School of Mathematical and Physical Sciences

Connection Formulations of Cosmology and Unimodular Gravity

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Thesis Summary

In this thesis we examine how chiral connection formulations of gravity can be applied in the fields of quantum cosmology and unimodular gravity. Chiral connection formulations are reformulations of general relativity (GR) in which the central dynamical field is a gauge connection à la Yang–Mills, as opposed to a metric tensor or a tetrad. Here, the fields are assumed to be complex, and we require further reality conditions to get solutions for Lorentzian GR. These formulations are derived from the (chiral) Plebański formulation by integrating out variables. Unimodular gravity refers to formulations in which the cosmological constant arises as an integration constant, which can be achieved by fixing the value of the metric determinant with a dynamical constraint. We construct actions for chiral connection formulations of unimodular gravity. We then derive canonical formulations for these actions, yielding constrained Hamiltonian systems whose constraint algebras resemble somewhat modified versions of the constraint algebra from GR. Following this, we examine the classical dynamics of the spatially homogeneous Bianchi I and IX models within a certain chiral connection formulation. We focus on approaches to implementing reality conditions, including an approach where they are treated as second class constraints in Dirac's formalism. We also see how one can derive Lorentzian solutions from Euclidean signature solutions through a kind of Wick rotation. Finally, we examine the quantum cosmology of a homogeneous and isotropic FLRW type spacetime from the perspective of Krasnov's pure connection formulation of GR. We derive an established result for a two-point function from a novel perspective.

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Declaration

Chapter 1

This chapter contains a non-technical introduction with no novel results.

Chapter 2

This chapter contains only review content with no novel results.

Chapter 3

This chapter is based on [46] which was written in collaboration with Steffen Gielen.

Chapter 4

This chapter is the basis for [44] which was finalised and published after the initial submission of this thesis in collaboration with Steffen Gielen. This chapter is partially review, and partially a presentation of novel results. Specifically, sections 4.1 and 4.3 consist of review content, while section 4.2 contains novel results.

Chapter 5

This chapter is mostly based on [45] which was written in collaboration with Steffen Gielen. Section 5.2 doesn't appear in [45], and consists of review content.

Chapter 6

Elements of this chapter have been adapted from the conclusions sections of [45, 46].

Contents

Conventions

Indices:

- Spacetime tensor indices: $\mu, \nu, \rho, \ldots = 0, 1, 2, 3$
- Spatial tensor indices: $a, b, c, \ldots = 1, 2, 3$
- Internal Lorentz indices: $I, J, K, \ldots = 0, 1, 2, 3$
- Internal $SO(3)$ Lie algebra indices: $i, j, k, \ldots = 1, 2, 3$

Unless otherwise specified.

$\mathbf{S}^k V$: k^{th} symmetric power of the vector space V

$\Lambda^k V$: k^{th} antisymmetric power of the vector space V

g_{IJ} : flat metric on $\mathbb{R}^4$, $g_{IJ} = \delta_{IJ}$ or $g_{IJ} = \eta_{IJ} = \text{diag}(-1, 1, 1, 1)$

$\delta^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_k} = k!\, \delta^{[\mu_1}_{\nu_1} \ldots \delta^{\mu_k]}_{\nu_k}$: Generalised Kronecker delta

ϵ^{ijk} : totally antisymmetric tensor over $\mathbb{R}^3$ satisfying $\epsilon^{123} = +1$

ϵ^{IJKL} : totally antisymmetric tensor over $\mathbb{R}^4$ satisfying $\epsilon^{0ijk} = -\epsilon^{ijk}$

ϵ_{IJKL} : lowered form of ϵ^{IJKL} w.r.t g_{IJ}

A rank (p, q) *tensor density* of weight $W \in \mathbb{Z}$ transforms under coordinate transformations via:

$$\det\left(\frac{\partial x'}{\partial x}\right)^W A'^{\mu_1 \ldots \mu_p}{}_{\nu_1 \ldots \nu_q}(x') = \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}} \cdots \frac{\partial x'^{\mu_p}}{\partial x^{\rho_p}} \frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\sigma_q}}{\partial x'^{\nu_q}} A^{\rho_1 \ldots \rho_p}{}_{\sigma_1 \ldots \sigma_q}(x)\ .$$

$\tilde{\epsilon}^{\mu\nu\rho\sigma}$: totally antisymmetric $(4, 0)$ tensor density of weight $+1$ satisfying $\tilde{\epsilon}^{0123} = +1$.

$\underset{\sim}{\epsilon}_{\mu\nu\rho\sigma}$: totally antisymmetric $(0, 4)$ tensor density of weight -1 satisfying $\underset{\sim}{\epsilon}_{0123} = +1$.

$\varepsilon_{\mu\nu\rho\sigma} = \epsilon_{IJKL}\, e^I_\mu e^J_\nu e^K_\rho e^L_\sigma$: tensor components of the volume form ε associated to e^I_μ

$\varepsilon^{\mu\nu\rho\sigma}$: volume form components $\varepsilon_{\mu\nu\rho\sigma}$ raised with the inverse metric $g^{\mu\nu} = e^\mu_I e^\nu_J\, g^{IJ}$

Metric signature parameter: $\sigma = \pm 1$, $\sqrt{\sigma} = 1, i$

$P^{\pm}{}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}\left(\delta^{[\rho}_\mu \delta^{\sigma]}_\nu \pm \frac{1}{2\sqrt{\sigma}}\varepsilon_{\mu\nu}{}^{\rho\sigma}\right)$: (anti) self dual projector on 2-forms

$P^{\pm}{}_{KL}{}^{IJ} = \frac{1}{2}\left(\delta^{[I}_K \delta^{J]}_L \pm \frac{1}{2\sqrt{\sigma}}\epsilon^{IJ}{}_{KL}\right)$: (anti) self dual projector on bivectors

Natural units:

The speed of light in vacuum and Planck's reduced constant $\hbar$ are normalised to 1.

$\ell_P = \sqrt{8\pi G}$ is the *reduced Planck length*, where G is Newton's gravitational constant

Chapter 1

Introduction

This thesis focuses on reformulations of general relativity (GR) and their applications in quantum cosmology. A comprehensive history of reformulations of GR can be found in the recent textbook [69]; a brief history will be provided here also. Equally, a history of quantum gravity approaches can be found in [15, 21, 91]. Quantum gravity (QG) will not be a major focus in this thesis, so only a short introduction will be provided here. Introductions to quantum cosmology can be found in [16, 110].

Generally speaking, a reformulation of GR is an alternative theory, possibly of different kinds of mathematical objects, which produces equations whose solutions are in correspondence with the solutions of the Einstein field equations. That is, one must be able to construct a metric tensor from the fields of the reformulation that satisfies the Einstein field equations. Since the inception of GR in the early twentieth century, many reformulations have been presented based on a wide variety of geometric principles. The Palatini formulation proposes a slight modification of GR by taking the metric and connection (Christoffel symbols) to be independent fields at the level of the action, so that their compatibility arises as a field equation [84, 102]. Following on from this, the Eddington–Schrödinger formulation [37] solves the (vacuum) *Einstein condition* $R_{\mu\nu} \propto g_{\mu\nu}$ for the metric in terms of the Ricci tensor at the level of the Einstein-Hilbert action, leading to a *pure connection* formulation of GR. This formulation only exists for a non-zero cosmological constant, and the construction of the action comes with certain ambiguities which prevent the theory from being well defined in the general case [69]. This will be a general theme surrounding pure connection formulations of GR in which the only independent field is some kind of connection.

There are a number of formulations which come under the umbrella of the *tetrad* formalism

[80]. These follow the Palatini philosophy, where the metric tensor is now replaced with a 4-tuple of linearly independent 1-forms called the tetrad, and the Christoffel symbols are replaced with an $SO(1,3)$ *spin-connection*. This is in line with Cartan's approach to differential geometry via *moving frames* [62]. One can begin with the Holst action [57] or the Einstein-Cartan action [69] to derive the dynamics of GR in terms of the tetrad and spin connection. The metric tensor is constructed from the tetrad such that the tetrad constitutes an orthonormal basis of 1-forms. An aspect of this formulation that distinguishes it from the standard metric formulation is the appearance of a local 'internal symmetry'; the tetrad and spin connection transform under local internal Lorentz transformations. In fact, this internal symmetry allows one to couple fermionic matter such as Dirac spinors, which was not possible in the metric formulation.

In both the metric and tetrad approaches to GR, the dynamics are prescribed by equations involving the curvature. However, one can alternatively encode the dynamics of GR into different elements of the geometry of spacetime. For example one can study *teleparallel gravity* [1, 7, 69], a formulation in which one assumes a flat connection with non-vanishing torsion. The dynamics of GR arise here from the non-trivial torsion instead of the curvature. Another formulation, *non-metricity gravity* (also called *symmetric teleparallel gravity*) [81, 93], assumes a flat and torsion free connection which is not compatible with the metric/tetrad such that the dynamics of GR arise from the non-metricity of the connection. Then GR, teleparallel, and non-metricity gravity form a trinity of equivalent formulations all built off distinct geometric principles. While all of these theories reproduce the same classical dynamics, one can ask how they compare at the quantum level. In fact, this is a major motivation for studying reformulations such as these, and will be a recurring theme in this thesis.

The *ADM formalism* [8, 9, 10, 101] – named for Arnowitt, Deser and Misner – is a canonical formulation of GR where one foliates the spacetime manifold into a 1-parameter family of spatial 3-manifolds on which the dynamical fields are the induced 3-metric and its associated extrinsic curvature, which are then canonically conjugate. One arrives at a constrained Hamiltonian system of the kind introduced by Dirac [36, 52], where the Hamiltonian is a (weighted) sum over constraints. The ADM formalism provides a starting point for canonical quantisation of GR. One constructs an operator representation of the classical Poisson algebra in which the constraints become operators that annihilate the so-called *physical states*. This leads to the famous *Wheeler-DeWitt* equation, which is unsolvable in the general case. Furthermore, we arrive at the equally famous *problem of time* [6] which comes from the observation that the right-hand-side of the Schrödinger equation – $\hat{H}|\psi\rangle$ – vanishes on physical states, since the Hamiltonian is pure constraint. Many developments have been made over the last half-century in the field of canonical quantum gravity which at least partially alleviate some of the issues discovered through this formalism.

Of central interest in this thesis, one can formulate the equations of GR in terms of the *Hodge dual* [41], an automorphism on the space of differential 2-forms induced by the metric. For Lorentzian metrics, the Hodge dual has a pair of eigenspaces corresponding to eigenvalues $\pm i$ called the *self-dual* and *anti-self-dual* spaces respectively. The lowered Riemann tensor, with four covariant indices, decomposes into various dual-dual components. Then in vacuum, the Einstein condition is equivalent to setting some of these dual-dual components to zero. This is the geometric principle underlying the Plebański formulation of GR [29, 71, 89]. Of particular note is the *chiral* Plebański formulation where the metric/tetrad is now replaced with a triad of complex valued 2-forms, and where the Christoffel/spin connection is replaced with a complex $SO(3)$ connection à la Yang-Mills. The field equations come from the *Plebański action* via standard variational methods. In this formulation, the metric is secondary and constructed from the complex 2-forms via the construction due to Urbantke [104], such that those 2-forms become a basis of self-dual 2-forms. This formulation has a local symmetry corresponding to internal (complex) $SO(3)$ rotations of the fields; it is a *diffeomorphism invariant gauge theory* [66, 70]. Since the fields are initially complex valued, one must apply further *reality conditions* on top of the field equations, resulting in an Urbantke metric which is Lorentzian (perhaps with a global factor of $\pm i$). The Plebański formulation admits further formulations derived by repeatedly *integrating out* variables from the Plebański action [30, 61, 66]. Of particular note is a *chiral pure connection* formulation whose only independent field is the complex Yang-Mills connection [72, 33]. As with the Eddington-Schrödinger theory, this formulation is not well defined in the general case as the action involves the square root of a complex matrix. One also has access to a *Euclidean signature* (where the metric is positive or negative definite, but not necessarily flat) version of the Plebański theory in which all of the fields are real valued, and no reality conditions are required.

Through the canonical analysis of Plebański gravity [3, 4, 28, 61, 86, 90], one recovers the Ashtekar variables [11, 17, 57], albeit with some nuances regarding the complexity of the variables in Lorentzian signature. These variables are central to the canonical loop quantisation scheme [12, 13, 21, 91, 94, 100]. Here, one uses *spin-networks* – 4-valent graphs whose edges are labelled by the spin representations of $SU(2)$ and whose vertices are labelled by equivariant maps between the spin representations called *intertwiners* – to construct gauge invariant functions on the space of connections called *spin-network states*. These spin-network states span the algebra of square integrable function(al)s of connections, allowing one to formally define a Hilbert space of connection wavefunctions. Any given spin-network can be interpreted as being dual to a complex of tetrahedra, such that each edge has a 2D dual face and each vertex sits at the center of a tetrahedron. In this way, a spin-network defines a discretisation of space, and one defines hermitian operators corresponding to the volume and the areas of the faces of the tetrahedra. While implementing the constraints corresponding to the local $SU(2)$ gauge symmetry and the spatial diffeomorphism symmetry is well understood here, there is no universally accepted general implementation of the *Hamiltonian constraint* – corresponding to time reparametrisations – on spin-network states. In any case, one arrives at an alternative formulation of the Wheeler-DeWitt equation which is only slightly more tractable.

An alternative route to quantum gravity starting from the Plebański formulation is through *spin foam* models [15, 18, 88, 92]. While LQG provides a framework for canonical quantisation of GR in terms of the Ashtekar variables, spin foam models provide a covariant *path-integral* approach to quantising these variables. It is common in spin foam investigations to work in Euclidean signature so that all of the variables are real valued, and so that the path integral $\int \mathcal{D}A\,\mathcal{D}B\; e^{-S[A,B]}$, where S is an action for Euclidean signature GR in connection variables, is exponentially suppressed instead of oscillatory as it would be in Lorentzian signature (where one replaces $e^{-S} \to e^{iS}$). To evaluate such a path-integral, one considers a limit of discretisations of spacetime generated from spin foams: complexes formed of 2D surfaces labelled by spin representations of $SU(2)$ joined along edges labelled by intertwiners. Specifically, each spin foam is dual to a discretisation of spacetime, similar to how each spin-network is dual to a discretisation of space. This discretisation due to spin foams allows one to formally define the path integral, and also allows one to approximately evaluate transition amplitudes between initial and final spin-network states – which can be viewed as forming the boundary of the spin foam. While this is certainly a great step forward, one should note that evaluating these transition amplitudes analytically is prohibitively difficult, and evaluating them numerically is very computationally expensive.

Quantum cosmology refers to a number of related approaches in cosmology. The approach of greatest relevance to this thesis can be seen in [40, 51, 59, 78, 110], which involves restricting an action for GR (in some formulation) to spatially homogeneous and isotropic fields, and then using that restricted action as a starting point to define a quantum theory. The resulting *minisuperspace* action is much simpler than the general action, typically resembling an action for a classical particle. Here, one attains the quantum dynamics either from the reduced Wheeler-DeWitt equation, taking the form of a first or second order ODE, or via path integral methods, with both approaches yielding the same results [47]. One of the central aims of this field is to compute transition amplitudes between initial and final (current day) states of the universe, described in terms of different kinds of boundary data [39, 105]. From these investigations we get the Hartle–Hawking *no boundary proposal* [50], as well as the Vilenkin quantum tunneling proposal [106, 107].

The term *unimodular gravity* describes a collection of formulations of gravity that adhere to a set of common principles [5]. The earliest investigations of unimodular gravity originated from a simple (partial) gauge fixing of GR where one fixes the value of the metric determinant, typically to ± 1, in order to simplify computations [38, 85]. These early approaches inspired the development of new formulations in which the metric determinant is fixed at the level of the action by means of a dynamical constraint [25, 26, 103]. In these formulations, the symmetry group is reduced to 'volume preserving' diffeomorphisms which leave the value of the metric determinant unchanged. A significant aspect of these theories is that the value of the cosmological constant is not prescribed in the action; rather the cosmological constant arises as an integration constant such that these theories contain all solutions corresponding to all its possible values. This opens up discussion regarding the so-called *cosmological constant problem* [63, 95]. An alternative unimodular formulation introduced in [53] achieves the same result for the cosmological constant while retaining the full diffeomorphism symmetry from GR. Here, one promotes the cosmological constant to an independent field which is constrained to be constant by means of a Lagrange multiplier-like field which also parametrises the metric determinant. A noteworthy aspect of this approach is that the metric determinant is equal to the 4-divergence of a vector density, and hence the 4-volume of any spacetime domain is given by an integral over its 3D boundary. In fact, one can use this property to define a kind of 'volume time' for globally hyperbolic spacetimes. The equivalence of GR and unimodular gravity is fairly well understood – and universally agreed upon – at the classical level, however the equivalence at the quantum level is still debated [31, 76, 42, 74].

We conclude this introduction with an outline of the contents of this thesis. Chapter 2 will provide the reader with the necessary technical background to engage with the subsequent chapters. This includes short technical introductions to: the underlying structure of diffeomorphism invariant gauge theories; formulations of gravity such as the tetrad formalism, Plebański gravity, pure connection gravity, and unimodular gravity; and Dirac's formalism for constrained Hamiltonian dynamics. Chapter 3, based on [46], outlines how one can apply the principles of unimodular gravity in Plebański gravity; we present new actions – alongside some established actions – for *unimodular Plebański* formulations of gravity. In addition, we present unimodular analogues of the chiral connection actions which come from integrating out variables from the Plebański action, such as Krasnov's pure connection action [72]. Adding to this, there is some discussion on *unimodular clocks* mirroring [53]. In chapter 4, containing some unpublished novel results, we provide the canonical formulations for the unimodular actions introduced in [46], as well as a somewhat original presentation of the already well-studied canonical formulation of Plebański gravity. In particular, we present a novel canonical formulation of the *preferred volume* unimodular Plebański theory. Here, one arrives at a constraint algebra distinct from the algebra for GR, reflecting the reduction of the symmetry group to volume preserving diffeomorphisms in this theory. Additionally, one attains a Hamiltonian which is not pure constraint, and has a non-(weakly)-vanishing component. For completeness, we also review the unimodular formulation from [96]. Chapter 5, based on [45], contains an investigation of chiral connection gravity in the spatially homogeneous setting. Specifically, we examine models with Bianchi I and IX type symmetries in diagonal variables starting from the *first order* chiral connection action, which comes from integrating out the 2-form fields from the Plebański action [30, 55, 61]. We examine approaches to implementing the reality conditions, which have multiple solution branches, including an approach where they are treated as (second class) constraints in Dirac's formalism. We also see how the various Lorentzian signature solutions can be attained from Euclidean signature solutions via a kind of *Wick rotation*. Following this, we examine the quantum cosmology of homogeneous and isotropic models starting from chiral connection actions, using a path integral approach [39, 40, 47]. We derive an established result, an analytic expression for a two-point function with connection boundary data [59], from a novel perspective. To conclude, we conduct the same path integral computation instead starting from a unimodular chiral connection action introduced earlier [46], deriving a slightly modified result.

Chapter 2

Technical Background

2.1 Differential Forms and Gauge Theories

2.1.1 The exterior differential algebra

Our treatment of differential forms closely follows the textbook [23] which provides a thorough introduction to the exterior differential algebra, along with an exploration of its applications in algebraic topology. One can also see [58, 80] for alternative perspectives. A short review of the exterior differential algebra is provided here.

Let $\mathcal{M}$ be a smooth manifold of dimension $D+1$, and let x^μ be local coordinates where lowercase Greek indices of the kind $\mu, \nu, \ldots$ take values $0, 1, \ldots, D-1, D$. Let dx^μ denote the standard coordinate basis of covector fields. We define a formal pairwise product on these basis elements called the *wedge* product (or *exterior* product), denoted $dx^\mu \wedge dx^\nu$, that is associative, C^∞-linear, and alternating:

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu \ . \tag{2.1}$$

In some texts, for example [23, 69], the wedge symbol $\wedge$ is suppressed and the product is written simply $dx^\mu dx^\nu$. In this thesis, we always include the wedge symbol. For any positive integer $k \leq D+1$, a differential k-form $\omega \in \Omega^k(\mathcal{M})$ is an object of the kind

$$\omega = \frac{1}{k!}\, \omega_{\mu_1 \ldots \mu_k}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} \ , \tag{2.2}$$

where $\omega_{\mu_1 \ldots \mu_k} = \omega_{[\mu_1 \ldots \mu_k]}$ are component functions. We identify the space of k-forms with the space of totally antisymmetric $(0, k)$ tensor fields via

$$\omega = \omega_{\mu_1 \ldots \mu_k}\, dx^{\mu_1} \otimes \ldots \otimes dx^{\mu_k} \ . \tag{2.3}$$

Then $\omega_{\mu_1 \dots \mu_k}$ are the *tensor components*. Furthermore, we define the space of 0-forms to be the ring of smooth functions over $\mathcal{M}$, so that $\Omega^0(\mathcal{M}) = C^\infty(\mathcal{M})$. The wedge product defined on the basis elements dx^μ linearly extends to a product of arbitrary k and ℓ forms ω and θ, yielding a $(k+\ell)$-form $\omega \wedge \theta$ satisfying

$$\omega \wedge \theta = (-)^{k\ell} \theta \wedge \omega \ . \tag{2.4}$$

The tensor components of the wedge product are given by

$$(\omega \wedge \theta)_{\mu_1 \dots \mu_{k+\ell}} = \frac{(k+\ell)!}{k!\,\ell!}\, \omega_{[\mu_1 \dots \mu_k}\, \theta_{\mu_{k+1} \dots \mu_{k+\ell}]} \ . \tag{2.5}$$

Interior product

Given a vector field X, the interior product of a k-form ω w.r.t X is defined

$$i_X \omega = \frac{1}{(k-1)!}\, \omega_{\mu\, \mu_1 \dots \mu_{k-1}} X^\mu\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{k-1}} \ . \tag{2.6}$$

That is, the interior product is the insertion of a vector field into the first slot of a k-form, treated as a totally antisymmetric covariant tensor.

Exterior derivative and cohomology

The *exterior derivative* is an $\mathbb{R}$-linear map $d : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ which is defined first on 0-forms (functions) by

$$df = \partial_\mu f\, dx^\mu \ , \tag{2.7}$$

and then on $0 < k$ forms by

$$d\omega = \frac{1}{k!}\, d\omega_{\mu_1 \dots \mu_k} \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} \ . \tag{2.8}$$

The exterior derivative is nilpotent of order 2: for any k-form ω, acting twice with the exterior derivative yields 0, $d^2\omega = d(d\omega) = 0$. Furthermore, for any k-form ω and any ℓ-form θ we have

$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-)^k \omega \wedge d\theta \ . \tag{2.9}$$

A k-form ω is called *closed* if $d\omega = 0$, and *exact* if there exists some $(k-1)$-form τ such that $\omega = d\tau$. All exact forms are closed, but there may exist closed forms which are not exact. The extent to which the space of closed k-forms, denoted $Z^k(\mathcal{M})$, differs from the space of exact k-forms, denoted $B^k(\mathcal{M})$, is measured by their quotient $H^k(\mathcal{M}) = Z^k(\mathcal{M})/B^k(\mathcal{M})$ which is called the *De Rham Cohomology* of order k. The Cohomologies $H^k(\mathcal{M})$ are topological invariants of the manifold $\mathcal{M}$.

Integration and pull-back

It is straightforward to check that the maximal form degree is equal to the dimension of the manifold. In our case, the highest order forms are $\Omega^{D+1}(\mathcal{M})$ which are called *top-forms*. Top-forms can be integrated over submanifolds $\mathcal{U} \subset \mathcal{M}$ of maximal dimension. In order to define integration for top-forms, we need to choose a preferred ordering of the coordinates $(x^0, \ldots, x^D)$ called an *orientation*. It is a straightforward exercise to see that any top-form Ω can be written as

$$\Omega = \tilde{\Omega}\, dx^0 \wedge \ldots \wedge dx^D \ , \tag{2.10}$$

where $\tilde{\Omega}$ is a scalar density of weight $+1$, and where the coordinate 1-forms dx^μ are ordered according to the orientation. This decomposition is unique. The integral of the top-form Ω over the submanifold $\mathcal{U}$ is defined

$$\int_{\mathcal{U}} \Omega = \int_{\mathcal{U}} d^{D+1}x\, \tilde{\Omega} \ . \tag{2.11}$$

Strictly speaking, integration on manifolds is only well defined if the manifold is orientable. That is, if there exists a covering of coordinate charts such that the Jacobians of the chart transition maps are strictly positive. The need for this restriction becomes especially pronounced in the - not at all rare - cases where it is impossible to cover the manifold with a single chart. In these cases, the integral must be constructed chart-wise. Famous examples of non-orientable manifolds are the *Möbius band*, and its higher dimensional counterpart the *Klein bottle*. We need not worry about this aspect of the theory since all of the manifolds we examine in this thesis will be orientable.

Let $\mathcal{N}$ be a smooth manifold with local coordinates y^α, and let $\phi : \mathcal{M} \to \mathcal{N}$ be a smooth map. ϕ induces a map ϕ^* called the *pull-back* which sends a k-form $\omega \in \Omega^k(\mathcal{N})$ to a k-form $\phi^*\omega \in \Omega^k(\mathcal{M})$. The pull-back is defined via

$$\phi^* \left(f\, dy^{\alpha_1} \wedge \ldots \wedge dy^{\alpha_k} \right) = (f \circ \phi)\, \frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}} \cdots \frac{\partial y^{\alpha_k}}{\partial x^{\mu_k}}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} \ . \tag{2.12}$$

$\partial y^\alpha / \partial x^\mu$ is the Jacobi matrix of the map ϕ w.r.t the local coordinates x^μ and y^α on $\mathcal{M}$ and $\mathcal{N}$ respectively. We have the following theorem which relates integration with the pull-back.

Theorem 2.1.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be a pair of smooth manifolds, let $\phi : \mathcal{M} \to \mathcal{N}$ be a smooth map, and let ω be a top-form on $\mathcal{N}$. Furthermore, let $\mathcal{U} \subset \mathcal{M}$ be a submanifold such that the image $\phi(\mathcal{U}) \subset \mathcal{N}$ is a submanifold of maximal dimension. Then*

$$\int_{\phi(\mathcal{U})} \omega = \int_{\mathcal{U}} \phi^* \omega \ . \tag{2.13}$$

To conclude, we have the following important theorem which relates integration with the exterior derivative.

Theorem 2.1.2 (Stokes' theorem on manifolds)**.** *Let $\mathcal{M}$ be a smooth manifold of dimension $D+1$ with boundary $\partial\mathcal{M}$, which we allow to be empty, and let $\omega \in \Omega^D(\mathcal{M})$ be a differential form whose degree is one fewer than the dimension of $\mathcal{M}$. Then*

$$\int_{\mathcal{M}} d\omega = \int_{\partial\mathcal{M}} \omega \ . \tag{2.14}$$

These theorems are well established and proofs can be found in any sufficiently comprehensive textbook on differential geometry/topology. For examples, see [23, 80].

2.1.2 Differential forms in gauge theory

Let $SO(3)$ be the group of real valued 3×3 matrices h with unit determinant $\det h = 1$, and satisfying $hh^T = \mathbb{I}$. $SO(3)$ is a compact and simply connected Lie group whose underlying manifold is real projective 3-space $\mathbf{RP}^3$. The associated Lie algebra $\mathfrak{so}(3)$ is the algebra of antisymmetric real valued 3×3 matrices. The Lie bracket is the commutator of matrices, $[X,Y] = XY - YX$. One can construct a basis τ_i on $\mathfrak{so}(3)$ via

$$\tau_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} , \quad \tau_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \quad \tau_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{2.15}$$

This basis set satisfies $[\tau_i, \tau_j] = \epsilon_{ij}{}^k \tau_k$ where lowercase Latin indices $i, j, k, \ldots$ take values $1, 2, 3$ and are raised and lowered with the Kronecker deltas δ^{ij}, δ_{ij}. Note, the raising and lowering of these indices is purely aesthetic. The Lie bracket of a pair $X, Y \in \mathfrak{so}(3)$ can be expressed in index form by

$$[X,Y]^i = \epsilon_{jk}{}^i X^j Y^k \ . \tag{2.16}$$

The complexified Lie algebra $\mathfrak{so}(3)_{\mathbb{C}} \cong \mathfrak{so}(3) \oplus i\mathfrak{so}(3)$ can be constructed as the free complex Lie algebra generated from the basis set τ_i. In practice, we allow the components X^i to be complex valued. Then $\mathfrak{so}(3)_{\mathbb{C}}$ consists of antisymmetric 3×3 matrices with complex entries. The matrix exponential is a map from the space of arbitrary square matrices to the group of invertible matrices defined

$$\exp : \mathrm{End}(n) \to GL(n) \ , \ A \ \mapsto \ e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \ . \tag{2.17}$$

The matrix exponential maps $\mathfrak{so}(3)$ into $SO(3)$, and maps $\mathfrak{so}(3)_{\mathbb{C}}$ into $SO(3,\mathbb{C})$ – the group of complex 3×3 matrices with unit determinant and whose inverses are given by their transposes.

A differential k-form ω can be thought of as an assignment to each point $x \in \mathcal{M}$, of a multilinear map $\omega_x : T_x\mathcal{M} \times \overset{k}{\cdots} \times T_x\mathcal{M} \to \mathbb{R}$. One can extend the notion of a differential form by changing the codomain of this map from $\mathbb{R}$ to some other vector space. In particular, one could choose the codomain to be $\mathfrak{so}(3)$ or $\mathfrak{so}(3)_{\mathbb{C}}$, or any other Lie algebra. The result is a *Lie algebra valued k-form.* Any Lie algebra valued k-form $B \in \Omega^k(\mathcal{M}) \otimes \mathfrak{so}(3)_{\mathbb{C}}$ can be written in the form

$$B = B^i \otimes \tau_i \ , \tag{2.18}$$

where B^i are complex valued k-forms called the component k-forms. We define a graded Lie bracket on $\mathfrak{so}(3)_{\mathbb{C}}$ valued forms by

$$[B, C]^i = \epsilon^i{}_{jk} B^j \wedge C^k \ , \tag{2.19}$$

where B and C are a pair of a k-form and an ℓ-form valued in $\mathfrak{so}(3)_{\mathbb{C}}$ respectively, and $[B, C]$ is a $(k+\ell)$-form valued in $\mathfrak{so}(3)_{\mathbb{C}}$. The Lie bracket satisfies

$$[B, C] = -(-)^{k\ell}\,[C, B] \ . \tag{2.20}$$

One can check that this definition of the Lie bracket is independent of the choice of basis τ_i.

Gauge transformations

We are building towards using these Lie algebra valued differential forms as fields in $SO(3,\mathbb{C})$ gauge theories. As such, we need to prescribe how these fields should transform under the action of local $SO(3,\mathbb{C})$ gauge transformations. There are a number of ways to do this, each producing a different kind of gauge theory. For our purposes, we define the action of a local $SO(3,\mathbb{C})$ gauge transformation, generated by a smooth group valued function, $U(x) \in SO(3,\mathbb{C})$, on an $\mathfrak{so}(3)_{\mathbb{C}}$ valued k-form B via

$$U : B \ \mapsto \ U^{-1} B U \ . \tag{2.21}$$

A Lie algebra valued form that transforms in this way is said to transform in the *adjoint representation.* Let U be a group valued function generated via exponentiation by $U = e^{\varepsilon\alpha}$ where $\alpha \in C^\infty(\mathcal{M}) \otimes \mathfrak{so}(3)_{\mathbb{C}}$ is a smooth function valued in the Lie algebra, and where ε is a small parameter. Then $e^{\varepsilon\alpha}$ is approximately given by $e^{\varepsilon\alpha} \approx \mathbb{I} + \varepsilon\alpha$, and the local gauge transformation induced by $e^{\varepsilon\alpha}$ is given approximately by

$$B \overset{\sim}{\mapsto} B + \varepsilon[B, \alpha] \;, \tag{2.22}$$

where we ignore terms above linear order in ε. The right hand side (RHS) here has the form $B + \varepsilon\, \delta_\alpha B$ where $\delta_\alpha B = [B, \alpha]$ is the *infinitesimal form* of the gauge transformation. More formally, we would construct a 1-parameter subgroup via $g_\lambda = e^{\lambda\alpha}$, then the infinitesimal form of the transformation is computed via

$$\delta_\alpha B = \left.\frac{d}{d\lambda}\right|_{\lambda=0} g_\lambda^{-1} B g_\lambda \overset{!}{=} [B, \alpha] \;. \tag{2.23}$$

We may write the infinitesimal gauge transformations in index form as

$$\delta_\alpha B^i = \epsilon^i{}_{jk} B^j \alpha^k \;. \tag{2.24}$$

Connections

A *connection form* is an $\mathfrak{so}(3)_\mathbb{C}$ valued 1-form $A \in \Omega^1(\mathcal{M}) \otimes \mathfrak{so}(3)_\mathbb{C}$. The connection form induces a graded derivation on Lie algebra valued forms called the *exterior covariant derivative*, denoted D_A. More precisely, D_A acts on $\mathfrak{so}(3)_\mathbb{C}$ valued forms which transform in the adjoint representation under gauge transformations. The action of D_A is given explicitly by

$$D_A B^i = dB^i + \epsilon^i{}_{jk} A^j \wedge B^k \;. \tag{2.25}$$

Note that D_A increases the form degree by 1. The connection transforms under the action of local $SO(3, \mathbb{C})$ gauge transformations by

$$U : A \;\mapsto\; U^{-1} A U + U^{-1} dU \;. \tag{2.26}$$

The infinitesimal form of this transformation is given by

$$\delta_\alpha A^i = D_A \alpha^i \;. \tag{2.27}$$

It is straightforward to confirm that $D_A B^i$ transforms in the adjoint representation under the action of local $SO(3, \mathbb{C})$ gauge transformations

$$\begin{aligned} \delta_\alpha D_A B^i &= d\delta_\alpha B^i + \epsilon^i{}_{jk}\, \delta_\alpha A^j \wedge B^k + \epsilon^i{}_{jk} A^j \wedge \delta_\alpha B^k \\ &\overset{!}{=} \epsilon^i{}_{jk}\, D_A B^j\, \alpha^k \;. \end{aligned} \tag{2.28}$$

Hence D_A provides us with an extension of the exterior derivative that is compatible with gauge transformations. This derivative can, and will, be used to construct actions and equations for physical theories such that they transform correctly under gauge transformations. Acting twice with the exterior covariant derivative yields

$$D_A D_A B^i = \epsilon^i{}_{jk} F^j \wedge B^k \ . \tag{2.29}$$

Here, F^i is an $\mathfrak{so}(3)_{\mathbb{C}}$ valued 2-form called the *curvature*, which expands as

$$F^i = dA^i + \frac{1}{2}\epsilon^i{}_{jk} A^j \wedge A^k \ . \tag{2.30}$$

The curvature 2-form transforms in the adjoint representation under the action of local $SO(3,\mathbb{C})$ gauge transformations

$$\delta_\alpha F^i = \epsilon^i{}_{jk} F^j \alpha^k \ . \tag{2.31}$$

Then the curvature 2-form provides us with a first order derivative of the connection that is compatible with local gauge transformations. Again, this will be an indispensable building block when constructing physical theories.

Lie algebra tensors

So far, we have examined differential forms valued in $\mathfrak{so}(3)_{\mathbb{C}}$. We would now like to extend our constructions to differential forms valued in tensors over $\mathfrak{so}(3)_{\mathbb{C}}$. These are objects of the kind

$$Q = Q^{i_1 \ldots i_p}{}_{j_1 \ldots j_q} \otimes \tau_{i_1} \otimes \ldots \otimes \tau_{i_p} \otimes \tau^{j_1} \otimes \ldots \otimes \tau^{j_q} \ , \tag{2.32}$$

where $Q^{i_1 \ldots i_p}{}_{j_1 \ldots j_q}$ are complex k-forms, and where τ^i denote the *dual basis* on $\mathfrak{so}(3)^*_{\mathbb{C}}$ satisfying $\tau^i(\tau_j) = \delta^i_j$. In the case of τ_i and τ^i, the raising and lowering of indices is not superficial. The map $\mathfrak{so}(3)_{\mathbb{C}} \to \mathfrak{so}(3)^*_{\mathbb{C}}$ defined $X^i\tau_i \mapsto X_i\tau^i$ is a linear isomorphism akin to the musical isomorphisms $\flat$ and $\sharp$ of Riemannian geometry.

A form $Q^i{}_j$ valued in order (1,1) tensors over $\mathfrak{so}(3)_{\mathbb{C}}$ that transforms in the (composite) adjoint representation under the action of local gauge transformations has infinitesimal transformations given by

$$\delta_\alpha Q^i{}_j = \epsilon^i{}_{kl}\, Q^k{}_j\, \alpha^l - \epsilon^k{}_{jl}\, Q^i{}_k\, \alpha^l \ . \tag{2.33}$$

One can extrapolate the rule for transforming forms valued in Lie algebra tensors of higher order. The exterior covariant derivative of a form $C^i{}_j$ is given by

$$D_A Q^i{}_j = dQ^i{}_j + \epsilon^i{}_{kl}\, A^k \wedge Q^l{}_j + \epsilon_{jk}{}^l\, A^k \wedge Q^i{}_l \ . \tag{2.34}$$

These extensions of the of δ_α and D_A are compatible with the raising and lowering of Lie algebra indices $i, j, k, \ldots$ by the Kronecker deltas.

2.2 Einstein-Cartan Gravity

2.2.1 Palatini Formalism

The reformulations that we will explore in this thesis share a common key principle. This is the principle of separating the metric and the connection at the level of the action. The Riemann curvature tensor, which directly provides the information about the curvature of spacetime that we need to model the effects of gravity, depends on the metric only through the connection. That is, one can write $\mathrm{Riem}(g) = \mathrm{Riem}(\Gamma(g))$. In GR, the connection Γ is chosen to be the unique torsion-free and metric compatible connection. This is the *Levi-Civita* connection connection which can be written explicitly in terms of the metric components and their derivatives via the famous formula

$$\Gamma^{\nu}{}_{\rho\mu} = \frac{1}{2} g^{\nu\sigma} \left(\partial_\rho g_{\sigma\mu} + \partial_\mu g_{\rho\sigma} - \partial_\sigma g_{\rho\mu} \right) . \tag{2.35}$$

This restriction on the connection is a modelling assumption. It is common practice within theoretical physics to weaken certain modelling assumptions to see what happens to the theory. The motivation for this comes from the general sense among physicists that the best theories tend to be the simplest, with the fewest assumptions. In this case, we would like the metric and connection variables to be independent at the level of the action. One begins with the *bare* Einstein-Hilbert action, without the Gibbons–Hawking–York term [43], with cosmological constant Λ given by

$$S_{\mathrm{EH}}[g] = \frac{1}{16\pi G} \int d^4x \, \sqrt{-g} \, (R - 2\Lambda) \ . \tag{2.36}$$

One then explicitly writes the Ricci scalar R as a contraction of the inverse metric with the Ricci tensor $R_{\mu\nu}(\Gamma)$ taken as a function of the connection (which is now independent of the metric). This yields the *Palatini formulation* [84] with action

$$S_{\mathrm{Pal}}[g, \Gamma] = \frac{1}{16\pi G} \int d^4x \, \sqrt{-g} \, (g^{\mu\nu} R_{\mu\nu}(\Gamma) - 2\Lambda) \ , \tag{2.37}$$

where G is Newton's gravitational constant. We maintain that the connection is torsion-free at the level of the action, so that $\Gamma^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu}$. Variations of (2.37) with respect to the metric provide the familiar Einstein equations,

$$\frac{\delta S_{\mathrm{Pal}}}{\delta g_{\mu\nu}} = 0 \quad : \quad R_{\mu\nu} - \frac{1}{2} g^{\alpha\beta} R_{\alpha\beta} \, g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \ . \tag{2.38}$$

Furthermore, variations with respect to the connection provide

$$\frac{\delta S_{\rm Pal}}{\delta \Gamma^{\rho}{}_{\mu\nu}} = 0 \quad : \quad \nabla_\rho(\sqrt{-g}\, g^{\mu\nu}) - \delta^\nu_\rho\, \nabla_\sigma(\sqrt{-g}\, g^{\mu\sigma}) = 0 \ , \tag{2.39}$$

which can be shown to be equivalent to the metric compatibility condition $\nabla_\rho g_{\mu\nu} = 0$. Hence the Palatini formulation is equivalent to Einstein's GR at the level of the field equations. One should note that the separation of the metric and the connection may give rise to extra quantum fluctuations corresponding to non-metricity that otherwise wouldn't appear in quantum extensions of GR. Furthermore, the Palatini action contains only up to first derivatives in the fields, which is in contrast to the Einstein-Hilbert action which contains up to second derivatives in the metric. Consequently, the Palatini action has a well defined variational principle and no extra boundary term, such as the Gibbons–Hawking–York term from GR [43, 113], is required. More specifically, variations of the bare Einstein–Hilbert action (2.36) w.r.t the metric produce a surface term containing derivatives of the metric variations $\nabla_\mu \delta g_{\nu\rho}$ which cannot be fixed on the boundary. Then the addition of a boundary term to the bare action is necessary to cancel out these troublesome terms and make the variational principle well defined. In contrast, variations of the Palatini action w.r.t the connection produce a surface term containing no such derivatives.

2.2.2 The solder form and the spin connection

A comprehensive exploration of the following constructions can be found in [62]. The *solder form*, often also called the *tetrad*, is a smooth map e that assigns to each point $x \in \mathcal{M}$ a linear isomorphism $e_x : T_x\mathcal{M} \to \mathbb{R}^4$ from the tangent space over x to Minkowski space $(\mathbb{R}^4, \eta)$. Here η, with indices η_{IJ}, is the Minkowski metric with signature $(-+++)$. Alternatively, one can interpret the solder form as a 4-tuple of 1-forms

$$e^I = e^I_\mu\, dx^\mu \ . \tag{2.40}$$

The indices $I, J, \ldots = 0, 1, 2, 3$ are called internal indices; they index over the components of a 4-vector in the local copy of Minkowski space associated – or 'soldered' – to the tangent space $T_x\mathcal{M}$ by e. One can use the solder form to pull-back the Minkowski inner product onto each tangent space on $\mathcal{M}$, yielding a metric tensor over M given by

$$g(X, Y) = \langle e(X), e(Y)\rangle_\eta \quad \Leftrightarrow \quad g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ} \ . \tag{2.41}$$

Informally, the solder form is the 'square root' of the metric. This is seen at the level of the metric volume factor

$$\sqrt{|g|} = |\det e| \ , \tag{2.42}$$

where $\det e$ denotes the determinant of the 4×4 matrix with components e^I_μ. The metric constructed from e is invariant under *local Lorentz transformations*, which are given by

$$e^I_\mu \;\mapsto\; (\Lambda^{-1})^I{}_J\, e^J_\mu \;, \tag{2.43}$$

where $\Lambda^I{}_J$ is a matrix in $O(1,3)$ whose value varies smoothly across $\mathcal{M}$.

In a more formal introduction to this topic, one would say that the solder form constitutes a vector bundle isomorphism between the tangent bundle and a vector bundle $E \overset{\Pi}{\to} \mathcal{M}$ whose fibers are copies of Minkowski space. Furthermore, there is a right action by the Lorentz group $O(1,3)$ on the fibers of the vector bundle $E \overset{\Pi}{\to} \mathcal{M}$ in the form of the local Lorentz transformations. From this perspective, the spin connection can be taken to be an $O(1,3)$ affine connection on the vector bundle $E \overset{\Pi}{\to} \mathcal{M}$. Locally, this is an $\mathfrak{so}(1,3)$ valued 1-form

$$\omega^I{}_J = \omega_\mu{}^I{}_J\, dx^\mu \;, \tag{2.44}$$

where we take $\mathfrak{so}(1,3)$ to be the algebra of real 4×4 matrices $A^I{}_J$ satisfying the raised antisymmetry condition $A^I{}_K\, \eta^{KJ} = A^{IJ} = -A^{JI}$, whose Lie bracket is the commutator of matrices. The spin connection transforms under local Lorentz transformations via

$$\omega^I{}_J \;\mapsto\; (\Lambda^{-1})^I{}_K\, \omega^K{}_L\, \Lambda^L{}_J + (\Lambda^{-1})^I{}_K\, d\Lambda^K{}_J \;. \tag{2.45}$$

The curvature of the connection is an $\mathfrak{so}(1,3)$ valued 2-form given by

$$R^I{}_J = d\omega^I{}_J + \omega^I{}_k \wedge \omega^K{}_J \;. \tag{2.46}$$

This curvature transforms under local Lorentz transformations via

$$R^I{}_J \;\mapsto\; (\Lambda^{-1})^I{}_K\, R^K{}_L\, \Lambda^L{}_J \;. \tag{2.47}$$

Riemannian gravity

Instead of using the solder form to identify each tangent space with a copy of Minkowski space, one can instead identify each tangent space with a copy of Euclidean space $(\mathbb{R}^4, \delta)$ where $\delta(U,V) = \delta_{IJ}\, U^I V^J$ is the Euclidean inner product on $\mathbb{R}^4$. The pull-back of this inner product across the solder form induces a Riemannian metric on M. This is a metric, $g_{\mu\nu} = e^I_\mu e^J_\nu\, \delta_{IJ}$, with signature $(++++)$. In this case, the metric is invariant under local $O(4)$ transformations, $e^I_\mu \mapsto R^I{}_J e^J_\mu$. The spin connection is an $\mathfrak{so}(4)$ valued 1-form $\omega^I{}_J$ satisfying $\omega^{IJ} = -\omega^{JI}$ where internal indices are raised and lowered with the Kronecker deltas, δ^{IJ}, δ_{IJ}.

2.2.3 Einstein-Cartan Action

Einstein-Cartan theory is a reformulation of GR in the language of solder forms and spin connections. For original literature, one should see [32]. For more modern explorations of this formulation, as a precursor to formulations of quantum gravity, one can see [21, 13, 100]. In Einstein-Cartan theory, the metric and the connection – which are taken to be independent at the level of the action à la Palatini – are replaced by the solder form e and the spin-connection ω. The Einstein-Cartan action reads

$$S[e,\omega] = \frac{1}{32\pi G}\int \epsilon_{IJKL}\, e^I \wedge e^J \wedge \left(R^{KL} - \frac{\Lambda}{6} e^K \wedge e^L\right) \ . \tag{2.48}$$

ϵ_{IJKL} is a totally antisymmetric symbol which only takes the values $0, \pm 1$. We choose our conventions such that the raised form ϵ^{IJKL} always satisfies $\epsilon^{0ijk} = -\epsilon^{ijk}$. Then by lowering with the Minkowski metric, or the Kronecker delta, we compute $\epsilon_{0ijk} = \epsilon_{ijk}$ (Lorentzian), or $\epsilon_{0ijk} = -\epsilon_{ijk}$ (Riemannian). Here, Λ is the cosmological constant. The action in this form can be found in [69]. This action is diffeomorphism invariant, and invariant under local Lorentz, or $O(4)$, transformations. In this way, one might consider Einstein-Cartan theory to be a diffeomorphism invariant gauge theory. Although, this is not how it is typically presented in the literature. The field equations are computed via standard variational methods yielding

$$de^I + \omega^I{}_J \wedge e^J = 0 \ , \tag{2.49a}$$

$$\epsilon_{IJKL}\, e^J \wedge \left(R^{KL} - \frac{\Lambda}{3} e^K \wedge e^L\right) = 0 \ . \tag{2.49b}$$

The first of these (2.49a) is *Cartan's first structure equation*. The left hand side (LHS) of (2.49a) defines a quantity called the torsion, denoted $\Theta^I = de^I + \omega^I{}_J \wedge e^J$. Informally, the torsion measures the extent to which the local frame induced by e^I 'twists' as it is parallelly transported along a curve by the spin connection. Then (2.49a) states that the solder form has vanishing torsion w.r.t the connection ω. (2.49a) has a unique solution for the connection in terms of the solder form $\omega(e)$. Furthermore, this solution can be expanded as

$$\omega_\mu{}^I{}_J = e^I_\alpha\, \Gamma^\alpha{}_{\beta\mu}\, e^\beta_J + e^I_\alpha\, \partial_\mu e^\alpha_J \ , \tag{2.50}$$

where $\Gamma^\alpha{}_{\beta\mu}$ are the Christoffel symbols for the Levi-Civita connection associated to the metric $g_{\mu\nu} = e^I_\mu e^J_\nu g_{IJ}$ with $g_{IJ} = \eta_{IJ}$ or $g_{IJ} = \delta_{IJ}$. Additionally, $e^\mu_I = g^{\mu\nu} g_{IJ} e^J_\nu$ is the inverse of the solder form, satisfying $e^I_\mu e^\mu_J = \delta^I_J$ and $e^I_\mu e^\nu_I = \delta^\mu_\nu$. This relationship between the spin-connection and the Levi-Civita connection yields a further relationship between the spin curvature and the Riemann curvature which reads

$$R^I{}_{J\mu\nu} = e^I_\alpha e^\beta_J R^\alpha{}_{\beta\mu\nu} \ . \tag{2.51}$$

The proof is by direct computation. Using these relationships as a dictionary to translate between the Einstein-Cartan variables and the usual metric GR variables, one can show that the second field equation (2.49b) is equivalent to the vacuum Einstein field equations (with cosmological constant Λ),

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \ . \tag{2.52}$$

The proof is again by direct computation, and can be found in the supplementary material (2.A.1).

2.3 Plebański Gravity

2.3.1 Self-dual 2 forms

The presence of a metric tensor on a 4-manifold $\mathcal{M}$ induces a linear isomorphism on the space of 2-forms $\Omega^2(\mathcal{M})$ called the *Hodge dual*. The Hodge dual of a 2-form A, denoted $\star A$, is defined

$$\star A_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}A_{\rho\sigma} \ . \tag{2.53}$$

$\varepsilon_{\mu\nu\rho\sigma}$ are the components of the metric volume form in some chosen orientation. The Hodge dual satisfies

$$\star^2 - \sigma = 0 \ , \tag{2.54}$$

where σ denotes the signature of the metric. That is $\sigma = +1$ for metrics with Euclidean signature - all positive or all negative eigenvalues - and $\sigma = -1$ for metrics with Lorentzian signature. Hence, the Hodge dual decomposes the space of 2-forms into a direct product $\Omega^+(\mathcal{M}) \oplus \Omega^-(\mathcal{M})$ of eigenspaces corresponding to eigenvalues $+\sqrt{\sigma}$ and $-\sqrt{\sigma}$ called the *self-dual* (SD) and *anti-self-dual* (ASD) spaces respectively. In the case of Lorentzian signature metrics, which will be of primary concern in this thesis, we only have access to this decomposition if we extend the space of real 2-forms to the space of complex valued 2-forms $\Omega^2(\mathcal{M}, \mathbb{C})$. We may define projectors $P^{\pm}$ that map any (complex) 2-form onto their self-dual and anti-self-dual parts via,

$$P^{\pm}{}_{\mu\nu}{}^{\rho\sigma} = \frac{1}{2}\left(\delta_\mu^{[\rho}\delta_\nu^{\sigma]} \pm \frac{1}{2\sqrt{\sigma}}\varepsilon_{\mu\nu}{}^{\rho\sigma}\right) \ . \tag{2.55}$$

Then any 2-form decomposes as $A = P^+A + P^-A$.

Decomposition of the complex Lorentz algebra

Let g be a real metric tensor with either Euclidean signature, or Lorentzian signature. In the context of this thesis, a metric tensor is said to have *Euclidean signature* if it has all positive or all negative Eigenvalues. A Euclidean signature metric is not required to be flat. Similarly, a metric tensor is said to have Lorentzian signature if one of the Eigenvalues has the opposite sign to the other three, $(-+++)$ or $(+---)$. Continuing on, let

$$\varepsilon = \frac{1}{24}\,\epsilon_{IJKL}\, e^I \wedge e^J \wedge e^K \wedge e^L \tag{2.56}$$

be the volume form. We maintain our conventions, $\epsilon^{0ijk} = -\epsilon^{ijk}$, from the previous section. We recall that the solder form can be viewed as a map that assigns to each point $x \in \mathcal{M}$ a linear isomorphism $e_x : T_x\mathcal{M} \to \mathbb{R}^4$ that couples the tangent space with a copy of flat space. Then the solder form induces a map $\Lambda^2\, T^*_x\mathcal{M} \to \Lambda^2\mathbb{R}^4$ from the space of 2-forms over x to the space of *bivectors* over $\mathbb{R}^4$,

$$e_x : B_{\mu\nu} \;\mapsto\; B^{IJ} = e^{I\mu} e^{J\nu} B_{\mu\nu}\ . \tag{2.57}$$

In the Lorentzian signature, this naturally extends to a map from the space of complex valued 2-forms over x to the complex bivectors $\Lambda^2\mathbb{C}^4$. The bivectors inherit the Hodge dual from the 2-forms via

$$\star B^{IJ} = \frac{1}{2}\epsilon^{IJ}{}_{KL} B^{KL}\ , \tag{2.58}$$

along with the self-dual and anti-self-dual projectors,

$$P^{\pm}{}_{KL}{}^{IJ} = \frac{1}{2}\left(\delta^{[I}_{K}\delta^{J]}_{L} \pm \frac{1}{2\sqrt{\sigma}}\,\epsilon^{IJ}{}_{KL}\right) \tag{2.59}$$

which have the following compatibility conditions,

$$\varepsilon_{\mu\nu}{}^{\rho\sigma} e^I_\rho e^J_\sigma = \epsilon^{IJ}{}_{KL}\, e^K_\mu e^L_\nu \quad , \quad P^{\pm}{}_{\mu\nu}{}^{\rho\sigma} e^I_\rho e^J_\sigma = P^{\pm}{}_{KL}{}^{IJ}\, e^K_\mu e^L_\nu\ . \tag{2.60}$$

In the Euclidean signature setting, the real bivectors $\Lambda^2\mathbb{R}^4$ form a representation of $\mathfrak{so}(4)$, while in the Lorentzian signature setting the complex bivectors $\Lambda^2\mathbb{C}^4$ form a representation of the complex Lorentz algebra $\mathfrak{so}(1,3)_{\mathbb{C}}$. In both cases, the Lie bracket is given by

$$[B, C]^{IJ} = B^I{}_K C^{KJ} - C^I{}_K B^{KJ}\ . \tag{2.61}$$

The self-dual and anti-self-dual subspaces are each isomorphic to the real or complex rotation algebras, $\mathfrak{so}(3)$ or $\mathfrak{so}(3)_{\mathbb{C}}$ respectively. We have the following decompositions,

$$\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \quad , \quad \mathfrak{so}(1,3)_{\mathbb{C}} = \mathfrak{so}(3)_{\mathbb{C}} \oplus \mathfrak{so}(3)_{\mathbb{C}}\ . \tag{2.62}$$

To see this, note that the Lie bracket (2.61) of a pair of SD bivectors is again SD, and the bracket of a pair of ASD bivectors is ASD. Furthermore, the bracket of an SD bivector with an ASD one is vanishing. This is summarised as

$$[\Lambda^+\mathbb{F}^4, \Lambda^+\mathbb{F}^4] \subseteq \Lambda^+\mathbb{F}^4 \ , \quad [\Lambda^-\mathbb{F}^4, \Lambda^-\mathbb{F}^4] \subseteq \Lambda^-\mathbb{F}^4 \ , \quad [\Lambda^+\mathbb{F}^4, \Lambda^-\mathbb{F}^4] = \{0\} \ . \tag{2.63}$$

Here, $\mathbb{F} = \mathbb{R}$ in the Euclidean signature setting, and $\mathbb{F} = \mathbb{C}$ in the Lorentzian signature setting. Hence the bivectors, and therefore $\mathfrak{so}(4)$ and $\mathfrak{so}(1,3)_{\mathbb{C}}$, decompose as a direct sum of a pair of commuting Lie sub-algebras. A generic bivector contains six independent degrees of freedom. However, a self-dual bivector only contains 3 independent degrees of freedom corresponding to the components $0i$ for $i = 1, 2, 3$. The dependence of the ij components on the $0i$ components is given by

$$B^{ij} = -\sqrt{\sigma}\, \epsilon^{ij}{}_k B^{0k} \ . \tag{2.64}$$

We perform a change of basis on the space of self-dual bivectors via

$$\begin{aligned} B^{IJ} \ \mapsto \ B^i &= 2\sqrt{\sigma} B^{0i} \\ &= \sqrt{\sigma}\, B^{0i} - \frac{1}{2}\epsilon^i{}_{jk} B^{jk} \ . \end{aligned} \tag{2.65}$$

Under this change of basis, we see that the Lie bracket of a pair of self-dual bivectors becomes

$$[B, C]^i = 2\sqrt{\sigma}\, [B, C]^{0i} \stackrel{!}{=} \epsilon^i{}_{jk} B^j C^k \ . \tag{2.66}$$

Therefore the Lie sub algebra of self-dual bivectors is isomorphic to real or complexified $\mathfrak{so}(3)$. One can show a similar result for the anti-self-dual bivectors. Furthermore, we have the following useful result for contractions,

$$B_{IJ}\, P^{+}_{KL}{}^{IJ}\, C^{KL} = B_i C^i \ . \tag{2.67}$$

That is, the contraction of a pair of self-dual bivectors - note that the SD and ASD projectors square to themselves - is equivalent to the contraction of their $\mathfrak{so}(3)$ representatives. These constructions provide us with a bridge between the Einstein-Cartan formulation and formulations of diffeomorphism invariant gauge theories, such as the ones introduced in section 2.1.

Self-dual part of the solder form

The self-dual part of a solder form e^I is a real or complex $\mathfrak{so}(3)$ valued 2-form Σ^i defined

$$\Sigma^i = 2\sqrt{\sigma}\, P^{+}_{IJ}{}^{0i}\, e^I \wedge e^J = \sqrt{\sigma}\, e^0 \wedge e^i - \frac{1}{2}\epsilon^i{}_{jk}\, e^j \wedge e^k \ . \tag{2.68}$$

Σ^i constitute a basis of the self-dual 2-forms w.r.t the metric $g = \sigma e^0 \otimes e^0 + \sum_{i=1}^3 e^i \otimes e^i$, and the volume form $\varepsilon = (\sqrt{\sigma}/6)\, \Sigma^i \wedge \Sigma_i = (1/24)\, \epsilon_{IJKL}\, e^I \wedge \ldots \wedge e^L$. One can raise the second tensor index of $\Sigma^i_{\mu\nu}$ using the inverse metric $g^{\mu\nu}$ yielding a linear endomorphism on each tangent space $\Sigma^{i\ \nu}_{\mu}$ satisfying the composition law,

$$\Sigma^{i\ \rho}_{\mu}\Sigma^{j\ \nu}_{\rho} = -\delta^{ij}\delta^{\nu}_{\mu} + \epsilon^{ij}{}_{k}\Sigma^{k\ \nu}_{\mu} \ . \tag{2.69}$$

From this we may compute the full contraction to be $\Sigma^{i\mu\nu}\Sigma^{j}_{\mu\nu} = 4\delta^{ij}$. The proof is by direct computation and can be found in the supplementary material, (2.A.2).

2.3.2 Reformulations of the Einstein Condition

What follows is a condensed review of the recent textbook [69], where a more thorough exposition of these constructions can be found. One can also see [71]. Proofs for especially significant results can be found in the supplementary material at the end of this chapter, (2.A.2). A spacetime $(\mathcal{M}, g)$ is called *Einstein*, or is said to satisfy the *Einstein condition*, if it satisfies $R_{\mu\nu} = kg_{\mu\nu}$ for some constant k. Informally, we may say that the Ricci tensor is 'proportional' to the metric, and we may write $R_{\mu\nu} \propto g_{\mu\nu}$. The Einstein condition only constrains the Ricci (trace) part of the Riemann tensor; it doesn't constrain the Weyl (trace-free) part of the Riemann tensor. An Einstein manifold still has all of the propagating degrees of freedom. Solutions of the vacuum Einstein equations - with cosmological constant - are Einstein, and the proportionality constant is the cosmological constant. The Einstein condition can be reformulated in terms of the Hodge dual. To see this, consider the lowered Riemann tensor $R_{\mu\nu\rho\sigma} = g_{\mu\lambda}R^{\lambda}{}_{\nu\rho\sigma}$ which has the following index symmetries,

$$R_{\mu\nu\rho\sigma} = R_{[\mu\nu]\,[\rho\sigma]} = R_{\rho\sigma\mu\nu} \ . \tag{2.70}$$

One can interpret the Riemann tensor as an element of the symmetric tensor product space $\Omega^2(\mathcal{M}) \otimes_S \Omega^2(\mathcal{M})$. The Hodge dual acts on the Riemann tensor from both the left and the right via

$$(\star R)_{\mu\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta\rho\sigma} \quad , \quad (R\star)_{\mu\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\rho\sigma}{}^{\alpha\beta}R_{\mu\nu\alpha\beta} \ . \tag{2.71}$$

The Einstein condition may be equivalently stated as

$$(\star R)_{\mu\nu\rho\sigma} = (R\star)_{\mu\nu\rho\sigma} \ . \tag{2.72}$$

The proof is by direct computation and can be found in the supplementary material (2.A.2). One can extract the various dual-dual parts of the Riemann tensor via

$$(P^{\pm}RP^{\pm})_{\mu\nu\rho\sigma} = P^{\pm\ \alpha\beta}_{\mu\nu}\, R_{\alpha\beta\gamma\delta}\, P^{\pm\ \gamma\delta}_{\rho\sigma} \ . \tag{2.73}$$

One can show that the SD-SD and ASD-ASD parts of the Riemann tensor decompose as

$$(P^+RP^+)_{\mu\nu\rho\sigma} = P^{+\,\alpha\beta}_{\mu\nu}\left(C_{\alpha\beta\gamma\delta} + \frac{R}{6}\, g_{\alpha\,[\gamma}g_{\delta]\,\beta}\right) P^{+\,\gamma\delta}_{\rho\sigma} \;, \tag{2.74a}$$

$$(P^-RP^-)_{\mu\nu\rho\sigma} = P^{-\,\alpha\beta}_{\mu\nu}\left(C_{\alpha\beta\gamma\delta} + \frac{R}{6}\, g_{\alpha\,[\gamma}g_{\delta]\,\beta}\right) P^{-\,\gamma\delta}_{\rho\sigma} \;. \tag{2.74b}$$

Here, $C_{\mu\nu\rho\sigma}$ denotes the Weyl tensor which is given in 4 dimensions by

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{R}{3}\, g_{\mu[\rho}g_{\sigma]\nu} \;. \tag{2.75}$$

The proof of this is by direct computation, and can be found in [69]; it is omitted here and in the supplementary material for brevity. In fact, one can show that the SD-SD and ASD-ASD parts of the Riemann tensor are the only components with dependence on the Weyl tensor. That is, the SD-ASD and ASD-SD parts only have dependence on the Ricci tensor. In the case of a Riemann tensor which is derived from a real Lorentzian metric, we have the following relations,

$$(P^+RP^+)_{\mu\nu\rho\sigma} = \overline{(P^-RP^-)}_{\mu\nu\rho\sigma} \quad , \quad (P^+RP^-)_{\mu\nu\rho\sigma} = \overline{(P^-RP^+)}_{\mu\nu\rho\sigma} \;. \tag{2.76}$$

In the general case, the Riemann tensor expands as a sum $R_{\mu\nu\rho\sigma} = \sum (P^\pm RP^\pm)_{\mu\nu\rho\sigma}$ over the four dual-dual parts. One can substitute this expansion into the reformulated Einstein condition (2.72) to get

$$0 = \star R - R\star = 2i(P^+RP^- - P^-RP^+) \quad \Leftrightarrow \quad P^+RP^- = P^-RP^+ = 0 \;. \tag{2.77}$$

To get the equality with zero on the RHS of the equivalence arrow '$\Leftrightarrow$', one can operate on both sides of $P^+RP^- = P^-RP^+$ with $-i\,\star$, from the left or the right, yielding a further equation $P^+RP^- = -P^-RP^+$. The system of equations is solved when both P^+RP^- and P^-RP^+ are vanishing. This yields as further reformulation of the Einstein condition. We may write this reformulation concisely as

$$(P^+RP^-)_{\mu\nu\rho\sigma} = 0 \;. \tag{2.78}$$

This particular reformulation will play an important role in our exposition of the the *Plebański formulation*, in the following.

2.3.3 Chiral Plebański Action

The Chiral Plebański action reads

$$S[A,\Sigma,M,\mu] = \frac{1}{8\pi G\sqrt{\sigma}} \int \Sigma_i \wedge F^i - \frac{1}{2} M^{ij} \Sigma_i \wedge \Sigma_j + \frac{1}{2}\mu\,(\operatorname{tr} M - \Lambda)\ , \tag{2.79}$$

where:

- A^i is an $SO(3)$ connection form with curvature $F^i = dA^i + \frac{1}{2}\epsilon^i{}_{jk} A^j \wedge A^k$ and exterior covariant derivative D_A which acts on $\mathfrak{so}(3)$ valued 2-forms, which transform in the adjoint representation, $D_A B^i = dB^i + \epsilon^i{}_{jk} A^j \wedge B^k$,
- Σ^i is an $\mathfrak{so}(3)$ valued 2-form that transforms in the adjoint representation under local $SO(3)$ gauge transformations,
- M^{ij} is an $\mathbf{S}^2\mathfrak{so}(3)$ valued function which transforms in the composite adjoint representation under local $SO(3)$ gauge transformations,
- μ is a 4-form,
- Λ is the cosmological constant.

In the Euclidean signature setting, where $\sigma = \sqrt{\sigma} = 1$, the fields are restricted to be real valued. While in the Lorentzian signature setting, where $\sigma = -1$ and $\sqrt{\sigma} = i$, the fields are taken to be complex valued so that A^i is an $SO(3,\mathbb{C})$ connection, and so on. The fields transform under local $SO(3)$ or $SO(3,\mathbb{C})$ gauge transformations; the infinitesimal forms are given by

$$\delta_\alpha A^i = D_A \alpha^i\ , \quad \delta_\alpha \Sigma^i = \epsilon^i{}_{jk} \Sigma^j \alpha^k\ , \quad \delta_\alpha M^{ij} = \epsilon^i{}_{kl} M^{kj} \alpha^l + \epsilon^j{}_{kl} M^{ik} \alpha^l\ , \quad \delta_\alpha \mu = 0\ . \tag{2.80}$$

The action is invariant under local, real or complex, $SO(3)$ gauge transformations. Furthermore, the action is manifestly diffeomorphism invariant. Hence the chiral Plebański theory is a diffeomorphism invariant gauge theory. The field equations are computed via standard variational methods;

$$\textit{Chiral zero torsion condition} \quad : \quad D_A \Sigma^i = 0\ , \tag{2.81a}$$

$$\textit{Chiral Einstein condition} \quad : \quad F^i = M^{ij} \Sigma_j\ , \tag{2.81b}$$

$$\textit{Metricity constraint} \quad : \quad \Sigma^i \wedge \Sigma^j = \mu \delta^{ij}\ , \tag{2.81c}$$

$$\textit{Chiral trace equation} \quad : \quad \operatorname{tr} M = \Lambda\ . \tag{2.81d}$$

In the Lorentzian signature, the variables are taken to be complex valued. The solutions of the Plebański field equations (2.81) are not in one to one correspondence with the solutions of real Lorentzian GR. In general, solutions of (2.81) correspond to solutions of the complexified Einstein field equations. We will investigate these solutions later on. The solution space of (2.81) does contain solutions which correspond to solutions of real Lorentzian GR. In order to locate these, we require further conditions on the variables. These are the *reality conditions* which come in two types;

$$\textit{Wedge type} \quad : \quad \Sigma^i \wedge \overline{\Sigma^j} = 0 \ , \tag{2.82a}$$

$$\textit{Trace type} \quad : \quad \mathrm{Re}(\Sigma_i \wedge \Sigma^i) = 0 \ . \tag{2.82b}$$

Note that the reality conditions are algebraic equations: they don't contain any derivatives. In this case, one can interpret these conditions as constraints on the initial data on some 3D hypersurface embedded in spacetime. Elaborating on this idea, note that the Plebański field equations are first order in derivatives. Then, by prescribing some notion of a $3+1$ space and time split, one can immediately write the field equations as time evolution equations with the form $\dot{A}^i_a(\boldsymbol{x},t) = \mathcal{F}(A^i_a(\boldsymbol{x},t),\ldots)$, and so on. The reality conditions are, in general, not preserved by this time evolution. As a consequence, there will be further constraints on the fields coming from the first (and possibly higher) order 'time' derivatives of the reality conditions. We will examine this in further detail in our application of these models to highly symmetric spacetimes, such as Bianchi I and IX, later on in this thesis.

2.3.4 Equivalence with GR

In what follows, we make clear the relationships between the field equations (2.81) of the Plebański theory, and corresponding equations in the standard metric and tetrad formulations of GR.

The Metricity Constraint

The field equation (2.81c), which we call the *metricity constraint*, and the trace type reality condition (2.82b) allow us to write

$$\Sigma^i \wedge \Sigma^j = -2i\,\delta^{ij}\varepsilon_\Sigma \ , \tag{2.83}$$

where ε_Σ is a real 4-form and where the factor of 2 on the RHS is conventional. When this condition is satisfied, and when the wedge type reality condition (2.82a) is satisfied also, there exists a real solder form e^I such that Σ^i decomposes either as

$$\Sigma^1 = ie^0 \wedge e^1 - e^2 \wedge e^3 \ , \quad \Sigma^2 = ie^0 \wedge e^2 - e^3 \wedge e^1 \ , \quad \Sigma^3 = \pm \left(ie^0 \wedge e^3 - e^1 \wedge e^2 \right) \ , \quad (2.84)$$

or as

$$\Sigma^1 = e^0 \wedge e^1 - ie^2 \wedge e^3 \ , \quad \Sigma^2 = e^0 \wedge e^2 - ie^3 \wedge e^1 \ , \quad \Sigma^3 = \pm \left(e^0 \wedge e^3 - ie^1 \wedge e^2 \right) \ . \quad (2.85)$$

The proof is somewhat lengthy, can be found in [69]. In each of the four cases one can construct a real Lorentzian metric tensor via $g = \eta_{IJ} e^I \otimes e^J$ where η_{IJ} has signature $(-+++)$. Then $\varepsilon_\Sigma = e^0 \wedge \ldots \wedge e^3$ is a volume form for this metric. Furthermore, from the definition of ε_Σ we see that $\mu = -2i\,\varepsilon_\Sigma$. Hence the 4-form μ encodes the metric volume form. In the first case (2.84), Σ^i constitute a basis of the self-dual 2-forms w.r.t g and ε_Σ. However, in the second case (2.85), Σ^i constitute a basis of the anti-self-dual forms w.r.t g and ε_Σ. In the following, we will show that the metric g constructed in this way satisfies the vacuum Einstein equations $R_{\mu\nu} = \pm\Lambda g_{\mu\nu}$ or $R_{\mu\nu} = \pm i\Lambda g_{\mu\nu}$ depending on whether Σ^i expands as (2.84) or as (2.85).

Alternatively, one can construct a metric immediately from Σ^i via the Urbantke formula, which is given by

$$g_\Sigma(X,Y)\,\varepsilon_\Sigma = -\frac{i}{6}\,\epsilon_{ijk}\,i_X\Sigma^i \wedge i_Y\Sigma^j \wedge \Sigma^k \quad : \quad \varepsilon_\Sigma = \frac{i}{6}\Sigma^i \wedge \Sigma_i \ . \quad (2.86)$$

This metric satisfies $R_{\mu\nu} = \Lambda(g_\Sigma)_{\mu\nu}$ in all cases, with the caveat that the metric itself now expands as $g_\Sigma = \pm\eta_{IJ} e^I \otimes e^J$ or as $g_\Sigma = \pm i\,\eta_{IJ} e^I \otimes e^J$ depending on whether Σ^i expands as (2.84) or as (2.85). That is, one cannot a priori fix the signature or the overall reality of g_Σ. The Urbantke construction is such that Σ^i are always self-dual w.r.t g_Σ and ε_Σ.

In order to show the equivalence of the Plebański theory and real Lorentzian GR, we will first work in the case where the 2-forms Σ^i expand as (2.84) with $\Sigma^3 = +(ie^0 \wedge e^3 - e^1 \wedge e^2)$. That is, Σ^i expand as the self-dual part of a solder form, (2.68). Once we have shown equivalence in this case, we will see that the equivalence in the other cases follows as a natural consequence. It is a straight forward, albeit rather tedious, exercise in algebra to show that when the Σ^i expands as the self-dual part of a solder form e^I, (2.68), the Urbantke metric expands as $g_\Sigma = \eta_{IJ} e^I \otimes e^J$. Hence the Urbantke metric g_Σ coincides with the metric g constructed immediately from the solder form in this case.

The chiral zero torsion condition

The *chiral zero torsion* condition (2.81a) is in correspondence with the zero torsion condition between the solder form e^I and the spin connection ω^{IJ}, (2.49a), in the Einstein-Cartan formulation. To see this, consider the following theorem, which is independent of the choice of signature $\sigma = \pm 1$.

Theorem 2.3.1. *Let Σ^i be the self-dual part of a solder form e^I, $\Sigma^i = 2\sqrt{\sigma}\, P^{+\,0i}_{IJ}\, e^I \wedge e^J$. Then the chiral zero torsion condition $D_A \Sigma^i = 0$ is uniquely solved by the self-dual part of the unique zero torsion spin connection ω^{IJ} associated to e^I, $A^i = 2\sqrt{\sigma}\, P^{+\,0i}_{IJ}\, \omega^{IJ}$. Furthermore, F^i is the self-dual part of the curvature R^{IJ}, $F^i = 2\sqrt{\sigma}\, P^{+\,0i}_{IJ}\, R^{IJ}$.*

The proof can be found in the supplementary material, (2.A.2). Hence, the field equation (2.81a) is solved uniquely by the self-dual part of the unique spin connection ω^{IJ} that has vanishing torsion with e^I, and F^i is the self-dual part of its curvature R^{IJ},

$$A^i = i\omega^{0i} - \frac{1}{2}\epsilon^i{}_{jk}\,\omega^{jk} \ , \quad F^i = iR^{0i} - \frac{1}{2}\epsilon^i{}_{jk}\,R^{jk} \ . \tag{2.87}$$

The chiral Einstein condition

The field equation (2.81b), the *chiral Einstein condition*, is equivalent to the Einstein condition $R_{\mu\nu} = \frac{1}{4}R g_{\mu\nu}$. To see this, note that (2.81b) tells us that F^i expands as a linear combination of self-dual 2-forms Σ^i and hence is self-dual. From theorem 2.3.1, one can reconstruct the self-dual part of curvature of the spin connection $P^{+}_{KL}{}^{IJ} R^{KL}$ via

$$P^{+\,0i}_{IJ} R^{IJ} = \frac{1}{2\sqrt{\sigma}}\, F^i \quad , \quad P^{+\,ij}_{IJ} R^{IJ} = -\frac{1}{2}\epsilon^{ij}{}_k F^k \ , \tag{2.88}$$

with $\sqrt{\sigma} = i$. One sees that F^i is self-dual as a 2-form if and only if $P^{+}_{KL}{}^{IJ} R^{KL}$ is self-dual as a 2-form. Contracting the 2-form indices with the anti-self-dual projector $P^{-}_{\mu\nu}{}^{\rho\sigma}$, which is constructed from the volume form ε_Σ with tensor indices raised and lowered via the metric g, yields 0. Therefore

$$0 = P^{+}_{KL}{}^{IJ} R^{KL}{}_{\rho\sigma} P^{-}_{\mu\nu}{}^{\rho\sigma} \overset{!}{=} e^{I\rho} e^{J\sigma} (P^+ R P^-)_{\rho\sigma\mu\nu} \ . \tag{2.89}$$

We see that (2.81b) can be equivalently formulated as $P^+ R P^- = 0$, which is the reformulation of the Einstein condition given in (2.78).

The chiral trace equation

We deal now with the final equation (2.81d), the *chiral trace equation*. This equation is equivalent to the trace part of the Einstein equation $R = 4\Lambda$, where $R = R^{\mu\nu}{}_{\mu\nu}$ denotes the Ricci scalar derived from the metric g. To see this, one fully contracts both sides of $F^i_{\mu\nu} = M^{ij}\Sigma_{j\mu\nu}$ with the raised form $\Sigma_i^{\mu\nu}$ to get

$$4\,\mathrm{tr}\,M = \Sigma_i^{\mu\nu} F^i_{\mu\nu} = 2e^\mu_I e^\nu_J\, P^{+\ IJ}_{KL}\, R^{KL}{}_{\mu\nu} \stackrel{!}{=} 2P^{+\ \mu\nu}_{\rho\sigma} R^{\rho\sigma}{}_{\mu\nu} = R\,. \tag{2.90}$$

Note, in the penultimate equality, marked $\stackrel{!}{=}$, we have used the compatibility condition (2.60), and in the final equality we have used the Bianchi identity $\varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 0$. In short, the trace of the matrix M is proportional to the Ricci scalar via

$$\mathrm{tr}\,M = \frac{1}{4}R\,. \tag{2.91}$$

Theorem 2.3.2. *Let Σ^i be the self-dual part of a solder form e^I, $2\sqrt{\sigma}\,P^{+\ 0i}_{IJ}\,e^I \wedge e^J$, and let A^i be the self-dual part of the unique zero torsion spin connection ω^{IJ} associated to e^I, $A^i = 2\sqrt{\sigma}\,P^{+\ 0i}_{IJ}\,\omega^{IJ}$. Furthermore, let $F^i = f^{ij}\Sigma_j$ for some matrix f^{ij}. Then $\mathrm{tr}\,f = \frac{1}{4}R$ where R is the Ricci scalar derived from the metric $g = \sigma\, e^0 \otimes e^0 + \sum_{i=1}^3 e^i \otimes e^i$.*

Hence we see the equivalence of (2.81d) with $R = 4\Lambda$. In addition, the SD-SD part of the Weyl tensor can be recovered from the trace-free part of M^{ij}, which we denote ψ^{ij}, via

$$(P^+CP^+)_{\mu\nu\rho\sigma} = \frac{1}{2}\psi_{ij}\,\Sigma^i_{\mu\nu}\Sigma^j_{\rho\sigma}\,. \tag{2.92}$$

To summarise, in the case where the metricity constraint (2.81c) and the reality conditions (2.82) result in an expansion of Σ^i as in (2.84) where Σ^3 takes the positive sign, the chiral zero torsion condition (2.81a) tells us that A^i is the self-dual part of the unique zero torsion spin connection ω^{IJ} associated to e^I. Furthermore, F^i is the self-dual part of the spin curvature R^{IJ}. The chiral Einstein condition (2.81b) is equivalent to the Einstein condition $R_{\mu\nu} \propto g_{\mu\nu}$ for the metric $g = \eta_{IJ}e^I \otimes e^J$. Then the chiral trace equation (2.81d) specifies the proportionality constant, yielding $R_{\mu\nu} = \Lambda g_{\mu\nu}$. We recover the full Einstein field equations in vacuum with cosmological constant Λ. This is consolidated within the following lemma,

Lemma 2.3.1. Let Σ^i decompose as the self-dual part of a tetrad, $\Sigma^i = 2\sqrt{\sigma}\,P^{+\ 0i}_{IJ}\,e^I \wedge e^J$. Furthermore, let Σ^i satisfy the equations, $D_A\Sigma^i = 0$, $F^i = f^{ij}\Sigma_j$ and $\mathrm{tr}\,f = \lambda$ for some complex $SO(3)$ connection A^i, some complex matrix f^{ij}, and some constant λ. Then the metric defined $g = \sigma e^0 \otimes e^0 + \sum_{i=1}^3 e^i \otimes e^i$ satisfies $R_{\mu\nu} = \lambda g_{\mu\nu}$.

Equivalence in the other cases

Now we will see that we can use the equivalence of the Plebański-GR equivalence in the most simple case, (2.84) with positive sign in front of the expression for Σ^3, as a model to demonstrate the equivalence in the other cases. We will need the following lemma,

Lemma 2.3.2. Let Σ^i be a triple of 2-forms satisfying the metricity constraint (2.81c) and the reality conditions (2.82). Then either Σ^i expands as the self-dual part of a solder form e^I, $\Sigma^i = 2i\, P^{+\,0i}_{IJ} e^I \wedge e^J$, or $-\Sigma^i$ or $\pm i\Sigma^i$ does.

This is immediate in the case (2.84) where the given expression for Σ^3 has a positive sign. In the case (2.84) where the expression for Σ^3 has a negative sign, one can define a new solder form $\tilde{e}^I$ via $\tilde{e}^0 = -e^0$, $\tilde{e}^1 = e^1$, $\tilde{e}^2 = e^2$ and $\tilde{e}^3 = -e^3$, such that $-\Sigma^i = \tilde{\Sigma}^i$ where $\tilde{\Sigma}^i = 2i\, P^{+\,0i}_{IJ}\, \tilde{e}^I \wedge \tilde{e}^J$ is the self-dual part of the solder form $\tilde{e}^I$. For the cases (2.85), one may define the following solder forms,

$$(+): \quad \tilde{e}^0 = -e^0\ , \quad \tilde{e}^1 = e^1\ , \quad \tilde{e}^2 = e^2\ , \quad \tilde{e}^3 = e^3 \quad : \quad -i\Sigma^i = \tilde{\Sigma}^i\ , \tag{2.93a}$$

$$(-): \quad \tilde{e}^0 = e^0\ , \quad \tilde{e}^1 = e^1\ , \quad \tilde{e}^2 = e^2\ , \quad \tilde{e}^3 = -e^3 \quad : \quad i\Sigma^i = \tilde{\Sigma}^i\ . \tag{2.93b}$$

For each of the cases in (2.84) and (2.85) we have $\tilde{\Sigma}^i = \alpha\Sigma^i$ where $\alpha = \pm 1$ or $\alpha = \pm i$. The Plebański field equations (2.81), excluding the metricity constraint which we have already used to decompose Σ^i, may be written in terms of $\tilde{\Sigma}^i$ as

$$D_A\tilde{\Sigma}^i = 0\ , \quad F^i = \tilde{M}^{ij}\, \tilde{\Sigma}_j\ , \quad \mathrm{tr}\, \tilde{M} = \frac{\Lambda}{\alpha}\ , \tag{2.94}$$

where $\tilde{M}^{ij} = (1/\alpha) M^{ij}$. From lemma 2.3.1 we see that the metric $g = \eta_{IJ} \tilde{e}^i \otimes \tilde{e}^J = \eta_{IJ} e^I \otimes e^J$ satisfies $R_{\mu\nu} = (\Lambda/\alpha)\, g_{\mu\nu}$. Therefore, the metric g satisfies $R_{\mu\nu} = \pm\Lambda g_{\mu\nu}$ or $R_{\mu\nu} = \pm i\Lambda g_{\mu\nu}$.

Note, the metric g is also recovered by inserting $\tilde{\Sigma}^i$ into the Urbantke formula (2.86). One derives the relationship $g = \alpha g_\Sigma$, where we recall that g_Σ is the Urbantke metric generated from Σ^i. The Ricci tensor is invariant under constant rescalings of the metric, so g and g_Σ share the same Ricci tensor. Putting things together, one sees that the metric g_Σ satisfies $R_{\mu\nu} = \Lambda (g_\Sigma)_{\mu\nu}$ in all four cases.

2.3.5 Euclidean signature formulation

The Euclidean signature version of the chiral Plebański theory has the same action (2.79) and the same field equations (2.81), except where all of the field are taken to be real valued, and no reality conditions are required. Here, the metricity constraint (2.81c) alone implies the existence of a real tetrad e^I such that Σ^i expands as

$$\Sigma^1 = e^0 \wedge e^1 - e^2 \wedge e^3 \ , \quad \Sigma^2 = e^0 \wedge e^2 - e^3 \wedge e^1 \ , \quad \Sigma^3 = \pm \left(e^0 \wedge e^3 - e^1 \wedge e^2\right) \ . \qquad (2.95)$$

Then one may construct a Euclidean signature metric via $g_E = \delta_{IJ}\, e^I \otimes e^J$. This metric satisfies $R_{\mu\nu} = \pm\Lambda(g_E)_{\mu\nu}$, where the '$\pm$' is in correspondence with the $\pm$ which multiplies Σ^3 in the above expansion. Alternatively, one can construct a metric immediately from Σ^i via the *Euclidean Urbantke formula*, which is given by

$$g_\Sigma(X,Y)\, \varepsilon_\Sigma = -\frac{1}{6}\, \epsilon_{ijk}\, i_X\Sigma^i \wedge i_Y\Sigma^j \wedge \Sigma^k \quad : \quad \varepsilon_\Sigma = \frac{1}{6}\Sigma^i \wedge \Sigma_i \ . \qquad (2.96)$$

This metric is real valued and may have either signature $(++++)$ or $(----)$. in parallel with the Lorentzian version, this metric always satisfies $R_{\mu\nu} = \Lambda(g_\Sigma)_{\mu\nu}$. The proof of this final statement proceeds as in the Lorentzian signature case. Note that all the results there were true for both values of σ, so the analysis carries immediately over to the Euclidean signature case.

2.3.6 General complex formulation

One arrives at the general complex version of the Plebański theory by allowing the fields to be complex valued and omitting the reality conditions. In the general complex case, we have the following theorem,

Theorem 2.3.3. *Let Σ^i satisfy $\Sigma^i \wedge \Sigma^j = \delta^{ij}\mu$ for some complex 4-form μ. Then there exists a complex solder form e^I such that Σ^i expands as its self-dual part, $\Sigma^i = 2\sqrt{\sigma} P^{+\,0i}_{IJ}\, e^I \wedge e^J$.*

In this case, the value of σ is just a convention that we may choose. I.e., for any given Σ^i satisfying (2.81c), such a tetrad e^I exists for both values of σ. The proof is included in the supplementary material (2.A.2). As before, one can construct a metric from the tetrad via $g = \sigma\, e^0 \otimes e^0 + \sum_{i=1}^3 e^i \otimes e^i$. Alternatively, one can construct a metric immediately from Σ^i via the *generalised Urbantke formula* which is given by

$$g_\Sigma(X,Y)\, \varepsilon_\Sigma = -\frac{\sqrt{\sigma}}{6}\, \epsilon_{ijk}\, i_X\Sigma^i \wedge i_Y\Sigma^j \wedge \Sigma^k \quad : \quad \varepsilon_\Sigma = \frac{\sqrt{\sigma}}{6}\Sigma^i \wedge \Sigma_i \ . \qquad (2.97)$$

It is easily see that this version of the Urbantke formula recovers the Lorentzian (2.86) and Euclidean signature (2.96) versions when we fix $\sqrt{\sigma} = i$ and $\sqrt{\sigma} = 1$ respectively. When Σ^i expands as the self-dual part of the solder form e^I, we have $g = g_\Sigma$. For future reference, we write down the following theorem:

Theorem 2.3.4. *Let Σ^i decompose as the self-dual part of some solder form e^I. Then the Urbantke metric constructed as in (2.97) expands as $g_\Sigma = \sigma\, e^0 \otimes e^0 + \sum_{i=1}^{3} e^i \otimes e^i$, and the volume form expands as $\varepsilon_\Sigma = -\sigma\, e^0 \wedge \ldots \wedge e^3$.*

The proof is by direct calculation, and is omitted from this thesis. The metric g (and hence also g_Σ) satisfies $R_{\mu\nu} = \Lambda g_{\mu\nu}$. The proof once again proceeds as in the Lorentzian signature case.

2.3.7 Summary

To summarise, the Plebański formulation is written as a diffeomorphism invariant gauge theory with action (2.79) whose primary dynamical fields are a (complex) $SO(3)$ connection A^i and a triad of (complex) 2-forms Σ^i. Additionally, there are 'auxiliary' fields M^{ij} and μ which enforce certain constraints on the connection and 2-form triad. The action yields field equations (2.81) consisting of first order PDEs and algebraic constraints. One can construct a metric tensor via the generalised Urbantke formula (2.97), which is complex valued in general. When the Plebański fields satisfy the field equations (2.81), the constructed metric satisfies $R_{\mu\nu} = \Lambda g_{\mu\nu}$. Furthermore, the trace part of the matrix M^{ij} is proportional to the Ricci scalar of the constructed metric via $\operatorname{tr} M = \frac{1}{4}R$, and the trace-free part encodes the SD-SD part of the Weyl tensor. The 4-form μ is proportional to the volume form ε of the constructed metric, which is defined in the second equation in (2.97), such that $\mu = (2/\sqrt{\sigma})\,\varepsilon$. To obtain Lorentzian solutions, one imposes reality conditions (2.82) on top of the field equations. Consequently, the constructed metric is either Lorentzian, where both signatures $(-+++)$ and $(+---)$ are possible, or Lorentzian with an overall factor of $\pm i$. One might attempt to remedy this by redefining the constructed metric by $g_{\mu\nu} \mapsto g'_{\mu\nu} = \pm g_{\mu\nu}$ or $\pm i g_{\mu\nu}$, but this will also alter the form of the Einstein equations such that $R'_{\mu\nu} = \pm\Lambda g'_{\mu\nu}$ or $R'_{\mu\nu} = \pm i\Lambda g'_{\mu\nu}$. We see that the different solutions to the reality conditions are only consistent with particular values of Λ. For example, a purely imaginary metric tensor is only consistent with an imaginary Λ. Then the reality of Λ allows us to rule out certain solutions of the reality conditions. In parallel, Euclidean signature solutions are obtained by taking all of the fields to be real valued. Then the constructed metric is real with either positive or negative definite signature $(++++)$ or $(----)$ respectively.

2.4 Pure Connection Gravity

2.4.1 Chiral First Order Action

Starting from the chiral Plebański action (2.79) it is possible to consistently eliminate certain degrees of freedom and construct a new action over fewer independent fields. The first of these actions that we will examine is the *chiral first order action*, sometimes referred to as the *instanton representation* of Plebański gravity [61]. This formulation is discussed in detail in [69], with further discussion in the context of cosmological spacetimes in [55]. An early version was introduced in [30]. To get the first order action, we solve the field equation (2.81b) for Σ^i in terms of F^i and M^{ij}, and insert the result back into the action. That is, wherever Σ^i appears in (2.79) we replace it with $(M^{-1})^{ij}F_j$. The resulting action reads

$$S_{\mathrm{FO}}[A,M,\mu] = \frac{1}{16\pi G\sqrt{\sigma}}\int M^{-1}_{ij}F^i\wedge F^j + \mu\,(\mathrm{tr}\,M - \Lambda)\ . \tag{2.98}$$

We must check that this substitution doesn't change the dynamical content of the theory. To see this, denote $\Sigma^i_F := (M^{-1})^{ij}\,F_j$, then the field equations for the first order theory read

$$D_A\Sigma^i_F = 0\,,\qquad \Sigma^i_F\wedge\Sigma^j_F = \delta^{ij}\mu\ ,\quad \mathrm{tr}\,M = \Lambda\ . \tag{2.99}$$

We see that these field equations agree with the field equations of the Plebański theory (2.81) under the substitution $\Sigma^i \to \Sigma^i_F$. Therefore, eliminating Σ^i at the level of the action in this way doesn't change the dynamical content of the theory. Furthermore, using the Bianchi identity $D_AF^i = 0$, one sees that the first field equation $D_A\Sigma^i_F = 0$ can be written equivalently as $D_A(M^{-1})^{ij}\,F_j = 0$ which has only first order derivatives of M and A. Hence the theory is *first order*. Note, in order to eliminate the Σ^i 2-forms we must assume that the matrix M^{ij} is invertible at the level of the action. This assumption is not present in the full Plebański theory; the inverse of M^{ij} does not appear anywhere in the action (2.79) or its field equations (2.81). Consequently, only a restricted set of spacetimes are compatible with this first order formulation. For example, this formulation cannot describe Minkowski spacetime where $M^{ij} = 0$, which is otherwise well defined in the full Plebański theory (when one fixes $\Lambda = 0$).

The reality conditions for the first order theory are the reality conditions for the Plebański theory, given in (2.82), where Σ^i is replaced with Σ^i_F yielding

$$\textit{Wedge type}\quad :\quad F^i\wedge\overline{F^j} = 0\ , \tag{2.100a}$$

$$\textit{Trace type}\quad :\quad \mathrm{Re}\left(\Sigma^i_F\wedge\Sigma_{F\,i}\right) = 0\,. \tag{2.100b}$$

Notice that these conditions now contain first derivatives of the connection, whereas previously the reality conditions were algebraic conditions on the 2-forms Σ^i which could be imposed on some initial surface without any knowledge of the dynamics. Now, we should see the reality conditions as further conditions applied on top of the field equations. We must check that these conditions are compatible with the field equations.

2.4.2 Chiral Pure Connection Action

It is possible to eliminate the auxiliary fields M^{ij} and μ from the chiral first order action (2.98), yielding an action whose only independent field is the connection A^i. This process yields the *chiral pure connection action*. This action was first introduced by in [72], and further explored in [66, 70]. The derivation of this action starting from the first order action (2.98) and its field equations (2.99) can be seen in a number of steps. To begin with, one can write the metricity constraint of the first order theory, $\Sigma^i_F \wedge \Sigma^j_F = \delta^{ij}\mu$, in the form

$$F^i \wedge F^j = M^i{}_k M^{kj} \mu \, . \tag{2.101}$$

Let ε_X be a fixed, background, nowhere vanishing 4-form and let q be a function, and X^{ij} a matrix valued function such that $F^i \wedge F^j = X^{ij}\varepsilon_X$, and $\mu = q\varepsilon_X$. Then (2.101) can be written as

$$X = M^2 q \quad \Leftrightarrow \quad M = \frac{\sqrt{X}}{\sqrt{q}} \, . \tag{2.102}$$

$\sqrt{X}$ is the matrix square root satisfying $(\sqrt{X})^i{}_k(\sqrt{X})^{kj} = X^{ij}$. Much like the square root of a number, the square root of a matrix is not unique; there are multiple branches of matrix square root. At least one branch can be obtained via a series expansion given

$$\sqrt{X} = \sum_{k=0}^{\infty} (-)^k \, \frac{a_k}{k!} \, (\mathbb{I} - X)^k \quad : \quad a_k = \prod_{j=0}^{k} \left(\frac{1}{2} - j \right) \, , \tag{2.103}$$

subject to the condition that the matrix X^{ij} satisfies $\lim_{k\to\infty} \sup_{\ell \le k} \left\| (\mathbb{I} - X)^\ell \right\|^{\frac{1}{\ell}} < 1$, where $\|X\| = \sqrt{X_{ij}\overline{X^{ij}}}$ is a matrix norm. A more detailed exploration of the matrix square root can be found in [56]. Of particular interest to us will be the case where X^{ij} is taken to be diagonal, $X^{ij} = \chi^{(i)}\delta^{(i)j}$. In this case, the difficult problem of defining a square root for matrices is reduced to the much simpler problem of choosing a branch of the square root for complex numbers. We define $(\sqrt{X})^{ij} = \sqrt{\chi^{(i)}}\,\delta^{(i)j}$. Continuing on, one substitutes the expansion (2.102) into the field equation $\operatorname{tr} M = \Lambda$ and rearranges to get $\sqrt{q} = \Lambda^{-1} \operatorname{tr}\sqrt{X}$, which can be substituted back into (2.102) to get

$$M^{ij} = \frac{\Lambda}{\operatorname{tr}\sqrt{X}}\,(\sqrt{X})^{ij}\ . \tag{2.104}$$

One substitutes the expansion (2.104) into the first order action (2.98) to get the chiral pure connection action, which reads

$$S_{\rm PC}[A] = \frac{1}{16\pi G\sqrt{\sigma}\Lambda}\int \varepsilon_X\left(\operatorname{tr}\sqrt{X}\right)^2\ . \tag{2.105}$$

Note that there is a prefactor of $1/\Lambda$, and hence this action is only well defined when the Λ is non-zero. There exists a alternative formulation of pure connection gravity which allows for $\Lambda = 0$ that can be found in [86]. We will not discuss this formulation in this thesis. Variations of the action yield a single field equation given

$$D_A\Sigma^i_X = 0 \quad : \quad \Sigma^i_X = \frac{\operatorname{tr}\sqrt{X}}{\Lambda}(X^{-\frac{1}{2}})^{ij}F_j\ . \tag{2.106}$$

$X^{-\frac{1}{2}} = (\sqrt{X})^{-1}$ is the inverse of the matrix square root. It is a straightforward exercise to show that Σ^i_X satisfies the metricity constraint $\Sigma^i_X \wedge \Sigma^j_X = \delta^{ij}\mu_X$, where

$$\mu_X = \left(\frac{\operatorname{tr}\sqrt{X}}{\Lambda}\right)^2\varepsilon_X\ . \tag{2.107}$$

When we take both ε_X and Λ to be real valued, the reality conditions take the form

$$\textit{Wedge type} \quad : \quad F^i \wedge \overline{F^j} = 0\ , \tag{2.108a}$$

$$\textit{Trace type} \quad : \quad \operatorname{Re}\left\{\left(\operatorname{tr}\sqrt{X}\right)^2\right\} = 0\,. \tag{2.108b}$$

These are highly complex when seen as conditions on the connection A^i, which is a significant drawback of this formalism when applied to Lorentzian general relativity.

2.4.3 The reconstructed metric

In the Plebański theory the metric was constructed from Σ^i either via the Urbantke formula (2.97), which is well defined in all cases, or via the tetrad decompositions (2.84) and (2.85) in the Lorentzian signature, and (2.95) in the Euclidean signature. The Σ^i fields have been integrated out in both the first order and pure connection theories. However, they still exist as combinations of the remaining variables. In the first order theory we have Σ^i_F which is defined in the text immediately below (2.98), and in the pure connection theory we have Σ^i_X

which is defined in (2.106). By construction, both Σ^i_F and Σ^i_X satisfy all of the Plebański field equations (2.81). When we also impose the appropriate reality conditions, the metrics constructed from Σ^i_F and Σ^i_X satisfy the vacuum Einstein equations where the cosmological constant may be multiplied by ± 1 or $\pm i$.

2.4.4 Further Topics in Plebański and Pure Connection Gravity

To conclude this section on Plebański and chiral connection formulations of GR, we provide the reader with a handful of topics that will not be covered at any other point in this thesis, but are interesting nonetheless.

Coupling to matter

Matter coupling in the Plebański formulation at the level of the field equations is discussed in [71, 69]. Here, one adds extra terms to the chiral Einstein condition (2.81b) and the chiral trace equation (2.81d) which have dependence on the energy-momentum tensor $T^{\mu\nu}$. There is little discussion on matter coupling at the level of the action in the literature. However, one can see older literature such as [29, 30], which formulates the Plebański theory in terms of Penrose spinors [87], where the authors outline the use of the Urbantke metric to couple scalar fields. The authors also suggest that one can couple fermionic/spinorial matter using a tetrad that is generated from the Urbantke metric, though the such a procedure is not provided. For applications, one can see [2].

Deformations of GR

Modified gravity theories have been well studied in the metric language [82]. Here, the typical approach is to modify the Einstein-Hilbert action by introducing an arbitrary function $f(R)$ of the Ricci scalar to get $\propto \int d^4x \ \sqrt{|g|}\, f(R)$; these are called $f(R)$ theories [97]. Depending on the form of $f(R)$, one might find dynamics that are very similar to GR with only minor corrections terms, or theories that diverge radically from GR. There exist similar modifications of the Plebański action (2.79) which have the general form

$$S_{\text{Mod}}\left[A, \Sigma, M, \mu\right] = \frac{1}{8\pi G\sqrt{\sigma}} \int \Sigma_i \wedge F^i - \frac{1}{2} M_{ij} \Sigma^i \wedge \Sigma^j + \frac{\mu}{2} f(M)\,, \tag{2.109}$$

where $f(M)$ is a real or complex valued function of matrices. Here, one recovers the chiral zero torsion condition (2.81a) and Einstein condition (2.81b) as in the unmodified theory.

However, the metricity condition (2.81c) and the trace condition (2.81d) are respectively replaced by.

$$\Sigma_i \wedge \Sigma_j = \mu \frac{\partial f}{\partial M^{ij}} \quad , \quad f(M) = 0 \; . \tag{2.110}$$

The typical Plebański action (2.79) corresponds to $f(M) = \operatorname{tr} M - \Lambda$. Other choices for $f(M)$ will result in different dynamics. For further details, one can see [67, 68, 69].

Generalised pure connection theories

In section 2.4.2 we saw how one can derive Krasnov's pure connection action 2.105 by integrating out fields from the Plebański action 2.79. One can interpret this pure connection action as belonging to an extended family of pure connection actions each with the general form

$$S[A] = \frac{1}{16\pi G\sqrt{\sigma}} \int \varepsilon_X \, f(X) \; , \tag{2.111}$$

where $f(X)$ is a function of complex 3×3 matrices satisfying $f(O^T X O) = f(X)$ for any $SO(3, \mathbb{C})$ matrix O, and $f(\lambda X) = \lambda f(X)$ for any complex constant λ [66, 69, 70]. The characteristic function for GR is $f(X) = (\operatorname{tr} \sqrt{X})^2$; other choices for $f(X)$ lead to different theories with dynamics that may be wildly different from GR. In Euclidean signature, an interesting example is $f(X) = (\det(X))^{1/3}$, which also arises from a dimensional reduction of a theory of G_2-holonomy metrics in 7D [54].

2.5 Unimodular Gravity

Unimodular gravity is a term which refers to a somewhat broad range of reformulations of general relativity. Unimodular gravity was first conceived in the first half of the twentieth century as a restriction of full GR in which one would only consider metric tensors with determinant ± 1. This original approach can be seen in [38, 85]. In such a theory, one restricts the allowable coordinate transformations, or the allowable diffeomorphisms on the spacetime manifold, to only those which preserve the metric volume factor $\sqrt{|g|}$. In this way, the symmetry of the theory is reduced from the full group of diffeomorphisms to the subgroup of *volume preserving*, or *transverse* diffeomorphisms. These initial efforts would inspire further work such as [25, 26, 103], with a common theme of fixing the value of the metric volume factor at the level of the action. A more comprehensive introduction to the various formulations of unimodular gravity can be found in [5]. In addition, one can see [27, 31] for an exploration of the differences between GR and unimodular gravity at the classical and quantum levels. Our interest in unimodular gravity, besides an intrinsic interest in formulations of gravity, comes from the tendency of unimodular theories to eliminate the cosmological constant as a global degree of freedom. To see how this can occur, we will examine a particular action for unimodular gravity which can be found in [65, 112]. This action is given by

$$S_{\mathrm{PV}}\left[g,\alpha;\tilde{\mu}_0\right] = \frac{1}{8\pi G}\int d^4x \left[\frac{1}{2}\sqrt{|g|}\,R - \alpha\left(\sqrt{|g|} - \tilde{\mu}_0\right)\right] \,. \tag{2.112}$$

Here, α is a scalar field which acts as Lagrange multiplier enforcing a dynamical constraint which fixes the value of $\sqrt{|g|}$ to the background scalar density $\tilde{\mu}_0$. For emphasis, $\tilde{\mu}_0$ is not a dynamical field, its value is fixed at the level of the action. Consequently, one sees that the action (2.112) is no longer invariant under the full group of diffeomorphism, but rather only those which preserve $\tilde{\mu}_0$. These are the volume preserving/transverse diffeomorphisms mentioned prior. In this thesis, we shall refer to actions of this kind as *preferred volume* actions. Variations with respect to the metric $g_{\mu\nu}$ and the scalar α yield the field equations

$$R_{\mu\nu} = \frac{1}{4}Rg_{\mu\nu}\ , \quad R = 4\alpha\ , \quad \sqrt{|g|} = \tilde{\mu}_0\ . \tag{2.113}$$

The first equation here is the Einstein condition. One can contract both sides with ∇^μ to get $\nabla^\mu R_{\mu\nu} = \frac{1}{4}\nabla_\nu R$. Then one can make use of the contracted Bianchi identity, $\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R$, to arrive at

$$\nabla_\mu R = 0\ . \tag{2.114}$$

We see that the Ricci scalar is covariantly constant throughout spacetime. Furthermore, the second equation gives us $\nabla_\mu \alpha = 0$ also. Then α is a surrogate for the cosmological constant in this theory. Crucially, the theory doesn't specify what the constant value of the Ricci scalar should be. The cosmological constant has been removed as a parameter in the theory. However, in its place we have $\tilde{\mu}_0$, which specifies the value of the metric volume factor. One can argue that this opens up potential solutions to the *cosmological constant problem*, [108]. In addition, this formulation of unimodular gravity provides some insight into the *problem of time* in quantum gravity. In brief, the canonical representation of GR, whether via the ADM formalism [9] or via the Ashtekar variables [11], admits a Hamiltonian which is a sum of constraints. In the process of quantisation, one defines the Hilbert space of physical states to be the space whose states are annihilated by the operator representations of the classical constraints. This yields the Wheeler DeWitt equation in particular, [34]. Then the Schrodinger equation for physical states always produces zero on the RHS, and therefore the quantum theory is static. In contrast, the canonical representation of the preferred volume approach admits a Hamiltonian which has a definite, non-constraint, part. The action of the operator representation of this part of the Hamiltonian on the physical states may be non-trivial. Then we may have a non-trivial Schrodinger equation. We will examine these constructions in greater detail later in the section on unimodular formulations of Plebański gravity.

There exists an alternative approach, introduced in [53], which maintains general covariance while still eliminating the cosmological constant as a global DOF. The idea here is to promote the cosmological constant Λ to a scalar field λ, and to include a constraint term to enforce its constancy. We have the action

$$S_{\mathrm{HT}}\left[g, \lambda, \tilde{T}\right] = \frac{1}{8\pi G} \int d^4x \, \left[\frac{1}{2}\sqrt{|g|}\,(R - 2\lambda) - \tilde{T}^\mu \partial_\mu \lambda\right] \, . \tag{2.115}$$

$\tilde{T}^\mu$ is a vector density (of weight +1). In this thesis, we will refer to this approach as the *parametrised* approach, or alternatively as the *Henneaux-Teitelboim* approach. Variations of the action yield the field equations

$$R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} = 0 \, , \quad R = 4\lambda \, , \quad \sqrt{|g|} = \partial_\mu \tilde{T}^\mu \, , \quad \partial_\mu \lambda = 0 \, . \tag{2.116}$$

Again, we derive the conditions $\nabla_\mu R = 0$ and $\nabla_\mu \lambda = 0$ from the first and second equations. Then the final equation in the above is redundant. As in the preferred volume approach, we get the vacuum Einstein equations $R_{\mu\nu} = \lambda g_{\mu\nu}$ for some constant λ whose value is not specified by the theory. This formulation of unimodular gravity does come with its own pathologies however. One sees that metric volume factor is given by a divergence. Hence, given any integrable spacetime region $\mathcal{U} \subset \mathcal{M}$ with boundary $\partial\mathcal{U}$, which may be empty, the spacetime volume is given by a surface integral

$$\mathcal{V}_4(\mathcal{U}) = \int_{\partial\mathcal{U}} T \ , \tag{2.117}$$

where T is a 3-form with components $T_{\nu\rho\sigma} = \tilde{T}^{\mu} \underset{\sim}{\epsilon}_{\mu\nu\rho\sigma}$. In particular, if the boundary of this spacetime region is empty, then its volume is necessarily zero. On the other hand, this property of the parametrised approach can be of some use to us. In particular, we may use this property to define a clock function which provides a gauge invariant notion of time. This is achieved, in the context of globally hyperbolic spacetimes $\mathcal{M} = \mathbb{R} \times \mathcal{S}$, by computing the spacetime 4-volume of the region connecting a pair of constant time hypersurfaces. Further exposition is given in [53], and we will return to this idea later in our discussion of unimodular formulations of Plebański gravity.

In summary, we use the term *unimodular gravity* quite liberally in this thesis. Here, unimodular gravity refers to theories which produce the Einstein condition without explicitly specifying the constant value of the Ricci scalar. Although, this is not necessarily how the term is used in the literature.

2.6 Constrained Hamiltonian Dynamics

In this section, we will review Dirac's formalism for constrained Hamiltonian dynamics. This formalism will see heavy use in the main body if this thesis. This section will mainly serve to be a condensed review of the textbook [52], which contains a thorough exploration of this topic. A more applied introduction to this formalism can be found in [36].

Let $\mathcal{Q}$ be a smooth manifold of dimension N called the *configuration manifold*, and let q^n be local coordinates. These coordinates naturally extend to local coordinates (q^n, v^n) on the tangent bundle $T\mathcal{Q}$ such that for each vector $X \in T_{\boldsymbol{q}}\mathcal{Q}$ at a point $\boldsymbol{q} \in \mathcal{Q}$ we have

$$X = v^n(X) \left(\frac{\partial}{\partial q^n} \right)_{\boldsymbol{q}} . \tag{2.118}$$

Note, in the physics tradition, we abuse notation here by using the same labels for the coordinate functions q^n on $\mathcal{Q}$, and the first N coordinate functions q^n on $T\mathcal{Q}$. Let $L(q, v)$ be a smooth function on $T\mathcal{Q}$ called the Lagrangian, and define an action functional whose domain is the space of curves $q^n(t)$ on $\mathcal{Q}$ via

$$S[q] = \int dt \; L\left(q, \dot{q}\right) \ , \tag{2.119}$$

where $\dot{q}^n = dq^n/dt$ denotes the velocity of the curve. From this more geometric perspective, the Legendre transform should be understood as a map from the tangent bundle $T\mathcal{Q}$ to the cotangent bundle $T^*\mathcal{Q}$. We define local coordinates (q^n, p_n) on $T^*\mathcal{Q}$ such that for each covector $\alpha \in T^*_{\boldsymbol{q}}\mathcal{Q}$ at a point $\boldsymbol{q} \in \mathcal{Q}$ we have

$$\alpha = p_n(\alpha) \left(dq^n\right)_{\boldsymbol{q}} . \tag{2.120}$$

Going forward, we will now refer to the cotangent space $T^*\mathcal{Q}$ as *phase space*, and denote it by $\mathcal{P}$. The Legendre transform is given in terms of these coordinates by

$$\mathcal{L} : (q^n, v^n) \ \mapsto \ (q^n, p_n) = \left(q^n, \frac{\partial L}{\partial v^n} \right) . \tag{2.121}$$

The Legendre transform is invertible when the matrix of second partial derivatives of the Lagrangian w.r.t the 'velocities' v^n is non-degenerate;

$$\det \left(\frac{\partial^2 L}{\partial v^n \, \partial v^{n'}} \right) \neq 0 \ . \tag{2.122}$$

In this case, one can proceed to construct the Hamiltonian representation of this system in the usual way. In this section, we will be concerned with the situation where said matrix is not invertible, so we have

$$\text{Rank}\left(\frac{\partial^2 L}{\partial v^n \partial v^{n'}}\right) = N - M \ , \tag{2.123}$$

where $M < N$ is a positive integer. We assume for simplicity that the rank of this matrix is constant throughout $T\mathcal{Q}$, which may not be true in general. In this case, we expect there to be M independent constraints $\phi_m(q,p) = 0$ which follow immediately from the formula

$$p_n = \frac{\partial L}{\partial v^n} \ . \tag{2.124}$$

These constraints are called *primary constraints*. The constraint functions ϕ_m define a surface in phase space $\mathcal{S} \subset \mathcal{P}$ which is their zero locus, $\mathcal{S} = \{(q,p) \in \mathcal{P} : \phi_m(q,p) = 0, \forall m = 1, \ldots, M\}$. This surface contains all of the physically permissible states. One can construct a so-called *naive Hamiltonian*, denoted by H_0, in the usual way via

$$H_0 = \dot{q}^n p_n - L \ . \tag{2.125}$$

This Hamiltonian is also called the *canonical Hamiltonian* in the literature, for example in [52]. The most general physically permissible time evolution is generated by the equations

$$\dot{q}^n = \frac{\partial H_0}{\partial p_n} + u^m \frac{\partial \phi_m}{\partial p_n} \ , \tag{2.126a}$$

$$\dot{p}^n = -\frac{\partial H_0}{\partial q^n} - u^m \frac{\partial \phi_m}{\partial q^n} \ . \tag{2.126b}$$

Here u^m are a collection of arbitrary phase space functions, *Lagrange multipliers*. These time evolution equations come from variations of the canonical action $S_{\text{Can}} = \int dt \, (\dot{q}^n p_n - H_0)$ where we also have the added nuance that the variations δq^n and δp_n are not independent, but in fact satisfy certain conditions coming from variations of the constraints, $\delta \phi_m = 0$. one can see [52] for further details. We construct a Poisson bracket on phase space functions via

$$\{f, g\} = \frac{\partial f}{\partial q^n} \frac{\partial g}{\partial p_n} - \frac{\partial g}{\partial q^n} \frac{\partial f}{\partial p_n} \ . \tag{2.127}$$

Furthermore, we define an equivalence relation on phase space functions called *weak equality*, denoted $\approx$, such that a pair of phase space functions f, g are weakly equal $f \approx g$ if they take the same value on the constraint surface, $f|_{\mathcal{S}} = g|_{\mathcal{S}}$. In particular, a phase space function is weakly vanishing $f \approx 0$ if it vanishes on the constraint surface. Any weakly vanishing function can be expanded into the form $h^m \phi_m$ where h^m are a collection of phase space functions which are not uniquely determined in this expansion. One should see [52] for a proof of this statement.

The time evolution equations can be extended to arbitrary phase space functions f using the Poisson bracket via

$$\dot{f} = \{f, H_0\} + u^m\{f, \phi_m\} \ . \tag{2.128}$$

Note, we now take the over-dot $\dot{}$ to denote differentiation w.r.t the time parameter t, so that $\dot{q}^n = dq^n/dt$ and $\dot{p}_n = dp_n/dt$, and so on. One can construct a Hamiltonian via $H = H_0 + u^m\phi_m$, then the time evolution equations can be written compactly as

$$\dot{f} \approx \{f, H\} \ , \tag{2.129}$$

Weak equality is sufficient here, since we only concern ourselves with the values any phase space function takes on the constraint surface where all of the physically allowable states live.

Consistency conditions and secondary constraints

Time evolution should preserve the constraint surface so that physical states are evolved onto physical states. This requirement can be formalised as a collection of *consistency conditions* $\dot{\phi}_m \approx 0$ which expand as

$$\{\phi_m, H_0\} + u^{m'}\{\phi_m, \phi_{m'}\} \approx 0 \ . \tag{2.130}$$

For each m, the above condition may reduce to a condition that is independent of the Lagrange multipliers u^m, or it may result in a condition which restricts the allowable values of u^m. In the former case, we either have that $\dot{\phi}_m \approx 0$ results in a tautology $0 \approx 0$, or we have a condition of the kind $\chi(q, p) \approx 0$ where χ is a phase space function that is independent of the constraints ϕ_m. When such a χ appears, we make it a further constraint in our theory. A constraint that arises in this way is called as *secondary constraint*, in contrast to the primary constraints which are defined without reference to the equations of motion. Note, whenever we add secondary constraints to our theory, we should extend the definition of weak equality $\approx$ so that a pair of phase space functions are weakly equal if they coincide on the surface generated by both the primary and secondary constraints. χ should evolve consistently so that

$$\{\chi, H_0\} + u^m\{\chi, \phi_m\} \approx 0 \ . \tag{2.131}$$

Again, this condition is either identically satisfied, or it imposes restrictions on u^m, or it yields a further independent constraint $\chi'(q, p) \approx 0$ which should be added to our theory and tested for consistency. This process repeats until it terminates. At the conclusion, we should have a list of secondary constraints labelled by ϕ_k for $k = M + 1, \dots, M + K$ with

K being the total number of secondary constraints. For future convenience, we collect all of the primary and secondary constraint under a single label ϕ_j for $j = 1, \ldots, M + K$ such that the first M constraints in this list are primary, and the latter K are secondary. Then the conditions

$$\{\phi_j, H_0\} + u^m\{\phi_j, \phi_m\} \approx 0 \ , \tag{2.132}$$

can be solved for u^m, where $m = 1, \ldots, M$ only. Solutions take the form $u^m = U^m + V^m$ where U^m is a particular solution, and where V^m solves the homogeneous equations

$$\{\phi_j, \phi_m\}V^m \approx 0 \ . \tag{2.133}$$

In general, there are a collection of independent solutions ${V_a}^m$ which are indexed by $a = 1, \ldots, A$. Then V^m can be written as $v^a {V_a}^m$ where v^a are arbitrary phase space functions. With this, we construct the *Total Hamiltonian* as

$$H_T = H' + v^a \phi_a \ , \tag{2.134}$$

where $H' = H_0 + U^m \phi_m$ and where $\phi_a = {V_a}^m \phi_m$. The time evolution equations generated by this Hamiltonian via $\dot{f} \approx \{f, H_T\}$ preserve the constraint surface as required.

First class constraints and gauge transformations

A phase space function f is called *first class* if it has a weakly vanishing Poisson bracket with each constraint,

$$\{f, \phi_j\} \approx 0 \, , \tag{2.135}$$

and *second class* otherwise. One can show that the Poisson bracket of a pair of first class functions is again first class, and hence the first class functions form a closed Poisson algebra. It is easily seen that the quantities ϕ_a are first class, $\{\phi_a, \phi_j\} \approx 0$. The so-called *first class primary constraints* ϕ_a generate gauge transformations on the phase space via

$$\delta_\varepsilon q^n = \varepsilon^a \{q^n, \phi_a\} \ , \quad \delta_\varepsilon p_n = \varepsilon^a \{p_n, \phi_a\} \ . \tag{2.136}$$

ε^a are the gauge parameters. In fact, there may exist further permissible independent gauge transformations on the system. In a similar spirit to before, let $\gamma_a = {W_a}^j \phi_j$ where ${W_a}^j$ denote the independent solutions to the homogeneous equations

$$\{\phi_j, \phi_{j'}\}{W_a}^{j'} \approx 0 \ , \tag{2.137}$$

where a now takes values $a = 1, \ldots, A'$ with $A \leq A'$. It is straightforward to check that γ_a are first class. γ_a generate gauge transformations via

$$\delta_\varepsilon q^n = \varepsilon^a \{q^n, \gamma_a\} \ , \quad \delta_\varepsilon p_n = \varepsilon^a \{p_n, \gamma_a\} \ . \tag{2.138}$$

Note, the primary first class constraints ϕ_a produce gauge transformations of physical significance. That is, gauge transformations that appear at the level of the Lagrangian formalism. However, there may be constraints among γ_a which produce gauge transformations with no physical significance. Armed with this understanding of first class constraints and the gauge transformations they generate, we turn our attention back to the total Hamiltonian. We see that the total Hamiltonian has a definite part H', which we see is first class, as well as a part $v^a \phi_a$ which generates gauge transformations. We would like our time evolution to include all of the available gauge transformations. In pursuit of this, we define the *extended Hamiltonian* by

$$H_E = H' + v^a \gamma_a \ . \tag{2.139}$$

We allow this Hamiltonian to generate the time evolution equations in the usual way by

$$\dot{f} \approx \{f, H_E\} \ . \tag{2.140}$$

Second class constraints and the Dirac bracket

The constraint surface $\mathcal{S}$ is generated by the constraint set ϕ_j, but this generating set is not unique. There are alternative generating sets ϕ'_j which also yield $\mathcal{S}$ as their zero locus. Such generating sets may be constructed via

$$\phi'_j = M_j{}^{j'} \phi_{j'} \ , \tag{2.141}$$

where the matrix $M_j{}^{j'}$ is non-degenerate everywhere in phase space. In particular, there exists an alternative generating set ϕ'_j which decomposes as γ_a, χ_α for $a = 1, \ldots, A'$ and $\alpha = 1, \ldots, M{+}K{-}A'$. γ_a are exactly the first class constraints from before, and χ_α are second class constraints. Furthermore, χ_α are such that the matrix of brackets $C_{\alpha\beta} = \{\chi_\alpha, \chi_\beta\}$ is non-degenerate everywhere in phase space. The presence of second class constraints indicates that there is some redundancy in our parametrisation of the physical system. In short, we have more variables than we need. At the classical level, the presence of second class constraints is not an issue. However, problems arise when one attempts to transition to the quantum regime. We would like to remove the second class constraints at the classical level in order to make life easier for ourselves later when we attempt to construct a quantum mechanical representation of our system. We can make all of the constraints ϕ_j first class by adjusting the Poisson structure. Furthermore, we can do this while also preserving the first class constraint algebra and the time evolution equations. We construct a new Poisson bracket called the *Dirac bracket* via

$$\{f, g\}^* = \{f, g\} + \{f, \chi_\alpha\}C^{\alpha\beta}\{\chi_\beta, g\} \ , \tag{2.142}$$

where $C^{\alpha\beta}$ is the inverse of $C_{\alpha\beta}$ satisfying $C^{\alpha\gamma}C_{\gamma\beta} = \delta^\alpha_\beta$. The Dirac bracket has a number of important properties. The first of these is that $\{f, \chi_\alpha\}^* = 0$ for any phase space function f and any second class constraint χ_α. Another is that $\{f, \Gamma\}^* \approx \{f, \Gamma\}$ for any function f and any first class function Γ. Then one sees that all of the constraints γ_a, χ_α are first class w.r.t the Dirac bracket. Furthermore, the Dirac bracket preserves the time evolution equations, since $\dot{f} \approx \{f, H_E\} \approx \{f, H_E\}^*$. By the preceding arguments, we may replace the original Poisson bracket with the Dirac bracket, recovering the same physical dynamics. Under this new Poisson structure, all of the second class constraints simply generate trivial gauge transformations

$$\delta_\varepsilon q^n = \varepsilon^\alpha\{q^n, \chi_\alpha\}^* = 0 \ , \quad \delta_\varepsilon p_n = \varepsilon^\alpha\{p_n, \chi_\alpha\}^* = 0 \ . \tag{2.143}$$

2.7 Hamiltonian Field Theory

In Hamiltonian mechanics, the *dynamical variables* consisted of a finite collection of coordinate functions q^n, p_n on phase space $\mathcal{P}$, which we defined as the cotangent bundle over some finite dimensional configuration manifold. In canonical field theory, the dynamical variables now consist of 'canonically conjugate' pairs of fields over space $\phi^A(x), \pi_A(x)$ which evolve in time. Each point in our phase space consists of a particular configuration of our fields $\phi^A(x), \pi_A(x)$. We can treat $\phi^A(x), \pi_A(x)$ as coordinate labels over this infinite dimensional phase space. That is, for each choice $A = 1, \ldots, N$ and each choice $x \in \mathcal{U} \subseteq \mathbb{R}^3$ there is a freedom parametrised by $\mathbb{R}$ or $\mathbb{C}$, depending on whether the theory is of real or complex fields respectively, associated to the value of $\phi^A(x)$, and the same freedom again for the value of $\pi_A(x)$. The phase space in the field case is an (uncountable) infinite dimensional manifold. Despite the vast enlargement of the phase space which occurs when moving from the finite dimensional case to the field case, a lot of the structure we had in the finite dimensional case translates directly to field case with often only minor alterations required.

2.7.1 Primer on functional calculus

Roughly speaking a *functional* is the name given to a real or complex valued function whose domain is a function space. For an illustrative example, let $C([0,1],\mathbb{C})$ denote the ring of continuous complex valued functions over the interval $[0,1] \subset \mathbb{R}$. We we may construct a functional $F : C([0,1],\mathbb{C}) \to \mathbb{R}$ as $\psi \mapsto \int_0^1 dx\ |\psi(x)|^2$. In our discussion of Hamiltonian field theory, we will be interested in a particular class of functionals called *local functionals*. Consider a theory of real or complex valued fields z^A over a manifold $\mathcal{S}$ of dimension D with local coordinates x^a for $a = 1, \ldots, D$. In later applications we will fix $D = 3$, but we leave D general for now. In this context, a local functional has the form

$$F[z^A] = \int d^D x\ f\left(z^A, \partial_a z^A, \ldots, \partial_{a_1} \ldots \partial_{a_k} z^A, x\right)\ , \tag{2.144}$$

where f is an ordinary function - not a functional - of the field variables z^A and their spatial derivatives up to finite order. We also allow f to have explicit dependence on the integration variable x^a. We refer to f as a *non-integrated density*, or just a *density* for short. Let η^A be a particular collection of fields, we define an analogue of the usual directional derivative for a functional F in the 'direction' of η^A via

$$\delta_\eta F[z^A] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F[z^A + \varepsilon \eta^A]\ . \tag{2.145}$$

For local functionals, one can show

$$\delta_\eta F[z^A] = \int d^D x\ \eta^A(x) \frac{\delta F}{\delta z^A(x)}\ , \tag{2.146}$$

where $\delta F / \delta z^A(x)$ is the *Functional derivative* of F which is given by

$$\frac{\delta F}{\delta z^A(x)} = \frac{\partial f}{\partial z^A} - \partial_a \left(\frac{\partial f}{\partial\, \partial_a z^A}\right) + \ldots + (-)^k \partial_{a_1} \ldots \partial_{a_k} \left(\frac{\partial f}{\partial\, \partial_{a_1} \ldots \partial_{a_k} z^A}\right)\ . \tag{2.147}$$

Alternatively, the functional derivative can be computed via

$$\frac{\delta F}{\delta z^A(x)} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F[z^B + \varepsilon \delta^B_A\, \delta_x]\ , \tag{2.148}$$

where δ_x denotes the indefinite form of the Dirac delta distribution centered at x, so that $\delta_x(x') = \delta(x' - x)$. Let f be a non-integrated density and let $x \in \Sigma$ be a particular point. We define a functional which evaluates the function $f(z^A, \partial_a z^A, \ldots)$ at the point x by

$$f(x)[z^A] = f\left(z^A(x), \partial_a z^A(x), \ldots, \partial_{a_1} \ldots \partial_{a_k} z^A(x), x\right)\ . \tag{2.149}$$

A functional of this kind is sometimes called an *evaluation functional*, for example in [75]. Using (2.148), one computes the functional derivative of $f(x)[z^A]$ to be

$$\begin{aligned}\frac{\delta f(x)}{\delta z^A(x')} = \frac{\partial f}{\partial z^A}\delta(x-x') + \frac{\partial}{\partial x^a}\left(\frac{\partial f}{\partial\, \partial_a z^A}\,\delta(x-x')\right) + \dots \\ \dots + \frac{\partial^k}{\partial x^{a_1}\dots\partial x^{a_k}}\left(\frac{\partial f}{\partial\, \partial_{a_1}\dots\partial_{a_k} z^A}\,\delta(x-x')\right)\ .\end{aligned} \tag{2.150}$$

In the above, it should be understood that we insert $z^A(x), \partial_a z^A(x), \dots$ into the terms $\partial f/\partial z^A$ and $\partial f/\partial\, \partial_a z^A$ and so on for higher orders. One sees that this derivative contains Dirac deltas $\delta(x-x')$ and their spatial derivatives up to finite order. For some illustrative examples, consider a theory of a single scalar field ϕ. One computes the functional derivative of the functional $\phi \mapsto \phi(x)$ to be

$$\frac{\delta\phi(x)}{\delta\phi(x')} = \delta(x'-x)\ . \tag{2.151}$$

For a less trivial example, one computes the functional derivative of the k-fold spatial derivative functional $\phi \mapsto \partial_{a_1}\dots\partial_{a_k}\phi(x)$ to be

$$\frac{\delta}{\delta\phi(x')}\,\partial_{a_1}\dots\partial_{a_k}\phi(x) = \frac{\partial^k}{\partial x^{a_1}\dots\partial x^{a_k}}\delta(x-x')\ . \tag{2.152}$$

2.7.2 Legendre transform in the field setting

As in the finite dimensional case, we begin with an action functional

$$S[\phi^A] = \int_{\mathcal{M}} d^{D+1}x\ \mathcal{L}(\phi^A, \partial_\mu\phi^A)\ , \tag{2.153}$$

where $\phi^A(x)$ for $A = 1,\dots,N$ are a collection of fields over a *spacetime* manifold $\mathcal{M}$ of dimension $D+1$ with local coordinates x^μ for $\mu = 0,\dots,D$. $\mathcal{L}(\phi^A, \partial_\mu\phi^A)$ is a non-integrated density called the *Lagrangian density*. For simplicity, we assume that $\mathcal{M}$ admits a global coordinate chart $x^\mu = (t, x^a)$ where x^a (with $a = 1,\dots,D$) are called *spatial* coordinates, and $x^0 = t$ is called the *time* coordinate. Note that this decomposition into space and time may be unphysical. Decomposing $\mathcal{M}$ in this way allows us to represent a theory of fields in spacetime in terms of a dynamical theory of fields in space which evolve in time. To emphasise this perspective shift, we may rewrite the action as

$$S[\phi^A] = \int dt \int_{\mathcal{S}} d^D x\ \mathcal{L}\left(\phi^A, \partial_a\phi^A, \dot{\phi}^A\right)\ , \tag{2.154}$$

where the over dot $\dot{}$ is a shorthand for $\partial/\partial t$. As with the finite dimensional case, the Legendre transform consists of a variable transformation from configuration space $(\phi^A, \dot{\phi}^A)$

to phase space (ϕ^A, π_A) given explicitly by

$$(\phi^A, \dot{\phi}^A) \;\mapsto\; (\phi^A, \pi_A) = \left(\phi^A, \frac{\partial \mathcal{L}}{\partial \dot{\phi}^A}\right) \; . \tag{2.155}$$

We are primarily concerned with the case where this transformation is not invertible, so that

$$\text{Rank}\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{\phi}^A \, \partial \dot{\phi}^B}\right) = N - M \; , \tag{2.156}$$

for a positive integer $M < N$ which we assume takes a constant value throughout spacetime. In this case, the equations $\pi_A = \partial \mathcal{L} / \partial \dot{\phi}^A$ yield local conditions of the kind

$$\Phi_m(\phi^A, \partial_a \phi^A, \ldots, \pi_A, \partial_a \pi_A, \ldots) = 0 \; , \tag{2.157}$$

for $m = 1, \ldots, M$. These are the primary constraints, which appear in the field case as non-integrated densities. The naive Hamiltonian is constructed as

$$H_0 = \int_\Sigma d^D x \left[\dot{\phi}^A \pi_A - \mathcal{L}\right] \; , \tag{2.158}$$

which is a functional over the field variables ϕ^A, π_A with no explicit dependence on $\dot{\phi}^A$.

Functional Poisson bracket

A functional Poisson bracket extends the usual notion of the Poisson bracket from the finite dimensional theory into the field setting. The functional Poisson bracket is an antisymmetric bilinear map that sends a pair of local functionals F and G to a new local functional $\{F, G\}$ constructed as

$$\{F, G\} = \int_\Sigma d^D x \left[\frac{\delta F}{\delta \phi^A(x)} \frac{\delta G}{\delta \pi_A(x)} - \frac{\delta G}{\delta \phi^A(x)} \frac{\delta F}{\delta \pi_A(x)}\right] \; . \tag{2.159}$$

The functional Poisson bracket satisfies the following properties on evaluation functionals of the kind $f(x)[\phi^A, \pi_A] = f(\phi^A(x), \partial_a \phi^A(x), \ldots)$ introduced in (2.149):

$$\{\phi^A(x), \pi_B(x')\} = \delta^A_B \, \delta(x - x') \; , \tag{2.160a}$$

$$\{\frac{\partial}{\partial x^a} f(x), g(x')\} = \frac{\partial}{\partial x^a}\{f(x), g(x')\} \; , \tag{2.160b}$$

$$\{f(x), \int d^D x' \, g(x')\} = \int d^D x' \, \{f(x), g(x')\} \; , \tag{2.160c}$$

$$\{f(x), g(x') h(x'')\} = g(x') \, \{f(x), h(x'')\} + \{f(x), g(x')\} \, h(x'') \; . \tag{2.160d}$$

The constraint surface in field theory

In the field context, the constraint surface is the - infinite dimensional - surface in phase space consisting of all of the field configurations ϕ^A, π_A which satisfy the constraints $\Phi_m = 0$ everywhere across space. Any pair of local functionals F, G are called *weakly equal*, denoted $F \approx G$, if they are equal when their domains are restricted to the constraint surface. Equivalently, such a pair of functionals are weakly equal if (and only if) their exists a collection h^m of functions of the field variables and their derivatives such that

$$F - G = \int d^D x \; h^m \Phi_m \; . \tag{2.161}$$

At the level of the non-integrated densities, we have $f \approx g$ if and only if

$$f - g = h^m \Phi_m + h^{m\,a}\, \partial_a \Phi_m + \ldots + h^{m\,a_1 \ldots a_k}\, \partial_{a_1} \ldots \partial_{a_k} \Phi_m \; , \tag{2.162}$$

for some functions $h^m, h^{m\,a}, \ldots, h^{m\,a_1 \ldots a_k}$ of the variables and their derivatives. One can see [52] for a proof.

The time evolution equations for the field variables are given by

$$\dot{\phi}^A(x) = \{\phi^A(x), H_0\} + \int d^D x' \; u^m(x')\, \{\phi^A(x), \Phi_m(x')\} \; , \tag{2.163a}$$

$$\dot{\pi}_A(x) = \{\pi_A(x), H_0\} + \int d^D x' \; u^m(x')\, \{\pi_A(x), \Phi_m(x')\} \; . \tag{2.163b}$$

Here, $u^m(\phi^A, \partial_a \phi^A, \ldots)$ are an undetermined collection of functions of the field variables and their derivatives such that the contraction $u^m \Phi_m$ transforms as a scalar density of weight $+1$ under coordinate transformations. These u^m are in parallel with the Lagrange multipliers from the finite dimensional case. One can construct a Hamiltonian $H = H_0 + \int d^D x \; u^m \Phi_m$, then the evolution equation for an arbitrary functional F can be written compactly as

$$\dot{F} \approx \{F, H\} \; . \tag{2.164}$$

As in the finite dimensional case, we require that the constraint surface be preserved under time evolution. This gives rise to consistency conditions $\{\Phi_m, H\} \approx 0$. The consistency analysis in the field case proceeds in parallel with the finite dimensional case, so we omit those details here. At the conclusion of the consistency analysis, we have an extended constraint set Φ_j for $j = 1, \ldots, M + K$, consisting of the primary constraints when $j = 1, \ldots, M$, and secondary constraints when $j = M + 1, \ldots, M + K$. The definition of weak equality is adjusted so that a pair of functionals are weakly equal when they coincide on the total constraint surface generated by the primary and secondary constraints. Note that there exist alternative constraint sets Φ'_j generating the same constraint surface as Φ_j. Any such constraint set Φ'_j is related to Φ_j via a transformation of the kind

$$\Phi'_j = M_j{}^{j'}\Phi_{j'} + M_j{}^{j'a}\partial_a\Phi_{j'} + \cdots + M_j{}^{j'a_1\ldots a_k}\partial_{a_1}\ldots\partial_{a_k}\Phi_{j'} \ , \tag{2.165}$$

where $M_j{}^{j'}, M_j{}^{j'a}, \ldots, M_j{}^{j'a_1\ldots a_k}$ are functions of the field variables and their derivatives such that the matrix constructed as

$$M_j{}^{j'} - \partial_a M_j{}^{j'a} + \ldots + (-)^k\partial_{a_1}...\partial_{a_k}M_j{}^{j'a_1\ldots a_k} \ , \tag{2.166}$$

is non-degenerate everywhere in space. One can construct such an alternative constraint set which decomposes as $\Phi'_j = \Gamma_{\mathbf{a}}, \mathcal{X}_\alpha$ where $\Gamma_{\mathbf{a}}$ (with $\mathbf{a} = 1, \ldots, \mathcal{A}$) are first class and $\mathcal{X}_\alpha$ (with $\alpha = 1, \ldots, M + K - \mathcal{A}$) are second class, such that the linear operator defined by $\mathcal{O}_{\alpha\beta}(x, x') = \{\mathcal{X}_\alpha(x), \mathcal{X}_\beta(x')\}$ is non-degenerate everywhere throughout space for every configuration ϕ^A, π_A on the constraint surface. Hence there exists a Green's function $\Delta^{\alpha\beta}(x, x')$ satisfying

$$\int d^Dx''\, \Delta^{\alpha\gamma}(x, x'')\mathcal{O}_{\gamma\beta}(x'', x') = \delta^\alpha_\beta\, \delta(x - x') \ . \tag{2.167}$$

With this, one can construct a functional Dirac bracket with respect to which all of the constraints Φ'_j are first class by

$$\{F, G\}^* = \{F, G\} - \int d^Dx\, d^Dx'\, \{F, \mathcal{X}_\alpha(x)\}\, \Delta^{\alpha\beta}(x, x')\, \{\mathcal{X}_\beta(x'), G\} \ . \tag{2.168}$$

One can confirm the following properties by direct calculation:

1. $\{F, \mathcal{X}_\alpha(x)\}^* = 0$ for all functionals F and all second class constraints $\mathcal{X}_\alpha$,

2. $\{F, \mathcal{G}\}^* \approx \{F, \mathcal{G}\}$ for all functionals F and all *first class* functionals $\mathcal{G}$.

In parallel with the finite dimensional case, the Dirac bracket makes all of the constraints first class. Hence, all of the constraint are generators of gauge transformations on the field variables, albeit the constraints $\mathcal{X}_\alpha$ generate only trivial gauge transformations. Additionally, the Dirac bracket preserves the equations of motion and the first class constraint algebra. Hence, we may replace the standard functional Poisson bracket with the functional Dirac bracket in our theory without affecting the dynamics. As before, the primary motivation for constructing such a bracket to aid in the transition to the quantum theory.

The Extended Hamiltonian

We conclude our review of Hamiltonian field theory by outlining the construction of the *extended Hamiltonian*, which generates the most general permissible time evolution on the field variables. First, let $U^m(\phi^n, \partial_a\phi^n, \ldots)$ be a particular solution to the consistency conditions, satisfying

$$\{\Phi_j(x), H_0\} + \int d^D x \, \{\Phi_j(x), \Phi_m(x')\} \, U^m(x') \approx 0 \, . \tag{2.169}$$

Then one constructs a first class extension of the naive Hamiltonian via

$$H' = H_0 + \int d^D x \; U^m \Phi_m \, . \tag{2.170}$$

From here, the extended Hamiltonian is derived by further extending this first class Hamiltonian with the first class constraints $\Gamma_{\mathbf{a}}$ yielding

$$H_E = H' + \int d^D x \, v^{\mathbf{a}} \Gamma_{\mathbf{a}} \, . \tag{2.171}$$

The time evolution equation for an arbitrary functional F is given by

$$\dot{F} \approx \{F, H_E\} \, , \tag{2.172}$$

The constraint surface is preserved under this time evolution as desired.

2.A Supplementary Material: Chapter 2

2.A.1 Einstein-Cartan Gravity

Proof: Einstein field equations from (2.49b)

One can rewrite (2.49b) in the form,

$$\begin{aligned} 0 &= \epsilon_{IJKL}\, e^I_\mu e^J_\nu \left(\frac{1}{2}R_{\rho\sigma}{}^{KL} - \frac{\Lambda}{3} e^K_\rho e^L_\sigma\right)\varepsilon^{\lambda\nu\rho\sigma} \\ &= \epsilon_{IJKL}\, e^I_\mu e^J_\nu \left(\frac{1}{2}R_{\rho\sigma}{}^{\alpha\beta} e^K_\alpha e^L_\beta - \frac{\Lambda}{3} e^K_\rho e^L_\sigma\right)\varepsilon^{\lambda\nu\rho\sigma} \\ &= \frac{1}{2}R_{\rho\sigma}{}^{\alpha\beta}\varepsilon_{\mu\nu\alpha\beta}\,\varepsilon^{\lambda\nu\rho\sigma} - \frac{\Lambda}{3}\varepsilon_{\mu\nu\rho\sigma}\,\varepsilon^{\lambda\nu\rho\sigma} \\ &= \sigma\left(\frac{1}{2}R_{\rho\sigma}{}^{\alpha\beta}\,\delta^{\rho\sigma\lambda}_{\alpha\beta\mu} - 2\Lambda\,\delta^\lambda_\mu\right) \\ &= -2\sigma\left(R^\lambda{}_\mu - \frac{1}{2}R\delta^\lambda_\mu + \Lambda\delta^\lambda_\mu\right)\ . \end{aligned} \tag{2.173}$$

Multiplying the final line by $-1/2\sigma$, and lowering with the metric where appropriate gives the Einstein field equation,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0\ . \tag{2.174}$$

2.A.2 Plebański Gravity

Proof: Composition law (2.69)

The proof is immediate by directly substituting the expansion (2.68) into (2.69),

$$\begin{aligned} \Sigma^i_\mu{}^\rho\Sigma^j_{\rho\nu} &= \left(\sqrt{\sigma}\, e^0_\mu e^{i\rho} - \sqrt{\sigma}\, e^{0\rho} e^i_\mu - \epsilon^i{}_{kl}\, e^k_\mu e^{l\rho}\right)\left(\sqrt{\sigma}\, e^0_\rho e^j_\nu - \sqrt{\sigma}\, e^0_\nu e^i_\rho - \epsilon^j{}_{mn}\, e^m_\rho e^n_\nu\right) \\ &= -\delta^{ij}\left(\sigma\, e^0_\mu e^0_\nu + e^i_\mu e_{i\nu}\right) + \epsilon^{ij}{}_k\sqrt{\sigma}\left(e^0_\mu e^k_\nu - e^0_\nu e^k_\mu\right) - e^i_\mu e^j_\nu + e^j_\mu e^i_\nu \\ &= -\delta^{ij}\left(\sigma\, e^0_\mu e^0_\nu + e^i_\mu e_{i\nu}\right) + \epsilon^{ij}{}_k\left(\sqrt{\sigma}\, e^0_\mu e^k_\nu - \sqrt{\sigma}\, e^0_\nu e^k_\mu - \epsilon^k{}_{lm} e^l_\mu e^m_\nu\right) \\ &= -\delta^{ij} g_{\mu\nu} + \epsilon^{ij}{}_k\Sigma^k_{\mu\nu}\ . \end{aligned} \tag{2.175}$$

Note, we use $e^{0\rho}e^0_\rho = \sigma$, and $e^{i\rho}e^j_\rho = \delta^{ij}$, and $e^{0\rho}e^i_\rho = 0$.

Proof: Hodge dual reformulation of the Einstein condition

One can use the property of the Hodge dual $\star^2 = \sigma\mathbb{I}$ to rewrite $(\star R)_{\mu\nu\rho\sigma} = (R\star)_{\mu\nu\rho\sigma}$ as,

$$R_{\mu\nu\rho\sigma} = \frac{1}{\sigma}(\star R\star)_{\mu\nu\rho\sigma} = \frac{1}{4\sigma}\,\varepsilon_{\mu\nu}{}^{\alpha\beta}\varepsilon_{\rho\sigma}{}^{\gamma\delta}R_{\alpha\beta\gamma\delta}\ . \tag{2.176}$$

Contracting the first and third free indices on either side using the metric yields,

$$\begin{aligned} R^{\rho\mu}{}_{\rho\nu} &= \frac{1}{4\sigma}\,\varepsilon^{\rho\mu\alpha\beta}\varepsilon_{\rho\nu\gamma\delta}R_{\alpha\beta}{}^{\gamma\delta} \\ &\overset{!}{=} \frac{1}{4}\,\delta^{\mu}_{[\nu}\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta]}\,R_{\alpha\beta}{}^{\gamma\delta} \\ &= \frac{1}{2}\left(\delta^{\mu}_{\nu}R_{\alpha\beta}{}^{\alpha\beta} + R_{\nu\alpha}{}^{\alpha\mu} + R_{\alpha\nu}{}^{\mu\alpha}\right) \\ &= \frac{1}{2}\delta^{\mu}_{\nu}R - R^{\rho\mu}{}_{\rho\nu}\ . \end{aligned} \tag{2.177}$$

Note, we have used the identity $\varepsilon^{\rho\mu\alpha\beta}\varepsilon_{\rho\nu\gamma\delta} = \sigma\,\delta^{\mu}_{[\nu}\delta^{\alpha}_{\gamma}\delta^{\beta}_{\delta]}$ in the second equality marked $\overset{!}{=}$. Lowering indices with the metric where appropriate yields,

$$R_{\mu\nu} - \frac{1}{4}Rg_{\mu\nu} = 0\ . \tag{2.178}$$

Proof: Theorem 2.3.1

Let Σ^i be the self-dual part of a solder form e^I as in (2.68), and let $g = \sigma\,e^0\otimes e^0 + \sum_{i=1}^{3} e^i\otimes e^i$ be a metric with volume form $\varepsilon = -\sigma\,e^0\wedge\ldots\wedge e^3$. Tensor indices $\mu,\nu,\ldots$ are lowered and raised with $g_{\mu\nu}$ and its inverse. The chiral zero torsion condition $D_A\Sigma^i = 0$ can be written equivalently as

$$\begin{aligned} \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu\Sigma^i_{\rho\sigma} &= -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\epsilon^{i}{}_{jk}A^j_\nu\Sigma^k_{\rho\sigma} \\ &= -\epsilon^{i}{}_{jk}A^j_\nu(\star\Sigma^k)^{\mu\nu} = \sqrt{\sigma}\,\epsilon^{i}{}_{jk}(\Sigma^j)^{\mu\nu}A^k_\nu\ , \end{aligned} \tag{2.179}$$

which rearranges to

$$\frac{1}{2\sqrt{\sigma}}\,\varepsilon_\mu{}^{\nu\rho\sigma}\partial_\nu\Sigma^i_{\rho\sigma} = J_\Sigma\,A^i_\mu\ . \tag{2.180}$$

J_Σ is a linear map defined

$$J_\Sigma : A^i_\mu \ \mapsto\ \epsilon^{i}{}_{jk}\Sigma^{j\ \nu}_{\mu}A^k_\nu\ , \tag{2.181}$$

satisfying

$$J_\Sigma^2 = 2\mathbb{I} + J_\Sigma \quad\Leftrightarrow\quad J_\Sigma^{-1} = \frac{1}{2}\left(J_\Sigma - \mathbb{I}\right)\ . \tag{2.182}$$

Hence the inverse J_Σ^{-1} exists, and therefore the chiral zero torsion condition has a unique solution when Σ^i is the self-dual part of a solder form. Furthermore, one can write down a closed form expression for this solution,

$$A^i_\mu(\Sigma) = \frac{1}{4\sqrt{\sigma}}\left(\epsilon^{i}{}_{jk}\Sigma^j_{\mu\lambda}\varepsilon^{\lambda\nu\rho\sigma}\partial_\nu\Sigma^k_{\rho\sigma} - \varepsilon_\mu{}^{\nu\rho\sigma}\partial_\nu\Sigma^i_{\rho\sigma}\right)\ . \tag{2.183}$$

To complete the proof, it is a straightforward exercise to show that the chiral zero torsion condition is solved in this case by the self-dual part of the unique zero torsion spin connection ω^{IJ} associated to e^I which expands as

$$A^i = 2\sqrt{\sigma}\, P^{+\,0i}_{IJ}\, \omega^{IJ} = \sqrt{\sigma}\, \omega^{0i} - \frac{1}{2}\, \epsilon^i{}_{jk}\, \omega^{jk} \; . \tag{2.184}$$

Therefore, the self-dual part of the zero torsion spin connection is the unique solution in this case.

Proof: Proportionality of the Palatini term and the Ricci scalar

Consider the following:

$$\begin{aligned} \epsilon_{IJKL}\, e^I \wedge e^J \wedge R^{KL} &= \frac{1}{2} \epsilon_{IJKL} R^{KL}{}_{MN}\, e^I \wedge e^J \wedge e^M \wedge e^N \\ &\overset{!}{=} -\frac{1}{2} \epsilon_{IJKL}\, \epsilon^{IJMN} R^{KL}{}_{MN}\, e^0 \wedge \ldots \wedge e^3 \\ &= -2\sigma\, \delta^{[K}_I \delta^{L]}_J R^{IJ}{}_{KL}\, e^0 \wedge \ldots \wedge e^3 \\ &= 2R\, \varepsilon \; . \end{aligned} \tag{2.185}$$

The first equality comes from decomposing the 2-forms R^{IJ} in the tetrad basis such that $R^{IJ} = \frac{1}{2} R^{IJ}{}_{MN}\, e^M \wedge e^N$. The second equality marked $\overset{!}{=}$ arises from the identity

$$e^I \wedge e^J \wedge e^k \wedge e^L = -\epsilon^{IJKL}\, e^0 \wedge \ldots \wedge e^3 \; , \tag{2.186}$$

where the minus sign appears due to our convention $\epsilon^{0ijk} = -\epsilon^{ijk}$. The final equality comes from the definition $\varepsilon = -\sigma\, e^0 \wedge \ldots \wedge e^3$ from (2.56).

Proof: Theorem 2.3.3

Let Σ^i satisfy $\Sigma^i \wedge \Sigma^j = \delta^{ij} \mu$ for some complex valued 4-form μ. Define a pair, $\Sigma^{\pm} = \Sigma^1 \pm i \Sigma^2$. One sees that $\Sigma^{\pm}$ satisfy $\Sigma^+ \wedge \Sigma^+ = \Sigma^- \wedge \Sigma^- = 0$. Hence they can each be written as a product of a pair of 1-forms

$$\Sigma^+ = \theta^0 \wedge \theta^1 \quad , \quad \Sigma^- = \tilde{\theta}^2 \wedge \tilde{\theta}^3 \; . \tag{2.187}$$

Furthermore, $\Sigma^+ \wedge \Sigma^- = \theta^0 \wedge \theta^1 \wedge \tilde{\theta}^2 \wedge \tilde{\theta}^3 = 2\mu$. Then from $\Sigma^3 \wedge \Sigma^+ = \Sigma^3 \wedge \Sigma^- = 0$, we conclude that Σ^3 expands as

$$\Sigma^3 = \frac{1}{2} \left(O_{11}\, \theta^0 \wedge \tilde{\theta}^3 + O_{12}\, \theta^0 \wedge \tilde{\theta}^2 + O_{21}\, \theta^1 \wedge \tilde{\theta}^3 + O_{22}\, \theta^1 \wedge \tilde{\theta}^2 \right) \; , \tag{2.188}$$

for some complex coefficients O_{AB}. The condition $\Sigma^3 \wedge \Sigma^3 = \mu = \frac{1}{2}\theta^0 \wedge \theta^1 \wedge \tilde{\theta}^2 \wedge \tilde{\theta}^3$ yields a condition $O_{11}O_{22} - O_{12}O_{21} = 1$, and hence O_{AB} are the components of a matrix $O \in SL(2, \mathbb{C})$. We may define new 1-forms θ^2 and θ^3 via,

$$\begin{pmatrix} \theta^3 \\ \theta^2 \end{pmatrix} = O \begin{pmatrix} \tilde{\theta}^3 \\ \tilde{\theta}^2 \end{pmatrix} . \tag{2.189}$$

Then we have $\Sigma^- = \theta^2 \wedge \theta^3$, and $\Sigma^3 = \frac{1}{2}(\theta^0 \wedge \theta^3 + \theta^1 \wedge \theta^2)$. Finally, one can construct a complex tetrad e^I via,

$$\theta^0 = -e^1 - ie^2 , \quad \theta^1 = \sqrt{\sigma}\, e^0 - ie^3 , \quad \theta^2 = -i\sqrt{\sigma}\, e^0 + e^3 , \quad \theta^3 = ie^1 + e^2 , \tag{2.190}$$

such that $\Sigma^i = 2\sqrt{\sigma}\, P^{+\,0i}_{IJ}\, e^I \wedge e^J$ as desired.

Chapter 3

Unimodular Plebański Gravity

This chapter is based on [46] which was written in collaboration with Steffen Gielen.

In this chapter, we introduce *Unimodular Plebański Formulations* of GR: formulations of unimodular gravity (section 2.5) which are written in terms of the variables of the Plebański formulation of GR, introduced in (section 2.3). In brief, one observes that the *unimodular condition* $\partial_\mu R = 0$, coming from a Bianchi identity of the Riemann tensor, corresponds to an analogous condition in the Plebański variables given $\partial_\mu \operatorname{tr} M = 0$, also coming from a Bianchi identity. Then one can consider modified versions of the chiral connection actions (2.79), (2.98) and (2.105) which produce this unimodular condition without the equation that fixes $\operatorname{tr} M$ to a particular value, in analogy with the unimodular actions (2.115) and (2.115) in metric variables. One arrives at Plebański-type formulations where the cosmological constant is not fixed a priori, but can be interpreted as an integration constant. One of the formulations we present here was already explored in [95], while the others are novel.

3.1 The Unimodular Condition

For the purposes of this thesis, the central observation in the construction of unimodular formulations of general relativity was that the Einstein condition $R_{\mu\nu} = \frac{1}{4} R g_{\mu\nu}$ in combination with a contracted Bianchi identity $\nabla^\mu R_{\mu\nu} = \frac{1}{2} \nabla_\nu R$ already leads to the condition $\nabla_\mu R = 0$, which we will call the *unimodular condition*. Then the Ricci scalar is equal to some integration constant. In GR, the value of this constant is fixed to 4Λ by the trace part of the Einstein field equations, where Λ denotes the cosmological constant. In unimodular gravity, and in trace-free gravity, one drops the trace part of the Einstein field equations. In the Plebański formulation, the Einstein condition is given by $F^i = M^{ij} \Sigma_j$, and the trace part of the Einstein field equations is given by $\operatorname{tr} M = \Lambda$. In connection formulations we have the Bianchi identity $D_A F^i = 0$ which, in combination with the chiral Einstein condition (2.81b), gives

$$0 = D_A F^i = D_A \left(M^{ij} \Sigma_j \right) \stackrel{!}{=} D_A M^{ij} \wedge \Sigma_j \; . \tag{3.1}$$

Note, in the final equality, marked by '$\stackrel{!}{=}$' we have used the chiral zero torsion condition $D_A \Sigma^i = 0$ to eliminate the term $M^{ij} D_A \Sigma_j$ which would otherwise appear. One can write this in index form as

$$D_\mu M_{ij} \, \Sigma^j_{\nu\rho} + D_\nu M_{ij} \, \Sigma^j_{\rho\mu} + D_\rho M_{ij} \, \Sigma^j_{\mu\nu} = 0 \; . \tag{3.2}$$

From this point onwards, we will assume that Σ^i satisfies the metricity constraint (2.81c). Then theorem 2.3.3 guarantees the existence of a, likely complex valued, solder form e^I such that Σ^i expands as its self-dual part as in (2.68). Then one can use the Urbantke metric (2.97), which by theorem 2.3.4 expands as $g = \sigma \, e^0 \otimes e^0 + \sum_{i=1}^3 e^i \otimes e^i$, to raise the tensor indices of $\Sigma^i_{\mu\nu}$ yielding the raised form $\Sigma^{i\mu\nu}$ which satisfies the composition law (2.69). We fully contract the LHS of (3.2) with $\Sigma^{i\mu\nu}$ yielding,

$$\frac{1}{2} \Sigma^{i\mu\nu} \left(2 D_\mu M_{ij} \, \Sigma^j_{\nu\rho} + D_\rho M_{ij} \, \Sigma^j_{\mu\nu} \right) \stackrel{!}{=} \partial_\rho \operatorname{tr} M = 0 \; . \tag{3.3}$$

Note that the final equality in the above is independent of the solder form e^I and the metric g that it generates. This result is consolidated in the following theorem:

Theorem 3.1.1. *Let Σ^i satisfy $\Sigma^i \wedge \Sigma^j = \frac{1}{3} \delta^{ij} \Sigma^k \wedge \Sigma_k$, and $D_A \Sigma^i = 0$, and $F^i = M^{ij} \Sigma_j$, for some $SO(3)$ connection and some symmetric 3×3 matrix field M^{ij}. Then $\partial_\mu \operatorname{tr} M = 0$.*

This result holds in the real Lorentzian, Euclidean and general complex versions of the Plebański theory. This is clear since we have imposed no reality conditions on the fields. Recall from section 2.3 that in all said versions of Plebański gravity, $\operatorname{tr} M$ is proportional to the Ricci scalar of the metric g constructed from Σ^i. Specifically, $\operatorname{tr} M = (\alpha/4) R$, where $\alpha = \pm 1, \pm i$. Then we see that the condition $\partial_\mu \operatorname{tr} M = 0$ is equivalent to the unimodular condition $\partial_\mu R = 0$ in the metric formulation. Hence, a unimodular reformulation of Plebański gravity would be one that yields the chiral Einstein condition (2.81b), without providing the chiral trace equation (2.81d) which prescribes constant value of $\operatorname{tr} M$, and hence R. A natural question to ask is whether there exist actions that produce the first three Plebański field equations - the chiral zero torsion condition, the chiral Einstein condition, and the metricity constraint - without yielding the chiral trace equation. Such actions exist, and they will be the focus of this chapter. As in the metric formulation, there are two main schools of thought corresponding to fixing the value of the metric volume factor with a dynamical constraint, consequently reducing the overall symmetry of the theory, or promoting the cosmological constant to a field and enforcing its constancy with a dynamical constraint. These are the preferred volume, and parametrised/Henneaux-Teitelboim approaches discussed previously in section 2.5.

3.2 Preferred Volume Actions

In order to derive a preferred volume unimodular formulation of Plebański gravity, we must understand how the 'volume' presents itself in terms of the Plebański variables. By volume, we mean the volume form of the metric tensor. In the Plebański theory the metric is secondary and is constructed from the Σ^i 2-forms either via the Urbantke formula (2.97), or via the tetrad decomposition formulae (2.84), (2.85) and (2.95). In all cases the volume form ε of the constructed metric is proportional to the contraction $\Sigma^i \wedge \Sigma_i$. Additionally the metricity constraint (2.81c) gives us $\Sigma^i \wedge \Sigma_i \propto \mu$, and hence the 4-form μ encodes the metric volume form. Then one can both remove the constraint $\operatorname{tr} M = \Lambda$ as a dynamical equation, and fix the metric volume form ε by replacing the 4-form μ with a fixed (background) 4-form μ_0. This turns (2.79) into the *unimodular Plebański action* given by

$$S_{\rm PV}[A, \Sigma, M; \mu_0] = \frac{1}{\ell_P^2 \sqrt{\sigma}} \int \Sigma_i \wedge F^i - \frac{1}{2} M_{ij} \Sigma^i \wedge \Sigma^j + \frac{1}{2} \mu_0 \operatorname{tr} M \ , \tag{3.4}$$

where the constant term $\propto \int \mu_0 \Lambda$ has been dropped, and where ℓ_P denotes the *reduced Planck length* $\ell_P = \sqrt{8\pi G}$. The subscript 'PV' stands for preferred volume. This action yields field equations

$$D_A \Sigma^i = 0 \quad , \quad F^i = M^{ij} \Sigma_j \quad , \quad \Sigma^i \wedge \Sigma^j = \delta^{ij} \mu_0 \tag{3.5}$$

which, as per the discussion in section 3.1, correspond to the dynamical equations of unimodular gravity. We reiterate that Λ no longer appears at the level of the action or the field equations, and instead appears as an integration constant coming from $\partial_\mu \operatorname{tr} M = 0 \Leftrightarrow \operatorname{tr} M = \Lambda$. Therefore the theory contains all possible solutions for all possible values of Λ, including complex values of Λ. Instead of having full diffeomorphism symmetry, the action (3.4) is only invariant under 'volume-preserving' diffeomorphisms which leave μ_0 unchanged.

Theories of fewer fields I

As with the non-unimodular theory, one can obtain further actions in fewer variables by integrating out fields. One eliminates Σ^i by substituting $\Sigma^i = M^{-1}_{ij} F^j$ back into (3.4) to get a *unimodular first order (UFO) action* in terms of only A, M and μ_0 which reads

$$S_{\rm UFO}[A, M; \mu_0] = \frac{1}{\ell_P^2\sqrt{\sigma}} \int M^{-1}_{ij}\, F^i \wedge F^j + \mu_0 \,{\rm tr}\, M \,. \tag{3.6}$$

This is the unimodular analogue of the first order chiral connection theory (2.98), discussed in [45, 55, 69]. This action yields field equations

$$D_A \Sigma^i_F = 0 \quad , \quad \Sigma^i_F \wedge \Sigma^j_F = \delta^{ij}\mu_0 \quad : \quad \Sigma^i_F := (M^{-1})^{ij} F_j \,, \tag{3.7}$$

which are equivalent to (3.5). As with the non-unimodular first order theory (2.98), we now assume that the matrix M^{ij} is invertible at the level of the action. Consequently, the solution space of the UFO theory (3.6) is restricted compared to the solution space of the full unimodular Plebański theory (3.4). For example, flat (Minkowski) spacetime with $M^{ij} = 0$ is a solution of the full unimodular Plebański theory, but not a solution of the UFO theory.

Going further, one can also eliminate the matrix field M^{ij} from the theory through a process which closely mirrors the derivation of the pure connection approach from section 2.4.2. One defines a matrix field X^{ij} such that $F^i \wedge F^j = X^{ij}\mu_0$. Then the metricity constraint becomes

$$(M^{-1})^{ik} F_k \wedge (M^{-1})^{jl} F_l = \delta^{ij}\mu_0 \quad \Leftrightarrow \quad M^{-1} X M^{-1} = \mathbb{I} \,, \tag{3.8}$$

which can be rewritten as $M = \sqrt{X}$, subject to the ambiguities in defining a matrix square root discussed previously. Substituting this expression for M^{ij} into (3.6) leads to the *unimodular pure connection action* given by

$$S_{\rm UPC}[A; \mu_0] = \frac{1}{\ell_P^2\sqrt{\sigma}} \int \mu_0 \,{\rm tr}\, \sqrt{X} \,. \tag{3.9}$$

This action yields a single field equation

$$D_A \Sigma^i_X = 0 \quad : \quad \Sigma^i_X = \left(X^{-\frac{1}{2}}\right)^{ij} F_j \,. \tag{3.10}$$

By construction, Σ^i_X already satisfy $\Sigma^i_X \wedge \Sigma^j_X = \delta^{ij}\mu_0$. Then we recover the same field equations as in (3.5), except where the matrix M is replaced with $\sqrt{X}$. Hence the unimodular condition, which typically reads as $\partial_\mu \,{\rm tr}\, M = 0$, becomes $\partial_\mu \,{\rm tr}\, \sqrt{X} = 0$ in this formulation. One can interpret this unimodular pure connection action as a gauge fixing of the Krasnov's pure connection action (2.105). In particular, our action (3.9) corresponds to the gauge defined by ${\rm tr}\, \sqrt{X_{\rm PC}} = \Lambda$, where $(X_{\rm PC})^{ij}$ denotes the matrix appearing in (2.105) defined w.r.t some fixed nowhere vanishing 4-form ε_X. Such a gauge fixing is always possible since ε_X can be chosen arbitrarily.

3.3 Parametrised Actions

Alternatively, one can follow the parametrised approach proposed by Henneaux and Teitelboim [53], discussed previously in section 2.5. To recap, the cosmological constant is promoted to a dynamical field which is then is forced to be constant by means of a dynamical constraint mediated by a vector density $\tilde{\mathcal{T}}^\mu$ whose divergence $\partial_\mu \tilde{\mathcal{T}}^\mu$ also provides the metric volume $\sqrt{|g|}$. In metric variables, one arrives at the action (2.115), and one sees that the full diffeomorphism symmetry of GR is preserved in this approach. One can replace $\tilde{\mathcal{T}}^\mu$ by its dual 3-form T with components $T_{\mu\nu\rho} = \tilde{\mathcal{T}}^\lambda \underset{\sim}{\epsilon}_{\lambda\mu\nu\rho}$ for use in Plebański type formulations which are written in terms of differential forms. Then one can define an Henneaux–Teitelboim analogue of the Plebański action (2.79) by

$$S[\Sigma, A, M, \mu, \lambda, T] = \frac{1}{\ell_P^2\sqrt{\sigma}} \int \Sigma^i \wedge F_i - \frac{1}{2} M_{ij}\, \Sigma^i \wedge \Sigma^j + \frac{1}{2}\mu\,(\mathrm{tr}\, M - \lambda) + dT\,\lambda\ . \tag{3.11}$$

As with the metric case, the scalar field $\lambda(x)$ is made constant by the dynamical constraint $d\lambda = 0$ coming from variation w.r.t the 3-form T, allowing it to play the role of the cosmological constant. In addition, variation with respect to λ leads to $\mu = 2\,dT$, and hence the volume form of the constructed metric is determined in terms of T. in fact, one sees that μ is redundant and one can alternatively work with the simpler *parametrised unimodular Plebański action* given by

$$S_{\mathrm{HT}}\,[\Sigma, A, M, T] = \frac{1}{\ell_P^2\sqrt{\sigma}} \int \Sigma^i \wedge F_i - \frac{1}{2} M_{ij}\, \Sigma^i \wedge \Sigma^j + dT\,\mathrm{tr}\, M\ , \tag{3.12}$$

which can also be obtained by substituting $\mu_0 = dT$ into (3.4). This action was first discussed, without the potentially imaginary prefactor, in [95]. These actions are evidently diffeomorphism invariant and no longer depend on background fields. The field equations arising from (3.12) are

$$F^i = M^{ij}\Sigma_j\ , \quad D_A\Sigma^i = 0\ , \quad \Sigma^i \wedge \Sigma^j = 2\delta^{ij}\, dT\ , \quad d\,\mathrm{tr}\, M = 0\ . \tag{3.13}$$

In this formulation we get the condition $\partial_\mu\, \mathrm{tr}\, M = 0$ as a field equation, without requiring Bianchi identities or a non-degenerate volume form. For a non-degenerate volume form $\mu \propto dT$, we have seen already that the field equation $\partial_\mu\, \mathrm{tr}\, M = 0$ would be redundant. Notice that the volume form is now required to be globally exact, so by Stokes' theorem the volume of any spacetime region could be evaluated by integrating the 3-form T over its boundary. In particular, the total volume of a compact spacetime without boundary would be zero. This aspect of the theory might be more relevant in Euclidean signature setting where compact manifolds are of greater interest.

Consider a globally simple spacetime $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ for some 3-manifold $\mathcal{S}$, containing a 'cylindrical' submanifold $\mathcal{U}$ diffeomorphic to $[0,1] \times \mathcal{S}$ such that it is bounded by a pair of 3-manifolds $\mathcal{S}_0$ and $\mathcal{S}_1$ at the two ends. The spacetime volume of such a region satisfies

$$\mathcal{V}_4(\mathcal{U}) \propto \int_{\mathcal{U}} dT = \int_{\mathcal{S}_1} T - \int_{\mathcal{S}_0} T \; . \tag{3.14}$$

The integral of T over 3-dimensional hypersurfaces can be seen as defining a global "volume" time proportional to spacetime volume, as in [53]. To make this more explicit, one can introduce a 1-parameter embedding map of the kind $\gamma_t : \mathcal{S} \to \mathcal{M}$. Then one can formulate this volume time as a function of the embedding parameter t by

$$t_{\mathrm{Vol}}(t) \propto \int_{\gamma_t(\mathcal{S})} T - \int_{\gamma_0(\mathcal{S})} T \; , \tag{3.15}$$

The hypersurface $\gamma_0(\mathcal{S})$ at $t = 0$ is where this volume time begins, $t_{\mathrm{Vol}}(0) = 0$. We investigate volume time in greater detail in chapter 4, in the context of the canonical formulations of the unimodular Plebański actions introduced in this chapter.

Theories of fewer fields II

The elimination of fields in the parametrised approach proceeds in the same way as in the preferred volume form approach, except where we replace μ_0 with dT wherever it appears. This yields the *parametrised unimodular first order* (PUFO) and *parametrised unimodular pure connection* (PUPC) actions given respectively by

$$S_{\mathrm{PUFO}}[A, M, T] = \frac{1}{2\ell_P^2 \sqrt{\sigma}} \int M^{-1}_{ij} \, F^i \wedge F^j + dT \;\mathrm{tr}\, M \; , \tag{3.16}$$

$$S_{\mathrm{PUPC}}[A, T] = \frac{1}{\ell_P^2 \sqrt{\sigma}} \int dT \;\mathrm{tr} \sqrt{X_T} \; , \tag{3.17}$$

where the matrix X_T^{ij} is defined via $F^i \wedge F^j = dT \, X_T^{ij}$. This relationship between X_T and T implies a constraint on the variations δX_T and δT which is given by

$$dT \, \delta X_T^{ij} = D_A \delta A^i \wedge F^j + F^i \wedge D_A \delta A^j - d\delta T \, X_T^{ij} \; . \tag{3.18}$$

The field equations arising from these actions are the same as (3.7) and (3.10) – where μ_0 is replaced by dT and X^{ij} is replaced with X_T^{ij} – respectively, plus an additional equation $d\,\mathrm{tr}\, M = 0$ or $d\left(\mathrm{tr}\sqrt{X_T}\right) = 0$, which would anyway arise from the Bianchi identities when dT is non-degenerate.

3.4 Summary

To summarise, in this section we have presented new actions for Plebański-like theories which incorporate the principles of unimodular gravity: a fixed volume form and a cosmological constant that arises as an integration constant as opposed to a fundamental parameter. We began from the observation that the zero torsion condition (2.81a) already implies, through the use of a Bianchi identity for the exterior covariant derivative, that $\operatorname{tr} M$ is constrained to be constant. Then the trace condition $\operatorname{tr} M = \Lambda$ coming from variations w.r.t μ of the Plebański action (2.79) simply fixes the value of this integration constant. By removing $\operatorname{tr} M = \Lambda$ as a dynamical constraint, one arrives at a modified version of Plebański theory whose dynamics are equivalent to those of unimodular gravity. As in metric formulations, there are two main approaches to removing this constraint on $\operatorname{tr} M$. First, in line with the preferred volume approach, one can fix the value of the 4-form μ – which is a surrogate for the metric volume form – at the level of the action, yielding the desired result. The resulting theory is then only invariant under volume-preserving diffeomorphisms which leave this fixed volume form unchanged. Alternatively, one can take the approach of [53, 96] by promoting the cosmological constant to a field which is then constrained to be constant by a Lagrange multiplier-like field. This approach maintains the full diffeomorphism symmetry of GR, but the metric volume form becomes an exact 4-form, and the 4-volume of any spacetime region reduces to an integral over its 3D boundary. One can make use of the exactness of the volume form to define a kind of unimodular 'volume time' as in [53, 95]. Finally, we saw that one can derive unimodular analogues of the first order and pure connection actions in both approaches.

Chapter 4

Canonical Analysis of Unimodular Plebański Gravity

A condensed version of this chapter is found in the article [44] which was finalised and published in collaboration with Steffen Gielen following the initial submission of this thesis. The sections in this chapter can be broken down as follows. Section 4.1 consists primarily of a review of the canonical formulation of Plebański gravity in terms of the (complex) Ashtekar variables. Section 4.2 contains novel results regarding the canonical formulation coming from the unimodular action (3.4). Section 4.3 can be taken as a detailed review of [96] with some added discussion of topics from [53].

4.1 Canonical Plebański Gravity

The perspective of this chapter, in line with the ADM formalism [9] and many other canonical formulations of generally covariant theories, is to understand unimodular Plebański gravity as a theory of the dynamical evolution of the geometry of a 3-manifold. This is in opposition to the Lagrangian viewpoint where one examines the unchanging geometry of a 4-manifold. In this chapter, we will first construct the canonical formulation of Plebański gravity starting from the action (2.79) before proceeding to construct canonical formulations of the two unimodular versions with actions (3.4) and (3.12) introduced in the previous section. The canonical formulation of the Plebański theory has been well studied [3, 4, 28, 61, 86, 90]. The presentation here is somewhat novel; we establish fairly non-standard conventions which will ease the transition to the unimodular formulation later on.

We take the spacetime manifold $\mathcal{M}$ to be diffeomorphic to the product $\mathbb{R} \times \mathcal{S}$ where $\mathcal{S}$ is a connected smooth 3-manifold without boundary. For example, $\mathcal{S}$ may be diffeomorphic to the 3-sphere S^3, or $\mathbb{R}^3$. Furthermore, let $\gamma_t : \mathcal{S} \to \mathcal{M}$ be a (differentiable) 1-parameter family of embedding maps, and let x^a be coordinates on $\mathcal{S}$. We define coordinates $x^\mu = (t, x^a)$ on $\mathcal{M}$ such that the real 4-tuple (t', x'^a) corresponds to the point $\gamma_t(x^{-1}(\boldsymbol{x}')) \in \mathcal{M}$, where $x^{-1} : \mathbb{R}^3 \to \mathcal{S}$ denotes the inverse of the coordinate map x^a on $\mathcal{S}$. The embedding γ_t determines the foliation of the spacetime into constant 'time' hypersurfaces $\mathcal{S}_{t_0} := \gamma_{t_0}(\mathcal{S})$. In the general case, this time t is not physical time, but rather some abstract evolution parameter associated to the foliation of spacetime induced by γ_t.

When $\mathcal{S}$ is closed, for example if $\mathcal{S} = S^3$, we have $\int_{\mathcal{S}_{t_0}} d^3x\, \partial_a \tilde{W}^a = 0$ for each vector density of weight $+1$. However, if $\mathcal{S}$ is not closed, for example if $\mathcal{S} = \mathbb{R}^3$, the integral $\int_{\mathcal{S}_{t_0}} d^3x\, \partial_a \tilde{W}^a$ may be equal to some surface term which comes from taking a limit of surface integrals. In this case, one should require that all of the fields 'fall off' to zero sufficiently quickly such that this surface term is always vanishing. In what follows, the spatial integral of the divergence of any vector density will be vanishing.

With respect to these coordinates, the connection and the Plebański 2-forms decompose as

$$A^i = A^i_0 dt + A^i_a dx^a \quad , \quad \Sigma^i = \Sigma^i_{0a} dt \wedge dx^a + \frac{1}{2}\Sigma^i_{ab} dx^a \wedge dx^b \ , \tag{4.1}$$

and the curvature forms decompose as

$$F^i = F^i_{0a} dt \wedge dx^a + \frac{1}{2} F^i_{ab} dx^a \wedge dx^b \quad : \quad F^i_{0a} = \dot{A}^i_a - D_a A^i_0 \ . \tag{4.2}$$

The over dot $\dot{}$ is shorthand for the derivative in the direction of the t coordinate $\mathcal{L}_{\partial_t}$, and $D_a B^i = \partial_a B^i + \epsilon^i{}_{jk} A^j_a B^k$ is the spatial part of the gauge covariant derivative. We introduce the *densitised triad* $\tilde{E}^a_i$ defined as a weight $+1$ vector density valued in $\mathfrak{so}(3)^*_{\mathbb{C}}$, which encodes the 9 degrees of freedom contained in Σ^i_{ab} such that $\Sigma^i_{ab} = \tilde{E}^{ic} \underset{\sim}{\epsilon}_{abc}$. We will define two further quantities which will see use later on. First, a scalar density of weight $+2$ constructed as

$$\det(\tilde{E}^a_i) := \frac{1}{6} \underset{\sim}{\epsilon}_{abc} \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j \tilde{E}^c_k \ . \tag{4.3}$$

This is the determinant of the 3×3 matrix with components $\tilde{E}^a_i$. Secondly, we define an $\mathfrak{so}(3)_{\mathbb{C}}$ valued covector density of weight -1 via

$$\underset{\sim}{E}^i_a = \frac{\underset{\sim}{\epsilon}_{abc} \epsilon^{ijk} \tilde{E}^b_j \tilde{E}^c_k}{2 \det(\tilde{E}^a_i)} \ . \tag{4.4}$$

This is the matrix inverse of $\tilde{E}^a_i$ satisfying $\tilde{E}^a_i \underset{\sim}{E}^i_b = \delta^a_b$ and $\tilde{E}^a_j \underset{\sim}{E}^i_a = \delta^i_j$. Throughout this thesis, we assume that $\tilde{E}^a_i$ is non-degenerate, $\det(\tilde{E}^a_i) \neq 0$, so that $\underset{\sim}{E}^i_a$ is well defined. The 9 degrees of freedom contained in Σ^i_{0a} can be encoded into a pair of a vector field V^a and an $\mathbf{S}^2\mathfrak{so}(3)_{\mathbb{C}}$ valued weight +1 scalar density $\tilde{\varphi}^{ij}$ via

$$\Sigma^i_{0a}(\boldsymbol{V},\varphi) = -\left(\underset{\sim}{\epsilon}_{abc} V^b \tilde{E}^{ic} + \tilde{\varphi}^{ij} \underset{\sim}{E}_{ja}\right) . \tag{4.5}$$

This parametrisation is sufficiently non-degenerate so that $\Sigma^i_{0a} = 0$ identically only when $V^a = 0$ and $\tilde{\varphi}_{ij} = 0$ identically also. Furthermore, we parametrise $A^i_0(\alpha, \boldsymbol{V}) = \alpha^i + V^a A^i_a$ where α^i is an $\mathfrak{so}(3)_{\mathbb{C}}$ valued scalar. With these parametrisations for A^i_0 and Σ^i_{0a}, one computes

$$\begin{aligned}\Sigma_i \wedge \left(F^i - \frac{1}{2}M^{ij}\Sigma_j\right) = d^4x \Big[\dot{A}^i_a \tilde{E}^a_i + \alpha^i D_a \tilde{E}^a_i - V^a \left(F^i_{ab}\tilde{E}^b_i - A^i_a D_b \tilde{E}^b_i\right) \\ -\tilde{\varphi}_{ij}\left(\frac{F^i_{ab}\,\epsilon^{jkl}\tilde{E}^a_k\tilde{E}^b_l}{2\det(\tilde{E}^a_i)} - M^{ij}\right) - \partial_a\left(A^i_0 \tilde{E}^a_i\right)\Big] .\end{aligned} \tag{4.6}$$

Whenever d^4x appears without an integral sign, it denotes the coordinate 4-form given by $d^4x = dt \wedge dx^1 \wedge dx^2 \wedge dx^3$. It will be useful to decompose $\frac{1}{2}(\det(\tilde{E}^a_i))^{-1} F^i_{ab}\,\epsilon^{jkl}\tilde{E}^a_k\tilde{E}^b_l = \mathcal{R}^{ij} + \epsilon^{ijk}J_k$ where the symmetric part $\mathcal{R}^{ij}$ and the anti-symmetric part J_i are given by

$$\mathcal{R}^{ij} = \frac{F_{ab}{}^{(i}\epsilon^{j)kl}\tilde{E}^a_k\tilde{E}^b_l}{2\det(\tilde{E}^a_i)} \quad , \quad J_i = \frac{\tilde{E}^a_i F^j_{ab}\tilde{E}^b_j}{2\det(\tilde{E}^a_i)} . \tag{4.7}$$

Then one computes

$$\begin{aligned}\frac{1}{\ell_P^2\sqrt{\sigma}}\int \Sigma_i \wedge \left(F^i - \frac{1}{2}M^{ij}\Sigma_j\right) = \frac{1}{\ell_P^2\sqrt{\sigma}}\int d^4x \Big[\dot{A}^i_a \tilde{E}^a_i + \alpha^i D_a \tilde{E}^a_i \\ -V^a\left(F^i_{ab}\tilde{E}^b_i - A^i_a D_b \tilde{E}^b_i\right) - \tilde{\varphi}_{ij}\left(\mathcal{R}^{ij} - M^{ij}\right)\Big] .\end{aligned} \tag{4.8}$$

This term appears in the Plebański action (2.79), and in the unimodular actions (3.4) and (3.12). We parametrise the 4-form $\mu = -2\tilde{N}\,d^4x$ where $\tilde{N}$ is a scalar density of weight +1, and where the factor of -2 is a added for later convenience. Then adding the final constraint term in the Plebański action (2.79) gives us the *extended canonical Plebański action*

$$\begin{aligned}S'_{\text{Can}}\left[A, \tilde{E}, M, \alpha, \boldsymbol{V}, \tilde{\varphi}, \tilde{N}\right] = \frac{1}{\ell_P^2\sqrt{\sigma}}\int d^4x \Big[\dot{A}^i_a \tilde{E}^a_i + \alpha^i D_a \tilde{E}^a_i - V^a\left(F^i_{ab}\tilde{E}^b_i - A^i_a D_b \tilde{E}^b_i\right) \\ -\tilde{\varphi}_{ij}\left(\mathcal{R}^{ij} - M^{ij}\right) - \tilde{N}\left(\operatorname{tr} M - \Lambda\right)\Big] .\end{aligned} \tag{4.9}$$

We call this the 'extended' canonical action since the fields M^{ij} and $\tilde{\varphi}_{ij}$ are superfluous and can be eliminated at the level of the action. To see this, consider the field equations which arise from variations w.r.t M^{ij} and $\tilde{\varphi}_{ij}$ which read

$$\frac{\delta S'_{\text{Can}}}{\delta M^{ij}} = 0 \ \Leftrightarrow\ \tilde{\varphi}_{ij} = \tilde{N}\delta_{ij} \quad , \quad \frac{\delta S'_{\text{Can}}}{\delta \tilde{\varphi}_{ij}} = 0 \ \Leftrightarrow\ M^{ij} = \mathcal{R}^{ij} \ . \tag{4.10}$$

We see that M^{ij} and $\tilde{\varphi}_{ij}$ are fully determined in terms of the other fields in the theory. It is a straightforward exercise to check that the action given by

$$\begin{aligned} S_{\text{Can}}\left[A, \tilde{E}, \alpha, \boldsymbol{V}, \tilde{N}\right] = \frac{1}{\ell_P^2\sqrt{\sigma}} \int d^4x \Big[\dot{A}^i_a \tilde{E}^a_i + \alpha^i D_a \tilde{E}^a_i - V^a \left(F^i_{ab}\tilde{E}^b_i - A^i_a D_b \tilde{E}^b_i \right) \\ - \tilde{N}\left(\frac{\epsilon^{ij}{}_k F^k_{ab} \tilde{E}^a_i \tilde{E}^b_j}{2\det(\tilde{E}^a_i)} - \Lambda \right) \Big] \ , \end{aligned} \tag{4.11}$$

yields the same field equations as the extended canonical action (4.9) for the remaining variables. Note, the first term in the brackets to the right of $\tilde{N}$ in this expression is the explicit form of the trace of $\mathcal{R}^{ij}$, which we may also denote by $\mathcal{R} := \mathcal{R}^i{}_i$. This action over fewer variables is obtained by solving either of the field equations in (4.10) at the level of the action. We recall that this procedure is often referred to as 'integrating out' fields. In this text, we choose to fix $\tilde{\varphi}_{ij} = \tilde{N}\delta_{ij}$ at the level of the action. We interpret this as a restriction on our ansatz for Σ^i_{0a} so that it now reads

$$\Sigma^i_{0a}(\boldsymbol{V}, \tilde{N}) = -\left(\underset{\sim}{\epsilon}_{abc} V^b \tilde{E}^{ic} + \tilde{N} \underset{\sim}{E}{}^i_a \right) \ . \tag{4.12}$$

By doing this, we are restricting to 2-forms Σ^i which already satisfy the metricity constraint (2.81c). We use the action (4.11), which we will call the *canonical Plebański action*, as a starting point for developing a canonical formulation of Plebański gravity. The transition to the Hamiltonian picture is quite clear. We have a canonically conjugate pair A^i_a and $\tilde{E}^a_i$ with principal Poisson bracket

$$\{A^i_a(\boldsymbol{x}), \tilde{E}^b_j(\boldsymbol{x}')\} = \ell_P^2 \sqrt{\sigma}\, \delta^i_j \delta^b_a\, \delta^{(3)}(\boldsymbol{x} - \boldsymbol{x}') \ . \tag{4.13}$$

Furthermore, we have local constraints on the variables which read

$$\tilde{\mathcal{G}}_i = -D_a \tilde{E}^a_i \approx 0 \ , \tag{4.14a}$$

$$\tilde{\mathcal{D}}_a = F^i_{ab}\tilde{E}^b_i - A^i_a D_b \tilde{E}^b_i \approx 0 \ , \tag{4.14b}$$

$$\mathcal{H} = \frac{\epsilon^{ij}{}_k F^k_{ab} \tilde{E}^a_i \tilde{E}^b_j}{2\det(\tilde{E}^a_i)} - \Lambda \approx 0 \ . \tag{4.14c}$$

These constraints can be written in smeared form using the Lagrange multiplier fields $\alpha^i, V^a, \tilde{N}$ via

$$\mathcal{G}(\alpha) = \frac{1}{\ell_P^2\sqrt{\sigma}} \int d^3x\ \alpha^i \tilde{\mathcal{G}}_i \ , \quad \mathcal{D}(\boldsymbol{V}) = \frac{1}{\ell_P^2\sqrt{\sigma}} \int d^3x\ V^a \tilde{\mathcal{D}}_a \ , \quad \mathcal{H}(\tilde{N}) = \frac{1}{\ell_P^2\sqrt{\sigma}} \int d^3x\ \tilde{N}\mathcal{H} \ . \tag{4.15}$$

These constraints are first class; they satisfy the Poisson algebra

$$\begin{aligned}
&\{\mathcal{G}(\alpha),\mathcal{G}(\beta)\} = -\mathcal{G}\left([\alpha,\beta]\right) \ , \quad \{\mathcal{G}(\alpha),\mathcal{D}(\boldsymbol{V})\} = -\mathcal{G}(\mathcal{L}_{\boldsymbol{V}}\alpha) \ , \quad \{\mathcal{G}(\alpha),\mathcal{H}(\tilde{N})\} = 0 \ , \\
&\{\mathcal{D}(\boldsymbol{U}),\mathcal{D}(\boldsymbol{V})\} = \mathcal{D}\left([\boldsymbol{U},\boldsymbol{V}]\right) \ , \quad \{\mathcal{D}(\boldsymbol{V}),\mathcal{H}(\tilde{N})\} = \mathcal{H}(\mathcal{L}_{\boldsymbol{V}}\tilde{N}) \ , \\
&\{\mathcal{H}(\tilde{N}_1),\mathcal{H}(\tilde{N}_2)\} = \mathcal{D}^0\left(\boldsymbol{\mathcal{V}}(\tilde{N}_1,\tilde{N}_2)\right) \ ,
\end{aligned} \tag{4.16}$$

where the Lie derivative of a scalar density is given by $\mathcal{L}_{\boldsymbol{V}}\tilde{N} = \partial_a\left(V^a\tilde{N}\right)$, and where we define

$$\mathcal{D}^0(\boldsymbol{V}) = \frac{1}{\ell_P^2\sqrt{\sigma}}\int d^3x\ V^a F^i_{ab}\tilde{E}^b_i \quad , \quad \mathcal{V}^a(\tilde{N}_1,\tilde{N}_2) = \frac{\tilde{E}^a_i\tilde{E}^{ib}\left(\tilde{N}_2\,\partial_b\tilde{N}_1 - \tilde{N}_1\,\partial_b\tilde{N}_2\right)}{(\det(\tilde{E}^a_i))^2} \ . \tag{4.17}$$

Note that the smeared quantity $\mathcal{D}^0$ is not a new independent constraint, but rather a functional linear combination of the constraints $\mathcal{G}$ and $\mathcal{D}$. Furthermore, $\mathcal{V}^a$ is a vector field parametrised by a pair of weight +1 scalar densities. The canonical variables (4.13) satisfying primary constraints (4.14) are the *complex Ashtekar variables* [11].

The constraint $\mathcal{G}$, which we call the *Gauss constraint*, produces local $SO(3)$ gauge transformations on the variables via $\delta_\alpha\mathcal{O} = \{\mathcal{O},\mathcal{G}(\alpha)\}$ such that

$$\delta_\alpha A^i_a = D_a\alpha^i \quad , \quad \delta_\alpha\tilde{E}^a_i = -\epsilon^k{}_{ij}\tilde{E}^a_k\alpha^j \ . \tag{4.18}$$

The constraint $\mathcal{D}$, which we call the *diffeomorphism constraint*, produces spatial diffeomorphism transformations on the variables via $\delta_{\boldsymbol{V}}\mathcal{O} = \{\mathcal{O},\mathcal{D}(\boldsymbol{V})\}$ such that

$$\delta_{\boldsymbol{V}}A^i_a = V^b F^i_{ba} + D_a(V^b A^i_b) \quad , \quad \delta_{\boldsymbol{V}}\tilde{E}^a_i = \partial_b(V^b\tilde{E}^a_i) - \tilde{E}^b_i\partial_b V^a \ . \tag{4.19}$$

In each of the above, the RHS is the explicit form of the Lie derivative $\mathcal{L}_{\boldsymbol{V}}$ acting on that variable. The constraint $\mathcal{H}$, which we call the *Hamiltonian constraint*, generates transformations corresponding to reparametrisations of the embedding parameter t via $\delta_{\tilde{N}}\mathcal{O} = \{\mathcal{O},\mathcal{H}(\tilde{N})\}$ such that

$$\delta_{\tilde{N}}A^i_a \approx -\tilde{N}\Lambda \underaccent{\tilde}{E}^i_a + \tilde{N}\left(\frac{\epsilon^{ij}{}_k F^k_{ab}\tilde{E}^b_j}{\det(\tilde{E}^a_i)}\right) \quad , \quad \delta_{\tilde{N}}\tilde{E}^a_i = -D_b\left(\tilde{N}\,\frac{\epsilon_i{}^{jk}\tilde{E}^a_j\tilde{E}^b_k}{\det(\tilde{E}^a_i)}\right) \ , \tag{4.20}$$

where we use $\mathcal{H}\approx 0$ to simplify the left-most expression in the above. Dynamical evolution (evolution w.r.t t, which we also call *time evolution* throughout this thesis) is generated by the Hamiltonian

$$H_{\text{Ple}}(\alpha,\boldsymbol{V},\tilde{N}) = \mathcal{G}(\alpha) + \mathcal{D}(\boldsymbol{V}) + \mathcal{H}(\tilde{N}) \tag{4.21}$$

via the usual mechanism, $\dot{\mathcal{O}} \approx \{\mathcal{O},H_{\text{Ple}}\}$ for any functional $\mathcal{O}$ of the variables. This time evolution is pure gauge, which is typical of generally covariant systems. The fields $\alpha^i, V^a, \tilde{N}$ which appear in the action are completely unconstrained, and must be fixed by hand.

4.1.1 Holomorphic Hamiltonian systems, reality conditions, and the metric tensor

So far in this canonical analysis we have overlooked a key detail. This is the complexity of the dynamical fields. Recall from section 2.3 that the Plebański formulation exists for the Euclidean signature, the Lorentzian signature, and for 'complex gravity' where the metric tensor (or tetrad) is allowed to be complex valued. In the Euclidean signature, corresponding to $\sigma = \sqrt{\sigma} = 1$, one takes all of the fields to be real valued. Then there is no ambiguity in how to construct the Poisson bracket or how to derive the constraints from the action. In contrast, in the Lorentzian signature, corresponding to $\sigma = -1$ and $\sqrt{\sigma} = i$, one takes the fields to be complex valued. Then the canonical action (4.11) is manifestly complex valued. In general this can make it difficult, or impossible in some fringe cases, to construct the Hamiltonian representation. Luckily, the Lagrangian density for (4.11) is holomorphic in the fields, and hence we may construct a *holomorphic Hamiltonian system*. In this case, the Poisson bracket is holomorphic in both slots, and the constraints are holomorphic functions of the field variables and their spatial derivatives. This holomorphic Poisson bracket is not defined on the complex conjugates of the variables, and hence is not equipped to deal with non-holomorhic function(al)s of the variables. However, one can derive the time evolution equations for the complex conjugate variables by simply conjugating both sides of the regular time evolution equations via

$$(\dot{A}^i_a)^* = \big(\{A^i_a, H_{\text{Ple}}\}\big)^* \quad , \quad (\dot{\tilde{E}}^a_i)^* = \Big(\{\tilde{E}^a_i, H_{\text{Ple}}\}\Big)^* \ . \tag{4.22}$$

This is of particular use to us in our treatment of the reality conditions (2.82) in the canonical formulation. Here, the reality conditions are treated as extra *non-holomorphic* constraints on the initial data which, in general, are not preserved under time evolution. In order to have a consistent theory, we must impose further constraints on the initial data which come from the higher order time derivatives of the reality conditions. These constraints coming from the reality conditions and their time derivatives are not treated as constraints in the Dirac sense. We don't require that they form a closed Poisson algebra, and we don't allow them generate 'gauge transformations' on the variables. We simply require that they are altogether consistent under the time evolution generated from H_{Ple}. One can write the reality conditions (2.82) in the language of the canonical formulation. Using the restricted ansatz (4.12) for Σ^i_{0a}, we see that the 2-forms Σ^i decompose as

$$\Sigma^i = -\tilde{N} \underset{\sim}{E}{}^i_a \, dt \wedge w^a + \frac{1}{2} \tilde{E}^{ia} \underset{\sim}{\epsilon}_{abc} \, w^b \wedge w^c \ , \tag{4.23}$$

where we introduce a triad of 1-forms defined $w^a = V^a dt + dx^a$ for future convenience. From this, one reformulates the trace reality condition (2.82b) as

$$\mathrm{Re}\left(\Sigma^i \wedge \Sigma_i\right) = 0 \quad \Leftrightarrow \quad \mathrm{Re}\,\tilde{N} = 0\ . \tag{4.24}$$

Additionally, one reformulates the wedge reality condition (2.82a) as

$$\tilde{N}\left(\tilde{E}^{ia}(\underset{\sim}{E}{}^j_a)^* - (\tilde{E}^{ia})^* \underset{\sim}{E}{}^j_a\right) - \underset{\sim}{\epsilon}_{abc}\tilde{E}^{ib}(\tilde{E}^{jc})^*\left(V^a - (V^a)^*\right) = 0\ , \tag{4.25}$$

where we have used the trace condition $\mathrm{Re}\,\tilde{N} = 0$ to simplify the first term. We can derive further insights from this condition by examining the reconstructed metric. Recall that it is always possible to construct a metric tensor from Σ^i satisfying the vacuum Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$ via the (generalised) Urbantke formula (2.97). It is possible to write down a closed form expression for the Urbantke metric in terms of the fields present in the canonical formulation. In pursuit of this, one can define a solder form e^I via

$$e^0 = \frac{i\tilde{N}}{\sqrt{\sigma}\,(\det(\tilde{E}^a_i))^{1/2}}\,dt \quad , \quad e^i = i(\det(\tilde{E}^a_i))^{1/2}\underset{\sim}{E}{}^i_a\left(V^a dt + dx^a\right)\ . \tag{4.26}$$

such that Σ^i decomposes as its self-dual part (2.68). Then, by theorem 2.3.4, the Urbantke metric expands in terms of the solder form as $g = \sigma\, e^0 \otimes e^0 + e^i \otimes e_i$, and therefore as

$$\begin{aligned} g &= -\frac{\tilde{N}^2}{\det(\tilde{E}^a_i)}\,dt\otimes dt - \det(\tilde{E}^a_i)\left(V^a \underset{\sim}{E}{}^i_a \underset{\sim}{E}{}_{ib}\left(dt\otimes dx^b + dx^b \otimes dt\right) + \underset{\sim}{E}{}^i_a \underset{\sim}{E}{}_{ib}\,dx^a\otimes dx^b\right) \\ &= -\frac{\tilde{N}^2}{\det(\tilde{E}^a_i)}\,dt\otimes dt - \left(\det(\tilde{E}^a_i)\right)\underset{\sim}{E}{}^i_a \underset{\sim}{E}{}_{ib}\,w^a\otimes w^b\ . \end{aligned} \tag{4.27}$$

One can compare this form of the metric to the decomposition of the metric in the ADM formalism [9]. We borrow our conventions from the review [110]. In the ADM formalism, the metric has the form

$$g = \sigma\, w^0 \otimes w^0 + h_{ab}\, w^a \otimes w^b\ , \tag{4.28}$$

where we define $w^0 = \mathcal{N} dt$ where $\mathcal{N}$ is the *lapse function*, and where h_{ab} is the 3-metric induced on the spatial surfaces. By inspection we see

$$\mathcal{N} = \frac{i\tilde{N}}{\sqrt{\sigma}\,(\det(\tilde{E}^a_i))^{1/2}} \quad , \quad h_{ab} = -\det(\tilde{E}^a_i)\,\underset{\sim}{E}{}^i_a \underset{\sim}{E}{}_{ib}\ . \tag{4.29}$$

Then it is clear that $\tilde{N}$ is proportional to the metric volume factor $\sqrt{|g|}$ and encodes the lapse function, and V^a is the shift vector which should be taken to be real valued. Then the second term in the expression for the wedge reality condition (4.25) vanishes, leaving only $\tilde{N}\left(\tilde{E}^{ia}(\underset{\sim}{E}^j_a)^* - (\tilde{E}^{ia})^*\underset{\sim}{E}^j_a\right) = 0$ which rearranges to

$$\operatorname{Im}\left(\underset{\sim}{E}^i_a \underset{\sim}{E}_{ib}\right) = 0 \; . \tag{4.30}$$

Further manipulations yield $\operatorname{Im}\left[(\det(\tilde{E}^a_i))^2\right] = 0$, and hence $\det(\tilde{E}^a_i)$ must be purely real or imaginary valued. Then returning to the metric (4.27), the reality conditions $\operatorname{Re}\tilde{N} = 0$ and $\operatorname{Im}(\underset{\sim}{E}^i_a\underset{\sim}{E}_{ib}) = 0$ guarantee that g is either real Lorentzian, or real Lorentzian with an overall factor of $\pm i$ as expected. Note that the overall signature of the metric, $(-+++)$ vs $(+---)$, is controlled by the sign of $\det(\tilde{E}^a_i)$ which is not constrained dynamically or via the reality conditions. With this in mind, consider that one constructs the same tetrad (4.26) and the same metric (4.27) in the Euclidean signature formulation, except where all of the fields are taken to be real valued. In this case, we see that the tetrad is either purely real or purely imaginary valued depending on the sign of $\det(\tilde{E}^a_i)$. Our conventions are somewhat backwards here; the metric (4.27) is positive definite when $\det(\tilde{E}^a_i) < 0$ (real tetrad), and negative definite when $\det(\tilde{E}^a_i) > 0$ (imaginary tetrad). The notion of an imaginary tetrad may induce feelings of discomfort in some individuals, the author of this thesis for example. But note, the only way for a metric constructed as $\sum_{I=0}^3 e^I \otimes e^I$ to have negative definite signature is if each e^I is imaginary. Our inability to constrain the sign of $\det(\tilde{E}^a_i)$, and hence the signature of the metric, is a major aspect of this formulation that differentiates it from most other well studied reformulations of GR. We will examine this nuance in much greater detail in chapter 5 in the context of spatially homogeneous models such as Bianchi I and IX.

Recovering the Ricci scalar

Recall from the discussion in section 2.3, theorem 2.3.2 in particular, that the trace of the matrix field M^{ij} is proportional to the Ricci scalar for the metric constructed as in (4.27) via $\operatorname{tr} M = \frac{1}{4}R$. In the canonical theory, the field M^{ij} is replaced by the quantity $\mathcal{R}^{ij}$, and consequently $\operatorname{tr} M$ is replaced with the quantity

$$\mathcal{R} = \frac{\epsilon^{ij}{}_k F^k_{ab} \tilde{E}^a_i \tilde{E}^b_j}{2\det(\tilde{E}^a_i)} \tag{4.31}$$

which appears in the Hamiltonian constraint. Hence $\mathcal{R}$ is directly proportional to the Ricci scalar. Furthermore, one can show that the trace free part $\mathcal{R}^{ij}_{\mathrm{TF}} = \mathcal{R}^{ij} - \frac{1}{3}\delta^{ij}\mathcal{R}$ recovers the self-dual part of the Weyl tensor. This relationship between $\mathcal{R}$ and R will be significant in our investigation of the canonical formulation of unimodular plebański gravity later on.

4.1.2 Gauge transformations and Lagrange multipliers

In the canonical formulation, local gauge transformations on the dynamical fields are generated symplectically from the first class constraints. However, the Poisson structure doesn't prescribe the gauge transformations for the Lagrange multiplier fields $\alpha^i, V^a, \tilde{N}$ that appear in the Hamiltonian. It is possible to derive the gauge transformations for these Lagrange fields starting from the observation that the canonical action (4.11) should be invariant under arbitrary gauge transformations. There is some discussion on gauge transformations of the action in [47, 52]. The action transforms under an arbitrary gauge transformation via

$$\begin{aligned}\delta_\varepsilon S_{\text{Can}} &= \left[\int d^3x\ \tilde{E}^a_i \frac{\delta\Phi(\varepsilon)}{\delta\tilde{E}^a_i} - \Phi(\varepsilon)\right]_{t_0}^{t_1} \\ &\quad + \int_{t_0}^{t_1} dt \left[\Phi(\dot{\varepsilon}) + \{\Phi(\varepsilon), H_{\text{Ple}}\} - \mathcal{G}(\delta_\varepsilon\alpha) - \mathcal{D}(\delta_\varepsilon \boldsymbol{V}) - \mathcal{H}(\delta_\varepsilon\tilde{N})\right] ,\end{aligned} \tag{4.32}$$

where $\Phi(\varepsilon)$ denotes an arbitrary first class constraint with non-dynamical gauge parameter $\varepsilon(t,\boldsymbol{x})$. One can see appendix 4.A.1 for a derivation. Note that the boundary conditions $\varepsilon(t_0,\boldsymbol{x}) = \varepsilon(t_1,\boldsymbol{x}) = 0$ cause the first line in the above expansion vanish, leaving only the second line. Setting the second line to zero gives us equations that can be solved for the gauge transformations of the Lagrange multipliers, $\delta_\varepsilon\alpha^i, \delta_\varepsilon V^a, \delta_\varepsilon\tilde{N}$. As an illustrative example, consider the transformation of the canonical action generated from the constraint $\mathcal{D}$ with gauge parameter U^a which reads

$$\delta_{\boldsymbol{U}} S_{\text{Can}} = \int_{t_0}^{t_1} dt \left[\mathcal{G}\left(\mathcal{L}_{\boldsymbol{U}}\alpha - \delta_{\boldsymbol{U}}\alpha\right) + \mathcal{D}\left(\dot{\boldsymbol{U}} + [\boldsymbol{U},\boldsymbol{V}] - \delta_{\boldsymbol{U}}\boldsymbol{V}\right) + \mathcal{H}\left(\mathcal{L}_{\boldsymbol{U}}\tilde{N} - \delta_{\boldsymbol{U}}\tilde{N}\right)\right] . \tag{4.33}$$

We treat the constraints $\mathcal{G}, \mathcal{D}, \mathcal{H}$ as linearly independent formal symbols so that any arbitrary combination of the form $\mathcal{G}(\beta) + \mathcal{D}(\boldsymbol{U}) + \mathcal{H}(\tilde{Q})$ vanishes identically if and only if all of the smearing functions $\beta^i, U^a, \tilde{Q}$ vanish identically. Then to find the transformations $\delta_{\boldsymbol{U}}\alpha^i$ and so on, one sets the arguments of the constraints in (4.33) to zero yielding

$$\delta_{\boldsymbol{U}}\alpha^i = \mathcal{L}_{\boldsymbol{U}}\alpha^i , \quad \delta_{\boldsymbol{U}} V^a = \dot{U}^a + [\boldsymbol{U},\boldsymbol{V}]^a , \quad \delta_{\boldsymbol{U}}\tilde{N} = \mathcal{L}_{\boldsymbol{U}}\tilde{N} . \tag{4.34}$$

One can interpret the RHS of the second equation $\delta_{\boldsymbol{U}} V^a = \dot{U}^a + [\boldsymbol{U},\boldsymbol{V}]^a$ as the spatial part of a commutator of a pair of spacetime vector fields U^μ and V^μ such that $U^0 = 0$ and $V^0 = -1$. Note that the vector field V^μ is null w.r.t the 1-forms $w^a = V^a dt + dx^a$ introduced previously, $w^a_\mu V^\mu = 0$. Following the procedure outlined above for the remaining constraints

$\mathcal{G}$ and $\mathcal{H}$ with gauge parameters β^i and $\tilde{Q}$ respectively gives

$$\delta_\beta \alpha^i = \dot{\beta}^i + [\beta, \alpha]^i - \mathcal{L}_{\boldsymbol{V}} \beta \ , \quad \delta_\beta V^a = 0 \ , \quad \delta_\beta \tilde{N} = 0 \tag{4.35}$$

$$\delta_{\tilde{Q}} \alpha^i = -A^i_a \mathcal{V}^a(\tilde{Q}, \tilde{N}) \ , \quad \delta_{\tilde{Q}} V^a = \mathcal{V}^a(\tilde{Q}, \tilde{N}) \ , \quad \delta_{\tilde{Q}} \tilde{N} = \dot{\tilde{N}} - \mathcal{L}_{\boldsymbol{V}} \tilde{Q} \ , \tag{4.36}$$

where $\mathcal{V}^a$ is defined in (4.17). Again, we stress that the above transformations are only accurate when the gauge parameters $\beta^i, U^a, \tilde{Q}$ all vanish everywhere on the initial and final hypersurfaces corresponding to times t_0 and t_1.

4.1.3 Canonical first order and pure connection theories

Recall in the Lagrangian formulation of Plebański gravity we were able to integrate out fields to get the first order (FO) and pure connection (PC) actions (2.98) and (2.105) respectively. We saw that each action provided the same field equations, for the remaining variables, as the full Plebański action (2.79). Then it would be a fair guess that the canonical formulations of the FO and PC theories would be indistinguishable from the canonical formulation of the full theory. This turns out to be true, and can be confirmed as follows. First, inserting $A^i = A^i_0 \, dt + A^i_a dx^a$ and $\mu = -2\tilde{N} \, d^4x$ into the first order action (2.98) yields

$$\begin{aligned} S'_{\mathrm{FO}} \left[A, M, \alpha, \tilde{N}\right] = \frac{1}{\ell_P^2 \sqrt{\sigma}} \int d^4x \, \Bigg[& \frac{1}{2} \dot{A}^i_a M^{-1}_{ij} F^j_{bc} \, \tilde{\epsilon}^{abc} \\ & + A^i_0 \, D_a \left(\frac{1}{2} M^{-1}_{ij} F^j_{bc} \, \tilde{\epsilon}^{abc} \right) - \tilde{N} \left(\operatorname{tr} M - \Lambda \right) \Bigg] \ . \end{aligned} \tag{4.37}$$

It is clear from this expression that A^i_a should have conjugate momentum $\tilde{E}^a_i$ satisfying the primary constraint

$$\tilde{E}^a_i - \frac{1}{2} M^{-1}_{ij} F^j_{bc} \, \tilde{\epsilon}^{abc} = 0 \quad \Leftrightarrow \quad M^{ij} = \frac{F^i_{ab} \, \epsilon^{jkl} \tilde{E}^a_k \tilde{E}^b_l}{2 \det(\tilde{E}^a_i)} \ . \tag{4.38}$$

The equation on the RHS of the equivalence arrow '⇔' decomposes into symmetric and antisymmetric parts $M^{ij} = \mathcal{R}^{ij}$ and $J_i = 0$, where J_i is defined in (4.7). The antisymmetric part is equivalently formulated as $F^i_{ab} \tilde{E}^b_i = 0$. Then one can construct an extended action given by

$$\begin{aligned} S_{\mathrm{FOE}} \left[A, \tilde{E}, \ldots\right] = \frac{1}{\ell_P^2 \sqrt{\sigma}} \int d^4x \, \Big[& \dot{A}^i_a \tilde{E}^a_i + A^i_0 D_a \tilde{E}^a_i - V^a F^i_{ab} \tilde{E}^b_i \\ & - \tilde{\varphi}_{ij} \left(\mathcal{R}^{ij} - M^{ij} \right) - \tilde{N} \left(\operatorname{tr} M - \Lambda \right) \Big] \ , \end{aligned} \tag{4.39}$$

which produces the same dynamics as (4.38). By making the substitution $A^i_0 = \alpha^i + V^a A^i_a$, we see that this action is the same as the extended canonical Plebański action (4.9), and consequently the first order theory shares the same canonical formulation as the full Plebański theory. In short, the transition to the canonical theory requires us to re-insert the Σ^i fields which were integrated out to derive the first order theory. The same is true in the pure connection theory. The construction of the canonical formulation in this case is detailed in [33]. We again see the appearance of the conjugate momentum $\tilde{E}^a_i$ along with the Gauss, diffeomorphism and Hamiltonian constraints.

This concludes our review of the canonical formulation of Plebański gravity.

4.2 Canonical Preferred Volume Unimodular Plebański Gravity

We now present a novel canonical analysis of the action (3.4) originally introduced in [46]. Conveniently, the unimodular action (3.4) is very similar to the non-unimodular one (2.79). In particular, both actions share the term $\propto \int \Sigma_i \wedge \left(F^i - \frac{1}{2}M^{ij}\Sigma_j\right)$ which we have already formulated in terms of the canonical variables in (4.8). To complete the unimodular action, one adds the missing term $\propto \int \mu_0 \operatorname{tr} M$. In pursuit of this, we parametrise $\mu_0 = -2\tilde{N}_0\, d^4x$ where $\tilde{N}_0$ is a fixed weight +1 scalar density. As with the non-unimodular version, we fix $\tilde{\varphi}_{ij} = \tilde{N}_0\delta_{ij}$ in the expression for $\Sigma^i(\boldsymbol{V}, \tilde{\varphi})$ given in (4.5) in order to eliminate the redundant pair of fields M^{ij} and $\tilde{\varphi}_{ij}$. Then the fields are given in terms of the 3+1 split by

$$A^i = \alpha^i dt + A^i_a\, w^a \ , \tag{4.40a}$$

$$\Sigma^i = -\tilde{N}_0 \underset{\sim}{E}{}^i_a\, dt \wedge w^a + \frac{1}{2}\tilde{E}^{ia} \underset{\sim}{\epsilon}_{abc}\, w^b \wedge w^c \ , \tag{4.40b}$$

$$\mu_0 = -2\tilde{N}_0\, d^4x \ , \tag{4.40c}$$

where $w^a = V^a dt + dx^a$ is the triad of 1-forms introduced previously. Inserting the above expressions for the fields into (3.4) yields the *canonical unimodular Plebański action* which reads

$$S_{\mathrm{UCan}}\left[A, \tilde{E}, \alpha, \boldsymbol{V}; \tilde{N}_0\right] = \frac{1}{\ell_P^2\sqrt{\sigma}} \int d^4x \left[\dot{A}^i_a \tilde{E}^a_i + \alpha^i D_a \tilde{E}^a_i \right. \\ \left. - V^a\left(F^i_{ab}\tilde{E}^b_i - A^i_a D_b \tilde{E}^b_i\right) - \tilde{N}_0\, \frac{\epsilon^{ij}{}_k F^k_{ab} \tilde{E}^a_i \tilde{E}^b_j}{2\det(\tilde{E}^a_i)}\right] \ . \tag{4.41}$$

This action will be our starting point for developing the canonical formulation. Once again, the transition to the Hamiltonian viewpoint is clear. We have a canonically conjugate pair A^i_a and $\tilde{E}^a_i$ whose principal Poisson bracket is the same as in the non-unimodular formulation (4.13). In this case, we yield only the Gauss and Diffeomorphism constraints $\mathcal{G}$ and $\mathcal{D}$ from variations w.r.t α^i and V^a respectively. The Hamiltonian constraint doesn't appear since its Lagrange multiplier $\tilde{N}$ is now replaced with a field whose value is fixed. The action admits a naive Hamiltonian which reads

$$H^{(0)}(\alpha, \boldsymbol{V}) = \frac{1}{\ell_P^2 \sqrt{\sigma}} \int d^4x \; \tilde{N}_0 \, \frac{\epsilon^{ij}{}_k F^k_{ab} \tilde{E}^a_i \tilde{E}^b_j}{2 \det(\tilde{E}^a_i)} + \mathcal{G}(\alpha) + \mathcal{D}(\boldsymbol{V}) \; . \tag{4.42}$$

Note, this Hamiltonian has a non-constraint part, and hence the time evolution it generates is not pure gauge. The non-constraint part of this Hamiltonian, the integral term in the above expansion, has a non-weakly vanishing Poisson bracket with the diffeomorphism constraint. Then the consistency condition for the diffeomorphism constraint $\{\mathcal{D}(\boldsymbol{U}), H^{(0)}\} \approx 0$ yields a secondary constraint on the variables given by

$$\mathcal{K}_a = \partial_a \left(\frac{\epsilon^{ij}{}_k F^k_{bc} \tilde{E}^b_i \tilde{E}^c_j}{2 \det(\tilde{E}^a_i)} \right) \; , \tag{4.43}$$

with smeared form

$$\mathcal{K}(\tilde{\boldsymbol{T}}) = \frac{1}{\ell_P^2 \sqrt{\sigma}} \int d^4x \; \tilde{T}^a \mathcal{K}_a \; , \tag{4.44}$$

where $\tilde{T}^a$ is a weight $+1$ vector density. All together, the constraints $\mathcal{G}, \mathcal{D}, \mathcal{K}$ are first class. The pairwise Poisson brackets involving $\mathcal{K}$ are given by

$$\begin{aligned} &\{\mathcal{G}(\alpha), \mathcal{K}(\tilde{\boldsymbol{T}})\} = 0 \quad , \quad \{\mathcal{D}(\boldsymbol{V}), \mathcal{K}(\tilde{\boldsymbol{T}})\} = \mathcal{K}(\mathcal{L}_{\boldsymbol{V}} \tilde{\boldsymbol{T}}) \; , \\ &\{\mathcal{K}(\tilde{\boldsymbol{T}}), \mathcal{K}(\tilde{\boldsymbol{L}})\} = \mathcal{D}^0 \left(\boldsymbol{\mathcal{V}} \left(\partial_a \tilde{T}^a, \partial_b \tilde{L}^b \right) \right) \; . \end{aligned} \tag{4.45}$$

where $\mathcal{V}^a$ is defined in (4.17). The most general consistent time evolution is generated by the extended Hamiltonian which is given by

$$H_{\mathrm{PV}}(\alpha, \boldsymbol{V}, \tilde{\boldsymbol{T}}) = \frac{1}{\ell_P^2 \sqrt{\sigma}} \int d^4x \; \tilde{N}_0 \, \frac{\epsilon^{ij}{}_k F^k_{ab} \tilde{E}^a_i \tilde{E}^b_j}{2 \det(\tilde{E}^a_i)} + \mathcal{G}(\alpha) + \mathcal{D}(\boldsymbol{V}) + \mathcal{K}(\tilde{\boldsymbol{T}}) \; , \tag{4.46}$$

via the usual mechanism, $\dot{\mathcal{O}} \approx \{\mathcal{O}, H_{\mathrm{PV}}\}$ for any functional $\mathcal{O}$ of the field variables. The subscript 'PV' is short for preferred volume. We already understand the nature of the gauge transformations generated by the Gauss and diffeomorphism constraints. We would like to understand the nature of the gauge transformations generated by the new constraint $\mathcal{K}$. Integrating by parts, one sees

$$\{\mathcal{O}, \mathcal{K}(\tilde{\boldsymbol{T}})\} = -\{\mathcal{O}, \mathcal{H}(\partial_a \tilde{T}^a)\} \; , \tag{4.47}$$

For any arbitrary functional $\mathcal{O}$. From this we conclude that the gauge transformations generated from $\mathcal{K}$ are the same as the ones generated from $\mathcal{H}$ in the non-unimodular formulation, except where the gauge parameter $\tilde{N}$ is restricted to be a divergence $-\partial_a \tilde{T}^a$. That is, the gauge transformations generated from $\mathcal{K}$ are a restricted set of time reparametrisation transformations. Then the reduction of the symmetry group from the non-unimodular formulation to the unimodular one is made apparent here. The gauge transformations of the Lagrange multipliers $\alpha^i, V^a, \tilde{T}^a$ can be computed following the procedure outlined in section 4.1.2, yielding:

	α^i	V^a	$\tilde{T}^a$
$\mathcal{G}(\beta) : \delta_\beta$	$\dot{\beta}^i + [\beta, \alpha]^i - \mathcal{L}_{\boldsymbol{V}} \beta^i$	0	0
$\mathcal{D}(\boldsymbol{U}) : \delta_{\boldsymbol{U}}$	$\mathcal{L}_{\boldsymbol{U}} \alpha^i$	$\dot{U}^a + [\boldsymbol{U}, \boldsymbol{V}]^a$	$\mathcal{L}_{\boldsymbol{U}} \tilde{T}^a - \tilde{N}_0 U^a$
$\mathcal{K}(\tilde{\boldsymbol{L}}) : \delta_{\tilde{\boldsymbol{L}}}$	$A^i_a \mathcal{V}^a(\partial_b \tilde{L}^b, \tilde{N}_0)$	$-\mathcal{V}^a(\partial_b \tilde{L}^b, \tilde{N}_0)$	$\dot{\tilde{L}}^a - \mathcal{L}_{\boldsymbol{V}} \tilde{L}^a$

One can construct a metric tensor via the generalised Urbantke formula (2.97) which reads

$$g = -\frac{(\tilde{N}_0)^2}{\det(\tilde{E}^a_i)} \, dt \otimes dt - (\det(\tilde{E}^a_i)) \, \underset{\sim}{E}^i_a \underset{\sim}{E}_{ib} \, w^a \otimes w^b \, . \tag{4.48}$$

We derive this expression by first decomposing the expression for Σ^i given in (4.40b) as the self-dual part of the tetrad which is given by

$$e^0 = \frac{i\tilde{N}_0}{\sqrt{\sigma} \, (\det(\tilde{E}^a_i))^{1/2}} \, dt \quad , \quad e^i = i(\det(\tilde{E}^a_i))^{1/2} \underset{\sim}{E}^i_a \, w^a \, . \tag{4.49}$$

Then, by theorem 2.3.4, the Urbantke metric expands as $\sigma \, e^0 \otimes e^0 + e^i \otimes e_i$. We see that this metric is almost identical to the metric tensor constructed in the non-unimodular theory (4.27), except where the lapse surrogate $\tilde{N}$ has been replaced with the fixed background scalar density $\tilde{N}_0$. One computes the metric volume factor to be $\sqrt{|g|} = \tilde{N}_0$, and hence we see that the metric volume is fixed as expected. In the Lorentzian signature, the wedge reality condition (2.82a) takes the same form as in the non-unimodular formulation: $\mathrm{Im}\,(\underset{\sim}{E}^i_a \underset{\sim}{E}_{ib}) = 0$, and consequently $\mathrm{Im}\left[(\det(\tilde{E}^a_i))^2\right] = 0$ (section 4.1.1). However, the trace condition (2.82b) now reduces to $\mathrm{Re}\,\tilde{N}_0 = 0$. To see this, one simply inserts the expression for Σ^i in (4.40b) into the trace condition (2.82b). From this we see that Lorentzian solutions only exist when the background scalar density $\tilde{N}_0$ is purely imaginary. In contrast, Euclidean signature solutions only exist when $\tilde{N}_0$ is real valued. In this case, one takes all of the dynamical fields and Lagrange multipliers to be real also.

Dynamical unimodular condition

We would like to recover the covariant unimodular condition $\partial_\mu R = 0$ from our canonical formulation. As in the non-unimodular formulation, theorem 2.3.2 tells us that the quantity $\mathcal{R} = \frac{1}{2}(\det(\tilde{E}^a_i))^{-1}\epsilon^{ij}{}_k F^k_{ab}\tilde{E}^a_i\tilde{E}^b_j$ is directly proportional to the Ricci scalar R of this reconstructed metric. Hence, the constraints $\mathcal{K}_a$ gives us the spatial parts of the unimodular condition, $\partial_a\mathcal{R} \propto \partial_a R = 0$. To recover the temporal part, we can take the time derivative of $\mathcal{R}$ using the Hamiltonian H_{PV}. This yields

$$\dot{\mathcal{R}}(\boldsymbol{x}) = \{\mathcal{R}(\boldsymbol{x}), H_{\mathrm{PV}}\} \overset{!}{\approx} 0 \; . \tag{4.50}$$

Then we recover the full condition $\partial_\mu\mathcal{R} \propto \partial_\mu R = 0$ as expected.

4.2.1 Summary

In summary, the canonical analysis of the unimodular formulation of Plebański gravity arising from the action (3.4) yields a constraint algebra which is distinct from the constraint algebra of GR written in terms of the Ashtekar variables. In particular, the typical Hamiltonian constraint is replaced with the constraints $\mathcal{K}_a$ in (4.43) which are computed as the three spatial derivatives of the usual Hamiltonian constraint. This in turn restricts the gauge transformations on the variables, mirroring the restriction of the symmetry group to volume preserving diffeomorphisms seen at the level of the action (3.4). Furthermore, the Hamiltonian which generates dynamical evolution on the variables has a definite (non-constraint) part in addition to the constraint part; the Hamiltonian doesn't necessarily vanish as it does in the non-unimodular theory. This has potentially interesting implications for the quantum theory, which will be discussed in the conclusions.

4.3 Canonical Parametrised Unimodular Plebański Gravity

As mentioned at the beginning of this chapter, the analysis presented in this section was originally carried out in [96]. The following constructions were derived independently by the author of this thesis in collaboration with Steffen Gielen, and have been adapted to more closely follow the presentation in [96].

In this section, we aim to derive a canonical formulation of parametrised unimodular Plebański gravity starting from the action (3.12). As in the previous section, we begin with the expression for the term $\propto \int \Sigma_i \wedge \left(F^i - \frac{1}{2}M^{ij}\Sigma_j\right)$ given in (4.8). Then to complete the action (3.12), one needs to formulate the missing term $\propto \int dT \,\mathrm{tr}\, M$ in the terms of the 3+1 split. One can parametrise the 3-form T by

$$\begin{aligned} T &= \tilde{\tau}\, w^1 \wedge w^2 \wedge w^3 - \frac{1}{2}\tilde{T}^a \underset{\sim}{\epsilon}_{abc}\, dt \wedge w^b \wedge w^c \\ &= \tilde{\tau}\, d^3x - \frac{1}{2}\left(\tilde{T}^a - \tilde{\tau} V^a\right) \underset{\sim}{\epsilon}_{abc}\, dt \wedge dx^b \wedge dx^c , \end{aligned} \tag{4.51}$$

where $\tilde{\tau}$ is a weight $+1$ scalar density, and $\tilde{T}^a$ is a weight $+1$ vector density as in the previous formulation. The exterior derivative of the 3-form T is computed to be

$$dT = d^4x \left[\dot{\tilde{\tau}} + \partial_a \left(\tilde{T}^a - \tilde{\tau} V^a\right)\right] . \tag{4.52}$$

Inserting this expression for dT into the action (3.12) yields

$$\begin{aligned} S'_{\mathrm{PUCan}}\left[A, \tilde{E}, M, \tilde{\tau}, \alpha, \boldsymbol{V}, \tilde{\varphi}, \tilde{\boldsymbol{T}}\right] = \frac{1}{\ell_P^2 \sqrt{\sigma}} \int d^4x \Big[& \dot{A}^i_a \tilde{E}^a_i + \dot{\tilde{\tau}}\,\mathrm{tr}\, M + \alpha^i D_a \tilde{E}^a_i \\ & -V^a \left(F^i_{ab}\tilde{E}^b_i - A^i_a D_b \tilde{E}^b_i - \tilde{\tau}\partial_a \,\mathrm{tr}\, M\right) \\ & -\tilde{\varphi}_{ij}\left(\mathcal{R}^{ij} - M^{ij}\right) - \tilde{T}^a \partial_a \,\mathrm{tr}\, M \Big] . \end{aligned} \tag{4.53}$$

On inspection, we now see an extra term $\propto \int d^4x\, \dot{\tilde{\tau}}\,\mathrm{tr}\, M$ which contributes to the symplectic part of the action. Then the field variables $\tilde{\tau}$ and $\mathrm{tr}\, M$ are canonically conjugate. We decompose $M^{ij} = \psi^{ij} + \frac{1}{3}\lambda\,\delta^{ij}$ where ψ^{ij} is the trace-free part, and were λ is a field such that $\mathrm{tr}\, M = \lambda$. In addition, we decompose the field $\tilde{\varphi}_{ij}$, which appears in the expression for $\Sigma^i_{0a}(\boldsymbol{V}, \tilde{\varphi})$ given in (4.5), such that $\tilde{\varphi}_{ij} = \tilde{\chi}_{ij} + \tilde{N}\delta_{ij}$ where $\tilde{\chi}_{ij}$ is a symmetric and trace-free $\mathbf{S}^2\mathfrak{so}(3)_{\mathbb{C}}$ valued scalar density of weight $+1$. With this, the naive action now becomes

$$\begin{aligned} S'_{\mathrm{PUCan}}\left[A, \tilde{E}, \ldots\right] = \frac{1}{\ell_P^2 \sqrt{\sigma}} \int d^4x \Big[& \dot{A}^i_a \tilde{E}^a_i + \dot{\tilde{\tau}}\lambda + \alpha^i D_a \tilde{E}^a_i \\ & - V^a \left(F^i_{ab}\tilde{E}^b_i - A^i_a D_b \tilde{E}^b_i - \tilde{\tau}\partial_a \Lambda\right) \\ & -\tilde{N}\left(\mathcal{R} - \lambda\right) - \tilde{\chi}_{ij}\left(\mathcal{R}^{ij}_{\mathrm{TF}} - \psi^{ij}\right) - \tilde{T}^a \partial_a \lambda \Big] . \end{aligned} \tag{4.54}$$

In the non-unimodular and preferred volume unimodular cases, we were able to integrate out the fields M^{ij} and $\tilde{\varphi}_{ij}$ in their entirety. In this case, we may only integrate out the trace free parts ψ^{ij} and $\tilde{\chi}_{ij}$ since the trace part of M^{ij} is now a dynamical field. Integrating out these fields proceeds as in the non-unimodular case, yielding the *canonical parametrised unimodular Plebański action* given by

$$
\begin{aligned}
S_{\text{PUCan}}\left[A,\tilde{E},\tilde{\tau},\lambda,\alpha,\boldsymbol{V},\tilde{N},\tilde{\boldsymbol{T}}\right] = \frac{1}{\ell_P^2\sqrt{\sigma}}\int d^4x\,\Big[&\dot{A}^i_a\tilde{E}^a_i+\dot{\tilde{\tau}}\lambda+\alpha^i D_a\tilde{E}^a_i \\
&-V^a\left(F^i_{ab}\tilde{E}^b_i - A^i_a D_b\tilde{E}^b_i - \tilde{\tau}\partial_a\Lambda\right) \\
&-\tilde{N}\left(\frac{\epsilon^{ij}{}_k F^k_{ab}\tilde{E}^a_i\tilde{E}^b_j}{2\det(\tilde{E}^a_i)}-\lambda\right)-\tilde{T}^a\partial_a\lambda\Big]\ .
\end{aligned}
\tag{4.55}
$$

The transformation into the Hamiltonian setting is then clear. We have dynamical fields $A^i_a,\tilde{E}^a_i,\tilde{\tau},\lambda$, which satisfy the principal Poisson brackets

$$
\{A^i_a(\boldsymbol{x}),\tilde{E}^b_j(\boldsymbol{x}')\} = \ell_P^2\sqrt{\sigma}\,\delta^i_j\delta^b_a\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')\ ,\quad \{\tilde{\tau}(\boldsymbol{x}),\lambda(\boldsymbol{x}')\} = \ell_P^2\sqrt{\sigma}\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')\ , \tag{4.56}
$$

with all other unrelated combinations vanishing. The fields $\alpha^i, V^a, \tilde{N}, \tilde{T}^a$ are Lagrange multipliers enforcing constraints

$$
\tilde{\mathcal{G}}_i = -D_a\tilde{E}^a_i \approx 0\ , \tag{4.57a}
$$

$$
\tilde{\mathcal{D}}'_a = F^i_{ab}\tilde{E}^b_i - A^i_a D_b\tilde{E}^b_i - \tilde{\tau}\partial_a\lambda \approx 0\ , \tag{4.57b}
$$

$$
\mathcal{H}' = \frac{\epsilon^{ij}{}_k F^k_{ab}\tilde{E}^a_i\tilde{E}^b_j}{2\det(\tilde{E}^a_i)} - \lambda \approx 0\ , \tag{4.57c}
$$

$$
\mathcal{J}_a = \partial_a\lambda \approx 0\ , \tag{4.57d}
$$

whose smeared forms are constructed as

$$
\begin{aligned}
\mathcal{G}(\alpha) &= \frac{1}{\ell_P^2\sqrt{\sigma}}\int d^3x\ \alpha^i\tilde{\mathcal{G}}_i \quad ,\quad \mathcal{D}'(\boldsymbol{V}) = \frac{1}{\ell_P^2\sqrt{\sigma}}\int d^3x\ V^a\tilde{\mathcal{D}}'_a\ , \\
\mathcal{H}'(\tilde{N}) &= \frac{1}{\ell_P^2\sqrt{\sigma}}\int d^3x\ \tilde{N}\mathcal{H}' \quad ,\quad \mathcal{J}(\tilde{\boldsymbol{T}}) = \frac{1}{\ell_P^2\sqrt{\sigma}}\int d^3x\ \tilde{T}^a\mathcal{J}_a\ .
\end{aligned}
\tag{4.58}
$$

This constraint set is first class. The three constraints $\mathcal{G},\mathcal{D}',\mathcal{H}'$ satisfy the same Poisson algebra as their non-unimodular counterparts, given in (4.16). The remaining Poisson brackets involving $\mathcal{J}$ are given by

$$
\begin{aligned}
\{\mathcal{G}(\alpha),\mathcal{J}(\tilde{\boldsymbol{T}})\} = 0 \quad &,\quad \{\mathcal{D}(\boldsymbol{V}),\mathcal{J}(\tilde{\boldsymbol{T}})\} = \mathcal{J}(\mathcal{L}_{\boldsymbol{V}}\tilde{\boldsymbol{T}}) \\
\{\mathcal{H}(\tilde{N}),\mathcal{J}(\tilde{\boldsymbol{T}})\} = 0 \quad &,\quad \{\mathcal{J}(\tilde{\boldsymbol{T}}),\mathcal{J}(\tilde{\boldsymbol{L}})\} = 0\ .
\end{aligned}
\tag{4.59}
$$

The Hamiltonian for the theory is a sum over constraints which reads

$$
H_{\text{HT}}(\alpha,\boldsymbol{V},\tilde{N},\tilde{\boldsymbol{T}}) = \mathcal{G}(\alpha)+\mathcal{D}'(\boldsymbol{V})+\mathcal{H}'(\tilde{N})+\mathcal{J}(\tilde{\boldsymbol{T}})\ . \tag{4.60}
$$

The time evolution equations are generated via the usual mechanism, $\dot{\mathcal{O}} \approx \{\mathcal{O}, H_{\text{HT}}\}$ for any functional $\mathcal{O}$ of the field variables. Once again, this time evolution is pure gauge, reflecting the fact that this formulation of unimodular gravity is generally covariant. As before, the Gauss and diffeomorphism constraints $\mathcal{G}$ and $\mathcal{D}'$ respectively generate local (complex) $SO(3)$ gauge transformations and spatial diffeomorphisms on the variables. Note, the diffeomorphism constraint is extended with an extra term $-\tilde{\tau}\partial_a\lambda$ to account for the extra variables in this theory. The Gauss constraint is not extended here since the new variable $\tilde{\tau}$ and λ have no $\mathfrak{so}(3)_{\mathbb{C}}$ indices, and therefore transform trivially under local gauge transformations. The Hamiltonian constraint $\mathcal{H}'$ generates precisely the same time coordinate reparametrisation transformations on the variables A^i_a and $\tilde{E}^a_i$ as in the non-unimodular theory. The transformation due to $\mathcal{H}'$ has no effect on λ, but it does induce a shift $\tilde{\tau} \mapsto \tilde{\tau} + \delta\tilde{\tau}$ where $\delta\tau = -\tilde{N}$ when $\tilde{N}$ is small. The constraint $\mathcal{J}$ tells us that λ is constant is on each constant time hypersurface, and the dynamical equation $\dot{\lambda} = \{\lambda, H_{\text{HT}}\} \overset{!}{\approx} 0$ tells us that the value of λ doesn't change from hypersurface to hypersurface. So in aggregate we see that λ is constant over $\mathcal{M}$, which is sensible as it is a stand-in for the cosmological constant in this theory. We would like to interpret the transformations of $\tilde{\tau}$ and λ due to $\mathcal{H}'$ as time reparametrisations. In this case, λ doesn't transform under the action of $\mathcal{H}'$ since it is constant and hence its value at any given coordinate time is independent of the choice of time coordinate. We will justify the shift $\delta\tilde{\tau} = -\tilde{N}$ as resulting from a time reparametrisation in the forthcoming discussion on unimodular volume time. The constraint $\mathcal{J}$ only acts non-trivially on $\tilde{\tau}$, where it induces a shift $\delta\tilde{\tau} = -\partial_a\tilde{T}^a$ when $\tilde{T}^a$ is small. The transformations due to $\mathcal{J}$ correspond to a symmetry of the action (3.12) where the 3-form T is shifted by a closed 3-form, $T \mapsto T + \theta$ where $d\theta = 0$. To see this explicitly, we need the gauge transformations for the Lagrange multipliers which are computed following the procedure outlined in section 4.1.2 yielding:

	α^i	V^a	$\tilde{N}$	$\tilde{T}^a$
$\mathcal{G}(\beta) \quad : \delta_\beta$	$\dot{\beta}^i + [\beta, \alpha]^i - \mathcal{L}_{\boldsymbol{V}}\beta^i$	0	0	0
$\mathcal{D}'(\boldsymbol{U}) \quad : \delta_{\boldsymbol{U}}$	$\mathcal{L}_{\boldsymbol{U}}\,\alpha^i$	$\dot{U}^a + [\boldsymbol{U}, \boldsymbol{V}]^a$	$\mathcal{L}_{\boldsymbol{U}}\,\tilde{N}$	$\mathcal{L}_{\boldsymbol{U}}\,\tilde{T}^a$
$\mathcal{H}'(\tilde{Q}) \quad : \delta_{\tilde{Q}}$	$-A^i_a\mathcal{V}^a(\tilde{Q}, \tilde{N})$	$\mathcal{V}^a(\tilde{Q}, \tilde{N})$	$\dot{\tilde{Q}} - \mathcal{L}_{\boldsymbol{V}}\tilde{Q}$	0
$\mathcal{J}(\tilde{\boldsymbol{L}}) \quad : \delta_{\tilde{\boldsymbol{L}}}$	0	0	0	$\dot{\tilde{L}}^a - \mathcal{L}_{\boldsymbol{V}}\tilde{L}^a$

With this we compute the infinitesimal transformation δT due to $\mathcal{J}(\tilde{\boldsymbol{L}})$, starting from the expression for T given in (4.51), to be

$$\delta T = -\partial_a \tilde{L}^a \, d^3x - \frac{1}{2}\left[\dot{\tilde{L}}^a - \partial_d\left(V^d \tilde{L}^a - V^a \tilde{L}^d\right)\right] \underset{\sim}{\epsilon}_{abc} \, dt \wedge dx^b \wedge dx^c \; . \tag{4.61}$$

Evaluating the exterior derivative $d\delta T$ reveals that δT is closed,

$$d\delta T = d^4x \left[-\partial_a \dot{\tilde{L}}^a + \partial_a \dot{\tilde{L}}^a - \partial_a \partial_b \left(V^b \tilde{L}^a - V^a \tilde{L}^b\right)\right] = 0 \; . \tag{4.62}$$

For emphasis, we have used the transformation $\delta\tilde{\tau} = \{\tilde{\tau}, \mathcal{J}(\tilde{\boldsymbol{L}})\} = -\partial_a \tilde{L}^a$ as well as the transformations $\delta_{\tilde{\boldsymbol{L}}} V^a$ and $\delta_{\tilde{\boldsymbol{L}}} \tilde{T}^a$ from the above table to achieve this result. This was shown in [53], except where the authors formulate the Lagrangian symmetry as a shift of the vector density $\tilde{\mathcal{T}}^\mu$, which appears in the action (2.115), by some $\tilde{\theta}^\mu$ satisfying $\partial_\mu \tilde{\theta}^\mu = 0$. One recovers Henneaux and Teitelboim's vector density via $\tilde{\mathcal{T}}^\mu = \frac{1}{6}\tilde{\epsilon}^{\mu\nu\rho\sigma} T_{\nu\rho\sigma}$ with components $\tilde{\mathcal{T}}^0 = \tilde{\tau}$ and $\tilde{\mathcal{T}}^a = \tilde{T}^a - \tilde{\tau} V^a$.

The Hamiltonian constraint $\mathcal{H}'$ provides us with $\mathcal{R} = \lambda$, where we remind the reader that $\mathcal{R} = \frac{1}{2}(\det(\tilde{E}^a_i))^{-1} \epsilon^{ij}{}_k F^k_{ab} \tilde{E}^a_i \tilde{E}^b_j$ is the trace of $\mathcal{R}^{ij}$ defined in (4.7). Additionally we recall that $\mathcal{R}$ is proportional to the Ricci scalar for the constructed metric, whether one uses the Urbantke formula (2.97) or the tetrad decomposition formulae in (2.84), (2.85) and (2.95). Hence we see that the Ricci scalar is equal to an integration constant. Hence, the theory contains all possible solution for all possible values of λ. This mirrors the discussions from section 2.5 and chapter 3.

4.3.1 Unimodular volume time

In section 2.5 we briefly mentioned how one could construct a clock function on a globally simple spacetime $\mathbb{R} \times \mathcal{S}$ by integrating the metric volume form over a cylindrical 4D region $\sim [0,1] \times \mathcal{S}$ which is bounded by a pair of constant time hypersurfaces on either end. We will explore this idea in greater depth here. First, constructing the Urbantke metric (2.97) for the canonical parametrised formulation (4.55) yields precisely the metric (4.27) and the corresponding tetrad (4.26) from the non-unimodular theory. The volume form ε, constructed either as in (2.97) or equivalently as $\varepsilon = -\sigma\, e^0 \wedge \ldots \wedge e^3$, can be written as

$$\varepsilon = -\sqrt{\sigma}\, \tilde{N} \, d^4x \; . \tag{4.63}$$

In Lorentzian signature, the reality conditions (2.82b) and (2.82a) can be formulated as $\mathrm{Re}\, \tilde{N} = 0$ and $\mathrm{Im}\,(\underset{\sim}{E}{}^i_a \underset{\sim}{E}{}_{ib}) = 0$ respectively, with the latter yielding the further condition

$\text{Im}\left[(\det(\tilde{E}^a_i))^2\right] = 0$ as a direct consequence. These come from the discussion in section 4.1.1. Then it is clear that the volume form ε is real valued in Lorentzian signature where $\sqrt{\sigma} = i$. Furthermore, the volume form is manifestly real in Euclidean signature where $\sqrt{\sigma} = 1$ and where $\tilde{N}$ is taken to be real valued. Hence in the Lorentzian and Euclidean signature regimes, any clock we construct by integrating ε will be real valued.

A significant result in the parametrised approach, whether in metric [53] or connection variables [96], is that the volume form is exact. One can recover this result from the canonical theory also. To see this, consider the equation of motion for $\tilde{\tau}$ computed via $\dot{\tilde{\tau}} = \{\tilde{\tau}, H_{\text{HT}}\}$ which is given by

$$\dot{\tilde{\tau}} = -\tilde{N} - \partial_a\left(\tilde{T}^a - \tilde{\tau}V^a\right) . \tag{4.64}$$

Rearranging this expression yields $-\tilde{N} = \dot{\tilde{\tau}} + \partial_a\left(\tilde{T}^a - \tilde{\tau}V^a\right)$, which we insert into the expression for the volume form ε given in (4.63) to get

$$\varepsilon = \sqrt{\sigma}\, d^4x\left[\dot{\tilde{\tau}} + \partial_a\left(\tilde{T}^a - \tilde{\tau}V^a\right)\right] \overset{!}{=} \sqrt{\sigma}\, dT , \tag{4.65}$$

where the final equality marked '$\overset{!}{=}$' uses the expression for the exact 4-form dT given in (4.52). Hence, we recover the exactness of the metric volume from.

We define the 'volume time' t_{Vol} between hypersurfaces corresponding to $t = t_0$ and $t = t_1$ to be the spacetime volume of the 4D cylindrical region which is bounded by them, as in (3.14) and (3.15). Computing this spacetime volume by evaluating the integral of $\varepsilon = \sqrt{\sigma}\, dT$ over such a spacetime region yields

$$t_{\text{Vol}}(t_0, t_1) = \sqrt{\sigma}\left(\int_{t=t_1} d^3x\ \tilde{\tau} - \int_{t=t_0} d^3x\ \tilde{\tau}\right) . \tag{4.66}$$

Then we have a geometric interpretation for the field $\tilde{\tau}$ which is conjugate to the cosmological 'constant' field λ: $\tilde{\tau}$ encodes the volume time between constant t hypersurfaces. This volume time is certainly not gauge invariant; it depends on the time coordinate which can be chosen arbitrarily. The transformations on the variable $\tilde{\tau}$ generated from the constraint $\mathcal{H}'$ produce exactly the time reparametrisation transformations on t_{Vol} that we would expect. These are

$$\begin{aligned}\delta t_{\text{Vol}}(t_0, t_1) &\overset{!}{=} -\sqrt{\sigma}\left(\int_{t=t_1} d^3x\ \tilde{N} - \int_{t=t_0} d^3x\ \tilde{N}\right) \\ &\overset{!!}{=} \sqrt{\sigma}\left(\int_{t=t_1} d^3x\ \dot{\tilde{\tau}} - \int_{t=t_0} d^3x\ \dot{\tilde{\tau}}\right) = \frac{\partial t_{\text{Vol}}}{\partial t_0} + \frac{\partial t_{\text{Vol}}}{\partial t_1} .\end{aligned} \tag{4.67}$$

In the first equality marked by '$\overset{!}{=}$' we use the shift $\delta\tilde{\tau} = \{\tilde{\tau}, \mathcal{H}'\} = -\tilde{N}$, and in the second equality marked by '$\overset{!!}{=}$' we use the equation of motion (4.64). However, the gauge transformations generated from the other constraints have no affect on t_{Vol}. To show that t_{Vol} provides a suitable clock function, consider the following. First, it is clear from the definition (4.66) that

$$t_{\text{Vol}}(t_0, t_1) = t_{\text{Vol}}(t_0, t_2) + t_{\text{Vol}}(t_2, t_1) \ . \tag{4.68}$$

Furthermore t_{Vol} is monotonic w.r.t increasing $|t_1 - t_0|$ when $\int d^3x\ \tilde{\tau}$ is monotonic, which corresponds to choosing $\tilde{N}$ such that $\int d^3x\ \tilde{N}$ is either non-negative or non-positive. Note that t_{Vol} is allowed to take negative values; we see $t_{\text{Vol}}(t_0, t_1) = -t_{\text{Vol}}(t_1, t_0)$. Hence this time function is sensitive to the ordering of events. One could define a 'duration' function $|t_{\text{Vol}}(t_0, t_1)|$ which is always positive, except when $t_0 = t_1$ where it vanishes. Although, this duration function would no longer satisfy the additive property (4.68), but would satisfy a triangle inequality instead.

4.A Supplementary Material: Chapter 4

4.A.1 Gauge transformations of the action

Derivation of equation (4.32)

In the non-field setting, consider a canonical action $S[q,p,\lambda]$ over dynamical variables q^i, p_i given by

$$S[q,p,\lambda] = \int dt \left[\dot{q}^i p_i - H(q,p) - \lambda^\alpha G_\alpha(q,p)\right] \ , \tag{4.69}$$

where λ^α are Lagrange multipliers enforcing constraints $G_\alpha(q,p) \approx 0$ which we assume are first class. In addition, where $H(q,p)$ is the definite, non-constraint, part of the Hamiltonian which we also assume to be first class, $\{H, G_\alpha\} \approx 0$. We consider the action of a gauge transformation generated via $\delta_\varepsilon f = \varepsilon^\beta \{f, G_\beta\}$ where $\varepsilon^\beta(t)$ is the non-dynamical gauge parameter. We assume the linearised gauge transformation operator δ_ε commutes with differentiation and integration w.r.t time t. We compute $\delta_\varepsilon S$ via

$$\begin{aligned}
\delta_\varepsilon S &= \left[\varepsilon^\beta \frac{\partial G_\beta}{\partial p_i} p_i\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \left[-\delta_\varepsilon q^i \dot{p}_i + \dot{q}^i \delta_\varepsilon p_i - \delta_\varepsilon H - \delta_\varepsilon \lambda^\alpha G_\alpha - \lambda^\alpha \delta_\varepsilon G_\alpha\right] \\
&= \left[\varepsilon^\beta \frac{\partial G_\beta}{\partial p_i} p_i\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \Bigg[- \varepsilon^\beta \left(\frac{\partial G_\beta}{\partial p_i}\dot{p}_i - \frac{\partial G_\beta}{\partial q^i}\dot{q}^i\right) - \varepsilon^\beta \{H, G_\beta\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \delta_\varepsilon \lambda^\alpha G_\alpha - \lambda^\alpha \{G_\alpha, G_\beta\}\varepsilon^\beta \Bigg] \qquad (4.70) \\
&= \left[\varepsilon^\beta \frac{\partial G_\beta}{\partial p_i} p_i\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \left[-\varepsilon^\beta \dot{G}_\beta - \varepsilon^\beta \{H, G_\beta\} - \delta_\varepsilon \lambda^\alpha G_\alpha - \lambda^\alpha \{G_\alpha, G_\beta\}\varepsilon^\beta\right] \\
&= \left[\varepsilon^\beta \frac{\partial G_\beta}{\partial p_i} p_i - \varepsilon^\beta G_\beta\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \left[\dot{\varepsilon}^\beta G_\beta + \{\varepsilon^\beta G_\beta, H + \lambda^\alpha G_\alpha\} - \delta_\varepsilon \lambda^\alpha G_\alpha\right] \ .
\end{aligned}$$

Extending this formula into the field setting is straightforward, and yields (4.32).

Chapter 5

Chiral Connection Cosmology

In the previous chapter, we explored the canonical formulation of the Plebański theory, and its unimodular modifications, in the most general setting. We recover the Ashtekar variables [11] in both cases, albeit the constraint structure is somewhat modified in the unimodular formulations to reflect the different symmetries of those theories compared to the standard theory. Formally, these variables best lend themselves to the spin-network quantisation scheme of LQG [21, 91, 100], where computations tend to be intractable in general. A common approach in quantum gravity investigations is to restrict ones attention to models which have a high degree of symmetry, and consequently fewer degrees of freedom. In particular, one can examine *spatially homogeneous* models where the fields are invariant under a certain group of translations on each spatial surface. Here, the spatial dependence of the field variables $A^i_a(t, \boldsymbol{x}), \tilde{E}^a_i(t, \boldsymbol{x})$ 'drops out' leaving us with a canonical theory over non-field variables with only t dependence. Hence the overall number of degrees of freedom reduces from an uncountable infinity to a finite number. This is the primary tenet of the field of *quantum cosmology.* In this section, we will examine spatially homogeneous models of chiral connection gravity from the covariant and canonical viewpoints. Specifically, we examine homogeneous models which are also *diagonal*; a further restriction on the variables which allows us to circumvent certain obstacles which arise in non-diagonal models. In terms of the diagonal variables, one can more easily write down and classify the different solution branches of the reality conditions in Lorentzian signature. We focus on different approaches for implementing the reality conditions, including an approach where they are treated as dynamical constraints à la Dirac [36]. Finally, we restrict further to *isotropic* variables resulting in a *minisuperspace action* which can be 'quantised' via a path integral approach, in line with [35, 39, 40, 47]. Ultimately, one finds a particularly simple form for a cosmological two-point function with connection boundary data consistent with the result in [79], albeit from a novel viewpoint.

This chapter is based on the paper [45] which was written in collaboration with Steffen Gielen; the presentation here diverges significantly in some areas, especially w.r.t notation and sign conventions for some definitions. Section 5.2 of this thesis doesn't appear in [45]; it consists of a review of spatially homogeneous models, including some speculative remarks.

5.1 Spatially Homogeneous Models

In this chapter, we once again consider spacetime manifolds of the form $\mathcal{M} = \mathbb{R}\times\mathcal{S}$ where the spatial manifold $\mathcal{S}$ is a connected 3-dimensional Lie group. In this case, for each $s \in \mathcal{S}$ there is a left translation map defined $\ell_s : \mathcal{S} \to \mathcal{S}, q \mapsto sq$ which is a diffeomorphism. We define a *spatially homogeneous*, or just *homogeneous*, connection to be one that is invariant under the pull-back of all such left translation maps. To construct a homogeneous connection, one can define a globally independent and non-vanishing triad of left translation invariant 1-forms θ^α called a *Cartan frame*. Such a Cartan frame satisfies the *Maurer–Cartan equation*

$$d\theta^\alpha + \frac{1}{2} f^\alpha{}_{\beta\gamma}\, \theta^\beta \wedge \theta^\gamma = 0 \ , \tag{5.1}$$

where $f^\alpha{}_{\beta\gamma}$ are the structure constants of the Lie algebra associated to the group action on $\mathcal{S}$ in some basis. Note, we use lowercase Greek indices from the beginning of the alphabet $\alpha, \beta, \ldots = 1, 2, 3$ to count over the components of a Cartan frame. A homogeneous connection may be constructed as

$$A^i = A^i_0(t)\, dt + A^i_\alpha(t)\, \theta^\alpha \ , \tag{5.2}$$

with curvature 2-forms given by

$$F^i = \left(\dot{A}^i_\alpha - \epsilon^i{}_{jk} A^j_\alpha A^k_0\right) dt \wedge \theta^\alpha + \frac{1}{2} F^i_{\alpha\beta}\, \theta^\alpha \wedge \theta^\beta \ , \tag{5.3}$$

where we define $F^i_{\alpha\beta} = -A^i_\gamma\, f^\gamma{}_{\alpha\beta} + \epsilon^i{}_{jk} A^j_\alpha A^k_\beta$. In general, local $SO(3,\mathbb{C})$ gauge transformations do not preserve the homogeneity of the connection. We call the restricted set of $SO(3,\mathbb{C})$ gauge transformations that do preserve the homogeneity of the connection *homogeneous gauge transformations*. These are generated by group valued functions $U(t) \in SO(3,\mathbb{C})$ which depend only on the time coordinate via

$$A \ \mapsto \ A' = U^{-1}\left(A_0\, U + \dot{U}\right) dt + \left(U^{-1} A_\alpha\, U\right) \theta^\alpha \ . \tag{5.4}$$

By solving the condition $\dot{U} = -A_0 U$ for $U(t)$, one can construct a family of homogeneous gauge transformations that send the A^i_0 components of the connection to zero. Such solutions have the form

$$U(t) = \mathcal{P}\exp\left(-\int_0^t d\lambda\ A_0(\lambda)\right) U_0\ , \tag{5.5}$$

where $U_0 \in SO(3,\mathbb{C})$ is some constant matrix, and where $\mathcal{P}\exp$ denotes the path ordered exponential defined

$$\begin{aligned}\mathcal{P}\exp\left(-\int_0^t d\lambda\ A_0(\lambda)\right) = \mathbb{I} - \int_0^t d\lambda\ A_0(\lambda) + \int_0^t d\lambda_1 \int_0^{\lambda_1} d\lambda_2\ A_0(\lambda_1)A_0(\lambda_2) + \ldots \\ (-)^k \int_0^t d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \ldots \int_0^{\lambda_{k-1}} d\lambda_k\ A_0(\lambda_1)\ldots A_0(\lambda_k) + \ldots\end{aligned} \tag{5.6}$$

Then one can always bring the connection into the form $A^i = A^i_\alpha\, \theta^\alpha$ under the action of a homogeneous gauge transformation, which amounts to a fixing of the internal $SO(3)$ gauge symmetry.

The central aim of this chapter is to understand the dynamics of chiral connection theories for spatially homogeneous models. To do this, we construct the most general homogeneous ansatzes for the fields $A^i, \Sigma^i, M^{ij}, \mu$, and substitute them into the Plebański action (2.79) to get a *minisuperspace* action over fewer degrees of freedom. The *symmetric criticality principle* [83] implies that, for example in the case $\mathcal{S} = S^3 \cong SU(2)$ which we will examine later in this chapter, the compactness of the group guarantees that the restriction to $SU(2)$ translation-invariant connections commutes with the variational principle. Later on, we will also make a further restriction to *diagonal connections*, which will be defined in a future section. In this case, the restriction does not appear to be due to invariance under some group of transformations, so the symmetric criticality principle does not apply in an obvious way. We can check by hand that the restriction to diagonal connections commutes with the variational principle. That is, the field equations that we obtain from the minisuperspace action are equivalent to the equations that we get when we substitute our homogeneous and diagonal ansatzes into the field equations of the full theory.

5.2 Canonical Spatially Homogeneous Plebański Gravity

We now pursue the most general canonical formulation for spatially homogeneous Plebański gravity. We have already seen how to construct a spatially homogeneous connection. This definition for spatial homogeneity extends naturally to the other fields present in the Plebański formulation. Any differential form, or Lie algebra valued differential form, will be called spatially homogeneous if it is invariant under the pull-back of any left translation map on $\mathcal{S}$. Constructing such fields is simply done. In the general case, we would construct differential forms as linear combinations of the basis elements dx^μ, or their wedge products, with coefficients which are functions of space and time. In the homogeneous case, we construct differential forms as linear combinations of the basis elements dt and θ^α, or their wedge products, with coefficients which are functions of time only. With this in mind, we proceed to construct homogeneous versions of the fields from the Plebański formulation. We begin by constructing homogeneous Σ^i 2-forms via

$$\Sigma^i = \Sigma^i_{0\alpha}(t)\, dt \wedge \theta^\alpha + \frac{1}{2}\Sigma^i_{\alpha\beta}(t)\, \theta^\alpha \wedge \theta^\beta \; . \tag{5.7}$$

The components $\Sigma^i_{\alpha\beta}$ contain only 9 degrees of freedom which can be encoded into an object of the kind E^α_i with Lie algebra and Cartan frame indices via $\Sigma^i_{\alpha\beta}(t) = \epsilon_{\alpha\beta\gamma}\, E^{i\gamma}(t)$. In this section $\epsilon_{\alpha\beta\gamma}$ and $\epsilon^{\alpha\beta\gamma}$ are totally antisymmetric tensors over the Cartan frame indices satisfying $\epsilon_{123} = \epsilon^{123} = 1$. This is the homogeneous version of the densitised triad $\tilde{E}^a_i$ from the non-homogeneous formulation. From this triad, we construct two further quantities via

$$\det(E^\alpha_i) = \frac{1}{6}\epsilon_{\alpha\beta\gamma}\, \epsilon^{ijk}\, E^\alpha_i E^\beta_j E^\gamma_k \; , \tag{5.8a}$$

$$(E^{-1})^i_\alpha = \frac{\epsilon_{\alpha\beta\gamma}\, \epsilon^{ijk} E^\beta_j E^\gamma_k}{2\det(E^\alpha_i)} \; , \tag{5.8b}$$

which are the determinant and the inverse of the matrix with components E^α_i respectively. The temporal part $\Sigma^i_{0\alpha}$ also contains 9 degrees of freedom which can be encoded into a pair of a homogeneous vector field $V^\alpha(t)$ and a symmetric 3×3 matrix $\varphi^{ij}(t)$ via

$$\Sigma^i_{0\alpha}(\boldsymbol{V}, \varphi) = -\epsilon_{\alpha\beta\gamma} V^\beta E^{i\gamma} - \varphi^{ij}(E^{-1})_{j\alpha} \; . \tag{5.9}$$

In this context, we define a homogeneous vector field to be one that is invariant under the push-forward of any left translation map on $\mathcal{S}$. These are typically called *left invariant vector fields* in differential geometry literature [58]. Any homogeneous vector field $\boldsymbol{V}$ may be expanded as $V^\alpha(t)\, \xi_\alpha$ where ξ_α are a triad of vector fields which are dual to θ^α such that $\theta^\alpha(\xi_\beta) = \delta^\alpha_\beta$. The vector fields ξ_α form a basis of the Lie algebra of $\mathcal{S}$ and satisfy $[\xi_\alpha, \xi_\beta] = f^\gamma{}_{\alpha\beta}\, \xi_\gamma$. The parametrisation in (5.9) is in direct parallel with the parametrisation (4.5) in the full (non-homogeneous) theory. Also in parallel with the full theory, we parametrise the temporal part of the connection by $A^i_0 = \alpha^i + V^\alpha A^i_\alpha$, where $\alpha^i(t)$ is a Lie algebra valued scalar. To make the matrix field M^{ij} homogeneous, we simply restrict it to be a function of time only, $M^{ij}(t)$. In general, a scalar field is homogeneous only if it takes a constant value on each constant time hypersurface. Finally, the homogeneous version of the the top-form μ has the form $\mu = -2n(t)\, dt \wedge \varepsilon_\theta$ where $\varepsilon_\theta = \theta^1 \wedge \theta^2 \wedge \theta^3$ is a fiducial volume form on each constant time hypersurface. The function $n(t)$ transforms under time reparametrisations such that

$$n(t)\, dt = n'(t')\, dt' \ . \tag{5.10}$$

We may now construct an action for spatially homogeneous Plebański gravity; one inserts the homogeneous ansatzes for the fields into the Plebański action (2.79) to get

$$\begin{aligned} S'_{\mathrm{Hom}}\left[A, E, M, \alpha, \boldsymbol{V}, \varphi, n\right] = \frac{U_\theta}{\ell_P^2 \sqrt{\sigma}} \int dt \Big[\dot{A}^i_\alpha E^\alpha_i - \alpha^i \epsilon^k{}_{ji} A^j_\alpha E^\alpha_k - V^\alpha f^\gamma{}_{\beta\alpha} A^i_\gamma E^\beta_i \\ -\varphi_{ij} \left(\mathcal{R}^{ij}_H - M^{ij} \right) - n \left(\mathrm{tr}\, M - \Lambda \right) \Big] \ , \end{aligned} \tag{5.11}$$

where $\mathcal{R}^{ij}_H$ is the homogeneous version of the quantity $\mathcal{R}^{ij}$ from the non-homogeneous formulation, given by

$$\mathcal{R}^{ij}_H = \frac{F_{\alpha\beta}{}^{(i} \epsilon^{j)kl} E^\alpha_k E^\beta_l}{2 \det(E^\alpha_i)} \ . \tag{5.12}$$

In order to arrive at this form of the action, we notice that the action separates into a product of an integral over space and an integral over time, $S_{\mathrm{Hom}} = \left(\int \varepsilon_\theta \right) \int dt\ L(t)$. At this stage, we can simply evaluate the spatial integral which yields a constant factor $U_\theta = \int \varepsilon_\theta$ in front of the time integral. One must carefully consider the domain of this spatial integral. In the case where $\mathcal{S}$ is a compact Lie group, for example in the case of Bianchi IX models where we choose $\mathcal{S} = SU(2)$, we may evaluate this spatial integral over all of $\mathcal{S}$ to get finite U_θ. However, when $\mathcal{S}$ is not compact, for example in the case of Bianchi I models where $\mathcal{S} = \mathbb{R}^3$, the spatial integral evaluated over all of $\mathcal{S}$ will be divergent. In these cases, we need to choose some representative *fiducial cell* $\mathcal{U}$, which is a compact 3D submanifold on which the spatial integral is finite. For the case of Bianchi I, we may choose a rectangular fiducial cell $\mathcal{U} = \{(y^1, y^2, y^3) : 0 \leq y^a \leq l^a\}$ where $0 < l^a$ are unphysical side lengths.

One observes the following Lemma:

Lemma 5.2.1. The restriction of the Plebański action to spatially homogeneous fields, as outlined above, commutes with the variational principle only for Bianchi *class A* models satisfying $f^\beta{}_{\alpha\beta} = 0$ (i.e., with a symmetric matrix $\mathcal{C}^{\alpha\beta} := f^\alpha{}_{\gamma\delta}\, \epsilon^{\beta\gamma\delta}$).

That is, the dynamical equations coming from the restricted Plebański action (5.11) are equivalent to the equations (2.81) coming from the full Plebański action, where one inserts the homogeneous ansatzes, only if the group $\mathcal{S}$ is Bianchi class A [20, 21]. The proof can be found in the supplementary material, 5.A.1.

Returning to the action (5.11), consider the equations of motion arising from variations w.r.t the variables M^{ij} and φ^{ij} which read

$$\frac{\delta S_{\text{Hom}}}{\delta M^{ij}} = 0 \;\Leftrightarrow\; \varphi_{ij} = n\delta_{ij} \quad , \quad \frac{\delta S_{\text{Hom}}}{\delta \varphi_{ij}} = 0 \;\Leftrightarrow\; M^{ij} = \mathcal{R}_H^{ij} \; . \tag{5.13}$$

At the level of the equations of motion, the variables M^{ij} and φ_{ij} are redundant. One can 'integrate out' these fields by solving either of the two above equations at the level of the action. This yields an action in fewer variables which reads

$$S_{\text{Hom}} = \frac{U_\theta}{\ell_P^2 \sqrt{\sigma}} \int dt \left[\dot{A}^i_\alpha E^\alpha_i - \alpha^i \epsilon^k{}_{ji} A^j_\alpha E^\alpha_k + V^\alpha f^\gamma{}_{\beta\alpha} A^i_\gamma E^\beta_i - n\left(\mathcal{R}_H - \Lambda\right) \right] , \tag{5.14}$$

where $\mathcal{R}_H = (\mathcal{R}_H)^i{}_i = \frac{1}{2}(\det(E^\alpha_i))^{-1} \epsilon^{ij}{}_k F^k_{\alpha\beta} E^\alpha_i E^\beta_j$ denotes the trace of $\mathcal{R}^{ij}_H$. It is a straightforward exercise to check that this action yields the same equations of motion as (5.11). From here, the transition to the Hamiltonian picture is clear. We have a pair of dynamical variables A^i_α and E^β_j which have Poisson bracket

$$\{A^i_\alpha, E^\beta_j\} = \ell_P^2 \sqrt{\sigma}\, U_\theta^{-1}\, \delta^i_j \delta^\beta_\alpha \; . \tag{5.15}$$

In addition, we have primary constraints on the variables $\mathcal{G}^{(H)}_i, \mathcal{D}^{(H)}_\alpha, \mathcal{H}^{(H)}$ defined via

$$U_\theta^{-1} \ell_P^2 \sqrt{\sigma}\, \mathcal{G}^{(H)}_i = \epsilon^k{}_{ji} A^j_\alpha E^\alpha_k \approx 0 \; , \tag{5.16a}$$

$$U_\theta^{-1} \ell_P^2 \sqrt{\sigma}\, \mathcal{D}^{(H)}_\alpha = -f^\gamma{}_{\alpha\beta}\, A^i_\gamma E^\beta_i \approx 0 \; , \tag{5.16b}$$

$$U_\theta^{-1} \ell_P^2 \sqrt{\sigma}\, \mathcal{H}^{(H)} = \frac{\epsilon_i{}^{jk} F^i_{\alpha\beta} E^\alpha_j E^\beta_k}{2\det(E^\alpha_i)} - \Lambda \approx 0 \; , \tag{5.16c}$$

with closed Poisson algebra given by

$$\begin{aligned} &\{\mathcal{G}^{(H)}_i, \mathcal{G}^{(H)}_j\} = -\epsilon^k{}_{ij}\, \mathcal{G}^{(H)}_k \; , \quad \{\mathcal{G}^{(H)}_i, \mathcal{D}^{(H)}_\alpha\} = 0 \; , \quad \{\mathcal{G}^{(H)}_i, \mathcal{H}^{(H)}\} = 0 \; , \\ &\{\mathcal{D}^{(H)}_\alpha, \mathcal{D}^{(H)}_\beta\} = f^\gamma{}_{\alpha\beta}\, \mathcal{D}^{(H)}_\gamma \; , \quad \{\mathcal{D}^{(H)}_\alpha, \mathcal{H}^{(H)}\} = 0 \; . \end{aligned} \tag{5.17}$$

All of the above expressions/equations for the constraints and their Poisson brackets are valid independent of the choice of structure constants $f^\alpha{}_{\beta\gamma}$. However, certain choices for $f^\alpha{}_{\beta\gamma}$ may cause the total number of independent constraints to be reduced. The most severe example of this occurs for models with Bianchi I symmetry where the structure constants vanish $f^\alpha{}_{\beta\gamma} = 0$. Here, the constraint functions $\mathcal{D}^{(H)}_\alpha$ are identically zero, and hence the theory has 3 fewer first class constraints. This is a weakness of this formulation of homogeneous Plebański gravity. To see why, we first examine the gauge transformations generated by the constraints (5.17). $\mathcal{G}_i^{(H)}$ generate homogeneous $SO(3,\mathbb{C})$ gauge transformations with infinitesimal forms $\delta_\alpha \mathcal{O} = \alpha^i \{\mathcal{O}, \mathcal{G}_i^{(H)}\}$ such that

$$\delta_\alpha A^i_\beta = \epsilon^i{}_{jk} A^j_\beta \alpha^k \quad , \quad \delta_\alpha E_i^\beta = -\epsilon^k{}_{ij} E_k^\beta \alpha^j \; . \tag{5.18}$$

In the cases where $\mathcal{D}_\alpha^{(H)}$ are non-vanishing and independent, they generate transformations with infinitesimal forms $\delta_{\boldsymbol{V}} \mathcal{O} = V^\alpha \{\mathcal{O}, \mathcal{D}_\alpha^{(H)}\}$ such that

$$\delta_{\boldsymbol{V}} A^i_\alpha = -V^\beta f^\gamma{}_{\beta\alpha} A^i_\gamma \quad , \quad \delta_{\boldsymbol{V}} E_i^\alpha = V^\beta f^\alpha{}_{\beta\gamma} E_i^\gamma \; . \tag{5.19}$$

For a geometric interpretation of these transformations, note that one can use the identity $\mathcal{L}_{\xi_\beta} \theta^\alpha = -f^\alpha{}_{\beta\gamma} \theta^\gamma$ to compute $\mathcal{L}_{\boldsymbol{V}}(A^i_\alpha \theta^\alpha) = -V^\beta f^\gamma{}_{\beta\alpha} A^i_\gamma \, \theta^\alpha$. Then one sees that the first equation in (5.19) yields the components of the Lie derivative $\mathcal{L}_{\boldsymbol{V}} A^i$ w.r.t the frame θ^α. Furthermore, the second equation in (5.19) yields the components of the Lie derivative $\mathcal{L}_{\boldsymbol{V}}(E_i^\alpha \xi_\alpha)$ w.r.t the dual frame ξ_α. That is, (5.19) provides the infinitesimal form of the diffeomorphism generated by the homogeneous vector field V^α via exponentiation. One can show that diffeomorhisms generated in this way are *right translations*, which have the form $r_s : \mathcal{S} \to \mathcal{S}, q \mapsto qs$. Hence right translations on $\mathcal{S}$ are a gauge symmetry in this theory. Alternatively, one could view the transformations generated by $\mathcal{D}^{(H)}$ as a *change of frame transformations* (CoF). To see this we can write $(A^i_\alpha + \delta A^i_\alpha)\theta^\alpha = A^i_\alpha \, (\theta^\alpha + \delta\theta^\alpha)$ where δA^i_α is given by the first equation in (5.19), and where $\delta\theta^\alpha = -V^\beta f^\alpha{}_{\beta\gamma} \, \theta^\gamma$. Furthermore, one can show that $\delta\theta^\alpha$ satisfies the linearised Maurer-Cartan equation

$$^3 d \, \delta\theta^\alpha + f^\alpha{}_{\beta\gamma} \, \delta\theta^\beta \wedge \theta^\gamma = 0 \; , \tag{5.20}$$

where $^3 d$ denotes the restriction of the exterior derivative to the spatial surfaces, defined such that $^3 df = 0$ for any function $f(t)$ of t only, and such that $^3 d\theta^\alpha$ recovers (5.1). Hence a change of frame with infinitesimal form $\theta^\alpha \mapsto \theta^\alpha + \delta\theta^\alpha$ preserves the structure constants $f^\alpha{}_{\beta\gamma}$. The frame transformations generated in this way are a restricted set of all possible CoF transformations; they are the ones generated via right translations, $\theta^\alpha \mapsto \theta'^\alpha = r_s^* \, \theta^\alpha$. Finally, $\mathcal{H}^{(H)}$ generates time reparametrisation transformations $\delta_n \mathcal{O} = n\{\mathcal{O}, \mathcal{H}^{(H)}\}$, as in the general canonical formulation. Also in analogy with the full theory, dynamical evolution is pure gauge and is generated from the Hamiltonian

$$H_{\text{Hom}}(\alpha, \boldsymbol{V}, n) = \alpha^i \mathcal{G}_i^{(H)} + V^\alpha \mathcal{D}_\alpha^{(H)} + n\mathcal{H}^{(H)} \tag{5.21}$$

via the usual mechanism $\dot{\mathcal{O}} = \{\mathcal{O}, H_{\text{Hom}}\}$.

Consider again models with Bianchi I symmetry (with $f^{\alpha}{}_{\beta\gamma} = 0$) where the constraints $\mathcal{D}^{(H)}_{\alpha}$ don't appear, and hence don't generate gauge transformations on the variables. Here, the connection and triad variables A^i_α and E^α_i have 18 degrees of freedom, which are then reduced to $18 - 2 \times (3+1) = 10$ gauge independent degrees of freedom by the constraint structure. This is in stark contrast to models with Bianchi IX symmetry where the constraints $\mathcal{D}^{(H)}_{\alpha}$ do appear, and are independent. In this case, the number of gauge independent degrees of freedom is $18 - 2 \times (3+3+1) = 4$, which is the expected amount. I.e., we appear to have too many independent degrees of freedom in the Bianchi I model. This can be seen from the Kasner solutions [64, 98], exact analytic solutions for the Bianchi I model, where one obtains only 4 independent integration constants corresponding to the 4 physical degrees of freedom that one expects. This is a fairly significant inconsistency in this formulation that seems to limit its usefulness. One might expect there to be further symmetry transformations on the variables which, by some subtle mechanism, don't appear in this restricted Hamiltonian formulation. On the other hand, one could attempt to argue that these extra degrees of freedom are not superfluous, as in [24] which makes reference to the treatment of Bianchi models in [19]. It is currently unclear to the author what is the correct perspective here.

To gain more insight into this dilemma, one could instead examine the non-diagonal Bianchi I model as a restriction of the solution space of the general canonical theory outlined in the previous chapter in section 4.1. Here, one assumes the coordinates $x^\mu = (t, x^a)$ on $\mathcal{M}$ are adapted to the product structure $\mathcal{M} = \mathbb{R} \times \mathcal{S}$. The approach here will be to impose further conditions on the initial data given by $\partial_a A^i_b = 0$ and $\partial_a \tilde{E}^b_i = 0$, and then to find the most general restriction of the Lagrange multipliers $\alpha^i, V^a, \tilde{N}$ (or arbitrary gauge parameters) so that these conditions are preserved under time evolution (or arbitrary gauge transformations). We don't implement these extra conditions as (second class) dynamical constraints on the variables in Dirac's formalism. One finds $\partial_a \alpha^i = 0$ and $\partial_a \tilde{N} = 0$ as expected. However, the permissible shift vectors V^a are given by $V^a = U^a{}_b\, x^b$ for some time dependent matrix $U^a{}_b(t)$. I.e., the shift vectors compatible with this initial data are the ones whose second order spatial derivatives vanish. With this initial data, the constraints $\tilde{\mathcal{D}}_a$ are vanishing. However, the gauge transformations they generate are non-trivial:

$$\delta_{\boldsymbol{V}} A^i_a = A^i_b\, U^b{}_a \quad , \quad \delta_{\boldsymbol{V}} \tilde{E}^a_i = -\left(U^a{}_b - \delta^a_b \operatorname{tr} U\right) \tilde{E}^b_i \, . \tag{5.22}$$

When one restricts $\operatorname{tr} U = 0$, these are just the infinitesimal forms of $SL(3)$ transformations on the coordinate frame dx^a. If x^a are Cartesian coordinates on $\mathcal{S} = \mathbb{R}^3$, then dx^a is a Cartan frame and these transformations are the frame changes which are missing from our homogeneous Hamiltonian formalism. This provides further evidence for the 'missing symmetries' hypothesis.

Further investigation is needed to resolve the inconsistencies that appear here. In the cases of the Bianchi I and IX models, one can avoid this issue by further restricting to diagonal variables (defined in section 5.3). The restriction to diagonal variables is by far the most common approach for spatially homogeneous models in the literature [22, 60]; there is comparably much less discussion on non-diagonal models. As far as the author is aware, there is no clear consensus on the canonical descriptions of non-diagonal Bianchi models in connection or metric variables in the literature. As such, our investigations into Bianchi I and IX type models in sections 5.3 and 5.4 respectively will also make use of this restriction to diagonal variables.

Reality conditions and the metric

The following short discussion on the constructed metric and the reality conditions is largely unaffected by the issues with the canonical theory discussed above. To derive the reality conditions in terms of the homogeneous canonical variables, we substitute the homogeneous expression for Σ^i given by

$$\Sigma^i = -\left(\epsilon_{\alpha\beta\gamma} V^\beta E^{i\gamma} + n\,(E^{-1})^i_\alpha\right) dt \wedge \theta^\alpha + \frac{1}{2} E^{i\alpha} \epsilon_{\alpha\beta\gamma}\, \theta^\beta \wedge \theta^\gamma \,, \tag{5.23}$$

which was constructed in section 5.2 , into the reality conditions (2.82) to get

$$\mathrm{Re}\, n = 0 \quad , \quad \mathrm{Im}\left[(E^{-1})^i_\alpha (E^{-1})_{i\beta}\right] = 0 \,, \tag{5.24}$$

where the latter of these yields $\mathrm{Im}\left[(\det(E^\alpha_i))^2\right] = 0$ as a direct consequence. In order to arrive at the second condition in the above, we assume that V^α is real valued. In parallel with the full theory, V^α plays the role of the shift vector; it is the Lagrange multiplier associated to the constraint $\mathcal{D}^{(H)}$ which generates (a restricted set of) spatial diffeomorphism transformations on the variables. To make this more apparent, one can construct a metric g_H using the Urbantke formula (2.97) which reads

$$g_{\mathrm{Hom}} = -\frac{n^2}{\det(E^\alpha_i)}\, dt \otimes dt - \det(E^\alpha_i)(E^{-1})^i_\alpha (E^{-1})_{i\beta} \left(V^\alpha dt + \theta^\alpha\right) \otimes \left(V^\beta dt + \theta^\beta\right) \ . \tag{5.25}$$

Then V^α is a spatially homogeneous analogue of the shift vector, and n encodes the lapse function. This discussion very closely mirrors the discussion in section 4.1.1. In general, one recovers elements of the homogeneous theory from the full canonical theory via the following variable substitutions

$$\begin{aligned} A^i_a(\boldsymbol{x},t) &= A^i_\beta(t)\,\theta^\beta_a(\boldsymbol{x}) \quad , \quad \tilde{E}^a_i(\boldsymbol{x},t) = \det\left(\theta^\alpha_b(\boldsymbol{x})\right) E^\beta_i\,\xi^a_\beta(\boldsymbol{x}) \ , \\ V^a(\boldsymbol{x},t) &= V^\beta(t)\,\xi^a_\beta(\boldsymbol{x}) \quad , \quad \tilde{N}(\boldsymbol{x},t) = \det\left(\theta^\alpha_b(\boldsymbol{x})\right) n(t) \ , \end{aligned} \tag{5.26}$$

where $\det\left(\theta^\alpha_b\right)$ denotes the determinant of the 3×3 matrix with components θ^α_b.

5.3 Diagonal Bianchi IX model

In this section we study the *diagonal Bianchi IX* model in the context of the chiral first order theory defined by (2.98). From section 4.1.3 we recall the canonical formulation for the first order theory is the same as for the full Plebański theory. Then both actions would be equally valid starting points for our investigation here. We choose to start from the first order action since, from a certain point of view, it is a simpler action for dealing with highly symmetric models, such as the homogeneous and diagonal models we wish to study in this section. In short, the first order action has fewer independent variables than the full Plebański action (2.79), and doesn't involve any square roots as in the pure connection theory (2.105) which could complicate the analysis unnecessarily. The (diagonal) Bianchi IX model is discussed in section 6.7 of [69] and our analysis will initially follow the presentation there. In what follows, we work exclusively in the Lorentzian signature, with $\sqrt{\sigma} = i$, where the fields are complex valued an need to be constrained via the reality conditions (2.100). We will discuss additional solutions to the reality conditions (2.82) beyond the physically most relevant ones appearing in [69]. Additionally, We will construct a Hamiltonian formalism in which the reality conditions and their associated consistency conditions can be viewed as (second class) constraints in Dirac's formalism for constrained Hamiltonian systems.

The spacetime manifold of the Bianchi IX model is assumed to have the form $\mathcal{M} = \mathbb{R}\times S^3$, and we require the fields to be spatially homogeneous as described in section 5.1. As per the discussion in section 5.2, we also assume the fields are diagonal in order to avoid the ambiguities that appear for some non-diagonal models. For the connection, this means there exists a combination of a homogeneous gauge transformation and a change of frame that brings it into the form $A^i = C^{(i)}(t)\,\delta^{(i)}_\alpha\,\theta^\alpha$ (no sum over i). Diagonal connection models are discussed in the context of loop quantum cosmology in [22]. In general, the requirement that the connection be diagonal is a further restriction on the variables, and we will need to confirm that this restriction commutes with the variational principle.

We can normalise a Cartan frame θ^α so that it satisfies $d\theta^\alpha = -\sqrt{k}\,\epsilon^\alpha{}_{\beta\gamma}\,\theta^\beta \wedge \theta^\gamma$ where k is a positive constant; this k corresponds to the spatial curvature parameter commonly used in cosmology. In the context of this section, Cartan frame indices $\alpha, \beta, \ldots$ are raised and lowered with the Kronecker deltas $\delta^{\alpha\beta}, \delta_{\alpha\beta}$. Note that the Lie algebra associated to the Cartan frame, $\mathfrak{su}(2)$, is isomorphic to the internal gauge algebra, $\mathfrak{so}(3)$, up to complexification. Then the gauge indices $i, j, \ldots$ and the frame indices $\alpha, \beta, \ldots$ are of the same type, and we can, and will, use them interchangeably. Our homogeneous and diagonal connection ansatz, and the resulting expression for the curvature, are

$$A^i = iU_\theta^{-1/3} C^i\,\theta^i\ , \quad F^1 = iU_\theta^{-1/3}\dot{C}^1\,dt \wedge \theta^1 - U_\theta^{-2/3}\left(i\kappa C^1 + C^2C^3\right)\theta^2 \wedge \theta^3 \quad \text{etc.}\,, \tag{5.27}$$

where $\kappa := 2U_\theta^{1/3}\sqrt{k}$ is a convenient shorthand, and where 'etc.' indicates that one can derive further valid expressions by cyclically permuting the indices $i, j, k = 1, 2, 3$. For example, one can derive the expression for F^2 using the sequence of index swaps $1 \to 2, 2 \to 3, 3 \to 1$. The 4-form valued matrix $F^i \wedge F^j$ is diagonal with non-vanishing entries

$$F^1 \wedge F^1 = 2iU_\theta^{-1}\dot{C}^1\left(i\kappa C^1 + C^2C^3\right)\varepsilon_\theta \wedge dt \quad \text{etc.} \tag{5.28}$$

In the first order action (2.98), $F^i \wedge F^j$ contracts fully with the inverse matrix M^{-1}_{ij}. Since all of the off-diagonal entries of M^{-1}_{ij} vanish in this contraction, and don't appear anywhere else in the action, we can take M^{ij} to be diagonal. We parametrise $M^{ij} = \ell_P^{-2}\,M^i\,\delta^{ij}$ where $M^i(t)$ are complex scalars and where the factor ℓ_P^{-2} sets the units of M^{ij} to $(\text{length})^{-2}$ which is required to make the action dimensionless. In our conventions, coordinates and the connection and curvature forms are all dimensionless. The 4-form μ is parametrised as $\mu = 2iU_\theta^{-1}\ell_P^4\,\rho\,\varepsilon_\theta \wedge dt$ where $\rho(t)$ transforms as in (5.10) under time reparametrisations. Note that powers of U_θ are chosen to make U_θ drop out of the dynamical formalism. Substituting these expressions for F^i, M^{ij}, μ into the first order action (2.98) yields the (intermediate) homogeneous and diagonal (HD) action given by

$$S'_{\text{HD}}\left[C, M, \rho\right] = \int dt\left[\left(\frac{i\kappa C^1 + C^2C^3}{M^1}\,\dot{C}^1 + \text{permutations}\right) - \rho\left(\ell_P^2\Lambda - \textstyle\sum_{i=1}^3 M^i\right)\right]\ , \tag{5.29}$$

This action is only dependent on the fiducial volume U_θ through the quantity $\kappa = 2U_\theta^{1/3}\sqrt{k}$. However, k can be chosen arbitrarily allowing us to fix κ to be whatever positive value we wish. In this way the action, and its dynamics, is (are) independent of the fiducial volume. We can bring the action into a more convenient form by introducing a new triple of complex scalars $P_i(t)$ to replace $M^i(t)$ using the variable redefinitions

$$M_1 = \frac{i\kappa C^1 + C^2C^3}{P_1} \quad \text{etc.} \tag{5.30}$$

In terms of these new variables, the reduced action (5.29) becomes

$$S_{\text{HD}}\left[C,P,\rho\right]=\int dt\left[P_i\,\dot{C}^i-\rho\left(\ell_P^2\Lambda-\left(\frac{C^2C^3}{P_1}+\text{permutations}\right)-i\kappa\sum_{i=1}^{3}\frac{C^i}{P_i}\right)\right]\,. \tag{5.31}$$

This action is in Hamiltonian form $S_{\text{HD}}=\int dt\left[\dot{C}^iP_i-\rho\,\mathcal{H}_{\text{HD}}\right]$ where $\mathcal{H}_{\text{HD}}$, given by

$$\mathcal{H}_{\text{HD}}=\ell_P^2\Lambda-\left(\frac{C^2C^3}{P_1}+\text{permutations}\right)-i\kappa\sum_{i=1}^{3}\frac{C^i}{P_i}\,, \tag{5.32}$$

is the homogeneous and diagonal restriction of the Hamiltonian constraint $\mathcal{H}$ given in (4.14) from the full canonical theory presented in chapter 4. The Euler-Lagrange equations are computed to be

$$\dot{C}^1=\rho\,\frac{\partial\mathcal{H}_{\text{HD}}}{\partial P_1}=\rho\left(\frac{i\kappa C^1+C^2C^3}{(P_1)^2}\right)\quad\text{etc.}\,, \tag{5.33a}$$

$$\dot{P}_1=-\rho\,\frac{\partial\mathcal{H}_{\text{HD}}}{\partial C^1}=\rho\left(\frac{i\kappa}{P_1}+\frac{C^3}{P_2}+\frac{C^2}{P_3}\right)\quad\text{etc.}\,, \tag{5.33b}$$

with variations of the action (5.31) w.r.t ρ yielding a constraint $\mathcal{H}_{\text{HD}}=0$. To check that these Euler-Lagrange equations are in agreement with the field equations (2.99) coming from the first order action (2.98), we substitute our homogeneous and diagonal expressions for A^i and the other fields into the field equations (2.99) and rearrange to find equations of motion for the diagonal variables C^i, P_i, ρ which should be consistent with (5.33). We begin by constructing

$$\Sigma_F^i=\left(M^{-1}\right)^{ij}F_j=U_\theta^{-2/3}\ell_P^2\left(\frac{iU_\theta^{1/3}P_1\dot{C}^1}{i\kappa C^1+C^2C^3}\,dt\wedge\theta^1-P_1\,\theta^2\wedge\theta^3\right)\quad\text{etc.} \tag{5.34}$$

Then the condition $\Sigma_F^i\wedge\Sigma_F^j=\delta^{ij}\mu$ reduces to three equations

$$\Sigma_F^1\wedge\Sigma_F^1=2iU_\theta^{-1}\ell_P^4\left(\frac{(P_1)^2\,\dot{C}^1}{i\kappa C^1+C^2C^3}\right)\varepsilon_\theta\wedge dt=2iU_\theta^{-1}\ell_P^4\,\rho\,\varepsilon_\theta\wedge dt\quad\text{etc.}\,, \tag{5.35}$$

which yield three first order equations of motion

$$\dot{C}^1=\rho\,\frac{i\kappa C^1+C^2C^3}{(P_1)^2}\quad\text{etc.} \tag{5.36}$$

Hence we recover (5.33a). Furthermore, substituting (5.36) into (5.34) yields simpler expressions for Σ_F^i given by

$$\Sigma_F^1=U_\theta^{-2/3}\ell_P^2\left(\frac{iU_\theta^{1/3}\rho}{P_1}\,dt\wedge\theta^1-P_1\,\theta^2\wedge\theta^3\right)\quad\text{etc.} \tag{5.37}$$

Then the conditions $D_A\Sigma_F^i = 0$ expand as

$$\begin{aligned} D_A\Sigma_F^1 &= d\Sigma_F^1 + A^2 \wedge \Sigma_F^3 - A^3 \wedge \Sigma_F^2 \\ &= U_\theta^{-2/3}\ell_P^2\,\rho\left(\frac{i\kappa}{P_1} + \frac{C^3}{P_2} + \frac{C^2}{P_3} - \frac{\dot{P}_1}{\rho}\right) dt \wedge \theta^2 \wedge \theta^3 \quad \text{etc.}\ , \end{aligned} \tag{5.38}$$

which yield equations

$$\dot{P}_1 = \rho\left(\frac{i\kappa}{P_1} + \frac{C^3}{P_2} + \frac{C^2}{P_3}\right) \quad \text{etc.} \tag{5.39}$$

Hence we recover (5.33b). Finally, the constraint $\operatorname{tr} M = \Lambda$ becomes

$$\ell_P^2\Lambda - \left(\frac{C^2C^3}{P_1} + \text{permutations}\right) - i\kappa\sum_{i=1}^{3}\frac{C^i}{P_i} = 0\ , \tag{5.40}$$

which is the constraint $\mathcal{H}_{\rm HD} = 0$ arising from variations of the action (5.31) w.r.t ρ. Therefore the restriction to spatially homogeneous and diagonal variables commutes with the variatonal principle, and we may use the minisuperspace action (5.31) as a starting point to investigate the dynamics of homogeneous and diagonal models. From (5.31) we can immediately construct a (holomorphic) Hamiltonian system. We have a conjugate pair C^i, P_i with Poisson bracket $\{C^i, P_j\} = \delta^i_j$ which naturally extends to holomorphic functions of the complex variables C^i and P_i via

$$\{f, g\} = \frac{\partial f}{\partial C^i}\frac{\partial g}{\partial P_i} - \frac{\partial g}{\partial C^i}\frac{\partial f}{\partial P_i}\ , \tag{5.41}$$

where $\partial/\partial C^i$ and $\partial/\partial P_i$ denote complex partial derivatives (*Wirtinger derivatives*). In this context, a complex valued function of many complex inputs is called holomorphic if it has no explicit dependence on the conjugate variables, so that $\partial f/\partial \overline{C}^i = 0$ and $\partial f/\partial \overline{P}_i = 0$. The system has a single constraint $\mathcal{H}_{\rm HD} \approx 0$ which generates gauge transformations via $\delta C^i = \nu\{C^i, \mathcal{H}_{\rm HD}\}$ and $\delta P_i = \nu\{P_i, \mathcal{H}_{\rm HD}\}$ where $\nu(t)$ is the non-dynamical gauge parameter. The time evolution equations for this system are generated from the Hamiltonian defined $H_{\rm HD} = \rho\mathcal{H}_{\rm HD}$ via

$$\dot{C}^i = \{C^i, H_{\rm HD}\} \quad , \quad \dot{P}_i = \{P_i, H_{\rm HD}\}\ , \tag{5.42}$$

which recover (5.33) as expected. One sees that this time evolution is just the gauge transformation generated from $\mathcal{H}_{\rm HD}$ with gauge parameter $\rho(t)$. In parallel with the full theory, the constraint $\mathcal{H}_{\rm HD}$ generates time reparametrisation transformations on the variables, and time evolution is pure gauge.

One can construct a metric tensor by substituting the expression for Σ^i given in (5.37) into the Lorentzian Urbantke formula (2.86) to get

$$g_{\mathrm{HD}} = -\frac{\ell_P^2\,\rho^2}{P_1P_2P_3}\,dt\otimes dt + \ell_P^2 U_\theta^{-2/3} P_1P_2P_3 \sum_{i=1}^{3}\frac{1}{(P_i)^2}\,\theta^i\otimes\theta^i\ . \tag{5.43}$$

Additionally, one can construct a tetrad e^I via

$$e^0 = \frac{\ell_P\,\rho}{\sqrt{P_1P_2P_3}}\,dt \quad , \quad e^i = \frac{\ell_P\,\sqrt{P_1P_2P_3}}{U_\theta^{1/3}P_i}\,\theta^i\ , \tag{5.44}$$

such that Σ^i, as given in (5.37), decomposes as its self-dual part. Note, one must prescribe a procedure for evaluating the square root term $\sqrt{P_1P_2P_3}$ on a case by case basis. One can derive the metric (5.43) from the more general homogeneous metric tensor given in (5.25) using the substitutions

$$A^i_\alpha = iU_\theta^{-1/3}C^{(i)}\delta^{(i)}_\alpha \quad , \quad E^\alpha_i = -U_\theta^{-2/3}\ell_P^2\,P_{(i)}\,\delta^\alpha_{(i)} \quad , \quad n = iU_\theta^{-1}\ell_P^4\,\rho\ , \tag{5.45}$$

as well as the gauge fixing $V^\alpha = 0$. In terms of our homogeneous and diagonal variables, the reality conditions (2.82) become

$$\operatorname{Im}\rho = 0 \quad , \quad \operatorname{Im}\left[(P_i)^2\right] = 0\ , \tag{5.46}$$

where the latter condition applies for each $i = 1, 2, 3$. Note, deriving the second equation in the above from the wedge type condition as formulated in (2.100) requires us to make use of the equation of motion for $\dot{C}^i$ given in (5.33a). We see that ρ, which is the lapse surrogate in this model, must take only real values, and each P_i must be either real or purely imaginary valued. In total, these reality conditions split into $2^3 = 8$ solution branches which are characterised by the reality of each P_i. Many of these branches are equivalent under index relabelling, so there are really only 4 distinct branches. Of these, two branches yield a real valued $P_1P_2P_3$, and hence a real valued metric (5.43). These are the branches corresponding to all real P_i, and 1 real and 2 imaginary P_i. The remaining two branches yield an imaginary $P_1P_2P_3$ and an imaginary metric (5.43). These are the all imaginary P_i, and the 2 real and 1 imaginary P_i branches. In the branches which produce a real valued metric, the overall signature, $(-+++)$ vs $(+---)$, is controlled by the sign of $P_1P_2P_3$. So we have access to real Lorentzian solutions with both mostly positive and mostly negative signature. This is familiar from our previous discussion in the general case in section 4.1.1. Also of note, the various branches of the reality conditions place restrictions on the value of the cosmological constant Λ (or vice versa). In the branches corresponding to real $P_1P_2P_3$, and hence a real metric, the Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$ only have non-trivial solutions when Λ is real valued. Equivalently, the branches corresponding to imaginary $P_1P_2P_3$ are only consistent with an imaginary Λ. Then fixing Λ to be real, as is reasonable in physical applications, already allows us to discount half of the solution branches. Note, this inconsistency of certain branches of the reality conditions with certain values of Λ is independent of the Urbantke metric, which is not fundamental in these constructions, and can be observed at the level of the Hamiltonian constraint (5.32).

In the all real and all imaginary P_i branches, we recover a real Lorentzian metric, with an overall factor of $\pm i$ in the latter case, whose timelike direction is aligned with the coordinate t, which is the typical setup for cosmological applications. However in the mixed reality branches, the form of the metric can be quite atypical. For example, consider the branch characterised by $P_1 = ip_1, P_2 = ip_2, P_3 = p_3$ where p_i are real valued (1 real 2 imaginary P_i). The metric (5.43) becomes real Lorentzian with its timelike direction aligned with θ^3,

$$\tilde{g}_{\mathrm{HD}} = \frac{\ell_P^2 \rho}{p_1p_2p_3} dt \otimes dt + \ell_P^2 U_\theta^{-2/3} p_1p_2p_3 \left(\frac{1}{(p_1)^2}\theta^1 \otimes \theta^1 + \frac{1}{(p_2)^2}\theta^2 \otimes \theta^2 - \frac{1}{(p_3)^2}\theta^3 \otimes \theta^3 \right) . \quad (5.47)$$

That is, the surfaces of homogeneity are timelike w.r.t this metric, and the direction of dynamical evolution is spacelike.

5.3.1 Consistency of the reality conditions

As previously mentioned in section 4.1.1, the reality conditions are, in general, not preserved under time evolution; we need to impose further conditions on the initial data coming from the time derivatives of the reality conditions. Computing these further conditions in the general case is highly non-trivial. However, these computations are significantly more tractable for the homogeneous and diagonal models being studied in this section. To carry out this consistency analysis, one could work in terms of the complex variables C^i, P_i and their complex conjugates $\overline{C}^i, \overline{P}_i$. This is not the approach we will take here. Instead, we proceed by first transforming our holomorphic Hamiltonian system into an equivalent system over a phase space of real variables. Then we examine the consistency of the reality conditions within this real Hamiltonian framework. The virtues of transforming our complex system into a real one will become more evident later on in this chapter, when we investigate the consistency of the reality conditions (for a second time) from an alternate perspective where we treat them as dynamical constraints à la Dirac. For now, we continue our investigation by treating the reality conditions, and their secondary conditions, solely as conditions on the initial data which need to be preserved under time evolution. We begin by writing the complex variables in terms of real variables such that

$$C^i = b^i + ic^i \quad , \quad P_i = p_i - iq_i \quad , \quad \rho = u - iv \; . \tag{5.48}$$

Substituting these expressions into the homogeneous and diagonal action (5.31) allows us to decompose $S_{\rm HD} = S_{\rm Re} + iS_{\rm Im}$ where $S_{\rm Re}$ and $S_{\rm Im}$ are a pair of real actions over the real variables b^i, c^i, p_i, q_i, u, v which read

$$S_{\rm Re} = \int dt \left[\dot{b}^i p_i + \dot{c}^i q_i - u\mathcal{H}_{\rm Re} - v\mathcal{H}_{\rm Im}\right] \; , \tag{5.49a}$$

$$S_{\rm Im} = \int dt \left[-\dot{b}^i q_i + \dot{c}^i p_i - u\mathcal{H}_{\rm Im} + v\mathcal{H}_{\rm Re}\right] \; , \tag{5.49b}$$

where $\mathcal{H}_{\rm Re}$ and $\mathcal{H}_{\rm Im}$ are the real and imaginary parts of the constraint $\mathcal{H}_{\rm HD}$ which expand in terms of the real variables as

$$\mathcal{H}_{\rm Re} = \ell_P^2 \,{\rm Re}(\Lambda) - \sum_{i=1}^{3} \frac{p_i S^i - q_i T^i}{(p_i)^2 + (q_i)^2} \; , \tag{5.50a}$$

$$\mathcal{H}_{\rm Im} = \ell_P^2 \,{\rm Im}(\Lambda) - \sum_{i=1}^{3} \frac{p_i T^i + q_i S^i}{(p_i)^2 + (q_i)^2} \; , \tag{5.50b}$$

where $S^i(c,b)$ and $T^i(c,b)$ are defined

$$S^1 = -\kappa\, c^1 - c^2c^3 + b^2b^3 \quad \text{etc.} \quad , \quad T^1 = \kappa\, b^1 + b^2c^3 + b^3c^2 \quad \text{etc.} \tag{5.51}$$

Then we have a pair of actions $S_{\rm Re}$ and $S_{\rm Im}$ from which we can generate dynamical theories over the real variables. We find that both actions generate precisely the same dynamics on the real variables. This is a consequence of a more general result for complex valued actions with holomorphic Lagrangians:

Theorem 5.3.1. *Let $S[Q] = \int dt\; L(Q,\dot{Q})$ be an action over complex variables Q^i with a holomorphic Lagrangian $L(Q,\dot{Q})$. Let $Q^i = x^i \pm iy^i$ for real variables x^i and y^i, and let the action decompose as $S[Q] = S_{\rm Re}[x,y] + iS_{\rm Im}[x,y]$ so that $S_{\rm Re}$ and $S_{\rm Im}$ are real valued actions over real variables. Then the Euler-Lagrange equations coming from $S_{\rm Re}$ are equivalent to the Euler-Lagrange equations from $S_{\rm Im}$.*

The proof is based on section 5 of [73] and can be found in appendix 5.A.2. Then both actions $S_{\rm Re}$ and $S_{\rm Im}$ are real actions of real variables which each reproduce the full dynamics of the complex action (5.31). Furthermore, both actions are in Hamiltonian form allowing us to immediately construct their associated Hamiltonian systems.

Both systems have a phase space consisting of the variables b^i, c^i, p_i, q_i. The system coming from the real part of the action, which we will call the *real sector*, has Poisson bracket satisfying $\{b^i, p_j\}_{\text{Re}} = \{c^i, q_j\}_{\text{Re}} = \delta^i_j$, and vanishing on all unrelated combinations. The system coming from the imaginary part of the action, which we call the *imaginary sector*, has Poisson bracket satisfying $-\{b^i, q_j\}_{\text{Im}} = \{c^i, p_j\}_{\text{Im}} = \delta^i_j$, and vanishing on unrelated combinations. Both sectors have a pair of constraints $\mathcal{H}_{\text{Re}} \approx 0$ and $\mathcal{H}_{\text{Im}} \approx 0$ satisfying $\{\mathcal{H}_{\text{Re}}, \mathcal{H}_{\text{Im}}\}_{\text{Re}} = \{\mathcal{H}_{\text{Re}}, \mathcal{H}_{\text{Im}}\}_{\text{Im}} = 0$. The time evolution equations in the real and imaginary sectors are generated by the Hamiltonians

$$H_{\text{Re}} = u\mathcal{H}_{\text{Re}} + v\mathcal{H}_{\text{Im}} \ , \tag{5.52a}$$

$$H_{\text{Im}} = u\mathcal{H}_{\text{Im}} - v\mathcal{H}_{\text{Re}} \tag{5.52b}$$

respectively via $\dot{\mathcal{O}} = \{\mathcal{O}, H_\bullet\}_\bullet$, where '$\bullet$' is a place holder for either 'Re' or 'Im.' By theorem 5.3.1 both sets of time evolution equations are equivalent.

With these constructions at our disposal, we are ready to investigate the consistency of the reality conditions for this highly symmetric model. As an illustrative example, we investigate their consistency in the 'all real P_i' branch – the most interesting branch for cosmological applications – where we have the constraints $v = 0$ and $q_i = 0$ (for all i) on the real variables. The first of these, $v = 0$, is trivially preserved under time evolution since v is a Lagrange multiplier field which can be chosen arbitrarily. The second of these, $q_i = 0$, does produce further constraints via

$$\dot{q}_1 = u\{q_1, \mathcal{H}_{\text{Re}}\}_{\text{Re}} = -u\left(\frac{\kappa}{p_1} + \frac{c^3}{p_2} + \frac{c^2}{p_3}\right) = 0 \quad \text{etc.} \tag{5.53}$$

While fixing $u = 0$ is a valid solution of the above, this would lead to trivial dynamics: $\dot{\mathcal{O}} = 0$ for functions $\mathcal{O}$ of the variables. Instead, we set the term inside the brackets to zero, yielding further conditions on the variables given by

$$c^i - \Pi^i(p) = 0 \quad : \quad \Pi^1(p) := -\frac{\kappa p_2 p_3}{2}\left(-\frac{1}{(p_1)^2} + \frac{1}{(p_2)^2} + \frac{1}{(p_3)^2}\right) \quad \text{etc.} \tag{5.54}$$

Then one sees that the constraints on the initial data given by $v = 0$ and $q_i = 0$ and $c^i = \Pi^i(p)$ are all together preserved under time evolution. For a geometric interpretation of these secondary conditions, recall that the connection A^i is the self-dual part of a Lorentzian spin connection ω^{IJ}. From (2.65), we get the decomposition

$$A^i = iK^i - \Gamma^i \quad : \quad \Gamma^i = \tfrac{1}{2}\epsilon^i{}_{jk}\,\omega^{jk} \ , \quad K^i = \omega^{0i} \ , \tag{5.55}$$

where Γ^i is the Levi-Civita connection on the constant t hypersurfaces, corresponding to the physical triad $e^i = U_\theta^{-1/3}\ell_P \dfrac{\sqrt{|p_1p_2p_3|}}{p_i}\theta^i$, and where K^i encodes the extrinsic curvature of the constant t surfaces w.r.t the spacetime metric (5.43). One computes

$$K^i = U_\theta^{-1/3} b^i \, \theta^i \quad , \quad \Gamma^i = U_\theta^{-1/3}\Pi^i(p)\,\theta^i \; , \tag{5.56}$$

revealing that the quantities $\Pi^i(p)$ we derived via the consistency analysis in (5.54) encode the Levi-Civita connection of the induced metric on the constant t surfaces, and the variables b^i encode the extrinsic curvature of the constant t surfaces.

In the 1 real and 2 imaginary P_i solution branch mentioned previously, which is now characterised by constraints $v = 0$ and $p_1 = p_2 = q_3 = 0$ on the real variables, we find the following secondary conditions on the initial data

$$b^1 + \frac{\kappa\, q_2 p_3}{2}\left(\frac{1}{(q_1)^2} - \frac{1}{(q_2)^2} + \frac{1}{(p_3)^2}\right) = 0 \; , \tag{5.57a}$$

$$b^2 + \frac{\kappa\, p_3 q_1}{2}\left(-\frac{1}{(q_1)^2} + \frac{1}{(q_2)^2} + \frac{1}{(p_3)^2}\right) = 0 \; , \tag{5.57b}$$

$$c^3 + \frac{\kappa q_1 q_2}{2}\left(\frac{1}{(q_1)^2} + \frac{1}{(q_2)^2} + \frac{1}{(p_3)^2}\right) = 0 \; . \tag{5.57c}$$

Reality conditions as dynamical constraints

So far in this thesis, we have taken the approach of treating the reality conditions (2.82) exclusively as conditions on the initial data, which in general are not preserved under time evolution and need to be supplemented with further conditions coming from their time derivatives. In this approach, the reality conditions are not 'dynamical constraints'. That is, they don't come from variations of the action, and they don't generate transformations on the variables via the Poisson structure. One can take the opposite approach by implementing the reality conditions as constraints at the level of the action. In the general case, doing so drastically increases the complexity of the constraint algebra, making canonical analysis an intractable problem. However, in highly symmetric models such as the homogeneous and diagonal models we have studied in this section, this approach is far more accessible. For example, if one chooses to work in the all real P_i solution branch, one can extend the diagonal Bianchi IX action (5.31) with a constraint term to get

$$S[C,P,\rho,\beta] = \int dt \left[\dot{C}^i P_i - \rho\,\mathcal{H}_{\text{HD}} - \sum_{i=1}^{3}\beta^i\,\text{Im}(P_i)\right] \; . \tag{5.58}$$

Then the constraints $\text{Im}(P_i) = 0$ arise dynamically from variations w.r.t the complex Lagrange multiplier fields $\beta^i(t)$. We need not implement the constraint $\text{Im}(\rho) = 0$ at the level of the action since, as previously noted, ρ has no equation of motion coming from the variational principle and can therefore be chosen arbitrarily. A question that might arise is why we don't implement the more general form of the wedge reality condition given by $\text{Im}\left[(P_i)^2\right] = 0$? In this case, one wouldn't have to fix a particular solution branch from the beginning. To see why this is problematic, we can rewrite this condition in terms of the real variables as $p_i q_i = 0$ (no sum over i). The phase space hypersurface defined by these conditions is only piece-wise smooth; there are degenerate points at $q_i = p_i = 0$. On this surface, the dynamical theory is only well defined away from these degenerate points, on the 'pieces' corresponding to the individual branches. Hence we are forced into choosing a particular branch regardless. We proceed with our investigation by decomposing the action (5.58) into real and imaginary parts $S = S'_{\text{Re}} + iS'_{\text{Im}}$ such that

$$S'_{\text{Re}} = \int dt \left[\dot{b}^i p_i - \dot{c}^i q_i - u\mathcal{H}_{\text{Re}} - v\mathcal{H}_{\text{Im}} - U^i q_i\right] = S_{\text{Re}} - \int dt\ U^i q_i\ , \tag{5.59a}$$

$$S'_{\text{Im}} = \int dt \left[-\dot{b}^i q_i - \dot{c}^i p_i - u\mathcal{H}_{\text{Im}} + v\mathcal{H}_{\text{Re}} - V^i q_i\right] = S_{\text{Im}} - \int dt\ V^i q_i\ , \tag{5.59b}$$

where we decompose β^i into real and imaginary parts via $\beta^i = U^i + iV^i$, and where $S_{\text{Re}}, S_{\text{Im}}$ are given in (5.49) and $\mathcal{H}_{\text{Re}}, \mathcal{H}_{\text{Im}}$ are given in (5.50). The Hamiltonian theory which is generated from S'_{Re} is the same as the one generated from S_{Re} except with further primary constraints on the variables given by $q_i = 0$; the same is true for S'_{Im} w.r.t S_{Im}. The time evolution equations for the real and imaginary sectors, generated from S'_{Re} and S'_{Im} respectively, are given by

$$\text{Real sector} \quad : \quad \dot{\mathcal{O}} = \{\mathcal{O}, u\mathcal{H}_{\text{Re}} + U^i q_i\}_{\text{Re}}\ , \tag{5.60a}$$

$$\text{Imaginary sector} \quad : \quad \dot{\mathcal{O}} = \{\mathcal{O}, u\mathcal{H}_{\text{Im}} + V^i q_i\}_{\text{Im}}\ , \tag{5.60b}$$

where we implement the trace type condition by setting $v = 0$. The action in (5.58) is no longer holomorphic, the Lagrangian has explicit dependence on the complex conjugate variables. Hence theorem 5.3.1 doesn't apply here, and we cannot guarantee that these two sets of evolution equations are equivalent. In general, they are not. One must choose one sector of the theory to work in from the start. Furthermore, if one wishes to get the 'correct' constrained dynamics in the end, one must choose the correct sector of the theory. The rule is as follows: one must work in the real sector for branches of the reality conditions yielding a real valued $P_1P_2P_3$, and one must work in the imaginary sector for branches yielding imaginary $P_1P_2P_3$. The precise formal reasoning for this rule is still unclear to the author; it was derived via exhaustive trial and error. This all simplifies somewhat if we set the cosmological constant to be real valued from the start, which is a reasonable restriction for cosmological applications. Then the only consistent branches of the reality conditions are the ones yielding real $P_1P_2P_3$, which require us to work exclusively in the real sector of the theory.

As an illustrative example, we work through the constraint analysis in the all real P_i branch, which we recall is the most physically interesting branch as it produces a real Lorentzian metric via (5.43) whose surfaces of homogeneity are timelike. Working in the real sector of the theory, one computes the following time derivatives for the constraints:

$$\dot{q}_1 = -u\left(\frac{\kappa}{p_1} + \frac{c^3}{p_2} + \frac{c^2}{p_3}\right) \quad \text{etc. }, \tag{5.61a}$$

$$\dot{\mathcal{H}}_{\text{Re}} = U^1\left(\frac{\kappa}{p_1} + \frac{c^3}{p_2} + \frac{c^2}{p_3}\right) + \text{permutations }, \tag{5.61b}$$

$$\dot{\mathcal{H}}_{\text{Im}} = -b^1\left(\frac{U^3}{p_2} + \frac{U^2}{p_3}\right) + \text{permutations }. \tag{5.61c}$$

We have already seen that the consistency conditions $\dot{q}_i \approx 0$ yield further conditions on the variables given by $c^i = \Pi^i(p)$ where $\Pi^i(p)$ is given in (5.54). When we add these as secondary constraints on the variables, the consistency condition $\dot{\mathcal{H}}_{\text{Re}} \approx 0$ is identically satisfied. Furthermore, the consistency condition $\dot{\mathcal{H}}_{\text{Im}} \approx 0$ simply constrains (one of) the Lagrange multipliers U^i, and doesn't generate any further conditions on the variables. Following Dirac's method, [36, 52], we check the secondary constraints for consistency also. One computes

$$\frac{d}{dt}\left(c^i - \Pi^i\right) \approx U^i \ , \tag{5.62}$$

hence there are no further constraints on the variables, but we do have the further conditions $U^i = 0$ on Lagrange multipliers. It is an exercise in algebra to see that the constraint $\mathcal{H}_{\text{Im}}$ vanishes identically on the phase space hypersurface defined by $q_i = 0$ and $c^i = \Pi^i(p)$. Then $\mathcal{H}_{\text{Im}}$ is redundant and can be removed from the constraint set without harm, leaving us with constraints $\mathcal{H}_{\text{Re}}, q_i, c^i - \Pi^i$. We see that $\mathcal{H}_{\text{Re}}$ is first class, and $\chi_\alpha = q_i, c^i - \Pi^i$ are second class, where we briefly suspend the use of indices $\alpha, \beta, \ldots$ as Cartan-frame indices so that we can use them to index over second class constraints. Moreover, the 6×6 matrix defined $\{\chi_\alpha, \chi_\beta\}_{\text{Re}}$ is non-degenerate on the constraint surface, so there are no further first class constraints to be found from functional linear combinations of χ_α. One can construct a Dirac bracket

$$\{f,g\}_{\rm DB} := \{f,g\}_{\rm Re} - \{f,\chi_\alpha\}_{\rm Re}\, M^{\alpha\beta}\, \{\chi_\beta, g\}_{\rm Re}\ , \tag{5.63}$$

where $M^{\alpha\beta}$ denotes the inverse matrix satisfying $M^{\alpha\gamma}\{\chi_\gamma,\chi_\beta\}_{\rm Re} = \delta^\alpha_\beta$. This Dirac bracket is fully characterised by the matrix

$$\left\{X^n, X^{n'}\right\}_{\rm DB} = \begin{pmatrix} 0 & A & \mathbb{I} & 0 \\ -A^{\rm T} & 0 & 0 & 0 \\ -\mathbb{I} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\ , \tag{5.64}$$

where $X^n = (b^i, c^i, p_i, q_i)$ denotes the variables in general. This matrix is written in block form such that each entry corresponds to a 3×3 matrix. Here, 0 denotes the empty matrix, $\mathbb{I}$ denotes the identity matrix, and the matrix A^{ij} is given by

$$A^{ij} = \frac{\partial \Pi^i}{\partial p_j}\ . \tag{5.65}$$

One can confirm that A^{ij} is symmetric by direct calculation. We recall from section 2.6 that the Dirac bracket always vanishes when one of the entries is a second class constraint, and (weakly) recovers the original bracket, $\{\cdot,\cdot\}_{\rm Re}$ in this case, when one of the entries is a first class constraint. Then replacing the original bracket with the Dirac bracket in our Hamiltonian theory preserves the dynamics whilst causing the constraint algebra to become closed.

Reduction of phase space

The second class constraints reveal that the conjugate pair c^i, q_i is redundant. Our goal now is to remove these superfluous variables from the theory, which we will achieve through the following constructions. First, given any arbitrary function of the variables $f(b,c,p,q)$, we define its restriction to the *second class surface*, where χ_α are vanishing, such that $f|_{\chi_\alpha=0}(b,p) := f(b,\Pi(p),p,0)$. Then the Dirac bracket satisfies

$$(\{f,g\}_{\rm DB})|_{\chi_\alpha=0} = \{f|_{\chi_\alpha=0}\,,\, g|_{\chi_\alpha=0}\}_{\rm DB}\ , \tag{5.66}$$

for any pair of functions f, g of the phase space variables. Informally, we might say that the Dirac bracket 'commutes' with the second class constraints so that imposing the second class constraints prior to evaluating the Dirac bracket yields the same answer as imposing them after evaluating the Dirac bracket. The proof and further discussion can be found in [52]. Next, from the matrix in (5.64) we see that $\{b^i, p_j\}_{\rm DB} = \{b^i, p_j\}_{\rm Re} = \delta^i_j$ and consequently

$$\{f|_{\chi_\alpha=0}\,,\, g|_{\chi_\alpha=0}\}_{\rm DB} = \{f|_{\chi_\alpha=0}\,,\, g|_{\chi_\alpha=0}\}_{\rm Re}\ . \tag{5.67}$$

That is, the Dirac bracket $\{\cdot,\cdot\}_{\rm DB}$ agrees with the real bracket $\{\cdot,\cdot\}_{\rm Re}$ when restricted to the second class surface, which can also be confirmed by direct computation via

$$\begin{aligned}\{f|_{\chi_\alpha=0}, g|_{\chi_\alpha=0}\}_{\rm Re} &\overset{!}{=} \left[\frac{\partial f}{\partial b^i}\left(\frac{\partial g}{\partial c^j}\frac{\partial \Pi^j}{\partial p_i} + \frac{\partial g}{\partial p_i}\right) - \frac{\partial g}{\partial b^i}\left(\frac{\partial f}{\partial c^j}\frac{\partial \Pi^j}{\partial p_i} + \frac{\partial f}{\partial p_i}\right)\right]_{\chi_\alpha=0} \\ &= \left[\frac{\partial f}{\partial b^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial b^i}\frac{\partial f}{\partial p_i} + \frac{\partial \Pi^j}{\partial p_i}\left(\frac{\partial f}{\partial b^i}\frac{\partial g}{\partial c^j} - \frac{\partial g}{\partial b^i}\frac{\partial f}{\partial c^j}\right)\right]_{\chi_\alpha=0} \\ &= (\{f,g\}_{\rm DB})|_{\chi_\alpha=0}\ .\end{aligned} \tag{5.68}$$

With this result, we may eliminate the redundant variables c^i and q_i to get a theory over a lower dimensional phase space with dynamical variables b^i, p_i only. In practice, we simply restrict all of our structure to the second class surface. The second class constraints χ_α vanish trivially due to this restriction, and the real part of the Hamiltonian constraint reduces to $(\mathcal{H}_{\rm Re})|_{\chi_\alpha=0} = \mathcal{H}_{(-)} + \kappa^2\mathcal{H}_{\rm BG}$, where

$$\mathcal{H}_{(\pm)} = \ell_P^2\Lambda \pm \left(\frac{b^2b^3}{p_1} + \frac{b^3b^1}{p_2} + \frac{b^1b^2}{p_3}\right)\ , \quad \mathcal{H}_{\rm BG} = \frac{1}{2}\sum_{i=1}^{3}\left(\frac{p_1p_2p_3}{2(p_i)^4} - \frac{(p_i)^2}{p_1p_2p_3}\right)\ , \tag{5.69}$$

We denote this single remaining constraint by $\mathcal{H}_{\rm Lor} = \mathcal{H}_{(-)} + \kappa^2\mathcal{H}_{\rm BG}$, where the subscript 'Lor' is short for 'Lorentzian.' Then the Hamiltonian given by $u\mathcal{H}_{\rm Lor}$ generates the time evolution equations in the usual way via $\dot{\mathcal{O}} = \{\mathcal{O}, u\mathcal{H}_{\rm Lor}\}_{\rm Re}$.

The decomposition of $\mathcal{H}_{\rm Lor}$ illustrates the effect of background curvature on the dynamical evolution of the spatial geometry. The 'background' term $\mathcal{H}_{\rm BG}$ generates correction (for lack of a better word) terms in the dynamical equations which arise due to the curvature of $\mathcal{S}$, whose strength is controlled by the magnitude of the curvature parameter κ (equivalently k). Then as $\kappa \to 0$, these correction terms vanish and we arrive at a theory in which the surfaces of homogeneity are (topologically) flat. We shall examine this limit in greater detail in section 5.4 on diagonal Bianchi I models.

One can check that (5.69) represents the Hamiltonian constraint of the Lorentzian Bianchi IX model in general relativity with cosmological constant $\pm\Lambda$ (with the sign depending on the sign of the product $p_1p_2p_3$), for example by starting from Ashtekar variables (see Appendix

5.A.3) or from the metric representation.

This reduced Hamiltonian theory is generated from the canonical action given by

$$S_{\text{Reduced}}\left[b,p,\rho\right] = \int dt \left[\dot{b}^i p_i - \rho\left(\mathcal{H}_{(-)} + \kappa^2\,\mathcal{H}_{\text{BG}}\right)\right] \,, \tag{5.70}$$

which is obtained from the first order action (2.98) via the substitutions

$$A^i = U_\theta^{-1/3}\left(-\Pi^i(p) + ib^i\right)\theta^i \,, \tag{5.71a}$$

$$M^{11} = \frac{\kappa\left(-\Pi^1(p) + ib^1\right) - \left(-\Pi^2(p) + ib^2\right)\left(-\Pi^3(p) + ib^3\right)}{\ell_P^2\, p_1} \quad \text{etc.} \,, \tag{5.71b}$$

$$\mu = 2iU_\theta^{-1}\ell_P^4\,\rho\,\varepsilon_\theta \wedge dt \,, \tag{5.71c}$$

where we set the off-diagonal elements of M^{ij} to zero, and where ρ is now taken to be real valued (we now use ρ in place of its real part $u = \text{Re}(\rho)$ for future convenience).

A similar analysis can be repeated for the branches characterised by 1 real and 2 imaginary P_i, which are the only other branches consistent with a real valued Λ. These are all related via cyclic index permutations, so we only need to perform this constraint analysis once, for example for the branch defined by primary constraints $p_1 = 0$, $p_2 = 0$, $q_3 = 0$. These primary constraints lead to the secondary constraints given in (5.57). Moreover, the Hamiltonian constraint on the reduced phase space, consisting of canonically conjugate variables c^1, c^2, b^3 and q_1, q_2, p_3, is given by

$$\begin{aligned}\widetilde{\mathcal{H}}_{\text{Lor}} = \ell_P^2\Lambda + \frac{c^2 b^3}{q_1} + \frac{b^3 c^1}{q_2} + \frac{c^1 c^2}{p_3} \\ + \frac{\kappa^2}{2}\left(-\frac{q_1 q_2}{2(p_3)^3} - \frac{q_2 p_3}{2(q_1)^3} - \frac{q_1 p_3}{2(q_2)^3} - \frac{q_1}{q_2 p_3} - \frac{q_2}{q_1 p_3} + \frac{p_3}{q_1 q_2}\right) \,.\end{aligned} \tag{5.72}$$

This case then also leads to a well-defined Hamiltonian system, which can be studied in its own right. The corresponding metric is the one given in (5.47), albeit with some variable relabelling $p_i \leftrightarrow q_i$, which gives timelike surfaces of homogeneity and a spacelike direction dt of dynamical evolution. Hence this is not a typical cosmological Bianchi model. From the perspective of the Euclidean theory we will study later, it can be seen as a 'Wick rotation' in a spatial, rather than timelike direction.

5.4 Diagonal Bianchi I model

A somewhat simpler homogeneous model is obtained by assuming that the homogeneous submanifolds $\mathcal{S}$ in the decomposition $\mathcal{M} = \mathbb{R} \times \mathcal{S}$ are diffeomorphic to flat $\mathbb{R}^3$. Then the group of isometries acting on each of these leaves is an Abelian group of translations. This is the Bianchi I model, which was studied for a generalised class of pure connection theories of gravity in [55]. The Cartan frame $\underline{\theta}^i$ for a model with Bianchi I symmetry satisfies $d\underline{\theta}^i = 0$. One could choose $\underline{\theta}^i = dx^i$ in Cartesian coordinates x^i on $\mathbb{R}^3$ for example. We construct a fiducial volume form by $\varepsilon_{\underline{\theta}} = \underline{\theta}^1 \wedge \underline{\theta}^2 \wedge \underline{\theta}^3$. In contrast to the Bianchi IX case, integrating this volume form over all of $\mathcal{S} = \mathbb{R}^3$ will produce an infinite total volume. Hence, as discussed in section 5.2, we restrict to a compact fiducial cell $\mathcal{U}_0 \subset \mathcal{S}$ so that the fiducial volume defined $U_{\underline{\theta}} = \int_{\mathcal{U}_0} \varepsilon_{\underline{\theta}}$ is finite. We are still assuming a diagonal connection $A^i \propto C^i(t)\,\underline{\theta}^i$ (no sum over i), such that

$$A^i = iU_{\underline{\theta}}^{-1/3} C^i\,\underline{\theta}^i \quad , \quad F^1 = iU_{\underline{\theta}}^{-1/3} \dot{C}^1\,dt \wedge \underline{\theta}^1 - U_{\underline{\theta}}^{-2/3} C^2 C^3\,\underline{\theta}^2 \wedge \underline{\theta}^3 \quad \text{etc.} \tag{5.73}$$

We again take the matrix M^{ij} to be diagonal with non-vanishing entries given by

$$M^{11} = \frac{C^2 C^3}{\ell_P^2 P_1} \quad \text{etc.}\,, \tag{5.74}$$

and we parametrise the 4-form as $\mu = 2i\,U_{\underline{\theta}}^{-1} \ell_P^4\,\rho\,\varepsilon_{\underline{\theta}} \wedge dt$ where ρ is the same complex scalar from the diagonal Bianchi IX model (5.31), transforming under time reparametrisations according to $\rho(t)\,dt = \rho'(t')\,dt'$ as in (5.10). Inserting all these fields into the first order action (2.98) yields

$$S^0_{\text{HD}}\,[C, P, \rho] = \int dt \left[P_i\,\dot{C}^i - \rho \left(\ell_P^2 \Lambda - \frac{C^2 C^3}{P_1} - \frac{C^1 C^3}{P_2} - \frac{C^1 C^2}{P_3} \right) \right] \,, \tag{5.75}$$

which is precisely the action (5.31) for the diagonal Bianchi IX model where we set $\kappa = 0$. One can check that no elements of the analysis of the Bianchi IX model in section 5.3 become singular or non-degenerate when we set $\kappa = 0$. Then we can take over all results obtained there for the Bianchi IX model and set $\kappa = 0$ to get the corresponding results for the Bianchi I model. The global topological differences between S^3 and $\mathbb{R}^3$ play no role at the level of (5.75).

The reality conditions still imply that $\text{Im}(\rho) = 0$ and $\text{Im}\left[(P_i)^2\right] = 0$ for each i, with the latter condition once again defining various solution branches. However, the secondary conditions derived in section 5.3, (5.54) or (5.57) for example, simplify greatly. For instance, for the all real P_i case where $q_i = 0$ for all i, we found $c^i = \Pi^i(p)$ in (5.54). When we set $\kappa = 0$, these reduce to $c^i = 0$ as the expressions for $\Pi^i(p)$ all contain an overall factor of κ. This is as we would expect, since c^i represents the Levi-Civita connection on the spatial hypersurfaces which are flat. The reality conditions and secondary constraints hence become the simplest possible form of second-class constraints, requiring both variables of the conjugate pair c^i, q_i to vanish. These constraints can be "solved" by simply removing these variables from the theory. The Bianchi I model hence admits a particularly simple reduction to a real Lorentzian theory: we simply demand that all of the dynamical variables appearing in (5.75) are real valued.

Kasner solutions from connection variables

This Bianchi I model is simple enough to be solved analytically. The equations of motion coming from (5.75) are

$$\dot{C}^1 = \rho \frac{C^2 C^3}{(P_1)^2} \quad \text{etc.} \,, \quad \dot{P}_1 = \rho \left(\frac{C^3}{P_2} + \frac{C^2}{P_3} \right) \quad \text{etc.} \tag{5.76}$$

We can solve the latter three equations algebraically for C^i to find

$$C^1 = \frac{P_2 P_3}{2\rho} \left(-\frac{\dot{P}_1}{P_1} + \frac{\dot{P}_2}{P_2} + \frac{\dot{P}_3}{P_3} \right) \quad \text{etc.} \tag{5.77}$$

The Hamiltonian constraint, the expression which multiplies ρ in the integrand of (5.75), then becomes

$$\ell_P^2 \Lambda + \frac{P_1 P_2 P_3}{4\rho^2} \left(\frac{(\dot{P}_1)^2}{(P_1)^2} - 2\frac{\dot{P}_2 \dot{P}_3}{P_2 P_3} + \text{permutations} \right) = 0 \,. \tag{5.78}$$

We may rewrite this in the gauge $\rho^2 = P_1 P_2 P_3$ and writing $P_i(t) = \exp[s_i(t)]$ as

$$\ell_P^2 \Lambda + \frac{1}{4} \left(\dot{s}_1^2 + \dot{s}_2^2 + \dot{s}_3^2 - 2\dot{s}_1 \dot{s}_2 - 2\dot{s}_1 \dot{s}_3 - 2\dot{s}_2 \dot{s}_3 \right) = 0 \,. \tag{5.79}$$

The dynamical equations for C^i become, using (5.77),

$$-\ddot{s}_1 + \ddot{s}_2 + \ddot{s}_3 - \dot{s}_1 \left(\dot{s}_2 + \dot{s}_3 \right) + \dot{s}_2^2 + \dot{s}_3^2 = 0 \quad \text{etc.} \,, \tag{5.80}$$

that is, three first-order differential equations for the quantities $\dot{s}_i$. The sum of these three equations can be written as

$$\ddot{s}_1 + \ddot{s}_2 + \ddot{s}_3 + \frac{1}{2} \left(\dot{s}_1 + \dot{s}_2 + \dot{s}_3 \right)^2 = 6\ell_P^2 \Lambda \tag{5.81}$$

using the constraint (5.79). This is a Riccati equation for the quantity $\sum_i \dot{s}_i$, with solution

$$\dot{s}_1 + \dot{s}_2 + \dot{s}_3 = 2\sqrt{3\ell_P^2\Lambda}\,\coth\left(\sqrt{3\ell_P^2\Lambda}\,(t-t_0)\right)\ . \tag{5.82}$$

Furthermore, by adding two of the equations (5.80) and using the constraint again, we find

$$\ddot{s}_1 + \frac{1}{2}\dot{s}_1(\dot{s}_1 + \dot{s}_2 + \dot{s}_3) = 2\ell_P^2\Lambda \quad \text{etc.} \tag{5.83}$$

which can now be solved to get

$$\dot{s}_i = a_i\,\text{cosech}\left(\sqrt{3\ell_P^2\Lambda}\,(t-t_0)\right) + 2\sqrt{\frac{\ell_P^2\Lambda}{3}}\,\coth\left(\sqrt{3\ell_P^2\Lambda}\,(t-t_0)\right) \tag{5.84}$$

where the a_i are integration constants satisfying $a_1+a_2+a_3=0$. Substituting these solutions into (5.79) then gives a second constraint,

$$a_1a_2 + a_1a_3 + a_2a_3 = -\ell_P^2\Lambda\ . \tag{5.85}$$

Integrating and exponentiating the solutions (5.84) finally yields

$$P_i(t) = \tilde{P}_i\left[\tanh\left(\frac{\sqrt{3\ell_P^2\Lambda}}{2}\,(t-t_0)\right)\right]^{r_i}\sinh^{2/3}\left(\sqrt{3\ell_P^2\Lambda}\,(t-t_0)\right) \tag{5.86}$$

with the rescaled exponents r_i now satisfying $\sum_i r_i = 0$ and $r_1r_2 + r_1r_3 + r_2r_3 = -\frac{1}{3}$, and where $\tilde{P}_i$ are further integration constants. Substituting these solutions for $P_i(t)$ into the homogeneous and diagonal metric formula (5.43), except where θ^α is replaced with $\underline{\theta}^i$, reproduces the Bianchi I solution with cosmological constant originally given by Kasner in [64] (see also [98]).

5.5 Relation to the Euclidean theory

So far in our investigation of homogeneous and diagonal connection gravity we have only looked at real Lorentzian spacetimes. We recall that the Plebański theory is equally well defined in the Euclidean signature. Furthermore, the fields are all taken to be real in the Euclidean signature, so there is no need to impose any reality conditions, and one can avoid the pathologies that come with ensuring their consistency. At least in highly symmetric models, we expect there to be some correspondence between solutions in the Euclidean and Lorentzian signature. In particular, we anticipate a kind of 'Wick rotation' which transforms Euclidean solutions into Lorentzian ones by making certain variables imaginary. These Wick rotations become especially useful when applied in path integral approaches to quantisation [51].

We now study a diagonal Bianchi IX model in the Euclidean signature in analogy to the Lorentzian discussion of section 5.3. This investigation will be markedly shorter than the one in section 5.3, as there are no reality conditions to be made consistent in this case. The homogeneous and diagonal connection ansatz and expression for the curvature replacing (5.27) are now

$$A^i = U_\theta^{-1/3} C^i \, \theta^i \quad , \quad F^1 = U_\theta^{-1/3} \dot{C}^1 \, dt \wedge \theta^1 - U_\theta^{-2/3} \left(\kappa C^1 - C^2 C^3 \right) \theta^2 \wedge \theta^3 \quad \text{etc.} \tag{5.87}$$

Furthermore, we parametrise the matrix M^{ij} and the 4-form μ such that

$$M^{11} = \frac{\kappa C^1 - C^2 C^3}{\ell_{\mathrm{P}}^2 P_1} \quad \text{etc.} \; , \quad \mu = 2 U_\theta^{-1} \ell_P^4 \, \rho \, \varepsilon_\theta \wedge dt \; , \tag{5.88}$$

where we set the off-diagonal components of M^{ij} to zero. Inserting these fields into the first order action (2.98), with $\sqrt{\sigma} = 1$, yields

$$S_{\mathrm{HDE}}\left[C, P, \rho\right] = \int dt \left[P_i \, \dot{C}^i - \rho \left(\ell_{\mathrm{P}}^2 \Lambda + \left(\frac{C^2 C^3}{P_1} + \text{permutations} \right) - \kappa \sum_{i=1}^{3} \frac{C^i}{P_i} \right) \right] . \tag{5.89}$$

This action is manifestly real valued, so there is no need to further decompose the variables into real and imaginary parts as we did in the Lorentzian case. We can immediately construct a real Hamiltonian system with canonically conjugate phase space variables C^i, P_i, and Hamiltonian constraint

$$\mathcal{H}_{\mathrm{Euc}} = \ell_P^2 \Lambda + \left(\frac{C^2 C^3}{P_1} + \text{permutations} \right) - \kappa \sum_{i=1}^{3} \frac{C^i}{P_i} \approx 0 \; , \tag{5.90}$$

which generates the time evolution equations via $\dot{\mathcal{O}} = \{\mathcal{O}, \rho \mathcal{H}_{\mathrm{Euc}}\}$.

In order to clarify the correspondence between this theory and the real Lorentzian theory arising from the action (5.70), we will need a certain geometrically motivated change of dynamical variables. Following the same procedure as in section 5.3, one constructs a Euclidean signature metric

$$g_{\mathrm{Euc}} = \frac{\ell_{\mathrm{P}}^2 \, \rho^2}{P_1 P_2 P_3} \, dt \otimes dt + U_\theta^{-2/3} \ell_{\mathrm{P}}^2 \, P_1 P_2 P_3 \sum_{i=1}^{3} \frac{1}{(P_i)^2} \, \theta^i \otimes \theta^i \; , \tag{5.91}$$

which is the analogue of the metric (5.43) which appears in the Lorentzian (and general complex) theory. Note, we obtain a metric of positive definite signature for $P_1 P_2 P_3 > 0$ and negative definite signature for $P_1P_2P_3 < 0$. This spacetime metric induces a 3-metric on the constant t slices with corresponding orthonormal triad

$$e^i = U_\theta^{-1/3} \ell_P \, \frac{\sqrt{|P_1P_2P_3|}}{P_i} \, \theta^i \ . \tag{5.92}$$

The Levi-Civita connection Γ^i on the constant t slices, corresponding to this triad, is computed to be $\Gamma^i = U_\theta^{-1/3} \Pi^i(P)\, \theta^i$, where $\Pi^i(P)$ is given in (5.54). This is the same expression for Γ^i as in the Lorentzian signature (5.56), which is expected as the spacetime metrics (5.43) and (5.91) induce the same 3-metric on the constant t hypersurfaces.

In the Euclidean signature Plebański theory, the connection A^i is the self-dual part of an SO(4) spin connection ω^{IJ}. From (2.65), the connection decomposes as $A^i = K^i - \Gamma^i$ where K^i and Γ^i are (respectively) the extrinsic curvature and the Levi-Civita connection on the constant t hypersurfaces, as defined in (5.55). One can isolate the extrinsic curvature, which is the dynamical part of the spin connection for spatially homogeneous theories, via $K^i = A^i + \Gamma^i$. This amounts to a change of variables given by

$$b^i = C^i + \Pi^i(p) \quad , \quad p_i = P_i \ , \tag{5.93}$$

where b^i encodes the extrinsic curvature via $K^i = U_\theta^{-1/3}\, b^i\, \theta^i$. The new variables b^i, p_i are clearly canonically conjugate, so this change of variables is a canonical transformation. In fact, this canonical transformation relating extrinsic curvature and the $SO(3)$ connection A^i is also the basis of the Ashtekar–Barbero formulation of Lorentzian general relativity [100]. Then in terms of b^i, p_i, the Hamiltonian constraint (5.90) becomes $\mathcal{H}_{\rm Euc} = \mathcal{H}_{(+)} + \kappa^2 \mathcal{H}_{\rm BG}$ where $\mathcal{H}_{(+)}$ and $\mathcal{H}_{\rm BG}$ are defined in (5.69).

Wick rotations

We are now in a position to define a kind of Wick rotation (i.e., a transformation involving a replacement of real by purely imaginary variables) between this theory and the different Lorentzian theories corresponding to the various branches of the reality conditions.

Note, the complex transformation $b^i \mapsto b'^i = ib^i$ maps the Euclidean constraint $\mathcal{H}_{\rm Euc} = \mathcal{H}_{(+)} + \kappa^2 \mathcal{H}_{\rm BG}$ to the Lorentzian constraint $\mathcal{H}_{\rm Lor} = \mathcal{H}_{(-)} + \kappa^2 \mathcal{H}_{\rm BG}$ while changing the symplectic structure by an overall constant factor, $\{\cdot,\cdot\} \mapsto i\{\cdot,\cdot\}$. Absorbing this constant into the Lagrange multiplier, we may conclude that if $b^i(t), p_i(t), \rho(t)$ defines a solution to the Euclidean theory then $ib^i(t), p_i(t), i\rho(t)$ formally defines a solution to the Lorentzian theory. This map is invertible such that any Lorentzian solution likewise maps into a Euclidean one.

In fact, the Euclidean theory can be connected to all the other Lorentzian solution branches by similar "Wick rotations". For instance, the Lorentzian Hamiltonian constraint (5.72), corresponding to a case in which the surfaces of homogeneity are timelike, is obtained from the Euclidean Hamiltonian constraint $\mathcal{H}_{\rm Euc}$ after a transformation $b^3 \mapsto ib^3$, $p_1 \mapsto ip_1$, $p_2 \mapsto ip_2$ with all other variables unchanged. Again, Such a transformation has $\{\cdot,\cdot\} \mapsto i\{\cdot,\cdot\}$, and we hence see that any Euclidean solution also maps onto a Lorentzian solution of this kind.

One may then take the view, in particular when looking at quantisation, that the fundamental definition of the chiral connection formulation of general relativity should be the Euclidean theory. Lorentzian solutions emerge from such a theory for purely imaginary boundary conditions, with the different branches we found earlier corresponding to different types of variables chosen as real or imaginary. In the case of cosmological solutions, where the transformations we have discussed are defined unambiguously, this starting point might be preferable over dealing with the full complex theory and its reality conditions. As in the conventional metric formulation, it seems much less clear in which sense such a correspondence extends to the full theory.

5.6 Isotropic Models and Quantum Cosmology

In section 5.3 we saw that the Hamiltonian dynamics of the diagonal Bianchi IX model, for solutions corresponding to real Lorentzian signature metrics for which the homogeneous hypersurfaces are spacelike, can be defined in terms of a real phase space consisting of canonically conjugate variables b^i, p_i, and a Hamiltonian constraint $\mathcal{H}_{\rm Lor} = \mathcal{H}_{(-)} + \kappa^2\,\mathcal{H}_{\rm BG}$ as in (5.69). The equations of motion are generated from the Hamiltonian $\rho\mathcal{H}_{\rm Lor}$, where ρ is a Lagrange multiplier related to the time reparametrisation symmetry of the model. The expressions for the self-dual connection and Urbantke metric in terms of these dynamical variables are

$$A^i = U_\theta^{-1/3}\left(ib^i - \Pi^i(p)\right)\,\theta^i\ , \tag{5.94}$$

$$g_{\rm Lor} = -\frac{\ell_{\rm P}^2\,\rho^2}{p_1p_2p_3}\,dt\otimes dt + U_\theta^{-2/3}\ell_{\rm P}^2\,p_1p_2p_3\,\sum_{i=1}^{3}\frac{1}{(p_i)^2}\,\theta^i\otimes\theta^i\ , \tag{5.95}$$

where $\Pi^i(p)$ is defined in (5.54). We now want to discuss the specific case where the metric is also isotropic, i.e., of FLRW form. We can see that this case corresponds to $p_1 = p_2 = p_3$ for all t. One can check that this restriction only evolves consistently if we also restrict the connection variables such that $b^1 = b^2 = b^3$. Hence, starting from the fields appearing in the (real) Bianchi IX model, we can make the substitutions $b^i \to b/3$ and $p_i \to p$ where $b(t)$ and $p(t)$ are real scalars. This yields a connection and curvature

$$A^i = U_\theta^{-1/3}\left(\frac{ib}{3} + \frac{\kappa}{2}\right)\theta^i \ , \quad F^1 = iU_\theta^{-1/3}\,\frac{\dot{b}}{3}\,dt\wedge\theta^1 - U_\theta^{-2/3}\,\frac{b^2+K}{9}\,\theta^2\wedge\theta^3 \quad \text{etc.} \tag{5.96}$$

where $K := (3\kappa/2)^2 = 9U_\theta^{2/3}k > 0$ is a rescaled spatial curvature parameter. The 4-form valued matrix $F^i\wedge F^j$ is now proportional to δ^{ij}, so we can also take $M^{ij}\propto\delta^{ij}$. In fact, we see that applying our isotropic substitutions to the expressions for M^{ij} given in (5.71) yields

$$M^{ij} = \frac{b^2+K}{9\,\ell_{\rm P}^2\,p}\,\delta^{ij} \ . \tag{5.97}$$

In this section, we will choose a parametrisation for the 4-form μ that is different from the one in (5.71), given by

$$\mu = 2iU_\theta^{-1}\ell_P^2\left(\frac{p}{\Lambda}\right)\tilde{\rho}\,\varepsilon_\theta\wedge dt \ . \tag{5.98}$$

The Lagrange multiplier $\tilde{\rho}(t)$ is related to ρ, appearing in the real Lorentzian Bianchi IX action (5.70), via $\tilde{\rho} = \frac{\ell_P^2\Lambda}{p}\rho$. We choose this alternative parametrisation for future convenience. One obtains an action for an FLRW spacetime either by substituting these isotropic expressions for the fields A^i, M^{ij}, μ into the first order action (2.98), or from the Bianchi IX reduced action (5.70) via the substitutions $b^i\to b/3$ and $p_i\to p$. In either case, the result is the action given by

$$S_{\rm Iso}\left[b,p,\tilde{\rho}\right] = \int dt\left(\dot{b}p - \tilde{\rho}\left(p - \frac{b^2+K}{3\ell_P^2\Lambda}\right)\right) \ , \tag{5.99}$$

with Euler–Lagrange equations

$$\dot{b} = \tilde{\rho} \quad , \quad \dot{p} = 2\tilde{\rho}\,\frac{b}{3\ell_P^2\Lambda} \ , \tag{5.100}$$

which are the same equations that one obtains from (5.33a) and (5.33b) after the substitutions $C^i = \frac{c}{3} - \frac{i\kappa}{2}$, $P_i = p$ and $\rho = \frac{p}{\ell_P^2\Lambda}\tilde{\rho}$. Hence the restriction to isotropy commutes with the variational principle. The Urbantke metric now takes the explicit form

$$g_{\rm Iso} = -\frac{\tilde{\rho}^2}{\Lambda^2\,p}\,dt\otimes dt + U_\theta^{-2/3}\ell_P^2\,p\sum_{i=1}^{3}\theta^i\otimes\theta^i \ . \tag{5.101}$$

This parametrisation of the $k>0$ FLRW metric is well-known in quantum cosmology, see, e.g., the expression given in [47],

$$ds^2 = -\frac{N^2(t)}{q(t)}\,dt^2 + q(t)\,d\Omega_3^2 \tag{5.102}$$

and subsequent discussion of the path integral quantisation given there. (Note the variable q used in [47] corresponds to our p, whereas the variable p used there is conjugate to q, i.e., corresponds to the connection b in our notation).

The definition of a path integral for this system depends in general on the choice of boundary conditions; see the discussion in [35]. The path integral in [47] corresponds to the case where one wants to keep the spatial metric fixed at the initial and final times, so that in our variables p would be fixed. One is then interested in an amplitude, or two-point function, of the form

$$G(p_f|p_i) = \int \mathcal{D}b\,\mathcal{D}p\,\mathcal{D}\tilde{\rho}\;\exp\left[i\int_{t_i}^{t_f} dt\left(\dot{b}p - \tilde{\rho}\left(p - \frac{b^2+K}{3\ell_P^2\Lambda}\right)\right)\right]\,, \tag{5.103}$$

where the allowed paths for p must start at p_i and end at p_f. Due to the gauge symmetry of the system under time reparametrisations, this integral is ill-defined as it is given above. To remedy this, one can follow the standard approach proposed in [47] (and also followed in [39]) where one implements a gauge fixing, such as $\dot{\tilde{\rho}} = 0$, by adding a term to the action. To ensure that the final result does not depend on the gauge choice, one can then introduce additional anticommuting ghost fields to make the action invariant under a global Becchi–Rouet–Stora–Tyutin (BRST) symmetry. The path integral over the ghost fields can be done explicitly and leads to a new expression for the gauge-fixed path integral which reads

$$G(p_f|p_i) = \int d\tilde{\rho}\,(t_f - t_i)\int \mathcal{D}b\,\mathcal{D}p\;\exp\left[i\int_{t_i}^{t_f} dt\left(\dot{b}p - \tilde{\rho}\left(p - \frac{b^2+K}{3\ell_P^2\Lambda}\right)\right)\right]\,. \tag{5.104}$$

Note that there is no longer a functional integral over the Lagrange multiplier $\tilde{\rho}$, just an ordinary integral whose definition depends on what kind of two-point function one is interested in [48]. In the case where one is interested in solutions to the canonical Wheeler–DeWitt equation that may be interpreted as physical wavefunctions, one possible integration contour for $\tilde{\rho}$ is over the entire real line, leading to the usual "no-boundary" solutions expressed in terms of Airy functions [47].

In the context of a pure connection approach to gravity, it seems more natural to specify the connection variable b at the initial and final times and instead try to compute the path integral given by

$$G(b_f|b_i) = \int d\tilde{\rho}\,(t_f - t_i)\int \mathcal{D}b\,\mathcal{D}p\;\exp\left[i\int_{t_i}^{t_f} dt\left(\dot{b}p - \tilde{\rho}\left(p - \frac{b^2+K}{3\ell_P^2\Lambda}\right)\right)\right]\,, \tag{5.105}$$

using the same gauge-fixing method as before. This is now rather straightforward, since the p path integral may be defined through a time-slicing where one divides the interval $[t_i, t_f]$

into N segments with equal widths $\delta t = \frac{t_f - t_i}{N}$. One specifies the $N+1$ end-points of these segments by $t_j = t_i + j\delta t$ for $j = 0, \ldots, N$, and one denotes the values of b at these points by $b_j = b(t_j)$. One localises the momentum p at the midpoint of each section such that $p_{j-\frac{1}{2}} = p\left(t_i + (j - \frac{1}{2})\delta t\right)$ for $j = 1, \ldots, N$. With respect to this discretisation, integrals are approximated as

$$\int_{t_i}^{t_f} dt\, f(b,p) \sim \sum_{j=1}^{N} f\left(b_j, p_{j-\frac{1}{2}}\right) \delta t \,, \tag{5.106}$$

and derivatives are approximated as $\dot{b}(t_j) \sim \frac{b_j - b_{j-1}}{\delta t}$; these are only well defined for $j > 0$. With this, one can write the path integral w.r.t p as limit of discrete approximations which reads

$$\begin{aligned} \int \mathcal{D}p\, \exp\left[i \int_{t_i}^{t_f} dt \left(p\dot{b} - \tilde{\rho}\, p\right)\right] &= \lim_{N\to\infty} \prod_{j=1}^{N} \int \frac{dp_{j-\frac{1}{2}}}{2\pi}\, \exp\left[ip_{j-\frac{1}{2}}(b_j - b_{j-1} - \tilde{\rho}\,\delta t)\right] \\ &= \lim_{N\to\infty} \prod_{j=1}^{N} \delta(b_j - b_{j-1} - \tilde{\rho}\,\delta t) \,, \end{aligned} \tag{5.107}$$

where $b_0 = b_i$ and $b_N = b_f$. Substituting this result into the gauge fixed path integral (5.105) yields the following equivalent expressions for $G(b_f|b_i)$:

$$\begin{aligned} &\lim_{N\to\infty} \int d\tilde{\rho}\, (t_f - t_i) \int db_1 \ldots db_{N-1}\, \exp\left[i\tilde{\rho} \sum_{m=1}^{N} \delta t \left(\frac{b_m^2 + K}{3\ell_P^2 \Lambda}\right)\right] \prod_{j=1}^{N} \delta(b_j - b_{j-1} - \tilde{\rho}\,\delta t) \\ &= \lim_{N\to\infty} \int d\tilde{\rho}\, (t_f - t_i)\, \delta(b_f - b_i - \tilde{\rho}(t_f - t_i))\, \exp\left[i\tilde{\rho} \sum_{m=1}^{N} \delta t \left(\frac{(b_i + m\tilde{\rho}\,\delta t)^2 + K}{3\ell_P^2 \Lambda}\right)\right] \\ &= \lim_{N\to\infty} \exp\left[\frac{i}{N}(b_f - b_i) \sum_{m=1}^{N} \frac{(b_i + \frac{m}{N}(b_f - b_i))^2 + K}{3\ell_P^2 \Lambda}\right] \,, \end{aligned} \tag{5.108}$$

where the remaining delta function in the second line enforces the condition $\tilde{\rho} = \frac{b_f - b_i}{t_f - t_i}$, and where the chosen contour for $\tilde{\rho}$ is the real line. In the limit as $N \to \infty$, the argument of the exponential in the final line changes from a sum into an integral. To evaluate this integral, one defines a quantity $\delta b_N = \frac{b_f - b_i}{N}$, which clearly vanishes in the limit $N \to \infty$, such that

$$G(b_f|b_i) = \lim_{N\to\infty} \exp\left[i \sum_{m=1}^{N} \delta b_N \left(\frac{(b_i + m\,\delta b_N) + K}{3\ell_P^2 \Lambda}\right)\right] = \exp\left[i \int_{b_i}^{b_f} db\, \frac{b^2 + K}{3\ell_P^2 \Lambda}\right] \tag{5.109}$$

which then leads to the final form of the two-point function

$$G(b_f|b_i) = \exp\left[\frac{i}{3\ell_P^2\Lambda}\left(\frac{1}{3}(b_f^3 - b_i^3) + K(b_f - b_i)\right)\right] . \tag{5.110}$$

Hence, the path integral with b boundary conditions can be evaluated analytically, and yields a simple result of pure plane wave form, which may be written as

$$G(b_f|b_i) = \psi_{\rm CS}(b_f)\,\overline{\psi_{\rm CS}}(b_i) \quad : \quad \psi_{\rm CS}(b) := \exp\left[\frac{i}{3\ell_P^2\Lambda}\left(\frac{1}{3}b^3 + Kb\right)\right] , \tag{5.111}$$

where $\psi_{\rm CS}(b)$ is the unique (up to normalisation) solution to the Wheeler–DeWitt equation

$$i\frac{d}{db}\psi(b) = -\frac{b^2 + K}{3\ell_P^2\Lambda}\psi(b) \tag{5.112}$$

corresponding to the classical Hamiltonian constraint appearing in (5.99) as the term which multiplies $\tilde{\rho}$. $\psi_{\rm CS}$ can be seen as the restriction of a "Chern–Simons state" that can be defined in more general situations [111] to homogeneous and isotropic Universes (see also [79] for generalisations of the Chern–Simons state). The state can be seen as related to the Hartle–Hawking or Vilenkin wavefunctions of the metric formulation via a kind of Fourier transform [78].

We see that the two-point function $G(b_f|b_i)$ with connection boundary data is straightforward to obtain, as also shown in [59] following a very similar calculation. The result has an interesting, and somewhat novel, interpretation from the perspective of the chiral pure connection formulation of general relativity (2.105). Notice that starting from the classical action (5.99), one may "integrate out" the field p by substituting the solution to the Hamiltonian constraint,

$$p = \frac{b^2 + K}{3\ell_P^2\Lambda} , \tag{5.113}$$

back into the Lagrangian. This yields

$$S_{\rm Iso} = \int_{t_i}^{t_f} dt\left(\frac{b^2 + K}{3\ell_P^2\Lambda}\,\dot{b}\right) = \frac{1}{3\ell_P^2\Lambda}\left[\frac{1}{3}b^3 + Kb\right]_{t_i}^{t_f} . \tag{5.114}$$

That is, the Lagrangian is a total derivative and the action becomes a pure boundary term. Hence there is no equation of motion for b. This is expected as (5.100) already shows that *any* function $b(t)$ is a solution in a certain gauge, given by $\dot{b} = \tilde{\rho}$.

It is easy to see that the reduced action (5.114) is simply the reduction of the chiral pure connection action (2.105) to a homogeneous and isotropic connection. Starting from

$$F^i \wedge F^j = 2i\,\frac{\dot{b}\,(b^2 + K)}{27U_\theta}\,\delta^{ij}\,\varepsilon_\theta \wedge dt , \tag{5.115}$$

we may fix a top-form $\varepsilon_X = 2i\,\varepsilon_\theta \wedge dt$, therefore fixing the matrix X^{ij} appearing in (2.105) to be

$$X^{ij} = \frac{\dot{b}\,(b^2 + K)}{27 U_\theta}\,\delta^{ij}\ . \tag{5.116}$$

This leads to a pure connection action

$$S_{\mathrm{PC}}[b] = \frac{1}{\ell_P^2 \Lambda} \int \left(\mathrm{tr}\,\sqrt{X}\right)^2 \varepsilon_X = \frac{1}{3\ell_P^2 \Lambda} \int dt\ \dot{b}\,(b^2 + K)\ , \tag{5.117}$$

in agreement with (5.114). Hence for FLRW geometries, the chiral pure connection action is a pure boundary term.

An attempt to define a path integral directly at the Lagrangian level for the pure connection formulation would lead to an expression

$$G(b_f|b_i) = \int \mathcal{D}b\ \exp\left(i S_{\mathrm{PC}}[b]\right)\ . \tag{5.118}$$

However, since $S_{\mathrm{PC}}[b]$ is a pure boundary term, one would again be left with a divergent integration over redundant, gauge-equivalent configurations, which needs gauge fixing to be well-defined. Any such gauge fixing will turn the integration over b into a constant factor leading to our previous result (5.110). Then in the case of homogeneous isotropic connections, the pure connection path integral has an immediate exact definition given in terms of the classical action given by

$$G(b_f|b_i) = \exp\left(i S_{\mathrm{PC}}(b_f, b_i)\right)\ . \tag{5.119}$$

Analytic continuation of the two-point function

So far we have computed a two-point function with connection boundary data for a homogeneous and isotropic Lorentzian spacetime, $G(b_f|b_i)$ as in (5.110). One can analytically extend this two-point by simply allowing the two inputs to be complex valued. Through this extension, one can compute transition amplitudes for boundary connection states that are Lorentzian, Euclidean, or even generally complex.

Recall that the variable b – which is the isotropic analogue of b^i from section 5.3 – encodes the extrinsic curvature, which is the real part of the self-dual connection A^i in the Lorentzian case, as in (5.55). One can introduce a more generic homogeneous and isotropic connection variable $\tilde{C}$ such that

$$A^i = U_\theta^{-1/3}\,\frac{\tilde{C}}{3}\,\theta^i\ . \tag{5.120}$$

In the Lorentzian case, one can compare this expression with the expression for A^i in (5.96) to get a relationship between $\tilde{C}$ and b which reads

$$\tilde{C} = ib + 3U_\theta^{1/3}\sqrt{k} \quad \Leftrightarrow \quad b = -i\left(\tilde{C} - 3U_\theta^{1/3}\sqrt{k}\right) , \tag{5.121}$$

where is k the background curvature parameter as in section 5.3. Then replacing b_i and b_f with their equivalent expressions in terms of $\tilde{C}$ in (5.110) yields the more generic two-point function

$$\widetilde{G}\left(\tilde{C}_f|\tilde{C}_i\right) = \exp\left[-\frac{1}{3\ell_P^2\Lambda}\left(\frac{1}{3}\left(\tilde{C}_f^3 - \tilde{C}_i^3\right) - 3U_\theta^{1/3}\sqrt{k}\left(\tilde{C}_f^2 - \tilde{C}_i^2\right)\right)\right] . \tag{5.122}$$

On the other hand, our original formula (5.110) is sufficient for dealing with Lorentzian and/or Euclidean signature boundary data. To see this, consider a transition from a Euclidean signature initial state to a Lorentzian signature final state such that

$$(A^j)|_{t=t_i} = U_\theta^{-1/3}\left(\frac{b_i}{3} + \frac{\kappa}{2}\right)\theta^j \longrightarrow (A^j)|_{t=t_f} = U_\theta^{-1/3}\left(\frac{ib_f}{3} + \frac{\kappa}{2}\right)\theta^j , \tag{5.123}$$

where b_i, b_f are both real valued. Then the corresponding amplitude can be computed as $G(-ib_i, b_f)$.

One may now discuss particular choices of b_f and b_i, in particular "no-boundary" conditions for b_i. Recall that the general idea as pioneered by Hartle and Hawking is that the initial state of the Universe would be described by a Euclidean 4-sphere appearing to emerge from zero size (but such that the resulting geometry is actually regular). Such an initial condition is often interpreted as corresponding to zero scale factor, or $p = 0$ in our notation. However, it has been suggested (e.g., in [35], building on earlier work such as [77]) that an initial condition should rather be put on the connection to distinguish between different semiclassical saddle point solutions, and in particular single out the "Hartle–Hawking" over the "Vilenkin" solution. Given the constraint (5.113), a no-boundary initial condition on the connection corresponds to $b = \pm i\sqrt{K}$, equivalently $b = \pm 3iU_\theta^{1/3}\sqrt{k}$. With such an initial condition the two-point function (5.110) becomes

$$\begin{aligned} G(b_f|b_i) = \psi_{\rm CS}(b_f)\,\overline{\psi_{\rm CS}}(\pm i\sqrt{K}) &= \exp\left(\mp\frac{2K^{3/2}}{9\ell_P^2\Lambda}\right)\psi_{\rm CS}(b_f) \\ &= \exp\left(\mp\frac{6U_\theta\, k^{3/2}}{\ell_P^2\Lambda}\right)\psi_{\rm CS}(b_f) \end{aligned} \tag{5.124}$$

so that, depending on the choice of sign, we get either exponential suppression (*à la* Vilenkin) or exponential enhancement (*à la* Hartle–Hawking). Inserting the usual choices of $k = 1$ and $U_\theta = 2\pi^2$, this factor is $\exp\left(\mp\frac{12\pi^2}{\ell_P^2\Lambda}\right)$, consistent with the literature [40].

5.6.1 Unimodular quantum cosmology

One can apply the same homogeneous and isotropic restriction to the parametrised unimodular version of the first order action (3.16), and examine the resulting transition amplitudes from the perspective of unimodular connection gravity. In practice, one simply inserts the expressions for the connection A^i and matrix M^{ij} given in (5.96) and (5.97) respectively, as well as the parametrisation

$$T = -2iU_\theta^{-1}\ell_P^2\,\Pi\,\varepsilon_\theta \tag{5.125}$$

for the 3-form T, into the action (3.16) to get

$$S_{\mathrm{PIso}}[b,p,\Pi] = \int dt\left[\dot{b}p + \dot{\Pi}\left(\frac{b^2+9U_\theta^{2/3}k}{3\ell_P^2 p}\right)\right] . \tag{5.126}$$

The real scalar $\Pi(t)$ is the homogeneous analogue of the variable $\tilde{\tau}$ appearing in the general canonical action (4.55). The resulting Euler-Lagrange equations are

$$\dot{b} = \dot{\Pi}\left(\frac{b^2+9U_\theta^{2/3}k}{3\ell_P^2\,p^2}\right) \quad , \quad \dot{p} = \dot{\Pi}\,\frac{2b}{3\ell_P^2\,p} \quad , \quad \frac{d}{dt}\left(\frac{b^2+3U_\theta^{2/3}k}{3\ell_P^2\,p}\right) = 0\ . \tag{5.127}$$

From the Hamiltonian perspective, we can equip the variable λ with conjugate momentum λ satisfying the primary constraint

$$\lambda - \frac{b^2+9U_\theta^{2/3}k}{3\ell_P^2\,p} = 0\ , \tag{5.128}$$

which is the usual Hamiltonian constraint for the FLRW universe with cosmological constant $\lambda = \Lambda$, as in (5.113). One passes to an extended canonical action given by

$$S'_{\mathrm{PIso}}[b,p,\Pi,\lambda,\nu] = \int dt\left[\dot{b}p - \dot{\lambda}\Pi - \nu\left(p - \frac{b^2+9U_\theta^{2/3}k}{3\ell_P^2\,\lambda}\right)\right] , \tag{5.129}$$

where $\nu(t)$ is a real Lagrange multiplier field which enforces a rearranged version of the primary constraint (5.128). This action agrees with the standard minisuperspace action for gravity with cosmological constant (5.99), where the cosmological constant has been promoted to a dynamical field conjugate to Π. This addition of a new global degree of freedom associated to the cosmological constant is familiar from all unimodular extensions of general relativity.

The goal now is to derive amplitudes corresponding to an initial state defined by the pair b_i, λ_i, and a final state defined by b_f, λ_f. As before, one follows the procedure outlined in [47] to gauge fix the time reparametrisation symmetry, leading to an expression for the unimodular two-point function given by

$$G(b_f, \lambda_f | b_i, \lambda_i) = \int d\nu \; (t_f - t_i) \int \mathcal{D}b\, \mathcal{D}p\, \mathcal{D}\lambda\, \mathcal{D}\Pi \; \exp\left(iS'_{\text{PIso}}[b, p, \Pi, \lambda, \nu]\right) \;, \tag{5.130}$$

where the integration contour for ν is the real line. From the structure of (5.129), one may rewrite the above expression for the two-point function as

$$G(b_f, \lambda_f | b_i, \lambda_i) = \int \mathcal{D}\Pi\, \mathcal{D}\lambda \; \exp\left(-i \int dt \; \Pi \dot{\lambda}\right) G_\lambda(b_f | b_i) \;, \tag{5.131}$$

where $G_\lambda(b_f|b_i)$ is the same as $G(b_f|b_i)$ in (5.110), except where the cosmological constant Λ is replaced with the variable λ. Then one can evaluate the path integrals w.r.t λ and Π using the time slicing procedure defined in section 5.6, such that $\lambda_j = \lambda(t_i + j\delta t)$ and $\Pi_{j-\frac{1}{2}} = \Pi\left(t_i + (j - \frac{1}{2})\delta t\right)$. In brief, evaluating the $\Pi_{j-\frac{1}{2}}$ integrals yields a chain of deltas

$$\delta(\lambda_N - \lambda_{N-1})\, \delta(\lambda_{N-1} - \lambda_{N-2}) \ldots \delta(\lambda_2 - \lambda_1)\, \delta(\lambda_1 - \lambda_0) \tag{5.132}$$

in the integrand, where $\lambda_0 = \lambda_i$ and $\lambda_N = \lambda_f$. Then integrating over $\lambda_1, \ldots, \lambda_{N-1}$ collapses this chain into a single delta $\delta(\lambda_f - \lambda_i)$, yielding the final result

$$G(b_f, \lambda_f | b_i, \lambda_i) = \delta(\lambda_f - \lambda_i)\, G_{\lambda_f}(b_f | b_i) \;. \tag{5.133}$$

5.7 Summary

We define spatial homogeneity such that the fields are invariant under the action of a certain group of translations on each spatial slice. Then one can parametrise the fields by finitely many free functions with only time dependence, which then become the dynamical variables for our homogeneous formulation. We further restricted our attention to diagonal theories in order to avoid certain inconsistencies which seem to appear when one attempts a formulation in terms of non-diagonal variables (section 5.2). In particular, we examined the Bianchi IX and I models with closed (S^3) and flat ($\mathbb{R}^3$) spatial surfaces respectively. Beginning with the Bianchi IX model, we derived a holomorphic Hamiltonian system starting from the first order chiral connection action (2.98) to describe the dynamics of the homogeneous and diagonal variables.

An important aspect of chiral connection formulations (in Lorentzian signature) is that all fields are initially complex-valued, and Lorentzian solutions are obtained by imposing reality conditions. In this case, the reality conditions admit a handful of solution branches, some corresponding to a Lorentzian Urbantke metric (either of signature $(-+++)$ or $(+---)$) and some corresponding to an imaginary Urbantke metric. The latter cases would require extra care in defining a Lorentzian metric, but for real (and non-zero) cosmological constant Λ they do not have any consistent solutions to the field equations, and we can ignore them. The reality conditions of the first order theory (2.100) are not holomorphic, and also contain time derivatives of the dynamical variables. To deal with the latter issue, one simply uses the dynamical equations to eliminate the time derivatives leaving only algebraic conditions on the variables. To deal with the former issue, we reformulated our holomorphic Hamiltonian system as a pair of equivalent real Hamiltonian systems, starting from either the real or imaginary part of the full complex action. In each of the 4 solution branches, one can treat the reality conditions and their secondary consistency conditions as (second class) dynamical constraints in Dirac's formalism. Then one can derive the Dirac bracket and the associated reduced phase space structure. The different branches of the reality conditions can reproduce the usual Bianchi models, in which the homogeneous surfaces are spacelike, or models in which these surfaces are timelike and the direction of dynamical evolution is spacelike. Focusing on the physically more relevant first case, we explicitly recovered the Lorentzian Bianchi IX model [109]. The analysis was then repeated for the Bianchi I model, with similar results as well as analytical solutions reproducing the known Kasner solutions [14, 64].

The restriction to the Bianchi I and IX models is significantly easier in the Euclidean signature version of the first order connection theory where all of the fields are real valued and no further reality conditions are required. Starting from solutions in Euclidean signature, one can apply complex transformations on the variables in order to derive the various Lorentzian solutions. These transformations may be interpreted as a type of Wick rotation, suggesting that the Euclidean theory may be a more natural starting point for defining the theory especially in the quantum regime, where reality conditions will be a major obstacle. The signature ambiguity however persists as the Euclidean signature Urbantke metric can have signature $(++++)$ or $(----)$.

In the last section, we examined the isotropic limit of the Bianchi IX model which corresponds to the FLRW spacetime with closed spatial surfaces. This restriction leaves only a single dynamical pair of variables subject to a first class constraint so that the one remaining degree of freedom is pure gauge. We showed that the two-point function between states with connection boundary data, defined by a suitable path integral, reduces to e^{iS} where the action S, as in (5.114), is a pure boundary term. This result has been derived before [79] but we gave it a somewhat novel classical interpretation: we saw that in the FLRW model one can explicitly integrate out all variables apart from the connection, and obtain a classical action which is a pure boundary term. This action is the reduction of the pure connection formulation of [72] to the FLRW model which hence has an immediate, trivial, quantum definition as a path integral.

5.A Supplementary Material: Chapter 5

5.A.1 Canonical Spatially Homogeneous Plebański Gravity

Proof: Lemma 5.2.1

The spatially homogeneous ansatzes from section 5.2 are given by

$$A^i = A^i_0 dt + A^i_\alpha \theta^\alpha \ , \tag{5.134a}$$

$$\Sigma^i = \Sigma^i_{0\alpha}\, dt \wedge \theta^\alpha + \frac{1}{2} E^{i\alpha} \epsilon_{\alpha\beta\gamma}\, \theta^\beta \wedge \theta^\gamma \ , \tag{5.134b}$$

$$\mu = -2n\, dt \wedge \varepsilon_\theta \ , \tag{5.134c}$$

such that the fields $A^i_\alpha, E^\alpha_i, M^{ij}, A^i_0, \Sigma^i_{0\alpha}, n$ have only t dependence. For simplicity, we will not assume any particular forms for A^i_0 and $\Sigma^i_{0\alpha}$ as we do in the main text. Since $A^i_0, \Sigma^i_{0\alpha}$ are Lagrange multiplier fields, the result we derive here also applies for any non-degenerate parametrisation of $A^i_0, \Sigma^i_{0\alpha}$ by alternative Lagrange multiplier fields (e.g., α^i, V^α, n as in the main text). Substituting these ansatzes into the Plebański action (2.79) yields a restricted action

$$S = \frac{U_\theta}{\ell_p^2 \sqrt{\sigma}} \int dt \left[\dot{A}^i_\alpha E^\alpha_i - A^i_0 \epsilon^k{}_{ji} A^j_\alpha E^\alpha_k + \frac{1}{2} \epsilon^{\alpha\beta\gamma} \Sigma_{i\,0\alpha} F^i_{\beta\gamma} - M_{ij} \Sigma^i_{0\alpha} E^{j\alpha} - n\, (\mathrm{tr}\, M - \Lambda) \right] \ , \tag{5.135}$$

with Euler-Lagrange equations

$$\dot{A}^i_\alpha = \epsilon^i{}_{jk} A^j_\alpha A^k_0 + M^{ij} \Sigma_{j\,0\alpha} \ , \tag{5.136a}$$

$$\dot{E}^\alpha_i = -\epsilon^k{}_{ij} A^j_0 E^\alpha_k - \frac{1}{2} \Sigma_{i\,0\beta} (f^\alpha{}_{\gamma\delta} \epsilon^{\beta\gamma\delta}) + \epsilon^k{}_{ij} \epsilon^{\alpha\beta\gamma} A^j_\beta \Sigma_{k\,0\gamma} \ , \tag{5.136b}$$

$$\Sigma^{(i}_{0\alpha} E^{j)\alpha} + n\delta^{ij} = 0 \ , \tag{5.136c}$$

$$\epsilon^k{}_{ji} A^j_\alpha E^\alpha_k = 0 \ , \tag{5.136d}$$

$$\frac{1}{2} \epsilon^{\alpha\beta\gamma} F^i_{\beta\gamma} - M^{ij} E^\alpha_j = 0 \ , \tag{5.136e}$$

$$\mathrm{tr}\, M - \Lambda = 0 \ . \tag{5.136f}$$

We derive equations on the variables by inserting our ansatzes (5.134a) into the Plebański field equations (2.81), and compare them to (5.136) coming from variations of the restricted action. The equation $F^i = M^{ij}\Sigma_j$ from (2.81b) yields

$$F^i_{0\alpha} = \dot{A}^i_\alpha - \epsilon^i{}_{jk} A^j_\alpha A^k_0 = M^{ij}\Sigma_{j\,0\alpha}\,, \tag{5.137a}$$

$$F^i_{\alpha\beta} = M^{ij} E^\gamma_j \epsilon_{\alpha\beta\gamma}\,, \tag{5.137b}$$

in agreement with (5.136a) and (5.136e) respectively. The equation $D_A \Sigma^i = 0$ from (2.81a) yields

$$\dot{E}^{i\alpha} + \epsilon^i{}_{jk} A^j_0 E^{k\alpha} + \frac{1}{2}\Sigma^i_{0\beta}(f^\beta{}_{\gamma\delta}\epsilon^{\alpha\gamma\delta}) - \epsilon^i{}_{jk}\epsilon^{\alpha\beta\gamma} A^j_\beta \Sigma^k_{0\gamma} = 0\,, \tag{5.138a}$$

$$\epsilon^i{}_{jk} A^j_\alpha E^\alpha_k - E^{i\alpha} f^\beta{}_{\alpha\beta} = 0\,, \tag{5.138b}$$

which respectively agree with (5.136b) and (5.136d) *only* when the matrix defined $\mathcal{C}^{\alpha\beta} := f^\alpha{}_{\gamma\delta}\epsilon^{\beta\gamma\delta}$ is symmetric, which is equivalent to $f^\beta{}_{\alpha\beta} = 0$. This is the criterion which distinguishes the *class A* models in the Bianchi classification of 3D Lie groups [20]. See [21] for details. The metricity condition $\Sigma^i \wedge \Sigma^j = \mu\delta^{ij}$ from (2.81c) immediately reduces to (5.136c), and $\operatorname{tr} M = \Lambda$ from (2.81d) is evidently the same as (5.136f). Hence, the restriction of the Plebański action (2.79) to homogeneous spacetimes commutes with the variational principle only for Bianchi class A models where $\mathcal{C}^{\alpha\beta}$ is symmetric.

5.A.2 Diagonal Bianchi IX model

Proof: Theorem 5.3.1

Consider an action of the form

$$S[Q] = \int dt\; L\left(Q, \dot{Q}\right) \tag{5.139}$$

where $Q^i(t)$ are complex variables and the Lagrangian $L(Q, \dot{Q})$ is also complex valued. We assume that L is holomorphic so that

$$\frac{\partial L}{\partial \overline{Q}^i} = 0\,, \quad \frac{\partial L}{\partial \dot{\overline{Q}}^i} = 0\,. \tag{5.140}$$

Decomposing the complex variables into their real and imaginary parts as $Q^i = x^i + iy^i$ allows us to decompose the Lagrangian as $L(Q, \dot{Q}) = L_{\mathrm{Re}}(x, y, \dot{x}, \dot{y}) + iL_{\mathrm{Im}}(x, y, \dot{x}, \dot{y})$ where L_{Re} and L_{Im} are real valued functions of real variables. The holomorphic conditions (5.140) are equivalent to the Cauchy–Riemann equations

$$\frac{\partial L_{\mathrm{Re}}}{\partial x^i} = \frac{\partial L_{\mathrm{Im}}}{\partial y^i}\,, \quad \frac{\partial L_{\mathrm{Im}}}{\partial x^i} = -\frac{\partial L_{\mathrm{Re}}}{\partial y^i}\,, \quad \frac{\partial L_{\mathrm{Re}}}{\partial \dot{x}^i} = \frac{\partial L_{\mathrm{Im}}}{\partial \dot{y}^i}\,, \quad \frac{\partial L_{\mathrm{Im}}}{\partial \dot{x}^i} = -\frac{\partial L_{\mathrm{Re}}}{\partial \dot{y}^i}\,. \tag{5.141}$$

For a functional F over real variables $z^a(t)$ defined $F[z] = \int dt\ f(z, \dot{z})$ for some function f, the functional derivatives are

$$\frac{\delta F}{\delta z^a} = \frac{\partial f}{\partial z^a} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{z}^a}\right) . \tag{5.142}$$

Using this expansion one obtains functional form of the Cauchy-Riemann equations

$$\frac{\delta S_{\mathrm{Re}}}{\delta x^i} = \frac{\delta S_{\mathrm{Im}}}{\delta y^i} \quad , \quad \frac{\delta S_{\mathrm{Im}}}{\delta x^i} = -\frac{\delta S_{\mathrm{Re}}}{\delta y^i} , \tag{5.143}$$

where $S_{\mathrm{Re}} = \int dt\ L_{\mathrm{Re}}$ and $S_{\mathrm{Im}} = \int dt\ L_{\mathrm{Im}}$ are the real and imaginary parts of the action respectively. The Euler-Lagrange equations coming from the real and imaginary parts of the action are given by

$$\text{Real part} \quad : \quad \frac{\delta S_{\mathrm{Re}}}{\delta x^i} = 0 , \quad \frac{\delta S_{\mathrm{Re}}}{\delta y^i} = 0 , \tag{5.144a}$$

$$\text{Imaginary part} \quad : \quad \frac{\delta S_{\mathrm{Im}}}{\delta x^i} = 0 , \quad \frac{\delta S_{\mathrm{Im}}}{\delta y^i} = 0 . \tag{5.144b}$$

From (5.143) we then see that the Euler-Lagrange equations (5.144a) and (5.144b) are equivalent. This proof borrows heavily from section 5 of [73].

5.A.3 Hamiltonian constraint for the Lorentzian Bianchi IX model

Declaration

What follows was written by Steffen Gielen during the preparation of [45], *based off an earlier version written by the author of this thesis, and is presented here exactly as it appears in* [45] *with only minor formatting and notation changes.*

Here we show explicitly that (5.69) reproduces the dynamics of the Lorentzian Bianchi IX model in general relativity, as discussed in [109] in the context of Ashtekar–Barbero formulation. This formulation is based on an $SU(2)$ connection defined on each spatial hypersurface in a given foliation as $A^i = \Gamma^i[e] + \beta K^i$ where Γ^i is the torsion-free (Levi-Civita) connection 1-form associated to a given triad e^i, K^i is the extrinsic curvature 1-form, and β is a free parameter generally taken to be real. The Bianchi IX ansatz made in [109] is

$$A^i = U_\theta^{-1/3}\,\tilde{c}^i\,\tilde{\theta}^i \quad , \quad E_i = U_\theta^{-2/3}\pi_i\,\sqrt{\tilde{q}}\,\tilde{\xi}_i \tag{5.145}$$

where E_i is the "densitised triad" conjugate to A^i, $\tilde{\xi}_i$ is a co-frame of vector fields dual to the frame $\tilde{\theta}^i$, and $\sqrt{\tilde{q}}$ is the volume element associated to the "fiducial metric" $\tilde{q} := \tilde{\theta}^i \otimes \tilde{\theta}_i$.

The frame $\tilde{\theta}^i$ is assumed to satisfy $d\tilde{\theta}^i - \frac{1}{r_o}\,\epsilon^i{}_{jk}\,\tilde{\theta}^j \wedge \tilde{\theta}^k = 0$. The variables $\tilde{c}^i$ and π_i then satisfy

$$\{\tilde{c}^i, \pi_j\} = \beta\,\ell_P^2\,\delta^i_j\,. \tag{5.146}$$

Recall that on the branch of the reality conditions that leads to (5.69), the self-dual connection of the chiral connection theory can be written as $A^i = U_\theta^{-1/3}(ib^i - c^i)\,\theta^i$ with c^i fixed by (5.54) also corresponding to the Levi-Civita connection on spatial hypersurfaces. We can relate this complex connection to the Ashtekar–Barbero connection by setting $\beta = -i$, the choice made in the original Ashtekar formulation [11]. Our Cartan frame θ^i is also defined with the opposite orientation, so in order to relate to [109] we need to identify $\theta^i = -\tilde{\theta}^i$ with $\sqrt{k} = 1/r_o$.

With $\beta = -i$, the expression for the (pure gravity) Hamiltonian given in [109] becomes

$$N\mathcal{C}_H = \frac{N}{\ell_P^2\sqrt{|\pi_1\pi_2\pi_3|}}\left[\pi_1\pi_2\tilde{c}_1\tilde{c}_2+\pi_1\pi_3\tilde{c}_1\tilde{c}_3+\pi_2\pi_3\tilde{c}_2\tilde{c}_3+\kappa\left(\pi_1\pi_2\tilde{c}_3+\pi_2\pi_3\tilde{c}_1+\pi_1\pi_3\tilde{c}_2\right)\right] \tag{5.147}$$

where the orientation factor is given by $\varepsilon = +1$. We now need to rewrite this Hamiltonian in terms of the variables used in the main text. First of all, we can identify

$$\tilde{c}^i = -ib^i + \Pi^i(p) \quad , \quad \pi_i = \ell_P^2\,p_i \tag{5.148}$$

where the first equation follows from our identification of the real and imaginary parts of the self-dual connection, and the second is a rescaling to ensure canonical Poisson brackets $\{b^i, p_j\} = \delta^i_j$ (which may also be obtained from equating the Urbantke metric with the physical metric 'q' defined in [109]). Finally, using the Urbantke metric (5.43), we can write the lapse N as $N = \ell_P|p_1p_2p_3|^{-1/2}\rho$ in terms of the Lagrange multiplier ρ. We then find

$$\begin{aligned} N\mathcal{C}_H = \rho\,\mathrm{sgn}(p_1p_2p_3)\Bigg[&-\frac{b^1b^2}{p_3} - \frac{b^1b^3}{p_2} - \frac{b^2b^3}{p_1} \\ &+\frac{\kappa^2}{2}\left(\frac{p_1p_2}{2p_3^3} + \frac{p_1p_3}{2p_2^3} + \frac{p_2p_3}{2p_1^3} - \frac{p_1}{p_2p_3} - \frac{p_2}{p_1p_3} - \frac{p_3}{p_1p_2}\right)\Bigg] \end{aligned} \tag{5.149}$$

which equals (apart from the overall sign) the purely gravitational part of the Hamiltonian $\rho\,\mathcal{H}_{\rm Lor} = \rho\left(\mathcal{H}_{(-)} + \kappa^2\,\mathcal{H}_{\rm BG}\right)$ in (5.69). The contribution of the cosmological constant replaces the matter Hamiltonian for a massless scalar field appearing in [109], specifically

$$\mathcal{H}_{matt} = \frac{N}{2}\,\frac{p_T^2}{\sqrt{|q|}} \quad \to \quad \frac{N}{\ell_P^2}\Lambda\,\sqrt{|q|} \tag{5.150}$$

which means adding an extra term

$$\frac{N}{\ell_P^2}\Lambda\,\sqrt{|\pi_1\pi_2\pi_3|} = \rho\,\ell_P^2\Lambda \tag{5.151}$$

to (5.149). This is exactly the Λ term found in (5.69), which shows that the Hamiltonian dynamics recover those of the Lorentzian Bianchi IX model with a given Λ for $p_1p_2p_3 > 0$. For the opposite sign, from (5.149) one would expect to see a relative minus sign which is not seen in (5.69). This discrepancy is explained by the fact that in that case the Urbantke metric (5.43) has signature $(+---)$, but satisfies the Einstein equations with the same Λ; it is hence equivalent to a solution with signature $(-+++)$ but cosmological constant $-\Lambda$. The Ashtekar formulation of general relativity assumes signature $(-+++)$ throughout and can match the sign only for $p_1p_2p_3 > 0$.

Chapter 6

Conclusions

In chapter 3 we demonstrated how one can incorporate the core principles of unimodular gravity – a fixed volume form and a cosmological constant which appears as an integration constant as opposed to a fundamental parameter – into the (chiral) Plebański formulation. Our constructions mirror closely what is done in the metric approach; one either introduces a background field μ_0 which fixes the value of the volume form, or one promotes Λ to a field constrained to be constant by a new Lagrange multiplier-like field which then also determines the volume form. These unimodular formulations of Plebański gravity yield solutions for all possible values of Λ, including complex values. Then in Lorentzian signature, one can no-longer discount the solutions of the reality conditions yielding a complex valued metric, as these would correspond to some complex value of Λ which we now permit. Following a similar procedure to the one outlined in section 2.4.2, one can also derive unimodular analogues of the chiral first order and pure connection formulations.

Recall that the Plebański formulation and its descendants are the starting point for the construction of spin foam models for general relativity [15, 88]. Then one interesting question is whether one could now construct spin foam models for unimodular gravity starting from these unimodular Plebański formulations. In this case, one should expect a different implementation of the simplicity constraint, which here involves the preferred volume form. On a more foundational level, a key aspect of unimodular formulations is that they provide a globally valid time coordinate defined by evaluating the 4-volume of spacetime regions. Then one has access to a Schrödinger-like approach to quantisation [103]. Quantisation of such models, whether through spin foams or otherwise, could hence be studied in terms of (potentially unitary) evolution in unimodular time.

There is also the question of how to incorporate matter into these formulations. In the formulations in which the 2-forms Σ^i are dynamical variables, for instance (3.4) and (3.12), one may construct a metric tensor via the Urbantke formula (2.97) which can be used to couple scalar and tensor fields in the usual way, as in [99]. The coupling of fermionic matter is discussed in brief in [29]. In the cases where Σ^i are not independent variables, as in (3.6) or (3.9) for example, one reconstructs Σ^i via $\Sigma^i = (M^{-1})^{ij} F_j$ or $\Sigma^i = (X^{-1/2})^{ij} F_j$ respectively. Then one constructs the Urbantke metric from these reconstructed 2-forms, and proceeds to couple matter via the same approach. Hence we see that consistent coupling of (bosonic) matter fields can be achieved for all the unimodular actions listed in chapter 3. However, the complexity of the expression for the Urbantke metric increases as we descend the hierarchy from Plebański-like to first order and ultimately pure connection formulations.

In chapter 4 we continued our exploration of these new unimodular formulations of Plebański gravity through canonical analysis. We began with a review of the canonical analysis of the standard Plebański theory, in which we recover the (complex) Ashtekar phase space and constraint structure. This was supplemented with further discussion on the reality conditions and the construction of the metric. In fact, we provide an alternate proof for the reality of the Urbantke metric (up to a global imaginary factor) in terms of the canonical variables. Following this, we derived the canonical formulation of the preferred volume unimodular Plebański theory (3.4). In this case, we arrive at a modified constraint algebra where the usual Hamiltonian constraint is replaced with its 3 spatial derivatives. In total, the gauge transformations generated by this modified algebra are a restricted set of the transformations generated in the full theory. This mirrors the restriction of the symmetry group to volume-preserving diffeomorphisms at the level of the (preferred volume) action. What's more, we derive a Hamiltonian which has a definite, non-constraint, part in addition to the usual constraint/vanishing part. Hence dynamical evolution is no longer pure gauge in this formulation. Then in principle, one could attempt a quantisation of this theory along the lines of [103]. Of course, one would still need to deal with the reality conditions in Lorentzian signature, which would introduce fairly significant complexities in the transition to the quantum theory.

In chapter 5, we investigated aspects of spatially homogeneous models in the context of chiral connection gravity theories. Through these investigations, we emphasised various conceptual and technical differences of these formulations when compared with approaches based on a metric or a (Ashtekar–Barbero) connection and tetrad. To summarise, we began with an attempt to construct a general canonical formulation for spatially homogeneous models. We discovered that such a construction is poorly defined for models with Bianchi I type symmetry – as well as other models – as the homogeneous analogue of the diffeomorphism constraint vanishes, and consequently we end up with too many independent degrees of freedom. Following the general trend of the literature, we restricted to diagonal models in order to avoid this issue. Starting from the first order chiral connection action (2.98), we inserted spatially homogeneous ansatzes for the fields to get a reduced action resembling the action for a classical particle in complex space. From this, we constructed a holomorphic Hamiltonian system describing the dynamics. To implement the reality conditions, which we saw decomposed into 4 distinct solution branches, we formulated our holomorphic Hamiltonian system as a pair of equivalent real Hamiltonian systems which we called the real and imaginary sectors. Using these, we implemented the reality conditions as second class constraints in Dirac's formalism; we saw that each branch of the reality conditions was only compatible with one of the two sectors. Ultimately, we were able to construct dimensionally reduced Hamiltonian systems describing Bianchi I and IX models, in agreement with [14, 109]. Additionally, we derived a canonical formulation for the diagonal Bianchi IX model in the Euclidean signature setting where the variables are real valued from the start, and there is no need for reality conditions. We saw that one can define complex transformations on the variables (Wick rotations) that map solutions of the Euclidean formulation to solutions of the various Lorentzian formulations corresponding to the different branches of the reality conditions. In the final parts of chapter 5, we investigated the quantum cosmology of homogeneous and isotropic spacetimes from the perspective of the chiral first order and pure connection formulations. Specifically, we constructed minisuperspace actions by inserting homogeneous and isotropic field ansatzes into the actions (2.98) and (2.105) which we used to compute transition amplitudes with connection boundary data using canonical path integral methods [39, 47]. Starting from the pure connection action, one derives a minisuperpsace action which is purely a boundary term. Then the two-point function is immediate, and we recover the result in [59] from a new perspective. Concluding, we also performed the same path integral computation starting from the unimodular formulation (3.16), recovering a slightly modified result.

As we have discussed, an interesting nuance of the chiral connection formulations of gravity is the inability to fix the signature of the metric throughout. Even in a specific solution branch of the reality conditions, one obtains two types of solutions with different metric signature, unlike in metric-based formulations or the Ashtekar–Barbero approach where the metric signature is fixed. Dynamical signature change within a solution would require passing through a surface with degenerate metric and divergent curvature, which we may see as a classical endpoint of a given solution. But in any case, this implies that Λ in our original action (2.98) represents the cosmological constant of general relativity only *up to sign*: the Urbantke metric in the chiral connection framework satisfies the Einstein equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$, but its signature cannot be fixed, and a change of signature $g_{\mu\nu} \to -g_{\mu\nu}$ could be absorbed in $\Lambda \to -\Lambda$. In the homogeneous and isotropic sector, we always find both de Sitter and anti-de Sitter solutions. This fact has interesting implications if we wanted to fix the value of Λ in (2.98) by comparing with observation; even observing accelerated expansion would only determine the magnitude of Λ, not its sign. These conclusions are somewhat reminiscent of arguments in favour of the emergence of expanding solutions for negative Λ in quantum cosmology [49], but here appear already in the classical theory. They seem to be a feature only of chiral connection formulations of gravity, which is *essential* for the existence of a "pure connection" formulation in which all other variables have been integrated out. Indeed, in approaches with fixed metric signature (5.113) would usually be a condition on $|p|$ rather than p (or p would be fixed to be positive, unlike what we observe here), so that we would not obtain (5.114).

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