# Microlocal techniques for perturbative algebraic quantum field theory in the functional formalism

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PhD

University of York
Mathematics

September 2024

# Abstract

In this thesis, we study the functional formalism for perturbative Algebraic Quantum Field Theory (pAQFT), with a focus on *microcausal* functionals. These functionals are required to have a restricted singular structure, that ensures that the Poisson bracket and the quantum $\star$-product can be rigorously defined on them. However, we identify several inconsistencies in their treatment in the literature. The most significant of these is the fact that the Poisson bracket of two microcausal functionals can fail to be continuous, implying that the Poisson bracket does not close on this class of functionals.

A secondary goal of this thesis is the construction of tools that allow one to prove homotopical statements within the functional formalism, in the context of the Batalin-Vilkovisky (BV) formalism. The main result we present here is a prescription for lifting a retract of field complexes to a retract of the functionals on those complexes, as an extension of the results presented in [52]. However, these new tools do not extend to graded microcausal functionals, as demonstrated by an explicit counterexample. This presents a significant obstruction to proving the time-slice axiom for these functionals, raising doubts about their ability to properly account for local dynamics.

To address these problems, we introduce the class of *equicausal* functionals. These functionals address the identified inconsistencies by satisfying a stricter set of microlocal conditions. We show that equicausal functionals are closed under both the homotopical tools we define and the $\star$-product. The introduction of equicausal functionals enables us to recover several key results previously inaccessible due to the limitations of the microcausal framework, thereby reinforcing their significance for both the BV-formalism and pAQFT as a whole.

# Contents

# Acknowledgements

This thesis marks the end of a chapter of my life that was immensely fulfilling, but at the same time incredibly difficult. I am very grateful of all the people who supported me along the way, academically or otherwise.

First, I would like to thank my parents and my sister, for being the best family anyone could ever wish for. Part of this PhD was performed from their kitchen table, where I was waiting for COVID restrictions to lift enough to be able to move to the York, and I am grateful to have had a home there during these turbulent times.

On the academic front, I would like to thank my supervisors Chris Fewster and Eli Hawkins for all their help and guidance along the way. Eli was always ready to provide the big-picture insights to which I could put the technical details. Despite our differences in mathematical style, I think we were able to use our complimentary strengths to our advantage to obtain some really nice results. Chris I would like to thank in particular for keeping the PhD project on the rails during difficult times, and for technical discussions on distribution theory that would be difficult to hold with anyone else. Next to them, I would also like to thank my former supervisor and long term collaborator, Kasia Rejzner. I am glad that we were able to work through our differences, and I look forward to work with you in the future.

The vibrant social environment at the department of mathematics has made this whole experience that much more enjoyable, and I would like to thank all the wonderful people here. First and foremost, I would like to thank Maximilian Ruep and Samuel Crawford for being my academic brothers when I first arrived at the department. The reading groups and discussions we had together were a formative part of this whole experience. Special thanks also to Simon Hart, for being a great roommate and a fellow boardgame-aficionado.

Furthermore, I'd like to thank: Alice Dell'Arciprete, for being a great laugh at the pub. Ambroise Grau, for writing the thesis template that I am using right now, and for bringing cheese from France. Art'om Zolotarjov, for finding obscure publications through legal means. Ben Graham, for sending me Polish memes and making cocktails at my birthday party. Ben Mills, for having killer calves. Beth Dixon, for delving too deeply and too greedily. Cordelia Webb, for being a good sport upon losing the Sunday wars, and for your general wisdom. Daan Janssen, my brother in AQFT, I would like to thank for being a friend throughout this entire journey. I am very grateful that we were able to discover this field together. Dingjia Cao, for many a tense game of De Crew. Dorsa Hatefi, for climbing together and being a vibrant presence around the office. Special thanks to digital nomad Edoardo d'Angelo, who despite not actually being a student at this university became a part of the fixtures here. I look forward to frequenting Genoa without having to pay for accommodation. Georgia Schneider, for inviting me to your 7'th birthday party in the Lake District. Harkan Kirk Karakaya, for going to that dive bar in Cardiff together. Jamie Smith, for being the only person around that actually plays boardgames well. Jenny Winstone, for lending me books about dragons and feminism. We are pack. Kuntal Sengupta and Shashaank Khanna, for hosting a truly unforgettable Diwali party. Levent Dasar, for being a great office-mate and for your hot takes. Mihail Hurmozov, for opening my eyes to the fact that boardgames can be pirated. Nick Bryant, for your near unlimited, highly entertaining drama, and your knowledge of trains. Philip Liu, also for being a great office-mate, I appreciate your unique view of the world. Ricky Li, for your encyclopedic knowledge of eggs. Sam Engleman, for knowing stuff about geometry when I needed it. Sam Hill for spending a lot of money on Arkham Horror so that I didn't have to. Simen Bruinsma as well, thank you for spending lots of money on Arkham Horror so that I didn't have to. Zane Hassoun, for bringing some American flair to the office. Despite your best efforts I will not put you in a disacknowledgements section.

And last, but definitely not least, I'd like to thank Ross Ratcliffe for being an all-round outstanding friend. I love your do-first, think-second approach to life, and I treasure the countless pub-trips, dinners and boardgame nights we had together. I'll miss you buddy.

A special place in my heart is also reserved for the Red Goat climbing wall, for being unapologetically old-school. Also many thanks to all the wonderful people there, and in particular to Clement Degut and Zoë Ingold for many an outdoor adventure on the weekend.

Thank you to C&S sourdough for baking the only decent bread I could find around here.

Next to all these new friends, I would also like to thank my old friends back home, who kept in touch throughout the years that I've been away. I look forward to spending time with you again when I return. In particular, a special thank you to Bart Stroucken for the many hours of procrastination playing online boardgames, and to Stefan Driessen for talking about academic life in general.

And then finally, of course, I would like to thank my wonderful fiancée Sandra. This thesis marks, next to the end of my PhD, also the end of an overly long period of long-distance in our relationship. I am so grateful for your unwavering love and support during these difficult years, and I am so happy that we came through them together. I could not have done it without you.

# Author's declaration

I declare that the work presented in this thesis, except where otherwise stated, is based on my own research carried out at the University of York and has not been submitted previously for any degree at this or any other university. Sources are acknowledged by explicit references. Part of the work presented here is based on a joint paper [52] with Eli Hawkins and Kasia Rejzner, titled 'A novel class of functionals for perturbative algebraic quantum field theory', which is currently under review with the journal Communications in Mathematical Physics. When using work from this shared publication, a careful accounting of intellectual ownership is given.

# — 1 —

# Introduction

Quantum Field Theory (QFT) is a fundamental tool of modern theoretical physics, providing a framework for understanding the fundamental forces of nature and the behaviour of subatomic particles. Initially developed to unify quantum mechanics and special relativity, QFT has been remarkably successful in describing the dynamics of particles and fields. The Standard Model of particle physics, which is a quantum field theory at heart, is one of the most accurate physical theories ever developed by mankind, reaching staggering levels of precision when confronted with experiment.

Despite its successes, several profound mathematical challenges persist in the foundations of QFT. A major difficulty arises from the fact that field theories typically possess infinitely many degrees of freedom, meaning that most objects encountered are inherently distributional. When these distributional objects are combined, the result is often a formal infinity. In the latter half of the 20th century, significant efforts were made to address such issues, culminating in the development of techniques now collectively known as *renormalisation*.

Renormalisation generally follows a three-step process: first, a *regulator* is introduced to make the initially infinite quantity finite. Next, the divergent part of the quantity, which becomes problematic as the regulator is removed, is subtracted. Finally, the regulator is removed, leaving behind a finite remainder. While these methods are widely used in theoretical physics, they are often *ad hoc* and lack universal applicability. As such, they do not provide consistent, well-defined results in all contexts, especially in curved spacetime.

## 1.1 ALGEBRAIC QUANTUM FIELD THEORY

Algebraic Quantum Field Theory (AQFT), based on the Haag-Kastler framework developed in [49], offers an alternative approach to the traditional path integral formalism. It emphasises locality and causality as the fundamental principles of quantum field theory, relegating other concepts to a secondary role. The primary object in AQFT is a net of local algebras defined over a spacetime $\mathcal{M}$. This consists of an assignment

$$\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$$

of an algebra of observables to spacetime regions $\mathcal{O} \subset \mathcal{M}$. This net is required to satisfy several axioms, chief amongst them that of Einstein causality: If $\mathcal{O}_1$ and $\mathcal{O}_2$ are causally separated regions, then we require the commutation relation

$$[\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0.$$

Crucially, the definition of an AQFT does not include a specific choice of state or Hilbert space. Instead, the framework abstracts away from particular representations of quantum fields, focusing rather on their algebraic relations. Different states or representations may correspond to distinct physical realisations, but the underlying algebraic structure remains the same.

This abstraction makes AQFT particularly well-suited for addressing quantum field theory in curved spacetimes, where the notion of a vacuum state becomes ambiguous, see e.g. [39]. In the 1970s, Fulling realised that representations corresponding to different, physically reasonable states are generally **not** unitarily equivalent, see [44]. This realisation highlights a challenge for the traditional Hilbert space formalism, where comparing distinct states becomes technically awkward. An arbitrary choice of state in defining the theory can lead to inequivalent, and thus physically distinct, descriptions of the quantum field.

Over the decades, AQFT has produced landmark results in both physics and mathematics, including the rigorous treatment of Hawking radiation and black hole thermodynamics [74], and the clarification of the theory of superselection sectors via the DHR-formalism [31, 32]. We refer the reader to Haag's monograph [48] for an extensive review. Despite these advances, significant mathematical challenges in QFT remain unresolved.

## 1.2 MICROLOCAL ANALYSIS AND THE HADAMARD CONDITION

While AQFT clarifies the properties that observables in quantum field theory should possess, it does not address which states on these observables are physically reasonable. The Hadamard condition provides a criterion to address this. It was first introduced by Adler, Lieberman and Ng in [1], and then further studied by Fulling, Sweeney, Wald, and Narcowich in [45, 43]. Broadly, the Hadamard condition stipulates that the singular structure of the two-point function in curved spacetimes should resemble that of the vacuum state in flat spacetime. This requirement is motivated by the idea that these singularities reflect the high-energy, short length scale behaviour of the theory, which should be agnostic of the large-scale structure of spacetime.

Initially aimed at calculating renormalised quantities by so-called 'point splitting' regularisation, this condition was formulated in terms of an explicit form of the two-point function in terms of geodesic distance. However, the Hadamard condition proved challenging to define unambiguously, which led to several refinements until Kay and Wald provided a definition in [60], that was thought to be rigorous for 3 decades until Moretti pointed out some more inconsistencies, and solutions to those, in [65]. Shortly after the definition of Kay and Wald, Radzikowski provided a radical reformulation of the concept in [66], using tools from the field of microlocal analysis. This publication represents a pivotal moment in the history of AQFT, marking the beginning of an incredibly fruitful cross-disciplinary development, significantly advancing our understanding of quantum fields in curved spacetimes.

Microlocal analysis is a sophisticated framework developed to address various issues in the study of partial differential equations (PDEs) and their solutions. The foundational work in this area is largely attributed to Duistermaat and Hörmander, whose seminal contributions [57, 33] have been crucial in shaping the modern understanding of the subject. Their approach provides powerful tools for studying the behaviour of the singularities of solutions to PDEs, by examining their properties in both spatial and frequency domains.

A central concept in microlocal analysis is the notion of a *wave front set*, which characterises the singularities of distributions in terms decay properties of their Fourier transform. It captures not only the points where a distribution is singular, but also the direction in frequency-space in which its Fourier transform diverges.

This concept is instrumental in understanding the local and global behaviour of solutions to PDEs. For instance, Hörmander's celebrated propagation of singularities theorem describes how the singularities of a distributional solution to a differential equation can be understood in terms of the singularities of its initial data, and the bicharacteristics of the differential operator. This result is an essential tool for resolving many problems in modern treatments of AQFT.

Radzikowski's reformulation of the Hadamard condition represents a significant advancement in the field. By leveraging modern techniques in microlocal analysis, Radzikowski provided a more robust and general framework for understanding the condition, facilitating its application to a broader range of quantum field theories. This led to microlocal analysis becoming a core technique in modern treatments of AQFT. Recently, techniques from quantum field theory have also been finding their way back to the theory of PDE's, see e.g. [29].

## 1.3 PERTURBATIVE ALGEBRAIC QUANTUM FIELD THEORY

Another shortcoming of the AQFT framework is that it is distinctly lacking in examples. Importantly, interacting models in AQFT have only been constructed in highly symmetric, lower dimensional toy models, which leaves a gap in dealing with more realistic models from particle physics.

Perturbative Algebraic Quantum Field Theory (pAQFT) was developed to address this shortcoming, incorporating ideas from perturbative quantum field theory into the AQFT framework. Drawing inspiration from the microlocal techniques introduced by Radzikowski, pAQFT emerged at the close of the 20th century through contributions from Brunetti, Dütsch, Hollands, Fredenhagen, Wald, and others. Reviews of this framework can be found in [34] and [67].

In pAQFT, the quantisation process is carried out using formal deformation quantisation of a free field theory, upon which interactions are introduced perturbatively. This means that most expressions in the theory are formal power series in Planck's constant $\hbar$ and the coupling constants of the interacting theory. While this perturbative expansion mirrors traditional methods in particle physics, pAQFT distinguishes itself by ensuring full mathematical rigour at every stage.

Renormalisation in pAQFT is handled using the Epstein-Glaser method developed in [35], which offers a systematic and rigorous approach that does not require a choice of regulator, as it initially only considers quantities defined at non-coincident points. The key objective is the definition of time-ordered products, which are central to perturbative quantum field theory. As these products are well-defined for points in spacetime that do not overlap, there is no ambiguity at this point in the process.

Next, a set of renormalisation conditions is introduced to ensure a consistent result. Finally, techniques from distribution theory are used to extend the time-ordered products to include overlapping points, while ensuring that the renormalisation conditions continue to hold. The choice of extension is generally not unique, leading to the introduction of the Stueckelberg-Petermann group, which captures the arbitrary choices involved in the renormalisation process, see e.g. [17]. Importantly, this method is fully covariant, and is therefore particularly well-suited for the study of interacting quantum field theory in curved spacetimes.

## 1.4 THE BV FORMALISM

Another novel introduction to the AQFT framework is the Batalin-Vilkovisky (BV) formalism, introduced in [7]. This formalism is a powerful tool in quantum field theory and algebraic geometry, designed to handle the quantisation of gauge theories. In the presence of gauge redundancies, the path-integral formalism runs into trouble, as it requires one to integrate a constant quantity over the infinite dimensional gauge orbits. To remedy this problem, Faddeev and Popov introduced the notion of a 'ghost' field in [36], which is an additional field that is used to restrict the path integral to the gauge-fixed surface.

Building upon this idea, Becchi, Rouet, Stora, and Tyutin introduced what is now known as BRST quantisation. In their framework, ghost fields are treated in the language of homological algebra, allowing for a consistent and systematic treatment of the redundancies in a gauge theory. The BRST formalism is essential in simplifying gauge-fixed quantum field theory, but it has limitations when applied to more complicated gauge theories, and gauge symmetries that do not close off-shell, such as is the case for gravity.

The BV formalism generalises and extends BRST by introducing a dual set of fields, called *antifields*, for each field in the theory. These antifields encode the dynamics of the theory in a homological structure as well, putting the dynamics and the gauge redundancies at the same level. This approach allows for more complicated gauge theories to be dealt with, like effective quantum gravity as in [19]. We refer the reader to [21] for a comprehensive introduction, and to [53] for a treatment aimed at perturbative quantum field theory.

The integration of the BV formalism into pAQFT was carried out by Fredenhagen and Rejzner in [41, 40]. Their work extended the applicability of pAQFT to gauge theories in curved spacetimes, incorporating the BV formalism to handle the gauge symmetries and their quantisation systematically. This fusion of techniques forms the basis for much of the work presented in this thesis.

Another recent development in the BV-formalism that has served as inspiration for this work is the development of 'homotopical' AQFT by Benini, Schenkel and others, see e.g. [12] for a review. Their framework is more in line with the original axiomatic treatment of Haag and Kastler, but explores its generalisation in the context of homotopy theory and higher categorical structures. They take seriously the idea that the algebra of observables of a quantum field theory with gauge redundancies is a differential graded object, and that the physical information 'lives' in the 0'th (co)homology. As such, quasi-isomorphic theories should be considered to describe the same physical theory. This loosening of the idea of equality allows for a more general, systematic discussion on what it means to be a gauge theory.

In a similar spirit is the work of Costello and Gwilliam [23, 24] on factorisation algebras, which provides a modern and geometrically-inclined approach to quantum field theory. Their framework interprets the basic object in quantum field theories as a factorisation algebra, which is an analogue of a net of algebras. Similar to the approach taken in homotopical AQFT, this formalism builds on ideas from the BV framework but extends it by introducing higher categorical methods and techniques from derived algebraic geometry. In particular, their approach focuses on the local-to-global behaviour of observables in a theory, which becomes a more nebulous matter in the context of gauge theories.

## 1.5 MOTIVATION FOR THIS WORK

One of the key motivations for the present work is the development of the 'functional formalism' for pAQFT, introduced by Brunetti, Fredenhagen, and Ribeiro in [20]. This approach models observables as smooth functionals on the configuration space of a given field theory, which are typically non-polynomial. These observables become relevant when discussing non-perturbative interacting classical field theory models as in [20], where the propagator becomes a smooth function of the reference configuration. On top of this, there are models of interest, like the Sine-Gordon theory, where non-polynomial observables play a role. A similar point can be made for general relativity, where quantities like the square root of the metric determinant are key observables.

In defining algebraic structures on these functionals, derivatives of the field functionals are paired by way of certain propagators of the field theory. However, the derivatives of functionals are distributional in nature, meaning that the Poisson bracket and quantum product of arbitrary functionals are naively ill-defined. To overcome these issues, the *microcausal* functionals are introduced. These are functionals whose derivatives are required to have a restricted singular structure, related to the causal structure of the spacetime (hence the name 'micro-causal'). By restricting the singularities in this way, the problematic pairings involved in defining algebraic operations become well-defined.

This notion is effectively a generalisation of techniques that were already used in the literature for polynomial functionals, tracing back to early publications in pAQFT like [18] and [56], where they were known under variations of the name 'extended algebra of Wick polynomials'. The core aim of this algebra is to give a robust class of observables that contains local observables, but that is also large enough to allow for operations that break locality by contracting with propagators, which are needed in order to define the quantum product and renormalise.

The microcausal functionals render the pairings in the Poisson bracket and quantum product well-defined. However, surprisingly, we were able to show that they do not close under these operations, as the resulting functionals can be discontinuous. A central aim of this thesis is to address this issue by refining the notion of microcausal functional. This is done by introducing more sophisticated microlocal techniques to ensure that the singular structures of functionals are incorporated in a more robust fashion.

Additionally, this work expands the tools available for studying homotopical data within the pAQFT framework, within the functional formalism. By extending the work of Fredenhagen and Rejzner, we offer new ways to handle models in the BV-formalism, leading to a more rigorous and versatile framework for gauge theories in both flat and curved spacetimes. In this way, the present work not only resolves some longstanding issues with the use of microcausal functionals, but also provides rigorous tools for incorporating higher structures into the formalism. This opens up exciting possibilities for future developments in both pAQFT and AQFT, particularly in the study of gauge theories and other models where the interplay between local and global properties of fields is crucial.

## 1.6 OUTLINE OF THE THESIS

This thesis is structured as follows. In Chapter 2 we discuss some background material that is used throughout this text. We start by recalling some concepts from linear functional analysis. In particular, we give a self-contained treatment of the concept of equicontinuous families of linear maps, as these play a central role in our discussion in Chapter 6. We then go on to discuss some basic definitions and results from the field of microlocal analysis. We take a black-box approach here, and focus mainly on the tools that the field provides for pairings of distributions satisfying wavefront set conditions. We do not discuss more PDE-adjacent tools, such as the propagation of singularities theorem and pseudo-differential operators, as we will not explicitly employ them. Finally, we give a brief introduction to the theory of non-linear functionals on locally convex vector spaces, and discuss how these have been applied within the functional formalism for pAQFT.

Chapter 3 is an extension to the section on microlocal analysis in the preceding chapter. We need some advanced results from the literature on spaces of distributional sections in later parts of this thesis, that are only available in the context of scalar valued distributions on subsets of Euclidean space. We close this gap in the literature here, by giving the general statements that we need, and showing how they can be derived from the specialised case for which a proof is available. This part of the text contains most of the technical detail on microlocal analysis, which we will not need until Chapter 6. For this reason, we advise the reader to skip this chapter on a first read, and come back to the results as they are called for later.

We then move on to the main part of the novel material presented in this thesis. In Chapter 4 we discuss two explicit counterexamples that exhibit some undesirable features of the microcausal functionals. The first of these shows that the Poisson bracket of two microlocal functionals can fail to be a continuous functional. It appeared roughly in this shape already in my joint paper with Eli Hawkins and Kasia Rejzner [52]. The second counterexample shows that microcausal functionals do not close under the homotopy operator that we introduce later in Chapter 5. Chronologically, this counterexample precedes the first one, and highlights a difficulty in proving the time-slice axiom for the microcausal multivector fields in [41].

After that, we move to discuss homotopical tools in the functional formalism in Chapter 5. This is based on the homotopy operator presented in [52], where it appeared in the proof of exactness of the Koszul complex. We give a generalisation here to allow for any deformation retract of locally convex cochain complexes. Along the way, we discuss graded functionals as a generalisation of the (polynomial) graded symmetric algebra, and show how to extend several standard operations on them to this more general structure. We close by discussing some examples and ongoing problems that we intend to solve in a follow-up publication to [52].

We bring all the previous streams together to define the graded equicausal functionals in Chapter 6. These are a more robust variant of the microcausal functionals, and are aimed at solving the problems exhibited in Chapter 4. We start by giving a generalisation of the definition given in [52], and show that they are stable under the tools developed in Chapter 5. We then give some examples of two classes of functionals that are equicausal: the local functionals and the microcausal polynomials, which are the two main ingredients that are used in practical applications. We then close the chapter by giving a proof of the closure of the $\star$-product on the graded equicausal functionals. With future work in mind, we give it in the most general setting that we can phrase.

Finally, in Chapter 7, we summarise the key contributions of this thesis. We also address the broader implications of this work for algebraic quantum field theory, particularly in gauge theories. Finally, we outline some open questions and possible directions for future research, including further applications of homotopical methods and potential refinements of the equicausal framework.

# — 2 —

# Background material

In this chapter, we gather some background results that will be used throughout this text. We will refrain from giving proofs of results, unless they are non-standard and a good reference is not available.[1] Parts of this section has been taken from the background section in my joint publication with Eli Hawkins and Kasia Rejzner in [52], which was one of my contributions to that work.

## 2.1 LINEAR FUNCTIONAL ANALYSIS

We gather some general definitions and results from the theory of linear functional analysis in this section. This will serve as a basis for the study of *non-linear* functionals in Section 2.3. The main reference we use for this section is the book by Trèves [72], but there are many other good options where the reader may find the material presented here.

This section is structured as follows: We start by recalling some elementary definitions, before moving to discuss the notion of equicontinuity in some detail, as that features significantly throughout this thesis. Closely related to that is the Banach-Steinhaus theorem, which we state in its most general form. After that, we move to consider several categorical constructions in the setting of functional analysis, and close by giving some examples of some standard spaces used in distribution theory. Experts in these topics will be safely able to skip many parts of this section.

[1] To the best of the author's knowledge

### 2.1.1 Elementary concepts in functional analysis

The main object of study in functional analysis are topological vector spaces. Throughout this thesis, $\mathbb{K}$ will denote either the real or the complex numbers. When it is clear from the context which one we mean, or if there is no difference in the validity of the statement made, we suppress it in the notation.

**Definition 2.1.1.** *A real (complex)* ***topological vector space*** *(TVS) is a vector space $V$ over $\mathbb{R}$ ($\mathbb{C}$), endowed with a Hausdorff topology such that addition and scalar multiplication are continuous maps*

$$V \times V \xrightarrow{+} V,$$
$$\mathbb{K} \times V \xrightarrow{\cdot} V,$$

*with respect to the product topology.*

From the definition, it follows that the topology of a TVS is determined completely by the set of neighbourhoods of zero, as translation by a fixed element in $V$ is an automorphism. The Hausdorff requirement is not standard in every text, but can always be imposed by taking a quotient with respect to the closure of $\{0\} \subset V$.

A TVS is called **locally convex** if there exists a basis of neighbourhoods of zero consisting of convex sets. Equivalently, $V$ is locally convex if for every neighbourhood of zero $U \subset V$, there is a convex $\tilde{U} \subset U$ which is also a neighbourhood of zero. In practical applications, every space one uses in functional analysis is locally convex. This is due to their close relation to seminorms, which are a convenient way to define topological vector spaces.

**Definition 2.1.2.** *A* ***seminorm*** *on a vector space $V$ is a function $p : V \to \mathbb{R}^+$ such that, for all $x, y \in V$, $\lambda \in \mathbb{C}$,*

$$p(x + y) \leq p(x) + p(y),$$
$$p(\lambda x) = |\lambda| p(x).$$

If we have a collection of seminorms $\{p_i\}_{i \in I}$ on a vector space $V$, where $I$ is not required to be finite or even countable, then we can endow $V$ with a locally convex vector topology by defining a *subbase* for the topology at zero[2] to be

$$\{p_i^{-1}([0, \varepsilon)) \subset V \mid i \in I, \epsilon > 0\}.$$

[2]By this we mean the topology consisting of translates of finite intersections of elements of the subbase.

Alternatively, if $J \subset I$ is a finite subset, then we can define a new seminorm by

$$p_J(x) = \max_{i \in J} p_i(x).$$

A *base* for the topology of $V$ at zero is then given by

$$\{p_J^{-1}([0, \varepsilon)) \subset V \mid J \subset I \text{ finite}, \epsilon > 0\},$$

as taking the intersection of finitely many elements of the subbase is equivalent to taking the maximum of the appropriate seminorms.

We will call this the topology *generated* by the family $\{p_i\}_{i \in I}$. In general, different families of seminorms can generate the same topology. In this topology any continuous seminorm $p$ on $V$, not necessarily from the generating set, can be majorised by a seminorm of the form $Cp_J$, for some $C > 0$ and $J \subset I$ finite. It is a standard fact that a TVS is locally convex iff it can be generated by a family of seminorms, by use of the so-called *Minkowski functionals*, see e.g. Proposition 7.5 and following discussion in [72].

A map $f : V \to W$ between locally convex spaces is continuous iff for every continuous seminorm $q$ of $W$, there is a continuous seminorm $p$ of $V$ such that

$$q(f(x)) \leq p(x) \quad \forall x \in V. \tag{2.1}$$

In particular, a linear functional $f : V \to \mathbb{C}$ is continuous iff there is a continuous seminorm $p$ of $V$ such that

$$|f(x)| \leq p(x) \quad \forall x \in V. \tag{2.2}$$

In practice, it is convenient to work with complete spaces, i.e. spaces where every Cauchy sequence has a limit. But not every operation one performs with topological vector spaces respects completeness. A standard trick to deal with this kind of situation is to pass over to the completion of the space. If $V$ is a topological vector space, then a completion of $V$ is a complete topological vector space $\hat{V}$, together with a dense embedding $i : V \to \hat{V}$. If $f : V \to W$ is a continuous linear map into a complete space, then there is a unique continuous linear map $\hat{f} : \hat{V} \to W$ that extends $f$, i.e. $\hat{f} \circ i = f$, see e.g. Theorem 5.1 in [72]. From this fact, it follows that completions are unique up to isomorphism. Existence of completions is shown in Theorem 5.1 of [72]. The completion of a locally convex space is again locally convex.

#### 2.1.1.1 Bornology and spaces of continuous linear maps

A notion which is related to topology is that of *bornology*. Where topology deals with fundamental systems of *open* set, bornology treats fundamental systems of *bounded* sets. We give some general facts from the theory of bornology here that will prove useful in this text, and refer the interested reader to [55] for an in-depth treatment.

**Definition 2.1.3.** *Let $V$ be a vector space. A* ***vector bornology*** *on $V$ is a set $\mathcal{B} \subset \mathcal{P}V$, the power set of $V$, such that*

- *$\mathcal{B}$ covers $V$, i.e. $\bigcup \mathcal{B} = V$,*
- *$\mathcal{B}$ is closed under inclusion, i.e. $A \subset B \subset V$ and $B \in \mathcal{B} \implies A \in \mathcal{B}$,*
- *$\mathcal{B}$ is closed under finite union, i.e. $\{B_i\}_{i=1}^n \subset \mathcal{B} \implies \bigcup_{i=1}^n B_i \in \mathcal{B}$,*
- *$\mathcal{B}$ is closed under addition, i.e. $A, B \in \mathcal{B} \implies A + B \subset \mathcal{B}$,*
- *$\mathcal{B}$ is closed under scalar multiplication, i.e. $B \in \mathcal{B}$, $\lambda \in \mathbb{C} \implies \lambda B \in \mathcal{B}$,*
- *$\mathcal{B}$ is closed under taking disked hulls, i.e. $B \subset \mathcal{B} \implies \bigcup_{|\lambda| \leq 1} \lambda B \in \mathcal{B}$.*

*We call $\mathcal{B}$ a* ***convex vector bornology*** *if in addition, it is closed under forming convex hulls:*

$$B \in \mathcal{B} \implies CH(B) = \{\lambda x + (1 - \lambda)y \in V \mid x, y \in B, \lambda \in [0, 1]\} \in \mathcal{B}$$

*A vector space $V$ equipped with a (convex) vector bornology is called a* ***(locally convex) bornological space.***

If $(V, \mathcal{B}_V)$ and $(W, \mathcal{B}_W)$ are two bornological spaces then a linear map $\Phi : V \to W$ is called **bounded** (or bornological) if $\Phi(B) \subset \mathcal{B}_W$ for all $B \in \mathcal{B}_V$.[3] If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two vector-bornologies on $V$, then we say that $\mathcal{B}_1$ is **weaker** than $\mathcal{B}_2$ if $\mathcal{B}_1 \subset \mathcal{B}_2$.

*Example* 2.1.4. If $V$ is any vector space, then setting $\mathcal{B} = \{B \subset V \mid B \text{ finite}\}$ defines a vector bornology, which we call the **finite bornology** on $V$. If $V$ is endowed with a vector topology, then we call a set $B$ precompact (or relatively compact) if its closure is compact. We can then set $\mathcal{B} = \{B \subset V \mid B \text{ precompact}\}$ to obtain the **precompact** bornology. We note that the compact subsets of $V$ do not form a bornology, as they are not closed under inclusion.

*Example* 2.1.5. A vector topology on a vector space $V$ naturally induces a vector bornology on $V$, the **von Neumann bornology**: We define $B \subset V$ to be von

[3]Note the difference with continuous maps, which are defined in terms of *inverse* images.

Neumann bounded if, for any neighbourhood of zero $U \subset V$, there exists a $\rho \in \mathbb{R}^+$ such that $B \subset \rho U$. In this case, we will also say that $B$ is *absorbed* by $U$. *When we refer to a bounded set of a topological vector space without further specification, we will mean it in this sense.* In contrast to open sets, the image, rather than the inverse image, of a bounded set by a continuous map is bounded. If $V$ is locally convex, then $B \subset V$ is bounded iff $p(B) \subset \mathbb{R}$ is bounded for all continuous seminorms $p$ of $V$.

Precompact sets are always bounded, so that the precompact bornology of a TVS is weaker than its von Neumann bornology. The converse is not true in general. Spaces for which this holds are called semi-Montel. For example, if $\Omega \subset \mathbb{R}^n$ is an open set then $C^\infty(\Omega)$, whose topology we define in Section 2.1.4, is semi-Montel. This is a consequence of the Arzelà-Ascoli theorem, see e.g. Theorem 14.4 in [72].

The main interest of bornology for our purposes is that a bornology induces a topology on spaces of continuous linear maps. Let $V$ and $W$ be TVS's, then we denote the spaces of continuous linear maps from $V$ to $W$ by $L(V, W)$. If $\mathcal{B}$ is a bornology on $V$, weaker than the von Neumann bornology on $V$, then the sets

$$\mathfrak{U}(B, U) = \{\Phi \in L(V, W) \mid \Phi(B) \subset U\}$$

where $B \in \mathcal{B}$ and $U \subset W$ a neighbourhood of zero, define a basis of neighbourhoods at zero for a topology on $L(V, W)$, see e.g. Definition 32.1 in [72].

We recall the definition of a **net** in a topological space $X$, which is a generalisation of the notion of sequence. A **directed set** $\Lambda$ is a preordered set that is upwards directed: For any two $\lambda, \mu \in \Lambda$, there exist a $\nu \in \Lambda$ such that $\lambda \leq \nu$ and $\mu \leq \nu$. A net in $X$ over $\Lambda$ is an assignment of an element $f_\lambda$ in that space to indices $\lambda \in \Lambda$. A net **converges** to a point $f \in X$ if for all open neighbourhoods $U$ of $f$, the exist a $\lambda_0 \in \Lambda$ such that

$$\lambda \geq \lambda_0 \implies f_\lambda \in U.$$

Of course, if $\Lambda$ is given by the natural numbers with their usual ordering, this is nothing but a sequence, and a sequence converges as a net if and only if it converges as a sequence. Crucially, however, nets are better able to resolve the topology on a space than sequences are when looking at non-metric spaces.

In the case of the space $L(V, W)$ with the topology defined by a bornology $\mathcal{B}$ on $V$, a net $f_\lambda$ converges to zero iff, for all $B \in \mathcal{B}$ and neighbourhoods of zero $U \subset W$

$$f_\lambda(B) \subset U$$

for $\lambda$ larger than some $\lambda_0 \in \Lambda$. For this reason, this topology is referred to as the *topology of uniform convergence on sets in* $\mathcal{B}$.

The most important examples of this construction are the weak and strong duals of a TVS $V$. The dual $V^*$ of $V$ is given by all continuous linear functionals on $V$. The strong dual $V'$ is the TVS defined by endowing $V^*$ with the topology of uniform convergence on bounded sets of $V$. The weak dual $V'_\sigma$ is given by endowing it with the topology of uniform convergence on finite sets, i.e. using the finite bornology. Convergence in this topology is nothing more than pointwise convergence.

More generally, if $V$ and $W$ are topological vector spaces, then we denote by $L_b(V, W)$ the set $L(V, W)$ with the topology induced by the von Neumann bornology on $V$, by $L_\sigma(V, W)$ the topology induced by the finite bornology, and by $L_c(V, W)$ the topology induced by the pre-compact bornology.

### 2.1.2 Equicontinuous sets and the Banach-Steinhaus theorem

The set $L(V, W)$ allows for several natural choices of bornology. We described several topologies on this space above, and we can take the von Neumann bornology with respect to any of them. The most relevant one for us is the one induced by $L_b(V, W)$: If $H \subset L(V, W)$ is bounded in this sense, then we can find, for all $B \subset V$ bounded and $U \subset W$ open, $\rho > 0$ such that

$$H \subset \rho\mathfrak{U}(B, U) = \mathfrak{U}(B, \rho U),$$

which is equivalent to $H(B) = \{f(v) | f \in H, v \in B\}$ being bounded in $W$. Hence, $H$ is bounded in $L_b(V, W)$ if and only if it maps bounded sets to bounded sets. We will say that $H$ is *strongly bounded* if this is the case. Similarly, we say that $H$ is *weakly bounded* if it is bounded in the von Neumann bornology of $L_\sigma(V, W)$. This is the case if and only if it maps points in $V$ to bounded sets in $W$, which might occur as $H$ is not required to be finite itself.

Another bornology which is of fundamental importance in the following.

**Definition 2.1.6.** *A set $H \subset L(V, W)$ is* ***equicontinuous*** *if, for any neighbourhood of zero $U \subset W$, there is a neighbourhood of zero $\tilde{U} \subset V$ such that*

$$f(\tilde{U}) \subset U \quad \forall f \in H.$$

Alternatively phrased, $H \subset L(V,W)$ is equicontinuous if, for any neighbourhood of zero $U \subset W$,

$$\bigcap_{f \in H} f^{-1}(U) \subset V$$

is a neighbourhood of zero,[4] which is not immediate as $H$ might contain infinitely many elements. An equicontinuous set of mappings is thus a collection of mappings that are 'continuous at the same rate'. For this reason, we will also somewhat abusively say that '$H$ is equicontinuous from $V$ to $W$'. The equicontinuous subsets of $L(V,W)$ form a bornology. The image of a bounded set in $V$ by an equicontinuous family is bounded in $W$, so that the equicontinuous bornology is in general weaker than the strong bornology.

In the case where $V$ and $W$ are locally convex there is a simple characterisation of equicontinuous sets: A set $H \subset L(V,W)$ is equicontinuous iff there is, for every continuous seminorm $q$ of $W$, a continuous seminorm $p$ of $V$, such that

$$q(f(x)) < p(x) \quad \forall\, x \in V,\ f \in H. \tag{2.3}$$

In particular, $H \subset V'$ is equicontinuous iff there exists a continuous seminorm $p$ of $V$, such that

$$|f(x)| < p(x) \quad \forall\, x \in V,\ f \in H. \tag{2.4}$$

These statements should be compared to equations (2.1) and (2.2).

The analogy to continuous maps goes further: If $H$ is equicontinuous from $V$ to $W$ and $L$ is equicontinuous from $W$ to $X$, then their composition is defined as

$$L \circ H = \{g \circ f : V \to X \mid g \in L\,,\ f \in H\}, \tag{2.5}$$

and this is equicontinuous from $V$ to $X$. In particular, by taking either family to be a set containing a single continuous map, this implies that equicontinuous sets are stable under pullback and pushforward by continuous maps.

As an illustration of the central importance of equicontinuous sets in functional analysis, we recall the following fact, see e.g. Proposition 36.1 in [72]:

**Proposition 2.1.7.** *The topology of a topological vector space $V$ is equivalent to the topology of uniform convergence on equicontinuous subsets of $V'$, where we view $V$ as a subset of $(V')'$ by means of the evaluation map*

$$\mathrm{ev} : V \to (V')',$$

[4] We do not require that neighbourhoods of zero are open themselves, merely that they contain an open set that contains zero.

*defined by* $\mathrm{ev}_v(f) = f(v)$.

For lack of a better place, we discuss here the notion of *hypocontinuity* of a bilinear map. It often happens in functional analysis that seemingly innocuous bilinear operations fail to be jointly continuous. For example, if the pairing of a TVS with its dual

$$\langle \_\, , \_\rangle : V' \times V \to \mathbb{C},$$

is jointly continuous, then it follows that $V$ is a normed space. As there are many interesting spaces in functional analysis that are not normed, we must contend with the fact that discontinuous multilinear maps are pervasive in functional analysis.

A slightly weaker condition that most bilinear maps of interest do satisfy is the following:

**Definition 2.1.8.** *Let $V, W$ and $X$ be topological vector spaces, a bilinear map $\Phi : V \times W \to X$ is* ***hypocontinuous*** *if, whenever $A \subset V$ and $B \subset W$ are bounded sets, the sets*

$$\{\Phi(v, \_),\ |\ v \in A\} \subset L(W, X),$$
$$\{\Phi(\_, w),\ |\ w \in B\} \subset L(V, X),$$

*are equicontinuous.*

The three bornologies on spaces of linear maps that we described (equicontinuous, strong and weak) are generally distinct. However, they agree when $V$ is *barrelled*. A barrel $T \subset V$ is a set that is

- absorbing, i.e. for all $x \in V$ there is a $\lambda > 0$ such that $x \in \lambda T$,
- convex,
- balanced, i.e. $|\lambda| \leq 1 \implies \lambda T \subset T$,
- closed.

A TVS $V$ is barrelled if all barrels are neighbourhoods of zero. In practice, many spaces of interest in analysis, such as Fréchet spaces, are barrelled. We recall the Banach-Steinhaus theorem, see e.g. Theorem 33.1 in [72].

**Theorem 2.1.9** (Banach-Steinhaus theorem)**.** *If $V$ is barrelled and $W$ is locally convex, then the weak, strong and equicontinuous bornologies on $L(V, W)$ coincide.*

We note the following corollary of the Banach-Steinhaus theorem, which is Theorem 41.2 in [72]:

**Theorem 2.1.10.** *If $V$ and $W$ are barrelled and $X$ is locally convex, then every separately continuous bilinear map $V \times W \to X$ is hypocontinuous.*

### 2.1.3 Some categorical constructions involving topological vector spaces

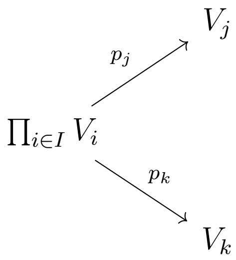
We recall some standard categorical constructions with locally convex topological vector spaces. This section is not meant to give a full overview of the field of category theory, but rather to highlight how some categorical constructions that we use in this thesis can be explicitly realised, as well as describe their topology. We assume the basic concepts of category, functor, (co-)limit, and adjunction to be known. We refer the reader to the literature for a more pedagogical introduction, for example in [64].

We shall take $\mathsf{LCTVS}$ to be the category whose objects are locally convex topological vector spaces, and whose morphisms are continuous linear maps.[5] This category is closely linked to $\mathsf{VS}$, the category of vector spaces with linear maps as morphisms. There is an obvious forgetful functor $\mathsf{LCTVS} \to \mathsf{VS}$. Conversely, if $V$ is a vector space, then we can endow it with the indiscrete topology (which is a locally convex vector topology), which defines a left adjoint to the forgetful functor. Similarly, we can endow $V$ with its discrete locally convex vector topology, which is the one defined by *all* seminorms on $V$. This defines a right adjoint to the forgetful functor $\mathsf{LCTVS} \to \mathsf{VS}$. As such, limits and colimits in $\mathsf{LCTVS}$ are, as vector spaces, equal to their counterparts in $\mathsf{VS}$.

If we have some collection of objects $\{V_i\}_{i \in I}$ in $\mathsf{VS}$, where $I$ is an index set (not necessarily finite or even countable), then their product $\prod_{i \in I} V_i$ is the limit of the

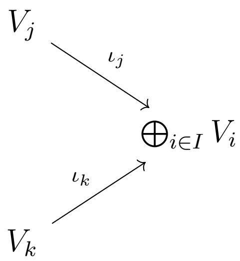
[5] We shall refrain from using the word homomorphism for them, as that is usually taken to mean a continuous map $f : V \to W$ such that the inverse $f^{-1} : \mathrm{Im}(f) \to V/\ker f$ is continuous as well. In particular, injective homomorphisms are topological embeddings, which is too strict a notion for our purposes. It is the content of the open mapping theorem that, between certain types of spaces, e.g. Fréchet spaces, a continuous linear map is automatically a homomorphism. We refer the reader to Chapter 17 of [72] for an in-depth discussion.

diagram in VS with no non-trivial arrows




Concretely, it can be realised as the Cartesian product of all the $V_i$, endowed with pointwise addition and scalar multiplication. The maps $p_i$ are the obvious projection maps.

The direct sum $\bigoplus_{i\in I} V_i$ is the colimit of the same diagram:




It can be realised as

$$\bigoplus_{i\in I} V_i = \{v \in \prod_{i\in I} V_i \,|\, v_i = 0 \text{ for all but finitely many } i\},$$

with the obvious injection maps $\iota_j : V_j \to \bigoplus_{i\in I} V_i$.

More complicated (co-)limits of diagrams in VS can be derived from these two basic constructions. Let $\mathcal{I}$ be a small category that encodes the shape of the diagram we are interested in. Concretely, it consists of a *set* of objects $\mathrm{Obj}(\mathcal{I})$, and a *set* of morphisms $\mathrm{Mor}(\mathcal{I})$ between objects. We denote by $\mathrm{Mor}(j, k)$ the morphisms from $j$ to $k$. There are of course identity morphisms from any object to itself, but we will suppress them in the notation when considering such 'diagram categories'. A diagram in VS of shape $\mathcal{I}$ is then a functor $X$ from $\mathcal{I}$ to VS.

As an example, we might take the objects of $\mathcal{I}$ to be given by the natural numbers, and have a single non-trivial morphism from $n$ to $m$ if $n \leq m$. Schematically we write this as

$$(\mathbb{N}, \leq) = (0 \to 1 \to 2 \to 3 \to \ldots),$$

where the unique map for $n < m$ is given by the composition

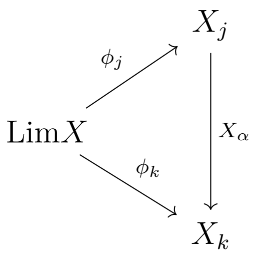
$$n \to n + 1 \to \ldots \to m.$$

A diagram of shape $(\mathbb{N}, \leq)$ is then explicitly given by a collection $\{V_i\}_{i\in\mathbb{N}}$ of vector spaces, together with linear maps $f_i : V_i \to V_{i+1}$:

$$V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} V_3 \xrightarrow{f_3} \ldots$$

Diagrams of this shape feature frequently in functional analysis, where they are used to form 'direct limits'.

The category of vector spaces allows for limits and colimits of all diagrams. Concretely, if $X : \mathcal{I} \to \mathsf{VS}$ is a diagram of shape $\mathcal{I}$, the limit is the universal element in the diagram




where $j, k \in \mathrm{Obj}(\mathcal{I})$ and $\alpha \in \hom(j, k)$. The limit can be explicitly realised by

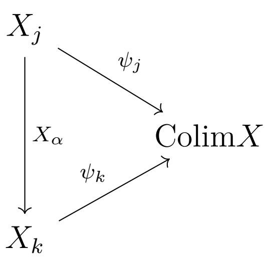
$$\mathrm{Lim}X = \left\{ v \in \prod_{i\in\mathrm{Obj}(\mathcal{I})} X_i \,\middle|\, X_\alpha \circ p_j(v) = p_k(v) \text{ for all } \alpha \in \mathrm{Mor}(j, k) \right\}.$$

The maps $\{\phi_j\}_{j\in I}$ are given by composing

$$\phi_j : \mathrm{Lim}X \to \prod_{i\in\mathrm{Obj}(\mathcal{I})} X_i \xrightarrow{p_j} X_j,$$

where the first map is the inclusion map.

Similarly, the colimit is the universal element in the diagram




The colimit can be explicitly realised by

$$\mathrm{Colim}X = \left( \bigoplus_{i\in\mathrm{Obj}(\mathcal{I})} X_i \right) / \sim$$

where we quotient by the relations

$$\iota_j(v) \sim \iota_k \circ X_\alpha(v),$$

for all $\alpha \in \mathrm{Mor}(j,k)$. The universal cone is given by composing

$$\psi_j : X_j \xrightarrow{\iota_j} \bigoplus_{i \in \mathrm{Obj}(\mathcal{I})} X_i \xrightarrow{\pi} \mathrm{Colim} X,$$

where $\pi$ is the quotient map.

We now go over to locally convex spaces, so that the entries of a diagram are LCTVSs, and the morphisms are continuous linear maps. We endow $\mathrm{Lim}X$ with the initial topology corresponding to the maps

$$\phi_i : \mathrm{Lim}X \to X_i.$$

for all $i \in \mathrm{Obj}(\mathcal{I})$. Concretely, this is the vector topology generated by sets of the form $\phi_i^{-1}(U_i)$ where $U_i \subset X_i$ is an open set. It is locally convex because all the $X_i$ are. In terms of seminorms, a generating system for this topology is given by all seminorms of the form $p \circ \phi_i$, where $i \in \mathrm{Obj}(\mathcal{I})$ and $p$ is a continuous seminorm on $X_i$.

The topology of $\mathrm{Colim}(X)$ is slightly more involved, as endowing the colimit with the final topology stemming from the universal cone is *not* guaranteed to yield a locally convex vector topology.[6] For this reason, we consider the finest *locally convex vector* topology on $\mathrm{Colim}X$ such that all maps

$$\psi_i : X_i \to \mathrm{Colim}X$$

are continuous. Explicitly, this topology is generated by *convex* sets $U \subset \mathrm{Colim}X$ such that $\psi_i^{-1}(U) \subset X_i$ is open for all $i \in \mathrm{Obj}(\mathcal{I})$, see e.g. Theorem 10.5 [73]. Equivalently, this topology is generated by all seminorms $p$ on $\mathrm{Colim}X$ such that $p \circ \psi_i$ is continuous for all $i \in \mathrm{Obj}(\mathcal{I})$.

We show an equicontinuous variant of the universal properties.

**Proposition 2.1.11.** *Let $X$ be a diagram in* LCTVS *of shape $\mathcal{I}$, and let $L$ be some set of indices. If $(V, \{\mu_l\}_{l \in L})$ is a collection of cones over $X$ such that, for all $i \in \mathrm{Obj}(\mathcal{I})$, the family of maps*

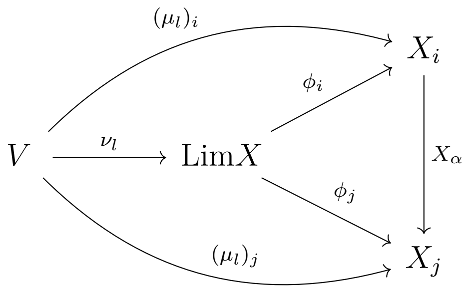
$$\{(\mu_l)_i : V \to X_i \,|\, l \in L\}$$

[6] A counterexample can be found in Chapter II, § 4, Exercise 15 of [13]

*is equicontinuous, then there is a unique equicontinuous family of linear maps*

$$\nu_l : V \to \mathrm{Lim}X,$$

*so that the diagram*




(2.6)

*commutes for all* $l \in L$, $i, j \in \mathrm{Obj}(\mathcal{I})$ *and* $\alpha \in \mathrm{Mor}(i, j)$.

*Proof.* Existence of a unique family of linear maps $\nu_l$ so that the diagram (2.6) commutes follows from the universal property of the limit as a vector space, so that it remains to be shown that it is an equicontinuous family.

Let $U \subset \mathrm{Lim}X$ be a neighbourhood of zero. Without loss of generality, we may assume that it is of the form

$$U = \bigcap_{i \in J} \phi_i^{-1}(U_i),$$

for $J \subset \mathrm{Obj}(\mathcal{I})$ *finite* and $U_i \subset X_i$ neighbourhoods of zero, as sets of that form are a basis for the topology at zero on $\mathrm{Lim}X$. By assumption of equicontinuity of the families $(\mu_l)_i$, there exists, for all $i \in J$, a neighbourhood of zero $V_i \subset V$ such that

$$V_i \subset (\mu_l)_i^{-1}(U_i) \quad \forall l \in L.$$

It follows that

$$\bigcap_{i \in J} V_i \subset \bigcap_{i \in J} (\mu_l)_i^{-1}(U_i) = \bigcap_{i \in J} \nu_l^{-1}\left(\phi_i^{-1}(U_i)\right) = \nu_l^{-1}(\bigcap_{i \in J} \phi_i^{-1}(U_i)) = \nu_l^{-1}(U) \quad \forall l \in L,$$

so that $\nu_l$ is an equicontinuous family. □

There is a dual statement for the colimit.

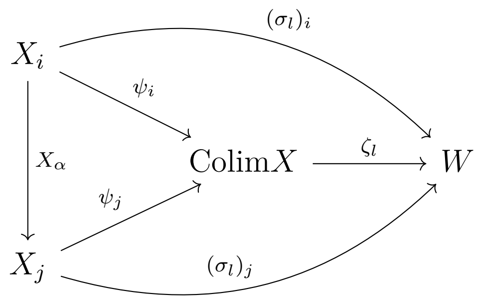
**Proposition 2.1.12.** *Let* $X$ *be a diagram in* LCTVS *of shape* $\mathcal{I}$*, and let* $L$ *be some set of indices. If* $(W, \{\sigma_l\}_{l \in L})$ *is a collection of cones under* $X$ *such that, for all* $i \in \mathrm{Obj}(\mathcal{I})$*, the family of maps*

$$\{(\sigma_l)_i : X_i \to W \mid l \in L\}$$

*is equicontinuous, then there is a unique equicontinuous family of linear maps*

$$\zeta_l : \mathrm{Colim} X \to W,$$

*so that the diagram*




(2.7)

*commutes for all* $l \in L$, $i, j \in \mathrm{Obj}(\mathcal{I})$ *and* $\alpha \in \mathrm{Mor}(i, j)$.

*Proof.* Again, existence of linear maps $\zeta_l$ making the diagram (2.7) commute follows from the universal property of colimits of vector spaces. Let $U \subset W$ be a neighbourhood of zero. As $W$ is locally convex, we may assume that $U$ is convex, so that also $\zeta_l^{-1}(U)$ is convex. But then

$$\bigcap_{l \in L} \zeta_l^{-1}(U)$$

is a neighbourhood of zero, as it is convex and

$$\psi_i^{-1}\left(\bigcap_{l \in L} \zeta_l^{-1}(U)\right) = \bigcap_{l \in L} (\sigma_l)_i^{-1}(U) \quad \forall i \in \mathrm{Obj}(\mathcal{I}),$$

which is a neighbourhood of zero in $X_i$ by assumption of equicontinuity of $(\sigma_l)_i$. □

Specialised to equicontinuous families of a single member, his shows that LCTVS is complete and cocomplete. In fact, this statement can be used to show (co-)completeness of the category of locally convex topological vector spaces, where morphisms are given by equicontinuous sets of linear mappings. As we have no explicit use for this category in what follows, we do not give a precise statement here.

### 2.1.4 Some standard spaces from distribution theory

We dress the abstract discussion from the previous sections by discussing some standard spaces from distribution theory. Throughout this section, $\Omega$ will denote an open set of $\mathbb{R}^m$ for some $m$, $M$ will denote a fixed manifold[7] of dimension m, and $E$ will denote a rank $k$ vector bundle over $M$. Many field theories of interest, such as Dirac fields, are inherently complex valued, but real valued fields are also studied. For this reason, we hold off on fixing a convention on whether vector bundles are real or complex valued, and phrase all statements in terms of $\mathbb{K}$. In any case, none of the statements in this chapter or the next depend on this choice.

We will use the notation for spaces of smooth functions due to Laurent Schwartz

$$\mathcal{E}(\Omega) = \{\varphi : \Omega \to \mathbb{K} \,|\, \varphi \text{ is smooth}\}.$$

If $K \subset \Omega$ is compact and $n \in \mathbb{N}$, then we can define a seminorm on $\mathcal{E}(\Omega)$ by

$$p_{n,K}(\varphi) = \sup_{|\alpha| \leq k} \sup_{x \in K} |\partial^\alpha \varphi(x)|.$$

These seminorms define the topology for $\mathcal{E}(\Omega)$, which turns it into a Fréchet space, i.e. a metrisable complete LCTVS.

The *test functions* are the compactly supported elements of $\mathcal{E}(\Omega)$

$$\mathcal{D}(\Omega) = \{\chi \in \mathcal{E}(\Omega) \,|\, \operatorname{supp} \chi \text{ is compact}\}.$$

Contrary to what one might expect, we do *not* endow this with the subspace topology, as the dual of $\mathcal{D}(\Omega)$ with that topology is the set of compactly supported distributions, rather than the set of all distributions, which is what not we aim to achieve by this definition.

Rather, if $K \subset \Omega$ is compact, which we will write as $K \Subset \Omega$, then we define the space

$$\mathcal{E}(K) = \{\chi \in \mathcal{E}(\Omega) \,|\, \operatorname{supp} \chi \subset K\},$$

endowed with the subspace topology inherited from $\mathcal{E}(\Omega)$. Clearly $\mathcal{D}(\Omega)$ is the union of these spaces over $K \Subset \Omega$. Furthermore, if $K \subset \tilde{K} \Subset \Omega$, then $\mathcal{E}(K)$ embeds in $\mathcal{E}(\tilde{K})$. Hence, we can exhibit $\mathcal{D}(\Omega)$ as a colimit of vector spaces by

$$\mathcal{D}(\Omega) = \mathrm{Colim}_{K \Subset \Omega} \mathcal{E}(K).$$

We embed it with the final topology stemming from this diagram.

The reason for this somewhat convoluted definition is to ensure that a sequence is convergent in $\mathcal{D}(\Omega)$ if and only if there is some $K \Subset \Omega$ such that the sequence

[7] We take manifolds to be second countable as a convention.

converges in $\mathcal{E}(K)$, meaning that there is a uniform bound on the supports of the elements in the sequence, and that the functions and all their derivatives converge uniformly on $K$. This notion of convergence, or pseudo-topology, was originally considered by Schwartz,[8] who defined a distribution to be a map

$$u : \mathcal{D}(\Omega) \to \mathbb{C}$$

such that

$$|u(\chi_j)| \xrightarrow{j\to\infty} 0,$$

whenever $\chi_j$ is a sequence in $\mathcal{D}(\Omega)$ converging to zero in this sense. This definition predates the modern interpretation of distributions as topologically continuous maps on $\mathcal{D}(\Omega)$, and the topology on $\mathcal{D}(\Omega)$ was designed specifically to recover this earlier characterisation.

Similarly, we denote by $\mathcal{E}(M;E)$ the space of smooth sections of $E$, and by $\mathcal{D}(M;E)$ the space of compactly supported smooth sections. If $U \subset M$ admits a coordinate chart that trivialises $E$, then there is an isomorphism of vector spaces

$$\mathcal{E}(U;E|_U) \cong \mathcal{E}(\Omega)^k$$

for $\Omega$ an open subset of some Euclidean space, and we endow $\mathcal{E}(U;E|_U)$ with the topology induced by this isomorphism.[9] We endow $\mathcal{E}(M,E)$ with the initial topology induced by the restriction maps to $\mathcal{E}(U;E|_U)$. As we may choose a countable atlas to induce this topology, $\mathcal{E}(M;E)$ is again a Fréchet space. The space $\mathcal{D}(M;E)$ is topologised analogously. We refer the reader to 1.1.4 in [70] for a detailed discussion.

The **dual** of $E$ is the bundle over $M$ with

$$(E^*)_x = (E_x)^* \quad \forall x \in M,$$

whose transition functions are the adjoints of the transition functions of $E$. Hence there is an isomorphism

$$E^* \otimes E \to \mathbb{K} \times M,$$

the trivial line bundle over $M$. We also introduce the density bundle $D_M$ of $M$, which is the line bundle over $M$ whose transition functions are given by the absolute

[8]We hold off on giving a precise citation for this definition, as the history of the notion of distribution is quite nebulous. The formalised version available today is largely due to Schwartz, but the concept of 'generalised function' has been around for longer. We refer the reader to [63] for an extensive review of the history of distribution theory.

[9]It is readily checked that this definition does not depend on a choice of trivialisation and coordinate chart.

value of the determinant of the Jacobian of the coordinate transform in question. Hence, the sections of this bundle are the natural objects that can be integrated on a manifold

$$\int_M : \mathcal{D}(M; D_M) \to \mathbb{C}$$

without an *a priori* choice of measure on $M$. It is isomorphic to the tensor product of the orientation bundle of the manifold with the top exterior power of the cotangent bundle $\Lambda^m T^*M$. Hence, if an orientation is implicitly chosen on $M$ (as is the case when $M$ is assumed to be a spacetime for example), then this bundle is isomorphic to $\Lambda^m T^*M$.

The **functional dual** of $E$ is then defined as

$$E^! = E^* \otimes D_M. \tag{2.8}$$

Because of the presence of the density bundle, a section of $E$ and a section of $E^!$ can be integrated against each other in a completely covariant fashion, so long as their supports have compact overlap. For later reference, we note that $D_M \otimes D_M^*$ is the trivial line bundle over $M$, so that

$$E^{!!} = (E^* \otimes D_M)^* \otimes D_M \cong E^{**} \otimes D_M^* \otimes D_M \cong E.$$

We implement this equivalence implicitly throughout this text.

The distributional sections of $E$ are defined by

$$\mathcal{D}'(M; E) := \mathcal{D}(M; E^!)'.$$

Similarly, we define

$$\mathcal{E}'(M; E) := \mathcal{E}(M; E^!)'.$$

We recall that both these spaces are equipped with their strong dual topology. The spaces of scalar distributions $\mathcal{D}'(M)$ and $\mathcal{E}'(M)$ are defined with respect to the trivial bundle $M \times \mathbb{R}$, which we suppress in the notation. It is a standard fact that $\mathcal{E}'(M; E)$ injects continuously into $\mathcal{D}'(M; E)$, and that its image is given by the distributional sections of compact support.

The reason for including densities into the formalism is that we can view $\varphi \in \mathcal{E}(M; E)$ as a distribution, by defining, for $\alpha \in \mathcal{D}(M; E^!)$

$$\langle \varphi, \alpha \rangle = \int_M \langle \varphi(x), \alpha(x) \rangle dx,$$

which is well-defined as the integrand is density-valued. Hence we have a continuous injection $\mathcal{E}(M;E) \to \mathcal{D}'(M;E)$, without having to select an arbitrary top-form. Similarly, $\mathcal{D}'(M;E)$ injects continuously into $\mathcal{E}'(M;E)$.

The spaces $\mathcal{E}(M,E)$ and $\mathcal{D}(M,E)$ are barrelled, being a Fréchet space resp. an LF-space, see e.g. Proposition 33.2 in [72]. Hence the Banach-Steinhaus theorem applies to their duals. One important consequence of this fact that we will use in the sequel is the following, see e.g. Proposition 34.6 and corollaries in [72]:

**Proposition 2.1.13.** *A sequence in $\mathcal{D}'(M,E)$ or $\mathcal{E}'(M,E)$ converges with respect to the respective strong topology if and only if it converges with respect to the respective weak topology.*

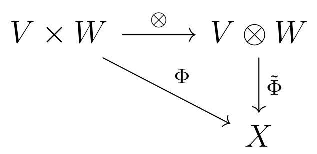
As a consequence, maps into these spaces from a second countable domain, such as $\mathbb{R}$, are strongly continuous as soon as they are weakly continuous, because there sequential continuity is equivalent to topological continuity. Our main use of this fact will be in the study of smooth curves of distributions below.

#### *2.1.4.1 Tensor products and the Schwartz kernel theorem*

We briefly recall some facts about tensor products that we will need in this text, without lingering on the details of topological tensor products. If $V$ and $W$ are vector spaces, then their tensor product is a vector space $V \otimes W$, together with a bilinear map

$$\otimes : V \times W \to V \otimes W.$$

It is required to satisfy the universal property that, for $\Phi : V \times W \to X$ a *bilinear* map, there exists a unique *linear* map $\tilde{\Phi}$ such that



is commutative. This universal property uniquely defines the tensor product up to linear isomorphism.[10]

[10] We will not have need of an explicit model for the tensor product, but one can be given by the space of formal sums of the form

$$\sum_{i=1}^{n} v_i \otimes w_i \quad v_i \in V,\, w_i \in W,$$

quotiented by the relations that implement bilinearity.

If $V$ and $W$ are LCTVSs, then we can endow their tensor product with a plethora of different topologies. The most straightforward one is the *projective* topology, which is nothing but the final locally convex topology with respect to the map $\otimes$. It has the universal property that, if $\Phi : V \times W \to X$ is bilinear and continuous, then its lift $\tilde{\Phi}$ is also continuous. We denote the tensor product endowed with this topology by $V \otimes_\pi W$. As this space is rarely complete, it is customary to complete it to the completed projective tensor product $V \hat{\otimes}_\pi W$.

The projective topology is, in a sense, the strongest reasonable locally convex topology that can be put on $V \times W$. Analogously, there is also a weakest topology satisfying certain nice properties (which we will not state), which is called the *injective* topology and is denoted by $V \otimes_\epsilon W$. This is a weaker topology than the projective one, and hence its completion $V \hat{\otimes}_\epsilon W$ is a bigger space.

A space $V$ is called **nuclear** if these completions match for any locally convex $W$, i.e. if

$$V \hat{\otimes}_\pi W \cong V \hat{\otimes}_\epsilon W =: V \hat{\otimes} W,$$

so that there is only one 'reasonable' tensor product for these spaces. The spaces of sections we discussed in this subsection are nuclear, as are their duals, see e.g. Theorem 51.5 and corollary in [72]. The topic of nuclear vector spaces is extensive, and it would take us too far afield to discuss its intricacies. We rather content ourselves by stating some results from this theory that are relevant to the work presented in this thesis, and refer the interested reader to Part III of [72] for details in the context of Euclidean spaces, and to [70] for the extensions to manifolds and bundles.

If $E_1 \to M$ and $E_2 \to N$ are two vector bundles, then their exterior tensor product bundle $E_1 \boxtimes E_2 \to M \times N$ is defined as a bundle whose fibre over $(x, y) \in M \times N$ is given by $(E_1)_x \otimes (E_2)_y$. We note that the functional dual commutes with the exterior tensor product, i.e.

$$(E_1 \boxtimes E_2)^! \cong E_1^! \boxtimes E_2^!,$$

from the fact that the density bundles satisfy $D_{M\times N} \cong D_M \boxtimes D_N$. We employ this isomorphism implicitly throughout this text.

There is a tensor product map

$$\otimes : \mathcal{E}(M, E_1) \times \mathcal{E}(N, E_2) \to \mathcal{E}(M \times N, E_1 \boxtimes E_2),$$

defined by

$$(\varphi \otimes \psi)(x, y) = \varphi(x) \otimes \psi(y) \in (E_1)_x \otimes (E_2)_y. \tag{2.9}$$

Reusing the notation $\otimes$ for this map would be slightly abusive, were it not for the fact that the map induced by the universal property of the tensor product extends to an isomorphism

$$\mathcal{E}(M;E_1)\,\hat{\otimes}\,\mathcal{E}(N;E_2) \cong \mathcal{E}(M\times N;E_1\boxtimes E_2).$$

The tensor product can be extended to distributional sections, see e.g. Theorem 40.4 in [72]. It is given as a map

$$\otimes : \mathcal{D}'(M;E_1)\times\mathcal{D}'(N;E_2)\to\mathcal{D}'(M\times N;E_1\boxtimes E_2), \tag{2.10}$$

satisfying the defining property

$$u\otimes v(\varphi\otimes\psi) = u(\varphi)\cdot v(\psi), \tag{2.11}$$

for $\varphi\in\mathcal{D}(M;E_1^!)$ and $\psi\in\mathcal{D}(N;E_2^!)$. Again, the use of the tensor product symbol is justified by the fact that this map extends to an isomorphism

$$\mathcal{D}'(M;E_1)\,\hat{\otimes}\,\mathcal{D}'(N;E_2) \cong \mathcal{D}'(M\times N;E_1\boxtimes E_2). \tag{2.12}$$

A similar statement holds for compactly supported distributional sections:

$$\mathcal{E}'(M;E_1)\,\hat{\otimes}\,\mathcal{E}'(N;E_2) \cong \mathcal{E}'(M\times N;E_1\boxtimes E_2). \tag{2.13}$$

These facts are shown in Theorem 51.6 and 51.7 in [72] for scalar functions on Euclidean domains. The extension to vector bundles can be found in Lemma 1.4.7 in [70].

As a powerful consequence of nuclearity of these spaces, we have the Schwartz kernel theorem, see e.g. Theorem 51.7 in [72] for the scalar result, and Theorem 1.5.1 in [70] for the general statement:

**Theorem 2.1.14** (Schwartz kernel theorem)**.** *If $E_1\to M$ and $E_2\to N$ are two vector bundles, then there is an isomorphism*

$$\mathcal{D}'(N\times M;E_2\boxtimes E_1^!) \cong L_b(\mathcal{D}(M;E_1),\mathcal{D}'(N;E_2)), \tag{2.14}$$

*given by mapping $K\in\mathcal{D}'(N\times M;E_2\boxtimes E_1^!)$ to*

$$\mathcal{K}:\varphi\mapsto(\psi\mapsto K(\psi\otimes\varphi)). \tag{2.15}$$

As a note on nomenclature, the bidistributions $K$ are called 'kernels'. The Schwartz kernel theorem states that any map from functions to distribution possesses a unique kernel so that the map can be expressed in the form (2.15).

## 2.2 MICROLOCAL ANALYSIS

Techniques from the field of microlocal analysis allow one to perform operations with distributions that are naively ill-defined, and as such have proven indispensable in the study of AQFT, where distributional objects are ubiquitous. In this section, we discuss some elementary concepts in the field and introduce the Hörmander spaces that play an important role in our discussion. We postpone the thorough treatment of operations on these spaces to Chapter 3, as we will need to generalise them to the case of interest to us first.

We refer the reader to the literature for more details: For introductory treatments, we recommend [14], as well as chapter 4 in [2] for an approach tailored to AQFT. The standard reference for this subject is [58], as well as subsequent volumes of that text. For a more hands-on introduction treating pseudo-differential operators, we recommend the excellent lecture notes by Hintz [54].

### 2.2.1 Elementary concepts in microlocal analysis

We first give an exposition of the relevant concepts on open domains $\Omega \subset \mathbb{R}^n$, and explain how to lift these concepts to manifolds and vector-bundles in chapter 3. The central notion in microlocal analysis is the wavefront set of a distribution. This is a sharpening of the notion of singular support, in that it not only describes *where* a distribution is singular, but also in which direction in Fourier space. It makes precise the idea that sharper localisation in space comes at the cost of a wider Fourier transform. To motivate the definition, we record the following standard proposition.

**Proposition 2.2.1.** *A distribution $u \in \mathcal{E}'(\Omega)$ is smooth iff*

$$\hat{u}(\xi) = \mathcal{O}(\langle \xi \rangle^{-N}) \qquad \text{as } \xi \to \infty \quad \forall N \in \mathbb{N}, \tag{2.16}$$

*where a hat denotes the Fourier transform and*

$$\langle \xi \rangle = \sqrt{1 + |\xi|^2}$$

*is the* Japanese bracket.

The Japanese bracket should be viewed as a smoothed out version of the absolute value. Its use is not standard in every text on microlocal analysis, other options are to use $(1 + |\xi|)$ or just $|\xi|$. These approaches of course give identical statements, because all these quantities are asymptotically equivalent. We favour the Japanese bracket because it is smooth on the whole of $\mathbb{R}^n$, and because we can take negative powers of it without issue. If equation (2.16) holds, we will say that $\hat{u}$ is *of rapid decay.*

In what follows, a **cone** $\Gamma \subset \Omega \times (\mathbb{R}^n \setminus \{0\})$ is a set such that

$$(x, \xi) \in \Gamma \implies (x, \lambda\xi) \, \forall \, \lambda > 0.$$

**Definition 2.2.2.** *Let $\Omega \subset \mathbb{R}^n$ be non-empty and open, and let $u \in \mathcal{D}'(\Omega)$. A pair $(x, \xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ is a* ***regular directed point*** *of $u$, if there exist*

- *A conic neighbourhood $V \subset \mathbb{R}^n \setminus \{0\}$ of $\xi$,*
- *A test function $f \in \mathcal{D}(\mathbb{R}^n)$ such that $f(x) \neq 0$,*

*such that, for any $N \in \mathbb{N}$,*

$$\sup_{\xi \in V} \langle \xi \rangle^N |\widehat{fu}(\xi)| < \infty,$$

*The* ***wavefront set*** $\mathrm{WF}(u)$ *is the complement in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ of the set of regular directed points of $u$.*

Whilst $u$ might not be compactly supported in this definition, multiplying by the test function $f$ enforces that $fu$ is compactly supported, and hence a tempered distribution, so that the Fourier transform is well-defined. On top of this, by taking test functions with increasingly small support, one is able to accurately distinguish between the singular behaviour at different points in $\Omega$.

Because the condition to be in the complement of the wavefront set is phrased in terms of conic *neighbourhoods*, the wavefront set of a distribution is a closed cone in $\Omega \times (\mathbb{R}^n \setminus \{0\})$. We will see below, in Theorem 2.2.4, that the wavefront set behaves like a set of non-zero covectors under a coordinate transformation. Hence, in order to have consistent notation later on, we introduce here the dotted cotangent bundle of a manifold $M$ by

$$\dot{T}^*M = \{(x, \xi) \in T^*M \mid \xi \neq 0\}$$

and we identify $\Omega \times (\mathbb{R}^n \setminus \{0\}) \cong \dot{T}^*\Omega$. Throughout this thesis we use the notation,

$$\Gamma_0 = \Gamma \cup \underline{0} \subset T^*M, \tag{2.17}$$

for $\Gamma \subset \dot{T}^*M$ *any* cone, where $\underline{0}$ is the zero-section of $T^*M$.

We will be interested in spaces of distributions with prescribed wavefront sets. If $\Gamma \subset \dot{T}^*\Omega$ is a closed cone then we can introduce the following set of distributions:

$$\mathcal{D}'_\Gamma(\Omega) = \{u \in \mathcal{D}'(\Omega) \mid \mathrm{WF}(u) \subset \Gamma\}.$$

It is a theorem of Hörmander that distributions exist whose wavefront set is precisely $\Gamma$, so that there are non-trivial[11] inclusions

$$\mathcal{E}(\Omega) \to \mathcal{D}'_\Gamma(\Omega) \to \mathcal{D}'(\Omega), \tag{2.18}$$

see Theorem 8.1.4 in [58]. In [28], this space is topologised as follows:

**Definition 2.2.3.** *Let $\Gamma \subset \dot{T}^*\Omega$ be a closed cone. We define the following seminorms on $\mathcal{D}'_\Gamma(\Omega)$:*

$$P_{\chi,N,V}(u) = \sup_{\xi \in V} \langle \xi \rangle^N |\widehat{\chi u}(\xi)|,$$
$$P_B(u) = \sup_{f \in B} |u(f)|,$$

*where $\chi \in \mathcal{D}(\Omega)$, $N \in \mathbb{N}$, $V \subset \mathbb{R}^n \setminus 0$ a closed cone with $(\mathrm{supp}(\chi) \times V) \cap \Gamma = \emptyset$ and $B \subset \mathcal{D}(\Omega)$ bounded. The **normal topology** on $\mathcal{D}'_\Gamma(\Omega)$ is the topology generated by these seminorms for all admissible choices of $\chi, N, V$ and $B$.*

Experts in microlocal analysis will recognize the first class of seminorms as those featuring in the Hörmander pseudotopology, of which the normal topology is a sharpening. In particular, convergence with respect to the normal topology implies convergence with respect to the Hörmander pseudotopology. We will refer to the spaces $\mathcal{D}'_\Gamma(M)$ with the normal topology as **Hörmander spaces**.

The topology on $\mathcal{D}'_\Gamma(\Omega)$ is called the *normal* topology due to the fact that it turns $\mathcal{D}'_\Gamma(\Omega)$ into a *normal space of distributions*, which means that the injections in (2.18) are dense and continuous, see Lemma 5 and the following discussion in [28].

### 2.2.2 Hörmander's pullback theorem

Several operations one can perform using functions can be extended to distributions under suitable assumptions on wavefront sets. We recall Hörmander's pullback theorem, which is of fundamental importance to manipulate these spaces:

[11] If $\Gamma$ is non-trivial.

**Theorem 2.2.4.** *Let $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^m$ be open domains, and let $f : \Omega_1 \to \Omega_2$ be a smooth function. We define the set of normals of $f$ to be*

$$\mathcal{N}_f = \{(f(x), \xi) \subset \dot{T}^*\Omega_2 \mid df_x^* \xi = 0\},$$

*where $df_x^*$ is the adjoint of the derivative of $f$ at $x$*

$$df_x : T_x\Omega_1 \to T_{f(x)}\Omega_2.$$

*If $\Gamma \subset \dot{T}^*\Omega_2$ is a closed cone with $\Gamma \cap \mathcal{N}_f = \emptyset$, then the pullback map*

$$f^* : \mathcal{E}(\Omega_2) \to \mathcal{E}(\Omega_1)$$

*admits a unique continuous extension*

$$f^* : \mathcal{D}'_\Gamma(\Omega_2) \to \mathcal{D}'_{f^*\Gamma}(\Omega_1),$$

*where*

$$f^*\Gamma = \{(x, df_x^*\xi) \in T^*\Omega_1 \mid (f(x), \xi) \in \Gamma\}$$

Note that the assumption $\Gamma \cap \mathcal{N}_f = \emptyset$ implies that $f^*\Gamma$ does not intersect the zero section. The original theorem, given in Theorem 8.2.4 in [58], is phrased in terms of convergence with respect to the Hörmander pseudotopology only. This stronger version was proved in Proposition 5.1 of [15].

The preceding proposition allows one to extend the definition of wavefront set to manifolds in a covariant way. Suppose that $M$ is a manifold, and that $(U, \kappa)$ is a coordinate chart on $M$. That is to say, $U$ is an open subset of $M$ and $\kappa : U \to \mathbb{R}^n$ maps $U$ diffeomorphically onto a domain in Euclidean space. This induces a pushforward map on the cotangent bundle of $U$

$$\kappa_* = (\kappa^{-1})^* : \dot{T}^*U \to \dot{T}^*(\kappa(U)).$$

Similarly, $\kappa$ induces a pushforward of distributions

$$\kappa_* = (\kappa^{-1})^* : \mathcal{D}'(U) \to \mathcal{D}'(\kappa(U)),$$

abusively indicated by the same notation. We then say that $(x, \xi) \in \dot{T}^*U$ is in the wavefront set of $u \in \mathcal{D}'(M)$ if and only if

$$\kappa_*(x, \xi) \in \mathrm{WF}(\kappa_*(u|_U))$$

for some coordinate chart $(U,\kappa)$ around $x$.

If we have two coordinate functions $\kappa_1$ and $\kappa_2$ on $U$, then $\kappa_2 \circ \kappa_1^{-1}$ is a diffeomorphism between $\kappa_1(U)$ and $\kappa_2(U)$, so that, for any $u \in \mathcal{D}'(U)$,

$$(\kappa_2 \circ \kappa_1^{-1})^* \circ \kappa_{2*} u = \kappa_{1*} u.$$

Hence it follows from Theorem 2.2.4 that $(\kappa_2 \circ \kappa_1^{-1})^*$ is a bijection between the wavefront sets of $\kappa_{1*}u|_U$ and $\kappa_{2*}u|_U$, so that we can determine the wavefront set of $u$ in any coordinate chart.

Similarly, if $\vec{u}$ is a distributional section of $\Omega \times \mathbb{R}^k$, i.e. a $k$ component vector $(u_1, \ldots, u_k)$ of distributions on $\Omega$, then we define

$$\mathrm{WF}(\vec{u}) \coloneqq \bigcup_i \mathrm{WF}(u_i).$$

If $E$ is a vector bundle of rank $k$ over $M$ then a trivialisation over $U \subset M$, i.e. an isomorphism of bundles

$$\phi : E|_U \xrightarrow{\sim} U \times \mathbb{R}^k,$$

induces an isomorphism

$$\phi_* : \mathcal{D}'(U; E|_U) \xrightarrow{\sim} \mathcal{D}'(U)^k.$$

If $u \in \mathcal{D}'(M;E)$, then we define $(x,\xi)$ to be an element of $WF(u)$ if and only if it is an element of $\mathrm{WF}(\phi_* u|_U)$ for some trivialisation $(U,\phi)$ around $x$. As transition functions are given by matrices of smooth functions, this definition is independent of the trivialisation chosen. We denote by $\mathcal{D}'_\Gamma(M)$ (resp. $\mathcal{D}'_\Gamma(M;E)$) the vector space of all distributions on $M$ (resp. distributional sections of $E$) with wavefront set contained in $\Gamma \subset \dot{T}^*M$.

We note here that we could have also defined a 'vector-valued wavefront set', rather than a scalar one. This alternate approach is known as the polarisation set and was introduced by Dencker in [30], and gives a finer description of the singular structure of distributional sections. As we will not need this greater resolution, we will not delve any deeper into this concept.

### 2.2.3 Hörmander spaces with respect to open cones

The multiplication of functions also has an extension to distributions, given that we take their wavefront sets into account. We introduce, for any cone $\Gamma$, the reflection

$$\Gamma' = \{(x,-\xi) \in \dot{T}^*M \mid (x,\xi) \in \Gamma\}.$$

We caution the reader that some authors use a prime to indicate reflection of the second covector when discussing the kernel of an operator between spaces of functions, but we will not use it in this way. The following is Hörmander's multiplication theorem, see e.g. Theorem 8.2.10 in [58]. The extension to work with the normal topology was given in Theorem 6.1 of [15].

**Proposition 2.2.5.** *Let $\Omega \subset \mathbb{R}^n$ be an open domain. If $\Gamma_1$ and $\Gamma_2$ are closed cones in $\dot{T}^*\Omega$ such that $\Gamma_1 \cap \Gamma_2' = \emptyset$, then the multiplication operator*

$$\cdot : \mathcal{E}(\Omega) \times \mathcal{E}(\Omega) \to \mathcal{E}(\Omega),$$

*allows for a unique hypocontinuous extension*

$$\cdot : \mathcal{D}'_{\Gamma_1}(\Omega) \times \mathcal{D}'_{\Gamma_2}(\Omega) \to \mathcal{D}'_{\Gamma_1+\Gamma_2}(\Omega).$$

*where the sum of cones is defined by*

$$\Gamma_1 + \Gamma_2 = \Gamma_1 \cup \Gamma_2 \cup \{(x, \xi_1 + \xi_2) \subset \dot{T}^*M \mid (x, \xi_1) \in \Gamma_1 \text{ and } (x, \xi_2) \in \Gamma_2\}. \quad (2.19)$$

We note that $\Gamma_1 \cap \Gamma_2' = \emptyset$ if and only if $\Gamma_1 + \Gamma_2$ does not intersect the zero section.

If $\Lambda \subset \dot{T}^*\Omega$ is an *open* cone, then we can define the family of distributions

$$\mathcal{E}'_\Lambda(\Omega) = \{v \in \mathcal{E}'(\Omega) \,|\, \mathrm{WF}(v) \subset \Lambda\}.$$

These families play a central role in defining microlocal constraints in the functional formalism for field theory. By the previous proposition, we can multiply $u \in \mathcal{D}'_{\Lambda'^c}(\Omega)$ and $v \in \mathcal{E}'_\Lambda(\Omega)$ to obtain a compactly supported distribution. We can evaluate it on the function that is 1 everywhere to define a pairing:

$$\langle v, u \rangle = v \cdot u(1). \quad (2.20)$$

Of course, if $v$ were an element of $\mathcal{D}(\Omega)$, then this is nothing more than the evaluation of $u$ on $v$. As such, this description gives a way of defining evaluation of distributions on other distributions. As multiplication by $v$ and evaluation on 1 are continuous maps, $v$ defines an element of the dual of $\mathcal{D}'_{\Lambda'^c}(\Omega)$. It is shown in Proposition 7 of [28] that any element of the dual of $\mathcal{D}'_{\Lambda'^c}(\Omega)$ is of this form.

We endow $\mathcal{E}'_\Lambda(\Omega)$ with the strong topology with respect to this duality. This topology is poorly behaved from the viewpoint of functional analysis, as was shown in detail in Section 6.4 of [28]. Most importantly, $\mathcal{E}'_\Lambda(\Omega)$ is not (even sequentially) complete. The completion of $\mathcal{E}'_\Lambda(\Omega)$ can be given in terms of the dual wavefront set, see [26, 27], but we will not discuss that concept so as not to leave the realm of wavefront sets which is the standard context used in the AQFT literature.

As $\mathcal{E}'_\Lambda(\Omega)$ is the dual of $\mathcal{D}'_{\Lambda'^c}(\Omega)$, we can consider equicontinuous subsets $H \subset \mathcal{E}'_\Lambda(\Omega)$. The following characterisation of such sets, which is Lemma 5.3 of [15], is of a fundamental importance to this work:

**Proposition 2.2.6.** *Let $\Lambda$ be an open cone inside $\dot{T}^*\Omega$. If $H \subset \mathcal{E}'_\Lambda(\Omega)$ is equicontinuous, then there exist a compact set $K \subset \Omega$ and a* ***closed*** *cone $\Xi \subset \Lambda|_K$ such that $H$ is a bounded subset of $\mathcal{D}'_\Xi(K)$.*

The equicontinuous subsets of $\mathcal{E}'_\Lambda(\Omega)$ form a proper subset of the bounded subsets. An explicit example of a bounded set which is not equicontinuous can be extracted from Theorem 27 in [28].[12] They construct a Cauchy sequence $(v_n) \subset \mathcal{E}'_\Lambda(\Omega)$ (which is a bounded set), so that the wavefront sets of $v_n$ 'converge' to contain a point on the boundary of $\Lambda$. As such $\bigcup_{n\in\mathbb{N}} \mathrm{WF}(v_n)$ can not be contained in a single closed cone inside $\Lambda$.

## 2.3 NONLINEAR FUNCTIONAL ANALYSIS

We now move to one of the main topics of this thesis: Smooth functionals on infinite dimensional spaces. As this notion is central to the theory we develop, we will gather some definitions and results in this section. Our main sources for this section is [50], but we also recommend Appendix A of [20] for a summary that is tailored to our situation.

### 2.3.1 CURVES INSIDE TOPOLOGICAL VECTOR SPACES

As a starting point for our definition of smooth maps on infinite dimensional spaces, we discuss the notion of differentiable curves. The definition of a (generally nonlinear!) continuous curve $c : \mathbb{R} \to V$ into a LCTVS $V$ is obvious. We call $c$ differentiable at $t_0 \in \mathbb{R}$ with derivative $c'(t_0)$ if

$$\frac{c(t_0 + s) - c(t_0)}{s} - c'(t_0) \xrightarrow{s\to 0} 0.$$

[12] We caution the reader that their construction is unpleasant in the extreme.

in $V$. We call $c$ a $C^1$ curve if it is differentiable at every point in $\mathbb{R}$, and $c'$ defines another continuous curve. $C^k$ curves and smooth curves are defined analogously.

Continuous curves in topological vector spaces can be integrated. Let $c : \mathbb{R} \to V$ be a continuous curve inside a TVS $V$. If $f \in V^*$, then $f \circ c : \mathbb{R} \to \mathbb{R}$ is continuous as well, and hence there exists a unique indefinite integral

$$\textstyle\int f \circ c : \mathbb{R} \to \mathbb{R},$$

which is a $C^1$ curve satisfying $(\int f \circ c)' = f \circ c$ and $(\int f \circ c)(0) = 0$. In Lemma 2.5 in [62], it is shown that we can perform the integral of $c$ inside $\hat{V}$, the completion of $V$. That is, there exists a unique $C^1$ curve

$$\textstyle\int c : \mathbb{R} \to \hat{V},$$

satisfying

$$\begin{aligned} &\textstyle\int c(0) = 0 \\ &(\textstyle\int c)' = c \\ &f \circ \textstyle\int c = \int f \circ c \quad \forall f \in V^*, \end{aligned} \tag{2.21}$$

where we extend $f$ to $\hat{V}$ using the same symbol. We record the following result, which is noticeably absent in their treatment.

**Proposition 2.3.1.** *Let $V$ and $W$ be topological vector spaces, $\Phi \in L(V, W)$ and $c : \mathbb{R} \to V$ a continuous curve. Then*

$$\Phi \circ \textstyle\int c = \int \Phi \circ c,$$

*where we extend $\Phi \in L(\hat{V}, \hat{W})$ using the same symbol.*

*Proof.* Let $f \in W^*$. As $f \circ \Phi \in V^*$, it follows from equation (2.21) that

$$f\left(\Phi \circ \textstyle\int c - \int \Phi \circ c\right) = 0 \quad \forall f \in W^*.$$

As $W^*$ separates the points of $\hat{W}$ by the Hahn-Banach theorem, the result follows. □

We can then define definite integrals between $a, b \in \mathbb{R}$ by

$$\int_a^b c(t)dt = (\textstyle\int c)(b) - (\int c)(a).$$

Definite integrals commute with continuous linear maps because indefinite integrals do. We register the following technical lemma, which is a more general version of Theorem 2.1.5 in [50].

**Lemma 2.3.2.** *Let $X$ be a topological space and $V$ a complete LCTVS. If $f : X \times [a, b] \to V$ is a continuous map for some $a \leq b$, then the map $\tilde{f}$ defined by*

$$\tilde{f}(x) = \int_a^b f(x, s)ds$$

*is continuous from $X \to V$.*

*Proof.* The statement in [50] is for $V$ a Fréchet space, but only completeness and local convexity are used in the proof. Hence it applies verbatim to this situation as well. □

### 2.3.2 Smooth functionals on locally convex topological vector spaces

After this lower dimensional prequel, we move to functionals on genuine infinite dimensional spaces. Throughout this section, $V$ and $W$ will be LCTVSs. By a **functional** we shall mean any function $F : V \to W$.[13] We note that many treatments, including [50], define functionals on open regions in $V$, rather than on the whole of $V$, with the aim of defining functionals on Fréchet (or more general) manifolds. As we will not go into these constructions in this work, we will make the simplifying assumption that all functionals are completely defined.

There are many inequivalent notions of smoothness available in the literature, and there is a rich literature available comparing different notions. We refer the reader to [16] for a historical overview. The main notion of smoothness we use is due to Bastiani [6], and will be our standard notion in this thesis. When we write differentiable, or smooth, without further specification, we will mean it in this way.

Let $v, \tilde{v} \in V$, and let $F : V \to W$. The directional derivative at $v$ in direction $\tilde{v}$ is

$$F^{(1)}(v)\{\tilde{v}\} := \frac{d}{ds}\Big|_{s=0} F(v + s\tilde{v}) = \lim_{s \to 0} \frac{1}{s} \left(F(v + s\tilde{v}) - F(v)\right),$$

whenever this limit exists. A functional is continuously differentiable, or $C^1$, if its directional derivatives exist at all points and in all directions and define a jointly continuous map

$$\begin{aligned} V \times V &\to W \\ (v, \tilde{v}) &\mapsto F^{(1)}(v)\{\tilde{v}\} \end{aligned}$$

[13] The terminology is somewhat confusing, but the idea is that $V$ is some space of real valued functions on some finite dimensional space, and that a functional is then a 'function of functions'.

The following is a variant of Lemma 3.3.1 in [50].

**Proposition 2.3.3.** *A functional $F : V \to W$ is $C^1$ if and only if there exists a continuous map $L : V \times V \times \mathbb{R} \to W$ satisfying*

$$L(v, \tilde{v}, t) = \frac{1}{t}\left(F(v + t\tilde{v}) - F(v)\right) \quad t \neq 0.$$

*In that case, we have that $F^{(1)}(v)\{\tilde{v}\} = L(v, \tilde{v}, 0)$.*

*Proof.* Clearly existence of a map $L$ with these properties implies that $F$ is $C^1$. Conversely, suppose that $F$ is $C^1$. It follows from the fundamental theorem of calculus on $\mathbb{R}$ that

$$F(v + t\tilde{v}) - F(v) = \int_0^1 F^{(1)}(v + st\tilde{v})\{t\tilde{v}\}ds = t\int_0^1 F^{(1)}(v + st\tilde{v})\{\tilde{v}\}ds.$$

Hence we can set

$$L(v, \tilde{v}, t) = \int_0^1 F^{(1)}(v + st\tilde{v})\{\tilde{v}\}ds,$$

which is continuous by Lemma 2.3.2. □

Similarly, the $n$-fold directional derivative is defined by

$$F^{(n)}(v)\{\tilde{v}_1, \dots, \tilde{v}_n\} = \lim_{s_i \to 0} \frac{1}{s_1 \dots s_n}\left(F\left(v + \sum_i s_i\tilde{v}_i\right) - F(v)\right),$$

whenever the limit exists, and call a functional $C^n$ if all these quantities exist and form a continuous map $V \times V^n \to W$. A similar characterisation to the one given in Proposition 2.3.3 holds in this scenario. A functional is called **(Bastiani) smooth** if it is $C^n$ to all orders.

We shall also need a notion of partial derivative in what follows. If $F : V_1 \times V_2 \to W$ is a functional of two variables, then we define its partial derivative with respect to $v_1$ by

$$\frac{\delta}{\delta v_1}F(v_1, v_2)\{\tilde{v}_1\} = \frac{d}{dt}\Big|_{t=0}F(v_1 + t\tilde{v}_1, v_2),$$

whenever these quantities are well-defined. Partial derivatives with respect to $v_2$ are defined similarly. As before, we say that the partial derivative is continuous if it defines a map that is jointly continuous in all variables. A functional $F$ has continuous derivatives with respect to the first argument $v_1$ if and only if there exists a continuous map

$$L : V_1 \times V_1 \times \mathbb{R} \times V_2 \to W,$$

satisfying

$$L(v_1, \tilde{v}_1, t, v_2) = \frac{1}{t}\left(F(v_1 + t\tilde{v}_1, v_2) - F(v_1, v_2)\right) \quad t \neq 0.$$

The proof is nearly identical to that of Proposition 2.3.3, so we do not spell out the details.

We summarise several facts about smooth functionals from Section I.3 in [50], all of which are proved there. If $F : V \to W$ is smooth then it follows that:

- $F$ is continuous.
- $F^{(n)} : V \times V^n \to W$ is multilinear in its last $n$ arguments, which we typographically indicate by putting all linear arguments in braces rather than parentheses.
- $F^{(n)}$ is the partial derivative of $F^{(n-1)}$ with respect to its nonlinear argument, i.e.

$$F^{(n)}(v)\{\tilde{v}_1, \ldots, \tilde{v}_n\} = \frac{\delta}{\delta v}\left(F^{(n-1)}(v)\{\tilde{v}_1, \ldots, \tilde{v}_{n-1}\}\right)\{\tilde{v}_n\}.$$

- The fundamental theorem of calculus holds:

$$F(v + \tilde{v}) - F(v) = \int_0^1 F^{(1)}(v + s\tilde{v})\{\tilde{v}\}ds.$$

- The chain-rule holds: If $F$ and $G$ are two composable smooth maps, then $G \circ F$ is also smooth and

$$(G \circ F)^{(1)}(v)\{\tilde{v}\} = G^{(1)}(F(v))\left\{F^{(1)}(v)\{\tilde{v}\}\right\}.$$

- If $V = V_1 \times V_2$ is a product of vector spaces, then $F$ is smooth iff all of its partial derivatives exist and are continuous.

We will need some non-standard results about smooth functionals in this work. As we are not aware of any detailed treatments of these results in the literature, we provide proofs for them here. The first of these results is a version of the Leibniz integral rule.

**Proposition 2.3.4** (Leibniz integral rule)**.** *Let $V$ and $W$ be LCTVSs, of which $W$ is complete. If $F : \mathbb{R} \times V \to W$ is a smooth functional, then the functional $G$ defined by*

$$G(v) = \int_a^b F(s, v)ds$$

*is smooth for any $a, b \in \mathbb{R}$. Furthermore, derivatives can be taken under the integral sign, i.e.*

$$G^{(n)}(v)\{\tilde{v}_1, \ldots, \tilde{v}_n\} = \int_a^b \frac{\delta^n}{\delta v^n} F(s, v)\{\tilde{v}_1, \ldots, \tilde{v}_n\}ds$$

*Proof.* As the partial derivative of $F$ with respect to $v$ exists and is continuous, there is a continuous map $L : \mathbb{R} \times V \times V \times \mathbb{R} \to W$ such that

$$L(s,v,\tilde{v},t) = \frac{1}{t}(F(s,v+t\tilde{v}) - F(s,v)),\ t \neq 0.$$

We find that, for $t \neq 0$

$$\begin{aligned}\frac{1}{t}(G(v+t\tilde{v}) - G(v)) &= \int_0^1 \frac{1}{t}(F(s,v+t\tilde{v}) - F(s,v))ds \\ &= \int_0^1 L(s,v,\tilde{v},t)ds \\ &=: \tilde{L}(v,\tilde{v},t).\end{aligned}$$

The map $\tilde{L}$ is continuous by Lemma 2.3.2. Hence it follows from Proposition 2.3.3 that $G$ is $C^1$, and that

$$G^{(1)}(v)\{\tilde{v}\} = \tilde{L}(v,\tilde{v},0) = \int_0^1 L(s,v,\tilde{v},0)ds = \int_0^1 \frac{\delta}{\delta v}F(s,v)\{\tilde{v}\}ds.$$

A similar argument shows that $G$ is $C^n$ to all orders, and that we may perform all derivatives under the integral sign. □

Secondly, we discuss alternative ways of viewing the derivatives of a functional $F : V \to W$. *A priori* $F^{(n)}$ is a jointly continuous map $V \times V^{\times n} \to W$, linear in the last $n$ arguments. Fixing $v \in V$, it defines a map

$$F^{(n)}(v) : (\tilde{v}_1, \ldots, \tilde{v}_n) \mapsto F^{(n)}(v)\{\tilde{v}_1, \ldots, \tilde{v}_n\} \tag{2.22}$$

that is $n$-linear and continuous. By definition, this is the same thing as a continuous linear map from the completed [14] projective tensor product $V^{\hat{\otimes}_\pi n}$ to $W$. We recall that $L_c(V^{\hat{\otimes}_\pi n}, W)$ denotes the space of all such maps, equipped with the topology of uniform convergence on compact sets.

As a general fact, if $Y$ and $Z$ are topological vector spaces, and $X$ is any topological space, then a continuous map $f : X \times Y \to Z$, linear in Y, induces another continuous map

$$\bar{f} : X \to L_c(Y,Z),$$

by

$$x \mapsto f(x,\_) \coloneqq (y \mapsto f(x,y))\,.$$

[14] As continuous maps can be uniquely extended to completions

This is nothing more than the observation that the topology on $L_c(Y,Z)$ is the restriction of the compact-open topology on $C^0(Y,Z)$, which has this 'exponential' property.

**Proposition 2.3.5.** *If $F : V \to W$ is a smooth functional and $n \in \mathbb{N}$, then the map*

$$\bar{F}^{(n)} : V \to L_c(V^{\hat{\otimes}_\pi n}, W),$$
$$v \mapsto F^{(n)}(v),$$

*defined by equation (2.22) is smooth.*

*Proof.* The partial derivative of $F^{(n)}$ with respect to $v$ equals $F^{(n+1)}$. Hence there exists a continuous map $L$

$$L : V \times V \times \mathbb{R} \times V^{\hat{\otimes}_\pi n} \to W,$$

linear in its last argument, such that for $t \neq 0$

$$L(v, \tilde{v}, t, \theta) = \frac{1}{t}\left(F^{(n)}(v + t\tilde{v})\{\theta\} - F^{(n)}(v)\{\theta\}\right),$$

with $L(v, \tilde{v}, 0, \theta) = F^{(n+1)}(v)\{\tilde{v} \otimes \theta\}$.

From the observation before this proposition, it follows that the map

$$\overline{L} : \begin{cases} V \times V \times \mathbb{R} & \to L_c(V^{\hat{\otimes}_\pi n}, W), \\ (v, \tilde{v}, t) & \mapsto L(v, \tilde{v}, t, \_), \end{cases}$$

is continuous, with

$$\frac{1}{t}\left(\bar{F}^{(n)}(v + t\tilde{v}) - \bar{F}^{(n)}(v)\right) = \bar{L}(v, \tilde{v}, t).$$

Hence it follows from Proposition 2.3.3 that $\bar{F}^{(n)}$ is $C^1$ with

$$\left(\bar{F}^{(n)}\right)^{(1)}(v)\{\tilde{v}\} = \overline{L}(v, \tilde{v}, 0) = L(v, \tilde{v}, 0, \_) = F^{(n+1)}(v)\{\tilde{v}, \_\}.$$

Higher orders are shown analogously. □

In what follows, we will drop the bar from the notation, and use $F^{(n)}$ for both maps interchangeably.

We close this section by mentioning another notion of smoothness which appears in the literature. A functional $F : V \to W$ is **conveniently** smooth if, whenever

$\gamma : \mathbb{R} \to V$ is a smooth curve, $F \circ \gamma : \mathbb{R} \to W$ is also smooth. This notion of smoothness is weaker than Bastiani smoothness, but is far easier to check as curves are easier to handle than general nonlinear functionals. It is a surprising fact, see Theorem 1 of [42], that this notion of smoothness coincides with Bastiani smoothness if $V$ is a metric space and $W$ is complete. In particular, when working with Fréchet spaces, these two notions coincide.

### 2.3.3 Functionals in AQFT

We now specialise to the case of interest for perturbative algebraic quantum field theory. This approach is known as the **functional formalism** for AQFT, and was mainly developed by Brunetti, Fredenhagen and Ribeiro in [20]. Subsequent works by Fredenhagen and Rejzner [41, 40] apply this formalism in the context of the BV formalism to describe gauge theories. Several mathematical intricacies on this formalism, as well as a rigorous characterisation of local functionals, were discussed in [16]. In this subsection, we take some modest background in Lorentzian geometry to be known, and refer the reader to e.g. [2] for an introduction to the relevant concepts.

We define a field theory on a spacetime $\mathcal{M}$ to be given by the following data. We assume that $\mathcal{M}$ is endowed with a vector bundle $E$ that describes the field content of the theory. The *off-shell configuration space* of the theory is given by the smooth sections of $E$. To abbreviate notation, we write $\mathcal{E} = \mathcal{E}(\mathcal{M}; E)$ in this section. We assume that $E$ is endowed with a smooth inner-product on its fibers, so that we have a pairing

$$\langle \_, \_ \rangle : \mathcal{E} \times \mathcal{E} \to C^\infty(\mathcal{M}),$$

which we use to define adjoints of differential operators on $\mathcal{E}$. Together with the volume form induced by the metric of $\mathcal{M}$, this induces an isomorphism between $E$ and $E^!$. The dynamics of the theory are given by a linear, formally self-adjoint Green hyperbolic operator

$$P : \mathcal{E} \to \mathcal{E},$$

as defined in [3]. Morally speaking, we view this operator as the linearisation of the Euler-Langrange equations corresponding to some local action.[15] The *on-shell*

[15]Really, $P$ should be mapping into $\mathcal{E}(\mathcal{M}; E^!)$, but we use the fiber inner-product to simplify our treatment.

*configuration space* is given by the solutions to the equation

$$P\varphi = 0,$$

which we will also call the solution space Sol.

We view the on-shell configurations as the physical ones, and impose that observables be given by smooth, complex valued[16] functionals on Sol. In this section, we will concern us with describing off-shell observables, by means of functionals on $\mathcal{E}$. We discuss the link to on-shell observables in detail in Chapter 5 in the context of the BV-formalism.

By definition of a Green hyperbolic operator, there are unique retarded and advanced Green's functions for $P$. Following Bär in [3], we view them as maps on sections with past or future compact support:[17]

$$\Delta^R : \mathcal{E}_{pc} \to \mathcal{E}_{pc}$$
$$\Delta^A : \mathcal{E}_{fc} \to \mathcal{E}_{fc}$$

that invert $P$ when restricted to those spaces, and that satisfy

$$\operatorname{supp} \Delta^{R/A} f \subset J^{\pm} \operatorname{supp} f.$$

Through the Schwartz-kernel Theorem, we can equivalently view $\Delta^{R/A}$ as 'matrix-valued' bidistributions

$$\Delta^{R/A} \in \mathcal{D}'(\mathcal{M}^2, E \boxtimes E^!).$$

We also define the 'commutator function' (also known under the name 'Pauli-Jordan function', or 'causal propagator') by

$$\Delta = \Delta^A - \Delta^R : \mathcal{D}(\mathcal{M}, E) \to \mathcal{E}(\mathcal{M}, E)$$

The prototypical example of a Green hyperbolic operator is a normally hyperbolic operator, such as the d'Alembertian $\Box = g^{\mu\nu} D_\mu D_\nu$, where $D_\mu$ is the covariant derivative defined with respect to the spacetime metric $g$. Greens' functions exist for

[16] Observables are always assumed to be complex valued, regardless of whether we are working with real or complex vector bundles. This can create some awkward situations that we comment on below.

[17] A set $K \subset \mathcal{M}$ is future/past compact if $J^{\pm}(x) \cap K$ is compact for all $x \in \mathcal{M}$.

this class of operators, as shown e.g. in [4]. One can use Hörmander's propagation of singularities theorem to show that the commutator function of such a theory satisfies

$$\mathrm{WF}(\Delta) = \{(x, y; \xi, \xi') \subset \dot{T}^*\mathcal{M}^2 \,|\, (x, \xi) \sim (y, -\xi')\}, \tag{2.23}$$

where $(x, \xi) \sim (y, -\xi')$ if the vector $g^{-1}(\xi, \_)$ is tangent to a **null** geodesic from $x$ to $y$, such that $-\xi'$ is the parallel transport of $\xi$ along this curve. This implies that $\xi$ and $\xi'$ are both null vectors, one future pointing and one past pointing. We refer the reader to Chapter 4 in [2] for a proof in the context of trivial bundles, and to Theorem A.5 of [69] for the general statement.

We choose to impose minimal constraints on the singular structure of the commutator function. The only feature in the wavefront set of $\Delta$ that is relevant to our constructions is that the two parts of the covectors in $WF(\Delta)$ are in 'opposite' directions, one along the forward light cone, and one along the backward light cone. Hence we impose, in addition to the fact that $P$ is Green hyperbolic, that the wavefront set of its commutator function is bounded by

$$WF(\Delta) \subset (V_+ \times V_-) \cup (V_- \times V_+)\,, \tag{2.24}$$

where $V_\pm$ denote the forward and backward light cones of $\mathcal{M}$. More general alternatives are discussed in [52] and [37].

Functionals on $\mathcal{E}$ have a notion of support on $\mathcal{M}$.

**Definition 2.3.6.** *The **spacetime support** of a functional $F : \mathcal{E} \to \mathbb{C}$ is defined through its complement as*

$$\begin{aligned}\mathrm{supp}(F)^c = \{x \in \mathcal{M} \,|\, \exists U \text{ open neighbourhood of } x \\ \text{such that } \varphi|_{U^c} = \tilde{\varphi}|_{U^c} \implies F(\varphi) = F(\tilde{\varphi})\}.\end{aligned}$$

Roughly speaking, the support of a functional is that region of $\mathcal{M}$ where $F$ is sensitive to perturbation. We mention the following characterisation of support from Lemma III.3 in [16]:

**Lemma 2.3.7.** *The support of a functional $F$ can be characterised as*

$$\mathrm{supp}(F) = \overline{\bigcup_{\varphi \in \mathcal{E}} \mathrm{supp}(F^{(1)}(\varphi))}$$

Note that, even though $\mathrm{supp}(F^{(1)}(\varphi))$ is compact[18] for all individual $\varphi \in \mathcal{E}$, the union over all $\varphi$ is in general non-compact. In most practical applications, one restricts to compactly supported functionals.

We explain the ramifications of Proposition 2.3.5 in the present context. As $\mathcal{E}^{\hat{\otimes} n} \cong \mathcal{E}(\mathcal{M}^n; E^{\boxtimes n})$ is a Montel[19] space, see e.g. Proposition 34.4 in [72], the compact-open topology and the strong topology on its dual are equivalent by Proposition 34.5 in [72]. This implies that:

$$L_c(\mathcal{E}^{\hat{\otimes} n}, \mathbb{C}) \cong L_b(\mathcal{E}^{\hat{\otimes} n}, \mathbb{C}) = \mathcal{E}(\mathcal{M}^n; E^{\boxtimes n})'^{\mathbb{C}} \cong \mathcal{E}'(\mathcal{M}^n; E^{!\boxtimes n})^{\mathbb{C}}.$$

Hence we can view $F^{(n)}$ as a smooth functional

$$F^{(n)} : \mathcal{E} \mapsto \mathcal{E}'(\mathcal{M}^n; E^{!\boxtimes n})^{\mathbb{C}}. \tag{2.25}$$

In what follows, we will suppress the complexification in the notation.

#### 2.3.3.1 *Some important classes of functionals*

The full class of all smooth functionals is usually too big for the purposes of field theory, so in practical applications one restricts attention to a smaller class of functionals. We describe two important classes in this section. For simplicity of notation, we work with a trivial vector bundle.

Local functionals are of fundamental importance in pAQFT because they describe self-interactions in field theory. Morally speaking, a local functional is a functional that can be written as an integral over spacetime of some function on the jet bundle:

$$F(\varphi) = \int f(j^k_x \varphi) dV_x,$$

for some $k \in \mathbb{N}$, where $j^k_x\varphi$ is the $k$-th jet-prolongation of $\varphi$ and $f$ is a smooth function on the jet bundle. For example, the interaction term in $\varphi^4$ theory is of the form

$$F(\varphi) = \int \chi(x)\varphi(x)^4 dV_x,$$

where $\chi \in \mathcal{D}(\mathcal{M})$ takes the role of an infrared cut-off.

In a recent paper by Brouder et al. [16] an alternative characterization of local functionals was given, which is the version that we will use:

[18] Due to the fact that $F^{(1)}(\varphi) \in \mathcal{E}'(\mathcal{M}; E^!)$

[19] I.e. barrelled and semi-Montel.

**Definition 2.3.8.** *A smooth functional $F : \mathcal{E} \to \mathbb{R}$ is* ***local*** *if it satisfies the following three properties:*

- *$F$ is additive, in the sense that*

$$F(\varphi_1 + \varphi_2 + \varphi_3) = F(\varphi_1 + \varphi_2) + F(\varphi_2 + \varphi_3) - F(\varphi_2),$$

  *whenever* $\mathrm{supp}(\varphi_1) \cap \mathrm{supp}(\varphi_3) = \emptyset$.
- *$F^{(1)}(\varphi) \in \mathcal{D}(\mathcal{M})$ for any $\varphi \in \mathcal{E}$, and furthermore,*
- *$F^{(1)}$ is implemented by a* ***smooth*** *map*

$$\nabla F : \mathcal{E} \to \mathcal{D}(\mathcal{M}),$$

*that is to say, for all $\psi \in \mathcal{E}$*

$$F^{(1)}(\varphi)\{\psi\} = \int \nabla F(\varphi)(x)\, \psi(x)\, dV_x. \tag{2.26}$$

They show in their Proposition V.5 that this implies that $F^{(n)}(\varphi)$ is supported on the thin diagonal $D = \{(x, \dots, x) \in \mathcal{M}^n \,|\, x \in \mathcal{M}\}$. We shall see in Chapter 6 that it is smooth along the diagonal, so that its wavefront set is contained in the conormal bundle of $D$,

$$ND = \left\{ (\mathbf{x}, \xi) \in \dot{T}^*\mathcal{M}^n \,\middle|\, \mathbf{x} \in D \text{ and } \langle \xi, X \rangle = 0 \quad \forall X \in T_x D \subset T_x\mathcal{M}^n \right\}.$$

The tangent bundle to $D$ is given by all vectors of the form $(x, \dots x; v, \dots v)$ for $v \in T_x\mathcal{M}$. Hence $(x, \dots x; \xi_1, \dots \xi_n) \in ND$ if and only if $\sum_{i=1}^n \xi_i = 0$.

In defining algebraic structures, like the Poisson bracket or the $\star$-product, we pair the derivatives of two functionals with a distribution. For example, the Poisson bracket of the Klein-Gordon theory is given by

$$\{F, G\}(\varphi) = \left\langle \Delta\, ,\, F^{(1)}(\varphi) \otimes G^{(1)}(\varphi) \right\rangle, \tag{2.27}$$

where $\Delta$ is the commutator function of the theory, which is a singular bidistribution. As the first derivative of a local functional is smooth and compactly supported, this pairing is well-defined. However, as $\Delta$ is not supported on the diagonal, the Poisson bracket of two local functionals will not be local in general.

As such, local functionals do not form a good basis for a Poisson algebra, because the physically motivated algebraic structure fails to close on them. It is for this reason that we would like to introduce a new class of functionals, which is small enough to allow the Poisson bracket and the $\star$-product to be well-defined, but at the same time large enough to contain all functionals that are of interest in perturbative treatments of QFT, i.e. the local functionals. The following definition is aimed at doing precisely that:

**Definition 2.3.9.** *We define a sequence of cones $\Gamma_n \subset \dot{T}^*\mathcal{M}^n$ by*

$$\Gamma_n = V_+^{\dot{\times}n} \cup V_-^{\dot{\times}n}.$$

*A functional $F : \mathcal{E} \to \mathbb{C}$ is **microcausal** if*

$$\mathrm{WF}\left(F^{(n)}(\varphi)\right) \cap \Gamma_n = \emptyset \quad \forall\, \varphi \in \mathcal{E}.$$

Differently put, $F$ is microcausal if

$$F^{(n)}(\varphi) \in \mathcal{E}'_{\Gamma_n^c}(M^n; E^{!\boxtimes n}) \quad \forall\, n \in \mathbb{N}, \varphi \in \mathcal{E},$$

noting that $\Gamma_n = \Gamma_n'$. Some authors include compact support in the definition of microcausal functionals, but we choose to disentangle the singular behaviour and the support properties so as to be more in line with [16].

Local functionals are microcausal, for if $F$ is local and $(x, \ldots x; \xi_1, \ldots \xi_n) \in V_+^n \setminus \underline{0}$, then certainly $\sum_{i=1}^n \xi_i \neq 0$, and similar for $V_-^n$.

Furthermore, because of the explicit form of the wavefront set of $\Delta$ that we assumed in equation (2.24), the Poisson bracket of two microcausal functionals is well-defined. Indeed, if $F$ and $G$ are microcausal functionals, then the wavefront sets of $F^{(1)}(\varphi)$ and $G^{(1)}(\varphi)$ do not intersect the lightcone by assumption. But the wavefront set of $\Delta$ contains only null covectors, so that the pairing in (2.27) is well-defined.

It is claimed in [20] that this bracket in fact closes on the microcausal functionals. This turns out to be false, which we prove in a joint paper with Eli Hawkins and Kasia Rejzner in [52]. In Chapter 4, we give explicit counterexamples that show that this class of functionals can behave in unexpected ways, and is therefore suboptimal for the purposes of field theory.

Our solution to these problems is to introduce a new class of functionals, which we have dubbed the *equicausal* functionals. This class is strictly smaller than the class of microcausal functionals, and is defined by introducing additional bounds on the singular structure of derivatives as we allow the configuration $\varphi \in \mathcal{E}$ to vary. The prime technical tool we use for this are the Hörmander spaces defined in Section

2.2, of which we give a generalisation in the next chapter that we will need for this purpose. We discuss the class of equicausal functionals and algebraic structures thereon in detail in Chapter 6.

# — 3 —

# Hörmander spaces on manifolds

The definition of the normal topology in [28] and the continuity of operations on these spaces shown in [15] form the basis for a large part of the technical details relating to the equicausal functionals defined in [52]. However, all these results are given with respect to domains in Euclidean space, whereas we need them in the context of manifolds and vector bundles.

Because we are not aware of this point being treated in detail elsewhere, we fill this gap in the literature here. We first show how to extend the normal topology to manifolds, and show that it defines a sheaf. This allows us to then reduce all theorems to the local model, where the relevant result holds. After that, we discuss the dual case of Hörmander spaces with respect to open cones. We end by discussing operations involving these spaces and their continuity properties. These results provide the groundwork for our treatment of equicausal functionals in Chapter 6.

## 3.1 SCALAR HÖRMANDER SPACES ON MANIFOLDS

Throughout this section $M$ will be a fixed manifold and $\Gamma \subset \dot{T}^*M$ a closed cone. As a convention, we take manifolds to be second countable and hence paracompact, so that every open cover has a locally finite subcover, with respect to which we can define partitions of unity.

If $U \subset M$ is a coordinate patch with coordinate functions $\kappa_1$ and $\kappa_2$, then it follows from the definition of wavefront set on manifolds in Section 2.2.2 that there are isomorphisms of vector spaces

$$\mathcal{D}'_{\Gamma|_U}(U) \cong \mathcal{D}'_{\kappa_{1*}\Gamma|_U}(\kappa_1(U)) \cong \mathcal{D}'_{\kappa_{2*}\Gamma|_U}(\kappa_2(U)).$$

By Theorem 2.2.4, the second isomorphism is topological. We endow $\mathcal{D}'_{\Gamma|_U}(U)$ with the topology induced by the first isomorphism, turning it into an isomorphism of locally convex spaces.

This defines the normal topology on coordinate charts. Restriction of distributions gives maps

$$R_U : \mathcal{D}'_\Gamma(M) \to \mathcal{D}'_{\Gamma|_U}(U),$$

and we endow $\mathcal{D}'_\Gamma(M)$ with the initial topology with respect to these maps as $U$ runs over all coordinate patches of $M$. We shall show that this turns $\mathcal{D}'_\Gamma(M)$ into a sheaf in LCTVS. We will need the following technical lemma on the normal topology.

**Lemma 3.1.1.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $\Upsilon \subset \dot{T}^*\Omega$ be a closed cone. If $\{U_i\}_{i\in I}$ is an open cover of $\Omega$, then the normal topology on $\mathcal{D}'_\Upsilon(\Omega)$ is equivalent to the initial topology with respect to the restrictions maps*

$$R_i : \mathcal{D}'_\Upsilon(\Omega) \to \mathcal{D}'_{\Upsilon|_{U_i}}(U_i).$$

*Proof.* As the maps $R_i$ are continuous for the normal topology, the normal topology is stronger than the initial one, by the universal property of initial topologies.

For the converse, we may assume without loss of generality that the cover $\{U_i\}$ is locally finite and select a partition of unity $\{\rho_i\}_{i\in I}$ with respect to this cover. Let $N \in \mathbb{N}$, $\chi \in \mathcal{D}(\Omega)$ and $V \subset \mathbb{R}^n \setminus 0$ an open cone with

$$(\operatorname{supp}\chi \times V) \cap \Upsilon = \emptyset.$$

As $\chi$ is compactly supported, $\rho_i\chi$ is non-zero only for a finite number of indices,[1] say $k$ of them. We estimate, for $u \in \mathcal{D}'_\Upsilon(\Omega)$

$$\begin{aligned} p_{N,\chi,V}(u) &= \sup_{\xi\in V} \langle\xi\rangle^N |\widehat{\chi u}(\xi)| \\ &\leq \sum_i \sup_{\xi\in V} \langle\xi\rangle^N |\widehat{\rho_i\chi u}(\xi)| \\ &\leq k \max_i p_{N,\rho_i\chi,V}(R_i u), \end{aligned}$$

and clearly $p_{N,\rho_i\chi,V}$ is a continuous seminorm on $\mathcal{D}'_{\Upsilon|_{U_i}}(U_i)$

Similarly, if $B \subset \mathcal{D}(\Omega)$ is bounded, then there is, by Proposition 14.6 in [72], a compact set $K \subset \Omega$ such that $\operatorname{supp}(f) \subset K$ for all $f \in B$. Hence $\rho_i B = 0$ for all but

[1] We recall that if $K \subset M$ is compact and $\{U_i\}_{i\in I}$ is a locally finite open cover of $M$, then only finitely many $U_i$ intersect $K$ non-trivially.

finitely many $i$, say $l$ of them. Again we estimate, for $u \in \mathcal{D}'_\Upsilon$

$$p_B(u) = \sup_{f\in B} |\langle u, f\rangle| \leq \sum_i \sup_{f\in B} |\langle u, \rho_i f\rangle| \leq l \max_i p_{\rho_i B}(R_i u).$$

As $\rho_i B \subset \mathcal{D}(U_i)$ is a bounded set, the $p_{\rho_i B}$ are seminorms of the normal topology of $\mathcal{D}'_{\Upsilon_{U_i}}(U_i)$.

Hence we conclude that the normal topology is weaker than the initial topology, meaning that they are in fact equivalent. □

**Proposition 3.1.2.** *If $M$ is a manifold and $\Gamma \subset \dot{T}^*M$ is a closed cone, then the assignment*

$$U \mapsto \mathcal{D}'_{\Gamma|_U}(U), \quad U \subset M \, open, \tag{3.1}$$

*together with the restriction maps of distributions, defines a sheaf of LCTVSs.*

*Proof.* Let $U \subset M$ be open and let $\{U_i \,|\, i \in I\}$ be a cover of $U$ by open subsets, which we may take to be locally finite without loss of generality, so that we may select a partition of unity $\{\rho_i\}$ subordinate to this cover. For any $i, j \in I$, the diagram

$$\begin{array}{ccccc} & & \mathcal{D}'_{\Gamma|_{U_i}}(U_i) & & \\ & \overset{R_i}{\nearrow} & & \overset{R_{ij}}{\searrow} & \\ \mathcal{D}'_{\Gamma|_U}(U) & & & & \mathcal{D}'_{\Gamma|_{U_i\cap U_j}}(U_i \cap U_j). \\ & \underset{R_j}{\searrow} & & \underset{R_{ji}}{\nearrow} & \\ & & \mathcal{D}'_{\Gamma|_{U_j}}(U_j) & & \end{array} \tag{3.2}$$

commutes, implying that the assignment in equation (3.1) defines a pre-sheaf. We have to show that $\mathcal{D}'_{\Gamma|_U}(U)$ is the limit of this diagram as we take all the elements of the cover into account, i.e. that

$$\mathcal{D}'_{\Gamma|_U}(U) \cong \mathrm{Lim}_{i\in I}\mathcal{D}'_{\Gamma|_{U_i}}(U_i) = \{(u_i) \in \prod_{i\in I} \mathcal{D}'_{\Gamma|_{U_i}}(U_i) \mid R_{ij}u_i = R_{ji}u_j\}. \tag{3.3}$$

The product of the restriction maps $\prod_{i\in I} R_i$ induces a linear map

$$\prod_{i\in I} R_i : \mathcal{D}'_{\Gamma|_U}(U) \to \mathrm{Lim}_{i\in I}\mathcal{D}'_{\Gamma|_{U_i}}(U_i).$$

We define a map

$$\Psi : \prod_{i\in I} \mathcal{D}'_{\Gamma|_{U_i}}(U_i) \to \mathcal{D}'_{\Gamma|_U}(U) \tag{3.4}$$

by setting, for $f \in \mathcal{D}(U; D_U)$

$$\langle \Psi(u_i)_{i\in I}, f\rangle = \sum_{i\in I} \langle u_i, \rho_i f\rangle, \tag{3.5}$$

where $D_U$ is the density bundle of $U$. As $f$ is compactly supported, only finitely many $\rho_i f$ are non-zero, so that this is a well-defined distribution on $U$. As the wavefront set is a local object that is stable under multiplication by smooth functions, we obtain that $\mathrm{WF}(\Psi(u_i)) \subset \Gamma|_U$. We claim that $\Psi$ is the inverse to $\prod_{i\in I} R_i$ once we restrict it to $\mathrm{Lim}_{i\in I}\mathcal{D}'_{\Gamma|_{U_i}}(U_i)$.

Indeed, we calculate, for $u \in \mathcal{D}'_{\Gamma|_U}(U)$ and $f \in \mathcal{D}(U; D_U)$,

$$\langle u, f\rangle = \sum_{i\in I}\langle u, \rho_i f\rangle = \sum_{i\in I}\langle R_i u, \rho_i f\rangle = \langle \Psi((R_i u)_{i\in I}), f\rangle = \left\langle \Psi \circ \left(\prod_{i\in I} R_i\right) u, f \right\rangle.$$

Similarly, if $(u_i) \in \mathrm{Lim}_{i\in I}\mathcal{D}'_{\Gamma_{U_i}}(U_i)$, then we calculate, for fixed $j \in I$ and $g \in \mathcal{D}(U_j; D_{U_j})$

$$\langle R_j \circ \Psi((u_i)_{i\in I}), g\rangle = \sum_{i\in I} \langle u_i, \rho_i g\rangle = \sum_{i\in I} \langle u_j, \rho_i g\rangle = u_j\left(\sum_{i\in I} \rho_i g\right) = \langle u_j, g\rangle,$$

where we used in the first step that $g$ is supported in $U_j$, and in the second step that $\operatorname{supp} \rho_i g \subset U_i \cap U_j$ for all $i \in I$, and that $u_i|_{U_i\cap U_j} = u_j|_{U_i\cap U_j}$. Hence $\prod_i R_i \circ \Psi$ is the identity.

Let now $V \subset U$ be a coordinate patch. Then there is a commutative diagram

$$\begin{array}{ccccc} & & \mathcal{D}'_{\Gamma|_{U_i}}(U_i) & & \\ & \overset{R_i}{\nearrow} & & \overset{R_{iV}}{\searrow} & \\ \mathcal{D}'_{\Gamma|_U}(U) & & & & \mathcal{D}'_{\Gamma|_{U_i\cap U_j}}(U_i \cap V). \\ & \underset{R_V}{\searrow} & & \underset{R_{Vi}}{\nearrow} & \\ & & \mathcal{D}'_{\Gamma|_V}(V) & & \end{array}$$

By definition of the normal topology on $\mathcal{D}'_{\Gamma|_{U_i}}(U_i)$, $R_i$ is continuous if and only if $R_{iV} \circ R_i = R_{Vi} \circ R_V$ is continuous for all coordinate regions $V$. But, by the previous lemma, $R_{Vi} \circ R_V$ is continuous if and only if $R_V$ is continuous. Hence the collections of maps $\{R_i\}$ and $\{R_V\}$ induce the same topology on $\mathcal{D}'_{\Gamma|_U}(U)$, so that the isomorphism in equation (3.3) is topological. □

For future reference, we register the following lemma.

**Lemma 3.1.3.** *If $\{U_i\}_{i\in I}$ is a locally finite cover of $M$ and $\{\rho_i\}_{i\in I}$ is a partition of unity subordinate to it, then the map $\Psi$ defined in equations (3.4) and (3.5) is continuous.*

*Proof.* By going over to a smaller cover if need be, we may assume that all the $U_i$ are coordinate patches. We suppress a choice of coordinates in the notation and identify $U_i$ with an open subset of $\mathbb{R}^n$.

Let $j \in I$ and let $C \subset \mathcal{D}(U_j)$ be bounded (with respect to the topology of $\mathcal{D}(U_j)$). It follows from Proposition 14.6 in [72] that there is a compact $K \subset U_j$ such that $\operatorname{supp} g \subset K$ for all $g \in C$, so that $\rho_i C \neq \{0\}$ for only finitely many $i$. We calculate

$$p_C(R_j \circ \Psi(u_i)) = \sup_{g\in C} |\langle R_j \circ \Psi(u_i), g\rangle| \leq \sum_{i\in I} \sup_{g\in C} |\langle u_i, \rho_i g\rangle| = \sum_{i\in I} p_{\rho_i C}(u_i).$$

Similarly, let $\chi \in \mathcal{D}(U_j)$, $N \in \mathbb{N}$, $V \subset \mathbb{R}^n \setminus 0$ a closed cone with $(\operatorname{supp}(\chi) \times V) \cap \Gamma|_{U_i} = \emptyset$. As $\chi$ is compactly supported, $\rho_i\chi = 0$ for all but finitely many indices, so that we may estimate

$$p_{\chi,N,V}(R_j \circ \Psi(u_i)) \leq \sum_{i\in I} p_{\rho_i\chi,N,V}(u_i).$$

From these two inequalities, we conclude that $R_j \circ \Psi$ is continuous for all $j \in I$, as the right-hand sides of these inequalities are continuous seminorms on $\prod_{i\in I} \mathcal{D}'_{\Gamma|_{U_i}}(U_i)$. Hence the same holds for $\Psi$ by the universal property of the initial topology on $\mathcal{D}'_\Gamma(M)$. □

## 3.2 FUNDAMENTAL OPERATIONS ON HÖRMANDER SPACES

Proposition 3.1.2 allows us to lift results about Hörmander spaces on Euclidean domains to manifolds. In this section, we discuss two of the fundamental operations of Hörmander spaces on manifolds, namely the pullback and the tensor product. The third fundamental operation, the push-forward, is treated in the next section as it is most naturally discussed in the bundle-valued case.

We recall from Theorem 2.2.4 that the set of normals of a smooth map $f : M \to N$ is defined as

$$\mathcal{N}_f = \{(f(x), \xi) \subset \dot{T}^*N \mid df_x^*\xi = 0\},$$

where $df_x^*$ is the adjoint of the derivative of $f$ at $x$

$$df_x : T_xM \to T_{f(x)}N.$$

**Proposition 3.2.1.** *Let $M$ and $N$ be manifolds and let $f \in C^\infty(M,N)$ be a smooth map. If $\Gamma \subset \dot{T}^*N$ is a closed cone with $\Gamma \cap \mathcal{N}_f = \emptyset$, then the pullback map*

$$f^* : \mathcal{E}(N) \to \mathcal{E}(M)$$

*admits a unique continuous extension*

$$f^* : \mathcal{D}'_\Gamma(N) \to \mathcal{D}'_{f^*\Gamma}(M).$$

*Proof.* We select an arbitrary atlas for $N$. By taking smaller charts if need be, we may assume that $M$ is endowed with an atlas such that the image under $f$ of any chart is contained in a coordinate patch of the atlas on $N$. If $(U,\kappa)$ is then a chart for $M$, and $(V,\psi)$ is a chart for $N$ such that $f(U) \subset V$, then the pullback theorem induces a continuous map by

$$\begin{array}{ccc} \mathcal{D}'_{\Gamma|_V}(V) & \overset{f^*_{V,U}}{\dashrightarrow} & \mathcal{D}'_{f^*\Gamma|_U}(U) \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{D}'_{\psi_*\Gamma|_V}(\psi(V)) & \xrightarrow{(\psi\circ f\circ\kappa^{-1})^*} & \mathcal{D}'_{\kappa_* f^*\Gamma|_U}(\kappa(U)) \end{array}$$

which is readily seen to be independent of the choices of coordinate functions $\psi$ and $\kappa$. Composing with restriction maps this induces maps

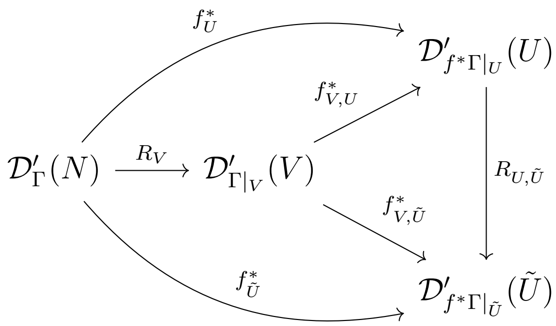
$$f^*_U : \mathcal{D}'_\Gamma(N) \xrightarrow{R_V} \mathcal{D}'_{\Gamma|_V}(V) \xrightarrow{f^*_{V,U}} \mathcal{D}'_{f^*\Gamma|_U}(U)$$

If $\tilde{V} \subset V$ also contains $f(U)$, then the diagram

$$\begin{array}{ccccc} & & \mathcal{D}'_{\Gamma|_V}(V) & & \\ & \overset{R_V}{\nearrow} & & \overset{f^*_{V,U}}{\searrow} & \\ \mathcal{D}'_\Gamma(N) & & \downarrow R_{V,\tilde{V}} & & \mathcal{D}'_{f^*\Gamma|_U}(U) \\ & \underset{R_{\tilde{V}}}{\searrow} & & \underset{f^*_{\tilde{V},U}}{\nearrow} & \\ & & \mathcal{D}'_{\Gamma|_{\tilde{V}}}(\tilde{V}) & & \end{array}$$

commutes. For the left triangle this is obvious. For the right triangle, this follows once coordinates on $V$, $\tilde{V}$ and $U$ are chosen. We conclude that $f^*_U$ does not depend on a choice of $V$.

Similarly, if $\tilde{U} \subset U$ is a smaller coordinate chart, then the diagram




commutes, so that the $f_U^*$ form a cone over the diagram defining $\mathcal{D}'_{f^*\Gamma}(M)$. The universal property of the limit then implies that it induces a unique continuous map $f^* : \mathcal{D}'_\Gamma(N) \to \mathcal{D}'_{f^*\Gamma}(M)$ lifting these maps. From the construction, it is clear that this map is an extension to the pullback map of functions. As $\mathcal{E}(N) \subset \mathcal{D}'_\Gamma(N)$ is dense,[2] it is necessarily unique. □

We already discussed the tensor product of distributions in Section 2.1.4. Here, we study its restriction to Hörmander spaces. To describe the wavefront set of the tensor product of two distributions, we introduce the following product of cones:

**Definition 3.2.2.** *Let $M$ and $N$ be manifolds, and $\Gamma_1 \subset \dot{T}^*M$ and $\Gamma_2 \subset \dot{T}^*N$ be cones (not necessarily closed). Their **dotted product** is defined by*

$$\Gamma_1 \dot{\times} \Gamma_2 \equiv (\Gamma_1 \times \Gamma_2) \cup (\underline{0} \times \Gamma_2) \cup (\Gamma_1 \times \underline{0}) \subset \dot{T}^*(M \times N), \tag{3.6}$$

*where $\underline{0}$ denotes the zero section in both $T^*M$ and $T^*N$.*

We note that this product can alternatively be written as

$$\Gamma_1 \dot{\times} \Gamma_2 = ((\Gamma_1)_0 \times (\Gamma_2)_0) \setminus \underline{0}$$

As the zero section of a cotangent bundle is closed, the dotted product of two closed cones is again closed. The product of open cones is not open in general. The wavefront set satisfies the relation

$$\mathrm{WF}(u \otimes v) \subset \mathrm{WF}(u) \dot{\times} \mathrm{WF}(v),$$

see e.g. Theorem 8.2.9 in [58]. This fact can be turned into a statement about Hörmander spaces.

[2] This is true in each coordinate patch, and $\mathcal{E}(N)$ is a sheaf in its own right. Hence the result follows from the fact that taking closures commutes with taking direct products.

**Proposition 3.2.3.** *Let $M$ and $N$ be manifolds, and let $\Gamma_1 \subset \dot{T}^*M$ and $\Gamma_2 \subset \dot{T}^*N$ be closed cones. The tensor product of distributions restricts to a hypocontinuous map*

$$\mathcal{D}'_{\Gamma_1}(M) \times \mathcal{D}'_{\Gamma_2}(N) \to \mathcal{D}'_{\Gamma_1 \dot{\times} \Gamma_2}(M \times N). \tag{3.7}$$

*Proof.* In the case that $M$ and $N$ are domains in a Euclidean space, this was shown in Theorem 4.6 in [15], and we again show how to lift their results to distributions on manifolds. Let $B \subset \mathcal{D}'_{\Gamma_2}(N)$ be bounded, and let $v \in B$. From the description of Hörmander spaces as sheaves, we see that the tensor product with $v$ is uniquely defined by the condition that the diagrams

$$\begin{array}{ccc}
\mathcal{D}'_{\Gamma_1}(M) & \xrightarrow{_\otimes v} & \mathcal{D}'_{\Gamma_1 \dot{\times} \Gamma_2}(M \times N) \\
\Big\downarrow R_U & & \Big\downarrow R_{U \times V} \\
\mathcal{D}'_{\Gamma_1|_U}(U) & \xrightarrow{_\otimes R_V(v)} & \mathcal{D}'_{(\Gamma_1 \dot{\times} \Gamma_2)|_{U \times V}}(U \times V)
\end{array}$$

commute for all coordinate neighbourhoods $U$ and $V$ in $M$ resp. $N$.

As $R_V(B)$ is a bounded set, the families of maps $\{\_ \otimes R_V(v) \,|\, v \in B\}$ are, by hypocontinuity of the tensor product on Euclidean domains, equicontinuous for all $U$ and $V$. It then follows from Proposition 2.1.11 that these families lift to an equicontinuous family in $L(\mathcal{D}'_{\Gamma_1}(M), \mathcal{D}'_{\Gamma_1 \dot{\times} \Gamma_2}(M \times N))$. The case where the roles of $M$ and $N$ are interchanged is identical. $\square$

All other operations that we use in this work can be derived from the pullback and the tensor product, and will be (hypo-)continuous as a result. For example, the multiplication of functions can be viewed as the pullback by the diagonal map

$$D : M \to M \times M,$$

of their tensor product:

$$f \cdot g(x) = (f \otimes g)(x, x) = D^*(f \otimes g)(x).$$

Hence we can extend the product to distributions $u$ and $v$ as the pullback of their tensor product, as long as we make sure that $WF(u \otimes v) \cap \mathcal{N}_D = \emptyset$, which is equivalent to imposing that $\mathrm{WF}(u) \cap \mathrm{WF}(v)' = \emptyset$.

## 3.3 HÖRMANDER SPACES OF DISTRIBUTIONAL SECTIONS

The results in the previous section admit straightforward generalisations to distributional sections of vector bundles. If $E$ is a rank $k$ vector bundle, then it follows from the definition of the wavefront set of distributional sections in subsection 2.2.2 that

$$\mathcal{D}'_{\Gamma|_U}(U;E|_U) \cong \mathcal{D}'_{\Gamma|_U}(U)^k,$$

whenever $U$ allows for a trivialisation of the bundle. Again, we endow the left-hand side with the topology induced by the right-hand side, which is independent of the trivialisation chosen, as transition functions of a vector bundle are invertible matrices of smooth functions over $U$. Again, we endow $\mathcal{D}'_\Gamma(M;E)$ with the initial topology with respect to all restriction maps

$$\mathcal{D}'_\Gamma(M;E) \xrightarrow{R_U} \mathcal{D}'_{\Gamma|_U}(U;E|_U).$$

Analogously to Proposition 3.1.2, this exhibits $\mathcal{D}'_\Gamma(M;E)$ as a sheaf by a straightforward generalisation of the argument given there.

We recall that, if $f: M \to N$ is any smooth map and $F \xrightarrow{\pi} N$ is a vector bundle, then the pullback of $f^*F$ is universal element (in the category of manifolds) of the diagram

$$\begin{array}{ccc} f^*F & \longrightarrow & F \\ \downarrow & & \downarrow \pi \\ M & \xrightarrow{f} & N, \end{array}$$

which is a vector bundle over $M$. The fibres of this bundle are given by

$$(f^*F)_x = F_{f(x)}.$$

We state the following results:

- If $f : M \to N$ is a smooth map and $F \to N$ is a vector bundle, then the pullback of sections
$$f^* : \mathcal{E}(N;F) \to \mathcal{E}(M;f^*F)$$
admits a unique continuous extension to a map
$$f^* : \mathcal{D}'_\Gamma(N;F) \to \mathcal{D}'_{f^*\Gamma}(M, f^*F), \tag{3.8}$$
so long as $\Gamma \cap \mathcal{N}_f = \emptyset$.

- If $E \to M$ and $F \to N$ are vector bundles, then the tensor product is a hypocontinuous map

$$\mathcal{D}'_{\Gamma_1}(M;E) \times \mathcal{D}'_{\Gamma_2}(N;F) \xrightarrow{\otimes} \mathcal{D}_{\Gamma_1 \dot{\times} \Gamma_2}(M \times N; E \boxtimes F). \tag{3.9}$$

- If $E_1, E_2$ are two bundles over $M$, and $\Gamma_1$ and $\Gamma_2$ cones with $\Gamma_1 \cap \Gamma_2' = \emptyset$, then the product of distributional sections is a hypocontinuous map

$$\mathcal{D}'_{\Gamma_1}(M;E_1) \times \mathcal{D}'_{\Gamma_2}(M;E_2) \xrightarrow{\cdot} \mathcal{D}'_{\Gamma_1+\Gamma_2}(M;E_1 \otimes E_2). \tag{3.10}$$

The first two statements can be proven analogously to Propositions 3.2.1 and 3.2.3 with minimal alterations, as their proofs rely mainly on the fact that $\mathcal{D}'_\Gamma(M)$ defines a sheaf. For the last statement, we used that the pullback of the bundle $E_1 \boxtimes E_2$ along the diagonal map $D : M \to M \times M$ is isomorphic to $E_1 \otimes E_2$.

### 3.3.1 Pushforward of distributions

As the final fundamental operation to round out our triumvirate, we discuss the pushforward of distributions along a map $f : M \to N$. This operation is somewhat less elegant than the others. It is most naturally defined on compactly supported distributions. However, the pushforward of an open cone (as defined below) is in general not open, unless $f$ is a submersion. It is therefore not well-defined on spaces of the form $\mathcal{D}'_\Gamma(M)$ if we are working with closed cones, and does not always map to a space of the form $\mathcal{E}'_\Lambda(M)$ when working with open cones. To make matters worse, whilst the pullback of a vector bundle is a well-defined quantity, the pushforward of a vector bundle only exists when $f$ is a finite covering. For these reasons, the pushforward is the author's least favourite fundamental operation of Hörmander spaces.

The push-forward of compactly supported distributions is the adjoint of the pullback map

$$f^* : \mathcal{E}(N;E^!) \to \mathcal{E}(M;f^*(E^!)),$$

so that

$$f_* : \mathcal{E}'(N;f^*(E^!)^!) \to \mathcal{E}'(M;E).$$

The bundle $f^*(E^!)$ is in general **not** isomorphic to $f^*(E)^!$ as the density bundles of $M$ and $N$ are *a priori* not related in any way.

This operation can be extended to non-compactly supported distributions $u$, so long as $f|_{\mathrm{supp}(u)}$ is proper. We state the following result, which is a bundle valued version of Theorem 6.3 in [15].

**Proposition 3.3.1.** *Let $M$ be a manifold, $F \to N$ a vector bundle and $f : M \to N$. If $C \subset M$ is a closed set such that $f|_C$ is proper*[3] *and $\Gamma \subset \dot{T}^*M$ is a closed cone over $C$, then the pushforward of distributions is a continuous map*

$$f_* : \mathcal{D}_\Gamma(C; f^*(E^!)^!) \to \mathcal{D}_{f_*\Gamma}(N; E),$$

*where*

$$\mathcal{D}_\Gamma(C; f^*(E^!)^!) = \left\{ u \in \mathcal{D}_\Gamma(M; f^*(E^!)^!) \,\middle|\, \operatorname{supp} u \subset C \right\},$$

*endowed with the subset topology, and the pushforward of $\Gamma$ is defined by*

$$f_*\Gamma = \left\{ (f(x), \eta) \in \dot{T}^*N \,\middle|\, (x, df_x^*\eta) \in \Gamma_0 \right\}. \tag{3.11}$$

We recall that $\Gamma_0 = \Gamma \cup \underline{0} \subset T^*M$.

*Proof.* The proof in [15] applies verbatim, once we exchange their pullback between scalar distributions on Euclidean domains with the one in equation (3.8). □

We unpack the statement in a specific case that we use several times throughout this thesis. Let $M$ be a manifold, $E \to N$ a vector bundle and $\pi : M \times N \to N$ the projection onto $N$. The pullback by $\pi$ of a bundle $F \to N$ is isomorphic to the exterior tensor product with the trivial line bundle $\mathbb{K} \times M \to M$:

$$(\pi^*F \to M \times N) \cong (\mathbb{K} \boxtimes F \to M \times N).$$

We use this to calculate

$$(\pi^*E^!)^! \cong (\mathbb{K} \boxtimes (E^* \otimes D_N))^* \otimes D_{M\times N} \cong D_M \boxtimes E$$

as $D_{M\times N} \cong D_M \boxtimes D_N$. If $\Gamma \subset \dot{T}^*M \times N$, then

$$\pi_*\Gamma = \left\{ \xi \in \dot{T}^*N \,|\, \exists \zeta \in \underline{0}_M \text{ such that } (\zeta, \xi) \in \Gamma \right\}. \tag{3.12}$$

If $u \in \mathcal{D}_\Gamma(D_M \times E)$ is 'vertically compact', i.e. the projection $\pi : \operatorname{supp}(u) \to N$ is a proper map, then the pushforward of $u$ is well-defined as a distribution on $N$.

[3] I.e. $C \cap f^{-1}(K)$ is compact for all compact $K \subset N$.

This pushforward corresponds with 'integrating out' the first variable of $u$, i.e. if $f \in \mathcal{D}(E^!)$, then it holds that

$$\pi_* u(f) = u(1 \otimes f), \tag{3.13}$$

as $\pi^* f = 1 \otimes f$.[4]

## 3.4 HÖRMANDER SPACES OF DISTRIBUTIONAL SECTIONS ON OPEN CONES

The Hörmander spaces with respect to open cones are of vital importance to applications in AQFT, as they are the spaces that can 'dodge' the singular structure of the propagators of a field theory. The present section is aimed at rigorously defining these spaces in the bundle-valued case, as well as discussing operations using these spaces and their continuity properties. We show that $\mathcal{E}'_\Lambda(E)$ can be viewed as the dual to $\mathcal{D}'_{\Lambda'^c}(E^!)$, which we use to define a topology on this space.

The two main operations, that we will need, using these spaces are the tensor product and partial evaluation, both of which we discuss in separate subsections. We close this section by discussing curves with equicontinuous image, which is a topic we will need when proving that equicausal functionals are conveniently smooth in Chapter 6.

Let $E \to M$ be a vector bundle, $\Lambda \subset \dot{T}^*M$ an open cone, $v \in \mathcal{E}'_\Lambda(M; E)$ and $u \in \mathcal{D}'_{\Lambda'^c}(M; E^!)$. Then the product of $u$ and $v$ is well-defined as a distributional section of $E \otimes E^! \cong D_M$. Hence we can once again evaluate it on the function that is identically 1 on $M$ to define a pairing

$$\langle v, u \rangle = v \cdot u(1).$$

As this pairing is continuous in $u$ for $v$ fixed, this implies that,

$$\mathcal{E}'_\Lambda(M; E) \subset (\mathcal{D}'_{\Lambda'^c}(M; E^!))',$$

and we endow $\mathcal{E}'_\Lambda(M; E)$ with the strong topology with respect to this duality.

**Proposition 3.4.1.** *If $E \to M$ is a vector bundle and $\Gamma \subset \dot{T}^*M$ a closed cone, then the dual of $\mathcal{D}'_\Gamma(M; E^!)$ is $\mathcal{E}'_{\Gamma^c}(M; E)$.*

[4] We use, somewhat abusively, the notation 1 for the function on $M$ that is 1 everywhere.

*Proof.* Let $v : \mathcal{D}'_\Gamma(M; E^!) \to \mathbb{C}$ be linear and continuous. As $\mathcal{E}(M; E^!)$ injects continuously into $\mathcal{D}'_\Gamma(M; E^!)$, $v$ is in particular a compactly supported distributional section. We select a locally finite covering of $M$ by coordinate patches $\{U_i\}_{i\in I}$ and a partition of unity $\{\rho_i\}_{i\in I}$ subordinate to that cover. Multiplication and extension by zero give continuous maps

$$m_{\rho_i} : \mathcal{D}'_\Gamma(U_i; E^!|_{U_i}) \to \mathcal{D}'_\Gamma(M; E^!).$$

Hence $v \circ m_{\rho_i}$ is a continuous linear functional on $\mathcal{D}'_\Gamma(U_i; E^!|_{U_i})$, and can therefore be identified with an element of $\mathcal{E}'_{\Gamma^c}(U_i; E_{U_i})$. Extending by zero (which does not change the wavefront set) we view it as an element of $\mathcal{E}'_{\Gamma^c}(M; E)$. As $v$ is compactly supported, only finitely many $\rho_i v$ are non-zero, so that

$$v = \left(\sum_{i\in I} \rho_i\right) \cdot v = \left(\sum_{i\in I} \rho_i v\right) \in \mathcal{E}'_{\Gamma^c}(M; E).$$

□

In fact, the argument in the preceding proof shows something stronger. If $H$ is an equicontinuous family of linear maps on $\mathcal{D}'_\Gamma(M; E^!)$, then $H \circ m_{\rho_i}$ is an equicontinuous family on $\mathcal{D}'_\Gamma(U_i; E)$, which is hence of the form given in Proposition 2.2.6. As $H$ is in particular an equicontinuous family of linear maps on $\mathcal{E}(M; E)$ we can bound the supports of elements of $H$ by a single compact subset of $M$, implying that $H \circ m_{\rho_i} \neq \{0\}$ for only finitely many indices. Hence we can sum these families together to obtain:

**Proposition 3.4.2.** *Let $\Lambda$ be an open cone inside $\dot{T}^*M$. If $H \subset \mathcal{E}'_\Lambda(M; E)$ is equicontinuous, then there exist a compact set $K \subset M$ and a **closed** cone $\Xi \subset \Lambda|_K$ such that $H$ is a bounded subset of $\mathcal{D}'_\Xi(K; E)$.*

In contrast to distributional sections, which naturally *restrict* to smaller subsets, compactly supported distributions *extend*. If $U \subset M$ is an open set, then the extension map

$$\mathrm{ext}_U : \mathcal{E}'(U; E|_U) \to \mathcal{E}'(M, E)$$

is defined as the pushforward along the inclusion $U \to M$. Somewhat more heuristically, we say that we 'extend by 0 outside $U$'. Clearly this operation does not enlarge wavefront sets, and defines linear maps

$$\mathrm{ext}_U : \mathcal{E}'_{\Lambda|_U}(U; E|_U) \to \mathcal{E}'_\Lambda(M; E).$$

These data exhibit a cosheaf:

**Proposition 3.4.3.** *Let $E$ be a vector bundle over $M$ and $\Lambda \subset \dot{T}^*M$ an open cone. The assignment*

$$U \mapsto \mathcal{E}'_{\Lambda|_U}(U; E|_U), \quad U \subset M \text{ open},$$

*together with the extension maps of compactly supported distributions, defines a cosheaf of LCTVSs.*

*Proof.* Multiplication by smooth function respects wavefront sets, so that the standard partition of unity argument showing that compactly supported *smooth* sections form a cosheaf *of vector spaces* applies to our situation, see e.g. Lemma 4.4.2 in [23].

Let $\{U_i\}_{i\in I}$ be a cover on $M$, we show that the topology on $\mathcal{E}'_\Lambda(M; E)$ is equivalent to the final locally convex topology induced by the maps

$$\mathcal{E}'_{\Lambda|_{U_i}}(U_i; E|_{U_i}) \xrightarrow{\text{ext}_i} \mathcal{E}'_\Lambda(M; E).$$

The topology of $\mathcal{E}'_\Lambda(M; E)$ is generated by seminorms of the form

$$p_B(v) = \sup_{u\in B} |\langle v, u\rangle|,$$

for $B \subset \mathcal{D}'_{\Lambda'^c}(M; E^!)$ bounded. If $\tilde{v} \in \mathcal{E}'_{\Lambda|_{U_i}}(U_i; E|_{U_i})$, then we can calculate

$$p_B(\text{ext}_i\, \tilde{v}) = \sup_{u\in B} |\langle \text{ext}_i\, \tilde{v}, u\rangle| = \sup_{u\in B} |\langle \tilde{v}, R_i u\rangle| = \sup_{\tilde{u}\in R_i B} |\langle \tilde{v}, \tilde{u}\rangle| = p_{R_i B}(\tilde{v}),$$

where we used in the second step that $\text{ext}_i$ is the adjoint of $R_i$. As $R_i B \subset \mathcal{D}'_{\Lambda'^c|_{U_i}}(U_i; E^!|_{U_i})$ is bounded for all $i$, this implies that the extension maps are continuous, so that the strong topology on $\mathcal{E}'_\Lambda(M)$ is stronger than the final one.

Conversely, suppose $p$ is a seminorm on $\mathcal{E}'_\Lambda(M)$, with the property that $p \circ \text{ext}_i$ is a continuous seminorm for all $i \in I$. Hence there are bounded sets $B_i \subset \mathcal{D}'_{\Lambda'^c|_{U_i}}(U_i; E^!|_{U_i})$ such that $p \circ \text{ext}_i \leq p_{B_i}$. We select a partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$, restricting to a smaller cover if need be, and we define

$$\Psi : \prod_{i\in I} \mathcal{D}'_{\Lambda'^c|_{U_i}}(U_i; E^!|_{U_i}) \to \mathcal{D}'_\Gamma(M; E^!)$$

as in equation (3.5), which is continuous by lemma 3.1.3.[5] We set

$$B = \Psi\left(\prod_{i\in I} \text{bal}(B_i)\right)$$

[5]The proof generalises straightforwardly to the bundle-valued case.

where $\mathrm{bal}(B_i) = \bigcup_{\lambda\in\mathbb{K},|\lambda|\leq 1} \lambda B_i$ is the balanced hull of $B_i$, which is bounded because $B_i$ is. Hence $B$ is bounded, as the bounded sets in a product of vector spaces are exactly the products of bounded sets, and $\Psi$ is continuous. We then calculate, for $v \in \mathcal{E}'_\Lambda(M;E)$

$$\begin{aligned} p(v) &= \sum_{i\in I} p(\rho_i v) \leq \sum_{i\in I} p_{B_i}(\rho_i v) = \sum_{i\in I} \mathrm{supp}_{g_i\in B_i} |\langle v, \rho_i g_i\rangle| \\ &\leq \mathrm{supp}_{g_i\in \mathrm{bal} B_i} \left|\sum_{i\in I}\langle v, \rho_i g_i\rangle\right| = p_B(v). \end{aligned}$$

Here we used the balanced hulls to turn all terms in the sum into positive real numbers, so that we may move the absolute value out of the sum. We may freely exchange sums and suprema because $\rho_i v = 0$ for all but finitely many $i \in I$, so that the sum is finite in every step. Hence we see that $p$ is continuous with respect to the strong topology, so that the strong topology is both weaker and stronger than the final one, and must hence be equivalent to it. □

### 3.4.1 Tensor products

We discuss the tensor product on these spaces. Note that, in contrast to the closed cone scenario, there is no 'smallest' space to map into, as the dotted product of open cones is not open in general.

**Theorem 3.4.4.** *Let $E \to M$ and $F \to N$ be vector bundles. If $\Lambda_1 \subset \dot{T}^*M$ and $\Lambda_2 \subset \dot{T}^*N$ are open cones and $\Lambda \subset \dot{T}^*(M \times N)$ is an open cone containing $\Lambda_1 \dot{\times} \Lambda_2$, then the tensor product of distributions restricts to a hypocontinuous map*

$$\mathcal{E}'_{\Lambda_1}(E) \times \mathcal{E}'_{\Lambda_2}(F) \to \mathcal{E}'_{\Lambda}(E \boxtimes F). \tag{3.14}$$

*Proof.* We first show a Euclidean version. Let $\Omega_1$ and $\Omega_2$ be Euclidean domains, $\Upsilon_i \subset \dot{T}^*\Omega_i$ two open cones, $\{K_l\}_{l\in\mathbb{N}}$ a compact exhaustion of $\Omega_1$ and $\Xi_l \subset \Upsilon_1|_{K_l}$ an exhausting sequence of **closed** cones of $\Upsilon_1$, i.e. $\Xi_l \subset \Xi_{l+1}$ and

$$\Upsilon_1 = \bigcup_{l\in\mathbb{N}} \Xi_l.$$

We refer the reader to Section 3.1 of [28] for an explicit construction of these exhaustions. We set

$$W_l := \mathcal{D}'_{\Xi_l}(K_l) \subset \mathcal{D}'_{\Xi_l}(\Omega_1), \tag{3.15}$$

endowed with the subspace topology, viewing $\Xi_l$ as a subset of $\dot{T}^*\Omega_1$. It was shown in Lemma 10 of [28] that

$$\mathcal{E}'_{\Upsilon_1}(\Omega_1) = \mathrm{Colim}(\ldots \to W_l \to W_{l+1} \to \ldots) \tag{3.16}$$

in LCTVS, where the maps $W_l \to W_{l+1}$ are the subset inclusions.

Let $v \in \mathcal{E}'_{\Upsilon_2}(\Omega_2)$ and write $\Gamma = \mathrm{WF}(v)$ and $K = \mathrm{supp}(v)$. The tensor product

$$W_l \xrightarrow{\_\otimes v} \mathcal{D}_{\Xi_l \dot{\times} \Gamma}(K_l \times K)$$

is continuous by Proposition 3.7, for all $l \in \mathbb{N}$. We obtain the following commutative diagram

$$\begin{array}{ccccc} W_l & \longrightarrow & W_{l+1} & \longrightarrow & \mathcal{E}'_{\Upsilon_1}(\Omega_1) \\ \downarrow{\scriptstyle \_\otimes v} & & \downarrow{\scriptstyle \_\otimes v} & & \downarrow{\scriptstyle \_\otimes v} \\ \mathcal{D}'_{\Xi_l \dot{\times} \Gamma}(K_l \times K) & \longrightarrow & \mathcal{D}'_{\Xi_{l+1} \dot{\times} \Gamma}(K_{l+1} \times K) & \longrightarrow & \mathcal{E}'_{\Upsilon}(\Omega_1 \times \Omega_2) \end{array}$$

for any $\Upsilon \supset \Upsilon_1 \dot{\times} \Upsilon_2$. It then follows from the universal property of the direct limit in equation (3.16) that the rightmost arrow in the diagram is continuous, as all the other maps are continuous.

Let now $\{U_i\}_{i\in I}$ be an open cover of $M$ of coordinate patches that trivialise $E$, and let $\{V_j\}_{j\in J}$ be a similar cover for $N$ with a partition of unity $\{\rho_j\}_{j\in J}$ subordinate to it. If $v \in \mathcal{E}'_{\Lambda_2}(N; F)$, then only finitely many $\rho_j v$ are non-zero, so that

$$\_ \otimes v = \sum_{j\in J} \_ \otimes \rho_j v : \mathcal{E}'_{\Lambda_1}(M; E) \to \mathcal{E}'_{\Lambda}(M \times N; E \boxtimes F).$$

We have, for all $i \in I$ and $j \in J$, commutative diagrams

$$\begin{array}{ccc} \mathcal{E}'_{\Lambda_1|_{U_i}}(U_i; E|_{U_i}) & \longrightarrow & \mathcal{E}'_{\Lambda_1}(M; E) \\ \downarrow{\scriptstyle \_\otimes \rho_j v} & & \downarrow{\scriptstyle \_\otimes \rho_j v} \\ \mathcal{E}'_{\Lambda|_{U_i \times V_j}}(U_i \times V_j; E|_{U_i} \boxtimes F|_{V_j}) & \longrightarrow & \mathcal{E}'_{\Lambda}(M \times N; E \boxtimes F) \end{array}$$

The left vertical arrow is continuous by the argument given above, implicitly using the trivialisation of $U_i$, so that it follows from the universal property of $\mathcal{E}'_{\Lambda_1}(M; E)$ that the right vertical arrow is continuous as well. We conclude that the tensor product by $v$ is continuous. Exchanging roles of $M$ and $N$, we have shown that the tensor product

$$\mathcal{E}'_{\Lambda_1}(M;E) \times \mathcal{E}'_{\Lambda_2}(N;F) \to \mathcal{E}'_{\Lambda}(M \times N; E \boxtimes F)$$

is separately continuous.

Both of these spaces are barreled: This holds in the Euclidean case by Proposition 28 in [28]. If $T \subset \mathcal{E}'_{\Lambda_1}(M;E)$ is a barrel, then $\mathrm{ext}_i^{-1}(T) \subset \mathcal{E}'_{\Lambda_1|_{U_i}}(U_i)$ is a barrel as well, as $\mathrm{ext}_i$ is injective and continuous. Hence $\mathrm{ext}_i^{-1}(T)$ is a neighbourhood of zero for all $i$. As $T$ is convex by assumption, it follows that $T$ is a neighbourhood of zero as well. The result then follows, as separately continuous bilinear maps on barreled spaces are hypocontinuous, see e.g. Theorem 41.2 in [72]. □

As the notation tends to get quite dense when working with these spaces, we suppress the manifolds in the notation in the remainder of this thesis, unless we want to make a specific point about the localisation of certain distributions. If $E \to M$ is any vector bundle, then we will write $\mathcal{E}(E)$ instead of $\mathcal{E}(M;E)$, and similar for all variations (distributional, compactly supported, etc...). We will **never** use this notation to indicate scalar functions or distributions on the total space of $E$ viewed as a manifold in its own right.

**Proposition 3.4.5.** *In the same situation as Theorem 3.4.4, if $H \subset \mathcal{E}'_{\Lambda_1}(E)$ and $L \subset \mathcal{E}'_{\Lambda_2}(F)$ are equicontinuous subsets, then $H \otimes L$ is an equicontinuous subset of $\mathcal{E}'_{\Lambda}(E \boxtimes F)$.*

*Proof.* By the characterization of equicontinuous sets in Proposition 3.4.2, there are closed cones $\Xi, \Delta$ and compact $K, Q$ such that $H \subset \mathcal{D}'_{\Xi}(K;E)$ and $L \subset \mathcal{D}'_{\Delta}(Q;F)$ are bounded subsets. As the image of a product of bounded sets by a hypocontinuous map is bounded, we find that $H \otimes L$ is bounded in $\mathcal{D}'_{\Xi \dot{\times} \Delta}(K \times Q; E \boxtimes F)$, and hence is equicontinuous as a subset of $\mathcal{E}'_{\Lambda}(M \times N; E \boxtimes F)$.

□

### 3.4.2 Partial evaluation

Somewhat dual to the tensor product discussed above, which increases the number of variables, we consider partial evaluation of distributions of multiple variables, which 'integrates out' variables.

Let $E \to M$ and $F \to N$ be vector bundles, and let $v \in \mathcal{E}'(E \boxtimes F)$. If $\varphi \in \mathcal{E}(E^!)$ then we can define the partial evaluation of $u$ on $\varphi$ to be the element of $\mathcal{E}'(F)$ defined by

$$\iota_\varphi u : \psi \mapsto u(\varphi \otimes \psi). \tag{3.17}$$

This operation can be extended to the case where $\varphi$ and $\psi$ are distributional sections, provided that a condition is satisfied by the wavefront sets of $u$, $\varphi$ and $\psi$.

Let $\Lambda \subset \dot{T}^*M \times N$ and $\Gamma \subset T^*M$ be two cones.[6] We define the partial contraction of $\Gamma$ with $\Lambda$ by

$$\Gamma \bullet \Lambda = \{\xi \in T^*N \,|\, \exists \zeta \in \Gamma_0 \text{ such that } (\zeta, \xi) \in \Lambda\},$$

where we recall the notation $\Gamma_0 = \Gamma \cup \underline{0}$. We note that $\Gamma \bullet \Lambda$ might intersect the zero section, even if $\Gamma$ does not.[7]

If $\Gamma$ is closed and $\Lambda$ is open, then $\Gamma \bullet \Lambda$ is an open subset of $T^*N$. Indeed, suppose that $(\xi_n)$ is a convergent sequence in $(\Gamma \bullet \Lambda)^c$ that converges to $\xi \in T^*N$. Then for all $\zeta \in \Gamma_0$, it holds that $(\xi_n, \zeta) \in \Lambda^c$. As $\Lambda$ is open (also as a set of $T^*N$, as it does not intersect the zero section), it follows that $(\xi, \zeta) \in \Lambda^c$ for all $\zeta \in \Gamma_0$, so that $\xi \in (\Gamma \bullet \Lambda)^c$.

We prove the following proposition on the partial evaluation of Hörmander spaces.

**Proposition 3.4.6.** *Let $E \to M$ and $F \to N$ be vector bundles. If $\Lambda \subset \dot{T}^*(M \times N)$ is an open cone and $\Gamma \subset \dot{T}^*M$ is a closed cone such that $(\Gamma' \bullet \Lambda) \cap \underline{0} = \emptyset$, then partial evaluation, as defined by equation (3.17), induces a bilinear map*

$$\mathcal{D}'_\Gamma(E^!) \times \mathcal{E}'_\Lambda(E \boxtimes F) \to \mathcal{E}'_{\Gamma' \bullet \Lambda}(F). \tag{3.18}$$

*Furthermore, if $B \subset \mathcal{D}'_\Gamma(E^!)$ is bounded and $H \subset \mathcal{E}'_\Lambda(E \boxtimes F)$ equicontinuous, then the family*

$$\iota_B H = \{\iota_\varphi u \,|\, \varphi \in B, u \in H\} \subset \mathcal{E}'_{\Gamma' \bullet \Lambda}(F)$$

*is equicontinuous.*

---

[6] Note that we allow $\Gamma$ to intersect the zero section.

[7] As a word of caution, we use the notation $\bullet$ to distinguish from the subtly different operation $\circ$ defined in Section 8.2 of [58], where a cone in two arguments acts on a cone in one argument, rather than the other way around. These notations are related by

$$\Gamma \bullet \Lambda = \Lambda^t \circ \Gamma_0,$$

where $\Lambda^t$ is the cone obtained by exchanging the two entries in $\Lambda$. To avoid confusion, we will not use the $\circ$ symbol when discussing contractions of cones.

*Proof.* Let $\pi : M \times N \to N$ be the projection map on $N$. It follows from equation (3.13) that we may write, for $u \in \mathcal{E}'_\Lambda(E \boxtimes F)$ and $\varphi$ and $\psi$ smooth,

$$\iota_\varphi u(\psi) = ((\varphi \otimes \mathbb{1}) \cdot u)\,(\mathbb{1} \otimes \psi) = \pi_*\,((\varphi \otimes \mathbb{1})u)\,(\psi),$$

where the pushforward is well-defined as $u$ is compactly supported. Hence partial evaluation can be viewed as a composition of a multiplication by $(\varphi \otimes \mathbb{1})$ with a pushforward along $\pi$.

We then note that

$$\Gamma' \bullet \Lambda = \pi_*(\Gamma \times \underline{0} + \Lambda),$$

from equations (2.19) and (3.12), and it is readily checked that

$$\Gamma' \bullet \Lambda \cap \underline{0} = \emptyset \Leftrightarrow (\Gamma \times \underline{0} + \Lambda) \cap \underline{0} = \emptyset$$

so that indeed the partial evaluation in equation (3.18) is well-defined, and lands in $\mathcal{E}'_{\Gamma'\bullet\Lambda}(F)$.

Put differently, this means that

$$\Gamma \dot{\times} (\Gamma' \bullet \Lambda)^{\prime c} \subset \Lambda^{\prime c},$$

so that the tensor product of distributions is a hypocontinuous map

$$\mathcal{D}'_\Gamma(E^!) \times \mathcal{D}'_{(\Gamma'\bullet\Lambda)^{\prime c}}(F^!) \to \mathcal{D}'_{\Lambda^{\prime c}}(E^! \boxtimes F^!).$$

The family $\iota_B H$ is hence given by the composition of the two equicontinuous families

$$\mathcal{D}_{(\Gamma'\bullet\Lambda)^{\prime c}}(F^!) \xrightarrow{\{\varphi\otimes\_\}_{\psi\in B}} \mathcal{D}_{\Lambda^{\prime c}}(E^! \boxtimes F^!) \xrightarrow{\{u(\_)\}_{u\in H}} \mathbb{K},$$

and is therefore equicontinuous itself. □

### 3.4.3 Curves with equicontinuous image

To close this chapter, we discuss smooth curves in $\mathcal{E}'(E)$ that map bounded intervals to equicontinuous sets of $\mathcal{E}'_\Lambda(E)$ for some open cone $\Lambda \in \dot{T}^*M$. Part of this material already appeared in the appendix to [52], and was one of my contributions to that publication.

We first prove a basic result about the Fourier transform of a curve of distributions.

**Lemma 3.4.7.** *If $u$ is a smooth curve $\mathbb{R} \to \mathcal{E}'(\mathbb{R}^n)$ for some $n \in \mathbb{N}$, then the Fourier transform of $u$ defines a smooth function*

$$\hat{u} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C},$$

*given by*

$$\hat{u} : (t, \xi) \mapsto \hat{u}_t(\xi).$$

*Proof.* For $\xi \in \mathbb{R}^n$, we define the oscillatory function

$$e_\xi(x) = \exp(-i\langle \xi, x\rangle),$$

where $\langle \_, \_\rangle$ denotes the Euclidean scalar product on $\mathbb{R}^n$. Suppose that $t_m \to t \in \mathbb{R}$ and $\xi_m \to \xi \in \mathbb{R}^n$, let $\epsilon > 0$ and define the set $B = \{e_{\xi_m} \,|\, m \in \mathbb{N}\} \subset \mathcal{E}(\mathbb{R}^n)$. As the map $\xi \mapsto e_\xi$ is continuous and $\{\xi_n \,|\, n \in \mathbb{N}\}$ is bounded, $B$ is bounded as well. Due to the fact that $u$ is continuous with respect to the strong topology, we can find a $\delta > 0$ such that

$$|s - t| < \delta \implies \sup_{f \in B} |u_s(f) - u_t(f)| < \epsilon/2.$$

Also, since $\hat{u}_t$ is a continuous function of $\xi$, there is a $\delta' > 0$ such that

$$|\xi - \zeta| < \delta' \implies |u_t(\xi) - u_t(\zeta)| < \epsilon/2.$$

Hence we find, for sufficiently large $n$

$$|\hat{u}_{t_n}(\xi_n) - \hat{u}_t(\xi)| \leq |\hat{u}_{t_n}(\xi_n) - \hat{u}_t(\xi_n)| + |\hat{u}_t(\xi_n) - \hat{u}_t(\xi)| < \epsilon,$$

This shows that $\hat{u}$ is continuous. To show smoothness, we calculate

$$\partial_t \hat{u}_t(\xi) = \partial_t u_t(e_\xi) = \widehat{\partial_t u_t}(\xi).$$

As $\partial_t u_t$ is a smooth curve in its own right, the previous argument implies that this is a jointly continuous function of $t$ and $\xi$. For the partial derivatives with respect to $\xi$, we calculate

$$\partial_{\xi_i} \hat{u}_t(\xi) = u_t(-i x_i e_\xi) = -i\widehat{x_i u_t}(\xi).$$

Where we have used the fact that $u_t$ is linear and continuous to move the differentiation inside $u_t$. Again, $x_i u$ is a smooth curve in its own right, and hence $\widehat{x_i u}$ is jointly continuous.

Hence $\hat{u}$ is a $C^1$ function, as both its partial derivatives exist and are continuous. Repeating these steps to higher order shows that $\hat{u}$ is smooth. □

**Lemma 3.4.8.** *Let $E \to M$ be a vector bundle and $\Lambda \subset \dot{T}^*M$ an open cone, and let*

$$u : \mathbb{R} \to \mathcal{E}'(E)$$

*be a smooth curve that maps bounded intervals to equicontinuous subsets of $\mathcal{E}'_\Lambda(E)$. If $I \subset \mathbb{R}$ is bounded, then there exists a closed cone $\Xi \subset \Lambda$ and a compact set $K \subset M$ such that*

$$u|_I : I \to \mathcal{D}'_\Xi(K; E) \tag{3.19}$$

*is a continuous map. Consequently, $u$ is continuous as a curve into $\mathcal{E}'_\Lambda(E)$.*

*Proof.* As $u(I)$ is an equicontinuous set, existence of $\Xi$ and $K$ such that $u|_I$ defines a map as in equation (3.19) is clear. We have to show that it is continuous.

Let $t_0 \in I$, and let $B \subset \mathcal{D}(E^!)$ be bounded. Then $B$ is also bounded as a subset of $\mathcal{E}(E^!)$. As u is smooth into $\mathcal{E}'(E)$, it follows that

$$p_B(u_t - u_{t_0}) = \sup_{f \in B} |u_t(f) - u_{t_0}(f)| \xrightarrow{t \to t_0} 0.$$

Similarly, let $\chi \subset \mathcal{D}(N)$ be supported in a coordinate patch,[8] let $N \in \mathbb{N}$ and let $V \subset \mathbb{R}^d$ be a closed cone such that

$$(\operatorname{supp} \chi \times V) \cap \Xi = \emptyset,$$

where $d$ is the dimension of $M$. We have to show that

$$\lim_{t \to t_0} p_{\chi,N,V}(u_t - u_{t_0}) = \lim_{t \to t_0} \sup_{\xi \in V} \langle \xi \rangle^N |\widehat{\chi u_t} - \widehat{\chi u_{t_0}}| = 0.$$

It is a known property of these seminorms that continuity in the seminorms $p_B$ together with boundedness in the seminorms $p_{\chi,N,V}$ implies continuity in these seminorms, see e.g. Definition 8.2.2 in [58] and the subsequent discussion. We present a proof of this fact for the convenience of the reader as we do not find the argumentation in that source very clear.

Let $\epsilon > 0$. As the family $\{u_t\}_{t \in I} \subset \mathcal{D}'_\Xi(K; E)$ is bounded, the constant

$$C = \sup_{t \in I} p_{\chi,N+1,V}(u_t)$$

is finite, and we have

$$\langle \xi \rangle^N |\widehat{\chi u_t}| \leq C \langle \xi \rangle^{-1} \quad \xi \in V,$$

[8]We suppress a choice of coordinates, as well as a choice of norm on the fibres of $E$ in the notation.

Hence there exist $R \in \mathbb{R}^+$ such that

$$|\xi| \geq R \implies \langle\xi\rangle^N |\widehat{\chi u_t}| \leq \epsilon/4 \quad \forall t \in I.$$

Because $\widehat{\chi u}$ is jointly smooth in the arguments $t$ and $\xi$ by Lemma 3.4.7, $u_t$ converges to $u_{t_0}$ uniformly on the compact set $\{\xi \in V \,|\, |\xi| \leq R\}$. Hence for $t$ sufficiently close to $t_0$

$$\sup_{\substack{\xi \in V \\ |\xi| \leq R}} \langle\xi\rangle^N |\widehat{\chi u_t} - \widehat{\chi u_{t_0}}| < \epsilon/2.$$

For such a $t$ we then have

$$\sup_{\xi \in V} \langle\xi\rangle^N |\widehat{\chi u_t} - \widehat{\chi u_{t_0}}| \leq \sup_{\substack{\xi \in V \\ |\xi| \leq R}} \langle\xi\rangle^N |\widehat{\chi u_t} - \widehat{\chi u_{t_0}}| + \sup_{\substack{\xi \in V \\ |\xi| \geq R}} \langle\xi\rangle^N \left(|\widehat{\chi u_t}| + |\widehat{\chi u_{t_0}}|\right) < \epsilon.$$

□

**Corollary 3.4.9.** *Let $u$ be as in the previous lemma. If $Z \in \mathcal{D}'_{\Lambda'^c}(E^!)$, then the map*

$$\iota_Z u : t \mapsto u_t(Z)$$

*is continuous $\mathbb{R} \to \mathbb{C}$. Furthermore, if $a \leq b$, then $\int_a^b u_t dt \in \mathcal{E}'_\Lambda(E)$, and*

$$\left(\int_a^b u_t dt\right)(Z) = \int_a^b u_t(Z) dt. \tag{3.20}$$

*Proof.* Let $I \subset \mathbb{R}$ be bounded and let $\Xi$ and $K$ be as above. We have that

$$(\Xi + \mathrm{WF}(Z)) \cap \underline{0} = \emptyset$$

by assumption on the wavefront set of $Z$. Hence we can exhibit $\iota_Z u|_I$ as the composition of continuous maps

$$I \mapsto \mathcal{D}'_\Xi(K; E) \xrightarrow{Z\cdot} \mathcal{D}'_{\Xi + \mathrm{WF}(Z)}(K; D_M) \xrightarrow{\pi_*} \mathbb{K},$$

where $\pi : M \to *$ is the trivial projection map. Hence $\iota_Z u|_I$ is continuous for all bounded intervals $I$, and hence the same holds for $\iota_Z u$ as continuity is a local property.

The integration statement now follows immediately from the fact that $\mathcal{D}'_\Xi(K; E)$ is complete, paired with the fact that evaluation on $Z$ is continuous on that space so that we may exchange that operation with the integral over $[a, b]$. □

**Lemma 3.4.10.** *Under the same assumptions as in Lemma 3.4.8, let*

$$u : \mathbb{R} \to \mathcal{E}'(E)$$

*be a smooth curve with the property that, for all $n \in \mathbb{N}$, the curve $\partial_t^n u$ maps bounded intervals to equicontinuous subsets of $\mathcal{E}'_\Lambda(E)$. If $I \subset \mathbb{R}$ is a bounded interval, then there exist a sequence of nested closed cones*

$$\Xi_0 \subset \Xi_1 \subset \ldots \subset \Lambda,$$

*and a compact set $K \subset M$ such that*

$$u|_I : I \to \mathcal{D}'_{\Xi_n}(K; E)$$

*is $C^n$. Consequently, $u$ is smooth as a curve into $\mathcal{E}'_\Lambda(E)$.*

It is unclear to the author at this point whether it is possible to choose a single cone that works for all orders of differentiation. Regardless, this result is strong enough for our purposes.

*Proof.* From Lemma 3.4.8, we get existence of closed cones $\tilde{\Xi}_n$ and compact sets $K_n \subset M$ such that

$$\partial_t^n u|_I : I \to \mathcal{D}'_{\tilde{\Xi}_n}(K_n; E)$$

is continuous. As $\{u_t\}_{t \in I} \subset \mathcal{E}'(E)$ is bounded, the sets $K_n$ may be taken to be uniform. Setting

$$\Xi_n = \bigcup_{i \leq n} \tilde{\Xi}_i$$

ensures that these cones are nested.

It remains to show differentiability. We proceed similarly as in the proof of Lemma 3.4.8. Let $t_0 \in I$ and let $B \subset \mathcal{D}(E^!)$ be bounded. As $u$ is smooth into $\mathcal{E}'(E)$ it is clear that

$$\lim_{t \to t_0} p_B \left( \frac{u_t - u_{t_0}}{t - t_0} - (\partial_t u)_{t_0} \right) = 0.$$

Similarly, let $\chi \subset \mathcal{D}(N)$ be supported in a coordinate patch,[9] let $N \in \mathbb{N}$ and let $V \subset \mathbb{R}^d$ be a closed cone such that

$$(\operatorname{supp} \chi \times V) \cap \Xi = \emptyset,$$

[9]Again, we suppress a choice of coordinates, as well as a choice of norm on the fibres of $E$ in the notation.

where $d$ is the dimension of $N$. We need to show that

$$\lim_{t \to t_0} p_{\chi,N,V}\left(\frac{u_t - u_{t_0}}{t - t_0} - (\partial_t u)_{t_0}\right) = 0.$$

By the discussion in the proof of Lemma 3.4.10, it suffices to show that

$$p_{\chi,N,V}\left(\frac{u_t - u_{t_0}}{t - t_0} - (\partial_t u)_{t_0}\right)$$

is bounded on $I \setminus \{t_0\}$. We set $H$ to be the closed convex hull of $\{(\partial_t u)_t\}_{t \in I}$, which is bounded in $\mathcal{D}'_{\Xi_1}(K; E)$ by the assumptions on $u$. It then follows from the mean value theorem, see e.g. Section 1.4 of [62], that

$$(u_t - u_{t_0}) \in (t - t_0)H \quad \forall t \in I,$$

from which the result follows.

We conclude that $u|_I$ is $C^1$ into $\mathcal{D}'_{\Xi_1}(K; E)$. By the same reasoning, we obtain that $\partial_t^n u$ is $C^1$ into $\mathcal{D}'_{\Xi_{n+1}}(K; E)$. The result now follows from induction. □

Finally, we discuss the case where $\Lambda$ is neither open nor closed. In that case, we can still define the *vector space* of distributions

$$\mathcal{E}'_\Lambda(E) = \{u \in \mathcal{E}'(E) \,|\, \mathrm{WF}(u) \subset \Lambda\}.$$

We will abusively(!) call a subset $H$ of that space equicontinuous if there exist a closed cone $\Xi \subset \Lambda$ and a compact $K \subset M$ such that

$$H \subset \mathcal{D}'_\Xi(K; E)$$

is a bounded set. We do not define any topology on this space, but rather note that this gives a definition of a bornological space.

The first statement in lemma 3.4.8 remains true in this more general scenario, as the proof does not depend on the fact that $\Lambda$ is open and makes no reference to the topology of $\mathcal{E}'_\Lambda(E)$. Similarly, the first statement of Lemma 3.4.10 remains valid. Finally, Proposition 3.4.5 remains true for any trio of cones $\Lambda_1, \Lambda_2$ and $\Lambda$ such that

$$\Lambda_1 \dot{\times} \Lambda_2 \subset \Lambda.$$

We will use these facts in Chapter 6 where we have the misfortune of encountering some cones that are neither open nor closed.

# — 4 —

# Pathological features of microcausal functionals

As already touched on at the end of Section 2.3.3, the class of microcausal functionals has several undesirable features that have thus far gone unnoticed in the literature. We give a full exposition with concrete counterexamples in this chapter. For ease of notation, we work with field configurations valued in a trivial complex bundle, so that $\mathcal{E}(M) = C^\infty(M, \mathbb{C})$, endowed with a normally hyperbolic operator $P$. We will also abbreviate $\mathcal{E}(M) = \mathcal{E}$ in this chapter when no confusion can arise.

The first problem is related to the Poisson bracket of two microcausal functionals. We recall that it is given by

$$\{F, G\}(\varphi) = \left\langle \Delta, F^{(1)}(\varphi) \otimes G^{(1)}(\varphi) \right\rangle, \tag{4.1}$$

which is well-defined when $F$ and $G$ are microcausal functionals. When attempting to calculate the derivative of $\{F, G\}$, it is tempting to use the Leibniz rule to conclude that

$$\{F, G\}^{(1)}(\varphi)\{\psi\} \overset{?}{=} \left\langle \Delta, F^{(2)}(\varphi)\{\psi, \_\} \otimes G^{(1)}(\varphi) + F^{(1)}(\varphi) \otimes \Delta G^{(2)}(\varphi)\{\psi, \_\} \right\rangle \tag{4.2}$$

The right-hand side of this equation is well-defined, and it is given by pairing $\psi$ with an element of $\mathcal{E}'_{\Gamma_1^c}$, due to the fact that $F$ and $G$ are microcausal, see e.g. Section 3 in [20]. However, despite the fact that there is an obvious, well-defined candidate for the derivative, it is not guaranteed that it gives the limit

$$\lim_{t \to 0} \frac{1}{t} \left( \{F, G\}(\varphi + t\psi) - \{F, G\}(\varphi) \right),$$

or indeed that the limit even exists. This turns out to not be true in general, which we support with a concrete counterexample in Section 4.3. Thus, whilst $\{F, G\}$ is well-defined as a functional $\mathcal{E} \to \mathbb{C}$, it is not guaranteed to be smooth, which is a problem as it means we can not take repeated Poisson brackets.

The second problem we discuss in this chapter relates to defining a homotopy operator on the Koszul complex of microcausal multivector fields. These are smooth functionals

$$X : \mathcal{E} \to \mathcal{E}(E^{!\boxtimes k}),$$

satisfying certain symmetry properties and some conditions on the singular structure of their derivatives, similar to the definition of microcausal functionals.

We denote by $\mathrm{Sol} = \mathrm{Ker}(P) \subset \mathcal{E}(M)$ the space of solutions to the field equations, which we view as the subset of 'physical configurations'. We assume for now that Sol admits a continuous projection $\pi_S : \mathcal{E} \to \mathrm{Sol}$,[1] which we will substantiate in more detail in Chapter 5. We denote the complementary projection by $\tilde{\pi}_S = \mathbb{1} - \pi_S$.

On the one hand, we can take an 'on-shell' approach and study the space of functionals defined on Sol, which we will denote by $\mathcal{F}_S$. This approach is somewhat cumbersome in practice, as the solution space is not a standard space of functions, so that we cannot appeal to results and tools from distribution theory when studying these functionals. On top of this, it is awkward to discuss the notion of localization in this framework; Solutions to the wave equations can not be 'locally perturbed', as solutions to the wave equation can not have compact support.

For these reasons, it is more convenient to take an 'off-shell' approach, and study functionals on the whole of $\mathcal{E}(M)$, identifying functionals that match on Sol. We denote by $\mathcal{F}(M)$ the space of all functionals on $\mathcal{E}(M)$, or just by $\mathcal{F}$ when no confusion can arise, and by $\mathcal{I}_S$ the ideal of functionals that vanish on Sol. We then have that

$$\mathcal{F}_S \cong \mathcal{F}/\mathcal{I}_S.$$

It is the aim of the Koszul complex, defined more concisely in the next chapter, to find a homological resolution of this quotient. This gives an off-shell description of the algebra of field observables, which is often more convenient than an on-shell description as the space of solutions to the field equations is not a standard space of functions. A key step in the proof that this is a resolution is the following. Given

[1] In general, any linear subspace of a vector space has a projection onto it, but the existence of a *continuous* projection is a non-trivial requirement.

$F \in \mathcal{F}$, we can pre-compose it with the projection $\pi_S$[2] to obtain another functional on $\mathcal{E}(M)$. It then follows from the fundamental theorem of calculus that

$$\begin{aligned}(F - F \circ \pi_S)(\varphi) =& \int_0^1 F^{(1)}(\pi_S(\varphi) + \lambda \tilde{\pi}_S \varphi)\{\tilde{\pi}_S \varphi\} d\lambda, \\ =& \left(\int_0^1 F^{(1)}(\pi_S(\varphi) + \lambda \tilde{\pi}_S \varphi) d\lambda\right)\{\tilde{\pi}_S \varphi\}, \\ \equiv& X(\varphi)\{\tilde{\pi}_S \varphi\}. \end{aligned} \tag{4.3}$$

Hence we see that the difference between $F$ and $\pi_S^* F$ at $u$ is the contraction of some vector field $X$ with $\tilde{\pi}_S$. If we denote, for $X$ a vector field, the functional

$$\delta X(\varphi) = X(\varphi)\{\tilde{\pi}_S \varphi\},$$

we see that we obtain a sequence

$$\mathcal{X}(M) \xrightarrow{\delta} \mathcal{F}(M) \xrightarrow{\pi_S^*} \mathcal{F}_S(M) \to 0,$$

which is exact. This is the start of the Koszul complex resolving the on-shell functionals. The prescription for $X$ in equation (4.3) is the starting point for defining a homotopy operator showing exactness of that resolution.

We will also want to use this resolution to define algebraic structure, like a Poisson bracket, on $\mathcal{F}_S(M)$. For this reason, we will have to restrict to a subcomplex, with prescribed bounds on the singular structure of the derivatives of functionals. In order to conclude that this operation does not change the homology, we should check that the homotopy operator we define respects this subcomplex.

It is a surprising fact that, for $F$ a microcausal functional, it is not guaranteed that $X$ is a microcausal vector field, so that our candidate homotopy-operator does not close on the microcausal subcomplex. This is due to the fact that an integral of the form

$$\int_0^1 F^{(1)}(\gamma_\lambda \varphi) d\lambda \in \mathcal{E}'(M) \tag{4.4}$$

does not respect wavefront structure in general, so that unwanted singularities arise when performing the integration. We will give a detailed counterexample illustrating this problem in Section 4.2.

The work presented in this chapter is based in part on my joint publication with Eli Hawkins and Kasia Rejzner in [52]. The construction in section 4.1 is based on

[2] Technically, we should also precompose with the inclusion map Sol $\to \mathcal{E}$ but we leave that map implicit in the notation.

Proposition 4.3 in that text, which was one of my contributions to that publication. The counterexample presented in section 4.3 is taken more or less verbatim from the paper, with several clarifications and changes of notation. Identifying the problem, constructing the counterexample and writing out the details were my personal contributions. The original counterexample, which is given at the start of Section 4.3 worked using curves of distributions. It was Eli's suggestion that we use not just regular curves, but regular maps on $\mathbb{R}^d$, which considerably simplified the counterexample.

## 4.1 REGULAR MAPS AND FUNCTIONALS

Our counterexamples in this chapter are all constructed on finite dimensional domains, and then 'lifted' to the realm of functionals in order to obtain the counterexamples in the functional formalism. To make matters even easier, we will only construct *regular* functionals.

**Definition 4.1.1.** *A compactly supported functional $F \in C^\infty(\mathcal{E} \to \mathbb{C})$ is called **regular** if*

$$F^{(n)}(\varphi) \in \mathcal{D}(M^n; D_{M^n}) \quad \forall n \in \mathbb{N}, \varphi \in \mathcal{E}(M),$$

*where $D_{M^n}$ is the density bundle of $M^n$.*

In particular, regular functionals are microcausal. Analogously, we define the notion of regular map on a finite dimensional domain

**Definition 4.1.2.** *A smooth map $A : \mathbb{R}^n \to \mathcal{E}'(M)$ is called **regular** if*

$$\partial^\alpha A(x) \in \mathcal{D}(M) \quad \forall x \in \mathbb{R}^n, \alpha \text{ multi-index},$$

*and the closure of $\bigcup_{x\in\mathbb{R}^n} \operatorname{supp} A(x)$ is compact.*

We stress that, due to the fact that we require $A$ to be smooth with respect to the topology of $\mathcal{E}'(M)$ only, it will in general not be smooth as a function on $\mathbb{R}^n \times M$. This should be contrasted with, the fact that

$$C^\infty(\mathbb{R}^n \times M) \cong C^\infty(\mathbb{R}^n, C^\infty(M)),$$

see e.g. Theorem 40.1 and corollary in [72].

We give an explicit example of a regular curve, for $n = 1$ and $M = \mathbb{R}$, that is not a jointly smooth function. Let $\chi \in \mathcal{D}(\mathbb{R})$ be a nontrivial test function, and consider the curve given by

$$A(t)(x) = \begin{cases} \chi(x/t) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Clearly, this is smooth in $x$ when $t$ is fixed, but it is not continuous as a function of two variables, as $A(t)(x)$ is constant along lines where $x$ is proportional to $t$.

To show that it is continuous as a curve of distributions, we calculate, for $f \in \mathcal{E}(M)$:

$$\langle A(t), f\rangle = \int_{\mathbb{R}} \chi(x/t) f(x) dx = t \int_{\mathbb{R}} \chi(x) f(xt) dx.$$

As we have that

$$\left| \int_{\mathbb{R}} \chi(x) f(xt) dx \right| \leq \left( \sup_{y \in t \operatorname{supp} \chi} |f(y)| \right) \cdot \left| \int_{\mathbb{R}} \chi(x) dx \right|,$$

which is bounded as $t$ goes to zero, it follows that

$$\lim_{t \to 0} \langle A(t), f\rangle = 0 \quad \forall f \in \mathcal{E}(M)$$

One can show differentiability to all orders by expanding $f$ in a Taylor series with remainder.

If we have a regular functional $F$ and a linear embedding $\iota : \mathbb{R}^n \to \mathcal{E}(M)$, then it follows from the chain rule that the map

$$F^{(1)} \circ \iota : \mathbb{R}^n \to \mathcal{E}'(M)$$

is regular. In fact, all regular maps can be written in this form.

**Proposition 4.1.3.** *Let $A : \mathbb{R}^n \to \mathcal{E}'$ be regular. There exist a linear embedding $\iota_A : \mathbb{R}^n \to \mathcal{E}$ and a regular functional $F_A$ on $\mathcal{E}$ such that*

$$A = F_A^{(1)} \circ \iota_A.$$

*Furthermore, if $B : \mathbb{R}^m \to \mathcal{E}'$ is another regular map, then $F_A, F_B, \iota_A$ and $\iota_B$ can be chosen so that $F_A^{(1)}$ is constant along the image of $\iota_B$ and vice versa:*

$$A(x) = F_A^{(1)} \left(\iota_A(x) + \iota_B(y)\right), \tag{4.5}$$

$$B(y) = F_B^{(1)}(\iota_A(x) + \iota_B(y)), \tag{4.6}$$

*for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.*

*Proof.* The first statement is a direct consequence of the first one if $B$ is trivial, so that it suffices to prove the more general statement.

We choose $\varphi_1, \ldots, \varphi_n, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_m \in \mathcal{E}$ linearly independent, satisfying

$$\operatorname{supp}(\varphi_i) \cap \left( \bigcup_{x \in \mathbb{R}^n} \operatorname{supp}(A(x)) \cup \bigcup_{y \in \mathbb{R}^m} \operatorname{supp}(B(y)) \right) = \emptyset,$$

and similar for $\tilde{\varphi}_j$, which is possible by the assumption of compact support on $A$ and $B$. This implies in particular that

$$\partial^\alpha A(x)\{\varphi_i\} = \partial^\alpha A(x)\{\tilde{\varphi}_i\} = 0, \tag{4.7}$$

for any multi-index $\alpha$, and similar for $B$. We define

$$\begin{aligned} \iota_A(x) &= \sum_{i=1}^n x_i \varphi_i, \\ \iota_B(y) &= \sum_{j=1}^m y_j \tilde{\varphi}_j. \end{aligned}$$

Through a Gram–Schmidt procedure, we choose $\beta_i, \tilde{\beta}_j \in \mathcal{D}(M)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$, satisfying

$$\begin{aligned} \beta_i(\varphi_k) &= \delta_{ik}, \\ \tilde{\beta}_j(\tilde{\varphi}_l) &= \delta_{jl}, \\ \beta_i(\tilde{\varphi}_l) &= 0 = \tilde{\beta}_j(\varphi_k), \end{aligned}$$

where we view the $\beta$'s as distributions. This defines a left inverse to $\iota_A$ through

$$\vec{\beta}(\varphi) = (\beta_i(\varphi))_{i=1}^n.$$

We define $\vec{\tilde{\beta}}$ similarly.

We then set

$$\begin{aligned} F_A(\varphi) &= A(\vec{\beta}\varphi)\{\varphi\}, \\ F_B(\varphi) &= B(\vec{\tilde{\beta}}\varphi)\{\varphi\}. \end{aligned}$$

These are both smooth functionals by the chain rule. We show that $F_A$ is regular, the case for $F_B$ being similar. The first derivative of $F_A$ is

$$F_A^{(1)}(\varphi)\{h\} = A(\vec{\beta}\varphi)\{h\} + \sum_{i=1}^m \left(\beta_i(h)\partial_i A\right)(\vec{\beta}\varphi)\{\varphi\}. \tag{4.8}$$

This implies that $F_A^{(1)}(\varphi) \in \mathcal{D}(M)$, as $h$ gets smeared against either $\beta_i$ or $A(\vec{\beta}\varphi)$, both of which are smooth functions. At higher order, an induction argument shows that

$$F_A^{(k)}(\varphi)\{h^{\otimes k}\} = \left[\left(\sum_{i=1}^{n} \beta_i(h)\partial_i\right)^k A\right](\vec{\beta}\varphi)\{\varphi\} + (k-1)\left[\left(\sum_{i=1}^{n} \beta_i(h)\partial_i\right)^{k-1} A\right](\vec{\beta}\varphi)\{h\}. \tag{4.9}$$

The case $k = 1$ is clear. Suppose that we have shown this for $k$, then taking a derivative of equation (4.9) with respect to $\varphi$ in the direction $g \in \mathcal{E}(M)$ gives

$$\begin{aligned} F_A^{(k+1)}(\varphi)\{h^{\otimes k} \otimes g\} = &\left[\left(\sum_{j=1}^{n} \beta_j(g)\partial_j\right)\left(\sum_{i=1}^{n} \beta_i(h)\partial_i\right)^k A\right](\vec{\beta}\varphi)\{\varphi\} \quad + \\ &\left[\left(\sum_{i=1}^{n} \beta_i(h)\partial_i\right)^k A\right](\vec{\beta}\varphi)\{g\} \quad + \\ (k-1)&\left[\left(\sum_{j=1}^{n} \beta_j(g)\partial_j\right)\left(\sum_{i=1}^{n} \beta_i(h)\partial_i\right)^{k-1} A\right](\vec{\beta}\varphi)\{h\} \end{aligned}$$

Setting $g = h$ proves the inductive step. We see that in equation (4.9) $h$ gets smeared against either $\beta_i$ or some partial derivative of $A$, both of which are smooth functions. Hence $F_A$ is a regular functional.

Finally, combining equations (4.7) and (4.8) with the definitions of $\iota_A$ and $\iota_B$, we see that

$$F_A^{(1)}(\iota_A(x) + \iota_B(y))\{h\} = A(x)\{h\} + \sum_{i=1}^{m} \beta_i(h)\partial_i A(x)\{\iota_A(x) + \iota_B(y)\} = A(x)\{h\},$$

which is what we set out to prove. □

## 4.2 INTEGRATION COUNTEREXAMPLE

In this section, we will provide an explicit example of a regular functional, that exhibits the failure of the integral in (4.4) to define an element of $\mathcal{E}'_{\Gamma_1^c}(E^!)$. The general strategy is to first find a smooth curve of distributions that is regular, but has a singular integral. We then lift that curve to the realm of functionals using the tools developed in the previous section.

### 4.2.1 Construction of curves of distributions

We gather here some general results regarding explicit constructions of curves of complex valued distributions (regular or otherwise). If $u : \mathbb{R} \to \mathcal{E}'(M)$ is a smooth curve of distributions, we will often use the shorthand notations

$$u_t := u(t)$$
$$u_t^{(n)} := (\partial_t^n u)(t)$$

We start our discussion by considering curves of distributions on $\mathbb{R}^n$, so that we have access to the Fourier transform. We recall that if $u \in \mathcal{E}'(\mathbb{R}^n)$, then its Fourier transform is a smooth function, given by

$$\hat{u}(\xi) = u(e_\xi)$$

where $e_\xi(x) = \exp(-ix \cdot \xi)$, see e.g. Proposition 7.1.14 in [58]. If we now have a curve of distributions, we can perform Fourier transforms at each individual $t$, to find an assignment $(t, \xi) \mapsto \hat{u}_t(\xi)$. We showed this map to be smooth in Lemma 3.4.7.

**Lemma 4.2.1.** *If $B \subset \mathcal{E}'(\mathbb{R}^n)$ is bounded, then there exists a compact set $K \subset \mathbb{R}^n$, a constant $C > 0$ and an $N \in \mathbb{N}$, such that $\bigcup_{u \in B} \operatorname{supp}(u) \subset K$ and*

$$|\hat{u}(\xi)| \leq C \langle \xi \rangle^N \quad \forall u \in B, \tag{4.10}$$

*Proof.* As discussed in Section 2.1.4, the Banach-Steinhaus theorem implies that bounded subsets of $\mathcal{E}'(\mathbb{R}^n)$ are equicontinuous. Hence there is a compact set $K$, a positive $C$ and $N \in \mathbb{N}$ such that

$$|u(f)| \leq C p_{K,N}(f) = C \sum_{|\alpha| \leq N} \sup_{x \in K} |\partial^\alpha f(x)| \quad \forall u \in B. \tag{4.11}$$

If $f \in \mathcal{E}(\mathbb{R}^n)$ with $f|_K = 0$, then

$$|u(f)| \leq 0 \quad \forall u \in B,$$

so that indeed $\operatorname{supp}(u) \in K$ for all $u \in B$. Furthermore, inserting $f = e_\xi$ in equation (4.11), we find that

$$\hat{u}(\xi) \leq C \sum_{|\alpha| \leq N} |\xi^\alpha| \leq C' \langle \xi \rangle^N$$

for some $C' > 0$. □

**Proposition 4.2.2.** *If $u$ is a smooth curve in $\mathcal{E}'(\mathbb{R}^n)$ and if $I \subset \mathbb{R}$ is a bounded interval, then there are constants $C_n > 0$ and $M_n \in \mathbb{N}$ such that*

$$|\partial_t^n \hat{u}(t,\xi)| \leq C_n \langle \xi \rangle^{M_n} \quad \forall t \in I, \tag{4.12}$$

*where $\hat{u} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is the smooth function defined in Lemma 3.4.7.*

*Conversely, if $\alpha \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ allows for bounds as in equation (4.12), then it defines a smooth curve in $\mathcal{D}'(\mathbb{R}^n)$ by*

$$u_t(f) \coloneqq (2\pi)^{-n} \int_{\mathbb{R}^n} \alpha(t,\xi) \hat{f}(-\xi) d\xi \quad \forall f \in \mathcal{D}(\mathbb{R}^n), \tag{4.13}$$

*i.e. by taking the inverse Fourier transform of $\alpha$ with respect to $\xi$.*

The converse statement is somewhat unsatisfying, as we obtain a smooth curve in $\mathcal{D}'(\mathbb{R}^n)$, rather than $\mathcal{E}'(\mathbb{R}^n)$. This can be remedied by putting extra requirements on $\alpha$ stemming from the Payley-Wiener theorem, the most important one being that $\alpha$ can be extended holomorphically in $\xi$. As we will employ $\alpha$ that have compact support properties, this would be somewhat cumbersome. In any case, we can always multiply the curve by a test function after performing the inverse Fourier transform to enforce compact support.

*Proof.* The required bounds in equation (4.12) follow from Lemma 4.2.1, as $u^{(n)}$ maps bounded intervals to bounded sets of $\mathcal{E}'(\mathbb{R}^n)$.

For the converse statement, suppose that $\alpha(t,x)$ is a smooth function satisfying the properties in the theorem, and define $u_t(f)$ by equation (4.13), which is well-defined as $\hat{f}$ is of rapid decay. To check that $u_t$ takes values in $\mathcal{D}'(\mathbb{R}^n)$, consider a sequence $f_m \to 0$ in $\mathcal{D}(\mathbb{R}^n)$. This means that there is a compact set $K$ containing the supports of all the $f_m$, and

$$p_{N,K}(f_m) = \sup_{x \in K} \sum_{|\alpha| \leq N} |\partial^\alpha f_m(x)| \xrightarrow{m \to \infty} 0 \quad \forall N \in \mathbb{N}.$$

By assumption on $\alpha$ there are, for fixed $t$, constants $C$ and $M$ so that

$$|\alpha(t,\xi)| \leq C \langle \xi \rangle^{2M} = C(1 + \xi \cdot \xi)^M.$$

We now use a trick from the theory of pseudo-differential operator theory to estimate

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \alpha(t,\xi)\hat{f}_m(-\xi)d\xi\right| &\leq C\left|\int\int \langle\xi\rangle^{2M} e^{ix\cdot\xi}\cdot f_m(x)dxd\xi\right|,\\ &= C\left|\int\int \langle\xi\rangle^{-2N}(1+\xi\cdot\xi)^{M+N} e^{ix\cdot\xi}\cdot f_m(x)dxd\xi\right|,\\ &= C\left|\int\int \langle\xi\rangle^{-2N}(1-\partial_x\cdot\partial_x)^{M+N}\left(e^{ix\cdot\xi}\right)\cdot f_m(x)dxd\xi\right|,\\ &= C\left|\int\int \langle\xi\rangle^{-2N} e^{ix\cdot\xi}\cdot(1-\partial_x\cdot\partial_x)^{M+N} f_m(x)dxd\xi\right|,\\ &\leq C\left(\int \langle\xi\rangle^{-2N}d\xi\right)\mathrm{Vol}(K)p_{M+N,K}(f_m),\end{aligned}$$

which is finite for $N$ sufficiently large. It follows that

$$u_t(f_m) \xrightarrow{m\to\infty} 0,$$

and hence that $u_t \in \mathcal{D}'(\mathbb{R}^n)$ for all $t \in \mathbb{R}$.

We have to show that this assignment is smooth. As discussed in Section 2.1.4, a sequence in $\mathcal{D}'(\mathbb{R}^n)$ converges strongly iff it converges weakly, so that it suffices to check that

$$t \mapsto u_t(f)$$

is smooth for fixed $f \in \mathcal{D}(\mathbb{R}^n)$. Fix a bounded interval $I$ and constants $C_1$ and $M_1$ for $\alpha$ as in equation (4.12). Then we estimate

$$|\partial_t\alpha(t,\xi)\hat{f}(-\xi)| \leq C_1\langle\xi\rangle^{M_1}|\hat{f}(-\xi)| \quad \forall t \in I,$$

which is an integrable function as $\hat{f}$ is of rapid decay. Hence we can use the Leibniz integral formula to conclude that

$$\partial_t u_t(f) = \partial_t\int \alpha(t,\xi)\hat{f}(-\xi)d\xi = \int \partial_t\alpha(t,\xi)\hat{f}(-\xi)d\xi,$$

so that $u$ is $C^1$. Repeating the argument to arbitrary high orders implies that $u$ is smooth. □

We turn to the question of the singular structure of integrals of distributional curves. Given a smooth curve in $\mathcal{E}'(\mathbb{R}^n)$ and a bounded interval $I \subset \mathbb{R}$, we would like to understand the relation between the wavefront set of $\int_I u_t dt$ and that of the individual $u_t$ for $t \in I$. If $u_t$ is defined through a function $\alpha$ as in (4.13), then we calculate, for $f \in \mathcal{D}(\mathbb{R}^n)$,

$$\int_I u_t(f)dt = (2\pi)^{-n}\int_I\int_{\mathbb{R}^n} \alpha(t,\xi)\hat{f}(-\xi)d\xi dt = (2\pi)^{-n}\int_{\mathbb{R}^n}\left(\int_I \alpha(t,\xi)dt\right)\hat{f}(-\xi)d\xi. \tag{4.14}$$

The use of Fubini's theorem is justified by the polynomial bounds on $\alpha$, paired with the fact that $\hat{f}$ is of rapid decay.

We note that, if $\alpha(t,\xi)$ is of rapid decay in $\xi$ for all $t \in I$, then $u_t$ will be a smooth function. However, if we do not impose that $u_t$ is **uniformly** of rapid decay for $t \in I$, then it is not guaranteed that $\int_I \alpha(t,\xi)dt$ is of rapid decay. We leverage this discrepancy in the following lemma.

**Lemma 4.2.3.** *There exists a regular curve of distributions $u : \mathbb{R} \to \mathcal{E}'(\mathbb{R}^n)$, supported within $[0,1]$, such that*

$$\int_0^1 u_t dt = \delta_0,$$

*the Dirac delta distribution on $\mathbb{R}^n$.*

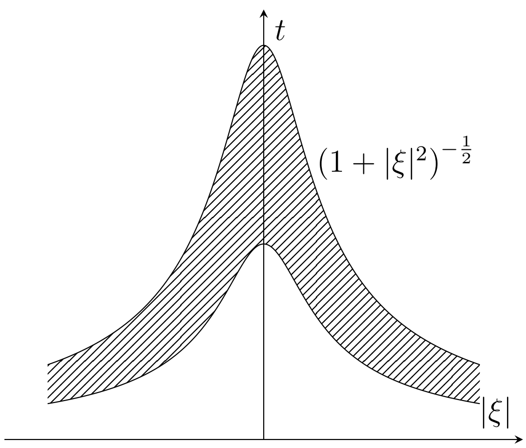
*Proof.* We select some positive non-zero $\chi \in \mathcal{D}(\mathbb{R})$, supported in $[\frac{1}{2}, 1]$, and normalised so that $\int_{\mathbb{R}} \chi(x)dx = 1$. We then set

$$\alpha(t,\xi) = \langle\xi\rangle\chi\left(\langle\xi\rangle t\right),$$

and we note that

$$\int_0^1 \alpha(t,\xi)dt = 1. \tag{4.15}$$

This function is supported for $t$ and $\xi$ such that $\sqrt{1+|\xi|^2}\cdot t \in [\frac{1}{2}, 1]$. In particular, for each fixed $\xi$, $\alpha$ is compactly supported in $t$, and vice versa.




Differentiating $\alpha$ with respect to $t$ yields an extra factor of $\langle\xi\rangle$, so that

$$|\partial_t^n \alpha(t,\xi)| \leq C_n \langle\xi\rangle^{n+1}. \tag{4.16}$$

Hence we may define a curve $\tilde{u}_t$ through equation (4.13). Furthermore, for each fixed $t$, the map $\xi \to \alpha(t,\xi)$ is compactly supported and hence of rapid decay, so that

$$\tilde{u}_t \in \mathcal{E}(\mathbb{R}^n) \quad \forall t \in \mathbb{R}.$$

Finally, we select $g \in \mathcal{D}(\mathbb{R}^n)$ which is 1 at 0, and define $u_t = g \cdot \tilde{u}_t$. By the above considerations, $u_t$ is a regular curve, and if $f \in \mathcal{E}(\mathbb{R}^n)$, then we find that

$$\int_0^1 u_t(f) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left( \int_0^1 \alpha(t,\xi)dt \right) \widehat{gf}(-\xi)d\xi = gf(0) = \delta_0(f)$$

by the Fourier inversion theorem. □

We give a general corollary to this result.

**Proposition 4.2.4.** *Let $w \in \mathcal{E}'(M)$. Then there exists a regular curve $v : \mathbb{R} \to \mathcal{E}'(M)$, supported within $[0,1]$, such that $w = \int_0^1 v_t dt$.*

*Proof.* By employing a partition of unity if necessary, we may assume that $w$ is defined on $\mathbb{R}^n$. Let $u_t$ denote the curve obtained in the previous lemma, and set $v_t = w * u_t$, which is a smooth curve as convolution is a continuous linear operation. As $u_t \in \mathcal{D}(M)$ for all $t$ and $w$ is compactly supported, it follows that $v_t$ is regular. Recalling that integrals commute with continuous operations, we calculate that

$$\int_0^1 v_t dt = w * \int_0^1 u_t dt = w * \delta_0 = w$$

□

### 4.2.2 IMPLICATIONS FOR RESOLUTION OF ON-SHELL FUNCTIONALS

We explain what the previous considerations mean for the resolution of the on-shell functionals as described at the start of this chapter. We recall that, if $F$ is a functional that vanishes on-shell, i.e. $\mathcal{F}|_S = 0$, then we showed in equation (4.3) that

$$F(\varphi) = X(\varphi)\{\tilde{\pi}_S \varphi\}$$

where $\tilde{\pi}_S$ is a projection on a complementary subspace to Sol. The vector field $X$ is defined by

$$X(\varphi) = \int_0^1 F^{(1)}(\pi_S(\varphi) + \lambda \tilde{\pi}_S \varphi)d\lambda.$$

We would like to understand how the singular structure of $X$ relates to that of $F$. As in the previous section, we show that, without further assumptions, $X$ can be singular, even if $F$ is a regular functional.

To see this, we select any $w \in \mathcal{E}'(M)$ and a curve $v$ as in Proposition 4.2.4, so that

$$w = \int_0^1 v_t dt.$$

Furthermore, we select a non-zero function $\beta \in \mathcal{D}(M)$, such that $\beta \circ \pi_S = 0$ as a distribution.[3] We define a functional as in Proposition 4.1.3, by

$$F(\varphi) = v_{\beta(\varphi)}(\varphi) \tag{4.17}$$

which we have shown to be regular.

Clearly, $F$ vanishes on Sol, due to the fact that $v_0 = 0$. We calculate

$$\beta(\pi_S\varphi + \lambda\tilde{\pi}_S\varphi) = \lambda\beta(\tilde{\pi}_S\varphi) = \lambda\beta(\pi_S\varphi + \tilde{\pi}_S\varphi) = \lambda\beta(\varphi),$$

and

$$F^{(1)}(\varphi) = v_{\beta(\varphi)}(\varphi)\beta(\_) + v_{\beta(\varphi)}(\_). \tag{4.18}$$

We can now calculate $X$ explicitly:

$$X(\varphi)(\_) = \left(\int_0^1 v_{\lambda\beta(\varphi)}(\pi_S\varphi + \lambda\tilde{\pi}_S\varphi)d\lambda\right)\beta(\_) + \int_0^1 v_{\lambda\beta(\varphi)}(\_)d\lambda$$

The first term here will be a smooth distribution, as the integral just provides some numerical factor. For the second term, we change our integration variable to $s = \lambda\beta(\varphi)$ and find

$$X(\varphi) \cong \beta(\varphi)^{-1}\int_0^{\beta(\varphi)} v_s ds,$$

where we have used $\cong$ to indicate equality up to smooth terms. We see that, for $\beta(\varphi) \geq 1$, we get

$$X(\varphi) \cong \frac{w}{\beta(\varphi)}.$$

As $w$ is arbitrary, apart from the fact that it needs to be compactly supported, it can have non-trivial wavefront set. Hence we see that we have constructed a regular

[3]This is always possible by use of the Hahn-Banach theorem: Pick some $\varphi_0 \in \mathcal{E}(M)$ such that $\varphi_0 \notin S$, and view $S \oplus \mathbb{C}\varphi_0$ as a subspace of $\mathcal{D}'(M)$. Then we set $\beta|_S = 0$ and $\beta(\varphi_0) = 1$. The Hahn-Banach theorem then implies that we can extend $\beta$ to an element of $(\mathcal{D}'(M))' \cong \mathcal{D}(M)$, see e.g. Proposition 34.4 and Proposition 36.9 in [72].

functional $F$, that vanishes on-shell, such that $F = \delta_S(X)$ for $X$ a non-regular vector field.

Of course, it is not clear whether there might exist some other vector field $\tilde{X}$, satisfying $F = \delta_S(\tilde{X})$, but such that $\tilde{X}$ **is** regular, allowing us to exchange $X$ for an equivalent vector field that is better behaved.

However, it is our aim to use the prescription in equation (4.3) to serve as a starting point for defining a homotopy operator in the next chapter. For this reason, it is a serious problem that this prescription can map a microcausal functional to a vector field that is not microcausal, as this obstructs the calculation of the homology of the microcausal version of the complex in an efficient way.

## 4.3 POISSON BRACKET COUNTEREXAMPLE

In the previous section, we showed that microcausal functionals are poorly behaved from a homological viewpoint, because they do not allow for certain homotopy operators to be well-defined. While this is definitely an awkward feature, it is not detrimental for the goal of defining models in pAQFT, it is merely unclear whether certain properties hold. In that light, the counterexample presented in this section is more serious, as it shows that the microcausal functionals do not close under the Poisson bracket or $\star$-product.

As the full counterexample is somewhat convoluted, we first present a toy counterexample, that nevertheless captures the most important points. We work on $\mathbb{R}$ rather than a general manifold, we consider only regular functionals, and we exchange the causal propagator for a $\delta$ distribution, so that

$$\{F, G\}_\delta(\varphi) = \int_{\mathbb{R}} F^{(1)}(\varphi)(x) G^{(1)}(\varphi)(x) dx.$$

As both $F^{(1)}(\varphi)$ and $G^{(1)}(\varphi)$ are test functions, this pairing is well-defined, but we shall show that it need not depend continuously on $\varphi \in \mathcal{E}(\mathbb{R})$. We first define a map

$$A : \mathbb{R} \to \mathcal{D}(\mathbb{R}),$$

by

$$A(\lambda)(x) = \begin{cases} \exp(-ix/\lambda)\chi(x) & \lambda \neq 0, \\ 0 & \lambda = 0. \end{cases} \tag{4.19}$$

for $\chi \in \mathcal{D}(\mathbb{R})$ some cutoff function. Clearly this map is discontinuous at $\lambda = 0$. However, if $f \in \mathcal{E}(\mathbb{R})$, then

$$\langle A(\lambda), f \rangle = \begin{cases} \widehat{\chi f}(\frac{1}{\lambda}) & \lambda \neq 0, \\ 0 & \lambda = 0. \end{cases}$$

As $\chi f$ is smooth and compactly supported, its Fourier transform is of rapid decay, so that $\lambda \mapsto \langle A(\lambda), f \rangle$ is a smooth map. This implies that $A$ is smooth with respect to the topology of $\mathcal{E}'(\mathbb{R})$.

We pick a reference configuration $\varphi_0 \in \mathcal{D}(\mathbb{R})$, normalised so that the $L^2$-norm $\|\varphi_0\|_2 = 1$, and such that $\operatorname{supp} \varphi_0 \cap \operatorname{supp} \chi = \emptyset$. We then define a pair of functionals by

$$F_\pm(\varphi) = \Big\langle A(\pm\langle \varphi_0, \varphi \rangle)\,,\, \varphi \Big\rangle.$$

We explicitly calculate its first derivative using the Leibniz rule:

$$\begin{aligned} F_\pm^{(1)}(\varphi)(x) =& A(\pm\langle \varphi_0, \varphi \rangle)(x) \\ & \pm \Big\langle \partial_\lambda A(\pm\langle \varphi_0, \varphi \rangle)\,,\, \varphi \Big\rangle \cdot \varphi_0(x), \end{aligned}$$

which is an element of $\mathcal{D}(\mathbb{R})$ for all $\varphi \in \mathcal{E}$. Inserting $\varphi = s\varphi_0$, we obtain

$$F_\pm^{(1)}(s\varphi_0) = A(\pm s).$$

We then calculate the bracket of $F_+$ with $F_-$ on the ray $\{s\varphi_0\}$:

$$\{F_+, F_-\}_\delta(s\varphi_0) = \int_{\mathbb{R}} A(s)(x)A(-s)(x)dx = \begin{cases} \int_{\mathbb{R}} \chi(x)^2 dx & s \neq 0, \\ 0 & s = 0, \end{cases}$$

which is clearly discontinuous if we take $\chi$ to be real and non-zero.

We move to the full counterexample. Again we simplify our discussion by working with regular functionals. Furthermore, we generalise the problem to work with any kind of distributional kernel, rather than just with the commutator function. That is, suppose that we have an element $W \in \mathcal{D}'(M \times M)$, where $M$ is any d-dimensional manifold (not necessarily a spacetime). We define a bracket of regular functionals as follows:

$$\{F, G\}_W(\varphi) = W(F^{(1)}(\varphi) \otimes G^{(1)}(\varphi)), \tag{4.20}$$

which is well-defined as both $F^{(1)}(\varphi)$ and $G^{(1)}(\varphi)$ are elements of $\mathcal{D}(M)$. The functional $\{F, G\}_W$ is in general poorly behaved:

**Theorem 4.3.1.** *The bracket* $\{F, G\}_W$ *is a smooth functional for all regular functionals* $F$ *and* $G$ *if and only if* $\mathrm{WF}(W) = \emptyset$.

We first simplify the problem by proving an analogous result about regular maps on finite dimensional domains.

**Lemma 4.3.2.** *If* $\mathrm{WF}(W) \neq \emptyset$*, then there exists a pair of regular maps*

$$A : \mathbb{R}^d \to \mathcal{E}'(M),$$
$$B : \mathbb{R}^d \to \mathcal{E}'(M),$$

*such that the map* $(x, y) \mapsto W(A(x) \otimes B(y))$ *is not continuous.*

*Proof.* Let $(x_1, x_2; \eta_1, \eta_2) \in \mathrm{WF}(W)$, and select coordinate charts defined on neighbourhoods $U_i$ of the $x_i$. We recall that the wavefront set on manifolds is defined with respect to an arbitrary choice of coordinates, and that all choices are equivalent. Hence we may identify the $U_i$ with open subsets of $\mathbb{R}^d$, and $T_{x_i}M$ with $\mathbb{R}^d$. In order not to overburden the notation, we make these identifications implicitly throughout this proof.

We recall the following functions on $U_1$, for $\xi_1 \in \mathbb{R}^d$

$$e_{\xi_1}(x) = \exp(-ix \cdot \xi_1).$$

We define $f_{\xi_2}$ similarly on $U_2$. As $(x_1, x_2; \eta_1, \eta_2) \in \mathrm{WF}(W|_{U_1 \times U_2})$, there are test functions $\chi_i \in \mathcal{D}(U_i)$ such that $\chi_i(x_i) \neq 0$, and such that

$$\Phi : \xi = (\xi_1, \xi_2) \mapsto W(\chi_1 e_{\xi_1} \otimes \chi_2 f_{\xi_2}),$$

is not of rapid decay on any conic neighbourhood of $(\eta_1, \eta_2)$, where we view $\chi_1 e_{\xi_1}$ and $\chi_2 f_{\xi_2}$ as functions on the whole of $M$ by extending by 0 outside of $U_i$.

In particular, $\Phi$ is not of rapid decay on $\mathbb{R}^{2d}$, so that there is $N \in \mathbb{N}$ such that $|\xi|^N \Phi(\xi)$ is unbounded. This in turn implies, by the binomial theorem, that there are $K, L \in \mathbb{N}$ such that $|\xi_1|^K |\xi_2|^L \Phi(\xi)$ is unbounded.

We set

$$A(\xi_1) = \begin{cases} |\xi_1|^{-K} \chi_1 e_{\xi_1/|\xi_1|^2} & \xi_1 \neq 0, \\ 0 & \xi_1 = 0, \end{cases}$$

and

$$B(\xi_2) = \begin{cases} |\xi_2|^{-L} \chi_2 f_{\xi_2/|\xi_2|^2} & \xi_2 \neq 0, \\ 0 & \xi_2 = 0. \end{cases}$$

Then the map $(\xi_1, \xi_2) \mapsto W(A(\xi_1), B(\xi_2))$ is unbounded on any neighbourhood of $(0,0)$, and hence can not be continuous at that point. The proof will therefore be complete once we show that $A$ and $B$ are regular maps. We show this for $A$, as the case for $B$ is similar.

We first check that $A$ defines a smooth curve into $\mathcal{E}'(U_1)$. It suffices to check this in the weak topology, as sequences in $\mathcal{E}'(U_1)$ converge weakly if and only if they converge strongly. We calculate, for $\xi_1 \neq 0$ and $f \in \mathcal{E}(U_1)$ that

$$\Big\langle A(\xi_1)\,,\, f \Big\rangle = |\xi_1|^{-K} \widehat{\chi_1 f}\left(\frac{\xi_1}{|\xi_1|^2}\right). \tag{4.21}$$

We of course also have $\langle A(0), f\rangle = 0$. Let $\alpha$ be any multi-index, and let $\xi_i \neq 0$, we calculate

$$\partial^\alpha \Big\langle A(\xi_1)\,,\, f \Big\rangle = \sum_{\gamma+\delta=\alpha} \binom{\alpha}{\gamma} \partial^\gamma \left(|\xi_1|^{-K}\right) \cdot \partial^\delta \left(\widehat{\chi_1 f}\left(\frac{\xi_1}{|\xi_1|^2}\right)\right). \tag{4.22}$$

We can bound

$$\partial^\gamma \left(|\xi_1|^{-K}\right) = O(|\xi_1|^{-K-|\gamma|}) \text{ as } \xi_1 \to 0.$$

The second factor in equation (4.22) is a little more involved, as we need to work out the repeated derivatives. Using Faà di Bruno's formula, this leads to terms of the form

$$\prod_{i=1}^k \partial^{\epsilon_i} \left(\frac{\xi_1}{|\xi_1|^2}\right) \cdot \partial^{\epsilon_0} \widehat{\chi_1 f}\left(\frac{\xi_1}{|\xi_1|^2}\right) = \prod_{i=1}^k \partial^{\epsilon_i} \left(\frac{\xi_1}{|\xi_1|^2}\right) \cdot (-i)^{|\epsilon_0|} \widehat{x^{\epsilon_0}\chi_1 f}\left(\frac{\xi_1}{|\xi_1|^2}\right),$$

where $\sum |\epsilon_i| = |\delta|$ and $k = |\epsilon_0| \leq |\delta|$. We bound

$$\partial^{\epsilon_i} \frac{\xi_1}{|\xi_1|^2} = O(|\xi_1|^{-1-|\epsilon_i|}) \text{ as } \xi_1 \to 0,$$

to find

$$\prod_{i=1}^k \partial^{\epsilon_i} \left(\frac{\xi_1}{|\xi_1|^2}\right) = O(|\xi_1|^{-k-\delta}) = O(|\xi_1|^{-2\delta}) \text{ as } \xi_1 \to 0.$$

Finally, as $x^{\epsilon_0}\chi_1 f$ is compactly supported and smooth, its Fourier transform is of rapid decay, so that

$$\widehat{x^{\epsilon_0}\chi_1 f}\left(\frac{\xi_1}{|\xi_1|^2}\right) = O(|\xi_1|^P) \text{ as } \xi_1 \to 0 \;\forall P \in \mathbb{N}.$$

This implies that

$$\lim_{\xi_1 \to 0} \Big\langle \partial^\alpha A(\xi_1)\,,\, f \Big\rangle = 0, \tag{4.23}$$

for all multi-indices $\alpha$.

Clearly this implies that $\xi_1 \mapsto \langle A(\xi_1), f\rangle$ is continuous at $\xi_1 = 0$. To show that it is $C^1$, we note that it follows from the mean value theorem that

$$\Big\langle A(s\xi_1) - A(0)\,,\, f\Big\rangle = \Big\langle \nabla A(\eta)\,,\, f\Big\rangle \cdot s\xi_1$$

for some $\eta$ on the line connecting 0 to $\xi_1$. Hence we have the inequality

$$\left|\frac{\langle A(s\xi_1), f\rangle - \langle A(0), f\rangle}{s}\right| \leq \sup_{|\eta|\leq s\xi_1|} \left|\Big\langle \nabla A(\eta)\,,\, f\Big\rangle \cdot \xi_1\right| \xrightarrow{s\to 0} 0,$$

by equation (4.23), so that $\xi_1 \mapsto \langle A(\xi_1), f\rangle$ is continuously differentiable at $\xi_1 = 0$. Higher orders are shown analogously.

As $\mathcal{E}'(U_1)$ injects continuously into $\mathcal{E}'(M)$, we conclude that $A$ is a smooth into $\mathcal{E}'(M)$, with

$$\partial^\alpha A(0) = 0 \quad \forall \alpha,$$

which is in particular an element of $\mathcal{D}(M)$. It is clear from the definition that $\partial^\alpha A(\xi_1) \in \mathcal{D}(M)$ for all $\xi_1 \neq 0$, so that $A$ is a regular map. □

*Proof of Theorem 4.3.1.* If $\mathrm{WF}(W) \neq \emptyset$, then there exist regular maps $A, B$ as in the previous lemma. By Proposition 4.1.3, there are regular functionals $F_A$ and $F_B$ and embeddings $\iota_A, \iota_B : \mathbb{R}^d \to \mathcal{E}(M)$ so that

$$\begin{aligned} A(x) &= F_A^{(1)}(\iota_A(x) + \iota_B(y)), \\ B(y) &= F_B^{(1)}(\iota_A(x) + \iota_B(y)). \end{aligned}$$

Hence it follows that

$$\{F_A, F_B\}_W \circ (\iota_A + \iota_B) : (x, y) \mapsto W(A(x), B(y))$$

is not continuous. As $\iota_A + \iota_B$ is continuous, this implies that $\{F_A, F_B\}_W$ can not be continuous.

Conversely, suppose that $\mathrm{WF}(W) = \emptyset$, which means that $W \in \mathcal{E}(M^2)$. The bracket of two regular functionals $F, G$ can be viewed as the following composition of smooth maps:

$$\mathcal{E}(M) \xrightarrow{F^{(1)}\times G^{(1)}} \mathcal{E}'(M) \times \mathcal{E}'(M) \xrightarrow{\otimes} \mathcal{E}'(M^2) \xrightarrow{W} \mathbb{C},$$

where we use that the tensor product of distributions is continuous, see Section 2.1.4, and that $F^{(1)}$ and $G^{(1)}$ are smooth functionals into $\mathcal{E}'(M)$ by Proposition 2.3.5. □

We stress that the counterexamples presented in this chapter are not a consequence of the precise form of the singular structure imposed on the derivatives of microcausal functionals, or of the fact that we work with open cones, rather than closed ones. Rather, they are a consequence of the pointwise fashion in which the singular structure is imposed in the definition of microcausal functional. Indeed, our counterexample required only regular functionals, which shows that the derivatives of functionals need not have **any** singularities in order for these pathological features to arise. We would expect similar problems to arise when working with 'refined' notions of wavefront set, such as Sobolev or Besov wavefront sets, so long as the topology of those spaces is not suitably taken into account.

— 5 —

# Graded functionals

In this chapter, we establish a theory of functionals on a graded configuration space. This builds on the work by Fredenhagen and Rejzner in [40, 41], where it was shown how to implement the BV formalism in pAQFT. The main result presented in this chapter is Theorem 5.6.3, which gives a prescription on how to lift retracts of field complexes to retracts of algebras of graded functionals. A special case of this appeared already in [52]. In this chapter we endeavour to give a more general version, which provides a tool for proving that the differential graded algebras used in [41] give resolutions of the algebra of functionals on the physical configuration space. Sharpening and applying these tools is work in progress with Eli Hawkins and Kasia Rejzner.

We return to the example discussed at the start of Chapter 4. Suppose that we are interested in describing functionals on the space of solutions of a differential operator $P$, acting on the space of functions $C^\infty(M)$. We denote the space of solutions by

$$\mathrm{Sol} = \ker(P).$$

From a homological viewpoint, we can describe Sol as the homology in degree 0 of the cochain complex

$$C := \left(0 \to C^\infty(M) \xrightarrow{P} C^\infty(M) \to 0\right)$$

concentrated in degrees 0 and 1. We will show how, under some technical assumptions, this feature can be lifted to the realm of functionals, so that the functionals on Sol are resolved by 'the functionals on $C$'.

For this purpose, we should define what we mean by functionals on a graded space of configurations. The approach taken in e.g. [23] and [9] is to consider the symmetric algebra of the dual (or of the smooth elements in the dual) of the graded space of configurations. As we are interested in bona-fide functionals, i.e. not just in polynomials, the graded symmetric tensor algebra is insufficient for this purpose. The approach taken in [41] is to rather look at functionals valued in the symmetric algebra with respect to the fields in non-zero degree only. This way, they obtain two kinds of coordinates; In the degree zero coordinates, general smooth behaviour is allowed. In the others, only polynomial behaviour is allowed. We largely follow this second approach, but give a description which is more convenient for our purposes.

## 5.1 SOME CONCEPTS FROM HOMOLOGICAL ALGEBRA

We use this section to recall some basic concepts from homological algebra, and set out some notation and conventions. We refer the interested reader to [75] for a pedagogical introduction to these concepts.

A **graded vector space** is a vector space $V$ that allows for a split into graded pieces:

$$V = \bigoplus_{i \in \mathbb{Z}} V_i.$$

The elements of $V_i$ are called **homogenous** elements of degree $i$. We call $V$ **bounded** if only finitely many $V_i$ are non-zero.

A linear map $\eta : V \to W$ between two different graded vector spaces is said to be of **degree** $n$ if

$$\eta(V_i) \subset W_{i+n},$$

i.e. if raises the degree by $n$. We denote the space of all such maps by $\mathrm{Hom}^n(V, W)$, and we denote the internal Hom space by

$$\mathrm{Hom}(V, W) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}^n(V, W).$$

We note that it this is strictly smaller than the space of all linear maps from $V \to W$, as that is isomorphic to the product $\prod_{n \in \mathbb{Z}} \mathrm{Hom}^n(V, W)$. This is a point of awkwardness when discussing graded locally convex spaces, as the direct sum can be more awkward to work with in practice. However, these notions coincide when $V$ and $W$ are bounded.

As a special case, we can view $\mathbb{C}$ as a graded vector space supported in degree 0. The **graded linear dual** of $V$ is then defined to be $V^* = \mathrm{Hom}(V, \mathbb{C})$. We note that with this definition $(V^*)_i \cong (V_{-i})^*$, i.e. dualising flips the grading.

A **cochain complex** is a graded vector space endowed with a linear map $d$ of degree 1, called the differential, such that $d^2 = 0$. We will on occasion use the notation $d_i = d|_{V_i}$ if we want to focus on a specific graded piece. Dually, a **chain complex** is a graded vector space endowed with a linear map of degree $-1$ satisfying the same relation. These two notions are of course related by suitably relabelling the objects. Applications of homotopy theory to field theory are usually phrased in terms of cochain complexes, which is a convention to which we adhere.[1]

The **cohomology** $H^\bullet(V)$ of a cochain complex $(V, d)$ is the graded vector space given by

$$H^i(V) = \operatorname{Ker} d_i / \operatorname{Im} d_{i-1}.$$

The elements of $\ker d_i$ are called **closed**, and the elements of $\operatorname{Im} d_{i-1}$ are called **exact**. As such, the cohomology measures the failure of the closed elements to be exact. If all the cohomology spaces are trivial, then we call the complex $(V, d)$ **exact**.

If $(V, d_V)$ and $(W, d_W)$ are cochain complexes then $\operatorname{Hom}(V, W)$ allows for a natural differential by setting, for $f \in \operatorname{Hom}^n(V, W)$

$$\partial f = d_W \circ f - (-1)^n f \circ d_V,$$

and extending linearly to $\operatorname{Hom}(V, W)$. It is a straightforward check that $\partial^2 = 0$ from the fact that $d_V$ and $d_W$ are differentials.

A **morphism of cochain complexes** (also called a cochain morphism or cochain map) between $V$ and $W$ is an element $f \in \operatorname{Hom}^0(V, W)$ such that

$$\partial f = d_W \circ f - f \circ d_V = 0.$$

Because of this relation, a chain morphism induces a well-defined map on the cohomology spaces by

$$H^\bullet(f)[v] = [fv].$$

This exhibits that taking cohomology is a functor from the category of cochain complexes with cochain morphisms to the category of graded vector spaces with linear maps of degree zero.

The general philosophy of homological algebra is that a cochain complex is an 'extended model' of the object you want to study, which you re-obtain when passing

[1] Our convention of lower indices indicating the grading is homological rather than cohomological. However, as we use superscripts to indicate both duals and repeated tensor products, we feel that using a weird convention for our indices is the lesser of two evils as it reduces notational clutter significantly.

to cohomology. For example, if $W$ is a vector space, then a **resolution** of $W$ is a cochain complex $(V, d)$ such that

$$H^n(V) \cong \begin{cases} W & n = 0, \\ 0 & n \neq 0. \end{cases}$$

Usually the pieces of the complex $V$ are chosen to be easier to handle than the space $W$ itself. As such, working with a resolution is a trade-off, where one chooses to work with many simpler objects, rather than with one complicated object.

It is customary to identify morphisms that match in cohomology. Two cochain morphisms $f_1$ and $f_2$ from $V$ to $W$ are called quasi-equivalent if

$$H^\bullet(f_1) = H^\bullet(f_2).$$

If this is the case, we shall write $f_1 \sim f_2$. A cochain map $f \in \mathrm{Hom}^0(V, W)$ is called a **quasi-isomorphism** if $H^\bullet(f)$ is an isomorphism of graded vector spaces. A **quasi-inverse** to $f$ is a map $g \in \mathrm{Hom}^0(W, V)$ so that $H^\bullet(g)$ is the inverse to $H^\bullet(f)$.

In practice, showing that two maps are quasi-equivalent is most easily done by finding a **chain homotopy**. This is a map $\eta \in \mathrm{Hom}^{-1}(V, W)$ satisfying

$$f_1 - f_2 = \partial\eta.$$

This map is a sort of witness to the fact $f_1$ and $f_2$ are quasi-equivalent, as $H^\bullet(\partial\eta) = 0$. In this case we write $f_1 \overset{\eta}{\sim} f_2$. A special case of a quasi-equivalence that comes up often is a retract, see e.g. [25].

**Definition 5.1.1.** *Let $(V, d_V)$ and $(W, d_W)$ be cochain complexes. A strong deformation retract of $V$ onto $W$ is a diagram*

$$W \underset{\iota}{\overset{\pi}{\leftrightarrows}} V \circlearrowleft \eta$$

*where $\pi$ and $\iota$ are morphisms of cochain complexes, $\eta$ is of degree $-1$, and*

$$\pi \circ \iota - \mathbb{1}_W = 0,$$
$$\iota \circ \pi - \mathbb{1}_V \overset{\eta}{\sim} 0.$$

*Furthermore, the maps $\pi, \iota$ and $\eta$ satisfy the conditions*

$$\pi \circ \eta = 0$$
$$\eta \circ \iota = 0$$
$$\eta \circ \eta = 0$$

The last three conditions are sometimes called the *side-conditions*. They can always be satisfied by changing $\pi, \iota$ and $\eta$, so long as they satisfy the first two conditions.

### 5.1.1 The graded symmetric algebra

The tensor product of two graded vector spaces can be endowed with a natural grading, by imposing that degrees should be additive on taking the tensor product. Explicitly, for $V$ and $W$ graded vector spaces, we set

$$(V \otimes W)_i = \bigoplus_{n+m=i} V_n \otimes W_m.$$

We use this tensor product to define the graded symmetric algebra of a graded vector space $V$.

The $n$-fold tensor power is defined by

$$T^n V = V^{\otimes n},$$

and we set $T^0 V = \mathbb{C}$. The full tensor algebra is then defined as the direct product

$$TV = \prod_{n \in \mathbb{N}} T^n V.$$

We say that elements of $T^n V$ are of **weight** $n$, to distinguish this grading from the one inherent to $V$. A direct sum is more customary in most treatments, but we prefer the direct product as it is more convenient in the context of the completed topological tensor product below. This unusual definition can cause problems when discussing operations involving $TV$ that mix different weights, as infinite summations might arise. We will comment on this when the need arises. [2]

The product of this algebra is nothing but the tensor product

$$\cdot : T^n V \times T^m V \to T^{n+m} V,$$

linearly extended to the whole of $TV$. We denote this with a dot, or simply by concatenation, to avoid overburdening the $\otimes$ symbol, which we reserve to indicate tensor products of vector spaces.

[2] We note that this means that our symmetric tensor algebra is technically speaking not a graded vector space as defined in this text, as it is a direct product of graded pieces rather than a direct sum. If we take the viewpoint that a graded vector space is really just a collection of graded pieces, this is inconsequential.

To pass to the graded symmetric algebra on $V$, we impose a graded commutation rule. If $v_1, v_2 \in V$ are of homogeneous degree, the we impose that

$$v_1 \cdot v_2 = (-1)^{|v_1||v_2|} v_2 \cdot v_1. \tag{5.1}$$

More generally, there is an action of $S_n$, the symmetric group of $n$ elements, on $T^n V$ by

$$\sigma(v_1 \cdot \ldots \cdot v_n) = \epsilon_\sigma v_{\sigma^{-1}(1)} \cdot \ldots \cdot v_{\sigma^{-1}(n)}.$$

Here $\epsilon_\sigma \in \{-1, 1\}$ is called the graded signature of $\sigma$ on $v_1, \ldots, v_n$, which is fixed by equation (5.1) together with a factorisation of $\sigma$ into transpositions of adjacent elements.

We then set

$$\mathrm{Sym}(V) = \prod_{n \in \mathbb{N}} \mathrm{Sym}^n(V) = \prod_{n \in \mathbb{N}} (V^{\otimes n})_{S_n},$$

where the lower index $S_n$ denotes coinvariants, i.e. we quotient by the action of $S_n$ at every weight. The product on this algebra is the one induced by the product on $TV$, which we also denote by a dot.

Let $\eta \in \mathrm{Hom}^n(V, V)$, we can extend the action of this map to $\mathrm{Sym}(V)$ by the graded Leibniz rule. If $v \in V = \mathrm{Sym}^1(V)$ then we set $\mathrm{der}(\eta)v = \eta v$. If $v, w \in V$ are homogeneous then we impose

$$\mathrm{der}(\eta)(v \cdot w) = \mathrm{der}(\eta)v \cdot w + (-1)^{n|v|} v \cdot \mathrm{der}(\eta)w. \tag{5.2}$$

As $V$ generates $\mathrm{Sym}(V)$, this rule defines $\mathrm{der}(\eta)$ uniquely as a map of degree $n$ on $\mathrm{Sym}(V)$.

If $\eta$ and $\zeta$ are two graded morphisms of $V$, of degree $m$ and $n$ respectively, then their **graded commutator** is

$$[\eta, \zeta] = \eta \circ \zeta - (-1)^{mn} \zeta \circ \eta.$$

The operation of turning a graded map into a derivation respects graded commutators, i.e.

$$\mathrm{der}([\eta, \zeta]) = [\mathrm{der}(\eta), \mathrm{der}(\zeta)].$$

This fact is easily checked on $V \subset \mathrm{Sym}(V)$, and can then be extended to higher weights using the graded Leibniz rule.

Consequently, if $(V, d)$ is a cochain complex, then $(\mathrm{Sym}(V), \mathrm{der}(d))$ is also a cochain complex, as

$$\mathrm{der}(d)^2 = \frac{1}{2}[\mathrm{der}(d), \mathrm{der}(d)] = \frac{1}{2}\,\mathrm{der}([d, d]) = 0.$$

As the differential der($d$) respects the product of Sym($V$), these data exhibit a differential graded algebra. If $f : V \to W$ is a cochain map, then taking $n$-fold tensor products defines maps

$$f^{\otimes n} : \mathrm{Sym}^n(V) \to \mathrm{Sym}^n(W).$$

One checks using the graded Leibniz rule that this is a chain map. We define Sym($f$) as the direct product of these maps, which is an algebra morphism, so that Sym defines a functor from cochain complexes to differential graded algebras.

Finally, we discuss the coalgebra structure of the symmetric algebra. This is a collection of linear maps

$$\Delta : \mathrm{Sym}^n(V) \to \prod_{p+q=n} \mathrm{Sym}^p(V) \otimes \mathrm{Sym}^q(V)$$

satisfying $\Delta(v) = v \otimes \mathbb{1} + \mathbb{1} \otimes v$, and which a homomorphism with respect to the product given on homogeneous terms by

$$(v_1 \otimes w_1) \cdot (v_2 \otimes w_2) = (-1)^{|w_1||v_2|}(v_1 v_2 \otimes w_1 w_2).$$

Roughly speaking, this map labels all the ways one might 'undo' the multiplication of Sym($V$). These maps will be of interest below because their adjoints together define a product on the *dual* of Sym($V$).

If $v_1, \ldots, v_n \in V$ are homogeneous, then the coproduct can be explicitly expressed as

$$\Delta(v_1 \ldots v_n) = \sum_{k+l=n} \sum_{\sigma \in \mathrm{Sh}(k,l)} \epsilon_\sigma v_{\sigma_1} \ldots v_{\sigma_k} \otimes v_{\sigma_{k+1}} \ldots v_{\sigma_{k+l}},$$

where Sh($k, l$) denotes the $(k, l)$ shuffles, i.e. those permutations of $(k + l)$ with $\sigma_1 < \ldots < \sigma_k$ and $\sigma_{k+1} < \ldots < \sigma_{k+l}$. The sign factor $\epsilon_\sigma \in \{-1, 1\}$ can be worked out by adding a factor $(-1)^{|v_i||v_j|}$ whenever adjacent $v_i$ and $v_j$ are exchanged. Explicitly, it is given by $(-1)^N$, where

$$N = \sum_{i=1}^{l} |v_{\sigma_{k+i}}| \cdot \sum_{\substack{j > \sigma_{k+i} \\ j \in \sigma\{1,\ldots,k\}}} |v_j|. \tag{5.3}$$

Due to our unusual definition of the symmetric tensor algebra as a direct product of the pieces of homogeneous weight, this does **not** define a map

$$\mathrm{Sym}(V) \to \mathrm{Sym}(V) \otimes \mathrm{Sym}(V)$$

at this point, because direct products and tensor products of vector spaces can not be interchanged when dealing with infinite dimensional spaces. We remedy this situation below by going over to completed tensor products.

### 5.1.2 Homotopy theory of topological vector spaces

The concepts exhibited in this section so far allow for extensions to more analytical settings. There does not appear to be a clear consensus in the literature on the 'right' approach. Notably, we refer the reader to Section 7.1 in [23] for an approach using convenient vector spaces. We take a pragmatic approach, and only develop the theory that we use in this work.

From this point on, we require all vector spaces to be LCTVSs, and all maps to be continuous linear maps. Graded locally convex spaces and locally convex cochain complexes are defined in the obvious fashion. We assume that all graded locally convex spaces are bounded, so as to circumvent awkward situations where direct products and sums do not match as we are working with a finite index set. The field complexes that we study later all satisfy this requirement. Notably, the symmetric tensor algebra does not, unless it is defined on a cochain complex supported in degree 0, but we will treat this case separately.

The strong dual of a graded locally convex space $V$ is then the graded locally convex space given by

$$V' \cong \bigoplus_{i\in\mathbb{Z}} (V')_i \cong \bigoplus_{i\in\mathbb{Z}} (V_{-i})'.$$

If $(V, d)$ is a cochain complex then dualising the differential gives a differential $d^*$ on $V'$. Taking the dual of $d$ reverses the order, but because we also flip the grading when going over to $V'$, $d^*$ is still of degree $+1$.

The construction of the symmetric algebra is more subtle in this context, as we need to choose a topology on the tensor product, which we also want to complete. The most tractable option, which is the one we use, is the projective tensor product.[3] Explicitly, we define

$$T^n V = (V^{\hat{\otimes}_\pi n})$$
$$\mathrm{Sym}^n(V) = (T^n V)_{S_n},$$

where the action of the symmetric group extends by density of $V^{\otimes n}$ in $V^{\hat{\otimes}_\pi n}$ and continuity of the group action. The tensor algebra and graded symmetric tensor algebra are again defined by the direct product of all the weight $n$ pieces.

[3] All our examples in field theory are phrased in terms of nuclear vector spaces. In that case, there is a unique completed tensor product that we can use. We do not restrict to this setting at this point however.

The completed projective tensor product commutes with taking quotients, i.e. if $\pi_i : V_i \to W_i$ are two quotient maps, i.e. surjective, open and continuous, then $\pi_1 \otimes \pi_2 : V_1 \hat{\otimes}_\pi V_2 \to W_1 \hat{\otimes}_\pi W_2$ is also a quotient map, see e.g. Section 15.2 in [59]. As such the completed projective tensor product respects the quotient step used in defining the graded symmetric tensor product. We can hence define the product on $\mathrm{Sym}(V)$ by lifting the diagram

$$\begin{array}{ccc} V^{\otimes_\pi n} \otimes_\pi V^{\otimes_\pi m} & \xrightarrow{\quad\cdot\quad} & V^{\otimes_\pi n+m} \\ \downarrow{\scriptstyle \pi_m \otimes \pi_n} & & \downarrow{\scriptstyle \pi_{n+m}} \\ (V^{\otimes_\pi n})_{S_n} \otimes_\pi (V^{\otimes_\pi m})_{S_m} & \xrightarrow{\cdot} & (V^{\otimes_\pi n+m})_{S_{n+m}} \end{array}$$

to the respective completions. As $\pi_n \otimes \pi_m$ is a quotient map, the bottom horizontal arrow is continuous because the top one is. Hence the product extends to a continuous map

$$(V^{\hat{\otimes}_\pi n})_{S_n} \times (V^{\hat{\otimes}_\pi m})_{S_m} \xrightarrow{\otimes} (V^{\hat{\otimes}_\pi n})_{S_n} \hat{\otimes}_\pi (V^{\hat{\otimes}_\pi m})_{S_m} \xrightarrow{\cdot} (V^{\hat{\otimes}_\pi n+m})_{S_{n+m}},$$

where the first map is the algebraic tensor product followed by the inclusion into the completed tensor product.

The completed projective tensor product does not commute with direct sums in general, unless one works with spaces that satisfy specific prerequisites[4]. However, the situation is much better for the direct product: If $\{V_i\}$ is a collection of locally convex spaces and $W$ is another locally convex space, then it always holds that

$$\left(\prod_{i\in I} V_i\right) \hat{\otimes}_\pi W \cong \left(\prod_{i\in I} V_i \hat{\otimes}_\pi W\right),$$

by continuous extension of the map

$$(v_i) \otimes w \to (v_i \otimes w)$$

see Section 15.4 in [59]. This is the prime reason why we defined the symmetric tensor algebra with respect to the direct product, as we can define the product by the composition of continuous maps

$$\mathrm{Sym}(V) \otimes \mathrm{Sym}(V) \to \mathrm{Sym}(V) \hat{\otimes}_\pi \mathrm{Sym}(V) \cong \prod_{m,n\in\mathbb{N}} \mathrm{Sym}^n(V) \hat{\otimes}_\pi \mathrm{Sym}^m(V)$$
$$\to \prod_{m,n\in\mathbb{N}} \mathrm{Sym}^{m+n}(V) \to \prod_{p\in\mathbb{N}} \mathrm{Sym}^p(V).$$

[4]These are so called gDF-spaces, see e.g. Section 15.5 in [59]

The first two steps are just inclusion and the isomorphism referenced above. The third step is given by the graded symmetric multiplication. The last step is given by summing together all terms where $m + n = p$ (which is a finite set for fixed $p$).

The co-multiplication is now given as the map

$$\begin{aligned}\Delta : \mathrm{Sym}(V) = \prod_{n\in\mathbb{N}} \mathrm{Sym}^n(V) &\to \prod_{n\in\mathbb{N}} \prod_{m+p=n} \mathrm{Sym}^m(V) \,\hat{\otimes}_\pi\, \mathrm{Sym}^p(V), \\ &\cong \prod_{m,p\in\mathbb{N}} \mathrm{Sym}^m(V) \,\hat{\otimes}_\pi\, \mathrm{Sym}^p(V) \cong \mathrm{Sym}(V) \,\hat{\otimes}_\pi\, \mathrm{Sym}(V),\end{aligned}$$

where the second map is the comultiplication in weight $n$ defined above and extended by continuity to the relevant completions, and the third map is a relabelling of the direct products.

## 5.2 DUAL OF THE SYMMETRIC ALGEBRA

After this review, we discuss the dual of $\mathrm{Sym}(V)$, as that is the algebra that we are interested in in practice. As we are not aware of a place where these constructions are performed in detail, we give a fairly explicit exposition in this section.

In what follows we will view the summands of $V$ to give the graded field content of the theory, such as ghosts and antifields. In modelling graded observables, we should look at functions on these field variables that have suitable symmetry properties to account for the graded nature of the fields. Dualising changes the direct product to a direct sum, so that

$$(TV)' = \bigoplus_{n=0}^{\infty} (V^{\hat{\otimes}_\pi n})'.$$

It follows from the universal property of the projective tensor product that $(V^{\hat{\otimes}_\pi n})'$ is linearly isomorphic to the space of continuous $n$-linear maps on $V$, which is an identification that we make implicitly. As such, $(TV)'$ can be viewed as the space of polynomials on $V$. As $\mathrm{Sym}(V)$ is a quotient of $TV$, its dual is a subspace of $(TV)'$, namely precisely those multilinear maps that are invariant under the action of the permutation group:

$$\mathrm{Sym}(V)' \cong \bigoplus_{n=0}^{\infty} (V^{\hat{\otimes}_\pi n})'^{S_n}.$$

We make this identification implicitly throughout this work. Consequently, $\mathrm{Sym}(V)'$ is the space of graded symmetric polynomials on $V$.

We turn $(TV)'$ into an algebra by mapping $\Phi \in (T^nV)'$ and $\Psi \in (T^mV)'$ to the weight $n+m$ multilinear map

$$\Phi \otimes \Psi(v_1, \ldots, v_{n+m}) = \Phi(v_1, \ldots, v_m)\Psi(v_{m+1}, \ldots, v_{m+n}). \tag{5.4}$$

If $V$ is a space of sections of a vector bundle, then this is nothing but the tensor product of multi-variable distributions.

We turn $\mathrm{Sym}(V)'$ into an algebra by dualising the comultiplication on $\mathrm{Sym}(V)$:

$$\odot : \mathrm{Sym}(V)' \otimes \mathrm{Sym}(V)' \to \Big(\mathrm{Sym}(V)\,\hat{\otimes}_\pi\,\mathrm{Sym}(V)\Big)' \xrightarrow{\Delta^*} \mathrm{Sym}(V)' \tag{5.5}$$

where the first map is the inclusion of the tensor product of duals into the dual of projective tensor products, see e.g. Proposition 43.6 in [72]. Explicitly, if $\Phi \in \mathrm{Sym}^n(V)'$ and $\Psi \in \mathrm{Sym}^m(V)'$ as continuous multilinear maps, and $v_1, \ldots, v_{n+m} \in V$ are homogeneous, then

$$\Phi \odot \Psi(v_1, \ldots, v_{n+m}) = \sum_{\sigma \in \mathrm{Sh}(m,n)} \epsilon_\sigma \Phi(v_{\sigma_1}, \ldots, v_{\sigma_m})\Psi(v_{\sigma_{m+1}}, \ldots, v_{\sigma_{m+n}}), \tag{5.6}$$

where $\epsilon_\sigma$ is defined in equation (5.3). We stress that we view $\mathrm{Sym}(V)$ as a subspace of $TV$, but not as a subalgebra.

**Lemma 5.2.1.** *If $V$ is a graded vector space and $\eta \in \mathrm{Hom}^p(V, V)$, then $\mathrm{der}(\eta)^*$ is a graded derivation on $\mathrm{Sym}(V)'$ with respect to the product $\odot$, so that $\mathrm{der}(\eta)^* = \mathrm{der}\,(\eta^*)$.*

*Proof.* Let $\Phi \in \mathrm{Sym}^n(V)'$ and $\Psi \in \mathrm{Sym}^m(V)'$ be homogeneous. This means that $\Phi$ has a definite degree $|\Phi|$, and since $\Phi$ maps into $\mathbb{C}$ it has the property that $\Phi(v_1, \ldots, v_n) = 0$ unless $\sum_i |v_i| + |\Phi| = 0$, and similar for $\Psi$. We have to show that

$$\mathrm{der}(\eta)^*(\Phi \odot \Psi) = (\mathrm{der}(\eta)^*\Phi) \odot \Psi + (-1)^{p|\Phi|}\Phi \odot (\mathrm{der}(\eta)^*\Psi)\,. \tag{5.7}$$

Let $v_1, \ldots, v_{n+m} \in V$ be homogeneous such that $\sum_i |v_i| + p + |\Phi| + |\Psi| = 0$, because otherwise all terms calculated below will be zero for degree reasons, trivially satisfying the requirements.

We calculate

$$\begin{aligned}
&\Big(\Big(\mathrm{der}(\eta)^*\Phi\Big) \odot \Psi\Big)(v_1, \ldots, v_{n+m}) \\
&\quad= \sum_{\sigma \in \mathrm{Sh}(m,n)} \epsilon_\sigma \left(\mathrm{der}(\eta)^*\Phi\right)(v_{\sigma_1}, \ldots, v_{\sigma_n})\Psi(v_{\sigma_{n+1}}, \ldots, v_{\sigma_{n+m}}), \\
&\quad= \sum_{\sigma \in \mathrm{Sh}(m,n)} \sum_{l=1}^{n} \epsilon_\sigma (-1)^{p\sum_{k=1}^{l-1}|v_{\sigma_k}|}\Phi(v_{\sigma_1}, \ldots, \eta v_{\sigma_l} \ldots, v_{\sigma_n})\Psi(v_{\sigma_{n+1}}, \ldots, v_{\sigma_{n+m}}).
\end{aligned}$$

Note that there are no signs for moving $\eta$ past $\Phi$ and $\Psi$, as the pullback $\mathrm{der}(\eta)^*$ acts, by definition, only on the arguments $v_1, \ldots v_{n+m}$. Similarly, we calculate

$$\begin{aligned}
&(-1)^{p|\Phi|}\Big(\Phi \odot (\mathrm{der}(\eta)^*\Psi)\Big)(v_1, \ldots, v_{n+m}) \\
&\qquad = \sum_{\sigma \in \mathrm{Sh}(m,n)} \epsilon_\sigma (-1)^{p|\Phi|} \Phi(v_{\sigma_1}, \ldots, v_{\sigma_n}) \left(\mathrm{der}(\eta)^*\Psi\right) (v_{\sigma_{n+1}}, \ldots, v_{\sigma_{n+m}}), \\
&\qquad = \sum_{\sigma \in \mathrm{Sh}(m,n)} \sum_{l=1}^{m} \epsilon_\sigma (-1)^{p\left(|\Phi| + \sum_{k=1}^{l-1} |v_{\sigma_{n+k}}|\right)} \Phi(v_{\sigma_1}, \ldots, v_{\sigma_n}) \\
&\qquad\qquad \cdot \Psi(v_{\sigma_{n+1}} \ldots, \eta v_{\sigma_{n+l}}, \ldots, v_{\sigma_{n+m}}).
\end{aligned}$$

In both these terms, the sign $\epsilon_\sigma = (-1)^N$ is given as in equation (5.3) by

$$N = \sum_{k=1}^{m} |v_{\sigma_{n+k}}| \cdot \sum_{\substack{j > \sigma_{n+k} \\ j \in \sigma\{1,\ldots,n\}}} |v_j|. \tag{5.8}$$

For the left-hand side of equation (5.7), we calculate

$$\begin{aligned}
\mathrm{der}(\eta)^*(\Phi \odot \Psi)(v_1, \ldots, v_{n+m}) &= \sum_{i=1}^{n+m} (-1)^{p \sum_{j<i} |v_j|} \Phi \odot \Psi(v_1, \ldots, \eta v_i, \ldots, v_n), \\
= \sum_{i=1}^{n+m} (-1)^{p \sum_{j<i} |v_j|} &\left( \sum_{\substack{\sigma \in \mathrm{Sh}(m,n) \\ i \in \sigma\{1 \ldots n\}}} \epsilon_{\sigma,i} \Phi(v_{\sigma_1}, \ldots, \eta v_i, \ldots, v_{\sigma_n}) \Psi(v_{\sigma_{m+1}}, \ldots, v_{\sigma_{m+n}}) \right. \\
&\left. + \sum_{\substack{\sigma \in \mathrm{Sh}(m,n) \\ i \notin \sigma\{1 \ldots n\}}} \epsilon_{\sigma,i} \Phi(v_{\sigma_1}, \ldots, v_{\sigma_n}) \Psi(v_{\sigma_{m+1}}, \ldots, \eta v_i, \ldots, v_{\sigma_{m+n}}) \right),
\end{aligned}$$

where $\epsilon_{\sigma,i}$ is obtained similar to $\epsilon_\sigma$, but with the degree of $v_i$ raised by $p$ in equation (5.8). These signs are hence related by

$$\epsilon_{\sigma,i} = \begin{cases} \epsilon_\sigma (-1)^{p \sum_{l \in \{n+1,\ldots,n+m\}, \sigma_l < i} |v_{\sigma_l}|} & \text{if } i \in \sigma\{1, \ldots, n\}, \\ \epsilon_\sigma (-1)^{p \sum_{k \in \{1,\ldots,n\}, \sigma_k > i} |v_{\sigma_k}|} & \text{if } i \in \sigma\{n+1, \ldots, n+m\}. \end{cases}$$

To check that the signs work out, let first $i \in \sigma\{1, \ldots n\}$. We note that

$$\begin{aligned}
\epsilon_{\sigma,i} \cdot (-1)^{p \sum_{j<i} |v_j|} &= \epsilon_{\sigma,i} \cdot (-1)^{p \sum_{k \in \{1,\ldots n\}, \sigma_k < i} |v_{\sigma_k}|} \cdot (-1)^{p \sum_{l \in \{n+1,\ldots,n+m\}, \sigma_l < i} |v_{\sigma_l}|}, \\
&= \epsilon_\sigma \cdot (-1)^{p \sum_{k \in \{1,\ldots n\}, \sigma_k < i} |v_{\sigma_k}|}.
\end{aligned}$$

This takes care of the first term. Similarly, for $i \in \sigma\{n+1,\dots,n+m\}$, it follows from the fact that we may assume that

$$\sum_{k=1}^{n} |v_{\sigma_k}| = -|\Phi|$$

that

$$\begin{aligned}\epsilon_{\sigma,i} \cdot (-1)^{p\sum_{j<i}|v_j|} =& \epsilon_\sigma \cdot (-1)^{p\sum_{k\in\{1,\dots,n\},\sigma_k>i}|v_{\sigma_k}|} \cdot (-1)^{p\sum_{k\in\{1,\dots,n\},\sigma_k<i}|v_{\sigma_k}|} \\ &\cdot (-1)^{p\sum_{l\in\{n+1,\dots,n+m\},\sigma_l<i}|v_{\sigma_l}|}, \\ =& \epsilon_\sigma \cdot (-1)^{p|\Phi|} \cdot (-1)^{p\sum_{l\in\{n+1,\dots,n+m\},\sigma_l<i}|v_{\sigma_l}|},\end{aligned}$$

which is what we needed to show. □

If $(V, d)$ is a cochain complex, then we endow $\mathrm{Sym}(V)'$ with the differential $\mathrm{der}(d)^* = \mathrm{der}(d^*)$. Lemma 5.2.1 then implies that $(\mathrm{Sym}(V)', \odot, \mathrm{der}(d)^*)$ has the structure of a differential graded algebra.

As $\mathrm{Sym}^n(V)$ is a subspace of $T^nV$, we can act on it by tensor products of graded linear maps. However, we might not end up with a suitably symmetric multilinear map afterwards. To remedy this, we can act with the graded symmetric product, which we now view as a map

$$\odot : (T^nV)' \to (\mathrm{Sym}^n(V))'.$$

This is nothing more than the unique continuous extension of the mapping

$$u_1 \otimes \dots \otimes u_n \to u_1 \odot \dots \odot u_n$$

for $u_1 \dots u_n \in V'$. This operation is not equal to the identity on multilinear maps that are already symmetric: If $u_1, u_2 \in V_0'$ then

$$\odot(u_1 \odot u_2) = \odot(u_1 \otimes u_2 + u_2 \otimes u_1) = 2u_1 \odot u_2,$$

and similar for higher weights. Hence we introduce combinatorical factors to define the symmetrisation operator as

$$\mathcal{S} = \sum_{n=0}^{\infty} \frac{1}{n!} \odot : \bigoplus_{n=0}^{\infty} (T^n(V))' \to \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(V)'.$$

**Lemma 5.2.2.** *If $\eta \in \mathrm{Hom}^m(V,V)$ and $\Phi \in \mathrm{Sym}^n(V)'$, then*

$$\mathrm{der}(\eta)^*\Phi = n\mathcal{S}(\eta^* \otimes \mathbb{1}^{\otimes n-1})\Phi =: n(\eta^* \odot \mathbb{1}^{\otimes n-1})\Phi. \tag{5.9}$$

*Proof.* Let $v_1, \ldots, v_n \in V$ be homogeneous. We calculate

$$\begin{aligned}(\eta^* \odot \mathbb{1}^{\otimes n-1})\Phi(v_1, \ldots, v_n) &= \sum_{i=1}^{n} \frac{1}{n}(-1)^{|v_i|\sum_{j<i}|v_j|}\Phi(\eta v_i, v_1, \ldots, \hat{v}_i, \ldots, v_n), \\ &= \sum_{i=1}^{n} \frac{1}{n}(-1)^{m\sum_{j<i}|v_j|}\Phi(v_1, \ldots, \eta v_i, \ldots, v_n), \\ &= \frac{1}{n}\,\mathrm{der}(\eta)^*\Phi(v_1, \ldots, v_n).\end{aligned}$$

Again, there are no signs for moving $\eta$ through $\Phi$ as the adjoint $\eta^*$ acts directly on the arguments $v_1, \ldots, v_n$. □

As elements of $\mathrm{Sym}(V)'$ are multilinear functionals, we can define partial evaluation on elements of $\mathrm{Sym}(V)$.

**Definition 5.2.3.** *Let $\psi, \omega \in \mathrm{Sym}(V)$, and let $\Phi \in \mathrm{Sym}(V)'$. We define the* ***partial evaluation*** *of $\Phi$ on $\psi$ by*

$$(\iota_\psi\Phi)(\omega) = \Phi(\psi \cdot \omega).$$

Differently put, $\iota_\psi$ is the adjoint of the multiplication operator

$$m_\psi(\omega) = \psi \cdot \omega.$$

Partial evaluation has a nice commutation relation with derivations.

**Lemma 5.2.4.** *If $\psi \in \mathrm{Sym}(V)$ and $\eta \in \mathrm{Hom}(V,V)$, then*

$$[\iota_\psi, \mathrm{der}(\eta^*)] = \iota_{\mathrm{der}(\eta)\psi}.$$

*Proof.* Without loss of generality, we may assume that $\psi$ and $\eta$ are homogeneous. Let $\omega \in \mathrm{Sym}(V)$. It follows from the Leibniz rule that

$$\mathrm{der}(\eta)(\psi \cdot \omega) = (\mathrm{der}(\eta)\psi) \cdot \omega + (-1)^{|\eta||\psi|}\psi \cdot (\mathrm{der}(\eta)\omega).$$

Put differently, this shows the relation

$$[\mathrm{der}(\eta), m_\psi] = m_{\mathrm{der}(\eta)\psi},$$

on $\mathrm{Sym}(V)$. The result then follows upon taking the adjoint on both sides, noting that it reverses the order in the graded Lie bracket. □

To close this section, we note that the first map in equation (5.5) is not continuous in general if we endow the domain with the projective topology. We mention one scenario in which it is, which is the one of interest to us. Suppose that all the $V_i$ are nuclear Fréchet spaces (such as spaces of smooth sections of vector bundles). The class of Fréchet spaces is stable under taking quotients by closed subspaces and taking countable direct products, see e.g. Chapter 10 in [72]. Similarly, the class of nuclear spaces is closed under these operations, see e.g. Proposition 50.1 in the same reference. We conclude that $\mathrm{Sym}(V)$ is a nuclear Fréchet space. Hence we can use Proposition 50.7 in [72] to conclude that

$$\mathrm{Sym}(V)' \hat{\otimes} \mathrm{Sym}(V)' \cong \left(\mathrm{Sym}(V) \hat{\otimes} \mathrm{Sym}(V)\right)',$$

so that the multiplication of $\mathrm{Sym}(V)'$ is continuous in this scenario. [5]

## 5.3 GRADED FUNCTIONALS

From here on out, we assume that all the $V_i$ are nuclear Fréchet spaces, so that the multiplication of the dual of $\mathrm{Sym}(V)$ is continuous. We write $\mathcal{A}$ for the *complex* valued graded symmetric polynomials on $V$. If $V$ is a complex vector space, then this just matches $\mathrm{Sym}(V)'$. If $V$ is real then it is the complexification of that space. In both cases, we endow it with the product $\odot$ defined in the previous section. We write $\mathcal{A}^n$ for the weight $n$ elements in $\mathcal{A}$.

As we would like to resolve algebras of bona-fide *smooth functionals* in degree 0, this algebra does not suffice, as it allows only polynomial behaviour in this degree. We therefore consider an enlargement of this algebra in terms of graded functionals. The approach taken in [41] is to define them as functionals

$$V_0 \to \bar{\mathcal{A}} = \mathrm{Sym}(\bar{V})'^{\mathbb{C}}$$

where

$$\bar{V} = \bigoplus_{i \neq 0} V_i.$$

In this way, the dependence on the degrees of freedom in degree 0 is allowed to be smooth, and not merely polynomial.

[5] We dropped the subscript $\pi$ on the tensor products as is customary when working with nuclear vector spaces.

We do not use this approach, as it makes the discussion of the action of graded morphisms on the field complex cumbersome; We would need to discriminate between cases where we map out of degree zero, into degree zero, and between non-zero degrees. This makes working with this definition needlessly hard, even if it is the most intuitive option. We therefore rephrase their definition into a form which is more suitable to our needs. The rephrasing of the definition in this section, as well as the subsequent results on derivations and retracts in the following sections, are novel material.

We proceed as follows; We allow functionals that map into $\mathcal{A}$, but impose that degree zero variables that show up in the 'graded part' of the functional should be contracted with the argument of the functional. We first develop some notation, and give a precise statement in Definition 5.3.1.

We write

$$\mathcal{A}_0^m \coloneqq \mathrm{Sym}^m(V_0)'^{\mathbb{C}}$$
$$\mathcal{A}_0 \coloneqq \bigoplus_m \mathcal{A}_0^m,$$

and we view both $\mathcal{A}_0$ and $\bar{\mathcal{A}}$ as subalgebras of $\mathcal{A}$. By isolating factors in degree 0, we can give isomorphisms

$$\mathcal{A} \cong \mathcal{A}_0 \mathbin{\hat{\odot}} \bar{\mathcal{A}} \cong \bigoplus_m (\mathcal{A}_0^m \mathbin{\hat{\odot}} \bar{\mathcal{A}}),$$

where by $\hat{\odot}$ we mean the completion of the image of $\odot$ on the relevant spaces.

Let $\varphi \in V_0$, we define the evaluation map

$$\mathrm{ev}_\varphi : \mathcal{A} \to \bar{\mathcal{A}},$$

with respect to this direct sum as

$$\mathrm{ev}_\varphi = \bigoplus_m \frac{\iota_{\varphi^{\cdot m}}}{m!},$$

i.e. we contract all variables of degree 0 with $\varphi$. To motivate the combinatorial factor, suppose that $u \in \left(V_0^{\hat{\otimes}_\pi m}\right)'$ is symmetric (and hence an element of $\mathcal{A}_0$). Then the functional

$$F_u = \varphi \mapsto \mathrm{ev}_\varphi u = \frac{1}{m!} u(\varphi^{\cdot m})$$

is the unique functional satisfying

$$F_u^{(n)}(0) = \begin{cases} u \text{ if } n = m, \\ 0 \text{ else.} \end{cases}$$

so that this map identifies functionals and distributions in the appropriate way.

If then $F \in C^\infty(V_0 \to \mathcal{A})$, we define its *contraction* to be the functional defined by

$$\mathrm{Ev}F(\varphi) = \mathrm{ev}_\varphi F(\varphi), \tag{5.10}$$

which is now valued in $\bar{\mathcal{A}}$.

**Definition 5.3.1.** *Let $V$ be a bounded graded vector space, of which every graded piece is a nuclear Fréchet space. The algebra of **graded functionals** on $V$ is given by*

$$\mathcal{F}(V) = C^\infty(V_0 \to \mathcal{A}) / \ker(\mathrm{Ev}), \tag{5.11}$$

*endowed with the product*

$$(F \odot G)(\varphi) \coloneqq F(\varphi) \odot G(\varphi). \tag{5.12}$$

The image of Ev are exactly the functionals into $\bar{\mathcal{A}}$, which is the algebra of graded functionals used in [41]. Hence we see that our approach is identical to theirs. It is readily checked that $\mathrm{ev}_\varphi$ is a homomorphism on $\mathcal{A}$, so that the product in equation (5.12) descends to the quotient space $\mathcal{F}(E)$. The product of two functionals is smooth as the product $\odot$ is continuous. The assumption that all $V_i$ are nuclear Fréchet spaces is mainly to ensure that the product of two graded functionals is again a smooth functional. We find it likely that this definition could be generalised to allow for more general spaces, but as this one is sufficient for our purposes we do not delve into this question here.

We prove the following relation between evaluation and derivation for future reference:

**Lemma 5.3.2.** *If $\varphi \in V_0$, $\eta \in \mathrm{Hom}^n(V, V)$ and $m \in \mathbb{N}$, then the relation*

$$\mathrm{ev}_\varphi[\mathrm{ev}_\varphi, \mathrm{der}(\eta^*)] = \frac{1}{(m-1)!} \mathrm{ev}_\varphi \circ \mathrm{der}(\eta^*_{-n}) \circ \iota_{\varphi \cdot m-1} \tag{5.13}$$

*holds on $\mathcal{A}_0^m \mathbin{\hat{\odot}} \bar{\mathcal{A}}$.*

*Proof.* In the case where $n = 0$, we split $\eta = \eta_0 + \bar{\eta}$, both of which we view as maps defined on the whole of $V$. By linearity, it holds that

$$\mathrm{der}(\eta^*) = \mathrm{der}(\eta_0^*) + \mathrm{der}(\bar{\eta}^*).$$

As $\bar{\eta}$ acts only on variables in non-zero degree, it is clear that $[\mathrm{ev}_\varphi, \mathrm{der}(\bar{\eta}^*)] = 0$. Similarly, as $\mathrm{ev}_\varphi$ contracts all variables in degree 0, $\mathrm{der}(\eta_0^*) \circ \mathrm{ev}_\varphi = 0$. It then follows from Lemma 5.2.4 that

$$\begin{aligned}\mathrm{ev}_\varphi[\mathrm{ev}_\varphi, \mathrm{der}(\eta^*)] &= \mathrm{ev}_\varphi \circ \mathrm{der}(\eta_0^*) = \frac{1}{m!}\iota_{\varphi^{\cdot m}} \circ \mathrm{der}(\eta_0^*)\\ &= \frac{m}{m!}\iota_{\eta_0\varphi}\iota_{\varphi^{\cdot m-1}} = \frac{1}{(m-1)!}\mathrm{ev}_\varphi \circ \mathrm{der}(\eta_0^*) \circ \iota_{\varphi^{\cdot m-1}}\end{aligned}$$

as required.

In the case where $n \neq 0$ we split $\eta = \eta_0 + \eta_{-n} + \bar{\eta}$, as $\eta_0$ now maps out of degree 0, and $\eta_{-n}$ maps into it. Again, it is clear that $[\mathrm{ev}_\varphi, \mathrm{der}(\bar{\eta})] = 0$ and $\mathrm{der}(\eta_{-n}^*) \circ \mathrm{ev}_\varphi = 0$. We calculate

$$\begin{aligned}\mathrm{ev}_\varphi[\mathrm{ev}_\varphi, \mathrm{der}(\eta_0^*)] &= \frac{1}{(m+1)!}\iota_{\varphi^{\cdot m+1}} \circ \mathrm{der}(\eta_0^*) - \frac{1}{m!}\iota_\varphi \circ \mathrm{der}(\eta_0^*) \circ \iota_{\varphi^{\cdot m}}\\ &= \left(\frac{m+1}{(m+1)!} - \frac{1}{m!}\right)\iota_{\eta_0\varphi}\iota_{\varphi^{\cdot m}} = 0.\end{aligned}$$

Similarly, we calculate

$$\begin{aligned}\mathrm{ev}_\varphi[\mathrm{ev}_\varphi, \mathrm{der}(\eta_{-n}^*)] &= \mathrm{ev}_\varphi \circ \mathrm{der}(\eta_{-n}^*) = \frac{1}{(m-1)!}\iota_{\varphi^{\cdot m-1}} \circ \mathrm{der}(\eta_{-n}^*)\\ &= \frac{1}{(m-1)!}\,\mathrm{der}(\eta_{-n}^*) \circ \iota_{\varphi^{\cdot m-1}} = \frac{1}{(m-1)!}\mathrm{ev}_\varphi\,\mathrm{der}(\eta_{-n}^*) \circ \iota_{\varphi^{\cdot m-1}},\end{aligned}$$

using the fact that $\eta_{-n}\varphi = 0$ so that we may freely exchange $\iota_{\varphi^{\cdot m-1}}$ and $\mathrm{der}(\eta_{-n}^*)$. □

## 5.4 MAPS ON SPACES OF GRADED FUNCTIONALS

We discuss maps on graded functionals. Our main strategy is to dualise maps defined on the complex to the algebras of functions on them. In particular, we want to extend the construction of derivations on the graded symmetric algebra to algebras of graded functionals. Let $F \in C^\infty(V_0 \to \mathcal{A})$. By Proposition 2.3.5, the derivative of $F$ can be viewed as a functional

$$F^{(1)} : V_0 \to L_c(V_0 \to \mathcal{A}).$$

As the bounded bornology coincides with the precompact bornology on nuclear spaces, see e.g. Proposition 50.2 in [72], we have

$$L_c(V_0, \mathcal{A}) \cong L_b(V_0, \mathcal{A}).$$

Furthermore, it follows from equation (50.18) in the same source that $L_b(V_0, \mathcal{A}) \cong V_0' \hat{\otimes} \mathcal{A}$. We can hence view the derivative of $F$ as a functional

$$F^{(1)} : V_0 \to V_0' \hat{\otimes} \mathcal{A}, \tag{5.14}$$

and similarly for higher order derivatives.

Suppose that $\eta \in \mathrm{Hom}^n(V, V)$. The $(-n)$'th component of $\eta$ is a continuous linear map

$$\eta_{-n} : V_{-n} \to V_0,$$

so that its adjoint defines a continuous linear map

$$\eta^*_{-n} : V_0' \to V_{-n}'$$

Composing this with the symmetrisation operator $\mathcal{S}$ defines a functional

$$(\eta^*_{-n} \odot \mathbb{1})F^{(1)} : V_0 \xrightarrow{F^{(1)}} V_0'\hat{\otimes}\mathcal{A} \xrightarrow{\eta^*_{-n}\otimes\mathbb{1}} V_{-n}'\hat{\otimes}\mathcal{A} \xrightarrow{\mathcal{S}} \mathcal{A}.$$

**Definition 5.4.1.** *Suppose that $V$ is a graded locally convex vector space, and let $\eta \in \mathrm{Hom}^n(V, V)$. The **functional derivation** induced by $\eta$*

$$\mathrm{Der}(\eta^*) : \mathcal{F}(V) \to \mathcal{F}(V)$$

*is the operator defined by*

$$\mathrm{Der}(\eta^*)F = (\eta^*_{-n} \odot \mathbb{1})F^{(1)} + \mathrm{der}(\eta)^*F. \tag{5.15}$$

It is readily checked that this defines a derivation with respect to the product on $\mathcal{F}(V)$. To make sure that this definition is well-posed, we should show that $\mathrm{Der}(\eta^*)$ respects the kernel of Ev.

**Proposition 5.4.2.** *If $F \in C^\infty(V_0 \to \mathcal{A})$ and $\eta \in \mathrm{Hom}^n(V, V)$, then*

$$\mathrm{Ev}\,\mathrm{Der}(\eta^*)\mathrm{Ev}F = \mathrm{Ev}\,\mathrm{Der}(\eta^*)F.$$

*Consequently,* $\mathrm{Der}(\eta^*)$ *maps* $\ker(\mathrm{Ev})$ *to itself.*

*Proof.* Without loss of generality, we may assume that

$$F : V_0 \to \mathcal{A}_0^m \hat{\odot} \bar{\mathcal{A}},$$

i.e. it has $m$ degree 0 factors, as every graded functional is a sum of terms of that form. We need to show that

$$L = \mathrm{ev}_\varphi \,\mathrm{Der}(\eta^*)\mathrm{ev}_\varphi F(\varphi) - \mathrm{ev}_\varphi \,\mathrm{Der}(\eta^*)F(\varphi) = 0,$$

for any $\varphi \in V_0$

We use the Leibniz rule to calculate

$$(\mathrm{Ev}F)^{(1)}(\varphi) = (\mathbb{1} \otimes \mathrm{ev}_\varphi)\, F^{(1)}(\varphi) + m\iota_{\varphi\cdot m-1}F(\varphi)$$

where we view $\iota_{\varphi\cdot m-1}F(\varphi)$ to be valued in $V_0'\hat{\otimes}\bar{\mathcal{A}}$. By Lemma 5.2.2, we can write

$$(\eta_{-n} \odot \mathbb{1})\iota_{\varphi\cdot m-1}F(\varphi) = \mathrm{der}(\eta^*_{-n}) \circ \iota_{\varphi\cdot m-1}F(\varphi).$$

Inserting this into the formula for $\mathrm{Der}(\eta^*)$ gives

$$\begin{aligned} L = \mathrm{ev}_\varphi[\mathrm{der}(\eta^*), \mathrm{ev}_\varphi]F(\varphi) \quad &+ \quad \mathrm{ev}_\varphi(\eta^*_{-n} \odot \mathrm{ev}_\varphi)F^{(1)}(\varphi) \quad + \\ \frac{m}{m!}\mathrm{ev}_\varphi \circ \mathrm{der}(\eta^*_{-n}) \circ \iota_{\varphi\cdot m-1}F(\varphi) \quad &- \quad \mathrm{ev}_\varphi(\eta^*_{-n} \odot \mathbb{1})F^{(1)}(\varphi). \end{aligned} \tag{5.16}$$

The first and third terms cancel by Lemma 5.3.2. The second and fourth terms also cancel; If $n = 0$, this is due to the fact that both terms equal

$$(\iota_{\eta_0\varphi} \odot \mathrm{ev}_\varphi)\, F^{(1)}(\varphi).$$

If $n \neq 0$, both terms equal

$$\left(\eta^*_{-n} \odot \mathrm{ev}_\varphi\right) F^{(1)}(\varphi).$$

We conclude that $L = 0$ as required. □

Our alternative definition of graded functionals, as opposed to the one used by Fredenhagen and Rejzner, is mainly aimed at simplifying the proof of the following proposition.

**Proposition 5.4.3.** *If $\eta$ and $\zeta$ are graded morphisms of $V$ of degree $m$ and $n$ respectively, then*

$$[\mathrm{Der}(\eta^*), \mathrm{Der}(\zeta^*)] = \mathrm{Der}([\eta^*, \zeta^*]) \tag{5.17}$$

*Proof.* Let $F \in C^\infty(V_0 \to \mathcal{A})$. We expand

$$\begin{aligned} \mathrm{Der}(\eta^*)\,\mathrm{Der}(\zeta^*)F = \odot\,(\eta^*_{-m} \otimes \zeta^*_{-n} \otimes \mathbb{1})F^{(2)} \;+\; \mathrm{der}(\eta^*) \circ (\zeta_{-n} \odot \mathbb{1})F^{(1)} \quad + \\ \mathrm{der}(\eta^*) \circ \mathrm{der}(\zeta^*)F \quad + \; (\eta^*_{-m} \odot \mathbb{1}) \circ (\mathrm{der}(\zeta^*)F)^{(1)} \end{aligned} \tag{5.18}$$

As derivatives of functionals are symmetric, the first term equals

$$\odot(\zeta^*_{-n} \otimes \eta^*_{-m} \otimes \mathbb{1})F^{(2)} = (-1)^{nm} \odot (\eta^*_{-m} \otimes \zeta^*_{-n} \otimes \mathbb{1})F^{(2)}.$$

and hence this term cancels in the graded commutator. For the second term, we use that $\mathrm{der}(\eta^*)$ is a derivation with respect to the product $\odot$ to compute

$$\mathrm{der}(\eta^*) \circ (\zeta^*_{-n} \odot \mathbb{1})F^{(1)} = \left(\eta^*_{-n-m} \circ \zeta^*_{-n} \odot \mathbb{1} + (-1)^{nm} \zeta^*_{-n} \odot \mathrm{der}(\eta^*)\right) F^{(1)} \tag{5.19}$$

When we take the graded commutator the third term in (5.18) combines to

$$[\mathrm{der}(\eta^*), \mathrm{der}(\zeta^*)]F = \mathrm{der}([\eta^*, \zeta^*])F.$$

Finally, the fourth term equals

$$\left(\eta^*_{-m} \odot \mathrm{der}(\zeta^*)\right) F^{(1)}.$$

When taking the graded commutator, this term cancels against the second term in equation (5.19) (exchanging the roles of $\eta$ and $\zeta$). It follows that

$$\begin{aligned}[\mathrm{Der}(\eta^*), \mathrm{Der}(\zeta^*)]F &= ([\eta^*, \zeta^*]_{-n-m} \odot \mathbb{1})\, F^{(1)} + \mathrm{der}([\eta^*, \zeta^*])F \\ &= \mathrm{Der}([\eta^*, \zeta^*])F.\end{aligned}$$

□

## 5.5 FIELD COMPLEXES

The situation of interest in field theory is to take $V_i$ to be a space of smooth sections of a vector bundle $E_i$ that describes the field content in grade $i$, over some manifold $M$. We take it to be trivial outside of degrees $i = k, \ldots, l$. Equivalently, we define the graded vector bundle $E = \bigoplus_{i=k}^{l} E_i$, and consider sections thereof. Throughout this section, we write $\mathcal{E} = \mathcal{E}(M, E)$, and $\mathcal{E}_i = \mathcal{E}(M, E_i)$, so that

$$\mathcal{E} \cong \bigoplus_{i=k}^{l} \mathcal{E}_i.$$

We call $\mathcal{E}$ the graded configuration space of the theory.

The space of monomials of weight $n$ in the field configurations is given by $\mathcal{A}^n = \mathrm{Sym}^n(\mathcal{E})'^{\mathbb{C}}$. By the isomorphism in 2.13, we may view this space as a subset of

$$\mathcal{E}'(M^n; E^{!\boxtimes n})^{\mathbb{C}},$$

namely precisely those sections that are invariant under the action of $S_n$. To condense notation somewhat, we suppress the complexification in the rest of the chapter, and take it to be understood that sections of $E^!$ are complex valued by definition. We abbreviate $\mathcal{F}(\mathcal{E}(M,E)) = \mathcal{F}(E)$ when no confusion can arise.

Using Proposition 5.4.3, we can now treat the case where we do not just consider a graded vector bundle but a *field complex*, i.e. a graded vector bundle together with a differential operator $d : \mathcal{E} \to \mathcal{E}$ of degree +1. Explicitly, this is a sequence of vector bundles and differential operators

$$\ldots \to \mathcal{E}_{-1} \xrightarrow{d_{-1}} \mathcal{E}_0 \xrightarrow{d_0} \mathcal{E}_1 \to \ldots$$

such that $d_{i+1} \circ d_i = 0$. A standard example is the de Rahm complex of a manifold $M$, where $\mathcal{E}_n = \mathcal{E}(M; \Lambda^n T^* M)$, and $d$ is the exterior derivative. The previous proposition allows us to define a differential on $\mathcal{F}(E)$ by $\delta = \mathrm{Der}(d^*)$, as

$$2\delta^2 = [\mathrm{Der}(d^*), \mathrm{Der}(d^*)] = \mathrm{Der}([d^*, d^*]) = 0.$$

As $\mathrm{Der}(d^*)$ is a derivation of $\mathcal{F}(E)$, we see that $(\mathcal{F}(E), \delta)$ has the structure of a differential graded algebra.

A graded functional has two notions of support on spacetime: The first was introduced already in Definition 2.3.6, straightforwardly generalised to functionals valued in any locally convex space. The second is to consider the supports of $F(\varphi)$ for $\varphi \in \mathcal{E}_0$, as this is valued in a space of distributional sections.

Concretely, let $n \neq 0$ and $\Phi_n \in \mathrm{Sym}^n(V)' \subset \mathcal{E}'(M^n; E^{\otimes n})$, so that $\operatorname{supp} \Phi_n \subset M^n$. As $\Phi_n$ is graded symmetric, the projection onto any of these $n$ variables is the same, and we set $\operatorname{supp}_M \Phi_n = \pi_1 \operatorname{supp} \Phi_n$. If $n = 0$, we take the convention that $\operatorname{supp}_M(\Phi_n) = \emptyset$. If $\Phi = \sum_{n=0}^m \Phi_n$ for $\Phi_n \in \mathrm{Sym}^n(V)'$, then we set

$$\operatorname{supp}_M(\Phi) = \bigcup_{n=0}^{m} \operatorname{supp}_M(\Phi_n).$$

Finally, we define the spacetime support of a graded functional to be the union of these two notions

$$\operatorname{Supp} F = \operatorname{supp} F \bigcup_{\varphi \in \mathcal{E}_0} \operatorname{supp}_M F(\varphi),$$

We denote by $\mathcal{F}_c(E)$ the compactly supported functionals on $\mathcal{E}(M, E)$. As the differential in a field complex is a differential operator, the compactly supported functionals form a subcomplex of $(\mathcal{F}(E), \delta)$.

We give some examples of how these structures come into play in practical applications in quantum field theory models. We explain how the three main examples in [41] can be phrased in terms of the definition of graded functionals given above.

*Example* 5.5.1 (Koszul complex of a differential operator). The Koszul complex was introduced by Koszul in [61] for the study of Lie algebra cohomology, and was later improved upon by Tate in [71] as it was realised to be a useful general construction. We refer the reader to [53] for a historical overview of the application of the Koszul complex in field theory. The form in which it is given here is the same as that of Fredenhagen and Rejzner in [67].

Suppose that $P$ is a linear differential operator between the sections of two vector bundles $E_1$ and $E_2$, over the same manifold $M$, and further suppose that this operator is surjective. This data induces a field complex of two terms by

$$\mathcal{E} = \left(0 \to \mathcal{E}(M; E_1) \xrightarrow{P} \mathcal{E}(M; E_2) \to 0\right),$$

supported in degrees 0 and 1. In the context of the BV-formalism, the fields in degree 0 are the physical fields, whereas the fields in degree 1 are called the 'anti-fields'. The cohomology of this cochain complex is supported in degree 0, and is nothing but the solution space $\ker(P)$. A typical graded functional on $\mathcal{E}$ is a sum of maps $F_n : \mathcal{E}(M; E_1) \to \mathcal{E}'_a(M^n; E_2^{!\boxtimes n})$, the space of antisymmetric sections of $E_2^!$ in $n$ variables. These are the multi-vector fields of [41].

Exhibiting the differential of this complex is straightforward. As the field complex has no differential in degree $-1$, the term involving the functional derivative of $F$ in equation (5.15) is absent in the formula for $\mathrm{Der}(P^*)$. If $F$ is valued in $\mathcal{E}'_a(M^n; E_2^{!\boxtimes n})$, then we can act by $P$ on the $i$-th variable:

$$P_i^* : \mathcal{E}'_a(M^n; E_2^{!\boxtimes n}) \to \mathcal{E}'(M^n; E_2^{!\boxtimes i-1} \boxtimes E_1^! \boxtimes E_2^{!\boxtimes n-i-1}),$$

so that $\operatorname{der} P^* = \sum(-1)^i P_i^*$. According to our prescription of contracting 'graded' degrees of freedom in degree 0 with the argument of the functional, paired with the fact that we act on completely antisymmetric distributions, we find that

$$\delta F(\varphi) = \iota_{P\varphi} F(\varphi),$$

where $\iota_{P\varphi}$ inserts $P\varphi$ in the first variable.

*Example* 5.5.2 (Chevally-Eilenberg complex). Somewhat dual to the previous example is the Chevally-Eilenberg complex. Whereas the Koszul complex is aimed at resolving functionals on a space of solutions of some differential equation, this complex is aimed at resolving the space of functionals that are invariant under (linearised) gauge redundancies. It was originally introduced by Chevally and Eilenberg in [22], and found applications in field theory by way of the BRST-complex of Becchi, Rouet, Stora and Tyutin. Again, we refer the reader to [53] for a historical overview.

We consider a field complex of the form

$$\mathcal{E} = \left(0 \to \mathcal{E}(M; E_1) \xrightarrow{d} \mathcal{E}(M; E_2) \to 0\right),$$

but this time supported in degrees $-1$ and 0. The fields in degree 0 are again identified as the physical fields, whereas the fields in degree $-1$ are the 'ghosts', which label the redundancy in the formulation of the theory on $\mathcal{E}(M; E_2)$. We assume that $d$ is injective, so that there is no 'residual' gauge freedom. In this scenario, the cohomology of this complex is again supported in degree 0, but this time it is given by the coinvariants

$$\mathcal{E}(M; E_2)/d\mathcal{E}(M; E_1).$$

As such, functionals on this space can be identified with functionals that are invariant under the action of the gauge transformations, i.e. that satisfy

$$F(\varphi + d\psi) = F(\varphi) \quad \forall \varphi \in \mathcal{E}_0, \psi \in \mathcal{E}_{-1}.$$

The differential on this complex is given by taking a functional derivative with respect to the degree zero field, acting by the adjoint of $d$, and then anti-symmetrising. Explicitly, if $F : \mathcal{E}(M; E_2) \to \bar{\mathcal{A}}^n$, $\varphi \in \mathcal{E}(M; E_2)$ and $\psi_1, \ldots \psi_{n+1} \in \mathcal{E}(M; E_1)$, then

$$\langle \delta F(\varphi), \psi_1 \wedge \ldots \wedge \psi_{n+1} \rangle = \sum_{i=1}^{n+1} (-1)^{i-1} \left\langle F^{(1)}(\varphi)\{d\psi_i\}, \psi_1 \wedge \ldots \hat{\psi_i} \ldots \wedge \psi_{n+1} \right\rangle$$

In particular, if $n = 0$, i.e. $F$ is valued in $\mathbb{C}$, then

$$\langle \delta F(\varphi), \psi \rangle = F^{(1)}(\varphi)\{d\psi\},$$

so that indeed $\delta F = 0$ iff $F$ is invariant, by the fundamental theorem of calculus.

*Example* 5.5.3. Our final example is the BV-complex for electromagnetism, which mixes the preceding two examples in order to describe on-shell, gauge invariant observables. This approach is named after Batalin and Vilkovisky [7], and is an extension of the BRST-complex which aims to treat dynamical information and gauge invariance on the same footing.

Let $\mathcal{M}$ be a spacetime, we define a field complex by

$$\mathcal{E} = \left(0 \to \Omega^0(\mathcal{M}) \xrightarrow{d} \Omega^1(\mathcal{M}) \xrightarrow{\delta d} \Omega^1(\mathcal{M}) \xrightarrow{\delta} \Omega^0(\mathcal{M}) \to 0\right),$$

supported in degrees $-1$ through 2. Here the $\Omega^p(M)$ are $p$-forms on $\mathcal{M}$, $d$ is the exterior derivative and $\delta = *^{-1}d*$ is the codifferential defined with respect to the Hodge dual defined by the metric on $\mathcal{M}$. The second operator is the Maxwell operator, that has redundancies labeled by 0-forms on $\mathcal{M}$. The field in degree $-1$ is the ghost, the fields in degree 0 are the physical fields, i.e. the vector potentials. The fields in degree 1 are the anti-fields, which can be identified with currents. Finally, the degree 2 fields are the antifields for the ghosts.

The cohomology in degree zero is the physical configuration space of the theory, i.e. the solutions to the equations of motion, modulo the trivial solutions. There is cohomology in other degrees as well. In degree $-1$, we have the solutions to the equation $dc = 0$, i.e. the locally constant 0-forms. The cohomology in degrees 1 and 2 are the 3rd and 4th de Rahm cohomology groups respectively.

This is somewhat unsatisfactory, as we would like to view this complex as a resolution for the functions on the physical configuration space. However, this interpretation is hampered by the fact that there are 'large gauge transformations' present in the complex, in the form of the locally constant ghost fields. Getting rid of this 'unwanted' cohomology by imposing suitable support conditions is work in progress

## 5.6 MORPHISMS BETWEEN DIFFERENT COMPLEXES AND RETRACTS

Suppose now that $\gamma \in \mathrm{Hom}^0(V, W)$ for $V, W$ graded locally convex spaces. This induces a pullback map

$$\mathrm{Sym}(\gamma)^* : \mathrm{Sym}(W)' \to \mathrm{Sym}(V)'.$$

**Definition 5.6.1.** *The **functional pullback** of $\gamma$ is the map*

$$\gamma^\sharp : \mathcal{F}(W) \to \mathcal{F}(V) \tag{5.20}$$

*defined by*

$$\gamma^\sharp F(\varphi) = \mathrm{Sym}(\gamma)^* F(\gamma\varphi). \tag{5.21}$$

We check, for $F : C^\infty(W_0 \to \mathcal{A}_W)$ and $\varphi \in V_0$ that

$$(\gamma^\sharp \mathrm{Ev} F)(\varphi) = \mathrm{Sym}(\gamma^*)\mathrm{ev}_{\gamma\varphi} F(\gamma\varphi) = \mathrm{ev}_\varphi \mathrm{Sym}(\gamma^*) F(\gamma\varphi) = (\mathrm{Ev}\gamma^\sharp F)(\varphi)$$

so that this definition is well-posed.

Furthermore, if $\rho : (W, d_W) \to (X, d_X)$ is another map of cochain complexes, then

$$(\rho \circ \gamma)^\sharp = \gamma^\sharp \circ \rho^\sharp,$$

i.e. $\mathcal{F}(\_)$ behaves contravariantly.[6]

The functional pullback and derivations interact in the following way.

**Lemma 5.6.2.** *Let $V$ and $W$ be locally convex graded spaces. If $\gamma \in \mathrm{Hom}^0(V, W)$, $\eta \in \mathrm{Hom}^n(V, V)$, $\zeta \in \mathrm{Hom}^m(W, W)$ and $F : V_0 \to \mathcal{A}^n$, then*

$$\begin{aligned}
\mathrm{Der}(\eta^*)\gamma^\sharp F(\varphi) &= ((\gamma \circ \eta)^* \odot \mathrm{Sym}(\gamma)^*)\,(nF + F^{(1)})(\gamma\varphi) \\
\gamma^\sharp\, \mathrm{Der}(\zeta^*) F(\varphi) &= ((\zeta \circ \gamma)^* \odot \mathrm{Sym}(\gamma)^*)\,(nF + F^{(1)})(\gamma\varphi)
\end{aligned}$$

*Proof.* Assume that $F : V_0 \to \mathcal{A}^n$ for convenience. We use Lemma 5.2.2 to calculate

$$\begin{aligned}
\gamma^\sharp\, \mathrm{Der}(\zeta^*) F(\varphi) &= \gamma^\sharp\Big((\zeta^* \odot \mathbb{1}^{\otimes n}) F^{(1)} + n\left(\zeta^* \odot \mathbb{1}^{\otimes n-1}\right) F\Big)(\varphi) \\
&= \left(\gamma^*\zeta^* \odot \gamma^{*\otimes n}\right) F^{(1)}(\gamma\varphi) + n\left(\gamma^*\zeta^* \odot \gamma^{*\otimes n-1}\right) F(\gamma\varphi)
\end{aligned}$$

The other case is proven analogously. □

As an immediate corollary, if $V$ and $W$ are cochain complexes and $\gamma$ is a cochain map, then it follows that $\gamma^\sharp$ is a cochain map.

The main result presented in this chapter is a prescription on how to lift a deformation retract of complexes to a retract of functionals, as a generalisation of the homotopy operator constructed in Proposition 3.2 of [52]:

[6] We do not substantiate this statement further as giving a precise domain and codomain is somewhat subtle. As stated above, if $V$ is nuclear and Fréchet, then $\mathcal{F}(V)$ is an algebra, but if we allow $V$ to be more general it is *a priori* unclear whether the product of two graded functionals is a smooth functional.

**Theorem 5.6.3.** *If $(V, d_V)$ and $(W, d_W)$ are cochain complexes admitting a strong deformation retract, as in Definition 5.1.1,*

$$W \underset{\iota}{\overset{\pi}{\leftrightarrows}} V \circlearrowleft \eta,$$

*then there exists an operator $H$ of degree $-1$ on $\mathcal{F}(V)$ such that*

$$\mathcal{F}(W) \underset{\pi^\sharp}{\overset{\iota^\sharp}{\leftrightarrows}} \mathcal{F}(V) \circlearrowleft H$$

*is a deformation retract.*

*Proof.* For $\lambda \in \mathbb{R}$, we define an operator of degree zero on $V$ by

$$\gamma_\lambda = \lambda(\mathbb{1} - \iota \circ \pi) + \iota \circ \pi.$$

As $\iota$ and $\pi$ are cochain morphisms, so is $\gamma_\lambda$. Consequently, $\gamma_\lambda^\sharp$ commutes with $\delta$. From the fact that $\pi \circ \iota = \mathbb{1}_W$, it follows that $\iota \circ \pi$ is a projection, so that

$$(\mathbb{1} - \iota \circ \pi)\gamma_\lambda = \lambda(\mathbb{1} - \iota \circ \pi) = \lambda \partial_\lambda \gamma_\lambda. \tag{5.22}$$

If now $F : V_0 \to \mathcal{A}^n$, then we use Lemma 5.6.2 to calculate

$$\begin{aligned}\gamma_\lambda^\sharp \operatorname{Der}((\mathbb{1} - \iota \circ \pi)^*) F(\varphi) &= \left(\lambda \partial_\lambda \gamma_\lambda^* \odot \gamma_\lambda^{*\otimes n}\right) F^{(1)}(\gamma_\lambda \varphi) \qquad + \\ &\qquad n\left(\lambda \partial_\lambda \gamma_\lambda^* \odot \gamma_\lambda^{*\otimes n-1}\right) F(\gamma_\lambda \varphi) \\ &= \lambda \partial_\lambda \left(\gamma_\lambda^\sharp F\right)(\varphi). \end{aligned} \tag{5.23}$$

Similarly, from $\eta \circ \iota = 0$, it follows that

$$\eta \circ \gamma_\lambda = \lambda \eta,$$

which implies that

$$\gamma_\lambda^\sharp \operatorname{Der}(\eta^*) F(\varphi) = \left(\lambda \eta^* \odot \gamma_\lambda^{*\otimes n}\right) F^{(1)}(\gamma_\lambda \varphi) + n\left(\lambda \eta^* \odot \gamma_\lambda^{*\otimes n-1}\right) F(\gamma_\lambda \varphi) = \mathcal{O}(\lambda).$$

We define $H$ by

$$HF = \int_0^1 \lambda^{-1} \gamma_\lambda^\sharp \operatorname{Der}(\eta^*) F d\lambda. \tag{5.24}$$

It follows from Proposition 2.3.4 that $HF$ is a smooth functional, and that we can perform functional derivatives inside the integral. We can move the pullbacks of differentials inside because they are continuous maps. Hence

$$\delta H F(\varphi) = \int_0^1 \lambda^{-1}\left(\gamma_\lambda^\sharp \delta \operatorname{Der}(\eta^*) F\right)(\varphi) d\lambda.$$

Now we combine Proposition 5.4.3, equation (5.23) and the fact that

$$[\eta, d] = \eta d + d\eta = (\mathbb{1} - \iota \circ \pi)$$

to calculate

$$\begin{aligned}(\delta H + H\delta)F &= \int_0^1 \lambda^{-1}\gamma_\lambda^\sharp \operatorname{Der}([d^*, \eta^*])F,\\ &= \int_0^1 \lambda^{-1}\gamma_\lambda^\sharp \operatorname{Der}((\mathbb{1} - \iota\circ\pi)^*)F,\\ &= \int_0^1 \partial_\lambda \gamma_\lambda^\sharp F d\lambda,\\ &= F - \pi^\sharp \circ \iota^\sharp F.\end{aligned}$$

To check the side conditions, we note that

$$H\pi^\sharp F = \int_0^1 \lambda^{-1}\gamma_\lambda^\sharp \operatorname{Der}(\eta^*)\pi^\sharp F d\lambda = 0,$$

as $\operatorname{Der}(\eta^*)\pi^\sharp = 0$ from the fact that $\pi \circ \eta = 0$. Similarly, it holds that $\iota^\sharp H = 0$.

Finally, we note that $\gamma_\lambda^\sharp \operatorname{Der}(\eta^*) = \operatorname{Der}(\eta^*)\gamma_\lambda^\sharp$ from the side conditions on the complex, so that

$$\begin{aligned}H \circ HF &= \int_0^1 \int_0^1 \gamma_\lambda^\sharp \operatorname{Der}(\eta^*)\gamma_\sigma^\sharp \operatorname{Der}(\eta^*) F d\sigma d\lambda\\ &= \int_0^1 \int_0^1 \gamma_\lambda^\sharp \operatorname{Der}([\eta,\eta]^*)\gamma_\sigma^\sharp F d\sigma d\lambda = 0\end{aligned}$$

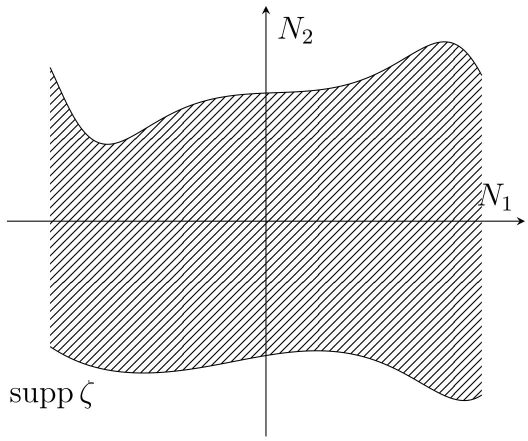
□

If $V$ is a field complex in this theorem, then we can also investigate the homotopical properties of the subcomplex of compactly supported functionals. We show that the homotopy operator defined above respects compactly supported functionals. We prepare the way for the proof with a technical lemma.

**Lemma 5.6.4.** *Let $E_1 \to N_1$ and $E_2 \to N_2$ be vector bundles. If $\zeta : \mathcal{E}(N_2; E_2) \to \mathcal{E}(N_1; E_1)$ is a continuous linear map, then the projection on the first coordinate of the support of the Schwartz kernel of $\zeta$*

$$\pi_1 : \operatorname{supp}(\zeta) \to N_1$$

*is a proper map, i.e. the preimage of a compact set by $\pi_1$ is compact.*




*Proof.* Let $x \in N$ and suppose that $k$ is the rank of $E_1$, so that we may identify $(E_1)_x \cong \mathbb{R}^k$ implicitly. The composition

$$\mathcal{E}(N_2; E_2) \xrightarrow{\zeta} \mathcal{E}(N_1; E_1) \xrightarrow{\mathrm{ev}_x} \mathbb{R}^k \xrightarrow{p_i} \mathbb{R}$$

is continuous and linear, and hence is a compactly supported distribution. We conclude that $\pi_1^{-1}(x)$ is compact for all $x \in N$. Furthermore, as $\pi_1$ is the restriction of a closed map to a closed set, it is a closed map. It now follows from a standard argument that $\pi_1$ is proper, which we provide for the convenience of the reader.

Let $K \subset N_1$ be compact and let $\{U_i\}_{i \in I}$ be an open cover of $\pi_1^{-1}(K)$. For every $x \in K$, there is a finite subset $I_x \in I$ such that $\{U_i\}_{i \in I_x}$ is an open cover of $\pi_1^{-1}(x)$. Then $\operatorname{supp}\zeta \setminus \cup_{i \in I_x} U_i$ is a closed set, so that

$$V_x = N_1 \setminus \pi_1 \left(\operatorname{supp}\zeta \setminus \cup_{i \in I_x} U_i\right)$$

is an open neighbourhood of $x$. As $K$ is compact, there is a finite number of points $x_1, \ldots, x_m$ such that $\{V_{x_j}\}_{j=1}^m$ is a cover of $K$. Finally, it follows that $\{U_i\}_{j \in [1,\ldots,m], i \in I_{x_j}}$ is a finite cover of $\pi_1^{-1}(K)$. □

In the same situation as the lemma, if $v \in \mathcal{E}'(N_1; E_1{}^!)$ then

$$\begin{aligned}\operatorname{supp}\zeta^* v \subset \operatorname{supp} v \circ \operatorname{supp}\zeta &:= \{y \in N_2 \mid \exists x \in \operatorname{supp}(v) \text{ such that } (x, y) \in \operatorname{supp}(\zeta)\},\\ &= \pi_2 \pi_1^{-1} \operatorname{supp} v.\end{aligned}$$

Hence, if we have a family of distributions $\{v_i\}_{i \in I} \subset \mathcal{E}'(N_1; E_1{}^!)$ such that $K = \cup_{i \in I} \operatorname{supp} v_i$ is compact, then it follows from the previous proposition that

$$\cup_{i \in I} \operatorname{supp}\zeta^* v_i \subset \pi_2 \pi_1^{-1}(K)$$

is also compact. This observation is the main technical result we need for the following proposition.

**Proposition 5.6.5.** *If $V$ is a cochain complex of topological vector spaces, and $(\mathcal{E}(M,E),d)$ is a field complex admitting a deformation retract*

$$V \underset{\iota}{\overset{\pi}{\leftrightarrows}} \mathcal{E}(M,E) \circlearrowleft \eta,$$

*then there exists an operator $H$ of degree $-1$ on $\mathcal{F}_c(E)$ such that*

$$\iota^\sharp\mathcal{F}_c(E) \underset{\pi^\sharp}{\overset{\iota^\sharp}{\leftrightarrows}} \mathcal{F}_c(E) \circlearrowleft H$$

*is a deformation retract.*

*Proof.* We check that the operator $H$ defined in equation (5.24) maps compactly supported functionals to compactly supported functionals. Suppose that $F : \mathcal{E}_0 \to \mathcal{A}^n$ is supported in $K$, and let $\varphi \in \mathcal{E}_0$. Then

$$\mathrm{Der}(\eta^*)F(\varphi) = (\eta^* \odot \mathbb{1})F^{(1)}(\varphi) + \mathrm{der}\,(\eta)^*F(\varphi).$$

both of these terms can be viewed as pullbacks as in Lemma 5.6.4 above, for the first term with $N_1 = N_2 = M^{n+1}$ and $E_1 = E_2 = E^{\boxtimes n+1}$, and for the second with $N_1 = N_2 = M^n$ and $E_1 = E_2 = E^{\boxtimes n}$. We conclude that $\mathrm{Der}(\eta^*)F$ is compactly supported. Similarly $\gamma_\lambda^\sharp\,\mathrm{Der}(\eta^*)F$ is compactly supported for every $\lambda$, and their supports can be bounded by a single compact set by the observation made before this proposition. Hence also

$$\mathrm{supp}\int_0^1 \lambda^{-1}\gamma_\lambda^\sharp\,\mathrm{Der}(\eta^*)Fd\lambda \subset \bigcup_{\lambda\in[0,1]} \mathrm{supp}\,\gamma_\lambda^\sharp\,\mathrm{Der}(\eta^*)F$$

is also compact.

Similarly, $\pi^\sharp \circ \iota^\sharp F$ is compactly supported, so that indeed

$$\pi^\sharp : \iota^\sharp\mathcal{F}_c(E) \to \mathcal{F}_c(E).$$

All algebraic relations hold because they hold in the non-compact case. □

### 5.6.1 Exactness of the Koszul complex

As an application of the machinery developed in this chapter, we show how the results obtained in Proposition 3.2 of [52] can be rephrased in this more streamlined language. Suppose that $E \to \mathcal{M}$ is a vector bundle with a Green-hyperbolic operator $P : \mathcal{E}(\mathcal{M};E) \to \mathcal{E}(\mathcal{M};E)$. As before, we define the field complex

$$\mathcal{E} = \left(0 \to \mathcal{E}(\mathcal{M};E) \xrightarrow{P} \mathcal{E}(\mathcal{M};E) \to 0\right),$$

supported in degrees 0 and 1. We construct a retract between $\mathcal{E}$ and $\mathrm{Sol} = \ker P$, viewing the latter as a cochain complex in degree 0, with trivial differential. Hence we need to construct a diagram

$$\begin{array}{ccc} & \overset{\alpha}{\curvearrowleft} & \\ \mathcal{E}(\mathcal{M};E) & \xrightarrow[P]{\quad} & \mathcal{E}(\mathcal{M};E) \\ \iota \uparrow\downarrow \pi & & \\ \mathrm{Sol} & & \end{array}$$

so that $\alpha \circ P = \mathbb{1} - \iota \circ \pi$, $P \circ \alpha = \mathbb{1}$, and $\pi \circ \iota = \mathbb{1}$. We take $\iota$ to be the subset inclusion. To define the other maps, we select a function $\theta \in C^\infty(M)$ with the property that

$$\{x \in \mathcal{M} \mid \theta(x) = 0\}$$

is future compact and

$$\{x \in \mathcal{M} \mid \theta(x) = 1\}$$

is past compact. As an example of such a function, we might select two Cauchy surfaces for $\mathcal{M}$, one lying strictly in the future of the other, and set $\theta$ equal to zero to the past and equal to one to the future, varying smoothly between the two surfaces. We view this function as a smoothed out version of a Cauchy surface, which makes a smooth cut of 'future' and 'past' on our spacetime.

We then define:

$$\alpha : \varphi \mapsto \Delta^A(1-\theta)\varphi + \Delta^R\theta\varphi,$$

which is well-defined due to the support properties of $\theta$.[7] It is a one-sided inverse to $P$, as it satisfies

$$P\alpha\varphi = (1-\theta)\varphi + \theta\varphi = \varphi.$$

[7] Recall that we view Green's functions as maps on sections with past or future compact support.

Using this map, we can define the projection $\pi$ onto the solution space to be the unique map satisfying $\iota \circ \pi = \mathbb{1} - \alpha \circ P$, as

$$P(\iota \circ \pi)\varphi = (P - P \circ \alpha \circ P)\varphi = (P - P)\varphi = 0.$$

The side conditions are easily checked in a similar fashion.

Hence it follows from Theorem 5.6.3 that the functionals on $\mathcal{E}$ give a resolution of the on-shell functionals

$$\mathcal{F}_S = C^\infty(\mathrm{Sol} \to \mathbb{R}).$$

The compactly supported functionals give a resolution of the compactly supported functionals on Sol.

Even though this result is not new, the method of proof is a lot more general than the one employed in [52]; The introduction of the homotopy operator in Theorem 5.6.3 reduces the task of finding resolutions of spaces of functionals to the task of finding resolutions of field complexes, which is a lot more tractable. Applying this machinery to different scenarios, for instance to the Green-hyperbolic complexes of Benini, Musante and Schenkel defined in [9], is work in progress with Eli Hawkins and Kasia Rejzner.

We close this chapter by mentioning that this resolution can be used to give a straightforward proof of the time-slice axiom for this theory. Let $\mathcal{N} \subset \mathcal{M}$ be a sub-spacetime that contains a Cauchy surface of $\mathcal{M}$. This implies that $\mathrm{Sol}_\mathcal{M} \cong \mathrm{Sol}_\mathcal{N}$, as the initial data to a solution in $\mathcal{M}$ can be constrained arbitrarily closed to any Cauchy surface of $\mathcal{M}$. Hence we also have a quasi-equivalence

$$\mathcal{F}(\mathcal{E}(\mathcal{M}; E)) \cong \mathcal{F}(\mathrm{Sol}_\mathcal{M}) \cong \mathcal{F}(\mathrm{Sol}_\mathcal{N}) \cong \mathcal{F}(\mathcal{E}(\mathcal{N}; E|_\mathcal{N})).$$

A detailed proof of the time-slice axiom using these tools is spelled out in Theorem 3.1 of [52].

# — 6 —

# Equicausal functionals

The machinery developed in the previous chapter allows us to study spaces of graded functionals that are of interest in the BV formalism. However, as stated before, this space of functionals is usually too large: When defining the Poisson bracket of classical field theory or the $\star$-product of quantum field theory, we need to pair our functionals through propagators of the wave operator defining the theory. These operations are generally ill-defined without proper care. Hence we would like to restrict to a subcomplex on which these operations are well-defined.

The approach taken in [40] was to consider graded microcausal functionals. However, the counterexamples we gave in Chapter 4 show that this is not an optimal choice; the algebraic structures defined on the microcausal functionals fail to close, and they are not stable under the homotopy operator defined in Theorem 5.6.3. We present here a new solution in terms of graded equicausal functionals, which is a strictly smaller class than the class of graded microcausal functionals. The main idea is to add the requirement that the derivatives of our functionals satisfy additional boundedness conditions as we allow the configuration to vary, for which the Hörmander spaces studied in Chapter 3 form the technical basis.

This definition was given already in my joint publication [52], specialised to the case of the Koszul complex. This definition was one of my contributions to this paper, as were the technical proofs exhibited in Sections 5, 6 and 7 and the appendices therein. Some of the material in this chapter is taken from this text, modified to work for general graded configuration spaces, rather than just for the Koszul complex.

## 6.1 DEFINITION AND BASIC PROPERTIES

Throughout this section $E \to \mathcal{M}$ denotes a fixed graded vector bundle over a spacetime $\mathcal{M}$. We recall the notations

$$\mathcal{A}^n = \mathrm{Sym}^n(\mathcal{E})'^{\mathbb{C}}$$
$$\mathcal{A} = \mathrm{Sym}(\mathcal{E})'^{\mathbb{C}} = \bigoplus_{n\in\mathbb{N}} \mathcal{A}^n$$

We identify $\mathcal{A}^n$ with a subset of $\mathcal{E}'(E^{!\boxtimes n})^{\mathbb{C}}$ throughout this chapter, as the symmetry properties of $\mathcal{A}^n$ are of lesser importance in the present discussion. As in the previous chapter, we suppress the complexification in the notation going forwards, and view all sections of exterior powers of $E^!$ as being complex.

In the definition of the microcausal functionals, we defined the sequence of cones $\Gamma_n \subset \dot{T}^*\mathcal{M}^n$ by

$$\Gamma_n = (V_+^{\times n} \cup V_-^{\times n}) \setminus \underline{0},$$

where $V_\pm$ denotes the forward/backward light cone bundle of $\mathcal{M}$. As a convention, we view zero covectors as null, so that $V_+ \cap V_- = \underline{0}$. If we are considering $n$ different spacetimes $\mathcal{M}_1 \dots \mathcal{M}_n$, then we abusively write

$$\Gamma_n = \left( \prod_{i=1}^n V_{+,\mathcal{M}_i} \cup \prod_{i=1}^n V_{-,\mathcal{M}_i} \right) \setminus \underline{0}, \tag{6.1}$$

when it is clear from the context which spacetimes we are referring to.

We register the following facts about these cones:

**Lemma 6.1.1.** *If $m, n, k \in \mathbb{N}$ and $k \leq m$, then*

$$\Gamma_m^c \dot{\times} \Gamma_n^c \subset \Gamma_{m+n}^c \tag{6.2}$$
$$\underline{0}_k \bullet \Gamma_m^c \subset \Gamma_{m-k}^c \tag{6.3}$$

*where $\underline{0}_k$ is the zero section of $\dot{T}^*\mathcal{M}^k$.*

We recall the definition of the partial contraction of cones $\Gamma \subset T^*M$ and $\Lambda \subset \dot{T}^*M \times N$:

$$\Gamma \bullet \Lambda = \{\xi \in T^*N \mid \exists \zeta \in \Gamma_0 \text{ such that } (\zeta, \xi) \in \Lambda\},$$

where $\Gamma_0 = \Gamma \cup \underline{0}$.

*Proof.* By taking the complement in $\dot{T}^*\mathcal{M}^n$ on both sides, the first inequality can be written as

$$(\Gamma_m^c \dot{\times} \Gamma_n^c) \cap \Gamma_{m+n} = \emptyset,$$

which is immediate from the definition of $\Gamma_i$.

For the second inequality, we write

$$\underline{0}_k \bullet \Gamma_m^c = \{\xi \in \dot{T}^*\mathcal{M}^{m-k} \mid \exists \zeta \in \underline{0}_k \subset T^*\mathcal{M}^k \text{ such that } (\zeta, \xi) \in \Gamma_m^c\},$$

which is clearly contained in $\Gamma_{m-k}^c$. □

We discuss a wish-list for properties a novel class of functionals should fulfil so that it does not exhibit the pathological features exhibited in Chapter 4. For simplicity, we state them for functionals $F : \mathcal{E}_0 \to \mathbb{C}$. To make sure that the wavefront sets work out in the definition of the Poisson bracket, we look for a subclass of the microcausal functionals, so that

$$F^{(n)} : \mathcal{E} \to \mathcal{E}'_{\Gamma_n^c}(E^{!\boxtimes n}),$$

with sufficient smoothness that Poisson brackets of allowed functionals are smooth and satisfy a Leibniz rule. Furthermore, we need these derivatives to be 'integrable' along curves $\gamma$ in $\mathcal{E}$:

$$\int_a^b F^{(n)} \circ \gamma(t) dt \in \mathcal{E}'_{\Gamma_n^c}(E^{!\boxtimes n}), \tag{6.4}$$

to ensure that the homotopy operator in equation (5.24) does not spawn additional singularities. Finally, we want the class to be sufficiently large, so that interesting functionals, i.e. the local ones, are elements of this space.

One solution one might attempt is to impose that the derivatives of $F$ define smooth maps into $\mathcal{E}'_{\Gamma_n^c}(E^{\boxtimes n})$. However, as this space is not complete, the integral in equation (6.4) would generally be valued in the completion of that space. This completion consists of the distributions whose Sobolev wavefront sets are all contained in $\Gamma_n^c$. A detailed study of this space, and its applications in AQFT, was performed in [27]. This approach offers several technical advantages, particularly that all integrals converge within the relevant space without additional requirements. Nevertheless, we have chosen not to take this route, as it introduces further technical complexity and departs from the framework of the smooth wavefront set, which remains the standard language among practitioners in AQFT.

Instead, we take the following, more bornological, approach:

**Definition 6.1.2.** *A functional $F : \mathcal{E}_0 \to \mathcal{A}^m$ is* ***equicausal*** *if, whenever $B \subset \mathcal{E}$ is a bounded set and $n \in \mathbb{N}$, the family*

$$F^{(n)}(B) \subset \mathcal{E}'(E^{!\boxtimes n+m})$$

*is an equicontinuous subset of $\mathcal{E}'_{\Gamma^c_{n+m}}(E^{!\boxtimes n+m})$. A functional $F : \mathcal{E}_0 \to \mathcal{A}$ is equicausal if it can be written as the sum of equicausal functionals of homogeneous weight.*

Equicausal functionals are stable under the Ev operation discussed in the previous chapter.

**Proposition 6.1.3.** *If $F$ is an equicausal functional, then* $\mathrm{Ev}F$ *is equicausal as well.*

*Proof.* For simplicity, we assume that $F$ is valued in $\mathcal{A}_0^{\hat{\odot} m} \hat{\odot} \bar{\mathcal{A}}^k$. In this scenario we have

$$\mathrm{Ev}F(\varphi) = \iota_{\varphi^{\otimes m}} F(\varphi)$$

Let $n \in \mathbb{N}$ and let $B \subset \mathcal{E}_0$ be bounded. By the Leibniz rule, the $n$'th derivative of $\mathrm{Ev}F$ is a sum of functionals of the form

$$\varphi \mapsto (\mathbb{1} \otimes \iota_{\varphi^{\otimes m-l}}) F^{(n-l)}(\varphi)$$

for $l \leq n$. As $F$ is equicausal, we have that

$$F^{(n-l)}(B) \subset \mathcal{E}'_{\Gamma^c_{m+k+n-l}}(E^{!\boxtimes m+k+n-l})$$

is an equicontinuous set. As $B^{\otimes m-l} \subset \mathcal{E}_0^{\otimes m-l}$ is bounded, it follows from Proposition 3.4.6 and equation (6.3) that

$$(\mathbb{1} \otimes \iota_{B^{\otimes m-l}}) \left( F^{(n-l)}(B) \right) \subset \mathcal{E}'_{\Gamma^c_{k+n}}(E^{!\boxtimes k+n})$$

is equicontinuous as well. □

This proposition immediately furnishes us with some examples of equicausal functionals. Recall (see e.g. [2]) that a Wick polynomial is a functional of the form

$$P(\varphi) = \sum_{n=0}^{k} \frac{1}{n!} u_n(\varphi^{\otimes n})$$

for $u_n \in \mathcal{E}'_{\Gamma^c_n}(E_0^{!\boxtimes n})$ fixed, which we take to be symmetric without loss of generality. Clearly, each $u_n$ is equicausal, when viewed as a constant functional. By the preceding proposition, so is $\mathrm{Ev}u_n$, and hence also

$$P = \sum_{n=0}^{k} \mathrm{Ev}u_n.$$

We conclude that Wick polynomials are equicausal.

The converse of Proposition 6.1.3 is not true; for instance, if $u \in \mathcal{E}'(E_0^!)$ then we can define a functional $G : \mathcal{E}_0 \to \mathcal{A}$ by

$$G(\varphi) = u - u(\varphi) \in \mathcal{E}'(E_0^!) \oplus \mathbb{C} \subset \mathcal{A}.$$

Clearly $G$ need not be equicausal as

$$G^{(0)}(0) = u,$$

which need not lie in $\mathcal{E}'_{\Gamma_1^c}(E_0^!)$. However, it is clear that $\mathrm{Ev}G = 0$ is equicausal. We hence define graded equicausal functionals as follows.

**Definition 6.1.4.** *A graded functional $[F] \in \mathcal{F}(E)$, as introduced in Definition 5.3.1, is **equicausal** if it has an equicausal representative, or equivalently if $\mathrm{Ev}F$ is equicausal, by Proposition 6.1.3. We denote the set of all graded equicausal functionals on $\mathcal{E}$ by $\mathcal{F}_{ec}(E)$, and the set of all compactly supported graded equicausal functionals by $\mathcal{F}_{c,ec}(E)$.*

It is not clear that this definition would achieve the first point on our wish list, as we have merely assumed a form of local boundedness rather than smoothness. However, we can show that the local boundedness implies that $F$ is conveniently smooth into this space, which will be strong enough for our purposes.

**Theorem 6.1.5.** *If $F \in C^\infty(\mathcal{E}_0 \to \mathcal{A}^k)$ is equicausal, then $F^{(n)}$ is a conveniently smooth functional*

$$\mathcal{E}_0 \to \mathcal{E}'_{\Gamma^c_{n+k}}(E^{!\boxtimes n+k}).$$

*Proof of Theorem 6.1.5.* We recall that a functional is conveniently smooth if it maps smooth curves to smooth curves, so let $\gamma : \mathbb{R} \to \mathcal{E}$ be smooth. We show that the curve $F^{(n)} \circ \gamma$ satisfies the hypotheses of Lemma 3.4.10 with $\Lambda = \Gamma^c_{n+k}$ , with respect to the bundle $E^{!\boxtimes n+k}$, as this implies that it is smooth into $\mathcal{E}'_{\Gamma^c_{n+k}}(E^{!\boxtimes n+k})$.

This reduces the task to showing that, for every $m \in \mathbb{N}$, $\partial_t^m(F^{(n)} \circ \gamma)$ maps bounded intervals of $\mathbb{R}$ to equicontinuous families in $\mathcal{E}'_{\Gamma^c_{n+k}}(E^{!\boxtimes n+k})$. We calculate, using Faà di Bruno's formula:

$$\partial_t^m \left(F^{(n)} \circ \gamma\right)(t) = \sum_{\pi \in P_m} F^{(n+|\pi|)}(\gamma_t) \left\{\bigotimes_{I \in \pi} \gamma_t^{(|I|)} \otimes \_\right\}$$

where the sum runs over all partitions $\pi$ of the set $(1, \dots, m)$. We can treat each of the terms separately, so that it suffices to consider the curve given by the contraction

$$\Phi_t := F^{(n+|\pi|)}(\gamma_t)\left\{\bigotimes_{I\in\pi}\gamma_t^{(|I|)}\otimes \_\right\} \in \mathcal{E}'_{\Gamma^c_{n+k}}(E^{!\boxtimes n+k})$$

for some fixed partition $\pi$. Let $J \subset \mathbb{R}$ be a bounded interval. As $\gamma$ is smooth the set

$$\left\{\bigotimes_{I\in\pi}\gamma_t^{(|I|)}\,\middle|\, t\in J\right\}\subset \mathcal{E}_0^{|\pi|}$$

is bounded. Furthermore,

$$\underline{0}_{|\pi|}\bullet\Gamma^c_{n+k+|\pi|}\subset\Gamma^c_{n+k},$$

by equation (6.3). Hence it follows from Proposition 3.4.6 that

$$\{\Phi_t\}_{t\in J}\subset\mathcal{E}'_{\Gamma^c_{n+k}}(E^{\boxtimes n+k})$$

is an equicontinuous family, and the result follows. □

We note in particular that the proof of this Theorem implies also that equation (6.4) is satisfied for equicausal functionals, as $\Phi_t$ satisfies the hypotheses of Lemma 3.4.8.

### 6.1.1 Some operations involving equicausal functionals

We discuss some of the operations that we defined on graded functionals in the last chapter in the context of equicausal functionals.

**Proposition 6.1.6.** *If $F$ and $G$ are equicausal functionals, then their product $F\odot G$, as defined in equation (5.12), is equicausal.*

*Proof.* We may assume that $F$ is of weight $k$ and $G$ is of weight $l$, as equicausal functionals are sums of terms of that form. We define maps, for $m, p \in \mathbb{N}$, by

$$\begin{aligned} F^{(m)}\odot G^{(p)} : \mathcal{E}_0 &\xrightarrow{F^{(m)}\otimes G^{(p)}} \left(\mathcal{E}_0'^{\hat\otimes m}\hat\otimes\mathcal{A}^k\right)\otimes\left(\mathcal{E}_0'^{\hat\otimes p}\hat\otimes\mathcal{A}^l\right)\\ &\longrightarrow \mathcal{E}_0'^{\hat\otimes m+p}\hat\otimes\left(\mathcal{A}^k\hat\otimes\mathcal{A}^l\right)\\ &\xrightarrow{\odot\otimes\odot}\mathcal{E}_0'^{\hat\odot m+p}\hat\otimes\mathcal{A}^{k+l}, \end{aligned}\qquad(6.5)$$

where the second map is an exchange of factors. If $\psi_1, \dots, \psi_n \in \mathcal{E}_0$, then it follows from the Leibniz rule with respect to the bilinear map $\odot$ that

$$\begin{aligned}(F \odot G)^{(n)}(\varphi)\{\psi_1, \dots, \psi_n\} \\ &= \sum_{m+p=n} \sum_{\sigma \in \mathrm{Sh}(m,p)} F^{(m)}(\varphi)\{\psi_{\sigma_1}, \dots, \psi_{\sigma_m}\} \odot G^{(p)}(\varphi)\{\psi_{\sigma_{m+1}}, \dots, \psi_{\sigma_{m+p}}\} \\ &= \sum_{m+p=n} F^{(m)} \odot G^{(p)}(\varphi)\{\psi_1, \dots, \psi_n\}\end{aligned}$$

so that

$$(F \odot G)^{(n)} = \sum_{m+p=n} F^{(m)} \odot G^{(p)}.$$

Now let $B \in \mathcal{E}_0$ be bounded. It follows from the fact that $F$ and $G$ are equicausal, combined with equation (6.2) and Lemma 3.4.5, that

$$F^{(m)}(B) \otimes G^{(p)}(B) \subset \mathcal{E}'_{\Gamma^c_{m+p+k+l}}(E^{!\boxtimes m+p+k+l})$$

is an equicontinuous family. The cones $\Gamma_i$ are completely symmetric, and the wavefront set is insensitive to factors of $-1$, so that the spaces $\mathcal{E}'_{\Gamma_i^c}$ are invariant under exchange of variables and the product of $\mathcal{A}$, and these operations respect equicontinuous sets. We conclude that $(F^{(m)} \odot G^{(p)})(B)$ is an equicontinuous set for all $m$ and $p$, and the result follows as the sum of equicontinuous sets is equicontinuous. □

Next, we study derivations and functional pullbacks. These will not respect equicausal functionals in full generality, as acting on distributions by the pullback of a map $\eta : \mathcal{E}(E) \to \mathcal{E}(E)$ changes the wavefront set, depending on the singular structure of $\eta^*$ (viewed as a distributional kernel).

**Definition 6.1.7.** *Let $E_1 \to \mathcal{N}_1$ and $E_2 \to \mathcal{N}_2$ be vector bundles over spacetimes $\mathcal{N}_1$ and $\mathcal{N}_2$.*[1] *A map $\eta : \mathcal{E}(E_2) \to \mathcal{E}(E_1)$ is called* ***causally compatible*** *if*

$$\mathrm{WF}(\eta) \subset \Xi := \{(\eta_1, \eta_2) \in (V_{\mathcal{N}_1,+} \times V_{\mathcal{N}_2,-}) \cup (V_{\mathcal{N}_1,-} \times V_{\mathcal{N}_2,+}) \cup \left(V^c_{\mathcal{N}_1} \times V^c_{\mathcal{N}_2}\right) \mid \eta_i \notin \underline{0}\}$$

*when $\eta$ is viewed as a kernel in $\mathcal{D}'(E_1 \boxtimes E_2^!)$.*

We note that, in particular

$$\Xi \subset ((V_{\mathcal{N}_1,+} \times V_{\mathcal{N}_2,+}) \cup (V_{\mathcal{N}_1,-} \times V_{\mathcal{N}_2,-}))^c,$$

[1] We do **not** mean these to be the degree 1 and 2 parts of a graded vector bundle $E$.

hence contracting with a causally stable operator does not reverse the direction of null covectors.

Differential operators on a spacetime $\mathcal{M}$ are causally compatible, as their kernels can be written as derivatives of $\delta$ distributions on the diagonal. The wavefront set of such a distribution is the conormal bundle of the diagonal $D \subset \mathcal{M} \times \mathcal{M}$

$$\mathrm{WF}(\delta_D) = ND = \{(x, x; \xi, -\xi) \in \dot{T}^*(\mathcal{M} \times \mathcal{M})\} \subset \Xi.$$

On top of this, propagators of normally hyperbolic operators are causally compatible. We recall that their wavefront sets are given by

$$\mathrm{WF}(\Delta^{A/R}) = \{(x, y; \xi, \xi') \subset \dot{T}^*\mathcal{M}^2 \,|\, x \in J^{\mp}(y) \text{ and } (x, \xi) \sim (y, -\xi')\} \cup ND,$$

where $(x, \xi) \sim (y, -\xi')$ if the vector $g^{-1}(\xi, \_)$ is tangent to a **null** geodesic from $x$ to $y$, such that $-\xi'$ is the parallel transport of $\xi$ along this curve, see e.g. Theorem A.5 in [69]. The wavefront sets of the Feynman and anti-Feynman propagators can be similarly bounded, see e.g. Theorem 4.5 in [66], and hence these are also causally compatible. The propagators of the Dirac equation have the same wavefront set, see e.g. Proposition A.7 in [69].

**Lemma 6.1.8.** *Let $n \in \mathbb{N}$ and $E_i \to \mathcal{N}_i$ be vector bundles over spacetimes for $i = 1, \ldots n$, and let $j \leq n$ with $\tilde{E}_j \to \tilde{\mathcal{N}}_j$ another such vector bundle. If $\eta : \mathcal{E}(E_j) \to \mathcal{E}(\tilde{E}_j)$ is causally compatible, then the action of $\eta$ on the $j$'th variable,*

$$\eta_j : \mathcal{E}\left(\boxtimes_{i=1}^{n} E_i\right) \to \mathcal{E}\left(\boxtimes_{i=1}^{j-1} E_i \boxtimes \tilde{E}_j \boxtimes \boxtimes_{i=j+1}^{n} E_i\right),$$

*admits a unique continuous extension*

$$\eta_j : \mathcal{D}'_{\Gamma_n}\left(\boxtimes_{i=1}^{n} E_i\right) \to \mathcal{D}'_{\Gamma_n}\left(\boxtimes_{i=1}^{j-1} E_i \boxtimes \tilde{E}_j \boxtimes \boxtimes_{i=j+1}^{n} E_i\right),$$

*where we extend the definition of $\Gamma_n$ as in equation (6.1).*

*Consequently, the pullback of $\eta_j$ defines a continuous map*

$$\eta_j^* : \mathcal{E}'_{\Gamma_n^c}\left(\boxtimes_{i=1}^{j-1} E_i^! \boxtimes \tilde{E}_j^! \boxtimes \boxtimes_{i=j+1}^{n} E_i^!\right) \to \mathcal{E}'_{\Gamma_n^c}\left(\boxtimes_{i=1}^{n} E_i^!\right),$$

*that maps equicontinuous sets to equicontinuous sets.*

*Proof.* We prove the $j = 1$ case only as all others are related to it by permuting variables. Let $u \in \mathcal{D}'_{\Gamma_n}(\boxtimes_{i=1}^n E_i)$. We denote by $\mathbb{R}_i$ the trivial line bundle over $\mathcal{N}_i$, and by $\tilde{\mathbb{R}}_1$ the trivial line bundle over $\tilde{\mathcal{N}_1}$. We denote by $\mathbb{1}$ the function that is constantly 1 on any of these spacetimes, and consider the distributions

$$\mathbb{1} \otimes u \in \mathcal{D}'_{\underline{0}_1 \times \Gamma_n}\left(\tilde{\mathbb{R}}_1 \boxtimes E_1 \boxtimes \bigboxtimes_{i=2}^n E_i\right),$$

and

$$\eta_1 \otimes \mathbb{1}^{\otimes n-1} \in \mathcal{D}'_{\Xi \times \underline{0}_{n-1}}\left(\tilde{E}_1 \boxtimes E_1^! \boxtimes \bigboxtimes_{i=2}^n \mathbb{R}_i\right).$$

These two distributions can be multiplied, as $\Xi \times \underline{0}_{n-1} + \underline{0}_1 \times \Gamma_n$ does not intersect the zero section. This product is valued in the bundle

$$\left(\tilde{\mathbb{R}}_1 \boxtimes E_1 \boxtimes \bigboxtimes_{i=2}^n E_i\right) \otimes \left(\tilde{E}_1 \boxtimes E_1^! \boxtimes \bigboxtimes_{i=2}^n \mathbb{R}_i\right) \cong \tilde{E}_1 \boxtimes D_{\mathcal{N}_1} \boxtimes \bigboxtimes_{i=2}^n E_i.$$

We would like to post-compose with the pushforward of the projection map

$$\pi : \tilde{\mathcal{N}_1} \times \mathcal{N}_1 \times \prod_{i=2}^n \mathcal{N}_i \to \tilde{\mathcal{N}_1} \times \prod_{i=2}^n \mathcal{N}_i,$$

to define the action of $\eta_1$ by

$$\eta_1 u = \pi_* \left(\left(\eta_1 \otimes \mathbb{1}^{\otimes n-1}\right) \cdot (\mathbb{1} \otimes u)\right). \tag{6.6}$$

We check that the pushforward is well-defined. Set $C = \operatorname{supp} \eta_1 \times \prod_{i=2}^n \mathcal{N}_i$, and let $K \subset \tilde{\mathcal{N}_1} \times \prod_{i=2}^n \mathcal{N}_i$ be compact. Without loss of generality, we may assume that $K = \prod_{i=1}^n K_n$ for all $K_i$ compact. If then $\pi_1 : \tilde{\mathcal{N}_1} \times \mathcal{N}_1 \to \tilde{\mathcal{N}_1}$ is the projection onto the first factor, then

$$\pi^{-1}(K) = K_1 \times \mathcal{N}_1 \times \prod_{i=2}^n K_i = \pi_1^{-1}(K_1) \times \prod_{i=2}^n K_i.$$

As $\pi_1^{-1}(K_1) \cap \operatorname{supp}(\eta_1)$ is compact by Lemma 5.6.4, we conclude that $\pi|_C$ is a proper map. As

$$\operatorname{supp}\left(\left(\eta_1 \otimes \mathbb{1}^{\otimes n-1}\right) \cdot (\mathbb{1} \otimes u)\right) \subset \operatorname{supp}\left(\eta_1 \otimes \mathbb{1}^{\otimes n-1}\right) = C,$$

the pushforward in equation (6.6) is well-defined and continuous.

Finally, we check that $\pi_*(\Xi \times \underline{0}_{n-1} + \underline{0}_1 \times \Gamma_n) \subset \Gamma_n$. Let $(\eta_1, \xi_2, \ldots, \xi_n)$ be an element of this cone, which means that there exist $(\eta_1, \eta_2) \in \Xi_0$ and $(\xi_1, \ldots, \xi_n) \in (\Gamma_n)_0$ such that $\eta_2$ and $\xi_1$ are covectors over the same point in $\mathcal{N}_1$, satisfying $\eta_2 + \xi_1 \in \underline{0}$.

- If $(\xi_1, \dots, \xi_n) \in \underline{0}$, then also $\eta_2 \in \underline{0}$, and hence $(\eta_1, \eta_2) \in \underline{0}$. But then $(\eta_1, \xi_2, \dots, \xi_n) \in \underline{0}$ which is impossible.
- If $(\xi_1, \dots, \xi_n) \in V_+^n$, then $\eta_2 \in V_{\mathcal{N}_1,-}$. Hence it follows that $\eta_1 \in V_{\tilde{\mathcal{N}}_1,+}$ so that $(\eta_1, \xi_2, \dots, \xi_n) \in \Gamma_n$.
- The case where $(\xi_1, \dots, \xi_n) \in V_-^n$ is analogous.

Finally, $\eta_1$ is continuous because it is the composition of continuous operations. □

**Corollary 6.1.9.** *Let $E_1 \to \mathcal{N}_1$ and $E_2 \to \mathcal{N}_2$ be graded vector bundles over spacetimes. If $\gamma : \mathcal{E}(E_1) \to \mathcal{E}(E_2)$ and $\eta : \mathcal{E}(E_2) \to \mathcal{E}(E_2)$ are causally compatible and $F \in \mathcal{F}_{ec}(E_2)$, then both* $\mathrm{Der}(\eta^*)F$ *and $\gamma^\sharp F$ are equicausal, so that*

$$\begin{aligned} \mathrm{Der}(\eta^*) : \mathcal{F}_{ec}(E_2) &\to \mathcal{F}_{ec}(E_2), \\ \gamma^\sharp : \mathcal{F}_{ec}(E_2) &\to \mathcal{F}_{ec}(E_1). \end{aligned}$$

*Proof.* This is clear for $\mathrm{Der}(\eta^*)F$ from the previous lemma, as that quantity is obtained by taking a derivative and acting by $\eta^*$ on the various variables, all of which are operations that preserve the relevant equicontinuous sets.

For $\gamma^\sharp F$, we assume that $F$ is of degree $m$, so that

$$(\gamma^\sharp F)^{(n)}(\varphi) = \gamma^{*\otimes n+m} F^{(n)}(\gamma\varphi).$$

If then $B \subset \mathcal{E}(E_1)$ is bounded, then also $\gamma B \subset \mathcal{E}(E_2)$ is bounded, so that

$$F^{(n)} \circ \gamma(B) \subset \mathcal{E}'_{\Gamma^c_{m+n}}(E_2^{!\boxtimes n+m})$$

is an equicontinuous set. The result then follows from applying the previous lemma $n+m$ times in succession. □

Consequently, if $(\mathcal{E}(M,E), d)$ is a field complex, then $(\mathcal{F}_{ec}, \mathrm{Der}(d^*))$ is a subcomplex of $(\mathcal{F}, \mathrm{Der}(d^*))$, as differential operators are causally compatible.

**Corollary 6.1.10.** *Let $V$ be a cochain complex of topological vector spaces and let $(\mathcal{E}(M,E), d)$ be a field complexes admitting a deformation retract*

$$V \underset{\iota}{\overset{\pi}{\leftrightarrows}} \mathcal{E}(M,E) \circlearrowleft \eta.$$

*If both $\eta$ and $\iota \circ \pi$ are causally compatible, then the homotopy operator $H$ defined in equation (5.24) maps $\mathcal{F}_{ec}(E)$ to itself.*

*Proof.* Suppose that $F : \mathcal{E}_0 \to \mathcal{A}^k$ is equicausal. Recall that $\gamma_\lambda$ was defined by

$$\gamma_\lambda = \lambda(\mathbb{1} - \iota \circ \pi) + \iota \circ \pi,$$

and that $H$ was defined by

$$HF = \int_0^1 \lambda^{-1} \gamma_\lambda^\sharp \operatorname{Der}(\eta^*) F d\lambda. \tag{6.7}$$

We note that $\gamma_\lambda$ is causally compatible for every $\lambda \in \mathbb{R}$.

If $B \subset \mathcal{E}_0$ is bounded then $\{\gamma_\lambda \varphi \in \mathcal{E}_0 \,|\, \lambda \in [0,1], \varphi \in B\}$ is bounded as $\gamma$ is jointly continuous in $\lambda$ and $\varphi$. If then $n \in \mathbb{N}$, it follows from the previous corollary that

$$\left\{ \lambda^{-1} \left[\gamma_\lambda^\sharp \operatorname{Der}(\eta^*) F\right]^{(n)} (\varphi) \,\middle|\, \varphi \in B, \lambda \in [0,1] \right\} \subset \mathcal{E}'_{\Gamma^c_{n+k}} \left(E^{!\boxtimes n+k}\right)$$

is equicontinuous, as the polynomial factors of $\lambda$ stemming from $\gamma_\lambda^\sharp$ are all smaller than 1. Hence there is a continuous seminorm $q$ of $\mathcal{D}'_{\Gamma_{n+k}}(E^{\boxtimes n+k})$ such that, for all $Z$ in that space

$$\left|\left\langle \lambda^{-1} \left[\gamma_\lambda^\sharp \operatorname{Der}(\eta^*) F\right]^{(n)} (\varphi), Z \right\rangle\right| \leq q(Z) \quad \forall \varphi \in B, \lambda \in [0,1],$$

and hence also

$$\left\langle (HF)^{(n)}(\varphi), Z \right\rangle \leq q(Z) \quad \forall \varphi \in B,$$

by Proposition 2.3.4 and Lemma 3.4.8. □

As a corollary, all statements about the Koszul complex that we have shown in Section 5.6.1 remain true when we restrict to the equicausal subcomplex; If we write

$$\mathcal{F}_{ec}(\mathrm{Sol}) = \iota^\sharp \mathcal{F}_{ec}(\mathcal{E}(E) \xrightarrow{P} \mathcal{E}(E)),$$

the on-shell equicausal functionals, then $\mathcal{F}_{ec}(\mathcal{E}(E) \xrightarrow{P} \mathcal{E}(E))$ is a resolution of that space. Similarly, the time-slice axiom holds for the algebra of equicausal multi-vector fields.

## 6.2 LOCAL FUNCTIONALS

We have already given a class of examples of equicausal functionals: The Wick polynomials. In this section, we give our main genuinely non-polynomial example, by way of the local functionals. The results presented in this section appeared already in [52] in an abridged version, which was one of my contributions to that work. The exposition given here contains all the technical details that were left out of the publication in the interest of brevity.

We shall take $E$ to be trivial in this section, so that $\mathcal{E} = C^\infty(\mathcal{M})$. The concept of graded local functionals is not completely unambiguous; The equivalence of the definition in terms of jet-bundles in [41] and one of the form given in 2.3.8 was conjectured in [16], but a precise statement is not given due to the absence of a clear formulation of the additivity property for graded functionals. To the best of our knowledge, this point has yet to be clarified in the literature, and we will not attempt to do so now.

We first prove an explicit criterion for equicausality.

**Proposition 6.2.1.** *Suppose that $F$ is a smooth functional on $\mathcal{E}$ such that for any $n \in \mathbb{N}$ and $\varphi \in \mathcal{E}$, there is a neighbourhood $V$ of $\varphi$, a closed cone $\Xi \subset \Gamma_n^c$ and a compact set $K \subset \mathcal{M}^n$ such that $F^{(n)}$ restricts to a continuous map*

$$F^{(n)}\Big|_V : V \to \mathcal{D}'_\Xi(K).$$

*Then $F$ is equicausal.*

*Proof.* Let $B \subset \mathcal{E}$ be bounded and let $n \in \mathbb{N}$. Without loss of generality, we may assume that $B$ is closed, and hence compact as $\mathcal{E}$ is Montel. For each $\varphi \in B$, pick a neighbourhood as in the proposition, and denote it by $V_\varphi$. By restricting $V_\varphi$ if necessary, we may assume they are all closed neighbourhoods. These $V_\varphi$ cover $B$, and hence we may choose a finite subset $\{\varphi_i\}_{i=1}^m \subset B$, such that the sets $V_i := V_{\varphi_i}$ still cover $B$.

By assumption, there exist $\Xi_i$ and $K_i$ so that

$$F^{(n)}\Big|_{V_i} : V_i \to \mathcal{D}'_{\Xi_i}(K_i)$$

is continuous. As $B \cap V_i$ is compact, being the intersection of a compact set with a closed set, $F^{(n)}(B \cap V_i)$ is compact in $\mathcal{D}'_{\Xi_i}(K_i)$, and hence bounded. Therefore, it is an equicontinuous subset of $\mathcal{E}'_{\Gamma_n^c}(\mathcal{M}^n)$ by Proposition 3.4.2. It follows that

$$F^{(n)}(B) = \bigcup_{i=1}^k F^{(n)}(B \cap V_i) \subset \mathcal{E}'_{\Gamma_n^c}(\mathcal{M}^n).$$

is equicontinuous as well. $\square$

Recall that we defined local functionals, following [16], as functionals $F : \mathcal{E} \to \mathbb{C}$ such that

- $F$ is *additive*, in the sense that

$$F(\varphi_1 + \varphi_2 + \varphi_3) = F(\varphi_1 + \varphi_2) + F(\varphi_2 + \varphi_3) - F(\varphi_2),$$

  whenever $\mathrm{supp}(\varphi_1) \cap \mathrm{supp}(\varphi_3) = \emptyset$.

- $F^{(1)}(\varphi) \in \mathcal{D}(\mathcal{M})$ for any $\varphi \in \mathcal{E}$, and furthermore,
- $F^{(1)}$ is implemented by a **smooth** map

$$\nabla F : \mathcal{E} \to \mathcal{D}(\mathcal{M}),$$

  meaning that

$$F^{(1)}(\varphi)\{\psi\} = \int_{\mathcal{M}} \nabla F(\varphi)(x)\, \psi(x)\, dV_x. \tag{6.8}$$

These three properties tightly constrain the form of the derivatives of local functionals.

**Proposition 6.2.2.** *If $F$ is a local functional, $l \in \mathbb{N}$ and $\varphi_0 \in \mathcal{E}$, then there is a neighbourhood $V \subset \mathcal{E}$ of $\varphi_0$ and $k \in \mathbb{N}$ such that, in any coordinate system of $\mathcal{M}$, the distributional kernel of $F^{(l)}$ takes the form*

$$F^{(l)}(\varphi)(x_1, \ldots, x_l) = \sum_{|\vec{\alpha}| \le k} f_{\vec{\alpha}}(\varphi)(x_1) \partial^{\vec{\alpha}} \left[\delta(x_2 - x_1) \ldots \delta(x_l - x_1)\right] \quad \forall \varphi \in V \tag{6.9}$$

*where $\vec{\alpha} = (\alpha_1, \ldots \alpha_l)$ is a vector of multi-indices, and the $f_{\vec{\alpha}}$ are smooth maps $V \to \mathcal{E}(K)$ for some compact $K \subset \mathcal{M}$.*

The proof rests on the following $n$-ary generalisation of equation (29) in [16].

**Lemma 6.2.3.** *Let $F$ be a functional whose first derivative is implemented by a map*

$$\nabla F : \mathcal{E}(\mathcal{M}) \to \mathcal{D}(\mathcal{M})$$

*as in equation (6.8). Then the $n$-th derivative of $F$ can be calculated as*

$$F^{(n)}(\varphi_0)\{\psi_1, \ldots, \psi_n\} = \int_{\mathcal{M}} \left((\nabla F)^{(n-1)}(\varphi_0)\{\psi_2, \ldots, \psi_n\}\right)(x)\, \psi_1(x) dV_x$$

*Proof.* Due to the fact that partial derivatives commute, we have that

$$\begin{aligned} F^{(n)}(\varphi_0)\{\psi_1,\dots,\psi_n\} &= \frac{d^n}{dt_1\dots dt_n} F\left(\varphi_0 + \sum_{i=1}^n t_i\psi_i\right)\Bigg|_{t_i=0}, \\ &= \frac{d^{n-1}}{dt_2\dots dt_n}\int_{\mathcal{M}} \nabla F\left(\varphi_0 + \sum_{i=2}^n t_i\psi_i\right)(x)\psi_1(x)dV_x\Bigg|_{t_i=0}. \end{aligned} \tag{6.10}$$

The proof will be finished once we show that we can move the derivatives with respect to $t_2,\dots,t_n$ inside the integral.

We denote the integrand by $J(t_2,\dots,t_n,x)$, which is a smooth map $\mathbb{R}^{n-1}\times\mathcal{M}\to\mathbb{R}$. Indeed, the composition

$$\bar{J}:(t_2,\dots,t_n)\mapsto \varphi_0+\sum_{i=2}^n t_i\psi_i \mapsto \nabla F\left(\varphi_0+\sum_{i=2}^n t_i\psi_i\right)\mapsto \nabla F\left(\varphi_0+\sum_{i=2}^n t_i\psi_i\right)\cdot\psi_1,$$

is a smooth map $\mathbb{R}^{n-1}\to\mathcal{E}(\mathcal{M})$. By Theorem 40.1 in [72], this implies that $J$ is smooth as well. By Proposition III.11 in [16], there is a neighbourhood $V$ of $\varphi_0$ such that

$$\operatorname{supp} F(\varphi)\subset K \qquad \forall\, \varphi\in V,$$

where $K\subset\mathcal{M}$ is some fixed compact set. We restrict all $t_i$ to lie within a compact interval $I$ containing zero, so that

$$\varphi_0+\sum_{i=2}^n t_i\psi_i\in V \qquad \forall\, t_i\in I.$$

Then $J$ is a smooth map on the compact set $I^{n-1}\times K$, so that its derivatives are bounded to all orders. Hence we may exchange derivatives and integration in equation (6.10). □

*Proof of Proposition 6.2.2.* As in Proposition III.11 of [16], we pick a neighbourhood of $\varphi_0$ such that $F|_V$ is supported within a compact set $K$, and of order bounded by $k\in\mathbb{N}$. By going to a coordinate system and using a partition of unity if necessary, we may assume that $K\subset\mathbb{R}^d$, where $d$ is the dimension of $\mathcal{M}$. We suppress a choice of coordinates in order to streamline the proof.

By the additive property of $F$, it follows from Proposition V.5 in [16] that the $n$'th derivative is supported on the thin diagonal of $K$, i.e. on the set

$$D_n=\{(x,x,\dots,x)\in K^n \mid x\in K\}\subset(\mathbb{R}^d)^n.$$

By a standard result in distribution theory, see e.g. Proposition 2.3.5 in [58], distributions supported on the diagonal are of the form (6.9) for $f_{\vec{\alpha}}(\varphi) \in \mathcal{E}'(\mathbb{R}^d)$ *a priori* arbitrary.

We show that they define smooth functions

$$f_{\vec{\alpha}} : V \to \mathcal{E}(K).$$

We contract the $x_i$ variables in equation (6.9) with oscillatory functions $e_{\xi_i}(x_i) = \exp(-i\xi_i \cdot x_i)$. Combining the fact that the map $\xi \mapsto e_\xi$ is smooth from $\mathbb{R}^d \to \mathcal{E}$ with Lemma 6.2.3, we obtain a smooth map

$$\Phi : \begin{cases} V \times (\mathbb{R}^d)^{n-1} & \to V \times \mathcal{E}^{n-1} & \to \mathcal{E}(K), \\ (\varphi, \xi_2, \ldots, \xi_n) & \mapsto (\varphi, e_{\xi_2}, \ldots, e_{\xi_n}) & \mapsto (\nabla F)^{(n-1)}(\varphi)\{e_{\xi_2}, \ldots, e_{\xi_n}\}. \end{cases}$$

recalling that $\mathcal{E}(K)$ is embedded in $\mathcal{D}(\mathcal{M})$, see e.g. Lemma 13.1 in [72].

Inserting the expression from equation (6.9), we obtain the explicit expression

$$\Phi(\varphi, \xi_2, \ldots, \xi_n) = \sum_{|\vec{\alpha}| \leq k} f_{\vec{\alpha}}(\varphi)(-i)^{|\vec{\alpha}|} \prod_{j=2}^{n} \xi_j^{\alpha_j} e_{\xi_j},$$

and we can extract

$$f_0(\varphi) = \Phi(\varphi, 0, \ldots, 0),$$

which is hence a smooth map $V \to \mathcal{E}(K)$. We proceed by induction on the order of the multi-indices: Suppose that we have shown that $f_\alpha$ is smooth for all $|\vec{\alpha}| \leq m$, so that

$$\Phi_m(\varphi, \xi_2, \ldots, \xi_n) = \sum_{|\vec{\alpha}| \leq m} f_{\vec{\alpha}}(\varphi) i^{|\vec{\alpha}|} \prod_{j=2}^{n} \xi_j^{\alpha_j} e_{\xi_j},$$

defines a smooth map into $\mathcal{E}(K)$. Let $|\vec{\beta}| = m+1$, we can extract

$$f_{\vec{\beta}}(\varphi) = \frac{i^{|\vec{\beta}|}}{\vec{\beta}!} \partial_{\vec{\xi}}^{\vec{\beta}} (\Phi - \Phi_m)(\varphi, 0, \ldots, 0),$$

which is hence a smooth map $V \to \mathcal{E}(K)$. □

**Theorem 6.2.4.** *Local functionals are equicausal.*

*Proof.* Let $F$ be a local functional, $\varphi_0 \in \mathcal{E}$, $n \in \mathbb{N}$ and let $V, k, K$, and $f_\alpha$ be as in the previous proposition. We denote by $ND_n$ the conormal bundle to the thin diagonal $D_n \subset \mathcal{M}^n$, and note that $WF(\partial^\alpha \delta_{D_n}) \in ND_n$ for any multi-index $\alpha$.

We showed already in section 2.3.3 that $Z_n \cap \Gamma_n = \emptyset$. Hence the composition

$$\varphi \mapsto f_\alpha(\varphi) \mapsto f_\alpha(\varphi)\partial^\alpha \delta_{D_n}$$

is a smooth map $V \to \mathcal{E}(K) \to \mathcal{D}'_{Z_n}(K)$. Summing over $\alpha$, we obtain that $F^{(n)}|_V$ is a smooth map $V \to \mathcal{D}_{Z_n}(K)$. The result then follows from Proposition 6.2.1. □

We hence obtain a hierarchy of functionals

$$\text{Local} \implies \text{Equicausal} \implies \text{Microcausal.}$$

An example of an equicausal functional that is not local is given by the Poisson bracket of two local functionals, as we will show in the next section. The pathological functionals constructed in Chapter 4 are examples of microcausal functionals that are not equicausal. Wick polynomials are equicausal, but not necessarily local.

## 6.3 CLOSURE UNDER ⋆-PRODUCT

To close out the technical part of the thesis, we show that graded equicausal functionals close under the $\star$ product of the quantum theory. We identify graded functionals as functionals $\mathcal{E}_0 \to \bar{\mathcal{A}}$ in this section as that simplifies notation considerably.[2]

We recall the definition of the $\star$-product of complex scalar field theory here to motivate the formulae. Let $\Delta$ be the commutator function of a scalar field theory, which we view as a kernel in $\mathcal{D}'(\mathcal{M} \times \mathcal{M})$ using the volume form of the spacetime $\mathcal{M}$. If $F, G$ are two equicausal functionals on $\mathcal{E} = C^\infty(M, \mathbb{C})$, then their first derivatives are distributional sections of $D_{\mathcal{M}}$. Hence we can define their Poisson bracket by pairing

$$\{F, G\}(\varphi) := \left\langle F^{(1)}(\varphi), \Delta G^{(1)}(\varphi) \right\rangle = \left\langle \Delta\, ,\, F^{(1)}(\varphi) \otimes G^{(1)}(\varphi) \right\rangle .$$

Due to Theorem 6.1.5, this is a conveniently smooth functional, and hence smooth as $\mathcal{E}$ is Fréchet and $\mathbb{C}$ is complete. Furthermore, one can show that it is again equicausal, see Section 7 in [52].

The aim of formal deformation quantisation is now to find an associative product $\star$ on $\mathcal{F}_{ec}(\mathcal{M})[[\hbar]]$, i.e. the formal power series in $\hbar$ with coefficients in equicausal

[2] Equivalently, we could view all expression as first being acted on by the contraction operator Ev.

functionals, such that

$$[F, G]_\star = F \star G - G \star F = i\hbar\{F, G\} + \mathcal{O}(\hbar^2)$$

The obvious first thing to try is the Moyal-Weyl product, given by

$$F \star G(\varphi) = \sum_n \frac{\hbar^n}{n!} \left\langle F^{(n)}(\varphi)\,, \left(\frac{i}{2}\Delta\right)^{\otimes n} G^{(n)}(\varphi) \right\rangle . \tag{6.11}$$

However, this product is ill-defined, as the commutator function is too singular. For instance, if $F$ and $G$ are local functionals, then their second derivatives are proportional to the delta distribution in two variables. Hence, to calculate their $\star$-product, we would need to calculate the square of $\Delta$. But it is readily checked from the expression of $\mathrm{WF}(\Delta)$ in 2.24 that this is not well-defined.[3]

To remedy this, we use a Hadamard two-point function. From a physical perspective, this corresponds to selecting a notion of positive energy, or positive frequency, which is not unambiguous on curved spacetimes. This choice essentially determines which modes of the field are associated with particles and which with antiparticles, thereby specifying a ground state for the theory.

Mathematically, this is encoded as a distributional bisolution to the equations of motion, i.e.

$$(P \otimes \mathbb{1})\Delta^+ = 0$$
$$(\mathbb{1} \otimes P)\Delta^+ = 0,$$

whose antisymmetric part equals $\frac{i}{2}\Delta$. Furthermore, it satisfies the *Hadamard condition*:

$$\mathrm{WF}(\Delta^+) \subset V_+ \times V_-. \tag{6.12}$$

This is sometimes called a *frequency splitting* of $\Delta$, as we have discarded all covectors along $V_-$ in the first argument.

Existence of Hadamard states for a wide class of theories has been shown in the literature. The notion of Hadamard state has been around since the work of Adler, Lieberman and Ng in [1], albeit in a now dated formulation, and existence results were given in specific cases by Fulling et al. in [45, 43]. Radzikowski reformulated the existing notion of Hadamard state into the modern microlocal formulation given here in [66], and showed that scalar theories admit Hadamard states. For the case of

[3] At least not through Hörmander's criterion for multiplication of distributions.

vector valued normally hyperbolic operators, we refer the reader to Sahlman and Verch in [68]. The case of Dirac fields is treated by Sanders in [69]. Finally, the Proca field is treated by Fewster and Pfenning in [38].[4]

We can write

$$\Delta^+ = \frac{i}{2}\Delta + H,$$

where $H$ is the symmetric part of $\Delta^+$.[5] The distribution $H$ is a symmetric bisolution, and we call a symmetric bisolution $H$ such that equation (6.12) is satisfied for $\frac{i}{2}\Delta + H$ a *Hadamard function*. We can add a smooth symmetric bisolution to the equations of motion to $H$ without changing the wavefront set of $\Delta^+$, which implies that the choice of two-point function is highly non-unique. We assume that such a choice has been made moving forwards.

If we substitute $\Delta^+$ for $\Delta$ in equation (6.11), one obtains a $\star$ product which is well-defined on equicausal functionals, and is closed on that set. We provide an explicit proof of this fact in [52], which we will not repeat as we prove a more general version in this section.

We note that this product can formally be written as

$$F \star G = \mu \exp\left(\hbar \left\langle \Delta^+, \frac{\delta}{\delta\varphi_1} \otimes \frac{\delta}{\delta\varphi_2} \right\rangle\right)(F \otimes G), \tag{6.13}$$

where $\mu$ is the operator that takes a functional in two variables, and evaluates it on the diagonal of $\mathcal{E} \times \mathcal{E}$. This 'exponential' form of the star product was studied in detail in [51]. Note that, in rewriting this explicit form, we use Theorem 6.1.5 to be able to conclude that derivatives can be performed one-by-one.

We discuss the most general version we think one might need, which also includes a commutator function that links components in non-zero degrees. The main motivation for this is the work on Green hyperbolic complexes of Benini, Musante and Schenkel in [9, 10], where they construct a Poisson algebra describing the observables on a Green-hyperbolic complex. We briefly discuss the main objects in their definition, and use those to define a $\star$-product on graded equicausal functionals.

We consider a **Free BV theory**, which is given by the following data: First off, it is a field complex $(\mathcal{E}(E), d)$ over a spacetime $\mathcal{M}$. Furthermore, there exists an inner product on the fibers of $E$, denoted by $\langle \_, \_ \rangle$ which is of degree $-1$, giving us a way

[4]This paragraph is by no means meant to give a historically accurate accounting of who proved existence in these cases first. We rather mean to highlight some sources that we feel give an insightful treatment.

[5]We follow the conventions of [67] with respect to factors of $i$

to pair configurations into functions. This inner product is required to be compatible with the differential, in the sense that for all homogeneous $\varphi, \psi \in \mathcal{E}(\mathcal{M}; E)$ with compact overlapping support

$$\int_{\mathcal{M}} (d\varphi, \psi) + (-1)^{|\varphi|}(\varphi, d\psi) dV_g = 0,$$

which can be viewed as a self-adjointness condition on $d$. Finally, there exists a Green's witness for the complex, which is a differential operator $W \in \mathrm{Hom}^{-1}(\mathcal{E}, \mathcal{E})$ such that $[d, W]$ is a Green hyperbolic operator. We denote the Green's functions with respect to this operator by $\Delta^{A/R}$. The advanced and retarded Green's homotopies are

$$\Lambda^{A/R} = \Delta^{A/R} \circ W,$$

and the Poisson bracket of the theory can be defined with respect to the advanced minus retarded homotopy

$$\Lambda = \Lambda^A - \Lambda^R$$

and the fiber inner product, by a similar formula as above. A $\star$-product can then be defined by a similar formula as for the Moyal-Weyl product above, as long as one restricts to regular functionals.

*A priori*, the kernel of $\Lambda$ is an element of

$$\mathcal{D}'(E \boxtimes E^!),$$

and it is of degree $-1$. For our purposes it will be more convenient to work with objects of degree 0, so we use the fiber metric on $E$ together with the volume form of $\mathcal{M}$ to give an isomorphism of bundles

$$\begin{aligned} E &\to E^!, \\ (e, x) &\mapsto (\langle e, \_\rangle \otimes dV_{g,x}, x) \in E^*_x \otimes (D_{\mathcal{M}})_x, \end{aligned}$$

which lowers degree by 1. We leave this isomorphism implicit in the notation, and view $\Lambda^{A/R}$ as elements of $\mathcal{D}(E^{\boxtimes 2})$ of degree 0 from here on out.

Similar to above, there is no reason to expect the kernel of $\Lambda$ to be sufficiently nice for the star product to be well-defined. We hence add another requirement onto the data of a free BV theory, namely that there exists a Hadamard function for $\Lambda$, which is now a degree 0 symmetric bidistribution $H \in D'(E \boxtimes E)$ satisfying

$$dH + Hd = 0$$

such that

$$\mathrm{WF}(\Lambda^+) := \mathrm{WF}\left(\frac{i}{2}\Lambda + H\right) \in V_+ \times V_-. \tag{6.14}$$

Again, we assume a choice of $H$ to have been made from here on out. As most examples of Green hyperbolic complexes in the literature are in some way derived from the case of a normally hyperbolic operator, we find it likely that Hadamard functions like this can be found for most cases of interest by appealing to the normally hyperbolic scenario. We will not delve into this question further at this point.

We take a bit of a detour before defining the $\star$ product with respect to $\Lambda^+$. We gave a proof of the closure of the $\star$-product of scalar field theory in [52] already. This proof treats all the contractions with propagators at once, which leads to the requirement to introduce several permutations to keep track of which variable gets paired with which. When allowing for graded functionals, more variables are introduced, and extending that proof would become needlessly messy.

Instead, we use an analogue of equation (6.13) to proceed step by step. We first take the tensor product of two equicausal functions, after which we 'mix' them together by contracting their distributional variables using $\Lambda^+$. After that, we restrict to the diagonal, and take the product of any leftover graded parts.

Hence we introduce a **graded bifunctional** on $\mathcal{E}(E)$ to be a smooth functional

$$L : \mathcal{E}(E_0) \times \mathcal{E}(E_0) \to \bar{\mathcal{A}} \hat{\otimes} \bar{\mathcal{A}},$$

where $E_0$ is the degree 0 part of $E$. We define an operator turning a graded bifunctional into a graded functional by

$$(\mu L)(\varphi) = \odot L(\varphi, \varphi).$$

Conversely, if $F$ and $G$ are graded functionals, then their tensor product defines a bifunctional by

$$F \otimes G(\varphi_1, \varphi_2) = F(\varphi_1) \otimes F(\varphi_2),$$

We note that we can write the product of two functionals, as defined in equation (5.12), as

$$F \odot G = \mu(F \otimes G).$$

We now want to define a notion of graded equicausal bifunctional, so that we can introduce contractions with $\Lambda^+$ as an intermediate step to define the $\star$-product. Furthermore, we require that the tensor product of two equicausal functionals is an equicausal bifunctional, and that $\mu$ maps equicausal bifunctionals to equicausal functionals.

For this purpose, we define a new family of cones by

$$\Upsilon_{n,m} = \left((V_-^n)^c \dot{\times} (V_+^m)^c\right) \cap \Gamma_{n+m}^c \tag{6.15}$$

Clearly, we have inclusions

$$\Gamma_n^c \dot{\times} \Gamma_m^c \subset \Upsilon_{n,m} \quad \forall n, m \neq 0, \tag{6.16}$$

and

$$\Upsilon_{n,m} \subset \Gamma_{n+m}^c \quad \forall n, m \in \mathbb{N}. \tag{6.17}$$

We introduce some notation to help us keep track of wavefront sets when contracting with $\Lambda^+$. Let $k \geq 2$ and $\Xi \subset \dot{T}^*\mathcal{M}^k$ and $\Gamma \subset \dot{T}^*\mathcal{M}^2$ be cones (not necessarily closed or open). If $1 \leq i < j \leq k$, then we define

$$\Gamma \bullet_{i,j} \Xi = \left\{ (\xi_1, \ldots \hat{\xi}_i \ldots \hat{\xi}_j \ldots \xi_k) \in T^*\mathcal{M}^{k-2} \,\middle|\, \exists (\xi_i, \xi_j) \in \Gamma_0 \right.$$
$$\left. \text{such that } (\xi_1, \ldots, \xi_n) \in \Xi \right\},$$

where a hat indicates an absent covector. This cone is closed when $\Gamma$ and $\Xi$ are. Hence, if we contract the $i$'th and $j$'th variables of a distribution $u \in \mathcal{E}'(E^{\boxtimes k})$ with $\Lambda^+$:

$$\left((\iota_{\Lambda^+})_{i,j} u\right)(x_1, \ldots \hat{x}_i \ldots \hat{x}_j \ldots x_n) = \int_{\mathcal{M}^2} \Lambda^+(x_i, x_j) u(x_1, \ldots x_n) dx_i dx_j$$

then the resulting distribution has wavefront set contained in $\mathrm{WF}(\Lambda^+)' \bullet_{i,j} \mathrm{WF}(u)$, provided that that cone does not intersect the zero-section, so that the contraction is well-defined. Recall that this contraction can be defined as multiplication by $\Lambda^+$ (with suitably permuted variables), followed by the pushforward of the projection

$$\pi_{i,j} : \begin{cases} \mathcal{M}^k \mapsto \mathcal{M}^{k-2} \\ (x_1, \ldots, x_k) \to (x_1, \ldots \hat{x}_i \ldots \hat{x}_j \ldots, x_k). \end{cases}$$

**Lemma 6.3.1.** *Let $n, m \neq 0$. If $1 \leq 1 \leq n$ and $n + 1 \leq j \leq n + m$, then*

$$\mathrm{WF}(\Lambda_+)' \bullet_{i,j} \Upsilon_{n,m} \subset \Upsilon_{n-1,m-1} \tag{6.18}$$

*Proof.* A covector $(\xi_1 \dots \hat{\xi}_i \dots \hat{\xi}_j \dots \xi_{n+m}) \in \mathrm{WF}(\Lambda_+)' \bullet_{i,j} \Upsilon_{n,m}$ arises by discarding the covectors $\xi_i, \xi_j$ from a covector $(\xi_1 \dots \xi_{n+m}) \in \Upsilon_{n,m}$, where either $(\xi_i, \xi_j) \in \underline{0}$, or $(\xi_i, \xi_j) \in \mathrm{WF}(\Lambda^+)' \subset V_- \times V_+$.

In the case where $(\xi_i, \xi_j) \in \underline{0}$, it holds that $(\xi_1 \dots \hat{\xi}_i \dots \hat{\xi}_j \dots \xi_{n+m}) \in \Gamma^c_{n+m}$, as we view zero covectors as both past and future pointing. If $(\xi_1, \dots, \xi_n) \in \underline{0}$ then also $(\xi_1, \dots \hat{\xi}_i \dots, \xi_n) \in \underline{0}$. Similarly, if $(\xi_1, \dots, \xi_n) \in (V_-^n)^c$ then it follows that $(\xi_1, \dots \hat{\xi}_i \dots, \xi_n) \in (V_-^{n-1})^c$. We conclude that

$$(\xi_1, \dots \hat{\xi}_i \dots, \xi_n) \in (V_-^{n-1})^c \cup \underline{0}$$
$$(\xi_{n+1}, \dots \hat{\xi}_j \dots, \xi_m) \in (V_+^{m-1})^c \cup \underline{0},$$

the second equation being derived similarly to the first one. Hence we conclude that

$$(\xi_1 \dots \hat{\xi}_i \dots \hat{\xi}_j \dots \xi_{n+m}) \in \Upsilon_{n-1,m-1}.$$

In the case where $\xi_i \in V_-$ and $\xi_j \in V_+$ we see that neither $n$ nor $m$ can be 1, as this would imply that either $\xi_1 \in V_- \cap V_-^c$ or $\xi_{n+1} \in V_+ \cap V_+^c$. We see that $(\xi_1, \dots \hat{\xi}_i \dots \xi_n) \in (V_-^{n-1})^c$ and is non-trivial. Similarly, $(\xi_{n+1}, \dots \hat{\xi}_j \dots \xi_{n+m}) \in (V_+^{n-1})^c$ is non-trivial. Hence not all covectors can be future-pointing null, and neither can they all be past-pointing null, so that again

$$(\xi_1 \dots \hat{\xi}_i \dots \hat{\xi}_j \dots \xi_{n+m}) \in \Upsilon_{n-1,m-1}.$$

□

An unsavoury feature of $\Upsilon_{n,m}$ is that it is neither open nor closed, as the zero section we add in the dotted product is closed. It is unclear to us whether a collection of open cones $\Upsilon_{n,m}$ satisfying equations (6.16), (6.17) and (6.18) exists.

This is not an insurmountable problem however. As already introduced in Chapter 3, we consider the *vector space* of distributions

$$\mathcal{E}'_{\Upsilon_{n,m}}(E^{!\boxtimes n+m}) = \{u \in \mathcal{E}'(E^{!\boxtimes n+m}) \,|\, \mathrm{WF}(u) \subset \Upsilon_{n,m}\},$$

and we call a subset $H$ of that space equicontinuous if there exist a closed cone $\Xi \subset \Upsilon_{n,m}$ and a compact $K \subset \mathcal{M}^{n+m}$ such that

$$H \subset \mathcal{D}'_\Xi(K; E^{!\boxtimes n+m})$$

is a bounded set.

**Definition 6.3.2.** *A graded bifunctional* $L : \mathcal{E}(E_0)\times\mathcal{E}(E_0) \to \mathcal{A}^k \hat{\otimes} \mathcal{A}^l$ *is* ***equicausal*** *if, for all* $B \in \mathcal{E}(E_0) \times \mathcal{E}(E_0)$ *bounded and* $n, m \in \mathbb{N}$,

$$\frac{\delta^{n+m}}{\delta^n_{\varphi_1}\delta^m_{\varphi_2}} L(B) \subset \mathcal{E}'_{\Upsilon_{n+k,m+l}}(E^{!\boxtimes m+n+k+l})$$

*is an equicontinuous family. A graded bifunctional* $L : \mathcal{E}(E_0) \times \mathcal{E}(E_0) \to \mathcal{A} \hat{\otimes} \mathcal{A}$ *is equicausal if it is the sum of equicausal bifunctionals of homogeneous weight.*

A set $B \subset \mathcal{E}(E_0) \times \mathcal{E}(E_0)$ is bounded iff there are bounded sets $B_1, B_2 \subset \mathcal{E}(E_0)$ such that $B \subset B_1 \times B_2$. Hence it follows from equation (6.16) and the first part of Proposition 3.4.5 that the tensor product of two equicausal functionals is an equicausal bifunctional.[6] Similarly, it follows from (6.17) that $\mu L$ is equicausal if $L$ is.

We define the equicausal elements of $\mathcal{A}^{\hat{\otimes}\, 2}$ by

$$(\mathcal{A}^k\hat{\otimes}\mathcal{A}^l)_{ec} = \left(\mathcal{A}^k\hat{\otimes}\mathcal{A}^l\right) \cap \mathcal{E}'_{\Upsilon_{k,l}}(E^{!\boxtimes k+l}),$$

and

$$(\mathcal{A}\hat{\otimes}\mathcal{A})_{ec} = \bigoplus_{k,l\in\mathbb{N}} (\mathcal{A}^k\hat{\otimes}\mathcal{A}^l)_{ec},$$

and similar for variants involving $\mathcal{A}_0$ and $\bar{\mathcal{A}}$.

If $\Phi \in (\bar{\mathcal{A}}^1\hat{\otimes}\bar{\mathcal{A}}^1)$ then we define the contraction of $\Phi$ with $\Lambda^+$ by

$$\begin{aligned} \mathcal{C}_{\Lambda^+}(\Phi) &= ((\iota_{\Lambda^+})_{1,2}\Phi)\, \mathbb{1} \otimes \mathbb{1}, \\ &= \left\langle \Lambda^+ , \Phi \right\rangle \mathbb{1} \otimes \mathbb{1} \in (\bar{\mathcal{A}}\hat{\otimes}\bar{\mathcal{A}})_{ec}, \end{aligned}$$

where $\mathbb{1} \in \mathcal{A}^0$ is the identity of the algebra $\mathcal{A}$. This operation is well-defined by the inclusion (6.18), and we extend it to the whole of $(\bar{\mathcal{A}}\hat{\otimes}\bar{\mathcal{A}})_{ec}$ by imposing the graded Leibniz rule

$$\mathcal{C}_{\Lambda^+}((\Phi_1 \odot \Phi_2) \otimes \Psi) = \mathcal{C}_\Lambda(\Phi_1 \otimes \Psi) \odot (\Phi_2 \otimes \mathbb{1}) + (-1)^{|\Phi_1||\Phi_2|}(\Phi_1 \otimes \mathbb{1}) \odot \mathcal{C}_\Lambda(\Phi_2 \otimes \Psi)$$

for $\Phi_1, \Phi_2 \in \bar{\mathcal{A}}$ homogeneous, and similar in the second argument.

**Proposition 6.3.3.** *If* $k, l \leq 1$ *and* $H \subset (\bar{\mathcal{A}}^k \hat{\otimes} \bar{\mathcal{A}}^l)_{ec}$ *is equicontinuous, then* $\mathcal{C}_{\Lambda^+} H \subset (\bar{\mathcal{A}}^{k-1} \hat{\otimes} \bar{\mathcal{A}}^{l-1})_{ec}$ *is equicontinuous as well.*

[6]This proposition is given in terms of an open cone, but only relies on the explicit form of equicontinuous sets. Hence the proof applies verbatim to this situation as well.

*Proof.* There is a closed cone $\Xi \subset \Upsilon_{k,l}$ and a compact $K \subset \mathcal{M}$ such that

$$H \subset \mathcal{D}'_{\Xi}(K^{k+l}; E^{!\boxtimes k+l}).$$

After contraction with $\Lambda^+$, we have

$$\mathcal{C}_{\Lambda^+} H \subset \mathcal{D}'_{\tilde{\Xi}}(K^{k+l-2}; E^{!\boxtimes k+l-2})$$

where

$$\tilde{\Xi} = \bigcup_{i=1}^{k} \bigcup_{j=k+1}^{k+l} \mathrm{WF}(\Lambda^+)' \bullet_{i,j} \Xi,$$

which is a closed cone. Furthermore, as multiplication by $\Lambda^+$ and pushforward by the projections $\pi_{i,j}$ are continuous maps, $\mathcal{C}_{\Lambda^+} H$ is bounded in that space. Hence we see that $\mathcal{C}_{\Lambda^+}$ maps equicontinuous sets to equicontinuous sets. □

On ungraded variables, we simply proceed by contracting without any extra factor of $-1$

$$(\iota_{\Lambda^+})_{1,2} : \left(\mathcal{A}_0 \hat{\otimes} \mathcal{A}_0 \hat{\otimes} \bar{\mathcal{A}} \hat{\otimes} \bar{\mathcal{A}}\right)_{ec} \to \left(\bar{\mathcal{A}} \hat{\otimes} \bar{\mathcal{A}}\right)_{ec}.$$

This operation respects equicontinuous families by a similar argument. We combine these two operations into a bi-derivation on graded equicausal bifunctionals $L$ by

$$\mathcal{B}_{\Lambda^+} L = (\iota_{\Lambda^+})_{1,2} \frac{\delta^2}{\delta\varphi_1 \delta\varphi_2} L + \mathcal{C}_{\Lambda^+} L \tag{6.19}$$

**Proposition 6.3.4.** *The operator $\mathcal{B}_{\Lambda^+}$ defined in equation (6.19) is an automorphism of the set of graded equicausal bifunctionals.*

*Proof.* Let $L \in C^\infty(\mathcal{E}(E_0) \times \mathcal{E}(E_0) \to \bar{\mathcal{A}}^k \hat{\otimes} \bar{\mathcal{A}}^l)$ be a graded equicausal bifunctional. We first show that $\mathcal{B}_{\Lambda^+} L$ is conveniently smooth, i.e. we show that if $\gamma$ in a smooth curve in $\mathcal{E}_0 \times \mathcal{E}_0$, then $L \circ \gamma$ is a smooth curve into $\bar{\mathcal{A}}^k \hat{\otimes} \bar{\mathcal{A}}^l \subset \mathcal{E}'(\bar{E}^{!\boxtimes k+l})$.

The curve $L \circ \gamma$ satisfies the prerequisites of Lemma 3.4.10, by a similar calculation as done in the proof of Theorem 6.1.5. Hence, if $I \subset \mathbb{R}$ is a bounded interval then there exists a sequence of nested closed cones $\Xi_n \subset \Upsilon_{k,l}$ and a compact set $K^{k+l} \subset \mathcal{M}^{k+l}$ such that

$$L \circ \gamma|_I : I \to \mathcal{D}'_{\Xi_n}(K^{k+l}; \bar{E}^{!\boxtimes k+l})$$

is $C^n$. If $1 \le i \le k$ and $k+1 \le j \le k+l$, then contracting variables $i$ and $j$ through $\Lambda^+$ gives a continuous map

$$(\iota_{\Lambda^+})_{i,j} : \mathcal{D}'_{\Xi_n}(K^{k+l}; \bar{E}^{!\boxtimes k+l}) \to \mathcal{D}'_{\mathrm{WF}(\Lambda^+)' \bullet_{i,j} \Xi_n}(K^{k+l-2}; \bar{E}^{!\boxtimes k+l-2}) \hookrightarrow \mathcal{E}'(\bar{E}^{!\boxtimes k+l-2}),$$

being the composition of multiplication by $\Lambda^+$ with a pushforward. It follows that $\mathcal{C}_{\Lambda^+} L \circ \gamma|_I$ is smooth into $\mathcal{E}'(\bar{E}^{!\boxtimes k+l-2})$, as it is $C^n$ for all $n \in \mathbb{N}$. A similar argument implies that

$$(\iota_{\Lambda^+})_{1,2} \frac{\delta^2}{\delta\varphi_1 \delta\varphi_2} L \circ \gamma|_I : I \to \mathcal{E}'(\bar{E}^{!\boxtimes k+l})$$

is smooth. As smoothness is a local property, these facts remain true if we remove the restriction to $I$.

We conclude that $\mathcal{B}_{\Lambda^+} L$ is a conveniently smooth functional into $\bar{\mathcal{A}}$. As this space is complete and $\mathcal{E}(E_0) \times \mathcal{E}(E_0)$ is a Fréchet space, it follows that $\mathcal{B}_{\Lambda^+} L$ is (Bastiani) smooth as well, and that

$$\frac{\delta^{n+m}}{\delta^n_{\varphi_1} \delta^m_{\varphi_2}} \mathcal{B}_{\Lambda^+} L = \mathcal{B}_{\Lambda^+} \frac{\delta^{n+m}}{\delta^n_{\varphi_1} \delta^m_{\varphi_2}} L$$

by the chain rule. As both $\mathcal{C}_{\Lambda^+}$ and $(\iota_{\Lambda^+})_{1,2}$ respect equicontinuous sets, $\mathcal{B}_{\Lambda^+} L$ is equicausal because $L$ is. □

We close by stating the main result of this chapter.

**Theorem 6.3.5.** *The ⋆-product of graded equicausal functionals, formally defined by*

$$F \star G = \mu \exp\left(\hbar \mathcal{B}_{\Lambda^+}\right) F \otimes G, \tag{6.20}$$

*is well-defined and closes on the space of equicausal functionals.*

*Proof.* The tensor product $F \otimes G$ is an equicausal bifunctional. It follows from Proposition 6.3.4 that we can act on it by $\mathcal{B}_{\Lambda^+}$ any number of times, and that the result will be an equicausal bifunctional. In particular, $\exp\left(\hbar \mathcal{B}_{\Lambda^+}\right) F \otimes G$ is a formal power series of equicausal bifunctionals. Finally, the product $\mu$ turns this into a power series of equicausal functionals. □

We conclude that the equicausal functionals achieve the objective we set out to accomplish: they encompass the local functionals and, unlike the microcausal functionals, are closed under the ⋆-product.

# — 7 —

# Conclusions

The central focus of this thesis has been the investigation of observables in perturbative algebraic quantum field theory (pAQFT). We adopted the principle that these observables should be described as smooth functionals on the space of physical configurations. This approach creates several technical challenges. As the space of physical configurations is not a standard space of functions, we use the BV-formalism to provide a resolution of this space. Additionally, the infinite-dimensional nature of the configuration space adds considerable complexity to the analysis. Many key elements in physical theories, such as propagators and local observables, are inherently distributional. To handle these rigorously, we relied on tools from microlocal analysis to properly define operations involving these objects.

The first part of this thesis, in Chapter 4, was devoted to presenting arguments that the customary treatment of these difficulties using *microcausal functionals* is inadequate. This integration of the theory of smooth functionals on locally convex spaces with microlocal analytical techniques is too superficial, and only works as intended for polynomial observables. A significant shortcoming of these functionals is their failure to provide a closed Poisson bracket. Since the primary aim of introducing these functionals is to construct a Poisson algebra that encompasses local functionals, this counterexample demonstrates that microcausal functionals are unsuitable for developing theories in pAQFT.

In the second part of the thesis, Chapter 5, we developed a theory of graded functionals, and provided some explicit tools to manipulate them. This builds on the work of Fredenhagen and Rejzner in [41, 40], as we give rigorous proofs that the algebras they construct provide resolutions of the physical algebras of observables. The primary tool we developed for this purpose is the homotopy operator constructed in Theorem 5.6.3, which allows one to rigorously prove homotopical results about graded functionals.

Finally, we set to the task of defining a new class of functionals, which implements the required microlocal constraints in a robust fashion. We achieved this goal by way of the equicausal functionals described in Chapter 6. The main improvement over the microcausal functionals is that these functionals take Hörmander's topology seriously. Their derivatives are not only *valued* in the appropriate space, they are also bounded in an appropriate sense. The main technical groundwork for this definition is the work by Brouder et al. in [14, 15] on Hörmander topologies and continuity of fundamental operations, which we generalised in Chapter 3 to be applicable to the scenario in which we need it. These functionals do perform as advertised: they are closed under all the maps constructed in Chapter 5;[1] they contain the local functionals; and the quantum $\star$-product closes on them.

We discuss some open questions and generalisations stemming from this work. The first is the extension of the results in Chapter 5 to more general scenarios. The data of a deformation retract is quite restrictive, as the maps involved need to satisfy multiple relations, making it somewhat hard to exhibit them in practice. Hence it would be nice to obtain a way to lift general quasi-isomorphisms between field complexes to algebras of graded functionals, as that would widen the range of applicability the tools developed here. An immediate application of such a result would be a rigorous proof of the time-slice axiom for *any* Green hyperbolic field theory, as it can be easily shown that the inclusion maps on the field complex are quasi-isomorphisms by a standard argument, similar to the one given in Section 5.6.1.

On top of that, we would like to apply these results to more examples. We treated the Koszul-complex in detail in [52]. However, the treatment of the Chevally-Eilenberg complex is awkward due to the presence of 'large gauge transformations', i.e. locally constant 0-forms. One might try to remove them by hand, and consider the field complex

$$\Omega^0_c(\mathcal{M}) \xrightarrow{d} \Omega^1(\mathcal{M}).$$

However, the extension maps on the AQFT defined by the functionals on a field complex are given by the functional pullback of the *restriction* maps of sections,

[1]Provided that the input for these maps is causally stable as in Definition 6.1.7.

which is not well-defined for compactly supported sections. Hence finding the right way to implement support conditions into the formalism is somewhat subtle. Related to this are the obstructions to the existence of Møller maps given by Benini et al. in [8], which hinge crucially on the way that these large gauge transformations are dealt with in the theory. These questions are part of an ongoing project with Kasia Rejzner and Eli Hawkins.

One of the inspirations for this work was the article [20] by Brunetti, Fredenhagen and Ribeiro, which, to our knowledge, is the first rigorous treatment of non-polynomial functionals within the context of perturbative Algebraic Quantum Field Theory (pAQFT).[2] Their main objective was to study quantum theories of non-linear wave equations using the Nash-Moser inverse function theorem. Given that this theorem yields only local solutions, their constructions are naturally set on Fréchet manifolds rather than Fréchet spaces. Hence it is natural to consider whether the equicausal functionals we introduce, as well as the homological tools developed in Chapter 5, can be extended to this broader setting.

Related to this is another technical hurdle that needs to be dealt with in their approach. In defining a Poisson-bracket for a non-linear theory, they studied families of differential operators $P_{\varphi_0}$ arising from the linearisation of the Euler-Lagrange equations around a reference configuration $\varphi_0$. Their main assumption on the action is that $P_{\varphi_0}$ is normally hyperbolic at each configuration, so that Green's functions $\Delta^{A/R}(\varphi_0)$ exist, as a function of the reference configuration. However, as should be clear from the counterexamples presented in Chapter 4, mere existence of the Green's function does not imply that they define a Poisson bracket valued in smooth functionals; some smoothness condition is required.

Using energy-estimate methods, as in [5], one can demonstrate that the Green's functions of a normally hyperbolic operator $P$, when viewed as elements of $\mathcal{D}'(E \boxtimes E^!)$, depend smoothly on the sub-principal terms of $P$. However, this level of smoothness is insufficient for our purposes, as we require smoothness into an appropriate Hörmander space in order to apply microlocal continuity results. Furthermore, transitioning to the quantum theory requires the selection of a two-point function that smoothly depends on the background configuration. We suspect that these challenges could be addressed using techniques from pseudo-differential operator theory.

Another important inspiration for this work is the program developed by Costello

[2] Although the article was formally published in 2019, preprints had been circulating for some time prior.

and Gwilliam in [23, 24], where they demonstrate how to quantise Euclidean field theories using factorisation algebras. Efforts have been made to adapt their framework to the Lorentzian setting, and to translate between quantisation via nets of algebras and via factorization algebras, see e.g. [46, 11, 47]. A central element of the analysis of Gwilliam and Rejzner is the time-ordering operator, which is constructed using renormalisation techniques, and is therefore closely linked to local functionals. It is, to the best of our knowledge, an open question whether this operator can be extended to a broader class of functionals. Investigating this possibility could potentially deepen the connections between the AQFT approach and the factorisation algebra formalism.

Closely related to this topic is the notion of *Green hyperbolic complexes* introduced by Benini, Musante, and Schenkel in [9], which we discussed briefly in Chapter 6. These complexes were conceived as a Lorentzian analogue to the elliptic complexes in Costello and Gwilliam's Euclidean framework in [23, 24]. However, we think that this notion is too general, as it does not ensure the existence of a suitable analogue of a two-point function as in equation (6.14). An interesting direction for future research would be to investigate additional constraints on Green hyperbolic complexes that guarantee the existence of such an object. In our opinion, solving this problem would complete the translation of Costello-Gwilliam's work into the Lorentzian regime, as it precisely pinpoints the most general initial data that the framework can handle.

In conclusion, this thesis has provided new insights into the structure of observables in pAQFT, particularly by refining the class of functionals used to incorporate microlocal constraints more rigorously. We have shown that microcausal functionals are insufficient for the demands of the theory, leading us to introduce and develop the class of equicausal functionals, which address these shortcomings. Furthermore, by applying a mix of homotopical and microlocal methods, we have demonstrated how these tools can be used to give a more robust treatment of quantum observables in the BV-formalism. Several open questions remain, particularly regarding the extension of our results to more general settings and the connection with other approaches to quantisation, such as factorisation algebras. Altogether, this thesis contributes to the ongoing development of pAQFT, offering new tools and perspectives that could be useful for both future theoretical explorations and practical applications.

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