Hilbert's fourteenth problem and finite generation ideals

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Abstract

Hilbert's fourteenth problem asks whether invariant rings under algebraic group actions are always finitely generated. There are a number of examples that have been constructed since the mid-20th century which demonstrate that this is not the case in general. This thesis is concerned with developing our understanding of these non-finitely generated invariant rings. This goal is ambitious, as by their nature these rings are difficult to work with and it is hard to build an intuition for what might be true in general. The difficulty of trying to develop a solid intuition from examples is exacerbated by the process of "removing symmetries," which relates some of the more well-understood invariant rings. A key construction we employ in order to better understand the structure of these counterexamples to Hilbert's problem is the finite generation ideal, consisting of invariants which make the invariant ring finitely generated after localisation.

We take a number of paths in order to achieve our aim, including computing the finite generation ideal for existing examples, constructing new counterexamples, and improving our understanding of both the process of removing symmetries and the finite generation ideal itself. Specifically, we first compute the finite generation ideal of a famous counterexample due to Daigle and Freudenburg. Next, we work on constructing new non-finitely generated invariant rings, focusing primarily on an example proposed by Maubach. We then investigate this process of removing symmetries on some new examples. Finally, we study the finite generation ideal in the setting of monomial algebras, with the intention of passing results obtained to SAGBI-bases; a form of generating set we employ to compute the finite generation ideal for invariant rings.

Contents

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Author's declaration

I declare that the work presented in this thesis, except where otherwise stated, is based on my own research carried out at the University of York and has not been submitted previously for any degree at this or any other university. Sources are acknowledged by explicit references. The work contained in Chapter 3 is based on a paper titled 'The finite generation ideal for Daigle and Freudenburg's counterexample to Hilbert's fourteenth problem' [22] for which I am the sole author. This paper has been accepted to be published in the journal Communications in Algebra.

— 1 —
Introduction

At the Paris conference of the International Congress of Mathematicians in 1900, Hilbert proposed a number of problems. Later in 1902 these were published as a complete list of 23 problems in the Bulletin of the American Mathematical Society. The work undertaken to solve these problems has shaped much of the course of mathematics across the 20th and 21st centuries.

In his fourteenth problem, Hilbert asks:

- Given $\mathbb{K}$ a field and L a subfield of $\mathbb{K}(x_1, \dots, x_n)$, the field of rational functions in n variables over $\mathbb{K}$, is $L \cap \mathbb{K}[x_1, \dots, x_n]$ finitely generated?

Nagata answered this question in the negative in [29], finding a counterexample to this problem in 1959. There is a special case where we have a positive answer: In 1954, Zariski [35] showed that if the transcendence degree of L over $\mathbb{K}$ is at most 2, then $L \cap \mathbb{K}[x_1, \dots, x_n]$ is finitely generated over $\mathbb{K}$.

A more specialised version of this problem, which was Hilbert's motivation for posing it, focuses on invariant rings: Given a group G acting on $\mathbb{K}^n$ and a polynomial ring R in n variables over an algebraically closed field $\mathbb{K}$, we say a polynomial is *G-invariant* if $f(x) = f(g \cdot x)$ for all $x \in \mathbb{K}^n$ and $g \in G$. R^G is then the invariant ring consisting of all such G-invariant polynomials. This specialised version of the problem asks:

- Is R^G always finitely generated as a $\mathbb{K}$-algebra?

In 1990 Roberts [31] produced another counterexample to Hilbert's fourteenth problem as the symbolic blow-up of a power series ring. This was later realised as the invariant ring of an action of the additive group $\mathbb{G}_a = (\mathbb{K}, +)$ on $\mathbb{A}^7$ by A'Campo-Neuen [1]. Nagata's counterexample arises as an invariant ring under an action of $G = \mathbb{G}_a^{13} \times \mathbb{G}_m^{15}$, where $\mathbb{G}_m = (\mathbb{K}, \cdot)$ is the multiplicative group. As with the general formulation of Hilbert's problem, this special case of the problem also has some cases where it holds true. This includes Hilbert's finiteness theorem, which states that if G is reductive, then R^G is finitely generated, see Theorem 2.2.2. We also have the Maurer-Weitzenböck theorem [33], which states:

- If $\mathbb{K}$ is a field of characteristic 0, and if $\mathbb{G}_a$ acts by linear transformations on $\mathbb{A}^n$, then $R^{\mathbb{G}_a}$ is finitely generated.

It is unknown whether this theorem generalises to all fields.

Many counterexamples to Hilbert's fourteenth problem have been found at this point. In addition to those above, further ones have been constructed by Freudenburg [17] and Daigle and Freudenburg [4], where they produce invariant rings under $\mathbb{G}_a$-actions which are not finitely generated in dimensions 6 and 5 respectively. Kuroda [25] generalised Roberts' counterexample. A family of non-finitely generated invariant rings are constructed in relation to work on the moduli space $\overline{M}_{0,n}$ by Castravet and Tevelev [3] and Doran, Giansiracusa, and Jensen [7]. Van den Essen, Kuroda, and Crachiola construct "a 'factory' that produces counterexamples to Hilbert's fourteenth problem," [16, § 2].

A consistent theme in all of these counterexamples is that beyond showing that they are non finitely generated, we understand little of the structure of these rings of invariants. This is not surprising, as many counterexamples like as Nagata's and the examples due to Castravet and Tevelev and Doran, Giansiracusa, and Jensen exist in high dimensions, and thus are difficult to work with. The examples of Roberts', Freudenburg and Daigle and Freudenburg are more tractable. However, these more well-understood examples are related: Daigle and Freudenburg's counterexample can be related to Roberts' counterexample by a process of "removing symmetries," [19, § 7.2]. Furthermore, there is a $\mathbb{K}$-algebra homomorphism from Freudenburg's example to Daigle and Freudenburg's which induces a surjective homomorphism on the invariant rings [32, §2]. Thus, any intuition we build from working on these examples may not generalise well to all counterexamples to Hilbert's fourteenth problem.

Our goal in this thesis is to further our understanding of counterexamples to Hilbert's fourteenth problem and the structure of non-finitely generated algebras in general. We focus on invariant rings under $\mathbb{G}_a$-actions. This is not too much of a restriction: Winkelmann [34] notes that a geometric restatement of Hilbert's fourteenth problem asks whether the ring of invariant functions is necessarily isomorphic to the ring of regular functions on some affine variety. When G is non-reductive Winkelmann showed that a $\mathbb{K}$-algebra occurs as the ring of invariants for some affine G-variety if and only if it is isomorphic to the algebra of regular functions on some quasi-affine variety. When the underlying quasi-affine variety is normal, this algebra corresponds to the invariant ring of a regular $\mathbb{G}_a$-action on a normal quasi-affine variety. Therefore, studying non-finitely generated invariant rings under $\mathbb{G}_a$-actions covers a broad class of counterexamples to Hilbert's fourteenth problem which arise as invariant rings. A useful property of working with additive group actions in characteristic 0 is that these are in one-to-one correspondence with locally nilpotent derivations [19, § 1], with the invariant ring corresponding to the kernel of the derivation.

A notable construction to help build our understanding of the underlying structure of these counterexamples is the *finite generation ideal*, introduced by Derksen and Kemper [6]. For a field $\mathbb{K}$, and R a $\mathbb{K}$-domain the finite generation ideal is:

$$\mathfrak{f}_R := \{f \in R \,|\, R_f \text{ is finitely generated over } \mathbb{K}\} \cup \{0\}.$$

Derksen and Kemper showed that $\mathfrak{f}_R$ is a radical ideal, and that if R is a subalgebra of a finitely generated algebra, then $\mathfrak{f}_R$ is non-zero. Thus $\mathfrak{f}_R$ is well placed as an object of study in order to better understand the structure of these counterexamples. Dufresne and Kraft [10, § 9] computed the finite generation ideal of Roberts' counterexample making use of a SAGBI-basis for the invariant ring. SAGBI-bases are generalisations of Gröbner bases to subalgebras; they were developed independently by both Robbiano and Sweedler [30] and Kapur and Madlener [24].

An alternative approach to understanding of these counterexamples can be taken by working with separating sets: Suppose V is an affine variety with coordinate ring R, and G is a group acting by automorphisms on R. A subset $S \subset R^G$ is a *separating set* if for any two points $x, y \in V$ and some invariant $f \in R^G$ with $f(x) \neq f(y)$, then there is some $g \in S$ with $g(x) \neq g(y)$. Derksen and Kemper [5, § 2.4], showed that if V is an affine variety and G a group acting on its coordinate ring R, then there is a finite separating set $S \subset R^G$. Thus, separating sets attempt to understand non-finitely generated invariant rings by means of finite subsets. Here we wish to take a more direct approach and as such we do not focus on separating sets in this thesis. We do however note that separating sets have been studied extensively, see, for example, [12, 10, 9, 5].

As stated above, we aim to further our understanding of counterexamples to Hilbert's fourteenth problem and the structure of non-finitely generated algebras in general. There are multiple paths to achieve this aim. To name a few: we can compute the finite generation ideal for existing known examples, construct new counterexamples to Hilbert's fourteenth problem to study, and develop our understanding of the finite generation ideal in general. Contained within this thesis is a computation of the finite generation ideal of Daigle and Freudenberg's example, study on a potential new counterexample to Hilbert's fourteenth problem proposed by Maubach [27] as well as work on methods of computing the finite generation ideal when restricted to polynomial rings generated by monomials. Additionally, we provide new approaches to showing that an invariant ring is finitely generated, and note of a number of interesting examples which help us to develop some new techniques and ideas.

In Chapter 2 we cover the relevant preliminaries for this thesis. This includes an introduction to locally nilpotent derivations and some related concepts such as the plinth ideal, degree functions and gradings. We give some simple but important examples, such as the Weitzenböck derivation. Then we introduce invariant rings, provide a proof that additive group actions are in one-to-one correspondence with locally nilpotent derivations; and show that the invariant ring of an additive group action equal to the kernel of a locally nilpotent derivation. We mostly follow [19] in our exposition here. With this in hand, using [5], we then cover the concepts of Gröbner and SAGBI-bases, which play a key role in computing the finite generation ideal in later chapters. Finally, using [6], we focus on the finite generation ideal and its properties, before giving a brief description of Roberts' counterexample and its finite generation ideal.

Chapter 3 comprises a paper which has been accepted by Communications in Algebra. As such the content contained within is completely self-contained. This chapter focuses on Daigle and Freudenburg's counterexample to Hilbert's fourteenth problem and culminates in a computation of its finite generation ideal. We adopt a strategy similar to Dufresne and Kraft in their work on Roberts' example in [10, § 9]. In the course of computing the finite generation ideal, we construct a generating set and show that it is a SAGBI-basis, which is key in allowing us to compute the finite generation ideal. In addition, we compute plinth variety, nullcone and fixed-point set for this example.

We cover Maubach's example in Chapter 4, which was conjectured by Maubach to be a counterexample to Hilbert's fourteenth problem in [27]. Our focus is on proving this conjecture, which we attempt to do by exploiting the structure of a subalgebra contained within the example, which is a copy of the Weitzenböck derivation in 4 variables. Specifically, we make use of a bi-grading on the polynomial ring which allows us to interpret the derivation as a linear map between the graded pieces of the ring. We then show that we can locate all elements of the kernel of the derivation purely by computing the difference between the domain and co-domain of the linear map. Currently, the proof we present is incomplete, since we are unable to finish an important induction argument. However, in the course of our development, we make note of how this example, if not finitely generated, differs from existing known counterexamples and does not fulfill the conditions of the non-finiteness criterion shown by Daigle and Freudenburg [4, § 7.2]. Furthermore, our results on the Weitzenböck derivation allow us to provide a new proof in section 4.2.1 that Daigle and Freudenburg's counterexample is not finitely generated. We expect the techniques in this chapter to generalise in many interesting directions.

Chapter 5 comprises a study of the process of "removing symmetries" from invariant rings in order to obtain new invariant rings. This is demonstrated in [19, § 7.2.3] where Daigle and Freudenburg's counterexample is obtained from Roberts' counterexample. It is achieved by introducing an action of $\mathbb{G}_m^3$ and S_3, where S_3 is the symmetric group on 3 points. A subgroup $H \subset \mathbb{G}_m^3$ is then taken for which $G := H \rtimes S_3$ and the $\mathbb{G}_a$-action of Roberts' example restricted to the invariant ring under this action is precisely the $\mathbb{G}_a$-action on Daigle and Freudenburg's example. Through this formulation it becomes clear that the example with symmetries removed is non-finitely generated, then so is the example without the symmetries removed. From work of Castravet and Tevelev and Doran, Giansiracusa, and Jensen in [3] and [7] respectively, there is a family of invariant rings connected to a moduli space $\overline{M}_{0,n}$ which is known to be not finitely generated for $n \geq 10$. We can remove the symmetries from these invariant rings, and in each case obtain a copy of the Weitzenböck derivation in n variables, which is finitely generated. Therefore we show that it is possible to remove symmetries and obtain an invariant ring which is finitely generated from one which is not. Furthermore, using our work in Chapter 4 on the Weitzenböck derivation we can determine when generalised cases of Daigle and Freudenburg's example are non-finitely generated. Kuroda [25], conjectures a condition to classify when some generalised forms of Roberts' counterexample are non-finitely generated. By combining his work with our own, we show we can remove symmetries from two generalised forms of Robert's counterexample, one finitely generated and the other not, and obtain the same invariant ring.

In Chapter 6, we shift focus away from invariant rings and derivations to instead study the finite generation ideal directly. Our aim here is to develop a deeper understanding of the structure of this ideal, as well as methods to compute it. We study subalgebras $R \subset \mathbb{K}[x_1, \dots, x_n]$ which are generated by monomials. As such, many of our results pass naturally to SAGBI-bases. First we work with $n = 2$ and make use of a function we term a *ratio function* in order to capture all the ways R can be non-finitely generated, then compute the finite generation ideal in these cases. We then generalise to $n \geq 3$, and show that finitely generated monomial subalgebras can be connected to polyhedral cones in $\mathbb{Q}^n$. A monomial corresponds to a point in $\mathbb{N}^n$ through its exponents, and multiplication of two monomials becomes vector addition. We show that R is finitely generated if and only if the semigroup in $\mathbb{N}^n$ generated by the points corresponding to the monomials in R is contained in a polyhedral cone in a suitably nice way. This result is already known as a generalisation of Gordan's lemma, and can be found in [2, p.53], we provide an alternate elementary proof of this. With this result in hand, we then return to our study of the finite generation ideal, which has much added complexity in at least 3 variables as demonstrated in Example 6.2.4. The connection to polyhedral cones affords us a geometric approach to understanding the finite generation ideal, and we can generalise our ratio functions from the $n = 2$ case as well. However, both approaches have their own issues. Starting with a semigroup in $\mathbb{N}^n$ and introducing a point in $\mathbb{Z}^n$ corresponding to the localisation of a monomial, it is difficult to deduce that this semigroup is sitting nicely in a polyhedral cone. Whereas with ratio functions, generalising to $n > 2$ requires us to construct multiple ratio functions that must work in tandem to capture the ways R can be non-finitely generated and compute the finite generation ideal. But it is not clear that this can be done in all cases.

Included in Chapters 4, 5 and 6 is a section which makes note of further points of study, highlights questions that have arisen in our exploration of these topics and conjectures some future results.

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Preliminaries

Throughout this thesis $\mathbb{K}$ is an algebraically closed field of characteristic 0. However some of the definitions we provide and results we obtain are true in greater generality.

2.1 LOCALLY NILPOTENT DERIVATIONS

Here we follow [19, § 1] to introduce locally nilpotent derivations and their associated concepts.

Definition 2.1.1. Let R be a commutative $\mathbb{K}$-domain, a *derivation* $D\colon R \longrightarrow R$ is a linear map satisfying the Liebniz rule, that is, for all $a, b \in R$:

$$D(ab) = D(a)b + aD(b).$$

By D^n, $n \geq 0$ we mean the n-fold composition of D with itself, where D^0 is the identity map. We say that a derivation is *locally nilpotent* if for all $a \in R$ there is some $n \geq 0$ such that $D^n(a) = 0$.

Definition 2.1.2. With R and D, as above, the *kernel* of a derivation is $R^D := \{x \in R \mid D(x) = 0\}$, and the *image* of a derivation is $D(R) := \{y \in R \mid y = D(x),\ x \in R\}$. The kernel of a derivation R^D is a subring of R. The *plinth ideal* of R is $\mathfrak{pl}(D) := R^D \cap D(R)$

Proposition 2.1.3. *The plinth ideal $\mathfrak{pl}(D)$ is an ideal of R^D.*

Proof. Note that for $a \in R^D$, $b \in R$, $D(ab) = aD(b)$, so D is an R^D-module endomorphism of R, and both R^D and $D(R)$ are R^D-submodules of R. Therefore, $R^D \cap D(R)$ is an R^D-submodule of R^D, an ideal of R^D. □

An element $s \in \mathfrak{pl}(D)$ which satisfies $D(s) = 1$ is called a *slice* of D, while any element $t \in \mathfrak{pl}(D)$ is called a *local slice* of D. In the literature, a local slice is also referred to as a pre-slice.

Definition 2.1.4. A *degree function* on R is any map $\deg : R \longrightarrow \mathbb{Z}_{\geq 0} \cup \{-\infty\}$ such that, for all $f, g \in R$ the following conditions hold:

1. $\deg(f) = -\infty$ if and only if $f = 0$,
2. $\deg(fg) = \deg(f) + \deg(g)$
3. $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$.

We understand that $(-\infty) + (-\infty) = -\infty$ and $(-\infty) + n = -\infty$ for all n.

We now introduce some important degree functions, but first we make clear some notation. Let $f \in \mathbb{K}[x_1 \ldots, x_n]$, if f is of the form $f = x_1^{a_1} \cdots x_n^{a_n}$, where $a_i \geq 0$ for all i, then we say that f is a *monomial.* A *polynomial* then is $f = \sum_{i=1}^{k} \lambda_i f_i$, where each f_i is a monomial and each $\lambda_i \in \mathbb{K} \setminus \{0\}$. We refer to λ_i as the *coefficient* of the monomial f_i and their combination $\lambda_i f_i$ is a *term* of f.

Example 2.1.5. Suppose that $R \subset \mathbb{K}[x_1, \ldots, x_n]$ is a $\mathbb{K}$-algebra, and D is a locally nilpotent derivation. Consider the function $\rho : R \longrightarrow \mathbb{Z}_{\geq 0} \cup \{-\infty\}$ defined as follows: For a monomial $m \in R$ we set

$$\rho(m) := \max\{i \in \mathbb{Z}_{\geq 0} \,|\, D^i(m) \neq 0, D^{i+1}(m) = 0\},$$

then for a polynomial $f \in R$ we define

$$\rho(f) := \max\{\rho(m) \,|\, m \text{ is a monomial of } f\}.$$

We set $\rho(0) := -\infty$. Then ρ then defines a degree function, and we call $\rho(f)$ the *ρ-degree* of f.

To see this, we note that for monomials $m_1, m_2 \in R$ it is clear that $\rho(m_1 m_2) = \rho(m_1) + \rho(m_2)$ and $\rho(m_1 + m_2) \leq \max\{\rho(m_1), \rho(m_2)\}$. For polynomials $f, g \in R$, it is clear that $\rho(f + g) \leq \max\{\rho(f), \rho(g)\}$. Now, let

$$R_i := \{h \in R \,|\, \rho(h) = i\}.$$

Suppose $\rho(f) = k, \rho(g) = l$, we can write $f = f_1 + f_2$ and $g = g_1 + g_2$, where $f_1 \in R_k$, $g_1 \in R_l$ and $\rho(f_2) < k, \rho(g_2) < l$. Then, as R is a domain, we must have $f_1 g_1 \neq 0$ and $\rho(f_1 g_1) = k + l$ as all monomials appearing in f_1 and g_1 have ρ-degree k and l respectively. We therefore conclude that $\rho(fg) = k + l$ since any combination of monomials in f_2 and g or f and g_2 will have ρ-degree less than $k + l$.

Example 2.1.6. Suppose R is a commutative $\mathbb{K}$-domain and $D : R \longrightarrow R$ is a locally nilpotent derivation. Then D induces a degree function ν_D on R, defined for $f \in R \setminus \{0\}$ as

$$\nu_D(f) := \min\{n \in \mathbb{Z}_{\geq 0} \,|\, D^{n+1}(f) = 0\},$$

additionally, we set $\nu_D(0) = -\infty$. A similar argument to the one used for ρ in Example 2.1.5 can be used to show that ν_D is also a degree function. These are similar degree functions, but an important distinction between these two is that given $f \in R$ with $D(f) = 0$, then we must have $\nu_D(f) = 0$, but $\rho(f)$ is not necessarily 0. Furthermore for any $f \in R$, $\nu_D(D(f)) = \nu_D(f) - 1$, whereas $\rho(D(f)) = \rho(f) - 1$ or $-\infty$.

Now suppose $R = \bigoplus_{i \in Z} R_i$, is a $\mathbb{Z}$-graded ring, each R_i is a $\mathbb{K}$-module, and $R_i R_j \subset R_{i+j}$ for all $i, j \in \mathbb{Z}$. Labelling this grading as ω, then we call an element $f \in B_i$ an *ω-homogeneous element* of R, and i is the *ω-degree* of f.

Definition 2.1.7. Let R be as above and $D : R \longrightarrow R$ a derivation. We say that D is an *ω-homogeneous* derivation if there is some $k \in \mathbb{Z}$ such that $D(R_i) \subset R_{i+k}$ for all $i \in \mathbb{Z}$. We say that k is the *ω-degree* of D, or simply $k = \deg(D)$, when there is no ambiguity.

For example both ν_D and ρ induce a $\mathbb{Z}$-grading on R and under this grading $D(R_i) \subset R_{i-1}$ for all i, so D is both ν_D-homogeneous and ρ-homogeneous, with degree -1 in both cases.

Note that if D is ω-homogeneous and $f \in R$, then we can write $f = \sum_{i \in \mathbb{Z}} f_i$, where $f_i \in R_i$, and $D(f) = 0$ if and only if $D(f_i) = 0$ for all i.

Definition 2.1.8. Given R, a commutative $\mathbb{K}$-domain, a *$\mathbb{Z}$-filtration* of R is a collection $\{R_i\}_{i \in \mathbb{Z}}$ of subsets of R satisfying the following properties:

1. Each R_i is a vector space over $\mathbb{K}$.
2. $R_j \subset R_i$ whenever $j \leq i$.
3. $R = \bigcup_{i \in \mathbb{Z}} R_i$.
4. $R_i R_j \subset R_{i+j}$ for all $i, j \in \mathbb{Z}$.

Additionally, a $\mathbb{Z}$-filtration is a *proper $\mathbb{Z}$-filtration* if it also satisfies:

5. $\bigcap_{i \in \mathbb{Z}} R_i = \{0\}$.
6. Given $a \in R_i \cap R_{i-1}^C$ and $b \in R_j \cap R_{j-1}^C$, then $ab \in R_{i+j} \cap R_{i+j-1}^C$,

where R_i^C is taken to be the complement of R_i in R. Any degree function on R will give rise to a proper $\mathbb{Z}$-filtration. Additionally, we can define filtrations with $\mathbb{Z}$ replaced by any ordered abelian semigroup.

Definition 2.1.9. Suppose $R = \bigcup_{i\in\mathbb{Z}} R_i$ is a proper $\mathbb{Z}$-filtration, we define the *associated graded algebra* $Gr(R)$ as follows: The $\mathbb{K}$-additive structure on $Gr(R)$ is given by

$$Gr(B) = \bigoplus_{n\in\mathbb{Z}} R_n/R_{n-1},$$

and for elements $a+R_{i-1}$ and $b+R_{j-1}$ belonging to R_i/R_{i-1} and R_j/R_{j-1} respectively, their product is the element of R_{i+j}/R_{i+j-1} defined by

$$(a + R_{i-1})(b + R_{j-1}) := ab + R_{i+j-1}.$$

Multiplication is then extended to all of $Gr(R)$ by the distributive law. By property 6, $Gr(R)$ is a commutative $\mathbb{K}$-domain. From property 5, given non-zero $a \in R$, the set $\{i \in \mathbb{Z} \,|\, a \in R_i\}$, has a minimum, which we denote $\iota(a)$. Then we define the map

$$\kappa : R \longrightarrow Gr(R),$$

which sends each non-zero $a \in R$ to its class in R_i/R_{i-1}, where $i = \iota(a)$, we also set $\kappa(0) := 0$.

If R is already $\mathbb{Z}$-graded, then R admits a filtration relative to which R and $Gr(R)$ are canonically isomorphic via κ. In particular, if $R = \bigoplus_{i\in\mathbb{Z}} R_i$, then a proper $\mathbb{Z}$-filtration is defined by $R_i = \bigoplus_{j\leq i} R_j$.

Definition 2.1.10. Suppose $R = \bigcup_{i\in\mathbb{Z}} R_i$ is a proper $\mathbb{Z}$-filtration and D is a derivation. We say that D *respects the filtration* if there is some integer t such that, for all $i \in \mathbb{Z}$, $D(R_i) \subset R_{i+t}$.

Returning to the degree function ρ from Example 2.1.5, for locally nilpotent D, it induces a proper $\mathbb{Z}$-filtration $R = \bigcup_{i\in\mathbb{Z}_{\geq 0}} R_i$ which D respects. We have $D(R_i) \subset R_{i-1}$ for $i \geq 1$ and $D(R_0) = 0$, where $R_i = \{f \in R \,|\, \rho(f) \leq i\}$. Similarly ν_D also induces a proper $\mathbb{Z}$-filtration, and in this case $R_0 = R^D$. We now move to discuss properties of certain locally nilpotent derivations:

Definition 2.1.11. Let D be a locally nilpotent derivation on $\mathbb{K}[x_0, \dots, x_n]$ given by:

$$D := a_0\frac{\partial}{\partial x_0} + \cdots + a_n\frac{\partial}{\partial x_n},$$

where each $a_i \in \mathbb{K}[x_0, \dots, x_n]$. Now:

- If each a_i is a term, by a slight abuse of notation to remain consistent with the literature, we say D is a *monomial derivation.*
- If, for all $i \geq 1$ we have that $a_i \in \mathbb{K}[x_0, \ldots x_{i-1}]$ and $a_0 \in \mathbb{K}$ then we say that D is *triangular.*

Example 2.1.12. Let $R_n := \mathbb{K}[x_0, x_1, \ldots, x_n]$ and consider the locally nilpotent derivation

$$D_n := x_0 \frac{\partial}{\partial x_1} + \cdots + x_{n-1} \frac{\partial}{\partial x_n},$$

then D_n is a triangular monomial locally nilpotent derivation. This derivation is called the *Weitzenböck derivation* and is well-known to be finitely-generated via the Maurer-Weitzenböck theorem [33]. However, the number of generators of $R_n^{D_n}$ grows quickly with n, and minimal generating sets are only reliably known for $n \leq 7$, [12].

In [26], Maubach completed the proof of the following result:

Theorem 2.1.13. *Let $R := \mathbb{K}[x_1, x_2, x_3, x_4]$ and suppose that D is a triangular, monomial locally nilpotent derivation on R, then R^D is finitely generated by at most 4 elements.*

Maubach's proof of this result provides an explicit algorithm to compute the generators of R^D, which we make use of.

Example 2.1.14. Let $S = \mathbb{K}[x, y, z, u]$ and let Δ be the Weitzenböck derivation on S in 4 variables. Using Maubach's algorithm we obtain that $S^\Delta = \mathbb{K}[x, \gamma, \delta, g]$ where

$$\begin{aligned}
\gamma &:= 2xz - y^2, \\
\delta &:= 3x^2u - 3xyz + y^3, \\
g &:= 9x^2u^2 - 18xyzu + 8x^3z + 6y^3u - 3y^2z^2.
\end{aligned}$$

Suppose $R = \mathbb{K}[x_1, \ldots, x_n]$ and D is a locally nilpotent derivation. Of interest to us is whether R^D is finitely generated. First we make clear some basic notions:

Definition 2.1.15. Let R be a commutative $\mathbb{K}$-domain, a set $G \subset R$ is a *generating set* for R if G generates R as a $\mathbb{K}$-algebra. That is, every $r \in R$ can be expressed as

$$r = \sum_{i=1}^{k} \lambda_i \prod_{g \in G} g^{i_g},$$

where $\lambda_i \in \mathbb{K}$ and all but finitely many $i_g = 0$ for all i. We say R is *finitely generated* if R has a finite generating set. R is *not finitely generated* if no finite generating set exists.

Definition 2.1.16. We call a generating set $G \subset R$ *minimal* if for all $g \in G$ we have $G \setminus \{g\}$ is not a generating set.

Definition 2.1.17. $R \subset \mathbb{K}[x_1, \ldots, x_n]$ is said to be *generated by monomials* if R has a generating set G with each $g \in G$ a monomial. We also call such an R a *monomial subalgebra.*

Remark. *If R is a finitely generated monomial subalgebra, then R has a finite monomial generating set. Indeed, if $\{g_1, \ldots, g_n\} \subset R$ is a generating set, and $H \subset R$ a monomial generating set, then each g_i is a combination of finitely many elements of H. Taking the finite collection of these finitely many elements we obtain a finite monomial generating set for R.*

Let R be a finitely generated $\mathbb{K}$-algebra, and D a locally nilpotent derivation on R. Essen [14] provides an algorithm to compute generators for R which completes in a finite number of steps if R^D is finitely generated.

2.2 GROUP ACTIONS AND INVARIANT RINGS

Locally nilpotent derivations are very closely related to invariant rings under additive group actions, and we introduce these here. If R is a $\mathbb{K}$-domain then let $V = \operatorname{Spec}(R)$ denote the corresponding scheme, which is an affine variety when R is affine. Conversely, given an affine variety V, we denote its coordinate ring by R. We denote affine k-space by $\mathbb{A}^k$, with coordinate ring the polynomial ring $\mathbb{K}[x_1, \ldots, x_k]$. Suppose that G is an algebraic group acting by morphisms on an affine variety V, then G acts by $\mathbb{K}$-algebra automorphisms on R as

$$(g \cdot f)(v) := f(g^{-1} \cdot v) \text{ for all } x \in V,\ f \in R.$$

The *invariant ring* for this action is

$$R^G := \{f \in R \,|\, g \cdot f = f \text{ for all } g \in G\}.$$

An element $f \in R^G$ is called an *invariant*. The *fixed point set* of the action is defined as

$$V^G := \{x \in V \mid g \cdot x = x \text{ for all } g \in G\}.$$

Definition 2.2.1. Let G be an algebraic group acting on two varieties V and W, a morphism $\varphi : V \longrightarrow W$ is called *G-equivariant* relative to these two actions if $\varphi(g \cdot v) = g \cdot \varphi(v)$ for all $g \in G$ and $v \in V$.

The invariant ring of a group action is not necessarily finitely generated, though if G satisfies certain properties then it must be. A group G is *reductive* if it contains no nontrivial connected normal unipotent subgroup, we then have [18]:

Theorem 2.2.2. *Hilbert's Finiteness theorem: Suppose $\mathbb{K}$ is a field, and G is a reductive algebraic $\mathbb{K}$-group acting by algebraic automorphisms on an affine $\mathbb{K}$-variety V, then the ring of invariants R^G is finitely generated over $\mathbb{K}$.*

Given an affine variety V and an algebraic group G acting on V, we can define a quotient $V/\!\!/G$. When G is reductive, the ring R^G is finitely-generated, so we get a corresponding variety $V/\!\!/G := \operatorname{Spec}(R^G)$. When G is not reductive, R^G may not be finitely generated. Nevertheless, we may still define $V/\!\!/G := \operatorname{Spec}(R^G)$ as an affine scheme, and the usual universal property still holds in the category of affine schemes, [28, p.3]. We record this in the following definition:

Definition 2.2.3. Given V, an affine variety, and G, an algebraic group acting on V, there is a morphism induced by the inclusion $R^G \subset R$:

$$\pi_V \colon V \to V/\!\!/G := \operatorname{Spec}(R^G),$$

we call this the *quotient morphism*. $V/\!\!/G$ is the *categorical quotient* in the category of affine schemes, satisfying the universal property that every G-invariant morphism from V to some affine scheme W factors uniquely through π_V.

Now suppose $G = \mathbb{G}_a = (\mathbb{K}, +)$ is the additive group. It is known that in this case we have a one-to-one correspondence between algebraic group actions and locally nilpotent derivations, [19, §1.5]. Let D be a derivation on R, first we note

Proposition 2.2.4. *If $f \in R$ is nonzero, then fD is a locally nilpotent derivation if and only if D is locally nilpotent and $f \in R^D$.*

Proof. It is clear that if D is locally nilpotent and $f \in R^D$ that fD is also locally nilpotent. Suppose fD is locally nilpotent, we may assume that $D \neq 0$. Recalling the degree function from Example 2.1.6, set $N = \nu_{fD}(f) \geq 0$, and suppose $g \in R \setminus \{0\}$. On the one hand, if $D^n(g) \neq 0$ for some $n \geq 1$ then $\nu_{fD}(D^n(g)) \geq 0$, and we have

$$\nu_{fD}(f \cdot D^n(g)) = \nu_{fD}\left((fD)(D^{n-1}(g))\right) = \nu_{fD}(D^{n-1}(g)) - 1,$$

whilst on the other hand we have

$$\nu_{fD}(f \cdot D^n(g)) = \nu_{fD}(f) + \nu_{fD}(D^n(g)) = N + \nu_{fD}(D^n(g)).$$

Therefore

$$\nu_{fD}(D^n(g)) = \nu_{fD}(D^{n-1}(g)) = -(N+1),$$

for all $n \geq 1$. This implies

$$\nu_{fD}(D^n(g)) = \nu_{fD}(g) - n(N+1),$$

which is impossible since ν_{fD} cannot take values in the negative integers. Thus, D must be locally nilpotent. To see that $f \in R^D$, note that $(fD)(f) \in f(R)$. Suppose that $\nu_D(f) \geq 1$ and let $a \in R$ satisfy $D(f) = a$, then $\nu_D(a) \geq 0$ and

$$\nu_D(f) - 1 = \nu_D(D(f)) = \nu_D(af) = \nu_D(a) + \nu_D(f) \geq \nu_D(f),$$

which is absurd. So $f \in R^{fD}$; and since R is a domain we must have $R^{fD} = R^D$. □

Furthermore, given a locally nilpotent derivation D on R, we can define an *exponential map* $\exp(D) : R \longrightarrow R$ as

$$\exp(D)(f) := \sum_{i \geq 0} \frac{1}{i!} D^i(f).$$

Proposition 2.2.5. *1.* $\exp(D)$ *defines a* $\mathbb{K}$*-automorphism of* R*.*

2. If $[D, E] = DE - ED = 0$ *for a locally nilpotent derivation* E*, then* $D + E$ *is also locally nilpotent and* $\exp(D + E) = \exp(D) \circ \exp(E)$*.*

3. The subgroup of $\mathbb{K}$*-automorphisms of* R *generated by locally nilpotent derivations is normal.*

Proof. Note that every function D^i is additive, hence $\exp D(f)$ is additive. Now suppose that $f, g \in R$ are nonzero, with $\nu_D(f) = m$ and $\nu_D(g) = n$. Then

$$\begin{aligned}
(\exp D)(f) \cdot (\exp D)(g) &= \left(\sum_{i=0}^{m} \frac{1}{1!} D^i(f)\right) \cdot \left(\sum_{i=0}^{n} \frac{1}{j!} D^j(g)\right) \\
&= \sum_{0 \le i+j \le m+n} \frac{1}{i!j!} D^i(f) D^j(g) \\
&= \sum_{0 \le i+j \le m+n} \frac{1}{(i+j)!} \binom{i+j}{j} D^i(f) D^j(g) \\
&= \sum_{0 \le t \le m+n} \frac{1}{t!} \left(\sum_{i+j=t} \binom{i+j}{j} D^i(f) D^j(g) \right) \\
&= \sum_{0 \le t \le m+n} \frac{1}{t!} D^t(fg) \\
&= (\exp D)(fg).
\end{aligned}$$

Thus $\exp D$ is an algebra homomorphism.

Now for part 2 fix $f \in R$ and $m \ge 0$ so that $D^m(f) = E^m(f) = 0$. Set $n = 2m$, since D and E commute

$$(D+E)^n(f) = \sum_{i+j=n} \binom{n}{i} D^i E^j(f) = 0,$$

as each term of this sum has either $i \ge m$ or $j \ge m$. Furthermore, using this same expansion for $(D+E)^n$, the proof that $\exp(D+E) = \exp(D) \circ \exp(E)$ now follows by mimicking the argument used above, proving 2. Additionally, by Proposition 2.2.4, $-D$ is also locally nilpotent, and so by part 2, it follows that

$$\exp(D) \circ \exp(-D) = \exp(-D) \circ \exp(D) = \exp(0) = I.$$

Therefore we conclude that $\exp D$ is an automorphism, completing the proof of part 1. Now part 3 follows from the observation that for a $\mathbb{K}$-automorphism α we have

$$\alpha(\exp D)\alpha^{-1} = \exp(\alpha D \alpha^{-1}),$$

and $\alpha D \alpha^{-1}$ is again locally nilpotent. □

Suppose $V = \mathrm{Spec}\,(R)$ is the corresponding affine variety and let $\mathrm{Aut}_{\mathbb{K}}(R)$ be the set of all $\mathbb{K}$-automorphisms of R. Combining these two propositions, given a locally nilpotent derivation D, we obtain a group homomorphism

$$\eta : (R^D, +) \longrightarrow \mathrm{Aut}_{\mathbb{K}}(R), \quad \eta(f) = \exp(fD).$$

If $D \neq 0$, then η is injective. Now restricting η to the subgroup $\mathbb{G}_a$, we obtain the algebraic representation $\eta : \mathbb{G}_a \hookrightarrow \mathrm{Aut}_{\mathbb{K}}(R)$. Geometrically, this means that D induces the faithful algebraic $\mathbb{G}_a$-action $\exp(tD)$ on X, $t \in \mathbb{K}$.

Now conversely, let $\mu : \mathbb{G}_a \times V \longrightarrow V$ be an algebraic $\mathbb{G}_a$-action over $\mathbb{K}$. Then μ induces a derivation $\mu'(0)$, where differentiation takes place relative to $t \in \mathbb{G}_a$. In this situation, we in fact have $D = (\exp(tD))'(0)$ and $\mu = \exp(t\mu'(0))$. We have therefore shown that there is a bijective correspondence between the locally nilpotent derivations of R and the set of all algebraic $\mathbb{G}_a$-actions on V, where D induces the action $\exp(tD)$ and the action μ induces the derivation $\mu'(0)$. Additionally, the kernel of the derivation coincides with the invariant ring of the corresponding action:

$$R^D = R^{\mathbb{G}_a},$$

as $D(f) = 0$ if and only if $\exp(tD)(f) = f$ for all $t \in \mathbb{K}$.

Example 2.2.6. Consider the Weitzenböck derivation D_n on R_n from Example 2.1.12, the corresponding group action on R_n is given by

$$t \cdot x_i = \sum_{j=0}^{i} \frac{t^j}{j!} x_{i-j}.$$

For $f \in R_n$ we have $t \cdot f = \exp(tD_n)f$.

Now, given an additive group action of $\mathbb{G}_a$ on V, we can ask if there is some $\mathbb{G}_m$-action on V commuting with our $\mathbb{G}_a$-action. If so, then $\mathbb{G}_m$ acts on $V /\!/ \mathbb{G}_a$ and hence induces a grading on $R^{\mathbb{G}_a}$, as well as on R, see for example [19, §10.2]. When $V \subset \mathbb{A}^k$ is affine, there is some maximal subtorus of the natural k-dimensional torus action on $\mathbb{A}^k$ that is $\mathbb{G}_a$-equivariant. Since $R^{\mathbb{G}_a} = R^D$ for some locally nilpotent derivation D, this grading is also induced on R^D, with the property that if $f \in R_{(a_1,\ldots,a_k)}$ some graded piece of R, then $D(f) \in R_{(a_1,\ldots,a_k)}$ also. If we combine this grading with the degree function ρ from Example 2.1.5 D then splits into linear maps

$$D : R_{(a_1,\ldots,a_k,b)} \longrightarrow R_{(a_1,\ldots,a_k,b-1)}.$$

2.3 GRÖBNER AND SAGBI-BASES

Here we introduce the concepts of monomial orderings and Gröbner/SAGBI bases, we follow [5, § 1.].

Definition 2.3.1. Let $M \subset \mathbb{K}[x_1, \ldots, x_n]$ be the set of all monomials. A total order " $>$ " on M is a *monomial ordering* if it satisfies the following conditions:

1. $m > 1$ for all $m \in M \setminus \{1\}$.
2. Given $t, m_1, m_2 \in M$, if $m_1 > m_2$, then $tm_1 > tm_2$.

A monomial ordering can also compare terms. Given a monomial ordering, for a nonzero polynomial f we can uniquely write $f = ct + r$ where $t \in M$ and $c \in \mathbb{K} \setminus \{0\}$ and every term of g is smaller than t with respect to " $>$ ". We write

$$LT(f) := ct, \quad LM(f) := t, \quad LC(f) := c,$$

for the *leading term, leading monomial* and *leading coefficient* of f. If $f = 0$, we consider all three to be 0.

Example 2.3.2. On $\mathbb{K}[x_1, \ldots, x_n]$, the *lexicographic* monomial ordering, " $>_{lex}$ " is defined in the following way: We set $x_1^{a_1} \cdots x_n^{a_n} > x_1^{b_1} \cdots x_n^{b_n}$ if $a_i > b_i$ for the smallest i with $a_i \neq b_i$. For example $x_1^j > x_2^k$ for all $j, k \geq 1$, and $LM(6x_1 + x_2x_3^3 + 2x_5) = x_1$. The (reverse) lexicographic monomial ordering sets $x_1^{a_1} \cdots x_n^{a_n} > x_1^{b_1} \cdots x_n^{b_n}$ if $a_i > b_i$ for the largest i with $a_i \neq b_i$. When we opt to use the lexicographic monomial ordering, we indicate which one we are using by writing either $x_1 < \cdots < x_n$ or $x_1 > \cdots > x_n$.

Now fix a monomial ordering on $\mathbb{K}[x_1, \ldots, x_n]$. For a set $S \subset \mathbb{K}[x_1, \ldots, x_n]$ of polynomials, we write

$$L(S) := (LM(g) \,|\, g \in S),$$
$$L_{alg}(S) := \mathbb{K}[LM(g) \,|\, g \in S].$$

$L(S)$ is the ideal generated by the leading monomials from S, which we call the *leading ideal* of S, and $L_{alg}(S)$ is the subalgebra generated by the leading monomials from S, which we call the *leading algebra* of S.

Definition 2.3.3. Let $I \subset \mathbb{K}[x_1, \ldots, x_n]$ be an ideal. A finite subset $\mathcal{G} \subset I$ is called a *Gröbner* basis of I if $L(I) = L(\mathcal{G})$.

Remark. *A Gröbner basis of an ideal I generates I as an ideal. If $I \setminus (\mathcal{G})$ is nonempty, then we can take $f \in I \setminus (\mathcal{G})$ with minimal leading monomial. Since $L(I) = L(\mathcal{G})$, there is some $g \in \mathcal{G}$ with $LM(g) = LM(f)$, and so $f - g \in I \setminus (\mathcal{G})$ has smaller leading monomial than f, a contradiction. Note also that $I = \mathbb{K}[x_1, \ldots, x_n]$ if and only if $\mathcal{G}$ contains a nonzero constant polynomial.*

We note Hilbert's basis theorem:

Theorem 2.3.4. *Hilbert's Basis Theorem If R is a Noetherian ring, then $R[x]$ is a noetherian ring.*

By Hilbert's basis theorem, all ideals in $\mathbb{K}[x_1, \cdots x_n]$ are finitely generated, thus Gröbner bases can always be taken to be finite.

Definition 2.3.5. Let $S \subset \mathbb{K}[x_1, \ldots, x_n]$ be a subalgebra. A finite subset $\mathcal{S} \subset S$ is called a *Subalgebra Analogue for Gröbner Bases of Ideals* or *SAGBI*-basis if $L_{alg}(S) = L_{alg}(\mathcal{S})$.

Remark. *Similarly to a Gröbner basis, a SAGBI-basis of a subalgebra S generates S as an algebra. SAGBI-bases were first constructed by Robbiano and Sweedler in [30] and Kapur and Madlener in [24] independently.*

Contrary to Gröbner bases, we do not require that SAGBI-bases be finite. Furthermore, even if $S \subset \mathbb{K}[x_1, \ldots, x_n]$ is finitely generated then it may not admit a finite SAGBI-basis. For example, the subalgebra $\mathbb{K}[x + y, xy, xy^2] \subset \mathbb{K}[x, y]$ does not admit a finite SAGBI-basis for any choice of monomial ordering [30, Example 1.20].

2.4 FINITE GENERATION IDEALS

In this section we define the finite generation ideal, first introduced by Derksen and Kemper [6, § 2.2], before covering its properties. We then cover some examples where the finite generation ideal has been computed.

Definition 2.4.1. Let R be a $\mathbb{K}$-domain, the *finite generation ideal* of R is defined as

$$\mathfrak{f}_R := \{g \in R \setminus \{0\} \mid R_g \text{ is finitely generated as a } \mathbb{K}\text{-algebra}\} \cup \{0\}.$$

Note that if R is finitely generated and $f \in R \setminus \{0\}$, then R_f is also finitely generated. Thus in this case $\mathfrak{f}_R = R$. Following [6, § 2.2], we show that the finite generation ideal has the following properties:

Lemma 2.4.2. *Let R be a $\mathbb{K}$-domain, then $\mathfrak{f}_R$ is a radical ideal and if R is a subalgebra of a finitely generated $\mathbb{K}$-domain S, then $\mathfrak{f}_R \neq 0$.*

First we note that the radical of an ideal $I \subset R$ is

$$\sqrt{I} := \{f \in R \,|\, f^n \in R \text{ for some } n \geq 0\},$$

an ideal is radical if it is equal to its radical. Additionally we require the following result, a proof of which can be found in [11, § 14.2]

Lemma 2.4.3. *Grothendieck's Generic Freeness Lemma*
Suppose that R is a Noetherian domain and S is a finitely generated R-algebra. If M is a finitely generated S-module, then there is an element $a \in R \setminus \{0\}$ such that $M[a^{-1}]$ is a free $R[a^{-1}]$-module.

Proof. Proof of Lemma 2.4.2 We begin by first showing that $\mathfrak{f}_R$ is a radical ideal. To do so, we first prove the following:

Claim. *If $f, g \in R \setminus \{0\}$ with $(f, g) = R$ and R_f and R_g finitely generated, then so is R and $R = R_f \cap R_g$.*

To show this, we write $R_f = \mathbb{K}[a_1, \ldots, a_r, f^{-1}]$ and $R_g = \mathbb{K}[b_1, \ldots, b_l, g^{-1}]$ with $a_i, b_j \in R$. Then $1 = sf + tg$ with $s, t \in R$. Let $z \in R_f \cap R_g$, then

$$z = \frac{a}{f^m} = \frac{b}{g^n}, \quad n, m \in \mathbb{N},\, a \in \mathbb{K}[a_1, \ldots, a_r, f],\, b \in \mathbb{K}[b_1, \ldots, b_l, g],$$

and so

$$z = z(sf+tg)^{m+n} = \sum_{i=1}^{m} \binom{m+n}{i} (sf)^i t^{m+n-i} g^{m-i} b + \sum_{i=m+1}^{m+n} \binom{m+n}{i} s^i f^{i-m} (tg)^{m+n-i} a.$$

Thus

$$R \subset R_f \cap R_g \subset \mathbb{K}[a_1, \ldots, a_r, b_1, \ldots, b_l, f, g, s, t] \subset R,$$

implying the claim.

Now suppose $f, g \in \mathfrak{f}_R$ are non-zero, then

$$R_{fg} = (R_f)_g,$$

is finitely generated, as R_f is finitely generated, so $fg \in \mathfrak{f}_R$. Now suppose that $f, g \in \mathfrak{f}_R$ with f, g and $f + g$ all non-zero. We have $(f, g)R_{f+g} = R_{f+g}$, and the algebras $(R_{f+g})_f = (R_f)_{f+g}$ and $(R_{f+g})_g = (R_g)_{f+g}$ are finitely generated. By the claim, R_{f+g} is finitely generated, so $f + g \in \mathfrak{f}_R$, hence $\mathfrak{f}_R$ is an ideal. This ideal is clearly radical since $R_{f^r} = R_f$ for every $f \in R$ and any positive integer r.

To prove that $\mathfrak{f}_R \neq 0$ when R is a subalgebra of a finitely generated $\mathbb{K}$-domain S, let $T \subset R$ be a finitely generated subalgebra with the property that T and R have the same quotient field. By Lemma 2.4.3, there is a nonzero $f \in T$ with S_f a free T_f-module. Let B be a basis of S_f over T_f, we can write

$$1 = \sum_{i=1}^{r} u_i e_i,$$

with $e_i \in B$ and $u_i \in T_f$ for all i. Since R_f and T_f have the same quotient field, it follows that the submodule $R_f \subset S_f$ is contained in

$$\bigoplus_{i=1}^{r} T_f e_i \cong T_f^r.$$

This shows that R_f is contained in a finitely generated T_f-module. Since T_f is a finitely generated algebra, R_f is finitely generated as a T_f-module. It then follows that R_f is a finitely generated algebra. □

When working with locally nilpotent derivations, we can show more about the finite generation ideal:

Proposition 2.4.4. *Let R be a commutative $\mathbb{K}$-domain, and let $D : R \longrightarrow R$ be a locally nilpotent derivation, then $\mathfrak{pl}(D) \subset \mathfrak{f}_R$.*

To show this, we follow [10, § 5]:

Definition 2.4.5. A $\mathbb{G}_a$-variety V is called a *trivial $\mathbb{G}_a$-bundle* if there is a $\mathbb{G}_a$-equivariant morphism

$$V \longrightarrow \mathbb{G}_a.$$

Equivalently, there is a $\mathbb{G}_a$-equivariant ismorphism

$$\mathbb{G}_a \times X \longrightarrow V,$$

where X can be identified with the orbit space $V/\mathbb{G}_a$.

In this case the quotient morphism $\pi : V \longrightarrow V/\mathbb{G}_a$ admits a section. If V is affine, then $V/\mathbb{G}_a = \text{Spec}\,(R)^{\mathbb{G}_a}$. We now prove Proposition 2.4.4:

Proof. of Proposition 2.4.4 Let $d \in \mathfrak{pl}(D)$ and suppose $p \in R$ satisfies $D(p) = d$, then $D\left(\frac{p}{d}\right) = 1$ and so the morphism

$$\frac{p}{d} : V_s \longrightarrow \mathbb{G}_a,$$

is $\mathbb{G}_a$-equivariant. To see this, let $x \in V_d$, and $\alpha \in \mathbb{G}_a$, then as

$$\alpha \cdot \frac{p}{d} = \exp(\alpha D)\left(\frac{p}{d}\right) = \frac{p}{d} + \alpha \cdot 1,$$

we have

$$\left(\alpha \cdot \frac{p}{d}\right)(x) = \left(\frac{p}{d} + \alpha \cdot 1\right)(x) = \left(\frac{p}{d}\right)(x) + \alpha = \alpha \cdot \left(\left(\frac{p}{d}\right)(x)\right).$$

where we have used that $d \in R^{\mathbb{G}_a}$. We conclude then that V_d is a trivial $\mathbb{G}_a$-bundle, and $V_d/\mathbb{G}_a = \operatorname{Spec}(R_d)^{\mathbb{G}_a}$ and, in particular, $R_d^{\mathbb{G}_a} = (R^{\mathbb{G}_a})_d$ is finitely generated. □

We now cover some examples where the finite generation ideal has been computed:

Example 2.4.6. Suppose $R = \mathbb{K}[x, xy, \ldots, xy^n, \ldots] \subset \mathbb{K}[x, y]$ is the graded subalgebra generated by monomials of the form xy^n for $n \geq 0$. Let $M := (x, xy, \ldots)$ be the maximal graded ideal of R. We show that $\mathfrak{f}_R = \sqrt{xR} = M$. Indeed, consider first R_x, then $x^{-1} \cdot (xy) = y \in R_x$ and we conclude that $R_x = \mathbb{K}[x, y]_x$. Now $\mathbb{K}[x, y]_x$ is finitely generated and thus $\sqrt{xR} \subset \mathfrak{f}_R$. For xy^n with $n \geq 1$, note that $(xy^n)^2 = x \cdot xy^{2n} \in \sqrt{xR}$, and we have shown $M \subset \sqrt{xR}$, but since $\mathfrak{f}_R \subset M$, we conclude the result.

Example 2.4.7. Roberts' Example Let $R := \mathbb{K}[x_1, x_2, x_3, y_1, y_2, y_3, z]$ and consider the locally nilpotent derivation

$$D := x_1^n \frac{\partial}{\partial y_1} + x_2^n \frac{\partial}{\partial y_2} + x_3^n \frac{\partial}{\partial y_3} + (x_1 x_2 x_3)^{n-1} \frac{\partial}{\partial z},$$

with corresponding group action:

$$t \cdot (a_1, a_2, a_3, b_1, b_2, b_3, c) = (a_1, a_2, a_3, b_1 + ta_1^n, b_2 + ta_2^n, b_3 + ta_3^n, c + t(a_1 a_2 a_3)^{n-1}).$$

The kernel, R^D of this derivation is non-finitely generated, and has a generating set $G = \{\beta_{i,n},\ u_{j,k} \mid i = 1, 2, 3,\ n \geq 0, j, k = 1, 2, 3,\ j < k\}$, where

$$\begin{aligned}
\beta_{i,n} &:= x_i v^n + \text{ terms of lower } v-\text{degree},\\
u_{1,2} &:= x_1^n y_2 - x_2^n y_1,\\
u_{1,3} &:= x_1^n y_3 - x_3^n y_1,\\
u_{2,3} &:= x_2^n y_3 - x_3^n y_2.
\end{aligned}$$

Kuroda [25] showed that this generating set G forms a SAGBI-basis for R^D using the lexicographic monomial ordering with $x_1 < x_2 < x_3 < y_1 < y_2 < y_3 < z$. Then using this SAGBI-basis, Dufresne and Kraft [10, § 9] showed that the finite generation ideal of Roberts' example is

$$\mathfrak{f}_R = \sqrt{x_1, x_2, x_3 R},$$

where $\beta_{i,n} \in \mathfrak{f}_R$ for all i, n whilst $u_{j,k} \notin \mathfrak{f}_R$ for all j, k.

— 3 —

Daigle and Freudenburg's counterexample

3.1 PREFACE

The work contained in this chapter arises from my paper 'The finite generation ideal for Daigle and Freudenburg's counterexample to Hilbert's fourteenth problem' [22]. As such the results within are entirely self-contained and the chapter has its own introduction and preliminaries. The material in Chapter 2 may aid the reader in providing greater depth to the preliminaries appearing in this chapter. This paper is on arXiv with number 2203.15569, and has been accepted for publication in Communications in Algebra.

This paper focuses on computing the finite generation ideal of Daigle and Freudenburg's counterexample to Hilbert's fourteenth problem, showing that it is the radical of an ideal generated by 3 elements, which each comprise the first element in an infinite family of generators for the invariant ring. In order to do so, we show that these 3 infinite families, together with an additional invariant, form a SAGBI-basis for the invariant ring.

3.2 INTRODUCTION

Let $\mathbb{K}$ be a field, and let $\mathbb{K}[x_1, x_2, \ldots, x_n]$ be a polynomial ring in n variables over $\mathbb{K}$, with $\mathbb{K}(x_1, x_2, \ldots, x_n)$ its field of fractions and L a subfield of $\mathbb{K}(x_1, x_2, \ldots, x_n)$. In his fourteenth problem, Hilbert asked whether the subalgebra $L \cap \mathbb{K}[x_1, x_2, \ldots, x_n]$ is finitely generated. In characteristic zero, this has been shown to not always be the case, with Nagata finding the first counterexample in 1959 [29]. Further examples have been found, for example by Roberts in 1990, [31]. In 1994, A'Campo-Neuen showed that Roberts' example arises as the invariant ring of a $\mathbb{G}_a$-action on $\mathbb{A}^7$ in [1], where $\mathbb{G}_a = (\mathbb{K}, +)$ is the additive group. The smallest known counterexample to Hilbert's problem which arises as an invariant ring was found by Daigle and Freudenburg in dimension five, [4]. Daigle and Freudenburg's example arises as an invariant ring of the following $\mathbb{G}_a$-action on $\mathbb{A}^5$ in characteristic zero:

$$\alpha \cdot (a, b, c, d, e) = \left(a, b + \alpha a^3, c + \alpha b + \frac{1}{2}\alpha^2 a^3, d + \alpha c + \frac{1}{2}\alpha^2 b + \frac{1}{6}\alpha^3 a^3, e + \alpha a^2\right).$$

The invariant ring of an additive group action is known to correspond to the kernel of a locally nilpotent derivation, see for example [19, §1.5]. Daigle and Freudenburg's $\mathbb{G}_a$-action corresponds to the kernel of the locally nilpotent derivation

$$D := x^3 \frac{\partial}{\partial s} + s \frac{\partial}{\partial t} + t \frac{\partial}{\partial u} + x^2 \frac{\partial}{\partial v},$$

on the polynomial ring $R := \mathbb{K}[x, s, t, u, v]$. There are many algorithms which have been developed to aid in computing kernels of locally nilpotent derivations which we exploit in this paper. Daigle and Freudenburg's example is also closely related to Roberts' example in dimension seven, and a counterexample found by Freudenburg in dimension six, [17]. It is possible to construct Daigle and Freudenburg's counterexample by "removing symmetries" from Roberts' example, [19, §7.2] and there is a $\mathbb{K}$-algebra homomorphism from Freudenburg's example to Daigle and Freudenburg's which induces a surjective homomorphism on the invariant rings, [32, §2].

Castravet and Tevelev [3] provide an infinite family of non-finitely generated $\mathbb{K}$-algebras, which Doran, Giansiracusa, and Jensen [7] realised as the ring of invariants of a $\mathbb{G}_a$-action on a polynomial ring. Additionally, Kuroda has generalised Robert's example in [25]. However, it remains a difficult task to construct counterexamples as invariant rings from $\mathbb{G}_a$-actions; and little is known about the structure of these invariant rings in general.

In this paper we determine the *finite generation ideal* of the invariant ring, R^D, defined below. That is, the radical ideal of elements $f \in R^D$ for which R^D_f is finitely generated, [6, §2]. Such an ideal can be understood to track how far a ring is from being finitely generated, with it being the ring itself when R^D is finitely generated. Dufresne and Kraft computed the finite generation ideal for Roberts' example in [10, §9] and our computation shows that the finite generation ideal is what would be expected by "removing symmetries" from Roberts' example. Preliminaries and some early results on Daigle and Freudenburg's example are covered in Sections 3.3. and 3.4 respectively.

In order to compute the finite generation ideal, we first construct a generating set for the $\mathbb{K}$-algebra generators of the invariant ring in Section 3.5 with useful properties. This requires us to construct three infinite families of invariants using a method similar to van den Essen in [13]. We then show that these families, together with an additional invariant, generate the invariant ring. In fact, we show that this generating set forms a SAGBI-basis for R^D; that is, a Subalgebra Analogue for a Gröbner Basis of Ideals, Definition 3.5.3. Our calculation of the SAGBI-basis uses an argument similar to that of Kuroda for Roberts' example, [25, §3]. SAGBI-bases were first constructed by Robbiano and Sweedler in [30] and Kapur and Madlener [24] independently. The properties of a SAGBI-basis and the relations between its elements are key to our computation of the finite generation ideal in Section 3.6, which comprises Theorem 3.6.1, our main result. We additionally show that the leading terms of these three infinite families generate the subalgebra generated by the leading terms of the finite generation ideal.

3.3 PRELIMINARIES

Throughout the following we fix $\mathbb{K}$ to be an algebraically closed field of characteristic zero. We will begin with a few preliminaries, followed by a discussion of the invariant ring of a group action. We follow parts of [14], [10] and [19]. Let R be a commutative $\mathbb{K}$-domain and recall that a derivation $D\colon R \to R$ is *locally nilpotent* if, for all $a \in R$, there is some $n \in \mathbb{N}$ for which $D^n(a) = 0$. We denote the kernel of the derivation by R^D and its image by $D(R)$. Note that R^D is a subring of R. We call an element $p \in R$ with $D(p) \in R^D$ and $D(p) \neq 0$ a *local slice* for D. The *plinth ideal* is defined as $\mathfrak{pl}(D) := R^D \cap D(R)$; it is simple to check that $\mathfrak{pl}(D)$ is an ideal of R^D, see for example [19, p.17].

Definition 3.3.1. Let R be a $\mathbb{K}$-domain, the *finite generation ideal* of R is defined as

$$\mathfrak{f}_R := \{g \in R \setminus \{0\} \mid R_g \text{ is finitely generated as a } \mathbb{K}\text{-algebra}\} \cup \{0\}.$$

Note that if R is finitely generated, then $\mathfrak{f}_R = R$. If R is a subalgebra of a finitely generated algebra, then $\mathfrak{f}_R$ is non-zero; additionally, $\mathfrak{f}_R$ is a radical ideal, [6, §2.2].

One can hence view the finite generation ideal as a form of measure of how far R is from being finitely generated by comparing $\mathfrak{f}_R$ to R. Of particular interest to us in this paper is $\mathfrak{f}_{R^D}$, where D is a locally nilpotent derivation.

If $R = \mathbb{K}[X_1, \ldots, X_n]$ we call an element of the form $X_1^{a_1} \cdots X_n^{a_n}$ a *monomial* and an element of the form $\alpha \cdot X_1^{a_1} \cdots X_n^{a_n}$, where $\alpha \in \mathbb{K} \setminus \{0\}$ is called a *term*; a *polynomial* is a sum of terms.

Definition 3.3.2. Suppose $R = \mathbb{K}[X_1, X_2, \ldots, X_n]$ and suppose that $D = a_1 \frac{\partial}{\partial X_1} + \cdots + a_n \frac{\partial}{\partial X_n}$ is a derivation on R. We say that D is a *monomial* derivation if each $a_i \in \mathbb{K}[X_1, X_2, \ldots, X_n]$ is a monomial; a derivation is called *triangular* if $a_1 \in \mathbb{K}$ and each $a_i \in \mathbb{K}[X_1, \ldots, X_{i-1}]$ for $2 \leq i \leq n$.

If R is a $\mathbb{K}$-domain then $V = \operatorname{Spec}(R)$ denotes the corresponding scheme, which is an affine variety when R is affine. We denote affine k-space by $\mathbb{A}^k$, with coordinate ring the polynomial ring $\mathbb{K}[x_1, \ldots, x_k]$. Suppose that G is an algebraic group acting on an affine variety V, then G acts by $\mathbb{K}$-algebra automorphisms on R as

$$g \cdot f(v) := f(g^{-1} \cdot v) \text{ for all } x \in V,\ f \in R.$$

The *invariant ring* for this action is

$$R^G := \{f \in R \,|\, g \cdot f = f \text{ for all } g \in G\}.$$

An element $f \in R^G$ is called an *invariant*. The *fixed point set* of the action is defined as

$$V^G := \{x \in V \,|\, g \cdot x = x \text{ for all } g \in G\}.$$

We focus on the case where $G = \mathbb{G}_a = (\mathbb{K}, +)$ is the additive group of the field $\mathbb{K}$. It is known that in this case we have a one-to-one correspondence between algebraic group actions and locally nilpotent derivations, [19, §1.5]. Let D be a locally nilpotent derivation on R, an element $\alpha \in \mathbb{G}_a$ acts on R as

$$\exp(\alpha D)(f) := \sum_{i=0}^{\infty} \frac{1}{i!} \alpha^i D^i(f).$$

Note that since D is locally nilpotent, for each $f \in R$ the displayed sum has only finitely many non-zero terms. Conversely a given $\mathbb{G}_a$-action $\rho : \mathbb{G}_a \times V \to V$ induces a derivation $\rho'(0)$, which can be shown to be locally nilpotent, see for example [19, §1.5]. Additionally, the invariant ring of the group action is equal to the kernel of the derivation, that is $R^{\mathbb{G}_a} = R^D$.

Given an affine variety V and an algebraic group G acting on V, we wish to define a quotient $V/\!\!/G$. When G is reductive, the ring R^G is finitely-generated, so we get a corresponding variety $V/\!\!/G := \mathrm{Spec}\,(R^G)$. When G is not reductive, R^G may not be finitely generated. Nevertheless, we may still define $V/\!\!/G := \mathrm{Spec}\,(R^G)$ as an affine scheme, and the usual universal property still holds in the category of affine schemes, [28, p.3]. We record this in the following definition:

Definition 3.3.3. Given V, an affine variety, and G, an algebraic group acting on V, there is a morphism induced by the inclusion $R^G \subset R$:

$$\pi_V \colon V \to V/\!\!/G := \mathrm{Spec}\,(R^G),$$

we call this the *quotient morphism*. $V/\!\!/G$ is the *categorical quotient* in the category of affine schemes, satisfying the universal property that every G-invariant morphism from V to some affine scheme W factors uniquely through π_V.

Given an additive group action of $\mathbb{G}_a$ on V, we can ask if there is some $\mathbb{G}_m$-action on V commuting with our $\mathbb{G}_a$-action. If so, then $\mathbb{G}_m$ acts on $V/\!\!/\mathbb{G}_a$ and hence induces a grading on $R^{\mathbb{G}_a}$, as well as on R, see for example [19, §10.2]. When $V \subset \mathbb{A}^k$ is affine, there is some maximal subtorus of the natural k-dimensional torus action on $\mathbb{A}^k$ that is $\mathbb{G}_a$-equivariant.

If X is a variety with an action of the additive group $\mathbb{G}_a$, we say that X is a *trivial* $\mathbb{G}_a$*-bundle* if there is a $\mathbb{G}_a$-equivariant morphism $X \to \mathbb{G}_a$. In this case we can identify $X/\!\!/\mathbb{G}_a$ with $X/\mathbb{G}_a$ and the quotient morphism $\pi_X : X \to X/\mathbb{G}_a$ admits a section. If X is affine, then $X/\mathbb{G}_a = \mathrm{Spec}\,(R)^{\mathbb{G}_a}$.

Suppose $V = \mathrm{Spec}\,(R)$ is an affine $\mathbb{K}$-variety, with G an algebraic group acting on V. Suppose also that $R = \bigoplus_{i\geq 0} R_i$ is a graded ring with $R_0 = \mathbb{K}$, and let z_0 be its homogeneous maximal ideal. This means that V admits an action of the multiplicative group $\mathbb{G}_m = \mathbb{K}^*$ with $z_0 \in \mathrm{Spec}\,(R)$ the unique closed orbit. We say that V is a *fix-pointed* G-variety with fixed point z_0 if this $\mathbb{G}_m$-action commutes with the G-action. Note that $V/\!\!/G$ is also fix-pointed, and we define the *nullcone* as $\mathcal{N}_V = \pi^{-1}(\pi(z_0))$. Thus, given an additive group action of $\mathbb{G}_a$ on an affine variety V, the nullcone can be defined using the induced $\mathbb{Z}^r$-grading on R.

Let D be a locally nilpotent derivation on an affine $\mathbb{K}$-domain R, with $V = \mathrm{Spec}\,(R)$. Suppose $x \in V$ is a fixed point under the corresponding $\mathbb{G}_a$-action and, for $f \in R$, let $n \in \mathbb{N}$ be such that $D^n(f) \neq 0$ and $D^{n+1}(f) = 0$. Then we have

$$f(x) = \exp(\alpha D)(f)(x) = f(x) + \alpha D(f)(x) + \cdots \frac{1}{n!}\alpha^n D^n(f)(x).$$

Letting α vary, we conclude that $D^i(f)(x) = 0$ for all $i \geq 1$, and in particular $D(f)(x) = 0$. Now suppose that for $x \in R$ we have $D(f)(x) = 0$ for all $f \in R$. Then for $f \in R$

$$\exp(\alpha D)(f)(x) = f(x) + \alpha D(f)(x) + \cdots \frac{1}{n!}\alpha^n D^n(f)(x) = f(x),$$

thus x is a fixed point. We have shown that:

$$V^{\mathbb{G}_a} = \{x \in V \mid D(f)(x) = 0 \text{ for all } f \in R\} = \{x \in V \mid f(x) = 0 \text{ for all } f \in D(R)\}.$$

Consider $\mathcal{P}_V := \mathcal{V}(\mathfrak{pl}(D))$, the *Plinth variety* of V. We have

$$\mathcal{P}_V = \{x \in V \mid D(f)(x) = 0 \text{ for all } f \in R \text{ with } D^2(f) = 0\},$$

and hence we have $V^{\mathbb{G}_a} \subset \mathcal{P}_V$. Given a $\mathbb{Z}^r_{\geq 0}$-grading on $R^{\mathbb{G}_a}$ induced by a $(\mathbb{G}_m)^r$-action, we define $R^{\mathbb{G}_a}_+ := \{f \in R^{\mathbb{G}_a} \mid \deg(f) \neq (0, 0, \ldots, 0)\}$, the maximal graded ideal of $R^{\mathbb{G}_a}$. Our definition of the nullcone may then be rewritten as

$$\mathcal{N}_V = \{x \in V \mid f(x) = 0 \text{ for all } f \in R^{\mathbb{G}_a}_+\}.$$

3.4 DAIGLE AND FREUDENBURG'S COUNTEREXAMPLE

We now construct Daigle and Freudenburg's counterexample. Let $V = \mathbb{A}^5$ and let $R := \mathbb{K}[x, s, t, u, v]$ be the polynomial ring over $\mathbb{K}$ in 5 variables. We consider the following locally nilpotent derivation on R

$$D := x^3\frac{\partial}{\partial s} + s\frac{\partial}{\partial t} + t\frac{\partial}{\partial u} + x^2\frac{\partial}{\partial v}. \tag{3.1}$$

This corresponds to a $\mathbb{G}_a$-action on R defined by

$$\exp(\alpha D)\cdot(x, s, t, u, v) = (x, s+\alpha x^3, t+\alpha s+\frac{1}{2}\alpha^2 x^3, u+\alpha t+\frac{1}{2}\alpha^2 s+\frac{1}{6}\alpha^3 x^3, v+\alpha x^2).$$

The $\mathbb{G}_a$-action on $\mathbb{A}^5$ commutes with the following $\mathbb{G}_m$-action

$$\lambda \cdot (x, s, t, u, v) := (\lambda \cdot x, \lambda^3 \cdot s, \lambda^3 \cdot t, \lambda^3 \cdot u, \lambda^2 \cdot v), \tag{3.2}$$

which induces a grading on R with $\deg(x) = 1$, $\deg(s) = \deg(t) = \deg(u) = 3$ and $\deg(v) = 2$. In the sequel when we refer to $f \in R$ as *homogeneous*, we mean homogeneous with respect to this grading. Likewise, for $f \in R$, the degree of f is the maximal degree of some term of f with respect to this grading. In our treatment of this example we will occasionally consider elements ordered by $\deg_v$ or $\deg_x$, which are defined for $f \in R$ as $\deg_x(f) :=\max\{n \mid f$ has a term of the form $\alpha \cdot x^n s^a t^b u^c v^d, \alpha \in \mathbb{K} \setminus \{0\}, a, b, c, d \in \mathbb{N}\}$ and similarly for v. A polynomial $f \in R$ has v-degree n if $\deg_v(f) = n$.

We also make use of the grading induced by the derivation D itself, which we refer to as the *ρ-grading*. It is defined first on the monomials in R with

$$\rho(m) := \{i \in \mathbb{Z}_{\geq 0} \mid D^i(m) \neq 0, D^{i+1}(m) = 0\}. \tag{3.3}$$

We then set, for $f \in R$, $\rho(f) := \max\{\rho(m) \mid m \text{ is a term of } f\}$. We set $\rho(0) := -\infty$. Elements homogeneous with respect to this grading will be called *ρ-homogeneous*. Observe that $\rho(x^a) = 0$, $\rho(s^b) = b$, $\rho(t^c) = 2c$, $\rho(u^d) = 3d$ and $\rho(v^e) = e$. Note that the ρ-grading is indeed a grading; set

$$R_n := \left\{ \sum_i \lambda_i x^a s^b t^c u^d v^e \mid \lambda_i \in \mathbb{K}, a \in \mathbb{N}, b + 2c + 3d + e = n \right\}.$$

Now, for $p \in R_i,\ q \in R_j$ non-zero, their product is non-zero, and all terms in pq are of the form mn, where $m \in R_i, n \in R_j$. We have

$$D^{i+j}(mn) = \sum_{l=0}^{i+j} \binom{i+j}{l} D^l(m) D^{i+j-l}(n) = \binom{i+j}{i} D^i(m) D^j(n) \neq 0,$$

whilst $D^{i+j+1}(mn) = D(D^i(m)D^j(n)) = D^{i+1}(m)D^j(n) + D^i(m)D^{j+1}(n) = 0$, so we conclude $pq \in R_{i+j}$.

Remark. *$\rho(2x^3t - s^2) = 2$ whilst $D(2x^3t - s^2) = 0$, so for $p \in R$, $\rho(p)$ can differ from the unique non-negative integer m with $D^m(p) \neq 0$ but $D^{m+1}(p) = 0$.*

Now let $S := \mathbb{K}[x, s, t, u]$, and define

$$\Delta := D|_S = x^3 \frac{\partial}{\partial s} + s \frac{\partial}{\partial t} + t \frac{\partial}{\partial u}.$$

Our notions of degree and ρ-degree restrict to $f \in S$. We observe that Δ is a triangular monomial derivation on $\mathbb{K}[x, s, t, u]$. By a result of Maubach, [26, § 3], we have that S^Δ is generated by at most four elements. Through an application of van den Essen's algorithm, [14, § 4], using the local slice $s \in S$ we find the following four generators of S^Δ:

$$\begin{aligned} \beta_0 &= x, \\ \gamma_0 &= 2x^3t - s^2, \\ \delta_0 &= 3x^6u - 3x^3st + s^3, \\ g\ &= 9x^6u^2 - 18x^3stu + 6s^3u + 8x^3t^3 - 3s^2t^2. \end{aligned} \tag{3.4}$$

Observe that $\beta_0^3, \gamma_0, \delta_0 \in \Delta(S)$, since

$$\begin{aligned} &\Delta(s) &&= x^3 = \beta_0^3, \\ &\Delta(3x^3u - st) &&= 2x^3t - s^2 = \gamma_0, \\ &\Delta(3x^3su - 4x^3t^2 + s^2t) &&= 3x^6u - 3x^3st + s^3 = \delta_0. \end{aligned}$$

We can compute the plinth variety, the fixed-point set and the nullcone for Daigle and Freudenburg's counterexample:

Lemma 3.4.1. *1.* $(\mathbb{A}^5)^{\mathbb{G}_a} = \mathcal{V}_{\mathbb{A}^5}(x, s, t)$.

2. $\mathcal{P}_{\mathbb{A}^5} = \mathcal{V}_{\mathbb{A}^5}(x, s)$.

3. $\mathcal{N}_{\mathbb{A}^5} = \mathcal{V}_{\mathbb{A}^5}(x, s)$.

Proof. 1. As shown above, we have

$$\left(\mathbb{A}^5\right)^{\mathbb{G}_a} = \{p \in \mathbb{A}^5 \mid D(f)(p) = 0 \text{ for all } f \in R\}.$$

Suppose $p = (p_1, p_2, p_3, p_4, p_5) \in (\mathbb{A}^5)^{\mathbb{G}_a}$ and $f \in R$, then

$$D(f)(p) = p_1^3 \frac{\partial f}{\partial s}(p) + p_2 \frac{\partial f}{\partial t}(p) + p_3 \frac{\partial f}{\partial u}(p) + p_1^2 \frac{\partial f}{\partial v}(p).$$

Clearly if $p = (0, 0, 0, p_4, p_5)$, then $D(f)(p) = 0$ for all $f \in R$. Conversely suppose at least one of p_1, p_2 or $p_3 \neq 0$, then one of $D(s)(p), D(t)(p), D(u)(p)$ is non-zero, so $(\mathbb{A}^5)^{\mathbb{G}_a} = \mathcal{V}_{\mathbb{A}^5}(x, s, t)$ as claimed.

2. By our observations above we have $\beta_0^3, \gamma_0, \delta_0 \in \mathfrak{pl}(D)$, so

$$\mathcal{V}_{\mathbb{A}^5}(\beta_0^3, \gamma_0, \delta_0) = \mathcal{V}_{\mathbb{A}^5}(x, s) \subset \mathcal{P}_{\mathbb{A}^5}.$$

Let $f \in \mathfrak{pl}(D)$, we will show that $f \in \mathcal{V}_{\mathbb{A}^5}(x, s)$. Suppose that f is homogeneous, we induce on the degree, n, induced by the $\mathbb{G}_m$-action introduced in equation 3.2. Note that there are no elements of degree 0 or 1 in the plinth ideal. The only elements in the plinth ideal of degree 2 or 3 are the monomials x^2 and x^3. Now suppose $f \in \mathfrak{pl}(D)$ has degree $n > 3$ and that $f = xp_1 + sp_2 + g$, where $p_1, p_2 \in \mathbb{K}[x, s, t, u, v]$ and $g \in \mathbb{K}[t, u, v]$. Observe that the partial derivatives $\frac{\partial}{\partial t}, \frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ all commute with the derivation D, and so for any $p \in R^D$, we have

$$0 = \frac{\partial}{\partial t}(D(p)) = D\left(\frac{\partial}{\partial t}(p)\right) = \frac{\partial}{\partial u}(D(p)) = D\left(\frac{\partial}{\partial u}(p)\right) = \frac{\partial}{\partial v}(D(p)) = D\left(\frac{\partial}{\partial v}(p)\right).$$

Hence $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v} \in R^D$ since $f \in R^D$. But note that these partial derivatives all have degree $n - 3$, hence by induction we have that there are $h_1, h_2, h_3, h_4, h_5, h_6 \in \mathbb{K}[x, s, t, u, v]$ with

$$\frac{\partial f}{\partial t} = xh_1 + sh_2, \quad \frac{\partial f}{\partial u} = xh_3 + sh_4, \quad \frac{\partial f}{\partial v} = xh_5 + sh_6. \tag{3.5}$$

However

$$\frac{\partial f}{\partial t} = x\frac{\partial p_1}{\partial t} + s\frac{\partial p_2}{\partial t} + \frac{\partial g}{\partial t}, \qquad \frac{\partial f}{\partial u} = x\frac{\partial p_1}{\partial u} + s\frac{\partial p_2}{\partial u} + \frac{\partial g}{\partial u}, \qquad \frac{\partial f}{\partial v} = x\frac{\partial p_1}{\partial v} + s\frac{\partial p_2}{\partial v} + \frac{\partial g}{\partial v},$$

where $\frac{\partial g}{\partial t}, \frac{\partial g}{\partial u}, \frac{\partial g}{\partial v} \in \mathbb{K}[t, u, v]$, hence these partial derivatives of g must all be zero by equation 3.5. Since we have assumed f is homogeneous, this implies that $g = 0$.

3. Recall that the nullcone is given by

$$\pi_{\mathbb{A}^5}^{-1}(\pi_{\mathbb{A}^5}(0)) = \mathcal{N}_{\mathbb{A}^5} = \left\{v \in \mathbb{A}^5 \mid f(v) = 0 \text{ for all } f \in R_+^{\mathbb{G}_a}\right\}.$$

Suppose $v = (v_1, v_2, v_3, v_4, v_5) \in \mathcal{N}_{\mathbb{A}^5}$, then for $\beta_0, \gamma_0 \in R_+^{\mathbb{G}_a}$, we have $\beta_0(v) = v_1 = 0$, $\gamma_0(v) = 2(0)^3 v_3 - v_2^2 = 0$, so $v = (0, 0, v_3, v_4, v_5)$ and $\mathcal{N}_{\mathbb{A}^5} \subset \mathcal{V}_{\mathbb{A}^5}(x, s)$. Now suppose that $v \in \mathcal{V}_{\mathbb{A}^5}(x, s)$, and let $f \in R_+^{\mathbb{G}_a}$, the above calculation for the plinth variety shows that $R_+^{\mathbb{G}_a} \subset (x, s)\mathbb{K}[x, s, t, u, v]$. Therefore we can write $f = xh_1 + sh_2$ with $h_1, h_2 \in \mathbb{K}[x, s, t, u, v]$ and so $f(v) = 0$, implying $v \in \mathcal{N}_{\mathbb{A}^5}$ and hence that $\mathcal{N}_{\mathbb{A}^5} = \mathcal{V}_{\mathbb{A}^5}(x, s)$. □

3.5 A SAGBI-BASIS FOR DAIGLE AND FREUDENBURG'S COUNTEREXAMPLE

3.5.1 Constructing three infinite families of invariants

We first show that R^D is not finitely generated, to do this we show that R^D has three infinite families of homogeneous invariants. We call the members of these families

β_i, γ_i and δ_i respectively, with $i \in \mathbb{N}$ corresponding to the v-degree of the invariant. Recall that S^Δ is generated $\beta_0, \gamma_0, \delta_0$ and g, defined in equation 3.4. We construct the β_i, γ_i and δ_i so that $\beta_i := \beta_0 v^i +$ terms of lower v-degree, and similarly for γ_i and δ_i. For $i = 1$ we must find, for example, some $f \in R$ so that $D(\beta_0 v + f) = x^3 + D(f) = 0$. This is a simple task for β_1, γ_1 and δ_1 since $x^2\beta_0, x^2\gamma_0, x^2\delta_0 \in \mathfrak{pl}(D)$, giving:

$$\begin{aligned}
\beta_1 &= xv - s, \\
\gamma_1 &= (2x^3t - s^2)v + x^2st - 3x^5u, \\
\delta_1 &= (3x^6u - 3x^3st + s^3)v - 3x^5su + 4x^5t^2 - x^2s^2t.
\end{aligned}$$

In general these invariants are difficult to construct, but we show that such invariants exist. Once this is accomplished, we construct a SAGBI-basis for R^D.

Definition 3.5.1. Let $\mathcal{M}$ be the set of all monomials in $\mathbb{K}[x_1, x_2, \ldots, x_n]$, A *monomial* ordering is a total order " $>$ " on $\mathcal{M}$ which satisfies the following conditions:

- $m > 1$ for all $m \in \mathcal{M} \setminus \{1\}$,
- $m_1 > m_2$ implies $bm_1 > bm_2$ for all $b, m_1, m_2 \in \mathcal{M}$.

We write $x_i >> x_j$ if $x_i > x_j^a$ for all $a \in \mathbb{Z}_{\geq 0}$. Given a non-zero polynomial $f \in \mathbb{K}[x_1, x_2, \ldots, x_n]$, we can write f uniquely as $f = cm + g$, where $m \in \mathcal{M}$, $c \in \mathbb{K} \setminus \{0\}$ and every monomial appearing as part of a term in g is smaller than m with respect to our ordering. We call cm the *leading term* of f, and write $\mathrm{LT}(f) = cm$. Additionally, m is the *leading monomial* of f, denoted $\mathrm{LM}(f)$.

We now define a Gröbner basis, we use [5, p. 10].

Definition 3.5.2. Fix a monomial ordering on $\mathbb{K}[x_1, x_2, \ldots, x_n]$ and let $S \subset \mathbb{K}[x_1, x_2, \ldots, x_n]$ be a set of polynomials. We write

$$\mathrm{L}(S) := (\mathrm{LM}(f) \mid f \in S),$$

for the ideal generated by the leading monomials from S, called the *leading ideal* of S. Now let $I \subset \mathbb{K}[x_1, x_2, \ldots, x_n]$ be an ideal, then a finite subset $\mathcal{G} \subset I$ is called a Gröbner basis for I if $\mathrm{L}(I) = \mathrm{L}(\mathcal{G})$.

It is well-known that a Gröbner basis of an ideal generates the ideal. An analogous concept for subalgebras also exists, called a SAGBI-basis

Definition 3.5.3. A *Subalgebra Analogue for Gröbner Bases of Ideals* or "SAGBI-basis" is defined as follows: Let " $>$ " be a monomial ordering on the polynomial ring $\mathbb{K}[x_1, x_2, \ldots, x_n]$. For a subalgebra $A \subset \mathbb{K}[x_1, x_2, \ldots, x_n]$, we write $\mathrm{L}_{alg}(A)$ for the algebra generated by all leading monomials of non-zero elements in A. A subset $\mathcal{S} \subset A$ is called a SAGBI-basis of A if $\mathrm{L}_{alg}(\mathcal{S}) = \mathrm{L}_{alg}(A)$.

Note also that a SAGBI-basis $\mathcal{S}$ of a subalgebra A also generates A as an algebra as in the case of a Gröbner basis.

In the sequel we use the lexicographic monomial ordering on $R = \mathbb{K}[x, s, t, u, v]$, which is defined so that $x^{e_1}s^{e_2}t^{e_3}u^{e_4}v^{e_5} > x^{f_1}s^{f_2}t^{f_3}u^{f_4}v^{f_5}$ if $e_i > f_i$ for the largest i for which we have $e_i \neq f_i$. For example $tv > v$ since both monomials have v exponent 1, u-exponent 0 but tv has t exponent 1 whilst v has t exponent 0. We also use the lexicographic monomial ordering on $S = \mathbb{K}[x, s, t, u]$ with $x < s < t < u$.

Definition 3.5.4. Let $\mathcal{S} = \{f_1, \ldots, f_m\} \subset \mathbb{K}[x_1, \ldots, x_n]$ be a finite set of polynomials.

1. A polynomial $p \in \mathbb{K}[x_1, \ldots, x_n]$ is said to be in *normal form* with respect to $\mathcal{S}$ if no term of p is divisible by the leading monomial of any $f \in \mathcal{S}$.
2. If $p, \tilde{p} \in \mathbb{K}[x_1, \ldots, x_n]$, $\tilde{p}$ is said to be a *normal form* of p with respect to $\mathcal{S}$ if $\tilde{p}$ is in normal form with respect to $\mathcal{S}$ and there are $h_1, \ldots, h_m \in \mathbb{K}[x_1, \ldots, x_n]$ with
$$p - \tilde{p} = \sum_{i=1}^{m} h_i f_i \quad \text{and } \mathrm{LM}(h_i f_i) \leq \mathrm{LM}(p) \text{ for all } i.$$

We now state the image membership algorithm, see van den Essen [15, §1.4].

Lemma 3.5.5 (Image membership algorithm)**.** *Let $S = \mathbb{K}[x_1, \ldots, x_n]$ be a finitely generated $\mathbb{K}$-algebra, and $\mathcal{D}$ a non-zero locally nilpotent derivation on S. Fix $a \in S$, and let m be the unique non-negative integer satisfying $\mathcal{D}^m(a) \neq 0$, $\mathcal{D}^{m+1}(a) = 0$. Let p be a local slice of $\mathcal{D}$, with $d := \mathcal{D}(p)$ and $s := p/d \in S[d^{-1}]$. Suppose $S^{\mathcal{D}} = \mathbb{K}[f_1, \ldots, f_l]$, put*
$$b' := \sum_{i=0}^{m} \frac{(-1)^i}{(i+1)!} \mathcal{D}^i(a) s^{i+1},$$
and set $q := d^{m+1}b'$. Define the ideal J_m in $\mathbb{K}[X, Y] := K[x_1, \ldots, x_n, y_1, \ldots y_l]$ as:
$$J_m := (y_1 - f_1, \ldots, y_l - f_l, d^{m+1}),$$
and choose on $\mathbb{K}[X, Y]$ a monomial ordering so that $x_i >> y_j$ for all i, j. Let $\mathcal{G}$ be a Gröbner basis of J_m. Let $\tilde{q}$ be the normal form of q with respect to $\mathcal{G}$. Then $a \in \mathcal{D}(S)$ if and only if $\tilde{q} \in \mathbb{K}[Y]$. Furthermore, if $\tilde{q} \in \mathbb{K}[Y]$, then $b := (q - \tilde{q}(f_i))/d^{m+1} \in S$

satisfies $\mathcal{D}(b) = a$. *The polynomial* $\tilde{q}(f_i)$ *is defined by replacing each* y_i *appearing in* $\tilde{q}$ *with* f_i.

Now we show the existence of the β_i, γ_i and δ_i:

Proposition 3.5.6. *For each* $n \in \mathbb{N}$, *there are invariants* $\beta_n, \gamma_n, \delta_n \in R^D$ *with leading terms* $\beta_0 v^n, \gamma_0 v^n$ *and* $\delta_0 v^n$.

A proof of Proposition 3.5.6 has been given by Tanimoto in [32, §2], making use of the relation between Daigle and Freudenburg's counterexample and Freudenburg's counterexample. Here we provide a direct proof.

Proof. We induce on the degree of v. Note that β_0, γ_0 and δ_0 have already been defined. For the sake of brevity we use η_i to denote either β_i, γ_i or δ_i whenever it is unnecessary to differentiate between them. Now suppose that for all $i \leq n$, we have defined

$$\eta_i = e_i^{(i)} v^i + e_{i-1}^{(i)} v^{i-1} + \cdots + e_0^{(i)} = e_0^{(0)} v^i + \binom{i}{i-1} e_0^{(1)} v^{i-1} + \cdots + e_0^{(i)},$$

where $e_j^{(i)} = \binom{i}{j} e_0^{(i-j)}$ and $0 \leq j \leq i \leq n$. Note also that $D(e_0^{(i)}) = \Delta(e_0^{(i)}) = -x^2 e_1^{(i)} = -x^2 \binom{i}{1} e_0^{(i-1)}$, since $D(\eta_i) = 0$. Now we define

$$f_{n+1}^{\eta} := (-1)^n \left(\eta_0 v^{n+1} - \binom{n+1}{1} \eta_1 v^n + \cdots + (-1)^n \binom{n+1}{n} \eta_n v \right).$$

We calculate $D(f_{n+1}^{\eta})$, using that $D(\eta_i) = 0$ for all $i \leq n$ and $i\binom{n+1}{i} = (n+1)\binom{n}{i-1}$:

$$\begin{aligned} D(f_{n+1}^{\eta}) &= (-1)^n \left(D \left(\eta_0 v^{n+1} - \binom{n+1}{1} \eta_1 v^n + \cdots + (-1)^n \binom{n+1}{n} \eta_n v \right) \right) \\ &= (-1)^n D(\eta_0) v^{n+1} + \cdots + (-1)^{2n} \binom{n+1}{n} D(\eta_n) v \\ &\quad + (-1)^n (n+1) D(v) \eta_0 v^n + \cdots + (-1)^{2n} \binom{n+1}{n} D(v) \eta_n \\ &= -(n+1) D(v) \left((-1)^{n-1} \left(\eta_0 v^n + \cdots + (-1)^{n-1} \binom{n}{n-1} \eta_{n-1} v \right) - \eta_n \right) \\ &= (n+1) D(v) (\eta_n - f_n^{\eta}). \end{aligned}$$

We now show that $(n+1)D(v)(\eta_n - f_n^{\eta})$ has v degree 0. To do so we calculate the coefficient of v-degree $n - j$ in this expression above for $0 \leq j < n$. Note that the first j terms $(-1)^{n-1}(\eta_0 v^n + \cdots + (-1)^{j-1}\binom{n}{j-1}\eta_{j-1}v^{n-j+1})$ appearing in f_n^η all have higher v-degree and so we may disregard them in our calculations. Examining the term $(-1)^{n+k-1}\binom{n}{k}\eta_k v^{n-k}$ for $k \geq j$ we find

$$(-1)^{n+k-1}\binom{n}{k}\eta_k v^{n-k} = (-1)^{n+k-1}\binom{n}{k}\left(e_k^{(k)}v^k + \cdots + e_0^{(k)}\right)v^{n-k},$$

so the coefficient of v-degree $n-j$ for $(-1)^{n+k-1}\binom{n}{k}\eta_k v^{n-k}$ is

$$(-1)^{n+k-1}\binom{n}{k}e_{k-j}^{(k)} = (-1)^{n+k-1}\binom{n}{k}\binom{k}{k-j}e_0^{(j)} = (-1)^{n+k-1}\binom{n}{j}\binom{n-j}{k-j}e_0^{(j)}.$$

Summing these coefficients, we obtain

$$\sum_{k=j}^{n}(-1)^{n+k-1}\binom{n}{j}\binom{n-j}{k-j}e_0^{(j)} = (-1)^{n+j-1}e_0^{(j)}\binom{n}{j}\left(\sum_{t=0}^{n-j}(-1)^t\binom{n-j}{t}\right) = 0,$$

since

$$\sum_{t=0}^{n}(-1)^t\binom{n}{t} = 0 \qquad \text{for } n \geq 1.$$

It remains to calculate the term of v-degree 0 in $(n+1)D(v)(\eta_n - f_n^\eta)$, but this is simply $(n+1)x^2e_0^{(n)}$, and so we have shown

$$D(f_{n+1}^\eta) = (n+1)D(v)(\eta_n - f_n^\eta) = (n+1)x^2e_0^{(n)}.$$

Therefore, if we can show $x^2e_0^{(n)} \in \Delta(S)$ then we may define $e_0^{(n+1)} := -(n+1)h$, where $D(h) = \Delta(h) = x^2e_0^{(n)}$ and set $\eta_{n+1} := f_{n+1}^\eta + e_0^{(n+1)}$. To achieve this, we use a method similar to van den Essen in [13] by considering the construction of the element b' in the image membership algorithm. In the notation of Lemma 3.5.5, we choose our local slice to be $p = s \in S$ so that $d = \Delta(s) = x^3$ and then

$$b' = \sum_{i=0}^{n}\frac{(-1)^i}{(i+1)!}\Delta^i\left(x^2e_0^{(n)}\right)\left(\frac{s}{x^3}\right)^{i+1} = \sum_{i=0}^{n}\frac{1}{i+1}\binom{n}{i}e_0^{(n-i)}s^{i+1}x^{-i-1}.$$

Note that $x^{n+1}b' \in S$ and $\Delta(x^{n+1}b') = x^{n+3}e_0^{(n)}$. Our aim is to show that there is some $h \in S^\Delta$ such that $b := (x^{n+1}b' - h)/x^{n+1} \in S$, so that $\Delta(b) = x^2e_0^{(n)}$. To achieve this, we must now treat the $b_0^{(n)}$, $c_0^{(n)}$ and $d_0^{(n)}$ separately due to their differing degrees and ρ-degrees, though the arguments are very similar. We show only the case for $c_0^{(n)}$. In this case f_{n+1}^γ is homogeneous of degree $2n+8$, and hence $x^{n+1}b'$

is homogeneous of degree $3n+9$. We now define $S_{(a,k)}$ to be the $\mathbb{K}$-vector space spanned by monomials of degree a and ρ-degree k, that is

$$S_{(a,k)} := \left\{ \sum_i \lambda_i m_i \in S \mid \lambda_i \in \mathbb{K},\ \deg(m_i) = a,\ \rho(m_i) = k \right\}.$$

Note that $x^{n+1}b' \in S_{(3n+9,n+1)}$ and

$$\beta_0 = x \in S_{(1,0)},\ s \in S_{(3,1)},\ t \in S_{(3,2)},\ u \in S_{(3,3)},\ \gamma_0 \in S_{(6,2)},\ \delta_0 \in S_{(9,3)},\ g \in S_{(12,6)}.$$

Additionally, note that if $f \in S_{(a,k)},\ g \in S_{(b,l)}$, then $fg \in S_{(a+b,k+l)}$. Define

$$M := \{f \in S_{(3n+9,n+1)} \mid \deg_x D(f) \geq n+1\},$$
$$N := S^{\Delta} \cap S_{(3n+9,n+1)},$$

so $x^{n+1}b' \in M$. Finally, we define $\pi : S \to S$, where $\pi(f)$ removes all terms of f of x-degree greater than or equal to $n+1$. We show that $\pi(N) = \pi(M)$.

Let $f \in N$, note that $x^{2n+8}s^{n+1} \in M$ and $\pi(x^{2n+8}s^{n+1}) = 0$. If we set $q := x^{2n+8}s^{n+1} + f$, then $D(q) = D(x^{2n+8}s^{n+1})$, so $q \in M$, and $\pi(q) = \pi(x^{2n+8}s^{n+1} + f) = \pi(f)$, giving us that $\pi(N) \subset \pi(M)$.

For $h \in M$, write

$$h = \sum \alpha_{a,b,c,d} x^a s^b t^c u^d, \qquad \text{where } a + 3(b+c+d) = 3n+9, \quad b + 2c + 3d = n+1,$$

then we find

$$D(h) = \sum (b\alpha_{a,b,c,d} + (c+1)\alpha_{a+3,b-2,c+1,d} + (d+1)\alpha_{a+3,b-1,c-1,d+1}) x^{a+3} s^{b-1} t^c u^d.$$

The condition that $\deg_x D(h) = a+3 \geq n+1$ gives us that

$$(b\alpha_{a,b,c,d} + (c+1)\alpha_{a+3,b-2,c+1,d} + (d+1)\alpha_{a+3,b-1,c-1,d+1}) = 0, \tag{3.6}$$

whenever $a+3 < n+1$. Now, if $a+3 < n+1$ using that $a + 3(b+c+d) = 3n+9$ and $b + 2c + 3d = n+1$ we find that $2b + c > n + 10$ meaning that we require $b > 0$. Suppose that $n = 3k$, equation 3.6 can be applied iteratively to show that any $\alpha_{l,p,q,r}$ can be expressed as a linear combination of $\alpha_{3k+1,b,c,d}$ where $l < 3k+1$ and $3(b+c+d) = 6k+9$. Similarly for $n = 3k+1$ and $n = 3k+2$ these can be expressed as a linear combination of $\alpha_{3k+2,b,c,d}$ and $\alpha_{3k+3,b,c,d}$ respectively, with $3(b+c+d) = 6k+12$ and $3(b+c+d) = 6k+15$ respectively also. Therefore the dimension of $\pi(M)$ for say $n = 3k$ is at most the number of solutions to $3(b+c+d) = 6k+15$ and

$b+2c+3d=3k+1$. Computing this dimension for each of these cases we find that these again split into sub-cases depending on $n \mod 6$. For $n=6k+i$, $0 \le i \le 5$, the number of solutions is $k+1$, hence $\dim(\pi(M)) \le k+1$.

Now for N, we know that $S^\Delta = \mathbb{K}[\beta_0, \gamma_0, \delta_0, g]$, so N is generated by $\beta_0^a \gamma_0^b \delta_0^c g^d$ where $a+6b+9c+12d = 3n+9$, as elements of N are homogeneous of degree $3n+9$ and ρ-degree $2b+3c+6d = n+1$. Counting the number of solutions to these equations we find again that these split mod 6, with $\frac{1}{2}(k+1)(k+2)$ solutions for $n=6k+i$, $i \in \{0,1,2,4\}$ and $\frac{1}{2}(k+2)(k+3)$ solutions for $n = 6k+3, 6k+5$.

Consider the case $n=6k$, the general solution for these equations is $(a,b,c,d) = (6y, 3k-3z, 1+2z-2y, y)$ with $0 \le y \le z \le k$ integers. But, since $\gamma_0^3 + \delta_0^2 = x^6 g$, we obtain, for example, that $\gamma_0^{3k}\delta_0 + \gamma_0^{3k-3}\delta_0^3 = x^6\gamma_0^{3k-3}\delta_0 g$. Using this relation we find that all solutions with $y \neq 0$ can be written as a linear combination of solutions with $y=0$, thus reducing the number of solutions to $k+1$. The same argument for other values of n mod 6 reduces the number of solutions to $k+1$ for $n=6k+i$, $i \in \{0,1,2,4\}$ and $k+2$ for $n=6k+3, 6k+5$. Returning to $n=6k$, we show that $\pi(N)$ has dimension $k+1$. Let

$$n_p := (2x^3t - s^2)^{3k-3p}(3x^6u - 3x^3st + s^3)^{1+2p}, \quad 0 \le p \le k,$$

and consider

$$\begin{aligned}(-1)^p \pi(n_p)|_{t=0,u=s/3} &= \pi\left(s^{6k-6p}(x^6 s + s^3)^{1+2p}\right) \\ &= \pi\left(\sum_{i=0}^{2p} \binom{2p}{i} x^{6i} s^{6k+3-2i}\right).\end{aligned}$$

Now since $n+1 = 6k+1$, π removes all monomial terms of x-degree $\ge 6k+1$, we therefore find

$$(-1)^p \pi(n_p)|_{t=0,u=s/3} = \sum_{i=0}^{min(2p,k)} \binom{2p}{i} x^{6i} s^{6k+3-2i}.$$

So the linear independence of the $\pi(n_p)$ follows once we show that

$$\det\left(\binom{2p}{i}\right)_{0 \le i,p \le k} \neq 0.$$

But this is a special case of Corollary 2 of Gessel and Viennot's article [20, p.301], with $a_i = 2i$ and $b_i = i$ for $i = 0,1,\ldots,k$. The cases for $n = 6k+1, 6k+2, 6k+4$ follow similarly. In the case of $n = 6k+5$, we let

$$n_p := (2x^3t - s^2)^{3k+4-3p}(3x^6u - 3x^3st + s^3)^{2p}, \quad 0 \le p \le k+1,$$

again we consider

$$\begin{aligned}(-1)^p \pi(n_p)|_{t=0,u=s/3} &= \pi\left(s^{6k+8-6p}(x^6 s + s^3)^{2p}\right) \\ &= \pi\left(\sum_{i=0}^{2p} \binom{2p}{i} x^{6i} s^{6k+8-2i}\right).\end{aligned}$$

Now $n+1 = 6k+6$, and hence π removes all monomial terms of x-degree $\geq 6k+6$, so

$$(-1)^p \pi(n_p)|_{t=0,u=s/3} = \sum_{i=0}^{min(2p,k)} \binom{2p}{i} x^{6i} s^{6k+8-2i}.$$

Considering only the first $k+1$ terms, as above, we find these $\pi(n_p)$ are linearly independent by Gessel and Viennot's result. Using that $\pi(N) \subset \pi(M)$, and $\dim(\pi(M)) \leq k+1$, we conclude that $\pi(N) = \pi(M)$. □

Remark. *Note that the choice of the $e_0^{(n)}$ is not unique in general. However, given say $b_0^{(n)}$ and some $\tilde{b}_0^{(n)}$, we must have $D(b_0^{(n)} - \tilde{b}_0^{(n)}) = 0$. As both are chosen to be homogeneous with respect to our gradings, they must differ by an element of $S^\Delta \cap S_{(2n+1,n)}$. This vector space is non-empty precisely when $n = 6k$, with basis xg^k in this case. Since $b_0^{(0)} = x$, we observe that the $b_0^{(n)}$ are unique for $n \leq 5$. For $c_0^{(n)}$ and $d_0^{(n)}$, we find that these are unique for $n \leq 3$ and $n \leq 2$ respectively.*

3.5.2 A generating set for R^D

Lemma 3.5.7. *The set of invariants*

$$\mathcal{S} := \{g, \beta_n, \gamma_n, \delta_n | \, n \in \mathbb{N}\},$$

generates R^D.

Proof. We prove this by induction on n, the degree of v. Namely we show that the set

$$\mathcal{S}_n := \{g, \beta_m, \gamma_m, \delta_m | \, m \leq n\},$$

generates $A_n = \{f \in R^D \,|\, \deg_v(f) \leq n\}$.

When $n = 0$, we are considering the elements of v-degree 0, but these are the invariants in $\mathbb{K}[x,s,t,u] = S$ where $D|_S = \Delta$, and we know S^Δ is generated by $\beta_0, \gamma_0, \delta_0$ and g, which is just $\mathcal{S}_0$. Now suppose that A_k is generated by $\mathcal{S}_k$ for all $k \leq n-1$, and let $f \in R^D$ be an invariant whose terms have v-degree at most n.

Without loss of generality we may assume that f is homogeneous with respect to our $\mathbb{G}_m$-grading and that

$$f = a_n v^n + a_{n-1}v^{n-1} + \cdots + a_0,$$

with $a_i \in K[x,s,t,u]$ for all i, and $a_n \neq 0$. Now $D(f) = 0$, meaning that

$$\begin{aligned} D(f) &= D(a_n v^n + a_{n-1}v^{n-1} + \cdots + a_0) \\ &= D(a_n)v^n + nx^2 a_n v^{n-1} + D(a_{n-1}v^{n-1} + \cdots + a_0) = 0. \end{aligned}$$

Now comparing v-degrees, we see that we must have $D(a_n) = 0$, but $a_n \in \mathbb{K}[x,s,t,u]$ and so $a_n \in S^{\Delta}$ which is generated by g, β_0, γ_0 and δ_0. So we may write

$$a_n = \beta_0 p_1 + \gamma_0 p_2 + \delta_0 p_3 + \lambda g^k,$$

for some $p_1, p_2, p_3 \in \mathbb{K}[\beta_0, \gamma_0, \delta_0, g]$, $\lambda \in \mathbb{K}$ and $k \geq 0$. But if we define

$$G = \beta_n p_1 + \gamma_n p_2 + \delta_n p_3,$$

then $D(G) = 0$ and G is generated by $\mathcal{S}_n$ as $\beta_n, \gamma_n, \delta_n \in \mathcal{S}_n$ and $p_1, p_2, p_3 \in \mathcal{S}_0 \subset \mathcal{S}_n$. We also have $D(G - f) = 0$, where

$$G - f = \lambda g^k v^n + b_{n-1}v^{n-1} + \cdots + b_0.$$

We show that no such invariant can exist unless $\lambda = 0$, in which case each term of $G - f$ has v-degree at most $n-1$ and therefore must be generated by $\mathcal{S}_{n-1}$ by induction. We can then conclude that f is generated by $\mathcal{S}_n$, proving the result.

If $\lambda \neq 0$ then we can take $\lambda = 1$ by re-scaling and our task reduces to showing that there is no invariant of the form

$$h = g^k v^n + b_{n-1}v^{n-1} + \cdots + b_0.$$

To do this we consider

$$\begin{aligned} D(h) =& D(g^k)v^n + (nx^2 g^k + D(b_{n-1}))v^{n-1} + (n-1)x^2 b_{n-1}v^{n-2} \\ &+ D(b_{n-2}v^{n-2} + \cdots + b_0). \end{aligned}$$

Considering the terms of v-degree $n-1$ we see that, to have $D(h) = 0$, we require that $D(b_{n-1}) = \Delta(b_{n-1}) = -nx^2 g^k$. In other words, we require that $-nx^2 g^k \in \Delta(S)$. We show that this is not the case for all $k \in \mathbb{N}$ by use of Lemma 3.5.5, the image membership algorithm. By choosing our local slice to be $p = s \in R$, with $d = \Delta(s) = x^3$, we compute the Gröbner basis of the ideal

$$J := (y_1 - \beta_0,\ y_2 - \gamma_0,\ y_3 - \delta_0,\ y_4 - g,\ x^3),$$

with the lexicographic monomial ordering chosen so that $u > t > s > x > y_4 > y_3 > y_2 > y_1$. Using computational software such as Maple, we are able to find that our Gröbner basis is then

$$\mathcal{G} = (y_1^3,\ y_2^3+y_3^2,\ x-y_1,\ sy_2+y_3,\ sy_3-y_2^2,\ s^2+y_2,\ 6y_2^2u-3y_3t^2-y_4s,\ 6y_3u+3y_2t^2-d).$$

Now, since $\Delta(x^2g) = 0$, we find that $b' = x^{-1}g^ks$ and hence in the notation of Lemma 3.5.5 we have $q = x^2g^ks$. The normal form of q with respect to this basis is $\tilde{q} = y_1^2y_4^ks \notin \mathbb{K}[y_1, y_2, y_3, y_4]$, therefore by the image membership algorithm, $x^2g^k \notin \Delta(S)$ for all $k \in \mathbb{N}$. □

3.5.3 Computing a SAGBI-basis

Now we will show that the set $\mathcal{S}$ forms a SAGBI-basis for our invariant ring $R^{\mathbb{G}_a}$. We follow a method similar to that used in [25, § 3], where a SAGBI-basis for Roberts' counterexample is computed. We recall from Definition 3.5.3 that for a subalgebra R, a SAGBI-basis of R is a subset $\mathcal{S} \subset R$ which satisfies $\mathrm{L}_{alg}(R) = \mathrm{L}_{alg}(\mathcal{S})$, where $\mathrm{L}_{alg}(R)$ denotes the algebra generated by the leading monomials of the elements in R. Note that for our chosen $\mathcal{S}$ we have

$$\mathrm{L}_{alg}(\mathcal{S}) = \mathbb{K}[xv^n, x^3tv^n, x^6uv^n, x^6u^2 \mid n \in \mathbb{N}].$$

We set

$$b_n := xv^n, \quad c_n := 2x^3tv^n, \quad d_n := 3x^6uv^n, \quad e := 9x^6u^2.$$

Also note that, as remarked above, since the $e_0^{(n)}$ can be determined uniquely for $n \leq 2$, for the invariants $\beta_n, \gamma_n, \delta_n$ we can write the terms of v-degree $n, n-1$ and $n-2$. Namely, we have:

$$\beta_n = xv^n - nsv^{n-1} + n(n-1)x^2tv^{n-2} + l.o.t,$$

$$\gamma_n = (2x^3t - s^2)v^n - n(-3x^5u + x^2st)v^{n-1} + n(n-1)(3x^4su - 2x^4t^2)v^{n-2} + l.o.t,$$

$$\delta_n = (3x^6u - 3x^3st + s^3)v^n - n(3x^5su + 4x^5t^2 - x^2s^2t)v^{n-1} \\ -n(n-1)(3x^7tu - 3x^4s^2u + x^4st^2) + l.o.t,$$

where *l.o.t.* refers to terms of lower v-degree. Recall from the proof of Lemma 3.5.7:

$$\mathcal{S}_N := \{\beta_i, \gamma_i, \delta_i, g \,|\, 0 \leq i \leq N\}.$$

Let B_N be the subalgebra of R^D generated by $\mathcal{S}_N$ for all $N \geq 0$.

Lemma 3.5.8. *For all $N \geq 0$ the subalgebra $\mathrm{L}_{alg}(B_N) \subset R$ is generated by $\mathrm{L}_{alg}(\mathcal{S}_N)$, hence $\mathcal{S}_N$ is a SAGBI-basis of B_N for all $N \in \mathbb{N}$. As $R^D = \bigcup_N B_N$, $\mathcal{S}$ is a SAGBI-basis for R^D.*

Proof. $\mathrm{L}_{alg}(\mathcal{S}_N) = \mathbb{K}[b_i, c_i, d_i, e \,|\, 0 \leq i \leq N]$. The relations between the b_i, c_i and d_i are generated by

$$\begin{aligned} b_n c_m - b_{n'} c_{m'} &= 0, \quad b_n b_m - b_{n'} b_{m'} = 0, \\ c_n d_m - c_{n'} d_{m'} &= 0, \quad c_n c_m - c_{n'} c_{m'} = 0, \\ b_n d_m - b_{n'} d_{m'} &= 0, \quad d_n d_m - d_{n'} d_{m'} = 0, \end{aligned}$$

where $n, m, n', m' \in \mathbb{N}$ satisfy $n + m = n' + m' \leq N$. We also have the relations involving e

$$d_m d_n - e \prod_{i=1}^{6} b_{m_i} = 0,$$

with $n + m = \sum_{i=1}^{6} m_i \leq N$. The relations between the b_i, c_i, d_i and e all arise by noting that in any relation the terms must have equal v, t and u-degree and so there must be an equal number of b_i, c_i and d_i terms on either side in any relation involving just these three families. The relations involving e arise from comparing x and u-degree.

We now show that when substituting in the polynomials $\beta_i, \gamma_i, \delta_i$ and g, in the relations above, the leading term of the result lies in $\mathrm{L}_{alg}(\mathcal{S}_N)$. By considering the first two terms of the β_i and γ_i and noting that $m - m' = n' - n$, we see that

$$\begin{aligned} &\mathrm{LT}(\beta_n \gamma_m - \beta_{n'} \gamma_{m'}) \\ &\quad = \mathrm{LT}\Big((xv^n - nsv^{n-1})((2x^3t - s^2)v^m - m(-3x^5u + x^2st)v^{m-1}) \\ &\qquad\qquad - (xv^{n'} - n'sv^{n'-1})((2x^3t - s^2)v^{m'} - m'(-3x^5u + x^2st)v^{m'-1})\Big) \\ &\quad = \mathrm{LT}\left((2(n' - n)x^3st - (n' - n)s^3 + (m - m')x^3st - 3(m - m')x^6u)v^{n+m-1}\right) \\ &\quad = \mathrm{LT}\left((m - m')(3x^6u - 3x^3st + s^3 + s^3)v^{n+m-1}\right). \end{aligned}$$

So $\mathrm{LT}(\beta_n\gamma_m - \beta_{n'}\gamma_{m'}) = -3(m - m')x^6uv^{n+m-1} = (m' - m)d_{n+m-1} \in \mathrm{L}_{alg}(\mathcal{S}_N)$, in fact we have shown that the coefficient of v-degree $n + m - 1$ is precisely δ_0. Next we have

$$\begin{aligned}
&\mathrm{LT}(\beta_n\delta_m - \beta_{n'}\delta_{m'}) \\
&\quad = \mathrm{LT}\Big((xv^n - nsv^{n-1})((3x^6su - 3x^3st + s^3)v^m - m(x^5su + 4x^5t^2 - x^2s^2t)v^{m-1}) \\
&\qquad\qquad - (xv^{n'} - n'sv^{n'-1})((3x^6u - 3x^3st + s^3)v^{m'} \\
&\qquad\qquad\qquad - m'(x^5su + 4x^5t^2 - x^2s^2t)v^{m'-1})\Big) \\
&\quad = \mathrm{LT}\left((4(m-m')x^6t^2 + (3n - 3n' + m' - m)x^3s^2t - (n'-n)s^4)v^{n+m-1}\right) \\
&\quad = \mathrm{LT}\left(((m-m')(4x^6t^2 + x^3s^2t - s^4))v^{n+m-1}\right).
\end{aligned}$$

Therefore $\mathrm{LT}(\beta_n\delta_m - \beta_{n'}\delta_{m'}) = 4(m-m')x^6t^2v^{n+m-1} = (m-m')c_0c_{n+m-1}$, and the coefficient of v-degree $n+m-1$ is precisely γ_0^2. By the same method, we find that

$$\mathrm{LT}(\gamma_n\delta_m - \gamma_{n'}\delta_{m'}) = 9(n'-n)x^{11}u^2v^{n+m-1} = (n'-n)eb_0^4b_{n+m-1},$$

and the coefficient of v-degree $n+m-1$ is precisely $(n'-n)\beta_0^4 g$. Similarly we have

$$\mathrm{LT}(\delta_n\delta_m - g\prod_{i=1}^{6}\beta_{m_i}) = -8x^9t^3v^{n+m} = -c_0^2c_n,$$

and the coefficient of v-degree n is precisely $-\gamma_0^3$. Note that this arises from the relation $\gamma_0^3 + \delta_0^2 = x^6 g$. Now we require the first three terms of the β_i, γ_i and δ_i to compute the remaining relations, as the terms of v-degree $m+n$ and $m+n-1$ are both zero. We find:

$$\begin{aligned}
&\mathrm{LT}(\beta_n\beta_m - \beta_{n'}\beta_{m'}) \\
&\quad = \mathrm{LT}\Big(\big(xv^2 - nsv + n(n-1)x^2t)(xv^2 - msv + m(m-1)x^2t) \\
&\qquad\qquad - (xv^2 - n'sv + n'(n'-1)x^2tv)(xv^2 - m'sv + m'(m'-1)x^2t\big)v^{n+m-4}\Big) \\
&\quad = \mathrm{LT}\left(((nm - n'm')s^2 + (n(n-1) + m(m-1))x^3s^2t)v^{n+m-2}\right) \\
&\quad = \mathrm{LT}\left(-(nm - n'm')(2x^3t - s^2)v^{n+m-2}\right), \qquad (3.7)
\end{aligned}$$

using that $n^2 + m^2 - n'^2 - m'^2 = -2(nm - n'm')$, thus $\mathrm{LT}(\beta_n\beta_m - \beta_{n'}\beta_{m'}) = -(nm - n'm')2x^3tv^{n+m-2} = -(nm - n'm')c_{n+m-2}$ and the coefficient of v-degree

$n+m-2$ is precisely $-(nm-n'm')\gamma_0$.

$$
\begin{aligned}
&\mathrm{LT}(\gamma_n\gamma_m - \gamma_{n'}\gamma_{m'})\\
&\quad = \mathrm{LT}\Big(\big(nm(x^2st-3x^5u)^2 - (n^2-n+m^2-m)\big)(2x^3t-s^2)(3x^4su-2x^4t^2)\\
&\qquad\quad - n'm'(x^2st-3x^5u)^2\\
&\qquad\quad + (n'^2-n'+m'^2-m')(2x^3t-s^2)(3x^4su-2x^4t^2)\big)v^{n+m-2}\Big)\\
&\quad = \mathrm{LT}\Big(\big((nm-n'm')(9x^{10}u^2-6x^7stu+x^4s^2t^2)\\
&\qquad\quad + (n^2+m^2-n'^2-m'^2)(6x^7stu-3x^4s^3u-4x^7t^3+2x^4s^2t^2)\big)v^{n+m-2}\Big)\\
&\quad = \mathrm{LT}\left((nm-n'm')x^4(9x^6u^2-18x^3stu+6s^3u+8x^3t^3-3s^2t^2))v^{n+m-2}\right).
\end{aligned}
\tag{3.8}
$$

So $\mathrm{LT}(\gamma_n\gamma_m - \gamma_{n'}\gamma_{m'}) = (nm-n'm')9x^{10}u^2v^{n+m-2} = (nm-n'm')eb_0^3b_{n+m-2}$ and we have shown that the coefficient of v-degree $n+m-2$ of this expression is precisely $(nm-n'm')\beta_0^4g$. Now finally we have

$$
\begin{aligned}
&\mathrm{LT}(\delta_n\delta_m - \delta_{n'}\delta_{m'})\\
&= \mathrm{LT}((nm-n'm')x^4(2x^3t-s^2)(9x^6u^2-18x^3stu+6s^3u+8x^3t^3-3s^2t^2))v^{n+m-2}).
\end{aligned}
\tag{3.9}
$$

We conclude:

$$\mathrm{LT}(\delta_n\delta_m - \delta_{n'}\delta_{m'}) = (nm-n'm')9x^13tu^2v^{n+m-2} = (nm-n'm')b_0^3c_0eb_{n+m-2},$$

and we have shown that the coefficient of v-degree $n+m-2$ of this expression is precisely $(nm-n'm')\beta_0^4\gamma_0 g$.

Since $\mathcal{S}_N$ generates B_N, and any combination of elements in $\mathcal{S}_N$ yields an element whose leading term lies in $\mathrm{L}_{alg}(\mathcal{S}_N)$, we conclude that $\mathcal{S}_N$ is a SAGBI-basis for B_N. □

3.6 THE FINITE GENERATION IDEAL

We maintain our notation for B_N and $\mathcal{S}_N$ introduced in the previous section. Our aim in this section will be to prove the following:

Theorem 3.6.1. *The finite generation ideal, $\mathfrak{f}_{R^D}$, is the radical of the ideal of R^D generated by β_0, γ_0 and δ_0; that is, $\mathfrak{f}_{R^D} = \sqrt{(\beta_0, \gamma_0, \delta_0)R^D}$. Additionally, $\mathcal{G} = \{\beta_i, \gamma_i, \delta_i \,|\, i \geq 0\} \subset \mathfrak{f}_{R^D}$ satisfies $L(\mathcal{G}) \subset L(\mathfrak{f}_{R^D})$.*

To prove the theorem, we first prove the following lemma and proposition, analogues of results proven in Dufresne and Kraft's paper [10, p.21] in order to compute the finite generation ideal of Roberts' example. For subalgebras $S_1 \subset S_2 \subset R$, we define the *conductor* as $[S_1 \colon S_2] := \{s \in S_2 \,|\, sS_2 \subset S_1\}$.

Lemma 3.6.2. *Fix an integer $N \in \mathbb{N}$:*

1. *If $f \in R^D$ and $\deg_v(f) \leq N$, then $f \in B_N$.*
2. *$(\beta_0, \gamma_0, \delta_0)B_{N+1} \subset B_N$.*
3. *$[B_N \colon B_{N+1}] \cap B_0 = (\beta_0, \gamma_0, \delta_0)B_0$.*

Proof. If $\deg_v(f) = 0$, then $D(f) = \Delta(f) = 0$ so $f \in S^\Delta$ which is generated by $\beta_0, \gamma_0, \delta_0$ and g. But $B_0 = \mathbb{K}[\mathcal{S}_0]$ where $\mathcal{S}_0 = \{\beta_0, \gamma_0, \delta_0, g\}$, so $f \in B_0$. Suppose that this result holds for all $f \in R^D$ with $\deg_v(f) \leq k$. Now suppose that $f \in R^D$ and $\deg_v(f) = k+1$, then $\mathrm{LT}(f)$ is a monomial in $\mathrm{L}_{alg}(S)$ of v-degree $k+1$, and hence there is some $\tilde{f} \in B_{k+1}$ with $\mathrm{LT}(f) = \mathrm{LT}(\tilde{f})$. But $\deg_v(f - \tilde{f}) < k+1$, so $f - \tilde{f} \in B_k \subset B_{k+1}$ by induction. Hence

$$f = \tilde{f} + (f - \tilde{f}) \in B_{k+1}.$$

This proves part 1.

For part 2, let $\eta_0 \in \{\beta_0, \gamma_0, \delta_0\}$ and consider $\eta_0 B_{N+1}$. Since $g, \beta_i, \gamma_i, \delta_i \in B_N$ for $0 \leq i \leq N$ we need only show that $\eta_0\beta_{N+1}, \eta_0\gamma_{N+1}, \eta_0\delta_{N+1} \in B_N$. Let $\mathrm{LT}(\eta_0) = e_0$. Now

$$\begin{aligned}
\mathrm{LT}(\eta_0\beta_{N+1}) &= e_0 b_{N+1} = e_1 b_N, \quad \deg_v(\eta_0\beta_{N+1} - \eta_1\beta_N) \leq N,\\
\mathrm{LT}(\eta_0\gamma_{N+1}) &= e_0 c_{N+1} = e_1 c_N, \quad \deg_v(\eta_0\gamma_{N+1} - \eta_1\gamma_N) \leq N,\\
\mathrm{LT}(\eta_0\delta_{N+1}) &= e_0 d_{N+1} = e_1 d_N, \quad \deg_v(\eta_0\delta_{N+1} - \eta_1\delta_N) \leq N.
\end{aligned}$$

Applying part 1 of this lemma in each case gives us that:

$$\begin{aligned}
\eta_0\beta_{N+1} &= \eta_1\beta_N + (\eta_0\beta_{N+1} - \eta_1\beta_N) \in B_N,\\
\eta_0\gamma_{N+1} &= \eta_1\gamma_N + (\eta_0\gamma_{N+1} - \eta_1\gamma_N) \in B_N,\\
\eta_0\delta_{N+1} &= \eta_1\delta_N + (\eta_0\delta_{N+1} - \eta_1\delta_N) \in B_N.
\end{aligned}$$

This proves part 2.

Finally for part 3, note that if $f \in [B_N\colon B_{N+1}] \cap B_0$, then $f\beta_{N+1}, f\gamma_{N+1}, f\delta_{N+1} \in B_N$. Therefore all three of $\mathrm{LT}(f)xv^{N+1}$, $\mathrm{LT}(f)2x^3tv^{N+1}$ and $\mathrm{LT}(f)3x^6u$ are elements of $\mathrm{L}_{alg}(B_N)$ which must each have at least two factors of the form b_i, c_i, d_i for some $0 \le i \le N$. Now $\mathrm{LT}(f)$, as a monomial in $L_{alg}(S_0)$ must therefore contain a factor b_0, c_0 or d_0, call this e_0. Then $\mathrm{LT}(f) = e_0\mathrm{LT}(\tilde{f})$ for some $\tilde{f} \in B_0$, giving $f - e_0\tilde{f} < f$ and our result follows by induction since we have shown $\beta_0, \gamma_0, \delta_0 \in [B_N, B_{N+1}]$ in part 2. □

Proposition 3.6.3. *Let $f \in R^D$ be a homogeneous invariant with $f \neq g^k$ for any $k \in \mathbb{N}$. Then $f \in \sqrt{(\beta_0, \gamma_0, \delta_0)R^D}$, and hence $\beta_i, \gamma_i, \delta_i \in \sqrt{(\beta_0, \gamma_0, \delta_0)R^D}$ for all $i \in \mathbb{N}$. Furthermore, $\sqrt{(\beta_0, \gamma_0, \delta_0)R^D}$ is generated by $\{\beta_i, \gamma_i, \delta_i\}_{i\in\mathbb{N}}$, with $R^D/\sqrt{(\beta_0, \gamma_0, \delta_0)R^D}$ a polynomial ring in one variable.*

To prove this result, we begin by first showing that $g \notin \sqrt{(\beta_0, \gamma_0, \delta_0)R^D}$. Suppose that

$$g^k = \beta_0 p_1 + \gamma_0 p_2 + \delta_0 p_3,$$

for some $k \in \mathbb{N}$. We have $\deg(g^k) = 12k$ and $\rho(g^k) = 6k$. Now we may suppose that $p_1 \in R^D$ is homogeneous, with degree $12k - 1$ and ρ-degree $6k$; but there is simply no invariant in R^D which has both this corresponding degree and ρ-degree. This same argument holds for the degrees and ρ-degrees of both p_2 and p_3. Therefore $g \notin \sqrt{(\beta_0, \gamma_0, \delta_0)R^D}$ as claimed.

We focus now on showing that β_i, γ_i and $\delta_i \in \sqrt{(\beta_0, \gamma_0, \delta_0)R^D}$. Since doing so for β_i requires showing that $\gamma_i \in \sqrt{(\beta_0, \gamma_0, \delta_0)R^D}$ we start with γ_i. We use the equation 3.8 from Lemma 3.5.8 to examine the expression $\gamma_i\gamma_j - \gamma_{i'}\gamma_{j'}$. The v-degree 0 part of this expression is $c_0^{(i)}c_0^{(j)} - c_0^{(i')}c_0^{(j')}$. Our goal will be to construct an invariant, ξ, which has the same v-degree 0 terms. In doing so we observe that $\gamma_i\gamma_j - \gamma_{i'}\gamma_{j'} - \xi$ is an invariant with no v-degree 0 terms, therefore $\gamma_i\gamma_j - \gamma_{i'}\gamma_{j'} - \xi = v\mu$ and $D(v\mu) = x^2\mu + vD(\mu) = 0$. Comparing the v-degree 0 terms of this expression, we find that μ has no terms of v-degree 0, and hence $\gamma_i\gamma_j - \gamma_{i'}\gamma_{j'} - \xi$ has no terms of v-degree 1. Continuing in this way, we see that we must conclude $\gamma_i\gamma_j - \gamma_{i'}\gamma_{j'} = \xi$. If we can show that $\xi \in (\beta_0, \gamma_0, \delta_0)R^D$ for all i, j, then we have in particular $(\gamma_i)^2 = \gamma_0\gamma_{2i} + \xi \in (\beta_0, \gamma_0, \delta_0)R^D$, giving $\gamma_i \in \sqrt{(\beta_0, \gamma_0, \delta_0)R^D}$ as required. In the following, we let C be the vector space whose basis is given by finite combinations of $b_0^{(i)}, c_0^{(i)}, d_0^{(i)}$ and g. Recall $b_0^{(i)}, c_0^{(i)}, d_0^{(i)}$ are the v-degree 0 components of β_i, γ_i and δ_i respectively.

Lemma 3.6.4. *Fix $n \in \mathbb{N}$, with $n \geq 2$, then $c_0^{(i)}c_0^{(j)} - c_0^{(i')}c_0^{(j')} = -\lambda^n_{i,i'}x^3gb_0^{(n-2)} + r(n,i,i')$, where $r(n,i,i') = b_0^{(0)}h_1 + c_0^{(0)}h_2 + d_0^{(0)}h_3$, $h_i \in C$ for $i = 1,2,3$, and $\lambda^n_{i,i'} := -(ij - i'j')$, $j = n-i$, $j' = n-i'$.*

Proof. We prove this result by induction. First, for $n = 2$ we have

$$\begin{aligned} c_0^{(0)}c_0^{(2)} - \left(c_0^{(1)}\right)^2 &= 12x^7stu - 6x^5s^3u - 8x^7t^3 + 4x^4s^2t^2 - 9x^{10}u^2 + 6x^7stu - x^4s^2t^2 \\ &= -9x^{10}u^2 + 18x^7stu - 6x^4s^3u - 8x^7t^3 + 3x^4s^2t^2 \\ &= x^4g. \end{aligned}$$

We also note that, as can be seen from equation 3.8, the coefficient of v-degree $n-2$ in the expression $\gamma_i\gamma_j - \gamma_{i'}\gamma_{j'}$ is precisely $\lambda^n_{i,i'}\beta_0^4 g$.

Now, suppose that the result holds for all pairs i, j with $i + j = n$, we consider

$$\begin{aligned} &D\left(c_0^{(i+1)}c_0^{(j)} - c_0^{(i'+1)}c_0^{(j')}\right) \\ &= -x^2\Big((i+1)c_0^{(i)}c_0^{(j)} + jc_0^{(i+1)}c_0^{(j-1)} - (i'+1)c_0^{(i')}c_0^{(j')} - j'c_0^{(i'+1)}c_0^{(j'-1)}\Big) \\ &= -x^2\Big((i+1)\left(c_0^{(i)}c_0^{(j)} - c_0^{(i')}c_0^{(j')}\right) - (i'-i)c_0^{(i')}c_0^{(j')} \\ &\qquad + j\left(c_0^{(i+1)}c_0^{(j-1)} - c_0^{(i'+1)}c_0^{(j'-1)}\right) - (j'-j)c_0^{(i'+1)}c_0^{(j'-1)}\Big) \\ &= -x^2\Big((i+1)\left(c_0^{(i)}c_0^{(j)} - c_0^{(i')}c_0^{(j')}\right) + j\left(c_0^{(i+1)}c_0^{(j-1)} - c_0^{(i'+1)}c_0^{(j'-1)}\right) \\ &\qquad - (i'-i)\left(c_0^{(i')}c_0^{(j')} - c_0^{(i'+1)}c_0^{(j'-1)}\right)\Big) \\ &= -x^2\Big((i+1)\left(\lambda^n_{i,i'}x^3gb_0^{(k-2)} + r(n,i,i')\right) \\ &\qquad + j\left(\lambda^n_{i+1,i'+1}x^3gb_0^{(n-2)} + r(n,i+1,i'+1)\right) \\ &\qquad - (i'-i)\left(\lambda^n_{i',i'+1}x^3gb_0^{(n-2)} + r(n,i',i'+1)\right)\Big) \\ &= -x^2\Big(x^3gb_0^{(n-2)}\left((i+1)\lambda^n_{i,i'} + j\lambda^n_{i+1,i'+1} - (i'-i)\lambda^n_{i',i'+1}\right) + (i+1)r(n,i,i') \\ &\qquad + jr(n,i+1,i'+1) - (i'-i)r(n,i',i'+1)\Big) \\ &= -x^2\Big((n-1)\lambda^{n+1}_{i+1,i'+1}x^3gb_0^{(n-2)} + (i+1)r(n,i,i') + jr(n,i+1,i'+1) \\ &\qquad - (i'-i)r(n,i',i'+1)\Big). \end{aligned}$$

If either $i = n$ or $i' = n$ we instead obtain either $(i'+1)r(n,n,i') + j'r(n,n,i'+1)$ or $(i+1)r(n,i,n) + jr(n,i+1,n)$ in place of the other $r(n,a,b)$. For all $n \geq 2$, and all a, b we claim that there is some $R(n+1,a,b) = b_0^{(0)}H_1 + c_0^{(0)}H_2 + d_0^{(0)}H_3$, $H_i \in C$ for $i = 1,2,3$ with $D(R(n+1,a,b)) = -x^2 r(n,a,b)$. Let $C_N \subset C$ be the vector space whose basis is given by finite combinations $e_0^{(a_1)} \cdots e_0^{(a_k)}$, where $\sum_{i=1}^k a_i = N$ and each e appearing is one of b, c or d, not necessarily all the same. Consider a term of $r(n,a,b)$, by which we mean an element of the form $\lambda e_0^{(0)} h$, with $\lambda \in \mathbb{K}$, $h = e_0^{(a_1)} \cdots e_0^{(a_k)} g^l \in C_N$. Note that the expression $c_0^{(i+1)}c_0^{(j)} - c_0^{(i'+1)}c_0^{(j')}$ is homogeneous of degree $2n+14$ and ρ-degree $n+5$, so $h \neq 0$ for $n \geq 2$. Additionally $h \neq g^l$ since this would give the degree of $r(n,a,b)$ as $\deg(e_0^{(0)}) + 12l$, and ρ-degree $\rho(e_0^{(0)}) + 6l$ which cannot be $2n+14$ and $n+5$ respectively for any choice of $e_0^{(0)}$. We write $h = g^l h'$, where $h' \in C_N$ for some N. Since $\Delta(e_0^{(N)}) = -x^2 N e_0^{(N-1)}$ we can consider Δ as a linear map

$$\Delta : C_{N+1} \longrightarrow x^2 C_N.$$

We now show that Δ is surjective, in which case we can find $H \in C_{N+1}$ such that $D(H) = -x^2 h$, and $D(\lambda e_0^{(0)} H) = -\lambda x^2 e_0^{(0)} h$. Repeating this process for all terms of $r(n,a,b)$ then gives us $R(n+1,a,b)$. To show Δ is surjective it is sufficient to show that for all $f = x^2 e_0^{(a_1)} \cdots e_0^{(a_k)} \in x^2 C_N$, we have $f \in \Delta(C_{N+1})$. We describe the process of constructing an element F with $\Delta(F) = f$.

Firstly we let

$$F_1 := -\frac{1}{a_1+1} e_0^{(a_1+1)} \cdots e_0^{(a_k)},$$

then $\Delta(F_1) = f + G_1$, where all terms of G_1 are of the form $-x^2 \lambda e_0^{(a_1+1)} \cdot e_0^{(b_2)} \cdots e_0^{(b_k)}$ with $\lambda \in \mathbb{K}$, $\sum_{i=2}^k b_i = N - a_1 - 1$. Now set

$$F_2 := F_1 + \sum \kappa e_0^{(a_1+2)} \cdot e_0^{(b_2)} \cdots e_0^{(b_k)}.$$

Note that $\Delta\left(e_0^{(a_1+2)} \cdot e_0^{(b_2)} \cdots e_0^{(b_k)}\right)$ contains precisely one term of the form $e_0^{(a_1+1)} \cdot e_0^{(b_2)} \cdots e_0^{(b_k)}$. The remaining terms are of the form $e_0^{(a_1+2)} \cdot e_0^{(c_2)} \cdots e_0^{(c_k)}$, where $\sum_{i=2}^k c_i = N - a_1 - 2$. Since this is the case we can choose κ appearing in F_2 so that $\Delta(F_2)$ contains no terms of the form $e_0^{(a_1+1)} \cdot e_0^{(b_2)} \cdots e_0^{(b_k)}$. Continuing in this way we find

$$D(F_{N-a_1-1}) = f + \omega e_0^{(a_1+\cdots+a_k-1)} \cdot e_0^{(0)} \cdots e_0^{(0)},$$

where $\omega \in \mathbb{K}$. Finally we define $F_{N-a_1} := F_{N-a_1-1} + \frac{\omega}{N} e_0^{(a_1+\cdots+a_k)} \cdot e_0^{(0)} \cdots e_0^{(0)}$ and observe that $D(F_{N-a_1}) = f$ as required.

Having constructed $R(n+1,a,b)$ for all n, a and b we note that

$$D\Big(c_0^{(i+1)}c_0^{(j)} - c_0^{(i'+1)}c_0^{(j')} - \lambda^{n+1}_{i+1,i'+1}x^3gb_0^{(n-1)} - (i+1)R(n+1,i,i') \\ - jR(n+1,i+1,i'+1) + (i'-i)R(n+1,i',i'+1)\Big) = 0.$$

Therefore this expression is an invariant of degree $2n+14$ and ρ-degree $n+5$. If we consider all homogeneous invariants with such degree and ρ-degree we find:

n	$S^\Delta \cap S_{(2n+14,n+5)}$
$6l$	0
$6l+1$	$\{\lambda x^4 g^{l+1} \mid \lambda \in \mathbb{K}\}$
$6l+2$	0
$6l+3$	$\{\lambda x^2(2x^3t - s^2)g^{l+1} \mid \lambda \in \mathbb{K}\}$
$6l+4$	$\{\lambda x(3x^6u - 3x^3st + s^3)g^{l+1} \mid \lambda \in \mathbb{K}\}$
$6l+5$	$\{\lambda (2x^3t - s^2)^2 g^{l+1} \mid \lambda \in \mathbb{K}\}$

Note that all of these elements are of the form $b_0^{(0)}h_1 + c_0^{(0)}h_2 + d_0^{(0)}h_3$, with $h_i \in C$ for all i. Thus we can write

$$c_0^{(i+1)}c_0^{(j)} - c_0^{(i'+1)}c_0^{(j')} - \lambda^{n+1}_{i+1,i'+1}x^3gb_0^{(n-1)} - (i+1)R(n+1,i,i') \\ - jR(n+1,i+1,i'+1) - (i'-i)R(n+1,i',i'+1) = \mu p,$$

with $\mu \in \mathbb{K}$ and $p \in S^\Delta \cap S_{(2n+14,n+5)}$. By setting

$$r(n+1,i,i') := \lambda^{n+1}_{i+1,i'+1}x^3gb_0^{(n-1)} - (i+1)R(n+1,i,i') \\ - jR(n+1,i+1,i'+1) - (i'-i)R(n+1,i',i'+1) - \mu p,$$

then $r(n+1,i,i')$ is of the form $b_0^{(0)}h_1 + c_0^{(0)}h_2 + d_0^{(0)}h_3$, $h_i \in C$ for $i = 1,2,3$ and we obtain the required result. □

Now using this proof we consider the following expression

$$(\gamma_n)^2 - \gamma_{2n}\gamma_0 - \lambda^{2n}_{n,2n}x^3g\beta_{2n-2} - T(2n,n,2n),$$

where $T(n,a,b) \in R^D$ is defined by replacing every $e_0^{(k)}$ in $r(n,a,b)$ by the corresponding $\eta_k \in R^D$ which has $e_0^{(k)}$ as its v-degree zero term. From this and the observation made above we see that

$$(\gamma_n)^2 = \gamma_{2n}\gamma_0 + \lambda^{2n}_{n,2n}x^3g\beta_{2n-2} + T(2n,n,2n) \in (\beta_0,\gamma_0,\delta_0)R^D.$$

We now prove a similar result for δ_n.

Lemma 3.6.5. *Fix* $n \in \mathbb{N}$*, with* $n \geq 2$*, then* $d_0^{(i)} d_0^{(j)} - d_0^{(i')} d_0^{(j')} = \lambda_{i,i'}^n x^3 (2x^3 t - s^2) g a_0^{(n-2)} + r(n,i,i')$*, where* $r(n,i,i') = b_0^{(0)} h_1 + c_0^{(0)} h_2 + d_0^{(0)} h_3$*,* $h_i \in C$ *for* $i = 1,2,3$*, and* $\lambda_{i,i'}^n := -(ij - i'j')$*,* $j = n - i$*,* $j' = n - i'$*.*

Proof. Firstly for $n = 2$, we have

$$d_0^{(0)} d_0^{(2)} - \left(d_0^{(1)}\right)^2 = x^4 (2x^3 t - s^2) g.$$

Now suppose that the formula holds for all pairs $i + j = n$, we have

$$\begin{aligned} D&\left(d_0^{(i+1)} d_0^{(j)} - d_0^{(i'+1)} d_0^{(j')}\right) \\ &= -x^2 \Big((n-1) \lambda_{i+1,i'+1}^{n+1} x^3 (2x^3 t - s^2) g a_0^{(n-2)} + (i+1) r(n,i,i') \\ &\qquad + j r(n, i+1, i'+1) - (i' - i) r(n, i', i'+1) \Big). \end{aligned}$$

If either $i = n$ or $i' = n$ we instead obtain either $(i'+1) r(n,n,i') + j' r(n,n,i'+1)$ or $(i+1) r(n,i,n) + j r(n,i+1,n)$ in place of the other $r(n,a,b)$. As before we show that there is some $R(n+1,a,b)$ with $D(R(n+1,a,b)) = r(n,a,b)$ for all $a, b \leq n$. Let $\lambda e_0^{(0)} h$ be a term of $r(n,a,b)$, $\lambda \in \mathbb{K}$, $h \in I$ and $e \in \{b,c,d\}$. Note that the expression $d_0^{(i+1)} d_0^{(j)} - d_0^{(i'+1)} d_0^{(j')}$ is homogeneous of degree $2n + 20$ and ρ-degree $n + 7$, so $h \neq 0$ for $n \geq 2$. Now $h \neq g^k$ for some $k \in \mathbb{N}$ since $\deg(e_0^{(0)}) + 12k$ and $\rho(e_0^{(0)}) + 6k$ cannot be $2n + 20$ and $n + 7$ for any choice of $e_0^{(0)}$ or k. Therefore we can write $h = f_0^{(l)} h'$ for some $f \in \{b,c,d\}, l \in \mathbb{N}$ and proceed as described in the proof of Lemma 3.6.4. Now since

$$\begin{aligned} D\Big(& d_0^{(i+1)} d_0^{(j)} - d_0^{(i'+1)} d_0^{(j')} - \lambda_{i+1,i'+1}^{n+1} x^3 (2x^3 t - s^2) g b_0^{(n-1)} - (i+1) R(n+1,i,i') \\ & - j R(n+1, i+1, i'+1) + (i' - i) R(n+1, i', i'+1) \Big) = 0, \end{aligned}$$

this expression is then an invariant of degree $2n + 20$ and ρ-degree $n + 7$. Considering all such elements we find:

n	$S^{\Delta} \cap S_{(2n+20,n+7)}$
$6l$	0
$6l+1$	$\{\lambda x^4 (2x^3 t - s^2) g^{l+1} \mid \lambda \in \mathbb{K}\}$
$6l+2$	$\{\lambda x^3 (3x^6 u - 3x^3 st + s^3) g^{l+1} \mid \lambda \in \mathbb{K}\}$
$6l+3$	$\{\lambda x^2 (2x^3 t - s^2)^2 g^{l+1} \mid \lambda \in \mathbb{K}\}$
$6l+4$	$\{\lambda x (2x^3 t - s^2)(3x^6 u - 3x^3 st + s^3) g^{l+1} \mid \lambda \in \mathbb{K}\}$
$6l+5$	$\{\lambda (2x^3 t - s^2)^3 g^{l+1} + \mu (3x^6 u - 3x^3 st + s^3)^2 g^{l+1} \mid \lambda, \mu \in \mathbb{K}\}$

Note that all such elements are of the form $b_0^{(0)}h_1 + c_0^{(0)}h_2 + d_0^{(0)}h_3$, $h_i \in C$. If we let

$$d_0^{(i+1)}d_0^{(j)} - d_0^{(i'+1)}d_0^{(j')} - \lambda_{i+1,i'+1}^{n+1}x^3(2x^3t - s^2)gb_0^{(n-1)} - (i+1)R(n+1,i,i') \\ - jR(n+1,i+1,i'+1) + (i'-i)R(n+1,i',i'+1) = \mu p,$$

with $\mu \in \mathbb{K}$ and $p \in S^{\Delta} \cap S_{(2n+14,n+5)}$, by setting

$$r(n+1,i,i') := \lambda_{i+1,i'+1}^{n+1}x^3(2x^3t - s^2)gb_0^{(n-1)} - (i+1)R(n+1,i,i') \\ - jR(n+1,i+1,i'+1) - (i'-i)R(n+1,i',i'+1) - \mu p,$$

then $r(n+1,i,i')$ is of the form $b_0^{(0)}h_1 + c_0^{(0)}h_2 + d_0^{(0)}h_3$, $h_i \in C$ for $i = 1,2,3$, and we obtain the required result. □

Now as before by using this proof we consider the following expression

$$(\delta_n)^2 - \delta_{2n}\delta_0 - \lambda_{n,2n}^{2n}x^3(2x^3t - s^2)g\beta_{2n-2} - T(2n,n,2n),$$

where $T(n,a,b) \in R^D$ is defined by replacing every $e_0^{(k)}$ in $r(n,a,b)$ by the corresponding $\eta_k \in R^D$ which has $e_0^{(k)}$ as its v-degree zero term. From this we see that the expression above has no v-degree 0 terms, and therefore the whole expression must be zero. Thus we have that $(\delta_n)^2 \in (\beta_0, \gamma_0, \delta_0)\, R^D$. Now all that remains is to show that $\beta_n \in \sqrt{(\beta_0,\gamma_0,\delta_0)R^D}$. Firstly, we proceed as we have before for γ_n and δ_n:

Lemma 3.6.6. *Fix $n \in \mathbb{N}$, with $n \geq 2$, then $b_0^{(i)}b_0^{(j)} - b_0^{(i')}b_0^{(j')} = \lambda_{i,i'}^{k}c_0^{(k-2)} + r(n,i,i')$, where*
$r(n,i,i') = b_0^{(0)}h_1 + c_0^{(0)}h_2 + d_0^{(0)}h_3$, $h_i \in C$ for $i = 1,2,3$, and $\lambda_{i,i'}^{n} := -(ij - i'j')$, $j = n - i$, $j' = n - i'$.

Proof. Firstly note that for $n = 2$

$$\left(b_0^{(1)}\right)^2 - b_0^{(2)}b_0^{(0)} = s^2 - 2x^3t = -c_0^{(0)}.$$

Now assuming that the result holds for all pairs $i + j = n$, we compute

$$D\left(b_0^{(i+1)}b_0^{(j)} - b_0^{(i'+1)}b_0^{(j')}\right) \\ = -x^2\Big((n-1)\lambda_{i+1,i'+1}^{n+1}c_0^{(n-2)} + (i+1)r(n,i,i') \\ + jr(n,i+1,i'+1) - (i'-i)r(n,i',i'+1)\Big).$$

If either $i = n$ or $i' = n$ we instead obtain either $(i'+1)r(n,n,i') + j'r(n,n,i'+1)$ or $(i+1)r(n,i,n) + jr(n,i+1,n)$ in place of the other $r(n,a,b)$. As before we show that there is some $R(n+1,a,b)$ with $D(R(n+1,a,b)) = r(n,a,b)$ for all $a,b \leq n$. Let $\lambda e_0^{(0)} h$ be a term of $r(n,a,b)$, $\lambda \in \mathbb{K}$, $h \in I$ and $e \in \{b,c,d\}$. Note that the expression $b_0^{(i+1)} b_0^{(j)} - b_0^{(i'+1)} b_0^{(j')}$ is homogeneous of degree $2n+3$ and ρ-degree $n+1$. Therefore $h \neq 0$ for $n \geq 2$, and $h \neq g^k$ for some $k \in \mathbb{N}$ as $\deg(e_0^{(0)}) + 12k$ and $\rho(e_0^{(0)}) + 6k$ cannot be $2n+3$ and $n+1$ for any choice of $e_0^{(0)}$ or k. Therefore we can write $h = f_0^{(l)} h'$ for some $f \in \{b,c,d\}$ and $l \in \mathbb{N}$, we can then proceed as described in the proof of Lemma 3.6.4. Now since

$$D\Big(b_0^{(i+1)} b_0^{(j)} - b_0^{(i'+1)} b_0^{(j')} - \lambda_{i+1,i'+1}^{n+1} c_0^{(n-1)} - (i+1)R(n+1,i,i') \\ - jR(n+1,i+1,i'+1) + (i'-i)R(n+1,i',i'+1)\Big) = 0,$$

this expression is then an invariant of degree $2n+3$ and ρ-degree $n+1$. Considering all such elements we find:

n	$S^{\Delta} \cap S_{(2n+3,n+1)}$
$6l$	0
$6l+1$	$\{\lambda(2x^3t - s^2)g^l \mid \lambda \in \mathbb{K}\}$
$6l+2$	0
$6l+3$	0
$6l+4$	0
$6l+5$	$\{\lambda x^2 g^{l+1} \mid \lambda \in \mathbb{K}\}$

Note that all such elements are of the form $b_0^{(0)} h_1 + c_0^{(0)} h_2 + d_0^{(0)} h_3$, $h_i \in C$, and so if we let

$$b_0^{(i+1)} b_0^{(j)} - b_0^{(i'+1)} b_0^{(j')} - \lambda_{i+1,i'+1}^{n+1} c_0^{(n-1)} - (i+1)R(n+1,i,i') \\ - jR(n+1,i+1,i'+1) + (i'-i)R(n+1,i',i'+1) = \mu p,$$

with $\mu \in \mathbb{K}$ and $p \in S^{\Delta} \cap S_{(2n+14,n+5)}$. By setting

$$r(n+1,i,i') := \lambda_{i+1,i'+1}^{n+1} c_0^{(n-1)} - (i+1)R(n+1,i,i') \\ - jR(n+1,i+1,i'+1) - (i'-i)R(n+1,i',i'+1) - \mu p,$$

then $r(n+1,i,i')$ is of the form $b_0^{(0)} h_1 + c_0^{(0)} h_2 + d_0^{(0)} h_3$, $h_i \in C$ for $i = 1,2,3$, and we obtain the required result. $\square$

Now as before by using this proof we consider the following expression

$$(\beta_n)^2 - \beta_{2n}\beta_0 - \lambda_{n,2n}^{2n}\gamma_{2n-2} - T(2n, n, 2n),$$

where $T(n, a, b) \in R^D$ is defined by replacing every $e_0^{(k)}$ by the corresponding $\eta_k \in R^D$ which has $e_0^{(k)}$ as its v-degree zero term. From this we see that the expression above has its v-degree 0 term as 0, and therefore the whole expression must be zero, as the expression is an invariant, and there is no invariant which is divisible by v. This means we have

$$(\beta_n)^2 = \beta_{2n}\beta_0 + \lambda_{n,2n}^{2n}\gamma_{2n-2} + T(2n, n, 2n),$$

with both $\beta_{2n}\beta_0, T(2n, n, 2n) \in (\beta_0, \gamma_0, \delta_0)R^D$. We then square our expression to obtain

$$(\beta_n)^4 = (\lambda_{n,2n}^{2n})^2(\gamma_{2n-2})^2 + p,$$

where $p \in (\beta_0, \gamma_0, \delta_0)R^D$. Using our relations for the γ_i calculated in Lemma 3.6.4 we then have

$$(\beta_n)^4 = (\lambda_{n,2n}^{2n})^2 \left((\gamma_{2n-2})^2 - \gamma_{2n}\gamma_0\right) + p + (\lambda_{n,2n}^{2n})^2\gamma_{2n}\gamma_0.$$

Each term on the right-hand side is in $(\beta_0, \gamma_0, \delta_0)R^D$ and hence we have $(\beta_n)^4 \in (\beta_0, \gamma_0, \delta_0)R^D$. This concludes our proof of Proposition 3.6.3 and we are finally able to prove Theorem 3.6.1.

Proof of Theorem 3.6.1. First we remark that $\beta_0, \gamma_0, \delta_0 \in \mathfrak{f}_{R^D}$ since $\mathfrak{pl}(D) \subset \mathfrak{f}_{R^D}$. Indeed, given $d \in \mathfrak{pl}(D)$, with $d = D(p)$ we have $D(\frac{p}{d}) = 1$ and the morphism

$$\frac{p}{d} : \mathbb{A}_d^5 \longrightarrow \mathbb{G}_a$$

is $\mathbb{G}_a$-equivariant. To see this, let $x \in \mathbb{A}_d^5$, and $\alpha \in \mathbb{G}_a$, then as $\alpha \cdot \frac{p}{d} = \exp(\alpha D)\left(\frac{p}{d}\right) = \frac{p}{d} + \alpha \cdot 1$ we have

$$\left(\alpha \cdot \frac{p}{d}\right)(x) = \left(\frac{p}{d} + \alpha \cdot 1\right)(x) = \left(\frac{p}{d}\right)(x) + \alpha = \alpha \cdot \left(\left(\frac{p}{d}\right)(x)\right).$$

Hence the affine open set $\mathbb{A}_d^5$ is a trivial $\mathbb{G}_a$-bundle, and $\mathbb{A}_d^5/\mathbb{G}_a = \mathrm{Spec}(\mathbb{K}[x, s, t, u, v]_d^{\mathbb{G}_a})$. Thus $\mathbb{K}[x, s, t, u, v]_d^{\mathbb{G}_a} = (\mathbb{K}[x, s, t, u, v]^{\mathbb{G}_a})_d$ is finitely generated.

Additionally, since $\mathfrak{f}_{R^D}$ is a radical ideal by [6, §2.2], we have $\sqrt{(\beta_0, \gamma_0, \delta_0)R^D} \subset \mathfrak{f}_{R^D}$. Now suppose that $f \in \mathfrak{f}_{R^D}$. Note that $R^D = B_0 + (\beta_n, \gamma_n, \delta_n)_{n\in\mathbb{N}}R^D$, so we may assume that $f \in B_0$. Since $(R^D)_f$ is finitely generated, we therefore have that $(R^D)_f = (B_N)_f$ for some $N \in \mathbb{N}$. Hence there is some $k > 0$ satisfying $f^k\beta_{N+1}, f^k\gamma_{N+1}, f^k\delta_{N+1} \in B_N$ and $f^k \in [B_N \colon B_{N+1}] \cap B_0 = (\beta_0, \gamma_0, \delta_0)B_0 \subset (\beta_0, \gamma_0, \delta_0)R^D$ by Lemma 3.6.2. Thus $\mathfrak{f}_{R^D} = \sqrt{(\beta_0, \gamma_0, \delta_0)R^D}$, completing the proof of the first statement of Theorem 3.6.1.

It remains to show that $\mathrm{L}(\mathcal{G}) \subset \mathrm{L}(\mathfrak{f}_{R^D})$. By Proposition 3.6.3 we know that $\mathcal{G} = \{\beta_i, \gamma_i, \delta_i \mid i \geq 0\}$ generates the finite generation ideal. Note that the leading monomials of these generators are the b_n, c_n and d_n, $n \in \mathbb{N}$ described in the proof of Lemma 3.5.8. We have shown that applying the relations of these monomials to the corresponding generators yields an element with a leading monomial lying in $L(\mathcal{G})$. Additionally, we have shown that applying the relations between the generators corresponding to these leading monomials and e, the leading monomial of g, yields an element with leading monomial lying in $L(\mathcal{G})$. Any element in $L(\mathfrak{f}_{R^D})$ is obtained as the leading monomial of some combination of elements in $\mathfrak{f}_{R^D}$ and elements in R^D, which is generated by $\mathcal{G} \cup \{g\}$. Since all such combinations yield an element whose leading monomial lies in $L(\mathcal{G})$, we conclude that $\mathrm{L}(\mathcal{G}) \subset \mathrm{L}(\mathfrak{f}_{R^D})$. □

— 4 —

Maubach's Conjecture

The goal of this chapter is to study an example of a locally nilpotent derivation on a commutative $\mathbb{K}$-domain constructed by Maubach and conjectured to be non-finitely generated and thus a counterexample to Hilbert's fourteenth problem. Our aim, which we do not quite achieve, is to show this example is indeed non-finitely generated. In the course of doing so we demonstrate how this example differs from existing counterexamples, which prevents us from simply using the same methods applied in those cases in order to show that it is not finitely generated. Furthermore, we show how our approach here can be applied to other examples to show whether or not they are finitely generated.

To try to accomplish this goal we build upon methods we developed in Chapter 3, namely use of a bi-degree in order to write the derivation as a series of linear maps between vector spaces consisting of all bi-homogeneous elements of a given degree. We then try to show that these linear maps are surjective whenever the dimension of the domain is at least as large as the dimension of the codomain and injective whenever the dimension of the codomain is at least as large as the dimension of the domain. We then show that if this is the case, then there is an infinite sequence of invariants which cannot be generated by any finite set of invariants which would then allow us to conclude that this example must be a counterexample to Hilbert's fourteenth problem.

4.1 MAUBACH'S EXAMPLE

As in previous chapters we suppose that $\mathbb{K}$ is an algebraically closed field of characteristic zero. Set $R = \mathbb{K}[x, y, z, u, w]$ and consider the locally nilpotent derivation

$$D := x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + z\frac{\partial}{\partial u} + u^2\frac{\partial}{\partial w}. \tag{4.1}$$

Our main goal in this chapter is to try to prove the following, conjectured by Maubach in [27, §5]:

Conjecture 4.1.1. *R^D is not finitely generated.*

We provide an almost complete proof of this result, with the completion of the proof of Conjecture 4.1.6 being the only remaining hurdle.

This example bears a striking similarity to Daigle and Freudenburg's counterexample, with the exception of the first and final term of the derivation. As with all locally nilpotent derivations, there is a corresponding $\mathbb{G}_a$-action on R defined by

$$\begin{aligned}
&\exp(\alpha D)\cdot(x, y, z, u, w)\\
&\quad= \Big(x, y + \alpha x, z + \alpha y + \frac{1}{2}\alpha^2 x, u + \alpha z + \frac{1}{2}\alpha^2 y + \frac{1}{6}\alpha^3 x,\\
&\qquad w + \alpha u^2 + \alpha^2 zu + \frac{1}{3}\alpha^3(yu + z^2) + \frac{1}{12}\alpha^4(xu + 3yz)\\
&\qquad\qquad + \frac{1}{60}\alpha^5(4xz + 3y^2) + \frac{1}{36}\alpha^6 xy + \frac{1}{252}x^2\Big).
\end{aligned}$$

Note that the derivation D commutes with the following $\mathbb{G}_m$-action:

$$\lambda \cdot (x, s, t, u, v) := (\lambda \cdot x, \lambda \cdot y, \lambda \cdot z, \lambda \cdot u, \lambda^2 \cdot w). \tag{4.2}$$

This action yields a grading κ on R with $\kappa(x) = \kappa(y) = \kappa(z) = \kappa(u) = 1$ and $\kappa(w) = 2$. The ρ-grading induced by the derivation yields

$$\rho(x) = 0, \quad \rho(y) = 1, \quad \rho(z) = 2, \quad \rho(u) = 3, \quad \rho(w) = 7.$$

For $f \in R$, the bi-degree of f is $\deg(f) := (a, b)$, where $a = \kappa(f)$ and $b = \rho(f)$. We write $R_{(a,b)} := \{f \in R, \mid \deg(f) = (a, b), f \text{ is bi-homogeneous}\} \cup \{0\}$. Note that $R_{(a,b)}$ is a vector space over $\mathbb{K}$ with basis given by the monomials of degree (a, b), we set $r_{a,b} := \dim\left(R_{(a,b)}\right)$. Note that if $f \in R_{(a,b)}$ then $D(f) \in R_{(a,b-1)}$ so we can realize our derivation D as a collection of linear maps $D : R_{(a,b)} \longrightarrow R_{(a,b-1)}$.

Consider $S := \mathbb{K}[x, y, z, u]$ and

$$\Delta := D|_S = x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + z\frac{\partial}{\partial u}.$$

The notions of degree defined above restrict to S and so we may define $S_{(a,b)}$ in the obvious way, and we let $s_{a,b} := \dim\left(S_{(a,b)}\right)$. As with D we can realize Δ as a collection of linear maps $\Delta : S_{(a,b)} \longrightarrow S_{(a,b-1)}$. We note that Δ is the Weitzenböck derivation, and from Example 2.1.14 we know S^Δ is finitely generated, with generators

$$\begin{aligned}
&x,\\
&\gamma := 2xz - y^2,\\
&\delta := 3x^2u - 3xyz + y^3,\\
&g := 9x^2u^2 - 18xyzu + 6y^3u + 8xz^3 - 3y^2z^2.
\end{aligned} \tag{4.3}$$

We note that

$$\deg(x) = (1, 0), \quad \deg(\gamma) = (2, 2), \quad \deg(\delta) = (3, 3), \quad \deg(g) = (4, 6). \tag{4.4}$$

Additionally, there is a relation between these generators:

$$\gamma^3 + \delta^2 = x^2 g. \tag{4.5}$$

For $f \in R_{(c,d)}$ we define the map

$$m_f : R_{(a,b)} \hookrightarrow R_{(a+c,b+d)},$$

where $m_f(g) := fg$ for all $g \in R_{(a,b)}$. This map is clearly injective as R is a domain, similarly so when replacing R with S.

Note that for a monomial $m = x^{j_1}y^{j_2}z^{j_3}u^{j_4}w^{j_5} \in R_{(a,b)}$ the j_i must satisfy:

$$\begin{aligned}
j_1 + j_2 + j_3 + j_4 + 2j_5 &= a,\\
j_2 + 2j_3 + 3j_4 + 7j_5 &= b,
\end{aligned}$$

and hence $r_{a,b}$ is the number of non-negative integer solutions to these equations. Observe that for $a, b \in \mathbb{N}$ and $i \le a/2, b/7$ there is an injection $m_{w^i} : S_{(a-2i,b-7i)} \hookrightarrow R_{(a,b)}$, the image consisting of all polynomials in $R_{(a,b)}$ which have w-degree precisely i. Thus, for $i \neq j$, $m_{w^i}\left(S_{(a-2i,b-7i)}\right) \cap m_{w^j}\left(S_{(a-2j,b-7j)}\right) = 0$ and we can write

$$R_{(a,b)} = \bigoplus_{i=0}^{min(a/2,b/7)} m_{w^i}\left(S_{(a-2i,b-7i)}\right),$$

so

$$r_{a,b} = \sum_{i=0}^{min(a/2,b/7)} s_{a-2i,b-7i}.$$

4.1.1 THE STRUCTURE OF $S_{(a,b)}$

Observe that for a monomial $m = x^{j_1}y^{j_2}z^{j_3}u^{j_4} \in S$, $\deg(m) = (j_1 + j_2 + j_3 + j_4, j_2 + 2j_3 + 3j_4)$, in particular we have

$$j_2 + 2j_3 + 3j_4 \leq 3(j_1 + j_2 + j_3 + j_4).$$

Hence $S_{(a,b)} = 0$ whenever $b > 3a$. Fixing a, we compute $s_{a,k}$ for all k. Recall that the monomials of bi-degree (a, k) form a basis for $S_{(a,k)}$, and so $s_{a,k}$ is the number of these monomials. Observe that if a monomial $m \in S$ has bi-degree (c, k) with $c \leq a$, then $x^{a-c}m$ is a monomial of bi-degree (a, k). Thus, to count the number of monomials in $S_{(a,k)}$ we can count all monomials $m = y^{j_2}z^{j_3}u^{j_4}$ of bi-degree (c, b) with $0 \leq c \leq a$ and then add them together. We write $T_{(a,b)}$ to represent the sub-vector space of $S_{(a,b)}$ spanned by the monomials of bi-degree (a, b) and $j_1 = 0$, that is:

$$T_{(a,b)} := \left\{ \sum_m \lambda_m m \in S_{(a,b)} \mid \deg(m) = (a, b),\ \deg_x(m) = 0 \right\}. \tag{4.6}$$

Note that when $b > 0$:

$$S_{(a,b)} = \bigoplus_{i=0}^{a} m_{x^{k-i}} \left(T_{(i,b)}\right),$$

we write $t_{a,b} := \dim \left(T_{(a,b)}\right)$. Naturally we obtain for $b > 0$:

$$s_{a,k} = \sum_{i=0}^{a} t_{i,b}. \tag{4.7}$$

We note that $s_{a,0} = 1$ for all $a \in \mathbb{N}$, as $S_{(a,0)}$ is spanned by the monomial x^a.

Consider $T_{(a,k)}$, it consists of monomials $m = y^{j_2}z^{j_3}u^{j_4}$ with

$$j_2 + j_3 + j_4 = a,$$
$$j_2 + 2j_3 + 3j_4 = k.$$

Rearranging, we obtain $j_3 + 2j_4 = k - a$ and $2j_2 + j_3 = 3a - k$ allowing us to observe that $t_{a,k} = 0$ unless $a \leq k \leq 3a$. Additionally we have a general solution to these equations

$$\left(\frac{3a-k}{2} - l, 2l, \frac{k-a}{2} - l \right),$$

where we require $0 \leq l \leq \min\left(\frac{k-a}{2}, \frac{3a-k}{2}\right)$. We observe then that

$$t_{a,k} = \begin{cases} \left\lfloor \dfrac{k-a+2}{2} \right\rfloor & a \leq k \leq 2a \\ \left\lfloor \dfrac{3a-k+2}{2} \right\rfloor & 2a \leq k \leq 3a \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

Note that this equation shows that $t_{a,k} = t_{a,4a-k}$ for $0 \leq k \leq 2a$. Indeed, there is a bijection $\psi : T_{(a,k)} \longrightarrow T_{(a,4a-k)}$ where $\psi(y^{i_1}z^{i_2}u^{i_3}) = y^{i_3}z^{i_2}u^{i_1}$.

We give $t_{a,k}$ for $1 \leq a \leq 5$, $0 \leq k \leq 15$ below.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 2 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 2 & 2 & 1 & 1 \end{pmatrix}$$

$s_{a,k}$ for $1 \leq a \leq 5$ and $0 \leq k \leq 15$ is then:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 3 & 3 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 4 & 4 & 5 & 4 & 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 & 4 & 5 & 6 & 6 & 6 & 6 & 5 & 4 & 3 & 2 & 1 & 1 \end{pmatrix}$$

Proposition 4.1.2. *1.* $s_{a,b} = s_{a,3a-b}$.

2. For $0 \leq b \leq 3a/2$, *we have* $s_{a,b} \geq s_{a,b-1}$.

3. For $3a/2 < b \leq 3a$, *we have* $s_{a,b} \leq s_{a,b-1}$.

Proof. For part 1, we consider $\phi : S \longrightarrow S$ defined on monomials by $\phi(x^{i_1}y^{i_2}z^{i_3}u^{i_4}) = x^{i_4}y^{i_3}z^{i_2}u^{i_1}$. Note that $\phi^2 = 1_S$, the identity on S. Now if $x^{i_1}y^{i_2}z^{i_3}u^{i_4} \in S_{(a,b)}$ we have that

$$i_1 + i_2 + i_3 + i_4 = a,$$
$$i_2 + 2i_3 + 3i_4 = b.$$

Observe that $\deg(x^{i_4}y^{i_3}z^{i_2}u^{i_1}) = (a, i_3 + 2i_2 + 3i_1)$, and

$$i_3 + 2i_2 + 3i_1 = 3(i_1 + i_2 + i_3 + i_4) - (i_2 + 2i_3 + 3i_4) = 3a - b,$$

hence $x^{i_4}y^{i_3}z^{i_2}u^{i_1} \in S_{(a,3a-b)}$. Therefore ϕ defines a bijection between $S_{(a,b)}$ and $S_{(a,3a-b)}$, proving 1.

For parts 2 and 3 we induct on a, observing that these results hold for $a = 1$ and 2, as shown in the matrix above. Now suppose these results hold for $a \le n$ and consider

$$s_{n+1,b} - s_{n+1,b-1} = s_{n,b} + t_{n+1,b} - s_{n,b-1} + t_{n+1,b-1}.$$

Focusing first on part 2, by equation 4.8 we have that when $0 \le b \le 3n/2$, $t_{n,b} \ge t_{n,b-1}$, and $s_{n,b} \ge s_{n,b-1}$ by induction also and hence the result holds for $0 \le b \le 3n/2$.

Now suppose that $3n/2 < b \le 3(n+1)/2$, then by induction $s_{n,b} \le s_{n,b-1}$ and $t_{n+1,b} \ge t_{n+1,b-1}$. We show that either:

1. $s_{n,b} = s_{n,b-1}$,
2. $s_{n,b} = s_{n,b-1} - 1$ and $t_{n+1,b} = t_{n+1,b-1} + 1$,

which yields the result in each case. Note b is one of $(3n+1)/2, (3n+2)/2$ or $3(n+1)/2$. Suppose first that $b = (3n+1)/2$ in this case we have by part 1 that since

$$3n - (3n+1)/2 = (3n-1)/2 = b - 1,$$

we find immediately that $s_{n,b} = s_{n,b-1}$ and the result holds.

Now when $b = (3n+2)/2$, part 1 gives that $s_{n,b} = s_{n,b-2}$, but now

$$s_{n,b-2} = s_{n-1,b-2} + t_{n,b-2}, \quad s_{n,b-1} = s_{n-1,b-1} + t_{n,b-1},$$

and $s_{n-1,b-2} \ge s_{n-1,b-1}$ by induction. From equation 4.8 we note that $t_{n,b-1} - t_{n,b-2} =$ 0 or 1. If either $s_{n-1,b-2} > s_{n-1,b-1}$ or $t_{n,b-1} - t_{n,b-2} = 0$, then $s_{n,b-2} \ge s_{n,b-1}$ and hence $s_{n,b-2} = s_{n,b-1}$ and we are done. Otherwise, $t_{n,b-1} - t_{n,b-2} = 1$ and $s_{n,b} + 1 = s_{n,b-1}$, however in this case from equation 4.8 we see that we must have $t_{n+1,b} = t_{n+1,b-1} + 1$, and hence $s_{n+1,b} = s_{n+1,b-1}$ as required.

Finally, when $b = 3(n+1)/2$ we have $s_{n,b} = s_{n,b-3}$ and $s_{n,b-1} = s_{n,b-2}$. But now since

$$s_{n,b-3} = s_{n-1,b-3} + t_{n,b-3}, \quad s_{n,b-2} = s_{n-1,b-2} + t_{n,b-2},$$

and by induction $s_{n-1,b-3} \ge s_{n-1,b-2}$ we find that if either $s_{n-1,b-3} > s_{n-1,b-2}$ or $t_{n,b-2} = t_{n,b-3}$, we have $s_{n,b-1} = s_{n,b-2} = s_{n,b-3} = s_{n,b}$ and we are done. So suppose

$t_{n,b-2} = t_{n,b-3} + 1$ and $s_{n-1,b-3} = s_{n-1,b-2}$, hence $s_{n,b-2} = s_{n,b-3} + 1$. But in this case we again see from equation 4.8 that we must have $t_{n+1,b} = t_{n+1,b-1} + 1$, and hence $s_{n+1,b} = s_{n+1,b-1}$.

For part 3, if $3(n+1)/2 < b \leq 3(n+1)$, then since we have shown $s_{n+1,3(n+1)-b} \leq s_{n+1,3(n+1)-b+1}$ we immediately find that $s_{n+1,b} \leq s_{n+1,b-1}$ using part 1. □

Proposition 4.1.3. *For all $a, b \in \mathbb{N}$ we have:*

1. *If $s_{a,b} \geq s_{a,b-1}$, then Δ is surjective.*
2. *If $s_{a,b} \leq s_{a,b-1}$, then Δ is injective.*

Proof. Recall that the kernel of Δ is known to be finitely generated by the four elements x, γ, δ and g given in equation 4.3. These elements are bi-homogeneous, and have bi-degrees $(1,0), (2,2), (3,3)$ and $(4,6)$ respectively. We also recall the relation 4.5 between these generators, namely $\gamma^3 + \delta^2 = x^2 g$. Up to an accounting for this relation, the dimension of the kernel of $D : S_{(a,b)} \longrightarrow S_{(a,b-1)}$ will correspond to the number of ways these generators can be combined so they have bi-degree (a, b). Specifically, if $f = x^{a_1}\gamma^{a_2}\delta^{a_3}g^{a_4} \in S^{\Delta}$ and $a_1 \geq 2, a_4 \geq 1$ we can write f as

$$f = x^{a_1-2}\gamma^{a_2+3}\delta^{a_3}g^{a_4-1} + x^{a_1-2}\gamma^{a_2}\delta^{a_3+2}g^{a_4-1},$$

so we need only count the number of ways these generators can be combined to have bi-degree (a, b) with a_4 kept minimal. In Proposition 4.1.2 we showed that whenever $1 \leq b \leq 3a/2$ we have $s_{a,b} \geq s_{a,b-1}$ and whenever $3a/2 \leq b \leq 3a$ we have $s_{a,b} \geq s_{a,b+1}$. Using 4.4, an element $f = x^{a_1}\gamma^{a_2}\delta^{a_3}g^{a_4} \in S^{\Delta}_{(a,b)}$ has

$$\deg(f) = (a_1 + 2a_2 + 3a_3 + 4a_4, 2a_2 + 3a_3 + 6a_4). \tag{4.9}$$

Note that

$$b = 2a_2 + 3a_3 + 6a_4 \leq \frac{3}{2}(a_1 + 2a_2 + 3a_3 + 4a_4) = \frac{3a}{2},$$

and so whenever $3a/2 \leq b \leq 3a$ $\Delta : S_{(a,b)} \longrightarrow S_{(a,b-1)}$ has no kernel and so is injective.

To show the map is surjective whenever $s_{a,b} \geq s_{a,b-1}$, we compute $s_{a,b} - s_{a,b-1}$ and show that this is equal to the dimension of the kernel of $\Delta : S_{(a,b)} \longrightarrow S_{(a,b-1)}$. To compute $s_{a,b} - s_{a,b-1}$ we return to our construction of $S_{(a,b)}$ from $T_{(a,b)}$, namely that

$$S_{(a,b)} = \bigoplus_{i=0}^{a} m_{x^{a-i}}\left(T_{(i,b)}\right).$$

Note that

$$\Delta(T_{(a,b)}) \subset T_{(a,b-1)} \oplus m_x(T_{(a-1,b)}).$$

Projection on to the second component is surjective: given a monomial $x^a y^b z^c u^d \in m_x(T_{(a-1,b)})$ its image in $T_{(a,b-1)} \oplus m_x(T_{(a-1,b)})$ is:

$$\Delta\Big(\frac{1}{b+1}x^{a-1}y^{b+1}z^c u^d\Big) = \Big(\frac{c}{b+1}x^{a-1}y^{b+2}z^{c-1}u^d + \frac{d}{b+1}x^{a-1}y^{b+1}z^{c+1}u^{d-1}, x^a y^b z^c u^d\Big),$$

hence $t_{a,b} \geq t_{a-1,b-1}$ for all a, b. Using equation 4.8, we find that:

$$t_{n,m} - t_{n-1,m-1} = \begin{cases} 1 & 2n \leq m \leq 3n \\ 0 & \text{otherwise} \end{cases}$$

We note that $2n \leq m \leq 3n$ is equivalent to $m/3 \leq n \leq m/2$ and thus,

$$\begin{aligned} s_{a,b} - s_{a,b-1} &= \sum_{i=0}^{a}(t_{i,b} - t_{i,b-1}) \\ &= \sum_{i=1}^{a}(t_{i,b} - t_{i-1,b-1}) + t_{0,b} - t_{a,b-1} \\ &= \left\lfloor \frac{b}{2} \right\rfloor - \left\lceil \frac{b}{3} \right\rceil + 1 - \max\left(0, \left\lfloor \frac{b-a+1}{2} \right\rfloor\right), \end{aligned}$$

where we have used that since $0 \leq b \leq 3a/2$, $t_{a,b-1} = \max\left(0, \left\lfloor \frac{b-a+1}{2} \right\rfloor\right)$ by equation 4.8. In fact this formula holds in more generality, since

$$\left\lfloor \frac{b-a+1}{2} \right\rfloor = \left\lfloor \frac{3a-b+1}{2} \right\rfloor,$$

this formula holds for all $0 \leq b \leq 3a$. Note however that from Proposition 4.1.2 we have $s_{a,b} - s_{a,b-1} = -(s_{a,3a-b+1} - s_{a,3a-b})$ and not $-(s_{a,3a-b} - s_{a,3a-b-1})$. Now we compute the dimension of the kernel. As remarked previously S^Δ is generated by x, γ, δ and g, elements of the kernel with degree (a, b) of the form $x^{a_1}\gamma^{a_2}\delta^{a_3}g^{a_4}$ satisfy equation 4.9. Suppose that b is even, then a solution to these equations is given by $(a-b, \frac{b}{2}, 0, 0)$. Note that if $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a solution to these equations; then both $(\alpha_1 + 2, \alpha_2 - 3, \alpha_3, \alpha_4 + 1)$ and $(\alpha_1, \alpha_2 - 3, \alpha_3 + 2, \alpha_4)$ are also. Hence, a general solution to these equations is

$$\left(a - b + 2y, \frac{b}{2} - 3x - 3y, 2x, y\right).$$

The dimension of the kernel will be the number of the positive integer solutions to these equations, hence we require $a - b + 2y \geq 0$, $\frac{b}{2} - 3x - 3y \geq 0$, $x \geq 0$ and $y \geq 0$. Additionally, since $x^2 g = \delta^2 + \gamma^3$ the solution $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a linear combination of $(\alpha_1 - 2, \alpha_2 - 3, \alpha_3, \alpha_4 - 1)$ and $(\alpha_1 - 2, \alpha_2, \alpha_3 - 2, \alpha_4 - 1)$ hence we need only count the solutions where α_1 is minimal. The number of solutions is then

$$\left\lfloor \frac{b}{6} \right\rfloor + 1 - \max\left(0, \left\lceil \frac{b-a}{2} \right\rceil\right).$$

Now suppose that b is odd, a solution to these equations is $(a - b, \frac{b-3}{2}, 1, 0)$ and the general solution is given by

$$\left(a - b + 2y, \frac{b-3}{2} - 3x - 3y, 1 + 2x, y\right).$$

The number of solutions in this case then is

$$\left\lfloor \frac{b-3}{6} \right\rfloor + 1 - \max\left(0, \left\lceil \frac{b-a}{2} \right\rceil\right).$$

Now we show that in all cases the number of these solutions corresponds exactly to $s_{a,b} - s_{a,b-1}$. First, observe that $\left\lceil \frac{b-a}{2} \right\rceil = \left\lfloor \frac{b-a+1}{2} \right\rfloor$ and a quick computation shows that

$$\left\lfloor \frac{b}{2} \right\rfloor - \left\lceil \frac{b}{3} \right\rceil = \begin{cases} \left\lfloor \frac{b}{6} \right\rfloor & b \text{ even} \\ \left\lfloor \frac{b-3}{6} \right\rfloor & b \text{ odd} \end{cases}$$

thus completing the proof. □

Now we would like to prove a version of this result for $R_{(a,b)}$, as doing so would allow us to determine the dimension of $R^D_{(a,b)}$ simply by computing $r_{a,b} - r_{a,b-1}$. Before we do so we first prove the following:

Proposition 4.1.4. *Suppose* $0 \leq b \leq \frac{3a}{2}$*, then the map*

$$D^{3a-2b} : S_{(a,3a-b)} \longrightarrow S_{(a,b)},$$

is a bijection.

Proof. By Proposition 4.1.2 we have that $s_{a,b} = s_{a,3a-b}$. In order to prove this proposition we first observe that if $f \in S^{\Delta}_{(a,k)}$, then $f \in D^{3a-2k}(S)$. To start, observe that

1. $D^3(u) = D^2(z) = D(y) = x$, with $u \in S_{(1,3)}$,
2. $D^2(3yu - 2z^2) = D(3xu - yz) = 2xz - y^2 = \gamma$, with $\overline{\gamma} := 3yu - 2z^2 \in S_{(2,4)}$,
3. $D^3(-3xu^2 + 3yzu - \frac{4}{3}z^3) = D^2(-3xzu + 3y^2u - yz^2) = D(3xyu - 4xz^2 + y^2z) = 3x^2u - 3xyz + y^3$, with $\overline{\delta} := -3xu^2 + 3yzu - \frac{4}{3}z^3 \in S_{(3,6)}$.

Since x, δ, γ and g generate S^Δ, we can assume without loss of generality that f is of the form

$$f = \lambda x^{i_1}\gamma^{i_2}\delta^{i_3}g^{i_4},$$

where $i_1 + 2i_2 + 3i_3 + 4i_4 = a$, $2i_2 + 3i_3 + 6i_4 = k$ and $\lambda \in \mathbb{K}^*$. We then define

$$\overline{f} = \lambda u^{i_1}\overline{\gamma}^{i_2}\overline{\delta}^{i_3}g^{i_4},$$

and note that $D^{3a-2k}(\overline{f}) = \mu\lambda f$, where $\mu \in \mathbb{K}^*$, hence $f \in D^{3a-2k}(S)$.

Now, suppose there are linearly independent invariants $f_1, \ldots, f_n \in S^\Delta$, of the form

$$f_i = x^{i_1}\gamma^{i_2}\delta^{i_3}g^{i_4},$$

where $\deg(f_i) = (a, k_i)$ and $k_i \leq b$ for all i. Then $D^{3a-k_i-b}(\overline{f_i}) \in S_{(a,b)} \setminus \{0\}$ for all i. These must be linearly independent since otherwise we would have without loss of generality

$$D^{3a-k_1-b}(\overline{f}_1) = \sum_{i=2}^{n} \lambda_i D^{3a-k_i-b}(\overline{f}_i).$$

Hence

$$f_1 = D^{3a-2k_1}(\overline{f}_1) = \sum_{i=2}^{n} \lambda_i D^{3a-2k_1}(\overline{f_i}),$$

and since $D(f_1) = 0$ we must have for the nonzero λ_i that $D^{3a-2k_1}(\overline{f}_i) = f_i$. But then we have a contradiction since the f_i are assumed to be linearly independent.

Recall from Proposition 4.1.2, we have for $0 \leq b \leq 3a/2$ that $s_{a,b} \geq s_{a,b-1}$, therefore we must have

$$s_{a,b} = \sum_{i=0}^{b} s^\Delta_{a,i},$$

where $s^\Delta_{a,i} \geq 0$ for all i. Thus, there are $s_{a,b}$ invariants $\{f_1, \ldots, f_{s_{a,b}}\} \subset S^\Delta$ which are of the form $f_i = x^{i_1}\gamma^{i_2}\delta^{i_3}g^{i_4}$, which together generate $S^\Delta_{(a,k)}$ for all $0 \leq k \leq b$ and the set

$$\left\{D^{3a-\rho(f_i)-b} \in S_{(a,b)} \mid i = 1, \ldots, s_{a,b}\right\},$$

forms a basis for $S_{(a,b)}$, consisting entirely of elements in $D^{3a-2b}(S)$, proving the result. □

Remark. *The basis constructed in the proof above makes it very clear when an element is in the image. For $f \in S_{(a,b)}$, we can write*

$$f = \sum_{i=1}^{k} \lambda_i D^{k_i} \left(u^{a_{i_1}} \overline{\gamma}^{ai_2} \overline{\delta}^{a_{i_3}} g^{a_{i_4}} \right),$$

then $f \in D^{\min_i(k_i)}(S) \setminus D^{\min_i(k_i)+1}(S)$. However this choice of basis does not respect multiplication very well, for example

$$D^2(u) \cdot u = yu = \frac{1}{5} \left(D^2(u^2) + \overline{\gamma} \right) \in S_{(2,4)}.$$

Proposition 4.1.5. *Suppose R is a $\mathbb{K}$-domain, and δ a locally nilpotent derivation on R. Suppose that $\alpha, \beta \in R$ satisfy $\delta^{k-1}(\alpha) \neq 0$, $\delta^k(\alpha) = 0$ and $\beta \in \delta^{k+1}(R)$ then we have $\alpha\beta \in D(R)$.*

Proof. Suppose that $\beta_1, \ldots, \beta_{k+1} \in R$ satisfy $\delta(\beta_i) = \beta_{i-1}$, and $\delta(\beta_1) = \beta$. Then

$$\delta \left(\alpha\beta_1 - \delta(\alpha)\beta_2 + \delta^2(\alpha)\beta_3 + \cdots + (-1)^{k-1}\delta^{k-1}(\alpha)\beta_k \right) = \alpha\beta,$$

as required. □

Having $\beta \in \delta^{k+1}(R)$ is not a necessary condition to have $\alpha\beta \in \delta(R)$, for example $zu = D\left(\frac{1}{2}u^2\right)$ but $z \notin D^4(S)$ and $u \notin D^3(S)$. To obtain a version of Proposition 4.1.3 for R a better understanding of when $\alpha\beta \in D(S)$ is required. As a special case, it is clear that when $\alpha = u^{a_{i_1}} \overline{\gamma}^{ai_2} \overline{\delta}^{a_{i_3}} g^{a_{i_4}} \notin D(S)$ and $\beta = u^{b_{i_1}} \overline{\gamma}^{bi_2} \overline{\delta}^{b_{i_3}} g^{b_{i_4}} \notin D(S)$, then their product, $\alpha\beta \notin D(S)$. However, it is not true in general that if $\alpha, \beta \notin D(S)$ then $\alpha\beta \notin D(S)$. For example, if $\alpha = \overline{\gamma} - 9D^2(u^2))$, $\beta = \overline{\delta} \notin D(S)$,

$$\alpha\beta = \frac{10}{3} D(u\gamma^2) - 5D^2(u^2\overline{\delta}) = D\left(\frac{10}{3} u\gamma^2 - 5D(u^2\overline{\delta}) \right) \in D(S).$$

As remarked previously, we would like to have a version of Proposition 4.1.3 for R, however the proof we have of this result is incomplete. Thus instead we state the result as a conjecture, provide an incomplete proof and outline the remaining steps required to complete it

Conjecture 4.1.6. *For all $a, b \in \mathbb{N}$ we have:*

1. *If $r_{a,b} \geq r_{a,b-1}$, then D is surjective.*
2. *If $r_{a,b} \leq r_{a,b-1}$, then D is injective.*

We begin our incomplete proof as follows: First, recall that

$$R_{(a,b)} = \bigoplus_{i=0}^{b/7} m_{w^i}(S_{(a-2i,b-7i)}),$$

and note that we can restrict D to the map

$$D : P^n_{(a,b)} := \bigoplus_{i=0}^{n} m_{w^i}(S_{(a-2i,b-7i)}) \longrightarrow P^n_{(a,b-1)} = \bigoplus_{i=0}^{n} m_{w^i}\left(S_{(a-2i,b-1-7i)}\right),$$

We write $P^{n,D}_{(a,b)}$ to mean the kernel of D restricted to $P^n_{(a,b)}$. Additionally, we let $p^n_{a,b} := \dim\left(P^n_{(a,b)}\right)$ and, by slight abuse of notation, we write $p^{n,D}_{a,b} := p^n_{a,b} - p^n_{a,b-1}$ and refer to this as the *difference* of $P^n_{(a,b)}$. For added convenience, we shall do the same for $s^{\Delta}_{a,b}$ and $r^D_{a,b}$. We prove the result by induction on n, showing that D is surjective whenever $p^n_{a,b} \geq p^n_{a,b-1}$ and injective whenever $p^n_{a,b} \leq p^n_{a,b-1}$. We note that this result is equivalent to showing

$$\dim\left(P^{n,D}_{(a,b)}\right) = \max\{0, p^{n,D}_{a,b}\},$$

By Proposition 4.1.3 and the observation that $P^0_{(a,b)} = S_{(a,b)}$ we note this result holds when $n = 0$. Suppose the result holds for all $k \leq n-1$, and suppose first that $p^{n-1,D}_{(a,b)} \geq 1$. Note that

$$P^n_{(a,b)} = P^{n-1}_{(a,b)} \oplus m_{w^n}\left(S_{(a-2n,b-7n)}\right),$$

Observe that $b \leq 3a/2$, implies that $b-7 \leq 3(a-2)/2$, hence if $s_{a,b} > s_{a,b-1}$ we must have $s_{a-2,b-7} > s_{a-2,b-8}$. Since $p^{n-1,D}_{(a,b)} \geq 1$, and $P^{n-1}_{(a,b)} = \bigoplus_{i=0}^{n-1} m_{w^i}\left(S_{(a-2i,b-7i)}\right)$ we must have $s_{a-2n+2,b-7n+7} > s_{a-2n+2,b-7n+6}$ and hence $s_{a-2n,b-7n} > s_{a-2n,b-7n-1}$.

Let $g_1, \ldots, g_k \in S_{(a-2n,b-7n)}$ generate $S^{\Delta}_{(a-2n,b-7n)}$. Consider $g_i u^2 w^{n-1} \in P^{n-1}_{(a,b-1)}$, since $D : P^{n-1}_{(a,b)} \longrightarrow P^{n-1}_{(a,b-1)}$ is surjective by induction, there is some $h_i \in P^{n-1}_{(a,b)}$ with $D(h_i) = g_i u^2 w^{n-1}$. Therefore, $D(g_i w^n - h_i) = 0$ for all $i = 1, \ldots, k$, where $g_i w^n - h_i \in P^n_{(a,b)}$. Now any invariant $\beta \in P^{n,D}_{(a,b)}$ with $\deg_w(\beta) = n$ can be written as $\beta = \beta_n w^n + \beta_{n-1} w^{n-1} + \cdots + \beta_0$ and

$$D(\beta) = D(\beta_n)w^n + (nu^2\beta_n + D(\beta_{n-1}))w^{n-1} + \cdots + u^2\beta_1 + D(\beta_0) = 0.$$

Therefore $\beta_n \in S_{(a-2n,b-7n)}$ has $D(\beta_n) = 0$. Since the g_i generate $S^D_{(a-2n,b-7n)}$, invariants of w-degree n in $P^n_{(a,b)}$ are the $\mathbb{K}$-linear span of the $g_i w^n - h_i$, so we conclude in this case

$$\dim\left(P^{n,D}_{(a,b)}\right) = s^{\Delta}_{a-2n,b-7n} + p^{n-1,D}_{a,b} = p^{n,D}_{a,b},$$

as required.

Now suppose that $p^{n-1,D}_{(a,b)} \leq 0$, which implies $s^{\Delta}_{a,b} \leq 0$. We note that for there to be some invariant $f \in P^{n,D}_{(a,b)}$ it is necessary that to have $p^{n-1,D}_{(a-2,b-7)} > 0$. Indeed, if

$$f = f_n w^n + f_{n-1} w^{n-1} + \cdots + f_0 \in P^{n,D}_{(a,b)},$$

then by re-scaling the coefficients we can construct

$$f' := f_n w^{n-1} + \frac{n-1}{n} f_{n-1} w^{n-2} + \cdots + \frac{1}{n} f_1 \in P^{n-1,D}_{(a-2,b-7)},$$

If $p^{n-1,D}_{a-2,b-7} < 0$ then

$$\max(0, p^{n,D}_{a,b} = p^{n-1,D}_{a-2,b-7} + s^{\Delta}_{a,b}) = 0,$$

satisfying the conditions of the result. Now, given some $g = g_{n-1} w^{n-1} + \cdots + g_0 \in P^{n-1,D}_{(a-2,b-7)}$, we construct

$$h := \frac{(-1)^{n-1}}{n-1!} g_{n-1} w^n + \frac{(-1)^{n-2}}{n-2!} g_{n-1} w^{n-2} + \cdots + g_0 w \in P^n_{(a.b)},$$

where $D(h) = g_0 u^2 \in S_{(a,b-1)}$. If $g_0 u^2 \in D(S_{(a,b)})$ with $D(r) = g_0 u^2$ then $D(h-r) = 0$ and we say that the invariant g *lifts* to an invariant in $P^n_{(a,b)}$.

Thus, in order to conclude the result we must show that if $P^{n-1,D}_{(a-2,b-7)}$ is generated by $f_1, \ldots, f_k$ and $s^D_{a,b} = l \leq 0$ then precisely $k+l$ linearly independent combinations of these invariants lift to invariants in $P^{n,D}_{(a,b)}$. It is clear that at least $k+l$ invariants lift to invariants in $P^{n,D}_{(a,b)}$ so it remains to show that at most $k+l$ invariants can lift.

This concludes the portion of the proof that has been completed. At this point we outline the remaining steps intended to complete the proof. First, suppose that $r_1, \ldots, r_k \in S_{(a-2,b-7)}$ are the w-degree 0 terms of $f_1, \ldots, f_k$ respectively. By Proposition 4.1.5, if the $r_i \in D^7(S_{(a-2,b)})$ for all i, then $r_i u^2 \in D(S_{(a,b)})$ for all i. We claim that in this case $l = 0$, meaning $\Delta : S_{(a,b)} \longrightarrow S_{(a,b-1)}$ is surjective.

With this in hand, we suppose that the $r_i \in D^6(S_{(a,b)})$ for all i and $r_1, \ldots, r_p \notin D^7(S_{(a,b)})$ whilst the remaining r_i are. Let $s_i \in S_{(a-2,b-1)}$ satisfy $D(s_i) = r_i$, the invariants $f_1, \ldots, f_p$ lift if and only if $x^2 s_i \in S_{(a,b)}$. Note that we have a commutative diagram:

$$\begin{array}{ccc} S_{(a-2,b)} & \xrightarrow{D} & S_{(a-2,b-1)} \\ {\scriptstyle m_x}\downarrow & & \downarrow{\scriptstyle m_x} \\ S_{(a,b)} & \xrightarrow{D} & S_{(a,b)} \end{array}$$

and so if $t_i \in S_{(a,b)}$ satisfies $D(t_i) = x^2 s_i$, we must have $x^2 \nmid t_i$ for $i = 1, \ldots, p$. Now by construction

$$s^{\Delta}_{a,b} = s^{\Delta}_{a-2,b} + t^{\Delta}_{a,b} + t^{\Delta}_{a-1,b},$$

and using 4.8 we find

$$t^{\Delta}_{a,b} + t^{\Delta}_{a-1,b} = \begin{cases} 1 & a-1 \leq b \leq 2a-2 \\ -1 & 2a+1 \leq 3a+1 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore,

$$D : T_{(a,b)} \oplus m_x(T_{(a-1,b)}) \longrightarrow T_{(a,b-1)} \oplus m_x(T_{(a-1,b-1)}) \oplus m_{x^2}(T_{(a-2,b-1)}).$$

Combining this with the above observations, we obtain that there is at most one combination of monomials in $T_{(a,b)} \oplus m_x(T_{(a-1,b)})$ which is divisible by x^2, occurring precisely when $s^{\Delta}_{a-2,b} = l - 1$. Therefore, either we have $s_{a-2,b-1} = l$, in which case $p = -l$ and only $r_{-l+1}, \ldots, r_k$ lift, yielding $k + l$ invariants. Or, $s_{a-2,b-1} = l - 1$, $p = -l + 1$ and only one combination of the $r_1, \ldots, r_{-l+1}$ lift, as well as $r_{-l+2}, \ldots, r_k$, yielding $k + l$ invariants once again.

The proof is then completed by performing a similar argument when supposing that all the $r_i \in D^k(S_{(a-2,b-7+k)})$ for $k = 5, \ldots, 0$, albeit with added complexity as we cannot obtain a commutative diagram like we have for the $k = 6$ case.

4.2 INTEGRAL SEQUENCES

Let $A \subset \mathbb{K}[x_1, \ldots x_{n-1}]$, and let d be a locally nilpotent derivation on A with A^d finitely generated. Set $B = A[x_n]$ and extend d to B by setting $d(x_n) := h \in A \setminus d(A)$. We have shown in Chapter 3 that the existence of a sequence $(g_i)_{i=0}^k \subset A$ with the property that $D(g_i) = hg_{i-1}$ and $D(g_0) = 0$ is equivalent to the existence of an invariant

$$g = g_0 x_n^k + \cdots + g_{k-1} x_n + g_k \in B^d,$$

Definition 4.2.1. We call such a sequence an *integral sequence of length* k, and g_0 the *base* of the sequence.

Daigle and Freudenburg define a similar notion in [4], which we now note to explain how ours differs: Suppose that $A = \oplus_{i\in\mathbb{N}} A_i$ is an $\mathbb{N}$-graded $\mathbb{K}$-domain and $d : A \longrightarrow A$ is a homogeneous locally nilpotent derivation. A sequence $\{a_n\}_{n=1}^{\infty}$ of nonzero homogeneous elements of A^D with the property that there is some integer $p > 0$, called the *period*, for which

$$a_{n+p} = a_n, \text{ for all } n \geq 1,$$

shall be called a *kernel sequence*. A sequence $\{b_n\}_{n=0}^{\infty}$ where each term is nonzero and homogeneous and satisfies

$$d(b_n) = a_n b_{n-1},$$

for all $n > 0$ is called an *integral sequence of* (A, d) *belonging* to $\{a_n\}_{n=1}^{\infty}$.

Comparing their definition to ours, we only allow for a 'kernel sequence' of period 1, with this being just h. However we only require that $h \notin d(A)$, as opposed to having $h \in A^d$ as well. Thus, referring to h as a 'kernel sequence' would be disingenuous.

We call an infinite sequence $(g_i)_{i\in\mathbb{N}}$ with $D(g_0) = 0$ and $D(g_i) = hg_{i-1}$ an *(infinite) integral sequence*. The existence of an integral sequence makes it possible to construct invariants in B of arbitrary x_n-degree k with leading term $g_0 x_n^k$. Now, suppose that there is a $\mathbb{Z}^r_{\geq 0}$-grading on B which commutes with the derivation d. If we consider this grading together with the ρ-grading we divide B into vector spaces $B_{(a_1,\ldots,a_k,b)}$ where d restricts to a linear map

$$d : B_{(a_1,\ldots,a_k,b)} \longrightarrow B_{(a_1,\ldots,a_k,b-1)},$$

where $a_i \in \mathbb{Z}_{\geq 0}$ for all i, $b \geq 1$. Suppose that h has degree $(c_1, \ldots, c_k, t)$, then if we have an integral sequence $(g_i)_{i\in\mathbb{N}} \subset A$, where $g_0 \in A^d$ has degree $(p_1, \ldots, p_k, q)$, then each g_i must have degree $(p_1 + ic_1, \ldots, p_k + ic_k, q + i(t + 1))$. If the linear maps

$$d : A_{(p_1+ic_1,\ldots,p_k+ic_k,q+i(t+1))} \longrightarrow A_{(p_1+ic_1,\ldots,p_k+ic_k,q+i(t+1)-1)},$$

are surjective for all $i \geq 0$, then the existence of the integral sequence $(g_i)_{i\in\mathbb{N}}$ is guaranteed, and so B is not finitely generated. Using Proposition 4.1.3, we can obtain a new proof that Daigle and Freudenburg's example is not finitely generated using this method:

4.2.1 Daigle and Freudenburg's example

In order to not conflict with the notation of this chapter we shall set $A = \mathbb{K}[x,s,t,u]$ and let d be the locally nilpotent derivation

$$d := x^3\frac{\partial}{\partial s} + s\frac{\partial}{\partial t} + t\frac{\partial}{\partial u}.$$

We set $B = A[v]$, and extend d to B by setting $d(v) = x^2$. As observed in chapter 3, A^d is finitely generated by $\beta_0 = x, \gamma_0 = 2x^3t - s^2, \delta_0 = 3x^3u - 3xst + s^3$ and $g = 9x^6u^2 - 18x^3stu + 8x^3t^3 + 6s^3u - 3s^2t^2$. Recall there is a grading arising from a $\mathbb{G}_m$-action on B which yields $\deg(x) = 1, \deg(s) = \deg(t) = \deg(u) = 3$ and $\deg(v) = 2$. The ρ-grading on B gives $\rho(x) = 0, \rho(s) = \rho(v) = 1, \rho(t) = 2, \rho(u) = 3$ allowing us to divide A and B in to vector spaces $A_{(a,b)}$ and $B_{(a,b)}$, $a, b \geq 0$.

Now $S_{(a,b)}$ is in bijection with and $A_{(3a,b)}, A_{(3a+1,b)}$ and $A_{(3a+2,b)}$ by identifying the monomial $x^{i_1}y^{i_2}z^{i_3}u^{i_4}$ with $x^{3i_1}s^{i_2}t^{i_3}u^{i_4}, x^{3i_1+1}s^{i_2}t^{i_3}u^{i_4}$ and $x^{3i_1+2}s^{i_2}t^{i_3}u^{i_4}$ respectively. Thus using Propositions 4.1.2 and 4.1.3 we immediately obtain

Lemma 4.2.2. *1. $A_{(3a+i,b)}$ is in bijection with $A_{(3a+i,3a-b)}$ for $i = 0,1,2$.*

2. If $0 \leq b \leq \frac{3a}{2}$, $\dim(A_{(3a+i,b)}) \geq \dim(A_{(3a+i,b-1)})$ and $D : A_{(3a+i,b)} \longrightarrow A_{(3a+i,b-1)}$ is surjective for $i = 0,1,2$.

3. If $\frac{3a}{2} \leq b \leq 3a$, $\dim(A_{(3a+i,b)}) \leq \dim(A_{(3a+i,b-1)})$ and $D : A_{(3a+i,b)} \longrightarrow A_{(3a+i,b-1)}$ is injective for $i = 0,1,2$.

Now, to construct an integral sequence $(g_i)_{i\in\mathbb{N}}$ with $g_0 = x$ and $d(g_i) = x^2g_{i-1}$ we must have $x^2g_i \in A_{(2(i+1)+1,i)}$ and using Lemma 4.2.2 we can guarantee $x^2g_i \in d(A)$ provided

$$\frac{3\left\lfloor\frac{2(i+1)+1}{3}\right\rfloor}{2} \geq i.$$

Now if $i = 3k$, then we have

$$\frac{3\left\lfloor\frac{2(3k+1)+1}{2}\right\rfloor}{2} = \frac{6k+3}{2} \geq 3k,$$

and similarly if $i = 3k+1$ we have

$$\frac{3\left\lfloor\frac{2(3k+2)+1}{2}\right\rfloor}{2} = \frac{6k+3}{2} \geq 3k+1,$$

and finally if $i = 3k+2$ we have

$$\frac{3\left\lfloor\frac{2(3k+3)+1}{2}\right\rfloor}{2} = 3k+3 \geq 3k+2.$$

Therefore Lemma 4.2.2 is sufficient to show the existence of such an integral sequence, and consequently the existence of the infinite family of invariants we called $(\beta_i)_{i\geq 0}$ in chapter 3. It is a simple matter to repeat this check for the invariants $2x^3t - s^2$ and $3x^6u - 3x^3st + s^3$ and obtain the other two infinite families $(\gamma_i)_{i\geq 0}$ and $(\delta_i)_{i\geq 0}$. Showing that no integral sequence can be constructed using any power of g then completes the construction of a generating set for B.

4.2.2 The non-finiteness criterion

We note and provide the proof for the non-finiteness criterion shown by Daigle and Freudenburg, originating first as Lemma 2.1 in [4], we use the proof from [19, pp.165-166]:

Lemma 4.2.3. *Let $R = \oplus_{i\in\mathbb{N}} R_i$ be a graded $\mathbb{K}$-domain with $R_0 = \mathbb{K}$ and δ a locally nilpotent homogeneous $\mathbb{K}$-derivation on R. Given homogeneous $\alpha \in R^\delta$ which is not in the image of δ, let $\overline{\delta}$ be the extension of δ to $R[T]$ defined by $\overline{\delta}(T) = \alpha$, where T is a variable over R. Suppose β_n is a sequence of non-zero elements of $R[T]^{\overline{\delta}}$ having leading T-coefficients $b_n \in R$. If $\deg(b_n)$ is bounded, but $\deg_T(\beta_n)$ is not bounded, $R[T]^{\overline{\delta}}$ is not finitely generated over $\mathbb{K}$.*

Proof. Let $M[T]$ be the extension to $R[T]$ of the maximal ideal $M = \oplus_{i>0} R_i$ of R. Recall from Definition 2.1.7 that the degree of a locally nilpotent derivation δ is d, where $\delta(R_i) \subset R_{i+d}$ for all $i \geq 0$. Setting $m = \deg\alpha - \deg\delta$, and for every integer n, define $R[T]_n = \sum_{i\in\mathbb{N}} R_{n-mi}T^i$. Then $\oplus_{n\in\mathbb{Z}} K[T]_n$ is a $\mathbb{Z}$-grading of $K[T]$, and $\overline{\delta}$ is homogeneous.

If $\varphi \in \ker\overline{\delta}$ is homogeneous, then $\varphi = \sum \varphi_i T^i$ for homogeneous $\varphi_i \in R$. Since $\overline{\delta} = 0$, it follows from the product rule that $\delta(\varphi_{i-1}) = -i\alpha\varphi_i$ for $i > 0$. Thus, $\varphi_i \notin \mathbb{K}^*$ for $i > 0$, since otherwise $\alpha = \delta(-i^{-1}\varphi_i^{-1}\varphi_{i-1}) \in \delta(R)$. So if $i > 0$, then $\varphi_i \in M$, since each φ_i is homogeneous. Since also $\varphi_0 \in \mathbb{K} + M$, we conclude $\varphi \in \mathbb{K} + M[T]$.

Now for a general element $\psi \in \ker\overline{\delta}$, write $\psi = \sum \psi_n$, where $\psi_n \in K[T]_n$. Since $\overline{\delta}$ is homogeneous, we conclude each $\psi_n \in \ker\overline{\delta}$ as well, meaning $\ker\overline{\delta} \subset \mathbb{K} + M[T]$.

Finally, suppose $\{\beta_n\} \subset \mathbb{K}[f_1, \ldots, f_N]$ for $f_i \in \ker\overline{\delta}$. Then $f_i \in \mathbb{K} + M[T]$ for each i, and so we can assume without loss of generality that each $f_i \in M[T]$. This implies that any monomial expression $f_{i_1} f_{i_2} \cdots f_{i_s} \in M^s[T]$. Note that every R-coefficient of an element of $M^s[T]$ has degree at least s. Since $\deg b_n$ is bounded, there is a finite set F of monomial expressions $f_{i_1} f_{i_2} \cdots f_{i_s}$ such that $\{\beta\} \subset \langle F\rangle$, the $\mathbb{K}$-linear span of F. However, the T-degrees in $\langle F\rangle$ are bounded, whereas the T-degrees for the sequence β_n are unbounded, a contradiction. Therefore, the sequence β_n is not contained in any finitely generated subset of $\ker\overline{\delta}$. □

Remark. *Note that the above proof still holds when α is simply taken to not be in the image of δ, as it makes no use of α being an invariant.*

Returning to Maubach's example, the existence of an invariant of w-degree n is equivalent to an integral sequence of length n, $(a_i)_{i=0}^n \subset S$ where $D(a_0) = 0$ and $D(a_i) = u^2 a_{i-1}$. Therefore, if $a_0 \in S_{(a,b)}$, then $a_i \in S_{(a+2i,b+7i)}$. Now if $i > 3a - b$ then we have $b + 7i > 3a + 6i = 3(a + 2i)$ and so $S_{(a+2i,b+7i)} = \{0\}$. Therefore a_0 can only have an integral sequence of length at most $3a - b$ and so there cannot be any infinite integral sequences in S. Consequently, if it is shown that R^D is not finitely generated, it shows that the non-finiteness criterion is only sufficient and not necessary when we have $\alpha \in R \setminus D(R)$.

4.3 COMPLETING THE PROOF OF CONJECTURE 4.1.1

In this section we complete the proof of Conjecture 4.1.1, assuming Conjecture 4.1.6. In order to do so, we show that for each $x^k \in S$, there is an integral sequence of maximal length $l(k)$, and hence an invariant of the form $\beta_k = x^k w^{l(k)} + l.o.t$ with $l(k)$ growing sufficiently quickly so that each β_k is not generated by the previous β_i $i \le k$. We then show that there cannot be any finite set of invariants which generate all β_k and therefore must conclude that R^D is not finitely generated.

In order to do this, we first make some observations

Proposition 4.3.1. *Suppose that $r_{a,b} > r_{a,b-1}$, let l be chosen so that*

$$R_{(a,b)} = \bigoplus_{i=0}^{l} m_{w^i}\left(S_{(a-2i,b-7i)}\right),$$

then there is an invariant f in $R_{(a,b)}$ with $\deg_w(f) = l$.

Proof. As observed in the proof of Proposition 4.1.6, we have that

1. $r^D_{a,b} = \sum_i^l s^\Delta_{a-2i,b-7i}$,
2. If $b < 3a/2$, then $b - 7i < 3(a - 2i)/2$ for all $i \ge 1$.

Since $r^D_{a,b} > 0$, we must therefore have that $s^\Delta_{a-2l,b-7l} = k > 0$. Now consider

$$D : P^{l-1}_{(a,b)} \longrightarrow P^{l-1}_{(a,b-1)},$$

suppose $p^{l-1,D}_{(a,b)} = l$, then $r^D_{(a,b)} = k + l$. If $l \geq 0$, then we must have k invariants with w-degree l, and if $l < 0$ we must have $k + l > 0$ invariants with w-degree l. In either case we have an invariant with w-degree l, completing the proof. □

With this in hand, we conclude:

Proposition 4.3.2. *Suppose $f \in S_{(a,b)}$ where $b \leq 6$ and $\Delta(f) = 0$. Then for any $k \in \mathbb{N}$ with $r^D_{a+2k,b+7k} \geq s^\Delta_{a,b}$, there is an invariant in R with leading term fw^k.*

Having made these observations it is now clear that the maximal length $l(k)$ of an integral sequence with base x^k is the largest integer such that $r^D_{k+2l(k),7l(k)} > 0$. Since we have that

$$r^D_{k+2l(k),7l(k)} = \sum_{i=0}^{l(k)} s^\Delta_{k+2i,7i},$$

and we recall from the proof of Proposition 4.1.3 that for all $0 \leq b \leq 3a$:

$$s^\Delta_{a,b} = \left\lfloor \frac{b}{2} \right\rfloor - \left\lceil \frac{b}{3} \right\rceil + 1 - \max\left(0, \left\lfloor \frac{b-a+1}{2} \right\rfloor\right).$$

Thus finding $l(k)$ becomes a computational task:

$$r^D_{k+2l(k),7l(k)} = \sum_{i=0}^{l(k)} \left(\left\lfloor \frac{7i}{2} \right\rfloor - \left\lceil \frac{7i}{3} \right\rceil + 1 - \max\left(0, \left\lfloor \frac{5i-k+1}{2} \right\rfloor\right)\right),$$

Note that when $0 \leq 7i \leq k + 2i - 1$, or equivalently $i \leq \frac{k-1}{5}$, then $t_{k+2i,7i-1} = 0$. Additionally, in order to have $r^D_{k+2l(k),7l(k)} < 0$, we certainly require that $7l(k) > \frac{3(k+2l(k))}{2}$, or equivalently, $l(k) > \frac{3k}{8}$. This allows us to rewrite the sum as follows:

$$\begin{aligned} r^D_{k+2l(k),7l(k)} &= \sum_{i=0}^{\lfloor \frac{k-1}{5} \rfloor} \left(\left\lfloor \frac{7i}{2} \right\rfloor - \left\lceil \frac{7i}{3} \right\rceil + 1 \right) \\ &+ \sum_{i=\lfloor \frac{k-1}{5} \rfloor + 1}^{l(k)} \left(\left\lfloor \frac{7i}{2} \right\rfloor - \left\lceil \frac{7i}{3} \right\rceil + 1 - \left\lfloor \frac{5i-k+1}{2} \right\rfloor \right). \end{aligned}$$

Due to the presence of the floor and ceiling functions this sum is difficult to compute directly without taking a considerable number of cases. Instead, we shall find both an upper and lower bound for $l(k)$ and show that any value lying between these ranges will satisfy the required conditions to show that R^D is non-finitely generated. First we obtain an lower bound. To do so we will minimise each element taken in the sum. Note that

$$\left\lfloor \frac{7i}{2} \right\rfloor - \left\lceil \frac{7i}{3} \right\rceil + 1 \geq \frac{7i-1}{2} - \frac{7i+2}{3} + 1 = \frac{7i-1}{6},$$

while

$$\left\lfloor \frac{7i}{2} \right\rfloor - \left\lceil \frac{7i}{3} \right\rceil + 1 - \left\lfloor \frac{5i-k+1}{2} \right\rfloor \geq \frac{7i-1}{6} - \frac{5i-k+1}{2} = \frac{3k-8i-4}{6}.$$

Additionally the sum is made minimal when $\left\lfloor \frac{k-1}{5} \right\rfloor = \frac{k-5}{5}$, leaving us with

$$\begin{aligned} r^D_{k+2l(k),7l(k)} &= \sum_{i=0}^{\frac{k-5}{5}} \frac{7i-1}{6} + \sum_{i=\frac{k}{5}}^{l(k)} \frac{3k-8i-4}{6} \\ &= \frac{k(7k-45)}{300} - \frac{(11k-20l(k)-20)(k-5l(k)-5)}{150}. \end{aligned}$$

Setting this expression equal to 0 we obtain that $l(k)$ is the largest integer satisfying

$$l(k) < \frac{3k-8}{8} \pm \frac{\sqrt{105k^2-360k}}{40},$$

and since we require $l(k) > \frac{3k}{8}$ for all k, we take the positive root. As we are searching for a lower bound we can simply use

$$l(k) = \frac{3k-8}{8} + \frac{\sqrt{105k^2-360k}}{40} - 1.$$

To find an upper bound with the properties we will need to finish the proof we have to be more precise. Note that we can rearrange this sum as

$$r_{k+2l(k),7l(k)} = \sum_{i=0}^{l(k)} \left\lfloor \frac{7i}{2} \right\rfloor - \sum_{i=0}^{l(k)} \left\lceil \frac{7i}{3} \right\rceil + \sum_{i=0}^{l(k)} 1 - \sum_{i=\left\lfloor \frac{k-1}{5} \right\rfloor+1}^{l(k)} \left\lfloor \frac{5i-k+1}{2} \right\rfloor.$$

Labelling each of these sums in order as $(a),(b),(c)$ and (d), we find that

$$\begin{aligned}
(a) &= \frac{1}{2}\left(7\left\lfloor\frac{l(k)-2}{2}\right\rfloor^2 + 21\left\lfloor\frac{l(k)-2}{2}\right\rfloor + 7\left\lfloor\frac{l(k)-1}{2}\right\rfloor^2 + 13\left\lfloor\frac{l(k)-1}{2}\right\rfloor + 20\right),\\
(b) &= \frac{-1}{2}\left(7\left\lfloor\frac{l(k)-3}{3}\right\rfloor^2 + 7\left\lfloor\frac{l(k)-2}{3}\right\rfloor^2 + 7\left\lfloor\frac{l(k)-1}{3}\right\rfloor^2\right.\\
&\qquad \left. + 21\left\lfloor\frac{l(k)-3}{3}\right\rfloor + 15\left\lfloor\frac{l(k)-2}{3}\right\rfloor + 11\left\lfloor\frac{l(k)-1}{3}\right\rfloor + 26\right),\\
(c) &= l(k)+1,\\
(d) &= -\left(\frac{5}{2}\left\lfloor\frac{1}{2}\left(l(k)+\left\lceil\frac{1-k}{5}\right\rceil - 1\right)\right\rfloor\left(\left\lfloor\frac{1}{2}\left(l(k)+\left\lceil\frac{1-k}{5}\right\rceil - 1\right)\right\rfloor + 1\right)\right.\\
&\qquad + \left\lfloor\frac{1}{2}\left(-k+5\left\lfloor\frac{k-1}{5}\right\rfloor + 6\right)\right\rfloor\left(\left\lfloor\frac{1}{2}\left(l(k)+\left\lceil\frac{1-k}{5}\right\rceil - 1\right)\right\rfloor + 1\right)\\
&\qquad + \frac{5}{2}\left\lfloor\frac{1}{2}\left(l(k)+\left\lceil\frac{1-k}{5}\right\rceil - 2\right)\right\rfloor\left(\left\lfloor\frac{1}{2}\left(l(k)+\left\lceil\frac{1-k}{5}\right\rceil\right)\right\rfloor\right)\\
&\qquad \left. + \left\lfloor\frac{1}{2}\left(-k+5\left\lfloor\frac{k-1}{5}\right\rfloor + 11\right)\right\rfloor\left(\left\lfloor\frac{1}{2}\left(l(k)+\left\lceil\frac{1-k}{5}\right\rceil\right)\right\rfloor\right)\right).
\end{aligned}$$

Note that (a) is maximal when $l(k) \equiv 0 \mod 2$, and the expression simplifies to

$$(a) \leq \frac{1}{4}l(k)(7l(k)+6),$$

similarly (b) is maximal when $l(k) \equiv 2 \mod 3$, and the expression simplifies to

$$(b) \leq -\frac{1}{6}(l(k)+1)(7l(k)+2),$$

Finally, we maximise (d) by supposing $k \equiv 1 \mod 5$ and minimising each term, we obtain that

$$\begin{aligned}
(d) \leq &-\frac{1}{40}(-k+5p-9)(-k+5p+1) - \frac{1}{5}(-k+5p+1)\\
&-\frac{1}{40}(-k+5p-9)(-k+5p+1) - \frac{9}{20}(-k+5p+1).
\end{aligned}$$

Taking these terms together and solving for $l(k)$ we find that $l(k)$ is the largest integer satisfying

$$l(k) < \frac{3k}{8} - \frac{3}{16} + \frac{\sqrt{420k^2+1500k+4705}}{80},$$

and since we seek an upper bound we can use

$$l(k) = \frac{3k}{8} - \frac{3}{16} + \frac{\sqrt{420k^2 + 1500k + 4705}}{80}.$$

Now in order to show that R^D is non-finitely generated we shall first show that the corresponding invariants $\beta_k = x^k w^{l(k)} + l.o.t \in R^D$ cannot be generated by a finite subset of these invariants before continuing to show that these invariants cannot be generated by any finite set of invariants. In order to do so we consider the function

$$\lambda : R \longrightarrow \mathbb{Q},$$

defined for monomials $m \in R_{(a,b)}$ as

$$\lambda(m) := \frac{b}{a+b},$$

we then extend this definition to all of R by setting

$$\lambda(f) := \max\{\lambda(m) \,|\, m \text{ is a term of } f\}.$$

We note that the function λ has the following properties:

1. For $f \in R$ and $c \in \mathbb{K}^*$, $\lambda(cf) = \lambda(f)$,
2. If $f, g \in R$ with $\lambda(f) \leq \lambda(g)$, then $\lambda(f) \leq \lambda(fg) \leq \lambda(g)$.

This function is an example of what we call a *ratio function*, and we explore the properties of such functions in Chapter 6.

Now our invariants $\beta_k \in R_{(k+2l(k),7l(k))}$ and hence using our upper and lower bounds

$$\frac{105k - 560 + 7\sqrt{105k^2 - 360k}}{175k - 720 + 9\sqrt{105k^2 - 360k}} \leq \lambda(\beta_k) \leq \frac{210k - 105 + 7\sqrt{420k^2 + 1500k + 4705}}{350k - 135 + 9\sqrt{420k^2 + 1500k + 4705}}.$$

Note that both the upper and lower bounds tend to the limit

$$\frac{105 + 7\sqrt{105}}{175 + 9\sqrt{105}},$$

from below for $k \geq 4$. We note that this is sufficient since we can compute directly that $l(1) = l(2) = 0$, $l(3) = 1$ and $l(4) = 2$ which then gives us that $\lambda(\beta_1) = \lambda(\beta_2) = 0$, whilst $\lambda(\beta_3) = \frac{7}{12}, \lambda(\beta_4) = \frac{14}{22}$ and both of these are below the limit, meaning the sequence $\lambda(\beta_k)$ tends to the limit from below also. Hence $(\lambda(\beta_k))_{k=0}^{\infty}$ and consequently $(\sup_{i \leq k}(\lambda(\beta_k))_{k=0}^{\infty}$ tend towards the same limit. Now by property 2 of the function λ, each β_j appearing in the sequence $(\sup_{i\leq k}(\lambda(\beta_k))_{k=0}^{\infty}$ cannot be generated by the β_i with $i \leq j$. Since $(\sup_{i\leq k}(\lambda(\beta_k))_{k=0}^{\infty}$ defines an infinite strictly increasing sequence, we must conclude that no finite subset of the β_k can generate all β_k.

To show that R^D is not finitely generated we need to show that there is no finite set of invariants $\{f_1, \ldots, f_N\} \subset R^D$ which can generate all β_k. If there were such a set, we can assume each $f_i \in R_{(a_i,b_i)}$. Now we must have

$$\beta_k = \sum_{i=1}^{t} \lambda_i f_1^{i_1} \cdots f_N^{i_N},$$

and at least one of these terms has w-degree $l(k)$. But, since the leading term of β_k is x^k, all f_i appearing in these terms must have leading term $x^{a_i}w^{b_i}$, for some $a_i \leq k$ and $b_i \leq l(k)$. Thus, we have a contradiction since then we must have $\lambda(f_i) \leq \lambda(\beta_{b_i})$ and there are infinitely many β_k with $\lambda(\beta_k) > \lambda(f_i)$ for all i.

4.4 FURTHER RESEARCH

Our work in this chapter to show that Maubach's example is non-finitely generated relies upon results which we would like to generalise. First, we recall Proposition 4.1.3, which demonstrated that the Weitzenböck derivation in 4 variables on S satisfies

$$\dim\left(S^{\Delta}_{(a,b)}\right) = \max\{s_{a,b} - s_{a,b-1}, 0\}.$$

Meaning $\Delta : S_{(a,b)} \longrightarrow S_{(a,b-1)}$ is surjective whenever the dimension of the domain is at least as large of the dimension as the codomain and is injective whenever that inequality is reversed. In general, suppose $R = \mathbb{K}[x_1 \ldots, x_n]$ is a polynomial ring and D a locally nilpotent derivation on R. Suppose there is a maximal k-dimensional action of the torus $(\mathbb{G}_m)^k$ on R which commuting with D and yielding a $\mathbb{Z}^k_{\geq 0}$-grading on R. Combining this with the ρ-grading we divide D into a series of linear maps

$$D : R_{(a_1,\ldots,a_k,b)} \longrightarrow R_{(a_1,\ldots,a_k,b-1)}.$$

Definition 4.4.1. With R and D as above, we say that they satisfy (†), if for all a_i, b we have

$$\dim\left(R^D_{(a_1,\ldots,a_k,b)}\right) = \max\left\{\dim(R_{(a_1,\ldots,a_k,b)}) - \dim(R_{(a_1,\ldots,a_k,b-1)}), 0\right\}. \qquad (†)$$

Recalling Example 2.1.12, we let $R_n = \mathbb{K}[x_0, \dots, x_n]$ and

$$D_n := x_0 \frac{\partial}{\partial x_1} + \cdots + x_{n-1} \frac{\partial}{\partial x_n},$$

the Weitzenböck derivation in n variables. There is a $\mathbb{G}_m$-action commuting with D_n given by

$$\lambda \cdot (x_0, \dots, x_n) := (\lambda x_0, \dots, \lambda x_n),$$

and this yields a grading on R_n with $\deg(x_i) = 1$ for all i. This is a maximal torus action on R_n which we can combine with our ρ-grading, where $\rho(x_i) = i$, and divide R_n into vector spaces $R_{n,(a,b)}$ spanned by the monomials of degree a and ρ-degree b. Now, similar to the Weitzenböck derivation in 4 variables, we have a bijection

$$\varphi : R_{n,(a,b)} \longrightarrow R_{n,(a,na-b)},$$

where $\varphi(x_0^{a_0} \cdots x_n^{a_n}) = x_0^{a_n} \cdots x_n^{a_0}$. By a similar argument to Proposition 4.1.2, we believe it is possible to show:

Conjecture 4.4.2. *1. For $0 \le b \le \frac{na}{2}$, we have $\dim(R_{n,(a,b)}) \ge \dim(R_{n,(a,b-1)})$.*

2. For $\frac{na}{2} < b \le na$, we have $\dim(R_{n,(a,b)}) \le \dim(R_{n,(a,b-1)})$.

Using this, we could then show:

Conjecture 4.4.3. *R_n and D_n have property (†).*

In our proof that $S = R_3$ and Δ satisfies (†) in Proposition 4.1.3, we use a generating set for S^Δ and compute the dimension of the kernel $S^\Delta_{(a,b)}$ as well as $s_{a,b} - s_{a,b-1}$ for all a, b, showing that these coincide. For general n we can take a similar approach, though the combinatorics become increasingly complex. The main difficulty posed in generalising our result is constructing a generating set for general $R_n^{D_n}$, and accounting for their relations in computing the dimension of the kernel $R_{n,(a,b)}^{D_n}$. In [8, § 3], Drensky and Gupta construct a generating set for $R_n^{D_n}$ in $\mathbb{K}[x_0, \dots, x_n][x_0^{-1}]$ which could potentially be converted to a generating set in $\mathbb{K}[x_0, \dots, x_n]$. Recalling Proposition 4.1.4, we claim a generalisation exists:

Conjecture 4.4.4. *Suppose $0 \le b \le \frac{na}{2}$, then*

$$(D_n)^{na-2b} : R_{n,(a,na-b)} \longrightarrow R_{n,(a,b)},$$

is a bijection.

With Conjecture 4.4.3 in mind, we are lead to the question:

Question 4.4.5. *Which locally nilpotent derivations D on $\mathbb{K}[x_1, \ldots, x_n]$ satisfy (†)?*

We believe that all triangular monomial derivations should satisfy (†); it would be interesting to investigate whether this is an equivalence.

Conjecture 4.1.6 is the only remaining piece needed to show in order to prove that Maubach's example is non-finitely generated, which states that Maubach's example satisfies property (†). With this attained, we then show in Section 4.3 that a sequence of invariants exist which cannot be finitely generated in order to conclude that Maubach's example is not finitely generated. To do so, we computed $r^D_{a,b} = r_{a,b} - r_{a,b-1}$ for certain values of a and b in order to determine how far invariants of the form x^k lifted to invariants of the form $x^k w^{l(k)}+$ terms of lower w-degree. We can repeat this step for any invariant $f \in S^\Delta$ to determine how far it could possibly lift in order to try an obtain a generating set for the whole of R. Note however, that if we are given say $f_1, f_2 \in S^\Delta_{(a-2,b-7)}$ and only one lifts to an invariant in $R^D_{(a,b)}$, it is not immediately clear how to determine which of these invariants lift. Any efforts to construct a generating set for R^D must accommodate for this.

Generalising this, we can ask the question:

Question 4.4.6. *Suppose R and D satisfy (†), can we use this to construct an algorithm to compute a generating set for R^D?*

There are already algorithms to compute a generating set of R^D which complete in a finite number of steps if R^D is finitely generated. This includes van den Essen's algorithm [14, § 4], which relies on the computation of a Gröbner basis. Use of a Gröbner basis greatly impacts computational speed as opposed to determining dimensions of vector spaces, as an algorithm using (†) would rely upon instead. However, as remarked above we would not find the generators explicitly, instead merely their degrees. One could also work with invariant rings of the form $R = S[v]$, where $S^{D|_S}$ is finitely generated, and $D(v) \in S$, in order to determine which invariants lift in v-degree.

To demonstrate the approach one might take, we return to $S^\Delta = \mathbb{K}[x, \gamma, \delta, g]$. We know that S and Δ satisfy property (†) and so we can attempt to construct a generating set for S^Δ using only $s^\Delta_{a,b}$. Firstly we have $s^\Delta_{1,0} = 1$, so there is an invariant $r_1 \in S_{(1,0)}$. Additionally $s^\Delta_{k,0} = 1$ for all k and we know that $r_1^k \in S_{(a,0)}$ for all $k \geq 1$, so there are no more generators to find here. Continuing by in this way, we next find an invariant $r_2 \in S^{\Delta}_{(2,2)}$ and we have $r_1^k r_2 \in S_{(2+k,2)}$, which constitutes all invariants with ρ-degree 2. Following this, we find $r_3 \in S^{\Delta}_{(3,3)}$ and this is the only new invariant required for ρ-degree 3. We finally obtain $r_4 \in S^{\Delta}_{(4,6)}$, and since $r_2^3, r_3^2, r_1^2 r_4 \in S^{\Delta}_{(6,6)}$ but $s^{\Delta}_{6,6} = 2$, this indicates the existence of a relation between these three invariants. What is tricky here is determining that $\{r_1, r_2, r_3, r_4\}$ is a generating set for S^{Δ}. It can be done combinatorially by an approach similar to the proof of Proposition 4.1.3, though in general we would prefer not to make a combinatorial check as each new element is added to our generating set.

— 5 —

Symmetries of Invariant rings

First, recall Roberts' example from Section 2.4, set $B := \mathbb{K}[x_1, x_2, x_3, y_1, y_2, y_3, z]$ with locally nilpotent derivation

$$\partial_n := x_1^n \frac{\partial}{\partial y_1} + x_2^n \frac{\partial}{\partial y_2} + x_3^n \frac{\partial}{\partial y_3} + (x_1 x_2 x_3)^{n-1} \frac{\partial}{\partial z},$$

where $n \geq 3$. As shown in [19, § 7.2.3], Daigle and Freudenburg's counterexample can be obtained from Roberts' counterexample by a process of 'removing symmetries' as follows:

Consider the action of $\mathbb{G}_m^3$ on $\mathbb{K}[x_1, x_2, x_3, y_1, y_2, y_3, z]$ defined as

$$(\lambda, \mu, \nu) \cdot (x_1, x_2, x_3, y_1, y_2, y_3, z) := \left(\lambda x_1, \mu x_2, \nu x_3, \lambda^n y_1, \mu^n y_2, \nu^n y_3, \lambda^{n-1} \mu^{n-1} \nu^{n-1} z\right).$$

This action commutes with ∂_n. Furthermore, there is an action of the symmetric group S_3 on B generated by

$$\sigma := (x_1, x_2, x_3)(y_1, y_2, y_3)(z), \quad \tau := (x_1, x_2)(x_3)(y_1, y_2)(y_3)(z). \tag{5.1}$$

This action also commutes with ∂_n. S_3 also acts on $\mathbb{G}_m^3$ via conjugation, with $\tau(\lambda, \mu, \nu)\tau = (\mu, \lambda, \nu)$ and $\sigma(\lambda, \mu, \nu)\sigma^{-1} = (\nu, \lambda, \mu)$. Therefore, we obtain an action of $\mathbb{G}_m^3 \rtimes S_3$ on B. Now consider the subgroup $H \leq \mathbb{G}_m^3$ defined by $\lambda\mu\nu = 1$, which is a 2-dimensional torus. The group $G := H \rtimes S_3$ acts on B also and the invariant ring of H is generated by the monomials

$$\mathbb{K}[x_1, x_2, x_3, y_1, y_2, y_3, z]^H = \mathbb{K}[x_1x_2x_3, x_1^n x_2^n y_3, x_1^n x_3^n y_2, x_2^n x_3^n y_1, x_1^n y_2 y_3, x_2^n y_1 y_3, x_3^n y_1 y_2, y_1 y_2 y_3, z].$$

If we let

$$x = x_1x_2x_3, \quad s = \frac{1}{3}\left(x_1^n x_2^n y_3 + x_1^n x_3^n y_2 + x_2^n x_3^n y_1\right),$$
$$t = \frac{1}{6}\left(x_1^n y_2 y_3 + x_2^n y_1 y_3 + x_3^n y_1 y_2\right), \quad u = \frac{1}{6} y_1 y_2 y_3, \quad v = z,$$

then, since H is normal in G, the invariant ring of the G-action is

$$B^G = (B^H)^{S_3} = \mathbb{K}[x, s, t, u, v],$$

Since G commutes with the action of ∂_n, ∂_n restricts to a locally nilpotent derivation of B^G, with

$$\partial_n(x) = 0, \quad \partial_n(s) = x^n, \quad \partial_n(t) = s, \quad \partial_n(u) = t, \quad \partial_n(v) = x^{n-1}.$$

When $n = 3$, this corresponds precisely to Daigle and Freudenburg's counterexample, allowing us to prove, [19, p. 168]:

Theorem 5.0.1. *The kernel of ∂_3 is not finitely generated.*

Proof. As ∂_3 is manifestly the same derivation as that introduced in Daigle and Freudenburg's counterexample, we can immediately conclude that $B^{G\times\mathbb{G}_a} = (B^G)^{\mathbb{G}_a}$ is not finitely generated. Now suppose that $B^{\mathbb{G}_a}$ were finitely generated, then the reductive group G acts on the variety $\operatorname{Spec}(B^{\mathbb{G}_a})$. By Hilbert's Finiteness Theorem 2.2.2, the invariant ring $(B^{\mathbb{G}_a})^G = B^{\mathbb{G}_a\times G}$ would be finitely generated, a contradiction. □

The focus of this chapter will be on this process of 'removing symmetries.' This process guarantees that when the subalgebra with the symmetries removed is non-finitely generated the subalgebra with the symmetries is also non-finitely generated. We demonstrate by means of a counterexample that the reverse is not guaranteed. Furthermore, we use this method and our observations on integral sequences to apply this method to generalised forms of Roberts' counterexample.

5.1 THE INVARIANT RING CONNECTED TO $\overline{M}_{0,n}$

Recall from our introduction that there is a family of non finitely generated invariant rings under a $\mathbb{G}_a$-action constructed by Doran, Giansiracusa, and Jensen [7] arising from a study of a moduli space called $\overline{M}_{0,n}$. In [3], Castravet and Tevelev show that $\overline{M}_{0,n}$ is not a so-called 'Mori-Dream' space for $n \geq 134$ while it is one for $n \leq 6$. This result is then improved in [21], and then again in [23], where it is shown that it is not a 'Mori-Dream' space for $n \geq 10$.

To construct the above rings as invariant rings, we let

$$R := \mathbb{K}\left[y_1, \ldots, y_{n-1}, x_I \mid I \subset \{1, \ldots, n-1\}, 1 \leq |I| \leq n-4\right].$$

The additive group $\mathbb{G}_a$ then admits an action for $n \geq 6$ defined by

$$x_I \mapsto x_I, \quad y_i \mapsto y_i + t \prod_{i \in I} x_I, \; t \in \mathbb{G}_a.$$

$\overline{M}_{0,n}$ is then not a 'Mori-Dream' space precisely when this invariant ring is non-finitely generated.

Now, the $\mathbb{G}_a$-action on R corresponds to the locally nilpotent derivation

$$D := \sum_{i=1}^{n-1} \left(\prod_{i \in I} x_I \right) \frac{\partial}{\partial y_i},$$

As notation we will write $J_i := \prod_{i \in I} x_I$, so that $D(y_i) = J_i$, let $T = \{1, \ldots, n-1\}$ and fix the notation $I \subset T$. Then for all $a \neq b$, there are invariants

$$J_b y_a - J_a y_b,$$

In fact, letting $J_{a,b} := \prod_{a \in I, b \notin I} x_I$ we have invariants

$$J_{b,a} y_a - J_{a,b} y_b.$$

Similar to Roberts' example, this example has a number of symmetries we can remove. There is a maximal torus action commuting with D which we construct as follows: Let

$$K := \sum_{1 \leq |I| \leq n-4} 1 = 2^{n-1} - \frac{n^2 - n}{2} - 2.$$

The $(\mathbb{G}_m)^K$-action on R given by

$$x_I \mapsto \lambda_I x_I, \quad y_i \mapsto \left(\prod_{i \in I} \lambda_I \right) y_i,$$

commutes with D. Furthermore, there is an action of S_{n-1} on R which commutes with D also. For $\sigma \in S_{n-1}$, we let

$$x_I \mapsto x_{\sigma(I)}, \quad y_i \mapsto y_{\sigma(i)}.$$

Additionally, S_{n-1} acts on $(\mathbb{G}_m)^K$ by conjugation, with $\sigma\lambda_I\sigma^{-1} = \lambda_{\sigma(I)}$. Now consider the subgroup $H \leq (\mathbb{G}_m)^K$ defined by

$$\prod_{1\leq|I|\leq n-4} \lambda_I^{|I|} = 1.$$

Now H is a $(K-1)$-dimensional torus, and we have an action of $G := H \rtimes S_{n-1}$ on R. The invariant ring of H is generated by monomials, with

$$R^H = \mathbb{K}\left[\prod_{i=1}^{n-1} J_i, \ldots, \left(\prod_{i\in A} J_i\right)\left(\prod_{j\in T\setminus A} y_j\right), \ldots, \prod_{i=1}^{n-1} y_i \,\middle|\, A \subset T,\, \sigma \in S_{n-1}\right].$$

Since H is normal in G, the invariant ring of the G-action on R is:

$$\begin{aligned} R^G &= (R^H)^{S_{n-1}} \\ &= \mathbb{K}\left[\prod_{i=1}^{n-1} J_i, \ldots, \sum_{A\subset T,|A|=k}\left(\prod_{i\in A} y_i\right)\left(\prod_{j\in T\setminus A} J_j\right), \ldots, \prod_{i=1}^{n-1} y_i \,\middle|\, k = 1, \ldots n-2\right]. \end{aligned}$$

Now let $X_0 := \prod_{i=1}^{n-1} J_i$, and for $i = 1, \ldots, n-1$ let

$$X_i := \frac{(n-i-1)!}{(n-1)!} \sum_{A\subset T,\,|A|=i}\left(\prod_{i\in A} y_i\right)\left(\prod_{j\in T\setminus A} J_j\right).$$

Therefore we have

$$R^G = (R^H)^{S_{n-1}} = \mathbb{K}[X_1, \ldots, X_{n-1}].$$

Since D commutes with the action of G, D restricts to a locally nilpotent derivation on R^G. We find that $D(X_0) = 0$, and $D(X_i) = X_{i-1}$ for $i = 1, \ldots, n-1$ and so on R^G, D becomes

$$D = X_0\frac{\partial}{\partial X_1} + \cdots + X_{n-2}\frac{\partial}{\partial X_{n-1}}.$$

Note that D is linear, so $(R^G)^D$ is finitely generated, however for $n \geq 10$, R^D is not finitely generated.

5.2 KURODA'S CONJECTURE

In [25], Kuroda considers generalisations of the locally nilpotent derivation which defines Roberts' counterexample. Namely, for vectors

$$\underline{a} = (a_1, a_2, a_3), \quad \underline{b} = (b_1, b_2, b_3), \quad \underline{c} = (c_1, c_2, c_3), \quad \underline{d} = (d_1, d_2, d_3) \in \mathbb{Z}^3_{\geq 0},$$

Kuroda sets

$$D := x_1^{a_1}x_2^{a_2}x_3^{a_3}\frac{\partial}{\partial y_1} + x_1^{b_1}x_2^{b_2}x_3^{b_3}\frac{\partial}{\partial y_2} + x_1^{c_1}x_2^{c_2}x_3^{c_3}\frac{\partial}{\partial y_3} + x_1^{d_1}x_2^{d_2}x_3^{d_3}\frac{\partial}{\partial z}.$$

We examine the cases of this generalisation where we can remove symmetries from this example to obtain a locally nilpotent derivation similar to Daigle and Freudenburg's example. Note first that the action of S_3 on $B = \mathbb{K}[x_1, x_2, x_3, y_1, y_2, y_3, z]$ defined in equation 5.1 commutes with this derivation. Suppose $\underline{a}, \underline{b}, \underline{c}$ and $\underline{d}$ satisfy the relation

$$k(\underline{a} + \underline{b} + \underline{c}) = l\underline{d} = kl\underline{e}, \tag{5.2}$$

where $k, l \in \mathbb{Z}_{\geq 0}$ and $\underline{e} = (e_1, e_2, e_3) \in \mathbb{Z}^3_{\geq 0}$. Now consider the $\mathbb{G}_m^3$-action on B

$$\begin{aligned}&(\lambda, \mu, \nu) \cdot (x_1, x_2, x_3, y_1, y_2, y_3, z) = \\ &\qquad (\lambda x_1, \mu x_2, \nu x_3, \lambda^{a_1}\mu^{a_2}\nu^{a_3}y_1, \lambda^{b_1}\mu^{b_2}\nu^{b_3}y_2, \lambda^{c_1}\mu^{c_2}\nu^{c_3}y_3, \lambda^{d_1}\mu^{d_2}\nu^{d_3}z).\end{aligned}$$

This action commutes with D and S_3 also acts on $\mathbb{G}_m^3$ via conjugation, similar to Roberts' original counterexample. Let $H \leq \mathbb{G}_m^3$ be the subgroup of $\mathbb{G}_m^3$ defined by $\lambda^{e_1}\mu^{e_2}\nu^{e_3} = 1$. The group $G := H \rtimes S_3$ acts on B also, and the invariant ring of H is given by

$$\begin{aligned}B^H = \mathbb{K}[&x_1^{e_1}x_2^{e_2}x_3^{e_3},\ x_1^{a_1+b_1}x_2^{a_2+b_2}x_3^{a_3+b_3}y_3,\ x_1^{a_1+c_1}x_2^{a_2+c_2}x_3^{a_3+c_3}y_2, \\ &x_1^{b_1+c_1}x_2^{b_2+c_2}x_3^{b_3+c_3}y_1,\ x_1^{a_1}x_2^{a_2}x_3^{a_3}y_2y_3,\ x_1^{b_1}x_2^{b_2}x_3^{b_3}y_1y_3,\ x_1^{c_1}x_2^{c_2}x_3^{c_3}y_1y_2,\ z].\end{aligned}$$

Now let

$$\begin{aligned}x &:= x_1^{e_1}x_2^{e_2}x_3^{e_3}, \\ y &:= \frac{1}{3}\left(x_1^{a_1+b_1}x_2^{a_2+b_2}x_3^{a_3+b_3}y_3 + x_1^{a_1+c_1}x_2^{a_2+c_2}x_3^{a_3+c_3}y_2 + x_1^{b_1+c_1}x_2^{b_2+c_2}x_3^{b_3+c_3}y_1\right), \\ z &:= \frac{1}{6}\left(x_1^{a_1}x_2^{a_2}x_3^{a_3}y_2y_3 + x_1^{b_1}x_2^{b_2}x_3^{b_3}y_1y_3 + x_1^{c_1}x_2^{c_2}x_3^{c_3}y_1y_2\right), \\ u &:= \frac{1}{6}\left(y_1y_2y_3\right), \\ v &:= y_4.\end{aligned}$$

As H is normal in G, the invariant ring of the G-action is

$$B^G = (B^H)^{S_3} = \mathbb{K}[x, y, z, u, v].$$

Since D commutes with the action of G, D restricts to a locally nilpotent derivation on B^G, with

$$D(x) = 0, \quad D(y) = x^l, \quad D(z) = y, \quad D(u) = z, \quad D(v) = x^k.$$

Using our observations made in Section 4.2, we can find conditions on k and l which guarantee that $(B^G)^D$, and hence B^D, is non-finitely generated. Consider the subalgebra $A = \mathbb{K}[x, y, z, u] \subset B^G$ and the locally nilpotent derivation $\Delta := D|_A$. Note that Δ is a triangular monomial derivation, and hence by Theorem 2.1.13 A^Δ is finitely generated, with generators

$$x, \quad 2x^l z - y^2, \quad 3x^{2l}u - 3x^l yz + y^3, \quad 9x^{2l}u^2 - 18x^l yzu + 8x^l z^3 + 6y^3u - 3y^2z^2.$$

Now if $k \geq l$, then $D(v - yx^{k-l}) = 0 \in (B^G)^D$, and $(B^G)^D$ is finitely generated. Thus, in order for $(B^G)^D$ to not be finitely generated, we certainly require that $k < l$. There is a $\mathbb{G}_m$-action on B^G commuting with D given by

$$\alpha \cdot (x, y, z, u, v) = (\alpha x, \alpha^l y, \alpha^l z, \alpha^l u, \alpha^k v),$$

which yields a grading on B^G with

$$\deg(x) = 1, \quad \deg(y) = l, \quad \deg(z) = l, \quad \deg(u) = l, \quad \deg(v) = k.$$

This, together with the ρ-grading, allows us to divide B^G and A into vector spaces $B^G_{(a,b)}$ and $A_{(a,b)}$ respectively, which are spanned by monomials with degree a and ρ-degree b. Recalling $S_{(a,b)}$, from Chapter 4, we note that for $i = 0, \dots, l-1$ there is a bijection

$$\begin{aligned} \phi : S_{(a,b)} &\longrightarrow A_{(la+i,b)}, \\ x^{t_1}y^{t_2}z^{t_3}u^{t_4} &\longmapsto x^{lt_1+i}y^{t_2}z^{t_3}u^{t_4}. \end{aligned}$$

Exactly as in Lemma 4.2.2, using Propositions 4.1.2 and 4.1.3, we can show:

Lemma 5.2.1. *1. $A_{(la+i,b)}$ is in bijection with $A_{(la+i,3a-b)}$ for $i = 0, \dots, l-1$.*

2. If $0 \leq b \leq \frac{3a}{2}$, $\dim(A_{(la+i,b)}) \geq \dim(A_{(la+i,b-1)})$ and $\Delta : A_{(la+i,b)} \longrightarrow A_{(la+i,b-1)}$ is surjective for $i = 0, \dots, l-1$.

3. If $\frac{3a}{2} \leq b \leq 3a$, $\dim(A_{(la+i,b)}) \leq \dim(A_{(la+i,b-1)})$ and $\Delta : A_{(la+i,b)} \longrightarrow A_{(la+i,b-1)}$ is injective for $i = 0, \dots, l-1$.

Now to show that $(B^G)^D$ is not finitely generated, using the non-finiteness criterion, it is sufficient to show the existence of an integral sequence $(f_i)_{i\geq 0} \subset A$. We find conditions for such a sequence to exist with $f_0 = x^{l-k}$, and $D(f_i) = x^k f_{i-1}$. Note that $x^k f_i \in A_{(l+ik,i)}$ and by Lemma 5.2.1 we can guarantee $x^k f_i \in D(A)$ provided

$$\frac{3\left\lfloor \frac{l+ik}{l} \right\rfloor}{2} \geq i,$$

for all $i \geq 0$. Note that

$$\frac{3\left\lfloor\frac{l+ik}{l}\right\rfloor}{2} \geq \frac{3\left(\frac{l+ik-l-1}{l}\right)}{2},$$

giving us

$$\frac{3\left(\frac{ik+1}{l}\right)}{2} \geq i,$$

and rearranging we obtain

$$k \geq \frac{2li-3}{3i}.$$

Now the right hand side tends to $\frac{2l}{3}$ from below as i tends to infinity, so we are guaranteed an integral sequence provided

$$\frac{2l}{3} \leq k < l.$$

Thus we have shown:

Proposition 5.2.2. *Let $S = \mathbb{K}[x, y, z, u, v]$ and let D be the locally nilpotent derivation*

$$D := x^l \frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + z\frac{\partial}{\partial u} + x^k\frac{\partial}{\partial v},$$

then if $\frac{2l}{3} \leq k < l$, S^D is not finitely generated.

Now returning to Kuroda's work in [25], we set $\underline{t}_{\alpha,\beta} = (\alpha_1 - \beta_1, \alpha_2 - \beta_2, \alpha_3, -\beta_3)$, where $\alpha, \beta \in \{\underline{a}, \underline{b}, \underline{c}, \underline{d}\}$. Now let $t^i_{\alpha,\beta}$ be the i'th entry of $\underline{t}_{\alpha,\beta}$ and define

$$\xi_a := \frac{t^1_{a,d}}{\min\{t^1_{a,b}, t^1_{a,c}\}}, \quad \xi_b := \frac{t^2_{b,d}}{\min\{t^2_{b,a}, t^2_{b,c}\}}, \quad \xi_c := \frac{t^3_{c,d}}{\min\{t^3_{c,b}, t^3_{c,a}\}},$$

For example, in Roberts' derivation we have $\xi_a = \frac{n-(n-1)}{\min\{n-0,n-0\}} = \frac{1}{n}$. As Theorem 1.4 in [25], Kuroda shows that if

$$\xi_{\underline{a}} + \xi_{\underline{b}} + \xi_{\underline{c}} \leq 1,$$

and each $t^i_{\alpha,\beta} > 0$ for all i and α, β distinct, then B^D is not finitely generated.

Furthermore, Kuroda also conjectures in [25], that if

$$\xi_{\underline{a}} + \xi_{\underline{b}} + \xi_{\underline{c}} > 1,$$

with each $t^i_{\alpha,\beta} > 0$ for all i and α, β distinct, then B^D is finitely generated.

If this conjecture is true, then it is possible to remove symmetries from both a non-finitely generated invariant ring, as well as a finitely generated one and obtain the same invariant ring. Consider

$$\underline{a} = (7,1,1), \quad \underline{b} = (1,7,1), \quad \underline{c} = (1,1,7), \quad \underline{d} = (5,5,5),$$

note that these vectors satisfy equation 5.2 with $k = 5$ and $l = 9$, and thus when removing symmetries we obtain an invariant ring $(B^G)^D \subset \mathbb{K}[x,y,z,u,v]$ with

$$D := x^9 \frac{\partial}{\partial y} + y \frac{\partial}{\partial z} + z \frac{\partial}{\partial u} + x^5 \frac{\partial}{\partial v},$$

Now $\xi_{\underline{a}} = \xi_{\underline{b}} = \xi_{\underline{c}} = \frac{1}{3}$, so $\xi_{\underline{a}} + \xi_{\underline{b}} + \xi_{\underline{c}} \leq 1$ and hence B^D is not finitely generated. Now suppose instead that

$$\underline{a} = (9,0,0), \quad \underline{b} = (0,9,0), \quad \underline{c} = (0,0,9), \quad \underline{d} = (5,5,5).$$

These vectors also satisfy equation 5.2 with $k = 5$ and $l = 9$, thus when removing symmetries we obtain the same invariant ring as above. Now $\xi_{\underline{a}} = \xi_{\underline{b}} = \xi_{\underline{c}} = \frac{4}{9}$, so $\xi_{\underline{a}} + \xi_{\underline{b}} + \xi_{\underline{c}} = \frac{4}{3} > 1$. Thus, if Kuroda's conjecture holds, then B^D is finitely generated and hence $(B^G)^D$ is finitely generated also. With this we are able to remove symmetries from both a finitely generated and non-finitely generated invariant ring and obtain the same invariant ring.

5.3 FURTHER RESEARCH

Our approach in this chapter is somewhat ad hoc, taking the procedure performed in [19, § 7.2] and applying it to different invariant rings. A natural continuation to this work would be to develop a more systematic approach to this process of "removing symmetries." To speculate on what this would entail, suppose we have a locally nilpotent derivation D on $R = \mathbb{K}[x_1, \ldots, x_n]$, and there is a maximal $\mathbb{G}_m^k$-action on R commuting with D. We propose that the process of "removing symmetries" should be as follows:

1. Find an action of S_l, the symmetric group on l points, on R which commutes with D for some l and acts on $\mathbb{G}_m^k$ by conjugation.
2. Find some $(k-1)$-dimensional subtorus of $\mathbb{G}_m^k$ given by $\prod_{i=1}^k \lambda_i^{a_i} = 1$ for some a_i. This yields an action of $G := H \rtimes S_l$ on R.

3. On the invariant ring R^G, D restricts to a locally nilpotent derivation on R^G and the maximal $\mathbb{G}_m^t$-action commuting with D has $t = 1$.

Essentially, this process would construct a polynomial ring $S \subset R$ on which there is only a $\mathbb{G}_m$-action commuting with D. Work needs to be done to show when this might be possible. It is also worth considering what purpose this process might serve. Firstly, we reduce the number of variables we are working with, making things more tractable. Additionally, working with just one grading makes it simpler to work with S using our approach from Chapter 4, especially if S satisfies (†). We are left with some interesting questions:

Question 5.3.1. *Suppose R^D is not finitely generated, but by "removing symmetries" we obtain a finitely generated S^D, what does this tell us about the structure of R^D?*

Question 5.3.2. *Suppose that we have R_1 and R_2 with locally nilpotent derivations D_1 and D_2 respectively, and that by "removing symmetries" we obtain S, with $D_1|_S = D_2|_S$. What does this tell us about the relation between R_1 and R_2?*

We already have an examples from this chapter which satisfy both questions, assuming Kuroda's conjecture for the second question. Thus, a further examination of these examples would be a good starting point to answer these questions.

— 6 —

Monomial Subalgebras

In Chapter 3 we computed a generating set for Daigle and Freudenburg's counterexample, showed that this generating set was a SAGBI basis and then used this to compute the finite generation ideal. Additionally we found that the finite generation ideal contained all but one element and was the radical of a finitely generated ideal. From this work there are questions which naturally arise about the finite generation ideal, such as:

- Are there methods computing the finite generation ideal which do not rely upon a SAGBI-basis?
- Is the finite generation ideal always the radical of a finitely generated ideal?
- Can we construct a non-finitely generated subalgebra S with $\mathfrak{f}_S$ finitely generated?

In order to approach these questions from the most general perspective, we do not wish to focus just on invariant rings. However, to tackle arbitrary subalgebras of polynomial rings would be a step too far at this stage, so we first focus on subalgebras which have generating sets consisting of monomials. The results we prove on monomials are then easily generalised to SAGBI-bases, though with some limitations.

In this chapter we focus on answering the following questions:

1. Suppose $R \subset \mathbb{K}[x_1, \ldots, x_n]$ is generated by monomials. Under what conditions can we conclude that R is finitely generated?
2. With R as above and non-finitely generated, what is $\mathfrak{f}_R$?
3. Suppose $R \subset \mathbb{K}[x_1, \ldots, x_n]$ be a subalgebra, with SAGBI-basis G. Under what conditions can we conclude that R has a finite SAGBI-basis?

4. With R as above not finitely generated, what is $\mathfrak{f}_R$?
5. Is the finite generation ideal always the radical of a finitely generated ideal?

For convenience of notation in this chapter we assume $0 \in \mathbb{N}$.

6.1 MONOMIAL ALGEBRAS

We recall Definitions 2.1.15, 2.1.17 and 2.1.16 from Chapter 2:

For a commutative $\mathbb{K}$-domain R, a set $G \subset R$ is a *generating set* for R if G generates R as a $\mathbb{K}$-domain. We say R is *finitely generated* if R has a finite generating set and R is *not finitely generated* if no finite generating set exists. A generating set $G \subset R$ *minimal* if for all $g \in G$ we have that $G \setminus \{g\}$ is not a generating set. A subalgebra $R \subset \mathbb{K}[x_1, \ldots, x_n]$ is said to be *generated by monomials* if R has a generating set G with each $g \in G$ a monomial. We also call such an R a *monomial subalgebra*. As previously remarked, a finitely generated monomial subalgebra has a finite monomial generating set.

6.1.1 Two variables

To tackle the questions posed at the start of the chapter, we first consider the problem in a much simplified situation. Let $R \subset \mathbb{K}[x, y]$ be a subalgebra of $\mathbb{K}[x, y]$ with the following properties:

- R is generated by monomials;
- $x \in R$.

We define a function $\lambda : \mathbb{K}[x, y] \longrightarrow \mathbb{Q}$ first on monomials by $\lambda(x^a y^b) := \frac{b}{a+b}$ and then for $f \in \mathbb{K}[x, y]$, we set $\lambda(f) = \max\{\lambda(m) \mid m \text{ is a term of } f\}$. Observe that the function λ has the following properties:

1. $\lambda(cf) = \lambda(f)$ for all $f \in \mathbb{K}[x, y]$, $c \in \mathbb{K} \setminus \{0\}$.
2. For $f, g \in \mathbb{K}[x, y]$ with $\lambda(f) \leq \lambda(g)$, then $\lambda(f) \leq \lambda(fg) \leq \lambda(g)$, with equality if and only if $\lambda(f) = \lambda(g)$.
3. A monomial $m \in \mathbb{K}[x, y]$ can be uniquely recovered given any two of $\lambda(m)$, $\deg_x(m)$ and $\deg_y(m)$. For example, if $\lambda(m) = \alpha$ and $\deg_y(m) = k$ then $\deg_x(m) = \frac{b - b\alpha}{\alpha}$.

Properties 1 and 3 are clear, and a simple calculation shows that λ satisfies property 2 also. For any subalgebra, $S \subset \mathbb{K}[x, y]$, we denote the restriction of λ to S as simply λ.

Definition 6.1.1. If $\lambda : \mathbb{K}[x, y] \longrightarrow \mathbb{Q}$ is a function satisfying the above properties, then we say it is a *ratio function*. For $f \in \mathbb{K}[x, y]$ we call $\lambda(f)$ the *ratio* of f.

For the remainder of this section we shall set $\lambda(x^a y^b) := \frac{b}{a+b}$ as above. For the subalgebra R, we define a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{Q}$ as follows: For each $k \geq 0$, let

$$\lambda_k := \max\{\lambda(x^a y^i) \,|\, x^a y^i \in R, i \leq k\}.$$

Note that this sequence is non-decreasing.

Proposition 6.1.2. *Suppose that $S \subset \mathbb{K}[x, y]$ is a subalgebra, and $(\lambda_k)_{k\in\mathbb{N}}$ is the sequence as above but defined on S, then, if $(\lambda_k)_{k\in\mathbb{N}}$ has an infinite strictly increasing subsequence, S is not finitely generated.*

Proof. Let $(\lambda_{k_i})_{i\in\mathbb{N}}$ be a strictly increasing subsequence of (λ_k). Suppose $G = \{g_1, g_2, \ldots, g_n\} \subset S$ is a generating set for S, and that the g_i have y-degree at most $N > 0$. Now choose $t \in \mathbb{N}$ so that $k_t > N$. By property 2 of λ, the polynomial $f \in S$ which has $\lambda(f) = \lambda_{k_t}$ cannot be expressed as a combination of the g_i, a contradiction. □

We now aim to show:

Lemma 6.1.3. *R is not finitely generated if and only if $(\lambda_k)_{k\in\mathbb{N}}$ has an infinite strictly increasing subsequence $(\lambda_{k_i})_{i\in\mathbb{N}}$.*

Before we show this, we first prove the following useful result:

Proposition 6.1.4. *Let $S \subset \mathbb{K}[x, y]$ be a monomial subalgebra, fix some $\alpha \in \mathbb{Q}$ and set*

$$S_\alpha := \mathbb{K}\Big[\{f \in S \,|\, \lambda(m) = \alpha, \textit{ for all terms } m \textit{ of } f\}\Big].$$

Then S_α is finitely generated.

Proof. Note that since S is a monomial subalgebra, then so is S_α. Suppose that $G \subset S_\alpha$ is a minimal monomial generating set, label the elements of G as g_i, $i \in \mathbb{N}$ and assume that they are ordered by their y-degree, so that g_i has y-degree b_i, and if $i \geq j$, then $b_i \geq b_j$. Now we consider the $b_i \in \mathbb{N}$. Note that for any subset $M \subset \mathbb{N}$ there is a finite subset $M' \subset M$ such that $\gcd(M) = \gcd(M')$. Thus, there is a finite subset G' of G satisfying

$$\gcd\left(\{b_j \mid g_j \in G'\}\right) = \gcd\left(\{b_i, \mid b_i \in G\}\right).$$

Write $G' = \{h_1, \dots, h_n\}$ so that each generator h_i has y-degree c_i. We claim that $\mathbb{K}[G']$ contains all but finitely many of the elements of G and hence, since G is minimal, G itself is finite.

We first aim to find some lower bound $b \in \mathbb{N}$ so that if $g \in G$ has y-degree $e \geq b$, then $g \in \mathbb{K}[G']$. Let d denote the greatest common divisor of the c_i so that there are $t_1, \dots, t_n \in \mathbb{Z}$ satisfying

$$t_1c_1 + \cdots + t_nc_n = d.$$

We can assume, up to relabelling, that $t_1, \dots, t_{n'} \leq 0$ and $t_{n'+1}, \dots, t_n \geq 0$ for some $n' < n$. Then, since $d \mid c_i$ for all i, we can write $c_1 + \cdots + c_{n'} = ld$ so that

$$c_1 + \cdots + c_{n'} = l(t_1c_1 + \cdots + t_nc_n).$$

Rearranging, we obtain

$$(1 - lt_1)c_1 + (1 - lt_2)c_2 + \cdots + (1 - lt_{n'})c_{n'} = lt_{n'+1}c_{n'+1} + \cdots + lt_nc_n. \tag{6.1}$$

We also have $d \mid e$ so we can write $e = e'd$ and $e' = kl + r$ for some $r < l$. Thus, $e = (kl + r)d = k(c_1 + \cdots + c_{n'}) + rd$ and hence, using equation 6.1:

$$\begin{aligned} e &= (kl + r)t_1c_1 + \cdots + (kl + r)t_{n'}c_{n'} + (kl + r)t_{n'+1}c_{n'+1} + \cdots + (kl + r)t_nc_n \\ &= (k + rt_1)c_1 + \cdots + (k + rt_{n'})c_{n'} + rt_{n'+1}c_{n'+1} + \dots rt_nc_n. \end{aligned}$$

Now, $rt_i \geq 0$ for $i \geq n' + 1$ and $k + rt_i \geq k + lt_i$ for $i \leq n'$ where l, t_i are fixed. Hence by choosing e, and hence k, sufficiently large we can guarantee $k + rt_i \geq 0$ for all $i \leq n'$. Now, when this occurs, as the h_i all have the same ratio, and as $h_1^{k+rt_1} \cdots h_{n'}^{k+rt_{n'}} \cdot h_{n'+1}^r \cdots h_n^r$ has y-degree e we conclude

$$g = h_1^{k+rt_1} \cdots h_{n'}^{k+rt_{n'}} \cdot h_{n'+1}^r \cdots h_n^r.$$

Thus we can choose our lower bound $b \in \mathbb{N}$ to be any integer which guarantees $k + rt_i \geq 0$ for all $i \leq n'$. The remaining generators in G must have their y-degree bounded, and since for a given y-degree there is only one monomial with ratio α, there must only be finitely many generators in $G \setminus G'$. $\square$

We can now prove Lemma 6.1.3:

Proof of Lemma 6.1.3. Proposition 6.1.2 shows that if $(\lambda_k)_{k\in\mathbb{N}}$ has a strictly increasing subsequence, then R is not finitely generated. We will now prove the converse by showing that if there is some $N \in \mathbb{N}$ for which $\lambda_k = \lambda_N = \alpha$ for all $k \geq N$, then R is finitely generated.

Let $H \subset R$ be a minimal monomial generating set for R. By Proposition 6.1.4, there are only finitely many $g \in H$ with $\lambda(g) = \alpha$, it remains to show that there are only finitely many $g \in H$ with $\lambda(g) < \alpha$. Suppose $G = \{g_1, \ldots, g_n\} \subset H$ generates all elements in R with ratio α. Suppose the g_i are ordered by y-degree with $g_i = x^{a_i}y^{b_i}$. Observe that, since $x \in R$, a minimal generating set for R will not contain any two generators with the same y-degree. Suppose that the greatest common divisor of the b_i is d and the greatest common divisor of the a_i is δ. As shown in the proof of Proposition 6.1.4, there is some constant constant $N \in \mathbb{N}$, such that for all $kd > N$, $(x^\delta y^d)^k$ can be expressed as a combination of the $g_i \in G$. There are clearly finitely many elements in H with y-degree at most N. For any generator $f = x^e y^c \in H$ with $\lambda(f) < \alpha$ and $\deg_y(f) > N$ we must have $ld < c < (l+1)d$ for some $ld \geq N$ Write $c = ld + r$ and $e = l\delta + s$; given any $g = x^{(l+t)\delta}y^{(l+t)d}$, $fg = x^{(2l+t)\delta+s}y^{(2l+t)d+r}$. Therefore f together with G generates all elements in R which have y-degree at least $2l$ and congruent to $r \mod (d)$.

We can repeat this argument for each $r = 1, \ldots d-1$ to obtain a finite generating set for all elements of R with y-degree above a finite bound and conclude that H must be finite. $\square$

Lemma 6.1.5. *Suppose that $R \subset \mathbb{K}[x,y]$ satisfies the properties described above, if R is not finitely generated then $\mathfrak{f}_R = \{f \in R \mid R_f \text{ is f.g.}\} = \sqrt{xR}$, and this is a maximal ideal of R.*

Proof. Identifying $\mathbb{K}[y] \subset \mathbb{K}[x,y] \subset \mathbb{K}[x,y]_x$, note that $R_x \cap \mathbb{K}[y]$ is finitely generated, being a subalgebra of $\mathbb{K}[y]$. Since $\mathfrak{f}_R$ is a radical ideal, it is sufficient to show that $\sqrt{xR}$ is a maximal ideal of R. To do so, we show that all generators of R lie in this radical. Now consider $g = x^a y^b \in R$, where $b \neq 0$, note that $\lambda(g) = \frac{b}{a+b}$. Since R is not finitely generated by Lemma 6.1.3, the sequence (λ_k) has a strictly increasing subsequence. In particular there is some $h = x^c y^d \in R$, $d \neq 0$ which has $\lambda(h) > \lambda(g)$. Now if we consider h^b and g^d, we see that both have the same y-degree, but since

$\lambda(h^b) = \lambda(h) > \lambda(g) = \lambda(g^d)$, we can write $g^d = x^i h^a$ for some $i > 0$. In other words, $g^d \in xR$ and hence $g \in \sqrt{xR}$, proving the result. □

Example 6.1.6. The subalgebra $R := \mathbb{K}[x, xy, xy^2, \dots,]$ discussed in Example 2.4.6 is non-finitely generated by Lemma 6.1.3. We have $\lambda(xy^n) = n/(n+1)$, which yields a strictly increasing sequence. We have also shown in Example 2.4.6 that $\mathfrak{f}_R = \sqrt{xR}$ and this is the maximal graded ideal of S

Example 6.1.7. Consider $R := \mathbb{K}[x, x^3y, x^5y^2, x^7y^3, \dots, x^{2n+1}y^n, \dots]$. We have $\lambda(x^{2n+1}y^n) = \frac{n}{3n+1}$ which tends to $\frac{1}{3}$ from below as n tends to infinity. Hence R is non-finitely generated. Recalling the non-finiteness criterion 4.2.3, if we let $S = \mathbb{K}[x]$ so that $R = S[y]$. By writing $\beta_n = x^{2n+1}y^n$, $\deg_x(\beta_n)$ is unbounded whilst R is non-finitely generated. This example does not arise as an invariant ring under a $\mathbb{G}_a$-action and so does not fit the conditions of the non-finiteness criterion. It does however demonstrate how there could exist non-finitely generatied invariant rings which do not satisfy the non-finiteness criterion.

We can relax some of the conditions on R in order to obtain similar results. Now suppose that $R \subset \mathbb{K}[x, y]$ has the following properties:

1. R is generated by monomials,
2. $x^r \in R$ for some $r \geq 0$.

Lemma 6.1.8. *With the conditions above, R is finitely generated if and only if $(\lambda_k)_{k\in\mathbb{N}}$ is eventually constant.*

Proof. Again, if a strictly increasing subsequence exists, then R is not finitely generated by Proposition 6.1.2.

Suppose that the sequence is eventually constant with maximal ratio α, and that $H \subset R$ is a minimal monomial generating set. Note that for a given y-degree there can be at most r generators with that y-degree. Let $G = \{g_1, \dots, g_n\} \subset H$ be a minimal monomial generating set for all elements in R of ratio α, which is finite by Proposition 6.1.4. Order the elements of G by y-degree with $g_i = x^{a_i}y^{b_i}$. Suppose that the greatest common divisor of the b_i is d and the greatest common divisor of the a_i is δ. Choose $N \in \mathbb{N}$ suitably large so that for all $kd > N$, there is an element of R with ratio α, $k \geq 0$. Similar to the proof of Lemma 6.1.3, we will show that, in addition to G and x^r, at most r monomials of the form $x^e y^{kd+s}$ for each $0 \leq s < d$ are required to generate all but finitely many elements of R.

Similar to Lemma 6.1.3, for a generator $f = x^e y^c \in H$ with $c \geq N$ there will be two elements $p = x^{j\delta}y^{jd}$ and $q = x^{(j+1)\delta}y^{(j+1)d}$, generated by G with $jd \leq c < (j+1)d$. Write $c = jd + s$, $0 \leq s < d$. Note that together with x^r, f generates all elements with y-degree c and x-degree $e + lr$, $l \geq 0$. As f is a generator, and H is minimal, we have $x^{e-lr}y^c \notin R$ whenever $e - lr \geq 0$. Now, for

$$g = x^{(k+h)\delta}y^{(k+h)d},\ k \geq N/d,\ h \in \mathbb{N},$$

we have

$$fg = x^{(k+h)\delta+e}y^{(k+j+h)d+s}.$$

So $f \cup G \cup x^r$ generates elements t with $\deg_y(t) = kd + s$, where $k \geq (N + j)$ and $0 \leq s < d$. By repeating this process, we observe that at most r generators with y-degree congruent to s modulo d are required to generate all elements with y-degree congruent to s modulo d and sufficiently large. Repeating this process then for each s modulo d, it is clear that we need at most rd generators to generate all elements of R with sufficiently large y-degree. Since the remaining elements in R will have bounded y-degree, we would require only finitely many more generators to generate all of R. □

Lemma 6.1.9. *With R as above, let $\mathbb{K}[x^{a_1}, \ldots, x^{a_n}] = R \cap \mathbb{K}[x]$, and let $G \subset R$ be a monomial generating set for R. Then, if R is not finitely generated $\mathfrak{f}_R = \sqrt{(x^{a_1}, \ldots, x^{a_n})R}$ and $G \subset \mathfrak{f}_R$.*

Proof. First we show that each $x^{a_i} \in \mathfrak{f}_R$. Consider $R_{x^{a_i}}$, we note that if $R_{x^{a_i}} \cap \mathbb{K}[x, y]$ is finitely generated then so is $R_{x^{a_i}}$. Let $g \in G$ with $\deg_y(g) \geq 1$, then $g = x^\alpha y^\beta$ and

$$\frac{g^{a_i}}{(x^{a_i})^\alpha} = \frac{x^{a_i\alpha}y^{a_i\beta}}{x^{a_i\alpha}} = y^{a_i\beta} \in R \cap \mathbb{K}[x, y],$$

and hence on $R_{x^{a_i}} \cap \mathbb{K}[x, y]$, $\lambda_{a_i\beta} = 1$, meaning the sequence $(\lambda_k)_{k\in\mathbb{N}}$ on $R_{x^{a_i}} \cap \mathbb{K}[x, y]$ is eventually constant, and so $R_{x^{a_i}} \cap \mathbb{K}[x, y]$ is finitely generated by Lemma 6.1.8. Thus $\sqrt{(x^{a_1}, \ldots, x^{a_n})} \subset \mathfrak{f}_R$ and if we show $g \in \sqrt{(x^{a_1}, \ldots, x^{a_n})}$ for all $g \in G$ then we obtain the desired result. Clearly, for $g \in G$, if $\deg_y(g) = 0$, then $g \in \sqrt{(x^{a_1}, \ldots, x^{a_n})}$, so suppose $g = x^\alpha y^\beta$ with $\beta \geq 1$. Now $\lambda(g) \leq \lambda_\beta$, and since $(\lambda_k)_{k\in\mathbb{N}}$ has a strictly increasing subsequence there is some $h = x^\gamma y^\delta$ with $\lambda(h) = \lambda_\delta > \lambda_\beta \geq \lambda(g)$. Now

$$g^{\delta a_1} = x^{a_1(\alpha\delta-\gamma\beta)}h^{a_1\beta} \in (x^{a_1}, \ldots, x^{a_n})R,$$

where $(\alpha\delta - \gamma\beta) > 0$ since $\lambda(g^\delta) < \lambda(h^\beta)$. □

With this in hand, we can in fact relax the statement of Lemma 6.1.3 further. Suppose now that $R \subset \mathbb{K}[x, y]$ is simply just generated by monomials. We define a ratio function $\mu : \mathbb{K}[x, y] \longrightarrow \mathbb{Q}$ as $\mu(x^a y^b) = \frac{a}{a+b}$ and extend to polynomials in the same way as for λ. We also define the sequence $(\mu_k)_{k\in\mathbb{N}}$ with $\mu_k := \max\{\mu(x^i y^b) \mid x^i y^b \in R, i \leq k\}$. Note that the results of Propositions 6.1.2 and 6.1.4 hold when replacing λ with μ by swapping x and y.

If there is no monomial $x^a y^b \in R$ with $a \leq k$, then we set $\mu_k = 0$, and similarly we set $\lambda_k = 0$ if there is no monomial with $b \leq k$. Additionally note that for monomials m we have $\lambda(m) = 1 - \mu(m)$, hence $\lambda(m) \geq \lambda(m')$ if and only if $\mu(m) \leq \mu(m')$, with $\lambda(m) = \lambda(m')$ if and only if $\mu(m) = \mu(m')$.

Theorem 6.1.10. *R is finitely generated if and only if both sequences $(\lambda_k)_{k\in\mathbb{N}}$, $(\mu_k)_{k\in\mathbb{N}}$ are eventually constant.*

Proof. Suppose, without loss of generality, that $(\lambda_k)_{k\in\mathbb{N}}$ has a strictly increasing subsequence. By Proposition 6.1.2, R is not finitely generated.

Now suppose that for some $N \in \mathbb{N}$ we have $\lambda_k = \lambda_N$ and $\mu_k = \mu_N$ for all $k \geq N$. Let $H \subset R$ be a minimal monomial generating set. Using Proposition 6.1.4, let

$$G_1 = \{g_1, \ldots, g_n\} \subset H, \quad G_2 = \{h_1, \ldots, h_m\} \subset H,$$

where G_1 generates all $f \in R$ with $\lambda(f) = \lambda_N$, and G_2 generates all $f \in R$ with $\mu(f) = \mu_N$. Let $g_i = x^{a_i} y^{b_i}$ and $h_i = x^{c_i} y^{d_i}$, set

$$\rho := \gcd(a_i), \quad r := \gcd(b_i),$$
$$\epsilon := \gcd(c_i), \quad \upsilon := \gcd(d_i).$$

Let $g^* = x^\rho y^r$, $h^* = x^\epsilon y^\upsilon$. Choose $M \in \mathbb{N}$ sufficiently large so that for all $kr \geq M$ and $l\epsilon \geq M$, we have $(g^*)^k, (h^*)^l \in R$.

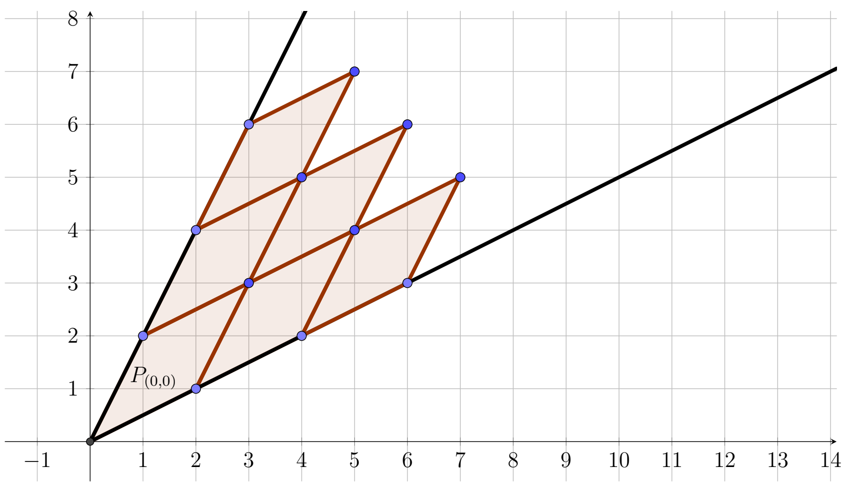
We can associate any monomial $x^\alpha y^\beta$ to the point $(\alpha, \beta) \in \mathbb{N}^2$. The monomials with ratios λ_N or μ_N can then be understood to be the points in $\mathbb{N}^2$ lying on the lines in $\mathbb{Q}^2$ with equations $y = \frac{r}{\rho}x$, $y = \frac{\epsilon}{\upsilon}$ respectively. The points in $\mathbb{N}^2$ corresponding to the monomials in R are therefore contained in a cone cut out by these ratios. Multiplying two monomials corresponds to addition of vectors in $\mathbb{N}^2$, thus the points corresponding to the monomials generated by $G_1 \cup G_2$ lie in the semigroup generated by the points $\{(a_1, b_1), \ldots (a_n, b_n), (c_1, d_1), \ldots, (c_m, d_m)\} \subset \mathbb{N}^2$.

Note that $G_1 \cup G_2$ generates three families of points:

- $(g^*)^k$, $k \geq M/d$, corresponding to $\{(k\rho, kr) \in \,|\, k \geq M/r\}$.
- $(h^*)^l$, $l \geq M/\epsilon$, corresponding to $\{(l\epsilon, lv) \,|\, l \geq M/\epsilon\}$.
- $(g^*)^k(h^*)^l$, $k \geq M/r$, $l \geq M/\epsilon$, corresponding to $\{(k\rho + l\epsilon, kr + lv) \,|\, k \geq M/r,\, l \geq M/\epsilon\}$.

The third family of points gives rise to a family of parallelograms $P_{(k,l)}$, $k \geq M/r, l \geq M/\epsilon$ with vertices

$$(k\rho + l\epsilon, kr + lv), ((k+1)\rho + l\epsilon, (k+1)r + lv),$$
$$(k\rho + (l+1)\epsilon, kr + (l+1)v), ((k+1)\rho + (l+1)\epsilon, (k+1)r + (l+1)v).$$



Note that within each parallelogram there are finitely many integer points, and this number is the same for each parallelogram; call it A.

Claim. *There is some $L \in \mathbb{N}$ and some finite set $G' \subset H$ such that $G_1 \cup G_2 \cup G'$ generates all monomials in R lying in all parallelograms $P_{(k,l)}$ with $k, l \geq L$*

To prove the claim, first consider some $m = x^e y^f \in R$ with $e, \geq M/r, f \geq M/\epsilon$, so m is one of the A points lying within some parallelogram $P_{(a,b)}$ say. If all such m are generated by $G_1 \cup G_2$, we are done, otherwise there are at most finitely many generators $p_1, \ldots p_s \in H$ such that $G_3 := \{p_1, \ldots, p_s\} \cup G_1 \cup G_2$ generates m. Now by construction, since m lies within $P_{(a,b)}$, $(g^*)^k(h^*)^l m$ lies within $P_{(a+k,b+l)}$ for $k \geq M/r$, $l \geq M/\epsilon$. Thus, for $q \in R$ a monomial corresponding to a point lying within some parallelogram $P_{(a,b)}$ for a, b sufficiently large, there are at most $A - 1$ possibilities for this point corresponding to q to not be generated by $G_1 \cup G_2 \cup G_3$.

Therefore, continuing this process and setting $G' = G_3 \cup \cdots \cup G_{A+2}$ we obtain the desired result for some suitably large $L \in \mathbb{N}$, thus proving the claim.

Now, given the claim, the point $(L\rho + L\epsilon, Lr + Lv) \in \mathbb{N}^2$ divides the cone into 4 parts by the lines

$$y = \frac{r}{\rho}x + L\left(v - \frac{r\rho}{\epsilon}\right), \quad y = \frac{v}{\epsilon}x + L\left(r - \frac{v\rho}{\epsilon}\right),$$

and $G_1 \cup G_2 \cup G'$ generates all monomials corresponding to points (a, b) with

$$b \geq \frac{v}{\epsilon}a + L\left(r - \frac{v\rho}{\epsilon}\right), \quad a \geq \frac{\rho}{r}b + L\left(\rho - \frac{v}{\epsilon}r\right).$$

There are are only finitely many possible points in the region bounded by

$$\frac{v}{\epsilon}a \leq b \leq \frac{v}{\epsilon}a + L\left(r - \frac{v\rho}{\epsilon}\right), \quad \frac{\rho}{r}b \leq a \leq \frac{\rho}{r}b + L\left(\rho - \frac{v\epsilon}{r}\right),$$

and hence only finitely many more monomials are required to generate all monomials corresponding to the points in this region. What remains are monomials corresponding to points satisfying either

$$b \geq \frac{v}{\epsilon}a + L\left(r - \frac{v\rho}{\epsilon}\right), \quad \frac{\rho}{r}b \leq a \leq \frac{\rho}{r}b + L\left(\rho - \frac{v\epsilon}{r}\right), \tag{6.2}$$

or

$$a \geq \frac{\rho}{r}b + L\left(\rho - \frac{v\epsilon}{r}\right), \quad \frac{v}{\epsilon}a \leq b \leq \frac{v}{\epsilon}a + L\left(r - \frac{v\rho}{\epsilon}\right). \tag{6.3}$$

Note that for a monomial corresponding to a point in case 6.2, for a given y-degree, there are at most $L\left(\rho - \frac{v\epsilon}{r}\right)$ monomials in R in this region. Similarly in case 6.3, for a given x-degree there are at most $L\left(r - \frac{v\rho}{\epsilon}\right)$ monomials in R in this region. To show that only finitely many elements of H are needed to generate all monomials in these regions we can proceed similarly to Lemma 6.1.8. Given some monomial $m = x^e y^f \in R$ in the region 6.2, write $f = lr + s$, $0 \leq s < r$, and $e = l\rho + \frac{\rho}{r}s + t$, $0 < t < L\left(\rho - \frac{v\epsilon}{r}\right)$. Then $(g^*)^k m$, for $k \geq M$ generates monomials corresponding to points (u, v) with $u = (l + k)\rho + \frac{\rho}{r}s + t$, and $v \equiv s \mod r$ for v sufficiently large. This can be done for all possible t and s, and since only finitely many elements of H are required to generate each m, we see that finitely many monomials are needed to generate every monomial in this region. The proof for region 6.3 follows exactly the same argument with the x and y-degrees swapped.

□

With this in hand we can now show:

Theorem 6.1.11. *Suppose $R \subset \mathbb{K}[x,y]$ is a non-finitely generated monomial subalgebra, and let $G \subset R$ be a monomial generating set. Then $G \subset \mathfrak{f}_R$ and $\mathfrak{f}_R$ is the radical of a finitely generated ideal.*

Proof. Note that if $R \subset \mathbb{K}[x,y]$ is a non-finitely generated monomial subalgebra then we have either:

1. Precisely one of the sequences $(\lambda_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}}$ has an infinite strictly increasing subsequence and the other is eventually constant.
2. Both sequences have an infinite strictly increasing subsequence

In case 1 suppose without loss of generality that $(\mu_n)_{k\in\mathbb{N}}$ is eventually constant with maximal ratio κ, and let $G = \{(x^\delta y^d)^{a_1}, \dots, (x^\delta y^d)^{a_m}\}$ be a generating set for all elements which have this maximal ratio κ. We will show that $\mathfrak{f}_R = \sqrt{GR} = \sqrt{(x^\delta y^d)^{a_i} R}$ and that $\mathfrak{f}_R$ is a maximal ideal of R.

Given an element $x^a y^b \in R$, since $(\lambda_n)_{k\in\mathbb{N}}$ has an infinite strictly increasing subsequence there is some element $x^\alpha y^\beta$ with $\lambda(x^a y^b) < \lambda(x^\alpha y^\beta)$, thus

$$y^{\beta a - \alpha b} = \frac{(x^\alpha y^\beta)^a}{(x^a y^b)^\alpha} \in R_{x^a y^b} \cap \mathbb{K}[x,y].$$

Now, if $\mu(x^a y^b) < \kappa$, then

$$x^{b a_1 \delta - a a_1 d} = \frac{(x^\delta y^d)^{a_1 b}}{(x^a y^b)^{a_1 d}} \in R_{x^a y^b} \cap \mathbb{K}[x,y].$$

Alternatively, we have $\mu(x^a y^b) = \mu(x^\delta y^d)$ and this remains the maximal ratio for κ on $R_{x^a y^b} \cap \mathbb{K}[x,y]$. In either case, $R_{x^a y^b} \cap \mathbb{K}[x,y]$ has both $(\lambda_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}}$ eventually constant, meaning $R_{x^a y^b}$ is finitely generated. Thus, $\mathfrak{f}_R$ is a maximal ideal.

Furthermore, if $x^a y^b \in R$ has $\mu(x^a y^b) = \kappa$, then $x^a y^b = (x^\delta y^d)^k$ for some k and hence $(x^a y^b)^{a_1} = (x^\delta y^d)^{a_1 k} \in (x^\delta y^d)^{a_1} R$, otherwise $\mu(x^a y^b) < \kappa$. Let $x^\alpha y^\beta \in R$ satisfy $\lambda(x^\alpha y^\beta) > \lambda(x^a y^b)$, then we have:

$$(x^a y^b)^\beta = x^{a\beta - b\alpha}(x^\alpha y^\beta)^b, \quad (x^a y^b)^{a_i \delta} = y^{b a_i \delta - a a_i d}(x^\delta y^d)^{a_i a},$$

where both $a\beta - b\alpha, b a_i \delta - a a_i d > 0$. Thus

$$(x^a y^b)^{\beta\alpha(b a_i \delta - a a_i d) + a_i \delta\beta(a\beta - b\alpha)} = (x^\alpha y^\beta)^{\alpha\beta(b a_i \delta - a a_i d)}(x^\delta y^d)^{a_i(a\beta - b\alpha)} \in (x^\delta y^d)^{a_i} R,$$

showing that $\mathfrak{f}_R = \sqrt{(x^\delta y^d)^{a_i} R}$.

In case 2 let $G \subset R$ be a generating set, suppose that $\lambda_k \neq 0$ but $\lambda_{k-1} = 0$ and $\mu_l \neq 0$, but $\mu_{l-1} = 0$. Let $x^p y^k, x^l y^q \in R$ be the corresponding elements with $\lambda(x^p y^k) = \lambda_k, \mu(x^l y^q) = \mu_l$. Note that we must have $q \geq k$ and $p \geq l$ and hence $\frac{p}{p+k} = \mu(x^p y^k) \geq \frac{l}{l+q} = \mu(x^l y^q)$. Let $x^a y^b \in R$ then, since both $(\lambda_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}}$ have infinite strictly increasing subsequences, there are $x^c y^d, x^e y^f \in R$ with $\lambda(x^c y^d) > \lambda(x^a y^b)$ and $\mu(x^e y^f) > \mu(x^a y^b)$. Hence we must have

$$y^{ad-bc} = \frac{(x^c y^d)^a}{(x^a y^b)^c}, \quad x^{eb-fa} = \frac{(x^e y^f)^b}{(x^a y^b)^f} \in R_{x^a y^b} \cap \mathbb{K}[x, y],$$

therefore $R_{x^a y^b} \cap \mathbb{K}[x, y]$, has both $(\lambda_k)_{k\in\mathbb{N}}, (\mu_k)_{k\in\mathbb{N}}$ eventually constant, so $R_{x^a y^b}$ is finitely generated and we must have $G \subset \mathfrak{f}_R$.

We now show that $\mathfrak{f}_R = \sqrt{(x^p y^k, x^l y^q)R}$. Note that if $\lambda(x^a y^b) = \lambda(x^p y^k)$ then we must have $x^a y^b \in \sqrt{x^p y^k R}$, and similarly if $\mu(x^a y^b) = \mu(x^l y^q)$ we must have $x^a y^b \in \sqrt{x^l y^q R}$. In either case, $x^a y^b \in \sqrt{(x^p y^k, x^l y^q)R}$. Since $\mu(x^p y^k) \geq \mu(x^l y^q)$, there are then three remaining cases for $x^a y^b$:

1. $\lambda(x^a y^b) > \lambda(x^p y^k)$ and $\mu(x^a y^b) > \mu(x^l y^q)$,
2. $\lambda(x^a y^b) < \lambda(x^p y^k)$ and $\mu(x^a y^b) > \mu(x^l y^q)$,
3. $\lambda(x^a y^b) > \lambda(x^p y^k)$ and $\mu(x^a y^b) < \mu(x^l y^q)$,

In case 1 we have $\lambda(x^a y^b) < \lambda(x^p y^k)$ and $\mu(x^a y^b) < \mu(x^l y^q)$, since then

$$(x^a y^b)^p = y^{pb-ka}(x^p y^k)^a, \quad (x^a y^b)^q = x^{aq-lb}(x^l y^q)^b,$$

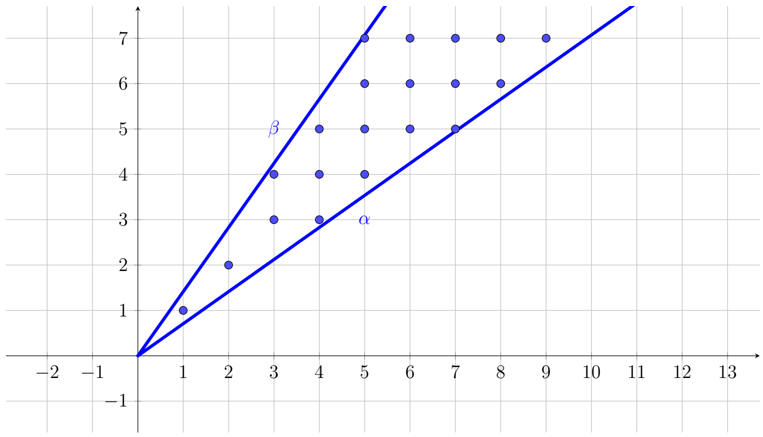
where both $pb - ka, aq - lb > 0$. Using this, we find

$$(x^a y^b)^{pk(aq-lb)+qp(pb-ka)} = (x^l y^q)^{ak(aq-lb)}(x^p y^k)^{pb(aq-lb)} \in (x^p y^k, x^l y^q)R.$$

Now in case 2, since $(\mu_n)_{n\in\mathbb{N}}$ has an infinite strictly increasing subsequence, there is some $x^c y^d \in R$ with $\mu(x^c y^d) > \mu(x^a y^b)$. Repeating the above calculation for case 1 replacing $x^p y^k$ with $x^c y^d$ we can show $x^a y^b \in \sqrt{x^l y^q R}$ and similarly for case 3 we find that $x^a y^b \in \sqrt{x^p y^k R}$. Thus, having dealt with all cases, we conclude that $\mathfrak{f}_R = \sqrt{(x^p y^k, x^l y^q)R}$, the radical of a finitely generated ideal, completing the proof. □

Example 6.1.12. Let $\alpha, \beta \in \mathbb{R}$ and let $R := \mathbb{K}\left[\left\{x^a y^b \,|\, \alpha < \frac{a}{b} < \beta\right\}\right]$. That is, R is generated by all monomials corresponding to points in $\mathbb{N}^2$ bounded by the lines

$y = \beta x$ and $y = \alpha x$ as shown in the following figure:



For $x^a y^b \in R$, we must have

$$\frac{\alpha}{1+\alpha} < \lambda(x^a y^b) = \frac{b}{a+b} < \frac{\beta}{1+\beta},$$
$$\frac{1}{1+\beta} < \mu(x^a y^b) = \frac{a}{a+b} < \frac{1}{1+\alpha}.$$

If we consider $(\lambda_k)_{k\in\mathbb{N}}$, this defines a sequence which converges to $\frac{\beta}{1+\beta}$ from below while $(\mu_k)_{k\in\mathbb{N}}$ converges to $\frac{1}{1+\alpha}$ from below also. Thus both $(\lambda_k)_{k\in\mathbb{N}}$ and $(\mu_k)_{k\in\mathbb{N}}$ have infinite strictly increasing sequences and R is non finitely generated.

If we let $R := \mathbb{K}\left[\left\{x^a y^b \,|\, \alpha < \frac{a}{b} \le \beta\right\}\right]$ and require $\beta = \frac{p}{q} \in \mathbb{Q}$, then $x^p y^q \in R$ has maximal $\lambda(x^p y^q) = \frac{q}{p+q} = \frac{\beta}{1+\beta}$ but $\frac{1}{1+\alpha}$ is not attained, hence R is not finitely generated.

Finally, if $R := \mathbb{K}\left[\left\{x^a y^b \,|\, \alpha \le \frac{a}{b} \le \beta\right\}\right]$ with $\alpha, \beta \in \mathbb{Q}$, then both maximal λ and μ are attained and R is finitely generated.

Definition 6.1.13. Let " $<$ " be a monomial ordering on $\mathbb{K}[x,y]$, we say that a function $\lambda^< : \mathbb{K}[x,y] \to \mathbb{Q}$ is an *ordered ratio function* if:

1. For $f \in \mathbb{K}[x,y]$ $\lambda^<(f) = \lambda^<(m)$ where m is the leading monomial of f.
2. For all $f \in \mathbb{K}[x,y]$, $i \ge 1$, $\lambda^<(f) = \lambda^<(f^i)$.
3. For $f, g \in \mathbb{K}[x,y]$ with $\lambda^<(f) \le \lambda^<(g)$, then $\lambda^<(f) \le \lambda^<(fg) \le \lambda^<(g)$.
4. A monomial $m \in \mathbb{K}$ can be uniquely recovered given any two of $\lambda^<(m)$, $\deg_x(m)$ and $\deg_y(m)$.

Lemma 6.1.14. *Let " $<$ " be a monomial ordering, $\lambda^<(x^a y^b) := \frac{b}{a+b}$ and $\mu(x^a y^b) := \frac{a}{a+b}$ ordered ratio functions and $R \subset \mathbb{K}[x,y]$ a subalgebra. Let $(\lambda_k^<)_{k\in\mathbb{N}}, (\mu_k^<)_{k\in\mathbb{N}}$ be* *sequences defined by* $\lambda_k^< := \max\{\lambda^<(f) | \deg_y(f) \leq k\}, \mu_k^< := \max\{\mu^<(f) | \deg_x(f) \leq k\}$. *Then R has a finite SAGBI-basis if and only if both sequences are eventually constant.*

Proof. Consider $L_{alg}(R)$, the algebra generated by the leading monomials of R. The ratio functions λ, μ are well-defined on this algebra, and since for $f \in R$, $\lambda^<(f) = \lambda^<(m)$ and $\mu^<(f) = \mu^<(m)$ where $m \in L_{alg}(R)$, the sequences $(\lambda_k^<)_{k\in\mathbb{N}}, (\mu_k^<)_{k\in\mathbb{N}}$ are the same on $L_{alg}(R)$. Thus by Theorem 6.1.10, $L_{alg}(R)$ is finitely generated if and only if these sequences are eventually constant. Additionally, given a generating set for $L_{alg}(R)$, the corresponding elements in R with these leading terms form a SAGBI-basis for R, which is finite if and only if the generating set for $L_{alg}(R)$ is finite, hence the result. □

6.1.2 Three or more variables

Throughout this subsection, we suppose that $R \subset \mathbb{K}[x_1, \ldots, x_n]$, $n \geq 3$ is a monomial subalgebra, unless stated otherwise. In order to obtain conditions on R which classify when it is finitely generated we provide a geometrical interpretation of our results obtained thus far. As noted previously, monomials $x_1^{a_1} \cdots x_n^{a_n} \in \mathbb{K}[x_1, \ldots, x_n]$ can be associated to vectors $(a_1, \ldots, a_n) \in \mathbb{N}^n \subset \mathbb{Q}^n$. Multiplication of two monomials corresponds to addition of their corresponding exponent vectors. Thus, the set of vectors corresponding to all monomials in a subalgebra of $\mathbb{K}[x_1, \ldots, x_n]$ have the structure of a semigroup S in $\mathbb{N}^n$ under addition.

Definition 6.1.15. A semigroup S is *affine* if it is finitely generated and isomorphic to a subsemigroup of $\mathbb{Z}^d$ for some $d \geq 0$. By finitely generated, we mean that there are $s_1, \ldots, s_m \in S$ such that for all $s \in S$ there are $a_i \in \mathbb{N}$ such that:

$$s = a_1 s_1 + a_2 s_2 + \ldots a_m s_m.$$

R is then finitely generated as a subalgebra if and only if the corresponding semigroup S is affine. Ratio functions also have a geometric interpretation. In two variables, the set of monomials which have a fixed ratio $\lambda \in \mathbb{Q}$ lie on lines in $\mathbb{Q}^2$. These functions can also be considered as functions from $\mathbb{N}^2 \setminus \{(0,0)\}$ to $\mathbb{Q}$. The existence of a maximal ratio corresponds to a line which bounds all points in the semigroup. For example, given the ratio function $\lambda(x^a y^b) = \lambda((a,b)) := b/(a+b)$, if λ_N is the maximal ratio, then for all $s = (s_1, s_2) \in S$ we must have $s_2/(s_1+s_2) \leq \lambda_N$, or equivalently $\lambda_N s_1 + (\lambda_N - 1)s_2 \geq 0$. When S is bounded by two lines it is contained within a *polyhedral cone*:

Definition 6.1.16. [2, p. 10] A *polyhedral cone* $C \subset \mathbb{Q}^n$, is the intersection of a finite number of half-spaces which have 0 on their boundary. Equivalently, a polyhedral cone is the non-negative span of a finite number of *vectors* $v_1, \ldots, v_m \in C$, meaning for all $v \in C$

$$v = \sum_{i=1}^{m} \lambda_i v_i, \quad \lambda_i \in \mathbb{Q},\ \lambda_i \geq 0.$$

A vector $v \in C$ is *extremal* if whenever we have $v_i = u_i + w_i$, with $u_i, w_i \in C$, we must have $u_i = \alpha_i v_i, w_i = \beta_i v_i,\ \alpha_i, \beta_i \in \mathbb{Q}$.

Theorem 6.1.17. [2, p. 11] *A polyhedral cone C is the non-negative span of a finite number of extremal vectors.*

Definition 6.1.18. Given a polyhedral cone C, a *set of extremal vectors* for C is a collection of vectors $\{v_1, \ldots, v_m\} \subset C$ whose non-negative linear span is all of C. Given an extremal vector v, the corresponding ray $R_v := \{\lambda v \in \mathbb{Q}^n \mid \lambda \in \mathbb{Q}\}$ is called an *extremal ray.*

Definition 6.1.19. A *polytope* $P \in \mathbb{Q}^n$ is the convex hull of a finite set of vectors $v_1, \ldots, v_m \in \mathbb{Q}^n$, that is

$$P := \left\{ \sum_{i=1}^{m} a_i v_i, \mid \sum_{i=1}^{m} a_i = 1, 0 \leq a_i \leq 1 \text{ for all } i \right\}.$$

A vector $v \in P$ is a *vertex* of P if it cannot be expressed as a convex combination of vectors in P. That is, if there are $w_i \in P$ such that

$$v = \sum_{i=1}^{k} a_i w_i,$$

where $\sum_{i=1}^{k} a_i = 1$ and $0 \leq a_i \leq 1$ for all i, then $k = 1$ and $w_1 = v$. Clearly the vertices of P are some subset of $\{v_1, \ldots, v_m\}$.

The result of Theorem 6.1.10 can therefore be restated for S in the following way:

Corollary 6.1.20. *Let $S \subset \mathbb{N}^2$ be a semigroup. Then, S is affine if and only if S is contained in a polyhedral cone, with half spaces defined by maximal ratios $\lambda_N, \mu_N \in \mathbb{Q}$, and there are extremal vectors $s, t \in S$ with $\lambda(s) = \lambda_N$ and $\mu(t) = \mu_N$.*

In this section, we prove an analogue to this result for dimension $n \geq 3$:

Theorem 6.1.21. *A monomial subalgebra $R \subset \mathbb{K}[x_1, \ldots, x_n]$ is finitely generated if and only if its corresponding semigroup $S \subset \mathbb{N}^n \subset \mathbb{Q}^n$ is contained in a polyhedral cone $C \subset \mathbb{Q}^n$ with a set of extremal vectors $v_1, \ldots, v_m$ where for each i, there is some $\mu_i \in \mathbb{Q}$ with $\mu_i v_i \in S$.*

This Theorem, together with the subsequent Corollary 6.1.25 we make upon completion of its proof, has been shown in [2, p. 53] using Hilbert's basis theorem. We provide an elementary proof.

Note that any subsemigroup of $\mathbb{N}^n$ is trivially contained in a polyhedral cone. Indeed, the cone defined by the half spaces $H_i := \{(a_1, \ldots, a_n) \in \mathbb{Q}^n \,|\, a_i \geq 0\}$ will contain all points in $\mathbb{N}^n$ and hence all possible subsemigroups of $\mathbb{N}^n$. When a semigroup S satisfies the conditions of Theorem 6.1.21 we say that it has an *associated polyhedral cone*. In order to prove this we first obtain some preliminary results:

Proposition 6.1.22. *Suppose $S \subset \mathbb{N}^n$ is a semigroup, $v \in S$ and $R := \{\lambda v \in \mathbb{Q}^n \,|\, \lambda \geq 0\}$ is the ray passing through v. Then the subsemigroup $T := S \cap R$ is affine.*

Proof. Let $v = (a_1, \ldots, a_n) \in S \subset \mathbb{N}^n$, there is some $\lambda \in \mathbb{Q}$ such that $\lambda v \in \mathbb{N}^n$ and $|v|$ is minimal. We claim that $R \cap \mathbb{N}^n = \{k(\lambda v) \,|\, k \in \mathbb{N}\}$. Indeed if $w \in R$ has $|w| = q|v| + |u|$, where $u = w - qv$ and $|u| < |v|$ we must have $u = 0$ since v was chosen minimally. Thus, $R \cap \mathbb{N}^n \cong \mathbb{N}$ and hence T is isomorphic to a subsemigroup of $\mathbb{N}$, hence is affine. □

This following lemma is known as Gordan's lemma, see [2, p.52].

Lemma 6.1.23. *Let $C \subset \mathbb{Q}^n$ be a polyhedral cone, and let $S = \mathbb{N}^n \cap C$, then S is affine.*

Proof. Suppose $v_1, \ldots, v_m \in C \subset \mathbb{Q}^n$ are a set of extremal vectors for C. We may assume, since we can rescale the vectors, that $v_i \in \mathbb{N}^n$ for all i. Now Let $v \in \mathbb{C} \cap \mathbb{N}^n$, then there are $\lambda_i \geq 0$ such that

$$v = \sum_{i=1}^{m} \lambda_i v_i,$$

since the $\lambda_i \in \mathbb{Q}$ we can rewrite this sum as

$$v = \sum_{i=1}^{m} a_i v_i + b_i v_i,$$

where $b_i \in \mathbb{N}$ and $0 \leq a_i < 1$. Note that since the $v_i \in \mathbb{N}^n$ we must have $\sum_{i=1}^{m} b_i v_i \in \mathbb{N}^n$ and hence

$$\sum_{i=1}^{m} a_i v_i \in \mathbb{N}^n.$$

Consider the set

$$A := \left\{ \sum_{i=1}^{m} a_i v_i \in \mathbb{N}^n \,|\, 0 \leq a_i < 1 \right\},$$

we note that A consists of all points within the polytope which is the convex hull of

$$\{0, v_1, \dots v_m, v_1 + v_2, \dots, v_{m-1} + v_m, \dots, v_1 + \cdots + v_m\}.$$

Clearly, we must have that $|A| < \infty$ and hence the set $A \cup \{v_1 \dots, v_m\}$ generates S over $\mathbb{N}$. □

Lemma 6.1.24. *Let $C \subset \mathbb{Q}^n$ be a polyhedral cone, and $v_1, \dots, v_m$ be a set of extremal vectors of C. Let $S \subset \mathbb{N}^n \cap C$ be a semigroup, with the property that for each i, there is some $\lambda_i \in \mathbb{Q}$ such that $\lambda_i v_i \in S$, then S is affine.*

Proof. To prove this result, we generalise the proof of Theorem 6.1.10 to n variables. We induct on n, with the aim to show that there is a generating set $G = G_1 \cup G_2$ for S with the property that each $g \in G_1$ lies on an extremal ray of C, and for each $s \in S$ we have $s = f + g$, where $g \in G_2$ and f lies in the semigroup generated by G_1. For $n = 1$ this result is trivial and for $n = 2$ this is the content of Theorem 6.1.10. Now, let R_i be the extremal ray corresponding to each v_i. By Proposition 6.1.22 each subsemigroup $R_i \cap S$ is affine and all elements in $R_i \cap S$ are of the form kw_i for some $w_i \in R \cap \mathbb{N}^n$ and $k \geq 0$. Suppose the generators of $R_i \cap S$ are $k_1^i w_i, \dots k_{j_i}^i w_i$, $k_l^i \in \mathbb{N}$. Let $d_i = \gcd(k_1^i, \dots, k_{j_i}^i)$. Let $U \subset S$ be the subsemigroup generated by $G_1 := \{k_1^i w_i, \dots k_{j_i}^i w_i \,|\, i = 1, \dots m\}$.

Now let $T \subset C$ be the semigroup with generators $d_i w_i$, $i = 1, \dots m$, note that T is not necessarily a subsemigroup of S. For $a \in T$ we define the polytope Q_a which is the convex hull of the vectors

$$\left\{ a, a + d_1 w_1, \dots a + d_n w_n, \dots, a + \sum_{i=1}^{m} d_i w_i \right\}.$$

Note that these polytopes cover the cone, meaning that for all $c \in C$, and hence all $s \in S$ there is some polytope Q_a which contains c. We set

$$A := \left\{ \sum_{i=1}^{m} a_i d_i w_i \in \mathbb{N}^n \mid 0 \le a_i \le 1 \right\},$$

as in the proof of Lemma 6.1.23 we remark that $|A| < \infty$. Note that since the v_i, and hence the $d_i w_i$, generate C, given any $g \in S$ we can write g as

$$g = \sum q_i d_i w_i + r_i d_i w_i,\ q_i \in \mathbb{N}, 0 \le r_i \le 1,$$

where $r := \sum_{i=1}^{n} r_i d_i w_i \in A$. We call r a *root vector* of g. Note that all $g \in S$ have at least one root vector. We claim that only finitely many generators are required to generate all points contained in polytopes of the form Q_{u+v}, where $v \in T$ and $u \in T$ are fixed vectors to be determined.

Since each of the subsemigroups $R_i \cap S$ are isomorphic to a subsemigroup of $\mathbb{N}$, there is some $M \in \mathbb{N}$ such that $(M+k)d_i w_i \in S$ for all $k \ge 0$. If $g \in S$ is contained in some polytope Q_w say, set $c_r = w + p$, where $p = \sum_{i=1}^{m} M d_i w_i$. Now $g + U$ contains a point in every polytope of the form Q_{c_r+v}, where $v \in T$. Write $g = w + r$, where r is a root vector of g. If $f \in g + U$, and lies in the polytope $Q_{(w+p)+v}$, then we must have $f = w + p + v + r$, where $w, p, v \in T$. Therefore all such f have r as a root vector. Now for each possible root vector r, of which there are a finite number since $|A| < \infty$, we can look for some $g_r \in S$ which has said root vector. If g_r exists, then by the argument above there is some vector c_r for which all $f \in S$ which have r as a root vector and are contained in polytopes of the form Q_{c_r+v}, $v \in T$, are generated by $g_r + U$. Otherwise there are no $f \in S$ with r as a root vector and we set $c_r = 0$. If we set

$$u := \sum_{r \in A} c_r,$$

then we require at most $|A| + \sum_{i=1}^{m} k^i_{j_i}$ generators to generate all points contained in polytopes of the form Q_{u+v}, where $v \in T$, as required. In fact more so is true, since the polytopes cover the cone, given any $w \in T$ lying in $u + C$, the cone shifted by u, no additional generators are required to generate all points in Q_w.

Let $H_1, \dots H_l \subset \mathbb{Q}^n$ denote the hyperplanes corresponding to the half spaces which define the cone C. To complete the proof we show that there are vectors $p^1_0 = 0, p^1_1, \dots, p^1_{r_1}, \dots, p^l_0 = 0, p^l_1, \dots, p^l_{r_l} \in \mathbb{Q}^n$ such that each point $s \in S \setminus (S \cap (u + C))$ is contained in $H_i + p^i_j$ for some i, j. With this in hand, we claim that by induction, only finitely many generators are required to generate all of $S \cap H_i + p^i_j$. We start by showing that $(C \setminus (u + C)) \cap \mathbb{N}^n$ can be covered by finitely many hyperplanes. For each H_i, we can describe H_i as

$$H_i := \left\{(p_1, p_2 \ldots, p_n) \in \mathbb{Q} \,|\, a^i_1 p_1 + a^i_2 p_2 + \cdots + a^i_n p_n = 0\right\},$$

where $a^i_j \in \mathbb{Q}$ for all i, j. Suppose that $a^i_j = \frac{p^i_j}{q^i_j}$, where if $p^i_j = 0$ we set $q^i_j = 1$, let $q_i := \mathrm{lcm}_j(q^i_j)$. Note that for $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$, we must have

$$a^i_1 p_1 + a^i_2 p_2 + \cdots + a^i_n p_n = \frac{s}{q_i},$$

where $s \in \mathbb{Z}$. Suppose for $u = (u_1, \ldots, u_n)$ we have that

$$a^i_1 u_1 + \cdots + a^i_n u_n = \frac{s_i}{q_i},$$

Now, if $s_i > 0$, for each $1 \leq j \leq s_i$ we take $p^i_j = (t_1, \ldots, t_n) \in \mathbb{Q}^n$ to be a vector satisfying

$$a^i_1 t_1 + \cdots + a^i_n t_n = \frac{j}{q_i},$$

and set $r_i = s_i$ If $s_i < 0$, we do the same for each $s_i \leq j \leq -1$ and set $r_i = -s_i$. Now given some $v \in (C \setminus (u + C)) \cap \mathbb{N}^n$ it must lie between some H_i and $H_i + u$ and hence lies in $H_i + p^i_j$ for some j, thus $(C \setminus (u + C)) \cap \mathbb{N}^n$ can be covered by finitely many hyperplanes. Additionally, since given any $b \in H_i + p^i_j$ we have $H_i + p^i_j = H_i + b$, we can assume that $p^i_j \in \mathbb{Z}^n$. Furthermore, if $p^i_j = (t_1, \ldots, t_n)$ we can choose p^i_j so that $t_i \leq 0$ for all i, so that whenever we have $b = c + p^i_j \in H_i + p^i_j \cap C$, it follows that $c \in H_i \cap C$.

Now for each i suppose that $\lambda^i_1 v^i_1, \ldots, \lambda^i_{a_i} v^i_{a_i} \in \{\lambda_1 v_1, \ldots, \lambda_m v_m\} \subset S \cap C$ are the extremal vectors contained in $H_i \cap C$. By induction the subsemigroup $S \cap H_i$ lying in the polyhedral cone $H_i \cap C$ is finitely generated with generating set $G^i = G^i_1 \cup G^i_2$. To show that S is finitely generated, the only remaining task is to show that finitely many elements are required generate all vectors in $(H_i + p^i_j) \cap S$ for all i, j. Note that for $p^i_j \neq 0$, $(H_i + p^i_j) \cap S$ is not a semigroup. Given $a \in H_i \cap S$ and $b \in (H_i + p^i_j) \cap S$, we must have $a + b \in (H_i + p^i_j) \cap S$. We claim that there is a finite subset $G \subset (H_i + p^i_j) \cap S$ with the property that each $b \in (H_i + p^i_j) \cap S$ can be written as $b = a + g$, where $g \in G$ and a lies in the semigroup generated by G. It is clear that a subset with this property exists, though may not necessarily be finite. Let $G \subset (H_i + p^i_j) \cap S$ be such a set which is minimal and let $g \in G$. We can write

$g = g' + p^i_j$, where $g' \in (H_i \cap C) \cap \mathbb{N}^n$. Consider the semigroup $U \subset (H_i \cap C) \cap \mathbb{N}^n$ generated by G^i_1 and $G' := \{g' \mid g' = g - p^i_j, g \in G\}$, by induction we must have that U is finitely generated and hence G' is finite as it was chosen to be minimal. Thus we conclude that finitely many generators are required to generate all points in $(H_i + p^i_j) \cap S$ for all i, j, completing the proof. □

Now we complete the proof of the theorem:

Proof of Theorem 6.1.21. Suppose that there is no polyhedral cone $C \subset \mathbb{Q}^n$ with extremal vectors $v_1, \ldots, v_n$ containing S with $\mu_i v_i \in S$ for some $\mu_i \in \mathbb{Q}$. Consider any finite subset $U := \{u_1, \ldots, u_k\} \subset S$, these generate a polyhedral cone

$$C_U := \left\{ \sum_{i=1}^{k} \lambda_k u_k \in \mathbb{Q}^n \mid \lambda_k \in \mathbb{Q}_{\geq 0} \right\},$$

where some subset of the vectors of U are the extremal vectors of C_U. Since S cannot be contained within C_U, we must therefore conclude that there is some $p \in S$ which is not a combination on the u_i and so no finite subset of S can generate S. □

Note that this theorem has an immediate generalisation, our proof uses that we are only able to take non-negative linear combinations of our points, but not that the points themselves have non-negative entries. Thus we obtain the following result for Laurent polynomials:

Corollary 6.1.25. *Suppose $R \subset \mathbb{K}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ is generated by monomials. Then R is finitely generated if and only if its corresponding semigroup $S \subset \mathbb{Z}^n \subset \mathbb{Q}^n$ is contained in a polyhedral cone $C \subset \mathbb{Q}^n$ with a set of extremal vectors $v_1, \ldots, v_m$ where for each i, there is some $\mu_i \in \mathbb{Q}$ with $\mu_i v_i \in S$.*

Remark. *Suppose that there are monomials $m_1, \ldots m_k \in R$ corresponding to points $a_1, \ldots, a_n \in \mathbb{Z}^n$ with the property that for all $q \in \mathbb{Q}^n$ we have*

$$q = \sum_{i=1}^{k} \lambda_i a_i,$$

where $\lambda_i \in \mathbb{Q}_{\geq 0}$ for all i. Then the proof of Theorem 6.1.21 can still be used, but here the polyhedral cone generated by the vectors $a_1, \ldots, a_n \in \mathbb{Z}^n$ is all of $\mathbb{Q}^n$.

As a second generalisation, note that Theorem 6.1.21 passes immediately to SAGBI bases.

Corollary 6.1.26. *Suppose $R \subset \mathbb{K}[x_1, \ldots, x_n]$ is a subalgebra and that " $<$ " is a monomial ordering on R. Then R has a finite SAGBI-basis if and only if the semigroup corresponding to the leading algebra $L_{alg}(R)$ has an associated polyhedral cone.*

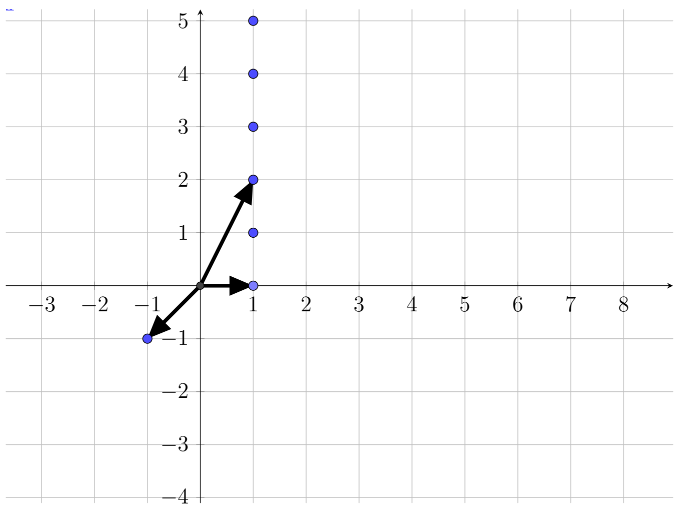
Now with Theorem 6.1.21 in hand, we dedicate the rest of this chapter to studying the finite generation ideal of non-finitely generated monomial subalgebras.

6.2 FINITE GENERATION IDEALS

Suppose $R \subset \mathbb{K}[x_1, \ldots, x_n]$ is a non-finitely generated monomial subalgebra. Corollary 6.1.25 re-contextualises our approach to determining whether a given monomial $m = x_1^{a_1} \cdots x_n^{a_n} \in R$ lies in $\mathfrak{f}_R$. When determining whether R_m is finitely generated, we can take a geometric approach. $m^{-1} \in R_m$ will correspond to the point $(-a_1, \ldots, -a_n)$ and using Corollary 6.1.25 we can determine whether R_m is finitely generated by determining whether the corresponding semigroup S, together with the negative ray

$$R_{-m} := \{k(-a_1, \ldots, -a_n) \in \mathbb{Z}^n \mid k \geq 1\} \subset \mathbb{Q}^n,$$

has an associated polyhedral cone. To demonstrate this, we return to Example 2.4.6 with $R = \mathbb{K}[x, xy, xy^2, \ldots,]$.



If we examine R_{xy}, the vectors corresponding to x, xy^2 and $(xy)^{-1}$, namely $(1, 0)$, $(1, 2)$ and $(-1, -1)$, generate a polyhedral cone which is all of $\mathbb{Q}^2$, and hence R_{xy} is finitely generated. Indeed, the same is true for all xy^n, $n \geq 1$ using the vectors corresponding to x, xy^{n+1} and $(xy^n)^{-1}$. When considering R_x however, the polyhedral cone which is associated to the semigroup corresponding to R_x is the half space:

$$H := \{(a, b) \in \mathbb{Q}^2 \mid b \geq 0\},$$

Note in this case the extremal vectors, $(1, 0)$ and $(-1, 0)$ do not generate the cone and we require at least one other vector, such as $(1, 1)$ to do so.

This example demonstrates a useful means of determining whether certain monomials are contained in the finite generation ideal. Suppose $R \subset \mathbb{K}[x_1, \dots, x_n]$ a non-finitely generated subalgebra, and $S \subset R$ is a finitely generated subalgebra of R. Suppose that the associated polyhedral cone of S is n-dimensional, meaning that the extremal points generating the cone generate $\mathbb{Q}^n$ as a vector space over $\mathbb{Q}$. Now suppose we localise by a monomial corresponding to a point contained in the interior of the cone, meaning not lying on any of the cone's faces. The localised vector, together with all of the extremal points generating the cone, will generate all of $\mathbb{Q}^n$ as a polyhedral cone. Hence, these monomials must lie in the finite generation ideal.

It remains unclear whether monomials which correspond to points lying on one of faces of the polyhedral cone associated to S, the semigroup corresponding to R, lie in the finite generation ideal also. To approach a solution, it quickly becomes necessary to understand the ways in which a subalgebra can fail to satisfy the conditions of Theorem 6.1.21. To do this, as in the two dimensional case we shall make use of ratio functions, which generalise as follows:

Definition 6.2.1. A function $\lambda : \mathbb{K}[x_1, \dots, x_n, y] \longrightarrow \mathbb{Q}$ is a *ratio function* if it satisfies following properties:

1. For $f \in \mathbb{K}[x_1, \dots, x_n, y]$, $\lambda(f) = \max\{\lambda(m) \mid m \text{ is a term of } f\}$.
2. $\lambda(cf) = \lambda(f)$ for all $f \in \mathbb{K}[x_1, \dots, x_n, y]$, $c \in \mathbb{K} \setminus \{0\}$.
3. Given $f, g \in \mathbb{K}[x_1, \dots, x_n, y]$ with $\lambda(f) \leq \lambda(g)$, then $\lambda(f) \leq \lambda(fg) \leq \lambda(g)$ with equality if and only if $\lambda(f) = \lambda(g)$
4. A monomial m can be uniquely recovered given any n of $\lambda(m)$, $\deg_{x_1}(m)$, $\deg_{x_2}(m), \dots, \deg_{x_n}(m)$.

As an extension to property 4, given ratio functions $\lambda_1, \dots, \lambda_k : \mathbb{K}[x_1, \dots, x_n] \longrightarrow \mathbb{Q}$ we say these functions are *compatible* if any monomial $m \in \mathbb{K}[x_1, \dots, x_n]$ can be uniquely recovered given any n of $\lambda_1(m), \dots, \lambda_k(m), \deg_{x_1}(m), \dots \deg_{x_n}(m)$.

Example 6.2.2. Let $\{i_1, \dots i_k\} \subset \{1, \dots, n,\}$ where $1 \leq k \leq n$, and consider the function $\lambda : \mathbb{K}[x_1, \dots, x_n] \longrightarrow \mathbb{Q}$ defined first on monomials as:

$$\lambda\left(x_1^{a_1} \cdots x_n^{a_n}\right) := \begin{cases} \frac{p_{i_1}a_{i_1} + \dots p_{i_k}a_{i_k}}{q_{i_1}a_{i_1} + \cdots + q_{i_k}a_{i_k}} & a_{i_j} \text{ not all } 0, \\ 0 & \text{otherwise.} \end{cases}$$

Where the $p_{i_j} \in \mathbb{N}, q_{i_j} \in \mathbb{N} \setminus \{0\}$ satisfy:

1. $0 \leq \frac{p_{i_j}}{q_{i_j}} \leq 1$.
2. If $p_{i_j} \neq 0$, then $\gcd(p_{i_j}, q_{i_j}) = 1$.
3. There is some $s, t \in \{1, \dots, k\}$ with $\frac{p_{i_s}}{q_{i_s}} \neq \frac{p_{i_t}}{q_{i_t}}$.
4. If $p_{i_k} = 0$ then $q_{i_j} = 1$.

We can extend λ to $f \in \mathbb{K}[x_1, \dots, x_n]$ by setting

$$\lambda(f) = \max\{\lambda(m) \,|\, m \text{ is a term of } f\}.$$

Then λ defines a ratio function on $\mathbb{K}[x_1, \dots, x_n]$: Clearly λ satisfies properties 1 and 2, and if

$$\frac{p_{i_1}a_{i_1} + \dots p_{i_k}a_{i_k}}{q_{i_1}a_{i_1} + \cdots + q_{i_k}a_{i_k}} = \alpha,$$

rearranging we have

$$(\alpha q_{i_1} - p_{i_1})a_{i_1} + \cdots + (\alpha q_{i_k} - p_{i_k})a_{i_k} = 0.$$

Since there is some s, t with $\frac{p_{i_s}}{q_{i_s}} \neq \frac{p_{i_t}}{q_{i_t}}$ we must have $(\alpha q_{i_j} - p_{i_j}) \neq 0$ for all j, and so a fixed ratio defines a hyperplane, and λ satisfies property 4. Suppose monomials $m = x_1^{a_1} \cdots x_n^{a_n}$ and $n = x_1^{b_1} \cdots x_n^{b_n}$ satisfy $\lambda(m) = \frac{p}{q} \leq \lambda(n) = \frac{r}{s}$, then

$$\frac{p + r}{q + s} - \frac{p}{q} = \frac{qr - ps}{q(q + s)} \geq 0,$$

since $qr - ps \geq 0$ and we also have

$$\frac{r}{s} - \frac{p + r}{q + s} = \frac{qr - ps}{s(q + s)} \geq 0.$$

We then conclude that

$$\lambda(m) = \frac{p}{q} \leq \lambda(mn) = \frac{p + r}{q + s} \leq \lambda(n) = \frac{r}{s},$$

with equality precisely when $p = r, q = s$. For general $f, g \in R$ where $\lambda(f) = \lambda(m)$, $\lambda(g) = \lambda(n)$, for monomials m, n appearing in f and g respectively, it is clear that

$\lambda(fg) = \lambda(mn)$ and so the property remains satisfied. We call this kind of ratio function a *linear ratio function.*

As in the 2 dimensional case, we can define an increasing sequence $(\lambda_k)_{k\in\mathbb{N}}$ with

$$\lambda_k := \max\{\lambda(f) \,|\, p_{i_1} \deg_{x_{i_1}}(f) + \cdots + p_{i_k} \deg_{x_{i_k}}(f) \leq k\}.$$

If $(\lambda_k)_{k\in\mathbb{N}}$ has an infinite strictly increasing subsequence, then R is non-finitely generated, we call this the *associated sequence* of λ.

When this sequence is eventually constant, then geometrically these ratio functions can be used to define a half-space with 0 on the boundary containing all monomials in R, where the maximal ratio defines the hyperplane boundary. Suppose $\alpha \in \mathbb{Q}$ is the maximal ratio, then for a monomial $x_1^{a_1} \cdots x_n^{a_n} \in R$ we must have

$$(\alpha q_{i_1} - p_{i_1})a_{i_1} + \cdots + (\alpha q_{i_k} - p_{i_k})a_{i_k} \geq 0.$$

Additionally, given a half-space with 0 on the boundary containing R, we can define a ratio function corresponding to it. Indeed, consider the half-space defined, for $q = (q_1, \ldots, q_n) \in \mathbb{Q}^n$ by

$$\frac{r_{i_1}}{s_{i_1}} q_{i_1} + \cdots + \frac{r_{i_k}}{s_{i_k}} q_{i_k} \geq 0,$$

where each $\frac{r_{i_j}}{s_{i_j}} \neq 0$. By setting $\alpha = \frac{1}{s_{i_1} \cdots s_{i_k}}$ we have:

$$\alpha\Big(r_{i_1} s_{i_2} \cdots s_{i_k} q_{i_1} + \cdots + r_{i_{k-1}} s_{i_1} \cdots s_{i_{k-2}} s_{i_k} q_{i_{k-1}} + r_{i_k} s_{i_1} \cdots s_{i_k}(s_{i_k} + 1) q_{i_k}\Big) \\ - r_{i_k} q_{i_k} \geq 0.$$

Rearranging, we obtain:

$$\frac{r_{i_k} q_{i_k}}{r_{i_1} s_{i_2} \cdots s_{i_k} q_{i_1} + \cdots + r_{i_{k-1}} s_{i_1} \cdots s_{i_{k-2}} s_{i_k} q_{i_{k-1}} + r_{i_k} s_{i_1} \cdots s_{i_{k-1}}(s_{i_k} + 1) q_{i_k}} \leq \alpha.$$

Thus, defining a ratio function μ on the monomials with

$$\mu(x_1^{a_1} \cdots x_n^{a_n}) := \\ \frac{r_{i_k} a_{i_k}}{r_{i_1} s_{i_2} \cdots s_{i_k} a_{i_1} + \cdots + r_{i_{k-1}} s_{i_1} \cdots s_{i_{k-2}} s_{i_k} a_{i_{k-1}} + r_{i_k} s_{i_1} \cdots s_{i_{k-1}}(s_{i_k} - 1) a_{i_k}},$$

the half-space is then defined by the maximal ratio α.

Example 6.2.3. Ratio functions need not be linear. Indeed, consider $\lambda : \mathbb{K}[x, y] \longrightarrow \mathbb{Q}$ defined on monomials by:

$$\lambda(x^a y^b) := \frac{p_1 a^2 + p_2 ab + p_3 b^2}{q_1 a^2 + q_2 ab + q_3 b^2},$$

where the p_i, q_i satisfy the conditions from Example 6.2.2. Additionally suppose that $\frac{p_1}{q_1} \leq \frac{p_2}{q_2} \leq \frac{p_3}{q_3}$, and extend λ to polynomials by taking the maximum value among all monomials in f. Then λ defines a ratio function on $\mathbb{K}[x, y]$. To see this, we note that it is clear that λ satisfies properties 1 and 2. For property 3, suppose that $\lambda(x^a y^b) \leq \lambda(x^c y^d)$, first we observe that

$$\begin{aligned}&\lambda(x^c y^d) - \lambda(x^a y^b)\\&\quad= \frac{-(ad - bc)\left((p_1 q_2 - p_2 q_1)ac + (p_1 q_3 - p_3 q_1)(ad + bc) + (p_2 q_3 - p_3 q_2)bd\right)}{(q_1 a^2 + q_2 ab + q_3 b^2)(q_1 c^2 + q_2 cd + q_3 d^2)} \geq 0.\end{aligned}$$

Since $p_i q_j - p_j q_i \leq 0$ for all $i < j$ and $a, b, c, d \geq 0$, we conclude that $ad - bc \geq 0$. Now by rearranging, we can obtain

$$\begin{aligned}&\lambda(x^{a+c} y^{b+d})\\&\quad= \frac{(p_1 a^2 + p_2 ab + p_3 b^2) + (p_1 c^2 + p_2 cd + p_3 d^2) + (2p_1 ac + p_2(ad + bc) + 2p_3 bd)}{(q_1 a^2 + q_2 ab + q_3 b^2) + (q_1 c^2 + q_2 cd + q_3 d^2) + (2q_1 ac + q_2(ad + bc) + 2q_3 bd)}.\end{aligned}$$

Claim. *Given $\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq \frac{a_3}{b_3}$, we have*

$$\frac{a_1}{b_1} \leq \frac{a_1 + a_2 + a_3}{b_1 + b_2 + b_3} \leq \frac{a_3}{b_3}.$$

To see this we simply have

$$\frac{a_1 + a_2 + a_3}{b_1 + b_2 + b_3} - \frac{a_1}{b_1} = \frac{(a_2 b_1 - a_1 b_2) + (a_3 b_1 - a_1 b_3)}{b_1(b_1 + b_2 + b_3)} \geq 0,$$

since $a_i b_j - a_j b_i \geq$ for $i > j$. The other inequality is similar. Now, given the claim, it is sufficient to show:

$$\alpha = \frac{p_1 a^2 + p_2 ab + p_3 b^2}{q_1 a^2 + q_2 ab + q_3 b^2} \leq \beta = \frac{2p_1 ac + p_2(ad + bc) + 2p_3 bd}{q_1 ac + q_2(ad + bc) + 2q_3 bd} \leq \gamma = \frac{p_1 c^2 + p_2 cd + p_3 d^2}{q_1 c^2 + q_2 cd + q_3 d^2}.$$

For the first inequality, we have

$$\beta - \alpha = \frac{-(ad - bc)\left((p_1 q_2 - p_2 q_1)a^2 + (p_1 q_3 - p_3 q_1)2ab + (p_2 q_3 - p_3 q_2)b^2\right)}{(2q_1 ac + q_2(ad + bc) + 2q_3 bd)(q_1 a^2 + q_2 ab + q_3 b^2)} \geq 0,$$

using $ad - bc \geq 0$, $p_i q_j - p_j q_i \leq 0$ for $i < j$ and $a, b, c, d \geq 0$. The second inequality is similar. Thus, λ satisfies property 3, leaving just property 4. This final property is satisfied, since for a fixed ratio $\alpha \in \mathbb{Q}$, we have

$$\frac{p_1 a^2 + p_2 ab + p_3 b^2}{q_1 a^2 + q_2 ab + q_3 b^2} = \alpha,$$

and rearranging we obtain

$$(\alpha q_1 - p_1)a^2 + (\alpha q_2 - p_2)ab + (\alpha q_3 - p_3)b^2 = 0.$$

Note that we must have $\frac{p_1}{q_1} \leq \alpha \leq \frac{p_3}{q_3}$, and hence $(\alpha q_1 - p_1) \geq 0$ and $(\alpha q_3 - p_3) \leq 0$. Solving this, we obtain

$$a = \frac{-(\alpha q_2 - p_2)b \pm \sqrt{((\alpha q_2 - p_2)^2 - 4(\alpha q_1 - p_1)(\alpha q_3 - p_3))\, b^2}}{2(\alpha q_1 - p_1)},$$

where $(\alpha q_2 - p_2)^2 - 4(\alpha q_1 - p_1)(\alpha q_3 - p_3) \geq 0$ by our observations. This equation can only have one solution with $a, b \geq 0$. This is because $(\alpha q_2 - p_2)^2 - 4(\alpha q_1 - p_1)(\alpha q_3 - p_3) \geq (\alpha q_2 - p_2)$, hence the numerator changes sign when changing the $+$ to $-$. We therefore conclude that λ defines a ratio function.

Remark. *We expect this form of ratio function to generalise to larger variables, and higher degrees of homogeneous polynomials in the numerator and denominator, under some restrictions similar to our requirement that* $\frac{p_1}{q_1} \leq \frac{p_2}{q_2} \leq \frac{p_3}{q_3}$*. However, the combinatorics required to show properties 3 and 4 hold grow in complexity.*

In two variables, we have shown that by looking at the associated sequences of only two fixed linear ratio functions, we can determine whether any given subalgebra $R \subset \mathbb{K}[x, y]$ is finitely generated. When working with 3 or more variables, this is no longer the case. In fact, it is not possible to fix any finite number of ratio functions in order to determine whether any given subalgebra $R \subset \mathbb{K}[x_1, \ldots, x_n]$ is finitely generated for $n \geq 3$. Geometrically this makes sense as in 2-dimensions a polyhedral cone can have only 2 faces, whereas in 3 or more dimensions a polyhedral cone can have any arbitrary number of faces. In order to obtain results on the finite generation ideal in $n \geq 3$ variables with an approach similar to that of Theorem 6.1.11, we would require at least $n - 1$ compatible ratio functions. In the proof of Theorem 6.1.11, when localising by a monomial, it is key that there are monomials with larger ratio than that monomial in order to show some power of x or y is in the localisation. With an increased number of variables, we would need an increased number of monomials, and hence ratio functions which yield monomials with larger ratios.

The following example is useful in demonstrating the added complexities that come with determining the finite generation ideal when working with 3 or more variables. Additionally, it provides a negative answer to the question of whether the finite generation ideal is always the radical of a finitely generated ideal.

Example 6.2.4. Consider the subalgebra

$$R := \mathbb{K}\left[x, y, xyz, x^3yz^2, \dots, x^{\frac{1}{2}n(n+1)}yz^n, \dots\right].$$

Note that $R_y \cong \mathbb{K}[x, y, y^{-1}, xz]$ and so $y \in \mathfrak{f}_R$. Let $g_n := x^{\frac{1}{2}n(n+1)}yz^n$ and consider R_{g_n}, if $m > n$ then

$$g_m/g_n = \frac{x^{\frac{1}{2}m(m+1)}yz^m}{x^{\frac{1}{2}n(n+1)}yz^n} = x^{\sum_{i=n+1}^m i}z^{m-n}.$$

We can consider just these elements as well as x as generators of a subalgebra of $\mathbb{K}[x, z]$; with the ratio function $\lambda(x^az^b) := \frac{b}{a+b}$, the generators then have ratio $\lambda(x^{\sum_{i=n+1}^m i}z^{m-n}) = \frac{m-n}{m-n+\sum_{i=n+1}^m i}$, which is strictly decreasing as m increases. Therefore the sequence λ_k is eventually constant and by Lemma 6.1.3 the subalgebra generated by these elements is finitely generated. Since there are only finitely many generators in R with z-degree less than n we conclude that R_{g_n} is finitely generated and hence $g_n \in \mathfrak{f}_R$.

Suppose that R_f is finitely generated, where $f \in \mathbb{K}[x]$ and let $G = \{f_1, \dots, f_n\}$ be a generating set. Note that if $h \in R_f$ and $\deg_y(h) = a$ and $\deg_z(h) = b$, then $\deg_y(hf^t) = a$ and $\deg_z(hf^t) = b$ for all $t \in \mathbb{Z}$. Let $N := \max_i(\deg_z(f_i))$ and let $m > N$, since G generates R_f we must have $g_m = \sum_j \lambda_j f_1^{a_1^j} \cdots f_n^{a_n^j}$. However, no term of this expression can have z-degree m and y-degree 1 since all f_i with $\deg_z(f_i) \geq 1$ must also have $\deg_y(f_i) \geq 1$. To obtain a term with z-degree m we require the product of at least 2 f_i with non-zero z-degree hence producing a term with y-degree at least 2. We therefore conclude that that R_f is not finitely generated and hence $\mathfrak{f}_R = (y, xyz, \dots)$.

We will now show that for all $n \geq 1, (g_n)^k \notin J_n := (x_1, x_2, g_1, \dots, \hat{g_n}, \dots)$. Note $(g_n)^k = x^{\frac{1}{2}nk(n+1)}y^kz^nk$ where $k \geq 1$. Since R is generated by monomials it is sufficient to show that $(g_n)^k$ cannot be written as $g_{i_1} \cdots g_{i_l}p$, where $p = x^ay^b$ for some $a, b \geq 0$ and at least one of the $i_j \neq n$. We require then that $g_{i_1} \cdots g_{i_l}$ has x-degree at most $\frac{1}{2}nk(n+1)$, y-degree at most k and z-degree at most nk. Therefore we must have $l \leq k$ and

$$\sum_{j=1}^{l} \frac{1}{2}i_j(i_j + 1) \leq \frac{1}{2}nk(n+1),$$

where $\sum_{j=1}^{l} i_j = nk$. Note that

$$\sum_{j=1}^{l} \frac{1}{2}i_j(i_j + 1) = \frac{1}{2}(nk + \sum_{j=1}^{l} i_j^2),$$

and so we require

$$\sum_{j=1}^{l} i_j^2 \leq n^2 k = \sum_{j=1}^{k} n^2.$$

Order the i_j so that $i_j = \frac{nk}{l} + d_j$ for $1 \leq j \leq t$, where $d_j \geq 0$ and $i_j = \frac{nk}{l} - c_j$ for $t < j \leq l$ where $c_j \geq 0$. Note that $(\frac{nk}{l} + d_j)^2 = \frac{(nk)^2}{l^2} + 2\frac{nkd_j}{l} + d_j^2$ whilst $(\frac{nk}{l} - c_j)^2 = \frac{(nk)^2}{l^2} - 2\frac{nkc_j}{l} + c_j^2$. Now

$$\sum_{i=1}^{l} i_j = \frac{n^2k^2}{l} + 2\frac{nk}{l}(d_1 + \cdots + d_t - c_{t+1} - \ldots c_l) + d_1^2 + \ldots d_t^2 + c_{t+1}^2 + \ldots c_l^2.$$

Observe that

$$\left(\frac{nk}{l} + d_1\right) + \cdots + \left(\frac{nk}{l} + d_t\right) + \left(\frac{nk}{l} - c_{t+1}\right) + \ldots \left(\frac{nk}{l} - c_l\right) = nk,$$

hence $\sum_{j=1}^{t} d_j = \sum_{j=t+1}^{l} c_j$. Thus

$$\sum_{i=1}^{k} i_j = \frac{n^2k^2}{l} + +d_1^2 + \ldots d_t^2 + c_{t+1}^2 + \ldots c_l^2$$

which is strictly larger than $n^2 k$ whenever $l < k$ or the d_j and c_j are not all 0. Therefore, we conclude that $(g_n)^k \notin J_n$ for all $k \geq 1$ and $g_n \notin \sqrt{J_n}$ for all n.

Suppose $\mathfrak{f}_R = \sqrt{I}$, where $I = (f_1, \ldots, f_m)$, then for all $n \in \mathbb{N}$ there is a n_k so that $(g_n)^{n_k} \in I$. We can therefore write each $g_n^{n_k}$ as

$$g_n^{n_k} = \sum_i \lambda_i f_1^{a_i^i} \cdots f_n^{a_n^i},$$

where $\lambda_i \in \mathbb{K} \setminus \{0\}, a_j^i \geq 0$ for all i, j. Consider the map $\phi : R \longrightarrow R/J_n \cong \mathbb{K}[g_n]$, since $(g_n)^k \notin J_n$ for all $k \geq 1$ we have that

$$g_n^{n_k} = \phi(g_n^{n_k}) = \phi\left(\sum_i \lambda_i f_1^{a_i^i} \cdots f_n^{a_n^i}\right) = \sum_i \lambda_i \overline{f}_1^{a_i^i} \cdots \overline{f}_n^{a_n^i} + J_n,$$

where $\overline{f}_j \in \mathbb{K}[g_n]$. We therefore must conclude that for all n, there is some j such that f_j contains a term of the form λg_n^l, $l \neq 0$, $\lambda \in \mathbb{K} \setminus \{0\}$. Since there are infinitely many such g_n, we conclude that there must be a polynomial f_j with infinitely many terms, a contradiction.

Consider the semigroup $S \subset \mathbb{Q}^3$ corresponding to this example, it is generated by the set

$$\mathcal{G} := \left\{(1,0,0), (0,1,0), \left(\frac{n(n+1)}{2}, 1, n\right), \ldots \;\middle|\; n \geq 1\right\}.$$

The points corresponding to the g_n all lie on the curve:

$$C := \left\{ (a,b,c) \in \mathbb{Q}^3 \,\middle|\, b = 1,\, a = \frac{c(c+1)}{2} \right\}.$$

Another property of this example is that it shows that we cannot construct multiple compatible ratio functions whose associated sequences have strictly increasing subsequences if we require all them to be linear. Indeed, consider the linear ratio function $\lambda : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow \mathbb{Q}$ defined by

$$\lambda(x^a y^b z^c) = \begin{cases} \frac{c}{b+c} & \text{one of } b, c \neq 0 \\ 0 & \text{otherwise,} \end{cases},$$

note that $\lambda(x) = \lambda(y) = 0$ and $\lambda(g_n) = \frac{n}{n+1}$. Hence the associated sequence $(\lambda_k)_{k\in\mathbb{N}}$ has an infinite strictly increasing subsequence. There is no other compatible linear ratio function whose associated sequence has a strictly increasing subsequence. Suppose $\mu : \mathbb{K}[x,y,z] \longrightarrow \mathbb{Q}$ were such a function, then to have μ compatible to λ, μ must take one of the following forms:

$$\mu(x^a y^b z^c) = \begin{cases} \frac{p_1 a + p_2 b + p_3 c}{q_1 a + q_2 b + q_3 c}, \\ \\ \begin{cases} \frac{p_1 a + p_2 b}{q_1 a + q_2 b} & \text{one of } a, b \neq 0, \\ 0 & \text{otherwise,} \end{cases} \\ \\ \begin{cases} \frac{p_1 a + p_3 c}{q_1 a + q_3 c} & \text{one of } a, c \neq 0, \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

If μ were not one of these forms, a monomial m could not be uniquely recovered from some fixed value of λ, μ and $\deg_y(m)$ or $\deg_z(m)$, as it would not be possible to recover $\deg_x(m)$. Furthermore since $x, y \in R$, we require $\frac{p_3}{q_3} > \frac{p_i}{q_i}$ for $i = 1, 2$ since otherwise $\mu(x)$ or $\mu(y)$ would be the maximal value of $(\mu_k)_{k\in\mathbb{N}}$, thus ruling out the second form μ could take. Now

$$\mu\left(x^{\frac{n(n+1)}{2}} y z^n\right) = \frac{p_1\left(\frac{n(n+1)}{2}\right) + p_2 + p_3 n}{q_1\left(\frac{n(n+1)}{2}\right) + q_2 + q_3 n},$$

converges to $\frac{p_1}{q_1}$ as n tends to infinity, and would still do so if we precluded $\frac{p_2}{q_2}$. Hence for any choice of p_i, q_i under these conditions, there is some $N \in \mathbb{N}$ with $\lambda(g_N) \geq \lambda(g_{N+k})$ for all $k \geq 0$ and so the associated sequence $(\mu_k)_{k\in\mathbb{N}}$ is eventually constant.

6.3 FURTHER RESEARCH

This chapter has prompted a number of questions and possible areas for further study. Our main goal in this chapter has been to develop our understanding of the finite generation ideal, with the hopes of applying this to invariant rings. The method we employ to compute the finite generation ideal relies upon construction of a SAGBI-basis and the results in this chapter on monomial generating sets pass easily to SAGBI-bases. From our work in 2 variables with monomials, a natural question to ask would be

Question 6.3.1. *Suppose $R \subset \mathbb{K}[x, y]$, has a non-finite SAGBI-basis, what can we say about $\mathfrak{f}_R$?*

Notably, there are finitely generated polynomial rings without finite SAGBI-bases, such as $\mathbb{K}[x + y, xy, xy^2]$, and any result would have to account for this. Similarly, once we obtain a further understanding of the finite generation ideal for monomials in at least 3 variables, we would then try to pass to SAGBI-bases as well.

Returning to ratio functions, there is still much development to be done to use these on algebras in 3 or more variables. We would like to obtain a result for $n \geq 3$ variables similar to Theorem 6.1.11, which leads us to the question:

Question 6.3.2. *Suppose $R \subset \mathbb{K}[x_1, \ldots, x_n]$ is a non-finitely generated subalgebra, are there $n - 1$ compatible ratio functions $\lambda^1, \ldots, \lambda^{n-1}$ whose associated sequences $(\lambda^j_k)_{k\in\mathbb{N}}$ have infinite strictly increasing subsequences?*

In Example 6.2.4, we showed that we could not construct multiple compatible linear ratio functions whose associated sequences had strictly increasing subsequences. We do, however, believe it is possible to show:

Conjecture 6.3.3. *Suppose $R \subset \mathbb{K}[x_1, \ldots, x_n]$ is a non-finitely generated monomial subalgebra, then there is some linear ratio function $\lambda : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow \mathbb{Q}$ whose associated sequence $(\lambda_k)_{k\in\mathbb{N}}$ has an infinite strictly increasing subsequence.*

Taking inspiration from Example 6.2.4, there is a simple way of constructing a number of non-finitely generated monomial subalgebras. Indeed, take any infinite subset $A \subset \mathbb{N}^n$ and consider

$$R = \mathbb{K}\left[x_1^{a_1} \ldots x_n^{a_n} x_{n+1} \,|\, (a_1, \ldots, a_n) \in A\right] \subset \mathbb{K}[x_1, \ldots, x_{n+1}],$$

this subalgebra is non-finitely generated for any such A. We can, for example, take A to be the the set of non-negative integer points on a curve or surface. For example, we could take $R = \mathbb{K}\left[x_1^a x_2^b x_3 \mid a, b \in \mathbb{N}\right]$, which is not finitely generated. Comparing this to Example 6.2.4, which relies on just one infinite sequence of invariants to be non-finitely generated, this example has an infinite number of infinite sequences. We propose that a new notion should be introduced to differentiate between these two types of non-finitely generated algebras, introducing some idea of dimension. The finite generation ideal is inadequate for this purpose, a simple computation shows that $\mathfrak{f}_R$ is maximal for the above example.

There are still more questions we have about the possible structure of the finite generation ideal. It is simple to construct a subalgebra with arbitrarily many generators not in the finite generation ideal, for example $\mathbb{K}[x_1, x_1x_2, x_1^2x_2, \ldots, x_3, \ldots, x_N]$ has $x_i \notin \mathfrak{f}_R$ for $i \geq 3$. But we can also ask:

Question 6.3.4. *Can we construct a non-finitely generated monomial subalgebra $R \subset \mathbb{K}[x_1, \ldots, x_n]$ with $\mathfrak{f}_R$ finitely generated?*

Finally, while obtaining a greater understanding of non-finitely generated monomial algebras is useful, we are really interested in invariant rings. As we have constructed many novel monomial algebras which break our expectations, we would like to demonstrate invariant rings, if they exist, which do the same. For example, as seen in Example 6.1.7, it is simple to demonstrate a monomial algebra which does not satisfy the non-finiteness criterion. Without completing the proof of Conjecture 4.1.6, we do not have an example of an invariant ring which does the same. Similarly in Example 6.2.4, we have a monomial algebra whose finite generation ideal is not the radical of a finitely generated ideal, we would like to know if it possible to construct an invariant ring with this property.

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Concluding remarks

The stated goal of this thesis is to deepen our understanding of non-finitely generated invariant rings, which notably are counterexamples to Hilbert's fourteenth problem. To achieve this, we honed in on the finite generation ideal. Our main result from Chapter 3 is that the finite generation ideal of Daigle and Freudenburg's counterexample is $\mathfrak{f}_R = \sqrt{(\beta_0, \gamma_0, \delta_0)R^D}$, which is the radical of a finitely generated ideal containing all but finitely many generators of R^D. Coupled with Dufresne and Kraft's similar results on Roberts' example in [10], this prompts us to question whether all finite generation ideals arising from invariant rings shared these properties. However, as demonstrated in Chapter 5, these examples are heavily related and may not be indicative of general invariant rings. In Chapter 4, we worked on constructing a new, unrelated example to test these questions on, and applied methods which show promise in being generalised to compute generating sets for more new examples. And in Chapter 6 we asked if these observed properties of the finite generation ideal hold in greater generality, using monomial algebras to do so as they are far easier to work with.

There are two significant obstacles to obtaining more general results on the structure of non-finitely generated invariant rings; firstly, the task of computing a generating set which is a SAGBI-basis; secondly, using this SAGBI-basis to compute the finite generation ideal. Our approach in Chapters 4 and 6 can be seen as attempts to mitigate these issues. By focusing on the ρ-grading in Chapter 4, which relies on the maximum value of ρ from amongst the monomials appearing in a polynomial, we bake in a way of building a generating set that can be made a SAGBI-basis relatively easily. Our focus in Chapter 6 on monomial subalgebras, with the goal of passing these results to SAGBI-bases builds our understanding of computing the finite generation ideal using a SAGBI-basis. There are myriad questions left to answer along the way, but it is our hope that this thesis helps to lay the foundation for overcoming these obstacles.

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