1 \subset \Delta$. Then, we calculate the matrix of $\left(\mathfrak{D}^{\tilde{\rho}}_{A_\Sigma}\right)_\gamma$ in the q-basis (4.35).
- In the q-basis, we have ϕ acting as a diagonal matrix with entries either 7 or -1, by Lemma 2.1.3, since q_i factors through either $V_{(0,0)} \cong \Lambda^0 \subset \Delta$, or $V_{(1,0)} \cong \Lambda^1 \subset \Delta$.

  Consequently, using (4.37), we calculate the matrix of $\left(\mathfrak{D}^{t}_{A_\Sigma}\right)_\gamma$ in the q-basis.
- We note that for $t = 1/3$, by (4.16), $\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2_\gamma$ should be a diagonal matrix in the q-basis, where the entries are either $-c_\gamma^{\mathfrak{spin}(7)} + \frac{1}{9}$ or $-c_\gamma^{\mathfrak{spin}(7)} - \frac{47}{9}$, by (4.40), which acts as a consistency check for our calculations.

  Throughout the calculations, we use (4.38) and (4.39) to calculate the Casimir eigenvalues.
- Finally, for $t = 0$ in the matrix of $\left(\mathfrak{D}^{t}_{A_\Sigma}\right)_\gamma$ in the q-basis, we calculate the desired eigenvalues of $\left(\mathfrak{D}^{0}_{A_\Sigma}\right)_\gamma$.

Eigenvalues from the representation $V_{(0,0,0)}$

We start with $V_\gamma = V_{(0,0,0)}$, the trivial representation of $Spin(7)$. Then, by Schur's lemma,

$$\mathrm{Hom}(V_\gamma, \Delta \otimes \mathfrak{spin}(7)_{\mathbb{C}})^{G_2} \cong \mathrm{Hom}(\mathbb{C}, \Delta \otimes \mathfrak{spin}(7)_{\mathbb{C}})^{G_2} \cong \mathrm{Hom}(\Delta, \mathfrak{spin}(7)_{\mathbb{C}})^{G_2}.$$

This space is one dimensional with a basis given by the map that factors through projections

$$q : \Delta \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}.$$

Now, when V_γ is the trivial representation, $\rho_{V_\gamma}(\mathrm{Cas}_{\mathfrak{m}}) = 0$ and $\rho_{\Delta\otimes V_\gamma}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_\Delta(\mathrm{Cas}_{\mathfrak{m}}) = 0$. Then, from (4.34) and (4.35), we have $\left(\mathfrak{D}^{\rho}_{A_\Sigma}\right)_\gamma = 0$ and $\left(\mathfrak{D}^{\tilde{\rho}}_{A_\Sigma}\right)_\gamma = 0$. Thus, $\left(\mathfrak{D}^{0}_{A_\Sigma}\right)_\gamma = -\frac{1}{2}\phi$. Now, by Lemma 2.1.3, ϕ acts as -1 on the space $\mathrm{Hom}(\Delta, \mathfrak{spin}(7)_{\mathbb{C}})^{G_2}$ since q factors through $V_{(1,0)} \cong \Lambda^1 \subset \Delta$. Hence, the eigenvalue of $\left(\mathfrak{D}^{0}_{A_\Sigma}\right)_\gamma$ is $\frac{1}{2}$.

Eigenvalues from the representation $V_{(1,0,0)}$

Let $V_\gamma = V_{(1,0,0)}$ be the standard representation of $Spin(7)$. The space $\mathrm{Hom}(\Delta \otimes V_{(1,0,0)}, \mathfrak{spin}(7)_{\mathbb{C}})^{G_2}$ $\cong \mathrm{Hom}(V_{(0,0,1)} \otimes V_{(1,0,0)}, \mathfrak{spin}(7)_{\mathbb{C}})^{G_2}$ is three dimensional. This follows from the facts that

$V_{(0,0,1)} \otimes V_{(1,0,0)} = V_{(0,0)} \oplus 2V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(2,0)}$, $\mathfrak{spin}(7)_{\mathbb{C}} = V_{(0,1,0)} = V_{(1,0)} \oplus V_{(0,1)}$, and Schur's lemma. Applying appropriate projection maps, the basis $q^{(i,j)(k,l)}_{(m,n)}$ of $\mathrm{Hom}(V_{(0,0,1)} \otimes V_{(1,0,0)}, \mathfrak{spin}(7)_{\mathbb{C}})^{G_2}$ is given by

Maps	Formula
$q_1 = q^{(0,0)(1,0)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(1,0,0)} \to V_{(0,0)} \otimes V_{(1,0)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt + v) \otimes (dt \wedge b + w) \mapsto adt \otimes \frac{1}{3}(w \lrcorner \phi) \mapsto \frac{1}{3}a(w \lrcorner \phi)$ $\mapsto \frac{1}{3}a(w \lrcorner \phi) \lrcorner \phi = aw$
$q_2 = q^{(1,0)(1,0)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(1,0,0)} \to V_{(1,0)} \otimes V_{(1,0)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(v + adt) \otimes (dt \wedge b + w) \mapsto v \otimes \left(\frac{1}{3}w \lrcorner \phi\right) \mapsto \left(v \wedge \frac{1}{3}(w \lrcorner \phi)\right) \lrcorner \phi$ $\mapsto \frac{1}{3}((v \wedge (w \lrcorner \phi)) \lrcorner \phi) \lrcorner \phi$
$q_3 = q^{(1,0)(1,0)}_{(0,1)}$	$V_{(0,0,1)} \otimes V_{(1,0,0)} \to V_{(1,0)} \otimes V_{(1,0)} \to V_{(0,1)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(v + adt) \otimes (dt \wedge b + w) \mapsto v \otimes \left(\frac{1}{3}w \lrcorner \phi\right) \mapsto \frac{1}{3}\pi_{14}(v \wedge (w \lrcorner \phi))$ $\mapsto \frac{1}{3}v \wedge (w \lrcorner \phi) - \frac{1}{9}((v \wedge (w \lrcorner \phi)) \lrcorner \phi) \lrcorner \phi$

To calculate the basis $p^{(i,j,k)}_{(m,n)}$ of $\mathrm{Hom}(V_{(0,0,1)} \otimes V_{(1,0,0)}, \mathfrak{spin}(7)_{\mathbb{C}})^{G_2}$ we use the following projection maps

$$V_{(0,0,1)} \otimes V_{(1,0,0)} \to V_{(0,0,1)}, \quad (adt + v) \otimes (dt \wedge b + w) \mapsto (adt + v) \lrcorner (dt \wedge b + w),$$

$$V_{(0,0,1)} \otimes V_{(1,0,0)} \to V_{(1,0,1)}, \quad (adt + v) \otimes (dt \wedge b + w) \mapsto \pi_{48}((adt + v) \wedge (dt \wedge b + w)).$$

where π_{48} is the projection $\Lambda^3(\mathbb{C}^8) \to \Lambda^3_{48}(\mathbb{C}^8)$. Then we apply appropriate projection formulae given in Appendix B.3.5 to project it further to $V_{(1,0)}$ or $V_{(0,1)}$. consequently, the basis $p^{(i,j,k)}_{(m,n)}$ is given by

Maps	Formula
$p_1 = p^{(0,0,1)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(1,0,0)} \to V_{(0,0,1)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt + v) \otimes (dt \wedge b + w) \mapsto aw - \frac{1}{3}((v \wedge (w \lrcorner \phi)) \lrcorner \phi) \lrcorner \phi$
$p_2 = p^{(1,0,1)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(1,0,0)} \to V_{(1,0,1)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt + v) \otimes (dt \wedge b + w) \mapsto aw + \frac{1}{18}((v \wedge (w \lrcorner \phi)) \lrcorner \phi) \lrcorner \phi$
$p_3 = p^{(1,0,1)}_{(0,1)}$	$V_{(0,0,1)} \otimes V_{(1,0,0)} \to V_{(1,0,1)} \to V_{(0,1)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt + v) \otimes (dt \wedge b + w) \mapsto -\frac{1}{3}v \wedge (w \lrcorner \phi) + \frac{1}{9}((v \wedge (w \lrcorner \phi)) \lrcorner \phi) \lrcorner \phi$

Finally, we write the basis $p^{(i,j,k)}_{(m,n)}$ in terms of $q^{(i,j)(k,l)}_{(m,n)}$ as

$$\begin{aligned} p_1 &= q_1 - q_2 \\ p_2 &= q_1 + \frac{1}{6}q_2 \\ p_3 &= -q_3. \end{aligned}$$

Now, $q^{(i,j)(k,l)}_{(m,n)}$ are eigenvectors of $-\rho_{\mathfrak{spin}(7)_\mathbb{C}}(\mathrm{Cas}_{\mathfrak{g}_2}) - \rho_\Delta(\mathrm{Cas}_\mathfrak{m}) - \rho_{V_{(1,0,0)}}(\mathrm{Cas}_\mathfrak{m})$ with eigenvalues

$$c^{\mathfrak{g}_2}_{(i,j)} + c^{\mathfrak{g}_2}_{(k,l)} - c^{\mathfrak{g}_2}_{(m,n)} - c^{\mathfrak{spin}(7)}_{(0,0,1)} - c^{\mathfrak{spin}(7)}_{(1,0,0)}.$$

Hence,

$$-\rho_{\mathfrak{spin}(7)_\mathbb{C}}(\mathrm{Cas}_{\mathfrak{g}_2}) - \rho_\Delta(\mathrm{Cas}_\mathfrak{m}) - \rho_{V_{(1,0,0)}}(\mathrm{Cas}_\mathfrak{m}) = \mathrm{diag}(15, 29/3, 15)$$

Moreover, $p^{(i,j,k)}_{(m,n)}$ are eigenvectors of $\rho_{\Delta\otimes V_{(1,0,0)}}(\mathrm{Cas}_{\mathfrak{spin}(7)})$ with eigenvalues $c^{\mathfrak{spin}(7)}_{(i,j,k)}$. Then the eigenvalues corresponding to the eigenvectors $p^{(0,0,1)}_{(1,0)}, p^{(1,0,1)}_{(1,0)}$ and $p^{(1,0,1)}_{(0,1)}$ are -7, $-\frac{49}{3}$, and $-\frac{49}{3}$ respectively, which implies, we have $\rho_{\Delta\otimes V_{(1,0,0)}}(\mathrm{Cas}_{\mathfrak{spin}(7)}) = \mathrm{diag}(-7, -49/3, -49/3)$ in the p-basis. Since q_1 factors through $V_{(0,0)} \cong \Lambda^0 \subset \Delta$, whereas q_2 and q_3 factor through $V_{(1,0)} \cong \Lambda^1 \subset \Delta$, , we note that in the basis q_1, q_2, q_3, the matrix $M_\gamma = \mathrm{diag}(1, -1, -1)$ and ϕ acts as the matrix $\mathrm{diag}(7, -1, -1)$. Consequently, by (4.37), the matrix of $\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma$ in the basis q_1, q_2, q_3 is given by

$$\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma = \begin{pmatrix} \frac{7}{2}(t-1) & 4 & 0 \\ \frac{2}{3} & 2 - \frac{1}{2}(t-1) & 0 \\ 0 & 0 & -2 - \frac{1}{2}(t-1) \end{pmatrix}.$$

We note that for $t = 1/3$, we have $\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2_\gamma = \mathrm{diag}(43/9, 43/9, 25/9)$, in the q-basis, which shows that our calculations have been consistent. Finally, for $t = 0$, the eigenvalues are given by $\frac{1}{6}(-3 + 2\sqrt{105}), \frac{1}{6}(-3 - 2\sqrt{105}), -\frac{3}{2}$.

Eigenvalues from the representation $V_{(0,0,1)}$

Let $V_\gamma = V_{(0,0,1)}$ be the spin representation of $Spin(7)$. The space
$\mathrm{Hom}(\Delta \otimes V_{(0,0,1)}, \mathfrak{spin}(7)_\mathbb{C})^{G_2} \cong \mathrm{Hom}(V_{(0,0,1)} \otimes V_{(0,0,1)}, \mathfrak{spin}(7)_\mathbb{C})^{G_2}$ is four dimensional. This follows from the facts that $V_{(0,0,1)} \otimes V_{(0,0,1)} = 2V_{(0,0)} \oplus 3V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(2,0)}$, $\mathfrak{spin}(7)_\mathbb{C} = V_{(0,1,0)} = V_{(1,0)} \oplus V_{(0,1)}$, and Schur's lemma. The basis $q^{(i,j)(k,l)}_{(m,n)}$ of the space $\mathrm{Hom}(V_{(0,0,1)} \otimes V_{(0,0,1)}, \mathfrak{spin}(7)_\mathbb{C})^{G_2}$ is given by

Maps	Formula
$q_1 = q_{(1,0)}^{(0,0)(1,0)}$	$V_{(0,0,1)} \otimes V_{(0,0,1)} \to V_{(0,0)} \otimes V_{(1,0)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(v + adt) \otimes (w + bdt) \mapsto adt \otimes w \mapsto aw \mapsto aw \lrcorner \phi$
$q_2 = q_{(1,0)}^{(1,0)(0,0)}$	$V_{(0,0,1)} \otimes V_{(0,0,1)} \to V_{(1,0)} \otimes V_{(0,0)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(v + adt) \otimes (w + bdt) \mapsto v \otimes bdt \mapsto bv \mapsto bv \lrcorner \phi$
$q_3 = q_{(1,0)}^{(1,0)(1,0)}$	$V_{(0,0,1)} \otimes V_{(0,0,1)} \to V_{(1,0)} \otimes V_{(1,0)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(v + adt) \otimes (w + bdt) \mapsto v \otimes w \mapsto (v \wedge w) \lrcorner \phi \mapsto ((v \wedge w) \lrcorner \phi) \lrcorner \phi$
$q_4 = q_{(0,1)}^{(1,0)(1,0)}$	$V_{(0,0,1)} \otimes V_{(0,0,1)} \to V_{(1,0)} \otimes V_{(1,0)} \to V_{(0,1)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(v + adt) \otimes (w + bdt) \mapsto v \otimes w \mapsto \pi_{14}(v \wedge w)$ $\mapsto \pi_{14}(v \wedge w) = (v \wedge w) - \frac{1}{3}((v \wedge w) \lrcorner \phi) \lrcorner \phi$

The basis $p_{(m,n)}^{(i,j,k)}$ of $\mathrm{Hom}(V_{(0,0,1)} \otimes V_{(0,0,1)}, \mathfrak{spin}(7)_{\mathbb{C}})^{G_2}$ is given by

Maps	Formula
$p_1 = p_{(1,0)}^{(1,0,0)}$	$V_{(0,0,1)} \otimes V_{(0,0,1)} \to V_{(1,0,0)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(v + adt) \otimes (w + bdt) \mapsto aw \lrcorner \phi - bv \lrcorner \phi + ((v \wedge w) \lrcorner \phi) \lrcorner \phi$
$p_2 = p_{(1,0)}^{(0,1,0)}$	$V_{(0,0,1)} \otimes V_{(0,0,1)} \to V_{(0,1,0)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(v + adt) \otimes (w + bdt) \mapsto 3aw \lrcorner \phi - 3bv \lrcorner \phi - ((v \wedge w) \lrcorner \phi) \lrcorner \phi$
$p_3 = p_{(0,1)}^{(0,1,0)}$	$V_{(0,0,1)} \otimes V_{(0,0,1)} \to V_{(0,1,0)} \to V_{(0,1)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(v + adt) \otimes (w + bdt) \mapsto v \wedge w - \frac{1}{3}((v \wedge w) \lrcorner \phi) \lrcorner \phi$
$p_4 = p_{(1,0)}^{(0,0,2)}$	$V_{(0,0,1)} \otimes V_{(0,0,1)} \to V_{(0,0,2)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(v + adt) \otimes (w + bdt) \mapsto aw \lrcorner \phi + bv \lrcorner \phi$

Finally, we write the basis $p_{(m,n)}^{(i,j,k)}$ in terms of $q_{(m,n)}^{(i,j)(k,l)}$ as

$$p_1 = q_1 - q_2 + q_3$$
$$p_2 = 3q_1 - 3q_2 - q_3$$
$$p_3 = q_4$$
$$p_4 = q_1 + q_2.$$

Now, $-\rho_{\mathfrak{spin}(7)_{\mathbb{C}}}(\mathrm{Cas}_{\mathfrak{g}_2}) - \rho_{\Delta}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_{V_{(1,0,0)}}(\mathrm{Cas}_{\mathfrak{m}}) = \mathrm{diag}(14, 14, 26/3, 14)$. Moreover, the eigenvalues corresponding to the eigenvectors $p_{(1,0)}^{(1,0,0)}, p_{(1,0)}^{(0,1,0)}, p_{(0,1)}^{(0,1,0)}$ and $p_{(1,0)}^{(0,0,2)}$ are $-8, -\frac{40}{3}, -\frac{40}{3}$, and -16 respectively. which implies, in p-basis,

$$\rho_{\Delta \otimes V_{(0,0,1)}}(\mathrm{Cas}_{\mathfrak{spin}(7)}) = \mathrm{diag}(-8, -40/3, -40/3, -16).$$

Since q_1 factors through $V_{(0,0)} \cong \Lambda^0 \subset \Delta$, whereas q_2, q_3 and q_4 factor through $V_{(1,0)} \cong \Lambda^1 \subset \Delta$, in the basis q_1, q_2, q_3, q_4, the matrix $M_\gamma = \mathrm{diag}(1,-1,-1,-1)$ and ϕ acts as the matrix $\mathrm{diag}(7,-1,-1,-1)$. Consequently, by (4.37), the matrix of $\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma$ in the basis q_1, q_2, q_3, q_4 is given by

$$\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma = \begin{pmatrix} -\frac{7(t-1)}{2} & 1 & -2 & 0 \\ 1 & -\frac{t-1}{2} & -6 & 0 \\ -\frac{1}{3} & -1 & -1-\frac{t-1}{2} & 0 \\ 0 & 0 & 0 & 1-\frac{t-1}{2} \end{pmatrix}$$

We note that for $t = 1/3$, we have $\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2_\gamma = \mathrm{diag}(64/9, 64/9, 64/9, 16/9)$, in the q-basis, which shows that our calculations have been consistent. Finally, for $t = 0$, the eigenvalues are given by $\frac{1}{6}(-3-8\sqrt{6}), \frac{1}{6}(-3+8\sqrt{6}), -\frac{5}{2}, \frac{3}{2}$.

Eigenvalues from the representation $V_{(0,1,0)}$

Let $V_\gamma = V_{(0,1,0)}$ be the adjoint representation of $Spin(7)$. The space $\mathrm{Hom}(\Delta \otimes V_{(0,1,0)}, \mathfrak{spin}(7)_\mathbb{C})^{G_2} \cong \mathrm{Hom}(V_{(0,0,1)} \otimes V_{(0,1,0)}, \mathfrak{spin}(7)_\mathbb{C})^{G_2}$ is five dimensional. This follows from the facts that $V_{(0,0,1)} \otimes V_{(0,1,0)} = V_{(0,0)} \oplus 3V_{(1,0)} \oplus 2V_{(0,1)} \oplus 2V_{(2,0)} \oplus V_{(1,1)}$, $\mathfrak{spin}(7)_\mathbb{C} = V_{(0,1,0)} = V_{(1,0)} \oplus V_{(0,1)}$, and Schur's lemma. The basis $q^{(i,j)(k,l)}_{(m,n)}$ of $\mathrm{Hom}(V_{(0,0,1)} \otimes V_{(0,1,0)}, \mathfrak{spin}(7)_\mathbb{C})^{G_2}$ is given by

Maps	Formula
$q_1 = q^{(0,0)(1,0)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(0,1,0)} \to V_{(0,0)} \otimes V_{(1,0)} \to V_{(1,0)} \to \mathfrak{spin}(7)_\mathbb{C}$ $(v + adt) \otimes (dt \wedge b + w) \mapsto adt \otimes b \mapsto ab \mapsto -a(w \lrcorner \phi) \lrcorner \phi$
$q_2 = q^{(0,0)(0,1)}_{(0,1)}$	$V_{(0,0,1)} \otimes V_{(0,1,0)} \to V_{(0,0)} \otimes V_{(0,1)} \to V_{(0,1)} \to \mathfrak{spin}(7)_\mathbb{C}$ $(v + adt) \otimes (dt \wedge b + w) \mapsto adt \otimes \pi_{14}(w) \mapsto a\pi_{14}(w) \mapsto aw - \frac{1}{3}a(w \lrcorner \phi) \lrcorner \phi$
$q_3 = q^{(1,0)(1,0)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(0,1,0)} \to V_{(1,0)} \otimes V_{(1,0)} \to V_{(1,0)} \to \mathfrak{spin}(7)_\mathbb{C}$ $(v + adt) \otimes (dt \wedge b + w) \mapsto v \otimes b \mapsto (v \wedge b) \lrcorner \phi$ $\mapsto (v \lrcorner ((w \lrcorner \phi) \lrcorner \phi) \lrcorner \phi$
$q_4 = q^{(1,0)(1,0)}_{(0,1)}$	$V_{(0,0,1)} \otimes V_{(0,1,0)} \to V_{(1,0)} \otimes V_{(1,0)} \to V_{(0,1)} \to \mathfrak{spin}(7)_\mathbb{C}$ $(v + adt) \otimes (dt \wedge b + w) \mapsto v \otimes b \mapsto \pi_{14}(v \wedge b) \mapsto \pi_{14}(v \wedge b)$ $= -(v \wedge (w \lrcorner \phi)) + \frac{1}{3}((v \wedge (w \lrcorner \phi)) \lrcorner \phi) \lrcorner \phi$
$q_5 = q^{(1,0)(0,1)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(0,1,0)} \to V_{(1,0)} \otimes V_{(0,1)} \to V_{(1,0)} \to \mathfrak{spin}(7)_\mathbb{C}$ $(v + adt) \otimes (dt \wedge b + w) \mapsto v \otimes \pi_{14}(w) \mapsto v \lrcorner \pi_{14}(w) \mapsto (v \lrcorner \pi_{14}(w)) \lrcorner \phi$ $= (v \lrcorner w - \frac{1}{3}v \lrcorner ((w \lrcorner \phi) \lrcorner \phi))) \lrcorner \phi$

The basis $p^{(i,j,k)}_{(m,n)}$ of $\mathrm{Hom}(V_{(0,0,1)} \otimes V_{(0,1,0)}, \mathfrak{spin}(7)_\mathbb{C})^{G_2}$ is given by

Maps	Formula
$p_1 = p^{(0,0,1)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(0,1,0)} \to V_{(0,0,1)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(v + adt) \otimes (dt \wedge b + w) \mapsto -a(w \lrcorner \phi) \lrcorner \phi + (v \lrcorner w) \lrcorner \phi$
$p_2 = p^{(1,0,1)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(0,1,0)} \to V_{(1,0,1)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(v + adt) \otimes (dt \wedge b + w) \mapsto 4a(w \lrcorner \phi) \lrcorner \phi - 7(v \lrcorner ((w \lrcorner \phi)) \lrcorner \phi) \lrcorner \phi$ $+3(v \lrcorner w) \lrcorner \phi$
$p_3 = p^{(1,0,1)}_{(0,1)}$	$V_{(0,0,1)} \otimes V_{(0,1,0)} \to V_{(1,0,1)} \to V_{(0,1)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(v + adt) \otimes (dt \wedge b + w) \mapsto aw - \frac{1}{3}(aw \lrcorner \phi) \lrcorner \phi + (v \wedge (w \lrcorner \phi))$ $-\frac{1}{3}((v \wedge (w \lrcorner \phi)) \lrcorner \phi) \lrcorner \phi$
$p_4 = p^{(0,1,1)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(0,1,0)} \to V_{(0,1,1)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ orthogonal to p_1 and p_2
$p_5 = p^{(0,1,1)}_{(0,1)}$	$V_{(0,0,1)} \otimes V_{(0,1,0)} \to V_{(0,1,1)} \to V_{(0,1)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ orthogonal to p_3

Then, we write the basis $p^{(i,j,k)}_{(m,n)}$ in terms of $q^{(i,j)(k,l)}_{(m,n)}$ as

$$\begin{aligned} p_1 &= q_1 + \frac{1}{3}q_3 + q_5 \\ p_2 &= -4q_1 - 6q_3 + 3q_5 \\ p_3 &= q_2 - q_4. \end{aligned}$$

In order to calculate p_4 and p_5, we calculate the norm of the q-basis element. By computing the matrix of q explicitly, and using $\|q\|^2 = \mathrm{Tr}(q^\dagger q)$, we find that

$$\|q_1\|^2 = \frac{63}{4},\ \|q_2\|^2 = 14,\ \|q_3\|^2 = \frac{189}{2},\ \|q_4\|^2 = 21,\ \text{and}\ \|q_5\|^2 = 84.$$

Now, p_4 factors through $V_{(1,0)}$, so must be a linear combination of q_1, q_3, q_5, and similar p_5 factors through $V_{(0,1)}$, so must be a linear combination of q_2 and q_4. Then, we use the fact that the p-basis and the q-basis are orthogonal, we find,

$$\begin{aligned} p_4 &= q_1 - \frac{1}{6}q_3 - \frac{1}{8}q_5 \\ p_5 &= q_2 + \frac{2}{3}q_4. \end{aligned}$$

Now, $-\rho_{\mathfrak{spin}(7)_{\mathbb{C}}}(\mathrm{Cas}_{\mathfrak{g}_2}) - \rho_{\Delta}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_{V_{(1,0,0)}}(\mathrm{Cas}_{\mathfrak{m}}) = \mathrm{diag}(61/3, 61/3, 15, 61/3, 29/3)$. Moreover, the eigenvalues corresponding to the eigenvectors $p^{(0,0,1)}_{(1,0)}, p^{(1,0,1)}_{(1,0)}, p^{(1,0,1)}_{(0,1)}, p^{(0,1,1)}_{(1,0)}$ and $p^{(0,1,1)}_{(0,1)}$

are
$-7, -\frac{49}{3}, -\frac{49}{3}, -23$ and -23 respectively. which implies,

$$\rho_{\Delta\otimes V_{(1,0,1)}}(\mathrm{Cas}_{\mathfrak{spin}(7)}) = \mathrm{diag}(-7, -49/3, -49/3, -23, -23).$$

Now, since q_1 and q_2 factors through $V_{(0,0)} \cong \Lambda^0 \subset \Delta$, whereas q_3, q_4 and q_5 factor through $V_{(1,0)} \cong \Lambda^1 \subset \Delta$, in the q-basis, the matrix M_γ is given by $\mathrm{diag}(1, 1, -1, -1, -1)$. and ϕ acts as the matrix $\mathrm{diag}(7, 7, -1, -1, -1)$. Consequently, by (4.37), the matrix of $\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma$ in the basis q_1, q_2, q_3, q_4, q_5 is given by

$$\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma = \begin{pmatrix} \frac{7}{2}(t-1) & 0 & -4 & 0 & -\frac{16}{3} \\ 0 & \frac{7}{2}(t-1) & 0 & 2 & 0 \\ -\frac{2}{3} & 0 & -2-\frac{1}{2}(t-1) & 0 & \frac{8}{3} \\ 0 & \frac{4}{3} & 0 & 2-\frac{1}{2}(t-1) & 0 \\ -1 & 0 & 3 & 0 & -\frac{1}{2}(t-1) \end{pmatrix}.$$

We note that for $t = 1/3$, we have $\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2_\gamma = \mathrm{diag}(121/9, 73/9, 121/9, 73/9, 121/9)$, in the q-basis, which shows that our calculations have been consistent. Finally, for $t = 0$, the eigenvalues are given by $\frac{1}{2}(-1-2\sqrt{17}), \frac{1}{6}(-3-2\sqrt{105}), \frac{1}{2}(-1+2\sqrt{17}), -\frac{7}{2}, \frac{1}{6}(-3+2\sqrt{105})$.

Eigenvalues from the representation $V_{(2,0,0)}$

Now, let us consider the irreducible $Spin(7)$ representation $V_\gamma = V_{(2,0,0)}$. The space $\mathrm{Hom}(\Delta \otimes V_{(2,0,0)}, \mathfrak{spin}(7)_\mathbb{C})^{G_2} \cong \mathrm{Hom}(V_{(0,0,1)} \otimes V_{(2,0,0)}, \mathfrak{spin}(7)_\mathbb{C})^{G_2}$ is two dimensional, since $V_{(0,0,1)} \otimes V_{(2,0,0)} = V_{(1,0)} \oplus V_{(0,1)} \oplus 2V_{(2,0)} \oplus V_{(3,0)} \oplus V_{(1,1)}$, $\mathfrak{spin}(7)_\mathbb{C} = V_{(0,1,0)} = V_{(1,0)} \oplus V_{(0,1)}$. The basis $q^{(i,j)(k,l)}_{(m,n)}$ of $\mathrm{Hom}(V_{(0,0,1)} \otimes V_{(2,0,0)}, \mathfrak{spin}(7)_\mathbb{C})^{G_2}$ is given by

$$q_1 = q^{(1,0)(2,0)}_{(1,0)} : V_{(0,0,1)} \otimes V_{(2,0,0)} \to V_{(1,0)} \otimes V_{(2,0)} \to V_{(1,0)} \to \mathfrak{spin}(7)_\mathbb{C}$$
$$q_2 = q^{(1,0)(2,0)}_{(0,1)} : V_{(0,0,1)} \otimes V_{(2,0,0)} \to V_{(1,0)} \otimes V_{(2,0)} \to V_{(0,1)} \to \mathfrak{spin}(7)_\mathbb{C}.$$

Now, the basis $p^{(i,j,k)}_{(m,n)}$ of $\mathrm{Hom}(V_{(0,0,1)} \otimes V_{(2,0,0)}, \mathfrak{spin}(7)_\mathbb{C})^{G_2}$ is given by

$$p_1 = p^{(1,0,1)}_{(1,0)} : V_{(0,0,1)} \otimes V_{(2,0,0)} \to V_{(1,0,1)} \to V_{(1,0)} \to \mathfrak{spin}(7)_\mathbb{C}$$
$$p_2 = p^{(1,0,1)}_{(0,1)} : V_{(0,0,1)} \otimes V_{(2,0,0)} \to V_{(1,0,1)} \to V_{(0,1)} \to \mathfrak{spin}(7)_\mathbb{C}.$$

Since the maps q_1 and q_2 are unique up to scale, we choose the basis $p^{(i,j,k)}_{(m,n)}$ in terms of $q^{(i,j)(k,l)}_{(m,n)}$ as

$$p_1 = q_1 \quad \text{and} \quad p_2 = q_2.$$

Now, in the q-basis, $-\rho_{\mathfrak{spin}(7)_{\mathbb{C}}}(\mathrm{Cas}_{\mathfrak{g}_2}) - \rho_\Delta(\mathrm{Cas}_{\mathfrak{m}}) - \rho_{V_{(1,0,0)}}(\mathrm{Cas}_{\mathfrak{m}}) = \mathrm{diag}(119/9, 167/9)$. The eigenvalues corresponding to the eigenvectors $p^{(1,0,1)}_{(1,0)}$ and $p^{(1,0,1)}_{(0,1)}$ are $-\frac{49}{3}$, and $-\frac{49}{3}$ respectively, which implies, in the p-basis, $\rho_{\Delta\otimes V_{(2,0,0)}}(\mathrm{Cas}_{\mathfrak{spin}(7)}) = \mathrm{diag}(-49/3, -49/3)$. Now, since q_1 and q_2 factor through $V_{(1,0)} \cong \Lambda^1 \subset \Delta$, in the q-basis, $M = \mathrm{diag}(-1,-1)$ and ϕ acts as the matrix $\mathrm{diag}(-1,-1)$. Consequently, by (4.37), the matrix of $\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma$ in the basis q_1, q_2 is given by

$$\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma = \begin{pmatrix} -\frac{14}{3} - \frac{t-1}{2} & 0 \\ 0 & \frac{10}{3} - \frac{t-1}{2} \end{pmatrix}.$$

We note that for $t = 1/3$, in the q-basis, we have $\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2_\gamma = \mathrm{diag}(169/9, 121/9)$, which shows that our calculations have been consistent. Finally, for $t = 0$, the eigenvalues are given by $-\frac{25}{6}, \frac{23}{6}$.

Eigenvalues from the representation $V_{(1,0,1)}$

Finally, we consider the irreducible $Spin(7)$ representation $V_\gamma = V_{(1,0,1)}$. The space $\mathrm{Hom}(\Delta \otimes V_{(1,0,1)}, \mathfrak{spin}(7)_{\mathbb{C}})^{G_2} \cong \mathrm{Hom}(V_{(0,0,1)} \otimes V_{(1,0,1)}, \mathfrak{spin}(7)_{\mathbb{C}})^{G_2}$ is seven dimensional, since $V_{(0,0,1)} \otimes V_{(1,0,1)} = 4V_{(2,0)} \oplus 4V_{(1,0)} \oplus 3V_{(0,1)} \oplus V_{(1,1)} \oplus V_{(3,0)} \oplus V_{(1,1)} \oplus V_{(0,0)}$, $\mathfrak{spin}(7)_{\mathbb{C}} = V_{(0,1,0)} = V_{(1,0)} \oplus V_{(0,1)}$. The basis $q^{(i,j)(k,l)}_{(m,n)}$ of $\mathrm{Hom}(V_{(0,0,1)} \otimes V_{(1,0,1)}, \mathfrak{spin}(7)_{\mathbb{C}})^{G_2}$ is given by

Maps	Formula
$q_1 = q^{(0,0)(1,0)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(1,0,1)} \to V_{(0,0)} \otimes V_{(1,0)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt + v) \otimes (dt \wedge b + w) \mapsto a(b \lrcorner \phi) \lrcorner \phi$
$q_2 = q^{(1,0)(1,0)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(1,0,1)} \to V_{(1,0)} \otimes V_{(1,0)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt + v) \otimes (dt \wedge b + w) \mapsto ((v \wedge (b \lrcorner \phi)) \lrcorner \phi) \lrcorner \phi$
$q_3 = q^{(1,0)(1,0)}_{(0,1)}$	$V_{(0,0,1)} \otimes V_{(1,0,1)} \to V_{(1,0)} \otimes V_{(1,0)} \to V_{(0,1)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt + v) \otimes (dt \wedge b + w) \mapsto v \wedge (b \lrcorner \phi) - \frac{1}{3}((v \wedge (b \lrcorner \phi)) \lrcorner \phi) \lrcorner \phi$
$q_4 = q^{(0,0)(0,1)}_{(0,1)}$	$V_{(0,0,1)} \otimes V_{(1,0,1)} \to V_{(0,0)} \otimes V_{(0,1)} \to V_{(0,1)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt + v) \otimes (dt \wedge b + w) \mapsto ab - \frac{1}{3}a(b \lrcorner \phi) \lrcorner \phi$
$q_5 = q^{(1,0)(0,1)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(1,0,1)} \to V_{(1,0)} \otimes V_{(0,1)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt + v) \otimes (dt \wedge b + w) \mapsto (v \lrcorner b) \lrcorner \phi + \frac{1}{3}((v \wedge (b \lrcorner \phi) \lrcorner \phi)) \lrcorner \phi$
$q_6 = q^{(1,0)(2,0)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(1,0,1)} \to V_{(1,0)} \otimes V_{(2,0)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt + v) \otimes (dt \wedge b + w) \mapsto \frac{1}{3}((v \lrcorner w) \lrcorner \phi) \lrcorner \phi + \frac{1}{6}((v \wedge (b \lrcorner \phi)) \lrcorner \phi) \lrcorner \phi$
$q_7 = q^{(1,0)(2,0)}_{(0,1)}$	$V_{(0,0,1)} \otimes V_{(1,0,1)} \to V_{(1,0)} \otimes V_{(2,0)} \to V_{(0,1)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt + v) \otimes (dt \wedge b + w) \mapsto v \lrcorner w - \frac{1}{3}((v \lrcorner w) \lrcorner \phi) \lrcorner \phi - \frac{1}{4}v \wedge (b \lrcorner \phi)$ $+\frac{1}{12}((v \wedge (b \lrcorner \phi)) \lrcorner \phi) \lrcorner \phi$

Now, the basis $p^{(i,j,k)}_{(m,n)}$ of $\mathrm{Hom}(V_{(0,0,1)} \otimes V_{(1,0,1)}, \mathfrak{spin}(7)_{\mathbb{C}})^{G_2}$ is given by

Maps	Formula
$p_1 = p^{(1,0,0)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(1,0,1)} \to V_{(1,0,0)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt+v) \otimes (dt \wedge b + w) \mapsto a(b \lrcorner \phi) \lrcorner \phi + ((v \lrcorner w) \lrcorner \phi) \lrcorner \phi - (v \lrcorner b) \lrcorner \phi$
$p_2 = p^{(0,1,0)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(1,0,1)} \to V_{(0,1,0)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt+v) \otimes (dt \wedge b + w) \mapsto a(b \lrcorner \phi) \lrcorner \phi + ((v \lrcorner w) \lrcorner \phi) \lrcorner \phi + 3(v \lrcorner b) \lrcorner \phi$
$p_3 = p^{(0,1,0)}_{(0,1)}$	$V_{(0,0,1)} \otimes V_{(1,0,1)} \to V_{(0,1,0)} \to V_{(0,1)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt+v) \otimes (dt \wedge b + w) \mapsto ab - \frac{1}{3}(a(b \lrcorner \phi) \lrcorner \phi + (v \lrcorner w) - \frac{1}{3}((v \lrcorner w) \lrcorner \phi) \lrcorner \phi$
$p_4 = p^{(0,0,2)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(1,0,1)} \to V_{(0,0,2)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt+v) \otimes (dt \wedge b + w) \mapsto -a(b \lrcorner \phi) \lrcorner \phi + (v \lrcorner b) \lrcorner \phi$ $+((v \lrcorner w) \lrcorner \phi) \lrcorner \phi - 2(v \lrcorner ((b \lrcorner \phi) \lrcorner \phi)) \lrcorner \phi$
$p_5 = p^{(1,1,0)}_{(0,1)}$	$V_{(0,0,1)} \otimes V_{(1,0,1)} \to V_{(1,1,0)} \to V_{(0,1)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ $(adt+v) \otimes (dt \wedge b + w) \mapsto 4v \wedge (b \lrcorner \phi) - 2((v \wedge (b \lrcorner \phi)) \lrcorner \phi) \lrcorner \phi + 6ab$ $-2(ab \lrcorner \phi) \lrcorner \phi - 2(v \lrcorner w) + \frac{2}{3}((v \lrcorner w) \lrcorner \phi) \lrcorner \phi$
$p_6 = p^{(1,0,2)}_{(1,0)}$	$V_{(0,0,1)} \otimes V_{(1,0,1)} \to V_{(1,0,2)} \to V_{(1,0)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ orthogonal to p_1, p_2 and p_4
$p_7 = p^{(1,0,2)}_{(0,1)}$	$V_{(0,0,1)} \otimes V_{(1,0,1)} \to V_{(1,0,2)} \to V_{(0,1)} \to \mathfrak{spin}(7)_{\mathbb{C}}$ orthogonal to p_3 and p_5

For p_5, we note that the representation $V_{(1,1,0)}$ is not a subspace of the exterior algebra. We cannot use the norm technique either, since we also need to calculate the expression of p_7 in terms of q-basis using the norm technique. But, we note that the irreducible decomposition of the $V_{(1,0,0)} \otimes V_{(0,1,0)}$ is $V_{(1,0,0)} \oplus V_{(0,0,2)} \oplus V_{(1,1,0)}$ which contains $V_{(1,1,0)}$. Whereas, $\Lambda^4(\mathbb{C}^8)$ has the decomposition $V_{(0,0,0)} \oplus V_{(1,0,0)} \oplus V_{(2,0,0)} \oplus V_{(0,0,2)}$ (See Appendix B.3.5). Hence we consider the following model of $V_{(1,1,0)}$. Consider the map

$$\theta : \Lambda^2(\mathbb{C}^8) \otimes \Lambda^2(\mathbb{C}^8) \to \Lambda^4(\mathbb{C}^8)$$
$$\omega \otimes \eta \mapsto \omega \wedge \eta.$$

Then,

$$V_{(1,1,0)} \cong \ker \theta\big|_{\Lambda^2_7(\mathbb{C}^8) \otimes \Lambda^2_{21}(\mathbb{C}^8)}.$$

Hence, for $\omega_7 \otimes \omega_{21} \in \Lambda^2_7(\mathbb{C}^8) \otimes \Lambda^2_{21}(\mathbb{C}^8)$, we have

$$\pi_{V_{(1,1,0)}}(\omega_7 \otimes \omega_{21}) = \pi_{\ker \theta}(\omega_7 \otimes \omega_{21}) = \omega_7 \otimes \omega_{21} - \frac{1}{12} E^{\mu\nu} \otimes E^{\mu\nu} \lrcorner (\omega_7 \wedge \omega_{21})$$

where E^μ is an orthonormal basis of $\Lambda^1(\mathbb{C}^8)$ and $E^0 = dt$. In order to express p_6 and p_7 in terms of q-basis elements, we calculate the norm of the q-basis element. By computing the matrix of q explicitly, and using $\|q\|^2 = \mathrm{Tr}(q^\dagger q)$, we find that $\|q_1\|^2 = 36$, $\|q_2\|^2 = 216$, $\|q_3\|^2 = 48$, $\|q_4\|^2 = 14$, $\|q_5\|^2 = 84$, $\|q_6\|^2 = 18$, and $\|q_7\|^2 = 63$. Then, we use the fact that p-basis and q-basis are orthogonal. Hence, the basis $p^{(i,j,k)}_{(m,n)}$ in terms of $q^{(i,j)(k,l)}_{(m,n)}$ is

$$
\begin{aligned}
p_1 &= q_1 - \frac{1}{6}q_2 - q_5 + 3q_6 \\
p_2 &= q_1 - \frac{9}{6}q_2 + 3q_5 + 3q_6 \\
p_3 &= \frac{1}{4}q_3 + q_4 + q_7 \\
p_4 &= -q_1 + \frac{7}{6}q_2 + q_5 + 3q_6 \\
p_5 &= \frac{7}{2}q_3 + 6q_4 - 2q_7 \\
p_6 &= q_1 + \frac{1}{6}q_2 + \frac{1}{7}q_5 - \frac{1}{3}q_6 \\
p_7 &= q_3 - \frac{12}{7}q_4 + \frac{4}{21}q_7.
\end{aligned}
$$

Now, in q-basis, $-\rho_{\mathfrak{spin}(7)_\mathbb{C}}(\mathrm{Cas}_{\mathfrak{g}_2}) - \rho_\Delta(\mathrm{Cas}_{\mathfrak{m}}) - \rho_{V_{(1,0,0)}}(\mathrm{Cas}_{\mathfrak{m}})$ is the matrix $\mathrm{diag}(70/3, 18, 70/3, 70/3, 38/3, 98/9, 146/9)$. Moreover, the eigenvalues corresponding to the eigenvectors $p^{(1,0,0)}_{(1,0)}, p^{(0,1,0)}_{(1,0)}, p^{(0,1,0)}_{(0,1)}, p^{(0,0,2)}_{(1,0)}, p^{(1,1,0)}_{(0,1)}, p^{(1,0,2)}_{(1,0)}$ and $p^{(1,0,2)}_{(0,1)}$ are $-8, -\frac{40}{3}, -\frac{40}{3}, -16, -24, -\frac{80}{3}$, and $-\frac{80}{3}$ respectively, which implies, in the p-basis,

$$
\rho_{\Delta\otimes V_{(1,0,1)}}(\mathrm{Cas}_{\mathfrak{spin}(7)}) = \mathrm{diag}(-8, -40/3, -40/3, -16, -24, -80/3, -80/3).
$$

Now, in the q-basis, since q_1 and q_4 factors through $V_{(0,0)}$, whereas q_2, q_3, q_5, q_6 and q_7 factor through $V_{(1,0)}$, the matrix $M_\gamma = \mathrm{diag}(1, -1, -1, 1, -1, -1, -1)$ and ϕ acts as the matrix $\mathrm{diag}(7, -1, -1, 7, -1, -1, -1)$. Consequently, by (4.37), the matrix of $\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma$ in the q-basis is

$$
\begin{pmatrix}
\frac{7}{2}(t-1) & 5 & 0 & 0 & \frac{7}{3} & -\frac{3}{2} & 0 \\
\frac{5}{6} & \frac{5}{2} - \frac{1}{2}(t-1) & 0 & 0 & -\frac{7}{6} & -\frac{1}{4} & 0 \\
0 & 0 & -\frac{5}{2} - \frac{1}{2}(t-1) & -\frac{7}{12} & 0 & 0 & \frac{21}{8} \\
0 & 0 & -2 & \frac{7}{2}(t-1) & 0 & 0 & -\frac{9}{2} \\
1 & -3 & 0 & 0 & \frac{1}{2}(t-1) & -\frac{3}{2} & 0 \\
-3 & -3 & 0 & 0 & -7 & -\frac{7}{6} - \frac{1}{2}(t-1) & 0 \\
0 & 0 & 2 & -1 & 0 & 0 & \frac{5}{6} - \frac{1}{2}(t-1)
\end{pmatrix}.
$$

We note that for $t = 1/3$, we have

$$
\left(\mathfrak{D}^{1/3}_{A_\Sigma}\right)^2_\gamma = \mathrm{diag}(148/9, 148/9, 100/9, 100/9, 148/9, 148/9, 100/9)
$$

in the q-basis, which shows that our calculations have been consistent. Finally, for $t = 0$, the eigenvalues are given by $\frac{1}{2}(-1-4\sqrt{5}), \frac{1}{6}(-3-4\sqrt{33}),\ \frac{1}{6}(1+4\sqrt{37}), \frac{1}{2}(-1+4\sqrt{5}), \frac{1}{6}(1-4\sqrt{37}), \frac{1}{6}(-3+4\sqrt{33}), -\frac{19}{6}$.

Main Result

Theorem 4.4.1. *The eigenvalues of the twisted Dirac operator $\left(\mathfrak{D}^0_{A_\Sigma}\right)_\gamma$ are*

1. $\frac{1}{2}$ *for* $V_\gamma = V_{(0,0,0)}$.
2. *For* $V_\gamma = V_{(1,0,0)}$,
$$\frac{1}{6}(-3+2\sqrt{105}),\ \frac{1}{6}(-3-2\sqrt{105}),\ -\frac{3}{2}.$$
3. *For* $V_\gamma = V_{(0,0,1)}$,
$$\frac{1}{6}(-3-8\sqrt{6}),\ \frac{1}{6}(-3+8\sqrt{6}),\ -\frac{5}{2},\ \frac{3}{2}.$$
4. *For* $V_\gamma = V_{(0,1,0)}$
$$\frac{1}{2}(-1-2\sqrt{17}),\ \frac{1}{6}(-3-2\sqrt{105}), \frac{1}{2}(-1+2\sqrt{17}),\ -\frac{7}{2},\ \frac{1}{6}(-3+2\sqrt{105}).$$
5. *For* $V_\gamma = V_{(2,0,0)}$
$$-\frac{25}{6},\ \frac{23}{6}.$$
6. *For* $V_\gamma = V_{(1,0,1)}$
$$\frac{1}{2}(-1-4\sqrt{5}),\ \frac{1}{6}(-3-4\sqrt{33}),\ \frac{1}{6}(1+4\sqrt{37}),$$
$$\frac{1}{2}(-1+4\sqrt{5}),\ \frac{1}{6}(1-4\sqrt{37}),\ \frac{1}{6}(-3+4\sqrt{33}),\ -\frac{19}{6}.$$

Corollary 4.4.2. *The eigenvalues of the twisted Dirac operator $\left(\mathfrak{D}^0_{A_\Sigma}\right)_\gamma$ in the interval $\left[-\frac{5}{2}, \frac{5}{2}\right]$ are $-\frac{5}{2}, -\frac{3}{2}, \frac{1}{2}, \frac{3}{2}$. In particular, the only eigenvalue in the interval $\left(\frac{1}{2}, \frac{5}{2}\right)$ is $\frac{3}{2}$ corresponding to the spin representation $V_{(0,0,1)}$.*

Chapter 5

The Space of Deformations of FNFN $Spin(7)$-Instanton

In this chapter, we shall study the deformations of AC $Spin(7)$-instantons on the AC $Spin(7)$-manifold $\mathbb{R}^8$, where the $Spin(7)$-instantons on $\mathbb{R}^8$ will converge to the canonical connection on S^7 at infinity. Fairlie–Nuyts [25] and Fubini–Nicolai [27] independently constructed these instantons on $\mathbb{R}^8$, and hence will be referred to as *FNFN $Spin(7)$-instanton* on $\mathbb{R}^8$.

5.1 FNFN $Spin(7)$-Instanton

In this section, we derive FNFN-instanton using homogeneous space techniques. The exact same result and similar approach can also be found in [51].

Let us consider the asymptotically conical $Spin(7)$-manifold $\mathbb{R}^8$ asymptotic to the nearly G_2 manifold $\Sigma = S^7$. We consider S^7 as a homogeneous nearly G_2 manifold $Spin(7)/G_2$. Then we have the canonical bundle $G_2 \to Spin(7) \to S^7$ (call this bundle P). Also consider the bundle $Spin(7) \to Spin(7) \times_{(G_2,\iota)} Spin(7) \to S^7$ (call this bundle Q) where $\iota : G_2 \hookrightarrow Spin(7)$ is the inclusion. This bundle is (bundle) isomorphic to the trivial bundle $Spin(7) \to S^7 \times Spin(7) \to S^7$. Explicitly, the isomorphism is given by

$$\begin{aligned} Spin(7) \times_{(G_2,\iota)} Spin(7) &\to S^7 \times Spin(7) \\ [(g_1, g_2)] &\mapsto ([g_1], g_1 g_2) \end{aligned} \tag{5.1}$$

Then, an action of $Spin(7)$ on $Spin(7) \times_{(G_2,\iota)} Spin(7)$ given by $g[(g_1, g_2)] = [(gg_1, g_2)]$ induces an action on $S^7 \times Spin(7)$ given by $g([g_1], g_1 g_2) = ([gg_1], gg_1 g_2)$.

We want to find all $Spin(7)$-invariant connections on Q. From Wang's theorem [59], we know that this corresponds to all G_2-equivariant linear maps $\mathfrak{m} \to \mathfrak{spin}(7)$, for the subspace $\mathfrak{m}$ defined in (4.26). That is, the set

$$\{\Lambda : \mathfrak{m} \to \mathfrak{spin}(7) : \Lambda \text{ is } G_2\text{-equivariant}\}.$$

Now,

$$\mathfrak{spin}(7) \otimes \mathbb{C} = V_{(0,1)} \oplus V_{(1,0)}$$

and hence, restricting to $\mathfrak{m}$, we have the decomposition

$$\mathfrak{m} \otimes \mathbb{C} = V_{(1,0)} \cong \mathbb{C}^7.$$

Recall *Schur's lemma*:

Lemma 5.1.1. *Let V, W be two irreducible representations of H. The space of H-equivariant maps $V \to W$ is*

$$\begin{cases} \{\varphi \cdot \mathrm{Id} : V \to W \,|\varphi \in \mathbb{C}\} & \text{if } V \cong W, \\ 0 & \text{if } V \ncong W. \end{cases}$$

Thus, we have all the G_2-equivariant linear maps $\Lambda : \mathfrak{m} \to \mathfrak{spin}(7)$, explicitly given by

$$\varphi \cdot \mathrm{id} : V_{(1,0)} \cong \mathfrak{m} \to V_{(1,0)} \hookrightarrow \mathfrak{spin}(7)$$

where the complex number φ is necessarily real because Λ is (the complexification of) a map between real vector spaces $\mathfrak{m} \to \mathfrak{m}$.

Now the basis I_A ($A = 1, \ldots, 21$) for $\mathfrak{spin}(7)$ (where $I_1, \ldots, I_7$ spans $\mathfrak{m}$) can be represented by left invariant vector fields $\widehat{E}_A$ on $Spin(7)$ and also by the dual basis $\hat{e}^A$ of left invariant 1-forms. Denote the natural projection map

$$\pi : Spin(7) \to Spin(7)/G_2$$
$$g \mapsto gG_2$$

of the principal bundle. Let U be a contractible open subset of $Spin(7)/G_2$. Then we choose a map $L : U \to Spin(7)$ such that $\pi \circ L = \mathrm{Id}_U$, i.e., L is a local section of the bundle $Spin(7) \to Spin(7)/G_2$. We put $e^A := L^* \hat{e}^A$. Then $\{e^a : a = 1, \ldots, 7\}$ form an orthonormal frame for $T^*(Spin(7)/G_2)$ over U.

For e^A, we have the Maurer–Cartan equations

$$de^a = -f^a_{ib}e^i \wedge e^b - \frac{1}{2}f^a_{bc}e^b \wedge e^c$$
$$de^i = -\frac{1}{2}f^i_{bc}e^b \wedge e^c - \frac{1}{2}f^i_{jk}e^j \wedge e^k. \tag{5.2}$$

With respect to this local trivialisation, a connection on the bundle Q over the nearly G_2-manifold $Spin(7)/G_2$ can be written as $A = e^i I_i + \varphi e^a I_a$ where $A_\Sigma = e^i I_i$, is the canonical connection, and $\varphi \in \mathbb{R}$.

Now consider the 8-dimensional manifold $\mathbb{R} \times Spin(7)/G_2$. Moreover, consider the projection $\pi : \mathbb{R} \times Spin(7)/G_2 \to Spin(7)/G_2$. We choose the metric $g_8 = (e^0)^2 + g_7$ where $e^0 = dt$ for t the coordinate of $\mathbb{R}$, and g_7 is the metric on $Spin(7)/G_2$. This metric is conformal to the flat metric on $\mathbb{R}^8$. We can describe a $Spin(7)$-invariant connection on $\pi^*Q \to \mathbb{R} \times Spin(7)/G_2$ using a function $\varphi : \mathbb{R} \to \mathbb{R}$. The local connection 1-form A is given by

$$A = e^i I_i + \varphi(t)e^a I_a = A_a e^a. \tag{5.3}$$

where, $A_a = e^i_a I_i + \varphi(t) I_a$. Here, $e^i = e^i_a e^a$, for functions $e^i_a : U \to \mathbb{R}$. Without loss of generality, we have taken $A_0 = 0$ (called *temporal gauge*), since we can always choose such a gauge. We note that the connection A on $\mathbb{R} \times S^7$ can be identified with a family $\{A_t : t \in \mathbb{R}\}$ of connections on S^7. The curvature of this connection is given by

$$F_A = F_{0a}e^0 \wedge e^a + \frac{1}{2}F_{bc}e^b \wedge e^c \tag{5.4}$$

where

$$F_{0a} = \frac{\partial A_a}{\partial t} = \frac{d\varphi}{dt} I_a.$$

Note that we can identify the cone $(0, \infty) \times S^7 \subset \mathbb{R}^8$ with the cylinder $\mathbb{R} \times S^7$ by the map $r = e^t$. The ASD instanton equation $\Phi \wedge F_A = -*F_A$ can be written as

$$F_A \lrcorner \Phi = -F_A. \tag{5.5}$$

Now, we have $\phi = \frac{1}{6}\phi_{abc}e^a \wedge e^b \wedge e^c$, $*\phi = \psi = \frac{1}{24}\psi_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d$, for $a, b, c = 1, \ldots, 7$, where ϕ_{abc} are structure constants of the octonions and $\psi_{pqrs} = \epsilon_{abcpqrs}\phi_{abc}$. We have already seen that we can write the structure constants of the octonions ϕ_{abc} in terms of the structure constants f_{abc} as $f_{abc} = -\frac{2}{3}\phi_{abc}$.

Then, we get

$$F_A \lrcorner \Phi = \left(F_{0a}e^0 \wedge e^a + \frac{1}{2}F_{bc}e^b \wedge e^c\right) \lrcorner \left(e^0 \wedge \phi + *\phi\right)$$

$$= \frac{1}{2}F_{bc}\phi_{abc}e^0 \wedge e^a + \frac{1}{2}F_{0a}\phi_{abc}e^b \wedge e^c + \frac{1}{4}F_{ad}\psi_{abcd}e^b \wedge e^c.$$

Then, from (5.5) we have two equations, $F_{0a} = \frac{3}{4}f_{abc}F_{bc}$ and $F_{bc} = -F_{0a}\phi_{abc} - \frac{1}{2}F_{ad}\psi_{abcd}$, but the first one implies the second. Hence, the ASD instanton equation (5.5) reduces to

$$F_{0a} = \frac{3}{4}f_{abc}F_{bc}. \tag{5.6}$$

Applying the Maurer–Cartan equations (5.2), we calculate dA_t, as well as $[A_t \wedge A_t]$, and the curvature is given by

$$F_{bc} = (\varphi^2 - 1)f^i_{bc}I_i + (\varphi^2 - \varphi)f_{abc}I_a, \quad F_{0a} = \frac{d\varphi}{dt}I_a. \tag{5.7}$$

Then, the ASD instanton equation (5.6) is equivalent to

$$\frac{d\varphi}{dt}I_a = \frac{3}{4}f_{abc}(\varphi^2 - 1)f^i_{bc}I_i + \frac{3}{4}f_{abc}(\varphi^2 - \varphi)f_{hbc}I_h.$$

Simplifying, we have the differential equation $\frac{d\varphi}{dt} = 2(\varphi^2 - \varphi)$. Solving, we get

$$\varphi = \frac{1}{1 + e^{2t+2C_1}} = \frac{1}{Cr^2 + 1} \tag{5.8}$$

for $C > 0$, using the substitution $r = e^t$.

We note that $\varphi(0) = 1$ defines a flat connection over S^7, and the corresponding trivialisation extends across the origin. Thus this connection on $\mathbb{R} \times S^7$ is nothing but the restriction of a flat connection on $\mathbb{R}^8$. Hence from the calculations above, and from the fact that the ASD instanton equations are conformally invariant, it follows that the connection A defined in (5.3), where φ is given in (5.8), is in fact an instanton on $\mathbb{R}^8$. We call this the *FNFN $Spin(7)$-instanton*. Clearly FNFN $Spin(7)$-instanton A is asymptotic to the canonical connection A_Σ with fastest rate of convergence -2, since $\varphi = O(r^{-2})$ as $r \to \infty$.

5.2 Index of the Twisted Dirac Operator

We want to calculate the index of the Dirac operator $\mathfrak{D}^-_A$ on $\mathfrak{S}(\mathbb{R}^8)$ twisted by the trivial bundle $\mathfrak{g}_P := \mathfrak{spin}(7) \times \mathbb{R}^8$ over $\mathbb{R}^8$. We use the *Atiyah–Patodi–Singer Index Theorem* for manifolds with boundaries, by relating the index of the Dirac operator $\mathfrak{D}^-_A$ on $\mathbb{R}^8$ with the index of the Dirac operator on a closed ball B^8_R of large enough radius R. Moreover, we consider the FNFN instanton to be an instanton on $\mathbb{R}^8$ and, for the purposes of calculating the eta-invariant appearing in the index theorem, on $\mathbb{R} \times S^7$.

Atiyah–Patodi–Singer Index Theorem

The Atiyah–Patodi–Singer index theorem is applicable when the manifold has non-empty boundary (for more details see [2], [3], [4], [24], [30]). Let M be a 8-manifold with non-empty boundary ∂M. Let $P \to M$ be a principal G-bundle and consider the negative Dirac operator $\mathfrak{D}_A^-$ acting on the bundle $E := \mathfrak{S}^-(X) \otimes \mathfrak{g}_P$ over M. Then the index of the operator $\mathfrak{D}_A^-$ requires topological information on the manifold M as well as analytic information on the boundary ∂M.

The Atiyah–Patodi–Singer index theorem for a manifold M with non-empty boundary ∂M has the form

$$\mathrm{Index}(\mathfrak{D}_A^-, M, \partial M) = I(M) + CS(\partial M) + \frac{1}{2}\eta(\partial M). \tag{5.9}$$

where $I(M)$ is an integral of characteristic classes over M and $\eta(\partial M)$ is the eta-invariant of the boundary. The Chern–Simons term $CS(\partial M)$ of the boundary arises when the manifold does not admit a product metric on the boundary. Moreover, the Dirac operator $\mathfrak{D}_A^-$ is subject to non-local boundary condition which will be explained later.

$\mathbb{R}^8$ with Cigar metric and Index

Let g_C be the asymptotically conical metric, i.e., the flat metric on $\mathbb{R}^8$. We define the metric $g_{CI} := \frac{1}{\varrho^2} g_C$, where ϱ is the radius function (2.30). Then $(\mathbb{R}^8, g_{CI})$ resembles a cigar (the reason g_{CI} is usually called a *cigar metric*).

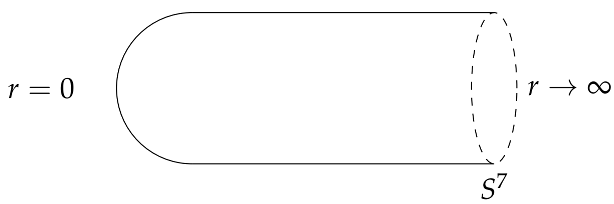


Figure 5.1: $\mathbb{R}^8$ with cigar metric.

In particular, for

$$\varrho(r) = \begin{cases} r & r > 1 \\ \frac{1}{2}(1 + r^2) & r \leq 1, \end{cases}$$

$(\mathbb{R}^8, g_{CI})$ is a hemisphere M_1 glued to a cylinder $M_2 = (1, \infty) \times S^7$ (see Figure 5.2).

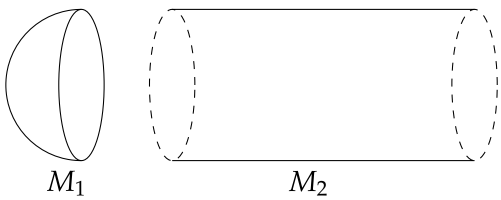


Figure 5.2: $\mathbb{R}^8$ as a hemisphere glued to a cylinder.

The weighted Sobolev space $W^{0,2}_\nu$ on $\mathbb{R}^8$ is defined by the norm

$$\|\eta\|_{W^{0,2}_\nu} := \left(\int |\varrho^\nu \eta|^2 \varrho^{-8}\, \mathrm{dvol}_C\right)^{1/2}.$$

The space $W^{0,2}_{CI}$ of L^2-functions on the cigar M is defined by

$$\|\eta\|_{W^{0,2}_{CI}} := \left(\int |\eta|^2\, \mathrm{dvol}_{CI}\right)^{1/2}.$$

Now, $\mathrm{dvol}_{CI} = \varrho^{-8}\, \mathrm{dvol}_C$. Hence,

$$\begin{aligned} W^{0,2}_\nu &\to W^{0,2}_{CI} \\ \eta &\mapsto \varrho^\nu \eta \end{aligned}$$

is an isomorphism. Similarly, we can extend this to an isomorphism $W^{k,2}_\nu \to W^{k,2}_{CI}$.

By conformal properties of Dirac operators, we have that the Dirac operator of g_{CI} is $\mathfrak{D}^-_{A,CI} = \varrho^{\frac{9}{2}} \mathfrak{D}^-_{A,C} \varrho^{-\frac{7}{2}}$ (where $\mathfrak{D}^-_{A,CI}$ and $\mathfrak{D}^-_{A,C}$ are the Dirac operators corresponding to cigar and conical metrics respectively.) Then we have the commutative diagram

$$\begin{tikzcd} W^{k,2}_{CI} \arrow[r, "\mathfrak{D}^-_{A,CI}"] \arrow[d, "\varrho^{-\frac{7}{2}}"'] & W^{k-1,2}_{CI} \\ W^{k,2}_{-\frac{7}{2}} \arrow[r, "\mathfrak{D}^-_{A,C}"] & W^{k-1,2}_{-\frac{9}{2}} \arrow[u, "\varrho^{\frac{9}{2}}"'] \end{tikzcd}$$

Since the vertical arrows are isomorphism, we have

$$\mathrm{Index}\left(\mathfrak{D}^-_{A,C} : W^{k,2}_{-\frac{7}{2}} \to W^{k-1,2}_{-\frac{9}{2}}\right) = \mathrm{Index}\left(\mathfrak{D}^-_{A,CI} : W^{k,2}_{CI} \to W^{k-1,2}_{CI}\right). \tag{5.10}$$

Let us define a function $\widetilde{\varphi} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{\varphi}(t) = \begin{cases} 1 & t < -T \\ \alpha', & -T < t < -\frac{T}{2} \\ \varphi(t), & -\frac{T}{2} < t < \frac{T}{2} \\ \alpha, & \frac{T}{2} < t < T \\ 0 & t > T. \end{cases} \tag{5.11}$$

where α is a smooth interpolation between its definition at $\frac{T}{2}$ and T and α' is that of between its definition at $-T$ and $-\frac{T}{2}$.

Then we have a connection

$$\widetilde{A} = e^i I_i + \widetilde{\varphi}(t) e^a I_a. \tag{5.12}$$

We note that

$$A - \widetilde{A} = (\varphi(t) - \widetilde{\varphi}(t)) e^a I_a \in \Omega^1(\mathfrak{g}_P). \tag{5.13}$$

Proposition 5.2.1. *Let $B^8_R := \{x \in \mathbb{R}^8 : |x| \leq R\}$ be 8-dimensional ball of radius R. Then for sufficiently large R, we have*

$$\operatorname{Index}\left(\mathfrak{D}^-_{\widetilde{A},CI}, \mathbb{R}^8, g_{CI}\right) = \operatorname{Index}\left(\mathfrak{D}^-_{\widetilde{A},CI}, B^8_R, g_{CI}\right).$$

Moreover, for sufficiently large T, we have

$$\operatorname{Index}\left(\mathfrak{D}^-_{\widetilde{A}}, \mathbb{R} \times S^7, g\right) = \operatorname{Index}\left(\mathfrak{D}^-_{\widetilde{A}'}, [-T, T] \times S^7, g\right).$$

where g is the cylindrical metric $g = dt^2 + g_{S^7}$.

Proof. Let $\eta : \mathbb{R}^8 \to \$(\mathfrak{g}_P)$ be a spinor such that $\mathfrak{D}^-_{\widetilde{A},CI}\eta = 0$ and $\eta \in L^2(\mathbb{R}^8, g_{CI})$. Now, $\mathfrak{D}^-_{\widetilde{A},CI} = E^0\left(\frac{\partial}{\partial t} - \mathfrak{D}_{\widetilde{A}_{t,\Sigma}}\right)$. For $t > \ln R$, since $\widetilde{\varphi}(t) = 0$, we have $\mathfrak{D}_{\widetilde{A}_{t,\Sigma}} = \mathfrak{D}_{A_\Sigma}$. Let $\lambda_n \in \operatorname{Spec}\mathfrak{D}_{A_\Sigma}$. Then, we have the Fourier expansion of η given by $\eta = \sum_{n\in\mathbb{Z}} e^{\lambda_n(t-\ln R)}\eta_n$ where $\eta_n \in \ker(\mathfrak{D}_{A_\Sigma} - \lambda_n)$. Hence, $\eta \in L^2$ implies $\eta_n = 0$ when $\lambda_n > 0$. So η can be written as a sum of eigenvectors η_n of Dirac operator on the boundary with negative eigenvalues. Hence η solves *Atiyah–Patodi–Singer boundary condition*.

Conversely, let $\eta : B^8_R \to \$(\mathfrak{g}_P)$ such that $\mathfrak{D}^-_{\widetilde{A},CI}\eta = 0$ and η solves Atiyah–Patodi–Singer boundary condition. We extend η to $\mathbb{R}^8$. On ∂B^8_R, $\eta = \sum_{n<0} \eta_n$, where $\lambda_n < 0$ if and only if $n < 0$. So, for $r > R$ (i.e., $t > \ln R$) we set $\eta = \sum_{n<0} e^{\lambda_n(t-\ln R)}\eta_n$. Then $\eta \in L^2(\mathbb{R}^8, g_{CI})$ and solves $\mathfrak{D}^-_{\widetilde{A},CI}\eta = 0$.

Hence, we have just proved that $\ker\left(\mathfrak{D}^-_{\widetilde{A},CI}, \mathbb{R}^8, g_{CI}\right) \cong \ker\left(\mathfrak{D}^-_{\widetilde{A},CI}, B^8_R, g_{CI}\right)$. Similarly, we can show that $\operatorname{coker}\left(\mathfrak{D}^-_{\widetilde{A},CI}, \mathbb{R}^8, g_{CI}\right) \cong \operatorname{coker}\left(\mathfrak{D}^-_{\widetilde{A},CI}, B^8_R, g_{CI}\right)$. Hence, we proved

$$\operatorname{Ind}\left(\mathfrak{D}^-_{\widetilde{A},CI}, \mathbb{R}^8, g_{CI}\right) = \operatorname{Ind}\left(\mathfrak{D}^-_{\widetilde{A},CI}, B^8_R, g_{CI}\right).$$

Finally, we prove that

$$\operatorname{Ind}\left(\mathfrak{D}^-_{A,CI}, \mathbb{R}^8, g_{CI}\right) = \operatorname{Ind}\left(\mathfrak{D}^-_{\widetilde{A},CI}, \mathbb{R}^8, g_{CI}\right).$$

Consider

$$\|\mathfrak{D}^-_{A,CI} - \mathfrak{D}^-_{\widetilde{A},CI}\| = \|(\varphi(t) - \widetilde{\varphi}(t))e^a I_a\| = \sup_{\eta \in L^2(\mathbb{R}^8, g_{CI})} \frac{\|(\varphi(t) - \widetilde{\varphi}(t))e^a I_a \eta\|_{L^2}}{\|\eta\|_{L^2}}. \tag{5.14}$$

Now,

$$\begin{aligned}
\|(\varphi(t) - \widetilde{\varphi}(t))e^a I_a \eta\|^2_{L^2} &= \int |(\varphi(t) - \widetilde{\varphi}(t))e^a I_a \eta|^2 \,\mathrm{dvol} \\
&\leq \sup(\varphi(t) - \widetilde{\varphi}(t))^2 \int |e^a I_a \eta|^2 \,\mathrm{dvol} \\
&\leq \sup(\varphi(t) - \widetilde{\varphi}(t))^2 \|e^a I_a\|^2 \int |\eta|^2 \,\mathrm{dvol} \\
&= \sup(\varphi(t) - \widetilde{\varphi}(t))^2 \|e^a I_a\|^2 \|\eta\|^2_{L^2}.
\end{aligned}$$

Hence, from (5.14), we have

$$\|\mathfrak{D}^-_{A,CI} - \mathfrak{D}^-_{\widetilde{A},CI}\| \leq \sup|\varphi(t) - \widetilde{\varphi}(t)| \|e^a I_a\|.$$

Hence, for all $\epsilon > 0$, there exists $R > 0$ such that $\|\mathfrak{D}^-_{A,CI} - \mathfrak{D}^-_{\widetilde{A},CI}\| < \epsilon$. Then the result follows from the fact that two Fredholm operators belonging to the same connected component of the space of all Fredholm operators have the same index, since the Fredholm index is continuous and integer-valued.

The second part of the theorem can be proved similarly. $\square$

The term $I(\mathbb{R}^8)$

The term $I(\mathfrak{D}^-_{A,CI}, \mathbb{R}^8, g_{CI})$ in (5.9) is given by

$$\begin{aligned}
I(\mathfrak{D}^-_{A,CI}, \mathbb{R}^8, g_{CI}) &= -\int_{\mathbb{R}^8} \widehat{A}(M)\,\mathrm{ch}(\mathfrak{g}_P \otimes \mathbb{C}) \\
&= -\int_{\mathbb{R}^8} \left(1 - \frac{1}{24} p_1(\mathbb{R}^8) + \frac{1}{5760}(7p_1(\mathbb{R}^8)^2 - 4p_2(\mathbb{R}^8))\right) \\
&\qquad \left(\dim \mathfrak{g} + p_1(\mathfrak{g}_P) + \frac{1}{12}\left(p_1(\mathfrak{g}_P)^2 - 2p_2(\mathfrak{g}_P)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{12}\int_{\mathbb{R}^8} \left(p_1(\mathfrak{g}_P)^2 - 2p_2(\mathfrak{g}_P)\right) + \frac{1}{24}\int_{\mathbb{R}^8} p_1(\mathbb{R}^8)p_1(\mathfrak{g}_P) \\
&\qquad\qquad - \frac{1}{5760}\dim\mathfrak{g}\int_{\mathbb{R}^8}(7p_1(\mathbb{R}^8)^2 - 4p_2(\mathbb{R}^8)) \\
&= -\frac{1}{12}\int_{\mathbb{R}^8} \left(p_1(\mathfrak{g}_P)^2 - 2p_2(\mathfrak{g}_P)\right) + \frac{1}{24}\int_{M_1} p_1(M_1)p_1(\mathfrak{g}_P) \\
&\quad + \frac{1}{24}\int_{M_2} p_1(M_2)p_1(\mathfrak{g}_P) - \frac{1}{5760}\dim\mathfrak{g}\int_{M_1}(7p_1(M_1)^2 - 4p_2(M_1)) \\
&\quad - \frac{1}{5760}\dim\mathfrak{g}\int_{M_2}(7p_1(M_2)^2 - 4p_2(M_2))
\end{aligned}
$$

where the Pontryagin classes p_i are given in terms of the curvature as

$$
\begin{aligned}
p_1(\mathfrak{g}_P) &= -\frac{1}{8\pi^2}\operatorname{tr}(F_A^2) \\
p_2(\mathfrak{g}_P) &= \frac{1}{128\pi^4}\left[\operatorname{tr}(F_A^2)^2 - 2\operatorname{tr}F_A^4\right]
\end{aligned}
$$

where the trace is taken over $\mathfrak{g}$.

Since M_1 is a hemisphere of S^8, the curvature of M_1 is the same as the curvature of S^8. Let E^μ, for $\mu = 0, 1, \ldots, 7$ be an orthonormal local frame for S^8. Then, the Riemann curvature is given by

$$
R = \frac{1}{2}R^\alpha_{\beta\gamma\lambda}E^\gamma \wedge E^\lambda
$$

where $R^\alpha_{\beta\gamma\lambda} = \delta^\alpha_\gamma\delta_{\beta\lambda} - \delta^\alpha_\lambda\delta_{\beta\gamma}$. Then, $\operatorname{tr}(R\wedge R) = 0$ and $\operatorname{tr}(R\wedge R\wedge R\wedge R) = 0$. Hence,

$$
p_1(M_1) = -\frac{1}{8\pi^2}\operatorname{tr}(R^2) = 0
$$

and

$$
p_2(M_1) = \frac{1}{128\pi^4}\left[\operatorname{tr}(R^2)^2 - 2\operatorname{tr}R^4\right] = 0.
$$

Consider the projection $\pi : \mathbb{R}\times S^7 \to S^7$. Then, $p_i(\mathbb{R}\times S^7) = \pi^* p_i(S^7)$. Now, if R' is the curvature of S^7, then,

$$
p_1(S^7) = -\frac{1}{8\pi^2}\operatorname{tr}(R'\wedge R') = 0
$$

and

$$
p_2(S^7) = \frac{1}{128\pi^4}\operatorname{tr}(R'\wedge R')^2 - 2\operatorname{tr}(R'\wedge R'\wedge R'\wedge R').
$$

But $\operatorname{tr}(R'\wedge R'\wedge R'\wedge R') = 0$ being an 8-form on 7-dimensional manifold S^7. Further, we have $\operatorname{tr}(R'\wedge R')\wedge\operatorname{tr}(R'\wedge R') = 0$ being an 8-form with only 7-coordinate functions, which means there must be repeating terms. Consequently,

$$
p_1(M_2) = 0, \quad \text{and} \quad p_2(M_2) = 0.
$$

Hence,

$$I(\not{\mathfrak{D}}^{-}_{A,CI}, \mathbb{R}^8, g_{CI}) = -\frac{1}{12}\int_{\mathbb{R}^8}\left(p_1(\mathfrak{g}_P)^2 - 2p_2(\mathfrak{g}_P)\right) = -\frac{1}{384\pi^4}\int_{\mathbb{R}^8} \operatorname{tr} F_A^4. \tag{5.15}$$

Eta Invariant of the Boundary

We calculate the eta-invariant of the twisted Dirac operator by relating it to the untwisted Dirac operator, whose eta-invariant is zero, using a spectral flow.

Spectral Flow

Recall the FNFN $Spin(7)$-instanton A given by (5.3), where $\varphi(t)$ is given by (5.8), can be identified with a family of connections $\{A_t : t \in \mathbb{R}\}$ on S^7. Then, we have a family of Dirac operators on S^7 twisted by the connections A_t given by

$$\not{\mathfrak{D}}_{A_{t,\Sigma}} = \not{\mathfrak{D}}_{A_\Sigma} + \varphi(t)e^a I_a.$$

Now, the curvature of the connection is given by (5.7) for which we note that $F_{bc} = 0$ for $\varphi(t) = 1$. Hence, A_t is a flat connection for $t = -\infty$. Since the underlying manifold is simply connected, this flat connection is the trivial connection (unique up to gauge). Hence corresponding to this connection, or equivalently, for $\varphi(t) = 1$, we have the untwisted Dirac operator $\mathcal{D}_\Sigma$, i.e.,

$$\mathcal{D}_\Sigma = \not{\mathfrak{D}}_{A_\Sigma} + e^a I_a. \tag{5.16}$$

We want to calculate the *spectral flow* of the family $\{\not{\mathfrak{D}}_{A_{t,\Sigma}}\}_{t\in\mathbb{R}}$, where spectral flow is the net number of eigenvalues flowing from negative to positive. First let us calculate the eigenvalues of the operator $e^a I_a$.

We note that the operator $e^a I_a$ acts fibre-wise: on $\Delta \otimes \mathfrak{spin}(7)$. Let e^μ, $\mu = 0, 1, \ldots, 7$ be a basis of Δ and I_A be a basis of $\mathfrak{spin}(7)$. Then,

$$(e^a I_a)\,(e^\mu \otimes I_A) = (e^a \cdot e^\mu) \otimes [I_a, I_A] = (E^a \otimes \operatorname{ad} I_a)(e^\mu, I_A)$$

where E^a is the matrix of Clifford multiplication with e^a, calculated using (2.8). Taking the Kronecker product of E^a and $\operatorname{ad} I_a$, we get the matrix of $e^a I_a$ whose eigenvalues are listed in the Table 5.1.

Now, let us plot the eigenvalues of the operators $\mathcal{D}_\Sigma$ and $\not{\mathfrak{D}}_{A_\Sigma}$ near zero respectively. The eigenvalues of $\mathcal{D}_\Sigma$ can be found in [8] and that of $\not{\mathfrak{D}}_{A_\Sigma}$ from Corollary 4.4.2.

Eigenvalues	Multiplicities
4	8
-4	7
$\frac{1}{3}\left(-3+\sqrt{33}\right)$	14
$\frac{1}{3}\left(3-\sqrt{33}\right)$	14
$\frac{1}{3}\left(-1+\sqrt{57}\right)$	27
$\frac{1}{3}\left(-1-\sqrt{57}\right)$	27
2	7
0	64

Table 5.1: Eigenvalues of $e^a I_a$ and corresponding multiplicities.

From the figure 5.3 below and the eigenvalues of $e^a I_a$, we have the complete description of the spectral flow. We note that the eigenvalue of $e^a I_a$ with the highest magnitude is 4. Again, from the figure 5.3, we see that the only possibility of having a non-zero spectral flow is the eigenvalue $1/2$ of $\mathfrak{D}_{A_\Sigma}$ flowing down to the eigenvalue $-7/2$ of $\mathcal{D}_\Sigma$. Since $1/2$ corresponds to eigenvalue of $\mathfrak{D}_{A_\Sigma}$ obtained from the trivial representation $V_{(0,0,0)}$ of $Spin(7)$, the eigenspinor η corresponding to eigenvalue $1/2$ belongs to the space $\mathrm{Hom}(V_{(0,0,0)}, \Delta \otimes \mathfrak{spin}(7)_{\mathbb{C}})^{G_2} \otimes V_{(0,0,0)} \subset L^2(Spin(7), \Delta \otimes \mathfrak{spin}(7)_{\mathbb{C}})^{G_2}$ in the decomposition (4.8). Now, we have the decomposition

$$\Delta \otimes \mathfrak{spin}(7)_{\mathbb{C}} \cong V_{(0,0,1)} \otimes V_{(0,1,0)} \cong V_{(0,0)} \oplus 3V_{(1,0)} \oplus 2V_{(0,1)} \oplus 2V_{(2,0)} \oplus V_{(1,1)}.$$

Hence by Schur's lemma, we have that $\eta \in \mathrm{Hom}(V_{(0,0,0)}, V_{(0,0)})^{G_2} \otimes V_{(0,0,0)}$ which is a subspace of $L^2(Spin(7), V_{(0,0)})^{G_2}$. Hence, in order to check whether a flow from the eigenvalue $1/2$ of $\mathfrak{D}_{A_\Sigma}$ flowing down to the eigenvalue $-7/2$ of $\mathcal{D}_\Sigma$ exists, we need to calculate the eigenvalue of $e^a I_a$ corresponding to the trivial subrepresentation $V_{(0,0)}$ of $\Delta \otimes \mathfrak{spin}(7)_{\mathbb{C}}$.

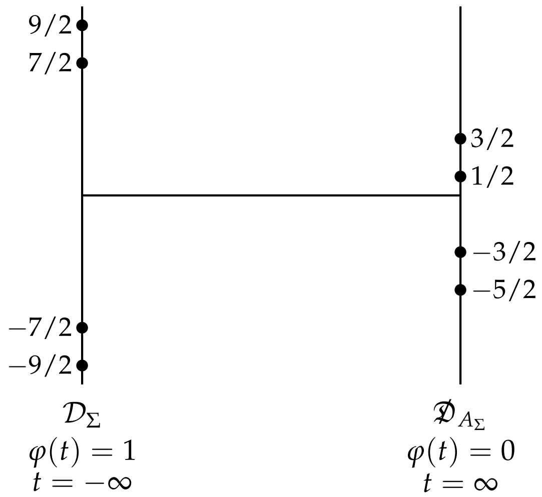


Figure 5.3: Spectral flow of the family $\{\not{\mathfrak{D}}_{A_{t,\Sigma}}\}_{t\in\mathbb{R}}$ on S^7.

Then, from the above table of eigenvalues of $e^a I_a$ and corresponding multiplicities, it is clear that since $\dim V_{(1,1)} = 64$, $V_{(1,1)}$ is the eigenspace of the eigenvalue 0. Similarly, the two copies of $V_{(2,0)}$ are the eigenspaces of the eigenvalues $\frac{1}{3}(-1 \pm \sqrt{57})$, the two copies of $V_{(0,1)}$ are the eigen spaces of the eigenvalues $\frac{1}{3}(\mp 3 \pm \sqrt{33})$, the three copies of $V_{(1,0)}$ are the eigenspaces of the eigenvalues $2, 4$ and -4 respectively, $V_{(0,0)}$ is the eigenspace of one eigenvalue 4. Thus we have a flow of the eigenvalue moving up to $9/2$ and not down to $-7/2$. Hence, there is no flow from the eigenvalue $1/2$ of $\not{\mathfrak{D}}_{A_\Sigma}$ to the eigenvalue $-7/2$ of $\mathcal{D}_\Sigma$, and hence, we have no flow of eigenvalues of $\mathcal{D}_\Sigma$ flowing up or down across 0 to the eigenvalues of $\not{\mathfrak{D}}_{A_\Sigma}$. Consequently, the spectral flow of the family $\{\not{\mathfrak{D}}_{A_{t,\Sigma}}\}_{t\in\mathbb{R}}$ is given by

$$\mathrm{sf}\left(\{\not{\mathfrak{D}}_{A_{t,\Sigma}}\}_{t\in\mathbb{R}}\right) = 0. \tag{5.17}$$

Eta Invariant of the Boundary

We recall that we can identify the family of Dirac operators $\{\not{\mathfrak{D}}_{A_{t,\Sigma}}\}_{t\in\mathbb{R}}$ on S^7 with a Dirac operator $\not{\mathfrak{D}}_A^-$ on the cylinder $\mathbb{R} \times S^7$, where the identification is given by

$$\not{\mathfrak{D}}_A^- = E^0 \cdot \left(\frac{d}{dt} - \not{\mathfrak{D}}_{A_{t,\Sigma}}\right).$$

Then, the index of the Dirac operator $\not{\mathfrak{D}}_A^-$ on the cylinder $\mathbb{R} \times S^7$ is precisely the negative of the spectral flow of the operator $\mathrm{sf}\left(\{\not{\mathfrak{D}}_{A_{t,\Sigma}}\}_{t\in\mathbb{R}}\right)$ (see [39] proposition 14.2.1). This follows from the fact that $\frac{d}{dt}$ and $\mathfrak{D}_{A_{t,\Sigma}}$ have opposite signs, and Clifford multiplication by E^0 is an isomorphism that does not affect index. Hence, from (5.17), we have

$$\operatorname{Ind}(\mathfrak{D}^-_A, \mathbb{R}\times S^7) = -\operatorname{sf}\left(\{\mathfrak{D}_{A_{t,\Sigma}}\}_{t\in\mathbb{R}}\right) = 0.$$

Now, from Proposition 5.2.1 applying Atiyah–Patodi–Singer index formula on the compact manifold with boundary $[-T,T]\times S^7$, we have

$$\operatorname{Ind}(\mathfrak{D}^-_A, \mathbb{R}\times S^7) = \operatorname{Ind}(\mathfrak{D}^-_{\widetilde{A}}, [-T,T]\times S^7) = I\left(\mathfrak{D}^-_{\widetilde{A}}, [-T,T]\times S^7\right) + \frac{1}{2}\eta(\partial([-T,T]\times S^7)).$$

However, we note that $\operatorname{Ind}(\mathfrak{D}^-_A, \mathbb{R}\times S^7)$ is independent of T, and hence taking $T\to\infty$, we have

$$\operatorname{Ind}(\mathfrak{D}^-_A, \mathbb{R}\times S^7) = I\left(\mathfrak{D}^-_{\widetilde{A}}, \mathbb{R}\times S^7\right) + \frac{1}{2}\eta(\partial(\mathbb{R}\times S^7)).$$

Now, from (5.13) and (5.15), we have

$$I\left(\mathfrak{D}^-_{\widetilde{A}}, \mathbb{R}\times S^7\right) = -\frac{1}{384\pi^4}\int_{\mathbb{R}^8}\operatorname{tr} F^4_{\widetilde{A}} = -\frac{1}{384\pi^4}\int_{\mathbb{R}^8}\operatorname{tr} F^4_{A} = I\left(\mathfrak{D}^-_A, \mathbb{R}\times S^7\right).$$

Moreover, since $\partial(\mathbb{R}\times S^7) = S^7 \amalg \overline{S^7}$, where $\overline{S^7}$ is S^7 with opposite orientation, we have

$$\eta(\partial(\mathbb{R}\times S^7)) = \eta(\mathcal{D}_\Sigma, \overline{S^7}) + \eta(\mathfrak{D}_{A_\Sigma}, S^7) = \eta(\mathfrak{D}_{A_\Sigma}, S^7) - \eta(\mathcal{D}_\Sigma, S^7) = \eta(\mathfrak{D}_{A_\Sigma}, S^7),$$

since, eta-invariant of $\mathcal{D}_\Sigma$ is zero, which follows from the fact that the metric and Levi-Civita connection of S^7 are invariant under an orientation-reversing isometry. We note that the orientation of S^7 corresponding to the operator $\mathfrak{D}_{A_\Sigma}$ is the same as the boundary S^7 of $\mathbb{R}^8$.

So, finally, we have

$$\begin{aligned}\frac{1}{2}\eta(\mathfrak{D}_{A_\Sigma}, S^7) &= \frac{1}{2}\eta(\partial(\mathbb{R}\times S^7)) = \operatorname{Ind}(\mathfrak{D}^-_A, \mathbb{R}\times S^7) - I(\mathfrak{D}^-_A, \mathbb{R}\times S^7)\\ &= \frac{1}{384\pi^4}\int_{\mathbb{R}\times S^7}\operatorname{tr} F^4_A. \end{aligned}\tag{5.18}$$

Index of the Twisted Dirac Operator

From (5.10) and Proposition 5.2.1, we have

$$\operatorname{Ind}_{-\frac{5}{2}}(\mathfrak{D}^-_A, \mathbb{R}^8, g) = \operatorname{Ind}(\mathfrak{D}^-_{A,CI}, \mathbb{R}^8, g_{CI}) = \operatorname{Ind}(\mathfrak{D}^-_{\widetilde{A},CI}, B^8_R, g_{CI}).$$

Since, B^8_R is a compact manifold with boundary, applying Atiyah–Patodi–Singer index formula,

$$\operatorname{Ind}_{-\frac{5}{2}}(\mathfrak{D}^-_A, \mathbb{R}^8, g) = I\left(\mathfrak{D}^-_{\widetilde{A},CI}, B^8_R, g_{CI}\right) + \frac{1}{2}\eta(\mathfrak{D}_{A_\Sigma}, \partial B^8_R).$$

Since $\mathrm{Ind}_{-\frac{5}{2}}(\not{\mathfrak{D}}^-_A, \mathbb{R}^8, g)$ is independent of R, taking $R \to \infty$, and from (5.15) and (5.18) we have

$$\begin{aligned}
\mathrm{Ind}_{-\frac{5}{2}}(\not{\mathfrak{D}}^-_A, \mathbb{R}^8, g) &= I(\not{\mathfrak{D}}^-_{\tilde{A},CI}, \mathbb{R}^8, g_{CI}) + \frac{1}{2}\eta(\not{\mathfrak{D}}_{A_\Sigma}, S^7) \\
&= -\frac{1}{384\pi^4}\int_{\mathbb{R}^8} \operatorname{tr} F_{\tilde{A}}^4 + \frac{1}{384\pi^4}\int_{\mathbb{R}\times S^7} \operatorname{tr} F_A^4 \\
&= -\frac{1}{384\pi^4}\int_{\mathbb{R}^8} \operatorname{tr} F_A^4 + \frac{1}{384\pi^4}\int_{\mathbb{R}\times S^7} \operatorname{tr} F_A^4 \\
&= 0. \qquad (5.19)
\end{aligned}$$

5.3 The Main Result

Finally, we have the main result on the deformations of FNFN $Spin(7)$-instanton.

Theorem 5.3.1. *The virtual dimension of the moduli space $\mathcal{M}(A_\Sigma, \nu)$ of FNFN $Spin(7)$-instanton with decay rate $\nu \in (-2, 0) \setminus \{-1\}$ is given by*

$$\text{virtual-dim}\,\mathcal{M}(A_\Sigma, \nu) = \begin{cases} 1 & \text{if } \nu \in (-2, -1) \\ 9 & \text{if } \nu \in (-1, 0). \end{cases} \qquad (5.20)$$

Proof. From (5.19) we have that the index of the Dirac operator $\not{\mathfrak{D}}^-_A$ corresponding to the rate $-5/2$ is zero. Moreover, from Corollary 4.4.2, we see that the only critical rates greater that $-5/2$ are -2 and -1, corresponding to the eigenvalues $1/2$ and $3/2$ respectively. Then, from the facts that the eigenspace of the eigenvalue $1/2$ is 1-dimensional and the eigenspace of the eigenvalue $3/2$ is 8-dimensional, the result follows from Theorem 3.1.11. □

Now, the two known types of deformations of the FNFN instanton on $\mathbb{R}^8$ are the translation and the dilation. It is clear that translation being 8-dimensional, should come from spin representation, whereas dilation being one dimensional, should come from the trivial representation.

From the fact that the eigenvalues of the twisted Dirac operator in the range $[1/2, 5/2]$ are $1/2$ and $3/2$, corresponding to the trivial and spin representations respectively, we should expect that the rate of dilation should be $1/2 - 5/2 = -2$ and that of translation should be $3/2 - 5/2 = -1$. This can be easily verified from the fact that the two deformations translation and dilation are given by $\iota_{\frac{\partial}{\partial x^i}} F_A$ and $\iota_{x^i\frac{\partial}{\partial x^i}} F_A$ respectively.

Chapter 6

Deformations of Clarke-Oliveira's Instanton on Bryant-Salamon Manifold

The aim of this chapter is to compute the deformations of Clarke–Oliveira's Instanton on the Bryant–Salamon $Spin(7)$-Manifold. The Bryant–Salamon $Spin(7)$-Manifold is the negative spinor bundle of S^4 which is an asymptotically conical manifold where the link is the squashed sphere $\frac{Sp(2)\times Sp(1)}{Sp(1)\times Sp(1)}$ (see [12]). Clarke and Oliveira in [16] have constructed instantons on this manifold. To calculate the deformations of the instanton we use the deformation theory of asymptotically conical $Spin(7)$-instantons developed in chapter 3.

6.1 Bryant–Salamon $Spin(7)$-Manifold

In this section, we derive the Bryant–Salamon metric using homogeneous space techniques, where we identify the link — the squashed 7-sphere — with the homogeneous space $\frac{Sp(2)\times Sp(1)}{Sp(1)\times Sp(1)}$.

6.1.1 The Squashed 7-Sphere

Friedrich–Kath–Moroianu–Semmelmann in [26] have classified all compact, simply connected homogeneous nearly G_2 manifolds. As homogeneous space, the squashed 7-sphere can be written as $\Sigma^7 := \frac{Sp(2)\times Sp(1)}{Sp(1)\times Sp(1)}$. Recall the groups

$$Sp(1) := \left\{a \in \mathbb{H} : aa^\dagger = 1\right\}, \quad Sp(2) := \left\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{H}, AA^\dagger = I\right\}$$

and corresponding Lie algebras

$$\mathfrak{sp}(1) := \left\{ x \in \mathbb{H} : x + x^\dagger = 0 \right\}, \quad \mathfrak{sp}(2) := \left\{ A = \begin{pmatrix} x & y \\ z & w \end{pmatrix} : x, y, z, w \in \mathbb{H}, A + A^\dagger = 0 \right\}.$$

Denote

$$Sp(1)_u := \left\{ \left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, 1 \right) : g \in Sp(1) \right\}, \quad Sp(1)_d := \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}, g \right) : g \in Sp(1) \right\}.$$

The corresponding Lie algebras are given by

$$\mathfrak{sp}(1)_u := \left\{ \left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, 0 \right) : x \in \mathfrak{sp}(1) \right\}, \quad \mathfrak{sp}(1)_d := \left\{ \left(\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}, x \right) : x \in \mathfrak{sp}(1) \right\}.$$

Then,

$$\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d = \left\{ \left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, y \right) : x, y \in \mathfrak{sp}(1) \right\}.$$

We have a decomposition of the Lie algebra $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ as

$$\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) = \mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d \oplus \mathfrak{m}.$$

We want to find $\mathfrak{m} = (\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d)^\perp$, where the orthogonality is with respect to the Killing form. Since $\mathfrak{m}$ is a representation of $Sp(1)_u \times Sp(1)_d$, we want to decompose $\mathfrak{m}$ into irreducible representations of $Sp(1)_u \times Sp(1)_d$.

Let W_i be the unique irreducible representation of $SU(2) \cong Sp(1)$ of dimension $(i+1)$. Then,

$W_0 \equiv$ Trivial representation ($\dim W_0 = 1$),

$W_1 \equiv$ Standard representation ($\dim W_1 = 2$),

$W_2 \equiv$ Adjoint representation ($\dim W_2 = 3$).

Let W_i^u be an irreducible representation of $Sp(1)_u$ and W_i^d be an irreducible representation of $Sp(1)_d$. Let us define $W_{(i,j)} := W_i^u \otimes W_j^d$, the irreducible representations of $Sp(1)_u \times Sp(1)_d$. Clearly, $\dim W_{(i,j)} = (i+1)(j+1)$. Then

$$\mathfrak{m} = W_{(1,1)} \oplus W_{(0,2)}. \tag{6.1}$$

Now, we want to find a basis for $\mathfrak{m}$. We note that $\mathfrak{m} \cong T_p\Sigma \cong V_p \oplus H_p \cong \operatorname{Im}\mathbb{H} \oplus \mathbb{H}$, where V_p is the vertical space and H_p is the horizontal space with dimensions 3 and 4 respectively, corresponding to the Hopf fibration $S^7 \to S^4$. Now,

$$\operatorname{Im}\mathbb{H} \cong \left\{ \left(\begin{pmatrix} 0 & 0 \\ 0 & -qz \end{pmatrix}, pz \right) : z \in \mathfrak{sp}(1) \right\},$$

where we determine $p = 3$ and $q = 2$ using the Killing form. We choose a basis

$$I_1 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix}, -3i \right), \; I_2 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 2j \end{pmatrix}, -3j \right), \; I_3 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 2k \end{pmatrix}, -3k \right). \tag{6.2}$$

Moreover,

$$\mathbb{H} = \left\{ \left(\begin{pmatrix} 0 & b \\ -b^{\dagger} & 0 \end{pmatrix}, 0 \right) : b \in \mathbb{H} \right\}$$

and, we choose a basis

$$I_4 = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0 \right), \; I_5 = \left(\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, 0 \right), \; I_6 = \left(\begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix}, 0 \right), \; I_7 = \left(\begin{pmatrix} 0 & -k \\ -k & 0 \end{pmatrix}, 0 \right). \tag{6.3}$$

Denote the dual basis of I_a by e^a for $a = 1, \ldots, 7$.

Then $I_1, \ldots, I_7$ together with

$$\begin{aligned} I_8 &= \left(\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, 0 \right), \; I_9 = \left(\begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, 0 \right), \; I_{10} = \left(\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, 0 \right), \\ I_{11} &= \left(\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, i \right), \; I_{12} = \left(\begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, j \right), \; I_{13} = \left(\begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}, k \right) \end{aligned} \tag{6.4}$$

form a basis of $\mathfrak{sp}(2) \times \mathfrak{sp}(1)$. Our objective is to calculate the $Sp(2) \times Sp(1)$-invariant metric g, three-form ϕ and $\psi = *\phi$ on Σ. We note that this corresponds to calculating the $Sp(1)_u \times Sp(1)_d$-invariant metric g, three-form ϕ and $\psi = *\phi$ on $\mathfrak{m}$. We consider an ansatz for ϕ given by

$$\phi = \alpha^3 e^{123} - \alpha\beta^2 (e^1 \wedge \omega_1 + e^2 \wedge \omega_2 + e^3 \wedge \omega_3) \tag{6.5}$$

where $\omega_1, \omega_2, \omega_3$ forms a basis for $\Lambda^2_+ \mathbb{H}^*$. Explicitly, we take $\omega_1 = e^{45} + e^{67}, \omega_2 = e^{46} - e^{57}, \omega_3 = e^{47} + e^{56}$. Then, we can write $\psi = *\phi$, the metric g and the volume form as

$$\psi = \frac{1}{6}\beta^4 (\omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3) - \alpha^2\beta^2 (e^{12} \wedge \omega_3 + e^{23} \wedge \omega_1 + e^{31} \wedge \omega_2)$$

$$g = \alpha^2 \sum_{i=1}^{3} e^i \otimes e^i + \beta^2 \sum_{j=4}^{7} e^j \otimes e^j$$

and $\mathrm{dvol} = \alpha^3 \beta^4 e^{1234567}$ respectively. Hence,

$$\begin{aligned} \phi &= \alpha^3 e^{123} - \alpha\beta^2 (e^{145} + e^{167} + e^{246} - e^{257} + e^{347} + e^{356}), \\ \psi &= \beta^4 e^{4567} - \alpha^2\beta^2 (e^{1247} + e^{1256} + e^{2345} + e^{2367} - e^{1346} + e^{1357}). \end{aligned}$$

It is easy to verify that ϕ is indeed $Sp(1)_u \times Sp(1)_d$-invariant. We need to find α and β such that the metric determined by ϕ is the squashed metric, which is a nearly parallel metric.

Now, from the Maurer–Cartan equation

$$de^a = -f^a_{ib} e^i \wedge e^b - \frac{1}{2} f^a_{bc} e^b \wedge e^c$$

and explicitly calculating the structure constants, we have

$$\begin{aligned}
& d\phi = 4\psi \\
& \Rightarrow \alpha^3 \left(\frac{2}{5}(-e^{1247} - e^{1256} + e^{1346} - e^{1357} - e^{2345} - e^{2367}) \right) \\
& \quad - \alpha\beta^2 \left(10(e^{1247} + e^{1256} - e^{1346} + e^{1357} + e^{2345} + e^{2367}) - \frac{12}{5} e^{4567} \right) \\
& = 4\beta^4 e^{4567} - 4\alpha^2\beta^2 (e^{1247} + e^{1256} + e^{2345} + e^{2367} - e^{1346} + e^{1357}) \\
& \Rightarrow 3\alpha\beta^2 = 5\beta^4 \text{ and } \frac{2}{5}\alpha^3 + 10\alpha\beta^2 = 4\alpha^2\beta^2 \\
& \Rightarrow \alpha = 3, \beta = \pm \frac{3}{\sqrt{5}}.
\end{aligned}$$

Hence,

$$g = 9 \sum_{i=1}^{3} e^i \otimes e^i + \frac{9}{5} \sum_{j=4}^{7} e^j \otimes e^j. \tag{6.6}$$

is the “squashed" metric on Σ^7. An orthonormal basis of $\mathfrak{m}$ is given by

$$\widehat{I}_1 = \frac{1}{3}\left(\begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix}, -3i \right),\ \widehat{I}_2 = \frac{1}{3}\left(\begin{pmatrix} 0 & 0 \\ 0 & 2j \end{pmatrix}, -3j \right),\ \widehat{I}_3 = \frac{1}{3}\left(\begin{pmatrix} 0 & 0 \\ 0 & 2k \end{pmatrix}, -3k \right).$$

$$\widehat{I}_4 = \frac{\sqrt{5}}{3}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0 \right),\ \widehat{I}_5 = \frac{\sqrt{5}}{3}\left(\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, 0 \right),$$

$$\widehat{I}_6 = \frac{\sqrt{5}}{3}\left(\begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix}, 0 \right),\ \widehat{I}_7 = \frac{\sqrt{5}}{3}\left(\begin{pmatrix} 0 & -k \\ -k & 0 \end{pmatrix}, 0 \right).$$

We denote the dual basis by $\widehat{e}^i$ for $i = 1, \ldots, 7$.

6.1.2 The Bryant–Salamon Metric

We just studied the squashed sphere $\Sigma^7 := \frac{Sp(2)\times Sp(1)}{Sp(1)\times Sp(1)}$ as a nearly G_2 manifold where the G_2-structure is given by (6.5).

Now, we consider $(0,\infty)\times\Sigma^7$ equipped with the $Spin(7)$-structure $\Phi = dr\wedge\phi+\psi$ where we consider α,β (and hence ϕ,ψ) as functions of r. The metric is given by

$$g = dr^2+\alpha(r)^2\sum_{i=1}^{3} e^i\otimes e^i+\beta(r)^2\sum_{j=4}^{7} e^j\otimes e^j. \tag{6.7}$$

The metric has holonomy $Spin(7)$ iff Φ is closed. Then, we have,

$$\frac{\partial\psi}{\partial r} = d_\Sigma\phi,$$

which implies,

$$\frac{d\beta^2}{dr} = \frac{6}{5}\alpha \tag{6.8}$$

$$\Rightarrow \frac{d\beta}{dr} = \frac{3\alpha}{5\beta}. \tag{6.9}$$

and

$$\frac{d\alpha}{dr} = \frac{25\beta^2-2\alpha^2}{5\beta^2} \tag{6.10}$$

Hence,

$$\frac{d\beta}{d\alpha} = \frac{3\alpha\beta}{25\beta^2-2\alpha^2}.$$

This is a homogeneous ordinary differential equation. The solution is $\beta^4(5\beta^2-\alpha^2)^3 = C$.

Now, with the initial condition $\alpha(0)=0$ and $\beta(0)=:\beta_0$, we have $\beta^4(5\beta^2-\alpha^2)^3=\beta_0^{10}$, and α,β are both strictly increasing for $r>0$. It can be shown that the metric (6.7) on $(0,\infty)\times\Sigma^7$ can be smoothly extended over $((0,\infty)\times\Sigma^7)\cup S^4$.

Now,

$$\beta^4(5\beta^2-\alpha^2)^3=\beta_0^{10}\Rightarrow \alpha^2=\left(5-(\beta_0\beta^{-1})^{\frac{10}{3}}\right)\beta^2. \tag{6.11}$$

Moreover,

$$dr^2=\left(\frac{dr}{d\beta}\right)^2 d\beta^2=\frac{25}{9}\frac{1}{5-(\beta_0\beta^{-1})^{\frac{10}{3}}}d\beta^2.$$

Moreover, from (6.8), We note that

$$\beta^2(r)=\beta_0^2+\frac{6}{5}\int_0^r\alpha(s)\,ds. \tag{6.12}$$

Hence, considering β as an independent variable, the metric (6.7) can be written as

$$g = \frac{25}{9}\frac{1}{5-(\beta_0\beta^{-1})^{\frac{10}{3}}}d\beta^2 + \left(5-(\beta_0\beta^{-1})^{\frac{10}{3}}\right)\beta^2\sum_{i=1}^{3} e^i \otimes e^i + \beta^2 \sum_{j=4}^{7} e^j \otimes e^j \tag{6.13}$$

which is the *Bryant–Salamon metric* on $((0,\infty)\times\Sigma^7)\cup S^4 \cong \mathcal{S}^-(S^4)$. Thus, $\mathcal{S}^-(S^4)$ is an asymptotically conical $Spin(7)$-manifold over the link squashed sphere with rate $-10/3$.

6.2 Clarke-Oliveira's Instanton

Consider the gauge group $Sp(1) \cong SU(2)$. Then we have three isotropy homomorphisms from $Sp(1)_u \times Sp(1)_d$ to $Sp(1)$, namely

$$\begin{aligned}
\lambda_0 : Sp(1)_u \times Sp(1)_d &\to Sp(1)\\
\left(\begin{pmatrix} g_1 & 0\\ 0 & g_2\end{pmatrix}, g_2\right) &\mapsto 1\\
\lambda_1 : Sp(1)_u \times Sp(1)_d &\to Sp(1)\\
\left(\begin{pmatrix} g_1 & 0\\ 0 & g_2\end{pmatrix}, g_2\right) &\mapsto g_1\\
\lambda_2 : Sp(1)_u \times Sp(1)_d &\to Sp(1)\\
\left(\begin{pmatrix} g_1 & 0\\ 0 & g_2\end{pmatrix}, g_2\right) &\mapsto g_2.
\end{aligned}$$

Consider the bundle $P_i = (Sp(2)\times Sp(1))\times_{\lambda_i} Sp(1)$ over $\Sigma^7 := \frac{Sp(2)\times Sp(1)}{Sp(1)\times Sp(1)}$. From Wang's theorem [59], it follows that the invariant connections on P_i correspond to the $Sp(1)\times Sp(1)$-equivariant homomorphisms

$$\Lambda_i : (\mathfrak{m}, \mathrm{ad}) \to (\mathfrak{sp}(1), \mathrm{ad}\circ\lambda_i).$$

Now,

$$\mathrm{ad}\circ\lambda_i : \mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d \to \mathrm{End}(\mathfrak{sp}(1)),$$

Then,

$$\mathrm{ad}\circ\lambda_0(X,Y)Z = \mathrm{ad}(0)Z = 0.$$

Hence, the map Λ_0 is equivalent to a map

$$W_{(1,1)} \oplus W_{(0,2)} \to W_{(0,0)},$$

and so, must be trivial. Moreover,

$$\mathrm{ad} \circ \lambda_1(X,Y)Z = \mathrm{ad}(X)Z = [X,Z].$$

Hence, the map Λ_1 equivalent to a map

$$W_{(1,1)} \oplus W_{(0,2)} \to W_{(2,0)}$$

is again trivial. Finally,

$$\mathrm{ad} \circ \lambda_2(X,Y)Z = \mathrm{ad}(Y)Z = [Y,Z].$$

Hence, the map Λ_2 can be described as follows,

$$W_{(1,1)} \oplus W_{(0,2)} \to W_{(0,2)}.$$

That is, by Schur's lemma, $\Lambda_2|_{W_{(1,1)}}$ is trivial map, whereas $\Lambda_2|_{W_{(0,2)}}$ is the map

$$\varphi \cdot \mathrm{Id} : W_{(0,2)} \to W_{(0,2)}$$

for some real number φ.

We note that for λ_0 we get the flat connection, and for λ_1, the canonical connection. Thus, these two cases fail to give us anything interesting. Hence, we ignore these two cases. We rename λ_2 to be λ, Λ_2 to be Λ and the corresponding bundle P_2 to be P.

Let us fix a basis $T_a,\ a = 1,2,3$ for $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$, where $T_a = -i\sigma_a$ and $\sigma_a,\ a = 1,2,3$ are the Pauli matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let us denote Ψ by the matrix of Λ, i.e., $\Lambda(I_a) = \Psi_{ac}T_c$, for $a,c = 1,2,3$.

In local coordinates, any $Sp(2) \times Sp(1)$-invariant connection on the bundle P over the nearly G_2-manifold Σ^7 can be written as

$$A = e^i \lambda(I_i) + e^a \Lambda(I_a)$$

where $i = 11, 12, 13$, and the basis elements I_A have been listed in (6.2). Now, we consider $(0,\infty) \times \Sigma^7$ equipped with the $Spin(7)$-structure $\Phi = dr \wedge \phi + \psi$. The metric is given by

$$g = dr^2 + \alpha(r)^2 \sum_{i=1}^{3} e^i \otimes e^i + \beta(r)^2 \sum_{j=4}^{7} e^j \otimes e^j.$$

We consider the normalised basis $\widetilde{e^a}$ where $\widetilde{e}^0 = e^0 = dr, \widetilde{e}^a = \alpha(r)e^a$ for $a = 1, 2, 3$ and $\widetilde{e}^b = \beta(r)e^b$ for $b = 4, 5, 6, 7$. For $a = 1, 2, 3$, we choose $e^{a+10} = \widetilde{e}^{a+10}$. Denote the dual of $\widetilde{e}^i$ by $\widetilde{I}_i$ for $i = 1, \ldots, 13$.

A connection 1-form on the bundle $\pi^*P \to (0,\infty) \times \Sigma^7$ for the projection $\pi : (0,\infty) \times \Sigma^7 \to \Sigma^7$ is given by $A = A_0\widetilde{e}^0 + A_a\widetilde{e}^a$ which yields the $Sp(2) \times Sp(1)$-invariant connection (see Appendix A) given by

$$A = \widetilde{e}^i\lambda(\widetilde{I}_i) + \widetilde{e}^a\Lambda(\widetilde{I}_a)$$

where $i = 11, 12, 13$. Now, for $a = 1, 2, 3$, we have $\Lambda(\widetilde{I}_a) = \widetilde{\Psi}_{ab}T_b$ where $\widetilde{\Psi}_{ab}(r) = \widetilde{\varphi}(r)\delta_{ab}$. Whereas, $\Lambda(\widetilde{I}_a) = \frac{1}{\alpha(r)}\Lambda(I_a) = \frac{1}{\alpha(r)}\Psi_{ab}T_b$ where $\Psi_{ab}(r) = \varphi(r)\delta_{ab}$. Then,

$$A = \widetilde{e}^{a+10}T_a + \widetilde{\varphi}(r)\widetilde{e}^aT_a \tag{6.14}$$

$$= e^{a+10}T_a + \varphi(r)e^aT_a, \tag{6.15}$$

for $a = 1, 2, 3$, where $\alpha(r)\widetilde{\varphi}(r) = \varphi(r)$. Here without loss of generality, we take the temporal gauge $A_0 = 0$. The curvature of this connection is given by

$$F_A = F_{0a}\widetilde{e}^0 \wedge \widetilde{e}^a + \frac{1}{2}F_{bc}\widetilde{e}^b \wedge \widetilde{e}^c$$

where

$$F_{0a} = \frac{\partial A_a}{\partial r} = \frac{d\widetilde{\varphi}(r)}{dr}T_a + \widetilde{\varphi}(r)\frac{\partial\alpha}{\partial r}\frac{1}{\alpha}T_a. \tag{6.16}$$

Now, the ASD instanton equation can be written as

$$F_{0a} = -\frac{1}{2}\phi_{abc}F_{bc}$$

where ϕ_{abc} are structure constants of the octonions. Applying the Maurer–Cartan equations,

$$de^a = -f^a_{ib}e^i \wedge e^b - \frac{1}{2}f^a_{bc}e^b \wedge e^c$$
$$de^i = -\frac{1}{2}f^i_{bc}e^b \wedge e^c - \frac{1}{2}f^i_{jk}e^j \wedge e^k$$

where f^C_{AB} are the structure constants for the basis I_A dual to e^A, i.e., $[I_A, I_B] = f^C_{AB}I_C$. Let $\widetilde{f}^C_{AB}$ are the structure constants for the basis $\widetilde{I}_A$ dual to $\widetilde{e}^A$. we have

$$(dA)_{bc} = -\widetilde{f}^{d+10}_{bc}T_d - \widetilde{\varphi}(r)\widetilde{f}^d_{bc}T_d$$

and

$$[A \wedge A]_{bc} = 4\widetilde{\varphi}(r)^2\epsilon_{dbc}T_d.$$

Hence,

$$F_{bc} = -\widetilde{f}_{bc}^{d+10} T_d - \widetilde{\varphi}(r)\widetilde{f}_{bc}^{d} T_d + 2\widetilde{\varphi}(r)^2 \epsilon_{dbc} T_d.$$

Hence the ASD instanton equation reduces to

$$2\frac{d\widetilde{\varphi}(r)}{dr} T_a + 2\widetilde{\varphi}(r)\frac{\partial\alpha}{\partial r}\frac{1}{\alpha} T_a = \phi_{abc}\widetilde{f}_{bc}^{d+10} T_d + \widetilde{\varphi}(r)\phi_{abc}\widetilde{f}_{bc}^{d} T_d - 2\widetilde{\varphi}(r)^2 \phi_{abc}\epsilon_{dbc} T_d. \tag{6.17}$$

From the values of the structure constants, simplifying, we get

$$\frac{d\widetilde{\varphi}(r)}{dr} - 12\left(\frac{5\beta^2 - \alpha^2}{5\alpha^2\beta^2}\right) + \left(\frac{35\beta^2 + 2\alpha^2}{5\alpha\beta^2}\right)\widetilde{\varphi}(r) + 2\widetilde{\varphi}(r)^2 = 0. \tag{6.18}$$

To solve the equation (6.18), we first simplify by the substitution $x := \alpha\widetilde{\varphi} + 3 = \varphi + 3$, which gives,

$$\dot{x} = -\frac{2}{\alpha}x\left(x - \left(5 - \frac{2\alpha^2}{5\beta^2}\right)\right). \tag{6.19}$$

Now, following the analysis done in [16] we use the substitution

$$y(r) = \frac{x(r)}{\alpha(r)^2}$$

$$\Rightarrow x = \alpha^2 y \tag{6.20}$$

$$\Rightarrow \dot{x} = \alpha^2\dot{y} + 2\alpha y\frac{25\beta^2 - 2\alpha^2}{5\beta^2} \tag{6.21}$$

where we have used (6.10). Substituting (6.20) and (6.21) in (6.19), we have

$$\dot{y} = -2\alpha y^2 \Rightarrow \frac{dy}{y^2} = -2\alpha(r)dr. \tag{6.22}$$

Now, we consider the initial condition,

$$y(0) = y_0. \tag{6.23}$$

Then, integrating (6.22) with the initial condition (6.23), we have

$$y(r) = \frac{1}{\frac{1}{y_0} + 2\int_0^r \alpha(r)dr} \Rightarrow x(r) = \frac{y_0\alpha^2}{1 + 2y_0\int_0^r \alpha(s)ds}.$$

Then, from (6.11) and (6.12), we have

$$\varphi(r) = \frac{\alpha^2}{\frac{1}{y_0} + 2\int_0^r \alpha(s)ds} - 3 = \frac{5\beta_0^2\beta^{\frac{4}{3}} - \beta_0^{\frac{10}{3}} - \frac{3}{y_0}\beta^{\frac{4}{3}}}{\beta^{\frac{4}{3}}\left(\frac{1}{y_0} + \frac{5}{3}(\beta^2 - \beta_0^2)\right)}. \tag{6.24}$$

Remark 6.2.1. We note that $\varphi(r) = 0$ corresponds to the canonical connection. We want to find the value of $\varphi(r)$ for which we have the flat connection. For flat connection, we can write $\varphi(r) = c$ where c is a constant. To find c we substitute $\varphi = c$ and $\frac{\alpha}{\beta} = \sqrt{5}$ in the above equation. Then, equation (6.18) implies $c = 0, -3$. It is easy to verify that $c = -3$ corresponds to the flat connection, which takes the form

$$\begin{aligned} A_0 &= e^{a+10}T_a - 3e^a T_a \\ &= \widehat{e}^{a+10}T_a - \widehat{e}^a T_a. \end{aligned} \tag{6.25}$$

for $a = 1, 2, 3$, where $\widehat{e}^a$ for $a = 1, \ldots, 7$ is an orthonormal basis for the metric (6.7), and $e^{a+10} = \widehat{e}^{a+10}$. Now, the underlying manifold being simply connected, the flat connection A_0 is the trivial connection (up to gauge).

Thus, we have a real 1-parameter family of $Spin(7)$-instantons which, following [16], we denote by A_{y_0}.

For $y_0 = 0$, the connection $A_{y_0=0}$ is a flat connection, whereas for $y_0 > 0$, A_{y_0} is irreducible. For $y_0 < 0$, the $Spin(7)$-instantons are only locally defined in a neighbourhood of S^4 [16].

As $y_0 \to \infty$, the instanton A_{y_0} and all its derivatives converge uniformly to an instanton $A_{\lim}$ [16].

The following proposition follows from the removable singularity theorem of Tao and Tian [57].

Proposition 6.2.2 ([16]). *The instanton A_{y_0} on $\Sigma^7 \times \mathbb{R} \cong \not{S}^-(S^4) \setminus S^4$ smoothly extends over the zero section S^4 (up to gauge) if and only if the curvature $F_{A_{y_0}}$ is bounded.*

Then, we have the following theorem.

Theorem 6.2.3 ([16]). *$\{A_{y_0}\}_{y_0 \in [0,\infty)}$ is a real 1-parameter family of $Spin(7)$-instantons on the trivial bundle $\not{S}^-(S^4) \times \mathbb{C}^2 \to \not{S}^-(S^4)$.*

Moreover, A_{lim} extends smoothly over S^4 and gives a $Spin(7)$-instanton on the (non-trivial) bundle $\pi^*(\not{S}^-(S^4)) \to \not{S}^-(S^4)$, for the projection map $\pi : \not{S}^-(S^4) \setminus S^4 \to S^4$.

Since, for large r, we have $\alpha = O(r)$ and $\beta = O(r)$, clearly $\varphi = O(r^{-2})$. Then, for the diffeomorphism $h : C(\Sigma) = \Sigma^7 \times \mathbb{R} \to \not{S}^-(S^4) \setminus S^4$ and projection $p : C(\Sigma^7) \to \Sigma^7$, we have,

$$\begin{aligned} |h^*(A_{y_0}) - p^*(A_\Sigma)|_{g_C} &= |\varphi(r) e^a T_a|_{g_C} \\ &= |\frac{1}{3}\varphi(r)\widehat{e}^a T_a|_{g_C} \\ &= \frac{1}{3}\frac{1}{r}|\varphi(r)\widehat{e}^a T_a|_{g_\Sigma} \end{aligned}$$

$$= O(r^{-2-1}).$$

where $A_\Sigma = e^{a+10}T_a$. Then, following definition 2.34, the fastest rate of convergence of Clarke–Oliveira's instanton is -2.

6.3 Eigenvalues of the Dirac Operators on Squashed 7-Sphere

In this section, using various representation theoretic and homogeneous space techniques, we calculate the eigenvalues of the untwisted and twisted Dirac operators on the squashed 7-sphere. The results will directly be used to find the critical rates of the negative twisted Dirac operators for Clarke–Oliveira's instanton and in the spectral flow analysis for the index of the Dirac operator.

6.3.1 Eigenvalue Bounds for the Twisted Dirac Operators on Squashed 7-Sphere

Let $V_{(a,b)}$ be the irreducible representations of $Sp(2)$ corresponding to the highest weight vector (a, b). Then,
$V_{(0,0)} \cong \mathbb{C}$ is the trivial representation,
$V_{(0,1)} \cong \mathbb{H}^2$ is the standard representation,
$V_{(1,0)}$ is the 5-dimensional representation under the isomorphism $Sp(2) \cong Spin(5)$.

Define $V_{(a,b,c)} := V_{(a,b)} \otimes W_c$ to be the irreducible representation of $Sp(2) \times Sp(1)$ and let $W_{(a,b)}$ be that of $Sp(1)_u \times Sp(1)_d$.

The Casimir eigenvalues of the Casimir operator (4.11) using the nearly G_2-metric (4.5) for $c^2 = 3/40$, are given by

$$\rho_{(a,b,c)}\left(\mathrm{Cas}_{\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)}\right) = c^{\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)}_{(a,b,c)}\,\mathrm{Id},$$
$$\rho_{(a,b)}(\mathrm{Cas}_{\mathfrak{sp}(1)_u\oplus\mathfrak{sp}(1)_d}) = c^{\mathfrak{sp}(1)_u\oplus\mathfrak{sp}(1)_d}_{(a,b)}\,\mathrm{Id}\,.$$

where,

$$c^{\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)}_{(a,b,c)} = -\frac{5}{9}(4a^2 + 2b^2 + 3c^2 + 4ab + 12a + 8b + 6c),$$
$$c^{\mathfrak{sp}(1)_u\oplus\mathfrak{sp}(1)_d}_{(a,b)} = -\frac{2}{9}(5a^2 + 3b^2 + 10a + 6b).$$

Eigenvalue Bounds

Since, for Clarke-Oliveira's Instanton, the fastest rate of convergence is -2, we consider the family of moduli spaces $\mathcal{M}(A_\Sigma, \nu)$ for $\nu \in (-2, 0)$. Recall the $Sp(1)$-bundle $P = (Sp(2) \times Sp(1)) \times_\lambda Sp(1)$ over Σ^7 corresponding to the isotropy homomorphism λ. Denote the adjoint vector bundle $\mathfrak{g}_P$. Then, we are interested in the eigenvalues of the twisted Dirac operator $\mathfrak{D}_{A_\Sigma}$ twisted by the bundle $\mathfrak{g}_P$, in the interval $\left(-2+\frac{5}{2}, 0+\frac{5}{2}\right) = \left(\frac{1}{2}, \frac{5}{2}\right)$.

Since $(\mathfrak{sp}(1)_\mathbb{C}, \mathrm{Ad} \circ \lambda) = W_{(0,2)}$, we have

$$\mathrm{Hom}\left(V_\gamma, \Delta \otimes \mathfrak{sp}(1)_\mathbb{C}\right)^{Sp(1)_u \times Sp(1)_d} = \mathrm{Hom}\left(V_\gamma, \Delta \otimes W_{(0,2)}\right)^{Sp(1)_u \times Sp(1)_d}.$$

Then, since $c_{(0,2)}^{\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d} = -16/3$, we calculate the eigenvalues of $\left(\mathfrak{D}_{A_\Sigma}^{1/3}\right)_\gamma^2$ to be $-c_\gamma^{\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)} + \frac{1}{9}$ with multiplicities $\dim \mathrm{Hom}\left(V_\gamma, \Delta \otimes W_{(0,2)}\right)^{Sp(1)_u \oplus Sp(1)_d}$.

Hence, we can restate theorem 4.1.3 as

Theorem 6.3.1. *Let $V_\gamma = V_{(a,b,c)}$ be an irreducible representation of $Sp(2) \times Sp(1)$. If*

$$L_\gamma := L_{(a,b,c)} := \sqrt{-c_{(a,b,c)}^{\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)} + \frac{1}{9}} - \frac{7}{6} > 0$$

then L_γ is a lower bound on the absolute values of the eigenvalues of $\left(\mathfrak{D}_{A_\Sigma}^0\right)_\gamma$.

Corollary 6.3.2. *Consider the irreducible representations of $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ given by*

$$V_{(0,0,0)},\ V_{(1,0,0)},\ V_{(0,0,1)},\ V_{(0,1,0)},\ V_{(1,0,1)},\ V_{(0,1,1)},\ V_{(0,2,0)},\ V_{(0,0,2)}.$$

If V_γ is not one of these irreducible representations, then the operator

$$\left(\mathfrak{D}_{A_\Sigma}^0\right)_\gamma : \mathrm{Hom}\left(V_\gamma, \Delta \otimes \mathfrak{sp}(1)_\mathbb{C}\right)^{\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d} \to \mathrm{Hom}\left(V_\gamma, \Delta \otimes \mathfrak{sp}(1)_\mathbb{C}\right)^{\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d}$$

has no eigenvalues in the interval $\left[-\frac{5}{2}, \frac{5}{2}\right]$.

6.3.2 Irreducible Representations of $Sp(2) \times Sp(1)$ and Bases

In order to calculate the eigenvalues the the Dirac operator on the squashed 7-sphere, we need to fix orthonormal bases for the $Sp(2) \times Sp(1)$ irreducible representations and the restrictions to the subspace $Sp(1)_u \times Sp(1)_d$.

Irreducible Representations of $Sp(2)$

For $g \in Sp(1)$, consider the embedding of $Sp(1)$ in $Sp(2)$ given by $\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. As irreducible representations of $Sp(2)$, we have the following decomposition in terms of $Sp(1)$-representations.

$$\begin{aligned}
&\text{Trivial representation } V_{(0,0)} \cong W_0 \text{ (1-dimensional)},\\
&\text{Standard representation } V_{(1,0)} \cong W_0 \oplus 2W_1, \text{ (5-dimensional)}\\
&\text{Vector representation } V_{(0,1)} \cong 2W_0 \oplus W_1, \text{ (4-dimensional)}\\
&\text{Adjoint representation } V_{(0,2)} \cong 3W_0 \oplus 2W_1 \oplus W_2 \text{ (10-dimensional)}.
\end{aligned}$$

We will use the following models for these representations.

- $\dim V_{(0,0)} = 1$. A basis can be taken as 1.
- $\dim V_{(1,0)} = 5$. We consider the model

  $$V^{\mathbb{R}}_{(1,0)} = \left\{ X \in Mat(2, \mathbb{H}) : X^{\dagger} = X, \operatorname{Tr} X = 0 \right\} = \left\{ \begin{pmatrix} x & h \\ h^{\dagger} & -x \end{pmatrix} : x \in \mathbb{R}, h \in \mathbb{H} \right\}.$$

  Then, $V_{(1,0)} = V^{\mathbb{R}}_{(1,0)} \otimes \mathbb{C}$. $Sp(2) \cong Spin(5)$ acts on $\mathbb{R}^5 \otimes \mathbb{C} \subset Cl(\mathbb{R}^5) \otimes \mathbb{C}$ by conjugation. By writing $h = a - ib - jc - kd$, we find a basis as

  $$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}.$$

- $\dim V_{(0,1)} = 4$. $V_{(0,1)} \cong \mathbb{C}^4 \cong \mathbb{H}^2$. $Sp(2)$ acts on $\mathbb{H}^2$ by matrix multiplication. Here the action of $\mathbb{C}$ on $\mathbb{H}^2$ is given by

  $$(a + ib) \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1(a + ib) \\ v_2(a + ib) \end{pmatrix}.$$

  for $a + ib \in \mathbb{C}$ and $v_1, v_2 \in \mathbb{H}$.

  A basis can be taken as

  $$K_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; K_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; K_3 = \begin{pmatrix} j \\ 0 \end{pmatrix}, \; K_4 = \begin{pmatrix} 0 \\ j \end{pmatrix},$$

  where,

  $$2W_0 = \operatorname{Span}\{K_2, K_4\}, \; W_1 = \operatorname{Span}\{K_1, K_3\}.$$

- $\dim V_{(0,2)} = 10$. Bases for the isotypical summands can be taken as

$$3W_0 = \mathrm{Span}\left\{ L_1 = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix},\ L_2 = \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix},\ L_3 = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} \right\},$$

$$2W_1 = \mathrm{Span}\left\{ L_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ L_5 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},\ L_6 = \begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix},\ L_7 = \begin{pmatrix} 0 & -k \\ -k & 0 \end{pmatrix} \right\},$$

$$W_2 = \mathrm{Span}\left\{ L_8 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix},\ L_9 = \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix},\ L_{10} = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

$Sp(2)$ acts adjointly on $V_{(0,2)}$.

Irreducible Representations of $Sp(2) \times Sp(1)$

As irreducible representations of $Sp(2) \times Sp(1)$, we have the following decomposition in terms of $Sp(1)_u \times Sp(1)_d$-representations.

$$\begin{aligned}
V_{(0,0,0)} &\cong W_{(0,0)}, \\
V_{(0,0,1)} &\cong W_{(0,1)}, \\
V_{(0,0,2)} &\cong W_{(0,2)}, \\
V_{(1,0,0)} &\cong W_{(0,0)} \oplus W_{(1,1)}, \\
V_{(1,0,1)} &\cong W_{(0,1)} \oplus W_{(1,0)} \oplus W_{(1,2)}, \\
V_{(0,1,0)} &\cong W_{(1,0)} \oplus W_{(0,1)}, \\
V_{(0,1,1)} &\cong W_{(0,0)} \oplus W_{(1,1)} \oplus W_{(0,2)}, \\
V_{(0,2,0)} &\cong W_{(1,1)} \oplus W_{(2,0)} \oplus W_{(0,2)}.
\end{aligned}$$

Basis of $V_{(0,0,2)}$

We choose an orthonormal basis $\{1 \otimes i, 1 \otimes j, 1 \otimes k\}$ of $W_{(0,2)} \cong V_{(0,0,2)}$.

Basis of $V_{(1,0,0)}$

For $V_{(1,0,0)}$, we choose the orthonormal basis given by,

$$J_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1,\ J_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes 1,\ J_3 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes 1,\ J_4 := \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix} \otimes 1,\ J_5 := \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} \otimes 1,$$

where $W_{(0,0)} = \mathrm{Span}\{J_1\} \subset V_{(1,0,0)}$ and $W_{(1,1)} = \mathrm{Span}\{J_2, J_3, J_4, J_5\} \subset V_{(1,0,0)}$.

Basis of $V_{(0,1,1)}$

First, we note that

$$\mathrm{Span}\{K_2 \otimes j - K_4 \otimes 1\} = W_{(0,0)}.$$

By identifying the highest weight vectors of $W_{(0,2)} \subset V_{(0,1,1)}$ and that of $W_{(0,2)} \subset V_{(0,2,0)}$, and similarly for $W_{(1,1)} \subset V_{(0,1,1)}$ and that of $W_{(1,1)} \subset V_{(0,2,0)}$, we fix an orthonormal basis of $V_{(0,1,1)}$, given by

$$\begin{aligned} M_1 &= i(K_2 \otimes j + K_4 \otimes 1), \\ M_2 &= (K_2 \otimes 1 + K_4 \otimes j), \\ M_3 &= i(K_2 \otimes 1 - K_4 \otimes j), \end{aligned}$$

and

$$\begin{aligned} M_4 &= (-K_1 \otimes j + K_3 \otimes 1), \\ M_5 &= i(K_1 \otimes j + K_3 \otimes 1), \\ M_6 &= (K_1 \otimes 1 + K_3 \otimes j), \\ M_7 &= i(K_1 \otimes 1 - K_3 \otimes j), \end{aligned}$$

together with

$$M_0 = (K_2 \otimes j - K_4 \otimes 1),$$

where $W_{(0,0)} = \mathrm{Span}\{M_0\} \subset V_{(0,1,1)}$, $W_{(0,2)} = \mathrm{Span}\{M_1, M_2, M_3\} \subset V_{(0,1,1)}$ and $W_{(1,1)} = \mathrm{Span}\{M_4, M_5, M_6, M_7\} \subset V_{(0,1,1)}$.

Basis of $V_{(0,2,0)}$

An orthonormal basis of $V_{(0,2,0)}$ is given by

$$W_{(0,2)} = \mathrm{Span}\left\{ L_1 := \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \otimes 1,\ L_2 := \begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix} \otimes 1,\ L_3 := \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} \otimes 1 \right\},$$

$$\begin{aligned} W_{(1,1)} = \mathrm{Span}\Bigg\{ & L_4 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes 1,\ L_5 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \otimes 1, \\ & L_6 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix} \otimes 1,\ L_7 := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -k \\ -k & 0 \end{pmatrix} \otimes 1 \Bigg\}, \end{aligned}$$

$$W_{(2,0)} = \mathrm{Span}\left\{ L_8 := \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \otimes 1,\ L_9 := \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix} \otimes 1,\ L_{10} := \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix} \otimes 1 \right\}.$$

6.3.3 Eigenvalues of the Untwisted Dirac operator

Consider the complex spinor bundle $\not{S}(\Sigma^7) = (Sp(2) \times Sp(1)) \times_{Sp(1)_u \otimes Sp(1)_d} \Delta$ where Δ is the spinor space. Since $\mathfrak{m} \cong W_{(1,1)} \oplus W_{(0,2)} \cong \Lambda^1, W_{(0,0)} \cong \Lambda^0$, and $\Delta \cong \Lambda^0 \oplus \Lambda^1$, we have

$$\Delta \cong W_{(0,0)} \oplus W_{(1,1)} \oplus W_{(0,2)}. \tag{6.26}$$

The canonical connection

$$\nabla^1 : L^2(Sp(2) \times Sp(1), \Delta)^{Sp(1)_u \otimes Sp(1)_d} \to L^2(Sp(2) \times Sp(1), \mathfrak{m}^* \otimes \Delta)^{Sp(1)_u \otimes Sp(1)_d}$$

can be written as

$$\nabla^1 \eta = e^a \otimes \rho_R(I_a)\eta,$$

where e^a is the basis of $\mathfrak{m}^*$ dual to I_a and $\eta \in L^2(Sp(2) \times Sp(1), \Delta)^{Sp(1)_u \otimes Sp(1)_d}$, and ρ_R is defined in (4.2). The corresponding untwisted Dirac operator $\mathcal{D}^1_\Sigma$ is given by

$$\mathcal{D}^1_\Sigma = I_a \cdot \rho_R(I_a). \tag{6.27}$$

Then, from, (2.15) and (6.27), we have a family of Dirac operators

$$\mathcal{D}^t_\Sigma = \mathcal{D}^1_\Sigma + \frac{(t-1)}{2}\phi \tag{6.28}$$

where for $t = 0$, we have $\mathcal{D}^0_\Sigma = \mathcal{D}_\Sigma$ (defined in Proposition 3.1.5).

Now,

$$L^2(\not{S}(\Sigma^7)) \cong L^2(Sp(2) \times Sp(1), \Delta)^{Sp(1)_u \otimes Sp(1)_d} \cong \bigoplus_{\gamma \in \widehat{Sp(2) \times Sp(1)}} \mathrm{Hom}\,(V_\gamma, \Delta)^{Sp(1)_u \otimes Sp(1)_d} \otimes V_\gamma. \tag{6.29}$$

Then, similar to the twisted case in section 4.1, for every $t \in \mathbb{R}$, the Dirac operator $\mathcal{D}^t_\Sigma$, restricted to $\mathrm{Hom}(V_\gamma, \Delta)^{Sp(1)_u \otimes Sp(1)_d} \otimes V_\gamma$ is given by

$$\mathcal{D}^t_\Sigma|_{\mathrm{Hom}(V_\gamma,\Delta)^{Sp(1)_u \otimes Sp(1)_d} \otimes V_\gamma} = \left(\mathcal{D}^t_\Sigma\right)_\gamma \otimes \mathrm{Id} \tag{6.30}$$

where $\left(\mathcal{D}^t_\Sigma\right)_\gamma : \mathrm{Hom}(V_\gamma, \Delta)^{Sp(1)_u \otimes Sp(1)_d} \to \mathrm{Hom}(V_\gamma, \Delta)^{Sp(1)_u \otimes Sp(1)_d}$ is the Dirac operator

$$\left(\mathcal{D}^t_\Sigma\right)_\gamma \eta = -I_a \cdot (\eta \circ \rho_{V_\gamma}(I_a)) + \frac{t-1}{2}\phi \cdot \eta. \tag{6.31}$$

Remark 6.3.3. We note that the untwisted Dirac operator $\mathcal{D}_\Sigma$ acting on the bundle $\not{S}(\Sigma^7)$ can be identified with the twisted Dirac operator $\not{\mathfrak{D}}_{A_0}$ acting the bundle $\not{S}(\Sigma^7) \otimes \mathfrak{g}_P$ twisted by flat connection A_0 (6.25) on the adjoint bundle $\mathfrak{g}_P$. Hence, the eigenvalues of $\mathcal{D}_\Sigma$ and $\not{\mathfrak{D}}_{A_0}$ are the same, a fact that will be used later in calculating the spectral flow of connection for the index calculation.

Eigenvalues from the representation $V_{(0,0,0)}$

$V_\gamma = V_{(0,0,0)} \cong W_{(0,0)}$. Then,

$$\mathrm{Hom}\left(V_\gamma, \Delta\right)^{Sp(1)_u \otimes Sp(1)_d} \cong \mathrm{Hom}\left(W_{(0,0)}, \Delta\right)^{Sp(1)_u \otimes Sp(1)_d}$$

is 1-dimensional. A basis is given by

$$q^{(0,0)} : V_{(0,0,0)} \to W_{(0,0)} \to \Delta$$

which factors through $\Lambda^0 \subset \Delta$. Now,

$$\left(\mathcal{D}^1_\Sigma\right)_\gamma = I_a \cdot \rho_{V^*_\gamma}(I_a) \equiv 0.$$

Hence,

$$\left(\mathcal{D}^0_\Sigma\right)_\gamma = \left(\mathcal{D}^1_\Sigma\right)_\gamma - \frac{1}{2}\phi = -\frac{1}{2}\phi.$$

Now, since $q^{(0,0)}$ factors through $W_{(0,0)} \subset \Lambda^0$, ϕ acts as 7, which follows from Lemma 2.1.3.

Proposition 6.3.4. *Let $V_\gamma = V_{(0,0,0)}$. Then the eigenvalue of $\left(\mathcal{D}^0_\Sigma\right)_\gamma$ is $-\frac{7}{2}$ with multiplicity* 1.

Eigenvalues from the representation $V_{(0,0,1)}$

$V_\gamma = V_{(0,0,1)} \cong W_{(0,1)}$. Then,

$$\mathrm{Hom}\left(V_\gamma, \Delta\right)^{Sp(1)_u \otimes Sp(1)_d} \cong \mathrm{Hom}\left(W_{(0,1)}, \Delta\right)^{Sp(1)_u \otimes Sp(1)_d}$$

is a 0-dimensional vector space, by Schur's lemma.

Eigenvalues from the representation $V_{(0,0,2)}$

$V_\gamma = V_{(0,0,2)} \cong W_{(0,2)}$. Then,

$$\mathrm{Hom}\left(V_\gamma, \Delta\right)^{Sp(1)_u \otimes Sp(1)_d} \cong \mathrm{Hom}\left(W_{(0,2)}, \Delta\right)^{Sp(1)_u \otimes Sp(1)_d}$$

is 1-dimensional. A basis is given by

$$q^{(0,2)} : V_{(0,0,2)} \to W_{(0,2)} \to \Delta$$

which factors through $\Lambda^1 \subset \Delta$.

Now, we identify $q^{(0,2)}$ with an invariant element of $W_{(0,2)} \otimes W_{(0,2)} \subset V_{(0,0,2)} \otimes \Delta$. We choose a basis $\{1 \otimes i, 1 \otimes j, 1 \otimes k\}$ of $W_{(0,2)} \subset V_{(0,0,2)}$, and the basis I_1, I_2, I_3 of $W_{(0,2)} \subset \Delta$. Then, we see that

$$q^{(0,2)} = (1 \otimes i) \otimes I_1 + (1 \otimes j) \otimes I_2 + (1 \otimes k) \otimes I_3.$$

Then,

$$\begin{aligned}\left(\mathcal{D}^1_\Sigma\right)_\gamma q^{(0,2)} &= I_a \cdot \rho_{V_\gamma^*}(I_a) q^{(0,2)} \\ &= I_a \cdot (1 \otimes i) \otimes I_a \cdot I_1 + I_a \cdot (1 \otimes j) \otimes I_a \cdot I_2 + I_a \cdot (1 \otimes k) \otimes I_a \cdot I_3 \\ &= 4(1 \otimes i) \otimes I_1 + 4(1 \otimes j) \otimes I_2 + 4(1 \otimes k) \otimes I_3 = 4q^{(0,2)}.\end{aligned}$$

Moreover, $q^{(0,2)}$ factors through $W_{(0,2)} \subset \Lambda^1$. Hence, from Lemma 2.1.3, ϕ acts as -1. Now,

$$\left(\mathcal{D}^t_\Sigma\right)_\gamma = \left(\mathcal{D}^1_\Sigma\right)_\gamma + \frac{t-1}{2}\phi.$$

Then,

$$\left(\mathcal{D}^{\frac{1}{3}}_\Sigma\right)^2_\gamma = \left(4 + \frac{\frac{1}{3}-1}{2}(-1)\right)^2 = (13/3)^2 = 169/9,$$

whereas,

$$\left(\mathcal{D}^{\frac{1}{3}}_\Sigma\right)^2_\gamma = -c^{\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)}_{(0,0,2)} + \frac{49}{9} = \frac{120}{9} + \frac{49}{9} = \frac{169}{9}$$

which shows the consistency of the calculation. Finally,

$$\left(\mathcal{D}^0_\Sigma\right)_\gamma = \left(\mathcal{D}^1_\Sigma\right)_\gamma - \frac{1}{2}\phi = 4 + \frac{1}{2} = \frac{9}{2}.$$

Proposition 6.3.5. *Let $V_\gamma = V_{(0,0,2)}$. Then the eigenvalue of $\left(\mathcal{D}^0_\Sigma\right)_\gamma$ is $\frac{9}{2}$ with multiplicity* 1.

Eigenvalues from the representation $V_{(1,0,0)}$

$V_\gamma = V_{(1,0,0)} \cong W_{(0,0)} \oplus W_{(1,1)}$. Then,

$$\operatorname{Hom}\left(V_\gamma, \Delta\right)^{Sp(1)_u \otimes Sp(1)_d} \cong \operatorname{Hom}\left(V_{(1,0,0)}, \Delta\right)^{Sp(1)_u \otimes Sp(1)_d}$$

is 2-dimensional. A basis is given by

$$q^{(0,0)} : V_{(1,0,0)} \to W_{(0,0)} \to \Delta$$

which factors through $\Lambda^0 \subset \Delta$, and

$$q^{(1,1)} : V_{(1,0,0)} \to W_{(1,1)} \to \Delta$$

which factors through $\Lambda^1 \subset \Delta$.

Now, we identify $q^{(0,0)}$ with an invariant element of $W_{(0,0)} \otimes W_{(0,0)} \subset V_{(1,0,0)} \otimes \Delta$. Thus,

$$q^{(0,0)} = 1 \otimes 1 = (J_1 \otimes 1) \otimes 1.$$

Hence,

$$I_a \cdot \rho_{V_{(1,0,0)}}(I_a) q^{(0,0)} = \frac{2\sqrt{5}}{3} q^{(1,1)}.$$

We identify $q^{(1,1)}$ with an invariant element of $W_{(1,1)} \otimes W_{(1,1)} \subset V_{(1,0,0)} \otimes \Delta$. Thus,

$$q^{(1,1)} = (J_2 \otimes 1) \otimes I_4 + (J_3 \otimes 1) \otimes I_5 + (J_4 \otimes 1) \otimes I_6 + (J_5 \otimes 1) \otimes I_7.$$

Hence,

$$\begin{aligned} \left(\mathcal{D}^1_\Sigma\right)_\gamma q^{(0,0)} &= I_a \cdot \rho_{V_\gamma^*}(I_a) q^{(0,0)} = \frac{2\sqrt{5}}{3} q^{(1,1)} \\ \left(\mathcal{D}^1_\Sigma\right)_\gamma q^{(1,1)} &= I_a \cdot \rho_{V_\gamma^*}(I_a) q^{(1,1)} = \frac{8\sqrt{5}}{3} q^{(0,0)} + 2q^{(1,1)}. \end{aligned}$$

Thus,

$$\left(\mathcal{D}^1_\Sigma\right)_\gamma = \begin{pmatrix} 0 & \frac{8\sqrt{5}}{3} \\ \frac{2\sqrt{5}}{3} & 2 \end{pmatrix}.$$

Moreover, $q^{(0,0)}$ factors through $W_{(0,0)} \subset \Lambda^0$ and $q^{(1,1)}$ factors through $W_{(1,1)} \subset \Lambda^1$. Hence, from Lemma 2.1.3, ϕ acts as $\operatorname{diag}(7, -1)$. Now,

$$\left(\mathcal{D}^t_\Sigma\right)_\gamma = \left(\mathcal{D}^1_\Sigma\right)_\gamma + \frac{t-1}{2}\phi = \begin{pmatrix} \frac{7}{2}(t-1) & \frac{8\sqrt{5}}{3} \\ \frac{2\sqrt{5}}{3} & 2 + \frac{1-t}{2} \end{pmatrix}.$$

We note that for $t = 1/3$, we have $\left(\mathcal{D}^{\frac{1}{3}}_\Sigma\right)_\gamma^2 = \operatorname{diag}(43/3, 43/3)$, which shows the consistency of the calculation, as $-c^{\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)}_{(1,0,0)} + \frac{49}{9} = \frac{80}{9} + \frac{49}{9} = \frac{129}{9}$. Finally, for $t = 0$, we have

$$\left(\mathcal{D}^0_\Sigma\right)_\gamma = \begin{pmatrix} -\frac{7}{2} & \frac{8\sqrt{5}}{3} \\ \frac{2\sqrt{5}}{3} & \frac{5}{2} \end{pmatrix}$$

and the eigenvalues are given by $\frac{1}{6}(-3 - 2\sqrt{161}), \frac{1}{6}(-3 + 2\sqrt{161})$. Thus we have the following proposition.

Proposition 6.3.6. *Let $V_\gamma = V_{(1,0,0)}$. Then the eigenvalues of $\left(\mathcal{D}^0_\Sigma\right)_\gamma$ are $\frac{1}{6}(-3 - 2\sqrt{161})$, $\frac{1}{6}(-3 + 2\sqrt{161})$ with multiplicity 1.*

Eigenvalues from the representation $V_{(1,0,1)}$

$V_\gamma = V_{(1,0,1)} \cong W_{(0,1)} \oplus W_{(1,0)} \oplus W_{(1,2)}$. Then,

$$\text{Hom}\left(V_\gamma, \Delta\right)^{Sp(1)_u \otimes Sp(1)_d} \cong \text{Hom}\left(V_{(1,0,1)}, \Delta\right)^{Sp(1)_u \otimes Sp(1)_d}$$

is a 0-dimensional vector space, by Schur's lemma.

Eigenvalues from the representation $V_{(0,1,0)}$

$V_\gamma = V_{(0,1,0)} \cong W_{(1,0)} \oplus W_{(0,1)}$. Then,

$$\text{Hom}\left(V_\gamma, \Delta\right)^{Sp(1)_u \otimes Sp(1)_d} \cong \text{Hom}\left(V_{(0,1,0)}, \Delta\right)^{Sp(1)_u \otimes Sp(1)_d}$$

is a 0-dimensional vector space, by Schur's lemma.

Eigenvalues from the representation $V_{(0,1,1)}$

$V_\gamma = V_{(0,1,1)} \cong W_{(0,0)} \oplus W_{(1,1)} \oplus W_{(0,2)}$. Then,

$$\text{Hom}\left(V_\gamma, \Delta\right)^{Sp(1)_u \otimes Sp(1)_d} \cong \text{Hom}\left(V_{(0,1,1)}, \Delta\right)^{Sp(1)_u \otimes Sp(1)_d}$$

is 3-dimensional. A basis is given by

$$q^{(0,0)} : V_{(0,1,1)} \to W_{(0,0)} \to \Delta$$

which factors through $\Lambda^0 \subset \Delta$,

$$q^{(1,1)} : V_{(0,1,1)} \to W_{(1,1)} \to \Delta$$

which factors through $\Lambda^1 \subset \Delta$, and

$$q^{(0,2)} : V_{(0,1,1)} \to W_{(0,2)} \to \Delta$$

which factors through $\Lambda^1 \subset \Delta$.

Now, we identify $q^{(0,0)}$ with an invariant element of $W_{(0,0)} \otimes W_{(0,0)} \subset V_{(0,1,1)} \otimes \Delta$. Thus,

$$q^{(0,0)} = M_0 \otimes 1.$$

We identify $q^{(1,1)}$ with an invariant element of $W_{(1,1)} \otimes W_{(1,1)} \subset V_{(0,1,1)} \otimes \Delta$. Thus,

$$q^{(1,1)} = M_4 \otimes I_4 + M_5 \otimes I_5 + M_6 \otimes I_6 + M_7 \otimes I_7,$$

and, we identify $q^{(0,2)}$ with an invariant element of $W_{(0,2)} \otimes W_{(0,2)} \subset V_{(0,1,1)} \otimes \Delta$. Thus,

$$q^{(0,2)} = M_1 \otimes I_1 + M_2 \otimes I_2 + M_3 \otimes I_3.$$

Then,

$$\left(\mathcal{D}^1_\Sigma\right)_\gamma q^{(0,0)} = \frac{\sqrt{5}}{3} q^{(1,1)} - \frac{5}{3} q^{(0,2)},$$

$$\left(\mathcal{D}^1_\Sigma\right)_\gamma q^{(1,1)} = \frac{4\sqrt{5}}{3} q^{(0,0)} - 3q^{(1,1)} - \frac{4\sqrt{5}}{3} q^{(0,2)},$$

and

$$\left(\mathcal{D}^1_\Sigma\right)_\gamma q^{(0,2)} = -5q^{(0,0)} - \sqrt{5} q^{(1,1)} + \frac{2}{3} q^{(0,2)}.$$

Hence,

$$\left(\mathcal{D}^1_\Sigma\right)_\gamma = \begin{pmatrix} 0 & \frac{4\sqrt{5}}{3} & -5 \\ \frac{\sqrt{5}}{3} & -3 & -\sqrt{5} \\ -\frac{5}{3} & -\frac{4\sqrt{5}}{3} & \frac{2}{3} \end{pmatrix}.$$

Moreover, $q^{(0,0)}$ factors through $W_{(0,0)} \subset \Lambda^0$, $q^{(1,1)}$ factors through $W_{(1,1)} \subset \Lambda^1$ and $q^{(0,2)}$ factors through $W_{(0,2)} \subset \Lambda^1$. Hence, from Lemma 2.1.3, ϕ acts as $\mathrm{diag}(7, -1, -1)$. Now,

$$\left(\mathcal{D}^t_\Sigma\right)_\gamma = \left(\mathcal{D}^1_\Sigma\right)_\gamma + \frac{t-1}{2}\phi = \begin{pmatrix} \frac{7}{2}(t-1) & \frac{4\sqrt{5}}{3} & -5 \\ \frac{\sqrt{5}}{3} & -3 + \frac{1-t}{2} & -\sqrt{5} \\ -\frac{5}{3} & -\frac{4\sqrt{5}}{3} & \frac{2}{3} + \frac{1-t}{2} \end{pmatrix}.$$

We note that for $t = 1/3$, we have $\left(\mathcal{D}^{\frac{1}{3}}_\Sigma\right)^2_\gamma = \mathrm{diag}(16, 16, 16)$, which shows the consistency of the calculation, as $-c^{\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)}_{(0,1,1)} + \frac{49}{9} = \frac{95}{9} + \frac{49}{9} = \frac{144}{9}$. Finally, for $t = 0$, we have

$$\left(\mathcal{D}^0_\Sigma\right)_\gamma = \begin{pmatrix} -\frac{7}{2} & \frac{4\sqrt{5}}{3} & -5 \\ \frac{\sqrt{5}}{3} & -\frac{5}{2} & -\sqrt{5} \\ -\frac{5}{3} & -\frac{4\sqrt{5}}{3} & \frac{7}{6} \end{pmatrix}$$

and the eigenvalues are given by $\frac{1}{6}(-3 - 8\sqrt{11}), \frac{1}{6}(-3 + 8\sqrt{11}), -\frac{23}{6}$. Thus we have the following proposition.

Proposition 6.3.7. *Let $V_\gamma = V_{(0,1,1)}$. Then the eigenvalues of $\left(\mathcal{D}^0_\Sigma\right)_\gamma$ are $\frac{1}{6}(-3 - 8\sqrt{11})$, $\frac{1}{6}(-3 + 8\sqrt{11})$, $-\frac{23}{6}$ with multiplicity* 1.

Eigenvalues from the representation $V_{(0,2,0)}$

$V_\gamma = V_{(0,2,0)} \cong W_{(1,1)} \oplus W_{(2,0)} \oplus W_{(0,2)}$. Then,

$$\mathrm{Hom}\,(V_\gamma, \Delta)^{Sp(1)_u \otimes Sp(1)_d} \cong \mathrm{Hom}\left(V_{(0,2,0)}, \Delta\right)^{Sp(1)_u \otimes Sp(1)_d}$$

is 2-dimensional. A basis is given by

$$q^{(0,2)} : V_{(0,2,0)} \to W_{(0,2)} \to \Delta$$

which factors through $\Lambda^1 \subset \Delta$, and

$$q^{(1,1)} : V_{(0,2,0)} \to W_{(1,1)} \to \Delta$$

which factors through $\Lambda^1 \subset \Delta$.

Now, we identify $q^{(0,2)}$ with an invariant element of $W_{(0,2)} \otimes W_{(0,2)} \subset V_{(0,2,0)} \otimes \Delta$. Thus,

$$q^{(0,2)} = L_1 \otimes I_1 + L_2 \otimes I_2 + L_3 \otimes I_3.$$

and, we identify $q^{(1,1)}$ with an invariant element of $W_{(1,1)} \otimes W_{(1,1)} \subset V_{(0,2,0)} \otimes \Delta$. Thus,

$$q^{(1,1)} = L_4 \otimes I_4 + L_5 \otimes I_5 + L_6 \otimes I_6 + L_7 \otimes I_7.$$

Hence,

$$\begin{aligned}
\left(\mathcal{D}^1_\Sigma\right)_\gamma q^{(0,2)} &= I_a \cdot \rho_{V_\gamma^*}(I_a) q^{(0,2)} = -\frac{8}{3} q^{(0,2)} + \sqrt{10} q^{(1,1)} \\
\left(\mathcal{D}^1_\Sigma\right)_\gamma q^{(1,1)} &= I_a \cdot \rho_{V_\gamma^*}(I_a) q^{(1,1)} = \frac{4\sqrt{10}}{3} q^{(0,2)} + 2 q^{(1,1)}.
\end{aligned}$$

Thus,

$$\left(\mathcal{D}^1_\Sigma\right)_\gamma = \begin{pmatrix} -\frac{8}{3} & \frac{4\sqrt{10}}{3} \\ \sqrt{10} & 2 \end{pmatrix}.$$

Moreover, $q^{(0,2)}$ factors through $W_{(0,2)} \subset \Lambda^1$ and $q^{(1,1)}$ factors through $W_{(1,1)} \subset \Lambda^1$. Hence, ϕ acts as $\mathrm{diag}(-1,-1)$. Now,

$$\left(\mathcal{D}^t_\Sigma\right)_\gamma = \left(\mathcal{D}^1_\Sigma\right)_\gamma + \frac{t-1}{2}\phi = \begin{pmatrix} -\frac{8}{3} + \frac{1-t}{2} & \frac{4\sqrt{10}}{3} \\ \sqrt{10} & 2 + \frac{1-t}{2} \end{pmatrix}.$$

We note that for $t = 1/3$, we have $\left(\mathcal{D}_{\Sigma}^{\frac{1}{3}}\right)_{\gamma}^{2} = \mathrm{diag}(169/9, 169/9)$, which shows the consistency of the calculation, as $-c_{(0,2,0)}^{\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)} + \frac{49}{9} = \frac{40}{3} + \frac{49}{9} = \frac{169}{9}$. Finally, for $t = 0$, we have

$$\left(\mathcal{D}_{\Sigma}^{0}\right)_{\gamma} = \begin{pmatrix} -\frac{13}{6} & \frac{4\sqrt{10}}{3} \\ \sqrt{10} & \frac{5}{2} \end{pmatrix}$$

and the eigenvalues are given by $\frac{9}{2}, -\frac{25}{6}$. Thus we have the following proposition.

Proposition 6.3.8. *Let $V_{\gamma} = V_{(0,2,0)}$. Then the eigenvalues of $\left(\mathcal{D}_{\Sigma}^{0}\right)_{\gamma}$ are $\frac{9}{2}$, $-\frac{25}{6}$ with multiplicity* 1.

Main Result

Theorem 6.3.9. *The eigenvalues of the untwisted Dirac operator $\left(\mathcal{D}_{\Sigma}^{0}\right)_{\gamma}$ are*

1. *For $V_{\gamma} = V_{(0,0,0)}$,*
$$-\frac{7}{2}.$$
2. *For $V_{\gamma} = V_{(0,0,2)}$,*
$$\frac{9}{2}.$$
3. *For $V_{\gamma} = V_{(1,0,0)}$,*
$$\frac{1}{6}(-3 - 2\sqrt{161}),\ \frac{1}{6}(-3 + 2\sqrt{161}).$$
4. *For $V_{\gamma} = V_{(0,1,1)}$*
$$\frac{1}{6}(-3 - 8\sqrt{11}),\ \frac{1}{6}(-3 + 8\sqrt{11}),\ \ -\frac{23}{6}.$$
5. *For $V_{\gamma} = V_{(0,2,0)}$*
$$\frac{9}{2},\ -\frac{25}{6}.$$

6.3.4 Eigenvalues of the Twisted Dirac Operator

We note that

$$\begin{aligned} \Delta \otimes \mathfrak{sp}(1)_{\mathbb{C}} &\cong \Delta \otimes W_{(0,2)} \\ &\cong [W_{(0,0)} \oplus W_{(1,1)} \oplus W_{(0,2)}] \otimes W_{(0,2)} \\ &\cong W_{(0,2)} \oplus [W_{(1,1)} \otimes W_{(0,2)}] \oplus [W_{(0,2)} \otimes W_{(0,2)}] \\ &\cong W_{(0,2)} \oplus [W_{(1,1)} \oplus W_{(1,3)}] \oplus [W_{(0,0)} \oplus W_{(0,2)} \oplus W_{(0,4)}]. \end{aligned} \tag{6.32}$$

Now,

$$L^2(\mathcal{S}(\Sigma)\otimes\mathfrak{g}_P)=\bigoplus_{\gamma\in\widehat{Sp(2)\times Sp(1)}}\mathrm{Hom}\left(V_\gamma,\Delta\otimes W_{(0,2)}\right)^{Sp(1)_u\otimes Sp(1)_d}\otimes V_\gamma$$

where $\mathfrak{g}_P$ is the adjoint bundle of the principal bundle $P=(Sp(2)\times Sp(1))\times_\lambda Sp(1)$ defined in section 6.2. Hence, for each γ, the operator $\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma$ defined in (4.10), acts on the space $\mathrm{Hom}\left(V_\gamma,\Delta\otimes W_{(0,2)}\right)^{Sp(1)_u\otimes Sp(1)_d}$.

Eigenvalues from the representation $V_{(0,0,0)}$

$V_\gamma=V_{(0,0,0)}\cong W_{(0,0)}$. Then,

$$\mathrm{Hom}\left(V_\gamma,\Delta\otimes W_{(0,2)}\right)^{Sp(1)_u\otimes Sp(1)_d}\cong\mathrm{Hom}\left(W_{(0,0)},\Delta\otimes W_{(0,2)}\right)^{Sp(1)_u\otimes Sp(1)_d}$$

is 1-dimensional. A basis is given by

$$q^{(0,0)}_{(0,2)(0,2)}:V_{(0,0,0)}\to W_{(0,0)}\to W_{(0,2)}\otimes W_{(0,2)}\to\Delta\otimes W_{(0,2)}$$

which factors through $\Lambda^1\subset\Delta$. Now,

$$\left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma=I_a\cdot\rho_{V_\gamma^*}(I_a)\equiv 0.$$

Hence,

$$\left(\mathfrak{D}^0_{A_\Sigma}\right)_\gamma=\left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma-\frac{1}{2}\phi=-\frac{1}{2}\phi.$$

Now, since $q^{(0,0)}_{(0,2)(0,2)}$ factors through $W_{(0,2)}\subset\Lambda^1$, ϕ acts as -1.

Proposition 6.3.10. *Let $V_\gamma=V_{(0,0,0)}$. Then the eigenvalue of $\left(\mathfrak{D}^0_{A_\Sigma}\right)_\gamma$ is $\frac{1}{2}$ with multiplicity* 1.

Eigenvalues from the representation $V_{(0,0,1)}$

$V_\gamma=V_{(0,0,1)}\cong W_{(0,1)}$. Then,

$$\mathrm{Hom}\left(V_\gamma,\Delta\otimes W_{(0,2)}\right)^{Sp(1)_u\otimes Sp(1)_d}\cong\mathrm{Hom}\left(W_{(0,1)},\Delta\otimes W_{(0,2)}\right)^{Sp(1)_u\otimes Sp(1)_d}$$

is a 0-dimensional vector space, by Schur's lemma.

Eigenvalues from the representation $V_{(0,0,2)}$

$V_\gamma = V_{(0,0,2)} \cong W_{(0,2)}$. Then,

$$\mathrm{Hom}\left(V_\gamma, \Delta \otimes W_{(0,2)}\right)^{Sp(1)_u \otimes Sp(1)_d} \cong \mathrm{Hom}\left(W_{(0,2)}, \Delta \otimes W_{(0,2)}\right)^{Sp(1)_u \otimes Sp(1)_d}$$

is 2-dimensional. A basis is given by

$$q^{(0,2)}_{(0,0)(0,2)} : V_{(0,0,2)} \to W_{(0,2)} \to W_{(0,0)} \otimes W_{(0,2)} \to \Delta \otimes W_{(0,2)}$$

which factors through $\Lambda^0 \subset \Delta$, and

$$q^{(0,2)}_{(0,2)(0,2)} : V_{(0,0,2)} \to W_{(0,2)} \to W_{(0,2)} \otimes W_{(0,2)} \to \Delta \otimes W_{(0,2)}$$

which factors through $\Lambda^1 \subset \Delta$.

Now, we identify $q^{(0,2)}_{(0,0)(0,2)}$ with an invariant element of $W_{(0,2)} \otimes W_{(0,0)} \otimes W_{(0,2)} \subset V_{(0,0,2)} \otimes \Delta \otimes W_{(0,2)}$. Thus,

$$q^{(0,2)}_{(0,0)(0,2)} = (1 \otimes i) \otimes 1 \otimes I_1 + (1 \otimes j) \otimes 1 \otimes I_2 + (1 \otimes k) \otimes 1 \otimes I_3.$$

Now, we identify $q^{(0,2)}_{(0,2)(0,2)}$ with an invariant element of $W_{(0,2)} \otimes W_{(0,2)} \otimes W_{(0,2)} \subset V_{(0,0,2)} \otimes \Delta \otimes W_{(0,2)}$. Thus,

$$q^{(0,2)}_{(0,2)(0,2)} = (1 \otimes i) \otimes (I_2 \otimes T_3 - I_3 \otimes T_2) + (1 \otimes j) \otimes (I_3 \otimes T_1 - I_1 \otimes T_3) + (1 \otimes k) \otimes (I_1 \otimes T_2 - I_2 \otimes T_1).$$

Then,

$$\left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma q^{(0,2)}_{(0,0)(0,2)} = I_a \cdot \rho_{V^*_\gamma}(I_a) q^{(0,2)}_{(0,0)(0,2)} = 2q^{(0,2)}_{(0,2)(0,2)},$$

and

$$\left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma q^{(0,2)}_{(0,2)(0,2)} = I_a \cdot \rho_{V^*_\gamma}(I_a) q^{(0,2)}_{(0,2)(0,2)} = 4q^{(0,2)}_{(0,0)(0,2)} + 2q^{(0,2)}_{(0,2)(0,2)}.$$

Thus,

$$\left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma = \begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix}$$

Moreover, $q^{(0,2)}_{(0,0)(0,2)}$ factors through $W_{(0,0)} \subset \Lambda^0$ and $q^{(0,2)}_{(0,2)(0,2)}$ factors through $W_{(0,2)} \subset \Lambda^1$. Hence, ϕ acts as $\mathrm{diag}(7, -1)$. Now,

$$\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma = \left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma + \frac{t-1}{2}\phi = \begin{pmatrix} \frac{7}{2}(t-1) & 4 \\ 2 & 2 + \frac{1-t}{2} \end{pmatrix}.$$

We note that for $t = 1/3$, we have $\left(\mathfrak{D}^{\frac{1}{3}}_{A_\Sigma}\right)^2_\gamma = \mathrm{diag}(121/9, 121/9)$, which shows the consistency of the calculation, as $-c^{\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)}_{(0,2,0)} + \frac{1}{9} = \frac{120}{9} + \frac{1}{9} = \frac{121}{9}$. Finally, for $t = 0$, we have

$$\left(\mathfrak{D}^{0}_{A_\Sigma}\right)_\gamma = \begin{pmatrix} -\frac{7}{2} & 4 \\ 2 & \frac{5}{2} \end{pmatrix}$$

and the eigenvalues are given by $\frac{1}{2}(-1-2\sqrt{17}), \frac{1}{2}(-1+2\sqrt{17})$. Thus we have the following proposition.

Proposition 6.3.11. *Let $V_\gamma = V_{(0,0,2)}$. Then the eigenvalues of $\left(\mathfrak{D}^{0}_{A_\Sigma}\right)_\gamma$ are $\frac{1}{2}(-1-2\sqrt{17})$, $\frac{1}{2}(-1+2\sqrt{17})$ with multiplicity* 1.

Eigenvalues from the representation $V_{(1,0,0)}$

$V_\gamma = V_{(1,0,0)} \cong W_{(0,0)} \oplus W_{(1,1)}$. Then,

$$\mathrm{Hom}\left(V_\gamma, \Delta \otimes W_{(0,2)}\right)^{Sp(1)_u \otimes Sp(1)_d} \cong \mathrm{Hom}\left(V_{(1,0,0)}, \Delta \otimes W_{(0,2)}\right)^{Sp(1)_u \otimes Sp(1)_d}$$

is 2-dimensional. A basis is given by

$$q^{(0,0)}_{(0,2)(0,2)} : V_{(1,0,0)} \to W_{(0,0)} \to W_{(0,2)} \otimes W_{(0,2)} \to \Delta \otimes W_{(0,2)}$$

which factors through $\Lambda^1 \subset \Delta$, and

$$q^{(1,1)}_{(0,2)(0,2)} : V_{(1,0,0)} \to W_{(1,1)} \to W_{(1,1)} \otimes W_{(0,2)} \to \Delta \otimes W_{(0,2)}$$

which factors through $\Lambda^1 \subset \Delta$.

Now, we identify $q^{(0,0)}_{(0,2)(0,2)}$ with an invariant element of $W_{(0,0)} \otimes W_{(0,2)} \otimes W_{(0,2)} \subset V_{(1,0,0)} \otimes \Delta \otimes W_{(0,2)}$. Thus,

$$q^{(0,0)}_{(0,2)(0,2)} = J_1 \otimes (I_1 \otimes T_1 + I_2 \otimes T_2 + I_3 \otimes T_3)$$

and, we identify $q^{(1,1)}_{(1,1)(0,2)}$ with an invariant element of $W_{(1,1)} \otimes W_{(1,1)} \otimes W_{(0,2)} \subset V_{(1,0,0)} \otimes \Delta \otimes W_{(0,2)}$. Thus,

$$\begin{aligned} q^{(1,1)}_{(1,1)(0,2)} &= (J_2 \otimes I_5 - J_3 \otimes I_4 + J_4 \otimes I_7 - J_5 \otimes I_6) \otimes T_1 \\ &\quad + (J_2 \otimes I_6 - J_4 \otimes I_4 - J_3 \otimes I_7 + J_5 \otimes I_5) \otimes T_2 \\ &\quad + (J_2 \otimes I_7 - J_5 \otimes I_4 + J_3 \otimes I_6 - J_4 \otimes I_5) \otimes T_3. \end{aligned}$$

Computing directly, we have,

$$\left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma q^{(0,0)}_{(0,2)(0,2)} = I_a \cdot \rho_{(1,0,0)}(I_a) q^{(0,0)}_{(0,2)(0,2)} = \frac{2\sqrt{5}}{3} q^{(1,1)}_{(1,1)(0,2)},$$

and

$$\left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma q^{(1,1)}_{(1,1)(0,2)} = I_a \cdot \rho_{(1,0,0)}(I_a) q^{(1,1)}_{(1,1)(0,2)} = \frac{8\sqrt{5}}{3} q^{(0,0)}_{(0,2)(0,2)} - \frac{2}{3} q^{(1,1)}_{(1,1)(0,2)}.$$

Hence,

$$\left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma = \begin{pmatrix} 0 & \frac{8\sqrt{5}}{3} \\ \frac{2\sqrt{5}}{3} & -\frac{2}{3}. \end{pmatrix}$$

Moreover, $q^{(0,0)}_{(0,2)(0,2)}$ factors through $W_{(0,2)} \subset \Lambda^1$ and $q^{(1,1)}_{(1,1)(0,2)}$ factors through $W_{(1,1)} \subset \Lambda^1$. Hence, ϕ acts as $\mathrm{diag}(-1,-1)$. Now,

$$\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma = \left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma + \frac{t-1}{2}\phi = \begin{pmatrix} \frac{1-t}{2} & \frac{8\sqrt{5}}{3} \\ \frac{2\sqrt{5}}{3} & -\frac{2}{3} + \frac{1-t}{2}. \end{pmatrix}.$$

We note that for $t = 1/3$, we have $\left(\mathfrak{D}^{\frac{1}{3}}_{A_\Sigma}\right)^2_\gamma = \mathrm{diag}(9,9)$, which shows the consistency of the calculation, as $-c^{\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)}_{(1,0,0)} + \frac{1}{9} = \frac{80}{9} + \frac{1}{9} = \frac{81}{9}$. Finally, for $t = 0$, we have

$$\left(\mathfrak{D}^0_{A_\Sigma}\right)_\gamma = \begin{pmatrix} \frac{1}{2} & \frac{8\sqrt{5}}{3} \\ \frac{2\sqrt{5}}{3} & -\frac{1}{6}. \end{pmatrix}$$

and the eigenvalues are given by $\frac{19}{6}, -\frac{17}{6}$. Thus we have the following proposition.

Proposition 6.3.12. *Let $V_\gamma = V_{(1,0,0)}$. Then the eigenvalues of $\left(\mathfrak{D}^0_{A_\Sigma}\right)_\gamma$ are $\frac{19}{6}$, $-\frac{17}{6}$ with multiplicity* 1.

Eigenvalues from the representation $V_{(1,0,1)}$

$V_\gamma = V_{(1,0,1)} \cong W_{(0,1)} \oplus W_{(1,0)} \oplus W_{(1,2)}$. Then,

$$\mathrm{Hom}\left(V_\gamma, \Delta \otimes W_{(0,2)}\right)^{Sp(1)_u \otimes Sp(1)_d} \cong \mathrm{Hom}\left(V_{(1,0,1)}, \Delta \otimes W_{(0,2)}\right)^{Sp(1)_u \otimes Sp(1)_d}$$

is a 0-dimensional vector space, by Schur's lemma.

Eigenvalues from the representation $V_{(0,1,0)}$

$V_\gamma = V_{(0,1,0)} \cong W_{(1,0)} \oplus W_{(0,1)}$. Then,

$$\mathrm{Hom}\left(V_\gamma, \Delta \otimes W_{(0,2)}\right)^{Sp(1)_u \otimes Sp(1)_d} \cong \mathrm{Hom}\left(V_{(0,1,0)}, \Delta \otimes W_{(0,2)}\right)^{Sp(1)_u \otimes Sp(1)_d}$$

is a 0-dimensional vector space, by Schur's lemma.

Eigenvalues from the representation $V_{(0,1,1)}$

$V_\gamma = V_{(0,1,1)} \cong W_{(0,0)} \oplus W_{(1,1)} \oplus W_{(0,2)}$. Then,

$$\mathrm{Hom}\left(V_\gamma, \Delta \otimes W_{(0,2)}\right)^{Sp(1)_u \otimes Sp(1)_d} \cong \mathrm{Hom}\left(V_{(0,1,1)}, \Delta \otimes W_{(0,2)}\right)^{Sp(1)_u \otimes Sp(1)_d}$$

is 4-dimensional. A basis is given by

$$q^{(0,0)}_{(0,2)(0,2)} : V_{(0,1,1)} \to W_{(0,0)} \to W_{(0,2)} \otimes W_{(0,2)} \to \Delta \otimes W_{(0,2)}$$

which factors through $\Lambda^1 \subset \Delta$,

$$q^{(1,1)}_{(1,1)(0,2)} : V_{(0,1,1)} \to W_{(1,1)} \to W_{(1,1)} \otimes W_{(0,2)} \to \Delta \otimes W_{(0,2)}$$

which factors through $\Lambda^1 \subset \Delta$,

$$q^{(0,2)}_{(0,0)(0,2)} : V_{(0,1,1)} \to W_{(0,2)} \to W_{(0,0)} \otimes W_{(0,2)} \to \Delta \otimes W_{(0,2)}$$

which factors through $\Lambda^0 \subset \Delta$, and

$$q^{(0,2)}_{(0,2)(0,2)} : V_{(0,1,1)} \to W_{(0,2)} \to W_{(0,2)} \otimes W_{(0,2)} \to \Delta \otimes W_{(0,2)}$$

which factors through $\Lambda^1 \subset \Delta$.

Now, we identify $q^{(0,0)}_{(0,2)(0,2)}$ with an invariant element of $W_{(0,0)} \otimes W_{(0,2)} \otimes W_{(0,2)} \subset V_{(0,1,1)} \otimes \Delta \otimes W_{(0,2)}$. Hence,

$$q^{(0,0)}_{(0,2)(0,2)} = M_0 \otimes (I_1 \otimes T_1 + I_2 \otimes T_2 + I_3 \otimes T_3).$$

We identify $q^{(1,1)}_{(1,1)(0,2)}$ with an invariant element of $W_{(1,1)} \otimes W_{(1,1)} \otimes W_{(0,2)} \subset V_{(0,1,1)} \otimes \Delta \otimes W_{(0,2)}$. Hence,

$$\begin{aligned} q^{(1,1)}_{(1,1)(0,2)} = {} & (M_4 \otimes I_5 - M_5 \otimes I_4 + M_6 \otimes I_7 - M_7 \otimes I_6) \otimes T_1 \\ & + (M_4 \otimes I_6 - M_6 \otimes I_4 - M_5 \otimes I_7 + M_7 \otimes I_5) \otimes T_2 \\ & + (M_4 \otimes I_7 - M_7 \otimes I_4 + M_5 \otimes I_6 - M_6 \otimes I_5) \otimes T_3. \end{aligned}$$

We identify $q^{(0,2)}_{(0,0)(0,2)}$ with an invariant element of $W_{(0,2)} \otimes W_{(0,0)} \otimes W_{(0,2)} \subset V_{(0,1,1)} \otimes \Delta \otimes W_{(0,2)}$. Hence,

$$q^{(0,2)}_{(0,0)(0,2)} = M_1 \otimes 1 \otimes T_1 + M_2 \otimes 1 \otimes T_2 + M_3 \otimes 1 \otimes T_3,$$

and, we identify $q^{(0,2)}_{(0,2)(0,2)}$ with an invariant element of $W_{(0,2)} \otimes W_{(0,2)} \otimes W_{(0,2)} \subset V_{(0,1,1)} \otimes \Delta \otimes W_{(0,2)}$. Hence,

$$q^{(0,2)}_{(0,2)(0,2)} = M_1 \otimes (I_2 \otimes T_3 - I_3 \otimes T_2) + M_2 \otimes (I_3 \otimes T_1 - I_1 \otimes T_3) + M_3 \otimes (I_1 \otimes T_2 - I_2 \otimes T_1),$$

A direct computation yields,

$$\left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma q^{(0,0)}_{(0,2)(0,2)} = \frac{\sqrt{5}}{3} q^{(1,1)}_{(1,1)(0,2)} + \frac{5}{3} q^{(0,2)}_{(0,0)(0,2)} + \frac{5}{3} q^{(0,2)}_{(0,2)(0,2)},$$

$$\left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma q^{(1,1)}_{(1,1)(0,2)} = \frac{4\sqrt{5}}{3} q^{(0,0)}_{(0,2)(0,2)} + q^{(1,1)}_{(1,1)(0,2)} + \frac{4\sqrt{5}}{3} q^{(0,2)}_{(0,0)(0,2)} - \frac{4\sqrt{5}}{3} q^{(0,2)}_{(0,2)(0,2)},$$

$$\left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma q^{(0,2)}_{(0,0)(0,2)} = \frac{5}{3} q^{(0,0)}_{(0,2)(0,2)} + \frac{\sqrt{5}}{3} q^{(1,1)}_{(1,1)(0,2)} + \frac{1}{3} q^{(0,2)}_{(0,2)(0,2)},$$

and

$$\left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma q^{(0,2)}_{(0,2)(0,2)} = \frac{10}{3} q^{(0,0)}_{(0,2)(0,2)} - \frac{2\sqrt{5}}{3} q^{(1,1)}_{(1,1)(0,2)} + \frac{2}{3} q^{(0,2)}_{(0,0)(0,2)} + \frac{1}{3} q^{(0,2)}_{(0,2)(0,2)}$$

Hence

$$\left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma = \begin{pmatrix} 0 & \frac{4\sqrt{5}}{3} & \frac{5}{3} & \frac{10}{3} \\ \frac{\sqrt{5}}{3} & 1 & \frac{\sqrt{5}}{3} & -\frac{2\sqrt{5}}{3} \\ \frac{5}{3} & \frac{4\sqrt{5}}{3} & 0 & \frac{2}{3} \\ \frac{5}{3} & -\frac{4\sqrt{5}}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Moreover, $q^{(0,0)}_{(0,2)(0,2)}$ factors through $W_{(0,2)} \subset \Lambda^1$, $q^{(1,1)}_{(1,1)(0,2)}$ factors through $W_{(1,1)} \subset \Lambda^1$, $q^{(0,2)}_{(0,0)(0,2)}$ factors through $W_{(0,0)} \subset \Lambda^0$ and $q^{(0,2)}_{(0,2)(0,2)}$ factors through $W_{(0,2)} \subset \Lambda^1$. Hence, ϕ acts as $\mathrm{diag}(-1,-1,7,-1)$. Now,

$$\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma = \left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma + \frac{t-1}{2}\phi = \begin{pmatrix} \frac{1-t}{2} & \frac{4\sqrt{5}}{3} & \frac{5}{3} & \frac{10}{3} \\ \frac{\sqrt{5}}{3} & 1+\frac{1-t}{2} & \frac{\sqrt{5}}{3} & -\frac{2\sqrt{5}}{3} \\ \frac{5}{3} & \frac{4\sqrt{5}}{3} & \frac{7}{2}(t-1) & \frac{2}{3} \\ \frac{5}{3} & -\frac{4\sqrt{5}}{3} & \frac{1}{3} & \frac{1}{3}+\frac{1-t}{2} \end{pmatrix}.$$

We note that for $t = 1/3$, we have $\left(\mathfrak{D}^{\frac{1}{3}}_{A_\Sigma}\right)^2_\gamma = \mathrm{diag}(32/3, 32/3, 32/3, 32/3)$, which shows the consistency of the calculation, as $-c^{\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)}_{(0,1,1)} + \frac{1}{9} = \frac{95}{9} + \frac{1}{9} = \frac{96}{9}$. Finally, for $t = 0$, we have

$$\left(\mathfrak{D}^0_{A_\Sigma}\right)_\gamma = \begin{pmatrix} \frac{1}{2} & \frac{4\sqrt{5}}{3} & \frac{5}{3} & \frac{10}{3} \\ \frac{\sqrt{5}}{3} & \frac{3}{2} & \frac{\sqrt{5}}{3} & -\frac{2\sqrt{5}}{3} \\ \frac{5}{3} & \frac{4\sqrt{5}}{3} & -\frac{7}{2} & \frac{2}{3} \\ \frac{5}{3} & -\frac{4\sqrt{5}}{3} & \frac{1}{3} & \frac{5}{6} \end{pmatrix}$$

and the eigenvalues are given by $\frac{1}{6}(-3-16\sqrt{2}), \frac{1}{6}(-3+16\sqrt{2}), \frac{1}{6}(1-8\sqrt{6}), \frac{1}{6}(1+8\sqrt{6})$. Thus we have the following proposition.

Proposition 6.3.13. *Let $V_\gamma = V_{(0,1,1)}$. Then the eigenvalues of $\left(\not{\mathfrak{D}}^0_{A_\Sigma}\right)_\gamma$ are $\frac{1}{6}(-3-16\sqrt{2})$, $\frac{1}{6}(-3+16\sqrt{2})$, $\frac{1}{6}(1-8\sqrt{6})$, $\frac{1}{6}(1+8\sqrt{6})$ with multiplicity* 1.

Eigenvalues from the representation $V_{(0,2,0)}$

$V_\gamma = V_{(0,2,0)} \cong W_{(1,1)} \oplus W_{(2,0)} \oplus W_{(0,2)}$. Then,

$$\mathrm{Hom}\left(V_\gamma, \Delta \otimes W_{(0,2)}\right)^{Sp(1)_u \otimes Sp(1)_d} \cong \mathrm{Hom}\left(V_{(0,1,1)}, \Delta \otimes W_{(0,2)}\right)^{Sp(1)_u \otimes Sp(1)_d}$$

is 3-dimensional. A basis is given by

$$q^{(1,1)}_{(1,1)(0,2)} : V_{(0,2,0)} \to W_{(1,1)} \to W_{(1,1)} \otimes W_{(0,2)} \to \Delta \otimes W_{(0,2)}$$

which factors through $\Lambda^1 \subset \Delta$,

$$q^{(0,2)}_{(0,0)(0,2)} : V_{(0,2,0)} \to W_{(0,2)} \to W_{(0,0)} \otimes W_{(0,2)} \to \Delta \otimes W_{(0,2)}$$

which factors through $\Lambda^0 \subset \Delta$, and

$$q^{(0,2)}_{(0,2)(0,2)} : V_{(0,2,0)} \to W_{(0,2)} \to W_{(0,2)} \otimes W_{(0,2)} \to \Delta \otimes W_{(0,2)}$$

which factors through $\Lambda^1 \subset \Delta$.

Now, we identify $q^{(1,1)}_{(1,1)(0,2)}$ with an invariant element of $W_{(1,1)} \otimes W_{(1,1)} \otimes W_{(0,2)} \subset V_{(0,2,0)} \otimes \Delta \otimes W_{(0,2)}$. Hence,

$$\begin{aligned} q^{(1,1)}_{(1,1)(0,2)} &= (L_4 \otimes I_5 - L_5 \otimes I_4 + L_6 \otimes I_7 - L_7 \otimes I_6) \otimes T_1 \\ &+ (L_4 \otimes I_6 - L_6 \otimes I_4 - L_5 \otimes I_7 + L_7 \otimes I_5) \otimes T_2 \\ &+ (L_4 \otimes I_7 - L_7 \otimes I_4 + L_5 \otimes I_6 - L_6 \otimes I_5) \otimes T_3. \end{aligned}$$

We identify $q^{(0,2)}_{(0,0)(0,2)}$ with an invariant element of $W_{(0,2)} \otimes W_{(0,0)} \otimes W_{(0,2)} \subset V_{(0,2,0)} \otimes \Delta \otimes W_{(0,2)}$. Hence,

$$q^{(0,2)}_{(0,0)(0,2)} = L_1 \otimes 1 \otimes T_1 + L_2 \otimes 1 \otimes T_2 + L_3 \otimes 1 \otimes T_3,$$

and, we identify $q^{(0,2)}_{(0,2)(0,2)}$ with an invariant element of $W_{(0,2)} \otimes W_{(0,2)} \otimes W_{(0,2)} \subset V_{(0,2,0)} \otimes \Delta \otimes W_{(0,2)}$. Hence,

$$q^{(0,2)}_{(0,2)(0,2)} = L_1 \otimes (I_2 \otimes T_3 - I_3 \otimes T_2) + L_2 \otimes (I_3 \otimes T_1 - I_1 \otimes T_3) + L_3 \otimes (I_1 \otimes T_2 - I_2 \otimes T_1),$$

Then, from direct computation, we have

$$I_a \cdot \rho_{(0,2,0)}(I_a) q^{(1,1)}_{(1,1)(0,2)} = -\frac{2}{3} q^{(1,1)}_{(1,1)(0,2)} - \frac{4\sqrt{10}}{3} q^{(0,2)}_{(0,0)(0,2)} + \frac{4\sqrt{10}}{3} q^{(0,2)}_{(0,2)(0,2)},$$

$$I_a \cdot \rho_{(0,2,0)}(I_a) q^{(0,2)}_{(0,0)(0,2)} = -\frac{\sqrt{10}}{3} q^{(1,1)}_{(1,1)(0,2)} - \frac{4}{3} q^{(0,2)}_{(0,2)(0,2)},$$

and

$$I_a \cdot \rho_{(0,2,0)}(I_a) q^{(0,2)}_{(0,2)(0,2)} = \frac{2\sqrt{10}}{3} q^{(1,1)}_{(1,1)(0,2)} - \frac{8}{3} q^{(0,2)}_{(0,0)(0,2)} - \frac{4}{3} q^{(0,2)}_{(0,2)(0,2)}.$$

Hence,

$$\left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma = \begin{pmatrix} -\frac{2}{3} & -\frac{\sqrt{10}}{3} & \frac{2\sqrt{10}}{3} \\ -\frac{4\sqrt{10}}{3} & 0 & -\frac{8}{3} \\ \frac{4\sqrt{10}}{3} & -\frac{4}{3} & -\frac{4}{3} \end{pmatrix}.$$

Moreover, $q^{(1,1)}_{(1,1)(0,2)}$ factors through $W_{(1,1)} \subset \Lambda^1$, $q^{(0,2)}_{(0,0)(0,2)}$ factors through $W_{(0,0)} \subset \Lambda^0$ and $q^{(0,2)}_{(0,2)(0,2)}$ factors through $W_{(0,2)} \subset \Lambda^1$. Hence, ϕ acts as $\operatorname{diag}(-1, 7, -1)$. Now,

$$\left(\mathfrak{D}^t_{A_\Sigma}\right)_\gamma = \left(\mathfrak{D}^1_{A_\Sigma}\right)_\gamma + \frac{t-1}{2}\phi = \begin{pmatrix} -\frac{2}{3} + \frac{1-t}{2} & -\frac{\sqrt{10}}{3} & \frac{2\sqrt{10}}{3} \\ -\frac{4\sqrt{10}}{3} & \frac{7}{2}(t-1) & -\frac{8}{3} \\ \frac{4\sqrt{10}}{3} & -\frac{4}{3} & -\frac{4}{3} + \frac{1-t}{2} \end{pmatrix}.$$

We note that for $t = 1/3$, we have $\left(\mathfrak{D}^{\frac{1}{3}}_{A_\Sigma}\right)^2_\gamma = \operatorname{diag}(121/9, 121/9, 121/9)$, which shows the consistency of the calculation, as $-c^{\mathfrak{sp}(2)\oplus\mathfrak{sp}(1)}_{(0,2,0)} + \frac{1}{9} = \frac{120}{9} + \frac{1}{9} = \frac{121}{9}$. Finally, for $t = 0$, we have

$$\left(\mathfrak{D}^0_{A_\Sigma}\right)_\gamma = \begin{pmatrix} -\frac{1}{6} & -\frac{\sqrt{10}}{3} & \frac{2\sqrt{10}}{3} \\ -\frac{4\sqrt{10}}{3} & -\frac{7}{2} & -\frac{8}{3} \\ \frac{4\sqrt{10}}{3} & -\frac{4}{3} & -\frac{5}{6} \end{pmatrix}$$

and the eigenvalues are given by $\frac{1}{2}(-1-2\sqrt{17})$, $\frac{1}{2}(-1+2\sqrt{17})$, $-\frac{7}{2}$. Thus we have the following proposition.

Proposition 6.3.14. *Let $V_\gamma = V_{(0,2,0)}$. Then the eigenvalues of $\left(\mathfrak{D}^0_{A_\Sigma}\right)_\gamma$ are $\frac{1}{2}(-1-2\sqrt{17})$, $\frac{1}{2}(-1+2\sqrt{17})$, $-\frac{7}{2}$. with multiplicity* 1.

Main Result

Theorem 6.3.15. *The eigenvalues of the twisted Dirac operator $\left(\mathfrak{D}^0_{A_\Sigma}\right)_\gamma$ are*

1. *For* $V_\gamma = V_{(0,0,0)}$,

$$\frac{1}{2}.$$

2. *For* $V_\gamma = V_{(0,0,2)}$,

$$\frac{1}{2}(-1-2\sqrt{17}),\ \frac{1}{2}(-1+2\sqrt{17}).$$

3. *For* $V_\gamma = V_{(1,0,0)}$,

$$\frac{19}{6},\ -\frac{17}{6}.$$

4. *For* $V_\gamma = V_{(0,1,1)}$

$$\frac{1}{6}(-3-16\sqrt{2}),\ \frac{1}{6}(-3+16\sqrt{2}),\ \frac{1}{6}(1-8\sqrt{6}),\ \frac{1}{6}(1+8\sqrt{6}).$$

5. *For* $V_\gamma = V_{(0,2,0)}$

$$\frac{1}{2}(-1-2\sqrt{17}),\ \frac{1}{2}(-1+2\sqrt{17}),\ -\frac{7}{2}.$$

Corollary 6.3.16. *The only eigenvalue of the twisted Dirac operator* $\mathfrak{D}^0_{A_\Sigma}$ *in the interval* $\left[-\frac{5}{2}, \frac{5}{2}\right]$ *is* $\frac{1}{2}$ *corresponding to the trivial representation* $V_{(0,0,0)}$.

6.4 Deformations of Clarke–Oliveira's Instanton

In this section we calculate the deformations of Clarke–Oliveira's Instanton and calculate the virtual dimensions of the moduli space. Following similar techniques as in chapter 5, we compute the index of the twisted Dirac operator $\mathfrak{D}^-_A : W^{k,2}_{-\frac{7}{2}} \to W^{k-1,2}_{-\frac{9}{2}}$ on $\mathcal{S}^-(S^4)$, where A is Clarke–Oliveira's Instanton, using the Atiyah–Patodi–Singer theorem and various other techniques.

6.4.1 Index of the Twisted Dirac Operator

Let g_C be the Bryant–Salamon metric (6.13) on $\mathcal{S}^-(S^4)$. We define the asymptotically cylindrical "cigar" metric $g_{CI} := \frac{1}{\varrho^2} g_C$, where ϱ is the radius function (2.30).

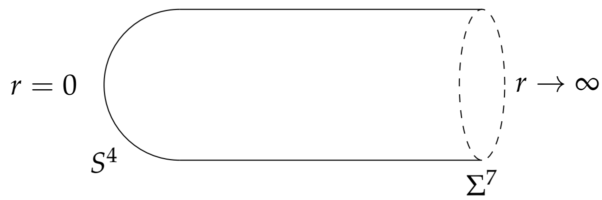


Figure 6.1: $\mathcal{S}^-(S^4)$ with cigar metric.

Then, as in Chapter 5, we have

$$\operatorname{Index}\left(\mathcal{D}^-_{A,C} : W^{k,2}_{-\frac{7}{2}} \to W^{k-1,2}_{-\frac{9}{2}}\right) = \operatorname{Index}\left(\mathcal{D}^-_{A,CI} : W^{k,2}_{CI} \to W^{k-1,2}_{CI}\right). \tag{6.33}$$

Throughout this section, We identify $\mathbb{R} \times \Sigma^7$ with $(0,\infty) \times \Sigma^7$ via $t = \ln r$ for $r \in (0,\infty)$ and $t \in \mathbb{R}$.

Define a function $\overline{\varphi} : \mathbb{R} \to \mathbb{R}$ by

$$\overline{\varphi}(t) = \begin{cases} -3 & t < -T \\ a', & -T < t < -\frac{T}{2} \\ \varphi(t), & -\frac{T}{2} < t < \frac{T}{2} \\ a, & \frac{T}{2} < t < T \\ 0 & t > T \end{cases} \tag{6.34}$$

where a is a smooth interpolation between its values at $\frac{T}{2}$ and T and a' is that of between its values at $-T$ and $-\frac{T}{2}$.

Consider the connection on $\mathcal{S}^-(S^4)$ given by

$$\overline{A} = A_\Sigma + \overline{\varphi}(t) e^a T_a. \tag{6.35}$$

This connection has the same limits at $t = \pm\infty$ as Clarke–Oliveira's instanton A.

We note that $\mathcal{S}^-(S^4)$ can be considered as the space $[0,\infty) \times (Sp(2) \times Sp(1))/\sim$ where

$$\begin{aligned} (r,g) &\sim (r, g \cdot h) \ \text{ for all } r > 0,\ h \in Sp(1)^2 \\ (0,g) &\sim (0, g \cdot h) \ \text{ for all } h \in Sp(1)^3. \end{aligned} \tag{6.36}$$

Proposition 6.4.1. *Let $K^8_R := [0,R) \times (Sp(2) \times Sp(1))/\sim$ be a compact 8-dimensional subset of $\mathcal{S}^-(S^4)$, where $R > 0$ and $\sim$ is defined as (6.36). Then for sufficiently large R, we have*

$$\operatorname{Index}\left(\mathcal{D}^-_{A,CI}, \mathcal{S}^-(S^4), g_{CI}\right) = \operatorname{Index}\left(\mathcal{D}^-_{\overline{A},CI}, K^8_R, g_{CI}\right),$$

and for sufficiently large T, we have

$$\text{Index}\left(\mathcal{D}^-_A, \mathbb{R}\times\Sigma^7, g\right) = \text{Index}\left(\mathcal{D}^-_{\overline{A}}, [-T,T]\times\Sigma^7, g\right).$$

where g is the cylindrical metric given by $g = dt^2 + g_{\Sigma^7}$.

The proof is similar to the proof of proposition 5.2.1.

Now, the index of the twisted Dirac operator $\mathcal{D}^-_{\overline{A},CI}$ on K^8_R is given by the Atiyah–Patodi–Singer index theorem, which states that

$$\text{Index}\left(\mathcal{D}^-_{\overline{A},CI}, K^8_R, g_{CI}\right) = I\left(\mathcal{D}^-_{\overline{A},CI}, K^8_R, g_{CI}\right) + \frac{1}{2}\eta(\mathcal{D}_{A_\Sigma}, \partial K^8_R). \tag{6.37}$$

By proposition 6.4.1, the index is independent of R, hence taking $R\to\infty$, we have,

$$\text{Index}\left(\mathcal{D}^-_{\overline{A},CI}, \not{S}^-(S^4), g_{CI}\right) = I\left(\mathcal{D}^-_{\overline{A},CI}, \not{S}^-(S^4), g_{CI}\right) + \frac{1}{2}\eta(\mathcal{D}_{A_\Sigma}, \Sigma^7). \tag{6.38}$$

where, the term $I(\mathcal{D}^-_{\overline{A},CI}, \not{S}^-(S^4), g_{CI})$ in (6.38) is given by

$$\begin{aligned}
I(\mathcal{D}^-_{\overline{A},CI}, \not{S}^-(S^4), g_{CI}) &= -\int_{\not{S}^-(S^4)} \widehat{A}(\not{S}^-(S^4))\,\mathrm{ch}(\mathfrak{g}_P\otimes\mathbb{C})\\
&= -\int_{\not{S}^-(S^4)}\left(1-\frac{1}{24}p_1(\not{S}^-(S^4)) + \frac{1}{5760}(7p_1(\not{S}^-(S^4))^2 - 4p_2(\not{S}^-(S^4)))\right)\\
&\qquad\left(\dim\mathfrak{g} + p_1(\mathfrak{g}_P) + \frac{1}{12}\left(p_1(\mathfrak{g}_P)^2 - 2p_2(\mathfrak{g}_P)\right)\right)\\
&= -\frac{1}{12}\int_{\not{S}^-(S^4)}(p_1(\mathfrak{g}_P)^2 - 2p_2(\mathfrak{g}_P)) + \frac{1}{24}\int_{\not{S}^-(S^4)} p_1(\not{S}^-(S^4))p_1(\mathfrak{g}_P)\\
&\quad - \frac{1}{5760}\dim\mathfrak{g}\int_{\not{S}^-(S^4)}(7p_1(\not{S}^-(S^4))^2 - 4p_2(\not{S}^-(S^4))),
\end{aligned} \tag{6.39}$$

where p_i denotes the ith Pontryagin classes.

6.4.2 Eta Invariant of the Boundary

We calculate the eta invariant $\eta(\mathcal{D}_{A_\Sigma}, \Sigma^7)$ of the twisted Dirac operator by first, relating it to the untwisted Dirac operator on the squashed sphere, and then, relating it to the untwisted Dirac operator on the round sphere, whose eta invariant is known to be zero.

Recall Clarke–Oliveira's instanton (6.14) with φ given by (6.24). The instanton can be identified with a family of connections $\{A_t : t\in\mathbb{R}\}$ on Σ^7, where $t = \ln r$. The family of Dirac operators twisted by the connections A_t is given by

$$\mathcal{D}_{A_{t,\Sigma}} = \mathcal{D}_{A_\Sigma} + \varphi(t)e^aT_a \tag{6.40}$$

$$= \not{\mathfrak{D}}_{A_\Sigma} + \frac{1}{3}\varphi(t)\hat{e}^a T_a, \tag{6.41}$$

where $\varphi(t)$ varies from -3 to 0. Then, from (6.25), (6.40) and remark 6.3.3, we note that, for $\varphi(t) = 0$, we have the Dirac operator $\not{\mathfrak{D}}_{A_\Sigma}$ twisted by the canonical connection, and for $\varphi(t) = -3$, we have the Dirac operator twisted by the flat connection, given by

$$\mathcal{D}_\Sigma = \not{\mathfrak{D}}_{A_\Sigma} - \hat{e}^a T_a. \tag{6.42}$$

We identify the family of Dirac operators $\{\not{\mathfrak{D}}_{A_{t,\Sigma}}\}_{t\in\mathbb{R}}$ on Σ^7 with the Dirac operator $\not{\mathfrak{D}}_A^-$ on the cylinder $\not{C} := \mathbb{R}\times\Sigma^7$, given by

$$\not{\mathfrak{D}}_A^- = dt\cdot\left(\frac{d}{dt} - \not{\mathfrak{D}}_{A_{t,\Sigma}}\right).$$

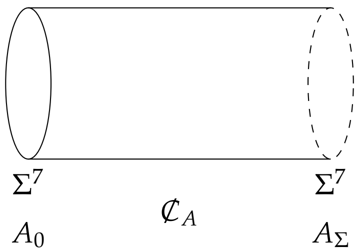


Figure 6.2: The cylinder $\not{C}$.

Then, from [39], the index of the Dirac operator $\not{\mathfrak{D}}_A^-$ on $\mathbb{R}\times\Sigma^7$ is

$$\mathrm{Ind}(\not{\mathfrak{D}}_A^-, \not{C}) = -\,\mathrm{sf}\left(\{\not{\mathfrak{D}}_{A_{t,\Sigma}}\}_{t\in\mathbb{R}}\right). \tag{6.43}$$

Now, from Proposition 6.4.1 and applying the Atiyah–Patodi–Singer index formula on $[-T,T]\times\Sigma^7$, we have

$$\mathrm{Ind}(\not{\mathfrak{D}}_A^-, \not{C}) = \mathrm{Ind}(\not{\mathfrak{D}}_A^-, [-T,T]\times\Sigma^7) = I\left(\not{\mathfrak{D}}_A^-, [-T,T]\times\Sigma^7\right) + \frac{1}{2}\eta(\partial([-T,T]\times\Sigma^7))$$

where the term $\eta(\partial([-T,T]\times\Sigma^7))$ is the eta invariant for the operator $\not{\mathfrak{D}}_A^-$ restricted to the submanifold $\partial([-T,T]\times\Sigma^7)$. By proposition 6.4.1, since $\mathrm{Ind}(\not{\mathfrak{D}}_A^-, \not{C})$ is independent of T, taking $T\to\infty$, we get,

$$\mathrm{Ind}(\not{\mathfrak{D}}_A^-, \not{C}) = I\left(\not{\mathfrak{D}}_A^-, \not{C}\right) + \frac{1}{2}\eta(\partial\not{C})$$

where $\eta(\partial\not{C})$ is the eta invariant for $\not{\mathfrak{D}}_A^-$ restricted to the submanifold $\partial\not{C}$.

Now, from (6.4.1), we have

$$I\left(\not{\mathfrak{D}}_A^-, \not{C}\right) = I\left(\not{\mathfrak{D}}_A^-, \mathbb{R} \times \Sigma^7\right)$$

Moreover, since $\partial \not{C} = \Sigma^7 \amalg \overline{\Sigma^7}$, where $\overline{\Sigma^7}$ is Σ^7 with opposite orientation, from (6.42) we have

$$\eta(\partial \not{C}) = \eta(\mathcal{D}_\Sigma, \overline{\Sigma^7}) + \eta(\not{\mathfrak{D}}_{A_\Sigma}, \Sigma^7) = \eta(\not{\mathfrak{D}}_{A_\Sigma}, \Sigma^7) - \eta(\mathcal{D}_\Sigma, \Sigma^7),$$

So, finally, we have

$$\begin{aligned}
\frac{1}{2}\eta(\not{\mathfrak{D}}_{A_\Sigma}, \Sigma^7) &= \frac{1}{2}\eta(\partial \not{C}) + \frac{1}{2}\eta(\mathcal{D}_\Sigma, \Sigma^7) \\
&= \mathrm{Ind}(\not{\mathfrak{D}}_A^-, \not{C}) - I(\not{\mathfrak{D}}_A^-, \not{C}) + \frac{1}{2}\eta(\mathcal{D}_\Sigma, \Sigma^7) \\
&= -\operatorname{sf}\left(\{\not{\mathfrak{D}}_{A_{t,\Sigma}}\}_{t\in\mathbb{R}}\right) - I(\not{\mathfrak{D}}_A^-, \not{C}) + \frac{1}{2}\eta(\mathcal{D}_\Sigma, \Sigma^7).
\end{aligned} \tag{6.44}$$

Now, we note that since the squashed sphere does not have an orientation reversing isometry, the eta-invariant of the untwisted Dirac operator $\mathcal{D}_\Sigma$ may not be zero. So, to find the eta-invariant of the untwisted Dirac operator on the squashed sphere, we relate it to that of the round sphere for which we know the eta-invariant of the untwisted Dirac operator is zero.

Consider the cylinder $C_\Sigma := \Sigma \times \mathbb{R}$, and for $t \in \mathbb{R}$, a family of Riemannian manifolds (Σ, g_t) where for $t = -\infty$ we have the squashed 7-sphere Σ^7 and for $t = \infty$ we have the round 7-sphere S^7.

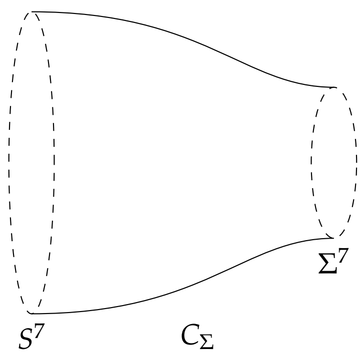


Figure 6.3: The cylinder C_Σ.

Consider corresponding family of untwisted Dirac operators $\{\mathcal{D}_{\Sigma,t}\}_{t\in\mathbb{R}}$ which we can identify with an untwisted Dirac operator $\mathcal{D}^-$ on the cylinder C_Σ. Then, using the result from [39], we have that the index of the Dirac operator on the cylinder C_Σ is the negative of the spectral flow of the family $\{\mathcal{D}_{\Sigma,t}\}_{t\in\mathbb{R}}$, i.e.,

$$\operatorname{Ind}(\mathcal{D}^-, C_\Sigma) = -\operatorname{sf}\left(\{\mathcal{D}_{\Sigma,t}\}_{t\in\mathbb{R}}\right). \tag{6.45}$$

Now, from Proposition 5.2.1 applying the Atiyah–Patodi–Singer index formula on $[-T,T]\times\Sigma^7$, we have

$$\operatorname{Ind}(\mathcal{D}^-, C_\Sigma) = I\left(\mathcal{D}^-, [-T,T]\times\Sigma^7\right) + \frac{1}{2}\eta(\mathcal{D}_\Sigma, \partial([-T,T]\times\Sigma^7)).$$

Since $\operatorname{Ind}(\mathcal{D}^-, C_\Sigma)$ is independent of T, taking $T\to\infty$, we get,

$$\operatorname{Ind}(\mathcal{D}^-, C_\Sigma) = I\left(\mathcal{D}^-, C_\Sigma\right) + \frac{1}{2}\eta(\partial(C_\Sigma)).$$

Moreover, since $\partial(C_\Sigma) = \Sigma^7 \amalg \overline{S^7}$, where $\overline{S^7}$ is S^7 with opposite orientation, we have

$$\eta(\partial(C_\Sigma)) = \eta(\mathcal{D}_\Sigma, \Sigma^7) - \eta(\mathcal{D}_\Sigma, S^7).$$

Hence, we have

$$\begin{aligned}
\frac{1}{2}\eta(\mathcal{D}_\Sigma, \Sigma^7) &= \frac{1}{2}\eta(\partial(C_\Sigma)) + \frac{1}{2}\eta(\mathcal{D}_\Sigma, S^7)\\
&= \operatorname{Ind}(\mathcal{D}^-, C_\Sigma) - I(\mathcal{D}^-, C_\Sigma) + \frac{1}{2}\eta(\mathcal{D}_\Sigma, S^7)\\
&= -\operatorname{sf}\left(\{\mathcal{D}_{\Sigma,t}\}_{t\in\mathbb{R}}\right) - I(\mathcal{D}^-, C_\Sigma) + \frac{1}{2}\eta(\mathcal{D}_\Sigma, S^7).
\end{aligned} \tag{6.46}$$

Then, substituting (6.46) in (6.44) and using the fact that $\eta(\mathcal{D}_\Sigma, S^7) = 0$, we have

$$\frac{1}{2}\eta(\not{\mathfrak{D}}_{A_\Sigma}, \Sigma^7) = -\operatorname{sf}\left(\{\not{\mathfrak{D}}_{A_{t,\Sigma}}\}_{t\in\mathbb{R}}\right) - I(\not{\mathfrak{D}}_A^-, \not{C}) - \operatorname{sf}\left(\{\mathcal{D}_{\Sigma,t}\}_{t\in\mathbb{R}}\right) - I(\mathcal{D}^-, C_\Sigma). \tag{6.47}$$

Spectral Flow of the Connection

We want to find the spectral flow of the family of Dirac operators (6.40). First we compute the eigenvalues of the operator $\widehat{e}^a T_a$ which acts fibre-wise, on $\Delta\otimes\mathfrak{sp}(1)$. Let e^μ, $\mu = 0,1,\ldots,7$ be a basis of Δ and T_a is a basis of $\mathfrak{sp}(1)$, for $a = 1,2,3$. Then,

$$\begin{aligned}
&(\widehat{e}^a T_a)\,(\widehat{e}^\mu\otimes T_b)\\
&= (\widehat{e}^a\cdot\widehat{e}^\mu)\otimes[T_a, T_b]\\
&= (\widehat{E}^a\otimes\operatorname{ad} T_a)(\widehat{e}^\mu\otimes T_b),
\end{aligned}$$

Eigenvalues	Multiplicities
4	4
-4	4
-2	8
2	8

Table 6.1: Eigenvalues of $\widehat{e}^a T_a$ and corresponding multiplicities.

where $\widehat{E}^a$ is the matrix given by Clifford multiplication with $\widehat{e}^a$. Similarly, we calculate the matrices $\operatorname{ad} T_a$. We get the matrix of $\widehat{e}^a T_a$ by taking the Kronecker product of $\widehat{E}^a$ and $\operatorname{ad} T_a$. The eigenvalues of $\widehat{e}^a T_a$ are given in table 6.1.

In figure 6.4, we plot the eigenvalues of the operators $\mathcal{D}_\Sigma$ and $\mathfrak{D}_{A_\Sigma}$ near zero, calculated in theorems 6.3.9 and 6.3.15.

Now, the the highest magnitude among the eigenvalues of $\widehat{e}^a T_a$ is 4. Hence, from (6.42), any flow from an eigenvalue of the twisted Dirac operator $\mathfrak{D}_{A_\Sigma}$ to an eigenvalue of the untwisted Dirac operator $\mathcal{D}_\Sigma$ can have a magnitude of maximum 4. Moreover, from the figure 6.4, it is evident that the only possible non-zero spectral flow would be a flow from the eigenvalue $1/2$ of $\mathfrak{D}_{A_\Sigma}$ to the eigenvalue $-7/2$ of $\mathcal{D}_\Sigma$, since any other flow of eigenvalues of $\mathfrak{D}_{A_\Sigma}$ to eigenvalues of $\mathcal{D}_\Sigma$ of opposite sign has magnitude greater than 4. Now, we recall that $1/2$ is an eigenvalue of $\mathfrak{D}_{A_\Sigma}$ that corresponds to the trivial representation $V_{(0,0,0)}$ of $Sp(2) \times Sp(1)$. Hence, in the decomposition (4.8), the eigenspinor η corresponding to eigenvalue $1/2$ belongs to the space $\operatorname{Hom}(V_{(0,0,0)}, \Delta \otimes \mathfrak{sp}(1)_{\mathbb{C}})^{\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d} \otimes V_{(0,0,0)} \subset L^2(Sp(2) \times Sp(1), \Delta \otimes \mathfrak{sp}(1)_{\mathbb{C}})^{\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d}$. Then, from the decomposition

$$\Delta \otimes \mathfrak{sp}(1)_{\mathbb{C}} \cong W_{(0,0)} \oplus 2W_{(0,2)} \oplus W_{(1,1)} \oplus W_{(1,3)} \oplus W_{(0,4)},$$

and by Schur's lemma, we have that $\eta \in \operatorname{Hom}(V_{(0,0,0)}, W_{(0,0)})^{\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d} \otimes V_{(0,0,0)}$ which is a subspace of $L^2(Sp(2) \times Sp(1), \Delta \otimes \mathfrak{sp}(1)_{\mathbb{C}})^{\mathfrak{sp}(1)_u \oplus \mathfrak{sp}(1)_d}$. Hence, we compute the eigenvalue of $\widehat{e}^a T_a$ corresponding to the trivial subrepresentation $W_{(0,0)}$ of $\Delta \otimes \mathfrak{sp}(1)_{\mathbb{C}}$.

A direct calculation shows that the eigenvalue is -4. Thus we have a flow of the eigenvalue moving up to $9/2$ and not down to $-7/2$. Thus, the spectral flow of the family $\{\mathfrak{D}_{A_{t,\Sigma}}\}_{t \in \mathbb{R}}$ is given by

$$\operatorname{sf}\left(\{\mathfrak{D}_{A_{t,\Sigma}}\}_{t \in \mathbb{R}}\right) = 0. \tag{6.48}$$

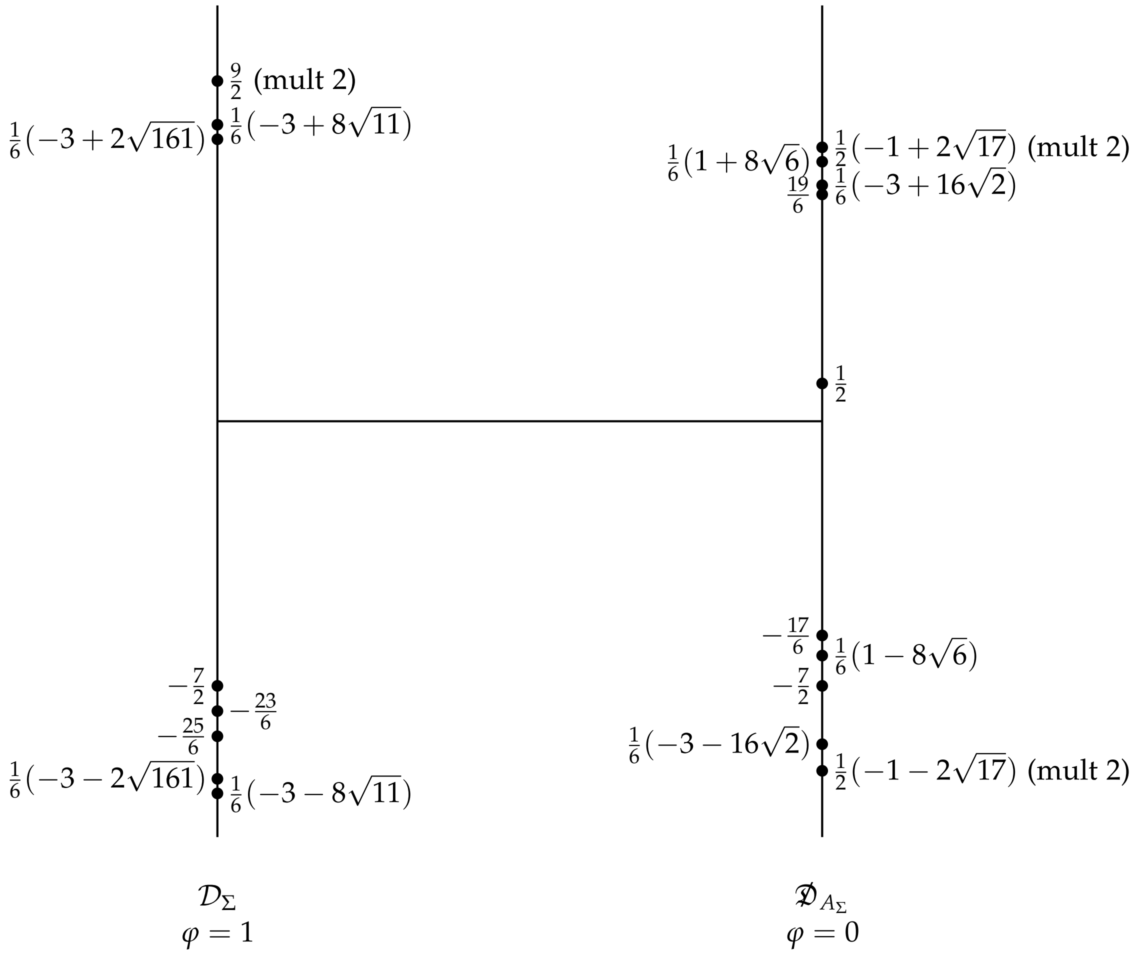


Figure 6.4: Spectral flow of the family $\{\not{\mathcal{D}}_{A_{t,\Sigma}}\}_{t\in\mathbb{R}}$ on Σ^7.

Spectral Flow of the Metric

Consider the Lichnerowicz–Weitzenböck formula for the family of Riemannian manifolds (Σ, g_t), given by

$$(\mathcal{D}_{\Sigma,t})^2 = \nabla^*_{\Sigma,t}\nabla_{\Sigma,t} + \frac{1}{4}s_t, \tag{6.49}$$

where s_t is the scalar curvature of the Riemannian manifold (Σ, g_t). Then, for the family of Dirac operators to have a nonzero spectral flow, the Dirac operator $\mathcal{D}_{\Sigma,t}$ should have a zero eigenvalue for some t, and it is only possible if the corresponding scalar curvature s_t is zero.

Following [37], we consider a family of metrics on Σ, given by

$$g(t) := a(t)^2(\eta_1^2 + \eta_2^2) + b(t)^2\eta_3^2 + c(t)^2\pi^* g_{S^4} \tag{6.50}$$

where $t \in \mathbb{R}$, $\pi : S^7 \to S^4$ is a Riemannian submersion, η_1, η_2, η_3 are one forms on S^7. Then, for this family, $a = b = c = 1$ corresponds to the round metric, and $a = b = \frac{1}{\sqrt{5}}, c = 1$ corresponds to the squashed metric.

Lemma 6.4.2. *[37] The Ricci curvature* $\mathrm{Ric}(t)$ *of the family* $g(t)$ *is given by*

$$2\left(2 - \frac{b^2}{a^2} + \frac{2a^4}{c^4}\right)(\eta_1^2 + \eta_2^2) + 2\left(\frac{b^4}{a^4} + \frac{2b^4}{c^4}\right)\eta_3^2 + 2\left(6 - \frac{2a^2 + b^2}{c^2}\right)\pi^* g_{S^4}. \tag{6.51}$$

It is interesting to note that the Ricci flow for the family $g(t)$ is well defined, and the only two critical points correspond to the round and squashed metrics respectively [37].

We can easily calculate the scalar curvature of this family to be

$$\begin{aligned}
&4\left(2 - \frac{b^2}{a^2} + \frac{2a^4}{c^4}\right)\frac{1}{a^2} + 2\left(\frac{b^4}{a^4} + \frac{2b^4}{c^4}\right)\frac{1}{b^2} + 8\left(6 - \frac{2a^2 + b^2}{c^2}\right)\frac{1}{c^2} \\
&= \frac{8}{a^2} - \frac{4b^2}{a^4} + \frac{8a^2}{c^4} + \frac{2b^2}{a^4} + \frac{4b^2}{c^4} + \frac{48}{c^2} - \frac{16a^2}{c^4} - \frac{8b^2}{c^4} \\
&= \frac{1}{a^4}(8a^2 - 2b^2) + \frac{1}{c^4}(48c^2 - 8a^2 - 4b^2).
\end{aligned}$$

For $a = b = c = 1$, we have the scalar curvature of the round metric, given by 42; and for $a = b = \frac{1}{\sqrt{5}}, c = 1$, we have the scalar curvature of the squashed metric, given by $378/5$.

Now, we consider a simpler family of Riemannian metrics

$$\tilde{g}(t) := a(t)^2(\eta_1^2 + \eta_2^2 + \eta_3^2) + \pi^* g_{S^4} \tag{6.52}$$

where $a(-\infty) = 1$ and $a(\infty) = 1/5$. The corresponding family of scalar curvatures is given by $\frac{6}{a^2} + 48 - 12a^2$, which we note to be always nonzero positive for $a \in \left[\frac{1}{\sqrt{5}}, 1\right]$.

Then, since a spectral flow of metrics, from the round sphere to the squashed sphere, being a topological invariant, does not depend on the path, proves that

$$\mathrm{sf}\left(\{\mathcal{D}_{\Sigma,t}\}_{t\in\mathbb{R}}\right) = 0. \tag{6.53}$$

Index of the Twisted Dirac Operator

From (6.48), (6.53) and (6.47), we have,

$$\frac{1}{2}\eta(\mathfrak{D}_{A_\Sigma}, \Sigma^7) = -I(\mathfrak{D}_A^-, \mathcal{C}) - I(\mathcal{D}^-, C_\Sigma). \tag{6.54}$$

To calculate $I(\mathfrak{D}^-_{A,CI}, \$^-(S^4), g_{CI})$ in (6.38), we split $\$^-(S^4)$ in two parts, $\mathcal{C}$, and the complement B_Σ.

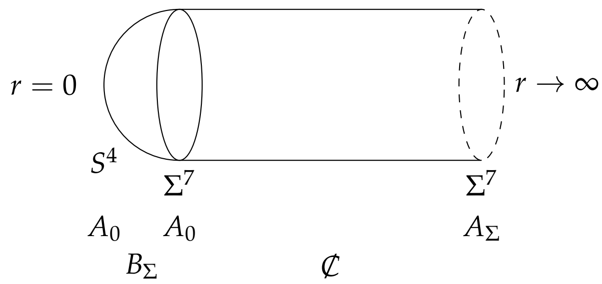


Figure 6.5: $\not{S}^-(S^4) = B_\Sigma \amalg \not{C}$.

Hence,

$$I(\not{\mathfrak{D}}^-_{\overline{A},CI}, \not{S}^-(S^4), g_{CI}) = I(\mathcal{D}^-, B_\Sigma) + I(\not{\mathfrak{D}}^-_{\overline{A}}, \not{C}). \tag{6.55}$$

Then, from (6.38), (6.55) and (6.54), we have,

$$\begin{aligned} \operatorname{Ind}(\not{\mathfrak{D}}^-_{\overline{A},CI}, \not{S}^-(S^4), g_{CI}) &= I(\mathcal{D}^-, B_\Sigma) - I(\mathcal{D}^-, C_\Sigma) \\ &= I(\mathcal{D}^-, B_\Sigma) + I(\mathcal{D}^-, \overline{C}_\Sigma) \\ &= I(\mathcal{D}^-, B_\Sigma \amalg \overline{C}_\Sigma) \\ &= I(\mathcal{D}^-, M_\Sigma), \end{aligned} \tag{6.56}$$

where $\overline{C}_\Sigma$ is C_Σ with opposite orientation, and the manifold M_Σ is diffeomorphic to $B_\Sigma \#_\partial \overline{C}_\Sigma$, where the boundary gluing is defined by $X \#_\partial Y := X \amalg Y / \partial X \sim \partial Y$.

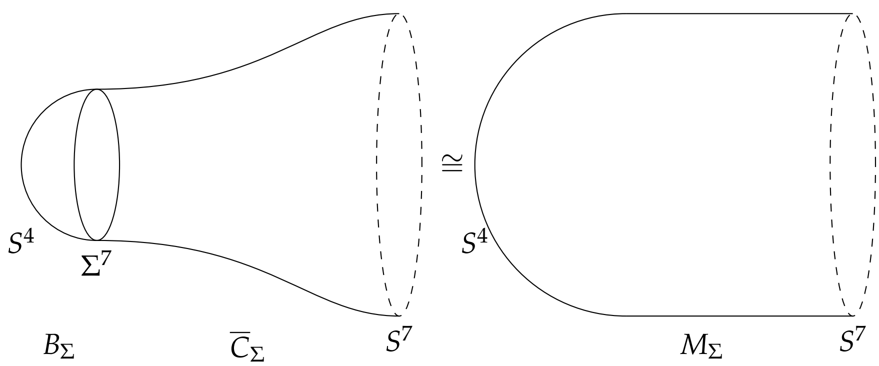


Figure 6.6: $B_\Sigma \#_\partial \overline{C}_\Sigma \cong M_\Sigma$.

Consider the 8-dimensional ball $D^8 := \{x \in \mathbb{R}^8 : |x| \leq R \text{ for some } R > 0\}$ equipped with the cigar metric g_{CI}. Then, we have the following lemma.

Lemma 6.4.3. *$M_\Sigma \#_\partial D^8$ is invariantly diffeomorphic to $\mathbb{H}P^2$.*

Proof. Recall that $\mathbb{H}P^2 \cong (\mathbb{H}^3 \setminus \{0\})/\sim$ where $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \sim \begin{pmatrix} x \\ y \\ z \end{pmatrix} \lambda$ for $\lambda \in \mathbb{H} \setminus \{0\}$. We note that there is a natural action of $Sp(2)$ on $\mathbb{H}P^2$, where, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(2)$, the matrix $\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$ acts on $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{H}P^2$. Since the orbits of the action of $Sp(2)$ on M_Σ are exactly S^4 and S^7, whereas, the orbits of the action of $Sp(2)$ on D^8 are exactly S^7 and $\{0\}$ respectively, it is enough to prove that the orbits of the action of $Sp(2)$ on $\mathbb{H}P^2$ are exactly $\{0\} \cong Sp(2)/Sp(2)$, $S^7 \cong Sp(2)/Sp(1)$ and $S^4 \cong Sp(2)/Sp(1)^2$. That is, the stabilizers of the action are $Sp(2), Sp(1), Sp(1)^2$ respectively.

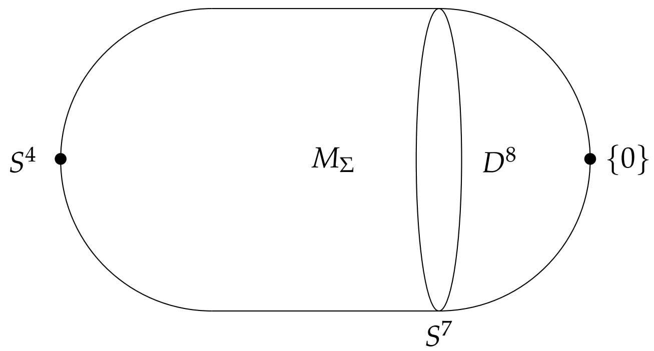


Figure 6.7: $M_\Sigma \#_\partial D^8 \cong \mathbb{H}P^2$.

Case 1. $z \neq 0$. Without loss of generality, we can take $z = 1$. Then, we have the following two subcases.

Subcase 1: $x = y = 0$. Then the stabilizer of $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{H}P^2$ is $Sp(2)$.

Subcase 2: x, y not both zero. Then, it is easy to calculate that the stabilizer of $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \in \mathbb{H}P^2$ is $Sp(1)$.

Case 2. $z = 0$. In this case, it is easy to calculate that the stabilizer of $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \in \mathbb{H}P^2$ for x, y not both zero, is $Sp(1) \times Sp(1)$.

$\square$

Let $a \in H^4(\mathbb{H}P^2, \mathbb{Z})$ be a generator such that $\int_{\mathbb{H}P^2} a^2 = 1$. Then, (see [48])

$$p_1(\mathbb{H}P^2) = 2a, \quad p_2(\mathbb{H}P^2) = 7a^2.$$

Hence,

$$\begin{aligned}
\operatorname{Ind}(\mathcal{D}^-, \mathbb{H}P^2) &= I(\mathcal{D}^-, \mathbb{H}P^2) \\
&= -\frac{1}{5760} \dim \mathfrak{g} \int_{\mathbb{H}P^2} (7p_1(\mathbb{H}P^2)^2 - 4p_2(\mathbb{H}P^2)) \\
&= -\frac{1}{5760} \dim \mathfrak{g} \int_{\mathbb{H}P^2} (7 \cdot 4a^2 - 4 \cdot 7a^2) \\
&= 0,
\end{aligned}$$

where the formula for the index is obtained from 6.39 by substituting $p_1(\mathfrak{g}_P) = p_2(\mathfrak{g}_P) = 0$, for the trivial connection. Hence,

$$I(\mathcal{D}^-, \mathbb{H}P^2) = I(\mathcal{D}^-, M_\Sigma) + I(\mathcal{D}^-_\Sigma, D^8) = 0.$$

But, we know that $p_1(D^8) = p_2(D^8) = 0$, which implies $I(\mathcal{D}^-, D^8) = 0$. Hence,

$$I(\mathcal{D}^-, M_\Sigma) = 0. \tag{6.57}$$

From (6.56), (6.57) and (6.33), we have

$$\operatorname{Ind}_{-\frac{5}{2}}(\not{\mathfrak{D}}^-_A, \not{\mathfrak{S}}^-(S^4), g) = \operatorname{Ind}(\not{\mathfrak{D}}^-_{A,CI}, \not{\mathfrak{S}}^-(S^4), g_{CI}) = 0. \tag{6.58}$$

6.4.3 Virtual Dimension of the Moduli Space

The main result on the deformations of Clarke–Oliveira's Instanton is given by the following theorem.

Theorem 6.4.4. *The virtual dimension of the moduli space $\mathcal{M}(A_\Sigma, \nu)$ of Clarke–Oliveira's instanton with decay rate $\nu \in (-2, 0)$ is given by*

$$\text{virtual-dim}\, \mathcal{M}(A_\Sigma, \nu) = 1. \tag{6.59}$$

Proof. The index of the Dirac operator $\not{D}_A^-$ corresponding to the rate $-5/2$ is zero, which follows from (6.58). Moreover, from Corollary 6.3.16, it follows that the only critical rate between $-5/2$ and 0 is -2, corresponding to the eigenvalue $1/2$. Then, the result follows from the fact that the eigenspace of the eigenvalue $1/2$ is 1-dimensional using Theorem 3.1.11. $\square$

Remark 6.4.5. We note that the deformation of the instanton described in theorem 6.4.4 comes from the parameter y_0 in the expression of $\varphi(r)$ (6.24) in Clarke–Oliveira's instanton (6.14).

Chapter 7

Obstructedness of AC $U(1)$ and $SU(2)$ Instantons on $\mathbb{R}^8$

In this chapter we investigate the existence of $U(1)$ and $SU(2)$ asymptotically conical $Sp(2) \times U(1)$-invariant instantons on $\mathbb{R}^8$. We show that with gauge groups $U(1)$ and $SU(2)$, no such invariant instantons exist. However, this result enables us to prove that any asymptotically conical $U(1)$- or $SU(2)$-instantons on $\mathbb{R}^8$ asymptotic to the flat connection on S^7 satisfying certain condition are obstructed.

$SU(2)$-instantons on $\mathbb{R}^8$ are already studied by Lewis [43]. He showed that there does not exist $SU(2)$-instanton on $\mathbb{R}^8$ which has finite energy. However, there is no a priori reason for any asymptotically conical $SU(2)$-instanton converging to a flat connection on S^7 to have finite energy.

7.1 $Sp(2) \times U(1)$-invariant metrics on $\mathbb{R}^8$

As a homogeneous space, the 7-sphere can be written as $\frac{Sp(2)\times Sp(1)}{Sp(1)\times Sp(1)}$. We note that the round metric on S^7 is not $Sp(2) \times Sp(1)$-invariant, which follows from the fact that $Sp(2) \times Sp(1)$ is not a subgroup of $Spin(7)$. It is $Sp(2)$-invariant, and we choose the maximal subgroup $Sp(2) \times U(1)$ of $Spin(7)$ containing $Sp(2)$ for which the round metric is invariant. Hence, we write the 7-sphere as the homogeneous space $\frac{Sp(2)\times U(1)}{Sp(1)\times U(1)}$. Recall the groups

$$Sp(1) := \left\{a \in \mathbb{H} : aa^\dagger = 1\right\}, \quad Sp(2) := \left\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{H}, AA^\dagger = I\right\}$$

and corresponding Lie algebras

$$\mathfrak{sp}(1) := \left\{x \in \mathbb{H} : x + x^\dagger = 0\right\}, \quad \mathfrak{sp}(2) := \left\{A = \begin{pmatrix} x & y \\ z & w \end{pmatrix} : x, y, z, w \in \mathbb{H}, A + A^\dagger = 0\right\}.$$

Now consider the embedding of $Sp(1)$ and $U(1)$ in $Sp(2) \times U(1)$ as

$$Sp(1) := \left\{\left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, 1\right) : g \in Sp(1)\right\}, \quad U(1) := \left\{\left(\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}, h\right) : h \in U(1)\right\}.$$

The corresponding Lie algebras are given by

$$\mathfrak{sp}(1) := \left\{\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, 0\right) : x \in \mathfrak{sp}(1)\right\}, \quad \mathfrak{u}(1) := \left\{\left(\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}, y\right) : y \in \mathfrak{u}(1)\right\}.$$

Then,

$$\mathfrak{sp}(1) \oplus \mathfrak{u}(1) = \left\{\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, y\right) : x \in \mathfrak{sp}(1), y \in \mathfrak{u}(1)\right\}.$$

We have a decomposition of the Lie algebra $\mathfrak{sp}(2) \oplus \mathfrak{u}(1)$ as

$$\mathfrak{sp}(2) \oplus \mathfrak{u}(1) = \mathfrak{sp}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{m}.$$

We want to find $\mathfrak{m} = (\mathfrak{sp}(1) \oplus \mathfrak{u}(1))^\perp$. But, since $\mathfrak{sp}(2) \oplus \mathfrak{u}(1)$ is not semi-simple, its Killing form is degenerate, so instead we use the Killing form of $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ and the projection $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1) \to \mathfrak{sp}(2) \oplus \mathfrak{u}(1)$ to choose

$$\mathfrak{m} = \left\{\left(\begin{pmatrix} 0 & b \\ -b^\dagger & 2(z_1 i + z_2 j + z_3 k) \end{pmatrix}, -3z_1 i\right) : b \in \mathbb{H}, z_1 i + z_2 j + z_3 k \in \mathfrak{sp}(1)\right\} \cong \mathfrak{sp}(1) \oplus \mathbb{H}. \tag{7.1}$$

Now, since $\mathfrak{m}$ is a representation of $Sp(1) \times U(1)$, we want to decompose $\mathfrak{m}$ into irreducible representations of $Sp(1) \times U(1)$.

Let V_i be the unique irreducible representation of $SU(2) \cong Sp(1)$ of dimension $(i+1)$. Then,

$V_0 \equiv$ Trivial representation ($\dim V_0 = 1$),
$V_1 \equiv$ Standard representation ($\dim V_1 = 2$),
$V_2 \equiv$ Adjoint representation ($\dim V_2 = 3$).

Let $\left(\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}, h\right) \in Sp(1) \times U(1)$ for $g \in Sp(1)$, $h \in U(1)$. It acts on an element of $\mathfrak{m}$ as

$$\left(\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}, h\right)\left(\begin{pmatrix} 0 & b \\ -b^\dagger & 2(z_1 i + z_2 j + z_3 k) \end{pmatrix}, -3z_1 i\right)\left(\begin{pmatrix} g^{-1} & 0 \\ 0 & h^{-1} \end{pmatrix}, h^{-1}\right)$$

$$= \left(\begin{pmatrix} 0 & gbh^{-1} \\ -hb^{\dagger}g^{-1} & 2h(z_1 i + z_2 j + z_3 k)h^{-1} \end{pmatrix}, -3hz_1 ih^{-1} \right).$$

This shows that $\mathfrak{m}$ contains three copies of the 1-dimensional irreducible representation and two copies of the 2-dimensional irreducible representation of $Sp(1)\times U(1)$. To explicitly write $\mathfrak{m}$ as a direct sum of irreducible representations of $Sp(1)\times U(1)$, we first need to calculate the weights of the irreducible representations of $U(1)$ appearing in the above expression. Let $b = b_0 + b_1 i + b_2 j + b_3 k$, $h = e^{i\theta} = \cos\theta + i\sin\theta$, $h^{-1} = e^{-i\theta} = \cos\theta - i\sin\theta$. Then, we consider the action

$$\begin{aligned} h\cdot b &= bh^{-1} \\ &= (b_0\cos\theta + b_1\sin\theta) + (b_1\cos\theta - b_0\sin\theta)i + (b_2\cos\theta - b_3\sin\theta)j + (b_3\cos\theta + b_2\sin\theta)k. \end{aligned}$$

Hence the matrix of the action is given by

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

whose eigenvalues are given by $e^{i\theta}, e^{-i\theta}, e^{i\theta}, e^{-i\theta}$.

Now, consider the action

$$\begin{aligned} & h(z_1 i + z_2 j + z_3 k)h^{-1} \\ &= (\cos\theta + i\sin\theta)(z_1 i + z_2 j + z_3 k)(\cos\theta - i\sin\theta) \\ &= z_1 i + (z_2\cos^2\theta - z_2\sin^2\theta - 2z_3\sin\theta\cos\theta)j + (z_3\cos^2\theta - z_3\sin^2\theta + 2z_2\sin\theta\cos\theta)k. \end{aligned}$$

Hence the matrix of the action is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & \sin 2\theta \\ 0 & -\sin 2\theta & \cos 2\theta, \end{pmatrix}$$

whose eigenvalues are given by $1, e^{2i\theta}, e^{-2i\theta}$.

Let W_j be the irreducible representation of $U(1)$ of weight j. Then, we have the expression of $\mathfrak{m}$ as

$$\mathfrak{m} \cong V_1\otimes W_1 \oplus V_1\otimes W_{-1} \oplus V_0\otimes W_0 \oplus V_0\otimes W_2 \oplus V_0\otimes W_{-2}.$$

Let us define $W_{(i,j)} := V_i \otimes W_j$, the irreducible representations of $Sp(1) \times U(1)$. Clearly, $\dim W_{(i,j)} = i+1$. Then

$$\mathfrak{m} = W_{(1,1)} \oplus W_{(1,-1)} \oplus W_{(0,0)} \oplus W_{(0,2)} \oplus W_{(0,-2)}. \tag{7.2}$$

Now, we want to find a basis for $\mathfrak{m}$. We note that $\mathfrak{m} \cong T_p\Sigma \cong V_p \oplus H_p \cong \operatorname{Im}\mathbb{H} \oplus \mathbb{H}$, where V_p is the vertical space and H_p is the horizontal space with dimensions 3 and 4 respectively. Now,

$$\operatorname{Im}\mathbb{H} = \left\{ \left(\begin{pmatrix} 0 & 0 \\ 0 & 2(z_1 i + z_2 j + z_3 k) \end{pmatrix}, -3z_1 i \right) : z_1 i + z_2 j + z_3 k \in \mathfrak{sp}(1), z_1 i \in \mathfrak{u}(1) \right\}.$$

So, we choose a basis

$$\overline{I}_1 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix}, -3i \right),\ \overline{I}_2 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 2j \end{pmatrix}, 0 \right),\ \overline{I}_3 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 2k \end{pmatrix}, 0 \right).$$

Now,

$$\mathbb{H} = \left\{ \left(\begin{pmatrix} 0 & b \\ -b^\dagger & 0 \end{pmatrix}, 0 \right) : b \in \mathbb{H} \right\}.$$

So, we choose a basis

$$\overline{I}_4 = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0 \right),\ \overline{I}_5 = \left(\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, 0 \right),\ \overline{I}_6 = \left(\begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix}, 0 \right),\ \overline{I}_7 = \left(\begin{pmatrix} 0 & -k \\ -k & 0 \end{pmatrix}, 0 \right).$$

Denote the dual basis of $\overline{I}_a$ by $\overline{e}^a$ for $a = 1, \dots, 7$.

Then $\overline{I}_1, \dots, \overline{I}_7$ together with

$$\overline{I}_8 = \left(\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, 0 \right),\ \overline{I}_9 = \left(\begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, 0 \right),\ \overline{I}_{10} = \left(\begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, 0 \right),\ \text{and}\ \overline{I}_{11} = \left(\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}, i \right).$$

form a basis of $\mathfrak{sp}(2) \times \mathfrak{u}(1)$. Our objective is to calculate the $Sp(2) \times U(1)$-invariant metrics g, three-form ϕ and $\psi = *\phi$ on S^7, i.e., $Sp(1) \times U(1)$-invariant metric g, three-form ϕ and $\psi = *\phi$ on $\mathfrak{m}$.

Expressions of ϕ, ψ and the metric g

We consider an ansatz for ϕ given by

$$\phi = \alpha \overline{e}^{123} - \beta(\overline{e}^1 \wedge \omega_1) - \gamma(\overline{e}^2 \wedge \omega_2 + \overline{e}^3 \wedge \omega_3)$$

where $\omega_1, \omega_2, \omega_3$ forms a basis for $\Lambda^2_+\mathbb{H}^*$. Explicitly, we take $\omega_1 = \overline{e}^{45} + \overline{e}^{67}, \omega_2 = \overline{e}^{46} - \overline{e}^{57}, \omega_3 = \overline{e}^{47} + \overline{e}^{56}$.

It is a routine matter to check ϕ is $Sp(1) \times U(1)$-invariant. To find the constants α, β, γ, we use the nearly G_2 condition $d\phi = 4\psi$.

For $\bar{e}^a$, we have the Maurer–Cartan equations

$$d\bar{e}^a = -f^a_{ib}\bar{e}^i \wedge \bar{e}^b - \frac{1}{2}f^a_{bc}\bar{e}^b \wedge \bar{e}^c.$$

Then, calculating the structure constants explicitly, we calculate,

$$\begin{aligned}
d\phi &= \alpha d\bar{e}^{123} - \beta d(\bar{e}^1 \wedge \omega_1) - \gamma d(\bar{e}^2 \wedge \omega_2 + \bar{e}^3 \wedge \omega_3) \\
&= \alpha\left(-\bar{e}^{1247} - \bar{e}^{1256} + \bar{e}^{1346} - \bar{e}^{1357} - \frac{2}{5}\bar{e}^{2345} - \frac{2}{5}\bar{e}^{2367}\right) \\
&\quad - \beta\left(-\frac{8}{5}\bar{e}^{2345} - \frac{4}{5}\bar{e}^{4567} - \frac{8}{5}\bar{e}^{2367} + 4\bar{e}^{1247} + 4\bar{e}^{1256} - 4\bar{e}^{1346} + 4\bar{e}^{1357}\right) \\
&\quad - \gamma\left(8\bar{e}^{2345} + 8\bar{e}^{2367} - 4\bar{e}^{4567}\right).
\end{aligned}$$

Consider the transformation,

$$\tilde{e}^i = \begin{cases} a\bar{e}^1 & i = 1 \\ b\bar{e}^i & i = 2, 3 \\ c\bar{e}^i & i = 4, 5, 6, 7. \end{cases}$$

We choose $\alpha = ab^2, \beta = ac^2, \gamma = bc^2$. Then

$$\begin{aligned}
\phi &= \alpha\bar{e}^{123} - \beta(\bar{e}^{145} + \bar{e}^{167}) - \gamma(\bar{e}^{246} - \bar{e}^{257} + \bar{e}^{347} + \bar{e}^{356}) \\
&= ab^2\bar{e}^{123} - ac^2(\bar{e}^{145} + \bar{e}^{167}) - bc^2(\bar{e}^{246} - \bar{e}^{257} + \bar{e}^{347} + \bar{e}^{356}) \\
&= \tilde{e}^{123} - \tilde{e}^{145} - \tilde{e}^{167} - \tilde{e}^{246} + \tilde{e}^{257} - \tilde{e}^{347} - \tilde{e}^{356}.
\end{aligned}$$

Corresponding ψ is given by

$$\begin{aligned}
\psi &= \tilde{e}^{4567} - \tilde{e}^{1247} - \tilde{e}^{1256} + \tilde{e}^{1346} - \tilde{e}^{1357} - \tilde{e}^{2345} - \tilde{e}^{2367} \\
&= c^4\bar{e}^{4567} - abc^2(\bar{e}^{1247} + \bar{e}^{1256} - \bar{e}^{1346} + \bar{e}^{1357}) - b^2c^2(\bar{e}^{2345} + \bar{e}^{2367}).
\end{aligned}$$

Now,

$$\begin{aligned}
d\phi &= \left(\frac{4}{5}\beta + 4\gamma\right)e^{4567} + (-\alpha - 4\beta)(\bar{e}^{1247} + \bar{e}^{1256} - \bar{e}^{1346} + \bar{e}^{1357}) + \left(-\frac{2}{5}\alpha + \frac{8}{5}\beta - 8\gamma\right)(\bar{e}^{2345} + \bar{e}^{2367}) \\
&= \left(\frac{4}{5}ac^2 + 4bc^2\right)\bar{e}^{4567} + (-ab^2 - 4ac^2)(\bar{e}^{1247} + \bar{e}^{1256} - \bar{e}^{1346} + \bar{e}^{1357}) \\
&\quad + \left(-\frac{2}{5}ab^2 + \frac{8}{5}ac^2 - 8bc^2\right)(\bar{e}^{2345} + \bar{e}^{2367}).
\end{aligned}$$

Then, $d\phi = 4\psi$ implies

$$\begin{aligned}\frac{4}{5}ac^2 + 4bc^2 &= 4c^4 \\ -ab^2 - 4ac^2 &= -4abc^2 \\ -\frac{2}{5}ab^2 + \frac{8}{5}ac^2 - 8bc^2 &= -4b^2c^2.\end{aligned}$$

Solving, we get

$$a = -5, \quad b = 2, \quad c = \pm 1,$$

and

$$a = 3, \quad b = \frac{6}{5}, \quad c = \pm\frac{3}{\sqrt{5}}.$$

Then the unique metric g and the volume form compatible with ϕ are given by

$$g = \sum_{i=1}^{3} \tilde{e}^i \otimes \tilde{e}^i + \sum_{j=4}^{7} \tilde{e}^j \otimes \tilde{e}^j,$$

and dvol $= \tilde{e}^{1234567}$ respectively. That is,

$$g = a^2\bar{e}^1 \otimes \bar{e}^1 + b^2 \sum_{i=2}^{3} \bar{e}^i \otimes \bar{e}^i + c^2 \sum_{j=4}^{7} \bar{e}^j \otimes \bar{e}^j, \tag{7.3}$$

and dvol $= ab^2c^4\bar{e}^{1234567}$.

Remark 7.1.1. We note that $a = 3, \ \ b = \frac{6}{5}, \ \ c = \pm\frac{3}{\sqrt{5}}$ corresponds to the squashed metric. Let us consider the inclusion $\iota : \mathfrak{sp}(2) \oplus \mathfrak{u}(1) \hookrightarrow \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$. We claim that the pullback of the squashed metric (6.6) is the metric (7.3) for $a = 3, \ \ b = \frac{6}{5}, \ \ c = \pm\frac{3}{\sqrt{5}}$. For that, it is enough to compare the coefficients of the pullback metric with $a = 3, \ \ b = \frac{6}{5}, \ \ c = \pm\frac{3}{\sqrt{5}}$. Consider the orthonormal basis element $\frac{5}{6}\bar{I}_2$ of $\mathfrak{m} \subset \mathfrak{sp}(2) \oplus \mathfrak{u}(1)$. Now consider the basis element I_2 of $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ and its dual e^2, defined in chapter 6. Then, the pullback of the orthonormal basis element $3e^2$ of $\mathfrak{m} \subset \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ paired with $\frac{5}{6}\bar{I}_2$ gives

$$\iota^*(3e^2)\left(\frac{5}{6}\bar{I}_2\right) = \iota^*(3e^2)\frac{5}{6}\left(\begin{pmatrix} 0 & 0 \\ 0 & 2j \end{pmatrix}, 0\right) = 3e^2\frac{5}{6}\left(\frac{2}{5}(I_2 + 3I_{12})\right) = 1,$$

which shows that $\iota^*(3e^2)$ is equals the orthonormal basis element $\frac{6}{5}\bar{e}^2$ for the metric (7.3). Similar calculations for $e^i, i = 1, 3, 4, \ldots, 7$ establish the claim.

Now, using the metric (7.3) with $a = -5, b = 2, c = 1$, which is the round metric, we normalise our basis for $\mathfrak{m}$, and denote the normalised bases by I_i and e^i for $\mathfrak{m}$ and $\mathfrak{m}^*$ respectively, where I_i and e^i are dual to each other. Hence, an orthonormal basis for $\mathfrak{m}$ is given by

$$I_1 = -\frac{1}{5}\left(\begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix}, -3i\right),\ I_2 = \left(\begin{pmatrix} 0 & 0 \\ 0 & j \end{pmatrix}, 0\right),\ I_3 = \left(\begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}, 0\right),$$

$$I_4 = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 0\right),\ I_5 = \left(\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, 0\right),\ I_6 = \left(\begin{pmatrix} 0 & -j \\ -j & 0 \end{pmatrix}, 0\right),\ I_7 = \left(\begin{pmatrix} 0 & -k \\ -k & 0 \end{pmatrix}, 0\right).$$

7.2 $U(1)$ Instantons on $\mathbb{R}^8$

Consider the gauge group $U(1)$. We want to construct invariant connections on $U(1)$ bundles, and the bundles are determined by their isotropy homomorphism. Now, we have two isotropy homomorphisms from $Sp(1) \times U(1)$ to $U(1)$, namely

$$\begin{aligned} \lambda_0 : Sp(1) \times U(1) &\to U(1) \\ \left(\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}, g_2\right) &\mapsto 1, \\ \lambda_1 : Sp(1) \times U(1) &\to U(1) \\ \left(\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}, h\right) &\mapsto h. \end{aligned}$$

Consider the bundle $P_i = (Sp(2) \times U(1)) \times_{\lambda_i} U(1)$ over $S^7 := \frac{Sp(2)\times U(1)}{Sp(1)\times U(1)}$. Then the invariant connections on P_i correspond to the $Sp(1) \times U(1)$-equivariant homomorphisms

$$\Lambda_i : (\mathfrak{m}, \mathrm{ad}) \to (\mathfrak{u}(1), \mathrm{ad} \circ \lambda_i).$$

where $\mathfrak{m} = W_{(1,1)} \oplus W_{(1,-1)} \oplus W_{(0,0)} \oplus W_{(0,2)} \oplus W_{(0,-2)}$. Now,

$$\mathrm{ad} \circ \lambda_i : \mathfrak{sp}(1) \oplus \mathfrak{u}(1) \to \mathrm{End}(\mathfrak{u}(1)),$$

Then,

$$\mathrm{ad} \circ \lambda_0(X, Y)Z = \mathrm{ad}(0)Z = 0.$$

Hence, by Schur's lemma, the map $\Lambda_0|_{W_{(0,0)}}$ is given by

$$\varphi \cdot \mathrm{Id} : W_{(0,0)} \to W_{(0,0)}$$

for a real number φ, whereas $\Lambda_0|_{W^{\perp}_{(0,0)}}$ is trivial. Moreover,

$$\mathrm{ad} \circ \lambda_1(X,Y)Z = \mathrm{ad}(Y)Z = [Y,Z] = 0.$$

Hence, again, by Schur's lemma, the map $\Lambda_1|_{W_{(0,0)}}$ is given by

$$\varphi \cdot \mathrm{Id} : W_{(0,0)} \to W_{(0,0)}$$

for a real number φ, whereas $\Lambda_1|_{W^{\perp}_{(0,0)}}$ is trivial. We note that we can choose I_1 to a be a basis of $W_{(0,0)}$. Then, for $i = 0, 1$, we have,

$$\Lambda_i(I_a) = \begin{cases} \varphi, & a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

7.2.1 $U(1)$ Instantons corresponding to λ_0

In local coordinates, we can write any $Sp(2) \times Sp(1)$-invariant connection on the bundle P_0 over the manifold $S^7 = \frac{Sp(2)\times U(1)}{Sp(1)\times U(1)}$ with round metric and gauge group $U(1)$ can be written as,

$$A = \lambda_0(e^i I_i) + e^a \Lambda_0(I_a) = \varphi e^1.$$

Now consider the 8-dimensional manifold $\mathbb{R} \times S^7$. We choose the metric $g_8 = (e^0)^2 + g_7$ where $e^0 = dt$ and t be the coordinate of $\mathbb{R}$ and note that this metric is conformal to the flat metric on punctured $\mathbb{R}^8$. The connection 1-form is given by $A = A_0 e^0 + A_a e^a$ which gives the $Sp(1) \times U(1)$-invariant connection

$$A = \varphi(t) e^1.$$

Here, without loss of generality, we take $A_0 = 0$. The curvature of this connection is given by

$$F_A = F_{0a} e^0 \wedge e^a + \frac{1}{2} F_{bc} e^b \wedge e^c$$

where

$$F_{01} = \frac{\partial A_1}{\partial t} = \dot{\varphi}(t).$$

Then, the ASD instanton equation $F_A \lrcorner \Phi = -F_A$ reduces to

$$F_{01} = -\frac{1}{2}\phi_{1bc} F_{bc}.$$

Now,

$$F_{bc} = (dA)_{bc} = -\varphi(t) f^1_{bc}.$$

Then,

$$\dot{\varphi}(t) = \frac{1}{2}\varphi(t)\phi_{1bc}f^1_{bc} = 2\varphi(t).$$

The solution is

$$\varphi(t) = Ce^{2t},$$

which shows the non-existence of irreducible $Sp(2) \times U(1)$-invariant asymptotically conical $U(1)$-instanton on $\mathbb{R}^8$ corresponding to λ_0.

7.2.2 $U(1)$ Instantons corresponding to λ_1

In local coordinates, we can write any $Sp(2) \times Sp(1)$-invariant connection on the bundle P_1 over S^7 with round metric and gauge group $U(1)$ as,

$$A = \lambda(e^i I_i) + e^a \Lambda(I_a) = e^{11} + \varphi e^1.$$

Similar to the previous case, on the manifold $\mathbb{R} \times S^7$, the $Sp(1) \times U(1)$-invariant connection is given by

$$A = e^{11} + \varphi(t)e^1.$$

The curvature of this connection is given by

$$F_A = F_{0a}e^0 \wedge e^a + \frac{1}{2}F_{bc}e^b \wedge e^c,$$

where

$$F_{01} = \frac{\partial A_1}{\partial t} = \dot{\varphi}(t).$$

Now,

$$F_{bc} = (dA)_{bc} = -f^{11}_{bc} - \varphi(t)f^1_{bc}.$$

Then, from the ASD equation $F_{01} = -\frac{1}{2}\phi_{1bc}F_{bc}$, we have

$$\dot{\varphi}(t) = \frac{1}{2}\phi_{1bc}f^{11}_{bc} + \frac{1}{2}\varphi(t)\phi_{1bc}f^1_{bc} = -\frac{6}{5} + 2\varphi(t).$$

The solution is

$$\varphi(t) = Ce^{2t} + \frac{3}{5},$$

which shows the non-existence of $Sp(2) \times U(1)$-invariant asymptotically conical $U(1)$-instanton on $\mathbb{R}^8$ corresponding to λ_1.

7.3 $SU(2)$ Instantons on $\mathbb{R}^8$

Consider the gauge group $Sp(1) \cong SU(2)$. We want to construct invariant connections on $SU(2)$-bundles. These bundles are determined by their isotropy homomorphisms. There are three isotropy homomorphisms from $Sp(1) \times U(1)$ to $Sp(1)$, namely

$$\begin{aligned}
&\lambda_0 : Sp(1) \times U(1) \to Sp(1)\\
&\left(\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}, g_2\right) \mapsto 1,\\
&\lambda_1 : Sp(1) \times U(1) \to Sp(1)\\
&\left(\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}, g_2\right) \mapsto g_1,\\
&\lambda_2 : Sp(1) \times U(1) \to Sp(1)\\
&\left(\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}, g_2\right) \mapsto g_2 \text{ as a subgroup embedding.}
\end{aligned}$$

Consider the bundle $P_i = (Sp(2) \times U(1)) \times_{\lambda_i} Sp(1)$ over $\Sigma := \frac{Sp(2)\times U(1)}{Sp(1)\times U(1)}$. Then from Wang's theorem, we know that the $Sp(2) \times U(1)$-invariant connections on P_i correspond to the $Sp(1) \times U(1)$-equivariant homomorphisms

$$\Lambda_i : (\mathfrak{m}, \mathrm{ad}) \to (\mathfrak{sp}(1), \mathrm{ad} \circ \lambda_i).$$

Now,

$$\mathrm{ad} \circ \lambda_i : \mathfrak{sp}(1) \oplus \mathfrak{u}(1) \to \mathrm{End}(\mathfrak{sp}(1)),$$

Then,

$$\mathrm{ad} \circ \lambda_0(X, Y)Z = \mathrm{ad}(0)Z = 0.$$

Hence, by Schur's lemma, the map $\Lambda_0|_{W_{(0,0)}}$ given by

$$W_{(0,0)} \to W_{(0,0)}$$

is an isomorphism, whereas $\Lambda_0|_{W_{(0,0)}^{\perp}}$ is trivial. Moreover,

$$\mathrm{ad} \circ \lambda_1(X, Y)Z = \mathrm{ad}(X)Z = [X, Z].$$

Hence, the map Λ_1 given by

$$W_{(1,1)} \oplus W_{(1,-1)} \oplus W_{(0,0)} \oplus W_{(0,2)} \oplus W_{(0,-2)} \to W_{(2,0)}$$

is trivial. Finally,

$$\mathrm{ad} \circ \lambda_2(X,Y)Z = \mathrm{ad}(Y)Z = [Y,Z].$$

Hence, the map Λ_2 restricted to $W_{(0,0)} \oplus W_{(0,2)} \oplus W_{(0,-2)}$ is the isomorphism

$$W_{(0,0)} \oplus W_{(0,2)} \oplus W_{(0,-2)} \to W_{(0,0)} \oplus W_{(0,2)} \oplus W_{(0,-2)}$$

whereas Λ_2 restricted to $W_{(1,1)} \oplus W_{(1,-1)}$ is trivial, by Schur's lemma.

Let us fix a basis $T_a,\ a = 1,2,3$ for $Sp(1) \cong SU(2)$, where $T_a = -i\sigma_a$ and $\sigma_a,\ a = 1,2,3$ are the Pauli matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

7.3.1 $SU(2)$ Instantons corresponding to λ_0

In local coordinates, any $Sp(2) \times Sp(1)$-invariant connection on the bundle P_0 over S^7 with round metric and gauge group $SU(2)$ can be written as,

$$A = \lambda_0(e^i I_i) + e^a \Lambda_0(I_a) = \varphi e^1 T_1.$$

Similar to the previous case, on the manifold $\mathbb{R} \times S^7$, the $Sp(1) \times U(1)$-invariant connection is given by

$$A = e^{11} + \varphi(t) e^1 T_1$$

The curvature of this connection is given by

$$F_A = F_{0a} e^0 \wedge e^a + \frac{1}{2} F_{bc} e^b \wedge e^c.$$

where

$$F_{01} = \frac{\partial A_1}{\partial t} = \dot{\varphi}(t) T_1.$$

Now,

$$F_{bc} = (dA)_{bc} = -\varphi(t) f^1_{bc} T_1.$$

Then, the instanton equation $F_{01} = -\frac{1}{2}\phi_{1bc}F_{bc}$ reduces to

$$\dot{\varphi}(t) = \frac{1}{2}\varphi(t)\phi_{1bc} f^1_{bc} = 2\varphi(t).$$

The solution is

$$\varphi(t) = Ce^{2t},$$

which shows the non-existence of irreducible $Sp(2) \times U(1)$-invariant asymptotically conical $SU(2)$-instanton on $\mathbb{R}^8$ corresponding to λ_0.

7.3.2 $SU(2)$ Instantons corresponding to λ_2

As $\dim \mathrm{Hom}(\mathfrak{m}, \mathfrak{sp}(1))^{Sp(1)\times U(1)} = 3$, we choose a basis τ_i, $i = 1, 2, 3$. Then, $\tau_i : \mathfrak{m} \to \mathfrak{sp}(1)$ acts on the basis $I_a, a = 1, \ldots, 7$ of $\mathfrak{m}$ by $\tau_i(I_b) = 0$ for $b = 4, \ldots, 7$ and $\tau_i(I_a) = \pm T_j$ where the sign and index j of $\pm T_j$ is determined by the $U(1)$-invariance. Then, we choose,

$$\begin{aligned}
\tau_1 &: (I_1, I_2, I_3) \mapsto (T_1, T_2, T_3),\\
\tau_2 &: (I_1, I_2, I_3) \mapsto (T_1, -T_3, T_2),\\
\tau_3 &: (I_1, I_2, I_3) \mapsto (T_1, -T_2, -T_3).
\end{aligned}$$

We can write,

$$\Lambda_2 = \varphi_1\tau_1 + \varphi_2\tau_2 + \varphi_3\tau_3.$$

Then, $\Lambda_2(I_a) = \varphi_1\tau_1(I_a) + \varphi_2\tau_2(I_a) + \varphi_3\tau_3(I_a) = \varphi_1\tau_{a1c}T_c + \varphi_2\tau_{a2c}T_c + \varphi_3\tau_{a3c}T_c$, i.e.,

$$\Lambda_2(I_a) = \varphi_b\tau_{abc}T_c.$$

Explicitly, τ_{abc} are given by

$$\tau_{111} = 1,\ \tau_{212} = 1,\ \tau_{313} = 1,\ \tau_{223} = -1,\ \tau_{322} = 1,\ \tau_{131} = 1.$$

Now, in local coordinates any connection on the bundle P_2 over the nearly G_2-manifold S^7 with round metric can be written as

$$\begin{aligned}
A &= e^i\lambda_2(I_i) + e^a\Lambda_2(I_a)\\
&= e^{11}T_1 + e^a\varphi_b\tau_{abc}T_c.
\end{aligned}$$

Then, on the 8-dimensional manifold $\mathbb{R}\times S^7$, the connection 1-form is given by $A = A_0e^0 + A_ae^a$ which gives the $Sp(1)\times U(1)$-invariant connection

$$A = e^{11}T_1 + e^a\varphi_b(t)\tau_{abc}T_c.$$

Here, we take $A_0 = 0$. The curvature of this connection is given by

$$F_A = F_{0a}e^0\wedge e^a + \frac{1}{2}F_{bc}e^b\wedge e^c$$

where

$$F_{0a} = \frac{\partial A_a}{\partial t} = \dot{\varphi}_b(t)\tau_{abc}T_c.$$

Applying the Maurer–Cartan equations (5.2) we have,

$$(dA)_{bc} = -f_{bc}^{11}T_1 - f_{bc}^{a}\varphi_p(t)\tau_{apq}T_q,$$

and

$$[A \wedge A]_{bc} = 4\varphi_p(t)\varphi_s(t)\tau_{bpa}\tau_{csr}\epsilon_{arq}T_q.$$

Hence,

$$F_{bc} = -f_{bc}^{11}T_1 + \left(2\varphi_p(t)\varphi_s(t)\tau_{bpa}\tau_{csr}\epsilon_{arq} - \varphi_p(t)f_{bc}^{a}\tau_{apq}\right)T_q.$$

Thus, the ASD instanton equation $F_{0a} = -\frac{1}{2}\phi_{abc}F_{bc}$ reduces to

$$\dot{\varphi}_b(t)\tau_{abc}T_c = \frac{1}{2}\phi_{abc}f_{bc}^{11}T_1 - \frac{1}{2}\phi_{abc}\left(2\varphi_p(t)\varphi_s(t)\tau_{bpd}\tau_{csr}\epsilon_{drq} - \varphi_p(t)f_{bc}^{d}\tau_{dpq}\right)T_q.$$

That is,

$$2\dot{\varphi}_b(t)\tau_{abc}T_c = \phi_{abc}f_{bc}^{11}T_1 - 2\phi_{abc}\varphi_p(t)\varphi_s(t)\tau_{bpd}\tau_{csr}\epsilon_{drq}T_q + \phi_{abc}\varphi_p(t)f_{bc}^{d}\tau_{dpq}T_q. \tag{7.4}$$

Simplifying, we have the system of ODE given by

$$\dot{\varphi}_1 = -\varphi_1\left(2\varphi_1 + 2\varphi_3 + \frac{24}{5}\right) \tag{7.5}$$

$$\dot{\varphi}_2 = -\varphi_2\left(2\varphi_1 + 2\varphi_3 + \frac{24}{5}\right) \tag{7.6}$$

$$\dot{\varphi}_1 + \dot{\varphi}_3 = -\frac{6}{5} - 2\varphi_1^2 - 2\varphi_2^2 + 2\varphi_1 + 2\varphi_3. \tag{7.7}$$

Now, we have the following four cases.

Case 1. $\varphi_1 = \varphi_2 = 0, \varphi_3 \neq 0$.
Then, the ODEs reduce to

$$\dot{\varphi}_3 = -\frac{6}{5} + 2\varphi_3. \tag{7.8}$$

The solution is $\varphi_3 = Ce^{2t} + \frac{3}{5}$.

Case 2. $\varphi_1 = 0, \varphi_2 \neq 0, \varphi_3 \neq 0$.
Then, the ODEs reduce to

$$\dot{\varphi}_2 = -\varphi_2\left(2\varphi_3 + \frac{24}{5}\right) \tag{7.9}$$

$$\dot{\varphi}_3 = -\frac{6}{5} - 2\varphi_2^2 + 2\varphi_3. \tag{7.10}$$

Substituting $a := \varphi_2(t), b := \varphi_3(t) - \frac{3}{5}$, we have,

$$\begin{aligned}\dot{a} &= -2a(b+3)\\ \dot{b} &= -2a^2 + 2b.\end{aligned}$$

The critical point is given by $(a,b) = (0,0)$. Near the critical point, the the linearised system is given by

$$\begin{aligned}\frac{da}{dt} &= -6a\\ \frac{db}{dt} &= 2b.\end{aligned}$$

The solutions are

$$x = C_1 e^{-6t},\ y = C_2 e^{2t},$$

which shows that the critical point is a saddle point. Hence, near the critical point $(0,0,3/5)$, the solution is given by

$$\varphi_1(t) = 0,\ \varphi_2(t) = C_1 e^{-6t},\ \varphi_3(t) = C_2 e^{2t} + \frac{3}{5},$$

for $C_1, C_2 > 0$.

Let $\varphi_3 + \frac{12}{5} =: y$ and $\varphi_2 =: x$. Then equations (7.9) and (7.10) become

$$\begin{aligned}\dot{x} &= -2xy\\ \dot{y} &= 2y - 2x^2 - 6.\end{aligned} \tag{7.11}$$

Case 3. $\varphi_1 \neq 0, \varphi_2 = 0, \varphi_3 \neq 0$.

Then, the ODEs reduce to

$$\dot{\varphi}_1 = -\varphi_1\left(2\varphi_1 + 2\varphi_3 + \frac{24}{5}\right) \tag{7.12}$$

$$\dot{\varphi}_1 + \dot{\varphi}_3 = -\frac{6}{5} - 2\varphi_1^2 + 2\varphi_1 + 2\varphi_3. \tag{7.13}$$

From (7.12) and (7.13), we have

$$\begin{aligned}&-2\varphi_1^2 - 2\varphi_1\varphi_3 - \frac{24}{5}\varphi_1 + \dot{\varphi}_3 = -\frac{6}{5} - 2\varphi_1^2 + 2\varphi_1 + 2\varphi_3\\ &\Rightarrow \dot{\varphi}_3 = -\frac{6}{5} + 2\varphi_3 + 2\varphi_1\varphi_3 + \frac{34}{5}\varphi_1\end{aligned} \tag{7.14}$$

Substituting $a := \varphi_1(t), b := \varphi_3(t) - \frac{3}{5}$, we have

$$\frac{da}{dt} = -6a - 2a^2 - 2ab$$
$$\frac{db}{dt} = 8a + 2b + 2ab.$$

The critical point is given by $(a, b) = (0, 0)$. Near the critical point, the the linearised system is given by

$$\frac{da}{dt} = -6a$$
$$\frac{db}{dt} = 8a + 2b.$$

The solutions are

$$x = C_1 e^{-6t},\ y = -C_1 e^{-6t} + (C_1 + C_3)e^{2t},$$

which shows that the critical point is a saddle point. Near the critical point $(0, 0, 3/5)$, the solution is given by

$$\varphi_1(t) = C_1 e^{-6t},\ \varphi_2(t) = 0,\ \varphi_3(t) = -C_1 e^{-6t} + (C_1 + C_2)e^{2t} + \frac{3}{5},$$

for $C_1, C_2 > 0$.

Let $\varphi_1 + \varphi_3 + \frac{12}{5} =: y$ and $\varphi_1 =: x$. Then equations (7.12) and (7.13) become

$$\begin{aligned} \dot{x} &= -2xy, \\ \dot{y} &= 2y - 2x^2 - 6. \end{aligned} \tag{7.15}$$

Case 4. $\varphi_1, \varphi_2, \varphi_3$ are all nonzero.

From (7.5) and (7.6), we have

$$\frac{\dot{\varphi}_1}{\varphi_1} = \frac{\dot{\varphi}_2}{\varphi_2} \Rightarrow \ln \varphi_2 = \ln \varphi_1 + \ln C \Rightarrow \varphi_2 = C'\varphi_1 \tag{7.16}$$

for $C \in (0, \infty)$. From (7.5) and (7.7), we have

$$\begin{aligned} &-2\varphi_1^2 - 2\varphi_1\varphi_3 - \frac{24}{5}\varphi_1 + \dot{\varphi}_3 = -\frac{6}{5} - 2\varphi_1^2 - 2\varphi_2^2 + 2\varphi_1 + 2\varphi_3 \\ &\Rightarrow \dot{\varphi}_3 = -\frac{6}{5} - 2C'^2\varphi_1^2 + 2\varphi_3 + 2\varphi_1\varphi_3 + \frac{34}{5}\varphi_1. \end{aligned} \tag{7.17}$$

Take $a := \varphi_1(t), b := \varphi_3(t) - \frac{3}{5}$. Then, from (7.5) and (7.17), we have

$$\frac{da}{dt} = -6a - 2a^2 - 2ab$$

$$\frac{db}{dt} = 8a - 2C'^2a^2 + 2b + 2ab.$$

The critical point is given by $(a, b) = (0, 0)$. Near the critical point, the the linearised system is given by

$$\frac{da}{dt} = -6a$$
$$\frac{db}{dt} = 8a + 2b.$$

The solutions are

$$x = C_1e^{-6t},\ y = -C_1e^{-6t} + (C_1 + C_3)e^{2t},$$

which shows that the critical point is a saddle point. Hence, near the critical point $(0, 0, 3/5)$, the solution is given by

$$\varphi_1(t) = C_1e^{-6t},\ \varphi_2(t) = C_2e^{-6t},\ \varphi_3(t) = -C_1e^{-6t} + (C_1 + C_3)e^{2t} + \frac{3}{5}$$

for $C_1, C_2, C_3 > 0$.

Let $\varphi_1 + \varphi_3 + \frac{12}{5} =: y, \varphi_1 =: x$ and $\varphi_2 =: C'x$ for $C' > 0$. Then equations (7.5) and (7.7) become

$$\dot{x} = -2xy,$$
$$\dot{y} = 2y - Cx^2 - 6 \tag{7.18}$$

for $C > 2$. The critical point is $(0, 3)$.

Curvature of the connection A at the critical point

We note that for all cases 2, 3 and 4, the critical points are the same, namely $(0, 0, 3/5)$. We want to calculate the curvature of the connection

$$A = e^{11}T_1 + e^a\varphi_b(t)\tau_{abc}T_c$$

at the critical point $(0, 0, 3/5)$. The curvature is

$$F_{bc} = -f^{11}_{bc}T_1 + \left(2\varphi_p(t)\varphi_s(t)\tau_{bpa}\tau_{csr}\epsilon_{arq} - \varphi_p(t)f^a_{bc}\tau_{apq}\right)T_q.$$

For $b, c = 1, 2, 3$

$$F_{bc} = -f^{11}_{bc}T_1 + \left(2\varphi_p(t)\varphi_s(t)\tau_{bpa}\tau_{csr}\epsilon_{arq} - \varphi_p(t)f^a_{bc}\tau_{apq}\right)T_q$$

and for $b, c = 4, 5, 6, 7$

$$F_{bc} = -f^{11}_{bc}T_1 - \varphi_p(t)f^a_{bc}\tau_{apq}T_q.$$

At the critical point, $F_{bc} = 0$. Hence, the connection $A = e^{11}T_1 + \frac{3}{5}e^1T_1$ is a flat connection on the link S^7.

Non-existence of closed orbits

The existence of closed orbits for the cases 2, 3, and 4 can be combined to the existence of closed orbits of the system of ODEs

$$\begin{aligned} \dot{x} &= -2xy \\ \dot{y} &= 2y - Cx^2 - 6 \end{aligned} \tag{7.19}$$

for $C \geq 2$, where $C = 2$ corresponds to cases 2 and 3, and $C > 0$ corresponds to case 4. The critical point is $(0, 3)$.

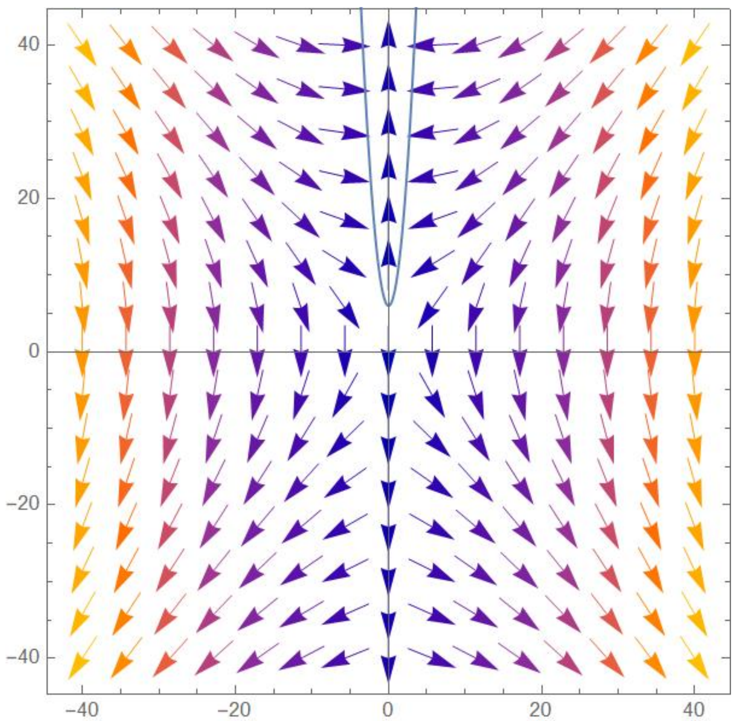


Figure 7.1: The direction field plot for the system 7.19.

Since an instanton is a path between the critical points, we note that existence of $SU(2)$-instanton corresponds to having a solution of the system (7.19) with boundary conditions $(x(t), y(t)) = (0, 3)$ at $t = -\infty$ and at $t = \infty$. From the direction field plot (7.1), we claim that such an orbit starting and ending at the critical point $(0, 3)$ cannot exist. To prove this, by inspecting the stable and unstable directions of the saddle point, it is clear that we just need to investigate the direction fields near the critical point for only the following cases.

Case 1. There exists t_0 such that $\dot{y}(t) > 0, y(t) > 3$ for all $t \leq t_0$: There are two sub-cases.
Sub-case 1: Let $t_0 < t_2$ such that $(x(t_0), y(t_0))$ is inside the parabola $2y - Cx^2 - 6 = 0$ and the

point $(x(t_2), y(t_2))$ is outside the parabola. Then there exists $t_1 \in (t_0, t_2)$ such that $(x(t_1), y(t_1))$ is on the parabola. Then, either $x(t_1) > 0, y(t_1) > 0$, which from the equation $\dot{x} = -2xy$ implies $\dot{x} < 0$; or $x(t_1) < 0, y(t_1) > 0$, which implies $\dot{x} > 0$, both leading to contradictions, since the signs of x and $\dot{x}$ should be the same for paths going out of the parabola from the inside.

Sub-case 2: For all $t > t_0$, the point $(x(t), y(t))$ is inside the parabola. Then, for all t $\dot{y} > 0$, so $y(t) > y(t_0) > 3$ for all $t > t_0$. Then, $y(t)$ cannot converge to 3 as $t \to \infty$.

Case 2. There exists t_0 such that $\dot{y}(t) < 0, y(t) < 3$ for all $t \leq t_0$: similar to Case 1, but here, all the directions fields are pointing downwards.

Thus, we have the following proposition.

Proposition 7.3.1.

1. *The only $Sp(2) \times U(1)$-invariant $SU(2)$-instanton on S^7 with round metric is the flat connection.*

2. *There are no irreducible $Sp(2) \times U(1)$-invariant $SU(2)$-instantons on $\mathbb{R}^8$ asymptotic to the flat connection on S^7.*

7.4 Obstructedness of $U(1)$ and $SU(2)$ Instantons on $\mathbb{R}^8$

Consider any asymptotically conical $U(1)$- or $SU(2)$-instanton on the trivial bundle over $\mathbb{R}^8$ asymptotic to the trivial connection A_Σ on S^7. Suppose, if possible, the instanton is unobstructed. Let us consider the Lie algebra of the Lie group $Sp(2) \times U(1) \ltimes \mathbb{R}^8$. The deformation complex of $Spin(7)$-instanton is given by (see [13])

$$0 \longrightarrow \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{d_A^7} \Omega^2_7(\mathfrak{g}_P) \longrightarrow 0. \tag{7.20}$$

We note that the cohomology group $H^1_{A,\nu} := \ker d^7_A / \operatorname{Im} d_A$ is isomorphic as a vector space to the deformation space $I(A,\nu)$ defined as in (3.19). We define a map

$$\begin{aligned} L : \mathrm{Lie}(Sp(2) \times U(1) \ltimes \mathbb{R}^8) &\to H^1_{A,\nu} \\ X &\mapsto [\iota_X F_A]\,. \end{aligned}$$

Clearly, we can think of X as a Killing field and hence determining a deformation $[\iota_X F_A] = [\mathcal{L}_X A]$ of the connection A. Then, we have the following proposition.

Proposition 7.4.1. *Under the assumption that*

$$-\int_{\mathbb{R}^8} \operatorname{Tr} F_A^4 < 3456\pi^4 \tag{7.21}$$

$\ker L$ *is a Lie subalgebra of* $\mathrm{Lie}(Sp(2) \times U(1) \ltimes \mathbb{R}^8)$ *isomorphic to* $\mathfrak{sp}(2) \oplus \mathfrak{u}(1)$.

Proof. It is a routine matter to show that $\ker L$ is a Lie subalgebra of $\mathrm{Lie}(Sp(2) \times U(1) \ltimes \mathbb{R}^8)$.

Now, from theorem 3.2.15, i.e., the deformation theory of AC instantons on $Spin(7)$-manifolds, we know that

$$\dim \mathcal{M}(A, \nu) = \operatorname{Ind} \mathfrak{D}_A^- = -\frac{1}{384\pi^4} \int_{\mathbb{R}^8} F_A^4 + \eta(\mathfrak{D}_{A_\Sigma}, S^7).$$

But, since A_Σ is the flat connection on S^7, we have $\eta(\mathfrak{D}_{A_\Sigma}, S^7) = 0$. Now, from the Rank-Nullity theorem, we have,

$$\ker L + \operatorname{rank} L = \dim \mathrm{Lie}(Sp(2) \times U(1) \ltimes \mathbb{R}^8) = 19. \tag{7.22}$$

Then, from (7.22) using the assumption (7.21), we have

$$\ker L = 19 - \operatorname{rank} L \geq 19 - \dim \mathcal{M}(A, \nu) > 19 - 9 = 10.$$

Now, if $\ker L \cong \mathrm{Lie}(H \ltimes U) \subset \mathrm{Lie}(Sp(2) \times U(1) \ltimes \mathbb{R}^8)$ for some non-trivial $U \subset \mathbb{R}^8$, then A has translational symmetries, and A cannot be asymptotically conical [23]. Since, $\mathfrak{sp}(2) \oplus \mathfrak{u}(1)$ is 11-dimensional, the only possibility is,

$$\ker L \cong \mathfrak{sp}(2) \oplus \mathfrak{u}(1).$$

□

As a direct consequence, we have the following proposition.

Proposition 7.4.2. *Any unobstructed asymptotically conical $U(1)$- or $SU(2)$-instanton on $\mathbb{R}^8$ asymptotic to the flat connection on S^7 satisfying (7.21) is $Sp(2) \times U(1)$-invariant.*

Then, proposition 7.3.1 implies the following main theorem.

Theorem 7.4.3. *There are no unobstructed irreducible asymptotically conical $U(1)$- or $SU(2)$-instantons on $\mathbb{R}^8$ asymptotic to the flat connection on S^7 satisfying (7.21).*

Appendix A

Gauge Theory on Homogeneous Spaces

In this chapter we briefly review the notions of canonical and invariant connections on homogeneous bundles, where we closely follow [50], [38], [54] and [22]. We conclude the chapter by discussing Wang's theorem, which has been integral in constructing instantons throughout the thesis.

A.1 Homogeneous Spaces and Homogeneous Bundles

We start with the following proposition.

Proposition A.1.1 ([38]). *Let G be a Lie group and H a closed subgroup of G. Then $M := G/H$ admits a structure of a real smooth manifold such that the transitive action of G on G/H given by*

$$\begin{aligned} L : G \times G/H &\to G/H \\ g_1, g_2H &\mapsto g_1 g_2 H \end{aligned}$$

is smooth. In particular, the canonical projection map $G \to G/H$ is smooth. Then, $M = G/H$ is called a homogeneous space.

Definition A.1.2.

1. A *homogeneous fibre bundle* over $M = G/H$ is a locally trivial fibre bundle $\pi : E \to M$ together with a G-action $\widetilde{L} : G \times E \to E$ which lifts the action $L : G \times M \to M$ on M. i.e.,
$$\pi(\widetilde{L}(g, y)) = L(g, \pi(y))$$
for all $g \in G$ and $y \in E$.

2. A *homogeneous principal bundle* over $M = G/H$ is a homogeneous fibre bundle $\pi : E \to M$ which is a principal bundle such that for each $g \in G$ the bundle map $\widetilde{L}_g := \widetilde{L}(g, \cdot)$ is a principal bundle homomorphism.

3. A *homogeneous vector bundle* over $M = G/H$ is a homogeneous fibre bundle $\pi : E \to M$ which is a vector bundle such that for each $g \in G$ the bundle map $\widetilde{L}(g, \cdot) : E \to E$ is a vector bundle homomorphism.

Example A.1.3. The canonical principal H-bundle is the bundle $H \to G \to G/H$. This is a homogeneous principal bundle under the action $\widetilde{L} : G \times G \to G$ which is the multiplication map.

A.2 Canonical Connection

We start by recalling the definition of Maurer–Cartan form.

Definition A.2.1. The *Maurer–Cartan form* $\theta \in \Omega^1(G, \mathfrak{g})$ is the $\mathfrak{g}$-valued 1-form on G such that

$$(\theta)_g(X_g) = (d_g L_{g^{-1}})(X_g) \in T_e G \cong \mathfrak{g}$$

for all $g \in G$ and $X_g \in T_g G$. Here $d_g L_{g^{-1}}$ is regarded as a linear map from $T_g G$ to $T_e G$.

Lemma A.2.2. *For the translation maps $L_g : G \to G$ and $R_g : G \to G$ defined by $L_g(h) = gh$ and $R_g(h) = hg$ respectively, we have $L_g^* \theta = \theta$ and $R_g^* \theta = \mathrm{Ad}_{g^{-1}} \circ \theta$ for all $g \in G$.*

Let H be a subgroup of G with Lie algebra $\mathfrak{h}$.

Definition A.2.3. A homogeneous bundle $H \to G \to G/H$ is called *reductive* if there exists a subspace $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$ (which implies $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$).

Theorem A.2.4.

1. *Let $H \to G \to G/H$ be a reductive principal bundle. Let $\theta \in \Omega^1(G, \mathfrak{g})$ be the Maurer–Cartan form such that with respect to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, we have $\theta = \theta_{\mathfrak{h}} \oplus \theta_{\mathfrak{m}}$ where $\theta_{\mathfrak{h}}$ and $\theta_{\mathfrak{m}}$ are the $\mathfrak{h}$ and $\mathfrak{m}$-components of θ respectively. Then $\theta_{\mathfrak{h}}$ defines a connection on the principal bundle $H \to G \to G/H$ called the canonical connection which is invariant by left translation of G (i.e., $L_g^* \theta_{\mathfrak{h}} = \theta_{\mathfrak{h}}$ for all $g \in G$).*

2. *Conversely, any connection on $H \to G \to G/H$ invariant by left translation of G (if exists) determines a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and the connection can be obtained as described by (1).*

3. *The curvature form F of the canonical connection $\theta_{\mathfrak{h}}$ is given by*

$$F = d\theta_{\mathfrak{h}} + \frac{1}{2}\theta_{\mathfrak{h}} \wedge \theta_{\mathfrak{h}} = -\frac{1}{2}(\theta_{\mathfrak{m}} \wedge \theta_{\mathfrak{m}})_{\mathfrak{h}},$$

i.e., the $\mathfrak{h}$-component of $-\frac{1}{2}(\theta_{\mathfrak{m}} \wedge \theta_{\mathfrak{m}})$.

It is important to note that the canonical connection on the principle bundle $H \to G \to G/H$ defines a canonical connection on the tangent bundle $T(G/H)$. This follows from the fact that $T(G/H) \cong G \times_H \mathfrak{m}$ is the associated vector bundle to $H \to G \to G/H$ by the representation $\rho : H \to GL(\mathfrak{m})$.

A.3 Invariant Connections and Wang's Theorem

Lemma A.3.1. *Let K, G be Lie groups and $H \subset G$ be a closed subgroup. Let $P \to G/H$ be a homogeneous principal K-bundle. Then there is a smooth homomorphism $\lambda : H \to K$, called the isotropy homomorphism, such that $P \cong G \times_{(H,\lambda)} K$ where the equivalence relation on $G \times K$ is given by*

$$(gh, k) \sim (g, \lambda(h)k)$$

for all $h \in H, g \in G, k \in K$.

Let $\mathfrak{k}$ be the Lie algebra of K. Let $X \in \mathfrak{k}$ and let ζ_X be the fundamental vector field on P generated by X. Further, let us denote the principal right action of $k \in K$ on P by R_k. Then $\theta \in \Omega^1(P, \mathfrak{k})$is a connection of P if

$$\theta(\zeta_X) = X$$
$$(R_k)^*\theta = \mathrm{Ad}_{k^{-1}} \circ \theta.$$

Let L_g be the left action of $g \in G$ on P. Since L_g is a bundle automorphism the pull back $L_g^*\theta$ is again a connection on P.

Definition A.3.2. A connection θ on P is said to be a *G-invariant connection* iff

$$(L_g)^*\theta = \theta$$

for all $g \in G$.

Consider an invariant connection θ on the homogeneous principal K-bundle $P := G \times_\lambda K \to G/H$. Define a map

$$\pi : G \to G \times_H K$$
$$g \to [(g, e)],$$

for the identity element $e \in K$. Consider the linear map

$$(\pi^*\theta - \lambda_*\theta_{\mathfrak{h}})|_e : T_eG \cong \mathfrak{g} \to \mathfrak{k}.$$

It can be proved that, $(\pi^*\theta - \lambda_*\theta_{\mathfrak{h}})|_e(X) = 0$ for all $X \in \mathfrak{h}$.

Define a linear map

$$\begin{aligned} \Lambda_\theta : \mathfrak{m} &\to \mathfrak{k} \\ X &\mapsto (\pi^*\theta - \lambda_*\theta_{\mathfrak{h}})|_e(X) \end{aligned}$$

for all $X \in \mathfrak{m}$. It is easy to check that Λ_θ is H-equivariant.

Thus, an invariant connection θ yields a linear map Λ_θ. The other direction is also true, and the complete result is given by Wang [59] as follows.

Theorem A.3.3 (*Wang's Theorem*)**.** *Let $\lambda : H \to K$ be a homomorphism. Consider the homogeneous principal K-bundle $P := G \times_\lambda K \to G/H$. Then there is a one-to-one correspondence between the G-invariant connections θ on P and linear maps*

$$\Lambda : (\mathfrak{m}, \mathrm{Ad}) \to (\mathfrak{k}, \mathrm{Ad} \circ \lambda)$$

as morphisms of H-representations.

Now, we introduce an orthonormal frame in order to present local expression of the connection.

We note that a basis I_A for $\mathfrak{g}$ can be represented by left invariant vector fields $\widehat{E}_A$ on G as well as by the dual basis $\hat{e}^A$ of left invariant 1-forms. Denote the natural projection map

$$\begin{aligned} p : G &\to G/H \\ g &\mapsto gH, \end{aligned}$$

of the principal bundle. Let U be a contractible open subset of G/H. Then consider a local section σ of the bundle $G \to G/H$, i.e., a map $\sigma : U \to G$ such that $p \circ \sigma = \mathrm{Id}_U$. We put $e^A := \sigma^*\hat{e}^A$. Then $\{e^a : a = 1, \ldots, \dim \mathfrak{m}\}$ form an orthonormal frame for $T^*(G/H)$ over U.

Then, with respect to this local trivialisation, the invariant connection is a local $\mathfrak{k}$-valued 1-form on G/H, and can be written as follows

$$e^i(d\lambda)(I_i) + e^a\Lambda(I_a),$$

for $a = 1, \ldots, \dim \mathfrak{m}$ and $i = (\dim \mathfrak{m}) + 1, \ldots, \dim \mathfrak{g}$. Here $e^i(d\lambda)(I_i)$ is local expression of the canonical connection.

Appendix B

Exceptional Holonomy Groups

In this chapter, we briefly review the important notions of Riemannian holonomy groups, Berger's classification of Riemannian holonomy groups, and the exceptional holonomy groups G_2 and $Spin(7)$. For the first part we closely follow [32] and for the holonomy groups G_2 and $Spin(7)$, we very closely follow [53].

B.1 Parallel Transport and Riemannian holonomy

B.1.1 Parallel Transports and Holonomy groups

Consider a vector bundle $\pi : E \to M$ on an orientable smooth manifold M and a connection ∇^E on E. Let $\gamma : [0,1] \to M$ be a smooth curve with $\gamma(0) = p$ and $\gamma(1) = q$. Then, we have the pullback connection $\gamma^*(\nabla^E)$ on the pullback bundle $\gamma^*(E) \to [0,1]$. It can be proved that for each $x \in E_p := \pi^{-1}(p) \subset E$, there is a unique section σ on $\gamma^*(E)$ with $\sigma(0) = x$, and $\gamma^*(\nabla^E)\sigma = 0$. Then the *parallel transport map* P_γ is defined by

$$\begin{aligned} P_\gamma : E_p &\to E_q \\ x &\mapsto \sigma(1). \end{aligned} \tag{B.1}$$

Now, we consider γ to be a piece-wise smooth loop based at p, that is, $\gamma(0) = \gamma(1) = p$. Then, the parallel transport map $P_\gamma : E_p \to E_p$ is an invertible linear map, and then, the set of all parallel transports P_γ for all piece-wise smooth loops γ based at p forms a group, which we call the *holonomy group* $\mathrm{Hol}_p(\nabla^E)$ of the connection ∇^E.

B.1.2 Riemannian holonomy

Now, consider an orientable Riemannian manifold (M, g) with the Levi–Civita connection ∇ on TM. Then, for $p \in M$, the *Riemannian holonomy group* $\mathrm{Hol}_p(g)$ is the holonomy group $\mathrm{Hol}_p(\nabla)$ of the Levi-Civita connection.

We note that

- $\mathrm{Hol}_p(g)$ is a closed subgroup of $SO(n)$.
- $\mathrm{Hol}_p(g)$ is independent of the base point p up to conjugation, and we denote it just by $\mathrm{Hol}(g)$.

B.1.3 Berger's classification of Riemannian holonomy groups

Consider a Riemannian product manifold $(M \times N, g \times h)$. That is, for $(p, q) \in M \times N$

$$(g \times h)|_{(p,q)} = g|_p + h|_q,$$

for $p \in M$ and $q \in N$. Then,

$$\mathrm{Hol}(g \times h) = \mathrm{Hol}(g) \times \mathrm{Hol}(h).$$

Definition B.1.1. A Riemannian manifold (M, g) is called *irreducible* if it not locally isometric to a Riemannian product manifold.

Consider an isometry $s_p : M \to M$ for any $p \in M$ such that $s_p(p) = p$ and $ds_p = -\mathrm{Id}$. Then the isometry s_p is called a *symmetry* at p.

Definition B.1.2. A *symmetric space* M is a homogeneous space with a symmetry s_p for every $p \in M$. A *locally symmetric space* is a Riemannian manifold locally isometric to a symmetric space.

A Riemannian manifold is locally symmetric if and only if $\nabla R = 0$ for Levi-Civita connection ∇ and Riemann curvature R.

Theorem B.1.3 (Berger's classification). *Let M be an orientable simply-connected n-dimensional Riemannian manifold where the Riemannian metric is irreducible and nonsymmetric (neither symmetric nor locally symmetric). Then we have only the following possibilities for holonomy groups.*

1. $\mathrm{Hol}(g) = SO(n)$,

2. $\mathrm{Hol}(g) = U(m)$ *for* $n = 2m$,
3. $\mathrm{Hol}(g) = SU(m)$ *for* $n = 2m$,
4. $\mathrm{Hol}(g) = Sp(m)$ *for* $n = 4m$,
5. $\mathrm{Hol}(g) = Sp(m) \cdot Sp(1) := Sp(m) \times_{\mathbb{Z}_2} Sp(1)$ *for* $n = 4m$,
6. $\mathrm{Hol}(g) = G_2$ *for* $n = 7$,
7. $\mathrm{Hol}(g) = Spin(7)$ *for* $n = 8$.

Remark B.1.4. Berger's classification has a deep relation with the existence of exactly four normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ over $\mathbb{R}$ of real dimensions $1, 2, 4$ and 8 respectively. The group $SO(m)$ acts on $\mathbb{R}^m$, $U(m)$ and $SU(m)$ act on $\mathbb{C}^m$, $Sp(m)$ and $Sp(m) \cdot Sp(1)$ act on $\mathbb{H}^m$, and G_2 acts on $\mathrm{Im}\,\mathbb{O}$ and $Spin(7)$ acts on $\mathbb{O}$.

B.2 Octonions and the Lie Group G_2

B.2.1 Cross Product

Definition B.2.1. A skew-symmetric bilinear map

$$\begin{aligned} V \times V &\to V \\ (u, v) &\mapsto u \times v \end{aligned}$$

is said to be a *cross product* if

1. $\langle u \times v, u \rangle = \langle u \times v, v \rangle = 0$
2. $|u \times v|^2 = |u|^2 |v|^2 - \langle u, v \rangle^2$

for all $u, v \in V$.

It can be proved that V admits a cross product if and only if the dimension of V is $0, 1, 3$, or 7. In dimension 0 and 1 the cross product is trivial. In dimension 3 it is unique up to sign determined by an orientation of V. In dimension 7 It is unique up to orthogonal isomorphism.

Definition B.2.2. Let $\dim V = 7$. The map $\phi : V \times V \times V \to \mathbb{R}$ defined by

$$\phi(u, v, w) := \langle u \times v, w \rangle$$

is called the *associative calibration* of $(V, \times)$.

It can be proved that $\phi \in \Lambda^3(V^*)$, i.e., ϕ is alternating.

B.2.2 Normed Algebras

Definition B.2.3. Let $(W, \langle \cdot\,, \cdot \rangle)$ be a finite-dimensional inner product space. Further, let $(W, \cdot)$ is a unital ring with respect to a product

$$\begin{aligned} \cdot : W \times W &\to W \\ (u, v) &\mapsto u \cdot w \end{aligned}$$

where the product $\cdot$ is compatible with the scalar multiplication of the vector space W. The W is called an *algebra*.

If moreover the norm on W satisfies

$$|uv| = |u||v|$$

for all $u, v \in W$, then W is called a *normed algebra*.

Remark B.2.4. For the normed algebra W, we can identify $\mathbb{R}$ with a subspace of W generated by the multiplicative identity 1. For $u \in W$ and $\lambda \in \mathbb{R}$, we prefer to write $u + \lambda$ for $u + \lambda 1$.

Definition B.2.5. The *conjugation* of an element of W is defined by the involution

$$\begin{aligned} W &\to W \\ u &\mapsto \overline{u} \end{aligned}$$

where $\overline{1} = 1$ and $\overline{u} = -u$ for $u \in 1^{\perp} = \{u \in W : \langle u, 1 \rangle = 0\}$. Combining, we can write

$$\overline{u} = 2\langle u, 1 \rangle - u$$

for all $u \in W$.

Definition B.2.6. The subspace $\mathbb{R}$ of W is considered as the *real part* of W. If V is the orthogonal complement of $\mathbb{R}$ in W, then V is the *imaginary part* of W.

If $u \in W$, the real and imaginary parts are defined by

$$\operatorname{Re} u := \langle u, 1 \rangle, \quad \operatorname{Im} u := u - \langle u, 1 \rangle$$

respectively.

Theorem B.2.7 ([53], Theorem 5.4)**.** *The one to one correspondence between Normed algebras and vector spaces equipped with cross products is given as follows.*

1. *If W is a normed algebra, then $V := 1^{\perp}$ is a subspace of W equipped with a cross product*

$$\begin{aligned} V \times V &\to V \\ (u,v) &\mapsto u \times v \end{aligned}$$

defined by

$$u \times v := uv + \langle u, v \rangle$$

for all $u, v \in V := 1^{\perp}$. Conversely,

2. *If $(V, \langle \cdot\ , \cdot \rangle)$ is a finite dimensional inner product space equipped with a cross product, then $W := \mathbb{R} \oplus V$ is a normed algebra. If $u = u_0 + u_1, v = v_0 + v_1 \in \mathbb{R} \oplus V$, then the product in W is defined by*

$$uv := u_0 v_0 - \langle u_1, v_1 \rangle + u_0 v_1 + v_0 u_1 + u_1 \times v_1$$

Here we identify $f \in \mathbb{R}$ with $(f, 0) \in \mathbb{R} \oplus V$ and $v \in V$ with $(0, v) \in \mathbb{R} \oplus V$.

The following corollary is due to Hurwitz.

Corollary B.2.8. *A normed algebra has dimension $1, 2, 4$ or 8 and is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ respectively.*

B.2.3 Octonions

The algebra $\mathbb{O} \cong \mathbb{R}^8$ is a non-associative algebra of real dimension 8. Let $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ is a basis of $\mathbb{O}$.

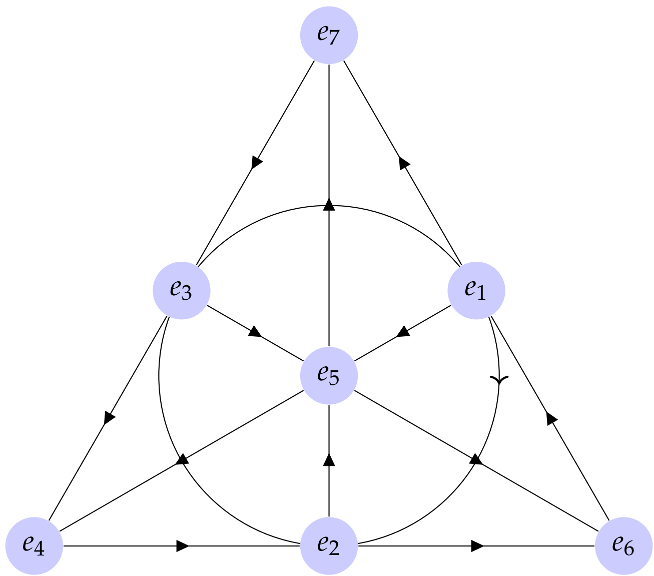
The multiplication table is given below.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	$-e_5$	e_4	$-e_7$	e_6
e_2	$-e_3$	-1	e_1	$-e_6$	e_7	e_4	$-e_5$
e_3	e_2	$-e_1$	-1	$-e_7$	$-e_6$	e_5	e_4
e_4	e_5	e_6	e_7	-1	$-e_1$	$-e_2$	$-e_3$
e_5	$-e_4$	$-e_7$	e_6	e_1	-1	$-e_3$	e_2
e_6	e_7	$-e_4$	$-e_5$	e_2	e_3	-1	$-e_1$
e_7	$-e_6$	e_5	$-e_4$	e_3	$-e_2$	e_1	-1

Table B.1: Multiplication table for Octonions.

Fano Plane

The multiplication table of octonions can be described by the diagram below, called the *Fano plane*.



Following the cyclic ordering of the diagram, we can clearly figure out the whole multiplication table.

Then Fano plane is the projective plane $\mathbb{Z}_2P^2$, consisting of lines through the origin in the vector space $\mathbb{Z}_2^3$ over the field $\mathbb{Z}_2$ of dimension 3.

Since each such line (being points themselves) contains a single nonzero element, the Fano plane can also be thought of the set consisting of the seven nonzero elements of $\mathbb{Z}_2^3$. Identifying the origin in $\mathbb{Z}_2^3$ with $1 \in \mathbb{O}$, we get a basis for the octonions.

Revisiting the octonionic product

Recall the basis $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ of the octonions $\mathbb{O}$. We note that

$$\{e_i, e_j\} = e_ie_j + e_je_i = -2\delta_{ij}$$

for $i, j = 1, \ldots, 7$. To be precise,

$$e_ie_j = C_{ijk}e_k - \delta_{ij}$$

where C_{ijk} is totally antisymmetric and $C_{ijk} = 1$ for

$$ijk = 123, 154, 176, 264, 257, 374, 365$$

(follows from Fano plane). C_{ijk} are called *the structure constants of the octonion algebra*.

B.2.4 Associative Calibrations

Definition B.2.9.

- Let V be a real vector space. A 3-form $\phi \in \Lambda^3 V^*$ is said to be *non-degenerate* if for every pair of linearly independent vectors $u, v \in V$, there exists a vector $w \in V$ such that $\phi(u, v, w) \neq 0$.
- An inner product $\langle \cdot\,, \cdot \rangle$ is said to be *compatible* with ϕ if the map

$$\begin{aligned} V \times V &\to V \\ (u, v) &\mapsto u \times v \end{aligned}$$

defined by $\phi(u, v, w) := \langle u \times v, w \rangle$ is a cross product.

Lemma B.2.10 ([53]). *Let V be a 7-dimensional real inner product space and $\phi \in \Lambda^3 V^*$. Then the following are equivalent.*

1. *ϕ is compatible with the inner product.*
2. *There is an orientation on V such that the volume form* vol $\in \Lambda^7 V^*$ *satisfies*

$$\iota_u \phi \wedge \iota_v \phi \wedge \phi = 6\langle u, v \rangle \,\mathrm{vol}$$

for all $u, v \in V$. The orientation is uniquely determined by ϕ.

Both conditions imply that ϕ is non-degenerate.

B.2.5 The Lie group G_2

Consider the associative calibration $\phi \in \Lambda^3(\mathbb{R}^7)^*$ defined by

$$\phi_0 = dx^{123} - dx^{145} - dx^{167} - dx^{246} + dx^{257} - dx^{347} - dx^{356}, \tag{B.2}$$

where $dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$. The *coassociative calibration* is the Hodge dual

$$\psi_0 := *\phi_0 = dx^{4567} - dx^{1247} - dx^{1256} + dx^{1346} - dx^{1357} - dx^{2345} - dx^{2367}.$$

The group G_2 is the stabilizer group of ϕ_0, i.e.

$$\begin{aligned} G_2 &:= \{g \in SO(7) : g^*\phi_0 = \phi_0\} \\ &= \{g \in SO(7) : gu \times gv = g(u \times v) \text{ for all } u, v \in \mathbb{R}^7\}. \end{aligned}$$

Theorem B.2.11 ([35]). *The group G_2 is the automorphism group* $\mathrm{Aut}(\mathbb{O})$.

Now let V be any 7-dimensional vector space and $\phi \in \Lambda^3 V^*$ be the associative calibration defined by $\phi(u,v,w) := \langle u \times v, w \rangle$. The group of automorphism of ϕ is

$$G(V,\phi) := \{g \in SO(V) : g^*\phi = \phi\}.$$

Then $G(V,\phi)$ is isomorphic to G_2.

Theorem B.2.12 ([53]). *The group G_2 is a 14-dimensional simple, connected, simply connected Lie group. The action of G_2 on S^6 is transitive and for every unit vector $u \in V$, the isotropy group $G_u := \{g \in G_2 : gu = u\}$ is isomorphic to $SU(3)$ and hence we have the fibration*

$$SU(3) \hookrightarrow G_2 \to S^6.$$

Theorem B.2.13 ([53]). *Let V be a 7-dimensional vector space and $\phi \in \Lambda^3 V^*$ be the associative calibration. There are orthogonal decompositions*

$$\begin{aligned}\Lambda^2 V^* &= \Lambda^2_7 \oplus \Lambda^2_{14}\\ \Lambda^3 V^* &= \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}\end{aligned}$$

where $\dim \Lambda^k_d = d$ *and*

$$\begin{aligned}\Lambda^2_7 &= \{u \lrcorner \phi : u \in V\} = \{w \in \Lambda^2 V^* : *(\phi \wedge w) = 2w\}\\ \Lambda^2_{14} &= \{w \in \Lambda^2 V^* : \psi \wedge w = 0\} = \{w \in \Lambda^2 V^* : *(\phi \wedge w) = -w\}\\ \Lambda^3_1 &= \langle \phi \rangle\\ \Lambda^3_7 &= \{u \lrcorner \psi : u \in V\}\\ \Lambda^3_{27} &= \{w \in \Lambda^3 V^* : \phi \wedge w = 0, \psi \wedge w = 0\}.\end{aligned}$$

Each Λ^k_d is an irreducible representation of G_2 and the representations Λ^2_7 and Λ^3_7 are both isomorphic to V. Λ^2_{14} is isomorphic to the Lie algebra of G_2, Λ^3_{27} is isomorphic to the space of traceless symmetric endomorphisms of V: the space $Sym^2_0(V)$.

B.3 The Lie group $Spin(7)$

B.3.1 Triple cross products

Definition B.3.1. Let $(W, \langle \cdot\, , \cdot \rangle)$ be a finite-dimensional inner product space. An alternating multi-linear map

$$W \times W \times W \to W$$

$$(u, v, w) \to u \times v \times w \tag{B.3}$$

is called a *triple cross product* if it satisfies

$$\langle u \times v \times w, u \rangle = \langle u \times v \times w, v \rangle = \langle u \times v \times w, w \rangle = 0 \tag{B.4}$$

and

$$|u \times v \times w| = |u \wedge v \wedge w| \tag{B.5}$$

for all $u, v, w \in W$.

When $u, v, w \in W$ are linearly dependent, we have $u \times v \times w = 0$.

Let (B.3) be a triple cross product. If $e \in W$ is a unit vector, then the subspace $V_e := e^{\perp}$ is equipped with a cross product

$$\begin{aligned} V_e \times_e V_e &\to V_e \\ u \times_e v &= u \times e \times v. \end{aligned}$$

Hence we conclude that $\dim V_e = 0, 1, 3$ or 7 and $\dim W = 1, 2, 4$ or 8.

Definition B.3.2. Let $\dim W = 8$ and W be equipped with a triple cross product. Then the map

$$\begin{aligned} \Phi : W \times W \times W \times W &\to \mathbb{R} \\ (x, u, v, w) &\to \langle x, u \times v \times w \rangle \end{aligned}$$

is an alternating 4-form, called the *Cayley calibration* of W.

We fix an orientation of W such that $\Phi \wedge \Phi > 0$.

Theorem B.3.3 ([53]). *Let* $\dim W = 8$ *and* W *be equipped with a triple cross product with Cayley calibration* $\Phi \in \Lambda^4 W^*$. *Let* $e \in W$ *be a unit vector.*

1. *Define the map* $\psi_e : W \times W \times W \times W \to \mathbb{R}$ *by*

$$\begin{aligned} \psi_e(u, v, w, x) := \langle e \times u \times v, e \times w \times x \rangle &- (\langle u, w \rangle - \langle u, e \rangle \langle e, w \rangle)(\langle v, x \rangle - \langle v, e \rangle \langle e, x \rangle) \\ &+ (\langle u, x \rangle - \langle u, e \rangle \langle e, x \rangle)(\langle v, w \rangle - \langle v, e \rangle \langle e, w \rangle). \end{aligned}$$

Then $\psi_e \in \Lambda^4 W^*$ *and* $\Phi = e^* \wedge \phi_e + \psi_e$ *where* e^* *is dual to* e *and* $\phi_e := \iota_e \Phi \in \Lambda^3 W^*$.

2. *The subspace $V_e := e^{\perp}$ is equipped with a cross product*

$$\begin{aligned} V_e \times V_e &\to V_e \\ (u,v) &\mapsto u \times_e v := u \times e \times v \end{aligned} \tag{B.6}$$

and $\phi_e|_{V_e}$ is the associative calibration of B.6 and $\psi_e|_{V_e}$ is the coassociative calibration of B.6.

3. *The inner product space W is a normed algebra with identity e, where the multiplication and conjugation are given by*

$$\begin{aligned} uv &:= u \times e \times v + \langle u,e\rangle v + \langle v,e\rangle u - \langle u,v\rangle e \\ \overline{u} &:= 2\langle u,e\rangle e - u. \end{aligned}$$

B.3.2 Cayley calibrations

Definition B.3.4. Let W be an 8-dimensional real inner product space.

- A 4-form $\Phi \in \Lambda^4 W^*$ is said to be *non-degenerate* if for all $u,v,w \in W$, linearly independent in W, there exists $x \in W$ such that

$$\Phi(u,v,w,x) \neq 0.$$

- The inner product $\langle \cdot\,,\cdot\rangle$ on W is said to be *compatible with* Φ if the map

$$\begin{aligned} W \times W \times W &\to W \\ (u,v,w) &\mapsto u \times v \times w \end{aligned}$$

  defined by

$$\langle x \times u \times v, w\rangle := \Phi(x,u,v,w)$$

  is a triple cross product.

- A 4-form $\Phi \in \Lambda^4 W^*$ is said to be a *Cayley form* if it admits a compatible inner product.

Example B.3.5. Let ϕ_0, defined in (B.2) be the associative calibration on $\mathbb{R}^7$ (with basis dx^i, $i = 1,\ldots,7$). Then $\mathbb{R}^8$ has the Cayley form

$$\Phi_0 = dx^0 \wedge \phi_0 + \psi_0$$

where $\psi_0 = *\phi_0$. We note that $\Phi_0 \wedge \Phi_0 = 14\,\mathrm{vol}$.

Lemma B.3.6. *[53] Let $(W, \langle \cdot\,,\cdot\rangle)$ be an inner product space and $\Phi \in \Lambda^4 W^*, \mathrm{vol} \in \Lambda^8 W^*$ are a 4-form and volume form respectively. Then the following are equivalent.*

1. *The inner product is compatible with Φ.*

2. *With volume form* vol $\in \Lambda^8 W^*$, *there is a unique orientation on W such that for all $u, v, w \in W$,*

$$\iota_v \iota_u \Phi \wedge \iota_v \iota_u \Phi \wedge \Phi = 6|u \wedge v|^2 \,\mathrm{vol}\,.$$

B.3.3 The Lie group $Spin(7)$

Let W be an 8-dimensional inner product space equipped with a positive triple product and $\Phi \in \Lambda^4 W^*$ be the Cayley calibration. We give W the orientation such that $\Phi \wedge \Phi > 0$. We note that Φ is self-dual with respect to the Hodge star operator. Recall the subspace $V_e := e^\perp \subset W$. Then

$$\begin{aligned}
\Phi &= e^* \wedge \phi_e + \psi_e, \\
\phi_e &:= \iota_e \Phi \in \Lambda^3 W^*, \\
\psi_e &:= *(e^* \wedge \phi_e) \in \Lambda^4 W^*.
\end{aligned}$$

Denote the group of automorphisms of Φ by

$$G(W, \Phi) := \{g \in GL(W) : g^* \Phi = \Phi\}.$$

Then $G(W, \Phi) \subset SO(W)$ and

$$G(W, \Phi) = \{g \in SO(W) : gu \times gv \times gw = g(u \times v \times w) \text{ for all } u, v, w \in W\}.$$

Denote $Spin(7) := G(\mathbb{R}^8, \Phi_0)$, where Φ_0 is the standard Cayley form on $\mathbb{R}^8$.

Lemma B.3.7 ([53])**.** *$G(W, \Phi)$ is isomorphic to $Spin(7)$ for all Cayley forms $\Phi \in \Lambda^4 W^*$.*

Theorem B.3.8 ([53])**.** *The group $Spin(7)$ is a 21-dimensional simple, connected, simply connected Lie group. If S^7 be the unit sphere in $\mathbb{R}^8$, then $Spin(7)$ acts transitively on the unit tangent bundle of S^7. Moreover, for every unit vector $e \in W$, the stabilizer group*

$$G_e := \{g \in Spin(7) : ge = e\}$$

is isomorphic to G_2, thus giving a fibration

$$G_2 \hookrightarrow Spin(7) \to S^7.$$

Theorem B.3.9 ([53])**.** *There are orthogonal decomposition*

$$\begin{aligned}
\Lambda^2 W^* &= \Lambda^2_7 \oplus \Lambda^2_{21} \\
\Lambda^3 W^* &= \Lambda^3_8 \oplus \Lambda^3_{48} \\
\Lambda^4 W^* &= \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35}
\end{aligned}$$

where $\dim \Lambda^k_d = d$ *and*

$$\begin{aligned}
\Lambda^2_7 &= \{w \in \Lambda^2 W^* : *(\Phi \wedge w) = 3w\} \\
\Lambda^2_{21} &= \{w \in \Lambda^2 W^* : *(\Phi \wedge w) = -w\} \\
\Lambda^3_8 &= \{u \lrcorner \Phi : u \in W\} \\
\Lambda^3_{48} &= \{w \in \Lambda^3 W^* : \Phi \wedge w = 0\} \\
\Lambda^4_1 &= \langle \Phi \rangle \\
\Lambda^4_7 &= \{\mathcal{L}_\xi \Phi : \xi \in \mathfrak{so}(W)\} \\
\Lambda^4_{27} &= \{w \in \Lambda^4 W^* : *w = w, w \wedge \Phi = 0, w \wedge \mathcal{L}_\xi \Phi = 0 \textit{ for all } \xi \in \mathfrak{so}(W)\} \\
\Lambda^4_{35} &= \{w \in \Lambda^4 W^* : *w = -w\}
\end{aligned}$$

(where $\mathcal{L}$ is the Lie derivative, for $\xi \in \mathfrak{so}(W)$, $\mathcal{L}_\xi \Phi \in \Lambda^4 W^*$ defined by $\mathcal{L}_\xi \Phi := \frac{d}{dt}|_{t=0} \exp(t\xi)^* \Phi$) Each Λ^k_d is an irreducible representation of $Spin(7)$.

B.3.4 A Few Identities

It is convenient to list a few identities involving ϕ and ψ we have used throughout the paper (For proof, see [53]).

Let $u, v \in \Lambda^1$, $w \in \Lambda^2$. Then,

1. $*(\psi \wedge u) = u \lrcorner \phi, \quad *(\phi \wedge u) = u \lrcorner \psi,$
2. $\phi \wedge (u \lrcorner \phi) = 2\psi \wedge u,$
3. $(u \lrcorner \phi) \lrcorner \psi = 2u \lrcorner \phi, \quad (u \lrcorner \psi) \lrcorner \psi = -4u,$
4. $(u \lrcorner \phi) \lrcorner \phi = 3u, \quad (u \lrcorner \psi) \lrcorner \phi = 0,$
5. $*(\phi \wedge u \wedge v) = v \lrcorner (u \lrcorner \psi),$
6. $(w \lrcorner \phi) \lrcorner \phi = *(\psi \wedge *(\psi \wedge w)) = w + *(\phi \wedge w),$
7. $(w \lrcorner \psi) \lrcorner \psi = *(\phi \wedge *(\phi \wedge w)) = 2w + *(\phi \wedge w).$

The following identities involving the structure constants of ϕ and ψ have been used frequently. Because of our convention, they differ from [34] or [35] in signs.

1. $\phi_{ijk}\phi_{abk} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja} + \psi_{ijab}$,
2. $\phi_{ijk}\psi_{abck} = -\delta_{ia}\phi_{jbc} - \delta_{ib}\phi_{ajc} - \delta_{ic}\phi_{abj} + \delta_{ja}\phi_{ibc} + \delta_{jb}\phi_{aic} + \delta_{jc}\phi_{abi}$,
3. $\psi_{ijkl}\psi_{abkl} = 4\delta_{ia}\delta_{jb} - 4\delta_{ib}\delta_{ja} + 2\psi_{ijab}$.

B.3.5 Irreducible Representations of G_2 and $Spin(7)$

First, we list the irreducible G_2-representations we came across in this thesis. Let $V_{(a,b)}$ be an irreducible representation of $\mathfrak{g}_2$ with highest weight (a, b).

G_2-reps	Dimensions	Modelled using	Also isomorphic to
$V_{(0,0)}$	1	$\Lambda^0(\mathbb{C}^7)$	$\Lambda^3_1(\mathbb{C}^7)$
$V_{(1,0)}$	7	$\Lambda^1(\mathbb{C}^7)$	$\Lambda^2_7(\mathbb{C}^7)$
$V_{(0,1)}$	14	$\Lambda^2_{14}(\mathbb{C}^7)$	
$V_{(2,0)}$	27	$\Lambda^3_{27}(\mathbb{C}^7)$	$\mathrm{Sym}^2_0(\mathbb{C}^7)$
$V_{(1,1)}$	64		
$V_{(0,2)}$	77		
$V_{(3,0)}$	77		

Finally, we list the irreducible $Spin(7)$-representations and the decompositions into irreducible G_2-representations. Let $V_{(a,b,c)}$ be an irreducible representation of $\mathfrak{spin}(7)$ with highest weight (a, b, c).

$Spin(7)$-reps	Dimensions	Isomorphic to	Decomposition into G_2-reps
$V_{(0,0,0)}$	1	$\Lambda^0(\mathbb{C}^8)$	$V_{(0,0)}$
$V_{(1,0,0)}$	7	$\Lambda^2_7(\mathbb{C}^8)$	$V_{(1,0)}$
$V_{(0,1,0)}$	21	$\Lambda^2_{21}(\mathbb{C}^8)$	$V_{(1,0)} \oplus V_{(0,1)}$

$V_{(0,0,1)}$	8	$\Lambda^1(\mathbb{C}^8)$	$V_{(0,0)} \oplus V_{(1,0)}$
$V_{(1,0,1)}$	48	$\Lambda^3_{48}(\mathbb{C}^8)$	$V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(2,0)}$
$V_{(2,0,0)}$	27	$\Lambda^4_{27}(\mathbb{C}^8)$	$V_{(2,0)}$
$V_{(0,0,2)}$	35	$\Lambda^4_{35}(\mathbb{C}^8)$	$V_{(0,0)} \oplus V_{(1,0)} \oplus V_{(2,0)}$
$V_{(0,1,1)}$	112		$V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(2,0)} \oplus V_{(1,1)}$
$V_{(2,0,1)}$	168		$V_{(1,1)} \oplus V_{(2,0)} \oplus V_{(3,0)}$
$V_{(1,1,0)}$	105		$V_{(0,1)} \oplus V_{(2,0)} \oplus V_{(1,1)}$
$V_{(1,0,2)}$	189		$V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(3,0)} \oplus V_{(2,0)} \oplus V_{(1,1)}$

Now, for G_2, we have the following projections.

$$\pi_7 : \Lambda^2(\mathbb{C}^7) \to \Lambda^2_7(\mathbb{C}^7), w \mapsto \frac{1}{3}w + \frac{1}{3}(w \lrcorner \psi) = \frac{1}{3}(w \lrcorner \phi) \lrcorner \phi,$$
$$\pi_{14} : \Lambda^2(\mathbb{C}^7) \to \Lambda^2_{14}(\mathbb{C}^7), w \mapsto \frac{2}{3}w - \frac{1}{3}(w \lrcorner \psi) = w - \frac{1}{3}(w \lrcorner \phi) \lrcorner \phi,$$
$$\pi_{27} : \Lambda^3(\mathbb{C}^7) \to \Lambda^3_{27}(\mathbb{C}^7), w \mapsto w + \frac{1}{4}(w \lrcorner \psi) \lrcorner \psi - \frac{1}{7}(w \lrcorner \phi)\phi.$$

For $Spin(7)$, we have the following projections.

$$\pi_7 : \Lambda^2(\mathbb{C}^8) \to \Lambda^2_7(\mathbb{C}^8), w \mapsto \frac{1}{4}(w + w \lrcorner \Phi),$$
$$\pi_{21} : \Lambda^2(\mathbb{C}^8) \to \Lambda^2_{21}(\mathbb{C}^8), w \mapsto \frac{1}{4}(3w - w \lrcorner \Phi),$$
$$\pi_{48} : \Lambda^3(\mathbb{C}^8) \to \Lambda^3_{48}(\mathbb{C}^8), w \mapsto w + \frac{1}{7}(w \lrcorner \Phi) \lrcorner \Phi,$$
$$\pi_{35} : \Lambda^4(\mathbb{C}^8) \to \Lambda^4_{35}(\mathbb{C}^8), w \mapsto \frac{1}{2}(w - *w).$$

Finally, we notice two important relations: if $dt \wedge a + v \in \Lambda^2_7(\mathbb{C}^8)$, where $a \in \Lambda^1(\mathbb{C}^7)$ and $v \in \Lambda^2(\mathbb{C}^7)$, then since, $(dt \wedge a + v) \lrcorner \Phi = 3(dt \wedge a + v)$, we have $v \lrcorner \phi = 3a$.

Moreover, if $dt \wedge b + w \in \Lambda^3_{48}(\mathbb{C}^8)$, where $b \in \Lambda^2(\mathbb{C}^7)$ and $w \in \Lambda^3(\mathbb{C}^7)$, since, $(dt \wedge b + w) \lrcorner \Phi = 0$, we have $b \lrcorner \phi = -w \lrcorner \psi$.

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