# Equivariant Minimal Surfaces in Complex Hyperbolic Spaces

*Cordelia Webb*

PhD

University of York
Mathematics

September 2023

# Abstract

An equivariant minimal surface in complex hyperbolic space $\mathbb{CH}^n$ is a triple consisting of a minimal immersion (possibly branched) of the Poincaré disc into $\mathbb{CH}^n$ and two representations of the fundamental group of a closed oriented surface: one a Fuchsian representation into the isometry group of the disc, and the other a reductive representation into the isometry group of $\mathbb{CH}^n$. We require the minimal immersion to be equivariant with respect to these two representations.

We see how such surfaces have both a corresponding Higgs bundle and a harmonic sequence via the non-abelian Hodge correspondence. This provides an original application of global harmonic sequence theory. In order to do this, we adapt known results about the moduli space of Higgs bundles, develop both the local and global theory for $\mathbb{CH}^n$ harmonic sequences and apply the latter to the former in a new context.

We consider the critical submanifolds fixed under the $\mathbb{C}^*$-action of the Higgs field. By considering their invariants, we fully classify these subspaces of Hodge bundles, and make clear links back to other geometric measures such as curvature. We also establish necessary conditions for stability of these as Higgs bundles.

Finally, we see how the rest of the moduli space can be understood in terms of a Hodge bundle and an extension, before linking this back to invariants from harmonic sequences. We consider the limits of the $\mathbb{C}^*$-action, describing some explicitly. Applying the concept of the isotropy order from harmonic sequences to Higgs bundles, we end by examining the existence of a Higgs bundle of a given isotropy order. The main results prove the existence of $\mathbb{CH}^3$ Higgs bundles of all possible isotropy order and then allow us to find topological conditions for the existence of $\mathbb{CH}^n$ Higgs bundles of a given isotropy order under certain assumptions.

# Contents

# Acknowledgments

First and foremost, I would like to thank Dr Ian McIntosh for all of his immeasurable help and support over the last four years, both on and offline!

I am very grateful to the wider PhD community in York and in particular the extended Millfield family and Sunday diners. Thank you all for letting me feed you.

Ambroise, thank you for this thesis template, your amazing proof-reading skills and so very much more.

Jenny, thanks for the walks, talks and everything else.

Finally, to my parents and sisters, thanks for never saying no and letting me prattle on.

# Author's declaration

I declare that the work presented in this thesis, except where otherwise stated, is based on my own research carried out at the University of York and has not been submitted previously for any degree at this or any other university. Sources are acknowledged by explicit references.

# Introduction

This thesis brings together the differential geometry from harmonic sequence theory with the holomorphic geometry which underpins the study of Higgs bundles, all in the context of minimal surfaces in complex hyperbolic spaces $\mathbb{CH}^n$. This provides a new application of global harmonic sequence theory where only the local theory has been applied before.

Analogously to how a real hyperbolic space can be modelled as the symmetric space of orthogonal groups, a complex hyperbolic space is given by the quotient of projective unitary groups. This is the non-compact dual to complex projective space $\mathbb{CP}^n$ with the group of orientation preserving isometries given by the real Lie group $PU(n, 1)$.

As any minimal surface has zero mean curvature and non-compact globally symmetric spaces have non-positive Gaussian curvature, there can be no compact minimal surfaces in a non-compact symmetric space. Instead, the relevant class of minimal surfaces to consider in such a symmetric space is the equivariant minimal surfaces.

First considered in this context by Loftin and McIntosh in [48] and [49], an equivariant minimal surface is an equivalence class of triples $(f, c, \rho)$ where $f : \mathcal{D} \to \mathbb{CH}^n$ is a minimal immersion of the Poincaré disc $\mathcal{D}$ which intertwines the action of a Fuchsian representation $c : \pi_1\Sigma \to \mathrm{Aut}(\mathcal{D})$ with an indecomposable representation $\rho : \pi_1\Sigma \to \mathrm{Isom}(\mathbb{CH}^n)$. Here, we take $\pi_1\Sigma$ to be the fundamental group of a closed oriented surface $\Sigma$ of genus $g$ at least two. To date, detailed work on such surfaces has only been carried out for $\mathbb{RH}^3$, $\mathbb{RH}^4$ and $\mathbb{CH}^2$ equivariant minimal surfaces by Loftin and McIntosh [48, 49, 50]

The space of conjugacy classes $[c]$ is given by the Teichmüller space $\mathcal{T}_g(\Sigma)$ of $\Sigma$

and Corlette's results [22] can be applied to show that, for a fixed conjugacy class $[c]$, every reductive representation $\rho$ has a unique equivariant harmonic map $f$. The moduli space of such representations $\rho$ is the character variety $\mathcal{R}(\pi_1\Sigma, PU(n,1))$ of $\pi_1\Sigma$. Overall, this means the moduli space of equivariant minimal surfaces $\mathcal{M}(\Sigma)$ acquires an analytic structure via an embedding into $\mathcal{T}_g(\Sigma) \times \mathcal{R}(\pi_1\Sigma, PU(n,1))$.

The non-abelian Hodge correspondence was developed in the late 1980s by Hitchin [43] and Donaldson [24] for rank 2 bundle over a compact Riemann surface, and extended to the general case by Corlette [22] and Simpson [58]. It shows that the character variety $\mathcal{R}(\pi_1\Sigma, G)$ of a semi-simple Lie group $G$ is homeomorphic to the moduli space of polystable $G$-Higgs bundles $\mathcal{M}_{Higgs}$. In our case, this means every $\mathbb{CH}^n$ equivariant minimal surface of fixed $[c]$ has a corresponding $PU(n,1)$-Higgs bundle $(E = V \oplus \underline{\mathbb{C}}, \Phi)$, where $V$ is a rank $n$ vector bundle, $\underline{\mathbb{C}}$ is the trivial line bundle over $\Sigma$ and $\Phi : E \to E \otimes K$ is the Higgs field for $K$, the canonical bundle.

This background is further explored in Chapter 1 where details of the moduli space of $PU(n,1)$-Higgs bundles are also presented. In particular, connected components of $\mathcal{M}_{Higgs}$ are indexed by the Toledo invariant $\tau(\rho) \in \frac{2}{n+1}\mathbb{Z}$ whose modulus is bounded by the genus of $\Sigma$.

We also detail wider results about $PU(2,1)$-Higgs bundles by Gothen [35] and Xia [65] alongside more general facts about $PU(n,1)$-bundles from Bradlow, Garcia-Prada and Gothen's paper [14]. Finally, we construct two models of the tangent space to the moduli space of $PU(n,1)$-Higgs bundles.

The non-abelian Hodge correspondence also identifies a harmonic map with every representation in the character variety. This allows us to show that every equivariant minimal surface also corresponds to a harmonic sequence: that is, to vector bundle flags over $\Sigma$ from which we can derive a set of line bundles.

Chapter 2 introduces harmonic sequences. Also developed in the 1980s and early 1990s, harmonic sequences in complex projective spaces were first studied globally by Eells and Wood [25]. Erdem and Glazebrook [27] generalised some of these global results to the $\mathbb{CH}^n$ case while Bolton and Woodward [10] further developed the local theory for complex projective space, defining two sets of global differential forms: $\Gamma_i$ and $U_{i,j}$. These development were key in understanding harmonic maps from surfaces to compact symmetric spaces.

Here, we build on Erdem and Glazebrook in adapting Eells and Wood's global results from $\mathbb{CP}^n$ to the (non-compact) $\mathbb{CH}^n$ case. We also follow Bolton and Woodward to develop the local theory for $\mathbb{CH}^n$ and extend the definitions of $\Gamma_i$ and

$U_{i,j}$.

Higgs bundles and harmonic sequences have previously been brought together in Baraglia's thesis [5] in the specific context of cyclic Higgs bundles within the Hitchin component for certain Lie groups. In Baraglia's case, the harmonic sequences consist of holomorphic line bundles which are part of the holomorphic Higgs bundle whereas, in our case, we don't have a harmonic sequence consisting of holomorphic line bundles but instead can only form holomorphic flags.

In §1.6, we will see there is a natural $\mathbb{C}^*$-action on $\mathcal{M}_{Higgs}$ given by scaling the Higgs field:

$$\begin{aligned} \mathbb{C}^* \times \mathcal{M}_{Higgs} &\to \mathcal{M}_{Higgs} \\ (z, [(E, \Phi)]) &\mapsto [(E, z\Phi)]. \end{aligned} \tag{1}$$

Higgs bundles fixed under the circle action, given by considering scaling by $e^{i\theta}$, form critical submanifolds of an ensuing proper, perfect Morse-Bott function

$$\begin{aligned} g : \mathcal{M} &\to \mathbb{R} \\ [(E, \Phi)] &\mapsto \|\Phi\|^2 := \int_X |\Phi|^2 vol. \end{aligned} \tag{2}$$

These submanifolds are shown to consist of Hodge bundles which further decompose into $(V \oplus \underline{\mathbb{C}}, (\phi_1, \phi_2))$ where $V = V_1 \oplus V_2$, $\phi_1 : \underline{\mathbb{C}} \to V_1 \otimes K$ and $\phi_2 : V_2 \to \underline{\mathbb{C}} \otimes K$.

In Theorem 2.10, we explicitly construct the bijection between $PU(n,1)$-Hodge bundles and $\mathbb{CH}^n$ superminimal surfaces. Such surfaces have finite harmonic sequences from which we can form an orthogonal frame similar to the Frenet frame of a holomorphic curve. The term superminimal was first used by Bryant in [17] and such maps were the focus of Eells and Wood's work [25] but they have not previously been linked to the Higgs bundle theory.

In Lemma 2.12, we consider necessary stability conditions for such Hodge bundles in terms of quantities from the harmonic sequence. This is not a sufficient result as it is hindered by $\Phi$-invariant subbundles being of higher rank. However, it does allow us to bound various topological invariants of our moduli space. We end this section by further linking the differential forms to geometric quantities such as the Kähler angle.

In Chapter 3, we use the theory of holomorphic chains to show that the connected submanifolds of $PU(n,1)$-Hodge bundles are classified by the rank and degree of the vector subbundle $V_1$, and then calculate their Morse index and critical values.

Proposition 3.2 shows how non-Hodge $PU(n,1)$-Higgs bundles $(V \oplus \underline{\mathbb{C}}, \Phi)$ can be given in terms of a Hodge bundle with $V = V_1 \oplus V_2$, and an extension

$$0 \longrightarrow V_1 \longrightarrow V(\alpha) \longrightarrow V_2 \longrightarrow 0. \tag{3}$$

Indeed the Hodge bundle is the limit of the $\mathbb{C}^*$-scaling of $\Phi$ as $|z| \to \infty$. We use this to consider the tangent space to the downward Morse flow from a given critical submanifold.

Next, we turn to the other limit as $|z| \to 0$ which begins to explore the boundary of our moduli space. This uses the Jordan-Hölder filtrations to generalise a result in [50] although a full result is not obtained due to stability conditions of higher rank vector bundles.

In the final chapter, we consider Higgs bundles of different isotropy orders. This is a well-known invariant of the harmonic sequence (see for example, [25, 10]) which relates to the orthogonality of the harmonic sequence. Building on the work of Wood [63], we define a family of functions $Q_i$ on $\mathcal{M}_{Higgs}$ which can be used to determine the isotropy order.

There are several further questions to be answered regarding these functions including relatively basic properties of $Q_i$ and how the isotropy order relates to the $\mathbb{C}^*$-scaling of the Higgs field. While some effort was made towards this latter problem, it was hindered by having to work on the level of individual Higgs bundles about which less is currently known.

Section 4.1 considers the function $Q_3$ in particular, which allows us to prove the existence of $\mathbb{CH}^3$ superconformal surfaces in Theorem 4.4. Previously, non-superminimal surfaces have been hard to find, particularly in non-compact spaces. However, we find the necessary and sufficient topological conditions for their existence.

In order to prove the existence of Higgs bundles of a more general isotropy order, we use $Q_3$ as a base case. By restricting our attention to the Higgs bundles with isotropy order at least $i-1$, we apply the implicit function theorem to iteratively build up a series of subspaces $\mathcal{U}_i$ consisting of Higgs bundles whose isotropy order is at least $i$. By applying certain assumptions, we prove Theorem 4.6 which enables us to calculate the codimensions of these spaces and hence find topological conditions for the existence of a Higgs bundles of a given isotropy order.

# — 1 —

# Background

We begin by detailing some of the standard notation following conventions found in [60, 37, 48].

For a holomorphic vector bundle $E$ over a compact oriented Riemann surface $M$ with genus $g \geq 2$ and a complex structure $J$, let $\mathcal{E}(U, E)$ be the space of sections of $E$ over $U$ and $\mathcal{E}^k(U) = \mathcal{E}(U, \wedge^k T^*_{\mathbb{C}} M)$ be the space of $k$-differential forms over a local open subset $U$ of $M$. Similarly, let $\mathcal{O}(U, E)$ denote the spaces of holomorphic sections.

Complexified cotangent spaces split into two eigenspaces with eigenvalues $\pm i$ which we denote $T^*_{\mathbb{C}} M = T^*_{1,0} M \oplus T^*_{0,1} M$. This carries over to differential forms and we write $\mathcal{E}^{p,q}(U) = \mathcal{E}(U, \wedge^{p,q} T^*_{\mathbb{C}} M)$ where $\wedge^{p,q} T^*_{\mathbb{C}} M$ is the subspace of $\wedge T^*_{\mathbb{C}} M$ generated by $u \wedge v$ for $u \in \wedge^p T^*_{1,0} M$ and $v \in \wedge^q T^*_{0,1} M$. In particular, there is a subset of holomorphic and antiholomorphic differential forms which we denote $\Omega^p(U) \subset \mathcal{E}^{p,0}(U)$ and $\overline{\Omega}^q(U) \subset \mathcal{E}^{0,q}(U)$ respectively.

The **canonical line bundle** $K$ is the highest exterior power of the holomorphic cotangent bundle: $K = \wedge T^*_{1,0} M$ for $M$ a Riemann surface. Note this means holomorphic sections of $K$ are holomorphic 1-forms: $\mathcal{O}(U, K) = \Omega^1(U)$.

## 1.1 SYMMETRIC SPACES

In order to understand $\mathbb{CH}^n$, we first consider symmetric spaces following Cheeger and Ebin [20] and Helgason [41].

A connected Riemannian manifold $M$ is **locally symmetric** if, for all points $p \in M$, there exists an isometric involution $\sigma_p$ such that $\sigma_p(p) = p$ and geodesics through $p$ are reversed. $M$ is **(globally) symmetric** if $\sigma_p$ extends to a global isometry $\sigma : M \to M$.

Taking $M$ to be globally symmetric, if $I(M)$ is the group of isometries on $M$, then $I(M)$ acts transitively on $M$ and $M \simeq G/H$ where $G$ is the connected component containing the identity of $I(M)$ and $H \subset G$ is the **isotropy (stabiliser) subgroup** of $p$.

Let $G$ be a connected Lie group and $H \subset G$ a closed subgroup, then $G/H = N$ is a space of cosets $\{gH\}$ with $\pi : G \to N$ such that $g \mapsto [gH]$. We call $G/H$ the **homogeneous space**. The isometry groups of symmetric spaces are Lie groups so all symmetric spaces are homogeneous.

In general, the tangent space to a homogeneous space $N$ at the point $[H]$ can be identified with the quotient of the corresponding Lie algebras so $T_{[H]}N \cong \mathfrak{g}/\mathfrak{h}$. $\mathrm{Ad}_H$ and $\mathrm{ad}_{\mathfrak{h}}$ are invariant on $\mathfrak{h}$ so act naturally on $\mathfrak{g}/\mathfrak{h}$ and, by considering the Lie bracket, we have $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$.

Moreover, for a symmetric space, as $\sigma^2 = \mathrm{Id}$ then $d\sigma^2 = \mathrm{Id}$ so $d\sigma$ has eigenvalues $\pm 1$ giving the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ where $\mathfrak{h}$ is the Lie algebra of $H$ and $\mathfrak{p}$ is $\mathrm{Ad}_H$-invariant. Any homogeneous space which can be decomposed like this is called **reductive** and has the properties that $\mathfrak{p} \cong \mathfrak{g}/\mathfrak{h} \cong T_{[H]}N$ and $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$.

Finally, it can be shown that $G/H$ is symmetric if and only if $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$.

A Riemannian globally symmetric space is **irreducible** if it cannot be written as the product of two or more symmetric spaces. Irreducible symmetric spaces are classified as one of three types: compact, non-compact or flat, with non-negative, non-positive or zero sectional curvature respectively.

As the involution $\sigma$ splits $\mathfrak{g}$ into $\mathfrak{h} \oplus \mathfrak{p}$, we define a Lie subalgebra $\mathfrak{g}_C = \mathfrak{h} + i\mathfrak{p} \subset \mathfrak{g}^{\mathbb{C}}$ where $\mathfrak{g}^{\mathbb{C}}$ is the complexification of the Lie algebra $\mathfrak{g}$. Note $\mathfrak{g}_C^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}$. This corresponds to a Lie group $G_C \subset G^{\mathbb{C}}$. The Killing form $B$ on $\mathfrak{g}$ induces a Killing form $\hat{B}$ on $\mathfrak{g}_C$ with $\hat{B}|\mathfrak{p} = -B|\mathfrak{p}, \hat{B}|\mathfrak{h} = B|\mathfrak{h}$ and $\hat{B}(\mathfrak{p}, \mathfrak{h}) = 0$.

Any real semisimple Lie group has at least one such involution: the **Cartan involution** $\sigma$ for which $B(X, \sigma Y)$ is negative definite. Any two involutions which satisfy this property are necessarily equivalent up to inner automorphism. This gives a unique decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ known as the **Cartan decomposition** where the corresponding Lie group $H$ is a maximal compact subgroup.

The **rank** of a symmetric space is the dimension of any maximal abelian subspace of $\mathfrak{p}$. Geometrically, this is the maximum dimension of a flat totally geodesic submanifold so is at least one.

## 1.2 COMPLEX HYPERBOLIC SPACE

We now let $\mathbb{C}^{n,1}$ denote $\mathbb{C}^{n+1}$ equipped with the Hermitian metric:

$$\langle u, v\rangle = \sum_{i}^{n} u_i \overline{v}_i - u_{n+1}\overline{v}_{n+1}. \tag{1.1}$$

Following the notation in [48], this enables us to decompose the non-zero elements of $\mathbb{C}^{n,1}$ into three subsets:

$$\begin{aligned} \mathbb{C}^{n,1}_{-} &= \left\{v \in \mathbb{C}^{n,1} \mid \langle v, v\rangle < 0\right\}, \\ \mathbb{C}^{n,1}_{0} &= \left\{v \in \mathbb{C}^{n,1}, v \neq 0 \mid \langle v, v\rangle = 0\right\}, \text{ and} \\ \mathbb{C}^{n,1}_{+} &= \left\{v \in \mathbb{C}^{n,1} \mid \langle v, v\rangle > 0\right\}. \end{aligned} \tag{1.2}$$

The standard projective map $\mathbb{P} : \mathbb{C}^{n,1}\backslash\{0\} \to \mathbb{CP}^n$ defines two non-zero vectors as equivalent if and only if they are non-zero complex scalar multiples of each other. By properties of the metric, the images of the above subsets under $\mathbb{P}$ remain disjoint.

The projective model of **complex hyperbolic space** defines $\mathbb{CH}^n$ as $\mathbb{PC}^{n,1}_{-}$ with boundary $\mathbb{PC}^{n,1}_{0}$.

Note that the group $U(n,1) \subset GL(n+1,\mathbb{C})$ preserves the metric given in equation (1.1) and acts transitively on $\mathbb{C}^{n,1}_{-}$. It has a maximal compact subgroup $U(n) \times U(1)$ and center $\hat{U}(1)$. We therefore take $PU(n,1)$ as the group of orientation preserving isometries of $\mathbb{CH}^n$ following [19]. As in [14], we then have that the following is an exact sequence:

$$1 \to \hat{U}(1) \to U(n,1) \to PU(n,1) \to 1. \tag{1.3}$$

Following the previous section, we can also view $\mathbb{CH}^n$ as the rank 1 non-compact symmetric space with $G = PU(n,1)$ and $H = P(U(n) \times U(1))$.

We have $\mathfrak{g} \simeq \mathfrak{su}(n,1)$ and the Cartan decomposition in this case has a block decomposition:

$$\begin{pmatrix} A & u \\ u^\dagger & a \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & u \\ u^\dagger & 0 \end{pmatrix} \tag{1.4}$$

where $A \in \mathfrak{u}(n)$, $a = -tr(A)$, $u \in \mathbb{C}^n$ and $u^\dagger$ is the Hermitian transpose.

Unlike real hyperbolic space, complex hyperbolic space does not have constant sectional curvature, although the holomorphic sectional curvature is constant [19, §2.2]. We normalise the holomorphic sectional curvature to -4 as in [48].

Note $PU(n,1)$ is a real Lie group [14]. More details on both complex hyperbolic geometry and $PU(n,1)$ can be found in [54] and [19].

## 1.3 EQUIVARIANT MINIMAL SURFACES

We now define equivariant minimal surfaces following Loftin and McIntosh in [48], [49] and [50]. Recall that we let $\Sigma$ be a smooth surface of genus $g \geq 2$.

First take $c : \pi_1(\Sigma) \to PU(1,1)$ as a **Fuchsian representation**, meaning it is discrete and faithful. In particular, $c$ is injective, $c(\pi(\Sigma))$ is a discrete subgroup of $PU(1,1)$ and $PU(1,1)/c(\pi(\Sigma))$ is compact.

As Goldman details in [32, §4.2], such a representation also arises as the holonomy of a hyperbolic structure on $\Sigma$: a hyperbolic structure is equivalent to a Riemannian metric of negative constant curvature and has an underlying conformal structure $\gamma$. Moreover, as $g \geq 2$, the uniformatisation theorem shows that this conformal structure is unique.

As detailed in [32, §5], the moduli space of conjugacy classes $[c]$ can then be identified with the **Teichmüller space** $\mathcal{T}_g$ of $\Sigma$.

Turning to the other parts of an equivariant minimal surface, we call a $G = PU(n,1)$ representative **reductive** if $\mathrm{Ad} \circ \rho : \pi_1\Sigma \to \mathrm{Aut}(\mathfrak{g})$ is completely reducible for Ad the adjoint representation of $PU(n,1)$ on $\mathfrak{g} = \mathfrak{pu}(n,1)$ (see, for example, [34]).

This enables us to define another useful space: the moduli space of reductive representations $\rho$ up to conjugacy is called the **character variety** $\mathcal{R}(\pi_1\Sigma, PU(n,1))$. The subset of irreducible representations is open ([14, §3]) and a reductive representation is called **indecomposable** if its image does not lie in a proper Lie subgroup ([48]).

To continue defining an equivariant minimal surface, let $\rho : \pi_1(\Sigma) \to PU(n,1)$ be an indecomposable representation and $f : \mathbb{CH}^1 \to \mathbb{CH}^n$ a harmonic and (weakly) conformal map. Taking $f$ to be conformal means $df_p : T_p\mathbb{CH}^1 \to T_p\mathbb{CH}^n$ is conformal for all $p \in \mathbb{CH}^1$ and **weakly conformal** if there exists a point $p \in \mathbb{CH}^1$ where $df_p$ is zero, in which case $p$ is called a **branch point** As $f$ is also harmonic, its image is minimal away from any branch points. Hence we can view $f$ as a minimal (possibly branched) immersion with isolated branch points (see [63]). In particular, the induced metric $f^*g$ on $\mathbb{CH}^1$, where $g$ is the metric on $\mathbb{CH}^n$, is conformally equivalent to the hyperbolic metric $\mu$ on $\mathbb{CH}^n$ so $f^*g = e^u\mu$ for some $u : \Sigma \to \mathbb{R}$ away from branch points.

This triple $(f, c, \rho)$ is called **equivariant** if they intertwine, i.e. $f \circ c(\delta) = \rho(\delta) \circ f$ for all $\delta \in \pi_1(\Sigma)$.

We only wish to consider such triples up to isometry so we have an equivalence relation: $(f, c, \rho) \sim (f', c', \rho')$ when there exists $\gamma \in PU(1,1)$ and $\alpha \in PU(n,1)$ such that for all $z \in \mathbb{CH}^1$,

$$f'(z) = \alpha f(\gamma^{-1} z), \quad c' = \gamma c \gamma^{-1}, \quad \rho' = \alpha \rho \alpha^{-1}. \tag{1.5}$$

This then gives equivalence classes $[f, c, \rho]$ and the set of all these classes form the **moduli space of equivariant minimal surfaces** $\mathcal{M}(\Sigma)$.

Corlette and Donaldson show in [22] and [24] that, for a fixed conjugacy class $[c]$, every reductive representation $\rho$ has a unique equivariant harmonic map $f$ into $\mathbb{CH}^n$ (also see [59]). If $\rho$ is decomposable, then its image lies in a proper subgroup of $PU(n,1)$ and the same $f$ can be equivariant with respect to several $\rho$: $\rho$ being indecomposable is equivalent to $f$ being linearly full. When $\rho$ is reductive, Corlette also gives that we can view $f$ as a minimal immersion $\Sigma \to \mathbb{CH}^n / \rho(\pi_1 \Sigma)$.

Taken altogether, this then means we have the following embedding:

$$\begin{aligned} \mathcal{M}(\Sigma) &\to \mathcal{T}_g \times \mathcal{R}(\pi_1 \Sigma, PU(n,1)) \\ [f, c, \rho] &\mapsto ([c], [\rho]). \end{aligned} \tag{1.6}$$

In particular, this embedding endows $\mathcal{M}(\Sigma)$ with a topology.

For the rest of this thesis, we fix $\Sigma$ with a conjugacy class $[c]$.

## 1.4 EQUIVARIANT MINIMAL SURFACES AND FLAT BUNDLES

Given an equivariant minimal surface with a fixed conjugacy class $[c]$, there is an associated flat bundle which is important for both the Higgs bundle and harmonic sequence perspectives that we go on to develop.

Following [22], [34] and [31] for the general case where $G$ is a real, reductive, connected Lie group, while applying ideas from [14] to the $G = PU(n,1)$ case, we show how given $\rho$, there is a corresponding flat bundle and then, from this, how we also have a harmonic map $f$. This forms the first part of the non-abelian Hodge correspondence.

We first follow [34] to define a (smooth) **principal $G$-bundle** $P$ on $\Sigma$ as a smooth fibre bundle $\pi : P \to \Sigma$ with a $G$-action which is free and transitive on each fibre.

### 1.4.1 Representations and Flat Connections

We wish to first consider representations $\mathrm{Hom}(\pi_1\Sigma, G) = \{\rho : \pi_1\Sigma \to G\}$. As $\pi_1\Sigma$ has $2g$ generators $\{A_i, B_i\}$, there is an induced embedding:

$$\begin{aligned} \mathrm{Hom}(\pi_1\Sigma, G) &\hookrightarrow G^{2g} \\ \rho &\mapsto (\rho(A_1), \rho(B_1), \cdots, \rho(A_g), \rho(B_g)), \end{aligned} \tag{1.7}$$

from which $\mathrm{Hom}(\pi_1\Sigma, G)$ inherits the subspace topology.

$G$ acts on $\mathrm{Hom}(\pi_1\Sigma, G)$ by conjugation: $g \cdot \rho = g\rho g^{-1}$. When $G$ is abelian, this action is trivial but more generally, if the $G$-action is not proper and $G$ is non-compact, then $\mathrm{Hom}(\pi_1\Sigma, G)/G$ may not be Hausdorff. Even if $G$ is compact and acts properly, if this action is not free, there may still be singularities in the quotient space [16].

To remedy this, we consider the subset $\mathrm{Hom}^{\mathrm{red}}(\pi_1\Sigma, G)$ of reductive representatives: those representations that are the direct sum of irreducible representation. By the definition we saw in §1.3, $\mathrm{Hom}^{\mathrm{red}}(\pi_1\Sigma, G)/G = \mathcal{R}(\pi_1\Sigma, G)$ is the **character variety**. Note it inherits the natural quotient topology.

Turning to the set of flat connections on a principal $G$-bundle $P$ over $\Sigma$, we let $\mathfrak{U}(P)$ be the set of all connections on $P$, and $D_0 \in \mathfrak{U}(P)$ be a fixed connection. Any connection $D \in \mathfrak{U}(P)$ can be written uniquely as $D = D_0 + \eta$ where $\eta \in \mathcal{E}^1(\mathrm{ad}P)$ for $\mathrm{ad}P$ the adjoint bundle to $P$. This means $\mathfrak{U}(P)$ is an affine space modelled on $\mathcal{E}^1(\mathrm{ad}P)$.

Let $\mathcal{F}(P) \subseteq \mathfrak{U}(P)$ be the space of flat connections on $P$. If $D_0$ is flat then the curvature with respect to $D$ is

$$\Theta(D) = (D_0 + \eta)(D_0 + \eta) = D_o\eta + \frac{1}{2}[\eta, \eta]. \tag{1.8}$$

Denoting the bundle of automorphisms of $P$ as $\mathrm{Aut}(P)$, the **gauge group** $\mathcal{G}$ is the automorphism group $\mathcal{E}^0(\Sigma, \mathrm{Aut}(P))$ and acts on $\mathfrak{U}(P)$ via conjugation $g \cdot D = gDg^{-1}$ and thus $\Theta(g \cdot D) = g\Theta(D)g^{-1}$. In particular, $\mathcal{F}(P)$ is preserved under gauge transformation enabling a moduli space $\mathcal{F}(P)/\mathcal{G}$ to be defined.

It is well known that a flat connection corresponds to a class of holonomy representations as detailed in [46, §2.10]. Conversely, letting $\tilde{\Sigma}$ be a universal covering space of $\Sigma$ with deck group $\pi_1\Sigma$, any $\rho \in \mathrm{Hom}(\pi_1\Sigma, G)$ acts on the trivial bundle over $\tilde{\Sigma}$ by:

$$\begin{aligned} \pi_1\Sigma \times (\tilde{\Sigma} \times G) &\to \tilde{\Sigma} \times G \\ (\gamma, \tilde{x}, v) &\mapsto (\gamma\tilde{x}, \rho(\gamma)v). \end{aligned} \tag{1.9}$$

As $\pi_1\Sigma$ acts properly and freely on $\tilde{\Sigma}$, we get a $G$-principal bundle $P_\rho$ over $\Sigma$:

$$P_\rho : (\tilde{\Sigma} \times G)/\pi_1\Sigma \to \tilde{\Sigma}/\pi_1\Sigma = \Sigma.$$

Since $P_\rho$ is a trivial bundle over $\Sigma$, it is flat and, as $\pi_1\Sigma$ has a discrete topology, this naturally defined flat connection is also flat over $\Sigma$.

A connection is called **reductive** if its holonomy representation is itself a reductive representation. We can then consider the subspace $\mathcal{F}(P)^{\mathrm{red}} \subseteq \mathcal{F}(P)$ of flat reductive representations.

If $D_1$ and $D_2$ are both flat connections corresponding to representations $\rho_1$ and $\rho_2$, then $[D_1] = [D_2]$ in $\mathcal{F}/\mathcal{G}$ if and only if there exists $g \in G$ such that $\rho_1 = g\rho_2 g^{-1}$. This then gives $\mathcal{R}(\pi_1\Sigma, G) \cong \mathcal{F}^{\mathrm{red}}/\mathcal{G}$ where $\mathcal{F}^{\mathrm{red}} \subset \mathcal{F}$.

In particular, considering the case of $G = PU(n,1)$ more closely, a flat $PU(n,1)$-bundle is equivalent to a projective equivalence class of smooth rank $n+1$ bundles. Moreover, we let

$$\Gamma = \langle A_i, B_i, J \mid \Pi_{i=1}^{g}[A_i, B_i] = J, [A_i, J] = 1 = [B_i, J]\rangle, \tag{1.10}$$

to form the **universal central extension**:

$$0 \to \mathbb{Z} \to \Gamma \to \pi_1\Sigma \to 0. \tag{1.11}$$

Note $\mathbb{Z}$ is isomorphic to the subgroup generated by $J$.

For a representation $\rho : \pi_1\Sigma \to PU(n,1)$, and recalling equation (1.3), Atiyah and Bott detail in [3, §6] how $\rho$ can be lifted to a (non-unique) representation $\hat{\rho} : \Gamma \to U(n,1)$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Gamma & \longrightarrow & \pi_1\Sigma & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \hat{\rho} & & \downarrow \rho & & \\ 0 & \longrightarrow & U(1) & \longrightarrow & U(n,1) & \longrightarrow & PU(n,1) & \longrightarrow & 0. \end{array} \tag{1.12}$$

Following [3] and [14], we can identify the space of reductive representations $\hat{\rho}$ with the moduli space of connections with constant central curvature on a $U(n,1)$ bundle over $\Sigma$.

### 1.4.2 Flat Connections and Harmonic Metrics

We next examine the link between connections and harmonic metrics for which we fix a metric on $\Sigma$. Again, we continue to primarily follow [34].

In a principal $G$-bundle $P$, we can consider a metric $h$ as a reduction of the structure group $G$ to a maximal compact subgroup $H \subset G$. This means we can view $h$ as a section $h : \Sigma \to P/H$. Fibres of this bundle are isomorphic to the symmetric space $G/H$ and the Cartan decomposition gives $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ with $\mathfrak{p} = T_V(P_H)$, the vertical tangent bundle of $P_H$.

Given a flat bundle $P$, we have:

$$P/H = \tilde{\Sigma} \times_\rho (G/H), \tag{1.13}$$

which means a metric $h$ corresponds to a map

$$\hat{h} : \tilde{\Sigma} \to G/H \tag{1.14}$$

such that $\hat{h}(\gamma \cdot x) = \gamma \cdot \hat{h}(x)$ for all $\gamma \in \pi_1\Sigma$ and $x \in \tilde{\Sigma}$.

This enables us to define the **energy** of a hermitian metric $h$ as

$$e(h) = \int_\Sigma \|d\hat{h}\|^2 \, vol. \tag{1.15}$$

Note this is well-defined as we have equipped $\Sigma$ with the conjugacy class $[c]$. Metrics in a flat $G$-bundles corresponding to critical points of the energy are called **harmonic**.

In terms of flat connections, let $i : P_H \hookrightarrow P$ be the principal $H$-bundle obtained by the reduction of structure group defined by $h$ and $D$ a flat connection on $P$. The Cartan decomposition gives $i^*D = D_H + \phi$ where $D_H$ is a connection on $P_H$ (so compatible with the metric $h$) and $\phi = dh \in \mathcal{E}^1(P_H, \mathfrak{p})$ is a section of $T_V(P_H)$.

We can calculate the critical points of the energy by taking a deformation of the metric $h$ as $h_t = \exp{(ts)}h$ where $s \in \mathcal{E}(\Sigma, P_H(\mathfrak{p}))$. We then have

$$\frac{d}{dt} e(h_t)\bigg|_{t=0} = \frac{d}{dt} \int_\Sigma \|dh_t\|^2 vol\bigg|_{t=0} = \langle dh, d_{D_H} s \rangle. \tag{1.16}$$

This means $h$ is a harmonic metric if and only if $d^*_{D_H}\phi = 0$ where $d^*$ is the adjoint operator of $d$ with respect to the fixed Kähler metric on $\Sigma$.

The curvature of $i^*D$ can be expressed in terms of $(D_H, \phi)$ as

$$\Theta(D_H) + d_{D_H}\phi + \frac{1}{2}[\phi, \phi] = 0. \tag{1.17}$$

Considering the $\mathfrak{h}$ and $\mathfrak{p}$ parts separately, we get:

$$\begin{aligned} d_{D_H}\phi &= 0, \\ \Theta(D_H) + \frac{1}{2}[\phi,\phi] &= 0, \text{ and} \\ d^*_{D_H}\phi &= 0, \end{aligned} \tag{1.18}$$

where we have also required $h$ to be harmonic for the third condition. A bundle $(P_H, D_H, \phi)$ which satisfies all three of these conditions is called a **harmonic bundle**.

Returning to flat connections, we have:

**Theorem 1.1** (Corlette-Donaldson Theorem - [34])**.** *A flat bundle $P \to \Sigma$ corresponding to a representation $\rho : \pi_1\Sigma \to G$ admits a harmonic metric if and only if $\rho$ is reductive.*

Let $\mathcal{U} \subset \mathcal{G}$ be the gauge group for $P_H$. As it preserves the metric, $\mathcal{U}$ is unitary. The Corlette-Donaldson theorem gives that for any projectively flat connection $D$ on $P$, there exists a gauge transformation $g \in \mathcal{G}$ such that $gD = D_H + \phi$ and $(P_H, D_H, \phi)$ is a harmonic bundle. Consequently,

$$\{(P_H, D_H, \phi) \mid (P_H, D_H, \phi) \text{ harmonic}\}/\mathcal{U} \cong \mathcal{F}/\mathcal{G}.$$

In the $PU(n,1)$ case, we want to consider bundles with constant central curvature $\lambda(\hat{\rho})$ which means the harmonic bundle equations we need consider are:

$$\begin{aligned} d^*_{D_H}\phi = 0 &= d_{D_H}\phi, \text{ and} \\ \Theta(D_H) + \frac{1}{2}[\phi,\phi] &= \lambda. \end{aligned} \tag{1.19}$$

## 1.5 HIGGS BUNDLES AND STABILITY

Still following [34], [14], [61] and [31], we now turn to Higgs bundles.

A **Higgs bundle** over a compact Riemann surface $\Sigma$ is a pair $(E, \Phi)$ where $E \to \Sigma$ is a holomorphic vector bundle, and $\Phi : E \to E \otimes K$ is a sheaf homomorphism known as the **Higgs field**, where $K$ is the canonical bundle. In particular, we can view $\Phi$ as an element of $H^0(\Sigma, \mathrm{End}(E) \otimes K)$.

For general vector bundles, we define the **slope** as $\mu(E) := \deg(E)/\mathrm{rank}(E)$ where $\deg(E)$ is the degree of $E$. $E$ is **stable** if $\mu(F) < \mu(E)$ for all proper subbundles $F$,

**semistable** if this is not a strict inequality and **polystable** if $E$ is isomorphic to the direct sum of stable vector bundles of equal slope.

A Higgs bundle $(E, \Phi)$ is subsequently stable if $\mu(F) < \mu(E)$ for all $\Phi$-invariant holomorphic subbundles $F \subseteq E$, where $F$ is **$\Phi$-invariant** if $\Phi(F) \subseteq F \otimes K$. Semistability and polystability are defined analogously. Note this is a weaker condition than vector bundle stability.

Hitchin showed in [43] that a Higgs bundle $(E, \Phi)$ is polystable if and only if it satisfies the (projectively flat) **Hitchin Equations**:

$$\begin{aligned} \overline{\partial}_E \Phi &= 0, \text{ and} \\ \Theta + [\Phi, \Phi^*] &= -i\mu I_E \omega, \end{aligned} \tag{1.20}$$

where $\mu = \mu(E)$, $I_E$ is the identity endomorphism of $E$ and $\omega$ is the Kähler form of $\Sigma$ equipped with conjugacy class $[c]$ and normalised to have volume $2\pi$. The right hand side of this second equation is known as the **central curvature** of $E$.

There are two descriptions of a $PU(n, 1)$-Higgs bundles which we will use.

In the first, we follow [34] to set $G = PU(n, 1)$ and recall from §1.2 that the maximal compact subgroup is $H = P(U(n) \times U(1))$. This gives the Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$. We can then take $(E', \Phi)$ to be a $PU(n, 1)$-Higgs bundle where $E' \to \Sigma$ is a holomorphic principal $H^{\mathbb{C}}$ bundle and $\Phi \in H^0(\Sigma, E(\mathfrak{p}^{\mathbb{C}}) \otimes K)$.

Note that as $H^{\mathbb{C}} = P(GL(n) \times GL(1))$, we also get an alternative description of a $PU(n, 1)$-Higgs bundle as a projective equivalence class of split bundles $V \oplus \underline{\mathbb{C}}$ and a Higgs field $\Phi$ interpreted as a holomorphic one-form with values in $\mathrm{Hom}(\underline{\mathbb{C}}, V) \oplus \mathrm{Hom}(V, \underline{\mathbb{C}})$.

As discussed in [14, §3.2], a $U(n, 1)$-Higgs bundle $(\hat{E}, \Phi)$ is a holomorphic $GL(n) \times GL(1)$-bundle $\hat{E} = V' \oplus L$, where $L$ is a holomorphic line bundle and $V'$ a rank $n$ holomorphic bundle, with an off-diagonal Higgs field $\Phi$ of the form $\Phi = \begin{pmatrix} 0 & \phi_1 \\ \phi_2 & 0 \end{pmatrix}$ where $\phi_1 \in H^0(\Sigma, \mathrm{Hom}(L, V) \otimes K)$ and $\phi_2 \in H^0(\Sigma, \mathrm{Hom}(V, L) \otimes K)$.

Following on from the end of §1.4.1, any flat $PU(n, 1)$-bundle lifts to a non-unique $U(n, 1)$-bundle $V' \oplus L$ with a projectively flat connection with constant central curvature $\lambda(\hat{\rho})$ in line with [31].

Two such $U(n, 1)$-bundles will result from a lift of the same $PU(n, 1)$-bundle if and only if there exists a line bundle with a unitary connection of constant curvature such that twisting one $U(n, 1)$-bundle by the line bundle, gives the second. We choose the $U(n,1)$ bundle twisted by $L^*$ meaning $E = V \oplus \underline{\mathbb{C}}$ where $V \simeq V' \otimes L^*$. Note that we then have $\deg(V) = \deg(E)$.

## 1.6 NON-ABELIAN HODGE CORRESPONDENCE

We can now consider the rest of the non-abelian Hodge correspondence and detail the association between the character variety and the moduli space of polystable Higgs bundles with the aid of [34], [61] and [31].

Let $E$ be the associated flat vector bundle to the principal bundle $P$. We first consider holomorphic structures on $E$. Recall that $E$ is equipped with a complex structure $J$ for which the bundle projection is holomorphic and there exist local holomorphic sections which can be locally trivialised.

Let $\overline{\partial}_E : \mathcal{E}^0(E) \to \mathcal{E}^{0,1}(E)$ be a linear operator satisfying a Leibniz rule:

$$\overline{\partial}_E(f\sigma) = (\overline{\partial}f)\sigma + f\overline{\partial}_E(\sigma) \tag{1.21}$$

for $f \in \mathcal{C}^\infty(\Sigma, \mathbb{C})$ and $\sigma \in \mathcal{E}^0(E)$. This extends to $\overline{\partial}^q : \mathcal{E}^{p,q}(E) \to \mathcal{E}^{p,q+1}(E)$ and, when $E$ is integrable, $\overline{\partial}^{q+1} \circ \overline{\partial}^q = 0$.

Moreover, giving an alternative proof to a result from Koszul–Malgrange in [47], Atiyah and Bott show in [3, p. 555] that, over a complex surface, the (0,1) part of any connection gives a holomorphic structure.

Alternatively, we have

**Theorem 1.2** (Newlander-Nirenberg theorem [4])**.** *Given a complex manifold $\Sigma$ and a smooth vector bundle $E$ with connection $A$ whose curvature is of type (1,1), then $E$ has a natural holomorphic structure with a unique (1,0) hermitian connection given by $A$.*

In particular, holomorphic transition functions can be defined by solving $\overline{\partial}_E s = 0$ where $s \in \mathcal{E}^0(U_i)$, so $\overline{\partial}_E$ gives $E$ a holomorphic structure.

On the other hand, two holomorphic structures are defined to be equivalent if the operators which they are given by are gauge equivalent. That is, there exists a $g \in \mathcal{G}$ such that $\overline{\partial}'_E = g^{-1}\overline{\partial}_E g$. For a given smooth bundle $E$, the set of holomorphic structures on $E$ therefore forms a moduli space itself: $\mathcal{D}(E)/\mathcal{G}$ where $\mathcal{D}(E)$ is the space of $\overline{\partial}_E$ operators on $E$. Furthermore, an arbitrary holomorphic bundle can be given by $(E, [\overline{\partial}_E])$, where $[\overline{\partial}_E]$ is a class of $G$-equivalent $\overline{\partial}$ operators.

With respect to a given operator $\overline{\partial}_0$, an arbitrary $\overline{\partial}_E = \overline{\partial}_0 + \Psi$ where $\Psi \in \mathcal{E}^{0,1}(\Sigma)$ is an endomorphism of $E$. $\mathcal{D}(E)$ is therefore an affine space. The gauge group $\mathcal{G}$ acts on $\mathcal{D}(E)$ by conjugation: $g \cdot \overline{\partial}_E = g\overline{\partial}_E g^{-1}$. As with representations, this space is generally non-Hausdorff but by restricting to stable holomorphic bundles, we overcome this problem.

We finally consider the moduli space of Higgs bundles $\mathcal{M}_{Higgs}$ given by $(E, \overline{\partial}_E, \Phi)$. Here, the $\mathcal{G}$ action is given by $g \cdot (\overline{\partial}_E, \Phi) \mapsto (g\overline{\partial}_E g^{-1}, g\Phi g^{-1})$. As $\overline{\partial}_E$ represents a holomorphic structure, to have a Hausdorff quotient space, as before, we require stable Higgs bundles. For further restriction, we need to understand the action on the Higgs field.

Bringing these ideas together, the **Hitchin-Simpson correspondence** [34, Thm 3.8] identifies the moduli space of Higgs bundles with the moduli space of harmonic metrics. In our case, the Hitchin-Simpson correspondence gives:

**Theorem 1.3** (Prop 3.9, [14]). *Let $(E, \Phi)$ be a $U(p,1)$-Higgs bundle with $E = V \oplus \underline{\mathbb{C}}$ and $\Phi = \begin{pmatrix} 0 & \phi_1 \\ \phi_2 & 0 \end{pmatrix}$. Then $(E, \Phi)$ is polystable if and only if it admits a harmonic hermitian metric such that $E = V \oplus \underline{\mathbb{C}}$ is an orthogonal decomposition and Hitchin's equations* (1.20) *are satisfied.*

Overall, given an equivariant minimal surface $(f, c, \rho)$, $[\rho] \in \mathcal{R}(\pi_1 X, PU(n,1))$ defines a projectively flat $PU(n,1)$-Higgs bundle where the Higgs field is Higgs field given by the $(1,0)$-part of the derivative of $f$. Therefore, for each choice of conjugacy class $[c]$, the non-abelian Hodge correspondence provides a homeomorphism between the character variety and the moduli space of polystable $PU(n,1)$-Higgs bundles over $\Sigma$ equipped with the conjugacy class $[c]$.

To be more explicit: given $[\rho]$, we have a class of projectively equivalent $\mathbb{C}^{n,1}$ bundles $E$ each equipped with a projectively flat connection $\nabla$. The immersion $f$ defines a line subbundle $L \subset E$ which further defines a splitting $E = L \oplus L^{\perp}$. This splitting then defines a bundle automorphism $\sigma$ such that $\sigma|L^{\perp} = 1$ and $\sigma|L = -1$, which further decomposes the connection $\nabla$ as $\nabla_E + \Psi$ for $\nabla_E = \frac{1}{2}(\nabla + \sigma\nabla\sigma)$ and $\Psi = \frac{1}{2}(\nabla - \sigma\nabla\sigma)$. The harmonic map equations, paired with projective flatness as in equation (1.19), then gives that $\overline{\partial}_E \Psi^{1,0} = 0$. So, if we take $\Psi^{1,0} = \Phi$, we get a Higgs bundle.

Conversely, given a semistable $PU(n,1)$-Higgs bundle $(E, \Phi)$ where $L \subset E$ is a line subbundle, we can fix the metric such that the $\mathbb{C}^{n,1}$ metric is negative definite on $L$. The line $L$ then determines a smooth section of the $\mathbb{CH}^n$ bundle $\mathbb{CH}^1 \times_\rho \mathbb{CH}^n$ equivalent to the $\rho$-equivariant map $f$ such that $\Phi = \partial f$. Indeed, as $f$ is $\rho$-equivariant, so is $df$ and we can decompose $df$ into four maps:

$$\begin{aligned} \partial f' &: T^{1,0}\Sigma \to T'\mathbb{CH}^n, & \partial f'' &: T^{1,0}\Sigma \to T''\mathbb{CH}^n, \\ \overline{\partial} f' &: T^{0,1}\Sigma \to T'\mathbb{CH}^n, & \overline{\partial} f'' &: T^{0,1}\Sigma \to T''\mathbb{CH}^n, \end{aligned} \tag{1.22}$$

with $\overline{\partial} f' = \overline{\partial f''}$ and similarly $\overline{\partial} f'' = \overline{\partial f'}$. We can then think of $df$ as a smooth section of $(T^{\mathbb{C}}\Sigma)^* \otimes f^{-1}\left(T^{\mathbb{C}}\mathbb{CH}^n/\rho\right)$ over $\Sigma$. This gives that $\partial f' \in H^0(K \otimes \mathrm{Hom}(\underline{\mathbb{C}}, V))$ and $\partial f'' \in H^0(K \otimes \mathrm{Hom}(V, \underline{\mathbb{C}}))$ so we can equate these with the components $(\phi_1, \phi_2)$ of a $PU(n,1)$-Higgs field.

Moreover, the connection $\nabla_E$ on $E$ induces a metric connection on $f^{-1}T\mathbb{CH}^n$ which agrees with the pullback of the Levi-Civita connection. As $(E, \Phi)$ satisfies the Hitchin equations (1.20), the Hitchin-Simpson correspondence gives that $f$ is harmonic. Meanwhile the semistability of $(E, \Phi)$ ensures that $f$ is linearly full.

## 1.7 MODULI SPACE OF $PU(n,1)$-HIGGS BUNDLES

We finally consider some known facts about the moduli space of $PU(n,1)$-Higgs bundles. Let $(E, \Phi)$ be a polystable $PU(n,1)$-Higgs bundle.

### 1.7.1 Toledo Components of $\mathcal{M}_{Higgs}$

Following the original definition of the Toledo invariant in [59], we recall that a representation $\rho$ defines a flat bundle $E_\rho$ over $\Sigma$ with fibre $\mathbb{CH}^n$ and a section equivalent to the equivariant map $f$. Letting $\Omega$ be the Kähler form on $\mathbb{CH}^n$, the form $f^*\Omega$ is fixed under the action of $\pi_1(\Sigma)$ on $\mathbb{CH}^1$ so descends to a form on $\Sigma$. We then define the **Toledo invariant** as:

$$\tau(\rho) = \frac{2}{\pi} \int_\Sigma f^*\Omega. \tag{1.23}$$

Recall that in §1.2, we fixed the holomorphic sectional curvature to be -4 which means here we follow the scaling convention in [48] and [14] rather than the original definition in [59] where the holomorphic sectional curvature is taken to be -1.

We now follow Goldman, Kapovich and Leeb [33] for an equivalent definition that will be useful.

Let $K$ be the canonical line bundle over $\mathbb{CH}^n$ and recall $\mathbb{CH}^n$ is a Kähler-Einstein manifold with Einstein coefficient $-2(n+1)$ (see [7, §1]). Being a Kähler manifold means the curvature of the canonical bundle is $iR$ where $R$ is the Ricci curvature, while being Einstein means $R = -2(n+1)\Omega$ (see, for example, [52, §13]). Together this means the curvature of $K$ is $\Theta(K) = -2i(n+1)\Omega$. Following [6] and [33], the pullback bundle $f^*K$ on the universal cover $\tilde{\Sigma}$ descends to a line bundle $L_\rho$ over $\Sigma$ with a curvature 2-form $\alpha_\rho$. In particular, $\deg(L_\rho) = \frac{1}{2\pi i}\int_\Sigma \alpha_\rho \in \mathbb{Z}$ (see, for example, [60, §3]).

Returning to the Toledo invariant, this means:

$$\tau(\rho) = \frac{2}{\pi}\int_\Sigma f^*\Omega = \frac{-1}{\pi i(n+1)}\int_\Sigma f^*\Theta(K) = -\frac{2}{n+1}\frac{1}{2\pi i}\int_\Sigma \alpha_\rho = -\frac{2}{n+1}\deg(L_\rho). \tag{1.24}$$

However, as in [48], we have $L_\rho \cong f^{-1}\left(T\mathbb{CH}^n/\rho\right) \cong \mathrm{Hom}(L,V)$ meaning $\deg(L_\rho) = \deg \mathrm{Hom}(L,V) = \deg V - n\deg L$.

We've then shown:

$$\tau := \tau(\rho) = -2\frac{\deg V - n\deg L}{n+1}, \tag{1.25}$$

which agrees with the definition in [14]. In our case, we take $L \simeq \underline{\mathbb{C}}$ meaning this can be further simplified to:

$$\tau = -\frac{2}{n+1}\deg V. \tag{1.26}$$

In particular, this gives that $\tau(\rho) \in \frac{2}{n+1}\mathbb{Z}$.

As detailed in [34], we can stratify the character variety by the degree of $E$ and, as $\deg(E) = \deg(V)$, we can now index these components by $\tau$. Indeed, Xia [66] proves that $\mathcal{M}_{Higgs}$ decomposes into connected components indexed by $\tau$.

We now show there is a bound on the values $\tau$ can take. Recall that a semistable $PU(n,1)$-Higgs bundle has a Higgs field which splits into two components $\phi_1 : \underline{\mathbb{C}} \to V \otimes K$ and $\phi_2 : V \to \underline{\mathbb{C}} \otimes K$. If $\Phi = 0$, then the Higgs bundle must lie in the component $\tau = 0$ as detailed in [14].

As in Lemma 3.24 in [14], $\mathrm{Im}\phi_2 \otimes K^{-1} \subset \underline{\mathbb{C}}$ so, assuming $\phi_2 \neq 0$, we have that $\mathrm{Im}\phi_2$ generates a subbundle $I$ (see [44]), which has rank 1. Similarly, let $N$ be the subbundle generated by $\ker\phi_2$ with rank $n-1$ then, as $\phi_2$ induces a nonzero section of $(V/N)^* \otimes \mathrm{Im}\phi_2 \otimes K$, we have:

$$\deg(N) \geq \deg(V) - 2(g-1). \tag{1.27}$$

Furthermore, $N$ is a $\Phi$-invariant subbundle so, by stability of $E$, we also have:

$$\deg(N) \leq \frac{n-1}{n+1}\deg(V). \tag{1.28}$$

Taking these together,

$$2(g-1) \geq \frac{2}{n+1}\deg(V) = \tau. \tag{1.29}$$

Applying similar reasoning for $\phi_1$, we get a **Milnor-Wood type inequality**:

$$0 \leq |\tau| \leq 2(g-1). \tag{1.30}$$

Milnor and Wood first proved a form of this inequality in [51] and [64] respectively, while the $\mathbb{CH}^n$ case was first shown by Domic and Toledo [23].

Xia [66] shows that there is exactly one connected component for each one of these values $\tau$, with Theorem 6.1 in [14] further giving that these components have a dimension of $1+(n+1)^2(g-1)$ unless $|\tau| = 2(g-1)$.

If $|\tau| = 2(g-1)$, then [14, Thm 3.32] gives that every such Higgs bundle is strictly semistable and decomposes into a polystable $\mathbb{CH}^1$-Higgs bundle of maximal $\tau$ and degree $2(1-g)$, and a polystable, rank $n-1$ vector bundle of degree $\deg(V)+2(g-1)$. The connected component is then of smaller dimension $2+(n^2-2n+5)(g-1)$.

### 1.7.2 The Hitchin Map and Nilpotent Cone

The **Hitchin map** on $\mathcal{M}_{Higgs}$ (see, for example, [29], [34]) is given by:

$$\begin{aligned} \chi : \mathcal{M}_{Higgs} &\to \bigoplus_{i=1}^{n+1} H^0(\Sigma, K^i) \\ [E,\Phi] &\mapsto \left(\mathrm{tr}(\Phi), \mathrm{tr}(\Phi^2), \ldots, \mathrm{tr}(\Phi^{n+1})\right) \end{aligned} \tag{1.31}$$

Hitchin [43] shows that this a proper map meaning the fibres are compact and, in particular, the fibre of the Hitchin map over 0,

$$\chi^{-1}(0) = \{[E,\Phi] | \chi([E,\Phi]) = 0\}, \tag{1.32}$$

is called the **nilpotent cone**.

In our equivariant minimal surfaces case, we can go further: as $\mathbb{CH}^n$ is a rank 1 symmetric space and we take $f$ to be conformal, then the corresponding Higgs bundle has $\Phi^2 = 0$. This means we have $\mathrm{tr}(\Phi^i) = 0$ for all $i \geq 0$ meaning that all such Higgs bundles are in the nilpotent cone.

### 1.7.3 The $\mathbb{C}^*$ Action, Hodge Bundles and Morse Theory

An important property of the Higgs moduli space (see [34] for example) is the $\mathbb{C}^*$ action given by:

$$\begin{aligned} \mathbb{C}^* \times \mathcal{M}_{Higgs} &\to \mathcal{M}_{Higgs} \\ (z, [(E, \Phi)]) &\mapsto [(E, z\Phi)]. \end{aligned} \tag{1.33}$$

With respect to the Kähler form $\omega$ on the moduli space $\mathcal{M}_{Higgs}$, [34] details how the action of the subgroup $S^1 \subset \mathbb{C}^*$ is Hamiltonian and, for $\xi$ the vector field generated by the action such that grad $g = i\xi$, there exists a moment map $g$ given by (up to scaling):

$$\begin{aligned} g : \mathcal{M}_{Higgs} &\to \mathbb{R} \\ [(E, \Phi)] &\mapsto \|\Phi\|^2 := \int_X |\Phi|^2 vol. \end{aligned} \tag{1.34}$$

As originally shown in [43], this is a proper, perfect Morse-Bott function, meaning the Hessian of $g$ is non-degenerate along the normal bundle to the connected critical submanifolds $N_\lambda \subset \mathcal{M}_{Higgs}$.

Observe that $S^1 \subset \mathbb{C}^*$ acts on the moduli space of solutions to the Hitchin equations (1.20) meaning that understanding the $\mathbb{C}^*$-action can be reduced to understanding the scaling of the Higgs field $\Phi$ by $z = e^t$ where $t \in \mathbb{R}$.

We see in [42] how the downward gradient flow $\Gamma_t(E)$ (i.e. of -grad $g$) on the moduli space from which we have the **stable** and **unstable manifolds** respectively:

$$\begin{aligned} N_\lambda^+ &= \{(E,\Phi) | \lim_{t\to\infty} \Gamma_t(E) \in N_\lambda\}, \text{ and} \\ N_\lambda^- &= \{(E,\Phi) | \lim_{t\to-\infty} \Gamma_t(E) \in N_\lambda\}. \end{aligned} \tag{1.35}$$

This gives a Morse stratification $\mathcal{M}_{Higgs} = \bigcup_\lambda N_\lambda^+$ which Kirwan [45] shows is equivalent to the **Bialynicki-Birula stratification**:

$$\mathcal{M}_{Higgs} = \bigcup_\lambda \left\{ (E,\Phi) \,\middle|\, \lim_{z\to 0} (E, z\Phi) \in N_\lambda \right\}. \tag{1.36}$$

As $g$ is proper and bounded below, it attains a minimum on each connected component of $\mathcal{M}_{Higgs}$. $\frac{|\tau|}{2}$ provides a lower bound for $g$ and [14, Prop 4.5] shows this is obtained if and only if $\phi_1 = 0$ or $\phi_2 = 0$. Note in particular, we only have $\|\Phi\|^2 = 0$ when $\tau = 0$.

We therefore want to better understand these critical submanifolds. It follows from Frankel's work [30] that both the critical Higgs bundles $(E,\Phi)$ and the adjoint Higgs bundle $(\mathrm{End}(E),\mathrm{ad}(\Phi))$ are fixed by the circle action so:

$$e^{i\theta}\cdot[(E,\Phi)] = [(E,e^{i\theta}\Phi)] = [(E,\Phi)], \tag{1.37}$$

for all values $\theta$. From this,

$$\left.\frac{d}{d\theta}\right|_{\theta=0} e^{i\theta}\cdot(E,\Phi) = (0,i\Phi) = (d_E\Psi,[\Psi,\Phi]), \tag{1.38}$$

where $\Psi$ is a unique infinitesimal automorphism of the Lie algebra. As $d_E\Psi = 0$, there is a holomorphic eigenbundle decomposition $\mathrm{End}(E) = \bigoplus E_\lambda$ with $E_\lambda = \ker(\Psi-\lambda)$ where $\lambda$ is the eigenvalue. Finally, as $[\Psi,\Phi] = i\Phi$, we get that $\Phi(E_\lambda)\subset E_{\lambda+i}\otimes K$ and the eigenvalues are integer multiples of $i$.

A holomorphic decomposition $E = \oplus_{i=1}^m E_i$ such that $\Phi(E_i)\subset E_{i+1}\otimes K$ and $E_{m+1} = 0$ is called a **length m Hodge bundle**.

In our $PU(n,1)$ case, $\Phi$ also splits into $\phi_1 : \underline{\mathbb{C}} \to V\otimes K$ and $\phi_2 : V\to\underline{\mathbb{C}}\otimes K$ which limits the possible lengths of Hodge bundles.

**Lemma 1.4.** *Every semistable $PU(n,1)$-Higgs bundle $(E = V\oplus\underline{\mathbb{C}},\Phi)$ fixed under the $\mathbb{C}^*$-action is either a length 2 or length 3 Hodge bundle.*

*Proof.* Following [14, Prop 4.10], each eigenbundle of the fixed point must be either a subbundle of $V$ or $\underline{\mathbb{C}}$. Moreover, $\Phi$ maps the eigenbundle $E_i$ to the consecutive eigenbundle $E_{i+1}$. Together with the definition of $PU(n,1)$-Higgs bundles, this means there must be at least two consecutive eigenbundles which $\Phi$ acts non-trivially between: one a subbundle of $V$ and one a subbundle of $\underline{\mathbb{C}}$.

If there exists consecutive eigenbundles which are subbundles of either $V$ or $\underline{\mathbb{C}}$, for ease we take their direct sum so we have:

$$0\to E^1 \xrightarrow{\Phi} E^2 \xrightarrow{\Phi} E^3 \to 0, \tag{1.39}$$

where we are suppressing the direct products with $K$ and $E^i$ are direct sums of consecutive eigenbundles that are either subbundles of $V$ or $\underline{\mathbb{C}}$. As $\Phi^2 = 0$ and $\mathrm{rank}(\underline{\mathbb{C}}) = 1$, we necessarily have that there can be at most three of these bundles.

If $E^1$ is a subbundle of $\underline{\mathbb{C}}$, then $E^2$ is a subbundle of $V$ which lies in the kernel of $\Phi$ meaning $E^3 = 0$ and $(E,\Phi) = (\underline{\mathbb{C}}\oplus V,\phi_1)$ is a length 2 Hodge bundle.

Similarly, if $E^3$ is a subbundle of $\underline{\mathbb{C}}$, then we must have $E^1 = 0$ and we get the length 2 Hodge bundle $(V \oplus \underline{\mathbb{C}}, \phi_2)$.

However, if $E^2$ is a subbundle of $\underline{\mathbb{C}}$, then both $\phi_1$ and $\phi_2$ are non-zero. We relabel $E^1 = V_2$ and $E^3 = V_2$ so we get a length 3 Hodge bundle $(V_2 \oplus \underline{\mathbb{C}} \oplus V_1, (\phi_1, \phi_2))$. □

Note that in the length 3 case, $\mathrm{Im}(\phi_1) \subseteq V_1$ and $V_1 \subseteq \ker \phi_2$, so we get $\phi_2\phi_1 = 0$ or, equivalently, $\mathrm{Tr}\ \Phi^2 = 0$. This also trivially holds in the length 2 case. All $PU(n,1)$-Higgs bundles with this property belong to the nilpotent cone (see equation (1.32)).

### 1.7.4 Tangent Space to the Moduli Space

In order to understand deformations of Higgs bundles, we now explore the tangent space to the full Higgs bundle moduli space at a smooth point $p = (E, \Phi)$ where $(E, \Phi)$ is a stable Higgs bundle and recall that $\mathcal{E}^k(U)$ is the space of $k$-differential forms over a local open subset $U$ of a Riemann surface.

Moreover, in §1.6, we saw that the Higgs moduli space is given by:

$$\mathcal{M}_{Higgs} = \{(A, \Phi) \text{ satisfying } (1.41)\} / C^\infty(\Sigma, H) \tag{1.40}$$

where

$$F_A + [\Phi, \Phi^*] = 0, \quad d''_A = 0. \tag{1.41}$$

and $H$ is the maximal compact group of $PU(n,1)$.

Adapting [61, Eq. 2.7], the infinitesimal structure of the moduli space is given by the deformation complex

$$\mathcal{E}^0(\mathfrak{h}^{\mathbb{C}}) \xrightarrow{D_1} \mathcal{E}^{0,1}(\mathfrak{h}^{\mathbb{C}}) \oplus \mathcal{E}^{1,0}(\mathfrak{m}^{\mathbb{C}}) \xrightarrow{D_2} \mathcal{E}^{1,1}(\mathfrak{m}^{\mathbb{C}}) \tag{1.42}$$

where the linearisation of the gauge action and $d''_A\Phi = 0$ gives:

$$D_1 B = (d''_A B, [\Phi, B])$$
$$D_2(\alpha, \varphi) = d''_A\varphi + [\alpha \wedge \Phi],$$

and $\mathfrak{h}$ and $\mathfrak{m}$ are the Lie algebra splitting of $PU(n,1)$ as in §1.1. Let $\mathcal{E}^k(\mathfrak{h}^{\mathbb{C}})$ be the space of smooth $k$-forms with value in the bundle of endomorphisms

$$\{\psi \in \mathrm{End}(V) \oplus \mathrm{End}(\underline{\mathbb{C}}) \mid \mathrm{Tr}(\psi) = 0\},$$

and $\mathcal{E}^k(\mathfrak{m}^{\mathbb{C}})$ correspond to endomorphisms $\psi \in \mathrm{Hom}(\underline{\mathbb{C}}, V) \oplus \mathrm{Hom}(V, \underline{\mathbb{C}})$.

This allows us to model the tangent space space of a smooth point as

$$T_p\mathcal{M}_{Higgs} \simeq \frac{\left\{(\alpha,\varphi) \in \mathcal{E}^1(\mathfrak{h}) \oplus \mathcal{E}^{1,0}(\mathfrak{m}^{\mathbb{C}}) \text{ satisfying } (1.44)\right\}}{\{(d_A B, [\Phi, B]) \mid B \in \mathcal{E}^0(\mathfrak{h})\}} \tag{1.43}$$

where

$$d''_A\varphi + [\alpha \wedge \Phi] = 0. \tag{1.44}$$

Alternatively, following on from the holomorphic principal $H^{\mathbb{C}}$ definition of a Higgs bundle seen in §1.5, we can consider the **Dolbeault moduli space** as constructed in [34, p. 13] as

$$\mathcal{M}_{Dol} = \left\{(\overline{\partial}_A, \Phi) \mid (\overline{\partial}_A, \Phi) \text{ polystable, } \overline{\partial}_A\Phi = 0\right\} / C^\infty(\Sigma, H^{\mathbb{C}}). \tag{1.45}$$

Here, the gauge action is given by:

$$g(t)\cdot(\overline{\partial}_A, \Phi) = (-\overline{\partial}_A g \cdot g^{-1}, g\Phi g^{-1}). \tag{1.46}$$

Differentiating this at $t = 0$ and setting $\dot{g}(0) = B$, we get:

$$T_p\mathcal{M}_{Dol} \simeq \frac{\left\{(\dot{A}'', \dot{\Phi}) \in \mathcal{E}^{0,1}(\mathfrak{h}^{\mathbb{C}}) \oplus \mathcal{E}^{1,0}(\mathfrak{m}^{\mathbb{C}})\right\}}{\{(d''_A B, [\Phi, B]) \mid B \in \mathcal{E}^0(\mathfrak{h}^{\mathbb{C}})\}}. \tag{1.47}$$

As we have a diffeomorphism $\mathcal{M}_{Dol} \to \mathcal{M}_{Higgs}$, for every $(\dot{A}'', \dot{\Phi}) \in T_p\mathcal{M}_{Dol}$, there exists $B \in \mathcal{E}^0(\mathfrak{h}^{\mathbb{C}})$ such that:

$$(\alpha'', \phi) = (\dot{A}'' + d''_A B, \Phi + [\Phi, B]) \tag{1.48}$$

satisfies (1.44).

We will be mostly interested in the $\mathbb{C}^*$ flow $(E, e^t\Phi)$ for which we can take the representation at $t = 0$ to be $(\dot{A}'', \dot{\Phi}) = (0, \Phi)$. This last expression then simplifies and we have a $B \in \mathcal{E}^0(\mathfrak{h}^{\mathbb{C}})$ such that

$$(\alpha'', \phi) = (d''_A B, \Phi + [\Phi, B]) \tag{1.49}$$

satisfies (1.44).

# — 2 —

# Harmonic Sequences

In this chapter we define harmonic sequences for our $\mathbb{CH}^n$ case and explore their link to $PU(n,1)$-Higgs bundles.

## 2.1 $\mathbb{CH}^n$ HARMONIC SEQUENCES

Continuing from the previous chapter, we take $(E,\Phi)$ to be a polystable $PU(n,1)$ Higgs bundle where $E = V \oplus \underline{\mathbb{C}}$. Let $\langle\cdot,\cdot\rangle$ be the $\mathbb{C}^{n,1}$ metric on $E$, $D : \mathcal{E}^0(E) \to \mathcal{E}^1(E)$ the Chern connection for the holomorphic structure $E = \mathbb{C} \oplus V$ and $\nabla = D + \Phi - \Phi^*$ the projectively flat $PU(n,1)$ connection on $E$. Here $\Phi^*$ is the adjoint operator to $\Phi$ with respect to $\langle\cdot,\cdot\rangle$ which has the opposite sign to the usual adjoint (e.g. [60, §2]) since $\langle\cdot,\cdot\rangle$ is negative definite on $\underline{\mathbb{C}}$.

We call a subbundle $\ell \subset E$ **definite** when $\ell \neq \underline{0}$ and $\langle\cdot,\cdot\rangle$ is definite on $\ell$. In this case, we can then take the orthogonal projections $\pi : E \to \ell$ and $\pi^\perp : E \to \ell^\perp$ such that $\pi + \pi^\perp = Id_E$. For a definite line subbundle $\ell \subset E$ we follow [10, 18, 63] to define:

$$\begin{aligned} A' &: \ell \to \ell^\perp \otimes K \\ A'' &: \ell \to \ell^\perp \otimes \overline{K} \end{aligned} \tag{2.1}$$

by:

$$\begin{aligned} A'\sigma &= \pi^\perp \nabla_Z \sigma dz \\ A''\sigma &= \pi^\perp \nabla_{\overline{Z}} \sigma d\overline{z} \end{aligned} \tag{2.2}$$

where $z$ is a local holomorphic coordinate on $\Sigma$, $Z = \partial/\partial z$ and $\sigma$ is any local smooth section. Note $A'$ and $A''$ are both independent of the choice of $z$ and $\sigma$ so are global definitions.

Given $\ell$ and $\ell^\perp$, we endow them with the holomorphic structures:

$$\begin{aligned}\overline{\partial}_\ell &= \pi\nabla'' : \ell \to \ell \otimes \overline{K} \\ \overline{\partial}_{\ell^\perp} &= \pi^\perp\nabla'' : \ell^\perp \to \ell^\perp \otimes \overline{K}.\end{aligned} \tag{2.3}$$

As stated in [62, Thm 2.1], we prove following:

**Lemma 2.1.** *$A'$ is holomorphic if and only if $A''$ is antiholomorphic with respect to the $\overline{\partial}_\ell$ and $\overline{\partial}_{\ell^\perp}$ holomorphic structures*

*Proof.* For $A'$ to be holomorphic, we require, for each local smooth section $\sigma$ of $\ell$ and local holomorphic vector field $Z$,

$$\pi^\perp\nabla_{\overline{Z}}A'(Z)\sigma - A'(Z)\pi\nabla_{\overline{Z}}\sigma = 0 \tag{2.4}$$

where $A'(Z) = \pi^\perp\nabla_Z$.

Since $\nabla$ is projectively flat, we can simplify this using $A'(Z) = \pi^\perp\nabla_Z$ to get:

$$\begin{aligned}\pi^\perp\nabla_{\overline{Z}}(\pi^\perp\nabla_Z\sigma) - \pi^\perp\nabla_Z\pi\nabla_{\overline{Z}}\sigma &= \pi^\perp((\nabla_{\overline{Z}}\nabla_Z - \nabla_{\overline{Z}}\pi\nabla_Z)\sigma) - \pi^\perp(\nabla_Z\nabla_{\overline{Z}}\sigma - \nabla_Z\pi^\perp\nabla_{\overline{Z}}\sigma) \\ &= \pi^\perp((\nabla_{\overline{Z}}\nabla_Z - \nabla_Z\nabla_{\overline{Z}})\sigma) - A''(\overline{Z})\pi\nabla_Z\sigma - \pi^\perp\nabla_Z A''(\overline{Z})\sigma \\ &= \pi^\perp\nabla_Z A''(\overline{Z})\sigma - A''(\overline{Z})\pi\nabla_Z\sigma\end{aligned}$$

Hence $A'$ is holomorphic if and only if this is 0, from which we get that $A''$ is antiholomorphic. ☐

Returning to our $PU(n,1)$ Higgs bundle $(V \oplus \underline{\mathbb{C}}, \Phi)$, let $\ell_0 = \underline{\mathbb{C}}$, hence $\ell^\perp = V$ meaning

$$A'_0 : \ell_0 \to V \otimes K \quad \text{and} \quad A''_0 : \ell_0 \to V \otimes \overline{K}. \tag{2.5}$$

Taking a global trivialising section $\sigma$ of $\ell_0$ we have

$$\begin{aligned}A'_0\sigma &: \pi_0^\perp(D'\sigma + \phi_1\sigma) = \phi_1\sigma \\ A''_0\sigma &: \pi_0^\perp(D''\sigma - \phi_2^*\sigma) = -\phi_2^*\sigma\end{aligned} \tag{2.6}$$

for $\pi_0 : E \to \underline{\mathbb{C}}$ and $\pi_0^\perp : E \to V$.

Hence we have $A'_0 = \phi_1$ and $A''_0 = -\phi_2^*$. Since $\phi_1$ is holomorphic and $\phi_2^*$ is antiholomorphic, we get well-defined line subbundles $\ell_1, \ell_{-1} \subset V$ such that:

$$\begin{aligned}\mathrm{Im}(A'_0) &= \mathrm{Im}(\phi_1) = \ell_1 \otimes K, \\ \mathrm{Im}(A''_0) &= \mathrm{Im}(\phi_2^*) = \ell_{-1} \otimes \overline{K}.\end{aligned} \tag{2.7}$$

Here Im denotes the saturation of the image sheaf: the smallest sheaf containing the image (complete definition can be found in [26]).

We can continue this process as the next result, similar to that stated in [62, Thm 2.2], shows:

**Lemma 2.2.** *Let $\ell_j \subset E$ be a definite line subbundle for which $A'_j : \ell_j \to \ell_j^\perp \otimes K$ is holomorphic so that $\ell_{j+1} = ImA'_j \otimes K$ is well-defined and assume $\ell_{j+1}$ is definite. Then*

$$A''_{j+1} : \ell_{j+1} \to \ell_{j+1}^\perp \otimes \overline{K} \tag{2.8}$$

*equals $-(A'_j)^*$.*

Note by Lemma 2.1, an immediate consequence of this is that

$$A'_{j+1} : \ell_{j+1} \to \ell_{j+1}^\perp \otimes K \tag{2.9}$$

is holomorphic.

*Proof.* Let $\sigma, \tau$ be local sections of $\ell_j, \ell_{j+1}$ respectively. Then:

$$\begin{aligned} \langle A'_j(Z)\sigma, \tau\rangle &= \langle \nabla_Z \sigma, \tau\rangle \\ &= -\langle \sigma, \nabla_{\overline{Z}} \tau\rangle \\ &= -\langle \sigma, A''_{j+1}(\overline{Z})\tau\rangle \end{aligned}$$

where we have used that $\langle \sigma, \tau\rangle = 0$ and $\nabla$ is a metric connection. □

Since $A'_0$ is holomorphic with definite image $\ell_1$, there is a line subbundle $\ell_2$ for which $A'_1 : \ell_1 \to \ell_2 \otimes K$ is holomorphic. Whenever the iteration continues, we produce $A'_j : \ell_j \to \ell_{j+1} \otimes K$ until $\ell_{j+1}$ fails to be definite.

Similarly, Lemma 2.2 gives that when $\ell_j$ is definite and $A''_j$ is antiholomorphic, we obtain $\ell_{j-1} \subset E$ for which $A''_j : \ell_j \to \ell_{j-1} \otimes \overline{K}$. This leads us to the definition of the harmonic sequence.

**Definition 2.3.** *If $\ell_j$ is definite and comes from $\ell_0$ by iteration of one of the above operations, then let $I \subset \mathbb{Z}$ be a subset such that $j \in I$ and $0 \in I$. We call the set $\{\ell_j : j \in I\}$ the **harmonic sequence** for the Higgs bundle $(E, \Phi)$.*

This is called a *harmonic* sequence because every $\ell_j$ in the sequence determines an equivariant harmonic map into either $\mathbb{CH}^n$ or the pseudo-Hermitian symmetric space $\mathbb{C}dS^n$, known as complex de Sitter space, as follows.

Let $E_\pm \subset E$ be the fibre subbundle consisting of $v \in E$ such that $\langle v, v\rangle = \pm 1$. Then using the metric $\langle\cdot,\cdot\rangle$, the fibres of $\mathbb{P}E_\pm$ are isometric to either $\mathbb{CH}^n$ or $\mathbb{C}dS^n$ for $E_-$ and $E_+$ respectively. It is well-known (e.g. [27, 25, 48]) that $\ell_j$ determines a harmonic section of $\mathbb{P}E_\pm$ precisely when $A_j'$ is holomorphic. Equally, $\ell_j$ gives an equivariant harmonic map into $\mathbb{CH}^n$ or $\mathbb{C}dS^n$.

Note that, for $\ell_0 = \underline{\mathbb{C}}$ and $\ell_0^\perp = V$, the equivariant harmonic map is $f : \mathbb{CH}^1 \to \mathbb{CH}^n$ as seen in §1.3.

For this equivariant harmonic map $f$, we then define the **k-th order Z osculating subbundle** $F_k \subset V$ as:

$$F_k = \mathrm{span}\{\nabla_Z^i \sigma : 0 < i \le k\} = \bigoplus_{i=1}^k \ell_i, \tag{2.10}$$

and **k-th order $\overline{\mathbf{Z}}$ osculating subbundle** as:

$$\overline{F}_k = \mathrm{span}\{\nabla_{\overline{Z}}^i \sigma : 0 < i \le k\} = \bigoplus_{i=1}^k \ell_{-i}, \tag{2.11}$$

for $\sigma$ a section of $\underline{\mathbb{C}}$. As $f$ is harmonic, then Lemma 2.2 shows that $\nabla_{\overline{Z}} \nabla_Z^i \sigma \in F_{i-1}$ and similar for the complex conjugate. In particular, $F_k$ is always a holomorphic bundle while $\overline{F}_k$ is antiholomorphic.

These osculating spaces form two flags:

$$\begin{aligned} &F_1 \subset \cdots \subset F_r = F_{r+1} \subset V, \text{ and} \\ &\overline{F}_1 \subset \cdots \subset \overline{F}_s = \overline{F}_{s+1} \subset V, \end{aligned} \tag{2.12}$$

where we take $r, s \in \mathbb{Z}$ to be smallest possible integers such that $F_r = F_{r+1}$ and $\bar{F}_s = \bar{F}_{s+1}$.

## 2.2 ISOTROPY ORDER

Now let us restrict our attention to $PU(n,1)$-Higgs bundles $(E, \Phi)$ in the nilpotent cone which are characterised by $\phi_2\phi_1 = 0$. For a global unitary section $\sigma$ of $\underline{\mathbb{C}} = \ell_0$,

$$\mathrm{Tr}(\phi_2\phi_1) = \langle \phi_2\phi_1\sigma, \sigma\rangle = \langle \phi_1\sigma, \phi_2^*\sigma\rangle. \tag{2.13}$$

We therefore see that $\ell_1$ is perpendicular to $\ell_{-1}$ meaning $\ell_1, \ell_0$ and $\ell_{-1}$ are mutually orthogonal by construction. This is a consequence of requiring $f$ in our equivariant minimal surface to be conformal.

**Lemma 2.4** (Prop. 2.4, [10]). *If there exists a subsequence $\ell_p, \cdots, \ell_{p+k}$ of $k+1$ consecutive mutually orthogonal subbundles of the harmonic sequence, then every $k+1$ consecutive bundles of the harmonic sequence are mutually orthogonal.*

*Proof.* Assume $\ell_p, \ell_{p+1}, ..., \ell_{p+k}$ are mutually orthogonal for $p, \ldots, p+k \subseteq I$, we then show that if $p-1 \in I$, then $\ell_{p-1}$ is orthogonal to $\ell_p, \ell_{p+1}, ..., \ell_{p+k-1}$, and if $p+k+1 \in I$, then $\ell_{p+k+1}$ is orthogonal to $\ell_{p+1}, \ell_{p+1}, ..., \ell_{p+k}$.

We first consider $p-1 \in I$ and let $\sigma$ and $\tau$ be local sections of $\ell_p$ and $\ell_r$ for $r \in \{p+1, \cdots, p+k-1\}$ respectively. Then we know $\langle \sigma, \tau \rangle = 0$ by assumption. Therefore:

$$\langle \nabla_{\overline{Z}} \sigma, \tau \rangle = -\langle \sigma, \nabla_Z \tau \rangle$$
$$\langle A''_p(\overline{Z})\sigma, \tau \rangle = -\langle \sigma, A'_r(Z)\tau \rangle$$

This last term is 0 because $\ell_p$ is orthogonal to $\ell_{r+1}$ by assumption. Furthermore, as $A''_p(\overline{Z})\sigma$ is a local section of $\ell_{p-1}$, we necessarily have that $\ell_{p-1}$ is perpendicular to $\ell_r$ for $r \in p+1, \cdots, p+k-1$. By construction, we also have that $\ell_{p-1}$ is perpendicular to $\ell_p$ meaning we have shown that $\ell_{p-1}$ is orthogonal to $\ell_p, \ell_{p+1}, ..., \ell_{p+k-1}$ as required.

On the other hand, we assume $p+k+1 \in I$ and now take $\sigma$ and $\tau$ be local sections of $\ell_{p+k}$ and $\ell_r$ for $r \in \{p+1, \cdots, p+k\}$ respectively. Again, $\langle \sigma, \tau \rangle = 0$ and we have:

$$\langle \nabla_Z \sigma, \tau \rangle = -\langle \sigma, \nabla_{\overline{Z}} \tau \rangle$$
$$\langle A'_{j+k}(Z)\sigma, \tau \rangle = -\langle \sigma, A''_r(\overline{Z})\tau \rangle = 0.$$

This last equality follows from $\ell_{p+k}$ being orthogonal to $\ell_{r-1}$ and, as before, this gives that $\ell_{p+k+1}$ is orthogonal to $\ell_{p+1}, \ell_{p+1}, ..., \ell_{p+k}$ as required. Inductively, we have that any $k$ consecutive bundles are mutually orthogonal. □

**Definition 2.5.** *Let $m+1$ be the maximum length of a consecutive sequence of mutually orthogonal subbundles in the harmonic sequence. We call $m$ the* ***isotropy order*** *of the harmonic sequence and of its corresponding Higgs bundle $(E, \Phi)$.*

Note that clearly $1 \leq m \leq n$ and, by restricting ourselves to Higgs bundles in the nilpotent cone where $\Phi \neq 0$, then we also have $m \geq 2$.

Following from the previous lemma, we also have:

**Corollary 2.6.** *If $(E, \Phi)$ is a Higgs bundle with isotropy order $m$ then there exists a subsequence $\ell_p, \cdots, \ell_{p+m}$ of the harmonic sequence consisting of mutually orthogonal subbundles containing $\ell_0$. Hence $\ell_j \subset V = \ell_0^\perp$ for $-m \leq j \leq m$, $j \neq 0$ and $j \in I$.*

## 2.3 SUPERMINIMAL HARMONIC SEQUENCES

We now want to consider the case of maximal mutual orthogonality. We prove the following as stated in [10, p. 368]:

**Proposition 2.7.** *Consider a harmonic sequence of isotropy order $m \geq 2$. If $A'_p = 0$ for some $p \in I$, then $0 \leq p \leq m$ and the harmonic sequence is finite, consisting of mutually orthogonal $\ell_j$ for $p - m \leq j \leq p$. Moreover, this holds if and only if $A''_{p-m} = 0$.*

*Proof.* We assume $A'_p = 0$. Consider any subbundle $\ell_j$ of the harmonic sequence with $j < p$. Let $\sigma$ and $\tau$ be local sections of $\ell_p$ and $\ell_{j+1}$ respectively. By assumption, we then have:

$$0 = \langle A'_p(Z)\sigma, \tau\rangle = \langle \sigma, A''_j(\overline{Z})\tau\rangle. \tag{2.14}$$

Therefore, for all $j \in I$, we have $\ell_j$ is perpendicular to $\ell_p$.

By construction $\ell_{p-1}$ is orthogonal to $\ell_p$ and $\ell_{p-2}$ and the previous paragraph gives that $\ell_{p-2}$ is perpendicular to $\ell_p$. Therefore $\ell_{p-2}, \ell_{p-1}, \ell_p$ are mutually orthogonal. By Lemma 2.4, we therefore have that $\ell_{p-3}, \ell_{p-2}, \ell_{p-1}$ are mutually orthogonal. The previous paragraph also gives that $\ell_{p-3}$ is perpendicular to $\ell_p$. Working iteratively like this, we get that $\ell_{p-j}, \cdots, \ell_p$ are mutually orthogonal for all $j \in I$.

By definition of the isotropy order, the harmonic sequence must therefore contain $\ell_{p-m}, \cdots, \ell_p$ and, by construction, we have $p \leq m$.

To show this is the entire harmonic sequence, we need to show that $A'_p = 0$ if and only if $A''_{p-m} = 0$. We let $\sigma$ and $\tau$ be local sections of $\ell_{p-m}$ and $\ell_p$ respectively, then

$$\langle A''_{p-m}(\overline{Z})\sigma, \tau\rangle = -\langle \sigma, A'_p(Z)\tau\rangle.$$

Assuming $A'_p = 0$, then $A''_{p-m} = 0$ or else $\ell_{p-m-1}$ is in the harmonic sequence contradicting the definition of the isotropy order. Conversely, if $A''_{p-m} = 0$, then we have $A'_p = 0$ or else $p + 1 \in I$. □

There are two special classes of harmonic sequence which have particular significance in the rest of this thesis. The terms defined below follow from those in [12]:

**Definition 2.8.** *The map corresponding to $\ell_0$ is **superminimal** if its corresponding harmonic sequence terminates. Equivalently, there exists a $p \in I$ such that $A'_p = 0$. We will also call the corresponding Higgs bundle $(E, \Phi)$ superminimal.*

*If the harmonic sequence is periodic, then both the corresponding map and the Higgs bundle are called* ***superconformal****.*

If we required **linearly full** $PU(n,1)$-Higgs bundles, then their harmonic sequences span the bundle. In this case, Proposition 2.7 shows that superminimal harmonic sequence have isotropy order $n+1$. Meanwhile linearly full superconformal Higgs bundles have isotropy order $n$.

Following our discussion after Definition 2.3, we also have that:

**Corollary 2.9.** *Every equivariant superminimal map into $\mathbb{CH}^n$ is either holomorphic, antiholomorphic or its harmonic sequence contains a holomorphic map into $\mathbb{C}dS^n$.*

We now can link Hodge bundles to harmonic sequences by showing that a superminimal harmonic sequence corresponds to a Hodge bundle.

**Theorem 2.10.** *For $f$ linearly full, the harmonic sequence given by an equivariant minimal surface $[f,c,\rho]$, is superminimal if and only if the $PU(n,1)$-Higgs bundle $(E,\Phi)$ corresponding to $[f,c,\rho]$ is a Hodge bundle.*

*Proof.* Suppose the Higgs bundle is a Hodge bundle. In the case of length 3 Hodge bundles, $V_1$ and $V_2$ are holomorphic subbundles and, by definition of $\Phi$, $\ell_1 \subseteq V_1$ and $\ell_{-1} \subseteq V_2$. As holomorphic bundles, they are both invariant under $\overline{\partial}_V$. Moreover, $\pi_{-1}^{\perp}\nabla_{\overline{Z}}\sigma_{-1} = \sigma_{-2}$ must be contained in $V_2$. Inductively, we have $\ell_i \subset V_2$ for all $i < 0$ and, as $V_1$ and $V_2$ have finite rank, there must exist some $p \in I$ such that $A'_p = 0$ as $V_1$ and $V_2$ are holomorphic subbundles. As $V$ is orthogonal to $\underline{\mathbb{C}}$, $\langle \sigma, \sigma \rangle > 0$ for any $\sigma \in \ell_0^{\perp}$ and the isotropy order must be maximal. By definition 2.8, we therefore have a superminimal harmonic sequence.

In the length 2 case, either $\phi_1 = 0$ or $\phi_2 = 0$. Note this means that $f$ is either holomorphic or antiholomorphic as seen in the $\mathbb{CH}^2$ case, and the harmonic sequence either has $\ell_1 = 0$ and $\ell_i \subset V$ for $i < 0$ or $\ell_{-1} = 0$ and $\ell_i \subset V$ for $i > 0$.

Conversely, suppose we have a superminimal harmonic sequence then we have maps given by $A'_i : F_i \to F_{i+1} \otimes K$, with $F_i = \overline{F}_{-i}$ for $i < 0$, as in the previous chapter. Then $\ell_0$, $F_p$ and $F_{p-n}$ are three orthogonal holomorphic subbundles for $0 \leq p \leq n$. Moreover, identifying $\phi_1 = A'_0$ and $\phi_2 = A'_{-1}$, then $\Phi(\sigma_0) = \pi_0 Z(\sigma_0) \subset F_p$ and $\pi_0 Z(\sigma_i) = 0$ for all $i \neq -1$, which together gives $\Phi(F_{p-n}) \subset \ell_0$, $\Phi(\ell_0) \subset F_p$ and $\Phi(F_p) = 0$. This therefore defines a Hodge bundle decomposition: $E \cong F_{p-n} \oplus \ell_0 \oplus F_p$. Moreover, as we have a superminimal sequence, the $\{\ell_i\}$ form a $PU(n,1)$-frame and we get that $E$ is a $PU(n,1)$-Higgs bundle. □

## 2.4 TOLEDO INVARIANT AND STABILITY

We continue this chapter by using previous results for Higgs bundles to bound the degree of line bundles within the context of the harmonic sequence. Let $d_i = \deg(\ell_i)$.

From the previous lemma, we have:

$$\deg(V_1) = \deg(F_p) = \sum_{i \geq 1} d_i, \text{ and } \deg(V_2) = \deg(F_{p-n}) = \sum_{i \leq -1} d_i. \tag{2.15}$$

Relating this back to the Toledo invariant $\tau$ in equation (1.25):

$$\sum_{i \in I} d_i = \frac{n+1}{2}\tau. \tag{2.16}$$

Recall the Milnor-Wood inequality (1.30):

$$0 \leq \left| \sum_{i \in I} d_i \right| \leq (n+1)(g-1). \tag{2.17}$$

As in [48], we also consider the points $q = f(z)$ where $\partial f''$ or $\partial f'$ vanishes. If $f$ is not holomorphic or anti-holomorphic, then these points are isolated. Moreover, with the prior identifications of $\phi_1 = \partial f'$ and $\phi_2 = \partial f''$, then we have that $\phi_1 = 0$ or $\phi_2 = 0$ on these points. Let $D_1$ and $D_2$ be the divisors defined by the zeroes of $\phi_1$ and $\phi_2$ respectively. Viewing $\phi_1$ and $\phi_2$ as non-zero global sections of the associated line bundles, we see that these are effective divisors.

Moreover, as $(E, \Phi)$ satisfies $\phi_2 \phi_1 = 0$, we have:

$$0 \longrightarrow K^{-1}(D_1) \xrightarrow{\phi_1} V \xrightarrow{\phi_2} K(-D_2) \longrightarrow 0. \tag{2.18}$$

Relating this back to the harmonic sequence, $\phi_1 : \ell_0 \to \ell_1 \otimes K$ and $\phi_2 : \ell_{-1} \otimes K^{-1} \to \ell_0$ so, as $D_i$ are effective divisors,

$$\begin{aligned} d_1 &\geq -2(g-1), \text{ and} \\ d_{-1} &\leq 2(g-1). \end{aligned} \tag{2.19}$$

This reflects Gothen's result in Proposition 3.2 in [35].

Recalling that our Higgs bundles are polystable by the non-abelian Hodge correspondence, then we can explore how the ensuing stability conditions impose further bounds on the degree of $V$ and hence on the degrees of the line bundles in the bundle sequence. Recall too that we take $\ell_0 = \underline{\mathbb{C}}$ so $d_0 = 0$.

We already know $(E, \Phi)$ is a stable Higgs bundle if every holomorphic $\Phi$-invariant subbundle $F$ satisfies $\mu(F) \leq \mu(E)$ as seen in §1.5.

For a length 2 Hodge bundle, either $\phi_1 = 0$ or $\phi_2 = 0$. In the case $\phi_1 = 0$ then $\ell_0$ is a $\Phi$-invariant bundle. Alternatively, $\phi_2 = 0$ and $V$ is $\Phi$-invariant. We get the stability conditions:

$$\begin{aligned} 0 &\leq \deg(E) \text{ if } \phi_1 = 0 \text{ or} \\ 0 &\geq \deg(E) \text{ if } \phi_2 = 0. \end{aligned} \tag{2.20}$$

In particular, as $\deg(E)$ depends on the Toledo invariant, we immediately get:

**Lemma 2.11.** *Let $(E, \Phi)$ be a stable length 2 Hodge bundle, then $\phi_1 = 0$ if and only if $\tau \leq 0$ while $\phi_2 = 0$ if and only if $\tau \geq 0$.*

If $\Phi = 0$, then $(V \oplus L, 0)$ is a global minimum of $\|\Phi\|^2$ and both of these must hold. This can only occur in the $\tau = 0$ component and the point $(V \oplus L, 0)$ is polystable.

More generally, as the definition of a stable Higgs bundle only regards holomorphic $\Phi$-invariant subbundle $F \subseteq E$, we now further explore the properties of such subbundles $F$ in the superminimal case.

We cannot find all $\Phi$-subbundles for $n \geq 3$ but we can find some to impose some necessary but not sufficient stability conditions.

Let $(E, \Phi)$ be a length 3 Hodge bundle then, by the Newlander-Nirenberg theorem, given a $\overline{\partial}$ operator, there is a unique holomorphic structure on the bundle. The harmonic sequence gives an orthogonal frame for $E$ and, with regards to the wider splitting $E = V_1 \oplus V_2 \oplus \underline{\mathbb{C}} \cong F_{p-n} \oplus F_p \oplus \ell_0$ from the proof of Theorem 2.10, we can form a representation of $\overline{\partial}_E$ with the only non-zero entries on the diagonal and superdiagonal:

$$\overline{\partial}_V = \begin{pmatrix} \overline{\partial}_1 & \alpha_1 & & & & & & & & \\ & \overline{\partial}_2 & \ddots & & & & & & & \\ & & \ddots & \alpha_{p-1} & & & & & & \\ & & & \overline{\partial}_p & 0 & & & & & \\ & & & & \overline{\partial}_{p-n} & \alpha_{p-n} & & & & \\ & & & & & \ddots & \ddots & & & \\ & & & & & & \overline{\partial}_{-2} & \alpha_{-2} & & \\ & & & & & & & \overline{\partial}_{-1} & 0 & \\ & & & & & & & & \overline{\partial}_{\underline{\mathbb{C}}} & \end{pmatrix}, \tag{2.21}$$

where $\overline{\partial}_i$ is the $\overline{\partial}$ operator on $\ell_i$ as generalised from (2.3) and $\alpha_i \in \mathcal{E}^{0,1}(\Sigma, \mathrm{Hom}(\ell_{i+1}, \ell_i))$.

With relation to this holomorphic structure, one way of constructing a holomorphic subbundle $F$ is to consider flags of sums of consecutive line bundles of the harmonic sequence starting from either $\ell_{p-n}$, $\ell_0$ or $\ell_1$ where $p = \mathrm{rank}(V_1)$.

Secondly, note that $\Phi$ only acts on $\ell_{-1}$ and $\ell_0$ meaning that any $\Phi$-invariant subbundle $F \subseteq E$ must necessarily either not contain $\ell_i$, or contain both $\ell_i$ and $\ell_{i+1}$ for $i = -1, 0$. Note that as $(E, \Phi)$ satisfies $\phi_2\phi_1 = 0$, we get $\Phi(\ell_1) = 0$.

We can now put these two ideas together to derive some necessary conditions for stability.

When $n = 2$, we can consider the $\Phi$-invariant bundles $\ell_1$ and $\ell_0 \oplus \ell_1$, which together give the stability conditions found in both [48] and [35], noting the difference in notation:

$$2d_1 < d_{-1}, \quad \text{and} \quad d_1 < 2d_{-1}. \tag{2.22}$$

Taken with equation (2.19), we have that:

$$\begin{aligned} -2(g-1) < d_1 < g-1, \text{ and} \\ -(g-1) < d_{-1} < 2(g-1). \end{aligned} \tag{2.23}$$

For $n = 3$, we first note that $\ell_{-1} \oplus \ell_0 \oplus \ell_1$ is a proper subbundle. Moreover, it is holomorphic when $p = 2$ so:

$$d_1 + d_{-1} < d_2. \tag{2.24}$$

As in the $n = 2$ case, we still have $\ell_1$ and $\ell_0 \oplus \ell_1$ as $\Phi$-invariant bundles provided $p \geq 1$ and thus:

$$3d_1 < \sum_{i \neq 1} d_i \text{ and } d_1 < \sum_{i \neq 1} d_i. \tag{2.25}$$

We can also consider these bundles with the direct sum of $\ell_{\pm 2}$ to give $\ell_1 \oplus \ell_{\pm 2}$ and $\ell_0 \oplus \ell_1 \oplus \ell_{\pm 2}$ giving:

$$\sum_{i \neq -1} d_i < d_{-1}. \tag{2.26}$$

Finally, we can consider $\ell_{\pm 2}$ itself. If $p = 1$, then $\ell_{-2}$ is holomorphic and

$$3d_{-2} < d_1 + d_{-1}. \tag{2.27}$$

However, if $p = 2$, then $\ell_2$ is instead an antiholomorphic bundle so does not give us any further stability condition.

Continuing inductively on $n$, we get the following necessary conditions for stability:

**Lemma 2.12.** *If $(E, \Phi)$ is a stable Higgs bundle with $p = \deg(V_1)$, then the following inequalities hold for all $1 \leq j \leq p$ and $k \neq 0, \pm 1$:*

$$\sum_{i=1}^{j} d_i < \frac{j}{2}\tau, \tag{2.28a}$$

$$\sum_{i=1}^{j} d_i < \frac{j+1}{2}\tau, \text{ and} \tag{2.28b}$$

$$\sum_{i=p-n}^{k} d_i < \frac{n-p+k+1}{2}\tau. \tag{2.28c}$$

*Proof.* We first consider $\Phi$-invariant subbundles containing $\ell_0 \oplus \ell_1$.

Letting $p \geq 1$, we can take the direct sum of consecutive line bundles $\ell_i$ to get successively larger subbundles. Starting from $\ell_1$ and $\ell_0$ respectively, we get:

$$\begin{aligned} (n+1)\sum_{i=1}^{j} d_i &< j \sum_{i \in I} d_i, \text{ and} \\ (n+1)\sum_{i=1}^{j} d_i &< (j+1)\sum_{i \in I} d_i \end{aligned} \tag{2.29}$$

for $1 \leq j \leq p$.

Alternatively, for $p \leq n-2$, we consider the flag $G_k = F_{p-n} \cap F_k^{\perp}$ for some integer $p - n < k < -1$. These flags are contained within $\ker \phi_2$ and their stability gives:

$$(n+1)\sum_{i=p-n}^{k} d_i < (n-p+k+1)\sum_{i \in I} d_i. \tag{2.30}$$

We can combine these two constructions, building a holomorphic subbundle as the direct sum of line bundles from the holomorphic end of the harmonic sequence and from $\ell_1$. However, the constraint we get in this case is implied by the previous conditions.

We next consider subbundles which include $\ell_{-1} \oplus \ell_0 \oplus \ell_1$. The simplest of these is the bundle $\ell_{-1} \oplus \ell_0 \oplus \ell_1$ itself, however this is only holomorphic if $p = n-1$. In general, we wish to consider the flag $\bigoplus_{i=p-n}^{1} \ell_i$ (recalling that $p \leq n$ by definition). In this case,

$$(n+1)\sum_{i=p-n}^{1} d_i < (n-p+2)\sum_{i \in I} d_i. \tag{2.31}$$

We can further generalise this by considering $\bigoplus_{i=p-n}^{j} \ell_i$ for $1 \leq j < p$ to get:

$$(n+1)\sum_{i=p-n}^{j} d_i < (n-p+j+1)\sum_{i \in I} d_i. \tag{2.32}$$

Recalling that $\sum_{i \in I} d_i$ is the degree of the total bundle which can be written in terms of the Toledo invariant, we get the above result, where we have combined equations (2.30) and (2.32). □

In this discussion, we have only considered holomorphic Φ-invariant subbundles as flags formed from the harmonic sequence. When $n = 2$, this is the only source of such subbundles and hence the subsequent stability conditions are both necessary and sufficient. However, for $n > 2$, there are different sources of holomorphic subbundles of $\ker \Phi$ whose stability we cannot currently address. In this case, the above conditions are only necessary.

Nonetheless, we can still use this result to bound the degrees of $V_1$ and $V_2$ in a length 3 Hodge bundle. Using equation (2.28a) and knowing that $V_1 \oplus V_2 = V$, if we set $j = p$, we get:

$$\begin{aligned} \deg(V_1) &< \frac{p}{2}\tau, \\ \deg(V_2) &> -\frac{n+p+1}{2}\tau. \end{aligned} \tag{2.33}$$

We also know $V_2 = F_{-2} \oplus \ell_{-1}$ so using equations (2.28c) and (2.23), we also have:

$$\begin{aligned} \deg(V_1) &> \frac{p-2n}{2}\tau - 2(g-1), \\ \deg(V_2) &< \frac{n-p-1}{2}\tau + 2(g-1). \end{aligned} \tag{2.34}$$

Note that, if $\tau \neq 0$, and by the additivity of the degree, both $V_1$ and $V_2$ are unstable as vector bundles as vector bundle stability requires all subbundles to have smaller slope, and not just the Φ-invariant ones.

## 2.5 DETERMINING THE ISOTROPY ORDER

We now return to considering the case where $(E, \Phi)$ is not superminimal and has isotropy order $m < n + 1$. From Lemma 2.4, we then have that:

$$\ell_{-1} \xrightarrow{A'_{-1}} \ell_0 \xrightarrow{A'_0} \ell_1 \xrightarrow{A'_1} \cdots \xrightarrow{A'_{m-2}} \ell_{m-1} \xrightarrow{A'_{m-1}} \cdots \tag{2.35}$$

consist of mutually orthogonal subbundles where we are suppressing the tensor product with the cannonical bundle $K$ at each step. In particular, $\ell_j \subset \ell_0^\perp = V$ for $j = -1, 1, \ldots, m$ so $\ell_m$ is in our harmonic sequence.

For any $\sigma \in \Gamma(\ell_0)$ such that $\langle \sigma, \sigma \rangle = -1$, we can then define

$$Q_m = -\langle A'_{m-1} \cdots A'_0 \sigma, A''_0 \sigma \rangle \tag{2.36}$$

as a global smooth section of $K^{m+1}$.

Note this agrees, up to the sign, with the definition of the differential $U_{m,-1}$ given in Bolton-Woodward [10]:

$$U_{i,j} = u_{i,j} dz^{i-j}, \qquad \text{where } u_{i,j} = \begin{cases} \frac{\langle \sigma_i, \sigma_j \rangle}{\langle \sigma_j, \sigma_j \rangle} & \text{if } i, j \in I, \\ 0 & \text{else.} \end{cases} \tag{2.37}$$

where $\sigma_i \in \Gamma(\ell_i)$. These $U$-invariants let us conclude more about orthogonality of the harmonic sequence as, if $\sigma_i$ is orthogonal to $\sigma_j$, then $U_{i,j} = 0$. In particular, if the harmonic sequence is superminimal, then $U_{i,j} = 0$ for all $i, j \in I$. We therefore have:

**Proposition 2.13.** *When the isotropy order is at least $m$, $Q_{m+1} = 0$ if and only if the isotropy order is at least $m + 1$.*

*Proof.* As seen above, we have $A'_{m-1} \cdots A'_0 \sigma = \sigma_m \in \ell_m$ and, from Lemma 2.2, $A''_0 \sigma = \phi_2^* \sigma = \sigma_{-1} \in \ell_{-1}$ meaning

$$Q_m = -\langle \sigma_m, \sigma_{-1} \rangle. \tag{2.38}$$

This is then 0 if and only if $\ell_m$ is orthogonal to $\ell_{-1}$ which, by definition of the isotropy order and under the assumption that the isotropy order is at least $m$, means the isotropy order is at least $m + 1$. $\square$

We can also consider a further equivalent definition which does not explicitly use the metric on $E$:

$$Q_{m+1} = \operatorname{Tr}\left(\phi_2 A'_{m-1} \cdots A'_1 \phi_1\right). \tag{2.39}$$

This follows since $\ell_m \subset V$, $A'_0 = \phi_1$ and $(A''_0)^* = \phi_2$, as we've already seen.

For $\mathbb{CH}^3$, we can now recover the isotropy order $m$ of a Higgs bundle. If $A'_1 \phi_1 \subseteq \ker \Phi$, then $m \geq 3$ and if the Higgs bundle is a Hodge bundle, we have $m = 4$. If $A'_1 \phi_1 \nsubseteq \ker \Phi$, we necessarily have $m = 2$.

More generally, we get the following:

**Corollary 2.14.** *We have* $A'_{k-2}\cdots A'_1\phi_1\sigma_0 \subseteq \ker\phi_2$ *if and only if* $k < m$*, where* $m$ *is the isotropy order of the corresponding harmonic sequence. In other words, the isotropy order* $m$ *is determined by the maximum length complete holomorphic flag* $F_1 \subset \ldots \subset F_{m-1}$ *which lies entirely in* $\ker\phi_2$*.*

Note that if a sequence is superconformal, then this gives $F_{n-1} = \ker\phi_2$ because $\phi_2 \neq 0$ so the nullity of $\phi_2$ is $n-1$.

Provided, $\ell_0, \ldots, \ell_m$ are defined for the harmonic sequence, then we already know that $A'_{m-1}\cdots A'_1\phi_1 : \ell_0 \to \ell_m \otimes K^m$ is holomorphic regardless of the isotropy order. However, since $\ell_{m-1}$ is non-trivial, $\ell_m$ itself is not a holomorphic subbundle meaning we cannot show that $Q_{m+1}$ is holomorphic this way.

To show that $Q_{m+1}$ is a holomorphic differential, we want to be able to express it using higher order derivatives.

With an abuse of notation, we let

$$D : \mathcal{E}^0(\mathrm{Hom}(\underline{\mathbb{C}}, V)) \to \mathcal{E}^1(\mathrm{Hom}(\underline{\mathbb{C}}, V)) \tag{2.40}$$

denote the metric connection induced by $D$ on $E$ which preserves $\underline{\mathbb{C}}$ and $V$. Let $\delta$ be the connection on $K^j$ induced by the Levi-Civita connection for the Käher metric fixed on $\Sigma$. Together, these give the connection:

$$D_j : \mathcal{E}^0(\mathrm{Hom}(\underline{\mathbb{C}}, V) \otimes K^j) \to \mathcal{E}^1(\mathrm{Hom}(\underline{\mathbb{C}}, V) \otimes K^j), \tag{2.41}$$

whose $(1,0)$ component defines:

$$D'_j : \mathcal{E}^0(\mathrm{Hom}(\underline{\mathbb{C}}, V) \otimes K^j) \to \mathcal{E}^0(\mathrm{Hom}(\underline{\mathbb{C}}, V) \otimes K^{j+1}). \tag{2.42}$$

**Lemma 2.15.** *For* $1 \leq j \leq m-1$*,*

$$D'_j(D'_{j-1}\cdots(D'_1\phi_1)\cdots)) \in \mathcal{E}^0(Hom(\underline{\mathbb{C}}, V) \otimes K^{j+1}),$$

*and, for a global unitary section of* $\underline{\mathbb{C}}$*,*

$$D'_j(D'_{j-1}\cdots(D'_1\phi_1)\cdots))\sigma \equiv A'_j\cdots A'_1\phi_1\sigma \mod F_j \otimes K^{j+1}. \tag{2.43}$$

For notational simplicity, we write:

$$(D')^j\phi_1 = D'_j(D'_{j-1}\cdots(D'_1\phi_1)\cdots)). \tag{2.44}$$

*Proof.* It is enough to prove the congruence which we will do by induction.

For a local coordinate $z$, we write $\phi_1 = \phi_1(z)dz$. Then:

$$\begin{aligned}(D_1'\phi_1)\sigma &= (D'\phi_1(z))\,\sigma dz + \phi_1(z)\delta' dz\\ &= (D'(\phi_1(z)\sigma) - \phi_1(z)D'\sigma)\,dz + \phi_1(z)\delta' dz\\ &\equiv \pi^{\perp} D'\left(\phi_1(z)\sigma\right) \mod F_1 \otimes K^2\\ &\equiv A_1'\phi_1\sigma \mod F_1 \otimes K^2\end{aligned}$$

where we have used orthogonality properties.

Now assume:

$$(D')^r\phi_1\sigma \equiv A_r' \cdots A_1'\phi_1 \mod F_r \otimes K^{r+1}, \tag{2.45}$$

which can be expressed locally as:

$$A_r'(z)\cdots A_1'(z)\phi_1(z)dz^{r+1} \subset F_{r+1} \otimes K^{r+1}.$$

For simplicity, we let $\beta_r = A_r' \cdots A_1'\phi_1$, then we have:

$$\begin{aligned}(D')^{r+1}\phi_1\sigma &= \left(D_{r+1}'\beta_r(z)dz^{r+1}\right)\sigma\\ &= \left(D'\left(\beta_r(z)\sigma\right) - \beta_r(z)D'\sigma\right)dz^{r+1} + \beta_r(z)\sigma\delta dz^{r+1}\\ &\equiv \pi_{r+1}^{\perp}D'(\beta_r\sigma) \mod F_{r+1} \otimes K^{r+2}\end{aligned}$$

since $D' : \mathcal{E}^0(F_j) \to \mathcal{E}^0(F_{j+1} \otimes K)$.

Overall, we therefore have that:

$$(D')^{r+1}\phi_1\sigma \equiv A_{r+1}' \cdots A_1'\phi_1\sigma \mod F_{r+1} \otimes K^{r+2}$$

for all $1 \leq j \leq m-1$. □

**Corollary 2.16.** *For isotropy order $m$,*

$$Q_{m+1} = Tr\left(\phi_2(D')^{m-1}\phi_1\right) \in H^0(K^{m+1}) \tag{2.46}$$

*Proof.* As we know $F_{m-1} \subset \ker\phi_2$, then by Lemma 2.15 and equation (2.39), we have:

$$Q_{m+1} = \mathrm{Tr}\left(\Phi_2 A_{m-1}' \cdots A_1'\Phi_1\right) = \mathrm{Tr}\left(\Phi_2(D')^{m-1}\Phi_1\right).$$

□

Note that the original definition of $Q_{m+1}$ given in (2.36) required that the harmonic sequence has length at least $m+1$ but this last definition holds for any Higgs bundle. In particular, for each $r \geq 0$, this definition gives functions on the nilpotent cone $\mathcal{N}$ so:

$$Q_{r+1} : \mathcal{N} \to \mathcal{E}^0(K^{r+1}). \tag{2.47}$$

It will later be useful to have defined:

$$\mathcal{U}_m = \{(E, \Phi) \in \mathcal{N} \mid Q_j((E, \Phi)) = 0 \text{ for } j \leq m\}. \tag{2.48}$$

This is the set of all Higgs bundles of isotropy order at least $m$. Recalling that any Higgs bundles in the nilpotent cone has isotropy order at least 2 meaning $Q_1 = Q_2 = 0$, we can work iteratively with the maps:

$$\begin{aligned} Q_{i+1} : \qquad \mathcal{U}_i &\to H^0(K^{i+1}) \\ (E, \Phi) &\mapsto \operatorname{Tr}\left(\phi_2 (D')^{i-1} \phi_1\right). \end{aligned} \tag{2.49}$$

## 2.6 METRICS, CURVATURE AND KÄHLER ANGLES

Harmonic sequences can also be described locally as in [10] for $\mathbb{CP}^n$; in this section, we use this local description to link to other common geometric quantities.

Recalling that $\sigma_0$ is a nonzero local holomorphic section of $\ell_0$, we then define:

$$\sigma_{i+1} = A_i'(\sigma_i) = \pi_i^\perp (\nabla_Z \sigma_i), \quad i \in I, \tag{2.50}$$

where $\pi_i^\perp$ is projection onto $\ell_i^\perp$. This gives a holomorphic section of $\ell_{i+1}$ since $A_i$ is a holomorphic map.

By considering projection onto $\ell_i^\perp$ and $\ell_i$ and recalling $\sigma_i$ is holomorphic, we obtain:

$$Z\sigma_i = \sigma_{i+1} + \frac{\langle Z\sigma_i, \sigma_i \rangle}{\langle \sigma_i, \sigma_i \rangle} \sigma_i = \sigma_{i+1} + Z \log \left| \langle \sigma_i, \sigma_i \rangle \right| \sigma_i. \tag{2.51}$$

Before considering the complex conjugate, we first define the $\mathbb{R}$-valued quadratic differentials:

$$\Gamma_i = \gamma_i |dz|^2, \qquad \text{where } \gamma_i = \begin{cases} \frac{\langle \sigma_{i+1}, \sigma_{i+1} \rangle}{\langle \sigma_i, \sigma_i \rangle} & \text{if } i \in I, \\ 0 & \text{else.} \end{cases} \tag{2.52}$$

This is invariant under the choice of $\sigma_i \in \Gamma(\ell_i)$ as indicated by Bolton and Woodward [10], and hence globally defined.

By definition of $A'_i$, $\pi^{\perp}_{i-1}\nabla_Z = \sigma_i$ and, as $-A''_i$ is the adjoint of $A'_{i-1}$, we have that $\overline{Z}\sigma_i \subset \ell_i \oplus \ell_{i-1}$. However, by the definition of $A''$, $\overline{Z}\sigma_i \subset \ell_i^{\perp}$ and therefore:

$$\begin{aligned}\overline{Z}\sigma_i &= \frac{\left\langle \overline{Z}\sigma_i, \sigma_{i-1}\right\rangle}{\langle \sigma_{i-1}, \sigma_{i-1}\rangle}\sigma_{i-1} \\ &= -\frac{\langle \sigma_i, Z\sigma_{i-1}\rangle}{\langle \sigma_{i-1}, \sigma_{i-1}\rangle}\sigma_{i-1} \\ &= -\frac{\langle \sigma_i, \sigma_i\rangle}{\langle \sigma_{i-1}, \sigma_{i-1}\rangle}\sigma_{i-1} \\ &= -\gamma_{i-1}\sigma_{i-1}.\end{aligned} \tag{2.53}$$

We therefore have that, for $i \in I$, $\ell_i$ is holomorphic if and only if $\Gamma_{i-1} = 0$, and is antiholomorphic if and only if $\Gamma_i = 0$.

As $\nabla$ is projectively flat in the $\mathbb{CH}^n$ case, $[Z, \overline{Z}] = \Lambda(Z, \overline{Z})I_{n+1}$ where $\Lambda(Z, \overline{Z})$ is a scalar 2-form and $I_{n+1}$ is the identity. Therefore:

$$\overline{Z}Z \log |\sigma_i|^2 = \gamma_i - \gamma_{i-1} - \Lambda(Z, \overline{Z}). \tag{2.54}$$

By considering the case $i = 0$ with $\ell_0 = \underline{\mathbb{C}}$ then $\langle \sigma_0, \sigma_0\rangle^2 = 1$ and we get:

$$\Lambda = \gamma_{-1} - \gamma_0. \tag{2.55}$$

Analogously to Bolton and Woodward's **unintegrated Plucker formula**, equation (2.54) also gives:

$$Z\overline{Z} \log |\gamma_i| = \gamma_{i+1} - 2\gamma_i + \gamma_{i-1}. \tag{2.56}$$

This last formula means that given any two consecutive $\Gamma$-invariants, $\Gamma_i$ can be determined for all $i \in I$. Moreover, recalling from [60] and [37], we can find the first Chern class as:

$$c_1(\ell_i) = -\frac{1}{2\pi i}\int_\Sigma Z\overline{Z} \log |\sigma_i|^2 \, d\overline{z} \wedge dz. \tag{2.57}$$

We therefore see that the $\Gamma$-invariants determine the degrees of the line bundle of the harmonic sequence.

As detailed in [10], given a line bundle $L$ over $\Sigma$, there exists a function $f_L : \Sigma \to \mathbb{CH}^n$ such that $L = f_L^*(\ell_{Taut})$. This function $f_L$ then induces a metric $g_L$ on $\Sigma$ such that:

$$g_L(X, Y) = Re\langle df_L(X), df_L(Y)\rangle,$$

where $X$, $Y$ are tangent vectors to $\Sigma$.

For the harmonic sequence, we let $f_i$ be the map corresponding to the line $\ell_i$ and can then calculate, for $i \in I$ and by equation (2.52) and (2.37),

$$|df_i(Z)|^2 = \frac{|A_i(\sigma_i)|^2}{|\sigma_i|^2} = \gamma_i,$$
$$|df_i(\overline{Z})|^2 = \frac{|\overline{A}_i(\sigma_i)|^2}{|\sigma_i|^2} = \gamma_{i-1},$$
$$\langle df_i(Z), df_i(\overline{Z})\rangle = \frac{\langle \sigma_{i+1}, -\gamma_{i-1}\sigma_{i-1}\rangle}{\langle \sigma_i, \sigma_i\rangle} = -u_{i+1,i-1}.$$

We've already seen that this last equation is 0 when $f$ is conformal.

Returning to the metric $g_i$, if we take $X = Z + \overline{Z}$ and $Y = Z - \overline{Z}$, then:

$$g_i(X,X) = \gamma_{i-1} + \gamma_i - 2\ \mathrm{Re}\ u_{i+1,i-1}, \tag{2.58}$$
$$g_i(Y,Y) = \gamma_{i-1} + \gamma_i + 2\ \mathrm{Re}\ u_{i+1,i-1}, \tag{2.59}$$
$$g_i(X,Y) = 2\ \mathrm{Im}\ u_{i+1,i-1}. \tag{2.60}$$

When $f$ is conformal, this then simplifies to give:

$$g_i = (\gamma_{i-1} + \gamma_i)\,|dz|^2 = \Gamma_{i-1} + \Gamma_i. \tag{2.61}$$

Given this and following Wells [60] together with the scaling used in [11], we can consider the curvature of $\ell_i$ as:

$$\kappa_i = \frac{-2}{\gamma_{i-1} + \gamma_i} Z\overline{Z}\log(\gamma_{i-1} + \gamma_i). \tag{2.62}$$

In particular, if $f_i$ is holomorphic, then by equation (2.56), we have $\kappa_i = \frac{4\gamma_i - 2\gamma_{i+1}}{\gamma_i}$. Similarly, $\kappa_i = \frac{4\gamma_{i-1} - 2\gamma_{i-2}}{\gamma_{i-1}}$ if $f_i$ is antiholomorphic.

The **Kähler angle** $\boldsymbol{\theta_i} \in [0, \pi]$ of $f_i$ is a function on the complement of the singular set of $f_i$ in $\Sigma$ with:

$$f^*\omega = \cos(\theta_i) v_g, \tag{2.63}$$

where $\omega$ is the Kähler form on $\mathbb{CH}^n$ and $v_g$ is the volume form with respect to the above metric $g_i$. Chern and Wolfson [21] show this is a measure of the failure of $f_i$ to be holomorphic. If we take the local coordinate $z = x + iy$ and recall that $J$ is the complex structure on $\mathbb{CH}^n$, then $\theta_i$ is the angle between $Jdf_i(X)$ and $df_i(Y)$. In particular,

$$\begin{aligned} \cos(\theta_i) &= \frac{g_i(JX,Y)}{\sqrt{g_i(X,X)g_i(Y,Y)-g_i(X,Y)^2}} \\ &= \frac{\gamma_i-\gamma_{i-1}}{\sqrt{(\gamma_i+\gamma_{i-1})^2-4|u_{i+1,i-1}|^2}}. \end{aligned} \tag{2.64}$$

When $f$ is conformal, we get that $\theta_i = 0$ if $f_i$ is holomorphic and similarly, $\theta_i = \pi$ when $f_i$ is antiholomorphic. In both of these cases, $\ell_i$ is a complex submanifold corresponding to either the complex or anti-complex points respectively as in [67].

In the conformal case, we can also rearrange equation (2.64) to get:

$$\tan^2\left(\frac{\theta_i}{2}\right) = \frac{\gamma_{i-1}}{\gamma_i} = \frac{\Gamma_{i-1}}{\Gamma_i}. \tag{2.65}$$

Note here that $\Gamma_i$ and $\Gamma_{i-1}$ now determine and are determined by the metric $g_i$ and Kähler angle $\theta_i$. In particular, if we consider $f = f_0$,

$$\begin{aligned} g_0 &= (\gamma_{-1}-\gamma_0)\,|dz|^2 = \Lambda(Z,\overline{Z})|dz|^2, \text{ and} \\ \tan^2\left(\frac{\theta_0}{2}\right) &= \frac{\gamma_{-1}}{\gamma_0}, \end{aligned} \tag{2.66}$$

where this second equation is analogous to that seen in the $\mathbb{CH}^2$ case in [48, p.16].

# — 3 —

# Superminimal Equivariant Minimal Surfaces

Recall that the function $g$ given in equation (1.34) is a Morse-Bott function whose critical points form critical submanifolds consisting of Hodge bundles. We now classify the corresponding superminimal equivariant surfaces and study their properties.

Throughout this chapter, we consider sets of critical points:

$$N_{p,d} = \{(E, \Phi) : (E, \Phi) \text{ stable Hodge, } \operatorname{rank}(V_1) = p \text{ and } \deg(V_1) = d\}. \tag{3.1}$$

## 3.1 HOLOMORPHIC CHAINS

Before continuing to look at the critical submanifolds, we briefly develop the theory of holomorphic chains as in [1] in order to understand the connected components of Hodge bundles.

A **holomorphic chain C of length m** consists of $m$ holomorphic vector bundles $\mathcal{E}_j$ with homomorphisms $\phi_i : \mathcal{E}_{i+1} \to \mathcal{E}_i$. These can be parametrised by their **type** $t = (r_1, \ldots, r_m, \delta_1, \ldots, \delta_m)$ where $r_i = \operatorname{rank}(\mathcal{E}_i)$ and $\delta_i = \deg(\mathcal{E}_i)$. A **subchain** $C' \subseteq C$ consists of subsheaves $\mathcal{F}_j \subseteq \mathcal{E}_j$ such that $\phi_i(\mathcal{F}_{i+1}) \subseteq \mathcal{F}_i$.

We will be considering the case of length 3 chains which have the form:

$$C : \mathcal{E}_3 \xrightarrow{\phi_2} \mathcal{E}_2 \xrightarrow{\phi_1} \mathcal{E}_1. \tag{3.2}$$

The **$\alpha$-degree** of a chain with $\alpha = (\alpha_m, \ldots, \alpha_1)$ is:

$$\deg_\alpha(C) = \sum_{j=1}^{m} (\delta_j + \alpha_j r_j). \tag{3.3}$$

This gives a corresponding definition of $\alpha$-**slope** $\mu_\alpha$ as $\mu_\alpha(C) = \frac{\deg_\alpha(C)}{r}$ where $r$ is the total rank $r = \sum r_i$. Following the definitions from vector bundles, a holomorphic chain $C$ is $\alpha$-**stable** if $\mu_\alpha(C') < \mu_\alpha(C)$ for all non-trivial subchains $C'$. $\alpha$-**semistablity** and $\alpha$-**polystablity** follow as in the vector bundle case.

**Lemma 3.1.** *Let $E = V_1 \oplus \underline{\mathbb{C}} \oplus V_2$ with Higgs field $\Phi = (\phi_1, \phi_2)$ be a length 3 $PU(n,1)$-Hodge bundle and let $C : V_2 \otimes K^{-1} \xrightarrow{\phi_2} \underline{\mathbb{C}} \xrightarrow{\phi_1} V_1 \otimes K$ be a holomorphic chain. The Higgs bundle $(E, \Phi)$ is stable if and only if $C$ is a $\alpha$-stable holomorphic chain for $\alpha = (-2(g-1), 0, 2(g-1))$.*

The first implication of this proof is adapted from [15] and the whole proof is similar to that of Proposition 3.5 in [2].

*Proof.* Let $C$ be an $\alpha$-stable chain, then we show the corresponding Higgs bundle $E$ is stable. Indeed if we first consider the $\alpha$-slope of $C$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and noting that $r_2 = \mathrm{rank}(\underline{\mathbb{C}}) = 1$, we get:

$$\begin{aligned} \mu_\alpha(C) &= \mu\left((V_2 \otimes K^{-1}) \oplus \underline{\mathbb{C}} \oplus (V_1 \otimes K)\right) + \frac{r_1\alpha_1 + \alpha_2 + r_3\alpha_3}{n+1} \\ &= \mu(E) + \frac{r_1(2g-2+\alpha_1) + \alpha_2 + r_3(-2g+2+\alpha_3)}{n+1}. \end{aligned} \tag{3.4}$$

Let $C' : F_3 \otimes K^{-1} \xrightarrow{\phi_2} F_2 \xrightarrow{\phi_1} F_1 \otimes K$ be a subchain of $C$. We first want to show that the subbundle $F = F_1 \oplus F_2 \oplus F_3$ is $\Phi$-invariant. Noting that $F_1 \subseteq V_1$ and $\Phi(V_1) = \underline{0}$, then, by definition of a holomorphic subchain, we require the image of $\phi_i$ to be included in $F_i$, meaning in particular that:

$$\Phi(F) = \Phi(F_3) \oplus \Phi(F_2) \oplus \Phi(F_1) \subseteq (F_2 \otimes K) \oplus (F_1 \otimes K) \oplus \underline{0} \subseteq F \otimes K \tag{3.5}$$

as required.

We next want to consider the stability of $F$. As $C$ is $\alpha$-stable, $\mu_\alpha(C') \leq \mu_\alpha(C)$. Setting $s_i = \mathrm{rank}(F_i) \leq r_i$, then by equation (3.4), we obtain:

$$\begin{aligned} &\mu(F) + \frac{s_1(2g-2+\alpha_1) + s_2\alpha_2 + s_3(-2g+2+\alpha_3)}{s_1+s_2+s_3} \\ &\qquad \leq \mu(E) + \frac{r_1(2g-2+\alpha_1) + \alpha_2 + r_3(-2g+2+\alpha_3)}{n+1}. \end{aligned} \tag{3.6}$$

For $\alpha = (-2(g-1), 0, 2(g-1))$, $F$ is a subbundle of $E$ such that $\mu(F) \leq \mu(E)$ as this simplifies to:

$$\mu(F) \leq \mu(E). \tag{3.7}$$

On the other hand, given a stable bundle $E$ with corresponding chain $C$ and a $\Phi$-invariant subbundle $F \subset E$, we consider the projection $p_1 : F \to V_1$. If we set $\mathrm{Im}(p_1) = F_1$ and $\ker(p_1) = G_1$, then we get a short exact sequence:

$$0 \longrightarrow G_1 \longrightarrow F \longrightarrow F_1 \longrightarrow 0. \tag{3.8}$$

From this, we have $\deg(F) = \deg(F_1) + \deg(G_1)$ and $\mathrm{rank}(F) = \mathrm{rank}(F_1) + \mathrm{rank}(G_1)$. Moreover, we note $G \subseteq \mathbb{C} \oplus V_2$. Similarly, we have the projection $p_2 : G_1 \to L$ to give $\mathrm{Im}(p_2) = F_2 \subseteq \mathbb{C}$, $F_3 = \ker(p_2|G_1) \subseteq V_2$ and the short exact sequence:

$$0 \longrightarrow F_3 \longrightarrow G_1 \longrightarrow F_2 \longrightarrow 0. \tag{3.9}$$

Using the same reasoning as above, we then have that:

$$\begin{aligned} \deg(F) &= \deg(F_1) + \deg(F_2) + \deg(F_3) \text{ and} \\ \mathrm{rank}(F) &= \mathrm{rank}(F_1) + \mathrm{rank}(F_2) + \mathrm{rank}(F_3). \end{aligned} \tag{3.10}$$

$F_3$ is necessarily a subbundle of $V_2$ and, as $F$ is $\Phi$-invariant, $\Phi(F_3) \subseteq \Phi(V_2) \subseteq L$ so it follows that $\Phi(F_3) \subseteq F_2$. This similarly holds for $F_2$ and $F_1$, meaning we can define a subchain $C' : F_3 \otimes K^{-1} \to F_2 \to F_1 \otimes K$.

For stability, $F$ is a stable subbundle and again taking $\alpha = (-2(g-1), 0, 2(g-1))$, we have bundle stability which implies chain stability as above. □

Note the previous proof also holds if either $V_1$ or $V_2$ are zero such as in the case of length 2 Hodge bundles. There is therefore a bijection between $PU(n,1)$-Hodge bundles $V_2 \oplus \mathbb{C} \oplus V_1$ and holomorphic chains of type;

$$t = (p, 1, n-p, \deg(V_1) - 2p(g-1), 0, \deg(V_2) + 2(g-1)). \tag{3.11}$$

As $\deg(V_2) = \frac{n+1}{2}\tau - \deg(V_1)$, within a given Toledo component, this type $t$ therefore only depends on $p = \mathrm{rank}(V_1)$ and $d = \deg(V_1)$:

$$t(p,d) = \left(p, 1, n-p, d - 2p(g-1), 0, \frac{n+1}{2}\tau - d\right). \tag{3.12}$$

Given a $\alpha$-semistable holomorphic chain, then [15] proves that there exists a Jordan-Hölder filtration:

$$0 \subsetneq C_1 \subsetneq \cdots \subsetneq C_q = C$$

of holomorphic subchains such that $\mu_\alpha(C_i) = \mu_\alpha(C)$ and $C_i/C_{i-1}$ is $\alpha$-stable for all $i \geq 1$. This gives a gradation:

$$Gr(C) = \oplus_{i=1}^q C_i/C_{i-1}. \tag{3.13}$$

Proposition 2.2 in [1] gives that this gradation does not depend on the Jordan-Hölder filtration. Two holomorphic chains are S-equivalent if their gradations are equivalent. Wentworth [61] shows that Higgs bundles have a similar filtration with equal slope and stable quotients. Moreover, the moduli space of semistable Higgs bundles identifies the orbits of a given Higgs bundle with its associated gradation. In particular, we can consider the moduli space of $\alpha$-semistable holomorphic chains of a given type as connected subsets of the Higgs moduli space.

Proposition 3.8 in [1] then gives that the moduli spaces of $\alpha$-stable holomorphic chains of a given type with $\alpha$ fixed has dimension:

$$(g-1)(2p^2+n^2-2np+n+1)+2d-\frac{n+1}{2}\tau+1. \tag{3.14}$$

Note that all of the Hodge bundles within $N_{p,d}$ correspond to holomorphic chains of the same type. In particular, these results show that $N_{p,d}$ is connected and of the dimension given in equation (3.14).

## 3.2 INVARIANTS OF THE CRITICAL SUBMANIFOLDS

We next wish to calculate the Morse indices of $N_{p,d}$. In order to do this, we consider the infinitesimal deformations of its elements. Biswas and Ramanan [8] show that these deformations are given by the hypercohomology groups of an associated complex, which we now develop. More on this and hypercohomology in general can be found in Appendix A.

In the case of Hodge bundles, we let $(E = V \oplus L, \Phi) \in N_{p,d}$, then its adjoint Higgs bundle $(E(\mathfrak{g}^{\mathbb{C}}), \mathrm{ad}\Phi)$ is also fixed under the $S^1$ action. Considering the complexified Cartan decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}$, we can make the following identifications $E(\mathfrak{g}^{\mathbb{C}}) = \mathrm{End}(E)$, $E(\mathfrak{h}^{\mathbb{C}}) = \mathrm{End}(V) \oplus \mathrm{End}(L)$ and $E(\mathfrak{p}^{\mathbb{C}}) = \mathrm{Hom}(V, L) \oplus \mathrm{Hom}(L, V)$, in order to then consider the deformation complex:

$$C^* : E(\mathfrak{h}^{\mathbb{C}}) \xrightarrow{ad(\Phi)} E(\mathfrak{p}^{\mathbb{C}}) \otimes K \to 0. \tag{3.15}$$

As further explained in the appendix and in [14], this has a corresponding long exact sequence:

$$\begin{array}{l} 0 \longrightarrow \mathbb{H}^0(C^*) \longrightarrow H^0\Big(E(\mathfrak{h}^{\mathbb{C}})\Big) \xrightarrow{ad\Phi} H^0\Big(E(\mathfrak{p}^{\mathbb{C}}) \otimes K\Big) \\ \qquad \swarrow \\ \mathbb{H}^1(C^*) \longrightarrow H^1\Big(E(\mathfrak{h}^{\mathbb{C}})\Big) \xrightarrow{-ad\Phi} H^1\Big(E(\mathfrak{p}^{\mathbb{C}}) \otimes K\Big) \\ \qquad \swarrow \\ \mathbb{H}^2(C^*) \longrightarrow 0 \end{array} \tag{3.16}$$

where $\mathbb{H}^1(C^*)$ is the space of infinitesimal deformations of $(E, \Phi)$.

As seen in [14], there are natural ad-invariant isomorphisms $\Big(E(\mathfrak{h}^{\mathbb{C}})\Big)^* \cong E\Big(\mathfrak{h}^{\mathbb{C}}\Big)$ and $\Big(E(\mathfrak{p}^{\mathbb{C}})\Big)^* \cong E\Big(\mathfrak{p}^{\mathbb{C}}\Big)$. Taking the Serre dual to

$$\mathrm{ad}(\Phi) : H^1\Big(E(\mathfrak{h}^{\mathbb{C}})\Big) \to H^1\Big(E(\mathfrak{p}^{\mathbb{C}}) \otimes K\Big) \tag{3.17}$$

therefore gives:

$$\mathrm{ad}(\Phi) : H^0\Big(E(\mathfrak{p}^{\mathbb{C}})\Big) \to H^0\Big(E(\mathfrak{h}^{\mathbb{C}}) \otimes K\Big). \tag{3.18}$$

A Higgs bundle is called **simple** when all endomorphisms are invertible scalars and so the kernel of $\mathrm{ad}(\Phi)$ in equation (3.18) is 0. If $(E, \Phi)$ is stable, then it is simple [53, Prop 4.3]. With equation (3.16), this shows that $\mathbb{H}^2(C^*) = 0$. Theorem 3.1 in [8] then shows that $(E, \Phi)$ is a smooth point of $\mathcal{M}_{Higgs}$.

Recall from equation (1.38) that fixed points of the $S^1$-action satisfy:

$$\frac{d}{d\theta}\bigg|_{\theta=0} e^{i\theta} \cdot (E, \Phi) = (0, i\Phi) = (d_E\Psi, [\Psi, \Phi]), \tag{3.19}$$

where $\Psi$ is a unique infinitesimal automorphism of the Lie algebra. As $\Psi \in E(\mathfrak{h})$, the eigenbundle decomposition $E(\mathfrak{g}^{\mathbb{C}}) = \bigoplus E(\mathfrak{g}^{\mathbb{C}})_k$ induced by the circle action is compatible with the complexified Cartan decomposition. Thus $E(\mathfrak{h}^{\mathbb{C}}) = \bigoplus E(\mathfrak{h}^{\mathbb{C}})_k$ and $E(\mathfrak{p}^{\mathbb{C}}) = \bigoplus E(\mathfrak{p}^{\mathbb{C}})_k$ with $\mathrm{ad}(\Phi) : E(\mathfrak{h}^{\mathbb{C}})_k \to E(\mathfrak{p}^{\mathbb{C}})_{k+1} \otimes K$ and we have deformation complexes:

$$C_k^* : E(\mathfrak{h}^{\mathbb{C}})_k \xrightarrow{ad(\Phi)} E(\mathfrak{p}^{\mathbb{C}})_{k+1} \otimes K \to 0. \tag{3.20}$$

Following [14], the decomposition of stable Hodge bundle into $V_2 \oplus \underline{\mathbb{C}} \oplus V_1 =: G_1 \oplus G_2 \oplus G_3$, induces a corresponding decomposition of $\mathrm{End}(E)$ into $U_k = \bigoplus_{i-j=k} \mathrm{Hom}(G_j, G_i)$ with eigenvalues $ik$ which is compatible with the Cartan decomposition of $\mathrm{End}(E)$ into $\bigoplus E(\mathfrak{h}^{\mathbb{C}})_k$ and $\bigoplus E(\mathfrak{m}^{\mathbb{C}})_k$.

In the length 2 case with $\phi_2 = 0$ so $V_1 = V$, we have:

$$E(\mathfrak{g}^{\mathbb{C}}) = \begin{pmatrix} 0 & \mathrm{Hom}(\underline{\mathbb{C}}, V) \\ 0 & 0 \end{pmatrix}_1 \oplus \begin{pmatrix} \mathrm{End}(V) & 0 \\ 0 & \mathrm{End}(\underline{\mathbb{C}}) \end{pmatrix}_0 \oplus \begin{pmatrix} 0 & 0 \\ \mathrm{Hom}(V, \underline{\mathbb{C}}) & 0 \end{pmatrix}_{-1}, \tag{3.21}$$

with eigenvalues $i, 0$ and $-i$ respectively. The alternative length 2 Hodge case with $V_2 = V$ defines eigenbundles in the same way but with eigenvalues $-i, 0$ and $i$ respectively.

Similarly, in the length 3 case with $E = V_1 \oplus V_2 \oplus L$ we have an eigendecomposition:

$$\begin{aligned} E(\mathfrak{g}^{\mathbb{C}}) = & \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{Hom}(V_1, V_2) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{-2} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{Hom}(L, V_2) \\ \mathrm{Hom}(V_1, L) & 0 & 0 \end{pmatrix}_{-1} \\ & \oplus \begin{pmatrix} \mathrm{End}(V_1) & 0 & 0 \\ 0 & \mathrm{End}(V_2) & 0 \\ 0 & 0 & \mathrm{End}(L) \end{pmatrix}_0 \\ & \oplus \begin{pmatrix} 0 & \mathrm{Hom}(V_2, V_1) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_2 \oplus \begin{pmatrix} 0 & 0 & \mathrm{Hom}(L, V_1) \\ 0 & 0 & 0 \\ 0 & \mathrm{Hom}(V_2, L) & 0 \end{pmatrix}_1, \end{aligned} \tag{3.22}$$

with eigenvalues $\pm 2i, \pm i$ and 0. Note both of these alternate between $E(\mathfrak{p}^{\mathbb{C}})$ and $E(\mathfrak{h}^{\mathbb{C}})$ as expected.

The bundle $(E, \Phi)$ is a local minimum if and only if $\mathbb{H}^1(C_k^*) = 0$ for all $k > 0$ which, we see from equation (3.16), occurs if and only if, for all $k > 0$, the adjoint map $\mathrm{ad}(\Phi) : E(\mathfrak{h}^{\mathbb{C}})_k \to E(\mathfrak{p}^{\mathbb{C}})_{k+1} \otimes K$ is an isomorphism. For length 3 Hodge bundles, $\mathrm{ad}(\Phi) : E(\mathfrak{h}^{\mathbb{C}})_2 \to E(\mathfrak{p}^{\mathbb{C}})_3 \otimes K = 0$ is not an isomorphism so $N_{p,d}$ cannot be local minimum for $0 < p < n$. However, in the length 2 case, we only need consider $C_{>0}^* : E(\mathfrak{h}^{\mathbb{C}})_1 = 0 \to E(\mathfrak{p}^{\mathbb{C}})_2 \otimes K = 0$. Equation (2.20) therefore gives that $(E, \Phi)$ is a local minimum for $\Phi \neq 0$ if and only if $(E, \Phi) \in N_{0,d}$ and $\tau > 0$ or $(E, \Phi) \in N_{n,d}$ and $\tau < 0$.

Otherwise, as $(E, \Phi)$ is a smooth critical point of $\mathcal{M}_{Higgs}$, then the first hypercohomology group of the complex is isomorphic to the $-k$ eigenspace of the Hessian of $g$ as given in [14, Prop 4.11], and so has dimension:

$$\begin{aligned} \dim \mathbb{H}^1(C_k^*) = (g-1) \Big( \mathrm{rank} \left( E(\mathfrak{h}^{\mathbb{C}})_k \right) + \mathrm{rank} \left( E(\mathfrak{p}^{\mathbb{C}})_{k+1} \otimes K \right) \Big) \\ + \deg \left( E(\mathfrak{p}^{\mathbb{C}})_{k+1} \otimes K \right) - \deg \left( E(\mathfrak{h}^{\mathbb{C}})_k \right). \end{aligned} \tag{3.23}$$

To explicitly calculate all other Morse indices, we recall that such indices are real dimensional, so $\operatorname{index}(N_{p,d}) = 2\dim \mathbb{H}^1(C_i^*)$ for $i > 0$. Moreover, as $E(\mathfrak{p}^{\mathbb{C}})_j = 0$ for $j$ even and $E(\mathfrak{h}^{\mathbb{C}})_j = 0$ for $j$ odd then, as our maximum eigenvalue is 2, the only subcomplex we need consider is $C_2^*$. It follows from (3.18) and(3.20) that $\mathbb{H}^1(C_2^*) \simeq H^1(\operatorname{Hom}(V_2, V_1))$. Overall, we have:

$$\begin{aligned}\operatorname{index}(N_{p,d}) &= 2\sum_{k>0} \dim \mathbb{H}^1(C_k^*) \\ &= 2\Big((g-1)\cdot \operatorname{rank}\Big(E(\mathfrak{h}^{\mathbb{C}})_2\Big) + 0 + 0 - \deg\Big(E(\mathfrak{h}^{\mathbb{C}})_2\Big)\Big) \\ &= 2\left((g-1)p(n-p) + (n-p)d - p\left(\frac{n+1}{2}\tau - d\right)\right) \\ &= 2\left((g-1)p(n-p) + nd - \frac{n+1}{2}p\tau\right),\end{aligned} \tag{3.24}$$

where the penultimate line applies well-known vector bundle identities as found in, for example, [13].

Note we also have $\mathbb{H}^0(C_2^*) \simeq H^0(\operatorname{Hom}(V_2, V_1))$ so, as described in [8], for stable bundles $H^0(\operatorname{Hom}(V_2, V_1)) = 0$.

This method can also be used to calculate the dimension of the critical submanifold as the fixed points of the $\mathbb{C}^*$-action corresponding to the nullity of the Hessian of $g$, or $\dim \mathbb{H}^1(C_0^*)$. This then agrees with the dimension count we found in equation (3.14).

We can also calculate the critical values of $\|\Phi\|^2$ associated to $N_{p,d}$. Following [31], we consider the short exact sequences:

$$\begin{aligned} &0 \longrightarrow V_1 \longrightarrow E \longrightarrow \underline{\mathbb{C}} \oplus V_2 \longrightarrow 0, \\ &0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \underline{\mathbb{C}} \oplus V_2 \longrightarrow V_2 \longrightarrow 0. \end{aligned} \tag{3.25}$$

We've already seen that in the superminimal case, both of these are trivial extensions, meaning we have block decompositions of both the connection and the Higgs field:

$$\nabla = \begin{pmatrix} \nabla_{V_1} & 0 & 0 \\ 0 & \nabla_{\underline{\mathbb{C}}} & 0 \\ 0 & 0 & \nabla_{V_2} \end{pmatrix}, \quad \text{and} \quad \Phi = \begin{pmatrix} 0 & \phi_1 & 0 \\ 0 & 0 & \phi_2 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.26}$$

We can consider Hitchin's equations (1.20) for this splitting to give:

$$\begin{aligned} \Theta(\nabla_{V_1}) + \phi_1 \wedge \phi_1^* &= -i\mu I_p \omega, \text{ and} \\ \Theta(\nabla_{V_1}) + \Theta(\nabla_{\underline{\mathbb{C}}}) + \phi_2 \wedge \phi_2^* &= -i\mu I_{p+1}\omega, \end{aligned} \tag{3.27}$$

where $\Theta$ denotes curvature. Taking the trace of this and integrating over $\Sigma$ with a factor of $\frac{i}{2\pi}$, we obtain:

$$\begin{aligned} d + \|\phi_1\|^2 &= \frac{p}{2}\tau, \text{ and} \\ d + \|\phi_2\|^2 &= \frac{p+1}{2}\tau. \end{aligned} \tag{3.28}$$

We then find that for any $(E, \Phi)$ within a given $N_{p,d}$:

$$\|\Phi\|^2 = \frac{2p+1}{2}\tau - 2d. \tag{3.29}$$

Note that if we consider length 2 bundles, then we have either $(p, d) = (0, 0)$ or $(p, d) = \left(n, \frac{n+1}{2}\tau\right)$, so the corresponding critical values are either $\frac{1}{2}\tau$ or $-\frac{1}{2}\tau$. These are both positive quantities as $\tau < 0$ if $p = n$ while $\tau > 0$ if $p = 0$ as seen in Lemma 2.11.

Finally, we can bound this function so, within a given $\tau$ component:

$$\frac{1}{2}\tau \leq \|\Phi\|^2 \leq \frac{2n+1}{2}\tau. \tag{3.30}$$

## 3.3 THE DOWNWARD MORSE FLOW IN THE NILPOTENT CONE

Having examined the critical submanifolds, we now consider the rest of the moduli space of $\mathbb{CH}^n$ equivariant minimal surfaces.

We recall that as $\mathbb{CH}^n$ is a rank 1 symmetric space, all $\mathbb{CH}^n$ equivariant minimal surfaces correspond to Higgs bundles in the nilpotent cone. Moreover, we have a Morse function given in equation (1.34) that corresponds to the $\mathbb{C}^*$-scaling of the Higgs field. As explained in [36], the nilpotent cone is stratified by the unstable manifolds $N^-_{p,d}$ of this downward Morse flow. Now, given a Hodge bundle $q = (E, \Phi)$ in $N_{p,d}$, we define:

$$N^-_{p,d}(q) = \{(E', \Phi') : \lim_{t\to\infty} (E', e^t\Phi') = q\} \tag{3.31}$$

for $N_{p,d}$ the critical submanifold indexed by the rank $p$ and degree $d$ of $V_1$ in $q$. By the Bialynicki-Birula theory [28], this is an affine subvariety isomorphic to the tangent space to the downward flow at $q$.

Furthermore, given a stable length 3 Hodge bundle $q = (V_1 \oplus V_2 \oplus \underline{\mathbb{C}}, \Phi)$, we can consider the short exact sequence:

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0. \tag{3.32}$$

From the Hitchin equations (1.20), we can find several hermitian metrics on $V$ which defines both a smooth splitting of this sequence, and an **extension class** $[\alpha]$ as in [31, §2.2], with:

$$\overline{\partial}_V = \begin{pmatrix} \overline{\partial}_{V_1} & \alpha \\ 0 & \overline{\partial}_{V_2} \end{pmatrix}, \tag{3.33}$$

where $[\alpha] \in H^1(\Sigma, \mathrm{Hom}(V_2, V_1))$. In particular, $[\alpha] = 0$ corresponds to the trivial holomorphic extension $V = V_1 \oplus V_2$ of the Hodge bundle $q$ itself.

**Proposition 3.2.** *Let $q = (E, \Phi)$ be a stable Hodge bundle of the form $E = \underline{\mathbb{C}} \oplus V_1 \oplus V_2$, with $\phi_1 : \underline{\mathbb{C}} \to V_1 \otimes K$ and $\phi_2 : V_2 \to K$.*

*For each extension class $[\alpha] \in H^1(\Sigma, Hom(V_2, V_1))$, there is a unique stable Higgs bundle $(E(\alpha), \Phi)$ given by $E(\alpha) = \underline{\mathbb{C}} \oplus V(\alpha)$ where $V(\alpha)$ is the extension:*

$$0 \longrightarrow V_1 \longrightarrow V(\alpha) \longrightarrow V_2 \longrightarrow 0. \tag{3.34}$$

*defined by $\alpha$ and $\Phi$ is as for $E = E(0)$. This gives a complex algebraic embedding of $H^1(\Sigma, Hom(V_1, V_2))$ into $\mathcal{N}$ which agrees with $N^-_{p,d}(q)$.*

*Proof.* Stability is a Zarikski open condition (see [9], [57]) so, for some open neighbourhood of $\alpha = 0$, we know $(E(\alpha), \Phi)$ is stable. Moreover, for any $\alpha \in H^1(\Sigma, \mathrm{Hom}(V_2, V_1))$, there is a value $t \in \mathbb{C}^*$ such that $t^{-2}\alpha$ lies in this open neighbourhood.

As in [48], there exists a constant gauge transformation $g(t)$ such that:

$$g(t) \cdot \left(E\left(t^{-2}\alpha\right), \Phi\right) = \left(E\left(\alpha\right), t\Phi\right) \tag{3.35}$$

where, for $p = \mathrm{rank}(V_1)$ and relative to the splitting $V_1 \oplus V_2 \oplus \underline{\mathbb{C}}$,

$$g(t) = \begin{pmatrix} tI_p & 0 & 0 \\ 0 & t^{-1}I_{n-p} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.36}$$

As this $\mathbb{C}^*$ action does not change the $\Phi$-invariant subbundles, it preserves the Higgs stability and therefore $(E(\alpha), t\Phi)$ is stable for all $\alpha \in H^1(\Sigma, \mathrm{Hom}(V_2, V_1))$ and $t \in \mathbb{R}$.

Furthermore, $(E(\alpha), \Phi) = (E(\alpha'), \Phi)$ if and only if there exists an isomorphism $f$ such that $f(E(\alpha)) = E(\alpha')$ and $f$ commutes with $\Phi$. However, as $\alpha$ determines $E(\alpha)$ up to common scaling of the extension map (3.34) and, by above, any scaling of this extensions acts non-trivially on $\Phi$, then we must have $\alpha = \alpha'$.

On the other hand, let $(E' = V' \oplus \underline{\mathbb{C}}, \Phi') \in N^-_{p,d}(q)$, i.e.

$$\lim_{t\to\infty}(E', t\Phi') = (E, \Phi), \tag{3.37}$$

for the Hodge bundle $q$.

As we saw just before (3.24), the downward Morse flow is given by the subcomplex $C_2^*$ and $\mathbb{H}^1(C_2^*) \simeq H^1(\mathrm{Hom}(V_2, V_1))$. The image of $H^1(\mathrm{Hom}(V_2, V_1))$ under the map to the extension bundles is open because it identifies with the tangent space as seen in §3.2. Moreover, this image covers $N^-_{p,d}(q)$ because $\mathbb{C}^*$ orbits of the latter intersect the open neighbourhood of $q$, and we saw above that this neighbourhood contains the entirety of any $\mathbb{C}^*$ orbit which intersects it.

We've therefore shown that any extension $V(\alpha)$ gives a stable Higgs bundle in $N^-_{p,d}(q)$ and all such elements of $N^-_{p,d}(q)$ arise in this manner as required. □

Note this proof just show existence and does not give an easy method for computing the corresponding Hodge bundle and extension class for an arbitrary stable Higgs bundle.

We can use this to calculate the critical values of $\|\Phi\|^2$ given in the previous section. As in the superminimal case, we have a block decomposition of the connection:

$$\nabla = \begin{pmatrix} \nabla_{V_1} & 0 & \alpha \\ 0 & \nabla_{\underline{\mathbb{C}}} & 0 \\ -\alpha^* & 0 & \nabla_{V_2} \end{pmatrix}. \tag{3.38}$$

Using this splitting as we did in equation (3.25), we have:

$$\begin{gathered} 0 \longrightarrow V_1 \longrightarrow E \longrightarrow \underline{\mathbb{C}} \oplus V_2 \longrightarrow 0, \\ 0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \underline{\mathbb{C}} \oplus V_2 \longrightarrow V_2 \longrightarrow 0. \end{gathered} \tag{3.39}$$

The Hitchin equations (1.20) now incorporate the extension $\alpha$:

$$\begin{gathered} \Theta(\nabla_{V_1}) - \alpha \wedge \alpha^* + \phi_1 \wedge \phi_1^* = -i\mu I_p \omega \text{ and} \\ \Theta(\nabla_{V_1}) + \Theta(\nabla_{\underline{\mathbb{C}}}) - \alpha \wedge \alpha^* + \phi_2 \wedge \phi_2^* = -i\mu I_{p+1}\omega. \end{gathered} \tag{3.40}$$

As in the superminimal case, we can take the trace, integrate over $\Sigma$ and scale to obtain:

$$\begin{aligned} d + \|\alpha\|^2 + \|\phi_1\|^2 &= \frac{p}{2}\tau, \text{ and} \\ d + \|\alpha\|^2 + \|\phi_2\|^2 &= \frac{p+1}{2}\tau. \end{aligned} \tag{3.41}$$

Therefore, for any Higgs bundle $(E, \Phi)$ in the nilpotent cone:

$$\|\Phi\|^2 = \frac{2p+1}{2}\tau - 2d - 2\|\alpha\|^2. \tag{3.42}$$

In particular, the trivial extension has the greatest energy within any given unstable manifold and the energy decreases along the downward Morse flow, as expected by [43].

## 3.4 TANGENT TO THE DOWNWARD MORSE FLOW

From now on, we fix a stable Hodge bundle $q = (V_1 \oplus V_2 \oplus \underline{\mathbb{C}}, \Phi)$ of length 3. For simplicity, we set $\mathcal{U} = N^-_{p,d}(q)$. By the previous proposition:

$$\mathcal{U} \simeq \left\{ (\hat{E}, \hat{\Phi}) \in \mathcal{N} \mid \lim_{t\to\infty} (\hat{E}, e^t\hat{\Phi}) = q \right\} \simeq H^1(\mathrm{Hom}(V_2, V_1)). \tag{3.43}$$

Our aim in this section is to give another interpretation of the tangent space $T_q\mathcal{U}$ in terms of deformations of the metric connection.

From §1.7.4, recall the complex given in 1.42:

$$\mathcal{E}^0(\mathfrak{h}^{\mathbb{C}}) \xrightarrow{D_1} \mathcal{E}^{0,1}(\mathfrak{h}^{\mathbb{C}}) \oplus \mathcal{E}^{1,0}(\mathfrak{m}^{\mathbb{C}}) \xrightarrow{D_2} \mathcal{E}^{1,1}(\mathfrak{m}^{\mathbb{C}}), \tag{3.44}$$

where:

$$\begin{aligned} D_1 B &= (d''_A B, [\Phi, B]) \\ D_2(\alpha, \varphi) &= d''_A \varphi + [\alpha \wedge \Phi] \end{aligned}$$

If we equip this complex with the hermitian metrics:

$$\begin{aligned} g_0(B, B) &= \int_\Sigma \mathrm{Tr}(BB^*)\omega \\ g_1((\beta, \varphi), (\beta, \varphi)) &= \frac{i}{2}\int_\Sigma \mathrm{Tr}(\beta^* \wedge \beta) + \mathrm{Tr}(\varphi \wedge \varphi^*), \end{aligned} \tag{3.45}$$

then we have an elliptic complex as this complex is exact [60].

The Hodge theorem for elliptic complexes (see [60, §4.5]) then gives that for a point $q$ in the Higgs moduli space $\mathcal{M}$:

$$T_q\mathcal{M} \simeq \ker(D_1^*) \cap \ker(D_2), \tag{3.46}$$

where $D_1^*$ is the adjoint of $D_1$ so:

$$g_1(D_1B, (\alpha, \varphi)) = g_0(B, D_1^*(\alpha, \varphi)). \tag{3.47}$$

This means that every coset in $\ker D_2/\mathrm{Im}D_1$ has a unique representation in $\ker D_1^*$.

**Lemma 3.3.** *Let $\beta \in \mathcal{E}^{0,1}(\mathfrak{h}^{\mathbb{C}})$ such that $d_A''\beta^* = 0$ and $[\Phi \wedge \beta] = 0$, then $(\beta, 0) \in \ker D_1^* \cap \ker D_2$.*

*Proof.* By assumption $(\beta, 0)$ satisfies $d_A''\varphi + [\beta \wedge \Phi] = 0$ as seen in (1.44), then $(\beta, 0) \in \ker D_2$.

We therefore just have to show that $D_1^*(\beta, 0) = 0$. Let $B \in \mathcal{E}^0(\mathfrak{h}^{\mathbb{C}})$, then:

$$\begin{aligned} g_0(B, D_1^*(\beta, 0)) &= g_1(D_1B, (\beta, 0)) \\ &= \frac{i}{2}\int_\Sigma \mathrm{Tr}(d_A''B \wedge \beta) \\ &= \frac{i}{2}\int_\Sigma d\mathrm{Tr}(B\beta^*) - \mathrm{Tr}(Bd_A''\beta^*) \\ &= 0 \end{aligned}$$

where we have used the definitions of the adjoint and $g_1$. As this holds for any $B$, we have that $D_1^*(\beta, 0) = 0$ as required. □

We now let $[\beta, 0] \in T_q\mathcal{M}$ denote the equivalence class of such $(\beta, 0)$.

**Lemma 3.4.** *Let $(\overline{\partial}_t, \Phi_t)$ be any curve of solutions to $\overline{\partial}_t\Phi_t = 0$ with tangent $(\beta, 0)$ at $t = 0$. Then the corresponding curve of Chern connections $A_t$ has $\dot{A}_0 = -\beta^* + \beta$.*

*Proof.* The pair $(A_t, \Phi_t)$ has a derivative $(\alpha, \varphi)$ which satisfies the linearisation of the Hitchin equation (1.20):

$$\begin{aligned} d_A\alpha + [\Phi \wedge \varphi^*] + [\varphi^* \wedge \Phi] &= 0 \\ d_A''\varphi + [\alpha'' \wedge \Phi] &= 0. \end{aligned} \tag{3.48}$$

Since $\overline{\partial}_A = A_t''$, and $(\beta, 0)$ is the tangent at $t = 0$, we have that $\alpha'' = \beta$ and $\dot{\Phi}_0 = 0$. Therefore $\alpha = \beta - \beta^* = \dot{A}_t$.

As $d''_A\beta^* = 0$ and $d''_A = -(d'_A)^*$ by definition, we have that $d'_A\beta = 0$ so $d_A(-\beta^* + \beta) = 0$.

Overall, this means $d_A\alpha = 0$ and the vector $(\alpha, 0)$ is tangent to the space of solutions to the Hitchin equations with $\alpha'' = \beta$. □

Returning to the tangent space, we have the following result:

**Lemma 3.5.** *At the length 3 Hodge bundle $q = (E, \Phi)$,*

$$T_q\mathcal{U} \simeq \frac{\{(\beta, 0) \mid \beta \in \mathcal{E}^{0,1}(Hom(V_2, V_1))\}}{\{(d''_\nabla B, 0) \mid B \in \mathcal{E}^0(Hom(V_2, V_1))\}}$$

*Proof.* By Proposition 3.2 and the Dolbeault isomorphism we have:

$$T_q\mathcal{U} \simeq H^1(\mathrm{Hom}(V_2, V_1)) \simeq H^{0,1}(\mathrm{Hom}(V_2, V_1)).$$

As we are on a Riemann surface, [38] gives that:

$$H^{0,1}(\mathrm{Hom}(V_2, V_1)) = \frac{\mathcal{E}^{0,1}(\mathrm{Hom}(V_2, V_1))}{d''_A\mathcal{E}^0(\mathrm{Hom}(V_2, V_1))}.$$

If $\beta \in \mathcal{E}^{0,1}(\mathrm{Hom}(V_2, V_1))$, we can write it in $\mathrm{End}(E)$ as

$$\beta = \begin{pmatrix} 0 & \beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{with} \quad \Phi = \begin{pmatrix} 0 & 0 & \phi_1 \\ 0 & 0 & 0 \\ 0 & \phi_2 & 0 \end{pmatrix}. \tag{3.49}$$

We then have that $[\Phi \wedge \beta] = 0$ so we have an embedding:

$$H^{0,1}(\mathrm{Hom}(V_1, V_2)) \to T_q\mathcal{M}$$

whose image is $T_q\mathcal{U}$. □

## 3.5 LIMITS OF $\mathbb{C}^*$-SCALING AS $t \to 0$

Continuing to look at the $\mathbb{C}^*$ action given by $t \cdot (E, \Phi) = (E, t\Phi)$, we now consider the limit as $t \to 0$ and compare our results with the $n = 2$ case in [50, Prop 4.3].

We take $(E = V \oplus \underline{\mathbb{C}}, \Phi)$ to be a stable Higgs bundle and, in the first instance, consider $E$ to be stable as a vector bundle. This is a stronger condition than Higgs bundle stability as we now require the slope $\mu$ of all subbundles $F$ of $E$, and not just $\Phi$-invariant subbundles, to satisfy:

$$\mu(F) \leq \mu(E). \tag{3.50}$$

**Lemma 3.6.** *Let* $(E = V \oplus \underline{\mathbb{C}}, \Phi)$ *be a* $PU(n,1)$*-Higgs bundle that is not a Hodge bundle with* $E$ *stable as a vector bundle. Then* $\lim_{t\to 0}(E, t\Phi)$ *is a length 2 Hodge bundle in either* $N_{n,\frac{n+1}{2}\tau}$ *if* $\tau > 0$ *or* $N_{0,0}$ *if* $\tau < 0$.

This is similar to the proof of Proposition 4.3 (ii) in [50].

*Proof.* We begin with the block decompositions

$$\overline{\partial}_E = \begin{pmatrix} \overline{\partial}_V & 0 \\ 0 & \overline{\partial}_{\underline{\mathbb{C}}} \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} 0 & \phi_1 \\ \phi_2 & 0 \end{pmatrix}, \tag{3.51}$$

along with the transformation

$$g_t = \begin{pmatrix} tI_n & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.52}$$

This then gives:

$$\lim_{t\to 0} g_t^{-1}\overline{\partial}_E g_t = \lim_{t\to 0} \begin{pmatrix} \overline{\partial}_V & 0 \\ 0 & \overline{\partial}_{\underline{\mathbb{C}}} \end{pmatrix} = \overline{\partial}_E, \tag{3.53}$$

and

$$\lim_{t\to 0} g_t^{-1}(t\Phi)g_t = \lim_{t\to 0} \begin{pmatrix} 0 & \phi_1 \\ t^2\phi_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \phi_1 \\ 0 & 0 \end{pmatrix}. \tag{3.54}$$

For this to be the limit, we need $(E, \phi_1)$ to be a stable length 2 Hodge bundle. As $\Phi(V) = 0$ in this case, the limit is a length 2 Hodge bundle (see [14, Prop 4.20]). Moreover, by equation (2.20), we have Higgs bundle stability if and only if $\tau > 0$, in which case $(E, \phi_1) \in N_{n,\frac{n+1}{2}\tau}$.

If $\tau < 0$, we can alternatively take:

$$g_t = \begin{pmatrix} t^{-1}I_n & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.55}$$

where similar calculations give the limit as $(E, \phi_2) \in N_{0,0}$. □

In terms of the equivariant minimal surface $(f, c, \rho)$, this means the limit is either holomorphic or antiholomorphic.

This result only considers $\tau \neq 0$ and we cover this remaining case in the next lemma. However, before that, we have a definition:

**Definition 3.7.** *For a semistable holomorphic bundle $\mathcal{E}$, the* ***Jordan-Hölder filtration****:*

$$0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_k = \mathcal{E} \tag{3.56}$$

*requires, for all $i > 0$, both that:*

1. *$\mathcal{E}_i/\mathcal{E}_{i-1}$ is stable and*
2. *$\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) = \mu(\mathcal{E})$.*

This is not a unique filtration. However, given a filtration, we can form the unique associated **graded bundle** $Gr(\mathcal{E}) = \bigoplus \mathcal{E}_i/\mathcal{E}_{i-1}$. In particular, the length $k$ of the filtration is fixed. Two bundles $\mathcal{E}$ and $\mathcal{F}$ with the same graded bundle $Gr(\mathcal{E}) = Gr(\mathcal{F})$ are called **S-equivalent**. More on this can be found in [55, 56].

The following useful result in [40] follows from the additivity of both degree and rank in exact sequences of vector bundles:

**Lemma 3.8.** *Given a short exact sequence of vector bundles:*

$$0 \to A \to B \to C \to 0, \tag{3.57}$$

*we have $\mu(A) < \mu(B)$ (resp. $=$, $>$) if and only if $\mu(A) < \mu(C)$ (resp. $=$, $>$) if and only if $\mu(B) < \mu(C)$ (resp. $=$, $>$).*

We now consider the $t \to 0$ limits in the case $\tau = 0$:

**Lemma 3.9.** *If $\tau = 0$ and $E$ is semistable as a vector bundle, then*

$$\lim_{t \to 0}(E, t\Phi) = (Gr(E), 0). \tag{3.58}$$

This develops the proof of Proposition 4.3 (i) in [50].

*Proof.* Taking $g_t = I_{n+1}$, we get that $\lim_{t\to 0}(E, t\Phi) = (E, 0)$. Moreover, $E$ is, by definition, S-equivalent to $Gr(E)$.

Note that when $\tau = 0$, as $E = V \oplus \underline{\mathbb{C}}$ and $\deg(\underline{\mathbb{C}}) = 0$, the Jordan-Hölder filtration has length $k \geq 2$.

If $k = 2$, we have one filtration of the form $\underline{\mathbb{C}} \subsetneq V \oplus \underline{\mathbb{C}} = E$. $V$ is then stable and has slope 0. Hence the limit is a polystable $PU(n, 1)$-Higgs bundle of the form $(V, 0) \oplus (\underline{\mathbb{C}}, 0)$.

Alternatively, if $k \geq 3$, there exists a filtration of the form:

$$W_1 \subsetneq W_2 \subsetneq \cdots \subsetneq W_{k-1} = V \subsetneq E \tag{3.59}$$

where $W_i/W_{i-1}$ is a stable bundle with slope 0 for all $2 \leq i \leq k-1$. $V$ is therefore strictly semistable and the limit is a (potentially polystable) semistable Higgs bundle. □

Note that if $E$ is a semistable vector bundle, then we necessarily have $0 = \deg(\underline{\mathbb{C}}) \leq \frac{1}{n+1}\deg(E)$ but $\deg(E) = -\frac{2}{n+1}\tau$ so $E$ cannot be a semistable vector bundle if $\tau > 0$.

For $\tau < 0$, finding the limit in general is hindered by not understanding stability conditions for higher rank vector bundles as seen in the previous chapter's stability results (see §2.4).

We finally turn to the possibility where $(E = V \oplus \underline{\mathbb{C}}, \Phi)$ is a semistable Higgs bundle such that $V$ is not semistable as a vector bundle. This means there exists a proper subbundle $W \subset V$ such that $\mu(W) > \mu(V)$. It follows that taking the sum of all subbundles with maximal slope gives the **maximal destabilising bundle** of the same slope.

We now return to the $t \to 0$ limit. Again, the higher rank of the $\Phi$-invariant subbundles means only a partial result can currently be obtained.

**Proposition 3.10.** *Let $(E = V \oplus \underline{\mathbb{C}}, \Phi)$ be a stable Higgs bundle which is not a Hodge bundle, such that $\tau \geq 0$ and $V$ is unstable as a vector bundle, whose maximal destabilising bundle is a line bundle $W$, such that $V/W$ is semistable. Then* $\lim_{t\to 0}(E, t\Phi)$ *is a length 3 Hodge bundle contained in $N_{n-1,\frac{n+1}{2}-\delta}$ with $\delta = \deg(W)$.*

This proof is similar to part (iv) of the proof of Proposition 4.3 in [50].

*Proof.* Given the maximal destabilising line bundle $W \subset V$, we can form the short exact sequence:

$$0 \to W \to V \to V/W \to 0, \tag{3.60}$$

and express $V$ as an extension of $W$ and $V/W$:

$$\overline{\partial}_E = \begin{pmatrix} \overline{\partial}_{V/W} & 0 & 0 \\ \beta & \overline{\partial}_W & 0 \\ 0 & 0 & \overline{\partial}_{\underline{\mathbb{C}}} \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} 0 & 0 & \Phi_{13} \\ 0 & 0 & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & 0 \end{pmatrix}, \tag{3.61}$$

for $\beta \in \mathcal{E}^{0,1}(X, \mathrm{Hom}(V/W, W))$. Note, both $W$ and $V/W$ are semistable as vector bundles and, by Lemma 3.8, $\mu(V/W_1) < \mu(V)$. By considering

$$g_t = \begin{pmatrix} tI_{n-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.62}$$

as before, we get:

$$\lim_{t\to 0} g_t^{-1}\overline{\partial}_E g_t = \lim_{t\to 0} \begin{pmatrix} \overline{\partial}_{V/W} & 0 & 0 \\ t\beta & \overline{\partial}_W & 0 \\ 0 & 0 & \overline{\partial}_{\underline{\mathbb{C}}} \end{pmatrix} = \overline{\partial}_E, \tag{3.63}$$

and

$$\lim_{t\to 0} g_t^{-1}(t\Phi)g_t = \lim_{t\to 0} \begin{pmatrix} 0 & 0 & \Phi_{13} \\ 0 & 0 & t\Phi_{23} \\ t\Phi_{31} & \Phi_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \Phi_{13} \\ 0 & 0 & 0 \\ 0 & \Phi_{32} & 0 \end{pmatrix}. \tag{3.64}$$

We therefore just need to show that $(W \oplus V/W \oplus \underline{\mathbb{C}}, (\Phi_{13}, \Phi_{32}))$ is a stable Hodge bundle.

As $W$ is a line bundle, the only non-trivial $\Phi$-invariant subbundles are subbundles of $V/W$ and $V/W \oplus \underline{\mathbb{C}}$. We've seen that both of these are semistable bundles such that $\mu(V/W) < \mu(V)$ meaning any subbundle $U$ of $V/W$ also necessarily has $\mu(U) < \mu(V)$. The limit as $t$ tends to 0 therefore lies in $N_{n-1,\frac{n+1}{2}\tau-\delta}$. $\square$

— 4 —

# Non-superminimal Equivariant Minimal Surfaces

In chapter 2 we defined the isotropy order $m$ as the maximal index such that $\ell_0, \ell_1, \ldots, \ell_m$ are mutually orthogonal. Equivalently, we can obtain $m$ as the maximal length complete holomorphic flag $F_1 \subset F_2 \subset \ldots \subset F_m$ where $F_k$ denotes the $k$-th order $Z$ osculating space as seen in §2.5. Recall too that $2 \leq m \leq n+1$. We also defined a function $Q_m$ in (2.49) which can be used to determine the isotropy order, and saw in §3.3 how all non-Hodge Higgs bundles lie in the downward Morse flow $\mathcal{U}$ of a critical submanifold $N_{p,d}$ consisting of Hodge bundles with $p = \operatorname{rank}(V_1)$ and $d = \deg(V_1)$.

## 4.1 THE DIFFERENTIAL $Q_3$

We now restrict our attention to $Q_3$. We use the implicit function theorem at a Hodge bundle $q \in \mathcal{U}$ to prove the existence of Higgs bundles of isotropy order 3.

As we saw in 3.2, any Higgs bundle in the nilpotent cone can be considered as an extension $(E(\alpha), \Phi)$ where $q = (E(0), \Phi)$ is a stable length 3 Hodge bundle.

Moreover, in Lemma 3.5, we identified the tangent space $T_q\mathcal{U}$ with the space of extensions $H^1(\operatorname{Hom}(V_2, V_1))$. By the Dolbeault isomorphism [37], $H^1(\operatorname{Hom}(V_2, V_1))$ is isomorphic to the space of harmonic $(0,1)$ forms $\mathcal{H}^{0,1}(\operatorname{Hom}(V_2, V_1))$. Using Serre duality and this last identification again, we have:

$$T_q\mathcal{U} \simeq \mathcal{H}^{0,1}(\operatorname{Hom}(V_2, V_1)) \simeq \mathcal{H}^{1,0}(\operatorname{Hom}(V_1, V_2)) \simeq H^0(\operatorname{Hom}(V_1, V_2) \otimes K) \qquad (4.1)$$
$$\beta \mapsto \beta^*.$$

**Lemma 4.1.** *At a Hodge bundle $q = (E, \Phi)$, the differential of $Q_3 : \mathcal{U} \to H^0(K^3)$ is*

$$\begin{aligned} dQ_3 : T_q\mathcal{U} &\to H^0(K^3) \\ \alpha &\mapsto -Tr(\phi_2\beta^*\phi_1) \end{aligned}$$

*where $\beta$ is the unique harmonic representative of its cohomology class $[\alpha]$ and hence, $\beta^* \in H^0(Hom(V_1, V_2) \otimes K)$.*

*Proof.* Since $Q_3$ is independent of the gauge used for $(D'', \Phi)$, we let $(D''_t, \Phi)$ be a curve tangent to $(\beta, 0)$ where $d''_A\beta^* = 0$. Now, by Lemma 3.4 we have:

$$\dot{D}_t = -\beta^* + \beta \quad \text{and} \quad \dot{\Phi} = 0. \tag{4.2}$$

Hence:

$$\begin{aligned} \frac{dQ_3}{dt} &= \mathrm{Tr}\left(\dot{\phi}_2 D'\phi_1\right) + \mathrm{Tr}\left(\phi_2\dot{D}'\phi_1\right) + \mathrm{Tr}\left(\phi_2 D'\dot{\phi}_1\right) \\ &= -\mathrm{Tr}(\phi_2\beta^*\phi_1). \end{aligned} \tag{4.3}$$

□

We needed the harmonic representative in order to ensure that $(\alpha, \varphi) = (-\beta^* + \beta, 0)$ is the tangent in the self-dual Yang Mills description of the tangent space as seen in Lemma 3.4:

$$\begin{aligned} d_A\alpha + [\Phi \wedge \varphi^*] + [\Phi^* \wedge \varphi] &= 0 \\ d''_A\varphi + [\beta \wedge \Phi] &= 0. \end{aligned} \tag{4.4}$$

Finally, we show that there is a collection of Hodge bundles at which $dQ_3$ is onto. Before returning to our first definition of $Q_3$ in 2.39, we recall from §2.4 and [48], the divisors $D_1$ and $D_2$ given by:

$$\ell_{-1} = K(-D_2) \quad \text{and} \quad \ell_1^* = K^{-1}(D_1). \tag{4.5}$$

**Lemma 4.2.** *Along $\mathcal{U}$, $Q_3 : \mathcal{U} \to H^0(K^3(-D_1 - D_2))$ where $D_1$ and $D_2$ are as defined in* (4.5)*.*

*Proof.* For $(E, \Phi) \in \mathcal{U}$, we have $Q_3(E, \Phi) = \mathrm{Tr}(\phi_2 A'_1\phi_1)$.

By definition of the harmonic sequence, $\phi_2 A'_1\phi_1$ is a holomorphic section of $\mathrm{Hom}(\ell_0, \ell_0) \otimes K^3$ with divisors of zeroes at least $D_1 + D_2$ meaning $Q_3(E, \Phi) \in H^0(K^3(-D_1 - D_2))$. □

As we have:

$$\begin{aligned} 0 \to K^{-1}(D_1) &\xrightarrow{\phi_1} V_1 \\ 0 \to W \to V_2 &\xrightarrow{\phi_2} K(-D_2) \to 0, \end{aligned} \tag{4.6}$$

for $\eta \in H^0(\mathrm{Hom}(V_1, V_2) \otimes K)$, we can define:

$$\begin{aligned} \gamma : H^0(\mathrm{Hom}(V_1, V_2) \otimes K) &\to H^0(K^3(-D_1 - D_2)) \\ \eta &\mapsto \mathrm{Tr}(\phi_2 \eta \phi_1). \end{aligned} \tag{4.7}$$

Note that $\mathrm{Tr}(\phi_2\eta\phi_1) \in H^0(\mathrm{Hom}(K^{-1}(D_1)), K(D_2)) \otimes K) = H^0(\mathrm{Hom}(K^3(-D_1 - D_2))$. We therefore have that $dQ_3$ is surjective if and only if $\gamma$ is surjective.

To establish when $\gamma$ is surjective, we consider two short exact sequences.

$$\begin{aligned} 0 \to W_2 \to V_2 &\xrightarrow{\phi_2} \ell_{-1} \to 0, \\ 0 \to W_1^* \to V_1^* &\xrightarrow{\phi_1^*} \ell_1^* \to 0. \end{aligned} \tag{4.8}$$

Together, these allow us to define a double complex of vector bundles:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(W_1, W_2) & \xrightarrow{\gamma_1} & \mathrm{Hom}(V_1, W_2) & \longrightarrow & \mathrm{Hom}(\ell_1, W_2) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}(W_1, V_2) & \xrightarrow{\gamma_1} & \mathrm{Hom}(V_1, V_2) & \xrightarrow{\delta_1} & \mathrm{Hom}(\ell_1, V_2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \delta_2} & \searrow^{\delta} & \downarrow{\scriptstyle \delta_2} & & \\ 0 & \longrightarrow & \mathrm{Hom}(W_1, \ell_{-1}) & \longrightarrow & \mathrm{Hom}(V_1, \ell_{-1}) & \xrightarrow{\delta_1} & \mathrm{Hom}(\ell_1, \ell_{-1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \tag{4.9}$$

As all squares commute, we have $\delta = \delta_2\delta_1$ and we define:

$$F := \ker \delta \simeq \frac{\ker \delta_1 \oplus \ker \delta_2}{\ker \delta_1 \cap \ker \delta_2} \simeq \frac{\mathrm{Hom}(V_1, W_2) \oplus \mathrm{Hom}(W_1, V_2)}{\mathrm{Hom}(W_1, W_2)}. \tag{4.10}$$

From $\delta$, we get a further exact sequence:

$$0 \to F \otimes K \to \mathrm{Hom}(V_1, V_2) \otimes K \to \mathrm{Hom}(\ell_1, \ell_{-1}) \otimes K \to 0, \tag{4.11}$$

which implies:

$$H^0(\mathrm{Hom}(V_1, V_2) \otimes K) \xrightarrow{\gamma} H^0(\mathrm{Hom}(\ell_1, \ell_{-1}) \otimes K) \to H^1(F \otimes K). \tag{4.12}$$

Therefore, $\gamma$ is surjective if and only if $H^1(F \otimes K) = 0$.

We now consider some special cases where this condition can be easily checked.

**Case 1: rank**$(V_2) = 1$

In this case, $V_2 \simeq K(-D_2)$ and $W_2 = \underline{0}$ so $F \simeq \mathrm{Hom}(W_1, K(-D_2)) \simeq \ker(\delta_1)$. By Serre duality, we have:

$$\begin{aligned} H^1(F \otimes K) &\simeq H^1(W_1^* \otimes K^2(-D_2)) \\ &\simeq H^0(W_1 \otimes K^{-1}(D_2)) \end{aligned} \tag{4.13}$$

Therefore $h^1(F \otimes K) = h^0(W_1 \otimes K^{-1}(D_2)) = 0$ provided:

$$\deg(W_1) < \deg(K) - \deg(D_2). \tag{4.14}$$

**Case 2: rank**$(V_1) = 1$

In this case, $V_1 \simeq K^{-1}(D_1)$ and $W_1 = \underline{0}$ so $F \simeq \mathrm{Hom}(K^{-1}(D_1), W_2) \simeq \ker(\delta_2)$. Similarly to above, we get:

$$H^1(F \otimes K) \simeq H^0(K(-D_1) \otimes W_2^*) \tag{4.15}$$

and $h^1((F \otimes K) = 0$ if and only if

$$\deg(W_2) > \deg(K) - \deg(D_1). \tag{4.16}$$

Note these two cases together complete the picture for $\mathbb{CH}^3$ Higgs bundles where $W_1$ and $W_2$ are necessarily line bundles.

**Lemma 4.3.** *Let $n = 3$ and $\mathcal{U}_3$ be the subset of $\mathcal{U}$ consisting of Higgs bundles of isotropy order at least 3. If $H^1(F \otimes K) = 0$, then $\mathcal{U}_3$ is a smooth manifold with dimension:*

$$h^1(Hom(V_2, V_1)) - h^0(K^3(-D_1 - D_2)) > 0. \tag{4.17}$$

*Proof.* By assumption $dQ_3$ is surjective so, by the implicit function theorem (e.g. in [37]), locally around the point $q$, $\mathcal{U}_3$ is a smooth submanifold and

$$\dim \mathcal{U}_3 = \dim \mathcal{U} - h^0(K^3(-D_1 - D_2)). \tag{4.18}$$

Moreover, we've seen that $\dim \mathcal{U} = h^1(\mathrm{Hom}(V_2, V_1))$ in Lemma 3.2. $\square$

We now calculate this dimension more explicitly in case 1 above, i.e. where $\mathrm{rank}(V_2) = 1$. Then, for $d_{-1} = \deg(\ell_{-1}) = \deg(V_2)$,

$$\begin{aligned} h^1(\mathrm{Hom}(V_2, V_1)) &= (n-1)(g-1) - \deg(V_1) + (n-1)\deg(V_2) \\ &= (n-1)(g-1) + \frac{n+1}{2}\tau + nd_{-1}, \end{aligned} \tag{4.19}$$

where we have used identities given in [39] and (1.26). We also have:

$$\begin{aligned} h^0(K^3(-D_1 - D_2)) &= 5(g-1) - \hat{d}_1 - \hat{d}_2 \\ &= (g-1) + d_1 + d_{-1}, \end{aligned} \tag{4.20}$$

where $\hat{d}_i = \deg(D_i)$ and, in the last line, we use an identification rising from (2.18). Overall, we therefore have:

$$\begin{aligned} \dim(\mathcal{U}_3) &= (n-1)(g-1) + \frac{n+1}{2}\tau + nd_{-1} - (g-1) - d_1 - d_{-1} \\ &= (n-2)(g-1) + \frac{n+1}{2}\tau + (n-1)d_{-1} - d_1. \end{aligned} \tag{4.21}$$

Restricting this result to the $\mathbb{CH}^3$ case, we get the following:

**Theorem 4.4.** *There exist superconformal $\mathbb{CH}^3$ Higgs bundles.*

*Proof.* In (3.14), we found the dimension of the critical submanifold in which a given length 3 Hodge bundle $q$ belongs, while we also calculated the Morse index in (3.24). In $\mathbb{CH}^3$, the maximal isotropy order is 4 so we therefore have:

$$\begin{aligned} \dim(\mathcal{U}_4) &= (g-1)(2p^2 - 6p + 13) + 2d - 2\tau + 1 \\ \dim(\mathcal{U}_3) &= (g-1) + 2\tau + 2d_{-1} - d_1 \\ \dim(\mathcal{U}_2) &= 2\left((g-1)p(3-p) + 3d - 2p\tau\right). \end{aligned} \tag{4.22}$$

This means the dimension of the subspace of Higgs bundles with strictly isotropy order 3, $U_3$ is:

$$\dim(U_3) = (g-1)(2p^2 - 6p + 12) + 2d - 4\tau + 1 - 2d_{-1} - d_1. \tag{4.23}$$

From (2.19) and Lemma 2.12, we can bound $d_{-1}$ $d_1$ respectively while (2.34) bounds $d$ so we have:

$$\dim(U_3) > 2(g-1)(p^2 - 3p + 2) + \frac{2p-21}{2}\tau + 1. \tag{4.24}$$

For $\mathbb{CH}^3$, a length 3 Hodge bundle can have either $p = 1$ or $p = 2$. In this former case, we have:

$$\dim(U_3) > -\frac{19}{2}\tau + 1, \tag{4.25}$$

while the latter gives:

$$\dim(U_3) > -\frac{17}{2}\tau + 1. \tag{4.26}$$

Recalling that the Toledo invariant for $\mathbb{CH}^3$ belongs in $\frac{1}{2}\mathbb{Z}$, then in either case, we have that Higgs bundles of isotropy order 3 exist in every Toledo invariant such that $-2(g-1) \leq \tau \leq 0$. □

## 4.2 HIGHER ISOTROPY ORDER

We now consider the case of general isotropy order. For ease, we begin by recalling a number of pertinent results. Recall that $q = (E, \Phi)$ is a length 3 Hodge bundle with $E = V_1 \oplus V_2 \oplus \mathbb{C}$ where $d = \deg(V_1)$ and $p = \operatorname{rank}(V_1)$.

1. In §2.1, we defined the holomorphic flag:

$$F_1 \subset \cdots \subset F_{n-1} = V_1$$

   where $F_k = \bigoplus_{j=1}^k \ell_j$. This enables us to consider the harmonic sequence as:

$$\ell_{-1} \xrightarrow{\phi_2} \ell_0 \xrightarrow{\phi_1} F_{n-1},$$

   where we have once again suppressed the tensor products with the canonical bundle $K$.

2. In equation (3.43), for a stable Hodge bundle $q = (E, \Phi)$, we defined the space of extension as:

$$\mathcal{U} = \left\{ (\hat{E}, \hat{\Phi}) \in \mathcal{N} \mid \lim_{t \to \infty} (\hat{E}, e^t \hat{\Phi}) = (E, \Phi) \right\}.$$

3. In Lemma 3.5, we showed that $T_q\mathcal{U} \simeq H^1(\operatorname{Hom}(V_2, V_1))$.
4. In Lemma 3.2, we found that the map $H^1(\operatorname{Hom}(V_2, V_1)) \to \mathcal{U}$ is an isomorphism of affine varieties.

5. From Lemma 4.1, we get that $dQ_3(\eta) = \mathrm{Tr}(\phi_2\eta^*\phi_1)$ where $\eta \in \mathcal{E}^{0,1}(\mathrm{Hom}(V_2, V_1))$ is the unique harmonic representative corresponding to $\alpha$ under the Dolbeault isomorphism.

We now generalise Lemma 4.1 before addressing the existence of Higgs bundles of higher isotropy order.

**Lemma 4.5.** *Let $\eta \in \mathcal{H}^{0,1}(Hom(V_2, V_1)) \simeq T_q\mathcal{U}$. If $dQ_j(\eta) = 0$ for all $3 \leq j \leq k$, then*

$$dQ_{k+1}(\eta) = -Tr(\phi_2\eta^* A'_{k-2} \dots A'_1\phi_1). \tag{4.27}$$

*Proof.* We prove this via an inductive argument.

From Corollary 2.14,

$$Q_4 = \mathrm{Tr}(\phi_2 D'_2 D'_1 \phi_1). \tag{4.28}$$

Let $D'(t)$ be the connection on the smooth bundle $\underline{\mathbb{C}} \oplus V_1 \oplus V_2$ which is the Chern connection for the metric on $V(t\eta)$. We then have $\dot{\Phi} = 0$ and $A(t) = D'(t) - D'(0) \in \mathcal{E}^1(\mathrm{End}(V, V))$ where $\dot{A}(0) = -\eta^*$ as seen in Lemma 3.4.

Hence, using $\mathrm{Hom}(\underline{\mathbb{C}}, V) = V$, we have $D'_j(t) = D'(t) \otimes \delta = (D'(0) + A) \otimes \delta$ where $\delta$ is the connection on $K^j$ induced by the Levi-Civita connection for the Käher metric fixed on $\Sigma$. We then have:

$$\begin{aligned}(D'_j(t) - D'_j(0))(v \otimes \sigma) &= (D'(0) + A)v \otimes \sigma + v \otimes \delta\sigma - D'(0)v \otimes \sigma - v \otimes \delta\sigma \\ &= Av \otimes \sigma.\end{aligned} \tag{4.29}$$

Therefore $D'_j(t) - D'_j(0) = A$ acting on $V \otimes K^j$ and:

$$\begin{aligned}\frac{d}{dt}Q_4([t\eta])\bigg|_0 &= \mathrm{Tr}(\phi_2\dot{A}(0)D'_1\phi_1) + \mathrm{Tr}(\phi_2 D'_2\dot{A}(0)\phi_1) \\ &= -\mathrm{Tr}(\phi_2\eta^* D'_1\phi_1) - \mathrm{Tr}(\phi_2 D'_2\eta^*\phi_1)\end{aligned} \tag{4.30}$$

By assumption $dQ_3([\eta]) = -\mathrm{Tr}(\phi_2\eta^* A'_1\phi_1) = 0$ which implies $\eta^* : F_1 \to \ker\phi_2 = V_1$. Since $\eta^* : V_1 \to V_2 \otimes K$ by definition, it follows that $\eta^*\phi_1 = 0$. In that case, $D'_2\eta^*\phi_1 = 0$ and $\eta^* D'_1\phi_1 = \eta^* A'_1\phi_1$. Therefore, as required,

$$dQ_4([\eta]) = -\mathrm{Tr}(\phi_2\eta^* A'_1\phi_1).$$

Now assume:

$$dQ_j([\eta]) = -\mathrm{Tr}(\phi_2\eta^* A'_{j-3} \dots A'_1\phi_1) = 0$$

for all $4 \leq j \leq k$, then we have:

$$\eta^* : F_{k-2} \to \ker \phi_2 = V_1$$

meaning, in particular, that $F_{k-2} \subset \ker \eta^*$.

Finally, we consider:

$$dQ_{k+1}([\eta]) = -\sum_{j=1}^{k-1} \mathrm{Tr}(\phi_2 D'_{k-1} \cdots D'_{j+1} \eta^* D'_{j-1} \cdots D'_1 \phi_1) \tag{4.31}$$

However, we know $\mathrm{Im} D'_j \cdots D'_1 \phi_1 \subset F_j \otimes K^{j+1}$ which means, by the inductive assumption:

$$\begin{aligned} dQ_{k+1}([\eta]) &= -\mathrm{Tr}(\phi_2 \eta^* D'_{k-1} \cdots D'_1 \phi_1) \\ &= -\mathrm{Tr}(\phi_2 \eta^* A'_{k-2} \cdots A'_1 \phi_1) \end{aligned}$$

where in the last line, we have used Lemma 2.15. □

In the previous section, we restricted the domain of $Q_3$ to $\mathcal{U}$ which consists of Higgs bundles with isotropy order at least 2. To generalise this argument, we first introduce some new notation.

Let

$$\mathcal{U}_k = \{ r \in \mathcal{U} \mid Q_j(r) = 0 \text{ for all } 2 \leq j \leq k \}. \tag{4.32}$$

We know $\mathcal{U}_2 = \mathcal{U}$ is a smooth submanifold containing the Hodge bundle $q$ and we have:

$$\mathcal{U} = \mathcal{U}_2 \supset \mathcal{U}_3 \supset \mathcal{U}_4 \supset \cdots. \tag{4.33}$$

To prove the existence of Higgs bundles of an isotropy order at least $m$, we need to show that $\dim \mathcal{U}_m > 0$. This can be found using an inductive argument that follows from the previous section. When $dQ_3(q)$ is surjective the implicit function theorem says $\mathcal{U}_3$ is locally smooth about $q$ and of dim $\ker(dQ_3(q))$. If this dimension is positive, we restrict $Q_4$ to $\mathcal{U}_3$ to consider $dQ_4(q) : T_q\mathcal{U}_3 \to H^0(K^4)$. By repeating the argument until the surjectivity of $dQ_{m+1}(q)$ fails, we determine the existence of Higgs bundles of a given isotropy order. We therefore have the following theorem:

**Theorem 4.6.** *Let $k \geq 1$ and $q = (E, \Phi)$ be a Hodge bundle with $\tau \geq 0$ and rank$(V_2) = 1$ such that, for $k \geq 2$,*

1. *$h^0(Hom(V_2, V_1/F_k)) = 0$ and $h^1(Hom(V_2, V_1/F_k)) > 0$*

*2.* $A_1', \cdots, A_{k-1}'$ *have no zeroes.*

*Then locally about* $q$ *and for* $1 \leq j \leq k$*, we have* $\mathcal{U} \supseteq \mathcal{U}_3 \supseteq \cdots \supseteq \mathcal{U}_{k+2}$ *with*

$$T_q\mathcal{U}_{j+2} \simeq H^1(Hom(V_2, V_1/F_j)).$$

As in the previous section, we wish to find an isomorphism $\gamma$ which is surjective if and only of $dQ_k$ is. This requires three technical lemmas before we prove Theorem 4.6.

**Lemma 4.7.** *If* $A_1', \ldots, A_{k-1}'$ *have no zeroes then* $H^0(Hom(V_2, V_1/F_k)) = 0$ *implies that* $H^0(Hom(V_2, V_1/F_j)) = 0$ *for* $1 \leq j \leq k$ *and* $rank(V_2) = 1$.

*Proof.* From the short exact sequence:

$$0 \to F_j/F_{j-1} \to V_1/F_{j-1} \to V_1/F_j \to 0, \tag{4.34}$$

we get the long exact sequence:

$$0 \to H^0(\mathrm{Hom}(V_2, F_j/F_{j-1})) \to H^0(\mathrm{Hom}(V_2, V_1/F_{j-1})) \to H^0(\mathrm{Hom}(V_2, V_1/F_j)) \to \cdots . \tag{4.35}$$

If $H^0(\mathrm{Hom}(V_2, V_1/F_k)) = 0$, then we have $H^0(\mathrm{Hom}(V_2, V_1/F_{k-1})) = 0$ if and only if $H^0(\mathrm{Hom}(V_2, F_k/F_{k-1})) = 0$ which we now show with induction.

By construction of $F_j$, $F_j/F_{j-1} \simeq \ell_j$ and, provided $A_j'$ has no zeroes for $1 \leq j \leq k$, then the harmonic sequence gives that $\ell_j \simeq K^{-j}(D_1)$. Moreover, as $\mathrm{rank}(V_2) = 1$, then $V_2 = K(-D_2)$. Therefore:

$$\deg(\mathrm{Hom}(V_2, F_j/F_{j-1})) = \deg(K^{-j-1}(D_1 + D_2)) < \deg(K^{-j}(D_1 + D_2)). \tag{4.36}$$

However, as $0 \neq \phi_j \in H^0(K(-D_j))$, then $\deg(K(-D_j)) \geq 0$ with equality if and only if $K(-D_j) \simeq \mathbb{C}$ which cannot happen as stability gives that $H^0(\mathrm{Hom}(V_2, V_1)) = 0$ as we saw in §3.2. Therefore, in the case $j = 1$, $H^0(\mathrm{Hom}(V_2, \ell_1)) = 0$ since $\mathrm{Hom}(V_2, \ell_1) \subseteq \mathrm{Hom}(V_2, V_1)$ is a holomorphic subbundle, so $H^0(K^{-2}(D_1 + D_2)) = 0$.

Overall this means, for $1 \leq j \leq k$,

$$\deg(K^{-j-1}(D_1 + D_2)) < \deg(K^{-2}(D_1 + D_2)) < 0 \tag{4.37}$$

and, by [39], we have $H^0(\mathrm{Hom}(V_2, F_j/F_{j-1})) = 0$. As this holds for all $1 \leq j \leq k$, induction then gives the desired results. □

Note that under the assumptions of the previous result, equation (4.35) gives that $H^0(\mathrm{Hom}(V_2, F_j/F_{j-1})) \simeq H^0(\mathrm{Hom}(V_2, V_1/F_{j-1}))$.

**Lemma 4.8.** *If $A'_1, \dots, A'_{k-1}$ have no zeroes and $h^0(Hom(V_2, V_1/F_k)) = 0$, then $h^1(Hom(V_2, V_1/F_k)) > 0$ implies $h^1(Hom(V_2, V_1/F_j)) > 0$ for $1 \le j \le k$.*

*Proof.* We can calculate:

$$\begin{aligned} h^1(\mathrm{Hom}(V_2, V_1/F_j)) &- h^1(\mathrm{Hom}(V_2, V_1/F_k)) \\ &= (n-1-j)(g-1) - \deg(V_1) + \deg(F_j) + (n-1-j)\deg(V_2) \\ &\quad - (n-1-k)(g-1) + \deg(V_1) - \deg(F_k) - (n-1-k)\deg(V_2) \\ &= (k-j)\left((g-1) + \deg(V_2)\right) + \sum_{i=k+1}^{j} \deg(K^i(-D_1)). \end{aligned} \tag{4.38}$$

By stability, $\deg(K^i(-D_1)) > 0$ for $i \ge 1$ and for $\tau > 0$, $\deg(V_2) > 0$ so this difference is positive as required. □

**Lemma 4.9.** *Using the identification $T_q\mathcal{U} \simeq H^1(Hom(V_2, V_1))$ and assuming both that $T_q\mathcal{U}_{j+2} = \ker(dQ_{j+2})_q$ and $rank(V_2) = 1$, then $[\eta] \in T_q\mathcal{U}_{j+2}$ if and only if $F_j \subset \ker \eta^*$.*

*Proof.* We saw in Lemma 4.5, that $[\eta] \in \ker(dQ_{i+2})_q$ for all $1 \le i \le j$ which means that $\eta^* \in H^0(\mathrm{Hom}(V_1, V_2) \otimes K)$ satisfies $\mathrm{Tr}(\phi_2 \eta^* A'_i \dots A'_1) = 0$ for $1 \le i \le j$. By definition, this is equivalent to $F_j \subset \ker \eta^*$ as required.

On the other hand, if $F_j \subset \ker \eta^*$, then $\mathrm{rank}(V_2) = 1$ implies $\eta^* A'_i \dots A'_1 = 0$ for all $1 \le i \le j$. But this is equivalent to $[\eta] \in \ker(dQ_{i+2})$ for all $1 \le i \le j$ as above and as required. □

We are now in a position to prove Theorem 4.6.

*Proof.* For $k = 1$, this is $Q_3$ and addressed in the previous section. This provides the base case for our induction.

We suppose it is true for $k = j$ and assume the conditions of the theorem hold of $k = j + 1$. From Lemma 4.9, we consider

$$\begin{aligned} (dQ_{j+3})_q : T_q\mathcal{U}_{j+2} &\to H^0(K^{j+3}(-D_1 - D_2)) \\ [\eta] &\mapsto \mathrm{Tr}(\phi_2 \eta^* A'_j \cdots A'_1 \phi_1) \end{aligned} \tag{4.39}$$

where $F_j \subset \ker \eta^*$. Note $F_j \subset \ker \eta^*$ happens if and only if $\eta^*$ lies in the kernel of the restriction map:

$$H^0(\mathrm{Hom}(V_1, V_2) \otimes K) \to H^0(\mathrm{Hom}(F_j, V_2) \otimes K). \tag{4.40}$$

This kernel equals the image of

$$0 \to H^0(\mathrm{Hom}(V_1/F_j, V_2) \otimes K) \to H^0(\mathrm{Hom}(V_1, V_2) \otimes K), \tag{4.41}$$

derived from the short exact sequence:

$$0 \to (V_1/F_j)^* \to V_1^* \to F_j^* \to 0. \tag{4.42}$$

Using the inductive assumption $T_q\mathcal{U}_{j+2} \simeq H^1(\mathrm{Hom}(V_2, V_1/F_j))$, which is in turn isomorphic to $H^0(\mathrm{Hom}(V_1/F_j, V_2) \otimes K)$ by Serre duality and as seen in (4.1). This total derivative is equivalent to the linear map

$$\begin{aligned} \gamma : H^0(\mathrm{Hom}(V_1/F_j, V_2) \otimes K) &\to H^0(\mathrm{Hom}(F_1, V_2) \otimes K^{j+1}) \\ \psi &\mapsto \psi A_j' \cdots A_1' \end{aligned} \tag{4.43}$$

using $A_j' \cdots A_1' : F_1 \to F_{j+1}/F_j \otimes K^j$.

From the short exact sequence dual to (4.34),

$$0 \to (V_1/F_{j+1})^* \to (V_1/F_j)^* \to (F_{j+1}/F_j)^* \to 0, \tag{4.44}$$

we get the long exact sequence:

$$\begin{array}{l} 0 \longrightarrow H^0(\mathrm{Hom}(V_1/F_{j+1}, V_2)) \longrightarrow H^0(\mathrm{Hom}(V_1/F_j, V_2))) \longrightarrow H^0(\mathrm{Hom}(F_{j+1}/F_j, V_2)) \\ \qquad\qquad \swarrow \\ H^1(\mathrm{Hom}(V_1/F_{j+1}, V_2)) \longrightarrow \cdots \end{array} \tag{4.45}$$

By assumption $A_j' \cdots A_1' : F_1 \to F_{j+1}/F_j \otimes K^j$ is an isomorphism so $\gamma$ is equivalent to

$$\hat{\gamma} : H^0(\mathrm{Hom}(V_1/F_j, V_2) \otimes K) \to H^0(\mathrm{Hom}(F_{j+1}/F_j, V_2) \otimes K) \tag{4.46}$$

and this is the map induced by the long exact sequence given in 4.35.

Hence $\hat{\gamma}$ is surjective as, by the inductive assumption $\underline{0} = H^0(\mathrm{Hom}(V_2, V_1/F_{j+1})) \simeq H^1(\mathrm{Hom}(V_1/F_{j+1}, V_2) \otimes K)^*$.

Also

$$T_q\mathcal{U}_{j+3} \simeq \ker \hat{\gamma} \simeq H^0(\mathrm{Hom}(V_2, V_1/F_{j+1}) \otimes K) \simeq H^1(\mathrm{Hom}(V_1/F_{j+1}, V_2)), \tag{4.47}$$

which concludes the induction as required. □

Provided $A'_1, \ldots, A'_{k-1}$ have no zeroes, then we can take:

$$\begin{aligned} \deg(F_k) &= \sum_{i=1}^{k} \deg(K^i(-D_1)) \\ &= \sum_{i=1}^{k} (2i(g-1) - \hat{d}_1) \\ &= k(k+1)(g-1) - k\hat{d}_1. \end{aligned} \tag{4.48}$$

A sufficient condition for the first assumption of Theorem 4.6 is that the degree of $\mathrm{Hom}(V_2, V_1/F_k))$ is negative. Therefore, we calculate:

$$\begin{aligned} \deg(\mathrm{Hom}(V_2, V_1/F_k)) &= (p-k)(\hat{d}_{-1} - 2(g-1)) + (p(p+1) - k(k+1))(g-1) + (k-p)\hat{d}_1 \\ &= (p-k)(\hat{d}_{-1} - \hat{d}_1) + (p^2 - p - k^2 + k)(g-1). \end{aligned} \tag{4.49}$$

For this to be negative, we therefore require:

$$(p^2 - p - k^2 + k)(g-1) < (p-k)(\hat{d}_1 - \hat{d}_{-1}). \tag{4.50}$$

The second assumption of Theorem 4.6, that each $A'_{k-1}$ has no zeroes, is therefore the main problem as everything else can be reduced to ensuring certain conditions involving topological quantities (such as $\hat{d}_1$ and $\hat{d}_{-1}$) can be found.

Alternatively, we could show that $Q_{k+2}$ has the same zeroes on $\mathcal{U}_{k+1}$. However, understanding the zeroes of $A'_1$ is already a challenge.

While Theorem 4.6 gives a method for calculating a dimension of $\mathcal{U}_k$ and hence determining the existence of Higgs bundles of a given isotropy, it predicts dimension 0 for high isotropy order relative to $n$ except for small $n$.

Under the assumptions of Theorem 4.6, we can find the codimension of $\mathcal{U}_{m+2}$ in $\mathcal{U}$ as:

$$\begin{aligned} \sum_{j=3}^{m+2} h^0(K^j(-D_1 - D_2)) &= \sum_{j=3}^{m+2} (j\deg(K) - \hat{d}_1 - \hat{d}_2 + 1 - g) \\ &= \left(\frac{(m+2)(m+3)}{2} - 3\right)\deg(K) + m(g - 1 - \hat{d}_1 - \hat{d}_2) \\ &= m\left((m+6)(g-1) - \hat{d}_1 - \hat{d}_2\right), \end{aligned} \tag{4.51}$$

which grows quadratically with $m$ whereas:

$$\begin{aligned}\dim \mathcal{U} &= h^1(\mathrm{Hom}(V_2, V_1)) \\ &= (n-1)(g-1) - \deg(V_1) + (n-1)\deg(V_2) \\ &= (n-1)(g-1) + \frac{n+1}{2}\tau + n\deg(V_2)\end{aligned} \tag{4.52}$$

grows linearly with $n$.

Moreover, for $A'_1, \ldots, A'_{m-1}$ to be invertible, $\ell_j \simeq K^{-j}(D_1)$ for all $j \leq m$ so $\deg(V_1) = m\left(-(m+1)(g-1) + \hat{d}_1\right)$ but we also know $\frac{n+1}{2}\tau + \deg(V_2) = -\deg(V_1)$ and $\tau < 2(g-1)$. So again, the quadratic growth of $m$ exceeds the linear growth $n$.

# — A —

# Spectral sequences and Hypercohomology

We here explore the deformation theory of general Higgs bundles from [37], [34] and [8] and as used in both §3.2 and §3.3.

From a given complex, we can also derive a cohomology complex $H^q(K^*)$ which has an associated filtration $F^pH^q(K^*)$ induced by the filtration $F^pK^*$. From this we get the **associated graded cohomology**

$$GH^*(K^*) = \bigoplus_{p,q} G^pH^q(K^*) \text{ where } G^pH^q(K^*) = \frac{F^pH^q(K^*)}{F^{p+1}H^q(K^*)}. \tag{A.1}$$

Consider a double complex $(K^{*,*}, d, \delta)$ with differentials $d : K^{p,q} \to K^{p+1,q}$ and $\delta : K^{p,q} \to K^{p,q+1}$ such that $d^2 = \delta^2 = 0$ and $d\delta = \delta d$. The **associated single complex** $(K^*, D)$ is given by $K^n = \bigoplus_{p+q=n} K^{p,q}$ and $D = (-1)^n d + \delta$.

There are two possible filtrations on $(K^*, D)$. The first is $(F^pK^*, D)$ where $F^pK^n = \bigoplus_{j\geq p} K^{j,n-j}$ which has the associated graded complex $GK^*$ where

$$GK^n = \bigoplus_p \frac{F^pK^n}{F^{p+1}K^n} \simeq \bigoplus_p K^{p,n-p} = K^n, \tag{A.2}$$

and the induced differential agrees with $\delta$ on $K^n$:

$$\begin{array}{cccccccccc}
GK^0 & \simeq K^0 = & F^0K^0 & = K^{0,0} & \supset & 0 & & & & \\
\downarrow{\scriptstyle \simeq\delta} & \downarrow & \downarrow{\scriptstyle D} & & & \downarrow & & & & \\
GK^1 & \simeq K^1 = & F^0K^1 & = K^{0,1} \oplus K^{1,0} & \supset & F^1K^1 & = K^{1,0} & \supset & 0 & \\
\downarrow{\scriptstyle \simeq\delta} & \downarrow & \downarrow{\scriptstyle D} & & & \downarrow{\scriptstyle D} & & & \downarrow & \\
GK^2 & \simeq K^2 = & F^0K^2 & = K^{0,2} \oplus K^{1,1} \oplus K^{2,0} & \supset & F^1K^2 & = K^{1,1} \oplus K^{2,0} & \supset & F^{2,2} = K^{2,0} & \supset \; 0 \\
\downarrow{\scriptstyle \simeq\delta} & \downarrow & \downarrow{\scriptstyle D} & & & \downarrow{\scriptstyle D} & & & \downarrow{\scriptstyle D} & \downarrow \\
\cdots & \cdots & \cdots & & & \cdots & & & \cdots & \cdots
\end{array} \tag{A.3}$$

Alternatively, the second filtration $(\hat{F}^q K^*, D)$ is given by $\hat{F}^q K^n = \bigoplus_{j \geq q} K^{n-j,j}$ with both associated graded complex $\hat{G}K^*$ such that

$$\hat{G}K^n = \bigoplus_q \frac{\hat{F}^q K^n}{\hat{F}^{q+1} K^n} \simeq \bigoplus_q K^{n-q,n} = K^n, \tag{A.4}$$

and an induced differential equivalent to $(-1)^n d$ on $K^n$:

$$\begin{array}{ccccccccccccc}
\hat{G}K^0 & \simeq K^0 = & \hat{F}^0 K^0 & = K^{0,0} & \supset & 0 & & & & & & \\
\downarrow{\scriptstyle \simeq d} & \downarrow & \downarrow{\scriptstyle D} & & & \downarrow & & & & & & \\
\hat{G}K^1 & \simeq K^1 = & \hat{F}^0 K^1 & = K^{1,0} \oplus K^{0,1} & \supset & \hat{F}^1 K^1 & = K^{0,1} & \supset & 0 & & & \\
\downarrow{\scriptstyle \simeq -d} & \downarrow & \downarrow{\scriptstyle D} & & & \downarrow{\scriptstyle D} & & & \downarrow & & & \\
\hat{G}K^2 & \simeq K^2 = & \hat{F}^0 K^2 & = K^{2,0} \oplus K^{1,1} \oplus K^{0,2} & \supset & \hat{F}^1 K^2 & = K^{1,1} \oplus K^{0,2} & \supset & \hat{F}^{2,2} & = K^{0,2} & \supset & 0 \\
\downarrow{\scriptstyle \simeq d} & \downarrow & \downarrow{\scriptstyle D} & & & \downarrow{\scriptstyle D} & & & \downarrow{\scriptstyle D} & & & \downarrow \\
\cdots & \cdots & \cdots & & & \cdots & & & \cdots & & & \cdots
\end{array} \tag{A.5}$$

Both filtrations also induce a filtration on the cohomology to give $F^p H^n(K^*, D)$ and $\hat{F}^q H^n(K^*, D)$. These in turn have associated graded cohomology

$$GH^n(K^*, D) = \bigoplus_p G^p H^n(K^*, D) = \oplus_p \frac{F^p H^n(K^*, D)}{F^{p+1} H^n(K^*, D)}, \tag{A.6}$$

and

$$\hat{G}H^n(K^*, D) = \bigoplus_q \hat{G}^q H^n(K^*, D) = \oplus_q \frac{\hat{F}^q H^n(K^*, D)}{\hat{F}^{q+1} H^n(K^*, D)}. \tag{A.7}$$

These can by computed by using the spectral sequences of the two filtrations. A **spectral sequence** is a sequence $\{E_r, d_r\}$ for $r \geq 0$ of double graded groups $E_r = \bigoplus_{p,q \geq 0} E_r^{p,q}$ together with the differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r^2 = 0$ and $H^*(E_r) = E_{r+1}$. In many cases, $E_r = E_{r+1}$ for all $r \geq R$ which is called the **limit group** $E_\infty$; we say the spectral sequence $\{E_r\}$ converges to $E_\infty$.

For the first filtration, we have $E_0^{p,q} = \frac{F^p K^{p+q}}{F^{p+1} K^{p+q}} \simeq K^{p,q}$ and $d_0 : E_0^{p,q} \to E_0^{p,q+1}$ induced by $D$ meaning $d_0 \simeq \delta$. As $E_{r+1} = H^*(E_r)$, $E_1^{p,q} = H^q(K^{p,*}, \delta)$ and the differential $d_1$ agrees with $(-1)^{p+q} d : H^q(K^{p,*}, \delta) \to H^q(K^{p+1,*}, \delta)$. As $D = (-1)^n d + \delta$ and $\delta = 0$ on $E_1$, then $E_2^{p,q} = H^p(H^q(K^{p,*}, \delta), d)$. The spectral sequence stabilizes at this term so $E_r = E_2$ for all $r \geq 2$. Overall, we therefore have

$$E^{p,q} : (K^{p,q}, \delta) \longrightarrow (H^q(K^{p,*}, \delta), (-1)^{p+q} d) \longrightarrow (H^p(H^q(K^{p,*}, \delta), d), d_2) \longrightarrow \cdots \tag{A.8}$$

By symmetry, $\hat{E}_0^{p,q} \simeq K^{p,q}$, $\hat{E}_1^{p,q} = H^p(K^{*,q}, d)$ and $\hat{E}_2^{p,q} = H^q(H^p(K^{*,q}, d), \delta)$.

This relates back to the associated grade cohomology complex as [37] shows that

$$\bigoplus_{p+q=n} E_2^{p,q} \simeq GH^n(K^*, D) \simeq \mathbb{H}^n(\mathcal{A}^*), \tag{A.9}$$

where $\mathbb{H}^n(\mathcal{A}^*)$ is the hypercohomology of a complex $\mathcal{A}^*$ as follows.

Consider a complex of sheaves $(\mathcal{A}^*, d)$ given by $\mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \xrightarrow{d} \cdots$ and set $K^{p,q} = \mathcal{C}^q(\mathcal{A}^p)$ to be the vector space of Cêch $q$-cochains of $\mathcal{A}^p$. This gives a double complex

$$\begin{array}{ccccccc} \mathcal{C}^0(\mathcal{A}^0) & \xrightarrow{d} & \mathcal{C}^0(\mathcal{A}^1) & \xrightarrow{d} & \mathcal{C}^0(\mathcal{A}^2) & \longrightarrow & \cdots \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\ \mathcal{C}^1(\mathcal{A}^0) & \xrightarrow{d} & \mathcal{C}^1(\mathcal{A}^1) & \xrightarrow{d} & \mathcal{C}^1(\mathcal{A}^2) & \longrightarrow & \cdots \\ \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\ \cdots & \xrightarrow{d} & \cdots & \xrightarrow{d} & \cdots & \longrightarrow & \cdots \end{array} \tag{A.10}$$

from which a single complex $(K^*, D)$ can then be defined as above: $K^n = \bigoplus_{p+q=n} K^{p,q}$ and $D = \delta + (-1)^n d$.

The **nth hypercohomology group** of the complex $(\mathcal{A}^*, d)$ is then defined as

$$\mathbb{H}^n(\mathcal{A}^*) = H^n(K^*, D) = \lim_{U} \frac{\ker D|_{K^n}}{DK^{n-1}}, \tag{A.11}$$

where the limit is taken over refinements of open covers.

Applying this to the moduli space of Higgs bundles with [34] and [8] as in §3.2, for a given $G$-Higgs bundle $(E, \Phi)$, its **deformation complex** $C^*(E, \Phi)$ is

$$E(\mathfrak{h}^{\mathbb{C}}) \xrightarrow{ad\Phi} E(\mathfrak{p}^{\mathbb{C}}) \otimes K \longrightarrow 0, \tag{A.12}$$

where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ is the Cartan decomposition.

Letting equation (A.12) be the complex of sheaves $(\mathcal{A}^*, d)$ with a double complex as given in equation (A.10), then, using the fact that $\mathcal{A}^p = 0$ for all $p \geq 2$, the corresponding single complex $(K^*, D)$ is given by:

$$\mathcal{C}^0(\mathcal{A}^0) \xrightarrow{D} \mathcal{C}^1(\mathcal{A}^0) \oplus \mathcal{C}^0(\mathcal{A}^1) \xrightarrow{D} \mathcal{C}^2(\mathcal{A}^0) \oplus \mathcal{C}^1(\mathcal{A}^1) \xrightarrow{D} \cdots \tag{A.13}$$

This gives a short exact sequence of complexes

$$0 \longrightarrow \mathcal{C}^*(\mathcal{A}^1)[1] \longrightarrow \mathcal{K}^* \longrightarrow \mathcal{C}^*(\mathcal{A}^0) \longrightarrow 0, \tag{A.14}$$

where $\mathcal{C}^*(\mathcal{A}^1)[1]$ is the $\mathcal{C}^*(\mathcal{A}^1)$ complex shifted by one step: $0 \to \mathcal{C}^0(\mathcal{A}^1) \to \mathcal{C}^2(\mathcal{A}^1) \to \cdots$. Applying the Snake lemma and recalling that in this case $d = \mathrm{ad}\Phi$, this gives a long exact sequence of cohomology groups as we saw in equation (3.16)

$$\begin{array}{l} 0 \longrightarrow \mathbb{H}^0(\mathcal{A}^*) \longrightarrow H^0(\mathcal{A}^0) \xrightarrow{ad\Phi} H^0(\mathcal{A}^1) \longrightarrow \mathbb{H}^1(\mathcal{A}^*) \\ \swarrow \\ H^1(\mathcal{A}^0) \xrightarrow{-ad\Phi} H^1(\mathcal{A}^1) \longrightarrow \mathbb{H}^2(\mathcal{A}^*) \longrightarrow \cdots \end{array} \tag{A.15}$$

From this we get that $\mathbb{H}^0(\mathcal{A}^*) \simeq \ker \mathrm{ad}\Phi(H^0(\mathcal{A}^0))$, which agrees with the spectral sequence identity given in equation (A.9) as we also have

$$E_2^{0,0} = H^0_{ad\Phi}(H^0(\mathcal{C}^0(\mathcal{A}^*), \delta)) = H^0_{ad\Phi}(H^0(\mathcal{A}^0)) = \ker \mathrm{ad}\Phi(H^0(\mathcal{A}^0)). \tag{A.16}$$

Similarly, we can use this to find an expression for $\mathbb{H}^1(\mathcal{A}^*) = E_2^{1,0} \oplus E_2^{0,1}$. We have

$$E_2^{0,1} = H^0_{ad\Phi}(H^1(\mathcal{C}^0(\mathcal{A}^*), \delta)) = H^0_{ad\Phi}(H^1(\mathcal{A}^0)) = \ker \mathrm{ad}\Phi(H^1(\mathcal{A}^0)), \tag{A.17}$$

and

$$E_2^{1,0} = H^1_{ad\Phi}(H^0(\mathcal{C}^1(\mathcal{A}^*), \delta)) = H^1_{ad\Phi}(H^0(\mathcal{A}^1)) = \frac{H^0(\mathcal{A}^1))}{\mathrm{Im}\ \mathrm{ad}\Phi(H^0(\mathcal{A}^0))}. \tag{A.18}$$

This is the same as that given by equation (A.13). More explicitly

$$\mathbb{H}^1(\mathcal{A}^*) = \frac{\{(s,t) \in \mathcal{C}^1(\mathcal{A}^0) \oplus \mathcal{C}^0(\mathcal{A}^1) | \delta s = 0, \delta t = [\phi, s]\}}{\{(\delta u, [\phi, u]) | u \in \mathcal{C}^0(\mathcal{A}^0)\}} =: \frac{Z_D^1}{B_D^1}. \tag{A.19}$$

We can show this is equivalent to the space of infinitesimal deformations of $(E, \Phi)$. Let $(\overline{E}, \overline{\Phi})$ be an infinitesimal deformation of $(E, \Phi)$ for $\overline{E}$ a principal $H^{\mathbb{C}}$ bundle over $\Sigma_c[\varepsilon] = \Sigma_c \times \mathrm{Spec}\ \mathbb{C}[\varepsilon]$ where $\mathbb{C}[\varepsilon] = \{a + b\varepsilon | a, b \in \mathbb{C}, \varepsilon^2 = 0\}$ is the **algebra of dual numbers** and $\overline{\Phi} \in H^0(\overline{E}(\mathfrak{m}^{\mathbb{C}}) \otimes K)$. Note $\mathrm{Spec}\ \mathbb{C}[\varepsilon] \cong T_0\mathbb{C}$.

If $E[\varepsilon] = E \times \mathrm{Spec}\ \mathbb{C}[\varepsilon]$ is taken to be the trivial deformation, then it has a automorphism $1 + s\varepsilon$ where $s$ is a section of $E(\mathfrak{h}^{\mathbb{C}})$. Moreover, if $u + v\varepsilon$ is a section of $E(\mathfrak{m}^{\mathbb{C}})[\varepsilon]$ then

$$\mathrm{Ad}(1 + s\varepsilon) \cdot (u + v\varepsilon) = u + (v + \mathrm{Ad}s \cdot u)\varepsilon. \tag{A.20}$$

Let $\{U_i\}$ be a **Leray cover** of $\Sigma_c$, that is a covering such that the Cech cohomology $H^k(|\tau|, \mathcal{S}) = 0$ for all $k > 0$ and all intersections of open sets $|\tau|$ (see [60]). We then have a projection $U_i[\varepsilon] \to U_i$ where $U_i[\varepsilon] = U_i \times \text{Spec } \mathbb{C}[\varepsilon]$. The pullback of $E$ along this map gives $E_i[\varepsilon]$. An infinitesimal deformation $\overline{E}$ can then be constructed by gluing $E_j[\varepsilon]$ to $E_i[\varepsilon]$ over $U_{ij}$ with $(1 + s_{ij}\varepsilon)$ where $(s,t) \in Z^1_D$. As $\delta s = 0$, this is a 1-cycle.

We can also construct a global section $\overline{\Phi}$ by taking $\overline{\Phi}_i = \Phi_i + t_i\varepsilon$. By equation (A.20),

$$Ad(1 + s_{ij}\varepsilon) \cdot (\Phi_i + t_i\varepsilon) = \Phi_i + ([s_{ij}, \Phi] + t_i)\varepsilon, \tag{A.21}$$

and as $\Phi_i = \Phi_j$ and $t_i - t_j = (\delta t)_{ij} = [\Phi, s_{ij}]$, then $\overline{\Phi}_j = Ad(1 + s_{ij}\varepsilon) \cdot \overline{\Phi}_i$.

On the other hand, given $(\overline{E}, \overline{\Phi})$ then over $U_i[\varepsilon]$, the bundle $\overline{E}$ has transition relations $r_{ij}(\varepsilon)$ and we can assume $r_{ij}(0) = 1$ meaning locally $\overline{E}_i \cong E_i[\varepsilon]$ and this construction can be reversed. Therefore every element of $Z^1_D$ corresponds to an infinitesimal deformation.

To finish, we still need to show that elements corresponding to the trivial deformation $(\overline{E}, \overline{\Phi}) \cong (E[\varepsilon], \Phi)$ lie in $B^1_D$. In this case, we require $s = \delta u$ for some $u \in \mathcal{C}^0(\mathcal{A}^0)$ but we have, $\text{Ad}(1 - u_i\varepsilon)\Phi_i = \Phi_i + t_i\varepsilon$, so $t = [\Phi, u]$ and $(s,t) = (\delta u, [\Phi, u]) \in B^1_D$.

# References


[1] L. Alvarez-Consul, O. Garcia-Prada, and A. H. W. Schmitt. “On the geometry of moduli spaces of holomorphic chains over compact Riemann surfaces”. In: *International Mathematics Research Papers* (2010).

[2] L. Álvarez-Cónsul and O. García-Prada. “Dimensional reduction, $SL(2, \mathbb{C})$-equivariant bundles and stable holomorphic chains”. In: *Internat. J. Math.* 12.2 (2001), pp. 159–201.

[3] M. F. Atiyah and R. Bott. “The Yang-Mills equations over Riemann surfaces”. In: *Philos. Trans. Roy. Soc. London Ser. A* 308.1505 (1983), pp. 523–615.

[4] M. F. Atiyah, N. J. Hitchin, and I. M. Singer. “Self-Duality in Four-Dimensional Riemannian Geometry”. In: *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 362.1711 (1978), pp. 425–461.

[5] D. Baraglia. “G2 geometry and integrable systems”. PhD thesis. University of Oxford, 2009.

[6] A. L. Besse. *Einstein Manifolds.* Berlin, Heidelberg, New York: Springer-Verlag, 1987. ISBN: 978-0-387-15279-0.

[7] O. Biquard. “Einstein deformations of hyperbolic metrics”. In: *Surveys in differential geometry* 6 (2001), pp. 235–246.

[8] I. Biswas and S. Ramanan. “An Infinitesimal Study of the Moduli of Hitchin Pairs”. In: *Journal of the London Mathematical Society* 49.2 (1994), pp. 219–231.

[9] I. Biswas and G. Schumacher. *Geometry of moduli spaces of Higgs bundles.* 2006. eprint: `math/0605589`.

[10] J. Bolton and L. M. Woodward. “Congruence theorems for harmonic maps from a Riemann surface into $\mathbb{CP}^n$ and $S^n$”. In: *Journal of the London Mathematical Society. Second Series* 45 (1992).

[11] J. Bolton and L. M. Woodward. “Minimal surfaces in $\mathbb{CP}^n$ with constant curvature and Kähler angle”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 112.2 (1992), pp. 287–296.

[12] J. Bolton, F. Pedit, and L. Woodward. “Minimal surfaces and the affine Toda field model”. In: *J. Reine Angew. Math.* 459 (1995), pp. 119–150.

[13] R. Bott and L. W. Tu. *Differential forms in algebraic topology.* Vol. 82. Graduate texts in mathematics. Springer-Verlag, 1995.

[14] S. Bradlow, O. Garcia-Prada, and P. Gothen. “Surface group representations and U(p,q)-Higgs bundles”. In: *Journal of Differential Geometry* 64 (2003), pp. 111–170.

[15] S. Bradlow, O. García-Prada, P. Gothen, and J. Heinloth. “Irreducibility of Moduli of Semi-Stable Chains and Applications to U(p, q)-Higgs Bundles”. In: *Geometry and Physics: Volume II* (Oct. 2018), pp. 455–470.

[16] S. Bradlow, G.-P. O., W. Goldman, and A. Wienhard. *Representations of Surface groups: Background material for AIM workshop.* 2007.

[17] R. L. Bryant. “Conformal and minimal immersions of compact surfaces into the 4-sphere”. In: *Journal of Differential Geometry* 17.3 (1982), pp. 455–473.

[18] F. Burstall and J. C. Wood. “The construction of harmonic maps into complex Grassmannians”. In: *Journal of Differential Geometry* 23 (1986), pp. 255–297.

[19] A. Cano, J. P. Navarrete, and J. Seade. *Complex Kleinian groups.* Vol. 303. Progress in Mathematics. Birkhäuser/Springer Basel AG, Basel, 2013.

[20] J. Cheeger and D. G. Ebin. *Comparison Theorems in Riemannian Geometry.* North-Holland Mathematics Library, 1975.

[21] S. S. Chern and J. G. Wolfson. “Minimal Surfaces by Moving Frames”. In: *American Journal of Mathematics* 105 (1983).

[22] K. Corlette. “Flat $G$-bundles with canonical metrics”. In: *J. Differential Geom.* 28.3 (1988), pp. 361–382.

[23] A. Domic and D. Toledo. “The Gromov norm of the Kaehler class of symmetric domains”. In: *Math. Ann.* 276.3 (1987), pp. 425–432. ISSN: 0025-5831,1432-1807.

[24] S. K. Donaldson. "Twisted Harmonic Maps and the Self-Duality Equations". In: *Proceedings of the London Mathematical Society* s3-55.1 (1987), pp. 127–131.

[25] J. Eells and J. Wood. "Harmonic Maps from Surfaces to Complex Projective Spaces". In: *Advances in Mathematics* 49 (1983).

[26] G. Ellingsrud and J. C. Ottem. *Introduction to Schemes*. 2018.

[27] S. Erdem and J. F. Glazebrook. "Harmonic Maps of Riemann Surfaces to Indefinite Complex Hyperbolic and Projective Spaces". In: *Proceedings of the London Mathematical Society* s3-47.3 (Nov. 1983), pp. 547–562.

[28] P. M. N. Feehan. *Bialynicki-Birula theory, Morse-Bott theory, and resolution of singularities for analytic spaces*. 2022. eprint: 2206.14710.

[29] C. Florentino, P. Gothen, and A. Nozad. "Homotopy type of moduli spaces of G-Higgs bundles and reducibility of the nilpotent cone". In: *Bulletin des Sciences Mathématiques* 150 (2019), pp. 84–101. ISSN: 0007-4497.

[30] T. Frankel. "Fixed Points and Torsion on Kahler Manifolds". In: *Annals of Mathematics* 70.1 (1959), pp. 1–8. ISSN: 0003486X.

[31] O. Garcia-Prada. "Differential Analysis on Complex Manifolds". In: 3rd ed. Vol. 65. Graduate Texts in Mathematics. Springer-Verlag, 2008. Chap. Moduli Spaces and Geometry Structures.

[32] W. M. Goldman. "Higgs Bundles and Geometric Structures on Surfaces". In: *The Many Facets of Geometry* (2008).

[33] W. M. Goldman, M. Kapovich, and B. Leeb. "Complex hyperbolic manifolds homotopy equivalent to a Riemann surface". In: *Communications in Analysis and Geometry* 9 (2001).

[34] P. B. Gothen. "Representations of surface groups and Higgs bundles". In: *Moduli spaces*. Vol. 411. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2014, pp. 151–178.

[35] P. B. Gothen. "Topology of U(2,1) representation spaces". In: *Bull. London Math. Soc.* 34.6 (2002), pp. 729–738.

[36] P. B. Gothen and R. A. Zúñiga-Rojas. "Stratifications on the nilpotent cone of the moduli space of Hitchin pairs". In: *Revista Matemática Complutense* 35.2 (July 2021), pp. 311–321.

[37] P. Griffiths and J. Harris. *Principles of Algebraic Geometry.* John Wiley & Sons Inc, 1978.

[38] R. Gunning. *Lectures on Riemann Surfaces.* Princeton Mathematical Notes, 1966.

[39] R. Gunning. *Lectures on Vector Bundles over Riemann Surfaces.* Princeton Mathematical Notes, 1967.

[40] G. Harder and M. S. Narasimhan. "On the cohomology groups of moduli spaces of vector bundles on curves". In: *Math. Ann.* 212 (1974/75), pp. 215–248.

[41] S. Helgason. *Differentiable Geometry, Lie Groups and Symmetric Spaces.* Academic Press, 1978.

[42] M. Hirsch. *Differential Topology.* Vol. 33. Graduate Texts in Mathematics. Springer-Verlag, Jan. 1994. ISBN: 978-1-4684-9451-8.

[43] N. J. Hitchin. "The self-duality equations on a Riemann surface". In: *Proc. London Math. Soc. (3)* 55.1 (1987), pp. 59–126.

[44] D. Huybrechts and M. Lehn. *The Geometry of Moduli Spaces of Sheaves.* 2nd ed. Cambridge Mathematical Library. Cambridge University Press, 2010.

[45] F. C. Kirwan. *Cohomology of Quotients in Symplectic and Algebraic Geometry. (MN-31), Volume 31.* Vol. 104. Princeton University Press, 1984. ISBN: 9780691083704.

[46] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry.* Vol. 1. Interscience Publishers, 1963.

[47] J. L. Koszul and B. Malgrange. "Sur certaines structures fibrées complexes". In: *Archiv der Mathematik* 9 (1958), pp. 102–109.

[48] J. Loftin and I. McIntosh. "Equivariant minimal surfaces in $\mathbb{CH}^2$ and their Higgs bundles". In: *Asian J. Math.* 23.1 (2019), pp. 71–106.

[49] J. Loftin and I. McIntosh. "The moduli spaces of equivariant minimal surfaces in $\mathbb{RH}^3$ and $\mathbb{RH}^4$ via Higgs bundles". In: *Geometriae Dedicata* (2019), pp. 325–351.

[50] I. McIntosh. "On the connected components of the moduli space of equivariant minimal surfaces in $\mathbb{CH}^2$." In: *Geom. Dedicata* 217.3 (2023), Paper No. 56, 19.

[51] J. Milnor. "On the existence of a connection with curvature zero." In: *Commentarii mathematici Helvetici* 32 (1957/58), pp. 215–223.

[52] A. Moroianu. *Lectures on Kähler Geometry*. London Mathematical Society Student Texts. Cambridge University Press, 2007.

[53] M. S. Narasimhan and C. S. Seshadri. "Stable and Unitary Vector Bundles on a Compact Riemann Surface". In: *Annals of Mathematics* 82.3 (1965), pp. 540–567.

[54] J. Parker. *Notes on Complex Hyperbolic Geometry*. 2003.

[55] M. Popa. *Lecture notes on Moduli of Vector Bundle*. Moduli Spaces and Arcs in Algebraic Geometry Summer School, University of Cologne. 2006.

[56] C. S. Seshadri. "Space of Unitary Vector Bundles on a Compact Riemann Surface". In: *Annals of Mathematics* 85.2 (1967), pp. 303–336. ISSN: 0003486X.

[57] C. T. Simpson. "Higgs bundles and local systems". In: *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* 75.1 (Dec. 1992), pp. 5–95.

[58] C. T. Simpson. "Constructing Variations of Hodge Structure Using Yang-Mills Theory and Applications to Uniformization". In: *Journal of the American Mathematical Society* 1.4 (1988), pp. 867–918. ISSN: 08940347, 10886834.

[59] D. Toledo. "Representations of surface groups in complex hyperbolic space". In: *Journal of Differential Geometry* 29 (1989).

[60] R. Wells. *Differential Analysis on Complex Manifolds*. Vol. 65. Graduate Texts in Mathematics. Springer-Verlag, 1973.

[61] R. A. Wentworth. "Higgs bundles and local systems on Riemann surfaces". In: *Geometry and quantization of moduli spaces*. Adv. Courses Math. CRM Barcelona. Birkhäuser/Springer, Cham, 2016, pp. 165–219.

[62] J. G. Wolfson. "Harmonic sequences and harmonic maps of surfaces into complex Grassmann manifolds". In: *Journal of Differential Geometry* 27.1 (1988), pp. 161–178.

[63] J. C. Wood. "Harmonic maps into symmetric spaces and integrable systems". In: *Harmonic Maps and Integrable Systems*. Ed. by A. P. Fordy and J. C. Wood. Vieweg+Teubner Verlag, 1994, pp. 29–55.

[64] J. W. Wood. “Bundles with Totally Disconnected Structure Group”. In: *Commentarii mathematici Helvetici* 46 (1971), pp. 257–273.

[65] E. Xia. “The Moduli of Flat PU(2,1) Structures on Riemann Surfaces”. In: *Pacific Journal of Mathematics* 195 (Aug. 2000).

[66] E. Xia. “The Moduli of Flat U(p, 1) Structures on Riemann Surfaces”. In: *Geometriae Dedicata* 97 (Mar. 2003), pp. 33–43.

[67] Y. Zhu. *The general definition of Kahler angle*. 2023. arXiv: `2303.17122`.