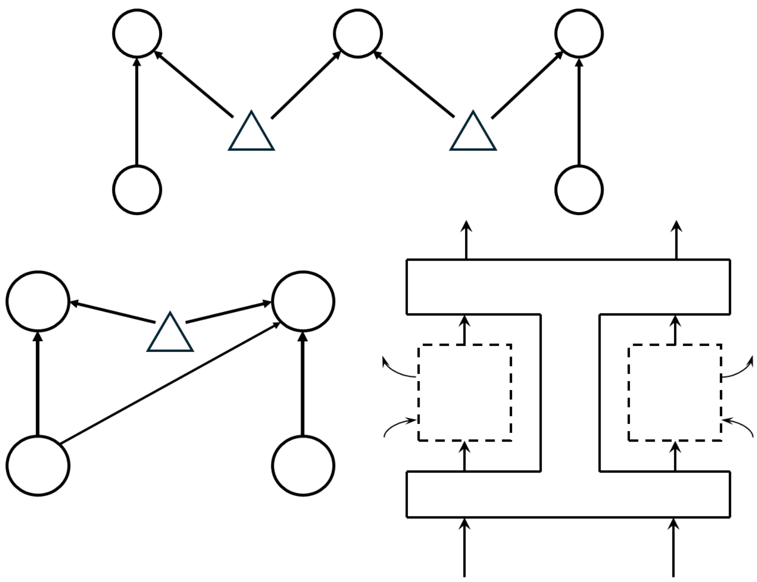
# Towards an Understanding of Quantum and Post-Quantum Correlations in Three Causal Settings

*Kuntal Sengupta*



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# Abstract

Understanding quantum and post-quantum correlations in various causal settings, is fundamental to singling out the first set from the latter. We present progress in this study in three directions: correlation self-testing of quantum theory, exploration of post-quantum correlations within an indefinite causal order, and certification of nonlocal quantum correlations in the absence of freedom of choice.

Correlation self-testing refers to identifying a set of tasks optimal performance of which, can only be achieved using quantum theory. The adaptive CHSH game, which was proposed as a candidate task, requires a theory to allow for entanglement swapping, if a post-classical performance is to be achieved. Its performance, however, was not explicitly tested in theories that do. In fact, a theory in which this task can be executed better than quantum theory, has also been proposed. We show that this theory does not violate Chained Bell inequalities and therefore can be ruled out. We present adaptive GHZ and adaptive Chained Bell Games and analyse its performance in various theories. Finally, we show that the existing theories that allow for entanglement swapping can also be ruled out using the adaptive CHSH game.

In the second part, we introduce an operational definition of superposition, consistent with quantum theory. We then construct a toy theory that admits superposition, under this definition, and can generate all non-signalling correlations in the Bell setting. We show, that even in an indefinite causal order, in particular, the switch setting, this theory can generate post-quantum correlations. We certify this using theory-independent techniques.

In the final part, we consider a relaxation of parameter independence in the Bell setting by allowing the parties to communicate with each other over a binary symmetric channel. We manage to show, that unless this channel is perfect, all quantum correlations cannot be reproduced using classical strategies. In addition, one way signalling is just as effective as two way signalling, for this channel model.

*To my parents, Rama and Pratul*

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# Author's declaration

I declare that the work presented in this thesis, except where otherwise stated, is based on my own research carried out at the University of York and has not been submitted previously for any degree at this or any other university. Sources are acknowledged by explicit references.

*“The true laboratory is the mind, where behind illusions, we uncover the laws of truth”*
*- Acharya Jagadish Chandra Bose*

# — 1 —

# Introduction

## 1.1 PROLOGUE

We have sought to understand the world around us for as far back as pre-history stretches. Perhaps even before, for octopuses can quickly learn to unscrew jars to access food. Clearly, there are underlying mechanisms that enable such an understanding of how objects like jars work. For the 21st-century physicist, for the most part, these mechanisms are mathematical models that describe various phenomena. A successful example can be found in the ability of the Hilbert space formalism of quantum theory to describe the microscopic world.

Although widely accepted, the Hilbert space formalism is rich in features that are highly counter-intuitive. Perhaps uncomfortably, it is purely based on mathematical assumptions, with little direct physical motivation to promote them. One might then question its wide acceptance. In the author's understanding, there are at least two reasons why: firstly, the predictions of this formalism, have been experimentally verified with remarkable accuracy. Therefore, to some, the physical significance of the underlying mathematical assumptions might hold feeble importance, as long as they keep on churning the wheels of technological progress. The second reason is slightly more subtle. There have been attempts to re-axiomatise quantum theory from first principles. At the core of each of these attempts, lie mathematical frameworks which are not fully physically motivated. Once these frameworks are assumed and some constraints (axioms) are imposed, quantum theory seems to emerge "naturally". Although these attempts are valuable in gaining insight into the structure of the theory, the physical meaning behind the axioms remains unclear. Therefore, choosing one mathematical formalism over another becomes somewhat subjective.

Apart from the predictions obtained from a mathematical model, understanding which other features of the model pertain to true physical phenomena, becomes relevant when seeking to uncover how Nature works, in the scale applicable to the model. For example, the aforementioned Hilbert space formalism of quantum theory is an intrinsically probabilistic model. If one now assumes that Nature is fundamentally probabilistic, one arrives at a large class of theories, many of which yield predictions that are not observed in experiments [81]. Why then these theories do not describe Nature, is a question that has been under scrutiny for the last few decades. One way to gain understanding is to take a top-down approach, and classify theories based on the permissible operations within them and what changes in predictions these operations bring in. One hopes that these classifications will point towards higher physical principles behind the various mathematical axioms. Generalised Probabilistic Theories(GPTs) [9] are an operational framework where these questions can be systematically phrased.

Quantum correlations generated by performing local measurements on parts of entangled states can generate correlations that cannot be explained by local physics, a phenomenon known as Bell nonlocality [11]. The large class of theories mentioned earlier share this feature as well, however, quantitatively by a larger amount than that achievable using quantum theory. What restrictions should one impose on set correlations, that are incompatible with local physics, such that they can be realised by quantum theory, has been a central topic of investigation in the foundations of quantum theory. Several interesting proposals [14, 58, 76, 71, 40] on how to constrain this set have provided deeper insights into the structure of quantum theory as well. However, for the most part, they single out a set of correlations slightly larger than what can be achieved using quantum theory [70].

An alternate and more objective approach was proposed [99, 98], where one looks for causal structures, in which the largest set of correlations that are incompatible with local physics are those that can be generated in quantum theory. This proposal came with a causal structure, in which, using entanglement swapping, one can generate quantum correlations that are non-realisable in most of the GPTs displaying stronger-than-quantum nonlocality; Primarily because they do not allow for entanglement swapping. However, examples of GPTs displaying such correlations and allowing for entanglement swapping also exist [93, 92]. Whether these theories, or other post-quantum theories allowing for entanglement swapping, can generate a larger set of correlations in the proposed causal structure, remains unexplored. A big portion of this thesis is dedicated to investigating this question.

An interesting feature of the Hilbert space formalism of quantum theory is superposition, which facilitates the description of operations that do not have a definite causal order [17]; The operations are in superposition of causal orders. Firstly, if elements of Nature can truly be in superposition of values that can be assigned to them, then their descriptions should not rely on the mathematical model used to describe them. In particular, if one departs from the Hilbert space formalism of quantum theory to a formalism that has equal predictive power, they should be able to formulate a consistent notion of superposition, and in fact indefinite causal order. In this thesis, we have tried to understand whether a consistent notion of superposition can be formulated in terms of the statistical properties of a theory and whether such a definition allows one to describe indefinite causal order.

The fact that quantum correlation are nonlocal, has immense applications in various information-theoretic tasks. To certify nonlocal correlations, however, one needs to justify locality. Miniaturisation of devices that produce nonlocal correlations, raises the question of whether the assumption of locality can still be justified. There is a vast technological interest in certifying quantum correlations nonlocal, that are generated in the absence of locality. However, whether quantum correlations generated under the assumption of locality rule out the presence of classical models that violate it remains less explored. Since, one does not always have access to the devices they are using, ruling out such classical model is a certificate of quantumness. In the final part of this thesis, we have tried to provide initial results towards this direction.

## 1.2 SYNOPSIS

Chapters in this thesis are collected into five parts. Part I consisting of Chapters 2, 3 and 4 provides the background for reading the main results. Chapters 5 and 6 in Part II, Chapter 7 in Part III, and Chapter 8 in Part IV constitute the main findings of this thesis. Part IV contains appendices to support the various claims made in the main text.

In Chapter 2, we present the framework of Generalised Probabilistic Theories (GPTs), highlighting the connection between Hilbert space formalism and probability state space formalism. In Chapter 4, we discuss compositional consistency in GPTs and introduce a necessary criterion, minimal $k$-preservibility, which transformations must satisfy to be compositionally consistent. In the short Chapter 4, we briefly overview notions of causal order, compatible causal structures and freedom of choice.

Next, in Chapter 5, we review the idea behind correlation self-testing, or simply self-testing, of theories and present new games that can be used for the purpose of self-testing of quantum theory. In Chapter 6, we present various GPTs, wherein, entanglement swapping is potentially possible. We show, however, that in most such GPTs, transformations underlying entanglement swapping are not compositionally consistent. Therefore, they are deemed invalid. A consequence of this is this is the ability to self-testing quantum theory against all such GPTs. Additionally, there are GPTs in which entanglement swapping transformations are compositionally consistent, but nevertheless, can also be ruled out. We conclude this chapter by presenting a connection between minimal $k$-preservibility and Tsirelson's bound, applicable to the GPTs we present.

Chapter 7 deals with the notion of superposition and indefinite causal order in GPTs, the latter of which is based around the quantum switch. We present a candidate definition of superposition for GPTs, consistent with the notion of quantum superposition. We then build a toy theory that admits superposition and show how, in that theory, it is possible to demonstrate indefinite causal order.

Finally, in Chapter 8, we show that it is possible to certify nonlocal non-signalling quantum correlations in the presence of arbitrarily large signalling, taking place over a binary symmetric channel. Additionally, using this channel, two way signalling cannot be used to gain any leverage over one way signalling when it comes to generating non-signalling correlations.

# Part I

# Preliminaries

# — 2 —

# Generalised Probabilistic Theories

## 2.1 INTRODUCTION

GPTs are a framework for operational theories, i.e., theories in which the basic building blocks are preparations of states, transformations of states and measurements on them. The earliest contribution to the development of GPTs is possibly due to Mackey [65] and Ludwig [63]. Further developments are due to Dähn [25, 26, 26], Stolz [95, 96], Davies and Lewis [29], Edwards [38, 35, 37, 34, 33, 36], Gudder [45], Mielnik [68, 67] and Ludwig [61, 62, 60], and a few years later by Kläy [56] and Wilce [100]. The most recent developments can be attributed to Popescu and Rohrlich [81] and Barrett [9]. A chronological read of the above shows the development of this framework over the last seven decades. In this chapter, we are going to use the presentation provided in [9].

The first building block, i.e., state preparation refers to setting up a system of interest in one of the configurations allowed in the theory. Transformation of state refers to evolving the system from one allowed state to another. Finally, measurement refers to the operation of deciphering the state of the system. In this chapter, we will only focus on state preparations and measurements. Let us denote by $s$ the state of a system and by $\mathcal{S}$ the set of all states the system could be in. $\mathcal{S}$ is referred to as the *state space* associated with the system. Now, let us assume that there exists a set of questions $M \coloneqq \{e_1, \cdots, e_n\}$, answers to which can fully describe $s$. In a deterministic theory, one of the answers to these $n$ questions will be yes and the rest will be no. In a probabilistic theory, the answer to each of these questions could be probabilistic. Therefore, let us associate to the answer $e_i[s]$, of the question $e_i$ on the state $s$, a probability $p_i$. Now, suppose that the state $s$ is a mixture $\lambda s_1 + (1-\lambda)s_2$ of two states $s_1$ and $s_2$ with $\lambda \in [0, 1]$. If $e_i[s_1] = p_{i,1}$ and $e_i[s_2] = p_{i,2}$, to respect convexity, we would like $\lambda p_{i,1} + (1-\lambda)p_{i,2} = p_i$, implying that the action of $e_i$ is convex-linear. For simplicity, we will assume linearity. The set $M$ is said to be a *measurement* if $\sum_i p_i = 1$. More precisely, $(\sum_{i=1}^{n} e_i)[s] = 1$ for every state $s \in \mathcal{S}$. The set of all questions is called the *effect space*.

One way to model state and effect spaces is to use real vector spaces. Here, a state space is identified as a subset of a real vector space and the effects are positive linear functional on the states. Below we formalise them concretely.

**Definition 2.1.1.** ***(State space:)*** *Let $\mathbb{V}$ be a finite dimensional real vector space and $\mathbb{V}^*$ its dual vector space. A state space is a compact convex subset of $\mathbb{V}$ such that there exists an element $u \in \mathbb{V}^*$ with the property that $\langle u, s\rangle$=1 for any $s \in \mathcal{S}$. The sub-normalised state space $\mathcal{S}_\leqslant$ of $\mathcal{S}$ is defined as the convex hull of $\mathcal{S}$ and the zero vector $v_0 \in \mathbb{V}$.*

The requirement of $\langle u, s\rangle = 1$ implies that every state is an element of a compact convex subset of a hyperplane in $\mathbb{V}$. We call this property of states as *normalisation*. Further, this entails that for $M$ to be a measurement, one requires $\sum_{i=1}^{n} e_i = u$.

**Definition 2.1.2.** ***(Effect space:)*** *Let $\mathbb{V}$ be a finite dimensional real vector space, $\mathbb{V}^*$ its dual vector space and $\mathcal{S}$ be a state space as defined in 2.1.1. The maximal effect space $\mathcal{E}_{\mathcal{S}}$ of $\mathcal{S}$ is a compact and convex subset of $\mathbb{V}^*$ defined as*

$$\mathcal{E}_{\mathcal{S}} := \{e \in \mathbb{V}^* \mid \langle e, s\rangle \in [0,1] \;\forall\; s \in \mathcal{S}\}.$$

*An effect space $\mathcal{E}$ is any subset of $\mathcal{E}_{\mathcal{S}}$ such that for every $e \in \mathcal{E}$, $u - e \in \mathcal{E}$, and $u, v_0^* \in \mathcal{E}$ where $v_0^*$ is the null effect with the property $\langle v_0^*, s\rangle = 0 \;\forall\; s \in \mathcal{S}$ and $u$ is called the unit effect.*

Now, a theory can have different types of systems. There should be a meaningful way of describing them both on their own and jointly. Let us label the various types of systems $\mathbf{A}_0, \mathbf{A}_1, \cdots$. A system in a given type can be in one of the various states allowed by that type. Let us associate to each type a state space: $\mathcal{S}_{\mathbf{A}_0}$ to type $\mathbf{A}_0$, $\mathcal{S}_{\mathbf{A}_1}$ to type $\mathbf{A}_1$ and so on, and denote their respective effect spaces by $\mathcal{E}_{\mathbf{A}_0}$, $\mathcal{E}_{\mathbf{A}_1}$ and so on. Given two state spaces $\mathcal{S}_{\mathbf{A}_i}$ and $\mathcal{S}_{\mathbf{A}_j}$, we denote by $\mathcal{S}_{\mathbf{A}_i} \boxtimes_{\mathbf{A}_i\mathbf{A}_j} \mathcal{S}_{\mathbf{A}_j}$ the state space corresponding to a system, which is a composite of a subsystem of type $\mathbf{A}_i$ and a subsystem of type $\mathbf{A}_j$. Here $\boxtimes_{\mathbf{A}_i\mathbf{A}_j}$ denotes a composition rule. This need not be thought of as a tensor product but rather a list of states allowed in the theory when one of the systems is of type $\mathbf{A}_i$ and the other of type $\mathbf{A}_j$. Additionally, there might exist several joint state spaces composed of $\mathcal{S}_{\mathbf{A}_i}$ and $\mathcal{S}_{\mathbf{A}_j}$, identified by the set of composition rules $\{\boxtimes^t_{\mathbf{A}_i\mathbf{A}_j}\}_t$. Note, that for every $(i,j)$, $t$ runs from 1 to $f(i,j)$, where $f$ is an integer function of $i$ and $j$. To completely describe a theory, one needs to specify the collection of all types of state spaces $\mathbf{S} := \{\mathcal{S}_{\mathbf{A}_i}\}_i$, their corresponding effect spaces $\mathbf{E} := \{\mathcal{E}_{\mathbf{A}_i}\}_i$ and the collection of composition rules $\boxtimes := \{\{\boxtimes^t_{\mathbf{A}_i,\mathbf{A}_j}\}_t\}_{(i,j)}$. With this, one can formally define a GPT as follows:

**Definition 2.1.3.** ***(GPT:)*** *Let $\mathbf{S}$ be the collection of state spaces $\{\mathcal{S}_{\mathbf{A}_i}\}_i$ in the vector spaces $\{\mathbb{V}_{\mathbf{A}_i}\}_i$ and $\mathbf{E}$ be the collection of the corresponding effect spaces $\mathbf{E} := \{\mathcal{E}_{\mathbf{A}_i}\}_i$ in the dual vector spaces $\{\mathbb{V}^*_{\mathbf{A}_i}\}_i$ and $\boxtimes$ is the collection of composition rules $\{\{\boxtimes^t_{\mathbf{A}_i,\mathbf{A}_j}\}_t\}_{(i,j)}$ where $i,j \in \mathbb{Z}_n$ and for each $(i,j)$, $t \in \mathbb{Z}_{f(i,j)}$, where $f$ is an integer function of $i$ and $j$. A GPT is the triple $(\mathbf{S}, \mathbf{E}, \boxtimes)$, where $\boxtimes$ is a composition rule with the property that, for every $t$,*

1. *$\mathcal{S}_{\mathbf{A}_i\mathbf{A}_j,t} := \mathcal{S}_{\mathbf{A}_i} \boxtimes^t_{\mathbf{A}_i\mathbf{A}_j} \mathcal{S}_{\mathbf{A}_j} \subset \mathbb{V}_{\mathbf{A}_i} \otimes \mathbb{V}_{\mathbf{A}_j}$ is a state space and $\mathcal{E}_{\mathbf{A}_i\mathbf{A}_j,t} := \mathcal{E}_{\mathbf{A}_i} \boxtimes^t_{\mathbf{A}_i\mathbf{A}_j} \mathcal{E}_{\mathbf{A}_j} \subset \mathbb{V}^*_{\mathbf{A}_i} \otimes \mathbb{V}^*_{\mathbf{A}_j}$ is the effect space of $\mathcal{S}_{\mathbf{A}_i\mathbf{A}_j,t}$, and for any $s_{\mathbf{A}_i\mathbf{A}_j,t} \in \mathcal{S}_{\mathbf{A}_i\mathbf{A}_j,t}$,*

$$\left(\mathsf{id}_{\mathbf{A}_i} \otimes e_{\mathbf{A}_j}\right)\left(s_{\mathbf{A}_i\mathbf{A}_j,t}\right) \in \mathcal{S}_{\mathbf{A}_{i\leqslant}} \quad \textit{and} \quad \left(e_{\mathbf{A}_j} \otimes \mathsf{id}_{\mathbf{A}_j}\right)\left(s_{\mathbf{A}_i\mathbf{A}_j,t}\right) \in \mathcal{S}_{\mathbf{A}_{j\leqslant}}$$

   *where $\mathsf{id}_{\mathbf{A}_i/\mathbf{A}_j} : \mathbb{V}_{\mathbf{A}_i/\mathbf{A}_j} \to \mathbb{V}_{\mathbf{A}_i/\mathbf{A}_j}$ is the identity map, $e_{\mathbf{A}_i/\mathbf{A}_j} \in \mathcal{E}_{\mathbf{A}_i/\mathbf{A}_j}$ and $\otimes$ is the tensor product,*

2. *for any collection of states $\{(r_k)_{\mathbf{A}_i}\}_{k=1}^m \subseteq \mathcal{S}_{\mathbf{A}_i}, \{(s_k)_{\mathbf{A}_j}\}_{k=1}^m \subseteq \mathcal{S}_{\mathbf{A}_j}$, the state*

$$\sum_{k=1}^{n} \lambda_k (r_k)_{\mathbf{A}_i} \otimes (s_k)_{\mathbf{A}_j} \in \mathcal{S}_{\mathbf{A}_i\mathbf{A}_j,t}$$

   *where $\lambda_k \geqslant 0 \;\forall\; k \;\in \{1, \cdots, n\}$, $\sum_{k=1}^{n} \lambda_k = 1$,*

3. *the effect space* $\mathcal{E}_{\mathbf{A}_i\mathbf{A}_j,t}$ *of* $\mathcal{S}_{\mathbf{A}_i\mathbf{A}_j,t}$, *is a subset of the maximal effect space* $\mathcal{E}_{\mathcal{S}_{\mathbf{A}_i\mathbf{A}_j,t}}$ *defined in* (2.1.2).

Note, that the definition of state space implies that the description of any state in the theory can be completely encoded in the entries of a finite real vector. Meaning, that state tomography can be performed using a finite set of measurements. The set of measurements with which state tomography can be performed are called *fiducial* measurements. Now for a bipartite system composed of subsystems of type $\mathbf{A}$ and $\mathbf{B}$, since its associated state space is a subset of $\mathbb{V}_\mathbf{A} \otimes \mathbb{V}_\mathbf{B}$, any bipartite state can be identified by local tomography. In addition, for any pair of effects $e_\mathbf{A}, e_\mathbf{B} \in \mathcal{E}$, and state $s_{\mathbf{AB}} \in \mathcal{S}_{\mathbf{AB}}$, since both $(\mathsf{id}_\mathbf{A} \otimes e_\mathbf{B})(s_{\mathbf{AB}})$ and $(e_\mathbf{A} \otimes \mathsf{id}_\mathbf{B})(s_{\mathbf{AB}})$ are valid sub-normalised states,

$$(e_\mathbf{A} \otimes e_\mathbf{B})(s_{\mathbf{AB}}) = (\mathsf{id}_\mathbf{A} \otimes e_\mathbf{B})(e_\mathbf{A} \otimes \mathsf{id}_\mathbf{B})(s_{\mathbf{AB}}) = (e_\mathbf{A} \otimes \mathsf{id}_\mathbf{B})(\mathsf{id}_\mathbf{A} \otimes e_\mathbf{B})(s_{\mathbf{AB}}) \in [0,1], \tag{2.1}$$

and hence all product effects $e_\mathbf{A} \otimes e_\mathbf{B}$ are elements of the bipartite effect space. Additionally, local actions of effects on respective subsystems always commute. From Definition 2.1.2, one can see that the only necessary restriction between a state and effect space pair is that the inner product between an effect and a state must be between 0 and 1. Although we have mentioned state space first, an alternative is to first define an effect space $\mathcal{E}$ and then choose an appropriate set $\mathcal{S} \subseteq \mathcal{S}_\mathcal{E}$ as the state space. Either way, for a fixed state space $\mathcal{S}$, $\mathcal{E}_\mathcal{S}$ is its maximal effect space and for a fixed effect space $\mathcal{E}$, $\mathcal{S}_\mathcal{E}$ is its maximal state space. We will use both these approaches. In Section 3.2, we will explain how this definition can be used to describe more parties.

## 2.2 COMPOSITE SYSTEMS

In the previous section, we saw that given two state spaces $\mathcal{S}_\mathbf{A}$ and $\mathcal{S}_\mathbf{B}$, a composition rule $\boxtimes_{\mathbf{AB}}$ specifies the composite state space $\mathcal{S}_{\mathbf{AB}}$. We present here two examples of composition rules that allow one to construct the joint state space, regardless of the types of the systems being composed together. These are the *minimal* and *maximal* tensor product compositions, formally defined below.

**Definition 2.2.1.** ***(Minimal and Maximal Tensor Products)*** *Let* $\mathcal{S}_\mathbf{A} \subset \mathbb{V}_\mathbf{A}$ *and* $\mathcal{S}_\mathbf{B} \subset \mathbb{V}_B$ *be two state spaces and let* $\mathcal{E}_\mathbf{A}$ *and* $\mathcal{E}_\mathbf{B}$ *be their respective effect spaces. Then*

- *the minimal tensor product of* $\mathcal{S}_\mathbf{A}$ *and* $\mathcal{S}_\mathbf{B}$ *is*

$$\mathcal{S}_\mathbf{A} \underset{\min}{\otimes} \mathcal{S}_\mathbf{B} := \mathrm{ConvHull}\{s_\mathbf{A} \otimes s_\mathbf{B} \mid s_\mathbf{A} \in \mathcal{S}_\mathbf{A}, s_\mathbf{B} \in \mathcal{S}_\mathbf{B}\},$$

- *the maximal tensor product of* $\mathcal{S}_\mathbf{A}$ *and* $\mathcal{S}_\mathbf{B}$ *is*

$$\mathcal{S}_\mathbf{A} \underset{\max}{\otimes} \mathcal{S}_\mathbf{B} := \{s_{\mathbf{AB}} \in \mathbb{V}_\mathbf{A} \otimes \mathbb{V}_\mathbf{B} \mid \langle e_\mathbf{A} \otimes e_\mathbf{B}, s_{\mathbf{AB}}\rangle \in [0,1],\ \forall\ e_\mathbf{A} \in \mathcal{E}_\mathbf{A}, e_\mathbf{B} \in \mathcal{E}_\mathbf{B}\}.$$

A similar definition for effect spaces is also possible. It is important to point out that the maximal effect space compatible with $\mathcal{S}_\mathbf{A} \underset{\min}{\otimes} \mathcal{S}_\mathbf{B}$ is $\mathcal{E}_\mathbf{A} \underset{\max}{\otimes} \mathcal{E}_\mathbf{B}$ and the maximal effect space compatible with $\mathcal{S}_\mathbf{A} \underset{\max}{\otimes} \mathcal{S}_\mathbf{B}$ is $\mathcal{E}_\mathbf{A} \underset{\min}{\otimes} \mathcal{E}_\mathbf{B}$. The maximal tensor product state space, as defined, is the largest set of bipartite states for which marginalisation to single system gives a valid state in the single system state space. Therefore, for any arbitrary composition $\boxtimes_{\mathbf{AB}}$ of state spaces, one gets the following chain of inclusions: $\mathcal{S}_{\mathbf{A}} \underset{\min}{\otimes} \mathcal{S}_{\mathbf{B}} \subseteq \mathcal{S}_{\mathbf{A}} \boxtimes_{\mathbf{AB}} \mathcal{S}_{\mathbf{B}} \subseteq \mathcal{S}_{\mathbf{A}} \underset{\max}{\otimes} \mathcal{S}_{\mathbf{B}}$. Although the definitions above are stated for bipartite systems, they can capture multipartite systems as well.

In the following, we illustrate how this framework allows discussions of classical, quantum and post-quantum theories.

## 2.3 EXAMPLE I: CLASSICAL PROBABILITY THEORY

Any theory in which all state spaces can be represented as simplices is referred to as *classical* probability theory. For instance, take a biased coin as the system of interest. The state of this coin describes the degree of bias it has. The only fiducial measurement we need to describe this state is tossing the coin and observing the outcome. To its event space $\{0, 1\}$, let us associate a probability density function $\mathrm{p}_{\mathrm{coin}}$ with the property $\mathrm{p}_{\mathrm{coin}}(0) = p$ and $\mathrm{p}_{\mathrm{coin}}(1) = 1 - p$ for some $p \in [0, 1]$. The extremal states of the coin correspond to the deterministic outcomes when either $p = 0$ or $p = 1$. Therefore, the probability state space corresponding to the state of a biased coin can be geometrically represented by a line segment in $\mathbb{R}^2$ with $(1, 0)$ and $(0, 1)$ as the extremal states. Similarly, for a certain event with 3 outcomes, the extremal states are:

$$s_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

which can be seen as the vertices of a triangle. In general, the state space for a $k$-outcome measurement can be geometrically represented by a $(k - 1)$-simplex. The min- and the max-tensor product of state spaces that can be represented by simplices coincide and is also a simplex [4, 7].

## 2.4 EXAMPLE II: NON-CLASSICAL PROBABILITY THEORIES

Theories with non-simplicial state spaces are referred to as *non-classical* probability theory. Below, we present three examples of non-classical probability theories, the last one being quantum theory.

### 2.4.1 Generalised Local Theory

A *generalised local theory* (GLT) refers to any GPT where the single system state space allows all probability distributions and in which every multipartite state is separable across all bipartitions [1] [9]. We provide two examples of non-classical GLTs that are relevant to this thesis. Consider the *gbit* state space $\mathcal{G}_{\mathrm{m}}^{\mathrm{n}}$ of a single system for which state tomography requires m fiducial measurements having n outcomes each, such that any valid probability distribution on these measurements and outcomes are permitted [2]. The min- tensor product of two such gbit state spaces

[1] Any classical probability theory is a GLT.

[2] In principle, one can have a theory where the numbers of outcomes depend on the choice of the corresponding measurements.

$\mathcal{G}^{\mathrm{n_A}}_{\mathrm{m_A}}$ and $\mathcal{G}^{\mathrm{n_B}}_{\mathrm{m_B}}$ is always a generalised local theory. The two examples are cases of this composition when i) $\mathrm{m_A} = \mathrm{m_B} = \mathrm{n_A} = \mathrm{n_B} = 2$ and when ii) $\mathrm{m_A} = \mathrm{m_B} = 3$ and $\mathrm{n_A} = \mathrm{n_B} = 2$.

When $\mathrm{m} = \mathrm{n} = 2$, the state space $\mathcal{G}^2_2$ can be characterised as the convex hull of four extremal deterministic states, in particular

$$s_1 = \begin{pmatrix} 1 \\ 0 \\ \hline 1 \\ 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 \\ 0 \\ \hline 0 \\ 1 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 \\ 1 \\ \hline 1 \\ 0 \end{pmatrix}, \quad s_4 = \begin{pmatrix} 0 \\ 1 \\ \hline 0 \\ 1 \end{pmatrix},$$

where we have used the notation $s = (\mathrm{p}(0|0)\ \mathrm{p}(1|0) \mid \mathrm{p}(0|1)\ \mathrm{p}(1|1))^T$ with $\mathrm{p}(a|x)$ denoting the probability of observing the outcome labelled $a$ when the fiducial measurement labelled $x$ is performed. For describing a bipartite state $\varsigma$, we will use the notation,

$$\varsigma = \left(\begin{array}{cc|cc} \mathrm{p}(0,0|0,0) & \mathrm{p}(0,1|0,0) & \mathrm{p}(0,0|0,1) & \mathrm{p}(0,1|0,1) \\ \mathrm{p}(1,0|0,0) & \mathrm{p}(1,1|0,0) & \mathrm{p}(1,0|0,1) & \mathrm{p}(1,1|0,1) \\ \hline \mathrm{p}(0,0|1,0) & \mathrm{p}(0,1|1,0) & \mathrm{p}(0,0|1,1) & \mathrm{p}(0,1|1,1) \\ \mathrm{p}(1,0|1,0) & \mathrm{p}(1,1|1,0) & \mathrm{p}(1,0|1,1) & \mathrm{p}(1,1|1,1) \end{array}\right) \tag{2.2}$$

where $\mathrm{p}(a, b|x, y)$ is the probability of getting the outcomes labelled $a$ and $b$ when the (fixed) fiducial measurements labelled $x$ and $y$ are performed on the respective subsystems. Although written as a matrix, this can be thought of as a way to write a 16-element vector.

The extremal states of the min-tensor product composition of two $\mathcal{G}^2_2$ state spaces are locally deterministic and can be calculated by taking the Kronecker product of $s_i$ with $s_j$ for $i, j \in \{1, 2, 3, 4\}$ [9]. As an example,

$$s_1 \otimes s_1 = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The resultant joint state space is a polytope characterised by the convex hull of these 16 states. We denote this state space as $\mathbb{H}^{[0]}_{(2,2)}$, where the 0 denotes the absence of any non-separable or entangled extremal state and $(2, 2)$ signifies $\mathrm{m} = \mathrm{n} = 2$. Any polytope can be characterised either by the convex hull of its extremal states (vertex description) or by a set of inequalities defining its facets (facet description) (see e.g. Section 2.2.4 of [13]). Assuming normalisation and non-signalling [3] conditions hold, $\mathbb{H}^{[0]}_{(2,2)}$ can be characterised by 24 facets, out of which 16 are positivity facets (i.e., corresponding to $\mathrm{p}(a, b|x, y) \geqslant 0\ \forall\ a, b, x, y$), justifying valid probabilities and the remaining 8 are

[3]Non-signalling conditions impose that the probability distributions realisable in a theory cannot signal from one party to another. For simplicity, in the representation (2.2), a state is non-signalling if the sum of the first two probabilities in each row (or column) equals the sum of the second two. We will discuss this in more detail in Section 2.7.

called CH facets. To list the CH facets, consider the following 4 vectors in $\mathbb{R}^{16}$:

$$
\begin{aligned}
e_{\mathrm{CH}_1} &= \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right), e_{\mathrm{CH}_2} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right), \\
e_{\mathrm{CH}_3} &= \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), e_{\mathrm{CH}_4} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right);
\end{aligned}
\tag{2.3}
$$

The 8 CH facets are given by $\{\langle e_{\mathrm{CH_i}}, \mathbf{x}\rangle \leqslant 1\}_{i=1}^4$ and $\{\langle e'_{\mathrm{CH_i}}, \mathbf{x}\rangle \leqslant 1\}_{i=1}^4$ where $e'_{\mathrm{CH}_i} := u - e_{\mathrm{CH_i}}$, $\mathbf{x} \in \mathbb{R}^{16}$ and the inner product is defined element-wise. Each CH facet inequality is saturated by 8 local deterministic states. Next, to find the effect polytope, let us first recall that for a vector $e \in \mathbb{V}^*$ to be an effect $e$, it must satisfy $\langle e, s\rangle \in [0,1]$ for any state $s$ in the state space (see Def. 2.1.3). Since we defined state spaces to be convex and compact, it sufficient to check whether $\langle e, s_i\rangle \in [0,1]$ for every extremal state $s_i$ of the state space. We denote the extremal states of $\mathbb{H}^{[0]}_{(2,2)}$ as Vert $\left[\mathbb{H}^{[0]}_{(2,2)}\right]$. The set of facet-defining inequalities of the effect polytope $\mathcal{E}_{\mathbb{H}^{[0]}_{(2,2)}}$ is then given by:

$$
\text{Facets}\left[\mathcal{E}_{\left[\mathbb{H}^{[0]}_{(2,2)}\right]}\right] := \left\{\mathbf{x}.s_{\text{vertex}} \geqslant 0 \mid s_{\text{vertex}} \in \text{Vert}\left[\mathbb{H}^{[0]}_{(2,2)}\right]\right\} \bigcup \left\{\mathbf{x}.s_{\text{vertex}} \leqslant 1 \mid s_{\text{vertex}} \in \text{Vert}\left[\mathbb{H}^{[0]}_{(2,2)}\right]\right\}; \tag{2.4}
$$

Finding the vertices of a polytope from its facets is called *vertex enumeration*. For this work, we have used PANDA [59] to solve all vertex enumeration problems. In the present case, we find that the effect polytope has 90 extremal effects (see Appendix B.1 for a full classification) of which 82 are separable effects and the remaining 8 are entangled effects of the form $e_{\mathrm{CH}_i}$ and $e'_{\mathrm{CH}_i}$ with $i = 1, \cdots, 4$. The 82 separable effects consist of the positivity effects and linear combinations of them. These 82 effects form the extremal effects of the largest effect space compatible with the maximal tensor product of two copies of $\mathcal{G}_2^2$; We will introduce this in the next section.

When $\mathrm{m}_{\mathbf{A}} = \mathrm{m}_{\mathbf{B}} = 3$ and $\mathrm{n}_{\mathbf{A}} = \mathrm{n}_{\mathbf{B}} = 2$, the extremal (deterministic) states of the state space $\mathcal{G}_3^2$ are

$$
\left(\begin{array}{c}1\\0\\\hline 1\\0\\\hline 1\\0\end{array}\right), \left(\begin{array}{c}1\\0\\\hline 1\\0\\\hline 0\\1\end{array}\right), \left(\begin{array}{c}1\\0\\\hline 0\\1\\\hline 1\\0\end{array}\right), \left(\begin{array}{c}1\\0\\\hline 0\\1\\\hline 0\\1\end{array}\right), \left(\begin{array}{c}0\\1\\\hline 1\\0\\\hline 1\\0\end{array}\right), \left(\begin{array}{c}0\\1\\\hline 1\\0\\\hline 0\\1\end{array}\right), \left(\begin{array}{c}0\\1\\\hline 0\\1\\\hline 1\\0\end{array}\right), \left(\begin{array}{c}0\\1\\\hline 0\\1\\\hline 0\\1\end{array}\right);
$$

There are 64 local deterministic states of the state space polytope $\mathbb{H}^{[0]}_{(3,2)}$ formed by taking the min-tensor product of two $\mathcal{G}_3^2$ state spaces. Alternatively, assuming normalisation and non-signalling conditions hold, $\mathbb{H}^{[0]}_{(3,2)}$ can be characterised by 36 positivity facets and 648 Bell facets [4]. These Bell facets can be categorised into two equivalence classes: the first containing 72 CH facets and the second containing 576 $I_{3322}$ facets [41, 78, 22]. In particular, consider the following two vectors in $\mathbb{R}^{36}$ in the condensed notation as (2.2):

$$\mathrm{F}_{\mathrm{CH}} = \left(\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \qquad \mathrm{F}_{\mathrm{I}_{3322}} = \frac{1}{3}\left(\begin{array}{cc|cc|cc} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right) \tag{2.5}$$

The CH facets are given by $\langle \mathrm{F}_{\mathrm{CH}}, \mathbf{x}\rangle \leqslant 1$ and the $\mathrm{I}_{3322}$ facets are given by $\langle \mathrm{F}_{\mathrm{I}_{3322}}, \mathbf{x}\rangle \leqslant 1$ where $\mathbf{x} \in \mathbb{R}^{36}$. The remaining elements for each class can be found by applying all relabelling symmetries to $\mathrm{F}_{\mathrm{CH}}$ and $\mathrm{I}_{3322}$ respectively, and then discarding duplicates. There are 32 extremal states that satisfy $\langle \mathrm{F}_{\mathrm{CH}}, s\rangle = 1$ and 32 extremal states that satisfy $\langle \mathrm{F}_{\mathrm{CH}}, s\rangle = 0$. On the other hand, there are 20 extremal state with $\langle \mathrm{F}_{\mathrm{I}_{3322}}, s\rangle = 1$, 28 with $\langle \mathrm{F}_{\mathrm{I}_{3322}}, s\rangle = 2/3$, 12 with $\langle \mathrm{F}_{\mathrm{I}_{3322}}, s\rangle = 1/3$ and 4 with $\langle \mathrm{F}_{\mathrm{I}_{3322}}, s\rangle = 0$. There are at most 18 extremal locals states that simultaneously saturate facets from each class.

We perform a vertex enumeration similar to the previous example to find that the effect polytope $\mathcal{E}_{\mathbb{H}^{[0]}_{(3,2)}}$ is given by the convex hull of 27968 extremal effects. A classification of these effects is provided in Table A.2 of Appendix A.

### 2.4.2 Box-World

Any GPT is said to be nonlocal if it is not a sub-theory of GLT. An example is the maximal tensor product of $\mathcal{G}^{\mathrm{n}_\mathbf{A}}_{\mathrm{m}_\mathbf{A}}$ and $\mathcal{G}^{\mathrm{n}_\mathbf{B}}_{\mathrm{m}_\mathbf{B}}$, also called *box-world* (BW). For the case of $\mathrm{m}_\mathbf{A} = m_\mathbf{B} = n_\mathbf{A} = n_\mathbf{B} = 2$, the extremal states include the 16 local deterministic states and 8 entangled states called PR boxes [80, 81, 9]. We denote this state space as $\mathbb{H}^{[8]}_{(2,2)}$ and list the 8 PR boxes:

$$\mathrm{PR}_1 = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right), \mathrm{PR}_2 = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right), \mathrm{PR}_3 = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right),$$

$$\mathrm{PR}_4 = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right), \mathrm{PR}'_1 = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right), \mathrm{PR}'_2 = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right),$$

[4] They are called Bell facets because they can witness whether state of the form $\varsigma$ can be written as $\sum_i \lambda_i r_i \otimes s_i$, where $\lambda_i \geqslant 0$ and $\sum_i \lambda_i = 1$; A consequence of Bell's theorem [11].

$$\mathrm{PR}_3' = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right), \mathrm{PR}_4' = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right);$$

Notice that the pairs $\mathrm{PR_i}$ and $\mathrm{PR_i'}$ are called *isotropically opposite*; Equal mixtures of them give the maximally mixed state.

Next, when $\mathrm{m}_{\mathbf{A}} = \mathrm{m}_{\mathbf{B}} = 3$ and $\mathrm{n}_{\mathbf{A}} = \mathrm{n}_{\mathbf{B}} = 2$, the state space polytope corresponding to the max-tensor product has 1408 extremal states, out of which 64 are local deterministic. Let us denote this polytope as $\mathbb{H}^{[1344]}_{(3,2)}$, where 1344 denotes the number of extremal entangled states in the state space. These entangled states can be classified into 4 relabelling classes in accordance with the discussion above [54]. We present one candidate state from each of these classes below :

$$\mathrm{N}_1 = \frac{1}{2}\left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 2 \end{array}\right), \mathrm{N}_2 = \frac{1}{2}\left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right),$$

$$\mathrm{N}_3 = \frac{1}{2}\left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right), \mathrm{N}_4 = \frac{1}{2}\left(\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right);$$

Each entangled state violates multiple $\mathrm{F_{CH}}$ and $\mathrm{F_{I_{3322}}}$ facets and each $\mathrm{F_{CH}}$ or $\mathrm{F_{I_{3322}}}$ has multiple entangled states violating them. Tables 2.1 and 2.2 below summarise this information.

| Class | # | $\#\mathrm{F_{CH}}$ | $\#\mathrm{F_{I_{3322}}}$ |
|---|---|---|---|
| $\mathrm{N}_1$ | 288 | 1 | 8 |
| $\mathrm{N}_2$ | 192 | 6 | 18 |
| $\mathrm{N}_3$ | 288 | 4 | 24 |
| $\mathrm{N}_4$ | 576 | 2 | 12 |

Table 2.1: Table above summarises "#" the number of elements in each class, "$\#\mathrm{F_{CH}}$" and "$\#\mathrm{F_{I_{3322}}}$" the number of $\mathrm{F_{CH}}$ and $\mathrm{F_{I_{3322}}}$ facets violated by an element of each class.

| Inequality | $\#\mathrm{N}_1$ | $\#\mathrm{N}_2$ | $\#\mathrm{N}_3$ | $\#\mathrm{N}_4$ |
|---|---|---|---|---|
| $\mathrm{F_{I_{3322}}}$ | 4 | 6 | 12 | 12 |
| $\mathrm{F_{CH}}$ | 4 | 16 | 16 | 16 |

Table 2.2: Table summarising the number of entangled states from each class violating a single facet.

The effect polytope of $\mathcal{E}_{\mathbb{H}^{[1344]}_{(3,2)}}$ has 248 extremal effects. All 248 of these effects are separable [90] and are extremal effects of $\mathcal{E}_{\mathbb{H}^{[0]}_{(3,2)}}$ as well. Just like the 2 input setting discussed earlier, these 248 effects can be classified into 7 relabelling classes. A classification of these effects can be found in [39] which we summarise in Table A.1 of Appendix A for completeness.

### 2.4.3 Qubit Quantum Theory

Qubit quantum theory is an example of a GPT where local tomography requires three fiducial measurements with two outcomes each. Below, we show how to connect the density matrix formalism of qubit quantum theory to the *probability state formalism*, where every state is represented by a table of probabilities (see (2.2)). In the density matrix formalism, a qubit is represented by a $2 \times 2$ density matrix, i.e., a positive semi-definite complex matrix with unit trace. The set of all such density matrices $\mathbb{D}(\mathbb{C}^2)$ is a strict subset of the real vector space $\mathbb{M}_{\mathrm{h}}(\mathbb{C}^2)$ of $2 \times 2$ Hermitian matrices. An effect or POVM element is represented by a positive semi-definite complex matrix $\Pi$ such that $0 \leqslant \mathrm{T}r[\rho\Pi] \leqslant 1$ for any $\rho \in \mathbb{D}(\mathbb{C}^2)$. We denote the set of all POVM elements as $\mathcal{E}_{\mathbb{D}(\mathbb{C}^2)}$ [5]. Since $\mathbb{D}(\mathbb{C}^2)$ and $\mathcal{E}_{\mathbb{D}(\mathbb{C}^2)}$ are both closed convex and compact, they form a well-defined state and effect space pair. The composite state space of two qubits is a subset of the real vector space $\mathbb{M}_{\mathrm{h}}(\mathbb{C}^2) \otimes \mathbb{M}_{\mathrm{h}}(\mathbb{C}^2)$. The composition rule $\tilde{\otimes}$ simply identifies this subset as the set of $4 \times 4$ density matrices. Compositions of multiple qubits can be understood similarly. With these, the GPT $\left(\mathbb{D}(\mathbb{C}^2), \mathcal{E}_{\mathbb{D}(\mathbb{C}^2)}, \tilde{\otimes}\right)$ describes qubit quantum theory [6] in the density matrix formalism. An analogous treatment is possible for quantum systems with higher (finite) dimensions.

The probability state formalism can be derived, from above, in the following way. Given a $2 \times 2$ density matrix, $\rho$, we first fix a set of fiducial measurements. A common choice is

$$\{M_x\}_{x \in \{0,1,2\}} \coloneqq \left\{ \left\{ \frac{\mathbb{I} + \sigma_x}{2}, \frac{\mathbb{I} - \sigma_x}{2} \right\} \right\}_{x \in \{0,1,2\}} \tag{2.6}$$

where

$$\sigma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

denote the Pauli matrices [7] and $\mathbb{I}$ is the identity matrix. The state tomography of $\rho$ with this choice of measurement then defines a convex bijection $\mathbb{D}(\mathbb{C}^2) \to \mathbb{R}^3$ which we embed in $\mathbb{R}^6$ as follows:

$$\rho \longmapsto \frac{1}{2} \left( \begin{array}{c} \mathrm{T}r\left[(\mathbb{I} + \sigma_0)\rho\right] \\ \mathrm{T}r\left[(\mathbb{I} - \sigma_0)\rho\right] \\ \hline \mathrm{T}r\left[(\mathbb{I} + \sigma_1)\rho\right] \\ \mathrm{T}r\left[(\mathbb{I} - \sigma_1)\rho\right] \\ \hline \mathrm{T}r\left[(\mathbb{I} + \sigma_2)\rho\right] \\ \mathrm{T}r\left[(\mathbb{I} - \sigma_2)\rho\right] \end{array} \right) = \left( \begin{array}{c} p(0|0) \\ p(1|0) \\ \hline p(0|1) \\ p(1|1) \\ \hline p(0|2) \\ p(1|2) \end{array} \right) =: \mathrm{p}_\rho, \tag{2.7}$$

[5] It is well known that the positive cone of complex square matrices is self-dual (see Example 2.24 of [13]). Therefore the positive cone generated by $\mathbb{D}(\mathbb{C}^2)$ is the same as the positive cone generated by $\mathcal{E}_{\mathbb{D}(\mathbb{C}^2)}$.

[6] Note that the minimal tensor product $\mathbb{D}(\mathbb{C}^2) \underset{\min}{\otimes} \mathbb{D}(\mathbb{C}^2)$ is a strict subset of $\mathbb{D}(\mathbb{C}^2)^{\tilde{\otimes}2}$, and describes the set of separable states in this formalism. Characterisation of the maximal tensor product is unknown.

[7] Conventionally, the Pauli matrices are denote by $\sigma_1$, $\sigma_2$ and $\sigma_3$. We have used non-standard notation here.

We call $p_\rho$ the *probability state* of $\rho$ and denote the set of all probability states obtained upon performing state tomography on qubits using Pauli measurements $\mathbb{P}[\mathbb{D}(\mathbb{C}^2)]$, the *probability state space* of a qubit. Note that since the set $\mathbb{D}(\mathbb{C}^2)$ is compact and convex, so is $\mathbb{P}[\mathbb{D}(\mathbb{C}^2)]$ via the convex bijection. The composite probability state space for two qubits can be similarly derived by performing local tomography with all pairs of fiducial measurements on all bipartite states in $\mathbb{D}(\mathbb{C}^2)^{\tilde{\otimes}2}$. We denote this joint state space as $\mathbb{P}[\mathbb{D}(\mathbb{C}^2)^{\tilde{\otimes}2}]$. A chain of inclusions $\mathbb{P}[\mathbb{D}(\mathbb{C}^2)] \underset{\min}{\otimes} \mathbb{P}[\mathbb{D}(\mathbb{C}^2)] \subset \mathbb{P}[\mathbb{D}(\mathbb{C}^2)^{\tilde{\otimes}2}] \subset \mathbb{P}[\mathbb{D}(\mathbb{C}^2)] \underset{\max}{\otimes} \mathbb{P}[\mathbb{D}(\mathbb{C}^2)]$ is invoked since probability states corresponding to entangled qubits are not necessarily separable, and $\mathbb{P}[\mathbb{D}(\mathbb{C}^2)^{\tilde{\otimes}2}]$ is not the maximal state space when only separable effects are considered.

It is important to stress that the Hilbert space and probability state formalism of quantum theory (or any physical theory) are equivalent in their statistical predictions. However, there are non-statistical features in the Hilbert space formalism, such as superposition, that do not have a direct analogue in the probability state formalism. It is then natural to ask whether such model-specific features pertain to any real description of Nature, or whether, they only serve as mathematical tools enabling the model to provide reasonably accurate predictions. We will discuss this in greater detail in Chapter 7.

## 2.5 ASIDE ON RELABELLINGS AND CHSH GAMES

### 2.5.1 Relabellings

So far, we have assumed that the probability representation of a state requires two ingredients: a fixed set of fiducial measurements and a fixed labelling of these measurements and their outputs. In a lab setting, these labels could correspond to a fixed colour coding of the switches at the input and LEDs at the output of some measurement apparatus. In general, an alternative colour coding, corresponding to the relabelling of the measurements and their outcomes, will give an alternate description of the same state. Note, that there are multiple ways of relabelling: relabelling the measurement choices, outcomes, or more generally, outcomes only for a particular choice of measurement, or in fact, any sequence of these. As an example, if a measurement previously labelled as 0 is now relabelled to 1 and vice versa, a state that gives rise to the probability table $\mathrm{PR}_1$ would instead give rise to $\mathrm{PR}_2$ and vice versa. For bipartite (in general multipartite) systems, there are two types of relabellings: i) local relabellings, i.e., relabelling the inputs and outputs for each subsystem and ii) global relabellings, i.e., relabelling the subsystems as well. The latter will be discussed further in Section 3.2.

An interesting observation is that the set of probability descriptions of extremal states resulting from relabelling the measurement choice instanced above coincides with the set of previously obtained probability descriptions. Even more strongly, this happens for any choice of relabelling and does not depend on the initial labels used. The list of probability descriptions that the state can be assigned to via relabellings is exactly 8 and coincides with the 8 probability tables $\mathrm{PR}_\mathrm{i}/\mathrm{PR}'_{\mathrm{i}]}$ above. Since with any fixed labelling, the 8 probability tables describe 8 distinct states, a relabelling operation that leaves the overall probability description of the state space unchanged, corresponds to an allowed reversible transformation on the state space. For instance, since relabelling a measurement 0 to 1 by both parties leaves the probability description of $\mathbb{H}^{[8]}_{(2,2)}$ unchanged, there is an allowed transformation on the state space that maps $s_1$ (the state that gives rise to $\mathrm{PR}_1$ under some fixed fiducial measurements) to $s_2$ (the state that gives rise to $\mathrm{PR}_2$ under the same fiducial measurements), with analogous maps on the other extremal states. Similarly, since all relabelling operations leave $\mathbb{H}^{[8]}_{(2,2)}$ invariant, there is a reversible transformation on the underlying state space corresponding to each of the relabellings. Therefore, these 8 entangled states are equivalent up to these reversible transformations, and their probability descriptions are equivalent up to relabelling operations. Hence, $\mathbb{H}^{[8]}_{(2,2)}$ contains two classes of extremal states up to relabelling: local deterministic states and PR boxes.

### 2.5.2 CHSH Games

There is an interesting connection between the probability tables PR above and CHSH games. CHSH games are played between two players who can share non-classical resources but are otherwise isolated in closed labs. The players are asked random questions labelled by random variables $X$ and $Y$ to which they need to provide answers labelled by random variables $A$ and $B$ respectively. The players win if their input-output bits satisfy the winning conditions of the game of interest. In the simplest scenario where $|X| = |Y| = |A| = |B| = 2$, the 8 winning conditions corresponding to the 8 given games are given by $a \oplus b = xy \oplus c_0 x \oplus c_1 y \oplus c_2$, where $c_0, c_1$ and $c_2$ are binary variables. Then the 8 vectors $\{\mathrm{CHSH}_i \coloneqq 1/2\mathrm{PR}_i\}_{i=1}^4$ and $\{\mathrm{CHSH}'_i \coloneqq 1/2\mathrm{PR}'_i\}_{\mathrm{i}=1}^4$ define the 8 CHSH games. The factor 1/4 arises since the binary questions $X$ and $Y$ are randomly chosen. The score of a correlation $\mathrm{p}(A,B|X,Y)$ in the game $\mathrm{CHSH}_i$ is given by $\langle 1/2\mathrm{PR}_i, \mathrm{p}(A,B|X,Y)\rangle$. For instance, since the correlation table obtained after performing the fiducial measurements on $\mathrm{PR}_1$ coincides with the probability table of $\mathrm{PR}_1$, one can see that

$$\mathrm{CHSH}_1[\mathrm{p}_{\mathrm{PR}_1}(A,B|X,Y)] = \langle 1/2\mathrm{PR}_1, \mathrm{p}_{\mathrm{PR}_1}(A,B|X,Y)\rangle =$$
$$\left\langle \left(\begin{array}{cc|cc} 1/4 & 0 & 1/4 & 0 \\ 0 & 1/4 & 0 & 1/4 \\ \hline 1/4 & 0 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 0 \end{array}\right), \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \right\rangle = 1. \tag{2.8}$$

The vectors $\mathrm{CHSH}_i$ and $\mathrm{CH_i}$ are related by an affine transformation, for the facets of the state space $\mathbb{H}^{[0]}_{(2,2)}$ can also be written as $\{\langle \mathrm{C_i}, \mathbf{x}\rangle \leqslant 3/4\}_{\mathrm{i}}$.

## 2.6 A HILBERT SPACE FORMALISM OF GLT AND BW

This section has been inspired from useful discussions with Stefan Weigert and Alaister Mansfield.

In the previous section, we saw that there is more than one way to mathematically model a theory. In particular, qubit quantum theory can be described in both the Hilbert space formalism and, as shown above, the probability state space formalism. In Section 2.4.1 and Section 2.4.2 we also saw the probability state space formalism of GLT and BW respectively. In this section, we will present a Hilbert space formalism of these two theories. We will stick to the case where state tomography requires three or two binary outcome fiducial measurements. We will first state the effect space and then work out the state space.

Let us start with the vector space of $2 \times 2$ Hermitian matrices. Let

$$\mathbb{E} := \text{ConvHull}\left\{\{\mathbf{0}, \mathbb{I}\} \bigcup \left\{\frac{\mathbb{I} \pm \sigma_i}{2}\right\}_{i=0}^{2}\right\} \tag{2.9}$$

be the effect space for a single system, where $\mathbf{0}$ is the zero matrix. The facet defining inequalities of the corresponding maximal state space $\mathcal{S}_{\mathbb{E}}$ can be written as

$$\text{Facets}\left[\mathcal{S}_{\mathbb{E}}\right] := \left\{\varrho \in \mathbb{M}_{\text{h}}\left(\mathbb{C}^2\right) \mid \text{Tr}\left[\rho e\right] \leqslant 1 \; \forall \; e \in \mathbb{E}\right\}; \tag{2.10}$$

Now, recall that since the Pauli matrices together with the identity matrix span the vectors space $\mathbb{M}_{\text{h}}\left(\mathbb{C}^2\right)$, any unit trace hermitian matrix can be expressed as

$$\varrho = \frac{\mathbb{I} + r_0\sigma_0 + r_1\sigma_1 + r_2\sigma_2}{2}, \tag{2.11}$$

where $r_i \in \mathbb{R} \; \forall \; i \in \{0, 1, 2\}$. The set of inequalities (2.10) above poses constraints on the ranges of $r_i$. Notice that,

$$\text{Tr}\left[\varrho\frac{\mathbb{I} \pm \sigma_i}{2}\right] = \frac{1 \pm r_i}{2}, \tag{2.12}$$

with which, the constraints above reduce to

$$\left\{0 \leqslant \frac{1 \pm r_i}{2} \leqslant 1\right\}_{i=0}^{2} = \left\{-1 \leqslant r_i \leqslant 1\right\}_{i=0}^{2}. \tag{2.13}$$

We can now construct 8 states by combining the various extremal values allowed for each $r_i$. These 8 states are:

$$\begin{pmatrix} 1 & \frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} + \frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} + \frac{i}{2} & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{1}{2} + \frac{i}{2} \\ \frac{1}{2} - \frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2} + \frac{i}{2} \\ \frac{1}{2} - \frac{i}{2} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{1}{2} - \frac{i}{2} \\ -\frac{1}{2} + \frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{2} - \frac{i}{2} \\ -\frac{1}{2} + \frac{i}{2} & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\frac{1}{2} + \frac{i}{2} \\ -\frac{1}{2} - \frac{i}{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{2} + \frac{i}{2} \\ -\frac{1}{2} - \frac{i}{2} & 1 \end{pmatrix};$$

Compactly,

$$\left\{\frac{\mathbb{I} \pm \sigma_0 \pm \sigma_1 \pm \sigma_2}{2}\right\}.$$

One can check that each of these states satisfies the inequalities (2.10) and since they are elements of $\mathbb{M}_{\text{h}}\left(\mathbb{C}^2\right)$, a theory in which the state space is the convex hull of these 8 states requires 3 fiducial measurements for state tomography. In addition, if these measurements are the projectors corresponding to the Pauli matrices, then a short calculation shows that the probability state space corresponding to $\mathcal{S}_{\mathbb{E}}$ is $\mathcal{G}_3^2$. In fact, the probability representation of the first state above is the first local deterministic state written out while characterising $\mathcal{G}_3^2$ and so on.

Bipartite state spaces can be constructed by assuming that state tomography of bipartite states can be done by performing local tomography of each subsystem. Therefore, instead of solving a set of constraints we can take each probability state and find a $4 \times 4$ unit trace hermitian matrix, such that if it were the bipartite state, on performing local tomography with the three Pauli measurements one would get back the probability state they started with. The extremal bipartite states for GLT are simply the Kronecker product of each of the states above. This gives us 64 states corresponding to the 64 local deterministic states mentioned earlier. For the extremal BW states, we provide the states to which $\mathrm{N}_1$, $\mathrm{N}_2$, $\mathrm{N}_3$ and $\mathrm{N}_4$ get mapped to in this process.

$$\mathrm{N}_1 \mapsto \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}-\frac{i}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}+\frac{i}{2} & 0 & 0 & 1 \end{pmatrix}$$

$$N_2 \mapsto \begin{pmatrix} 0 & -\frac{1}{4}+\frac{i}{4} & -\frac{1}{4}+\frac{i}{4} & \frac{1}{2}-\frac{i}{2} \\ -\frac{1}{4}-\frac{i}{4} & \frac{1}{2} & 0 & \frac{1}{4}-\frac{i}{4} \\ -\frac{1}{4}-\frac{i}{4} & 0 & \frac{1}{2} & \frac{1}{4}-\frac{i}{4} \\ \frac{1}{2}+\frac{i}{2} & \frac{1}{4}+\frac{i}{4} & \frac{1}{4}+\frac{i}{4} & 0 \end{pmatrix}$$

$$\mathrm{N}_3 \mapsto \begin{pmatrix} \frac{1}{2} & \frac{1}{4}-\frac{i}{4} & \frac{1}{4}-\frac{i}{4} & \frac{1}{2}-\frac{i}{2} \\ \frac{1}{4}+\frac{i}{4} & 0 & 0 & -\frac{1}{4}+\frac{i}{4} \\ \frac{1}{4}+\frac{i}{4} & 0 & 0 & -\frac{1}{4}+\frac{i}{4} \\ \frac{1}{2}+\frac{i}{2} & -\frac{1}{4}-\frac{i}{4} & -\frac{1}{4}-\frac{i}{4} & \frac{1}{2} \end{pmatrix}$$

$$\mathrm{N}_3 \mapsto \begin{pmatrix} 0 & \frac{1}{4}+\frac{i}{4} & 0 & \frac{1}{2}-\frac{i}{2} \\ \frac{1}{4}-\frac{i}{4} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{4}-\frac{i}{4} \\ \frac{1}{2}+\frac{i}{2} & 0 & -\frac{1}{4}+\frac{i}{4} & \frac{1}{2} \end{pmatrix}$$

Of special interest could be the Hilbert space state of the PR box. For this, let us assume that fiducial measurements are performed using Pauli $\sigma_i$ and $\sigma_j$ with $i, j \in \{0, 1, 2\}$, $i \neq j$ . The three Hilbert space states that the box $\mathrm{PR}_1$ gets mapped to for each $(i, j)$ are

$$\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}-\frac{i}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}+\frac{i}{2} & 0 & 0 & 1 \end{pmatrix}_{(0,1)}, \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \end{pmatrix}_{0,2}, \begin{pmatrix} 0 & -\frac{i}{4} & -\frac{i}{4} & 0 \\ \frac{i}{4} & \frac{1}{2} & \frac{1}{2} & \frac{i}{4} \\ \frac{i}{4} & \frac{1}{2} & \frac{1}{2} & \frac{i}{4} \\ 0 & -\frac{i}{4} & -\frac{i}{4} & 0 \end{pmatrix}_{(1,2)},$$

where the subscripts denote the choices of the Pauli measurements used.

## 2.7 THE SET OF CORRELATIONS

Recall that since local actions of effects on respective subsystems commute, when local measurements are performed on two halves of a composite system, the marginal distribution obtained from one half is the same whether calculated before or after the measurement on the other half. Consequentially, *non-signalling* correlations emerge which for bipartite systems can be phrased as follows.

$$\begin{aligned}\sum_b \mathrm{p}(a,b|x,y) = \sum_b \mathrm{p}(a,b|x,y') = \mathrm{p}(a|x) \quad \text{for all } a,x,y,y' \\ \sum_a \mathrm{p}(a,b|x,y) = \sum_a \mathrm{p}(a,b|x',y) = \mathrm{p}(b|y) \quad \text{for all } b,x,x',y\end{aligned} \tag{2.14}$$

In Section 2.4.3, we have introduced probability state spaces to phrase quantum theory. More generally, state spaces for any GPT can be described using probability tables. The interpretation here is that, although one might not have knowledge of the algebraic structure of state space, the probability state space gives a list of probability tables obtained when local tomography is performed on every state allowed in the theory. A potential confusion could be in the difference between a probability state space and the set of correlations generated from that state space. We attempt to clarify this here. Consider a probability state space $\mathcal{S}$ of a single system and its maximal effect space $\mathcal{E}_{\mathcal{S}}$. Further, consider the collection of all sets of $m$ measurements, each of which has $n$ outcomes, i.e.,

$$\mathcal{M}_{m,n} \coloneqq$$

$$\left\{ \left\{ \{e_{0|x}, \cdots, e_{a|x}, \cdots e_{n-1|x}\} \right\}_{x=0}^{m-1} \;\middle|\; e_{a|x} \in \mathcal{E}_{\mathcal{S}} \;\forall\; x \in \{0, \cdots, m-1\}, a \in \{0, \cdots, n-1\}, \sum_{a=0}^{n-1} e_{a|x} = u \right\}.$$

Let $X$ denote the random variable corresponding to the measurement choice $x$, $A$ denote the random variable corresponding to the outcome $a$ and $|Z|$ denote the number of values that a random variable $Z$ takes. Then, for some state $s \in \mathcal{S}$, the collection

$$\left\{ \mathrm{p}_s(A|X) \;\middle|\; |A| = n, |X| = m \right\}$$

denotes the set of all conditional probability distributions, or correlations, that can be generated when there are $m$ possible measurements that can be performed on $s$, each of which has $n$ outcomes. Here $|A|$ denotes the number of possible outcomes. A correlation set of the state space $\mathcal{S}$ can now be defined as:

$$\mathcal{C}_{(1,m,n)} \coloneqq \left\{ \mathrm{p}_s(A|X) \;\middle|\; |A| = n, |X| = m, s \in \mathcal{S} \right\},$$

where $(1, m, n)$ signifies the measurement setting and the number 1 highlights that we are talking about a single system. In the same way, a bipartite correlation set can be defined as:

$$\mathcal{C}_{(2,m,n)} \coloneqq \left\{ \mathrm{p}_\varsigma(A,B|X,Y) \;\middle|\; |A| = |B| = n, |X| = |Y| = m, \varsigma \in \mathcal{S}_{\mathbf{AB}} \right\}.$$

This can be generalised to more parties, i.e., to measurement settings $(p, m, n)$. A measurement setting for $p > 1$ is often referred to as a *Bell setting*. Note that the set of probability tables forming the description of a state space is always a subset of the set of correlations, when $m$ in the setting equals the number of fiducial measurements required to perform tomography, and $n$ equals the number of outcomes of those fiducial measurements. In this thesis, when $\varsigma$ is a bipartite probability state with $m = n = 2$, we will use $\mathrm{CHSH_i}[\varsigma]$ to denote the score in the game $\mathrm{CHSH_i}$ of the distribution obtained when fiducial measurements are performed on $\varsigma$. For example, $\mathrm{CHSH_1}[\mathrm{PR_1}] = 1$ or $\mathrm{CHSH_2}[\mathrm{PR_1}] = 1/2$.

Given a set of correlations, it is not always possible to uniquely identify the underlying state space. For example, assume that the underlying state space is the convex hull of $\mathbb{H}^{[0]}_{(2,2)}$ and $\mathrm{PR}_1$. Now, label the collection of fiducial measurements as $\{e_{0|0}, e_{1|0}\}$ and $\{e_{0|1}, e_{1|1}\}$, such that the correlation $\mathrm{p}_{\mathrm{PR}_1}(A, B|X, Y)$ obtained when this measurement is performed on $\mathrm{PR}_1$ is identical to the probability table of $\mathrm{PR}_1$. Now consider a relabelling of the same measurements such that the outcomes previously labelled as 0 are now relabelled to 1, and vice-versa, for the system labelled $\mathbf{A}$. The correlation $\mathrm{p}_{\mathrm{PR}_1}(A', B'|X, Y)$ obtained when the relabelled fiducial measurements are performed on $\mathrm{PR}_1$ would now be identical to the probability table of $\mathrm{PR}'_1$. Similarly, one can keep on relabelling the fiducial measurements and generate correlations that match the probability tables of the rest of the PR boxes. The fact that, for example, all PR boxes can be realised in the correlation space does not mean that there are states associated with each PR box in the state space.

# — 3 —

# Minimal $k$-Preservibility Criterion

Evolution of systems is an integral part of any theory. Once a collection of state spaces has been specified, we will assume that every GPT allows a system to evolve from a state in one state space to a state in the same or another state space. In this chapter, we will study the properties of maps that define such transformations.

Let us consider the evolution of a system from type $\mathbf{A}_0$ to type $\mathbf{A}_1$ and let the associated state spaces be $\mathcal{S}_{\mathbf{A}_0}$ and $\mathcal{S}_{\mathbf{A}_1}$ respectively. Further, let $\mathcal{T}$ be the map characterising this evolution. More precisely, $\mathcal{T}$ is a map from the vector space $\mathbb{V}_{\mathbf{A}_0}$ to the vector space $\mathbb{V}_{\mathbf{A}_1}$. Now, suppose that $\mathcal{T}$ maps the states $s_1, s_2 \in \mathcal{S}_{\mathbf{A}_0}$ to the states $r_1, r_2 \in \mathcal{S}_{\mathbf{A}_1}$ respectively. An experimenter flips a biased coin and prepares $s_1$ if she gets a Head and prepares $s_2$ if she gets a Tail. Let us assume that the probability of getting a Head is $p$ and the probability of getting a Tail is $1-p$, where $p \in [0,1]$. After the evolution happens, if the state were $s_1$, the evolved state will be $r_1$ and if the state were $s_2$, the evolved state will be $r_2$. However, if she forgets the outcome of the coin toss, from her perspective, the input state would be a convex mixture of $s_1$ and $s_2$, i.e., $ps_1 + (1-p)s_2$ and similarly, the expected output state should be the convex mixture $pr_1 + (1-p)r_2$. With this reasoning, the action of the map $\mathcal{T}$ needs to be convex-linear. More precisely,

$$\mathcal{T}\left(ps_1 + (1-p)s_2\right) = p\mathcal{T}(s_1) + (1-p)\mathcal{T}(s_2), \tag{3.1}$$

For simplicity, we will make a stronger assumption that the actions of all maps governing evolution of systems in any GPT are linear on the underlying vector spaces.

Linearity of maps, however, is not sufficient in order for them to characterise evolution of systems; Such maps should be able to suitably describe situations where the system is seen as a part of a larger composite, the rest of which is not undergoing any evolution. An introductory understanding of this property is the overarching theme of the following sections.

## 3.1 COMPLETE PRESERVIBILITY

Suppose a system in state $s \in \mathcal{S}_{\mathbf{A}_0}$ is seen as a subsystem of a larger composite in state $\varsigma \in \mathcal{S}_{\mathbf{A}_0} \boxtimes_{\mathbf{A}_0\mathbf{K}} \mathcal{S}_{\mathbf{K}}$. If a map $\mathcal{T}$ truly defines the evolution of the system in state $s$, it should consistently evolve the state $s$ to a state $r \in \mathcal{S}_{\mathbf{A}_1}$, without affecting the system type of the part it is not acting on. In other words, the evolved composite system must be in some state $\vartheta \in \mathcal{S}_{\mathbf{A}_1} \boxtimes_{\mathbf{A}_1\mathbf{K}} \mathcal{S}_{\mathbf{K}}$. A reasonable way to necessitate this is by requiring:

$$\mathcal{T}^{\mathbf{A}_0\to\mathbf{A}_1} \otimes \mathsf{id}^{\mathbf{K}}(\varsigma) \in (\mathcal{S}_{\mathbf{A}_1} \boxtimes_{\mathbf{A}_1\mathbf{K}} \mathcal{S}_{\mathbf{K}}), \tag{3.2}$$

where $\otimes$ represents a tensor product and $\mathsf{id}^{\mathbf{K}}$ is the identity map. Recall from the discussion leading to Definition 2.1.3, that the state space $\mathcal{S}_{\mathbf{K}}$ can also be compositions of subsystems of various types. The map $\mathcal{T}$ is said to be *$K$-preserving* if (3.2) holds. $\mathcal{T}$ is said to be *completely preserving* if it is $K$-preserving for every system type $\mathbf{K}$ allowed in the theory. In the following, we will see how these conditions manifest in some of the theories we have introduced in Chapter 2.

### 3.1.1 Complete Preservibility in Generalised Local Theory

In GLT, a composite state space is constructed using the minimal tensor product (see Definition 2.2.1) of single system state spaces. The various types of single system state spaces are solely given by the type of gbit state spaces allowed, i.e., the number of measurements with the corresponding number of outputs required to perform state tomography. Now, consider a certain composite formed of $(m+k)$ single systems, with types $\mathbf{A}_0, \cdots, \mathbf{A}_{m-1}$ and $\mathbf{K} \coloneqq \mathbf{K}_m, \cdots, \mathbf{K}_{m+k-1}$. Every extremal state $\varsigma$ in the composite state space can be written as

$$\varsigma = (s_0 \otimes s_1 \otimes \cdots \otimes s_{m-1}) \otimes \left(s'_m \otimes \cdots \otimes s'_{m+k-1}\right), \tag{3.3}$$

where $s_i \in \mathcal{G}_{\mathbf{A}_i}$ and $s'_j \in \mathcal{G}_{\mathbf{B}_j}$ for all $i \in \{0, \cdots, m-1\}$ and $j \in \{m, \cdots, m+k-1\}$. Now, let $\mathcal{T}$ define the evolution of systems from a state in the composite system labelled by $(\mathbf{A}_0, \cdots, \mathbf{A}_{m-1})$ to the composite system labelled by $(\mathbf{A}'_0, \cdots, \mathbf{A}'_{m-1})$. The action of $\mathcal{T}$ on the appropriate subsystems of $\varsigma$ can then be seen as

$$\begin{aligned} \mathcal{T}^{\mathbf{A}\to\mathbf{A}'} \otimes \mathsf{id}^{\mathbf{B}}(\varsigma) &= \mathcal{T}^{\mathbf{A}\to\mathbf{A}'}(s_0 \otimes s_1 \otimes \cdots \otimes s_{m-1}) \otimes \mathsf{id}^{\mathbf{K}}\left(s'_m \otimes \cdots \otimes s'_{m+k-1}\right) \\ &= \vartheta \otimes \left(s'_m \otimes \cdots \otimes s'_{m+k-1}\right) \end{aligned} \tag{3.4}$$

where $\vartheta$ is a state in the composite state space labelled by $(\mathbf{A}'_0, \cdots, \mathbf{A}'_{m-1})$. Finally, since $\mathcal{T}$ is linear on the underlying vector spaces, its action on a convex combination of extremal states can be understood similarly. The fact that we haven't specified anything about the composite system type labelled by $\mathbf{K}$, implies that every such map is completely preserving.

*Remark.* Here, 0-preserving means that the map is an effect of the state space it is acting on.

### 3.1.2 Complete Preservibility in Box-World

The single-system state spaces in BW are identical to the single-system state spaces of classical theory. Composite systems are described by the maximal tensor product of subsystems. It has been shown in [79] and independently in [12] that every $K$-preserving map is completely preserving. This implication might initially appear trivial, from these two examples, but it is important to stress that in quantum theory it does not hold.

### 3.1.3 Complete Preservibility in Quantum Theory

In the Hilbert space formalism of quantum theory, preservibility is equivalent to positivity since every density matrix is positive semi-definite. Here, the transpose map is positive but not completely positive. To see why, consider the action of the map on the qubit state space:

$$\mathrm{T} : \mathbb{D}\left(\mathbb{C}^2\right) \to \mathbb{D}\left(\mathbb{C}^2\right)$$

$$\rho \mapsto (\rho)^{\mathrm{T}} ,$$

where $(\rho)^{\mathrm{T}}$ is the matrix transpose of $\rho$. $(\rho)^{\mathrm{T}}$ is a valid density matrix, implying that T is positive. However, the action of T on a subsystem of a composite system does not always result in valid density matrices. For example,

$$\mathrm{T} \otimes \mathsf{id}\left(\Phi_+\right) = \mathrm{T} \otimes \mathsf{id} \left[ \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \right] = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ,$$

where $\Phi_+ := |\phi_+\rangle\langle\phi_+|$ with $|\phi_+\rangle := (|00\rangle + |11\rangle)/\sqrt{2}$. The $4 \times 4$ matrix on the right has negative eigenvalues and therefore is not a density matrix. The partial transpose has been used for witnessing entanglement, and it turns out that for bipartite qubit systems, positivity under the partial transpose is necessary and sufficient to conclude that a density matrix is not entangled [77, 51]. An extension of this to continuous variables, in particular Gaussian states has also been studied [91].

Necessary and sufficient conditions for complete positivity of maps have been extensively studied and criteria for maps to be completely positive have been proposed by Stinespring [94], Choi [19, 18] (see also [3]) and Kraus [57]. We refer the reader to [74] and [88] for a concise reading on the developments on this topic. Such a condition for any GPT is unknown. In the following sections, we restrict ourselves to only effects and present an easily verifiable necessary criterion for effects to be completely preserving.

## 3.2 MINIMAL K-PRESERVIBILITY

From this section and onwards, unless explicitly specified otherwise, we will assume that all single systems are of the same type and there is a single composition rule for every $k$-partite system with $k \geqslant 0$ i.e., for a given $k$, all $k$-partite systems are also of a single type. The preservibility discussion from above can then be simplified to the action of a map on subsystems of a composite which can now be numbered rather than being typed. In the following, we will first rephrase the preservibility requirements in terms of the number of systems and present our criterion thereafter.

So far, the only consistency condition we have put on state and effect spaces is that they should result in valid probabilities, i.e., their inner products are between 0 and 1. To build a full theory, we also need to consider composability of systems. In particular, effects corresponding to a given number of systems should also be compatible with states of larger systems, regardless of how they are arranged. More precisely, expanding on the first property of a composition rule $\boxtimes$ from

Definition 2.1.3, we need that for two state spaces $\mathcal{S}^{\boxtimes m}$ and $\mathcal{S}^{\boxtimes k}$ and effect space $\mathcal{E}^{\boxtimes m} \subseteq \mathcal{E}_{\mathcal{S}^{\boxtimes m}}$, every effect $\tilde{e} \in \mathcal{E}^{\boxtimes m}$ must have the property that

$$\frac{\mathsf{id}^{\otimes k_1} \otimes \tilde{e} \otimes \mathsf{id}^{\otimes k_2}\left(\varsigma\right)}{\left\langle u, \mathsf{id}^{\otimes k_1} \otimes \tilde{e} \otimes \mathsf{id}^{\otimes k_2}\left(\varsigma\right)\right\rangle} \in \mathcal{S}^{\boxtimes k}, \tag{3.5}$$

for any $\varsigma \in \mathcal{S}^{\boxtimes(m+k)}$ and $m, k, k_1, k_2 \in \mathbb{Z}_{\geqslant 0}$ with $k_1 + k_2 = k$. Here $\tilde{e}$ acts on any arbitrary $m$ subsystems out of the $m+k$ systems described by $\varsigma$ and $\mathcal{S}^0$ is the trivial system, i.e., the length 1 vector with entry 1. This requirement ensures that $\tilde{e}$ respects the composition $\boxtimes$ by preserving the state space structure even when not acting on any arbitrary $k$ subsystems. We call this property *k-preservibility* and say that $\tilde{e}$ is a *k-preserving* effect [1]. If $\tilde{e}$ is *k-preserving* for all $k \geqslant 0$, we say that $\tilde{e}$ is a completely preserving effect.

Given a rule $\boxtimes$, a complete characterisation of every state in $\mathcal{S}^{\boxtimes(m+k)}$ may not be straightforward, since there might be infinitely many extremal states[2]. Therefore, even showing that an effect $\tilde{e}$ is $k$-preserving is challenging. We thereby present a necessary condition for $\tilde{e}$ to be $k$-preserving that can be checked once all extremal states in $\mathcal{S}^{\boxtimes m}$ and $\mathcal{S}^{\boxtimes k}$ are known but a full list of extremal states in $\mathcal{S}^{\boxtimes(m+k)}$ is not. Recall from Definition. 2.2.1 that for two state spaces $\mathcal{S}^{\boxtimes m}$ and $\mathcal{S}^{\boxtimes k}$, one always has $\mathcal{S}^{\boxtimes m} \underset{\min}{\otimes} \mathcal{S}^{\boxtimes k} \subseteq \mathcal{S}^{\boxtimes(m+k)}$. Consequently, for any two states $r \in \mathcal{S}^{\boxtimes m}$ and $s \in \mathcal{S}^{\boxtimes k}$, their tensor product $r \otimes s$ is an element of $\mathcal{S}^{\boxtimes(m+k)}$. Therefore if $e$ is $k$-preserving, it must at least consistently act on any $m$ subsystems of $r \otimes s$. We call this necessary condition for $k$-preservability *minimal k-preservability* and formally define it below.

**Definition 3.2.1.** ***(Minimal k-Preservability)*** *Let $\mathcal{S}^{\boxtimes m}$ and $\mathcal{S}^{\boxtimes k}$ be a $m$- and $k$-partite state spaces and $r \otimes s$ be a state describing $m+k$ systems where $r \in \mathcal{S}^{\boxtimes m}$ describes a composite system labelled by $X_r := \{1, 2, \cdots, m\}$ and $s \in \mathcal{S}^{\boxtimes k}$ describes the composite system labelled by $X_s := \{m+1, m+2, \cdots, m+k\}$. Let $B_r \subseteq X_r$, $B_s \subseteq X_s$, $A_r = X_r \setminus B_r$ and $C_s = X_s \setminus B_s$. An effect $e \in \mathcal{E}_{\mathcal{S}^{\boxtimes m}}$ is said to be minimally k-preserving if*

$$\frac{\mathsf{id}^{A_r} \otimes e^{B_r \cup B_s} \otimes \mathsf{id}^{C_s}\left(r \otimes s\right)}{\left\langle u, \mathsf{id}^{A_r} \otimes e^{B_r \cup B_s} \otimes \mathsf{id}^{C_s}\left(r \otimes s\right)\right\rangle} \in \mathcal{S}^{\boxtimes k}$$

*for any $r \in \mathcal{S}^{\boxtimes m}$ and $s \in \mathcal{S}^{\boxtimes k}$ and any $B_r, B_s$ such that $|B_r| + |B_s| = m$, where $e^{B_r \cup B_s}$ denotes the action of $e$ on any arbitrary $m$ subsystems labelled by $B_r \cup B_s$.*

Requiring minimal 2-preservability puts constraints on the effect space, in the sense that not all elements of $\mathcal{E}_{\mathcal{S}}$ correspond to valid effects. It is natural to ask whether there is a simple set of sufficient conditions to test $k$- (more ambitiously complete) preservibility, however, we are not aware of one. Nevertheless, to illustrate the significance of this criterion, we provide below two examples of GPTs in which minimal 2-preservibility is broken.

[1] Remark: $K$-preservibility simplifies to $k$-preservibility when all single systems are of the same type and there is a single composition rule given the number of parties.

[2] Due to convexity of the effects, it is sufficient to check that an effect is minimal $k$-preserving on extremal states.

### 3.2.1 State space with 1 PR Box

For this example, we will consider a party swap symmetric GPT and focus on bipartite state and effect spaces. Let $e \in \mathcal{E}_{\mathcal{S}^{\boxtimes 2}}$ be an effect and $r$ and $s$ be two arbitrary states of $\mathcal{S}^{\boxtimes 2}$. Assume that $r$ is composed of 2 subsystems labelled by $\{1,2\}$ and $s$ is composed of 2 subsystems labelled by $\{3,4\}$. Since $e$ is a bipartite effect, for a party symmetric state space, it is sufficient to consider two different maps arising from $e$ depending on which subsystems of the state $r \otimes s$ it acts on:

$$\frac{e^{(1,2)} \otimes \mathsf{id}^{(3,4)} (r \otimes s)}{\left\langle u, e^{(1,2)} \otimes \mathsf{id}^{(3,4)} (r \otimes s) \right\rangle} \quad \text{and} \quad \frac{\mathsf{id}^{(1)} \otimes e^{(2,3)} \otimes \mathsf{id}^{(4)} (r \otimes s)}{\left\langle u, \mathsf{id}^{(1)} \otimes e^{(2,3)} \otimes \mathsf{id}^{(4)} (r \otimes s) \right\rangle}.$$

In both cases here, the identity acts on two systems. For $e$ to be minimally 2-preserving, we require that these two states are elements of $\mathcal{S}^{\boxtimes 2}$ for any choice of $r$ and $s$. From the definition of an effect space (see Def. 2.1.2), the first state is indeed an element of $\mathcal{S}^{\boxtimes 2}$. But the second one may not be. Denoting by $\Phi_e^{(23)}(r,s)$ the second state, minimally 2-preservability in this scenario corresponds to $\Phi_e^{(23)}(r,s)$ being an element of $\mathcal{S}^{\boxtimes 2}$ for all $r, s \in \mathcal{S}^{\boxtimes 2}$.

Our first example is the bipartite state space $\mathbb{H}^{[1]}_{(2,2)}[\mathrm{PR}_1]$, introduced in [93] to study entanglement swapping in GPTs. This state space is the convex hull of the 16 local deterministic states (vertices of $\mathbb{H}^{[0]}_{(2,2)}$ presented in Section 2.4.1) and the PR box $\mathrm{PR}_1$. There are effects in its maximal effect space $\mathcal{E}_{\mathbb{H}^{[1]}_{2,2}[\mathrm{PR}_1]}$, that are not minimally 2-preserving. In particular, none of the CH type effects that have an inner product in the range $[0,1]$ with every state of $\mathbb{H}^{[1]}_{(2,2)}[\mathrm{PR}_1]$ is minimally 2-preserving. As an example,

$$\frac{\mathsf{id} \otimes e_{\mathrm{CH}_4} \otimes \mathsf{id}\,(\mathrm{PR}_1 \otimes \mathrm{PR}_1)}{\langle u, \mathsf{id} \otimes e_{\mathrm{CH}_4} \otimes \mathsf{id}\,(\mathrm{PR}_1 \otimes \mathrm{PR}_1)\rangle} = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right) = \mathrm{PR}'_3$$

which is not in the state space. A similar calculation using other CH type effects will prove the rest of the claim.

### 3.2.2 Self-Dualised BW (Janotta)

The second example is the self- dualised version of BW presented by Janotta [52], the bipartite state space is given by the convex hull of all the local deterministic states and the 4 PR boxes

$$\mathrm{PR}_1 = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right), \mathrm{PR}'_1 = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right),$$

$$\mathrm{PR}'_3 = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right), \mathrm{PR}'_4 = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right).$$

The effect space is the convex hull of

$$e_{\mathrm{CH}_2} = \left(\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), e'_{\mathrm{CH}_2} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{array}\right),$$

$$\frac{2}{3}e'_{\mathrm{CH}_3} = \frac{2}{3}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{array}\right), \frac{2}{3}e'_{\mathrm{CH}_4} = \frac{2}{3}\left(\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right);$$

However, these extremal effects are not minimally 2-preserving, since

$$\frac{\mathsf{id} \otimes e_{\mathrm{CH}_2} \otimes \mathsf{id}\left(\mathrm{PR}_1 \otimes \mathrm{PR}'_3\right)}{\left\langle u, \mathsf{id} \otimes e_{\mathrm{CH}_2} \otimes \mathsf{id}\left(\mathrm{PR}_1 \otimes \mathrm{PR}'_3\right)\right\rangle} = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) = \mathrm{PR}_4,$$

$$\frac{\mathsf{id} \otimes e'_{\mathrm{CH}_2} \otimes \mathsf{id}\left(\mathrm{PR}_1 \otimes \mathrm{PR}'_1\right)}{\left\langle u, \mathsf{id} \otimes e'_{\mathrm{CH}_2} \otimes \mathsf{id}\left(\mathrm{PR}_1 \otimes \mathrm{PR}'_1\right)\right\rangle} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right) = \mathrm{PR}'_2,$$

$$\frac{\mathsf{id} \otimes (2/3)e'_{\mathrm{CH}_3} \otimes \mathsf{id}\left(\mathrm{PR}_1 \otimes \mathrm{PR}'_3\right)}{\left\langle u, \mathsf{id} \otimes e'_{\mathrm{CH}_3} \otimes \mathsf{id}\left(\mathrm{PR}_1 \otimes \mathrm{PR}'_3\right)\right\rangle} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right) = \mathrm{PR}'_2,$$

$$\frac{\mathsf{id} \otimes (2/3)e'_{\mathrm{CH}_4} \otimes \mathsf{id}\left(\mathrm{PR}'_3 \otimes \mathrm{PR}'_4\right)}{\left\langle u, \mathsf{id} \otimes e'_{\mathrm{CH}_4} \otimes \mathsf{id}\left(\mathrm{PR}'_3 \otimes \mathrm{PR}'_4\right)\right\rangle} = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) = \mathrm{PR}_4,$$

none of these effects are minimally 2-presrving.

## 3.3 FAILURE OF NO-RESTRICTION HYPOTHESIS(NRH)

The *no restriction hypothesis* states that there are no further restrictions between a state and effect space pair, other than that of the inner product. An interesting feature that emerges from our work is that NRH leads to inconsistencies when composite state spaces are considered. In particular, when an effect from the dual of the state space is performed on two halves of two bipartite systems, the resulting state may now lie outside the state space. Hence, the state spaces we consider can be seen as examples where restrictions further than requiring a valid inner product, in particular minimal $k$-preservibility, is needed to maintain consistency when composite systems are considered.

Previously, theories with restricted effect spaces have been considered [53]. In particular, it was shown that the maximal tensor product (as defined in 2.2.1) of single system state spaces with restricted effect spaces contains states whose marginals are not elements of the original state space. As a solution, the authors propose a generalised definition of the maximal tensor product. However, a physically motivated principle to demand restricted effect spaces was missing. In this thesis, we argue that since minimal $k$-preservibility coincides with complete preservibility for minimal tensor product theories, for any theory where the minimal tensor product state space is always a subset of the composite state space, minimal $k$-preservibility presents a physical principle for compositional consistency and necessitates the requirement to consider restricted effect spaces.

## 3.4 CONCLUSION

We have introduced a minimal necessary criterion for multipartite effects to be completely preserving. Our requirement can be seen as a natural generalisation of the NRH, since the NRH only demands effects to be 0-preserving. In quantum theory, all 0-preserving effects are completely preserving and therefore the NRH serves as a sufficiency condition for validity of effects. We have shown that in general theories this feature no longer holds and one might need to use the proposed generalised NRH. In addition, one can extend the notion of minimal $k$-preservibility of transformations between state spaces and ask which transformations are admissible in a theory. An interesting question there would be to look for transformations that although are minimally $k$-preserving but not completely $k$-preserving. We leave these investigations for future work.

Recently, in [27], the authors discuss complete preservibility in composite state spaces which are not constructed using the minimal or maximal tensor product rule. In particular, they refer to Janotta [52] where the state space considered was the convex hull of $\mathbb{H}^{[0]}_{(2,2)}$ and 4 PR boxes, out of which 2 PR boxes are isotropically opposite to each other and 2 PR boxes (not necessarily the other two) are symmetric under party swap. Naturally, the state space is not symmetric under party swap. In agreement with our results, they also find effects that are not minimally 2-preserving. Further, they state that the only possible compositions of two gbit state spaces, $\mathcal{G}_2^2$, which are completely preserving are $\mathbb{H}^{[0]}_{(2,2)}, \mathbb{H}^{[8]}_{(2,2)}$ and a party symmetric state space [3] of $\mathbb{H}^{[1]}_{(2,2)}[\mathrm{PR}_1]$ with a restricted effect space constructed from the convex hull of BW effects and only one entangled effect, i.e., the coupler $e_{\mathrm{pure}}$ only. Our results show that more state and effect space pairs are potentially completely 2-preserving. For instance, $\mathbb{H}^{[1]}_{(2,2)}[\mathrm{PR}_1]$ with a restricted effect space constructed from the convex hull of BW effects, the 9 coupling effects and their complementary effect. A noisier version of this example is also minimally 2-preserving and therefore potentially completely preserving. Other examples, to be covered in Chapter 6, include $\mathbb{H}^{[m]}_{\alpha,(2,2)}$ state spaces, with $m$ even, denoting the number of isotropically opposite noisy PR boxes present in the state space. Here the restricted effect spaces are constructed by taking the convex hull of all the extremal effects of $\mathcal{E}_{\mathbb{H}^{[m]}_{\alpha,(2,2)}}$, with the exception of the CH type effects.

[3] A state space is party swap symmetric if it the party swap relabelling maps aloowed states to allowed states.

## — 4 —

# Causal Order, Causal Structure and Freedom of Choice

This chapter presents a brief overview of the connections between the three concepts in its title, closely following the proposal of Colbeck and Renner [20].

## 4.1 CAUSAL ORDER

Any physical experiment must involve preparation of states, measurements on them and finally, generation of measurement outcomes. From our perception of the world around us, we expect there to be an order in which these operations occur, namely, measurement outcomes are observed after a state has been prepared and a measurement choice has been made. However, we do not necessitate that the measurement outcomes are always dependent on the other two.

One way to formulate this is by associating the random variable $\Lambda$ to state preparations, $X$ to measurement choices and $A$ to the observed outcomes. Since $X$ is a potential cause of $A$, we say that $A$ is in the *causal future* of $X$ and denote it by $X \dashrightarrow A$. Similarly, $\Lambda \dashrightarrow A$. Given a set, $\Gamma$, of random variables, a *causal order*, $(\Gamma, \dashrightarrow)$, is then a preorder relation $\dashrightarrow$ on $\Gamma$.

*Remark:* Instead, if we defined causal order using partial order, whenever $X \dashrightarrow A$ and $A \dashrightarrow X$, using antisymmetry, we will obtain $A = X$. However, there might be situations in which two distinct random variables precede each other. Therefore, we will use preorder relations for defining both causal order and causal structures.

## 4.2 CAUSAL STRUCTURES COMPATIBLE WITH A CAUSAL ORDER

Given a set of random variables, a *causal structure* represents the true causes of each random variable in the set. In the experiment described above, the experimenter is aware of the measurement choice and the outcomes observed but they need not know the state; The variables $A$ and $X$ are deemed observed, while $\Lambda$ is unobserved. A causal structure, $(\Gamma, \rightarrow)$, is then a preorder relation $\rightarrow$ on $\Gamma$, where $\rightarrow$ denotes true cause, such that at least one variable is observed. $(\Gamma, \rightarrow)$ can also be represented as a directed acyclic graph, where the random variables are the nodes, some of which are labelled as observed.

A casual structure $(\Gamma, \rightarrow)$ is said to be *compatible* with the causal order $(\Gamma, \dashrightarrow)$, if and only if for every pair of random variables $(A, X)$, if $X \dashrightarrow A$ then $A \not\rightarrow X$. This is illustrated in Figure 4.1.

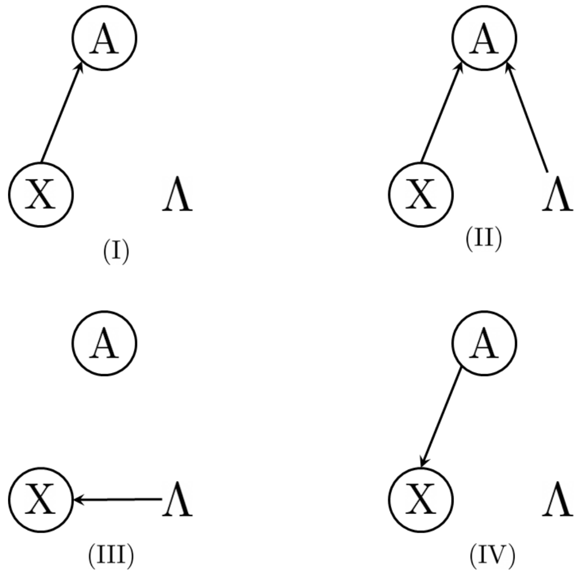


Figure 4.1: Causal structures with three random variables $A$, $X$ and $\Lambda$ where $\Lambda$ is unobserved. (I), (II) and (III) are compatible with the causal order described in Section 4.1, while (IV) is not.

## 4.3 FREEDOM OF CHOICE

In any experiment, such as the one above, we would ideally like the state preparation not to influence the measurement choice. In fact, it is reasonable to demand that the random variable $X$ is *free* and has no apparent cause. Therefore, it is independent of all variables outside its causal future. This assumption, *freedom of choice*, restricts the set of causal structures that are compatible with a given causal order. In Figure 4.1, the only compatible causal structures with $X$ free are (I) and (II). Note that (III) is not since $\mathrm{p}(X|\Lambda) \neq \mathrm{p}(X)$.

# Part II

# Self-Testing of Theories

# — 5 —

# Correlation Self-Testing of Theories

## 5.1 INTRODUCTION

Quantum theory is known to lead to nonlocal correlations between the observations that different, separated parties can make on a shared quantum system. This counter-intuitive phenomenon, known as Bell nonlocality, has implications for the foundations of quantum theory as well as for various applications, e.g. in quantum cryptography. These quantum correlations, even though nonlocal, are non-signalling, i.e., they are known to be compatible with special relativity, in the sense that they do not allow for superluminal transfer of information. However, it is also known that quantum theory does not allow for the most general non-signalling correlations [81, 80]. Thus, the question as to why the correlations of quantum theory are further restricted, has occupied researchers in the foundations of quantum theory in recent decades, developments of which we proceed to outline.

One way to approach this question is to start from the largest set of correlations compatible with special relativity, that can be realised by space-like separated parties, and then to identify a list of information-theoretic principles (ideally only one) that restrict this set to the set of quantum correlations. A few proposed principles include non-triviality of communication complexity [14], impossibility of nonlocal computation [58], information causality [76], macroscopic locality [71], and local orthogonality [40]. While these approaches provide us with insight into the properties of quantum correlations and reduce the set of allowed nonlocal correlations, none is known to single out the set of quantum correlations [70]. Additionally, principles naturally arising from information-theoretic requirements may or may not be deemed fundamentally "natural" with respect to our perception of Nature. Therefore, an objective approach would be to experimentally rule out theories that generate post-quantum correlations and then investigate a link to an underlying principle. *Correlation self-testing* of quantum theory [98, 99] takes this approach and asks whether there is an information-theoretic task that can only be optimally performed using quantum correlations. If such a task is found then the underlying information-theoretic requirement for optimally winning the task might point to a physical principle. For the rest of this thesis, we will use the term *self-testing* and correlation self-testing alternatively.

This chapter is presented as follows: in Sections 5.2 and 5.3 we review the idea behind self-testing and adaptive CHSH game. In Section 5.4, we present a GPT that is known to support entanglement swapping. In Section 5.6, we show that the GPT proposed in [30], that can perfectly win the adaptive CHSH game can be ruled out, for it cannot violate chained Bell inequalities. Finally, in Sections 5.8 and 5.6.1 we present two new games and analyse their performance against some GPTs, that produce post-quantum correlations.

## 5.2 CORRELATION SELF-TESTING OF PHYSICAL THEORIES

The setup of correlation self-testing is as follows: given a physical theory, $\mathcal{P}$, and a foil theory, $\mathcal{T}$, if $\mathcal{P}$ can produce correlations within a causal structure that cannot be produced by $\mathcal{T}$ in the same causal structure, then it is possible to devise an information processing task in which $\mathcal{P}$ outperforms $\mathcal{T}$. More generally, for a set of foil theories $\{\mathcal{T}_i\}_{i=1}^n$, suppose there is a set of tasks (ideally only one) that singles out $\mathcal{P}$ from the set of foil theories. Such a list of tasks is said to be a *correlation self-test* of $\mathcal{P}$ within $\{\mathcal{T}_i\}_{i=1}^n$. In particular, upon requiring a certain threshold for performance in each task, $\mathcal{P}$ can be singled out from the rest of the foil theories. The overarching ambition is to find a set of tasks that single out quantum theory within GPTs, and this would then point to higher principles for quantum theory.

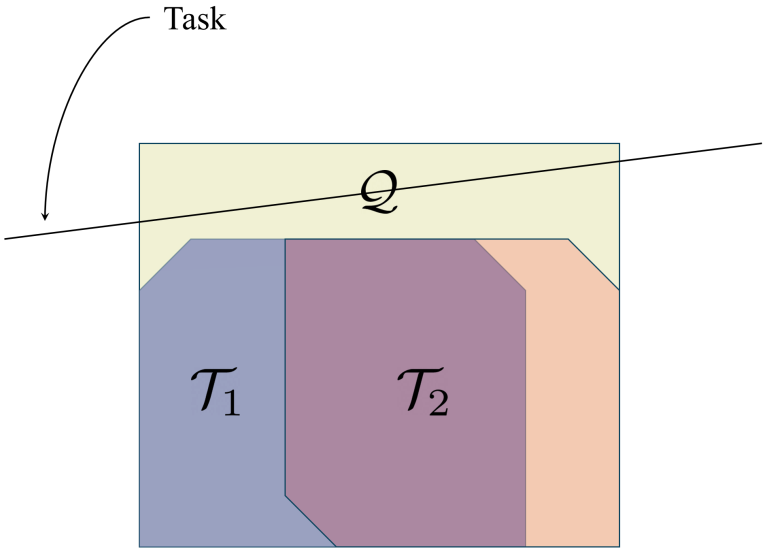


Figure 5.1: A pictorial representation of the high-level idea behind the Adaptive CHSH game. If there exists a causal structure within which the sets of correlations that can be generated in theories $\mathcal{T}_1$ and $\mathcal{T}_2$ are proper subsets of the set of quantum correlations, $\mathcal{Q}$, then an information-theoretic task whose optimal performance can only be achieved in quantum correlations (yellow region above the line).

## 5.3 THE ADAPTIVE CHSH (ACHSH) GAME

The Adaptive CHSH game was proposed as a candidate task for correlation self-testing of quantum theory [98, 99]. The task is as follows: in the bi-local causal structure displayed in Fig. 5.2,

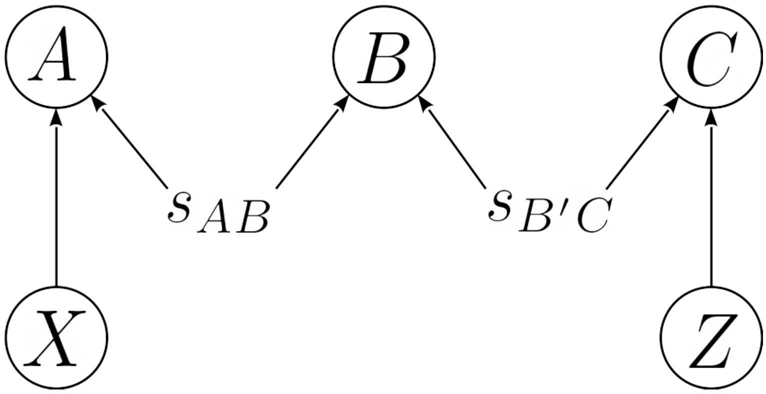


Figure 5.2: Causal structure for the Adaptive CHSH game. Bob shares the resource $s_{AB}$ with Alice and the resource $s_{B'C}$ with Charlie. A referee asks questions to Alice and Charlie labelled by random variables $X$ and $Z$ respectively. Bob performs a joint measurement on his share of resources, the outcomes of which are labelled by the random variable $B$. Alice and Charlie perform local measurements on their subsystems, the outcomes of which are labelled by random variables $A$ and $C$. The values of all the random variables determine the score in the game. There are no non-classical tripartite resources shared by all three parties (shared tripartite randomness is allowed).

three players Alice, Bob and Charlie indulge in a cooperative game, in which a referee asks Bob to randomly return two bits denoted by $B \in \{00, 01, 10, 11\}$. The referee then asks uniformly random binary questions denoted by $X$ and $Z$ to Alice and Charlie. Alice and Charlie need to provide binary answers $A$ and $C$, such that they win a game with winning conditions given by $a \oplus c = (x \cdot z) \oplus ((b_0 \oplus b_1) \cdot x) \oplus z \oplus b_0$, where $(b_0, b_1)$ denotes the two bit response $B$ of Bob. Here, $\oplus$ denotes addition modulo 2. In quantum theory, this game can be won with a maximum winning probability of $1/2\left(1 + 1/\sqrt{2}\right) \approx 0.85$ [98, 99]. For completeness, we present an optimal strategy below and direct the interested reader to [98, 99] for further reading on previous results.

### 5.3.1 Quantum Strategy in the ACHSH Game

The players, if allowed to use quantum theory, can use a strategy where Alice shares a two qubit maximally entangled state $\rho_{\mathbf{AB}}$ with Bob, and Charlie shares another two qubit maximally entangled state $\rho_{\mathbf{B'C}}$ with Bob. Then Bob performs a joint measurement in the Bell basis on his two qubits. This is an entanglement swapping operation and therefore, for each outcome of the measurement, Alice and Charlie will be left with a maximally entangled state. For example, if the Bell basis is denoted by

$$\begin{aligned} |\psi_{00}\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle_{\mathbf{BB'}} + |11\rangle_{\mathbf{BB'}}\right), \\ |\psi_{01}\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle_{\mathbf{BB'}} - |11\rangle_{\mathbf{BB'}}\right), \\ |\psi_{10}\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle_{\mathbf{BB'}} + |10\rangle_{\mathbf{BB'}}\right), \\ |\psi_{11}\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle_{\mathbf{BB'}} - |10\rangle_{\mathbf{BB'}}\right), \end{aligned} \tag{5.1}$$

then the resultant state with Alice and Charlie after the projection $|\psi_{00}\rangle$ is $\frac{1}{\sqrt{2}}\left(|00\rangle_{\mathbf{AC}} + |11\rangle_{\mathbf{AC}}\right)$ with an associated probability of 1/4, and so on. Further, denoting $|\theta\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$, Alice and Charlie execute the following operations:

- when $X = 0$, Alice measures in $\{|0\rangle, |\pi\rangle\}$ basis,
- when $X = 1$, Alice measures in $\{|\pi/2\rangle, |3\pi/2\rangle\}$ basis,
- when $Z = 0$, Charlie measures in $\{|\pi/4\rangle, |5\pi/4\rangle\}$ basis,
- when $Z = 1$, Charlie measures in $\{|3\pi/4\rangle, |7\pi/4\rangle\}$ basis,
- For each measurement, if the first element of the basis is obtained, the answer (A/C) is set to 0, otherwise to 1.

Using the notation 2.2, one then obtains

$$\mathrm{p}\left(A, C|X, Z, B = 00)\right) = \frac{1}{4}\left(\begin{array}{cc|cc} 1+\epsilon & 1-\epsilon & 1-\epsilon & 1+\epsilon \\ 1-\epsilon & 1+\epsilon & 1+\epsilon & 1-\epsilon \\ \hline 1+\epsilon & 1-\epsilon & 1+\epsilon & 1-\epsilon \\ 1-\epsilon & 1+\epsilon & 1-\epsilon & 1+\epsilon \end{array}\right), \tag{5.2}$$

where $\epsilon = 1/\sqrt{2}$. This helps the players win the CHSH game $a \oplus c = \overline{x} \cdot z$ with a score of $\frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$. Putting together winning scores for the other outcomes with their associated probabilities, the overall winning probability sums to $\frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$.

A key observation is that in order to achieve the score above, the parties perform an entanglement swapping operation. Moreover, this swapping operation is deterministic, i.e., when post-selecting on every outcome of Bob, Alice and Charlie are left with a maximally entangled state. Since Alice and Charlie are not allowed to share any non-classical resources, to exceed the classical bound of 3/4, a GPT must allow for entanglement swapping, which is the topic of our next section.

## 5.4 ENTANGLEMENT SWAPPING IN GPTS

Consider a scenario in which Bob shares two maximally entangled qubit pairs, one with Alice and one with Charlie. Bob then performs a joint measurement on his two qubits in the Bell basis. After post-selecting on any of Bob's outcomes, Alice's qubit and Charlie's qubit are maximally entangled. In such settings where Bob initially shares entangled states with Alice and Charlie and then jointly measures his systems, whenever Alice and Charlie's post-selected states are entangled for at least one of Bob's outcomes, we say that we have a *swapping* scenario. This idea can be extended to the landscape of GPTs. Let $\mathcal{S}$ be a bipartite state space and $s_{\mathbf{AB}}$ and $s_{\mathbf{B'C}}$ be two entangled states in $\mathcal{S}$. If for an effect $e \in \mathcal{E}$,

$$s_{\mathbf{AC}|e} = \frac{\mathsf{id}_{\mathbf{A}} \otimes e \otimes \mathsf{id}_{\mathbf{C}} \left(s_{\mathbf{AB}} \otimes s_{\mathbf{B'C}}\right)}{\langle u, \mathsf{id}_{\mathbf{A}} \otimes e \otimes \mathsf{id}_{\mathbf{C}} \left(s_{\mathbf{AB}} \otimes s_{\mathbf{B'C}}\right)\rangle} \tag{5.3}$$

is entangled, $e$ is called a *coupler*. Note that the effect $e$ must not be a convex combination of product effects, in particular, it must be entangled (non-separable). However, not all entangled effects are couplers for there might be entangled effects $\tilde{e} \in \mathcal{E}$ such that there is no pair of entangled states $s_{\mathbf{AB}}$ and $s_{\mathbf{B'C}}$ in the state space that can create a swapping scenario.

### 5.4.1 The House-like state space $\mathbb{H}^{[1]}_{(2,2)}$

Entanglement swapping is a key ingredient in achieving a post-classical score in the ACHSH game as shown in [98, 99]. Theories like BW generate correlations in the $(2, 2, 2)$ setting that can perfectly win CHSH games although they do not support entanglement swapping (see [89] for the case when the single-party state space is $\mathcal{G}^2_2$). In general, recall from Definition 2.2.1, theories in which the state space is formed by the maximal tensor product have the smallest effect space cone, with all effects being separable, in particular, there exists a trade-off between states and measurements [90]. In order for a theory to have entanglement swapping, it needs to allow entangled states and entangled effects, implying that the state space can neither be minimally nor maximally composed. In this regard, quantum theory lies in a sweet spot since for every quantum state $\rho$ and any number $t \in [0, 1]$, $t\rho$ is an allowed effect. Interestingly, in [93] the authors showed that with the state space $\mathbb{H}^{[1]}_{(2,2)}[\mathrm{PR}_1]$, characterised by the convex-hull of $\mathbb{H}^{[0]}_{(2,2)}$ and $\mathrm{PR}_1$, and its maximal effect space, one can demonstrate entanglement swapping. In particular, the effect space contains couplers. We have seen in Subsection 2.7 that in the $(2, 2, 2)$ setting, this state space generates all non-signalling correlations. Therefore, $\mathbb{H}^{[1]}_{(2,2)}[\mathrm{PR}_1]$ is a potential example where one might achieve a higher score in the ACHSH game as compared to quantum theory. Here we quickly revisit a variation of the example from [93] and flesh out all the coupling effects.

We consider the state space $\mathbb{H}^{[1]}_{(2,2)}[\mathrm{PR}_2]$. Solving the vertex enumeration problem for the effect polytope, we found that $\mathcal{E}_{\mathbb{H}^{[1]}_{(2,2)}}$ has 106 extremal effects, of which 82 are the extremal effects of $\mathbb{H}^{[8]}_{(2,2)}$. We call these 82 effects BW effects. Amongst the 24 non-BW effects, 9 effects are couplers. We can categorize them into two sets on whether the post-selected state with Alice and Charlie is extremal or not.

$$e_{\text{pure}} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -2/3 & 0 & 2/3 \\ \hline 0 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 0 \end{array}\right) = \quad 2/3 e_{\mathrm{CH}_2}. \tag{5.4}$$

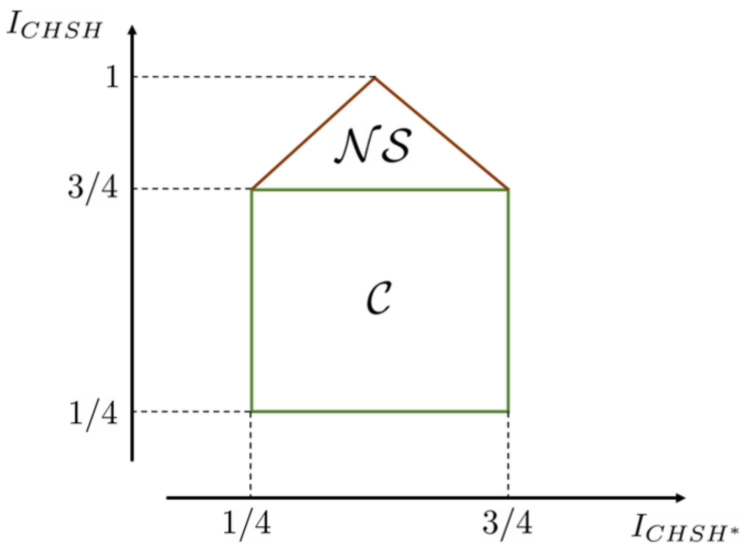


Figure 5.3: A two-dimensional slice of the set of correlations generated when fiducial measurements are performed on the states of the bipartite state space characterised by 16 local deterministic states and one PR box [93]. The vertical axes represent a CHSH inequality and the horizontal axes represent one of its symmetries obtained by relabelling the inputs. Local correlations, denoted by the square $\mathcal{C}$, satisfy $0 \leqslant \mathrm{F}_{\mathrm{CHSH}} \leqslant 3/4$ and $0 \leqslant \mathrm{F}_{\mathrm{CHSH}^*} \leqslant 3/4$.

$$E_{\text{noisy}} = \left\{ \begin{array}{l} \left(\begin{array}{cc|cc} 0 & 1/2 & 0 & 0 \\ 0 & -1/2 & 0 & 1/2 \\ \hline 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 1/2 \\ \hline 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \\ \hline 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \end{array}\right), \\ \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 1 \\ \hline 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 1/2 \\ \hline 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 1/2 \\ \hline 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right), \\ \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 1/2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 1/2 \\ \hline 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{array}\right); \end{array} \right\} \tag{5.5}$$

A small calculation shows that if Bob shared two instances of $\mathrm{PR}_2$, one with Alice and one with Charlie and performed a joint measurement $\{e_{\text{pure}}, u - e_{\text{pure}}\}$, then with a probability 1/3 the outcome corresponding to $e_{\text{pure}}$ occurs and the post-measurement state is :

$$\frac{\mathsf{id}_{\mathbf{A}} \otimes e_{\text{pure}} \otimes \mathsf{id}_{\mathbf{C}} \left((\mathrm{PR}_2)_{\mathbf{AB}} \otimes (\mathrm{PR}_2)_{\mathbf{B'C}}\right)}{\langle u, \mathsf{id}_{\mathbf{A}} \otimes e_{\text{pure}} \otimes \mathsf{id}_{\mathbf{C}} \left((\mathrm{PR}_2)_{\mathbf{AB}} \otimes (\mathrm{PR}_2)_{\mathbf{B'C}}\right)\rangle} = (\mathrm{PR}_2)_{\mathbf{AC}} \,. \tag{5.6}$$

Likewise, if the measurement was $\{e_{\text{noisy}}, u - e_{\text{noisy}}\}$ instead, where $e_{\text{noisy}}$ is the first element of $E_{\text{noisy}}$, a similar calculation shows that with probability 3/8, the outcome corresponding to $e_{\text{noisy}}$ occurs with the post-measurement state being

$$\frac{\mathsf{id}_{\mathbf{A}} \otimes e_{\text{noisy}} \otimes \mathsf{id}_{\mathbf{C}} \left((\text{PR}_2)_{\mathbf{AB}} \otimes (\text{PR}_2)_{\mathbf{B'C}}\right)}{\langle u, \mathsf{id}_{\mathbf{A}} \otimes e_{\text{noisy}} \otimes \mathsf{id}_{\mathbf{C}} \left((\text{PR}_2)_{\mathbf{AB}} \otimes (\text{PR}_2)_{\mathbf{B'C}}\right)\rangle} = \frac{2}{3}(\text{PR}_2)_{\mathbf{AC}} + \frac{1}{3}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)_{\mathbf{AC}} . \tag{5.7}$$

Notice that the local deterministic state L at the end of this equation satisfies $\text{CHSH}_2[\text{L}] = 3/4$. When a different effect from the set $E_{\text{noisy}}$ is used instead, we will get a similar decomposition with the corresponding local deterministic state $\text{L}_i$ also satisfying $\text{CHSH}_2[\text{L}_i] = 3/4$.

One might then wonder, how the theory above performs in the ACHSH game. This is precisely the topic in Chapter 6. However, it is worth mentioning that if more than one tensor products are allowed in a theory it is possible to perfectly win the ACHSH game. We briefly emphasise this in the following section.

## 5.5 THEORIES WITH MULTIPLE COMPOSITION RULES

One can perform deterministic entanglement swapping if more than one composition rule is allowed between single systems, such as the one in [8, 53]. Explicitly, one possibility is allowing BW compositions for bipartite states labelled by $\mathbf{AB_1}, \mathbf{B_2C}$ and between $\mathbf{AC}$ and allowing local composition between $\mathbf{B_1}, \mathbf{B_2}$. With this one can perfectly win the ACHSH game when Bob shares two copies of $\text{PR}_1$, one with Alice and another with Charlie and Bob performs a four-outcome joint measurement with appropriate CH-type effects. An alternate proposal can be found in [30].

For this thesis, we have avoided these types of constructions since requiring multiple composition rules is not a close analogy to quantum theory, for quantum theory only needs a single composition rule, namely, the tensor product of the underlying Hilbert spaces. These observations lead us to wonder whether there exists any GPT that can be described by a single composition rule and can perform perfect entanglement swapping. One such proposal has been presented in [30]. In the next section, we will show that although one can perfect deterministic entanglement swapping in this theory, it fails to generate all quantum correlations in the Bell causal structure.

## 5.6 OBLATE STABILIZER THEORY

The Oblate Stabilizer (OS) theory has been recently introduced in [30] as a toy theory within which one can perfectly win the ACHSH game. This provokes the claim that quantum theory cannot be self-tested against OS. We argue here that this is not the case. Below, we will first present the OS theory and show that although it can perfectly win the ACHSH game, there exist quantum correlations in the Bell causal structure that cannot be reproduced in OS. As a consequence, the Bell causal structure is sufficient to rule out OS, as it is in [30].

Let $|\tilde{x}_\pm\rangle\langle\tilde{x}_\pm| := (\mathbb{I}_2 \pm \sqrt[4]{2}\sigma_{\hat{X}})/2$, $|\tilde{y}_\pm\rangle\langle\tilde{y}_\pm| := (\mathbb{I}_2 \pm \sqrt[4]{2}\sigma_{\hat{Y}})/2$ and $|\tilde{z}_\pm\rangle\langle\tilde{z}_\pm| := (\mathbb{I}_2 \pm \sigma_{\hat{Z}})/2$, and let $R := \exp\left(-i\frac{\pi}{8}\sigma_{\hat{Z}}\right)$, where $i = \sqrt{-1}$. The single system state and effect spaces of OS are then defined as:

$$\mathbb{O}_{\mathbb{S}}{}^{(1)} := \text{ConvHull}\Big\{ |\tilde{x}_\pm\rangle\langle\tilde{x}_\pm|, |\tilde{y}_\pm\rangle\langle\tilde{y}_\pm|, |\tilde{z}_\pm\rangle\langle\tilde{z}_\pm| \Big\},$$

$$\mathbb{E}^{(1)} := \text{ConvHull}\Big\{ \mathbb{I}_2, \mathbf{O}, R\left(|\tilde{x}_\pm\rangle\langle\tilde{x}_\pm|\right)R^\dagger, R\left(|\tilde{y}_\pm\rangle\langle\tilde{y}_\pm|\right)R^\dagger, R\left(|\tilde{z}_\pm\rangle\langle\tilde{z}_\pm|\right)R^\dagger \Big\},$$

where $\mathbf{O}$ is the $2 \times 2$ null matrix. Now let $\Omega := \{\Phi_+, \Phi_-, \Psi_+, \Psi_-\}$ be the set density matrices corresponding to the four Bell states. Finally, consider the set

$$\mathrm{R}\Omega\mathrm{R}^\dagger := \Big\{ \left(R^m\right)(\omega)\left(R^m\right)^\dagger \ \mid \omega \in \Omega, m \in \mathbb{Z}_8, m \text{ odd} \Big\};$$

The bipartite state and effect spaces are then given as:

$$\mathbb{O}_{\mathbb{S}}{}^{(2)} := \text{ConvHull}\Big\{ \Big(\mathbb{O}_{\mathbb{S}}{}^{(1)} \underset{\text{min}}{\otimes} \mathbb{O}_{\mathbb{S}}{}^{(1)}\Big) \bigcup \mathrm{R}\Omega\mathrm{R}^\dagger \Big\},$$

$$\mathbb{E}^{(2)} := \text{ConvHull}\Big\{ \Big(\mathbb{E}^{(1)} \underset{\text{min}}{\otimes} \mathbb{E}^{(1)}\Big) \bigcup \mathrm{R}\Omega\mathrm{R}^\dagger \Big\},$$

Now, let us consider a scenario in which Alice and Bob share the entangled state

$$R\Phi_+R^\dagger = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2}e^{-\frac{i\pi}{4}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}e^{\frac{i\pi}{4}} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

in the Bell causal structure. Since there are six extreme effects in the single system effect space, let us assume that both Alice and Bob perform three binary measurements $|\tilde{x}_\pm\rangle\langle\tilde{x}_\pm|$, $|\tilde{y}_\pm\rangle\langle\tilde{y}_\pm|$ and $|\tilde{z}_\pm\rangle\langle\tilde{z}_\pm|$ each on the entangled state above. The resultant probability distribution is then

$$\mathrm{p}_{\mathbb{O}_{\mathbb{S}}} = \frac{1}{2}\left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 1 & 1/2 & 1/2 \\ 0 & 1 & 1 & 0 & 1/2 & 1/2 \\ \hline 0 & 1 & 0 & 1 & 1/2 & 1/2 \\ 1 & 0 & 1 & 0 & 1/2 & 1/2 \\ \hline 1/2 & 1/2 & 1/2 & 1/2 & 1 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 & 0 & 1 \end{array}\right). \tag{5.8}$$

One can show that choosing a different entangled state will generate probability distributions that are relabellings of $\mathrm{p}_{\mathbb{O}_{\mathbb{S}}}$. Next, we will show that this distribution cannot violate certain Bell inequalities, which are known to be violated by quantum correlations.

### 5.6.1 Chained Bell Inequalities

The *chained Bell inequalities* were introduced in [15] as an extension of the CHSH inequalities to more inputs. They are parameterised by the number of input settings $m$ for both parties. One way of expressing this inequality is

$$\sum_{\substack{x,y\in\{0,1,\cdots m-1\}\\ |2(y-x)+1|=1}} p\left(a\neq b|x,y\right)+p\left(a=b|x=0,y=n\right)\geqslant 1. \tag{5.9}$$

It is satisfied by every local correlation, while quantum correlations are known to exist that violate it [15]. For $m=3$, a rewriting of this inequality can be given by $\langle \mathrm{CB},\mathrm{p}\rangle \geqslant 1$, where,

$$\mathrm{CB} := \left(\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right) \tag{5.10}$$

There are 192 relabellings of CB which all correspond to variations of Inequality (5.9). Now, notice that any relabelling of CB must have six $(2\times 2)$ blocks with 1s in either the diagonal or off-diagonal entries. For any such relabelling, there will always be an odd number of blocks containing 1s in a diagonal that is different from the diagonal in which the rest of the blocks contain 1s. On the other hand, $\mathrm{p}_{\mathbb{O}_{\mathbb{S}}}$ has an even number of blocks that differ in the diagonals containing 1/2. Therefore, the smallest inner product between any relabelling of CB and any relabelling $\mathrm{p}_{\mathbb{O}_{\mathbb{S}}}$ must be 1. One relabelling of CB that indeed allows one to get to this minimum is

$$\left(\begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right).$$

One might try to use non-extremal effects to construct the measurements. However, since such effects will be convex mixtures of extremal effects, the resultant distribution will also be a convex mixture of extremal distributions. Since no extremal distribution can violate any chained bell inequality (relabellings of $\langle \mathrm{CB},\mathrm{p}\rangle \geqslant 1$), the non-extremal ones cannot either. Therefore, one can use these inequalities to correlation self-test quantum theory against the oblate stabilizer theory.

## 5.7 POSSIBLE EXTENSIONS TO THE ACHSH GAME

There are two primary directions in which one might want to generalise the idea behind the ACHSH game. The first is to extend this game to more parties. In quantum theory, it is possible to construct scenarios wherein one can perform deterministic entanglement swapping, regardless of the number of parties. If this is a feature innate only to theories that realise all quantum correlations in any causal structure, games involving multi-party deterministic swapping might be the key to singling out quantum correlations. The second direction is to stay within the bi-local causal structure 5.2 while allowing for more inputs. This would allow one to use more of the quantum state space.

## 5.8 THE ADAPTIVE GHZ (AGHZ) GAME

### 5.8.1 GHZ Games

The $k$-party GHZ game, introduced by Mermin [66], can be perfectly won within quantum theory and its quantum violation grows exponentially with the number of parties. For our discussion we will consider $k$ to be odd. In short, a referee asks each of the $k$-parties, $\mathbf{A}_0, \cdots \mathbf{A}_{k-1}$, a binary question $x_i \in \{0,1\}$ such that $(x_0, \cdots, x_{k-1})$ is uniformly distributed over bit strings of even parity. Each player then returns a binary answer $a_j \in \{0,1\}$. The players win if

$$(-1)^{\sum_{i=0}^{k-1} a_i + \frac{1}{2}\sum_{j=0}^{k-1} x_j} = 1.$$

In particular, if $(\sum_{j=0}^{k-1} x_j)/2$ is odd, then $\sum_{i=0}^{k-1} a_i$ is odd, and if $(\sum_{j=0}^{k-1} x_j)/2$ is even, then $\sum_{i=0}^{k-1} a_i$ is even. To win this game within quantum theory, the players can share the $k$-qubit GHZ state

$$\frac{|0\rangle^{\otimes k} + |1\rangle^{\otimes k}}{\sqrt{2}}. \tag{5.11}$$

Each player then performs the measurement $\{(\mathbb{I} + \sigma_{\hat{X}})/2, (\mathbb{I} - \sigma_{\hat{X}})/2\}$ when their question is 0 and $\{(\mathbb{I} + \sigma_{\hat{Y}})/2, (\mathbb{I} - \sigma_{\hat{Y}})/2\}$ when their question is 1. They record 0 if they get the first outcome and 1 if they get the second. This strategy allows the players to perfectly win the game stated above.

Now consider the projectors in the GHZ basis:

$$\mathrm{G}_{0\cdots0} := \frac{|0\cdots0\rangle + |1\cdots1\rangle}{\sqrt{2}}, \mathrm{G}_{0\cdots01} := \frac{|0\cdots01\rangle + |1\cdots10\rangle}{\sqrt{2}}, \cdots, \mathrm{G}_{01\cdots1} := \frac{|01\cdots1\rangle + |10\cdots0\rangle}{\sqrt{2}},$$

$$\mathrm{G}_{1\cdots0} := \frac{|0\cdots0\rangle - |1\cdots1\rangle}{\sqrt{2}}, \mathrm{G}_{1\cdots01} := \frac{|0\cdots01\rangle - |1\cdots10\rangle}{\sqrt{2}}, \cdots, \mathrm{G}_{11\cdots1} := \frac{|01\cdots1\rangle - |10\cdots0\rangle}{\sqrt{2}}.$$

If instead of $\mathrm{G}_{0\cdots0}$ the players started by sharing any of these states while using the same measurement strategy, then with the same questions from the referee there would exist another game that they can perfectly win. The rules of these games can be obtained by locally relabelling the outputs of the original game described above. Note that for $k$ parties, there are $2^k$ such relabellings. For each such relabelled game, there exists one of the $2^k$ projectors above that help in perfectly winning it. We don't provide a closed formula for these games here. We label a game $\mathcal{G}_{\underline{b}}$, where $\underline{b}$ denotes the subscript of the initial shared entangled state chosen to perfectly win it. Finally, note that since these games are related by relabellings, they have equal classical bounds.

### 5.8.2 AGHZ Game

The AGHZ game is motivated from the discussion in the previous section in addition to the fact that if each party starts by sharing the state $\phi_+ = (|00\rangle + |11\rangle)/\sqrt{2}$ with the referee, who then makes a $k$-partite GHZ basis measurement. This ensures that the post-selected states with the parties would exactly be the projectors observed by the referee. We now lay out the game elaborately.

The AGHZ game is a collaborative game played by $k > 1$ parties, with $k$ odd, labelled as $\mathbf{A}_0, \cdots \mathbf{A}_{k-1}$ and $\mathbf{B}$, who are set up in the star causal structure, depicted in Figure 5.4. A referee

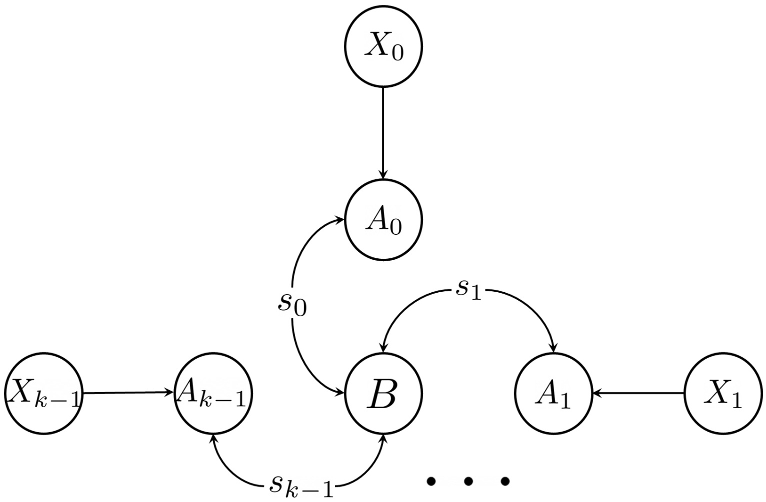


Figure 5.4: Causal structure for the Adaptive GHZ game. Bob shares the resource $s_i$ with $\mathbf{A}_i$ where $i \in \{0, \cdots, k-1\}$. A referee asks questions to $\mathbf{A}_i$, labelled by random variables $X_i$ and Bob performs a joint measurement on his share of resources, the outcomes of which are labelled by the random variable $B$. Parties $\mathbf{A}_i$ perform local measurements on their subsystems, the outcomes of which are labelled by random variables $A_i$. The values of all the random variables determine the score in the game. There are no non-classical $k$-partite resources shared by all three parties (shared $k$-partite randomness is allowed).

asks $\mathbf{B}$ to return $k$ bits $\underline{b} \coloneqq (b_0, b_1, \cdots, b_{k-1})$ denoted by the random variable $B$. The referee then asks binary questions denoted by random variables $X_0, \cdots, X_{k-1}$ to the $k$ parties, such that the string $\underline{x} \coloneqq (x_0, \cdots, x_{k-1})$ is uniformly distributed over bit strings of even parity. To these questions, the parties need to provide binary answers denoted by the random variables $A_0, \cdots, A_{k-1}$, in such a way that they win the version of the $k$-party GHZ game, $\mathcal{G}_{\underline{b}}$, mentioned in the previous section. The score in the game is given by

$$\mathcal{G}_{\mathrm{AGHZ}} \coloneqq \sum_{\underline{a},\underline{x},\underline{b} \in \{0,1\}^k} \mathrm{p}\left(\underline{a}, \underline{x}, \underline{b}\right) \delta\left(\underline{a}, \underline{x}, \underline{b}\right), \tag{5.12}$$

where $\underline{a} \coloneqq (a_0, \cdots, a_{k-1})$ and $\delta\left(\underline{a}, \underline{x}, \underline{b}\right)$ is 1 if the winning conditions are met, otherwise it is 0. This game can be won perfectly in quantum theory, as expected.

Now, for a concrete example, consider the case when $k = 3$. The winning conditions for the game are provided in Table 5.1. According to the strategy presented above, $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ share an instance of maximally entangled state $|\phi_+\rangle$ each with $\mathbf{B}$. $\mathbf{B}$ measures their 3 qubits on the following basis:

$$\mathrm{G}_{000} \coloneqq \frac{|000\rangle + |111\rangle}{\sqrt{2}}, \mathrm{G}_{001} \coloneqq \frac{|001\rangle + |110\rangle}{\sqrt{2}}, \mathrm{G}_{010} \coloneqq \frac{|010\rangle + |101\rangle}{\sqrt{2}}, \mathrm{G}_{011} \coloneqq \frac{|011\rangle + |100\rangle}{\sqrt{2}},$$
$$\mathrm{G}_{100} \coloneqq \frac{|000\rangle - |111\rangle}{\sqrt{2}}, \mathrm{G}_{101} \coloneqq \frac{|001\rangle - |110\rangle}{\sqrt{2}}, \mathrm{G}_{110} \coloneqq \frac{|010\rangle - |101\rangle}{\sqrt{2}}, \mathrm{G}_{111} \coloneqq \frac{|011\rangle - |100\rangle}{\sqrt{2}};$$

The state left with $\mathbf{A}_0, \mathbf{A}_1$ and $\mathbf{A}_2$, post-selecting on the outcome corresponding to $\mathrm{G}_{\underline{b}}$ is also $\mathrm{G}_{\underline{b}}$. On performing the said measurements, one can now verify that each of the GHZ games

| $x_0$ | $x_1$ | $x_2$ | $a_0 \oplus a_1 \oplus a_2$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | $(0,0,0)$ | $(0,0,1)$ | $(0,1,0)$ | $(0,1,1)$ | $(1,0,0)$ | $(1,0,1)$ | $(1,1,0)$ | $(1,1,1)$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |

Table 5.1: Winning conditions for Adaptive GHZ game for the case $k = 3$. The triples represent the bit string $\underline{b}$. For the inputs $(x_0, x_1, x_2)$ the parties need to output $(a_0, a_1, a_2)$ such that $a_0 \oplus a_1 \oplus a_2$ equals the entry corresponding to the row specified by the inputs and column specified by the game version $\underline{b}$. Here $\oplus$ denotes addition modulo 2.

corresponding to the bit string $\underline{b}$ as presented in Table 5.1 can be perfectly won. Now, since every outcome of the GHZ measurement occurs with a probability 1/8, the overall score in the game is 1.

*Remark.* An alternate way of playing this game in quantum theory is by allowing **B** to possess the three qubit state $G_{000}$. **B** then performs a Bell basis measurement, presented in (5.1), jointly on one of the qubits of $G_{000}$ and the qubit which is maximally entangled $\mathbf{A}_i$'s qubit. The outcome of each of these measurements is then stored in the set of bit pairs $(\underline{g}_0, \underline{g}_1, \underline{g}_2)$, where $\underline{g}_i := (g_{i,0}, g_{i,1})$ corresponds to the outcome obtained when the Bell measurement was performed on a qubit of $G_{000}$ and the qubit which is maximally entangled $\mathbf{A}_i$'s qubit. The bits string $\underline{b}$ can then be calculated as a function of $(\underline{g}_0, \underline{g}_1, \underline{g}_2)$. This strategy can be generalised to an arbitrary number of parties.

### 5.8.3 Classical and Box-World Strategies in ACB Game

The lack of entanglement swapping in both classical theory and BW limits one to perfectly win the AGHZ game in these theories. In any classical theory, each GHZ game can be won with a probability 3/4. Therefore, the overall score can also be upper-bounded by 3/4. In BW, the maximum score is also upper bounded by 3/4 since the parties $\{\mathbf{A}_i\}_i$ cannot share a maximally nonlocal box and upon the lack of any entanglement swapping operation are limited to classical strategies.

*Remark.* The case where $k$ is even can be formalised in the same way. For this, the list of questions asked by the referee is $2^k$. The idea behind the rest of the strategy remains the same. The case when $k = 2$ is equivalent to the ACHSH game.

## 5.9 ADAPTIVE CHAINED BELL (ACB) GAME

The Adaptive Chained Bell game is a generalisation of the ACHSH game to more input settings. It is based on the chained Bell inequalities mentioned in Section 5.6.1. Three parties, Alice Bob and Charlie are set up in the bi-local causal structure shown in Fig. 5.2. A referee asks Bob to randomly return two bits denoted by $B \in \{00, 01, 10, 11\}$. The referee then asks $m$ uniformly random questions to Alice and Charlie, denoted by random variables $X$ and $Z$, to which they need to return binary answers, denoted by random variables $A$ and $Z$ respectively, such that they satisfy the winning conditions of the game. If $b := (b_0, b_1)$ denotes the two bit response $B$ of Bob,

| $(b_0, b_1)$ | Winning Condition |
|---|---|
| $(0,0)$ | $a \neq c$ if $\lvert 2(z-x)+1 \rvert = 1$,<br>$a = c$ if $x = 0$ and $z = m-1$ |
| $(0,1)$ | $a \neq c$ if $\lvert 2(n-(x+z))-1 \rvert = 1$,<br>$a = c$ if $x = z = 0$ |
| $(1,0)$ | $a = c$ if $\lvert 2(n-(x+z))-1 \rvert = 1$,<br>$a \neq c$ if $x = z = 0$ |
| $(1,1)$ | $a = c$ if $\lvert 2(z-x)+1 \rvert = 1$,<br>$a \neq c$ if $x = 0$ and $z = m-1$ |

Table 5.2: Winning conditions of the Adaptive Chained Bell Game

the winning conditions are given in Table 5.2. With fixed strategies, if $\mathrm{p}(A, B, C, X, Z)$ is the probability distribution generated, the score of the game is given by

$$\mathcal{G}_{\mathrm{ACB}} := 1 - \frac{1}{4} \sum_{\substack{a, b_0, b_1, c \in \{0,1\} \\ x, z \in \{0, \cdots, m-1\}}} p\,(a, b, c, x, z)\,\delta\,(a, b, c, x, z)\,, \tag{5.13}$$

where $\delta = 1$ if the winning conditions in Table 5.2 are met and otherwise $\delta = 0$. When $m = 2$, this game is equivalent to the ACHSH game. Note, that since $X$ and $Z$ are uniformly distributed $p\,(a, b, c, x, z) = p\,(a, b, c|x, z)\,/4$.

Each of the winning conditions in Table 5.2 can be understood as a constraint corresponding to a chained Bell inequality. For instance, when $m = 4$,

$$\mathrm{CB}_{4,(0,0)} = \left(\begin{array}{cc|cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right), \mathrm{CB}_{4,(0,1)} = \left(\begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$\mathrm{CB}_{4,(1,0)} = \left(\begin{array}{cc|cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right), \mathrm{CB}_{4,(1,1)} = \left(\begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right);$$

Every local distribution $\mathrm{p_L}$ satisfies $\left\langle \mathrm{CB}_{4,(b_0,b_1)}, \mathrm{p_L} \right\rangle \geqslant 1$ for any $b_0, b_1 \in \{0, 1\}$. Therefore, if the inner product of a distribution with any of the vectors above is less than 1, Bell nonlocality is witnessed. Next, we provide a quantum strategy such that the ACB game can be won perfectly in the limit $m \to \infty$.

### 5.9.1 Quantum Strategy

If players have access to quantum theory, they can use entanglement swapping operations, just like the strategy in ACHSH game, and perform pre-strategised measurements. To get a high score, one needs to minimise the summation appearing on the right hand side of Equation (5.13). This can be achieved if the observables corresponding to the inputs $x$ and $z$ of Alice and Charlie are

$$\sin\left(\frac{x\pi}{m}\right)\sigma_{\hat{X}} + \cos\left(\frac{x\pi}{m}\right)\sigma_{\hat{Z}} \quad \text{and} \quad \sin\left(\frac{(2z+1)\pi}{2m}\right)\sigma_{\hat{X}} + \cos\left(\frac{(2z+1)\pi}{2m}\right)\sigma_{\hat{Z}} \tag{5.14}$$

respectively. When $b = (0,0)$, the post-measurement state with Alice and Charlie is $|\phi_+\rangle\langle\phi_+|$. On performing the measurements above, the probability of satisfying the first constraint is $2m\sin\left[\pi/(4m)\right]^2$. The probabilities of satisfying the other constraints with the respective post-selected states are all $2m\sin\left[\pi/(4m)\right]^2$, as well. Therefore, the final score in the game is

$$\mathcal{G}^{\mathcal{Q}}_{\text{ACB}} = 1 - \frac{1}{4}\frac{4\times 2m\sin\left[\pi/(4m)\right]^2}{4} = 1 - \frac{m\sin\left[\pi/(4m)\right]^2}{2}. \tag{5.15}$$

When $m = 2$, the score is $(1-1/\sqrt{2})/2$ coinciding with the score in the ACHSH game. In addition, as $m \to \infty$, $m\sin\left[\pi/(4m)\right]^2 \to 0$, implying , $\mathcal{G}^{\mathcal{Q}}_{\text{ACB}} \to 1$. However, we would ideally want to have a finite value for $m$ since the quantum-classical gap disappears as $m \to \infty$.

### 5.9.2 Classical and Box-World Strategies in ACB Game

Since the four conditions presented in Table 5.2 correspond to constraints of chained Bell inequalities, the smallest value of the summation achievable using classical strategies is 1. Therefore, the maximum score of the ACB game for any value of input size $m$ is strictly bounded above by $1-1/m$. The maximum achievable score in BW is $1-1/m$ as well, due to the lack of entanglement swapping and thereby only having access to local classical strategies. In addition, all theories that can be ruled out using the ACHSH game can also be ruled out by the ACB games since these games are equivalent when each party is restricted to only two input settings. Finally, the OS theory reviewed in Section 5.6 also cannot win this game better than classical theory since it cannot violate any chained Bell inequality. It is worth investigating whether there exist generalisations of this theory that can violate the chained Bell inequalities while preserving the feature of perfect entanglement swapping.

# — 6 —

# ACHSH Game in State Spaces with Restricted Relabelling Symmetries

This chapter contains our first set of main results: a large class of theories supporting entanglement swapping can be ruled out using the adaptive CHSH game. These theories produce post-quantum correlations, and some of them support entanglement swapping. The chapter is designed as follows: in Section 6.2 and 6.3, we provide an analytic construction for effect polytopes for any given noisy asymmetric state space in our model. In addition, we provide a complete list of the set of extremal effects for each such case and present a formula for calculating the number of extremal effects of such effect spaces. In Section 6.5, we check this consistency criteria against effects obtained in Sections 6.2 and 6.3 and present our first result showing that entanglement swapping cannot be performed in most of the state spaces in the model considered. Section 6.6 contains the main results stating that quantum theory can be successfully correlation self-tested using the ACHSH game against every state space considered. Finally, in Section 6.7 we present an interesting result drawing a connection between minimal $k$-preservibility and Tsirelson's bound for our model of state spaces. Throughout this chapter, we will assume that only a single copy of the state space is available to each party.

## 6.1 INTRODUCTION

We have shown in Section 2.7, that although both the bipartite gbit state spaces $\mathbb{H}^{[1]}_{(2,2)}$ and $\mathbb{H}^{[8]}_{(2,2)}$ generate all non-signalling correlations in the $(2, 2, 2)$ setting, only $\mathbb{H}^{[1]}_{(2,2)}$ supports the existence of couplers. In addition, recall that out of the state spaces that produce all non-signalling correlations, only $\mathbb{H}^{[8]}_{(2,2)}$ was tested for the purpose of correlation self-testing. These observations raise two natural questions: i) are there other bipartite gbit state spaces that support couplers?[1] ii) could such state spaces outperform quantum theory in the ACHSH game?

One way to try to answer these questions is to consider a bipartite composition gbit state spaces

[1]In [92], authors considered a noisy bipartite gbit state space model with two noisy entangled states. In the general case where the amounts of noise on them are not equal, the state space has only one maximally entangled state and supports couplers. When the noise is the same, there are two maximally entangled states that are isotropically opposite to each other, and the state space does not support couplers. In this thesis, we consider cases where the two maximally entangled states are not necessarily isotropically opposite.

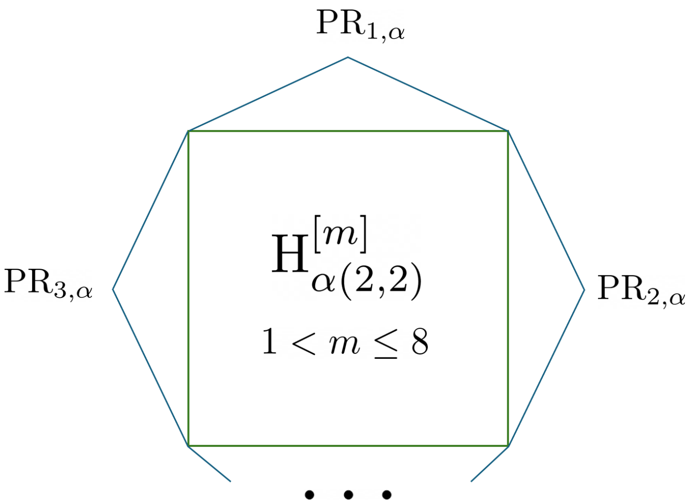


Figure 6.1: A pictorial representation of a state space characterised by the convex hull of the extremal states of $\mathbb{H}^{[0]}_{(2,2)}$ and the $m$ noisy PR boxes of the form presented in Equation (6.1).

characterised by the convex hull of $\mathbb{H}^{[0]}_{(2,2)}$ and $m$ noisy PR boxes of the form:

$$\mathrm{PR}_{i,\alpha_i} := \alpha_i \mathrm{PR}_i + (1-\alpha_i)\mathbb{U}, \tag{6.1}$$

where $\alpha_i \in [1/2, 1]$ and $1-\alpha_i$ denotes the amount of noise for the $i$th PR box and $\mathbb{U}$ is the maximally mixed state. In this work, we only consider the scenario where the amount of noise is the same on all of the PR boxes i.e., $\alpha_i = \alpha$, and denote such a state space by $\mathbb{H}^{[m]}_{\alpha(2,2)}$. Note that $\mathrm{PR}_{i,1/2}$ is local and therefore, $\mathbb{H}^{[m]}_{1/2(2,2)} = \mathbb{H}^{[0]}_{(2,2)}$. In the following, we will use the range of $\alpha$ as $(1/2, 1]$ unless specified otherwise. To investigate whether a state space of this form supports couplers, we need to find the extremal effects of its maximal effect space. Since any effect space is convex (see Definition 2.1.2), if couplers exist at least one of the extremal effects must be a coupler. Therefore, the search for couplers reduces to a check on the set of extremal effects only.

In the following, we first describe the effect polytope for state spaces with 1 noisy PR box and then show how to generalise to two or more noisy PR boxes. Inspired from the pictorial representation of $\mathbb{H}^{[1]}_{(2,2)}$ (see Fig. 5.3), we call a state space of the form $\mathbb{H}^{[m]}_{\alpha(2,2)}$ (see Fig. 6.1) "house with $m$ roofs". Note, that $\mathbb{H}^{[m]}_{\alpha(2,2)}$ is not a unique state space for a fixed $m, \alpha$.

## 6.2 BIPARTITE HOUSE COMPOSITIONS OF $\mathcal{G}_2^2$ AND THEIR EFFECT POLYTOPES

### 6.2.1 1 ROOF: $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$

The 8 PR boxes are equivalent up to relabelling symmetries (see Section 2.4.2); It suffices to work with any PR box while considering a house state space with 1 roof. The state space $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$ is characterised by 23 facets of which 16 are positivity facets and 7 are CH facets. These are the same as the facets of $\mathbb{H}^{[0]}_{(2,2)}$ with the exception of $\langle e_{\mathrm{CH}_2}.\mathbf{x}\rangle \leqslant 1$ [2]. Further, recall from Section 2.4.1 that

[2] Note that $\left\langle e_{\mathrm{CH}_2}.\mathrm{PR}_{2,\alpha}\right\rangle > 1$.

the maximal effect polytope of $\mathbb{H}^{[0]}_{(2,2)}$ has 90 extremal effects which includes 82 BW effects [9] and 8 entangled effects given by $\{e_{\mathrm{CH_i}}\}_{i=1}^{4}$ and $\{u-e_{\mathrm{CH_i}}\}_{i=1}^{4}$. The maximal effect polytope of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$ is a subset of the maximal effect polytope of $\mathbb{H}^{[0]}_{(2,2)}$, which is contained in the intersection of the half-spaces satisfying $\langle \mathbf{x}, \mathrm{PR}_{2,\alpha}\rangle \leqslant 1$ and $\langle \mathbf{x}, \mathrm{PR}_{2,\alpha}\rangle \geqslant 0$, where $\mathbf{x} \in \mathbb{R}^{16}$. Since $\langle e_{\mathrm{CH}_2} .\mathrm{PR}_{2,\alpha}\rangle > 1$, $e_{\mathrm{CH}_2}$ and $u - e_{\mathrm{CH}_2}$ cease to be valid effects of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$. The remaining 88 extremal effects of $\mathbb{H}^{[0]}_{(2,2)}$ are still valid effects for $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$ and in fact extremal. Additional extremal effects for this state space are vectors $\mathbf{x} \in \mathbb{R}^{16}$ that satisfy $\langle \mathbf{x}, \mathrm{PR}_{2,\alpha}\rangle = 0$ and $\langle \mathbf{x}, \mathrm{PR}_{2,\alpha}\rangle = 1$. We found that these come in 4 types up to relabelling. Below, we present a candidate effect of each type lying on the hyperplane $\langle \mathbf{x}, \mathrm{PR}_{2,\alpha}\rangle = 0$, and encourage the curious reader to Appendix B.1 for their derivation:

$$
\begin{aligned}
&\text{Type 1}: \quad \frac{1-\alpha}{\alpha} e_{\mathrm{CH}_2} + \left(1 - \frac{1-\alpha}{\alpha}\right) \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \\
&\text{Type 2}: \quad \frac{1-\alpha}{3\alpha-1} e_{\mathrm{CH}_2} + \left(1 - \frac{1-\alpha}{3\alpha-1}\right) \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \\
&\text{Type 3}: \quad \frac{3-\alpha}{3\alpha+1} e_{\mathrm{CH}_2} + \left(1 - \frac{3-\alpha}{3\alpha+1}\right) \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) =: e_{\mathrm{m},\alpha}, \\
&\text{Type 4}: \quad \frac{2}{3} e_{\mathrm{CH}_2} =: e_{\mathrm{p},\alpha}
\end{aligned}
\tag{6.2}
$$

On the hyperplane $\langle \mathbf{x}, \mathrm{PR}_{2,\alpha}\rangle = 0$, there are 12 effects of Type 1, 8 of Type 2, 8 of Type 3 and 1 of Type 4. Their complementary effects are also extremal effects and lie on the hyperplane $\langle \mathbf{x}, \mathrm{PR}_{2,\alpha}\rangle = 1$.

For a count, the maximal effect polytope of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$ is the convex hull of 146 extremal effects. These include 82 BW effects, 6 CH type effects, 29 effects satisfying $\langle \tilde{e}, \mathrm{PR}_{2,\alpha}\rangle = 1$ and 29 effects satisfying $\langle \tilde{e}, \mathrm{PR}_{2,\alpha}\rangle = 0$. Note that when $\alpha \to 1$, all the effects satisfying $\langle \tilde{e}, \mathrm{PR}_{2,\alpha}\rangle = 1$ from the first two types converge to deterministic effects, and their complementary effects $(u - \tilde{e})$ converge to the complimentary deterministic effects. This leaves us 106 extremal effect of $\mathbb{H}^{[1]}_{(2,2)}[\mathrm{PR}_2]$ in agreement with the Example from Section 5.4.

A natural question now is which of these extremal effects are couplers. A short calculation shows that

$$\mathrm{CHSH}_2\left[\frac{\mathsf{id}_{\mathbf{A}}\otimes e_{\mathbf{B}_1\mathbf{B}_2}\otimes \mathsf{id}_{\mathbf{C}}\left((\mathrm{PR}_{2,\alpha})_{\mathbf{AB}_1}\otimes(\mathrm{PR}_{2,\alpha})_{\mathbf{B}_2\mathbf{C}}\right)}{\left\langle u, \mathsf{id}_{\mathbf{A}}\otimes e_{\mathbf{B}_1\mathbf{B}_2}\otimes \mathsf{id}_{\mathbf{C}}\left((\mathrm{PR}_{2,\alpha})_{\mathbf{AB}_1}\otimes(\mathrm{PR}_{2,\alpha})_{\mathbf{B}_2\mathbf{C}}\right)\right\rangle}\right] = \begin{cases} \frac{\alpha+2}{4} & \text{if } e\in \text{Type 1} \\ \frac{\alpha(\alpha+10)-4}{20\alpha-8} & \text{if } e\in \text{Type 2} \\ \frac{5\alpha^2+2\alpha+4}{4(\alpha+2)} & \text{if } e\in \text{Type 3} \\ \frac{\alpha^2+1}{2} & \text{if } e\in \text{Type 4} \end{cases}. \tag{6.3}$$

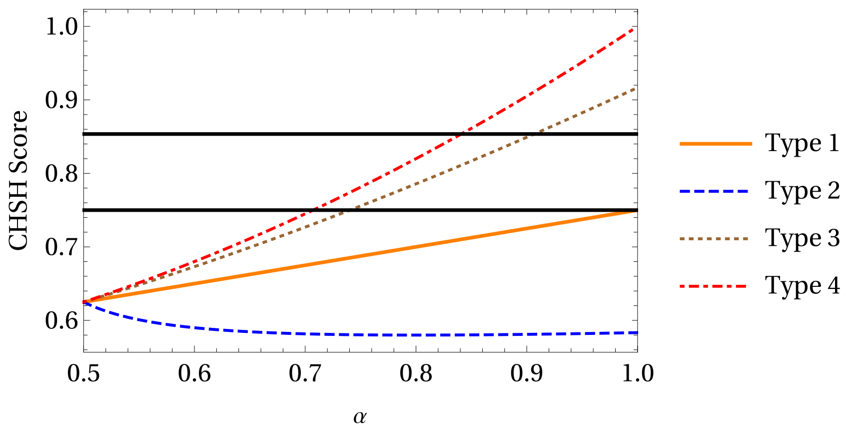


Figure 6.2: A plot of the CHSH scores of the renormalised states obtained when an effect $\tilde{e}$ from each of the four types is applied in the middle half of the 4-partite state $\mathrm{PR}_{2,\alpha}\otimes\mathrm{PR}_{2,\alpha}$. The red line is obtained when $\tilde{e}\in$ Type 4. The brown line is obtained for any $\tilde{e}\in$ Type 3. The yellow and blue lines are obtained when any $\tilde{e}$ is taken from Type 1 and Type 2 respectively. The straight horizontal black lines represent the classical score 3/4 and Tsirelson's bound.

When plotted against $\alpha$, as shown in Fig. 6.2, it is evident that only effects in Type 3 and Type 4 are couplers. In particular, effects in Type 3 are couplers in the range $(1+\sqrt{41})/10<\alpha\leqslant 1$. Similarly, the effect in Type 4 is a coupler in the range $1/\sqrt{2}<\alpha\leqslant 1$, which corresponds to the state spaces having nonlocality strictly more than Tsirelson's bound. Therefore, the only extremal effect which is a coupler in the range $1/\sqrt{2}<\alpha\leqslant(1+\sqrt{41})/10$ is the coupler of Type 4. Additionally, noting the CHSH values of the post-measurement states, one can also see that for the couplers of Type 3, the post-measurement state will have a CHSH value more than Tsirelson's bound when $\alpha>1/10(\sqrt{2}+\sqrt{2(1+20\sqrt{2})})$. Similarly, for the coupler from Type 4, this corresponds to $\alpha>1/\sqrt[4]{2}$. We report that no other extremal effect is a coupler.

Next, for the measurements $\{e_{\mathrm{p},\alpha}, u-e_{\mathrm{p},\alpha}\}$ and $\{e_{\mathrm{m},\alpha}, u-e_{\mathrm{m},\alpha}\}$, a straightforward calculation shows that if Bob shares a copy of the state $\mathrm{PR}_\alpha$, one with Alice and another with Charlie, then the probability of successful entanglement swapping can be expressed in terms of $\alpha$ as

$$p_{\text{success}} = \langle u, \mathsf{id} \otimes e \otimes \mathsf{id}\,(\text{PR}_\alpha \otimes \text{PR}_\alpha)\rangle = \begin{cases} \frac{1}{1+2\alpha} & \text{if } e = e_{\text{p}} \\ \frac{2+\alpha}{2+6\alpha} & \text{if } e = e_{\text{m}} \end{cases} \tag{6.4}$$

Note that when $\alpha = 1$, $e_{p,\alpha=1} = e_{\text{pure}}$ and $e_{m,\alpha=1} \in E_{noisy}$ with the success probability being 1/3 for $e_{\text{p}}$ and 3/8 for $e_{\text{m}}$ which matches with the example visited in Section 5.4 from [93].

### 6.2.2 2 Roofs: $\mathbb{H}^{[2]}_{\alpha(2,2)}[\text{PR}_{2,2'}]$ and $\mathbb{H}^{[2]}_{\alpha(2,2)}[\text{PR}_{12}]$

In this section, we consider state spaces with 2 noisy PR boxes and perform a similar analysis. There are $\binom{8}{2} = 28$ pairs of PR boxes. A pair of PR boxes $(\text{PR}_i, \text{PR}_j)$ is said to be equivalent to another pair $(\text{PR}_k, \text{PR}_l)$ if there exists a local relabelling operation R such that $\text{R}[\text{PR}_i] = \text{PR}_k$ and $\text{R}[\text{PR}_j] = \text{PR}_l$. We found that there are two classes of pairs of PR boxes. $(\text{PR}_1, \text{PR}_2)$ are an instance of the first class and $(\text{PR}_2, \text{PR}_2')$ are an instance of the second. Therefore, with 2 roof state spaces we have to do a two-part analysis to cover all possibilities. We will first look into state spaces where the pair is isotropically opposite and then investigate the other case.

Let us denote by $\mathbb{H}^{[2]}_{\alpha(2,2)}[\text{PR}_{2,2'}]$ the state space characterised by the convex hull of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\text{PR}_2]$ and the noisy PR box $\text{PR}'_{2,\alpha}$. This state space is characterised by 16 positivity facets and 6 Bell facets. In particular, $\langle u - e_{\text{CH}_2}, \mathbf{x}\rangle \leqslant 1$ which is a facet of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\text{PR}_2]$ is no longer a facet of $\mathbb{H}^{[2]}_{\alpha(2,2)}[\text{PR}_{2,2'}]$ [3]. The maximal effect space of $\mathbb{H}^{[2]}_{\alpha(2,2)}[\text{PR}_{2,2'}]$ is the subset of the maximal effect space of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\text{PR}_2]$ that is contained in the intersection of the half spaces satisfying $\langle \mathbf{x}, \text{PR}_{2',\alpha}\rangle \leqslant 1$ and $\langle \mathbf{x}, \text{PR}_{2',\alpha}\rangle \geqslant 0$. Using a similar technique as the case of 1 roof, we calculated all the extremal effects of the maximal effect space and found that there are no couplers. The extremal effects include 82 BW effects, 6 CH type effects, 12 effects shared by the hyperplanes $\langle \mathbf{x}, \text{PR}_{2,\alpha}\rangle = 0$ and $\langle \mathbf{x}, \text{PR}'_{2,\alpha}\rangle = 1$, 12 effects shared by the hyperplanes $\langle \mathbf{x}, \text{PR}_{2,\alpha}\rangle = 1$ and $\langle \mathbf{x}, \text{PR}'_{2,\alpha}\rangle = 0$ and 8 Type 2 effects lying on each of these four hyperplanes, making it a total of 144 extremal effects. We refer the reader to Appendix B.2 for more details.

Next, let us consider the second state space $\mathbb{H}^{[2]}_{\alpha(2,2)}[\text{PR}_{12}]$ where the two noisy PR boxes are not isotropically opposite to each other. We found that the only extremal effects of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\text{PR}_2]$ that cease to be valid are $e_{\text{CH}_1}$ and $e_{\text{CH}_{1'}}$. Following the same construction as above, we found that the extremal effects lying on the hyperplane $\langle \mathbf{x}, \text{PR}_{1,\alpha}\rangle = 0$ are exactly of the form Type 1,2,3 and 4. Consequently, their complementary effects lie on the hyperplane $\langle \mathbf{x}, \text{PR}_{1,\alpha}\rangle = 1$ and are extremal. These constitute the new extremal effects. This gives us the following count of the total number of extremal effects: 82 BW effects, 4 CH type effects and 29 effects each from the 4 hyperplanes, making it a total of 202 extremal effects. We refer the reader to Appendix B.3 for more details. Finally, note that since all the extremal effects from the previous state space are still extremal effects here, this state space does have couplers.

[3] Note that $\langle u - e_{\text{CH}_2}, \text{PR}'_{2,\alpha}\rangle > 1$.

### 6.2.3 General Algorithm for $m$ Roofs

Finally, let us focus on the general case, $\mathbb{H}^{[m]}_{\alpha(2,2)}$, which is the state space characterised by the convex hull of 16 local deterministic boxes and $m$ noisy PR boxes. For the sake of this discussion only, let us denote these $m$ PR boxes as $\mathrm{PR}_{i,\alpha}$ where $i \in \{1, \cdots, m\}$. To construct the effect polytope of $\mathbb{H}^{[m]}_{\alpha(2,2)}$, one can run the following sequence of steps:

**Step 1 :** Consider $m$ house state spaces with 1 roof each where the roofs are the noisy PR boxes of the state space of interest, i.e., house with $m$ roofs.

**Step 2 :** From the effect polytope described Section 6.2.1, find the effect polytope of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_i]$ for each $i$ via the identification of local relabelling symmetry. More precisely, if $\mathrm{R}_i$ is a relabelling operation such that $\mathrm{R}_i[\mathrm{PR}_2] = \mathrm{PR}_i$, then the extremal effects of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_i]$ are

$$\text{Extreme}\left[\mathcal{E}_{\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_i]}\right] = \left\{\mathrm{R}_i[e] \mid e \in \text{Extreme}\left[\mathcal{E}_{\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]}\right]\right\}$$

**Step 3 :** Define $V_E$ as the union of all the extremal effects found in each case, i.e.,

$$V_E \coloneqq \bigcup_{i=1}^{m} \text{Extreme}\left[\mathcal{E}_{\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_i]}\right].$$

**Step 4 :** Run through elements of $V_E$ and discard the ones that give an inner product outside $[0, 1]$ with any of the noisy $m$ PR boxes.

**Step 5 :** Denote all the remaining elements of $V_E$ as $\text{Extreme}\left[\mathcal{E}_{\mathbb{H}^{[m]}_{\alpha}}\right]$.

Note, that in Step 4, discarding suffices since all new effects that might arise are captured in Step 1. With this construction, one can also give a count of the number of extremal effects corresponding to any such state spaces. There are 90 extremal effects from $\mathbb{H}^{[0]}_{(2,2)}$. This includes 82 BW effects and 8 CH type effects. Now let us assume that in the state space $\mathbb{H}^{[m]}_{\alpha(2,2)}$ with $1/2 < \alpha < 1$ there are $t$ pairs of noisy PR boxes that are isotropically opposite to each other. Now, the addition of $(m - 2t)$ noisy PR boxes to the local effect polytope introduces $58(m - 2t)$ new extremal effects and eliminates $2(m - 2t)$ CH type effects. On the other hand $t$ pairs of isotropically opposite PR boxes introduce to the local effect polytope $56t$ new extremal effects and eliminates $2t$ CH type effects. Putting all these together one gets $90 + 56m - 58t$ extremal effects of the effect polytope. A similar analysis can be done when $\alpha = 1$ which leads us to the following formula for the total number of extremal effects of the effect polytope :

$$\left|\text{Extreme}\left[\mathcal{E}_{\mathbb{H}^{[m]}_{\alpha(2,2)}}\right]\right| = \begin{cases} 90 & \text{if } \alpha = 1/2 \\ 90 + 56m - 58t & \text{if } 1/2 < \alpha < 1 \\ 90 + 16m - 34t & \text{if } \alpha = 1. \end{cases} \tag{6.5}$$

The entire construction discussed in this section can be lifted to the case when the PR boxes have different amounts of noise on them. However, we refrain from that investigation in this thesis.

## 6.3 BIPARTITE HOUSE COMPOSITION OF $\mathcal{G}_3^2$ AND THEIR EFFECT POLYTOPES

In quantum theory, the smallest system, a qubit, needs three fiducial measurements to be characterised. Therefore, to draw a closer analogy with qubit quantum theory, we study bipartite state spaces of $\mathcal{G}_3^2$ and investigate the presence of couplers in them. We have looked at $\mathbb{H}^{[1]}_{(3,2)}$ state spaces constructed from the convex hull of 64 local deterministic states and 1 extremal entangled state. Since there are 4 classes of extremal entangled states we consider them separately. In Table 6.1 we provide a summary of the different facets of each state space and the number of extremal effects of their respective effect polytopes.

| Class | # $\mathrm{I_{CH}}$ *Facets* | # $\mathrm{I_{3322}}$ *Facets* | #Extreme Effects |
|---|---|---|---|
| $\mathbb{H}^{[1]}_{(3,2)}[\mathrm{N}_1]$ | 71 | 568 | 29486 |
| $\mathbb{H}^{[1]}_{(3,2)}[\mathrm{N}_2]$ | 66 | 558 | 41888 |
| $\mathbb{H}^{[1]}_{(3,2)}[\mathrm{N}_3]$ | 68 | 552 | 37376 |
| $\mathbb{H}^{[1]}_{(3,2)}[\mathrm{N}_4]$ | 70 | 564 | 32384 |

Table 6.1: Summary of the number of CH facets, $\mathrm{I_{3322}}$ facets and extremal effects for the state space $\mathbb{H}^{[1]}_{(3,2)}$. $\mathrm{N}_1, \mathrm{N}_2, \mathrm{N}_3$ and $\mathrm{N}_4$ are as defined in Section 2.4.2.

We will discuss the existence of couplers for such state spaces in Section 6.5.3.

## 6.4 MINIMAL 2-PRESERVIBILITY IN HOUSE STATE SPACES

### 6.4.1 CH Type Effects of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$

Recall from Section 6.2.1, that the CH type effects of the effect polytope $\mathcal{E}_{\mathbb{H}^{[1]}_{\alpha(2,2)}}$ for $\alpha \in (1/2, 1]$ are $e_{\mathrm{CH}_1}, e'_{\mathrm{CH}_1}, e_{\mathrm{CH}_3}, e'_{\mathrm{CH}_3}, e_{\mathrm{CH}_4}$ and $e'_{\mathrm{CH}_4}$, where $e'_{\mathrm{CH_i}} = u - e_{\mathrm{CH_i}}$. However, these effects are not minimally 2-preserving. A direct calculation shows that

$$\begin{aligned}
&\mathrm{CHSH}_{1'}\left[\frac{\Phi^{(23)}_{e_{\mathrm{CH}_1}}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})}{\left\langle u,\Phi^{(23)}_{e_{\mathrm{CH}_1}}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})\right\rangle}\right] = \mathrm{CHSH}_{1}\left[\frac{\Phi^{(23)}_{e_{\mathrm{CH}'_1}}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})}{\left\langle u,\Phi^{(23)}_{e_{\mathrm{CH}'_1}}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})\right\rangle}\right] \\
&= \mathrm{CHSH}_{4'}\left[\frac{\Phi^{(23)}_{e_{\mathrm{CH}_3}}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})}{\left\langle u,\Phi^{(23)}_{e_{\mathrm{CH}_3}}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})\right\rangle}\right] = \mathrm{CHSH}_{4}\left[\frac{\Phi^{(23)}_{e_{\mathrm{CH}'_3}}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})}{\left\langle u,\Phi^{(23)}_{e_{\mathrm{CH}'_3}}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})\right\rangle}\right] \\
&= \mathrm{CHSH}_{3'}\left[\frac{\Phi^{(23)}_{e_{\mathrm{CH}_4}}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})}{\left\langle u,\Phi^{(23)}_{e_{\mathrm{CH}_4}}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})\right\rangle}\right] = \mathrm{CHSH}_{3}\left[\frac{\Phi^{(23)}_{e_{\mathrm{CH}'_4}}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})}{\left\langle u,\Phi^{(23)}_{e_{\mathrm{CH}'_4}}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})\right\rangle}\right] \\
&= \frac{1}{2}(\alpha^2+1).
\end{aligned} \tag{6.6}$$

Since $(\alpha^2+1)/2 > 3/4$ when $\alpha > 1/\sqrt{2}$, no CH-type effect is minimally 2-preserving for $\alpha > 1/\sqrt{2}$ since in $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$ only $\mathrm{CHSH}_2$ can be violated. To check whether the state on systems 1 and 4 is an element of the state space, one can alternatively verify whether the list of inner products it generates with all the extremal effects are in the interval $[0,1]$ or not. For the current example, this boils down to checking this against the list of CH-type effects only since the non-trivial facets of the state space polytope are exactly the CH facets. However, in the above, we have calculated scores obtained in CHSH games with correlations obtained upon performing fiducial measurements on the states. This is because the winning criterion of these games, for local theories, can be written as inequalities that can be affinely transformed into the corresponding CH inequalities. Further, since we are ultimately going to investigate the ACHSH game, the use of CHSH games for our examples is useful to set the context.

### 6.4.2 Couplers of $\mathbb{H}^{[2]}_{\alpha(2,2)}[\mathrm{PR}_{1,2}]$

Recall from Section 6.2.2 that the state space $\mathbb{H}^{[2]}_{\alpha(2,2)}[\mathrm{PR}_{1,2}]$ contains couplers. However, these couplers are not minimally 2-preserving. To check this, note that:

$$\mathrm{CHSH}_{2'}\left[\frac{\Phi^{(23)}_{f_2(\alpha)}(\mathrm{PR}_{1,\alpha},\mathrm{PR}_{2,\alpha})}{\left\langle u,\Phi^{(23)}_{f_2(\alpha)}(\mathrm{PR}_{1,\alpha},\mathrm{PR}_{2,\alpha})\right\rangle}\right]=\mathrm{CHSH}_{1'}\left[\frac{\Phi^{(23)}_{f_1(\alpha)}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})}{\left\langle u,\Phi^{(23)}_{f_1(\alpha)}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})\right\rangle}\right]=\frac{1}{2}\left(\alpha^2+1\right),$$
$$\mathrm{CHSH}_{2'}\left[\frac{\Phi^{(23)}_{\tilde{e}(\alpha)}(\mathrm{PR}_{1,\alpha},\mathrm{PR}_{2,\alpha})}{\left\langle u,\Phi^{(23)}_{\tilde{e}(\alpha)}(\mathrm{PR}_{1,\alpha},\mathrm{PR}_{2,\alpha})\right\rangle}\right]=\mathrm{CHSH}_{1'}\left[\frac{\Phi^{(23)}_{\tilde{g}(\alpha)}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})}{\left\langle u,\Phi^{(23)}_{\tilde{g}(\alpha)}(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{2,\alpha})\right\rangle}\right]=\frac{5\alpha^2+2\alpha+4}{4(\alpha+2)}, \tag{6.7}$$

where $f_1(\alpha)$ and $\tilde{g}(\alpha)$ are the Type 4 and any Type 3 effect lying on the facet $\langle \mathbf{x},\mathrm{PR}_{1,\alpha}\rangle = 1$ and $f_2(\alpha)$ and $\tilde{e}(\alpha)$ are Type 4 and any Type 3 effects lying on the facet $\langle \mathbf{x},\mathrm{PR}_{2,\alpha}\rangle = 1$. This implies that none of the Type 3 and Type 4 effects lying on these hyperplanes are minimally 2 preserving and are therefore not valid. Note, however, that these become minimally 2 preserving exactly when they stop coupling i.e., $\alpha \leqslant (1+\sqrt{41})/10$ for Type 2 effects and $\alpha \leqslant 1/\sqrt{2}$ for Type 4 effects. As a result, there are no minimally 2-preserving extremal effects for this state space that are couplers. Additionally, the CH-type effects are not minimally 2-preserving either and a similar treatment as in the previous example shows that for a fixed $\alpha$, the maximum score in a $\mathrm{CHSH}_{\mathrm{i}\neq 1,2}$ game when a CH effect is applied in the middle half of two allowed PR boxes is $(\alpha^2+1)/2$.

## 6.5 MINIMALLY 2-PRESERVING COUPLERS OF PARTY SYMMETRIC STATE SPACES

### 6.5.1 Party Symmetric State Spaces with Restricted Relabelling

In this thesis, we will focus on bipartite house compositions of $\mathcal{G}_2^2$ and $\mathcal{G}_3^2$ that are party symmetric. A first classification can be based on the number of maximally entangled states (roofs) present in the state space. For compositions of $\mathcal{G}_2^2$, we will consider this varying from one to all eight noisy PR boxes. Next, for a given number of roofs, a second classification involves finding equivalence classes under relabelling. We say that two state spaces $\mathbb{H}^{[m]}_{\alpha(2,2)}$ and $\mathbb{H}'^{[m]}_{\alpha(2,2)}$ are equivalent if there exists a relabelling $R$ of probability tables that uniquely maps $\mathbb{H}^{[m]}_{\alpha(2,2)}$ to $\mathbb{H}'^{[m]}_{\alpha(2,2)}$, i.e., for every state $s' \in \mathbb{H}'^{[m]}_{\alpha(2,2)}$, there exists a state $s$ such that $R[s]=s'$.

We found that for $2 \leqslant m \leqslant 7$, one can find classes of party symmetric state spaces for every $m$. In the table below we give a count on the number of such classes for each choice $m$, and refer the reader to Appendix D for a full classification.

| # Roofs | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| # Classes | 2 | 2 | 4 | 2 | 2 | 1 |

Table 6.2: Table shows the number of equivalence classes of party symmetric states spaces with $m$ roofs.

When single systems are described by $\mathcal{G}_3^2$, party symmetric state spaces with one roof are characterised by the convex hull of 64 local deterministic states and one of the entangled states $\mathrm{N}_1, \mathrm{N}_2$ or $\mathrm{N}_3$. We will restrict to only one roof since the number of cases to consider with more is combinatorically large. In addition, we will only consider noiseless house compositions, i.e., state spaces of the form $\mathbb{H}^{[1]}_{\alpha=1(3,2)}[\mathrm{N_i}]$. We leave the treatment of more roofs and noisy state spaces for future work.

In the following, we consider bipartite house composition of systems whose single system state spaces are either $\mathcal{G}_2^2$ or $\mathcal{G}_3^2$. For each such state space, we calculate its effect polytope and search for effects which are minimally 2-preserving couplers.

### 6.5.2 Bipartite House Composition: $\mathbb{H}^{[m]}_{\alpha(2,2)}$

Recall from Section 3.2 that the state space $\mathbb{H}^{[1]}_{\alpha=1(2,2)}[\mathrm{PR}_2]$ allows the presence of couplers that are minimally 2-preserving [93]. This is not true for the state space $\mathbb{H}^{[2]}_{\alpha}[\mathrm{PR}_{1,2}]$. In fact, by using the ideas from the previous sections, we have shown case by case that none of the extremal effects of $\mathbb{H}^{[m]}_{\alpha(2,2)}$ with $1 < m \leqslant 8$ are minimally 2-preserving couplers. This, however, does not rule out that there might be non-extremal minimally 2-preserving effects that are couplers. It begs the question then of whether certain convex mixtures of extremal effects can be both minimally 2-preserving and couplers. Our next theorem says that these also do not exist.

**Theorem 1.** *Let $\tilde{e} \in \mathcal{E}_{\mathbb{H}^{[m]}_{\alpha(2,2)}}$ be an effect for a party symmetric bipartite state space $\mathbb{H}^{[m]}_{\alpha(2,2)}$ where $1 < m \leqslant 8$. Then the following does not simultaneously hold :*

- *$\tilde{e}$ is a coupler*
- *$\tilde{e}$ is minimally 2-preserving*

*Proof.* We refer the interested reader to Appendix E for the proof. □

We have also considered bipartite state spaces that do not have the party swap symmetry. For these, we have considered state spaces of the form $\mathbb{H}^{[m]}_{\beta(2,2)}$, where $\beta$ is a discrete variant of $\alpha$, varied between $1/2$ and $1$ with a step size of $1/30$ and $1 < m \leqslant 8$. We have numerical evidence that these state spaces do not support minimally 2-preserving couplers as well. Therefore, we conjecture that minimally 2-preserving couplers are only present in house state spaces with a single roof.

### 6.5.3 Bipartite House Composition: $\mathbb{H}^{[1]}_{(3,2)}$

Using PANDA, we found the maximal set of extremal effects for the state spaces $\mathbb{H}^{[1]}_{(3,2)}[\mathrm{N}_1]$, $\mathbb{H}^{[1]}_{(3,2)}[\mathrm{N}_2]$ and $\mathbb{H}^{[1]}_{(3,2)}[\mathrm{N}_3]$. From each of these maximal effect spaces, we filtered out every extremal effect which is not a minimally 2-preserving coupler and listed out the ones that are. We then classified the extremal effects present in each list into equivalent relabelling classes. Taking a class representative, we calculated the maximal violations of the $\mathrm{F}_{\mathrm{CH}}$ and $\mathrm{F}_{3322}$ inequalities and the corresponding probabilities of successful swapping.

$\mathbb{H}^{[1]}_{(3,2)}[\mathrm{N}_1]$ has 29786 extremal effects in its maximal effect space of which 28688 effects are minimally 2-preserving. Amongst these, there are 856 couplers which can be classified into 61 relabelling classes. 104 of these couplers are pure and fall into 15 of the 61 classes. We present a member of each of these 15 classes in Appendix C. We found that the maximum product of the probability of successful swapping and the inner product of the normalised state generated with $\mathrm{F}_{\mathrm{CH}}$ is 1/2. One of the couplers with which one can obtain this value is

$$f_{\mathrm{CH}_1} := \frac{2}{3}\left(\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Notice that the top left $4 \times 4$ block of $f_{\mathrm{CH}_1}$ is $e_{\mathrm{CH}_1}$. With this, the probability of a successful swap is 1/3 and the CHSH value is 1.

$\mathbb{H}^{[1]}_{(3,2)}[\mathrm{N}_2]$ has 41888 extremal effects in its maximal effect space. Of these, there are 19829 minimally 2-preserving effects. However, we found that none of these are couplers. The effect $f_{\mathrm{CH}_1}$, defined above, although extremal, turns out to be not minimally 2-preserving since the normalised state obtained after swapping is $\mathrm{N}_3$.

$\mathbb{H}^{[1]}_{(3,2)}[\mathrm{N}_3]$ has 37376 extremal effects in its maximal effect space. 35503 of these are minimally 2-preserving of which there are 2716 couplers that can be grouped into 78 relabelling classes. Only one of these classes contains pure couplers. Similar to the case of $\mathbb{H}^{[1]}_{(3,2)}[\mathrm{N}_1]$, the maximum product of the probability of successful swapping and the inner product of the normalised state generated with $\mathrm{F}_{\mathrm{CH}}$ is 1/2; One can get this value with $f_{\mathrm{CH}_1}$.

## 6.6 CORRELATION SELF-TESTING OF QUANTUM THEORY AGAINST PARTY SWAP SYMMETRIC STATE SPACES

In this section, we start laying out our main result showing that quantum theory can be correlation self-tested against any party swap symmetric state space of the form $\mathbb{H}^{[m]}_{\alpha(2,2)}$ and $\mathbb{H}^{[1]}_{(3,2)}$. To do this, we first introduce a Lemma and a Proposition and then provide proofs of the main claim in the following sections.

**Lemma 1.** *Let $A, B, X, Y$ be four random variables with $|A| = |B| = |X| = |Y| = 2$. Let* $\mathrm{CHSH} = \{\mathrm{CHSH}_i\}_{i=1}^{8}$ *be the 8 CHSH games in the Bell setting* $(2,2,2)$. *If* $\mathrm{CHSH}_i\,[\mathrm{p}(A,B|X,Y)] > 3/4$, *then* $\mathrm{CHSH}_j\,[\mathrm{p}(A,B|X,Y)] \leqslant 3/4$ *for all $j \neq i$.*

*Proof.* When a conditional probability distribution $\mathrm{p}(A,B|X,Y)$ scores less than 3/4 in a CHSH game, we can express it by $\mathrm{CHSH_i}\,[\mathrm{p}(A,B|X,Y)] \leqslant 3/4$, where $\mathrm{CHSH_i}$ is one of the following:

$$\mathrm{CHSH}_1 = \frac{1}{4}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right), \mathrm{CHSH}_2 = \frac{1}{4}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right), \mathrm{CHSH}_3 = \frac{1}{4}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

,

$$\mathrm{CHSH}_4 = \frac{1}{4}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right), \mathrm{CHSH}_5 = \frac{1}{4}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right), \mathrm{CHSH}_6 = \frac{1}{4}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

,

$$\mathrm{CHSH}_7 = \frac{1}{4}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right), \mathrm{CHSH}_8 = \frac{1}{4}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right);$$

There are 8 CHSH inequalities that can lead to 2 classes of pairs up to relabelling symmetry: $\{\mathrm{CHSH}_1, \mathrm{CHSH}_3\}$ and $\{\mathrm{CHSH}_1, \mathrm{CHSH}_5\}$. Therefore it suffices to prove the results for these pairs. Let us assume that $\mathrm{CHSH}_1[\mathrm{p}(A,B|X,Y)] > 3/4$. Note that the $2 \times 2$ blocks of $\mathrm{CHSH_{j\neq1}}$ are all different when $\mathrm{j} = 5$ whereas others are different from $\mathrm{CHSH}_1$ by two blocks. Let us collect them in two sets $\mathrm{CHSH}_5$ and $\mathrm{CHSH_{j\neq1,5}}$. First, let us assume that $\mathrm{CHSH}_3[\mathrm{p}(A,B|X,Y)] > 3/4$ as well. Then it follows that

$$\begin{aligned} (\mathrm{CHSH}_1 + \mathrm{CHSH}_3)[\mathrm{p}(A,B|X,Y)] &> \frac{3}{2} \\ \implies \frac{1}{4}\left(\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) + \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right)\right)[\mathrm{p}(A,B|X,Y)] &> \frac{3}{2} \qquad (6.8) \\ \implies 1 + 1 + 2\,(\mathrm{p}(0,0|0,0) + \mathrm{p}(1,1|0,0) + \mathrm{p}(0,0|1,0) + \mathrm{p}(1,1|1,0)) &> 6 \\ \implies (\mathrm{p}(0,0|0,0) + \mathrm{p}(1,1|0,0) + \mathrm{p}(0,0|1,0) + \mathrm{p}(1,1|1,0)) &> 2 \end{aligned}$$

On the other hand, for any conditional probability distribution $\mathrm{p}(A,B|X,Y)$, we have

$$\max_{\mathrm{p}\in\mathcal{P}} (\mathrm{p}(0,0|0,0) + \mathrm{p}(1,1|0,0) + \mathrm{p}(0,0|1,0) + \mathrm{p}(1,1|1,0)) \leqslant 2, \qquad (6.9)$$

which is a contradiction. A similar contradiction can be reached if any other element from the second set were chosen. Next, let us assume that $\mathrm{CHSH}_1[\mathrm{p}(A,B|X,Y)] > 3/4$ and $\mathrm{CHSH}_5[\mathrm{p}(A,B|X,Y)] > 3/4$, then

$$
\begin{aligned}
&(\text{CHSH}_1 + \text{CHSH}_5)[\text{p}(A,B|X,Y)] > \frac{3}{2} \\
\implies &\left(\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) + \left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right)\right)[\text{p}(A,B|X,Y)] > 6 \\
\implies &\left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right)[\text{p}(A,B|X,Y)] > 6 \\
\implies &4 > 6
\end{aligned} \tag{6.10}
$$

which is also a contradiction.

□

Although this result states a general property of conditional probabilities, when applied to $(2,2,2)$ Bell scenario, implies that every conditional probability distribution of the form $\text{p}(A,B|X,Y)$ can only win at most one out of 8 CHSH games, with a score of more than 3/4. Next, we prove an analytic upper bound of the maximum probability of winning the ACHSH game for any GPT characterized by the convex hull of local deterministic states and only one nonlocal state.

**Proposition 1.** *Let $\mathbb{H}^{[1]}_{\alpha,(\text{m},\text{n})}[\text{N}]$ be the convex hull of $\{\text{L}_1, \cdots, \text{L}_n, \text{N}_\alpha\}$, where $\{\text{L}_i\}_{i=1}^n$ are local deterministic states and $\text{N}_\alpha$ is a maximally entangled state. Let $E_{\text{coup}} \subset \text{Extreme}[\mathcal{E}_{\mathbb{H}^{[1]}_{\alpha,(\text{m},\text{n})}[\text{N}]}]$ be the subset of minimally 2-preserving couplers. Further, for $e \in E_{\text{coup}}$, let $p_{\text{succ}}(e) = \left\langle u, \Phi_e^{(2,3)}(\text{N}_\alpha, \text{N}_\alpha)\right\rangle$, $s_e = \Phi_e^{(2,3)}(\text{N}_\alpha, \text{N}_\alpha)/p_{\text{succ}}(e)$ and $\zeta_e$ be the maximum score of any distribution $\text{p}_{s_e}(A,B|X,Y)$, generated by $s_e$ in the $(2,2,2)$ Bell setting, in any CHSH game. Finally, let $p_{win}$ be the maximum probability of winning the ACHSH game in the state space $\mathbb{H}^{[1]}_{\alpha,(\text{m},\text{n})}[\text{N}]$. Then,*

$$
p_{win} \leqslant \begin{cases} \frac{3}{4} & \text{if} \quad E_{\text{coup}} = \emptyset \\ \frac{3}{4} + \max\limits_{e \in E_{\text{coup}}} p_{\text{succ}}(e)\left(\zeta_e - \frac{3}{4}\right) & \text{if} \quad E_{\text{coup}} \neq \emptyset \end{cases}. \tag{6.11}
$$

*Proof.* In the adaptive CHSH game, Bob performs a four outcome measurement $M = \{e_b\}_{b \in \{00,01,10,11\}}$. Corresponding to each outcome, Alice and Charlie need to win 4 different CHSH games labelled $\{\text{CHSH}_b\}_b$. To perform entanglement swapping, Bob shares two instances of the maximally entangled state, $\text{N}_\alpha$, one with Alice and one with Charlie. Since $e_{00}$ is minimally 2-preserving, $s_{e_{00}} \in \text{ConvHull}\{\text{L}_1, \text{L}_2, \cdots, \text{L}_n, \text{N}_\alpha\}$. Let,

$$
s_{e_{00}} = \sum_{j=1}^{n} \lambda_{00,j} L_j + \delta_i \text{N}, \tag{6.12}
$$

such that $\sum_{j=1}^n \lambda_{00,j} + \delta_i = 1$ and $\lambda_{00,j}, \delta_i \geqslant 0$ for all $j \in \{1, 2, \cdots, n\}$. Recall further that Alice and Charlie fix their measurements and do not change them throughout the run of the game. Since

$s_{e_{00}}$ admits the decomposition in Eq. (6.12), the probability distribution $\mathrm{p}_{s_{e_{00}}}(A,B|X,Z)$ obtained after Alice and Charlie measure the state $s_{e_{00}}$ can be expressed as

$$\mathrm{p}_{s_{e_{00}}}(A,B|X,Z)=\sum_{j=1}^{n}\lambda_{00,j}\mathrm{p}_{L_j}(A,B|X,Z)+\delta_i\mathrm{p}_{\mathrm{N}_\alpha}(\mathrm{A},\mathrm{B}|\mathrm{X},\mathrm{Z}), \tag{6.13}$$

where $\mathrm{p}_{L_j}(A,B|X,Z)$ is the distribution obtained if Alice and Charlie had measured the local deterministic state $L_j$, and $\mathrm{p}_{\mathrm{N}_\alpha}(A,B|X,Z)$ is the distribution obtained if Alice and Charlie had measured the entangled state $\mathrm{N}_\alpha$. Their objective is that $\mathrm{p}_{s_{e_{00}}}(A,B|X,Z)$ wins the CHSH game $\mathrm{CHSH}_{00}$. For this, the state $s_{e_{00}}$ must be entangled, i.e., $s_{e_{00}}\notin\mathrm{ConvHull}\{\mathrm{L}_1,\mathrm{L}_2,\cdots,\mathrm{L}_\mathrm{n}\}$; in other words, $e_{e_{00}}$ must be a coupler. Assume that $\mathrm{CHSH}_{00}[\mathrm{p}_{e_{00}}(A,B|X,Z)]>3/4$. Next, consider the state $s_{e_{01}}$ left with Alice and Charlie corresponding to the outcome of the effect $e_{01}$. From minimal 2-preservibility of $e_{01}$, $s_{e_{01}}$ will also have a decomposition as in Eq. (6.12) and since the measurements of Alice and Charlie are fixed, $s_{e_01}$ will generate a conditional probability distribution, $\mathrm{p}_{s_{e_{01}}}(A,B|X,Y)$, that admits a decomposition similar to Eq. (6.13). If $e_{01}$ is a coupler, the distribution $\mathrm{p}_{s_{e_{01}}}(A,B|X,Y)$ will win the game $\mathrm{CHSH}_{00}$ by an amount more than 3/4. However, this time Alice and Charlie need to win $\mathrm{CHSH}_{01}$ by an amount more than 3/4. Now, recall that by Lemma 1, no conditional distribution can simultaneously win two CHSH games by an amount more than 3/4. This implies that Alice and Charlie can only win $\mathrm{CHSH}_{01}$ by a score of at most 3/4. This argument can be extended to the remaining two post-selected states as well. The maximum winning probability is achieved if the measurement choice helps Alice and Charlie to have a score of 3/4 for the remaining games. Since $e_{00}$ is a dummy variable, replacing $e_{00}$ by $e$ gives the following upper bound on the winning probability

$$\begin{aligned}p_{\mathrm{win}}&\leqslant p_{\mathrm{succ}}(e)\zeta_e+(1-p_{\mathrm{succ}}(e))\frac{3}{4}\\&\leqslant\frac{3}{4}+p_{\mathrm{succ}}(e)\left(\zeta_e-\frac{3}{4}\right).\end{aligned} \tag{6.14}$$

This upper bound is maximised when the product $p_{\mathrm{succ}}(e)\,(\zeta_e-3/4)$ is maximised. When there are no minimally 2-preserving couplers, $\zeta_e$ can be at most 3/4. Putting these together, we get

$$p_{\mathrm{win}}\leqslant\begin{cases}\frac{3}{4} & \text{if}\quad E_{\mathrm{coup}}=\emptyset\\ \frac{3}{4}+\max\limits_{e\in E_{\mathrm{coup}}}p_{\mathrm{succ}}(e)\left(\zeta_e-\frac{3}{4}\right) & \text{if}\quad E_{\mathrm{coup}}\neq\emptyset\end{cases}. \tag{6.15}$$

. □

**Theorem 2.** *Let* $\mathbf{p}_{\mathbb{H}^{[1]}_{\alpha,(2,2)}}$ *and* $\mathbf{p}_{\mathbb{H}^{[1]}_{\alpha=1,(3,2)}}$ *denote the maximum winning probability of the ACHSH game in the party symmetric state spaces* $\mathbb{H}^{[1]}_{\alpha,(2,2)}$ *and* $\mathbb{H}^{[1]}_{\alpha=1,(3,2)}$ *respectively and let* $\mathbf{p}_{\mathcal{Q}}$ *denote Tsirelson's bound. Then the following hold:*

1. $\mathbf{p}_{\mathcal{Q}}>\mathbf{p}_{\mathbb{H}^{[1]}_{\alpha,(2,2)}}$,
2. $\mathbf{p}_{\mathcal{Q}}>\mathbf{p}_{\mathbb{H}^{[1]}_{\alpha=1,(3,2)}}$.

*Proof. 1.* For $\mathbb{H}^{[1]}_{\alpha,(2,2)}$ there is only one class to check since all the PR boxes are equivalent up to local relabelling. Going back to Section 6.2.1, we find that there are only two types of extremal effects that are minimally 2-preserving couplers: $e_{\mathrm{p},\alpha}$ and of $e_{\mathrm{m},\alpha}$. From Eq. (6.3) and Eq. (6.4), we get

$$\frac{3}{4}+p_{\mathrm{succ}}(e_{\mathrm{p},\alpha})\left(\zeta_{e_{\mathrm{p},\alpha}}-\frac{3}{4}\right)=\frac{\alpha(\alpha+3)+1}{4\alpha+2},\quad \frac{3}{4}+p_{\mathrm{succ}}(e_{\mathrm{m},\alpha})\left(\zeta_{e_{\mathrm{m}\alpha}}-\frac{3}{4}\right)=\frac{\alpha(5\alpha+17)+4}{24\alpha+8};$$

When $\alpha\in[1/2,1]$, both of these quantities are strictly less than $\mathbf{p}_{\mathcal{Q}}$. Both these functions are monotonically increasing in this range. When $\alpha=1$ they evaluate to $5/6$ and $13/16$ respectively. Using Proposition 1, the functions above can be seen as upper bounds to the winning score.

*2.* There are three classes of $\mathbb{H}^{[1]}_{\alpha=1,(3,2)}$ state spaces depending on whether the maximally entangled state is $\mathrm{N}_1,\mathrm{N}_2$ or $\mathrm{N}_3$. We discussed in Section 6.5.3 that $\mathbb{H}^{[1]}_{\alpha=1,(3,2)}[\mathrm{N}_2]$ does not have any minimally 2-preserving extremal effects that are couplers. Since the extremal coupling effects are not minimally 2-preserving, any convex combination of extremal effects will either be minimally 2-preserving and not coupling or would be coupling and not minimally 2-preserving. Therefore, the maximum score in the ACHSH game is upper bounded by the classical score of 3/4. For $\mathbb{H}^{[1]}_{\alpha=1,(3,2)}[\mathrm{N}_1]$ and $\mathbb{H}^{[1]}_{\alpha=1,(3,2)}[\mathrm{N}_3]$, the effect $f_{\mathrm{CH}_1}$ can be used to maximise the product of the probability of a successful swap and the CHSH score of the generated state. Since $f_{\mathrm{CH}_1}$ is pure coupler the CHSH score is 1 for both cases. The probability of a successful swap in both cases is 1/3. Therefore, using Proposition 1, we get a score of 5/6 in both cases. □

The theorem above proves that quantum theory can be correlation self-tested against single roof house state spaces considered in this chapter. In addition to this, we have considered noisier state spaces $\mathbb{H}^{[1]}_{\beta(3,2)}$ where $\beta$ is discretely varied between 1/2 and 1 with a step size of 1/30. For this, we have numerical evidence that the maximum score in the ACHSH game, calculated as per Proposition 1 is always strictly less than Tsirelson's bound. Therefore, we conjecture that quantum theory can be correlation self-tested against any state spaces of the form $\mathbb{H}^{[1]}_{\alpha(3,2)}$ where $\alpha\in[1/2,1]$.

Next, we provide a generalisation of the result showcasing that no party symmetric house state space $\mathbb{H}^{[m]}_{\alpha,(2,2)}$ allows one to win the ACHSH game better than Tsirelson's bound.

**Theorem 3.** *Let $\mathbb{H}^{[m]}_{\alpha(2,2)}$ be a party symmetric state space with $1\leqslant m\leqslant 8$ and $1/2\leqslant\alpha\leqslant 1$. Let $\mathbf{p}_{\mathbb{H}^{[m]}_{\alpha(2,2)}}$ be the maximum winning probability of the ACHSH game in $\mathbb{H}^{[m]}_{\alpha(2,2)}$ and $\mathbf{p}_{\mathcal{Q}}$ denotes Tsirelson's bound. Then,*

$$\mathbf{p}_{\mathcal{Q}}>\mathbf{p}_{\mathbb{H}^{[m]}_{\alpha}(2,2)} \tag{6.16}$$

*for any $m\in\{1,2,\cdots,8\}$ and any $\alpha\in[1/2,1]$.*

*Proof.* The proof has two cases :

- for $\mathbb{H}^{[1]}_{\alpha(2,2)}$, i.e., $m=1$, using Theorem 2 (part *1.*) we obtain $\mathbf{p}_{\mathcal{Q}}>\mathbf{p}_{\mathbb{H}^{[m]}_{\alpha(2,2)}}$
- when $1<m\leqslant 8$, from Theorem 1, we obtain that $\mathbf{p}_{\mathbb{H}^{[m]}_{\alpha(2,2)}}\leqslant 3/4$ since there are no minimally 2-preserving couplers.

□

An important observation is that this theorem holds even if multicopy nonlocality distillation is allowed. When $1<m\leqslant 8$, such distillations are not possible due to the absence of couplers. When $m=1$, from Theorem 2 one gets that the maximum score in the ACHSH game achievable

if nonlocal correlations are distilled to perfectly win one of the CHSH games is bounded by 5/6 which is less than Tsirelson's bound.

## 6.7 MINIMAL 2-PRESERVIBILITY AND TSIRELSON'S BOUND

Recall from Definition 2.1.3, that given a state space $\mathcal{S}$, the minimal tensor product composition $\mathcal{S}^{\overset{\otimes k}{\min}}$ is a subset of any composite state space $\mathcal{S}^{\boxtimes k}$. Similarly, for the effect space $\mathcal{E}$, the minimal tensor product composition $\mathcal{E}^{\overset{\otimes k}{\min}}$ is a subset of any composite effect space $\mathcal{E}' \subseteq \mathcal{E}_{\mathcal{S}^{\boxtimes k}}$. When $\mathcal{S}$ and $\mathcal{E}$ denote bipartite party swap symmetric state and effect spaces, then this requirement implies the following: i) for every pair of states $\sigma, \omega \in \mathcal{S}$, the states $\{\sigma^{(1,2)} \otimes \omega^{(3,4)}, \sigma^{(1,4)} \otimes \omega^{(2,3)}\}$ are valid states of the 4 partite state space $\mathcal{S}^{\overset{\otimes 2}{\min}}$ and ii) for every pair of effects $e, f \in \mathcal{E}$, the effects $\{e^{(1,2)} \otimes f^{(3,4)}, e^{(1,4)} \otimes f^{(2,3)}\}$ are valid effects of the 4 partite effect space $\mathcal{E}^{\overset{\otimes 2}{\min}}$.

Note, that a different way of stating minimal 2-preservibility, defined in Section 3.2, of an effect $f$ is requiring that for every effect $e$, the effects $\{e^{(1,2)} \otimes f^{(3,4)}, e^{(1,4)} \otimes f^{(2,3)}\}$ are valid. More strongly, a bipartite effect $f$ is minimally 2-preserving if and only if for every bipartite effect $e$, the effects $\{e^{(1,2)} \otimes f^{(3,4)}$ and $e^{(1,4)} \otimes f^{(2,3)}\}$ are valid 4-partite effects. Therefore, requiring both i) and ii) is equivalent to every bipartite effect $f \in \mathcal{E}$ being minimally 2-preserving.

Section 3.2 requiring both i) and ii) is equivalent to every effect $e \in \mathcal{E}$ being minimally 2-preserving. One outcome of our work is that in the case of house-like state spaces $\mathbb{H}^{[m]}_{\alpha(2,2)}$, requiring *i)* and *ii)* to simultaneously hold does not allow every effect in the maximal effect space $\mathcal{E}_{\mathbb{H}^{[m]}_{\alpha(2,2)}}$ to be minimal 2-preserving for every choice of $\alpha$. Interestingly, if one restricts the maximum nonlocality of $\mathbb{H}^{[m]}_{\alpha(2,2)}$ to be at most Tsirelson's bound, every effect in the maximal effect space becomes minimally 2-preserving. We make this more precise in the following theorem.

**Theorem 4.** *Let $\mathbb{H}^{[m]}_{\alpha}(2,2)$ be a party symmetric bipartite state space with $m < 8$ and let $\mathbf{p}_Q$ denote Tsirelson's bound. Then the following two statements are equivalent:*

1. *any $e \in \mathcal{E}_{\mathbb{H}^{[m]}_{\alpha}}$ minimally 2-preserving*

2. *$\max_i \mathrm{CHSH}_i[s] \leqslant \mathbf{p}_Q$ for any state $s \in \mathbb{H}^{[m]}_{\alpha}$*

*Proof.* ($1 \implies 2$) We split the proof in two cases, first when $m = 1$ and second when $1 < m < 8$. When $m = 1$, recall from Example 6.4.1 that for the state space $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$, the set of extremal effects that are not minimally 2-preserving are the CH-type effects. This was shown by the score of a $\mathrm{CHSH_{i\neq 2}}$ game of the distribution obtained after performing fiducial measurements on the state $\tilde{\Phi}^{(2,3)}_{e_{\mathrm{CH}_{j\neq 2,2'}}}(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{2,\alpha}) / \langle u, \cdot \rangle$ being $(\alpha^2 + 1)/2$, which in the range $1/\sqrt{2} < \alpha \leqslant 1$ is greater than 3/4 (maximum local score). Therefore, in order for these CH-type effects to be minimally 2-preserving, we require $\alpha \leqslant 1/\sqrt{2}$. This implies that the maximum CHSH score achievable in the state space is

$$\max_{\frac{1}{2} \leqslant \alpha \leqslant \frac{1}{\sqrt{2}}} \mathrm{CHSH}_2\left[\mathrm{PR}_{2,\alpha}\right] = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{\sqrt{2}}\right) = \mathbf{p}_Q. \tag{6.17}$$

Next, when $m = 2$, recall from Example 6.4.2, that for the state space $\mathbb{H}^{[2]}_{\alpha(2,2)}[\mathrm{PR}_{1,2}]$ the set of extremal effects that are not minimally 2-preserving are all the CH-type effects and all the couplers. In this case, one can see from Eq. (6.7) that for a given $\alpha$, the maximum score in a $\mathrm{CHSH}_{\mathrm{i}\neq 1,2}$ game is $(\alpha^2+1)/2$. Therefore, for all these effects to be minimally 2-preserving, we require that $\alpha \leqslant 1/\sqrt{2}$. This argument can be extended to $2 < m < 8$ without loss of generality.

($2 \implies 1$) When $\alpha \leqslant 1/\sqrt{2}$, there are no couplers and hence every effect is minimally 2-preserving. □

When $m = 8$, since any effect of the full effect space is minimally 2-preserving, no such tension arises for any amount of noise.

Our work provides an in-depth analysis of state space models where taking the dual of the state space is insufficient. A further restriction of compositional consistency is required. In fact, setting the collection of all elements in the dual of a state space as the effect space can be seen as a special case of minimal $k$-preservibility when $B_r = X_r$ in Definition 3.2.1. Our work shows that although when single systems are considered, i.e., $|X_r| = 1$, this is indeed sufficient, when more systems are involved, minimal $k$-preservibility (in fact $k$-preservibility) should be considered instead. Finally, for quantum theory, every element in the dual of the state space is also completely state space preserving, i.e., all POVM elements are completely positive. Theorem 4 addresses the restrictions needed on the house models if this feature needs to be carried over. Although the theorem finds a restriction on asymmetric state spaces, unlike quantum state spaces which are symmetric, this restriction being Tsirelson's bound is surprising.

## 6.8 CONCLUSION

We have shown that quantum theory can be correlation self-tested against a class of theories that allow post-quantum correlations. Further, these results hold even if the players are allowed to perform nonlocality distillation, since they can only distil up to the amount allowed by the state space and produce boxes that are valid states. However, to completely single out quantum correlations from BW correlations, one needs to look for all the theories whose state spaces are smaller than BW state spaces, thereby supporting entanglement swapping, but leaving enough scope to allow distillation of correlations such that all non-signalling correlations can be generated. Although one can arbitrarily truncate BW state spaces to construct state spaces of such theories, obtaining a complete list of such state spaces may not be straightforward. Here, we have only considered house-like compositions of $\mathcal{G}_3^2$ with single roofs. However, since there are multiple equivalence classes of maximally entangled states for this case, house state spaces with two or more roofs might enable us to outperform quantum theory. This is an unexplored area in this work due to heavy computational demands and is left for future work.

Recently, in [27], the authors discuss complete preservibility in composite state spaces which are not constructed using the minimal or maximal tensor product rule. In particular, they refer to Janotta [52] where the state space considered was the convex hull of $\mathbb{H}^{[0]}_{(2,2)}$ and 4 PR boxes, out of which 2 PR boxes are isotropically opposite to each other and 2 PR boxes (not necessarily the other two) are symmetric under party swap. Naturally, the state space is not symmetric under party swap. In agreement with our results, they also find effects that are not minimally 2-preserving. Further, they state that the only possible compositions of two gbit state spaces, $\mathcal{G}_2^2$, which are completely preserving are $\mathbb{H}^{[0]}_{(2,2)}, \mathbb{H}^{[8]}_{\alpha=1,(2,2)}$ and a party symmetric state space of $\mathbb{H}^{[1]}_{\alpha=1,(2,2)}[\mathrm{PR}_1]$ with a restricted effect space constructed from the convex hull of BW effects and only one entangled effect, i.e., the coupler $e_{\mathrm{pure}}$ only. Our results show that more state and effect space pairs are potentially completely 2-preserving. For instance, $\mathbb{H}^{[1]}_{\alpha=1,(2,2)}[\mathrm{PR}_1]$ with a restricted effect space constructed from the convex hull of BW effects, the 9 coupling effects and their complementary effect. A noisier version of this example is also minimally 2-preserving and therefore potentially completely preserving. Other examples include $\mathbb{H}^{[m]}_{\alpha,(2,2)}$ state spaces with $m$ even where the PR boxes are isotropically opposite and the restricted effect spaces are constructed by taking the convex hull of all the extremal effects of $\mathcal{E}_{\mathbb{H}^{[m]}_{\alpha,(2,2)}}$ with the exception of the CH-type effects.

The optimal strategy used in quantum theory in the ACHSH game involves a perfect entanglement swapping. In particular, post-selected on each of Bob's outcomes, Alice and Charlie's systems are maximally entangled. Note that one can mimic this feature if more than one composition rule is allowed, such as the one in [8] mentioned in Section 3.2. Explicitly, one possibility is allowing BW compositions for bipartite states labelled by $\mathbf{AB_1}, \mathbf{B_2C}$ and between $\mathbf{AC}$ and allowing local composition between $\mathbf{B_1}, \mathbf{B_2}$. With this one can perfectly win the ACHSH game when Bob shares two copies of $\mathrm{PR}_1$, one with Alice and another with Charlie and Bob performs a four outcome joint measurement with appropriate CH-type effects. As argued in Section 3.2, we have avoided these types of constructions since they require multiple composition rules, and hence not a close analogy to quantum theory, described by a single composition rule, namely, the tensor product of the underlying Hilbert spaces.

# Part III

# Superposition and Indefinite Causal Order

# — 7 —

# Superposition and Indefinite Causal Order in Generalised Probabilistic Theories

## 7.1 INTRODUCTION

General relativity(GR) and quantum theory(QT) have been shown to be successful at describing cosmic and atomic physics respectively. A long-standing quest is to formulate a higher theory that describes physics at all scales in a way that its descriptions of cosmic and atomic physics are equivalent to that of GR and QT respectively. To investigate such a theory, it might be useful to work in a framework in which features of both these theories can be expressed meaningfully. From an operational perspective, since QT is probabilistic, it is reasonable to assume that this higher theory might be probabilistic as well.

A minimal requirement for any such theory is to be able to describe operations and causal orderings amongst operations in every way permissible in GR and QT. If an operation causally precedes another, we say that there exists a definite causal ordering relating the two. It turns out that in both GR and QT, there are scenarios in which a lack of definite causal ordering is observed. In GR if two operations occur in regions that are space-like separated, then there is no definite causal order between them. On the other hand, in QT, the ordering between two operations can be in quantum superposition [17]. This lack of definite causal order in non-relativistic quantum theory has come to be known as *indefinite causal order* (ICO). Now, in a probabilistic theory, the lack of causal definiteness of events arising from GR can be modelled by assuming that operations performed in space-like separated regions commute. That of superposition of operations remains unknown. In this chapter, we try to take a step towards bridging this gap in the framework of GPTs.

Fundamental to causal superposition of operations is the notion of superposition itself. In QT, this notion is attributed to the fact that there exists a representation of pure states in which every pure state can be expressed as a linear combination of two other pure states. In an arbitrary probabilistic theory, such a representation need not exist. Therefore, one needs an operational understanding of superposition that can be used to check whether a probabilistic theory admits superposition or not. In this chapter, we present a candidate definition for superposition which captures the notion of quantum superposition and is inadmissible by classical probability theory. Additionally, we found that although composite systems of maximal theories respecting no-superluminal signalling [81] display superposition, their single systems do not.

This chapter is organised as follows: Sections 7.2 and 7.3, give an overview of developments in indefinite causal order in quantum theory and how it can be self-tested in certain cases. In Section 7.4, we propose an operational definition of superposition and show its consistency with quantum theory. We also show that GLT does not admit superposition, however, BW does. In Sections 7.5 and 7.6 we present a toy theory that admits superposition and then show how it can also display indefinite causal order, while keeping track of the underlying assumptions.

## 7.2 INDEFINITE CAUSAL ORDER IN QUANTUM THEORY

ICO in quantum theory has received a lot of attention in the past decade. Originally the idea behind ICO was proposed by Hardy to introduce a probabilistic framework for quantum gravity [50, 49]. Subsequently, a concrete example of a process displaying ICO, the *quantum switch*, was put forward by Chiribella [17]; We will discuss this in more detail in Section 7.3.1. The framework of ICO has so far been further developed by Colnaghi et al. [23] and Oreshkov et al. [72]. The use of ICO has also been shown to have theoretically improved various information-theoretic tasks. A handful, but by no means exhaustive, list of examples include communication complexity [46], reduced error quantum communication [28], metrology [101], quantum thermodynamics [16], and local implementation of nonlocal operations [43].

An important instalment in this direction is the development of causal inequalities, that are satisfied by every quantum process admitting a definite causal order [72, 2]. The quantum switch, however, does not violate these inequalities [2, 82]. Device-independent techniques have been proposed for correlation self-testing of ICO in the quantum switch [44, 64, 32]. The self-test introduced in [64] is not only device-independent but theory-independent as well, just like Bell inequalities. In the following, we will briefly recap the quantum switch and the inequality presented in [64].

## 7.3 QUANTUM SWITCH AND DRF INEQUALITY

### 7.3.1 Quantum Switch

The quantum switch [17] is a process in which the order of two operations $\mathcal{O}_{\mathcal{A}_1}$ and $\mathcal{O}_{\mathcal{A}_2}$ is controlled by a quantum state. Let us assume that if the control quantum state is a qubit, then $\mathcal{O}_{\mathcal{A}_1}$ precedes $\mathcal{O}_{\mathcal{A}_2}$ when the qubit is in state $|0\rangle\langle 0|$ and $\mathcal{O}_{\mathcal{A}_2}$ precedes $\mathcal{O}_{\mathcal{A}_1}$ when the qubit is in state $|1\rangle\langle 1|$. In particular, one obtains an entanglement between the state of the control qubit and the order in which the operations $\mathcal{O}_{\mathcal{A}_1}$ and $\mathcal{O}_{\mathcal{A}_2}$ are performed. In the case where the operations $\mathcal{O}_{\mathcal{A}_1}$ and $\mathcal{O}_{\mathcal{A}_2}$ represent unitary maps $U_1$ and $U_2$, the action of the quantum switch on them can be defined as

$$(U_1, U_2) \mapsto |0\rangle\langle 0|_C \otimes U_2 U_1 + |1\rangle\langle 1|_C \otimes U_1 U_2, \tag{7.1}$$

where $C$ represents the control qubit system [31]. When the control qubit is in a superposition of the states $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$, the ordering of the operations becomes superposed, exhibiting indefinite causal order. Next, we summarise the idea behind DI self-testing of ICO, as presented in [64].

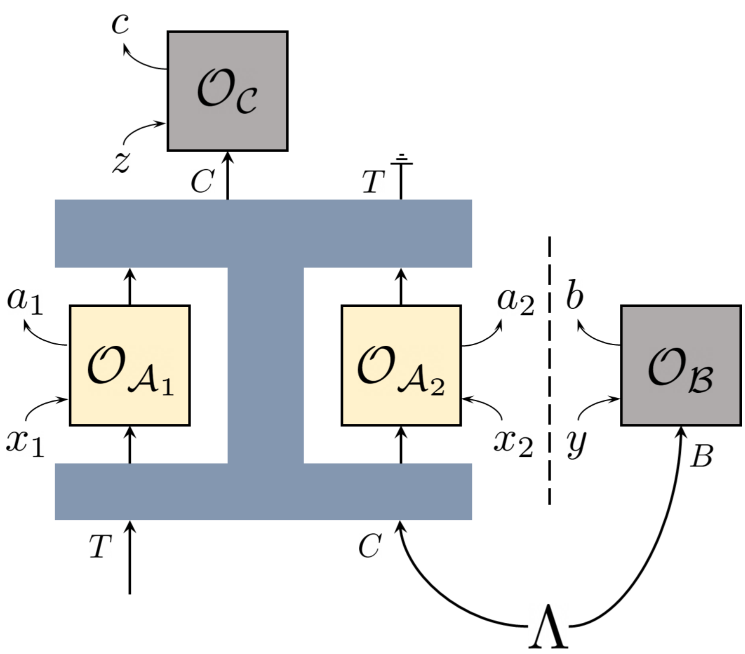


Figure 7.1: Abstraction of the setup considered in [64] to derive the DRF inequality 7.2. Blue region represents a process that implements the operations $\mathcal{O}_{\mathcal{A}_1}$ and $\mathcal{O}_{\mathcal{A}_2}$ (yellow) in the orders $\mathcal{O}_{\mathcal{A}_2} \prec \mathcal{O}_{\mathcal{A}_1}$ when the value of subsystem $C$ of $\Lambda$ is 0 and $\mathcal{O}_{\mathcal{A}_1} \prec \mathcal{O}_{\mathcal{A}_2}$ when the value of subsystem $C$ of $\Lambda$ is 1. $\mathcal{O}_\mathcal{C}$ is in the causal future of $\{\mathcal{O}_{\mathcal{A}_i}\}_i$ and $\mathcal{O}_\mathcal{B}$ is causally disjoint from $\{\mathcal{O}_{\mathcal{A}_i}\}_i$ , and $\mathcal{O}_\mathcal{C}$.

### 7.3.2 DRF Inequality

We split this discussion into two parts, first describing the various causal relations needed and second a possible justification for the implementation of these relations in space-time.

Given two random variables $A$ and $X$, $A \perp\!\!\!\perp X$ denotes that $A$ is independent of $X$. In a causal structure, $A$ is said to be in the *causal future* of $X$ if $X$ is a potential cause of $A$. $X$ is said to be *free* if it is independent of every random variable outside its causal future. Now, let us consider four operations $\mathcal{O}_{\mathcal{A}_1}, \mathcal{O}_{\mathcal{A}_2}, \mathcal{O}_\mathcal{B}$ and $\mathcal{O}_\mathcal{C}$. To each operation $\mathcal{O}$, let us assign a pair of random variables $(M, N)$ such that $N$ is in the causal future of $M$ (see Section 4.1). We assign $(X_1, A_1)$ to $\mathcal{O}_{\mathcal{A}_1}$, $(X_2, A_2)$ to $\mathcal{O}_{\mathcal{A}_2}$, $(Y, B)$ to $\mathcal{O}_\mathcal{B}$ and $(Z, C)$ to $\mathcal{O}_\mathcal{C}$. $\mathcal{O}_{\mathcal{A}_i} \prec \mathcal{O}_{\mathcal{A}_j}$, denotes that $(M_j, N_j)$ is in the causal future of $(M_i, N_i)$ and $\mathcal{O}_{\mathcal{A}_i} \times \mathcal{O}_{\mathcal{A}_j}$ denotes that neither $(M_i, N_i)$ nor $(M_j, N_j)$ is in the causal future of the other, i.e., they are *causally disjoint*. The causal relations can now be phrased in the following two assumptions:

*Assumption 1:* There is a random variable $\Lambda$ taking values $\lambda \in \{0, 1\}$. When $\lambda = 0$, $\mathcal{O}_{\mathcal{A}_1} \prec \mathcal{O}_{\mathcal{A}_2}$ and when $\lambda = 1$, $\mathcal{O}_{\mathcal{A}_2} \prec \mathcal{O}_{\mathcal{A}_1}$. In addition, $\mathcal{O}_{\mathcal{A}_i} \prec \mathcal{O}_\mathcal{C}$, $\mathcal{O}_\mathcal{B} \times \mathcal{O}_{\mathcal{A}_i}$ and $\mathcal{O}_\mathcal{B} \times \mathcal{O}_\mathcal{C}$ for any $i \in \{1, 2\}$.

*Assumption 2:* The random variables $X_1, X_2, Y, Z$ are free.

Figure 7.1 is one way to implement these causal relations with the justifications that the operations take place in four closed labs $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}$, and $\mathcal{C}$ respectively, $(M, N)$ represent random variables associated to the input and output of the operation $\mathcal{O}$, and that the following hold:

1. **Definite Causal Order (D):** All operations admit definite causal order, conditioned on $\Lambda$.

2. **Relativistic Causality (R):** If $\mathcal{O}_i \prec \mathcal{O}_j$, then the physical implementation of $\mathcal{O}_j$ is in the future lightcone of that of $\mathcal{O}_i$. If $\mathcal{O}_i \times \mathcal{O}_j$, then their physical implementations are space-like separated.
3. **Freedom of Choice (F):** The underlying theory allows for all input settings to be freely chosen.

Under these assumptions, the set of conditional probability distributions, $\mathrm{p}(A_1, A_2, C, B|X_1, X_2, Z, Y)$, can be characterised as a convex polytope, a facet of which is

$$\begin{aligned} p(b=0, a_2 =& x_1|y=0) + p\,(b=1, a_1 = x_2|y=0) \\ &+ p\,(b \oplus c = yz|x_1 = x_2 = 0) \leqslant \frac{7}{4}, \end{aligned} \tag{7.2}$$

where $\oplus$ denotes the modulo 2 operation. This inequality holds even if the random variable $\Lambda$ is correlated with the outcome represented by $B$, as depicted in Fig. 7.1. This inequality applies to theories in which the following assumptions can be physically justified: i) DRF, ii) the theory allows operations to be performed in a closed lab setting, i.e., the operations are unaffected by anything outside the respective labs they take place in and iii) classical theory is a sub-theory, in particular the classical random variable, $\Lambda$, can be modelled as a valid state. Within any theory, admitting these assumptions, whenever the order of operations between $\mathcal{O}_{\mathcal{A}_1}$ and $\mathcal{O}_{\mathcal{A}_2}$ is determined by $\Lambda$, every conditional distribution satisfies inequality (7.2). Therefore, this inequality presents a theory-independent (modulo i, ii and iii) constraint on conditional probabilities that respect the above assumptions.

In [64], the authors showed that if the order of the operations $\mathcal{O}_{\mathcal{A}_1}$ and $\mathcal{O}_{\mathcal{A}_2}$ were controlled by one subsystem of a bipartite maximally entangled state while the other subsystem is distributed to lab $\mathcal{B}$, it is possible to violate inequality (7.2). Since classical theory is a sub-theory of quantum mechanics, under the closed lab assumption, this violation implies that the DRF conditions do not simultaneously hold. If one further assumes that R and F hold, a device-independent violation of definite causal order is implied.

In the following, we summarise the quantum strategy presented in [64] that leads to such a violation.

### 7.3.3 Quantum Strategy in the Switch

Let us denote by

$$\sigma_{\hat{X}} \coloneqq \sigma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_{\hat{Y}} \coloneqq \sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_{\hat{Z}} \coloneqq \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the three Pauli matrices corresponding to some fixed orthogonal directions $\hat{X}, \hat{Y}$ and $\hat{Z}$ respectively. Operations $\mathcal{O}_{\mathcal{A}_i}$ are measure and prepare channels and operations $\mathcal{O}_{\mathcal{B}}$ and $\mathcal{O}_{\mathcal{C}}$ are measurements. The target system, $T$, is initially prepared in $|0\rangle\!\langle 0|$. One subsystem ($C$) of a maximally entangled state $\Phi_+$ is used as the control while the other subsystem ($B$) is distributed to lab $\mathcal{B}$. In lab $\mathcal{A}_i$, the incoming qubit is measured in the $\{|0\rangle\!\langle 0|, |1\rangle\!\langle 1|\}$ basis. The outcome is labelled $a_i$, and the state $|x_i\rangle\langle x_i|$ is prepared and sent off. In lab $\mathcal{C}$, the output control qubit is measured in the basis generated by the rank 1 projectors of $(\sigma_{\hat{Z}} + \sigma_{\hat{X}})/\sqrt{2}$ when $z = 0$ and of $(\sigma_{\hat{Z}} - \sigma_{\hat{X}})/\sqrt{2}$ when $z = 1$; Let us denote these projectors as $|\psi_{c|z}\rangle\langle\psi_{c|z}|$. The outcome is recorded as $c$. In the space-like separated lab $\mathcal{B}$, the distributed half of $\Phi_+$ is measured in the basis defined by the rank 1 projectors of $\sigma_{\hat{Z}}$ when $y = 0$ and $\sigma_{\hat{X}}$ when $y = 1$; Let us denote these projectors as $|\phi_{b|y}\rangle\langle\phi_{b|y}|$. The outcome is recorded in $b$. The elements of resultant distribution $\mathrm{p}(A_1, A_2, B, C|X_1, X_2, Y, Z)$ can be written as:

$$p\left(a_1, a_2, b, c|x_1, x_2, y, z\right) = \mathrm{Tr}\left[K\left(\Phi_+^{\mathbf{BC}} \otimes |0\rangle\langle 0|^{\mathbf{T}}\right) K^\dagger\right], \tag{7.3}$$

where $K := \langle\psi_{c|z}|^{\mathbf{C}}\langle\phi_{b|y}|^{\mathbf{B}}(|0\rangle\langle 0|^{\mathbf{C}} \otimes |x_2\rangle\langle a_2|x_1\rangle\langle a_1|^{\mathbf{T}} + |1\rangle\langle 1|^{\mathbf{C}} \otimes |x_1\rangle\langle a_1|x_2\rangle\langle a_2|^{\mathbf{T}})$. Now, note that when $y = 0$, the probability of getting either $b = 0$ or $b = 1$ is $1/2$. When $b = 0$, the post-selected control qubit is in the state $|0\rangle\langle 0|^C$ which implies $a_2 = x_1$. Similarly, when $b = 1$, the post-selected control qubit is in the state $|1\rangle\langle 1|^C$ which implies $a_1 = x_2$. Therefore, the first two terms of inequality (7.2) add up to 1. Next, when $x_1 = x_2 = 0$, the state of the control system is unaffected and therefore labs $\mathcal{B}$ and $\mathcal{C}$ can perform a Bell-test to get $\mathrm{p}\left(b \oplus c = yz|x_1 = x_2 = 0\right) = (1+1/\sqrt{2})/2$. As a consequence, inequality (7.2) is violated as the sum of all the conditional probabilities appearing in it is $1 + (1 + 1/\sqrt{2})/2 > 7/4$.

## 7.4 SUPERPOSITION IN GPTS

Textbook introduction to quantum superposition is attributed to the fact that certain linear combinations of pure states, when represented as vectors in $\mathbb{C}^d$, are also pure states. More precisely, for every pure state $|\phi\rangle$, there exists a pair of states $\{|\psi_1\rangle, |\psi_2\rangle\}$, a unique linear combination of which reproduces $|\phi\rangle$, i.e.,

$$\alpha|\psi_1\rangle + \beta|\psi_2\rangle = |\phi\rangle, \tag{7.4}$$

where $\alpha$ and $\beta$ are complex numbers. This notion of superposition cannot be generalised to arbitrary GPTs since it a priori depends on pure states being represented by vectors in $\mathbb{C}^d$. Indeed, even for quantum theory, this notion of superposition is solely dependent on its Hilbert space formalism. In a different formalism, for instance, if all states were represented by probability tables constructed from tomographic data, a clear understanding of quantum superposition is missing. To have an understanding of superposition that does not depend on the mathematical framework in which the underlying theory is phrased, one might want to take an operational approach and describe it in terms of the input-output statistics obtained upon performing suitable measurements.

Two attempts to address this were presented in [5] and [24]. In [5], superposition has been treated at equal footing as non-classicality. In particular, any theory with a non-simplicial [1] state space admits superposition. In [24], the property of superposition has only been explored for theories with infinitely many pure states. In this chapter, we take a slightly different approach by first looking at the statistical features of experimental outcomes that are traditionally associated with the presence of superposition in quantum theory and then characterise a minimal condition for a theory to display similar statistical behaviour.

[1] The state space of any classical probability theory can be described as a simplex.

For our first example, let us consider $|\phi\rangle = |0\rangle, |\psi_1\rangle = |+\rangle$ and $|\psi_2\rangle = |-\rangle$ with which one has

$$\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) = |0\rangle. \tag{7.5}$$

Looking at these states as elements of the state space, we see that for the state $|0\rangle\langle 0|$, there exists a unique measurement $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ such that the outcome of the measurement is deterministic, i.e., $\mathrm{T}r[|0\rangle\langle 0|.|0\rangle\langle 0|] = 1$ and $\mathrm{T}r[|1\rangle\langle 1|.|0\rangle\langle 0|] = 0$. Similarly, for the states $|+\rangle\langle +|$ and $|-\rangle\langle -|$, there exists a basis $\{|+\rangle\langle +|, |-\rangle\langle -|\}$ such that outcome of a measurement on these two states is deterministic. However, when $|0\rangle\langle 0|$ is measured in $\{|+\rangle\langle +|, |-\rangle\langle -|\}$ or $|+\rangle\langle +|/|-\rangle\langle -|$ is measured in $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ the probability of each outcome is $1/2$. Here, the probability of non-deterministic outcomes being $1/2$ is not crucial. For clarity, consider $|\phi\rangle = |0\rangle, |\psi_1\rangle = |\psi\rangle := \sqrt{2/3}|0\rangle + \sqrt{1/3}|1\rangle$ and $|\psi_2\rangle = |-\rangle$ with which one has

$$|0\rangle = \frac{\sqrt{3}}{1+\sqrt{2}}|\psi\rangle + \frac{\sqrt{2}}{1+\sqrt{2}}|-\rangle. \tag{7.6}$$

For the state $|\psi\rangle\langle\psi|$ there exists a unique measurement $\{|\psi\rangle\langle\psi|, \mathbb{1} - |\psi\rangle\langle\psi|\}$ which is deterministic with respect to the state $|\psi\rangle\langle\psi|$ but non-deterministic with respect to both $|0\rangle\langle 0|$ and $|-\rangle\langle -|$. In addition, $\mathrm{T}r[|0\rangle\langle 0|.|\psi\rangle\langle\psi|] = 2/3 \neq \mathrm{T}r[|0\rangle\langle 0|.|-\rangle\langle -|]$. To translate this to the framework of GPTs, we first note that to check whether a binary outcome measurement is deterministic with respect to a state or not, it is sufficient to check the probability corresponding to only one of the outcomes. Secondly, although in qudit quantum theory for every pair states $|\phi_1'\rangle\langle\phi_1'|$ and $|\phi_2'\rangle\langle\phi_2'|$ there exists a unique effect $|\phi_1'\rangle\langle\phi_1'|$ such that $\mathrm{T}r[|\phi_1'\rangle\langle\phi_1'|].|\phi_1'\rangle\langle\phi_1'| = 1$ and $|\phi_1'\rangle\langle\phi_1'|.|\phi_2'\rangle\langle\phi_2'| \in (0,1)$, this may not be true for an arbitrary GPT. In particular, there might be more than one extremal effect with this property. This discussion gives us a first requirement for a theory to have superposition: the state space of the theory must have three distinct extremal states $s$ and $\{r_1, r_2\}$ and three extremal effects $e_s$ and $\{f_{r_1}, f_{r_2}\}$ such that $\langle e_s, s\rangle = 1$ but $\langle e_s, r_{1/2}\rangle \in (0,1)$ and $\langle f_{r_1}, r_1\rangle = 1$ and $\langle f_{r_2}, r_2\rangle = 1$ but $\langle f_{r_{1/2}}, s\rangle \in (0,1)$. Since we are only considering pure states this requirement already distinguishes a superposition from a classical mixture, since for any classical mixture of $\alpha r_1 + (1-\alpha) r_2$, $\langle e_s, \alpha r_1 + (1-\alpha) r_2\rangle \in (0,1)$, where $\alpha \in [0,1]$. We now formalise our observations into an operational definition of superposition.

**Definition 7.4.1.** *Let $\mathcal{S}$ and $\mathcal{E}$ be a state and effect space pair of a GPT and denote by* $\mathrm{Extreme}[\mathcal{S}]$ *and* $\mathrm{Extreme}[\mathcal{E}]$ *the set of extremal states in $\mathcal{S}$ and extremal effects in $\mathcal{E}$ respectively. The GPT is said to admit superposition if there exists three distinct states $s, r_1, r_2 \in \mathrm{Extreme}[\mathcal{S}]$ and three effects $e_s, f_{r_1}, f_{r_2} \in \mathrm{Extreme}[\mathcal{E}]$, such that $\langle e_s, s\rangle = 1$, $\langle e_s, r_j\rangle \in (0,1)$, $\langle f_{r_j}, r_j\rangle = 1$ and $\langle f_{r_j}, s\rangle \in (0,1)$, for all $j \in \{1,2\}$.*

Note that the effects $e_s$ and $f_{r_j}$ can neither be the zero nor the unit effect. Further, notice that since the inner product between any extremal state and any extremal effect in a simplicial theory is either 0 or 1, classical theory cannot admit superposition. Finally, for qudit quantum theory, say with $d = 3$, although one might be able to represent a pure state as a linear combination of three other pure states, a superposition of two pure states is still a well-defined pure state. Definition 7.4.1 thus captures the minimal necessary requirements for a theory to admit superposition.

It is possible to satisfy the conditions stated in Definition 7.4.1 by mixed states lying on the boundary of the state space. One might urge that superposition should then be defined for mixed states as well. Indeed, there is no operational reason as to why only extremal states must possess superposition. We therefore do not impose a direct restriction on the type of states that might be described as superposition of other states. However, in this work, superposition can be associated to mixed states subject to the theory admitting superposition in accordance to Definition 7.4.1.

Next, let us look at the *gbit* state space $\mathcal{G}_2^2$ (see Section 2.4.1) consisting of the following extremal states:

$$\begin{pmatrix} 1 \\ 0 \\ \hline 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \hline 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \hline 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \hline 0 \\ 1 \end{pmatrix}. \tag{7.7}$$

in the notation $\mathrm{p}(A|X) := (p(0|0)p(1|0)|p(0|1)p(1|1))^T$, where $X$ and $A$ represent the random variables associated to the choices and outcomes of fiducial measurements. The maximal set of extremal effects for this state space is:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \tag{7.8}$$

Upon constructing a table of inner products between extremal states and effects, one can check that all the inner products are either 0 or 1. Therefore, the condition in Definition 7.4.1 cannot be met, implying that the single system state space does not admit superposition. In fact, no GLT admits superposition. It is then natural to ask whether there exists a GPT which admits superposition whilst having its single system state spaces being described by the gbits. We show that BW is an example of such a theory; To see this, consider the following collection of states :

$$\mathrm{PR}_1 := \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right), \mathrm{PR}_2 := \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right),$$

$$\mathrm{PR}_1' := \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right), \mathrm{PR}_2' := \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}\right),$$

and the collection of effects:

$$e_1 := \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right), e_2 := \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right),$$

$$e'_1 := \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), e'_2 := \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

from the bi-partite state and effect spaces. Taking the inner product by element-wise multiplication, we get $\langle e_i, \mathrm{PR}_i\rangle = 1$, $\langle e'_j, \mathrm{PR}'_j\rangle = 1$, $\langle e_i, \mathrm{PR}'_j\rangle = 1/2$ and $\langle e'_j, \mathrm{PR}_i\rangle = 1/2$ for any $i, j \in \{1, 2\}$. Furthermore, $1/2\mathrm{PR}_1 + 1/2\mathrm{PR}_2 = 1/2\mathrm{PR}'_1 + 1/2\mathrm{PR}'_2$, drawing resemblance to the quantum ensembles of states, $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$, discussed above. In the following two sections, we provide an example of a GPT that admits superposition, and show how one can use this theory to violate the DRF inequality (7.2) by an amount more than achievable in quantum theory, a first study of indefinite causal order in the framework of GPTs.

## 7.5 HEX-SQUARE THEORY

The derivation of inequality (7.2) assumes nothing more about the underlying theory than what has been stated in Section 7.3.2. Although quantum theory violates this inequality, there might be other theories in which the stated assumptions can be consistently made and that also violate inequality (7.2). In fact, such violations might be by an amount higher than what is achievable within quantum theory. Now, the part of inequality (7.2) whose algebraic limit cannot be met in quantum theory is $\mathrm{p}\,(b \oplus c = yz | x_1 = x_2 = 0)$. The condition $b \oplus c = yz$ is a nonlocal (CHSH) game, which can be won to its algebraic maximum by a PR box. Therefore, one way to outperform quantum theory in the violation of inequality (7.2) is to construct a theory where a PR box is a valid bipartite state and use one half of this state as the control while sharing the other half with lab $\mathcal{B}$.

For this, consider the state

$$\Phi_{\mathrm{PR}} := \frac{1+\sqrt{2}}{2}\Phi_+ + \frac{1-\sqrt{2}}{2}\Phi_-, \tag{7.9}$$

where $\Phi_- := |\phi_-\rangle\langle\phi_-|$ with $|\phi_-\rangle := (|00\rangle + |11\rangle)/\sqrt{2}$. When $\Phi_{\mathrm{PR}}$ is shared between two parties holding devices that can measure the observables $\{(\sigma_{\hat{X}} + \sigma_{\hat{Y}})/\sqrt{2}, (\sigma_{\hat{X}} - \sigma_{\hat{Y}})/\sqrt{2}\}$ and $\{\sigma_{\hat{X}}, \sigma_{\hat{Y}}\}$ respectively, they can generate PR correlations [1]. Further, recall that in the quantum strategy presented in Section 7.3.3, the observables measured in lab $\mathcal{C}$ were $(\sigma_{\hat{Z}} \pm \sigma_{\hat{X}})/\sqrt{2}$ and $\sigma_{\hat{Z}}$[2] and in lab $\mathcal{B}$ were $\{\sigma_{\hat{X}}, \sigma_{\hat{Z}}\}$. With this information, we construct here a bipartite theory where the only non-zero and non-unit extremal effects of one of the systems (system **C**) are the rank 1 projectors of $(\sigma_{\hat{X}} \pm \sigma_{\hat{Z}}), (\sigma_{\hat{X}} \pm \sigma_{\hat{Y}})$ and $\sigma_{\hat{Z}}$ , and that of the other system (system **B**) are the rank 1 projectors of $\sigma_{\hat{X}}, \sigma_{\hat{Y}}$ and $\sigma_{\hat{Z}}$. Recall from Section 2.6, that this effect space has already been studied in the context of a Hilbert space formalism of GLT.

Let us denote by $\mathcal{S}_{\mathbf{C}}$ and $\mathcal{S}_{\mathbf{B}}$ the state spaces for systems **C** and **B** respectively. $\mathcal{S}_{\mathbf{C}}$ is the set of all elements of $\mathbb{H}(\mathbb{C}^2)$ such that their Hilbert-Schmidt inner products with the rank 1 projectors of

[2] There is no direct measurement of $\sigma_{\hat{Z}}$. However, in the calculation of the probability using the formula in Equation (7.3), the action of the map $K$ on system **C** can be seen as a measure and prepare operation onto the eigen-basis of $\sigma_{\hat{Z}}$.

$(\sigma_{\hat{Z}} \pm \sigma_{\hat{X}})/\sqrt{2}, (\sigma_{\hat{X}} \pm \sigma_{\hat{Y}})/\sqrt{2}$ and $\sigma_{\hat{Z}}$ is a valid probability. Similarly, $\mathcal{S}_{\mathbf{B}}$ is the set of all elements of $\mathbb{H}(\mathbb{C}^2)$ such that the inner products with the rank 1 projectors of $\sigma_{\hat{X}}, \sigma_{\hat{Y}}$ and $\sigma_{\hat{Z}}$ are valid probabilities. For both systems, we consider $\mathbb{1}$ as the unit effect. Recall, that we can write any unit trace [3] Hermitian matrix $\varrho$ as

$$\varrho = \frac{\mathbb{1} + r_x\sigma_{\hat{X}} + r_y\sigma_{\hat{Y}} + r_z\sigma_{\hat{Z}}}{2} \tag{7.10}$$

where $r_x, r_y, r_z \in \mathbb{R}$. Our objective is to calculate the extremal states of systems **C** and **B**. To do this, we can first write the set of facet-defining inequalities for the state space polytope by exploiting the inner-product relation between states and effects:

$$\text{Facets}\,[\mathcal{S}] = \left\{ \text{Tr}[\varrho, e] \leqslant 1 \mid e \in \text{Extreme}[\mathcal{E}] \right\}; \tag{7.11}$$

Since the set of extremal effects for both systems is finite, the state spaces for both systems **C** and **B** can be characterised by finite lists of facets. For our problem, we enlist the intersection points of the hyperplanes defining the facets in variables $(r_x, r_y, r_z)$, and then check which of these intersection points satisfies all the facet inequalities for the given systems. In terms of elements in $\mathbb{H}(\mathbb{C}^2)$, the extremal states for systems **C** are then:

$$\begin{aligned} &\text{Extreme}\,[\mathcal{S}_{\mathbf{C}}] = \\ &\left\{ \frac{\mathbb{1} \pm \sqrt{2}\sigma_{\hat{X}}}{2}, \frac{\mathbb{1} \pm \sqrt{2}\sigma_{\hat{Y}} \pm \sigma_{\hat{Z}}}{2}, \frac{\mathbb{1} \pm r\sigma_{\hat{X}} \pm \sigma_{\hat{Y}} \pm \sigma_{\hat{Z}}}{2} \right\} \end{aligned} \tag{7.12}$$

where $r = \sqrt{2} - 1$. We encourage the reader to Appendix F for the derivation of Extreme $[\mathcal{S}_{\mathbf{C}}]$. We have already found in Section 2.6 that the extremal states of the maximal state space of system **B** are

$$\text{Extreme}\,[\mathcal{S}_{\mathbf{B}}] = \left\{ \frac{\mathbb{1} \pm \sigma_{\hat{X}} \pm \sigma_{\hat{Y}} \pm \sigma_{\hat{Z}}}{2} \right\}. \tag{7.13}$$

Figure 7.2 represents the structure of these state spaces $\mathcal{S}_{\mathbf{C}}$ and $\mathcal{S}_{\mathbf{B}}$ with respect to the real quantum state space (the Bloch-disc).

Since the cubic state space associated with system **B** is isomorphic to the gbit state space $\mathcal{G}_3^2$, system **B** does not admit superposition. However, we find that system **C** admits superposition. We make this precise in the following lemma.

**Lemma 2.** *If $\mathcal{S}_{\mathrm{C}}$ and $\mathcal{E}_{\mathrm{C}}$ are a state and effect space pair of a GPT, the GPT admits superposition.*

*Proof.* Take the collection of states $\{s_1, s_2, r_1, r_2\}$, where

$$s_1 := \frac{\mathbb{1} + \sqrt{2}\sigma_{\hat{X}}}{2}, \quad s_2 := \frac{\mathbb{1} - \sqrt{2}\sigma_{\hat{X}}}{2},$$

$$r_1 :=, \frac{\mathbb{1} + r\sigma_{\hat{X}} + \sigma_{\hat{Y}} + \sigma_{\hat{Z}}}{2} \quad r_2 := \frac{\mathbb{1} - r\sigma_{\hat{X}} - \sigma_{\hat{Y}} - \sigma_{\hat{Z}}}{2},$$

and effects :

$$f_1 := \frac{\mathbb{1} - (\sigma_{\hat{Z}} - \sigma_{\hat{X}})/\sqrt{2}}{2}, f_2 := \frac{\mathbb{1} + (\sigma_{\hat{Z}} - \sigma_{\hat{X}})/\sqrt{2}}{2},$$

[3] Since $\mathbb{1}$ is the unit effect, we require the states to be unit trace.

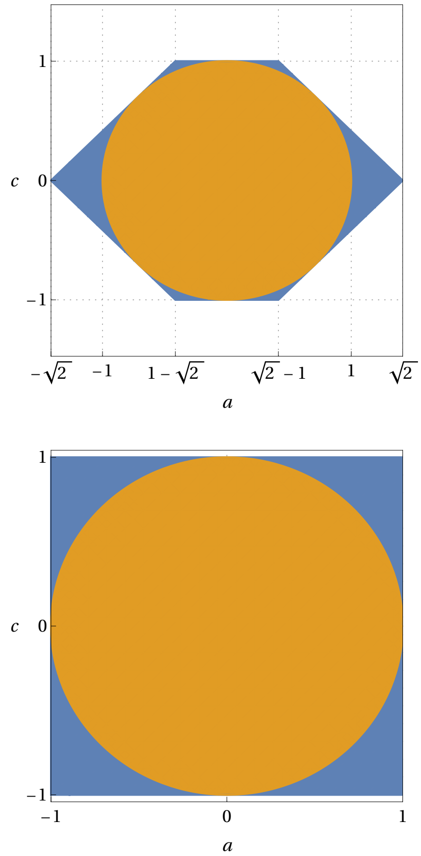


Figure 7.2: Projections on the $\hat{X} - \hat{Z}$ plane. (Top) Representation of the $\mathcal{S}_{\mathbf{C}}$ (blue) with respect to the quantum set (yellow). (Bottom) Representation of the $\mathcal{S}_{\mathbf{B}}$ (blue) with respect to the quantum set (yellow).

$$f_1' := \frac{\mathbb{1} - \sigma_{\hat{Z}}}{2}, \quad f_2' := \frac{\mathbb{1} + \sigma_{\hat{Z}}}{2},$$

One then gets $\langle f_i, s_i \rangle = 1$, $\langle f_j', s_j' \rangle = 1$, $\langle f_i, s_j' \rangle \in (0, 1)$ and $\langle f_j', s_i \rangle = 1/2$ for any $i, j \in \{1, 2\}$. Finally, $1/2 s_1 + 1/2 s_2 = 1/2 s_1' + 1/2 s_2'$, resembling quantum theory. □

### 7.5.1 Bipartite Systems

We are interested in the bipartite compositions of these state spaces. Any state in such a joint system must have the property that when marginalised to system **C** must be a valid state in the hexagonal state space and when marginalised to system **B** must be a valid state of the square state space. In particular, any unit trace $4 \times 4$ Hermitian matrix $\varsigma$ is a valid state as long as $Tr[(\Pi_{\mathbf{C}} \otimes \Pi_B)\varsigma]$ is a valid probability, where $\Pi_{\mathbf{C}}$ and $\Pi_{\mathbf{B}}$ are any effects from $\mathcal{E}_{\mathbf{C}}$ and $\mathcal{E}_{\mathbf{B}}$ respectively. For our work, it is sufficient to focus on only one bipartite state which is compatible with all product effects. In particular $\Phi_{\mathrm{PR}}$. Note, that $\Phi_{\mathrm{PR}}$ is not a quantum state since it has negative eigenvalues. Although this theory generates all non-signalling correlations in the $(2, 2, 2)$ Bell scenario, it is different from BW since one of the single system state spaces is not characterised by gbits. In the next section, we will see that this state can be used to display indefinite causal order.

## 7.6 INDEFINITE CAUSAL ORDER IN THE HEX-SQUARE THEORY

In this section, we present our main result: an example of a theory that can violate the DRF inequality by an amount more than that achievable using quantum theory. First, recall that we require the classical variable $\Lambda$ can be modelled as a valid state of the theory. For the quantum case, this was done by taking a classical mixture of the states $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$. The control being in the state $|0\rangle\langle 0|$ or $|1\rangle\langle 1|$ corresponds to $\Lambda$ being 0 or 1 respectively. First, note from (7.12) that both $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$ are states in $\mathcal{S}_{\mathcal{C}}$. Therefore, in our case, we stick to the same convention for the control. Secondly, since the hexagonal and square state spaces were constructed by requiring that all quantum operations required to demonstrate a violation of the DRF inequality (7.2) are allowed effects, we will stick to the quantum strategy here as well, with the exception of using a subsystem of $\Phi_{\mathrm{PR}}$ instead of $\Phi_+$ for controlling the causal order between the operations $\mathcal{O}_{\mathcal{A}_1}$ and $\mathcal{O}_{\mathcal{A}_2}$. Since system $\mathbf{C}$ admits superposition, it is natural to use it as the new control. With this, let us evaluate the values of the probabilities appearing in inequality (7.2). First, when $y = 0$ and $b = 0$, the post-selected sub-normalised state of the control is

$$\mathrm{Tr}_{\mathbf{B}}\left[(\mathsf{id}_{\mathbf{C}} \otimes |0\rangle\langle 0|_{\mathbf{B}})\,\Phi_{\mathrm{PR}}\right] = \frac{1}{2}|0\rangle\langle 0|_{\mathbf{C}}, \tag{7.14}$$

with 1/2 being the associated probability. Similarly, when $y = 0$, with a probability 1/2 one gets $b = 1$, with the post-selected state on the control being $|1\rangle\langle 1|_{\mathbf{C}}$. Hence, the first two terms in inequality (7.2) add up to 1. To calculate the third term, when $x_1 = x_2 = 0$, one gets

$$\begin{aligned}
&\mathrm{Tr}\left[K\left(\Phi_{\mathrm{PR}} \otimes |0\rangle\langle 0|\right)K^{\dagger}\right]_{x_1=x_2=0} = \\
&\frac{1+\sqrt{2}}{2}\mathrm{Tr}\left[K\left(\Phi_{+} \otimes |0\rangle\langle 0|\right)K^{\dagger}\right]_{x_1=x_2=0} + \\
&\frac{1-\sqrt{2}}{2}\mathrm{Tr}\left[K\left(\Phi_{-} \otimes |0\rangle\langle 0|\right)K^{\dagger}\right]_{x_1=x_2=0} = \\
&\mathrm{Tr}\left[\Pi_{\mathbf{CB}} \otimes \mathsf{id}_{\mathbf{T}}\left(\Phi_{\mathrm{PR}} \otimes |0\rangle\langle 0|_{\mathbf{T}}\right)\right]\delta_{a_1=a_2=0};
\end{aligned} \tag{7.15}$$

where $\Pi_{\mathbf{CB}} := |\psi_{c|z}\rangle\langle\psi_{c|z}|_{\mathbf{C}} \otimes |\phi_{b|y}\rangle\langle\phi_{b|y}|_{\mathbf{B}}$ . This implies that the operations inside the switch act as an identity on the control and target input systems when $x_1 = x_2 = 0$. Next, Charlie and Bob can measure $\Phi_{\mathrm{PR}}$ in their respective bases to generate the conditional probability distribution

$$\mathrm{p}(C, B|X, Z) = \left(\begin{array}{cc|cc}
\varepsilon_+/8 & \varepsilon_-/8 & 1/2 & 0 \\
\varepsilon_-/8 & \varepsilon_+/8 & 0 & 1/2 \\
\hline
\varepsilon_+/8 & \varepsilon_-/8 & 0 & 1/2 \\
\varepsilon_-/8 & \varepsilon_+/8 & 1/2 & 0
\end{array}\right) \tag{7.16}$$

where $\varepsilon_{\pm} = 2 \pm \sqrt{2}$. This distribution has a CHSH score of $(6+\sqrt{2})/8$. The three terms of inequality (7.2) therefore add up to $(14+\sqrt{2})/8$ which is bigger than the maximal violation achievable in quantum theory, i.e., $1 + (1 + 1/\sqrt{2})/2$. Under the assumptions taken in [64], this violation certifies indefinite causal order in a post-quantum theory. Although the Hex-Square theory can generate all Box-world correlations we do not see the algebraic maximal violation of inequality (7.2). This is because one needs to measure the observables $(\sigma_{\hat{X}} \pm \sigma_{\hat{Y}})/\sqrt{2}$ on system **C** and $\{\sigma_{\hat{X}}, \sigma_{\hat{Y}}\}$ on system **B**. If either $\sigma_{\hat{X}}$ or $\sigma_{\hat{Y}}$ is measured on system **B**, there are no outcomes, post-selecting on which a definite causal order is achieved between $\mathcal{O}_{\mathcal{A}_1}$ and $\mathcal{O}_{\mathcal{A}_2}$. Therefore, the sum of the first two terms in inequality (7.2) cannot be maximised.

## 7.7 DISCUSSION

We have shown the existence of indefinite causal order in post-quantum GPTs by introducing the Hex-Square theory. In particular, we have shown that it is possible to device-independently certify indefinite causal order in the Hex-Square theory. In addition, if one were to take the violation of the DRF inequality as a measure of indefinite causal order, in analogy to Bell inequalities and nonlocality, a larger than quantum violation of the DRF inequality in the Hex-Square theory would then suggest that quantum theory is neither the most nonlocal nor the most causally indefinite. To single out quantum theory from the other GPTs, one might then want to devise an information processing task in which an optimal performance is reached when one uses quantum correlations generated in an indefinite causal order. This might point towards a way towards possible axiomatisations of quantum theory.

The violation mentioned above is not the algebraic bound of the DRF inequality (7.2). An interesting avenue is to try to construct a theory that achieves this bound. Another possible direction is to check whether there are information processing tasks whose performances can only be enhanced by post-quantum theories in an indefinite causal order.

An interesting outcome of our work is that there exist non-classical theories which do not admit superposition. In particular, the minimal tensor product composition of gbit state spaces has been shown not to admit superposition. However, their maximal tensor product composition, i.e., box-world does. This is in contradiction to the notion of superposition presented in [5], in which any non-classical theory admits superposition.

# Part IV

# Nonlocality in the Presence of Noisy Channel

# — 8 —

# Quantum Correlations can be Certified Nonlocal with Arbitrarily Large Signalling

## 8.1 INTRODUCTION

Bell's theorem [11] concerns two parties, Alice and Bob, who are asked questions, $x$ and $y$, to which they need to return answers, $a$ and $b$ respectively. For simplicity, let us assume that $x, y, a, b \in \{0, 1\}$, and let us denote by $X, Y$ and $A, B$ the respective random variables corresponding to the inputs (questions) and outputs (answers). Since $a$ represents the output for the input $x$, the random variable $X$ is a potential cause for $A$, and therefore $A$ is in the causal future of $X$. Similarly, $B$ is in the causal future of $Y$. In addition, they are allowed to share classical randomness, denoted by the (hidden) random variable $\Lambda$. It is further assumed that $B$ and $\Lambda$ are not in the causal future of $X$, and $A$ and $\Lambda$ are not in the causal future of $Y$. With this causal ordering, one can find multiple causal structures that are compatible with it (see Section 4.2), some of which are presented in Figure 8.1.

The first assumption of the theorem is that of freedom of choice, stating that the random variables $X$ and $Y$ are free, i.e., they are independent of every random variable outside their respective causal futures. More precisely, the random variable $X$ is independent of $B, Y$ and $\Lambda$, and the random variable $Y$ is independent of $A, X$ and $\Lambda$. This assumption can be broken down into two parts: i) *measurement independence (MI):* $X$ and $Y$ are independent of $\Lambda$, ii) *parameter independence (PI):* $X$ (or $Y$) is independent of $B$ (or $A$). The second assumption of Bell's theorem is called *local determinism*, stating that the value of $A$ (or $B$) can be completely determined by the variables in its causal past, i.e., $X$ (or $Y$) and $\Lambda$. This means that $A$ and $B$ are conditionally independent. With these, one way of phrasing Bell's theorem is: *"there exist quantum correlations that are incompatible with the joint assumptions of freedom of choice and local determinism"*. This incompatibility can be demonstrated by first finding the facets of the set of conditional probability distributions $\mathrm{p}(A, B|X, Y)$ generated from the joint assumptions, and then showing the presence of a quantum correlation that violates one of the facet-defining inequalities. Such inequalities are called *Bell inequalities*.

Every conditional probability distribution $\mathrm{p}(A, B|X, Y)$ that satisfies the Bell inequalities has the property that for every $x, y, a, b$ and $\lambda$,

$$\mathrm{p}(a, b|x, y, \lambda) = \mathrm{p}(a|x, \lambda)\mathrm{p}(b|y, \lambda), \tag{8.1}$$

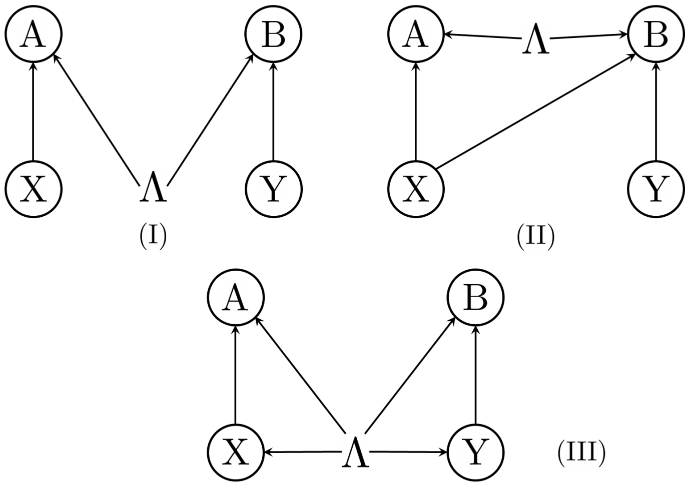


Figure 8.1: (I) Bell causal structure. (II) Relaxation of Parameter Independence. (III) Relaxation of Measurement Independence.

i.e., conditioned on $X, Y$ and $\Lambda$, the random variables $A$ and $B$ are uncorrelated. That quantum correlations violate Bell inequalities, testify that this statistical independence is not achieved in quantum theory. Additionally, from the freedom of choice assumption, one can infer that neither $A$ nor $B$ is the cause of the other. What is then the source of this correlation has been a topic under the limelight for several decades, dating back to a letter from von Neumann to Schrödinger [85]. A popular broadway is to adopt Reichenbach's principle[1] [86] and hypothesise that there is a common cause explanation for the correlations between $A$ and $B$ which can be found in the relaxation of freedom of choice.

In particular, one has to relax either (or both) MI or (and) PI. Given a model in which freedom of choice has been relaxed, two principal directions have been investigated. Firstly, how much measurement or parameter dependence is needed in order to violate Bell inequalities as much as violated by quantum theory. Secondly, how much of this dependence is needed to capture all quantum correlations (since there are post-quantum non-signalling distributions as well).

Various works have been presented to investigate how much of these assumptions need to be relaxed to account for violation of Bell inequalities. The amount of information about the measurement settings that needs to be transferred to the other party in order to violate Bell inequalities by Tsirelson's bound is 0.736 bits [75], developed further in [42]. To mention a few, all correlations generated by performing projective measurements on a singlet can be reproduced if Alice's input setting is correlated with the $\Lambda$, regardless of the size of the inputs of the two parties [10]. An upper bound on the amount of this correlation has also been proposed [47]. Additionally, one bit of classical communication is required to generate all correlations obtained by projective measurements on a singlet [97]. For two partially entangled qubits there is a 2 bits communication model [97]; However, it is still unknown whether this is optimal or whether one single bit would suffice. Relaxed Bell inequalities accounting for relaxations of the aforementioned

[1] *Reichenbach's Common Cause Principle:* If $A$ and $B$ are correlated, i.e., $\mathrm{p}(A \cap B) \neq \mathrm{p}(A)\mathrm{p}(B)$, either $A$ or $B$ is the cause of the other, or there is a *common cause* conditioned on which they are independent.

assumptions have also been studied [48] and then in [73]. In a slightly different direction, it has been found that arbitrarily small amount of measurement dependence is needed to demonstrate Bell nonlocality [84]. This has been followed up in [83]. Finally, relaxation of free will only for one party has been considered in [6]. This list captures the interest in relaxing the assumptions in the quest of understanding quantum Bell nonlocal correlations but is definitely not exhaustive.

The second line of query involving the detection of quantum correlations in the presence of strong relaxations of the assumptions has been answered in the context of MI [21, 87]. For the rest of this chapter, we will investigate the presence of quantum correlations in the presence of strong PI by considering communication of the inputs from one party to another over a binary symmetric channel. The chapter is designed as follows: in Section 8.2 we present a definition for one and two way signalling. Then, in Section 8.3 present the set of distributions that can be realised in the Bell setting, if signalling is allowed over a binary symmetric channel. In Section 8.4, we show that the set of non-signalling correlations that one way and two way signalling can reproduce are equal. Finally, in Section 8.5 we present our main result: quantum correlations can be certified nonlocal in the presence of arbitrarily strong signalling, when only one party is allowed to signal. In Section 8.6, we conjecture that the same holds when both the parties are allowed to signal.

## 8.2 RELAXATIONS OF PARAMETER INDEPENDENCE

Given a probability distribution,

$$\mathrm{p}(A,B|X,Y) = \left(\begin{array}{cc|cc} p(0,0|0,0) & p(0,1|0,0) & p(0,0|0,1) & p(0,1|0,1) \\ p(1,0|0,0) & p(1,1|0,0) & p(1,0|0,1) & p(1,1|0,1) \\ \hline p(0,0|1,0) & p(0,1|1,0) & p(0,0|1,1) & p(0,1|1,1) \\ p(1,0|1,0) & p(1,1|1,0) & p(1,0|1,1) & p(1,1|1,1) \end{array}\right),$$

in the $(2,2,2)$ setting, signalling relations can be verified by checking that the element-wise inner product of $\mathrm{p}(A,B|X,Y)$ with the vectors

$$\begin{aligned} \mathrm{NS}_0^{\mathbf{A}\to\mathbf{B}} &:= \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right), \quad \mathrm{NS}_1^{\mathbf{A}\to\mathbf{B}} := \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right), \\ \mathrm{NS}_0^{\mathbf{A}\leftarrow\mathbf{B}} &:= \left(\begin{array}{cc|cc} 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \mathrm{NS}_1^{\mathbf{A}\leftarrow\mathbf{B}} := \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right); \end{aligned} \tag{8.2}$$

is zero. A probability distribution p satisfying $\langle \mathrm{NS}_{i,\mathbf{A}\to\mathbf{B}}, \mathrm{p}\rangle = 0$, is non-signalling from Alice to Bob with respect to Bob's first measurement choice when $i=0$ and second measurement choice when $i=1$. If satisfying $\langle \mathrm{NS}_{j,\mathbf{A}\leftarrow\mathbf{B}}, \mathrm{p}\rangle = 0$, p is non-signalling from Bob to Alice, with respect to Alice's first measurement choice when $j=0$ and second measurement choice when $j=1$. These constraints are presented here with respect to the first outcomes. It is sufficient to do so since together with the normalisation condition, non-signalling with respect to the second outcomes is implied.

A non-zero inner product of p with at least one of the vectors in (8.2) implies a violation of parameter independence. Concisely, if p has a non-zero inner product with either $\mathrm{N}_x^{\mathbf{A}\to\mathbf{B}}$ or $\mathrm{N}_y^{\mathbf{A}\leftarrow\mathbf{B}}$ for $x, y \in \{0,1\}$, but not both, either Alice's input $x$ influences Bob's output $b$ or Bob's input $y$ influences Alice's output $a$. We call this *one way* signalling. On the other hand, if for some $x$ and $y$, p has a non-zero inner product with both $\mathrm{N}_x^{\mathbf{A}\to\mathbf{B}}$ and $\mathrm{N}_y^{\mathbf{A}\leftarrow\mathbf{B}}$, Alice's input $x$ influences Bob's output $b$ and simultaneously, Bob's input $y$ influences Alice's output $a$. We call this *two way* signalling. Given a set, $\mathbb{S}$, of conditional probability distributions that allows one way signalling, it is possible that signalling is allowed only from Alice to Bob (or Bob to Alice). We call this *one sided* one way signalling. If, on the other hand, $\mathbb{S}$ allows signalling in both directions but not simultaneously, we call it *genuinely* one way signalling. We formally define these notions below.

**Definition 8.2.1.** *Let $\mathbb{S}$ be a set of conditional probability distributions in the $(2,2,2)$ setting and let $\mathrm{NS}_0^{\mathbf{A}\to\mathbf{B}}, \mathrm{NS}_1^{\mathbf{A}\to\mathbf{B}}, \mathrm{NS}_0^{\mathbf{A}\leftarrow\mathbf{B}}$ and $\mathrm{NS}_1^{\mathbf{A}\leftarrow\mathbf{B}}$ be defined as in (8.2). If for some $\mathrm{p} \in \mathbb{S}$ and some $i, j \in \{0,1\}$, $\langle \mathrm{NS}_i^{\mathbf{A}\to\mathbf{B}}, \mathrm{p}\rangle \neq 0$ or $\langle \mathrm{NS}_j^{\mathbf{A}\leftarrow\mathbf{B}}, \mathrm{p}\rangle \neq 0$, $\mathbb{S}$ is*

1. *one sided one way(1S1W) signalling, if either $\langle \mathrm{NS}_i^{\mathbf{A}\to\mathbf{B}}, \mathrm{p}\rangle = 0$ for all $i \in \{0,1\}$ and for all $\mathrm{p} \in \mathbb{S}$, or $\langle \mathrm{NS}_j^{\mathbf{A}\leftarrow\mathbf{B}}, \mathrm{p}\rangle = 0$ for all $j \in \{0,1\}$ and for all $\mathrm{p} \in \mathbb{S}$,*
2. *genuinely one way(G1W) signalling, if there does not exist any $i, j \in \{0,1\}$ and any extremal $\mathrm{p} \in \mathbb{S}$ such that $\langle \mathrm{NS}_i^{\mathbf{A}\to\mathbf{B}}, \mathrm{p}\rangle \neq 0$ and $\langle \mathrm{NS}_j^{\mathbf{A}\leftarrow\mathbf{B}}, \mathrm{p}\rangle \neq 0$,*
3. *two way(2W) signalling, if for some $i, j \in \{0,1\}$ and some extremal $\mathrm{p} \in \mathbb{S}$, $\langle \mathrm{NS}_i^{\mathbf{A}\to\mathbf{B}}, \mathrm{p}\rangle \neq 0$ and $\langle \mathrm{NS}_j^{\mathbf{A}\leftarrow\mathbf{B}}, \mathrm{p}\rangle \neq 0$.*

Note that if $\mathbb{S}$ is 1S1W signalling it can also be seen as G1W signalling but not every G1W signalling set is 1S1W signalling. In addition, if a set is G1W it cannot be 2WS and vice-versa.

## 8.3 SIGNALLING WITH BINARY SYMMETRIC CHANNEL

Any experiment demonstrating a violation of Bell inequalities needs to justify how its physical implementation respects the assumptions leading to inequalities. One way to justify PI is to not allow signalling of one party's inputs to influence the other party's output. Therefore, one can justify relaxation of PI by allowing such signalling. In this chapter, we consider a modification of the Bell setup by introducing two identical *binary symmetric channels* to be used to signal Alice's input choice $x$ to Bob and Bob's input choice $y$ to Alice respectively. The action of this channel on a bit $z$ is defined as:

$$\mathcal{C}_p(z) \begin{cases} z & \text{with probability} \quad p \\ \overline{z} & \text{with probability} \quad 1-p \end{cases}, \tag{8.3}$$

where $p \in [0,1]$ can be seen as the amount of slack in the relaxation of PI. In particular, when $p = 1$, PI is completely relaxed, while when $p = 1/2$, PI is not relaxed at all. The case when $p = 0$ is symmetric to the case of $p = 1$ and therefore we will assume that the range of $p$ is $[1/2, 1]$, unless specified otherwise. In the next three subsections, we describe each of the three signalling models as obtained under the binary symmetric channels.

### 8.3.1 One Sided One Way (1S1W) Signalling

We assume here that the channel from Alice to Bob is active and the channel from Bob to Alice is inactive. Alice's output $a$ is then a function of her input $x$ and Bob's output is a function of his input $y$ and the noisy message $px + (1-p)\overline{x}$. We assume that although Bob might know that the channel is noisy, he has no information whether in a given round the bit was flipped or not. To characterise the set of all extremal probability distributions that the two parties can generate, let us first consider the case when $p = 1$. Since the set of distributions can be described as the convex hull of the deterministic strategies, it suffices to list the deterministic distributions. Given any $(x, y)$, Bob has two options for his outcome $b$. Since $x$ and $y$ can take one of two values, the total number of choices for outputting $b$ is $2^{(2\times 2)}$. In addition, for a given $x$, Alice has 2 choices of her output $a$, making the total number of possible choices $2^2$ . Therefore, the total number of deterministic strategies that Alice and Bob can realise is $2^2 \times 2^{(2\times 2)} = 64$. The set of 16 local deterministic non-signalling distributions is a subset of these deterministic strategies.

For a concrete example, consider the strategy where Alice outputs $a = x$ and Bob outputs $b = \overline{x} \wedge y$. When $p = 1$, they realise the distribution

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right).$$

This is a deterministic signalling distribution that can perfectly win one of the CHSH games. Now, if $p = 0$, Bob will always receive $\overline{x}$ and produce an output $b' = x \wedge y$. This would generate the deterministic signalling distribution

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right).$$

which is also signalling and can win one of the CHSH games perfectly. For an arbitrary $p$, a mixture of these distributions

$$p\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) + (1-p)\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 0 & 1-p & p \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & p & 1-p \end{array}\right)$$

is obtained. The rest 63 noisy strategies can be found by writing out $b$ as a deterministic function of $x$ and $y$ and then repeating the procedure above. Note, that when Alice and Bob realise the local deterministic non-signalling distributions, Bob's outcomes are independent of Alice's input choice. Therefore the 16 no signalling distributions are unaffected by the channel.

We found that these 64 distributions can be classified into 3 classes, up to equivalence of local relabelling symmetries. An element for each class is given below:

$$\mathrm{L} = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \mathrm{S}_1 := \left(\begin{array}{cc|cc} 1 & 0 & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1-p & p \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$\mathrm{S}_2 := \left(\begin{array}{cc|cc} p & 1-p & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline 1-p & p & 1-p & p \\ 0 & 0 & 0 & 0 \end{array}\right);$$

$\mathrm{S}_1$ represents a class of 32 elements and $\mathrm{S}_2$ represents a class of 16 elements. One can generate the members of a class by performing all local relabelling operations on the class candidate provided, followed by discarding distributions that involve signalling from Bob to Alice. Note, that $\mathrm{S}_1$ is signalling with respect to only one of Bob's input settings, while $\mathrm{S}_2$ is signalling for both of them. Let us denote this polytope as $\mathbb{S}^{A\to B}_{p,(2,2)}$. Note, that when $p = 1/2$, $\mathbb{S}^{A\to B}_{p=1/2,(2,2)} = \mathbb{H}^{[0]}_{(2,2)}$.

We varied $p$ in the interval $[4/5, 1]$ in 10 steps and for each step-size we generated the facets of the $\mathbb{S}^{A\to B}_{p,(2,2)}$. We then interpolated each facet to obtain the corresponding forms and classified them based on relabelling symmetries. We found that there are 13 inequivalent classes of facets. A candidate facet from each class is given as $\langle \mathrm{F}_i, \mathbf{x}\rangle \leqslant 1$, with the vectors $\mathrm{F}_i$ provided below.

$$\mathrm{F}_1 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \mathrm{F}_2 = \left(\begin{array}{cc|cc} 1 & -\frac{1-p}{2p-1} & 0 & 0 \\ 1 & -\frac{1-p}{2p-1} & 0 & 0 \\ \hline 0 & -\frac{p-1}{2p-1} & 0 & 0 \\ 0 & -\frac{p-1}{2p-1} & 0 & 0 \end{array}\right),$$

$$\mathrm{F}_3 = \left(\begin{array}{cc|cc} 1 & -\frac{-3p^3+4p^2-3p+1}{4p^3-4p^2+3p-1} & 0 & -\frac{p^3}{4p^3-4p^2+3p-1} \\ -\frac{-5p^3+5p^2-3p+1}{4p^3-4p^2+3p-1} & -\frac{-4p^3+3p^2-2p+1}{4p^3-4p^2+3p-1} & \frac{(p-1)^2p}{4p^3-4p^2+3p-1} & 0 \\ \hline 0 & -\frac{(p-1)^3}{4p^3-4p^2+3p-1} & 0 & -\frac{p^3-p^2}{4p^3-4p^2+3p-1} \\ 0 & -\frac{p^3-p^2}{4p^3-4p^2+3p-1} & 0 & -\frac{(p-1)^3}{4p^3-4p^2+3p-1} \end{array}\right),$$

$$\mathrm{F}_4 = \left(\begin{array}{cc|cc} -\frac{-3p^2+4p-2}{2p^2-3p+2} & \frac{(p-1)\left(5p^2-6p+2\right)}{(2p-1)(2p^2-3p+2)} & 0 & -\frac{(p-1)^2p}{(2p-1)(2p^2-3p+2)} \\ -\frac{-3p^2+4p-2}{2p^2-3p+2} & -\frac{-5p^3+9p^2-7p+2}{(2p-1)(2p^2-3p+2)} & 0 & -\frac{p^3}{(2p-1)(2p^2-3p+2)} \\ \hline 0 & -\frac{(p-1)p^2}{(2p-1)(2p^2-3p+2)} & 0 & -\frac{(p-1)p^2}{(2p-1)(2p^2-3p+2)} \\ 0 & -\frac{(p-1)^3}{(2p-1)(2p^2-3p+2)} & -\frac{p^2-p}{2p^2-3p+2} & -\frac{(p-1)\left(3p^2-3p+1\right)}{(2p-1)(2p^2-3p+2)} \end{array}\right),$$

$$\mathrm{F}_5 = \left(\begin{array}{cc|cc} 1 & -\frac{-3p^3+4p^2-3p+1}{4p^3-4p^2+3p-1} & 0 & -\frac{(p-1)^3}{4p^3-4p^2+3p-1} \\ -\frac{-3p^2+2p-1}{2p^2-p+1} & -\frac{-5p^3+5p^2-3p+1}{4p^3-4p^2+3p-1} & 0 & -\frac{p^3-p^2}{4p^3-4p^2+3p-1} \\ \hline 0 & -\frac{(p-1)^3}{4p^3-4p^2+3p-1} & 0 & -\frac{(p-1)^2p}{4p^3-4p^2+3p-1} \\ 0 & -\frac{p^3-p^2}{4p^3-4p^2+3p-1} & 0 & -\frac{p^3}{4p^3-4p^2+3p-1} \end{array}\right),$$

$$\mathrm{F}_6 = \left(\begin{array}{cc|cc} 1 & -\frac{-3p^3+4p^2-3p+1}{4p^3-4p^2+3p-1} & 0 & -\frac{(p-1)^3}{4p^3-4p^2+3p-1} \\ -\frac{-3p^2+2p-1}{2p^2-p+1} & -\frac{-5p^3+5p^2-3p+1}{4p^3-4p^2+3p-1} & 0 & -\frac{p^3-p^2}{4p^3-4p^2+3p-1} \\ \hline 0 & -\frac{(p-1)^3}{4p^3-4p^2+3p-1} & 0 & -\frac{(p-1)^2 p}{4p^3-4p^2+3p-1} \\ 0 & -\frac{p^3-p^2}{4p^3-4p^2+3p-1} & 0 & -\frac{p^3}{4p^3-4p^2+3p-1} \end{array}\right),$$

$$\mathrm{F}_7 = \left(\begin{array}{cc|cc} -\frac{-4p^2+3p-1}{3p^2-2p+1} & -\frac{-7p^3+10p^2-5p+1}{(2p-1)(3p^2-2p+1)} & 0 & -\frac{p^2-p}{3p^2-2p+1} \\ -\frac{-4p^2+3p-1}{3p^2-2p+1} & -\frac{-7p^3+8p^2-4p+1}{(2p-1)(3p^2-2p+1)} & 0 & 0 \\ \hline 0 & -\frac{(p-1)p^2}{(2p-1)(3p^2-2p+1)} & 0 & 0 \\ 0 & -\frac{(p-1)^3}{(2p-1)(3p^2-2p+1)} & -\frac{p^2-p}{3p^2-2p+1} & -\frac{p(2p-1)}{3p^2-2p+1} \end{array}\right),$$

$$\mathrm{F}_8 = \left(\begin{array}{cc|cc} 1 & -\frac{-5p^3+7p^2-4p+1}{(2p-1)(3p^2-2p+1)} & 0 & -\frac{p^2}{3p^2-2p+1} \\ -\frac{-4p^2+3p-1}{3p^2-2p+1} & \frac{(p-1)p^2}{(2p-1)(3p^2-2p+1)}+1 & 0 & 0 \\ \hline 0 & -\frac{(p-1)^3}{(2p-1)(3p^2-2p+1)} & 0 & -\frac{(p-1)p}{3p^2-2p+1} \\ 0 & -\frac{(p-1)p^2}{(2p-1)(3p^2-2p+1)} & 0 & 0 \end{array}\right),$$

$$\mathrm{F}_9 = \left(\begin{array}{cc|cc} 1 & -\frac{1-p}{2p-1} & 0 & 0 \\ \frac{(p-1)^2}{p(2p-1)} & 0 & -\frac{2-3p}{2p-1} & -\frac{2-3p}{2p-1} \\ \hline 0 & -\frac{p-1}{2p-1} & 0 & 0 \\ 0 & -\frac{p-1}{2p-1} & 0 & 0 \end{array}\right), \quad \mathrm{F}_{10} = \left(\begin{array}{cc|cc} 1 & \frac{(p-1)^3}{2p^3-3p^2+3p-1} & 0 & 0 \\ 1 & -\frac{-p^3+p^2-2p+1}{2p^3-3p^2+3p-1} & -\frac{p^2}{p^2-p+1} & 0 \\ \hline 0 & -\frac{(p-1)^3}{2p^3-3p^2+3p-1} & 0 & 0 \\ 0 & -\frac{(p-1)p^2}{2p^3-3p^2+3p-1} & 0 & \frac{(p-1)p}{p^2-p+1} \end{array}\right),$$

$$\mathrm{F}_{11} = \left(\begin{array}{cc|cc} 1 & -\frac{1-2p}{3p-1} & 0 & -\frac{p}{3p-1} \\ \frac{2(2p-1)}{3p-1} & \frac{2(2p-1)}{3p-1} & 0 & 0 \\ \hline 0 & -\frac{p-1}{3p-1} & 0 & 0 \\ 0 & 0 & 0 & -\frac{p-1}{3p-1} \end{array}\right), \quad \mathrm{F}_{12} = \left(\begin{array}{cc|cc} 1 & -\frac{1-2p}{3p-1} & 0 & 0 \\ 1 & 1 & -\frac{1-p}{3p-1} & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -\frac{p-1}{3p-1} & 0 & -\frac{p}{3p-1} \end{array}\right)$$

$$\mathrm{F}_{13} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & -\frac{1-p}{p} & 0 & -\frac{p-1}{p} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The number of facets in each class is summarised in Table 8.1. In the limit $p \to 1/2$, these facets converge to the 24 facets of $\mathbb{H}^{[0]}_{(2,2)}$. On the other hand, when $p = 1$, one might rightly guess that all non-signalling distributions can be realised by one way signalling. We formalise this in the following lemma.

| Class | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| # | 16 | 8 | 32 | 32 | 64 | 64 | 64 | 64 | 16 | 64 | 32 | 64 | 32 |

Table 8.1: Number of facets in each class, inequivalent up to relabelling symmetries, for $\mathbb{S}^{A\to B}_{p,(2,2)}$ (interpolated).

**Lemma 3.** $\mathbb{H}^{[8]}_{(2,2)} \subset \mathbb{S}^{A\to B}_{p=1,(2,2)}$.

*Proof.* It suffices to show that all 8 PR boxes are elements of the set $\mathbb{H}^{A\to B}_{p=1,(2,2)}$; First, we note that

$$\frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) = \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) + \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The remaining 7 PR boxes also admit similar decompositions. One can find them by applying appropriate local relabelling operations on both sides of the equality above and using linearity of relabelling operations to find the respective deterministic signalling distributions. □

Since the set of non-signalling quantum correlations is a subset of the set of all non-signalling correlations, one can generate all quantum correlations when $p = 1$. It is then natural to investigate a lower bound on $p$ such that one can still generate all non-signalling quantum correlations. We will discuss this in Section 8.5. Before that, we briefly present the sets of probability distributions arising from G1W signalling and 2WS signalling over binary symmetric channels.

### 8.3.2 Genuinely One Sided (G1S) Signalling

Let us assume here that during the generation of any probability distribution, either the channel from Alice to Bob is active or the channel from Bob to Alice is active. One can then use these two scenarios separately. The case where the channel from Alice to Bob is active gives us precisely $\mathbb{S}_p^{\mathbf{A}\to\mathbf{B}}$. Instead, if only Bob signalled to Alice, the set of distributions, $\mathbb{S}_p^{\mathbf{A}\leftarrow\mathbf{B}}$, would just be the party swap relabelling of the distributions in $\mathbb{S}_p^{\mathbf{A}\to\mathbf{B}}$. Therefore, the two sided one way signalling distributions can be obtained by first applying the party relabelling map to the extremal distributions of $\mathbb{S}_p^{\mathbf{A}\to\mathbf{B}}$ to obtain the extremal distributions of $\mathbb{S}_p^{\mathbf{A}\leftarrow\mathbf{B}}$ , followed by taking the convex hull of the union of these sets $\mathbb{S}_p^{\mathbf{A}\to\mathbf{B}}$ and $\mathbb{S}_p^{\mathbf{A}\leftarrow\mathbf{B}}$. Just like the one sided version, there are 3 equivalence classes of extremal distributions, candidates of which are $\mathrm{L}, \mathrm{S}_1$ and $\mathrm{S}_2$. There are 64 elements in the class represented by $\mathrm{S}_1$ and 32 elements in the class represented by $\mathrm{S}_2$. Full classes can be calculated by applying all relabelling operations on respective candidate distributions and then discarding duplicates. Since distributions that are signalling from Bob to Alice are now allowed, one gets the inclusion

$$\mathbb{S}_p^{\mathbf{A}\to\mathbf{B}} \subseteq \mathrm{ConvHull}\left[\mathbb{S}_p^{\mathbf{A}\to\mathbf{B}} \cup \mathbb{S}_p^{\mathbf{A}\leftarrow\mathbf{B}}\right] \tag{8.4}$$

with equality when $p = 1/2$. Together with Lemma 8.3.1, a further inclusion

$$\mathbb{H}^{[8]}_{(2,2)} \subset \mathrm{ConvHull}\left[\mathbb{S}_{p=1}^{\mathbf{A}\to\mathbf{B}} \cup \mathbb{S}_{p=1}^{\mathbf{A}\leftarrow\mathbf{B}}\right] \tag{8.5}$$

is implied.

### 8.3.3 Two Way (2W) Signalling

Finally, we assume here that on every round both the channels are active, i.e., Alice and Bob are allowed to simultaneously signal to each other. In particular, both Alice and Bob's outcomes are now functions of $x$ and $y$. More precisely, $a$ is a function of $x$ and $py + (1-p)\overline{y}$ and $b$ is a function of $y$ and $px + (1-p)\overline{x}$. Let us first consider the set of deterministic distributions that can be generated when $p = 1$. For each value of $x$ and $y$, Alice has two possible outcomes for $a$ and similarly, Bob has two possible choices for $b$, implying that there are $(2 \times 2)$ possible ways of choosing a pair $(a, b)$ for every $(x, y)$. Since there are 4 choices of pairings $(x, y)$, the total number of deterministic distributions that can be realised is $4^{(2\times 2)} = 256$. The set of 16 local non-signalling distributions, $\mathbb{S}_{p=1}^{[\mathbf{A}\to\mathbf{B}]}$ and ConvHull $\left[\mathbb{S}_{p=1}^{\mathbf{A}\to\mathbf{B}} \cup \mathbb{S}_{p=1}^{\mathbf{A}\leftarrow\mathbf{B}}\right]$ are all subsets of this set of 256 distributions. As an example, if $a = b = (x \oplus y)$, the distribution

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

will be generated. The inner products of this distribution with the vectors $\mathrm{NS}_{0,\mathbf{A}\to\mathbf{B}}$, $\mathrm{NS}_{1,\mathbf{A}\to\mathbf{B}}$, $\mathrm{NS}_{0,\mathbf{A}\leftarrow\mathbf{B}}$ and $\mathrm{NS}_{1,\mathbf{A}\leftarrow\mathbf{B}}$ are all non-zero, indicating two way signalling. Now if $p = 0$, one gets $a = (x \oplus \overline{y})$ and $b = (\overline{x} \oplus y)2$, giving the distribution

$$\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right),$$

which is also two way signalling. For any arbitrary $p \in [0, 1]$, the distribution

$$p\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) + (1-p)\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|cc} p & 0 & 1-p & 0 \\ 0 & 1-p & 0 & p \\ \hline 1-p & 0 & p & 0 \\ 0 & p & 0 & 1-p \end{array}\right)$$

is obtained. This distribution is local when $p = 1/2$, otherwise two way signalling. The rest of the 255 noisy strategies can be found in a similar way. These 256 distributions can be categorised into 6 inequivalent relabelling classes, representatives of which are:

$$\mathrm{L} = \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \mathrm{S}_1' = \left(\begin{array}{cc|cc} 1 & 0 & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1-p & p \\ 0 & 0 & 0 & 0 \end{array}\right), \mathrm{S}_2' = \left(\begin{array}{cc|cc} p & 1-p & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline 1-p & p & 1-p & p \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$\mathrm{S}_3' = \left(\begin{array}{cc|cc} 1 & 0 & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline p & 0 & 1-p & 0 \\ 1-p & 0 & 0 & p \end{array}\right), \mathrm{S}_4' = \left(\begin{array}{cc|cc} p & 1-p & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline 0 & p & 1-p & 0 \\ 1-p & 0 & 0 & p \end{array}\right),$$

$$\mathrm{S}_5' = \left(\begin{array}{cc|cc} p & 0 & 0 & 1-p \\ 0 & 1-p & p & 0 \\ \hline 0 & p & 1-p & 0 \\ 1-p & 0 & 0 & p \end{array}\right);$$

There are 64 elements in the second, fourth and fifth classes each, 32 in the third class and 16 in the sixth class. Note, that $S'_1$ is signalling with respect to only one of the measurement choices of one party, $S'_2$ is signalling with respect to both of the measurement choices of one party, $S'_3$ is signalling with respect to one of the input choices of one party and one of the input choices of the other, $S'_4$ is not signalling with respect to one of the input choices for one of the parties and finally, $S'_5$ is signalling with respect to both the inputs for both of the parties. Using the interpolation technique mentioned above, we found that there are 5 inequivalent classes of facets up to relabelling symmetries. A candidate facet of each class is given by $\langle F'_j, \mathbf{x}\rangle \leqslant 1$, with the vectors $F'_j$ listed below :

$$F'_1 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$F'_2 = \left(\begin{array}{cc|cc} \frac{P}{2P-1} & \frac{P}{2P-1} & \frac{P-1}{2P-1} & \frac{P-1}{2P-1} \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$F'_3 = \left(\begin{array}{cc|cc} 1 & -\frac{-3P^3+2P^2-2P+1}{4P^3-4P^2+3P-1} & 0 & \frac{(P-1)P^2}{4P^3-4P^2+3P-1} \\ -\frac{-3P^3+2P^2-2P+1}{4P^3-4P^2+3P-1} & -\frac{-2P^3+2P^2-2P+1}{4P^3-4P^2+3P-1} & -\frac{(P-1)^3}{4P^3-4P^2+3P-1} & 0 \\ \hline 0 & -\frac{(P-1)^3}{4P^3-4P^2+3P-1} & 0 & -\frac{-P^3+2P^2-P}{4P^3-4P^2+3P-1} \\ 0 & -\frac{(P-1)P^2}{4P^3-4P^2+3P-1} & -\frac{P}{2P^2-P+1} & -\frac{-P^3+2P^2-P}{4P^3-4P^2+3P-1} \end{array}\right),$$

$$F'_4 = \left(\begin{array}{cc|cc} 1 & -\frac{-P^2-P+1}{2P-1} & 0 & 0 \\ 1 & -\frac{-P^2-P+1}{2P-1} & 0 & 0 \\ \hline 0 & \frac{(P-1)^2}{2P-1} & 0 & 0 \\ -\frac{P^2}{2P-1} & 0 & -\frac{P^2-P}{2P-1} & -\frac{P^2-P}{2P-1} \end{array}\right),$$

$$F'_5 = \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ -\frac{1-2P}{3P-1} & 1 & 0 & -\frac{P-1}{3P-1} \\ \hline 0 & -\frac{1-P}{3P-1} & 0 & 0 \\ 0 & 0 & 0 & -\frac{P}{3P-1} \end{array}\right),$$

where $\mathbf{x} \in \mathbb{R}^{16}$. The number of facets in each class is summarised in Table 8.2.

| Class | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| # | 16 | 16 | 128 | 16 | 64 |

Table 8.2: Number of facets in each class, inequivalent up to relabelling symmetries, for $\mathbb{S}^{A\leftrightarrow B}_{p,(2,2)}$ (interpolated).

## 8.4 NON-SIGNALLING SUBSPACES OF SIGNALLING MODELS

Our main goal is to find the smallest $p$ in the various signalling models such that all non-signalling quantum correlations can be realised. Since the set of non-signalling quantum correlations, $\mathcal{Q}$, is a proper subset of all non-signalling correlations, it is worth understanding the set of all non-signalling distributions that can be generated from a given signalling model. This can be achieved by projecting the signalling sets of distributions down to the subspace of non-signalling distributions. In the following, three sections we provide the set of extremal distributions one obtains when each of the signalling sets is restricted to non-signalling.

### 8.4.1 Non Signalling Subspace of $\mathbb{S}_p^{\mathbf{A}\to\mathbf{B}}$

In the $(2,2,2)$ scenario, there are four non-signalling constraints. All non-signalling distributions lie on the intersection of the hyperplanes corresponding to these constraints. These four hyperplanes can be represented as

$$\left\{ \left\langle \mathrm{NS}_x^{\mathbf{A}\to B}, \mathbf{x} \right\rangle = 0 \right\}_x \text{ and } \left\{ \left\langle \mathrm{NS}_y^{\mathbf{A}\leftarrow B}, \mathbf{x} \right\rangle = 0 \right\}_y,$$

where $x, y \in \{0,1\}$ and $\mathrm{NS}_x^{\mathbf{A}\to B}$ and $\mathrm{NS}_y^{\mathbf{A}\leftarrow B}$ is defined in (8.2) and $\mathbf{x} \in \mathbb{R}^{16}$. In the probability tables used, there are 16 probabilities, knowing all of them is not necessary. The normalisation condition implies that within each block (fixed inputs for Alice and Bob), the probabilities sum up to 1. In the presence of normalisation, this reduces the dimension by 4. Now, if a probability distribution is non-signalling, then the sum of the first two probabilities in every row and column is the same as the sum of the last two probabilities. This further reduces the dimension by 4. Therefore, a signalling probability distribution must have true dimension between 9 and 12. In particular, the set containing this distribution can be seen as an embedding in $\mathbb{R}^d$ where $9 \leqslant d \leqslant 12$. Given such a set, one can look at its projection on the $d-1$ hyperplane described by one of the non-signalling constraints. If $d-1 \neq 8$, one can then further project it down by another non-signalling constraint, and continue until he dimension reaches 8. However, one might lose distributions if in the first step one projects down to $d-2$. With this intuition, we present a sequence of steps that one can use to get the non-signalling subspace of a set of signalling distributions.

- **Step 1:** Take the set of extremal distributions, $\text{Extreme}[\mathbb{S}]$, of a set of signalling distributions $\mathbb{S}$ and a non-signalling constraint vector, $\mathrm{NS}_i$, from (8.2).
- **Step 2:** Take the subset $\mathbb{S}^*$ of $\text{Extreme}[\mathbb{S}]$ such that for every $\mathrm{p} \in \mathbb{S}^*$, $\langle \mathrm{NS}_i, \mathrm{p} \rangle \neq 0$.
- **Step 3:** Collect all pairs $(\mathrm{p}_j, \mathrm{p}_k)$, such that $\mathrm{p}_{j/k} \in \mathbb{S}^*$.
- **Step 4:** For every pair $(\mathrm{p}_j, \mathrm{p}_k)$, define a line segment $l_{j,k,\theta} := \theta \mathrm{p}_j + (1-\theta)\mathrm{p}_k$ with $0 \leqslant \theta \leqslant 1$ and find $\theta'$ such that when $\theta = \theta'$, $\langle \mathrm{NS}_i, l_{j,k,\theta'} \rangle = 0$. Define the set of all such intersection points as $\mathrm{NS}_i[\mathbb{S}^*]$. Ignore cases when no solution exists.
- **Step 5:** Calculate the set
$$\left(\text{Extreme}[\mathbb{P}] \setminus \mathbb{P}^*\right) \bigcup \mathrm{NS}_i[\mathbb{P}^*].$$
- **Step 6:** If this set is signalling, repeat Step 1 through Step 4 starting with this set and a different non-signalling constraint $\mathrm{NS}_{j\neq i}$ from (8.2).

We will use this sequence of steps in the following two sub-sections to calculate the non-signalling sub-polytopes of the three signalling polytopes. We present the extremal distributions for each case in the beginning of the respective subsections so that the reader can skip the details of the calculation and move on to the next section.

### 8.4.2 Non-Signalling Subspace of One Way Signalling Polytope

There are two equivalence classes of extremal states defining the non-signalling sub-polytope of the state space $\mathbb{S}_p^{A\to B}$. One class is the set of local deterministic boxes. A representative of the other class is:

$$\mathrm{S}_{A\not\to B} \coloneqq \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & p & 1-p \\ 0 & 1 & 1-p & p \\ \hline 1 & 0 & 1-p & p \\ 0 & 1 & p & 1-p \end{array}\right) \tag{8.6}$$

There are 16 elements of this class, the rest 15 of which can be obtained by performing local relabelling operations on $\mathrm{S}_{A\not\to B}$. Notice, that the party-swap relabelling symmetry is not present. Finally, when $p = 1$, these states reduce to the 8 PR boxes. Derivation can be found in the Appendix G.

### 8.4.3 Non Signalling Subspace of ConvHull $\left[\mathbb{S}_p^{\mathbf{A}\to\mathbf{B}} \cup \mathbb{S}_p^{\mathbf{A}\leftarrow\mathbf{B}}\right]$

The non-signalling subspace for this case can be found in three steps. First, collect all the distributions for which the inner products with $\mathrm{NS}_0^{\mathbf{A}\to\mathbf{B}}$ and $\mathrm{NS}_1^{\mathbf{A}\to\mathbf{B}}$ are non-zero and restrict them. This is precisely the case considered in the last section. Then collect all the distributions for which the inner products with $\mathrm{NS}_0^{\mathbf{A}\leftarrow\mathbf{B}}$ and $\mathrm{NS}_1^{\mathbf{A}\leftarrow\mathbf{B}}$ are non-zero. Using the party relabelling symmetry all the reductions can be performed by taking each distribution obtained in the previous step and performing a party swap relabelling operation on them. Finally, one can take the union of the correlations obtained from these two steps along with the 16 local deterministic distributions. Up to equivalence of relabelling symmetries, there are two classes: class of 16 local deterministic correlations and class of 32 correlations of the form

$$\mathrm{S}_{A\not\leftrightarrow B} \coloneqq \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline p & 1-p & 1-p & p \\ 1-p & p & p & 1-p \end{array}\right). \tag{8.7}$$

### 8.4.4 Non-Signalling Subspace of $\mathbb{S}_p^{\mathbf{A}\leftrightarrow\mathbf{B}}$

We performed a step by step reduction, as presented in Subsection 8.4.2 and found that apart from the class of 16 local deterministic correlations, the only other class of distributions is exactly that of 32 elements of the form $\mathrm{S}_{A\not\leftrightarrow B}$ represented in (8.7). This implies that the non-signalling subspaces of the two sided one way signalling polytope and the genuinely one sided signalling polytopes are equal. Therefore all non-signalling correlations that can be generated by genuinely two way signalling can also be generated by two sided one way signalling. We formalise this statement in the theorem below.

**Theorem 5.** *The signalling polytopes* $\mathrm{ConvHull}\left[\mathbb{S}_p^{A\to B}\cup\mathbb{S}_p^{A\leftarrow B}\right]$ *and* $\mathbb{S}_p^{A\leftrightarrow B}$*, restricted to non-signalling, are equal for any* $p\in[1/2,1]$.

*Proof.* Proof for the case of $\mathrm{ConvHull}\left[\mathbb{S}_p^{A\to B}\cup\mathbb{S}_p^{A\leftarrow B}\right]$ is provided in Appendix G. Using the very same technique the case for $\mathbb{S}_p^{A\leftrightarrow B}$ has also been found. □

## 8.5 DETECTING NONLOCALITY AGAINST STRONG 1S1W SIGNALLING

In this section we will prove that the minimum value of $p$ one needs to generate all quantum correlations in the 1S1W signalling model is 1, suggesting that PI must be completely relaxed to describe correlations generated from quantum theory. To see this, let us consider the $(2,2,2)$ Bell setup where Alice and Bob share the bipartite qubit state

$$|\psi_\theta\rangle := \cos\theta|00\rangle + \sin\theta|11\rangle, \tag{8.8}$$

Note that $|\psi_\theta\rangle$ is maximally entangled when $\theta = z\pi/4$ where $z\in\mathbb{Z}\setminus\{0\}$. To check whether the set of all quantum correlations can be realised, it suffices to check whether the correlations on the boundary of this set can be realised. Since all distributions on the boundary of this set can be obtained by performing qubit measurements on (8.8) on the $\hat{Z}-\hat{X}$ plane [69] (a consequence of Jordan's Lemma [55]), we consider Alice's measurements $M_x$ and Bob's measurements $N_y$ to be the following projections:

$$|M_x\rangle := \cos\frac{\theta_x}{2}|0\rangle + \sin\frac{\theta_x}{2}|1\rangle \text{ and } |N_y\rangle := \cos\frac{\phi_y}{2}|0\rangle + \sin\frac{\phi_y}{2}|1\rangle, \tag{8.9}$$

which gives us the following projection valued measures:

$$\left\{\mathrm{M}_{0|x} := |M_x\rangle\langle M_x|, \mathrm{M}_{1|x} = \mathbb{I}-\mathrm{M}_{0|x}\right\} \text{ and } \left\{\mathrm{N}_{0|y} := |N_y\rangle\langle N_y|, \mathrm{N}_{1|y} = \mathbb{I}-\mathrm{N}_{0|y}\right\}. \tag{8.10}$$

Using Born's rule, the conditional probabilities are given as:

$$\mathrm{p}\left(a,b|x,y\right) = \mathrm{Tr}\left[\left(\mathrm{M}_{a|x}\otimes\mathrm{N}_{b|y}\right)\psi_\theta\right], \tag{8.11}$$

where $\psi_\theta := |\psi_\theta\rangle\langle\psi_\theta|$. We now show that for any $p\neq 1$, one can generate a non-signalling quantum correlation that lies outside the set $\mathbb{S}_{p\neq 1}^{A\to B}$.

**Theorem 6.** *Let* $\mathcal{Q}$ *be the set of non-signalling quantum correlations in the* $(2,2,2)$ *setting. For* $p\in[1/2,1)$, $\mathcal{Q}\not\subset\mathbb{S}_p^{\mathbf{A\to B}}$.

*Proof.* Consider the facet $\langle \mathrm{F}_{13},\mathbf{x}\rangle \leqslant 1$ of the polytope $\mathbb{S}_p^{\mathbf{A\to B}}$, where

$$\mathrm{F}_{13} = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & -\frac{1-p}{p} & 0 & -\frac{p-1}{p} \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Next, fix $\theta_0 = \pi/2$ and $\theta_1 = 0$, i.e., Alice measures the observables $\sigma_{\hat{X}}$ when $x = 0$ and $\sigma_{\hat{Z}}$ when $x = 1$. In addition, assume that $\phi_1 = \pi - \phi_0$. Let $\mathrm{p}_{(\theta,\phi_0)}$ be the correlation generated by this strategy. The inner product between $\mathrm{F}_{13}$ and $\mathrm{p}_{(\theta,\phi_0)}$ can be calculated to be

$$\left\langle \mathrm{F}_{13}, \mathrm{p}_{(\theta,\phi_0)} \right\rangle = \frac{p \sin 2\theta \sin \phi_0 + \cos \phi_0((1-2p)\cos 2\theta - p + 1) + p}{2p}. \tag{8.12}$$

Since $\theta$ and $\phi_0$ are the only free parameters, we need to maximize $\left\langle \mathrm{F}_{13}, \mathrm{p}_{(\theta,\phi_0)} \right\rangle$ with respect to $\theta$ and $\phi_0$. The partial derivatives of $\left\langle \mathrm{F}_{13}, \mathrm{p}_{(\theta,\phi_0)} \right\rangle$ with respect to $\theta$ and $\phi_0$ are

$$\begin{aligned} \frac{\partial \left\langle \mathrm{F}_{13}, \mathrm{p}_{(\theta,\phi_0)} \right\rangle}{\partial \theta} &= \cos 2\theta \sin \phi_0 + \frac{(2p-1)\sin 2\theta \cos \phi_0}{p} \\ \frac{\partial \left\langle \mathrm{F}_{13}, \mathrm{p}_{(\theta,\phi_0)} \right\rangle}{\partial \phi_0} &= \frac{p \sin 2\theta \cos \phi_0 + \sin \phi_0((2p-1)\cos 2\theta + p - 1)}{2p} \end{aligned} \tag{8.13}$$

Setting these two equations to 0 gives us the following conditional solutions for $\theta$ and $\phi_0$:

| $\theta$ | $\phi_0$ |
|---|---|
| $c_1\pi, c_1 \in \mathbb{Z}$ | $c_2 2\pi, c_2 \in \mathbb{Z}$ |
| $c_1\pi, c_1 \in \mathbb{Z}$ | $\pi + c_2 2\pi, c_2 \in \mathbb{Z}$ |
| $\frac{\pi + c_1 2\pi}{2}, c_1 \in \mathbb{Z}$ | $c_2 2\pi, c_2 \in \mathbb{Z}$ |
| $\frac{\pi + c_1 2\pi}{2}, c_1 \in \mathbb{Z}$ | $\pi + c_2 2\pi, c_2 \in \mathbb{Z}$ |
| $\frac{1}{2}\left(\tan^{-1}\left(\frac{g}{f}\right) + c_1 2\pi\right), c_1 \in \mathbb{Z}$ | $\tan^{-1}\left(\frac{g'}{f'-f''+f'''}\right) + c_2 2\pi, c_2 \in \mathbb{Z}$ |
| $\frac{1}{2}\left(\tan^{-1}\left(\frac{g}{f}\right) + c_1 2\pi\right), c_1 \in \mathbb{Z}$ | $\tan^{-1}\left(\frac{-g}{-f'+f''-f'''}\right) + c_2 2\pi, c_2 \in \mathbb{Z}$ |
| $\frac{1}{2}\left(\tan^{-1}\left(\frac{-g}{f}\right) + c_1 2\pi\right), c_1 \in \mathbb{Z}$ | $\tan^{-1}\left(\frac{-g'}{f'-f''+f'''}\right) + c_2 2\pi, c_2 \in \mathbb{Z}$ |
| $\frac{1}{2}\left(\tan^{-1}\left(\frac{-g}{f}\right) + c_1 2\pi\right), c_1 \in \mathbb{Z}$ | $\tan^{-1}\left(\frac{g'}{-f'+f''-f'''}\right) + c_2 2\pi, c_2 \in \mathbb{Z}$ |

Table 8.3: General solutions obtained after setting the identities in (8.13) to zero.

where

$$f := \frac{1-2p}{3p-1}, \quad g := \frac{\sqrt{p}\sqrt{5p-2}}{\sqrt{9p^2-6p+1}},$$

$$f' := \frac{2\sqrt{p}}{(5p-2)\sqrt{6p-2}\sqrt{9p^2-6p+1}}, \quad f'' := \frac{11p^{3/2}}{(5p-2)\sqrt{6p-2}\sqrt{9p^2-6p+1}},$$

$$f''' := \frac{15p^{5/2}}{(5p-2)\sqrt{6p-2}\sqrt{9p^2-6p+1}}, \quad g' := \frac{\sqrt{5p-2}}{\sqrt{6p-2}};$$

From the table above, consider the solution at row 5 with $c_1 = c_2 = 0$. Substituting back to Equation (8.12) gives us an inner product

$$\frac{\sqrt{p}\sqrt{(3p-1)^2}}{\sqrt{2}(3p-1)^{3/2}} + \frac{1}{2}.$$

When $p \in [1/2, 1)$, this expression is strictly positive and bigger than 1. When $p = 1$, the value of this expression is 1. Running through the rest of the solutions, one finds that this is indeed the global maxima. □

## 8.6 DETECTING NONLOCALITY AGAINST STRONG 2W SIGNALLING

The quantum strategy presented in the proof of Theorem 8.5 lies outside the polytope $\mathbb{S}_{p<1}^{\mathbf{A}\to\mathbf{B}}$. However, existence of quantum correlations outside the polytopes ConvHull $\left[\mathbb{S}_p^{A\to B}\cup\mathbb{S}_p^{A\leftarrow B}\right]$ and $\mathbb{S}_{p<1}^{\mathbf{A}\leftrightarrow\mathbf{B}}$ is unknown. For instance, the correlation mentioned above might lie inside these polytopes. Now, since the non-signalling subspaces of ConvHull $\left[\mathbb{S}_p^{A\to B}\cup\mathbb{S}_p^{A\leftarrow B}\right]$ and $\mathbb{S}_p^{\mathbf{A}\leftrightarrow\mathbf{B}}$ are equal, if a non-signalling quantum correlation lies outside ConvHull $\left[\mathbb{S}_p^{A\to B}\cup\mathbb{S}_p^{A\leftarrow B}\right]$, it must also lie outside $\mathbb{S}_{p<1}^{\mathbf{A}\leftrightarrow\mathbf{B}}$. Below, we present a conjecture based on numerical evidence that such correlations do exist.

**Conjecture 1.** *Let $\mathcal{Q}$ be the set of quantum correlations in the $(2,2,2)$ setting. For $p\in[1/2,1)$, $\mathcal{Q}\not\subset\mathbb{S}_p^{A\leftrightarrow B}$.*

**Numerical Evidence:** We considered the facet inequality $\langle \mathrm{F}_5',\mathbf{x}\rangle\leqslant 1$ of the polytope $\mathbb{S}_p^{A\leftrightarrow B}$ and the correlation $\mathrm{p}_{\theta,\{\theta_i\},\{\phi_j\}}$ generated by performing the measurements (8.10) on the bipartite state (8.8), where $i,j\in\{0,1\}$. The maximum value of the inner product between the facet vector $\mathrm{F}_5'$ at some $p=p'$ can be found by the following maximisation problem

$$\max_{\substack{-\pi\leqslant\theta\leqslant\pi\\ -\pi\leqslant\{\theta_i\}\leqslant\pi\\ -\pi\leqslant\{\phi_j\}\leqslant\pi}}\left\langle \mathrm{F}_5'|_{p=p'},\mathrm{p}_{\theta,\{\theta_i\},\{\phi_j\}}\right\rangle. \tag{8.14}$$

We used MATHEMATICA to calculate the value of this maximisation at $p=0.\overbrace{99\cdots9}^{30}$, and found the inner product, precise up to 40 decimal digits, to be

$$1.000000000000000000000000000000013787780312564251210456 2579,$$

with the angles

$$\theta=-1.385857778347348906140680174954229492016591993641211466 3033,$$

$$\theta_0=2.954347992445453529886918970098793343060939121696182892242,$$

$$\theta_1=0.930360804537414557619054330817584473368032700440680465 0056$$

$$\phi_0=\theta_1-\pi,\phi_1=\theta_0$$

Note that with the $\theta$ above, the state is not maximally entangled.

**Possible Route To Proof:** From the numerical data, one might assume that $\phi_0=\theta_1-\pi$ and $\phi_1=\theta_0$. Using this guess, the inner product between the facet vector $\mathrm{F}_5$ and the quantum correlation $\mathrm{p}_{(\theta,\theta_0,\theta_1)}$ is then a function of four variables $p,\theta,\theta_0$ and $\theta_1$. We need to find $\theta,\theta_0$ and $\theta_1$ as functions of $p$ such that the inner product is maximised. Setting the partial derivative $\partial\left\langle \mathrm{F}_5',\mathrm{p}_{\theta,\{\theta_i\},\{\phi_j\}}\right\rangle/\partial\theta$ to 0, one finds

$$\theta=\tan^{-1}\left[\frac{(p-1)(\cos 2\theta_0-\cos 2\theta_1)-4p\sin\theta_0\sin\theta_1}{4(2p-1)(\cos\theta_0+\cos\theta_1)}\right]+c\pi,c\in\mathbb{Z};$$

Note that the numerical value of $\theta$ presented can be generated by setting $c=0$ and substituting the rest of the angles and $p$ with their respective values stated above. Therefore, it might suffice to consider the case where $c = 0$ and substitute the expression for $\theta$ in the inner product function $\langle \mathrm{F}'_5, \mathrm{p}_{\theta,\{\theta_i\},\{\phi_j\}} \rangle$. The remaining steps are to calculate the partial derivatives of this updated inner product with respect to $\theta_0$ and $\theta_1$, set them to zero and obtain appropriate expressions for $\theta_0$ and $\theta_1$ in terms of $p$, such that the updated inner product is maximised.

## 8.7 OUTLOOK ON RESULTS

We have shown that arbitrarily large signalling is needed to realise all quantum correlations when the signalling is taking over a binary symmetric channel. The correlations that cannot be realised unless the channel is noiseless arise from performing qubit measurements on a partially entangled state. In addition, we have shown that the sets of non-signalling distributions that can be generated using one way and two way signalling are equal. As a result, non-signalling quantum correlations that cannot be realised using one way signalling remain non-realisable using two way signalling as well. Additionally, we have provided a technique to project any set of signalling correlations to the space of non-signalling distributions. These results have both practical and foundational implications which we highlight in the following two paragraphs.

The advent of quantum computers brings in the expectation of miniaturising them as well. If an on-chip quantum computer is to be made, that uses nonlocal correlations for information processing, one needs to certify that the assumptions behind Bell's theorem, in particular locality, are properly addressed. This might turn out not to be straightforward. However, if one manages to generate the quantum correlations we have presented above, they can certify that any relaxation of the assumptions which can be modelled as a binary symmetric channel, if exists, is limited. The family of quantum correlations parameterised by $p$ provides a strict upper bound on the amount of signalling that might be taking place. This might turn out to be a way to certify signalling thresholds in a device dependent manner. Another possible direction is to use these correlations to certify randomness in the presence of signalling. We leave these for future work.

To explain Bell nonlocality through a common-cause mechanism, one might propose that hidden signalling exists in the universe, though it remains inaccessible to us. Under this assumption, it is possible to simulate all quantum correlations. However, this idea also requires an additional assumption: fine-tuning. Specifically, while signalling can account for signalling distributions, it also allows non-signalling (PR) distributions, which cannot be realised in quantum theory. Our work adds a new dimension to the ways in which such signalling would need to be fine-tuned. In particular, if such signalling were taking place behind the scenes, it needs to be perfect, at least if modelled by binary symmetric channels.

# Part V

# Appendix

# — A —

# Classification of extremal effects of $\mathbb{H}^{[0]}_{(3,2)}$ and $\mathbb{H}^{[1344]}_{(3,2)}$

| | |
|---|---|
| $\left(\begin{array}{cc\|cc\|cc} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{array}\right)_{1}$ | $\left(\begin{array}{cc\|cc\|cc} 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{array}\right)_{36}$ |
| $\left(\begin{array}{cc\|cc\|cc} 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{array}\right)_{144}$ | $\left(\begin{array}{cc\|cc\|cc} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{array}\right)_{18}$ |
| $\left(\begin{array}{cc\|cc\|cc} 1&1&0&0&0&0\\ 0&0&0&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{array}\right)_{12}$ | $\left(\begin{array}{cc\|cc\|cc} 0&1&0&0&0&0\\ 1&1&0&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{array}\right)_{36}$ |
| $\left(\begin{array}{cc\|cc\|cc} 1&1&0&0&0&0\\ 1&1&0&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{array}\right)_{1}$ | – |

Table A.1: Effects of $\mathbb{H}^{[1344]}_{(3,2)}$ up to local relabelling. All effects are separable. Number beside an effect denotes the number of effects present in the class represented by that given effect.

| | | |
|---|---|---|
| Table A.1 Effects, 248 | $\frac{1}{3}\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}_{576}$ | $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{72}$ |
| $\frac{1}{5}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}_{2304}$ | $\frac{1}{3}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}_{2304}$ | $\frac{1}{3}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{2304}$ |
| $\frac{1}{3}\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{2304}$ | $\frac{1}{5}\begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{pmatrix}_{2304}$ | $\frac{1}{3}\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}_{2304}$ |
| $\frac{1}{3}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}_{1152}$ | $\frac{1}{5}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & -1 & 0 & 2 & 0 & 0 \end{pmatrix}_{2304}$ | $\frac{1}{5}\begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 2 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix}_{2304}$ |
| $\frac{1}{3}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}_{2304}$ | $\frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}_{2304}$ | $\frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}_{2304}$ |
| $\frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{576}$ | – | – |

Table A.2: Effects of $\mathbb{H}^{[0]}_{(3,2)}$ up to relabelling. Only the first 248 effects are separable. Number beside an effect denotes the number of effects present in the class represented by that given effect.

# — B —

# Construction of the Effect Polytopes

## B.1 CONSTRUCTION OF THE EFFECT POLYTOPE OF $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$

The extremal effects of the local state space $\mathbb{H}^{[0]}_{(2,2)}$ can be categorised into 8 equivalence classes based on relabelling symmetries. A representative from each class is given below:

$$e_0 \coloneqq \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), e_{\mathrm{ClassI}} \coloneqq \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), e_{\mathrm{ClassII}} \coloneqq \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

,

$$e_{\mathrm{ClassIII}} \coloneqq \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), e_{\mathrm{ClassIV}} \coloneqq \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), e_{\mathrm{ClassV}} \coloneqq \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

,

$$e_{\mathrm{CH}_2} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right), \quad u = \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

One can generate the rest of the effects from each class by applying all relabelling symmetries followed by discarding duplicates. There are 16 effects of Class I, 8 of Class II, 32 of Class III, 8 of Class IV, 16 of Class V and 8 CH type effects. These constitute the 90 extremal effects of the local state space. Note that effects of Class I have their complimentary pairs in Class V, for instance, $e_{\mathrm{ClassI}} + e_{\mathrm{ClassV}} = u$. For the rest of the classes, the complementary effects are self-contained in each class, apart from the zero effect $e_0$ whose complementary effect is the unit effect $u$. Further, when the CH type effects are removed from this list, one gets the extremal BW effects introduced in 2.4.2. In reverse, one can think that when all the PR boxes are added to $\mathbb{H}^{[0]}_{(2,2)}$, many effects of $\mathcal{E}_{\mathbb{H}^{[0]}_{(2,2)}}$ will give an inner product outside the interval $[0,1]$ and therefore cannot be deemed as valid effects of the BW state space $\mathbb{H}^{[8]}_{(2,2)}$. Upon removing those local effects the resultant effect space can be described by the convex hull of all but the CH type extremal effects of $\mathcal{E}_{\mathbb{H}^{[0]}_{(2,2)}}$.

| | Class I | | Class II | | Class III | | |
|---|---|---|---|---|---|---|---|
| $\langle\cdot,\mathrm{PR}_{2,\alpha}\rangle$ | $\frac{1-\alpha}{4}$ | $\frac{1+\alpha}{4}$ | $\frac{1-\alpha}{2}$ | $\frac{1+\alpha}{2}$ | $\frac{1-\alpha}{2}$ | $\frac{1+\alpha}{2}$ | $\frac{1}{2}$ |
| Count | 8 | 8 | 4 | 4 | 8 | 8 | 16 |

| | Class IV | Class V | | Class CH | | |
|---|---|---|---|---|---|---|
| $\langle\cdot,\mathrm{PR}_{2,\alpha}\rangle$ | $\frac{1}{2}$ | $\frac{3-\alpha}{4}$ | $\frac{3+\alpha}{4}$ | $\frac{1}{2}$ | $\frac{1+2\alpha}{2}$ | $\frac{1-2\alpha}{2}$ |
| Count | 8 | 8 | 8 | 6 | 1 | 1 |

Table B.1: Inner products between $\mathrm{PR}_{2,\alpha}$ and the extremal effects of $\mathbb{H}^{[0]}_{(2,2)}$ apart from the zero and the unit effect. When $\alpha > 1/2$, two CH type effects give inner products outside the interval $[0,1]$. All remaining extremal effects give inner products inside the interval $[0,1]$.

Here, we take a similar approach to find the extremal effects of the state space $\mathbb{H}^{[1]}_{\alpha(2,2)}$. In particular, the following sequence of steps are followed:

- **Step 1:** Consider the hyperplanes $\langle\mathbf{x},\mathrm{PR}_{2,\alpha}\rangle = 0$ and $\langle\mathbf{x},\mathrm{PR}_{2,\alpha}\rangle = 1$ and define

$$\mathcal{E}_{disc} := \left\{ e \in \text{Extreme}\left[\mathcal{E}_{\mathbb{H}^{[0]}_{(2,2)}}\right] \mid \langle e, \mathrm{PR}_{2,\alpha}\rangle \notin [0,1] \right\}$$

- **Step 2:** For each $e \in \mathcal{E}_{disc}$, construct a line segment $l_{e,w,f} := we + (1-w)f$, where $f \in \text{Extreme}\left[\mathcal{E}_{\mathbb{H}^{[0]}_{(2,2)}}\right]$. For each $l_{e,w,f}$, calculate $w'$ such that when $w = w'$ either $\langle l_{e,w',f}, \mathrm{PR}_{2,\alpha}\rangle = 0$ or $\langle l_{e,w',f}, \mathrm{PR}_{2,\alpha}\rangle = 1$ and store $l_{e,w',f}$ in $\mathcal{E}_0$ or $\mathcal{E}_1$ respectively.
- **Step 3:**
  1. Select an element from $\mathcal{E}_{0/1}$ and represent it as a convex combination of other elements from that set.
  2. If not possible, construct a hyperplane that separates this element from the rest of the set, showing that it is extremal.
- **Step 4:** Take the union of extremal elements of $\mathcal{E}_0, \mathcal{E}_1$ and the effects $\text{Extreme}\left[\mathcal{E}_{\mathbb{H}^{[0]}_{(2,2)}}\right] \setminus \mathcal{E}_{disc}$

After running these steps, we found that the extremal effects in the set $\mathcal{E}_1$ up to equivalence of relabelling symmetries are

$$e_1(\text{Type 1}) := \frac{1-\alpha}{\alpha} e_{\mathrm{CH}_2} + \left(1 - \frac{1-\alpha}{\alpha}\right) \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$e_{1'}(\text{Type 1}') := \frac{1-\alpha}{\alpha} e_{\mathrm{CH}_2} + \left(1 - \frac{1-\alpha}{\alpha}\right) \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$e_2(\text{Type 2}) \coloneqq \frac{1-\alpha}{3\alpha-1} e_{\mathrm{CH}_2} + \left(1 - \frac{1-\alpha}{3\alpha-1}\right) \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

$$e_3(\text{Type 3}) \coloneqq \frac{3-\alpha}{3\alpha+1} e_{\mathrm{CH}_2} + \left(1 - \frac{3-\alpha}{3\alpha+1}\right) \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\text{and} \qquad e_4(\text{Type 4}) \coloneqq 2/(1+2\alpha) e_{\mathrm{CH}_2};$$

The rest of the extremal effects can be found by applying all relabelling symmetries on each of the ones presented above and checking that the inner product with $\mathrm{PR}_{2,\alpha}$ is 1. The extremal effects in the set $\mathcal{E}_0$, up to equivalence of relabelling symmetries, are the complementary effects of the effects in $\mathcal{E}_1$. We present a working below with more details.

**Step 1:**

The only extremal effects of $\mathcal{E}_{\mathbb{H}^{[0]}_{(2,2)}}$ that give an inner product outside the interval $[0,1]$ with $\mathrm{PR}_{2,\alpha}$ are two CH type effects, see Table B.1. In particular,

$$e_{\mathrm{CH}_2} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \quad \text{and} \quad e'_{\mathrm{CH}_2} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right).$$

**Step 2:**

The addition of $\mathrm{PR}_{2,\alpha}$ introduces two hyperplanes through the polytope $\mathcal{E}_{\mathbb{H}^{[0]}_{(2,2)}}$, given by $\langle \mathbf{x}.\mathrm{PR}_{2,\alpha}\rangle = 0$ and $\langle \mathbf{x}.\mathrm{PR}_{2,\alpha}\rangle = 1$. The set $\mathcal{E}_{\mathbb{H}^{[1]}_{\alpha(2,2)}}[\mathrm{PR}_2]$ can be characterised as:

$$\mathcal{E}_{\mathbb{H}^{[1]}_{\alpha(2,2)}} = \left\{ e \in \mathcal{E}_{\mathbb{H}^{[0]}_{(2,2)}} \mid 0 \leqslant \langle e, \mathrm{PR}_{2,\alpha}\rangle \leqslant 1 \right\} \tag{B.1}$$

i.e., the set confined in the inner half-spaces of the two hyperplanes above. The extremal effects of this polytope can be collected in two groups based on whether they are lying on the hyperplanes or not. When $\alpha \in (1/2, 1)$, none of the extremal effects from Table B.1 lie on the hyperplanes and hence are extremal. To find the extremal effects lying on the hyperplanes, one can draw line segments between $e_{\mathrm{CH}_2}/e'_{\mathrm{CH}_2}$ and effects lying inside the hyperplanes and find all the points of intersection between the line segments and the hyperplanes. One can then find the convex hull of the set of intersection points on each hyperplane and find which points are extremal. The union of the extremal points found in this manner from each hyperplane constitute the remaining extremal effects of $\mathcal{E}_{\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]}$.

For the hyperplane $\langle \mathbf{x}.\mathrm{PR}_{2,\alpha}\rangle = 1$, we consider line segments of the form $w(\alpha)e_{\mathrm{CH}_2} + (1-w(\alpha))f$ where $f$ is an extremal local effect and calculate the weight $w(\alpha)$ such that the corresponding effect will lie on the hyperplane. Table B.2 summarises these weights alongside the extremal local effects such that the corresponding effect will lie on the hyperplane.

| | $\frac{1-\alpha}{\alpha}$ | $\frac{1-\alpha}{3\alpha-1}$ | $\frac{3-\alpha}{3\alpha+1}$ | $\frac{2}{1+2\alpha}$ | $\frac{3+\alpha}{1+5\alpha}$ | $\frac{1+\alpha}{3\alpha}$ | $\frac{1}{2\alpha}$ | $\frac{1+\alpha}{5\alpha-1}$ | 0 |
|---|---|---|---|---|---|---|---|---|---|
| Class I | – | – | 8 | – | 8 | – | – | – | – |
| Class II | 4 | – | – | – | – | 4 | – | – | – |
| Class III | 8 | – | – | – | – | 8 | 16 | – | – |
| Class IV | – | – | – | – | – | – | 8 | – | – |
| Class V | – | 8 | – | – | – | – | – | 8 | – |
| Class CH | – | – | – | – | – | – | 6 | – | – |
| zero | – | – | – | 1 | – | – | – | – | – |
| unit | – | – | – | – | – | – | – | – | 1 |

Table B.2: Table summaries weights $w(\alpha)$ on $e_{\mathrm{CH}_2}$ such that $\langle w(\alpha)e_{\mathrm{CH}_2} + (1-w(\alpha))f, \mathrm{PR}_{2,\alpha}\rangle = 1$, where $f$ is an extremal local effect. The numbers denote how many extremal local effects combine with $e_{\mathrm{CH}_2}$ with the corresponding weight to generate an effect on the hyperplane $\langle \mathbf{x}.\mathrm{PR}_{2,\alpha}\rangle = 1$.

**Step 3.1:**

Not all the effects deduced from this procedure are extremal. We found that all the effects corresponding to weights $(3+\alpha)/(1+5\alpha), (1+\alpha)/3\alpha, 1/2\alpha$ and $(1+\alpha)/(5\alpha-1)$ can be written as a convex sum of effects obtained from the first four rows of Table B.2. From each weight and class combination we pick one effect and show their convex decompositions below. To read the equalities below, recall that two effects are identical if the list of inner products it gives with the list of all extremal states in the state space are equal. For each of the effects appearing on the left and right of the equality, this is the case. To make the representation identical as well, one may apply non signalling moves on the effects. The following equalities, therefore, are up to non-signalling moves and should be read as such.

$$\begin{aligned}\frac{3+\alpha}{1+5\alpha}e_{\mathrm{CH}_2} + \frac{4\alpha-2}{1+5\alpha}e_{\mathrm{ClassI}} \overset{\mathrm{ns}}{=} \frac{2\alpha}{1+5\alpha}&\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha-1}{\alpha} & 0 & \frac{1-\alpha}{\alpha} \\ \hline 1-\frac{1-\alpha}{\alpha} & 0 & \frac{1-\alpha}{\alpha} & 0 \\ 0 & \frac{1-\alpha}{\alpha} & 0 & 1-\frac{1-\alpha}{\alpha}\end{array}\right) \\ +\frac{3\alpha+1}{5\alpha+1}&\left(\begin{array}{cc|cc} 0 & 0 & 1-\frac{3-\alpha}{3\alpha+1} & 0 \\ 0 & \frac{\alpha-3}{3\alpha+1} & 0 & \frac{3-\alpha}{3\alpha+1} \\ \hline 0 & 0 & \frac{3-\alpha}{3\alpha+1} & 0 \\ 0 & \frac{3-\alpha}{3\alpha+1} & 0 & 0\end{array}\right)\end{aligned} \tag{B.2}$$

To see that the effects on the right indeed arise from the first four rows, notice that

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha-1}{\alpha} & 0 & \frac{1-\alpha}{\alpha} \\ \hline 1-\frac{1-\alpha}{\alpha} & 0 & \frac{1-\alpha}{\alpha} & 0 \\ 0 & \frac{1-\alpha}{\alpha} & 0 & 1-\frac{1-\alpha}{\alpha}\end{array}\right) = \frac{1-\alpha}{\alpha}e_{\mathrm{CH}_2} + \left(1-\frac{1-\alpha}{\alpha}\right)\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1\end{array}\right) \quad \text{and}$$

$$\left(\begin{array}{cc|cc} 0 & 0 & 1-\frac{3-\alpha}{3\alpha+1} & 0 \\ 0 & \frac{\alpha-3}{3\alpha+1} & 0 & \frac{3-\alpha}{3\alpha+1} \\ \hline 0 & 0 & \frac{3-\alpha}{3\alpha+1} & 0 \\ 0 & \frac{3-\alpha}{3\alpha+1} & 0 & 0 \end{array}\right) = \frac{3-\alpha}{3\alpha+1} e_{\mathrm{CH}_2} + \left(1-\frac{3-\alpha}{3\alpha+1}\right)\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

where the local effect with two 1s and all 0s is from Class III and the one with a single 1 all 0s is from Class I. For the remaining, we omit this expanding for it can be noticed upon close inspection.

$$\frac{1+\alpha}{\alpha} e_{\mathrm{CH}_2} + \frac{2\alpha-1}{3\alpha} e_{\mathrm{ClassII}} \overset{\text{ns}}{=} \frac{1}{3}\left(\begin{array}{cc|cc} 0 & 0 & 1-\frac{1-\alpha}{\alpha} & 0 \\ 0 & \frac{\alpha-1}{\alpha} & 0 & 1 \\ \hline 0 & 0 & \frac{1-\alpha}{\alpha} & 0 \\ 0 & \frac{1-\alpha}{\alpha} & 0 & 0 \end{array}\right) + \frac{1}{3}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha-1}{\alpha} & 0 & \frac{1-\alpha}{\alpha} \\ \hline 1-\frac{1-\alpha}{\alpha} & 0 & \frac{1-\alpha}{\alpha} & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) + \frac{1}{3}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha-1}{\alpha} & 0 & \frac{1-\alpha}{\alpha} \\ \hline 0 & 0 & 1 & 0 \\ 0 & \frac{1-\alpha}{\alpha} & 0 & 1-\frac{1-\alpha}{\alpha} \end{array}\right) \tag{B.3}$$

$$\frac{1+\alpha}{\alpha} e_{\mathrm{CH}_2} + \frac{2\alpha-1}{3\alpha} e_{\mathrm{ClassIII}} \overset{\text{ns}}{=} \frac{1}{3}\left(\begin{array}{cc|cc} 0 & 1-\frac{1-\alpha}{\alpha} & 0 & 0 \\ 0 & \frac{\alpha-1}{\alpha} & 0 & \frac{1-\alpha}{\alpha} \\ \hline 1-\frac{1-\alpha}{\alpha} & 0 & \frac{1-\alpha}{\alpha} & 0 \\ 0 & \frac{1-\alpha}{\alpha} & 0 & 0 \end{array}\right) + \frac{1}{3}\left(\begin{array}{cc|cc} 0 & 0 & 1-\frac{1-\alpha}{\alpha} & 0 \\ 0 & \frac{\alpha-1}{\alpha} & 0 & 1 \\ \hline 0 & 0 & \frac{1-\alpha}{\alpha} & 0 \\ 0 & \frac{1-\alpha}{\alpha} & 0 & 0 \end{array}\right) + \frac{1}{3}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha-1}{\alpha} & 0 & \frac{1-\alpha}{\alpha} \\ \hline 0 & 0 & 1 & 0 \\ 0 & \frac{1-\alpha}{\alpha} & 0 & 1-\frac{1-\alpha}{\alpha} \end{array}\right) \tag{B.4}$$

$$\frac{1}{2\alpha} e_{\mathrm{CH}_2} + \frac{2\alpha-1}{2\alpha}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{\text{ns}}{=} \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1-\frac{1-\alpha}{\alpha} & 0 & 0 \\ 1-\frac{1-\alpha}{\alpha} & \frac{\alpha-1}{\alpha} & 0 & \frac{1-\alpha}{\alpha} \\ \hline 0 & 0 & \frac{1-\alpha}{\alpha} & 0 \\ 0 & \frac{1-\alpha}{\alpha} & 0 & 0 \end{array}\right) + \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha-1}{\alpha} & 0 & \frac{1-\alpha}{\alpha} \\ \hline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \tag{B.5}$$

$$\frac{1}{2\alpha}e_{\mathrm{CH}_2}+\frac{2\alpha-1}{2\alpha}e_{\mathrm{ClassIV}} \overset{\mathrm{ns}}{=} \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1-\frac{1-\alpha}{\alpha} & 0 & 0 \\ 0 & \frac{\alpha-1}{\alpha} & 0 & \frac{1-\alpha}{\alpha} \\ \hline 1-\frac{1-\alpha}{\alpha} & 0 & \frac{1-\alpha}{\alpha} & 0 \\ 0 & \frac{1-\alpha}{\alpha} & 0 & 0 \end{array}\right) + \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 0 & 1-\frac{1-\alpha}{\alpha} & 0 \\ 0 & \frac{\alpha-1}{\alpha} & 0 & \frac{1-\alpha}{\alpha} \\ \hline 0 & 0 & \frac{1-\alpha}{\alpha} & 0 \\ 0 & \frac{1-\alpha}{\alpha} & 0 & 1-\frac{1-\alpha}{\alpha} \end{array}\right) \tag{B.6}$$

$$\frac{1}{2\alpha}e_{\mathrm{CH}_2}+\frac{2\alpha-1}{2\alpha}e_{\mathrm{CH}_1} \overset{\mathrm{ns}}{=} \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 0 & 1-\frac{1-\alpha}{\alpha} & 0 \\ 0 & \frac{\alpha-1}{\alpha} & 0 & 1 \\ \hline 0 & 0 & \frac{1-\alpha}{\alpha} & 0 \\ 0 & \frac{1-\alpha}{\alpha} & 0 & 0 \end{array}\right) + \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha-1}{\alpha} & 0 & \frac{1-\alpha}{\alpha} \\ \hline 1-\frac{1-\alpha}{\alpha} & 0 & \frac{1-\alpha}{\alpha} & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \tag{B.7}$$

$$\frac{1+\alpha}{5\alpha-1}e_{\mathrm{CH}_2}+\frac{4\alpha-2}{5\alpha-1}\left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \overset{\mathrm{ns}}{=} \frac{3\alpha-1}{5\alpha-1}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha-1}{3\alpha-1} & 0 & \frac{1-\alpha}{3\alpha-1} \\ \hline 1-\frac{1-\alpha}{3\alpha-1} & 1-\frac{1-\alpha}{3\alpha-1} & \frac{1-\alpha}{3\alpha-1} & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) + \left(1-\frac{3\alpha-1}{5\alpha-1}\right)\left(\begin{array}{cc|cc} 0 & 0 & 1-\frac{1-\alpha}{\alpha} & 0 \\ 0 & \frac{\alpha-1}{\alpha} & 0 & \frac{1-\alpha}{\alpha} \\ \hline 0 & 0 & \frac{1-\alpha}{\alpha} & 0 \\ 0 & \frac{1-\alpha}{\alpha} & 0 & 1-\frac{1-\alpha}{\alpha} \end{array}\right) \tag{B.8}$$

Similar decompositions are possible for every other element arising from combinations of the last four non-zero weights and the various classes.

**Step 3.2:**

The effects arising from the first four rows, we claim, are extremal. First, let us collect them by rows and call them Type 1 for weight $\frac{1-\alpha}{\alpha}$, Type 2 for weight $\frac{1-\alpha}{3\alpha-1}$, Type 3 for weight $\frac{3-\alpha}{3\alpha+1}$ and Type 4 for weight $\frac{2}{1+2\alpha}$. Notice that when $\alpha = 1$, the Type 3 effects correspond to the noisy couplers and the Type 4 effect corresponds to the pure coupler. Now, to show that these effects are extremal, first recall that if a point on a polytope is extremal, the shape of the polytope will change if that point is removed and the new polytope is constructed from the convex hull of the remaining vertices. In essence, there will be a supporting hyperplane corresponding to a face of this new polytope that will witness the removed point (hyperplane separation theorem). For our purpose, we first collect all the effects from Table B.1 that satisfy $0 \leqslant \langle \tilde{e}, \mathrm{PR}_{2,\alpha} \rangle \leqslant 1$ and all the effects generated from Table B.2 and their complimentary effects lying on the hyperplane $\langle \mathbf{x}, \mathrm{PR}_{2,\alpha} \rangle = 0$. From this collection, we remove one effect from Type 1 and then perform a facet enumeration on the reduced set. This gives us a list of inequalities corresponding to the face defining supporting hyperplanes of the reduced polytope. We then filter out the hyperplane that witnesses the removed effect. For instance, consider

$$e_1 = \frac{1-\alpha}{\alpha} e_{\mathrm{CH}_2} + \left(1 - \frac{1-\alpha}{\alpha}\right) \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{and}$$

$$W_1 := -\frac{1}{2\alpha - 6} \left(\begin{array}{cc|cc} (\alpha-1) & -(\alpha-3) & (\alpha+1) & -(\alpha-1) \\ -(\alpha-3) & 3(\alpha-1) & -(\alpha-1) & 5\alpha-3 \\ \hline (\alpha+1) & -(\alpha-1) & -(\alpha-3) & (\alpha-1) \\ (\alpha-1) & (\alpha+1) & -3(\alpha-1) & 3\alpha-1 \end{array}\right);$$

one can verify that every effect, $f_1$, in the reduced polytope obtained after removing $e_1$, satisfies $\langle f_1.W_1 \rangle \leqslant 1$. However,

$$\langle e_1.W_1 \rangle = \frac{3\alpha(\alpha-2)+1}{\alpha(\alpha-3)}$$

which is 1 when $\alpha = 1/2$ or 1 but greater than one in the open interval $(1/2, 1)$. Since $e_1$ converges to $e_{\mathrm{CH}_2}$ as $\alpha \to 1/2$ and converges to the deterministic effect as $\alpha \to 1$, $W_1$ witnesses $e_1$. For a Type 2 effect, consider

$$e_2 = \frac{1-\alpha}{3\alpha-1} e_{\mathrm{CH}_2} + \left(1 - \frac{1-\alpha}{3\alpha-1}\right) \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{and}$$

$$W_2 := \frac{1}{22 - 10\alpha} \left(\begin{array}{cc|cc} 7-7\alpha & 7-\alpha & 3\alpha+3 & 8-8\alpha \\ 8-2\alpha & 4-4\alpha & 3-3\alpha & 8-2\alpha \\ \hline 9-3\alpha & 5-5\alpha & 7-\alpha & 5-5\alpha \\ 5-5\alpha & \alpha+5 & 7-7\alpha & 3\alpha+3 \end{array}\right)$$

with which, one gets $\langle f_2, W_2 \rangle \leqslant 1$ for any effect $f_2$ in the reduced polytope obtained after removing $e_2$ but

$$\langle e_2.W_2 \rangle = \frac{21\alpha^2 - 47\alpha + 14}{15\alpha^2 - 38\alpha + 11}$$

which is 1 when $\alpha = 1/2$ or 1 but bigger than one in the open interval $(1/2, 1)$. For a Type 3 effect, consider

$$e_3 = \frac{3-\alpha}{3\alpha+1} e_{\mathrm{CH}_2} + \left(1 - \frac{3-\alpha}{3\alpha+1}\right) \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{and}$$

$$W_3 := \frac{1}{4}\left(\begin{array}{cc|cc} -2 & 4 & \alpha+1 & 1-\alpha \\ 2\alpha-2 & -2 & -2 & 2\alpha-2 \\ \hline 2\alpha-2 & -2 & \alpha+1 & 1-\alpha \\ -2 & \alpha+1 & 1-\alpha & \alpha+1 \end{array}\right)$$

with which one has $\langle f_3.W_1\rangle \leqslant 1$ for any effect $f_3$ in the reduced polytope obtained after removing $e_3$ but

$$\langle e_3.W_3\rangle = \frac{(13-2\alpha)\alpha-1}{6\alpha+2}$$

which is 1 when $\alpha = 1/2$ but bigger than 1 otherwise. Finally, for Type 4, consider $e_4 = 2/(1+2\alpha)e_{\mathrm{CH}_2}$ and

$$W_4 := \left(\begin{array}{cc|cc} -\alpha-3 & \alpha-3 & \alpha-3 & -\alpha-3 \\ \alpha-3 & -\alpha-7 & -\alpha-3 & \alpha-3 \\ \hline \alpha-3 & -\alpha-11 & \alpha-3 & -\alpha-3 \\ -\alpha-3 & \alpha-3 & -\alpha-3 & \alpha-3 \end{array}\right);$$

one then has $\langle f_4.W_4\rangle \geqslant 0$ for any effect $f_4$ from the reduced polytope obtained after removing $e_4$ but

$$\langle e_4.W_4\rangle = 4 - \frac{8}{2\alpha+1}$$

which is 0 when $\alpha = 1/2$ but negative otherwise. We suppress the details of the rest of the witnesses for the remaining effects from these 4 types. For the hyperplane $\langle \mathbf{x}, \mathrm{PR}_{2,\alpha}\rangle = 0$ the extremal effects are exactly the complementary effects obtained above.

**Step 4:**

Taking the union we find that this effect polytope is the convex hull of 146 extremal effects. These include 82 BW effects, 6 CH type effects, 29 effects lying on the hyperplane $\langle \mathbf{x}, \mathrm{PR}_{2,\alpha}\rangle = 0$ and $\langle \mathbf{x}, \mathrm{PR}_{2,\alpha}\rangle = 1$ each.

## B.2 CONSTRUCTION OF THE EFFECT POLYTOPE OF $\mathbb{H}^{[2]}_{\alpha(2,2)}[\mathrm{PR}_{2,2'}]$

Next, take the state space $\mathbb{H}^{[2]}_{\alpha(2,2)}[\mathrm{PR}_{2,2'}]$ the state space characterised by the convex hull of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$ and the noisy PR box $\mathrm{PR}'_{2,\alpha}$. The addition of $\mathrm{PR}'_{2,\alpha}$ to $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$ introduces two hyperplanes through its effect polytope, given by $\left\langle \mathbf{x}, \mathrm{PR}'_{2,\alpha}\right\rangle = 1$ and $\left\langle \mathbf{x}, \mathrm{PR}'_{2,\alpha}\right\rangle = 0$. One can perform a similar analysis as shown in the previous section to check which of the extremal effects of the full effect polytope of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$ are still valid effects by ensuring that their inner products with $\mathrm{PR}'_{2,\alpha}$ is in the interval $[0,1]$. We found that the set of Type 1 and Type 1$'$ effects from the previous section are extremal and lie on the hyperplane $\left\langle \mathbf{x}, \mathrm{PR}'_{2,\alpha}\right\rangle = 0$. The only other class of extremal effect lying on this hyperplane is of the Type 2 form, a candidate of which is

$$\begin{pmatrix} \frac{2-4\alpha}{3\alpha-1} & \frac{2-4\alpha}{3\alpha-1} & 0 & 0 \\ 0 & -1 & 0 & \frac{1-\alpha}{3\alpha-1} \\ 0 & 0 & 1 & \frac{2-4\alpha}{1-3\alpha} \\ 0 & \frac{1-\alpha}{3\alpha-1} & \frac{2-4\alpha}{1-3\alpha} & \frac{2-4\alpha}{1-3\alpha} \end{pmatrix} = \frac{1-\alpha}{3\alpha-1}e_{\mathrm{CH}_2} + \frac{4\alpha-2}{3\alpha-1}\left(u - \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right);$$

More precisely, from Table B.3, we can ensure that the Type 1, Type 1′ and Type 2 effects of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$ are still valid effects but the effects in Type 3 and Type 4 are not. The extremal effects of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$ that are not lying on the hyperplanes $\langle \mathbf{x}, \mathrm{PR}_{2,\alpha}\rangle = 1$ and $\langle \mathbf{x}, \mathrm{PR}_{2,\alpha}\rangle = 0$ are still valid. To calculate the new effects generated on the hyperplanes $\langle \mathbf{x}, \mathrm{PR}'_{2,\alpha}\rangle = 1$ and $\langle \mathbf{x}, \mathrm{PR}'_{2,\alpha}\rangle = 0$, we can follow the algorithm described in the previous section. Note from Table B.3, that the Type 1 effects of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$ lie on the second hyperplane. Upon calculating all the effects lying on this hyperplane and filtering out the extremal effects as described in the previous section, we found that these Type 1 effects are still extremal. The remaining extremal effects are of the Type 2 form. In particular, they can be written as $(1-\alpha)/(3\alpha-1)e_{\mathrm{CH}_2} + (4\alpha-2)/(3\alpha-1)(u - f_V)$ where $f_V$ is a Class V effect with $\langle f_V, \mathrm{PR}_{2'}\rangle = 1$. Since there are 8 Class V effects that have an inner product of 1 with $\mathrm{PR}_{2'}$, there are 8 corresponding extremal effects of this form. The extremal effects on the hyperplane $\langle \mathbf{x}, \mathrm{PR}'_{2,\alpha}\rangle = 1$ can be calculated as the complements of the extremal effects on the hyperplane $\langle \mathbf{x}, \mathrm{PR}'_{2,\alpha}\rangle = 0$.

| | Type 1/1′ | Type 2 | Type 3 | Type 4 |
|---|---|---|---|---|
| $k(\alpha)$ | 0 | $\frac{1-2\alpha}{1-3\alpha}$ | $\frac{1-2\alpha}{3\alpha+1}$ | $\frac{1-2\alpha}{1+2\alpha}$ |
| $k(1/2)$ | 0 | 0 | 0 | 0 |
| $k(1)$ | 0 | 1/2 | −1/4 | −1/3 |

Table B.3: Inner product between extremal vectors $\tilde{e}$ from the four types and $\mathrm{PR}'_{2,\alpha}$. Here $k(\alpha) = \langle \tilde{e}, \mathrm{PR}'_{2,\alpha}\rangle$. Notice that Type 3 and Type 4 effects of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$ are no longer valid effects of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_{22'}]$ because of the negative inner product.

## B.3 CONSTRUCTION OF THE EFFECT POLYTOPE OF $\mathbb{H}^{[2]}_{\alpha(2,2)}[\mathrm{PR}_{1,2}]$

| | Type 1 ∪ Type 1′ | Type 2 | Type 3 | Type 4 |
|---|---|---|---|---|
| $\{k(\alpha)\}$ | $\left\{1-\alpha, \alpha, \frac{1}{2}\right\}$ | $\left\{\frac{\alpha^2-3\alpha+1}{1-3\alpha}, \frac{\alpha^2+2\alpha-1}{3\alpha-1}\right\}$ | $\left\{\frac{-\alpha^2+\alpha+1}{3\alpha+1}, \frac{\alpha^2+1}{3\alpha+1}\right\}$ | $\frac{1}{2\alpha+1}$ |
| $\{k(1/2)\}$ | 1/2 | 1/2 | 1/2 | 1/2 |
| $\{k(1)\}$ | $\{0, 1, 1/2\}$ | $\{1, 1/2\}$ | 1/4 | 1/3 |

Table B.4: Inner product between extremal vectors $\tilde{e}$ from the four types and $\mathrm{PR}_{1,\alpha}$. Here $k(\alpha) = \langle \tilde{e}, \mathrm{PR}_{1,\alpha}\rangle$ and the set $\{k(\alpha)\}$ runs over all effects from a given type. Since all the inner products are between 0 and 1 in the range $1/2 \leqslant \alpha \leqslant 1$, all extremal effects of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$ are also effects of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_{12}]$.

Next, let us consider the second state space $\mathbb{H}^{[2]}_{\alpha(2,2)}[\mathrm{PR}_{12}]$ where the two noisy PR boxes are not isotropically opposite to each other. Following the previous analysis, we construct Table B.4 to check the inner product between the extremal effects of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$ lying on the hyperplane $\langle \mathbf{x}, \mathrm{PR}_{2,\alpha}\rangle = 1$ and $\mathrm{PR}_{1,\alpha}$. From this table, it is clear that all the extremal effects of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$

have an inner product between 0 and 1 in the range $1/2 \leqslant \alpha \leqslant 1$ and therefore are valid effects of $\mathbb{H}^{[2]}_{\alpha(2,2)}[\mathrm{PR}_{12}]$ and in fact extremal and similarly the complementary effects. However, the effects of the local polytope $e_{\mathrm{CH}_1}$ and $e_{\mathrm{CH}_{1'}}$ are no longer valid. One can use the algorithm from Section B.1 to find that effects of the form Type 1,2,3 and 4 are new extremal effects on the hyperplane $\langle \mathbf{x}, \mathrm{PR}'_{2,\alpha} \rangle = 1$ and their complimentary effects on the hyperplane $\langle \mathbf{x}, \mathrm{PR}'_{2,\alpha} \rangle = 0$. The effect polytope of $\mathbb{H}^{[2]}_{\alpha(2,2)}[\mathrm{PR}_{12}]$ can be calculated by separately constructing the effect polytopes of $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_1]$ and $\mathbb{H}^{[1]}_{\alpha(2,2)}[\mathrm{PR}_2]$, taking their union and then discarding the extremal effects whose inner products with either $\mathrm{PR}_1$ or $\mathrm{PR}_2$ is outside the interval $[0, 1]$. In particular, these effects are $e_{\mathrm{CH}_1}, e'_{\mathrm{CH}_1}, e_{\mathrm{CH}_2}$ and $e'_{\mathrm{CH}_2}$.

# — C —

# Pure Couplers of $\mathbb{H}^{[1]}_{(3,2)}[\mathrm{N}_1]$

Class representatives of pure couplers. Subscripts denote the number of elements from each class.

$$\left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ \hline 0 & -\frac{2}{3} & 0 & 0 & 1 & \frac{2}{3} \\ 0 & -\frac{1}{3} & 0 & 0 & 1 & \frac{1}{3} \end{array}\right)_8, \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{2}{3} & 0 & 0 \\ \hline 0 & -\frac{2}{3} & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & -\frac{2}{3} & 0 & 0 & \frac{2}{3} & \frac{2}{3} \end{array}\right)_1, \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ \hline 0 & -\frac{1}{3} & 0 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{array}\right)_8$$

$$\left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ \hline 0 & -\frac{2}{3} & 0 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & -\frac{1}{3} & 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{array}\right)_8, \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & -\frac{1}{3} & \frac{1}{3} & 0 \\ \hline 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & -\frac{1}{3} & 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{array}\right)_{24}, \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{2}{3} & \frac{1}{3} & 0 \\ \hline 0 & -\frac{2}{3} & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & -\frac{2}{3} & 0 & 0 & \frac{2}{3} & \frac{2}{3} \end{array}\right)_8$$

$$\left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{2}{3} & 0 & 0 \\ \hline 0 & -\frac{2}{3} & 0 & 0 & 1 & \frac{2}{3} \\ 0 & -\frac{2}{3} & 0 & 0 & \frac{2}{3} & \frac{2}{3} \end{array}\right)_3, \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{2}{3} & 0 & 0 \\ \hline 0 & -\frac{1}{3} & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & -\frac{2}{3} & 0 & 0 & 1 & \frac{2}{3} \end{array}\right)_8, \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{2}{3} & 0 & 0 \\ \hline 0 & -\frac{2}{3} & 0 & 0 & 1 & \frac{2}{3} \\ 0 & -\frac{2}{3} & 0 & 0 & 1 & \frac{2}{3} \end{array}\right)_2$$

$$\left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{2}{3} & 0 & 0 \\ \hline 0 & -\frac{2}{3} & 0 & 0 & 1 & 1 \\ 0 & -\frac{2}{3} & 0 & 0 & 1 & \frac{2}{3} \end{array}\right)_1, \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} & -\frac{5}{6} & -\frac{2}{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{2}{3} & -\frac{1}{6} & 0 \\ \hline 0 & -\frac{5}{6} & 0 & 0 & 1 & \frac{5}{6} \\ 0 & -\frac{2}{3} & 0 & 0 & 1 & \frac{2}{3} \end{array}\right)_8, \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} & -\frac{5}{6} & -\frac{2}{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{2}{3} & 0 & 0 \\ \hline 0 & -\frac{5}{6} & 0 & -\frac{1}{6} & 1 & 1 \\ 0 & -\frac{2}{3} & 0 & 0 & 1 & \frac{2}{3} \end{array}\right)_8$$

$$\left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{2} & -\frac{2}{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{2}{3} & -\frac{1}{6} & 0 \\ 0 & -\frac{2}{3} & 0 & -\frac{1}{6} & 1 & \frac{5}{6} \\ \hline 0 & -\frac{2}{3} & 0 & 0 & \frac{5}{6} & \frac{2}{3} \end{array}\right)_8, \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{2} & -\frac{2}{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{2}{3} & 0 & 0 \\ \hline 0 & -\frac{5}{6} & 0 & -\frac{1}{6} & 1 & 1 \\ 0 & -\frac{2}{3} & 0 & 0 & \frac{5}{6} & \frac{2}{3} \end{array}\right)_8, \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{2}{3} & 0 & 0 \\ \hline 0 & -\frac{2}{3} & 0 & 0 & \frac{2}{3} & 1 \\ 0 & -\frac{2}{3} & 0 & 0 & 1 & \frac{2}{3} \end{array}\right)_1$$

# — D —

# Equivalence Classes of $m$ PR boxes

| 2-Roofs | | | | |
|---|---|---|---|---|
| Class | Description | Example | PS | Count |
| Class 1 | Isotropically opposite pairs | $\{\text{PR}_1, \text{PR}_1'\}$ | Yes | 4 |
| Class 2 | 1 party symmetric, 1 party asymmetric | $\{\text{PR}_1, \text{PR}_3\}$ | No | 16 |
| Class 3 | Party symmetric or asymmetric that are not isotropically opposite | $\{\text{PR}_1, \text{PR}_2\}$ | Yes | 8 |

Table D.1: Two Roofs

| 3-Roofs | | | | |
|---|---|---|---|---|
| Class | Description | Example | PS | Count |
| Class 1 | Two PR boxes isotropically opposite to each other. If they are party symmetric, the third is not and vice versa | $\{\text{PR}_1, \text{PR}_1', \text{PR}_3\}$ | No | 16 |
| Class 2 | Two PR boxes that are not isotropically opposite. If these two are party symmetric the third is party asymmetric and vice versa. | $\{\text{PR}_3, \text{PR}_4', \text{PR}_1\}$ | Yes | 32 |
| Class 3 | Either all party symmetric or party asymmetric. | $\{\text{PR}_1, \text{PR}_1', \text{PR}_2\}$ | Yes | 8 |

Table D.2: Three Roofs

| 4-Roofs | | | | |
|---|---|---|---|---|
| Class | Description | Example | PS | Count |
| Class 1 | Two isotropically opposite pairs. 1 pair party symmetric and 1 pair party asymmetric | $\{PR_1, PR'_1, PR_3, PR'_3\}$ | No | 4 |
| Class 2 | Three party asymmetric with one party symmetric/three party symmetric with one party asymmetric | $\{PR_3, PR_4, PR'_4, PR_1\}$ | No | 32 |
| Class 3 | 1 pair of party symmetric PR boxes and 1 pair of party asymmetric PR boxes. 1 pair isotropically opposite and 1 pair isotropically not opposite. | $\{PR_1, PR'_1, PR_3, PR'_4\}$ | Yes | 16 |
| Class 4 | 1 pair of party symmetric and 1 pair of party asymmetric. Pairs differ by the same diagonal block. | $\{PR_1, PR'_2, PR_3, PR'_4\}$ | Yes | 16 |
| Class 5 | 1 pair of party symmetric and 1 pair of party asymmetric. Pairs differ in different diagonal blocks. | $\{PR_1, PR_2, PR_3, PR'_4\}$ | Yes | 8 |
| Class 6 | All party symmetric/asymmetric | $\{PR_1, PR'_1, PR_2, PR'_2\}$ | Yes | 2 |

Table D.3: Four Roofs

| 5-Roofs | | | | |
|---|---|---|---|---|
| Class | Description | Example | PS | Count |
| Class 1 | 1 isotropically opposite pair party symmetric/asymmetric pair with three party asymmetric/symmetric | $\{PR_1, PR'_1, PR_3 PR'_3, PR_4\}$ | No | 16 |
| Class 2 | 1 isotropically non-opposite pair party symmetric/asymmetric pair with three party asymmetric/symmetric | $\{PR_1, PR_2, PR'_2, PR_3, PR'_4\}$ | Yes | 32 |
| Class 3 | All party symmetric/asymmetric and one party asymmetric/symmetric. | $\{PR_1, PR_3, PR'_3, PR_4, PR'_4\}$ | Yes | 8 |

Table D.4: Five Roofs

| 6-Roofs | | | | |
|---|---|---|---|---|
| Class | Description | Example | PS | Count |
| Class 1 | Three party symmetric PR boxes with three party asymmetric PR boxes | $\{\mathrm{PR}_1, \mathrm{PR}'_1, \mathrm{PR}_2, \mathrm{PR}_3\mathrm{PR}'_3, \mathrm{PR}_4\}$ | No | 16 |
| Class 2 | 1 isotropically opposite party symmetric/asymmetric pair with four party asymmetric/symmetric | $\{\mathrm{PR}_1, \mathrm{PR}'_1, \mathrm{PR}_3, \mathrm{PR}'_3, \mathrm{PR}_4, \mathrm{PR}'_4\}$ | Yes | 4 |
| Class 3 | 4 party asymmetric/symmetric PR boxes with two party symmetric/asymmetric PR boxes that are not isotropically opposite. or vice versa. | $\{\mathrm{PR}_1, \mathrm{PR}_2, \mathrm{PR}_3, \mathrm{PR}'_3, \mathrm{PR}_4, \mathrm{PR}'_4\}$ | Yes | 8 |

Table D.5: Six Roofs

| 7-Roofs | | | | |
|---|---|---|---|---|
| Class | Description | Example | PS | Count |
| Class 1 | All but one PR box | $\{\mathrm{PR}_1, \mathrm{PR}_2, \mathrm{PR}'_2, \mathrm{PR}_3, \mathrm{PR}'_3, \mathrm{PR}_4, \mathrm{PR}'_4\}$ | Yes | 8 |

Table D.6: Seven Roofs

# — E —

# Proof of Theorem 1

Let Extreme$[\mathcal{E}_{\mathbb{H}^{[m]}_{\alpha(2,2)}}]$ be the set of extremal effects of the effect polytope $\mathcal{E}_{\mathbb{H}^{[m]}_{\alpha(2,2)}}$ and $n$ denote the cardinality of Extreme$[\mathcal{E}_{\mathbb{H}^{[m]}_{\alpha(2,2)}}]$. Let us denote by $\underline{\mathbb{0}}$ and $\underline{\mathbb{1}}$ the vectors $(0\ 0\ \cdots 0)_{1\times n}$ and $(1\ 1\ \cdots 1)_{1\times n}$ respectively. Since the effects space is convex, any effect $e$ can be expressed as

$$e = \sum_{j=1}^{n} \mathrm{x}_j e_j \tag{E.1}$$

where $e_j$ is an extremal effect and $\mathrm{x}_j \in [0,1]$ such that $\sum_j^n \mathrm{x}_j = 1$. For $e$ to be minimally 2-preserving, we require that for any extremal effect $e_j$ and a pair of PR boxes $\mathrm{PR}_{k,\alpha}$ and $\mathrm{PR}_{l,\alpha}$ the inner product between $e_j$ and the sub-normalised state

$$\Phi_e(k,l) := \mathsf{id}^{(1,4)} \otimes e^{(2,3)} \left(\mathrm{PR}^{(1,2)}_{k,\alpha}, \mathrm{PR}^{(3,4)}_{l,\alpha}\right)$$

is non-negative. Since for every extremal effect $e_j$, the effect $u - e_j$ is also an extremal effect, this condition also implies that the above inner product cannot be more than 1. In other words, for an arbitrary pair of noisy PR boxes indexed by $(k,l)$, one requires that if $\underline{\mathbf{x}} \in \mathbb{R}^n_{\geqslant 0}$ represents the convex support of the effect $e$, then every entry of the vector,

$$M_{k,l}.\underline{\mathbf{x}}^T := \begin{pmatrix} \langle e_1, \Phi_{e_1}(k,l)\rangle & \langle e_1, \Phi_{e_2}(k,l)\rangle & \cdots & \langle e_1, \Phi_{e_n}(k,l)\rangle \\ \langle e_2, \Phi_{e_1}(k,l)\rangle & \langle e_2, \Phi_{e_2}(k,l)\rangle & \cdots & \langle e_2, \Phi_{e_n}(k,l)\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle e_n, \Phi_{e_1}(k,l)\rangle & \langle e_n, \Phi_{e_2}(k,l)\rangle & \cdots & \langle e_n, \Phi_{e_n}(k,l)\rangle \end{pmatrix}_{n\times n} . \begin{pmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \vdots \\ \mathrm{x}_n \end{pmatrix}_{n\times 1}, \tag{E.2}$$

must be non-negative. This can be viewed as the constraint:

$$-M_{k,l}.\underline{\mathbf{x}}^T \leqslant \underline{\mathbb{0}}^T \tag{E.3}$$

Additionally, since the vector $\underline{\mathbf{x}}$ represents the convex weights, one also needs the following convexity conditions to hold:

$$\underline{\mathbb{1}}.\underline{\mathbf{x}}^T \leqslant 1 \quad \text{and} \quad -\underline{\mathbb{1}}.\underline{\mathbf{x}}^T \leqslant -1. \tag{E.4}$$

With this one can define a *constraint* matrix $\mathbf{C}$ and a *bound* vector $\underline{\mathbf{b}}$ as:

$$\mathbf{C} := \begin{pmatrix} \underline{\mathbb{1}} \\ -\underline{\mathbb{1}} \\ -M_{1,1} \\ -M_{1,2} \\ \vdots \\ -M_{m,m-1} \\ -M_{m,m} \end{pmatrix}_{(m^2n+2)\times n} \quad \text{and} \quad \underline{\mathbf{b}} := \begin{pmatrix} 1 \\ -1 \\ \underline{\mathbb{0}}^T \\ \vdots \\ \underline{\mathbb{0}}^T \end{pmatrix}_{(m^2n+2)\times 1} \tag{E.5}$$

respectively. The effect $e$, if minimally 2- preserving, will satisfy $\mathbf{C}.\underline{\mathbf{x}}^T \leqslant \underline{\mathbf{b}}$. Next, since we are interested in finding minimally 2-preserving couplers, we would also like $e$ to satisfy

$$\mathrm{CHSH}_i\left[\tilde{\Phi}_e^{(2,3)}(\mathrm{PR}_{k,\alpha}, \mathrm{PR}_{l,\alpha})\right] > \frac{3}{4} \tag{E.6}$$

where $\mathrm{CHSH}_i$ is a CHSH game that can be won by an amount more than 3/4 by correlations obtained upon performing the fiducial measurements on the allowed noisy PR boxes and $\tilde{\Phi}_e^{(2,3)}(\mathrm{PR}_{k,\alpha}, \mathrm{PR}_{l,\alpha})$ is the normalised state $\Phi_e^{(2,3)}(\mathrm{PR}_{k,\alpha}, \mathrm{PR}_{l,\alpha})$. This is equivalent to requiring

$$\begin{aligned} &\mathrm{CHSH}_i\left[\Phi_e^{(2,3)}(\mathrm{PR}_{k,\alpha}, \mathrm{PR}_{l,\alpha})\right] > \frac{3}{4}\left\langle u, \Phi_e^{(2,3)}(\mathrm{PR}_{k,\alpha}, \mathrm{PR}_{l,\alpha})\right\rangle \\ \implies &\mathrm{CHSH}_i\left[\Phi_e^{(2,3)}(\mathrm{PR}_{k,\alpha}, \mathrm{PR}_{l,\alpha})\right] - \frac{3}{4}\left\langle u, \Phi_e^{(2,3)}(\mathrm{PR}_{k,\alpha}, \mathrm{PR}_{l,\alpha})\right\rangle > 0 \\ \implies &\left\langle \mathrm{CHSH}_i - \frac{3}{4}u, \Phi_e^{(2,3)}(\mathrm{PR}_{k,\alpha}, \mathrm{PR}_{l,\alpha})\right\rangle > 0 \\ \implies &\left\langle \mathrm{CHSH}_i - \frac{3}{4}u, \mathsf{id} \otimes \sum_j x_j e_j \otimes \mathsf{id}(\mathrm{PR}_{k,\alpha}, \mathrm{PR}_{l,\alpha})\right\rangle > 0 \\ \implies &\left(\left\langle \mathrm{CHSH}_i - \frac{3}{4}u, \Phi_{e_1}^{(2,3)}(k,l)\right\rangle, \cdots, \left\langle \mathrm{CHSH}_i - \frac{3}{4}u, \Phi_{e_n}^{(2,3)}(k,l)\right\rangle\right).\underline{\mathbf{x}}^T =: \underline{\mathbf{f}}_{k,l|i}.\underline{\mathbf{x}}^T > 0 \end{aligned} \tag{E.7}$$

We do not know whether there exists any effect at all such that for the choice of $k, l$ and $i$, $\underline{\mathbf{f}}_{k,l|i}.\underline{\mathbf{x}}^T > 0$. Therefore, one can alternatively look for a vector $\underline{\mathbf{x}}$ which maximises the value $\underline{\mathbf{f}}_{k,l|i}.\underline{\mathbf{x}}^T$. This can be done with the help of a Linear Program (LP) defined below:

$$\mathbb{P}_{k,l|i} := \begin{array}{ll} \underset{\underline{\mathbf{x}}\in\mathbb{R}^n}{\text{maximise:}} & \underline{\mathbf{f}}_{k,l|i}.\underline{\mathbf{x}}^T \\ \text{subject to:} & \mathbf{C}.\underline{\mathbf{x}}^T \leqslant \underline{\mathbf{b}} \\ & \underline{\mathbf{x}} \geqslant 0 \end{array} \tag{E.8}$$

The dual program is defined as:

$$\mathbb{D}_{k,l|i} := \begin{array}{ll} \underset{\underline{\mathbf{y}}\in\mathbb{R}^{|\underline{\mathbf{b}}|}}{\text{minimise:}} & \underline{\mathbf{b}}^T.\underline{\mathbf{y}} \\ \text{subject to:} & \mathbf{C}^T.\underline{\mathbf{y}} \geqslant \underline{\mathbf{f}}_{k,l|i}. \\ & \underline{\mathbf{y}} \geqslant 0 \end{array} \tag{E.9}$$

To prove that a minimally 2-preserving coupler exists, one needs to show that for at least one choice of $k', l'$ and $i'$,

$$\mathbb{P}_{k',l'|i'} = \mathbb{D}_{k',l'|i'} > 0. \tag{E.10}$$

Recall, that when $\alpha \leqslant 1/\sqrt{2}$, no extremal effects of party symmetric house state spaces are couplers. It therefore suffices to evaluate these LP pairs in the range $1/\sqrt{2} < \alpha \leqslant 1$. To analytically solve these primal and dual problems we have first solved them discretely and then interpolated the results. For each case, we considered the discretised effect polytope $\mathcal{E}_{\mathbb{H}^{[m]}_{\beta(2,2)}}$ where $\beta$ is discretely varied between 1/2 and 1 with a step-size 1/30. For every step-size we have then solved the primal and dual problem pairs and interpolated the corresponding primal and dual optimising vectors to find their analytic forms. We have then checked that these pair of vectors provide correct solutions to the analytic versions of the primal and dual problems and in addition, satisfy the analytic version of the constraints.

## E.1 2 ROOFS

There are 3 equivalent local relabelling classes of state spaces with 2 PR boxes. Out of these, party symmetric state spaces exist only in Class 1 and Class 3. However, since the PR boxes in Class 1 are isotropically opposite pairs, there are no couplers for this state space (see Section 6.2.2) and one therefore only needs to check for couplers for a state space in Class 2. In Table E.1, we focus on $\mathrm{CHSH}_1$ scores. The set of PR box pairs such that there exists an extremal effect in $\mathcal{E}_{\mathbb{H}^{[2]}_{\alpha(2,2)}[\mathrm{PR}_{2,3}]}$ for which a score of more than 3/4 can be achieved in the $\mathrm{CHSH}_1$ game are:

$$\{(\mathrm{PR}_{1,\alpha},\mathrm{PR}_{1,\alpha}),(\mathrm{PR}_{1,\alpha},\mathrm{PR}_{2,\alpha}),(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{1,\alpha})\}.$$

In the table below and all following tables, we will only consider pairs of noisy PR boxes for which such violations are possible using extremal effects. In addition we directly provide the effects as a convex combination of extremal effects and the vectors $\mathbb{C}^T.\mathbf{y}$. From this data one can figure the optimising vectors $\mathbf{x}$ and $\mathbf{y}$. For instance, the effect in the first row of Table E.1 is $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$. The underlying vectors $\mathbf{x}$ has all zero entries, with the exception of $\theta$ at the index position of $\mathrm{CH}_{1,\alpha}$ in (E.1), and $(1-\theta)$ at the index position of $u$ in (E.1). The corresponding entry of $\mathbb{C}^T.\mathbf{y}$ is $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$. Therefore underlying vector $\mathbf{y}$ has all zeros, with the exception of $\frac{1+2\alpha}{4}$ at the index corresponding to the column of $\mathbf{C}^T$, where the inner products of the sub-normalised states are taken with $e_{\mathrm{CH}_{1,\alpha}}$, in the block $-M_{2,2}$. This presentation style has been chosen to compress the data.

$$\theta := \frac{3\alpha(\alpha+1)}{4\alpha(\alpha+1)-2} \text{and } \theta' = \frac{2}{2\alpha^2+1}$$

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha},\mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{1,\alpha},\mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{2,\alpha},\mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |

Table E.1: $\mathrm{CHSH}_1$ (2 Roofs Class 3)

## E.2 3 ROOFS

### E.2.1 CLASS 2

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{3,4'}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,4'}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1}{2}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,4'}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{3,4'}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1}{2}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta' e_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{3,4'}}$ |

Table E.2: $\mathrm{CHSH}_1$ (3 Roofs Class 2)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,4'}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_2}\right]_{-M_{3,4'}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_2}\right]_{-M_{3,4'}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,4'}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta' e_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,3}}$ |

Table E.3: $\mathrm{CHSH}_3$ (3 Roofs Class 2)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta' e_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_2}\right]_{-M_{3,4'}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,4'}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_2}\right]_{-M_{3,4'}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,4'}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,3}}$ |

Table E.4: $\mathrm{CHSH}'_4$ (3 Roofs Class 2)

### E.2.2 Class 3

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |

Table E.5: $\mathrm{CHSH}_1$ (3 Roofs Class 3)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}'_{1,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}'_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |

Table E.6: $\mathrm{CHSH}'_1$ (3 Roofs Class 3)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{1,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |

Table E.7: $\mathrm{CHSH}_2$ (3 Roofs Class 3)

## E.3 4 ROOFS

### E.3.1 Class 3

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(PR_{1,\alpha}, PR_{3,\alpha})$ | $\theta e'_{CH_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{CH_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(PR_{1,\alpha}, PR'_{4,\alpha})$ | $\theta e_{CH_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{CH_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(PR_{3,\alpha}, PR_{1,\alpha})$ | $\theta e'_{CH_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{CH_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(PR_{3,\alpha}, PR'_{4,\alpha})$ | $\theta' e'_{CH_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{CH_2}\right]_{-M_{1,1}}$ |
| $(PR'_{4,\alpha}, PR_{1,\alpha})$ | $\theta e_{CH_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{CH_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(PR'_{4,\alpha}, PR_{3,\alpha})$ | $\theta' e_{CH_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{CH_2}\right]_{-M_{1,1}}$ |

Table E.8: $CHSH_1$ (4 Roofs Class 3)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(PR'_{1,\alpha}, PR_{3,\alpha})$ | $\theta e'_{CH_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{CH_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(PR'_{1,\alpha}, PR'_{4,\alpha})$ | $\theta e_{CH_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{CH_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(PR_{3,\alpha}, PR'_{1,\alpha})$ | $\theta e_{CH_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{CH_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(PR_{3,\alpha}, PR'_{4,\alpha})$ | $\theta' e_{CH_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{CH_2}\right]_{-M_{1,1}}$ |
| $(PR'_{4,\alpha}, PR'_{1,\alpha})$ | $\theta e_{CH_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{CH_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(PR'_{4,\alpha}, PR_{3,\alpha})$ | $\theta' e'_{CH_{2,}} + (1-\theta')u$ | $\frac{1}{2}\left[e_{CH_2}\right]_{-M_{1,1}}$ |

Table E.9: $CHSH'_1$ (4 Roofs Class 3)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(PR_{1,\alpha}, PR_{1,\alpha})$ | $\theta e'_{CH_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{CH_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $\left(PR_{1,\alpha}, PR'_{4,\alpha}\right)$ | $\theta' e'_{CH_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{CH_2}\right]_{-M_{1,1}}$ |
| $\left(PR'_{1,\alpha}, PR'_{1,\alpha}\right)$ | $\theta e'_{CH_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{CH_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $\left(PR'_{1,\alpha}, PR'_{4,\alpha}\right)$ | $\theta' e_{CH_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{CH_2}\right]_{-M_{1,1}}$ |
| $(PR_{3,\alpha}, PR_{3,\alpha})$ | $\theta e'_{CH_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{CH_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $\left(PR_{3,\alpha}, PR'_{4,\alpha}\right)$ | $\theta e_{CH_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{CH_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $\left(PR'_{4,\alpha}, PR_{1,\alpha}\right)$ | $\theta' e_{CH_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{CH_2}\right]_{-M_{1,1}}$ |
| $\left(PR'_{4,\alpha}, PR'_{1,\alpha}\right)$ | $\theta' e'_{CH_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{CH_2}\right]_{-M_{1,1}}$ |
| $\left(PR'_{4,\alpha}, PR_{3,\alpha}\right)$ | $\theta e_{CH_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{CH_{4,\alpha}}\right]_{-M_{1,2}}$ |

Table E.10: $CHSH_3$ (4 Roofs Class 3)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(PR_{1,\alpha}, PR_{1,\alpha})$ | $\theta e'_{CH_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{CH_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(PR_{1,\alpha}, PR_{3,\alpha})$ | $\theta' e_{CH_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{CH_2}\right]_{-M_{1,1}}$ |
| $\left(PR'_{1,\alpha}, PR'_{1,\alpha}\right)$ | $\theta e'_{CH_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{CH_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $\left(PR'_{1,\alpha}, PR_{3,\alpha}\right)$ | $\theta' e'_{CH_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{CH_2}\right]_{-M_{1,1}}$ |
| $(PR_{3,\alpha}, PR_{1,\alpha})$ | $\theta' e'_{CH_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{CH_2}\right]_{-M_{1,1}}$ |
| $\left(PR_{3,\alpha}, PR'_{1,\alpha}\right)$ | $\theta' e_{CH_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{CH_2}\right]_{-M_{1,1}}$ |
| $\left(PR_{3,\alpha}, PR'_{4,\alpha}\right)$ | $\theta e_{CH_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{CH_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $\left(PR'_{4,\alpha}, PR_{3,\alpha}\right)$ | $\theta e'_{CH_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{CH_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $\left(PR'_{4,\alpha}, PR'_{4,\alpha}\right)$ | $\theta e'_{CH_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{CH_{4,\alpha}}\right]_{-M_{1,2}}$ |

Table E.11: $CHSH'_4$ (4 Roofs Class 3)

### E.3.2 CLASS 4

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e'_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e'_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e'_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |

Table E.12: $\mathrm{CHSH}_1$ (4 Roofs Class 4)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e'_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e'_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |

Table E.13: $\mathrm{CHSH}'_2$ (4 Roofs Class 4)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e'_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e'_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{2,1}}$ |

Table E.14: $\mathrm{CHSH}_3$ (4 Roofs Class 4)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e'_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e'_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e'_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{2,1}}$ |

Table E.15: $\mathrm{CHSH}'_4$ (4 Roofs Class 4)

### E.3.3 Class 5

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\text{PR}_{1,\alpha}, \text{PR}_{1,\alpha})$ | $\theta e_{\text{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\text{PR}_{1,\alpha}, \text{PR}_{2,\alpha})$ | $\theta e_{\text{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\text{PR}_{1,\alpha}, \text{PR}_{3,\alpha})$ | $\theta e'_{\text{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\text{CH}_{4,\alpha}}\right]_{-M_{2,1}}$ |
| $(\text{PR}_{1,\alpha}, \text{PR}'_{4,\alpha})$ | $\theta e_{\text{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\text{PR}_{2,\alpha}, \text{PR}_{1,\alpha})$ | $\theta e_{\text{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\text{PR}_{2,\alpha}, \text{PR}_{3,\alpha})$ | $\theta e_{\text{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\text{PR}_{3,\alpha}, \text{PR}_{1,\alpha})$ | $\theta e'_{\text{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\text{CH}_{4,\alpha}}\right]_{-M_{2,1}}$ |
| $(\text{PR}_{3,\alpha}, \text{PR}_{3,\alpha})$ | $\theta e_{\text{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\text{PR}'_{4,\alpha}, \text{PR}_{1,\alpha})$ | $\theta e_{\text{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\text{PR}'_{4,\alpha}, \text{PR}_{2,\alpha})$ | $\theta e'_{\text{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\text{CH}_{4,\alpha}}\right]_{-M_{2,1}}$ |
| $(\text{PR}'_{4,\alpha}, \text{PR}_{3,\alpha})$ | $\theta e_{\text{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\text{PR}'_{4,\alpha}, \text{PR}'_{4,\alpha})$ | $\theta e_{\text{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |

Table E.16: $\text{CHSH}_1$ (4 Roofs Class 5)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\text{PR}_{1,\alpha}, \text{PR}_{2,\alpha})$ | $\theta e_{\text{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\text{PR}_{1,\alpha}, \text{PR}'_{4,\alpha})$ | $\theta e'_{\text{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\text{CH}_{4,\alpha}}\right]_{-M_{2,1}}$ |
| $(\text{PR}_{2,\alpha}, \text{PR}_{1,\alpha})$ | $\theta e_{\text{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\text{PR}_{2,\alpha}, \text{PR}_{2,\alpha})$ | $\theta e_{\text{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\text{PR}_{2,\alpha}, \text{PR}_{3,\alpha})$ | $\theta e'_{\text{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\text{CH}_{4,\alpha}}\right]_{-M_{2,1}}$ |
| $(\text{PR}_{2,\alpha}, \text{PR}'_{4,\alpha})$ | $\theta e_{\text{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\text{PR}_{3,\alpha}, \text{PR}_{1,\alpha})$ | $\theta e_{\text{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\text{PR}_{3,\alpha}, \text{PR}_{2,\alpha})$ | $\theta e'_{\text{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\text{CH}_{4,\alpha}}\right]_{-M_{2,1}}$ |
| $(\text{PR}_{3,\alpha}, \text{PR}_{3,\alpha})$ | $\theta e_{\text{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\text{PR}_{3,\alpha}, \text{PR}'_{4,\alpha})$ | $\theta e_{\text{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\text{PR}'_{4,\alpha}, \text{PR}_{2,\alpha})$ | $\theta e_{\text{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\text{PR}'_{4,\alpha}, \text{PR}'_{4,\alpha})$ | $\theta e_{\text{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\text{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |

Table E.17: $\text{CHSH}_2$ (4 Roofs Class 5)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{7,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |

Table E.18: $\mathrm{CHSH}_3$ (4 Roofs Class 5)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{7,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{2,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |

Table E.19: $\mathrm{CHSH}'_4$ (4 Roofs Class 5)

## E.4 5 ROOFS

### E.4.1 CLASS 2

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |

Table E.20: $\mathrm{CHSH}_1$ (5 Roofs Class 2)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,3}}$ |

Table E.21: $\mathrm{CHSH}_2$ (5 Roofs Class 2)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |

Table E.22: $\mathrm{CHSH}'_2$ (5 Roofs Class 2)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |

Table E.23: $\mathrm{CHSH}_3$ (5 Roofs Class 2)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e'_{\mathrm{CH}_{4,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e'_{\mathrm{CH}_{4,\alpha}}\right]_{-M_{1,3}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{3,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{3,\alpha}}\right]_{-M_{1,2}}$ |

Table E.24: $\mathrm{CHSH}'_4$ (5 Roofs Class 2)

### E.4.2 Class 3

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,3}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,3}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,3}}$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}_{4,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta' e_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $\theta' e_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,3}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta' e_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,3}}$ |

Table E.25: $\mathrm{CHSH}_1$ (5 Roofs Class 3)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,3}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{4,\alpha})$ | $\theta' e_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,3}}$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta' e_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |

Table E.26: $\mathrm{CHSH}_3$ (5 Roofs Class 3)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,3}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{4,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta' e_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,3}}$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta' e_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |

Table E.27: $\mathrm{CHSH}'_3$ (5 Roofs Class 3)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,3}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta' e_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,3}}$ |

Table E.28: $\mathrm{CHSH}_4$ (5 Roofs Class 3)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta' e_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,3}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta' e'_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e'_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta' e_{\mathrm{CH}_2} + (1-\theta')u$ | $\frac{1}{2}\left[e_{\mathrm{CH}_2}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,3}}$ |

Table E.29: $\mathrm{CHSH}'_4$ (5 Roofs Class 3)

## E.5 6 ROOFS

### E.5.1 Class 3

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |

Table E.30: $\mathrm{CHSH}_1$ (6 Roofs Class 3)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |

Table E.31: $\mathrm{CHSH}_2$ (6 Roofs Class 3)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |

Table E.32: $\mathrm{CHSH}_3$ (6 Roofs Class 3)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |

Table E.33: $\mathrm{CHSH}'_3$ (6 Roofs Class 3)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |

Table E.34: $\mathrm{CHSH}_4$ (6 Roofs Class 3)

| Pairs | Effects | $\mathbf{C}^T.\mathbf{y}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{1,\alpha}}\right]_{-M_{2,2}}$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{2,\alpha})$ | $\theta e_{\mathrm{CH}_{2,\alpha}} + (1-\theta)u$ | $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$ |

Table E.35: $\mathrm{CHSH}'_4$ (6 Roofs Class 3)

## E.6 7 ROOFS

The effects for each of these cases is $\theta e_{\mathrm{CH}_{1,\alpha}} + (1-\theta)u$ and every $\mathbf{C}^T.\mathbf{y}$ is $\frac{1+2\alpha}{4}\left[e_{\mathrm{CH}_{2,\alpha}}\right]_{-M_{1,1}}$.

| $\mathrm{CHSH}_1$ | $\mathrm{CHSH}_2$ | $\mathrm{CHSH}_{2'}$ | $\mathrm{CHSH}_3$ |
|---|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{1,\alpha})$ | $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{2,\alpha})$ | $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{3,\alpha})$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{1,\alpha})$ | $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}_{1,\alpha})$ | $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}'_{4,\alpha})$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}_{2,\alpha})$ | $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{4,\alpha})$ | $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}_{4,\alpha})$ |
| $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{3,\alpha})$ | $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}_{4,\alpha})$ | $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{2,\alpha})$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{4,\alpha})$ | $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{3,\alpha})$ | $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{2,\alpha})$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}'_{4,\alpha})$ | – | – | – |

| $\mathrm{CHSH}_{3'}$ | $\mathrm{CHSH}_4$ | $\mathrm{CHSH}_{4'}$ |
|---|---|---|
| $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}_{4,\alpha})$ | $(\mathrm{PR}_{1,\alpha}, \mathrm{PR}'_{4,\alpha})$ |
| $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{4,\alpha})$ | $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}_{3,\alpha})$ | $(\mathrm{PR}_{2,\alpha}, \mathrm{PR}'_{3,\alpha})$ |
| $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}'_{4,\alpha})$ | $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}'_{3,\alpha})$ | $(\mathrm{PR}'_{2,\alpha}, \mathrm{PR}_{3,\alpha})$ |
| $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}_{1,\alpha})$ | $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $(\mathrm{PR}_{3,\alpha}, \mathrm{PR}_{2,\alpha})$ |
| $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}'_{2,\alpha})$ | $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}_{2,\alpha})$ | $(\mathrm{PR}'_{3,\alpha}, \mathrm{PR}'_{2,\alpha})$ |
| $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{2,\alpha})$ | $(\mathrm{PR}_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ | $(\mathrm{PR}'_{4,\alpha}, \mathrm{PR}_{1,\alpha})$ |

Table E.36: Table describes pairs of noisy PR boxes relevant for the respective CHSH games.

## — F —

# Derivation of Hex State Space

| $\frac{\mathbb{1}+(\sigma_{\hat{X}}+\sigma_{\hat{Z}})/\sqrt{2}}{2}$ | $\frac{\mathbb{1}-(\sigma_{\hat{X}}+\sigma_{\hat{Z}})/\sqrt{2}}{2}$ | $\frac{\mathbb{1}+(\sigma_{\hat{X}}-\sigma_{\hat{Z}})/\sqrt{2}}{2}$ | $\frac{\mathbb{1}-(\sigma_{\hat{X}}-\sigma_{\hat{Z}})/\sqrt{2}}{2}$ | $\frac{\mathbb{1}+\sigma_{\hat{Z}}}{2}$ | $\frac{\mathbb{1}+(\sigma_{\hat{X}}+\sigma_{\hat{Y}})/\sqrt{2}}{2}$ | $\frac{\mathbb{1}-(\sigma_{\hat{X}}-\sigma_{\hat{Y}})/\sqrt{2}}{2}$ | $\frac{\mathbb{1}-\sigma_{\hat{Z}}}{2}$ |
|---|---|---|---|---|---|---|---|
| $\frac{1}{4}\left(\sqrt{2}r_x+\sqrt{2}r_z+2\right)$ | $\frac{1}{4}\left(-\sqrt{2}r_x-\sqrt{2}r_z+2\right)$ | $\frac{1}{4}\left(-\sqrt{2}r_x+\sqrt{2}r_z+2\right)$ | $\frac{1}{4}\left(\sqrt{2}r_x-\sqrt{2}r_z+2\right)$ | $\frac{r_z+1}{2}$ | $\frac{1}{4}\left(\sqrt{2}r_x+\sqrt{2}r_y+2\right)$ | $\frac{1}{4}\left(-\sqrt{2}r_x+\sqrt{2}r_y+2\right)$ | $\frac{1-r_z}{2}$ |

Table F.1: Inner products between the extremal effects of $\mathcal{E}_{\mathrm{C}}$ and the state (7.10).

One needs that every inner product listed in the entries of Table F.1 is between 0 and 1. This puts constraints on the ranges of the variables $r_x, r_y$ and $r_z$, which are provided in Table F.2 below. From the first two constraints, one gets the states

| |
|---|
| $r_x=-\sqrt{2}\wedge r_y=0\wedge r_z=0$ |
| $r_x=\sqrt{2}\wedge r_y=0\wedge r_z=0$ |
| $-\sqrt{2}<r_x\leqslant\frac{\sqrt{2}-2}{\sqrt{2}}\wedge\frac{-\sqrt{2}r_x-2}{\sqrt{2}}\leqslant r_y\leqslant\frac{\sqrt{2}r_x+2}{\sqrt{2}}\wedge\frac{-\sqrt{2}r_x-2}{\sqrt{2}}\leqslant r_z\leqslant\frac{\sqrt{2}r_x+2}{\sqrt{2}}$ |
| $\frac{\sqrt{2}-2}{\sqrt{2}}<r_x\leqslant 0\wedge\frac{-\sqrt{2}r_x-2}{\sqrt{2}}\leqslant r_y\leqslant\frac{\sqrt{2}r_x+2}{\sqrt{2}}\wedge -1\leqslant r_z\leqslant 1$ |
| $0<a\leqslant\frac{2-\sqrt{2}}{\sqrt{2}}\wedge\frac{\sqrt{2}a-2}{\sqrt{2}}\leqslant b\leqslant\frac{2-\sqrt{2}a}{\sqrt{2}}\wedge -1\leqslant c\leqslant 1$ |
| $\frac{2-\sqrt{2}}{\sqrt{2}}<r_x<\sqrt{2}\wedge\frac{\sqrt{2}r_x-2}{\sqrt{2}}\leqslant r_y\leqslant\frac{2-\sqrt{2}r_x}{\sqrt{2}}\wedge\frac{\sqrt{2}r_x-2}{\sqrt{2}}\leqslant r_z\leqslant\frac{2-\sqrt{2}r_x}{\sqrt{2}}$ |

Table F.2: Constraints on the variables $r_x, r_y$ and $r_z$ upon requiring every inner product listed in Table F.1 to be between 0 and 1.

From the third constraint onwards, one can set the inequalities to equalities, where possible and solve for the remaining states. We provide these states below:

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{1}{2}(r-i) \\ \frac{1}{2}(r+i) & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2}(r-i) \\ \frac{1}{2}(r+i) & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \frac{1}{2}(r+i) \\ \frac{1}{2}(r-i) & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2}(r+i) \\ \frac{1}{2}(r-i) & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \frac{1}{2}(-r-i) \\ \frac{1}{2}(-r+i) & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2}(-r-i) \\ \frac{1}{2}(-r+i) & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \frac{1}{2}(-r+i) \\ \frac{1}{2}(-r-i) & 0 \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2}(-r+i) \\ \frac{1}{2}(-r-i) & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}}i \\ -\frac{1}{\sqrt{2}}i & 1 \end{pmatrix}, \begin{pmatrix} 1 & \frac{1}{\sqrt{2}}i \\ -\frac{1}{\sqrt{2}}i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}}i & 1 \end{pmatrix}, \begin{pmatrix} 1 & -\frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}}i & 0 \end{pmatrix}$$

The group of first 4 states can be written as $\frac{\mathbb{1}\pm\sqrt{2}\sigma_{\hat{X}}}{2}$. The next 8 states can be written as $\frac{\mathbb{1}\pm\sqrt{2}\sigma_{\hat{Y}}\pm\sigma_{\hat{Z}}}{2}$. The last 4 states can be written as $\frac{\mathbb{1}\pm r\sigma_{\hat{X}}\pm\sigma_{\hat{Y}}\pm\sigma_{\hat{Z}}}{2}$.

# — G —

# Derivation of Non-Signalling Subspace of $\mathbb{S}_p^{\mathbf{A}\to\mathbf{B}}$

Next, we present the derivation of this subspace. Since in this model, Bob never signals to Alice, all distributions lie on the hyperplanes $\langle \mathrm{NS}_1, \mathbf{x}\rangle = 0$ and $\langle \mathrm{NS}_2, \mathbf{x}\rangle = 0$. Therefore, we only need to consider restrictions imposed by $\mathrm{NS}_3$ and $\mathrm{NS}_4$. We ran the steps above and calculated the set of intersection points on each of the two hyperplanes. In each case, we classified the distributions up to equivalence of relabelling operations. We first present a representative of the classes that we claim to be extremal and then present the representatives of the remaining classes as convex decompositions of the claimed extremals.

## G.1 RESTRICTION BY $\mathrm{NS}_3$:

We claim that there are three extremal distributions up to equivalence of relabelling symmetries;

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 1-p & p \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & p & 1-p \end{array}\right), \frac{1}{2}\left(\begin{array}{cc|cc} 1-p & p & 0 & 1 \\ p & 1-p & 1-p & p \\ \hline 1-p & p & p & 1-p \\ p & 1-p & 0 & 1 \end{array}\right), \frac{1}{2}\left(\begin{array}{cc|cc} 1-p & p & 1 & 0 \\ p & 1-p & 0 & 1 \\ \hline 1-p & p & 0 & 1 \\ p & 1-p & 1 & 0 \end{array}\right)$$

There are 16 elements in the first class, 32 in the second class and 8 in the third class. Elements of the rest of the classes are not extremal as they can be written as convex combinations of the distributions from the three classes above and the 16 non-signalling local deterministic distributions, as shown below.

$$\left(\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline \frac{1}{2} & \frac{1}{2} & 1-p & p \\ 0 & 0 & 0 & 0 \end{array}\right)_8 = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1-p & p \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & p \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)_8 = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

*G.1. Restriction by* $\mathrm{NS}_3$*:*

$$\left(\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 \\ \hline \frac{1}{2} & \frac{1}{2} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right)_{16} = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ \hline \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)_{4} = \frac{1}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \frac{1-p}{2} & \frac{p}{2} & \frac{1}{2} & 0 \\ \frac{p}{2} & \frac{1-p}{2} & \frac{1}{2} & 0 \end{array}\right)_{16} = \frac{1-p}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) + \frac{1-p}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$+\frac{p}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{p}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & \frac{1+p}{2} & \frac{1-p}{2} \\ 0 & 0 & 0 & 0 \\ \hline \frac{1-p}{2} & \frac{p}{2} & \frac{1}{2} & 0 \\ \frac{p}{2} & \frac{1-p}{2} & \frac{1-p}{2} & \frac{p}{2} \end{array}\right)_{32} = \frac{1-p}{2}\left(\begin{array}{cc|cc} 0 & 1 & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 1-p & p \end{array}\right) + \frac{1-p}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$+\frac{p}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{p}{2}\left(\begin{array}{cc|cc} 1 & 0 & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1-p & p \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & \frac{2-p}{2} & \frac{p}{2} \\ 0 & 0 & 0 & 0 \\ \hline \frac{1-p}{2} & \frac{p}{2} & \frac{1}{2} & 0 \\ \frac{p}{2} & \frac{1-p}{2} & \frac{p}{2} & \frac{1-p}{2} \end{array}\right)_{32} = \frac{1-p}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1-p & p \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & p & 1-p \end{array}\right) + \frac{1-p}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$
$$+\frac{p}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{p}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1-p & p \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & p & 1-p \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ \hline \frac{1-p}{2} & \frac{p}{2} & \frac{1}{2} & 0 \\ \frac{p}{2} & \frac{1-p}{2} & 0 & \frac{1}{2} \end{array}\right)_{16} = \frac{1-p}{2}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) + \frac{1-p}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$+\frac{p}{2}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{p}{2}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \frac{p}{2} & \frac{1-p}{2} & \frac{1}{2} & 0 \\ \frac{1-p}{2} & \frac{p}{2} & \frac{1}{2} & 0 \\ \hline \frac{1-p}{2} & \frac{p}{2} & \frac{1}{2} & 0 \\ \frac{p}{2} & \frac{1-p}{2} & \frac{1}{2} & 0 \end{array}\right)_{24} = \frac{1-p}{2}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{1-p}{2}\left(\begin{array}{cccc} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$+\frac{p}{2}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{p}{2}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \frac{1}{2} & \frac{1}{2} & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline \frac{1-p}{2} & \frac{p}{2} & \frac{1-p}{2} & \frac{p}{2} \\ \frac{p}{2} & \frac{1-p}{2} & \frac{1-p}{2} & \frac{p}{2} \end{array}\right)_{16} = \frac{1-p}{2}\left(\begin{array}{cc|cc} 0 & 1 & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 1-p & p \end{array}\right) + \frac{1-p}{2}\left(\begin{array}{cc|cc} 1 & 0 & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1-p & p \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$+\frac{p}{2}\left(\begin{array}{cc|cc} 0 & 1 & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1-p & p \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{p}{2}\left(\begin{array}{cc|cc} 1 & 0 & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1-p & p \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \frac{p}{2} & \frac{1-p}{2} & \frac{p}{2} & \frac{1-p}{2} \\ \frac{1-p}{2} & \frac{p}{2} & \frac{p}{2} & \frac{1-p}{2} \\ \hline \frac{1-p}{2} & \frac{p}{2} & \frac{1-p}{2} & \frac{p}{2} \\ \frac{p}{2} & \frac{1-p}{2} & \frac{1-p}{2} & \frac{p}{2} \end{array}\right)_{8} = \frac{1-p}{2}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & p & 1-p \\ \hline 1 & 0 & 1-p & p \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{1-p}{2}\left(\begin{array}{cc|cc} 0 & 1 & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 1-p & p \end{array}\right)$$

$$+\frac{p}{2}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & p & 1-p \\ \hline 0 & 1 & 1-p & p \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{p}{2}\left(\begin{array}{cc|cc} 1 & 0 & p & 1-p \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1-p & p \end{array}\right)$$

$$\left(\begin{array}{cc|cc} \frac{p}{2} & \frac{1-p}{2} & \frac{p}{2} & \frac{1-p}{2} \\ \frac{1-p}{2} & \frac{p}{2} & \frac{1-p}{2} & \frac{p}{2} \\ \hline \frac{1-p}{2} & \frac{p}{2} & \frac{1-p}{2} & \frac{p}{2} \\ \frac{p}{2} & \frac{1-p}{2} & \frac{p}{2} & \frac{1-p}{2} \end{array}\right)_{8} = \frac{1-p}{2}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{1-p}{2}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$+\frac{p}{4}\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) + \frac{p}{4}\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$+\frac{p}{4}\left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right)+\frac{p}{4}\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right)$$

The numbers at the subscripts denote the cardinality of the respective classes.

## G.2 ADDITIONAL RESTRICTION BY $\mathrm{NS}_4$:

We take all the extremal distributions obtained in the previous analysis and found that upon further restricting to the hyperplane $\langle \mathrm{NS}_4, \mathbf{x} = 0\rangle$, using the sequence of steps described above, there is only one class of extremal states with a cardinality of 16. A representative of this class is

$$\frac{1}{2}\left(\begin{array}{cc|cc} 1 & 0 & p & 1-p \\ 0 & 1 & 1-p & p \\ \hline 1 & 0 & 1-p & p \\ 0 & 1 & p & 1-p \end{array}\right).$$

In the following, we present one element from each extremal class. A convex decomposition, like the one obtained in the previous section, can also be obtained. However, an easier way of noticing that they are not extremal is by the fact that they are all local. The subscripts represent the number of elements present in each class.

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right)_8, \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ \hline 0 & \frac{1}{2} & \frac{1-p}{2} & \frac{p}{2} \\ 0 & \frac{1}{2} & \frac{p}{2} & \frac{1-p}{2} \end{array}\right)_{16}, \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \hline 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right)_4, \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \hline \frac{1}{2} & 0 & \frac{1-p}{2} & \frac{p}{2} \\ 0 & \frac{1}{2} & \frac{p}{2} & \frac{1-p}{2} \end{array}\right)_{16}$$

$$\left(\begin{array}{cc|cc} 0 & \frac{1}{2} & \frac{p}{2} & \frac{1}{6} \\ 0 & \frac{1}{2} & \frac{1}{6} & \frac{p}{2} \\ \hline 0 & \frac{1}{2} & \frac{1-p}{2} & \frac{p}{2} \\ 0 & \frac{1}{2} & \frac{p}{2} & \frac{1-p}{2} \end{array}\right)_8, \left(\begin{array}{cc|cc} \frac{1-p}{3} & \frac{p}{3} & 0 & \frac{1}{3} \\ \frac{p}{3} & \frac{2-p}{3} & \frac{1}{3} & \frac{1}{3} \\ \hline \frac{1-p}{3} & \frac{2(1-p)}{3} & \frac{1-p}{3} & \frac{2(1-p)}{3} \\ \frac{p}{3} & \frac{2p}{3} & \frac{p}{3} & \frac{2p}{3} \end{array}\right)_{32}, \left(\begin{array}{cc|cc} \frac{1-p}{3} & \frac{p}{3} & 0 & \frac{1}{3} \\ \frac{p}{3} & \frac{2-p}{3} & \frac{1}{3} & \frac{1}{3} \\ \hline \frac{p}{3} & \frac{1-p}{3} & 0 & \frac{1}{3} \\ \frac{1-p}{3} & \frac{1+p}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right)_{32},$$

$$\left(\begin{array}{cc|cc} \frac{1-p}{3} & \frac{p}{3} & \frac{p}{3} & \frac{1-p}{3} \\ \frac{p}{3} & \frac{2-p}{3} & \frac{1-p}{3} & \frac{1+p}{3} \\ \hline \frac{1-p}{3} & \frac{p}{3} & 0 & \frac{1}{3} \\ \frac{p}{3} & \frac{2-p}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right)_{32}, \left(\begin{array}{cc|cc} \frac{1-p}{3} & \frac{p}{3} & \frac{2-p}{6} & \frac{p}{6} \\ \frac{p}{3} & \frac{2-p}{3} & \frac{p}{6} & \frac{4-p}{6} \\ \hline \frac{p}{3} & \frac{1-p}{3} & \frac{p}{6} & \frac{2-p}{6} \\ \frac{1-p}{3} & \frac{1+p}{3} & \frac{2-p}{6} & \frac{2+p}{6} \end{array}\right)_{32}, \left(\begin{array}{cc|cc} \frac{1-p}{3} & \frac{p}{3} & \frac{p}{3} & \frac{1-p}{3} \\ \frac{p}{3} & \frac{2-p}{3} & \frac{2-p}{3} & \frac{p}{3} \\ \hline \frac{p}{3} & \frac{1-p}{3} & \frac{1-p}{3} & \frac{p}{3} \\ \frac{1-p}{3} & \frac{1+p}{3} & \frac{1+p}{3} & \frac{1-p}{3} \end{array}\right)_{32},$$

$$\left(\begin{array}{cc|cc} \frac{p}{3} & \frac{1-p}{3} & 0 & \frac{1}{3} \\ \frac{1-p}{3} & \frac{1+p}{3} & \frac{1}{3} & \frac{1}{3} \\ \hline \frac{2-p}{6} & \frac{1-p}{3} & \frac{1-p}{3} & \frac{2-p}{6} \\ \frac{p}{6} & \frac{1+p}{3} & \frac{p}{3} & \frac{2+p}{6} \end{array}\right)_{32}, \left(\begin{array}{cc|cc} \frac{p}{3} & \frac{p}{6} & \frac{p}{3} & \frac{p}{6} \\ \frac{p}{6} & \frac{3-2p}{3} & \frac{1-p}{3} & \frac{4-p}{6} \\ \hline \frac{p}{3} & \frac{1-p}{3} & 0 & \frac{1}{3} \\ \frac{p}{6} & \frac{4-p}{6} & \frac{1}{3} & \frac{1}{3} \end{array}\right)_{32}, \left(\begin{array}{cc|cc} \frac{1-p}{2} & \frac{p}{2} & 0 & \frac{1}{2} \\ \frac{p}{2} & \frac{1-p}{2} & \frac{1}{4} & \frac{1}{4} \\ \hline \frac{p}{4} & \frac{2-p}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{2-p}{4} & \frac{p}{4} & 0 & \frac{1}{2} \end{array}\right)_{16},$$

$$\left(\begin{array}{cc|cc} \frac{1-p}{2} & \frac{p}{2} & 0 & \frac{1}{2} \\ \frac{p}{2} & \frac{1-p}{2} & \frac{4-3p}{8} & \frac{3p}{8} \\ \hline \frac{3p}{8} & \frac{4-3p}{8} & \frac{1-p}{2} & \frac{p}{2} \\ \frac{4-3p}{8} & \frac{3p}{8} & \frac{p}{8} & \frac{p}{8} \end{array}\right)_{32}, \left(\begin{array}{cc|cc} \frac{1-p}{2} & 1-p & \frac{1-p}{2} & 1-p \\ 2p-1 & \frac{1-p}{2} & \frac{5p-3}{4} & \frac{1+p}{4} \\ \hline \frac{1-p}{2} & \frac{1-p}{3} & \frac{1-p}{2} & 1-p \\ 2p-1 & \frac{1-p}{2} & \frac{5p-3}{4} & \frac{1+p}{4} \end{array}\right)_{16},$$

$$
\left(\begin{array}{cc|cc}
\frac{1-p}{2} & \frac{p}{2} & \frac{1-p}{2} & \frac{p}{2} \\
\frac{p}{2} & \frac{1-p}{2} & \frac{p}{8} & \frac{4-p}{8} \\
\hline
\frac{3p}{8} & \frac{4-3p}{8} & \frac{4-3p}{8} & \frac{3p}{8} \\
\frac{4-3p}{8} & \frac{3p}{8} & 0 & \frac{1}{2}
\end{array}\right)_{32},
\left(\begin{array}{cc|cc}
\frac{p}{4} & \frac{2-p}{4} & \frac{1-p}{2} & \frac{p}{2} \\
\frac{2-p}{4} & \frac{p}{4} & \frac{p}{2} & \frac{1-p}{2} \\
\hline
\frac{p}{4} & \frac{2-p}{4} & \frac{13-2p}{28} & \frac{1+2p}{28} \\
\frac{2-p}{4} & \frac{p}{4} & \frac{2p+1}{28} & \frac{13+2p}{28}
\end{array}\right)_{16},
\left(\begin{array}{cc|cc}
\frac{p}{4} & 1-p & \frac{p}{4} & 1-p \\
\frac{p}{2} & \frac{p}{4} & \frac{p}{2} & \frac{p}{4} \\
\hline
\frac{3p}{8} & \frac{3p}{8} & \frac{3p}{8} & \frac{3p}{8} \\
\frac{3p}{8} & \frac{8-9p}{8} & \frac{3p}{8} & \frac{8-9p}{8}
\end{array}\right)_{12},
$$

$$
\left(\begin{array}{cc|cc}
\frac{p}{4} & \frac{p}{2} & \frac{3p}{8} & \frac{3p}{8} \\
1-p & \frac{p}{4} & \frac{8-9p}{8} & \frac{3p}{8} \\
\hline
\frac{8-9p}{8} & \frac{3p}{8} & \frac{8-7p}{8} & \frac{p}{8} \\
\frac{3p}{8} & \frac{3p}{8} & \frac{p}{8} & \frac{5p}{8}
\end{array}\right)_{8},
\left(\begin{array}{cc|cc}
\frac{3p}{8} & \frac{3p}{8} & 0 & \frac{3p}{4} \\
\frac{4-3p}{8} & \frac{4-3p}{8} & \frac{4-3p}{8} & \frac{4-3p}{8} \\
\hline
\frac{4-3p}{8} & \frac{4-3p}{8} & \frac{4-3p}{8} & \frac{4-3p}{8} \\
\frac{3p}{8} & \frac{3p}{8} & 0 & \frac{3p}{4}
\end{array}\right)_{8},
$$

$$
\left(\begin{array}{cc|cc}
\frac{3p-1}{4} & \frac{3(1-p)}{4} & \frac{1-p}{2} & \frac{p}{2} \\
\frac{3p-1}{4} & \frac{3(1-p)}{4} & \frac{1-p}{4} & \frac{1+p}{4} \\
\hline
\frac{3p-1}{4} & \frac{3(1-p)}{4} & \frac{1-p}{2} & \frac{p}{2} \\
\frac{3p-1}{4} & \frac{3(1-p)}{4} & \frac{1-p}{4} & \frac{1+p}{4}
\end{array}\right)_{8},
\left(\begin{array}{cc|cc}
\frac{3p}{8} & \frac{3p}{8} & \frac{p}{4} & \frac{p}{2} \\
\frac{3p}{8} & \frac{8-9p}{8} & 1-p & \frac{p}{4} \\
\hline
\frac{p}{4} & 1-p & \frac{8-9p}{8} & \frac{3p}{8} \\
\frac{p}{2} & \frac{p}{4} & \frac{3p}{8} & \frac{3p}{8}
\end{array}\right)_{8},
\left(\begin{array}{cc|cc}
\frac{3p-1}{4} & \frac{3-3p}{4} & \frac{1-p}{2} & \frac{p}{2} \\
\frac{3p-1}{4} & \frac{3-3p}{4} & \frac{p}{2} & \frac{1-p}{2} \\
\hline
\frac{3p-1}{4} & \frac{3-3p}{4} & \frac{1+p}{4} & \frac{1-p}{4} \\
\frac{3p-1}{4} & \frac{3-3p}{4} & \frac{1-p}{4} & \frac{1+p}{4}
\end{array}\right)_{16}
$$

$$
\left(\begin{array}{cc|cc}
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\hline
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4}
\end{array}\right)_{1};
$$

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