A Martingale Theory for a Class of Dynkin Games with Asymmetric Information

Submitted in accordance with the requirements for the degree of Doctor of Philosophy

The University of Leeds

School of Mathematics

September 2024

Author

Jacob Ethan Smith

Student number

200956155

Supervisors

Dr J. Palczewski

Prof. T. De Angelis

J. E. Smith

A Martingale Theory for a Class of Dynkin Games with Asymmetric Information

Under the supervision of
Dr. J. Palczewski
Prof. T. De Angelis

2024

PREFACE

They don't give out PhDs at the University of Leeds for nothing.

—DR. CHRISTOPHER BARKER

The study of Dynkin games has not abated since their original inception in 1968. Named—rather unsurprisingly—after their originator, Eugene Dynkin. Not only are these games fascinating in their own right, but they exist as a confluence of seemingly disparate fields of study. On the one hand, Dynkin games are a logical extension to the classical optimal stopping problem in that they exist firmly in the realm of stochastic control as two player zero-sum optimal stopping problems. On the other hand, however, they inherit the notions of game theory as we introduce the concepts of Nash equilibria, second mover advantage, and mixed strategies. The subject straddles the line between pure and applied mathematics, and in this text the reader will see the technical nuances required to rigorously analyse these games. Moreover, they will also see the links that exist to wider fields such as free-boundary problems and variational inequalities in PDEs, and the historical context which surrounds the current state of Dynkin games. Of course, anyone who is familiar with Dynkin games will be aware of the work done by Yuri Kifer, who was the first person to pose the Dynkin game as a viable option contract on the financial markets, as is often the fate of many stochastic optimal control problems.

As Dynkin games have developed, they have followed the typical trajectory of a problem in economic game theory, namely the posing of the standard, discrete-time case, followed by a generalisation to continuous-time, then shedding some of their more restrictive assumptions (such as Markovianity), and finally considering information asymmetry, resulting in the extension of the set of playable strategies from ‘pure strategies’ to so-called ‘mixed strategies’. In this work we focus on continuous-time, two-player, zero-sum Dynkin games with asymmetric information. We are by no means the first to broach this topic, but this work develops a martingale theory to fully characterise a class of these Dynkin games, a result which does not currently exist in the literature.

We would be remiss if we did not mention some of the foundational texts in the field, ones which have built much of the backbone of our understanding, and are excellent references for many of the ideas found in this work. First, we should mention two books written by Claude Dellacherie and Paul-André Meyer, *Probabilities and Potential A* and *B* ([DM78] and [DM82]). These books are cornerstones of probability theory, stochastic analysis, and martingale theory, and at many times during this work we have found ourselves pondering a problem that Dellacherie and Meyer had already solved back in the late 70’s and early 80’s. As a result, this pair of books sits prominently on the author’s desk. Secondly, a pair of books written by Ioannis Karatzas and Steven Shreve, *Brownian Motion and Stochastic Calculus* [KS98a], and *Methods of Mathematical Finance* [KS98b]. The former is a fantastic reference for stochastic calculus and martingale theory, and the latter contains an appendix which beautifully summarises the classical single-agent optimal stopping problem. Lastly, it would be a terrible omission to not include the lecture notes from Nicole El Karoui, [EK79]. Her work is a treatise on stochastic control, containing both a full analysis of the classical problem, akin to [KS98b], but also a full treatment of the general formulation too. This text contains a myriad of deep results and optimality conditions, and remains a real treasure of the field. One just needs to be able to read it in French.

This work is composed of two parts, Part I focuses on games with full information, that is where both players know everything there is to know about the game. Part II is concerned with games of asymmetric information, where players know differing amounts about the game. The outline of this text is as follows.

Chapter 1 is a general overview and a brief history of the development of, and methods involved in, stochastic control. We show the evolution of the control problem from single player, discrete-time, problems, through to continuous-time, and then into two-player games. We use this brief history to highlight the martingale approach to optimal stopping.

The work in Chapter 2 can effectively stand on its own, and it deals with non-Markovian games of optimal stopping. We analyse the paper by Lepeltier and Maingueneau [LM84], and this forms the backbone of the chapter, however, it is not merely a direct translation from the original French as we expand upon the details of the proofs and methodologies, and make large changes to some of the original proofs where they were either possible to do in a more efficient way, or where we were unable to properly verify them. We end the chapter with a novel result about the minimality of the Nash equilibrium found by Lepeltier and Maingueneau. This chapter is an incredibly important one as we formulate many of the tools that we will utilise throughout the rest of this work.

Chapter 3 is the first chapter of Part II and is where we introduce our problem. A critical concept in our study is that of 'randomised stopping times', for which a handful of definitions exist. We give these different definitions and then prove their equivalence. Our problem is a special case of the general problem found in [DAMP22], and we give both formulations here. In the penultimate section of the chapter we give a direct proof of the existence of a value and of a Nash equilibrium in our set-up. This proof is an adapted version of the one found in [DAMP22].The final section of the chapter is dedicated to a literature review and an in-depth discussion on the work done by Christine Grün [Grü13], as her problem is the same as ours, however, her approach is vastly different and we will obtain a wider selection of results.

Chapters 4 and 5 go hand in hand, and contain the baulk of the work developing the martingale theory for Dynkin games with asymmetric information. In order to characterise the games we develop necessary and sufficient conditions for the existence of both the value and of Nash equilibria. Specifically, in Chapter 4, we prove the existence of three processes based on the Nash equilibrium found at time zero in Chapter 3, more precisely a martingale, a supermartingale and a submartingale representing the evolution of the game when both players are playing optimally, and when only one is. We also prove that the Nash equilibria in subgames—assuming the game is still being played at that time—can be written in terms of the original Nash equilibrium at time zero using a method called 'truncation'. Finally, we then analyse the optimal strategies of both players, finding analogous results to traditional first entry times. Chapter 5 is then a relatively short chapter proving the sufficiency of the necessary conditions found in Chapter 4. We present two verification theorems.

We finish the main body of this work in Chapter 6 where we present an example of a rudimentary game where the randomness is contained only in the payoff at the terminal time. The simplicity of this allows us to explicitly calculate the values, the martingales and the optimal strategies, and helps to give a feel for how our theory works. We begin by posing the set-up of the game before presenting a chain of reasoning to deduce what the optimal strategies should be—recall that [DAMP22] only prove existence, and do not construct the optimal strategies, hence our need to deduce them—and then use our verification theorem to prove they are the correct choice. Finally, we extend the game by subtly varying the terminal payoff and show how the optimal strategies change, how the game becomes non-randomised, and where the uninformed player becomes indifferent to the informed players' actions.

Finally, we finish with the appendices. Appendix A contains some foundational results in Lebesgue-Stieltjes integral theory, specifically regarding the equivalence of measures, useful properties of the Lebesgue-Stieltjes integral, and finally, generalised inverses and time changes. Appendix B contains some additional basic results and background in probability theory and functional analysis.

* * *

In 2015, I came to the University of Leeds to study mathematics. I had just changed my university plans away from studying medicine and I had no idea what I wanted to do with mathematics, I only knew that it was the subject I enjoyed the most. On a whim, I chose the optional module *Financial Mathematics 1*, which was ran by a lecturer whose cursive *n*s looked like *m*s. In late January 2016, the module leader came into the lecture hall and enthusiastically told us that the Bank of Japan had just implemented a negative interest rate in an attempt to increase economic growth; I was hooked and knew that this was the field I wanted to be in. The module leader was Prof. Tiziano De Angelis.

After taking a year in industry in 2017, I returned to university with a drive to learn more about quantitative finance and took a course called *Stochastic Financial Modelling*, led by Dr. Jan

Palczewski. This module was my first experience of stochastic analysis and solidified my resolve to study this further. Nearly ten years on from my first introduction to mathematical finance—and the rabbit-hole I subsequently fell down—I sit at my desk finishing off this thesis, and I can not help but be thankful for the passion, patience, and guidance that both Jan and Tiziano have embodied, and for the opportunity to spend this time studying a field which I find genuinely fascinating, even if the process of a PhD can cloud that at times.

I would also like to thank Luis Mario Chaparro Jaquez for being one of the seemingly few voices of reason and normalcy, especially on those days when I'd had enough; I truly wish you well in everything that you do. I also need to thank everyone who I have met at the University of Leeds Fencing Club, and those that I have gone on to meet because of it; you have all kept me sane and have been the constant that has let me disconnect from the stress of academic work. Finally, I would like to thank both my Mum and Leo, both of whom have had to listen to my rants and stress for at least four years.

JACOB E. SMITH
MARCH 2024

Abstract

In this thesis we study a class of Dynkin games with asymmetric information where players' strategies are randomised stopping times. Information asymmetry is formulated as an exogenous random variable taking finitely many values, the outcome of which is only known by the minimiser. It is already known that for these types of games there exists both a value and a Nash equilibrium, however, these results are non-constructive and as such they provide no information regarding the properties of the value or the optimal strategies of the players.

Inspired by our additional work on classical results for games with full information, we introduce a martingale theory to derive a set of necessary and sufficient conditions for the existence of a value and a Nash equilibrium, characterising the game. By defining three processes—a martingale, a submartingale and a supermartingale—we are also able to characterise the optimal strategies of both players and their optimal subgame strategies. We finish this work with an example in which we use our necessary conditions to deduce the form of the optimal strategies, and then verify them to be optimal using our sufficient conditions.

CONTENTS

Notions of chance and fate are the preoccupations of men engaged in rash undertakings.

—CORMAC McCARTHY, *Blood Meridian*

PART I

GAMES WITH FULL INFORMATION

CHAPTER 1

STOCHASTIC CONTROL

1.1 INTRODUCTION

Stochastic control is a sprawling field with its roots in classical, deterministic optimal control problems. Historically we see the emergence of optimal control from the physical sciences, with a boom in the 1950s as the space race began, and engineers became interested in controlling systems governed by differential equations [FR75]. It is intuitive that optimisation problems should appear here, simply consider the classic problem of how much fuel does a rocket need to take off? Enough so that the mass of the rocket can reach its desired path through space? But what about the fuel needed to lift that mass of that fuel? Or conversely, what is the smallest amount of fuel needed to make a soft landing on the moon? The utility of these models led to their spread into the wider academic body too, appearing in economics and, somewhat inevitably, engineering.

A natural next step is to realise that most physical processes in the world are subject to random perturbations—unpredictable disturbances in a deterministic model—and from this the study of stochastic optimal control flourished. One of the most famous examples is the pricing of American options (financial market contracts). These are problems of timing—when should an investor stop and exercise their option to maximise their payoff when the underlying stock price is stochastic? The techniques developed from this study have spawned a whole subtopic in the field devoted to pricing options with increasingly more intricate payoff structures. In fact, the work we present here falls under what are called Dynkin games—a class of multi-agent optimal stopping problems—which were suggested, in [Kif00], as a basis of cancellable options.

To this day these systems are still studied from an analytic, differential equations based approach, but new techniques have been developed to utilise probabilistic methods (a key source being the lecture notes given by El Karoui [EK79]). Even though our work will use the latter, one can see the roots of the field more clearly through the former, and so in this chapter we will begin with a general introduction through this lens before moving towards the probabilistic methods.

A key moment in the development of the field was in 1957 with the release of Richard Bellman's book *Dynamic Programming* [Bel57]. Prior to this, standard methodologies existed to solve control problems, such as the use of calculus of variations or Pontryagin's maximum principle, but Bellman developed a new approach which focused on problems that could be broken down into sub-problems. The optimal solutions to these sub-problems are easier to find, and then by linking the results together one can obtain an optimal solution for the whole problem. Put another way this concept sounds much more obvious—his methodology relies on the fact that deviations from the optimal solution are suboptimal.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let us take a simple, discrete-time, N-step example from Bertsekas (Chapter 2, [Ber76]) that is governed by the following system,

$$x_{k+1} = f_k(x_k, u_k, w_k),$$

for $k \in \{0, 1, ..., N-1\}$ and given functions f_k. The state x_k is an element of a measurable space $(S_k, \mathscr{F}_k)$ for a σ-algebra $\mathscr{F}_k$, and $k \in \{0, 1, ..., N\}$. The control u_k belongs to a measurable space $(C_k, \mathscr{G}_k)$ for some σ-algebra $\mathscr{G}_k$, and $k \in \{0, 1, ..., N-1\}$, and the disturbances w_k are independent and identically distributed uniform random variables on $[0,1]$, i.e. $w_k \sim U[0,1]$, all for $k \in \{0,1,...,N-1\}$. Additionally, the control is sensibly restricted to a subspace $U_k(x_k) \subset C_k$ which depends on the state of the system. A control policy π is a finite sequence of functions $\pi = \{\mu_0, \mu_1, ...\mu_{N-1}\}$, where $\mu_k : S_k \to C_k$ such that $\mu_k(x_k) \in U_k(x_k)$, i.e. decisions as to how to act based on the current state of the system.

Given real valued functions (g_k) for $k \in \{0,1,...,N\}$, the aim is to find a control policy which will minimise the cost functional

$$J^\pi(x_0) = \mathbb{E}\left[g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, \mu_k(x_k), w_k)\right] \tag{1.1}$$

subject to the constraint

$$x_{k+1} = f_k(x_k, \mu_k(x_k), w_k), \quad \text{for } k \in \{0,1,...,N-1\},$$

where $\mathbb{E}$ is the expectation under $\mathbb{P}$.

The form of the cost functional J^π is very typical for optimal control problems, and can be seen as a running cost (the summation) accrued over the course of the problem, and a terminal cost, $g_N(x_N)$. In other words, we are looking for the infimum of $J^\pi(x_0)$ over all admissible control policies. Denote this set of control policies by Π, and we say that $J^*(x_0)$, defined as

$$J^*(x_0) := \inf_{\pi \in \Pi} J^\pi(x_0),$$

is the optimal value of the problem, and if an optimal control policy π^* exists, we write

$$J^*(x_0) = J^{\pi^*}(x_0) = \inf_{\pi \in \Pi} J^\pi(x_0).$$

This is a classic example of an optimal control problem. As Bertsekas points out, the optimisation is quite daunting since the problem requires us to minimise over Π, a set of sequences of functions of x_k, however, dynamic programming allows us to swap this to a sequence of optimisation problems, where, at each step, we optimise over the much nicer set $U_k(x_k)$ instead.

The key point of dynamic programming is that if $\{\mu_0^*, \mu_1^*, ...\mu_{N-1}^*\}$ is an optimal policy from time zero to time N, then $\{\mu_i^*, \mu_{i+1}^*, ...\mu_{N-1}^*\}$ must be optimal for the sub-problem from i to N, i.e.

minimising the functional

$$\mathbb{E}\left[g_N(x_N)+\sum_{k=i}^{N-1} g_k(x_k,\mu_k(x_k),w_k)\right].$$

The following proposition encapsulates the algorithm behind discrete-time dynamic programming.

Proposition 1.1.1. *(Page 50, [Ber76]) Let $J^*(x_0)$ be the optimal value of the cost functional (1.1), then $J^*(x_0)=J_0(x_0)$ where J_0 is given by the last step of the following dynamic programming algorithm, which proceeds backwards in time from period N to 0:*

$$J_N(x_N)=g_N(x_N) \quad \text{for all } x_N\in S_N,$$

$$J_k(x_k)=\inf_{u_k\in U_k(x_k)}\mathbb{E}[g_k(x_k,u_k,w_k)+J_{k+1}[f_k(x_k,u_k,w_k)]] \quad \text{for all } x_k\in S_k, \tag{1.2}$$

for $k\in\{0,1,...,N-1\}$. Furthermore, if $\mu_k^*(x_k)$ is measurable and minimises the right-hand side of (1.2) for each $x_k\in S_k$ and each k, the control law $\pi^*=\{\mu_0^*,\mu_1^*,...\mu_{N-1}^*\}$ is optimal.

In its purest form, this is what dynamic programming is, working backwards and constructing an optimal path, however, the precise method of this algorithm need not be strictly adhered to. We will see that dynamic programming looks slightly different in continuous-time, but the same idea of optimality over sub-periods remains.

One can see from this description that it is intentionally left vague in terms of the choice of strategies and cost functionals; a benefit is of course the wide use cases of this type of set-up, encapsulating the problems mentioned at the very beginning of this section. One extremely popular sub-problem of stochastic control is when we choose the space of controls to be stopping times, we call this the optimal stopping problem. As El Karoui points out in her lecture series, [EK79], since this problem is so widely studied, it can lead to a misconception that it is in someway a different problem altogether, which is far from the truth. The techniques developed to solve stochastic control problems also solve optimal stopping problems, with the additional benefit of clarifying the reasoning in an arguably simpler environment.

Before moving on to continuous-time problems, we will stay in the discrete set-up, but begin to introduce some of the common concepts in the probabilistic approaches to optimal control.

1.2 DISCRETE-TIME OPTIMAL STOPPING

In their book, *Optimal Stopping and Free-Boundary Problems* [PS06], Goran Peskir and Albert Shiryaev present an intuitive guide to optimal stopping problems, both in discrete and continuous-time, as well as from both a martingale and a Markovian approach. Here we will follow their section on discrete-time martingale problems because it will both provide links to the previous section, as well as develop a secondary methodology which will be widely used throughout this work—the concept of the essential supremum.

Something to be noted is that these problems are often symmetric in the choice of direction of optimisation. In the previous section we were minimising a cost functional, but an optimisation problem could equally focus on maximising a gain. Of course, the swap makes no material difference to the methodologies.

Let us fix a finite horizon N, which we can later consider going to infinity, and a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_n)_{0\le n\le N})$. Let $(G_n)_{0\le n\le N}$ be an $(\mathscr{F}_n)$-adapted processes (i.e. G_n is $\mathscr{F}_n$-measurable for each $n \in \{0,\ldots,N\}$), which we think of as the gain obtained if we stop at time n. We will denote by $\mathscr{T} = \mathscr{T}(\mathscr{F}_n)$ the set of all $(\mathscr{F}_n)$-stopping times, and further define the subset, for $0 \le n \le N$,

$$\mathscr{T}_n^N = \{\tau \in \mathscr{T} : n \le \tau \le N\}. \tag{1.3}$$

We are left with the following optimal stopping problem

$$V^* = \sup_{\tau\in\mathscr{T}} \mathbb{E}G_\tau. \tag{1.4}$$

The crux of the problem here is twofold; firstly we need to find a way to state the form of V^* as explicitly as possible, and secondly to find the form of the optimal stopping time τ^* which attains the supremum. Of course, there are conditions which need to be met to ensure that everything here is well defined, however, we omit them here as it would only detract from the clarity.

To move forward, we need to define the sub-problems akin to the previous section. Their form is intuitively clear, we must optimise over the period n to N using the definition in (1.3), and so the values functions take the form

$$V_n^N = \sup_{\tau\in\mathscr{T}_n^N} \mathbb{E}G_\tau. \tag{1.5}$$

The logic to solve this problem begins as follows. When $n = N$, we are forced to stop, and the payoff, call it S_N^N, is equal to G_N since there are no other choices. Next, we move backwards one step to $n = N-1$. Here we can either stop, where our payoff will be G_{N-1}, or we can continue, where our payoff will be the expected payoff at $n = N$ given the information we have at $n = N-1$; put more formally, we write $\mathbb{E}[S_N^N|\mathscr{F}_{N-1}]$. Clearly, if $G_{N-1} \geq \mathbb{E}[S_N^N|\mathscr{F}_{N-1}]$, it makes sense to stop at $n = N-1$, and if the reverse is true then we should continue. We argue the same way for each n, working backwards, and we obtain a stochastic process $(S_n^N)_{0\leq n\leq N}$ defined as follows

$$S_n^N = \begin{cases} \max\{G_n, \mathbb{E}[S_{n+1}^N|\mathscr{F}_n]\} & \text{for } n = N-1, ..., 1, 0, \\ G_N & \text{for } n = N. \end{cases}$$

The logic of this construction also seems to imply that, at any time $n \in \{0, ..., N\}$, we should endeavour to stop at the first moment that $S_k^N = G_k$ for $k \in \{n, ..., N\}$, meaning we suspect that the optimal stopping time, after time n, for the problem (1.5) will take the form

$$\tau_n^N = \inf\{n \leq k \leq N : S_k^N = G_k\}.$$

We call this the début time of the set $\{S^N = G\}$ after time n. Incidentally, this object is always well defined because we know this condition is always satisfied for $k = N$.

Nice parallels can be drawn between this algorithm and the one described in Proposition 1.1.1, especially with how both of the algorithms are defined. This alludes to what the dynamic programming method looks like in a stochastic setting. Contrast Proposition 1.1.1 against the following.

Theorem 1.2.1. *(Theorem 1.2, [PS06]) For all* $0 \leq n \leq N$*, we have:*

$$S_n^N \geq \mathbb{E}[G_\tau|\mathscr{F}_n] \quad \text{for each } \tau \in \mathscr{T}_n^N, \tag{1.6}$$

$$S_n^N = \mathbb{E}[G_{\tau_n^N}|\mathscr{F}_n]. \tag{1.7}$$

Moreover, if $0 \leq n \leq N$ *is given and fixed, then we have:*

- *the stopping time* τ_n^N *is optimal in (1.5).*

- *if* τ^* *is an optimal stopping time in (1.5), then* $\tau_n^N \leq \tau^*$ $\mathbb{P}$*-a.s.*

- *the sequence* $(S_k^N)_{n\leq k\leq N}$ *is the smallest supermartingale which dominates* $(G_k)_{n\leq k\leq N}$.

- *the stopped sequence* $(S_{k\wedge\tau_n^N}^N)_{n\leq k\leq N}$ *is a martingale.*

Remark 1.2.2. A quick remark to make about the theorem is how it relates back to the value. Taking the expectation on both sides of (1.6) and taking the supremum we get, $\mathbb{E}S_n^N \geq \sup_{\tau\in\mathscr{T}_n^N}\mathbb{E}G_\tau$, then taking expectation and bounding above by the supremum on (1.7) yields $\mathbb{E}S_n^N \leq \sup_{\tau\in\mathscr{T}_n^N}\mathbb{E}G_\tau$. In other words $\mathbb{E}S_n^N = V_n^N$; the sequence $(S_n^N)_{0\leq n\leq N}$ solves (1.5) in a stochastic sense, and taking expectation solves it in the mean-valued sense.

Remark 1.2.3. 'Domination' in this context is taken to be point-wise, $\mathbb{P}$-almost surely. Let I be an arbitrary index set. A process $(Y_t)_{t\in I}$ is said to dominate a process $(X_t)_{t\in I}$ if $\mathbb{P}(Y_t \geq X_t) = 1$ for all $t\in I$. Notice that the null set, on which this inequality fails, is dependent on t.

Much like Proposition 1.1.1, we see that the solution, V_0^N, to the original optimal stopping problem (1.4) is found recursively, hence $V_* = V_0^N$.

The optimal strategies also follow the same sub-problem optimality that we described in the previous section. Imagine that we have integers n and k such that $0\leq n\leq k\leq N$, and consider the event $\{\tau_n^N \geq k\}$, i.e. when we consider the sub-problem starting from n such that the optimal time to stop is after k. We see that $\tau_n^N = \tau_k^N$ and hence, if it was not optimal to stop at any of the times $\{n, n+1, ..., k-1\}$, then the same strategy is still optimal on $\{k, k+1, ..., N\}$. This is the same principle we described in the previous section when discussing optimal control policies $\{\mu_0^*, \mu_1^*, ...\mu_{N-1}^*\}$, and is the crux of the dynamic programming principle.

The final two conclusions of Theorem 1.2.1 are not immediately relevant to us right now, however, they play a very important role in the analysis of these problems. The third point in particular is leading us towards the concept of the 'Snell envelope', which we will now introduce.

The second method of approaching this problem is through the use of the essential supremum. In this context we are able to allow N to be infinite, a step we cannot do in the recursive method. Notice that equations (1.6) and (1.7) seem to imply that S_n^N should be some form of supremum of $\mathbb{E}[G_\tau|\mathscr{F}_n]$. The problem we run into here is that these are random variables and as a result the (in)equalities found so far actually only hold $\mathbb{P}$-almost surely. This means that the null set on which (1.6) and (1.7) do not hold may well depend on the choice of τ and with uncountably many stopping times, we find that a supremum over the set $\mathscr{T}_n^N$ is poorly defined. Alternatively, one can see directly that the supremum of an uncountable family of measurable functions (random variables) is not necessarily measurable itself, which is not ideal. To remedy this we must generalise our concept of a supremum to one that holds $\mathbb{P}$-almost surely. The following lemma is for discrete-time, but the continuous-time case is nearly identical.

Lemma 1.2.4. *(Lemma 1.3, [PS06]) Let $\{Z_\alpha : \alpha \in I\}$ be a family of random variables, defined on a probability space $(\Omega, \mathscr{G}, \mathbb{P})$, where the index set I can be arbitrary. Then there exists a countable subset J of I such that the random variable $Z^* : \Omega \to \overline{\mathbb{R}}$ defined by*

$$Z^* = \sup_{\alpha \in J} Z_\alpha,$$

satisfies the following two properties:

i) $\mathbb{P}(Z_\alpha \leq Z^*) = 1$ *for each $\alpha \in I$.*

ii) If $\tilde{Z} : \Omega \to \overline{\mathbb{R}}$ is another random variable satisfying i) in place of Z^*, then $\mathbb{P}(Z^* \leq \tilde{Z}) = 1$.

The random variable Z^* is called the essential supremum of $\{Z_\alpha : \alpha \in I\}$ (relative to $\mathbb{P}$) and is denoted by $Z^* = \operatorname{ess\,sup}_{\alpha \in I} Z_\alpha$. It is uniquely defined by properties i) and ii) up to a $\mathbb{P}$-null set.

Moreover, if the family $\{Z_\alpha : \alpha \in I\}$ is upwards directed in the sense that, for any α and β in I, there exists a γ in I such that $Z_\alpha \vee Z_\beta \leq Z_\gamma$ $\mathbb{P}$-almost surely, then the countable set $J = \{\alpha_n : n \geq 1\}$ can be chosen so that

$$Z^* = \lim_{n \to \infty} Z_{\alpha_n} \quad \mathbb{P}\text{-a.s.}$$

where $Z_{\alpha_1} \leq Z_{\alpha_2} \leq \dots$ $\mathbb{P}$-almost surely.

Remark 1.2.5. This lemma only defines the essential supremum, however, an analogous version exists called the essential infimum, and its properties and definitions are identical to what is presented above, except for the inequalities are reversed, and it being attained by a sequence of decreasing random variables when the family is downwards directed.

The essential supremum is vital and will appear regularly throughout this work, as will the concept of upwards (and downwards) directed families.

This new framework allows us to recast (1.6) and (1.7) as we first suspected. Let us fix $N = \infty$ and drop the superscript N from our notation. For all $n \geq 0$ we define the stochastic process

$$S_n = \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_n} \mathbb{E}[G_\tau | \mathscr{F}_n].$$

The sub-problem we are trying to solve is now phrased as

$$V_n = \sup_{\tau \in \mathcal{T}_n} \mathbb{E} G_\tau. \tag{1.8}$$

Moreover, we retain our candidate optimal stopping time

$$\tau_n = \inf\{k \geq n : S_k = G_k\},$$

where $\inf \emptyset = \infty$. Notice that we call τ_n a stopping time without proof; this is a consequence of the début theorem which we discuss briefly in Theorem 1.3.8. The process $(S_n)_{n \geq 0}$ is called the Snell envelope of $(G_n)_{n \geq 0}$. We are now in a position to see the equivalent of Theorem 1.2.1 under the essential supremum approach.

Theorem 1.2.6. *(Theorem 1.4, [PS06]) The following recurrent relation holds:*

$$S_n = \max\{G_n, \mathbb{E}[S_{n+1} | \mathcal{F}_n]\}$$

for all $n \geq 0$. Then, for all $n \geq 0$, we have:

$$S_n \geq \mathbb{E}[G_\tau | \mathcal{F}_n] \quad \textit{for each } \tau \in \mathcal{T}_n,$$
$$S_n = \mathbb{E}[G_{\tau_n} | \mathcal{F}_n].$$

Moreover, if $n \geq 0$ is given and fixed, then we have:

- *the stopping time τ_n is optimal in (1.8).*
- *if τ^* is an optimal stopping time in (1.8), then $\tau_n \leq \tau^*$ $\mathbb{P}$-a.s.*
- *the sequence $(S_k)_{k \geq n}$ is the smallest supermartingale which dominates $(G_k)_{k \geq n}$.*
- *the stopped sequence $(S_{k \wedge \tau_n})_{k \geq n}$ is a martingale.*

This result tells us that the essential supremum characterisation also obeys the recurrent relation we began with, moreover, S_n and τ_n solve the problem in a stochastic sense and, therefore, as mentioned earlier, leads to a solution to the original problem. Additionally, we also have a supermartingale characterisation of the solution, something that will become increasingly more important as we go on.

1.3 CONTINUOUS-TIME OPTIMAL STOPPING

The work we will present in later chapters will take place in continuous-time, and it is here where some extremely nice properties of optimal stopping problems become apparent. We will study a general, non-Markovian, game formulation since this is the most general setting we can utilise, however, as we will soon see, the 'martingale approach' to Dynkin games is built upon the theory of optimal stopping, and it is in this section that we will develop these foundational tools which will be present throughout this work.

An important remark is that we will consider the problem with a finite time-horizon. While this may seem restrictive, in the sense that we omit an entire strategy, i.e. not stopping at all, we can actually show that the two are effectively equivalent. We do this through a concept called the Alexandroff compactification (also known as the one-point compactification)(see Section 3.5 in [Eng89] or see Alexandroff's original paper [Ale24], although the latter is in German). A paraphrased version of Alexandroff's theorem (so as to not introduce technical nuance that is unneeded for our work) is as follows

Theorem 1.3.1. *(Alexandroff's theorem; Theorem 3.5.11, [Eng89]) Every non-compact, locally-compact space X has a compactification X^* with one-point remainder.*

This result tells us that, for example, we can add a single point to a non-compact, locally compact space X to obtain a new space X^* which is compact. Typically we write $X^* = X \cup \{\infty\}$; the addition of infinity. The half-line $[0,\infty)$ is clearly non-compact, but it is locally compact (i.e. every point x in $[0,\infty)$ has a compact neighbourhood), and so we can define $[0,\infty] := [0,\infty) \cup \{\infty\}$ as the one-point compactification of the half-line, i.e. $[0,\infty]$ is compact.

Later, when we come to study games, we will see that, in practice, when authors such as Lepeltier and Maingueneau, [LM84], or De Angelis, Merkulov and Palczewski, [DAMP22], consider games with an infinite time-horizon, they do so by defining their function or process 'at infinity'. For the example of a càdlàg stochastic process $(X_t)_{t\geq 0}$ on $[0,\infty]$, we insist that $X_{\infty-} := \lim_{t\to\infty} X_t$ exists, X_∞ is $\mathscr{F}_\infty$-measurable, and the σ-algebra $\mathscr{F}_\infty$ can potentially be different from $\mathscr{F}_{\infty-} = \sigma\big(\cup_{t\in[0,\infty)}\mathscr{F}_t\big)$. More generally, as long as we define the relevant properties of a stochastic process at infinity then the interval $[0,\infty]$ can essentially be treated as the interval $[0,T]$—one can imagine simply mapping from $[0,\infty]$ to $[0,T]$ using $t \mapsto T[1-(1+t)^{-1}]$, and from $[0,T]$ to $[0,\infty]$ using $t \mapsto T(T-t)^{-1} - 1$. We can therefore, without loss of generality, study finite-horizon problems as an infinite horizon is equivalent to a finite-horizon via this compactification procedure.

We can now begin our study of the martingale approach to optimal stopping. This theory, while extremely powerful, is in general inherently non-constructive, and revolves around the use of conditional expectations of the payoff, which is arguably harder to handle than the expected payoff itself. A key text in the formulation of this theory are the 1979 notes from El Karoui [EK79], from which we will be quoting heavily in this section, however, we will omit the relevant proofs. She provides a wide consideration of the martingale methods in both a regular case—the case we will present here—as well as a much more technical general case, and the Markovian case. As she points out, we lose nothing in presenting the regular case of optimal stopping since while it is less general, the lack of technical details stops the methodology from becoming confusing.

Let us begin with the basic set-up. El Karoui studies optimal stopping problems with an infinite time horizon, but recall from the beginning of this chapter that we can use Theorem 1.3.1 to study finite-horizon problems so long as the payoff process is adequately defined at infinity. To this end, we take a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t))$, whose filtration is assumed to satisfy the usual conditions. We take a gain process, $(G_t)_{t\geq 0}$, which is assumed to be $\mathscr{F}_t$-adapted, càdlàg, positive and bounded. Note that in order to consider a game with an infinite horizon, the idea of 'stopping at infinity' should be defined. Simply put, we extend our filtration to include the element $\mathscr{F}_\infty$, and then extend our set of stopping times $\mathscr{T}$ (denoting it with the same notation as Section 1.2, only in continuous-time, $\mathscr{T} = \mathscr{T}(\mathscr{F}_t)$) to $\mathscr{T} \cup \{+\infty\}$. Moreover we ensure that G_∞ is defined and that $G_{\infty-} := \lim_{t\to\infty} G_t$ exists. The problem we are trying to solve is then

$$V_0 = \sup_{\tau \in \mathscr{T}} \mathbb{E} G_\tau. \tag{1.9}$$

The vocabulary used in the general martingale method is specific, and it is worth providing some definitions of concepts that will appear throughout this work. Since we will mostly be dealing with families of random variables, we need to introduce the concepts of a $\mathscr{T}$-system, $\mathscr{T}$-supermartingale systems (a generalisation of a supermartingale) and, importantly, the concept of aggregation. Additionally, optionality will be prevalent throughout this section, as well as Chapter 2, and so we define it here too.

Definition 1.3.2. (Definition VI.4.1, [RW00]) The optional σ-algebra $\mathscr{O}$ of subsets of $\Omega \times [0,\infty)$ is defined to be the smallest σ-algebra on $\Omega \times [0,\infty)$ such that every adapted càdlàg process is

$\mathscr{O}$-measurable. A stochastic process $(X_t)_{t\geq 0}$ is said to be optional if $(X_t)_{t\geq 0}$ is $\mathscr{O}$-measurable as a map from $\Omega \times [0,\infty)$ to $\mathbb{R}$.

Remark 1.3.3. This is not the only way that $\mathscr{O}$ can be defined, in fact, it is equivalently generated by both the set of intervals $[\tau,\infty)$, where τ is a stopping time (see Theorem VI.4.5 attributed to Meyer, [RW00]), and also (under the usual conditions) by right-continuous and adapted processes (see Theorem IV.65, [DM78]). A full treatment of the concepts of optionality can be found in Section VI.4 [RW00].

Definition 1.3.4. A $\mathscr{T}$-system is a family $\{X(\tau), \tau \in \mathscr{T}\}$ of random variables, indexed by $\mathscr{T}$, which satisfies

i) $X(\tau) = X(\sigma)$ $\mathbb{P}$-a.s. on $\{\tau = \sigma\}$

ii) $X(\tau)$ is $\mathscr{F}_\tau$-measurable.

A $\mathscr{T}$-supermartingale system is a $\mathscr{T}$-system which also satisfies

i) $X(\tau)$ is integrable, i.e. $\mathbb{E}|X(\tau)| < \infty$, for all τ in $\mathscr{T}$,

ii) if τ and σ are two elements of $\mathscr{T}$, such that $\tau \leq \sigma$, then

$$\mathbb{E}[X(\sigma)|\mathscr{F}_\tau] \leq X(\tau) \quad \mathbb{P}\text{-a.s.} \tag{1.10}$$

The definitions of $\mathscr{T}$-submartingale systems and $\mathscr{T}$-martingale systems is the same, except (1.10) has the reverse inequality and equality respectively.

A $\mathscr{T}$-system is left-continuous in expectation if, for all increasing sequences, τ_n, of stopping times converging to τ

$$\mathbb{E}X(\tau_n) \to \mathbb{E}X(\tau).$$

An optional process $(Z_t)_{t\geq 0}$ aggregates the $\mathscr{T}$-system $\{X(\tau), \tau \in \mathscr{T}\}$ if, for all $\tau \in \mathscr{T}$

$$Z_\tau = X(\tau) \quad \mathbb{P}\text{-a.s.}$$

The aggregation of these martingale systems into stochastic processes is of the utmost importance, and the following result allows us to do so, while only requiring boundedness and right-continuity in expectation.

Proposition 1.3.5. *(Proposition 2.14, [EK79]) Let $\{X(\tau), \tau \in \mathscr{T}\}$ be a $\mathscr{T}$-supermartingale system which is right-continuous in expectation and bounded. There exists a unique, right-continuous and bounded supermartingale, $(X_t)_{t \geq 0}$, which aggregates $\{X(\tau), \tau \in \mathscr{T}\}$.*

Notice that we do not mention the fact that $(X_t)_{t\geq 0}$ is optional, as is expected from the definition of aggregation, however, since $(X_t)_{t\geq 0}$ is right-continuous and adapted, it is inherently optional (see Remark 1.3.3).

As mentioned before, our magic bullet in this theory is the Snell envelope—which we have touched on in the discrete-time case, but whose full potential is realised in continuous-time—which is intimately related to the 'maximal conditional gain'. Denote the following

$$\mathscr{T}_\tau = \{\sigma \in \mathscr{T} : \sigma \geq \tau\}.$$

Recall the definition of the essential supremum in Definition 1.2.4 and define

$$V(\tau) = \operatorname*{ess\,sup}_{\sigma \in \mathscr{T}_\tau} \mathbb{E}[G_\sigma | \mathscr{F}_\tau] \quad \mathbb{P}\text{-a.s.} \tag{1.11}$$

The family of random variables $\{V(\tau), \tau \in \mathscr{T}\}$ is the maximal conditional gain, and is not quite the Snell envelope, however, after describing some of its properties we can arrive at it.

Theorem 1.3.6. *(Theorem 2.12, [EK79]) The maximal conditional gain $\{V(\tau), \tau \in \mathscr{T}\}$, defined in (1.11), is a $\mathscr{T}$-supermartingale system. A necessary and sufficient condition for a stopping time τ^* to be optimal is that*

i) $V(\tau^*) = G_{\tau^*}$ $\quad \mathbb{P}$*-a.s.*

ii) $\{V(\tau \wedge \tau^*), \tau \in \mathscr{T}\}$ *is a* $\mathscr{T}$*-martingale system.*

One can show that the maximal conditional gain is right-continuous in expectation (see Lemma 2.13 [EK79]), and so an application of Proposition 1.3.5 allows us to aggregate.

Theorem 1.3.7. *(Theorem 2.15, [EK79]) Let $(G_t)_{t\geq 0}$ be an adapted, right-continuous, positive and bounded process. There exists a unique, right-continuous and bounded supermartingale which aggregates the maximal conditional gain, which we denote by $(V_t)_{t\geq 0}$. So we have*

$$V_\tau = \operatorname*{ess\,sup}_{\sigma \in \mathscr{T}_\tau} \mathbb{E}[G_\sigma | \mathscr{F}_\tau] \quad \mathbb{P}\text{-a.s.} \tag{1.12}$$

$(V_t)_{t\geq 0}$ is the smallest right-continuous supermartingale which dominates $(G_t)_{t\geq 0}$. We call this process the Snell envelope of $(G_t)_{t\geq 0}$.

Here we see the subtlety of the Snell envelope; it is the process which aggregates the maximal conditional gain. V_τ does take on the from of the essential supremum, but the equality is only up to a $\mathbb{P}$-null set. In other words, (1.12) does not define the process $(V_t)_{t\geq 0}$, it is a result relating the process to the family $\{V(\tau), \tau \in \mathscr{T}\}$.

Theorem 1.3.7 solves the optimal stopping problem (1.9) in a stochastic sense, but the assumptions made so far are often not enough on their own to guarantee the existence of an optimal stopping time in the sense of Theorem 1.3.6. Generally we need to assume additional regularity from the left for $(G_t)_{t\geq 0}$, but we will address this soon.

λ-OPTIMAL APPROACH

Often in practice, as we will see in Chapter 2, finding optimal stopping times can be difficult, and instead of trying to find them directly, we use an approximation approach. We introduce $\lambda \in [0,1)$, and the set $A^\lambda = \{(\omega,t) : G_t(\omega) \geq \lambda V_t(\omega)\}$. Denote by D_τ^λ the début time of A^λ after τ, i.e.

$$D_\tau^\lambda(\omega) := \inf\left\{t \geq \tau : (\omega,t) \in A^\lambda\right\} = \inf\{t \geq \tau : G_t(\omega) \geq \lambda V_t(\omega)\}.$$

D_τ^λ is in fact a stopping time. This is guaranteed by the début theorem which we present below followed by a chain of explanatory remarks as some notions have not been defined before, and the definition is not immediately applicable.

Theorem 1.3.8. *(Début Theorem, Theorem IV.50, [DM78]) Let $(\mathscr{F}_t)_{t\geq 0}$ be a filtration satisfying the usual conditions, and A be a progressive subset of $\Omega \times [0,\infty)$, then the début of A, $D^A(\omega) := \inf\{t \in [0,\infty) : (\omega,t) \in A\}$, is a stopping time.*

Remark 1.3.9. A subset $A \subset \Omega \times [0,\infty)$ is said to be progressive with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$, if the stochastic process $X_t(\omega) := \mathbb{1}_A(\omega,t)$ is progressively measurable (see Definition 1.1.11 in [KS98a] for the definition of progressive measurability).

Remark 1.3.10. Since both $(G_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ are right-continuous and adapted, Remark 1.3.3 tells us they are optional. Theorem VI.4.7 [RW00] tells us that optional processes are progressive, and so because A^λ is defined by the inequality of two progressive processes it must be a progressive set. Thus D_0^λ is a stopping time.

Remark 1.3.11. The début theorem is classically stated for t belonging to $[0,\infty)$, however, if we define $D^A_\tau(\omega) := \inf\{t \in [\tau,\infty) : (\omega,t) \in A\}$, then either by viewing the proof of Theorem IV.50, [DM78], or by recognising that for all $t < \tau$ we have $t < D^A_\tau(\omega)$ for all $\omega \in \Omega$, we can see that D^A_τ is still a stopping time.

The classical theory of optimal stopping gives us two very nice relations between the Snell envelope and the 'approximately optimal' stopping time D^λ_τ (see Proposition 2.16, [EK79], and Theorem 3, [Mai78]), namely that for any stopping time τ

$$V_\tau = \mathbb{E}\left[V_{D^\lambda_\tau}\middle|\mathscr{F}_\tau\right] \quad \mathbb{P}\text{-a.s.}$$

$$\lambda V_{D^\lambda_\tau} \le G_{D^\lambda_\tau} \quad \mathbb{P}\text{-a.s.}$$

For simplicity we will focus on the $\tau = 0$ case, i.e. the first début time of A^λ; we will denote $D^\lambda = D^\lambda_0$ and drop the subscript for clarity of notation. The first natural observation we make about the family $(D^\lambda)_{\lambda\in[0,1)}$ is that it is non-decreasing as λ goes to one. This is trivial to see; take $\lambda_1 < \lambda_2$, then because of the right-continuity of both $(G_t)_{t\ge0}$ and $(V_t)_{t\ge0}$, and the positivity of the latter, we have

$$G_{D^{\lambda_2}} \ge \lambda_2 V_{D^{\lambda_2}} > \lambda_1 V_{D^{\lambda_2}}.$$

Hence $D^{\lambda_2}(\omega)$ belongs to A^{λ_1} (more accurately the graph of $D^{\lambda_2}(\omega)$, denoted $[\![D^{\lambda_2}(\omega)]\!] := \{(\omega,t) : D^{\lambda_2}(\omega) = t\}$, belongs to A^{λ_1}), and so must be greater than its first entry time, i.e. $D^{\lambda_2} \ge D^{\lambda_1}$.

The second observation is to define the limit $\overline{D} = \lim_{\lambda\searrow 0} D^\lambda$. One can now immediately see why some form of left-regularity of $(G_t)_{t\ge0}$ is needed to obtain an optimal stopping time; Karatzas and Shreve [KS98b] use left-continuity, Peskir and Shiryaev [PS06] use left-continuity over stopping times, and El Karoui [EK79] uses left-continuity in expectation over stopping times; the latter of which is what we assume here.

Definition 1.3.12. A stochastic process, $(Y_t)_{t\ge0}$, is left-continuous in expectation over stopping times if, for every non-decreasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ converging to a limit τ, $\mathbb{E}Y_{\tau_n} \to \mathbb{E}Y_\tau$.

It turns out that this approximation approach leads to the existence of an optimal stopping time. As we have alluded to during this section, we need to assume further left-regularity for $(G_t)_{t\ge0}$, specifically that is is left-continuous in expectation over stopping times. This assumption is relatively unrestrictive, however, in Chapter 2 we will further relax the assumptions on the left-regularity of $(G_t)_{t\geq 0}$.

Theorem 1.3.13. *(Theorem 2.18, [EK79]) Let $(G_t)_{t\geq 0}$ be a positive, bounded and adapted process which is right-continuous over deterministic times, and left-continuous in expectation over stopping times. $(V_t)_{t\geq 0}$ is the right-continuous supermartingale which aggregates the maximal conditional gain, in the sense of (1.12). The stopping time $\overline{D}$ coincides with the stopping time $\tau_D := \inf\{t \geq 0 : V_t = G_t\}$ and is an optimal stopping time, that is to say*

$$\sup_{\tau\in\mathscr{T}} \mathbb{E}G_\tau = V_0 = \mathbb{E}G_{\tau_D}. \tag{1.13}$$

Remark 1.3.14. Recall that $(G_t)_{t\geq 0}$ needs to be positive, this is imperative for the multiplicative approximation method described above since we need to ensure that λV_t can lie below G_t. In practice, however, we can take $(G_t)_{t\geq 0}$ to be merely bounded (say, bounded below by $K \in \mathbb{R}$) since we can always define a new process $\tilde{G}_t := G_t + K$ which is positive. One can then see that the theory above (and in particular, equations (1.12) and (1.13)) is indifferent to linear scaling in this manner, as constants can be removed from expectations and suprema and cancelled, i.e.

$$\sup_{\tau\in\mathscr{T}} \mathbb{E}\tilde{G}_\tau = \tilde{V}_0 = \mathbb{E}\tilde{G}_{\tilde{\tau}_D} \quad \Longleftrightarrow \quad \sup_{\tau\in\mathscr{T}} \mathbb{E}[G_\tau + K] = V_0 + K = \mathbb{E}[G_{\tilde{\tau}_D} + K],$$

where $(\tilde{V}_t)_{t\geq 0}$ is the Snell envelope of $(\tilde{G}_t)_{t\geq 0}$, and

$$\tilde{\tau}_D = \inf\{t \geq 0 : \tilde{V}_t = \tilde{G}_t\} = \inf\{t \geq 0 : V_t = G_t\} = \tau_D.$$

One often finds optimal stopping results, such as the one above, with two additional results; the optimal stopping time τ_D is minimal, and the value process $(V_t)_{t\geq 0}$, when stopped at the optimal time τ_D, is a martingale. These follow on almost immediately from Theorem 1.3.13.

Corollary 1.3.15. *Let τ^* be another optimal stopping time, then $\tau_D \leq \tau^*$ $\mathbb{P}$-almost surely. Furthermore, the stopped process $(V_{t\wedge\tau_D})_{t\geq 0}$ is a martingale.*

Proof. First we prove the martingale property. It is sufficient to prove that $\mathbb{E}V_{t\wedge\tau_D} = V_0$ for all $t \geq 0$, since a supermartingale with constant expectation is a martingale (Problem 1.3.25, [KS98a]).

Since $(V_t)_{t\geq 0}$ is a supermartingale, the optional sampling theorem (Theorem 1.3.22, [KS98a]) tells us that $\mathbb{E}V_{t\wedge\tau_D} \leq V_0$ for all $t \geq 0$.

For the reverse inequality, by right-continuity of both $(G_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$, we know that $G_{\tau_D} = V_{\tau_D}$. Moreover, since τ_D is optimal we have that $V_0 = \mathbb{E}G_{\tau_D}$. Using equation (1.12), we see

$$\mathbb{E}V_{t\wedge\tau_D} = \mathbb{E}\left[\operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{t\wedge\tau_D}} \mathbb{E}[G_\sigma|\mathscr{F}_{t\wedge\tau_D}]\right] \geq \mathbb{E}G_{\tau_D}$$

where we choose the stopping time $\sigma = \tau_D$, since $\tau_D \geq t\wedge\tau_D$. Therefore, $V_0 \leq \mathbb{E}V_{t\wedge\tau_D}$, which concludes the argument.

Next, we prove minimality. Let τ^* to be another optimal stopping time. Theorem 1.3.6 tells us that, since τ^* is optimal and $(V_t)_{t\geq 0}$ aggregates the $\mathscr{T}$-system $\{V(\tau), \tau\in\mathscr{T}\}$, that $V_{\tau^*} = G_{\tau^*}$ $\mathbb{P}$-almost surely. Since $\tau_D := \inf\{t\geq 0 : V_t = G_t\}$, we must have $\tau_D \leq \tau^*$ $\mathbb{P}$-almost surely. ■

Remark 1.3.16. When taken together, Theorem 1.3.13 and Corollary 1.3.15 give the continuous-time analogue of Theorem 1.2.6. The exact continuous-time version of Theorem 1.2.6 exists in Peskir and Shiryaev's book (Theorem 2.2, [PS06]), but since they utilise different assumptions, we have not included it here.

ε-OPTIMAL APPROACH

Above, we have presented the 'multiplicative' method for approximately optimal stopping times, however, there does exist an 'additive' method. We typically distinguish between the two by saying λ-optimal for the former, and ε-optimal for the latter. We will utilise both methodologies in our subsequent work, and in this section we will demonstrate the additive method. The results described here are very similar to the λ-optimal section, however, they contain enough additional interest to warrant the explanation.

Our main reference here are the lecture notes by Lamberton, [Lam09], however, since these have not been published, and we will be utilising a slightly different formulation, we will present the required results in full. We will present this theory for positive processes $(G_t)_{t\geq 0}$, since we have already shown in Remark 1.3.14 how we can turn any bounded process into a positive one. It is worth mentioning that the assumptions made by Lamberton are slightly different to our own, so the results here are framed under our assumptions.

Recall that $(G_t)_{t\geq 0}$ is assumed to be $(\mathscr{F}_t)$-adapted, càdlàg, positive and bounded. The following theorem introduces the renowned Doob-Meyer decomposition. We look to Theorem 1.4.10 in [KS98a] for reference, however, their result is demonstrated for a submartingale. We choose to give it in terms of supermartingales as this is the more typical formulation that one sees the Doob-Meyer decomposition in, which can be seen in other books containing this theorem such as Dellacherie and Meyer, [DM82], and Protter [Pro04].

Before presenting this theorem, however, we require another classical definition from stochastic analysis, namely that of a 'class D' process.

Definition 1.3.17. (Remark V.15 b)) A stochastic process $(X_t)_{t\geq 0}$ is said to be class D if the set of random variables $\{X_\tau, \tau \in \mathscr{T}\}$ is uniformly integrable, i.e. if

$$\lim_{K\to\infty} \sup_{\tau\in\mathscr{T}} \mathbb{E}\left[|X_\tau| \mathbb{1}_{\{|X_\tau|>K\}}\right] = 0,$$

(see Definition 5.2 [Bal17] for uniform integrability).

Remark 1.3.18. Throughout this work we are often dealing with bounded processes, and so the class D property follows immediately.

Theorem 1.3.19. *Let $(X_t)_{t\geq 0}$ be a right-continuous supermartingale of class D. There exists a right-continuous, uniformly integrable martingale $(M_t)_{t\geq 0}$ and an integrable, non-decreasing, previsible, right-continuous process $(R_t)_{t\geq 0}$ with $R_0 = 0$ $\mathbb{P}$-almost surely, such that*

$$X_t = M_t - R_t \quad t \geq 0.$$

This decomposition is unique up to indistinguishability.

Next, we need a concept of ε-optimality is. Let τ be an $(\mathscr{F}_t)$-stopping time, and let $\varepsilon > 0$ be given. A stopping time $\sigma \in \mathscr{T}_\rho$ is said to be ε-optimal if $\mathbb{E}G_\sigma \geq \mathbb{E}V_\tau - \varepsilon$. Let us define the objects A^ε and D^ε_τ in a similar manner to A^λ and D^λ_τ, i.e. let $A^\varepsilon = \{(\omega,t) : G_t(\omega) \geq V_t(\omega) - \varepsilon\}$, and let D^ε_τ be the début time after τ of A^ε, i.e. $D^\varepsilon_\tau(\omega) = \inf\{t \geq \tau : G_t(\omega) \geq V_t(\omega) - \varepsilon\}$.

Since the process $(M_t)_{t\geq 0}$ in the Doob-Meyer decomposition is a martingale, we need only focus our attention on the process $(R_t)_{t\geq 0}$, and the following lemma (see Lemma 2.3.4, [Lam09]) gives us a very useful relation.

Lemma 1.3.20. *Let $(R_t)_{t\geq 0}$ be the non-decreasing process in the Doob-Meyer decomposition of the Snell envelope $(V_t)_{t\geq 0}$. We have, for any $t \geq 0$ and $\varepsilon > 0$, $R_{D^\varepsilon_t} = R_t$ $\mathbb{P}$-almost surely, and the processes $(R_{D^\varepsilon_t})_{t\geq 0}$ and $(R_t)_{t\geq 0}$ are indistinguishable.*

Proof. 1° From the definition of the Snell envelope, we know that

$$\mathbb{E}V_t = \sup_{\tau \in \mathscr{T}_t} \mathbb{E}G_\tau.$$

By the definition of the supremum, we can introduce a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}} \subset \mathscr{T}_t$ such that $\lim_{n\to\infty} \mathbb{E}G_{\tau_n} = \mathbb{E}V_t$. Since $(V_t)_{t\geq 0}$ dominates $(G_t)_{t\geq 0}$, and using the Doob-Meyer decomposition we have

$$\begin{aligned}
\mathbb{E}G_{\tau_n} \leq \mathbb{E}V_{\tau_n} &= \mathbb{E}M_{\tau_n} - \mathbb{E}R_{\tau_n} \\
&= \mathbb{E}M_t - \mathbb{E}R_{\tau_n} \\
&= \mathbb{E}V_t - \mathbb{E}[R_{\tau_n} - R_t],
\end{aligned}$$

where the second line follows from the martingale property. Rearranging and taking limits we see

$$0 \leq \lim_{n\to\infty} \{\mathbb{E}V_{\tau_n} - \mathbb{E}G_{\tau_n}\} = -\lim_{n\to\infty} \mathbb{E}[R_{\tau_n} - R_t],$$

since $(\tau_n)_{n\in\mathbb{N}}$ is chosen specifically so that $\mathbb{E}V_t - \lim_{n\to\infty} \mathbb{E}G_{\tau_n} = 0$. Moreover, because $(R_t)_{t\geq 0}$ is non-decreasing—as well as being positive by construction—we must have that

$$\lim_{n\to\infty} \{\mathbb{E}V_{\tau_n} - \mathbb{E}G_{\tau_n}\} = \lim_{n\to\infty} \mathbb{E}[R_{\tau_n} - R_t] = 0. \tag{1.14}$$

We can pass to a subsequence $(\tau_{n_k})_{k\in\mathbb{N}} \subset (\tau_n)_{n\in\mathbb{N}}$ and retain the convergence of the limits in (1.14) $\mathbb{P}$-almost surely ([Bre11], Theorem 4.9), i.e.

$$\lim_{k\to\infty} \left\{V_{\tau_{n_k}} - G_{\tau_{n_k}}\right\} = 0 \quad \mathbb{P}\text{-a.s.} \tag{1.15}$$

$$\lim_{k\to\infty} \left\{R_{\tau_{n_k}} - R_t\right\} = 0 \quad \mathbb{P}\text{-a.s.}. \tag{1.16}$$

For a given $\varepsilon > 0$, (1.15) implies that for k large enough, $G_{\tau_{n_k}} \geq V_{\tau_{n_k}} - \varepsilon$, i.e. $D_t^\varepsilon \leq \tau_{n_k}$, and therefore $R_{D_t^\varepsilon} \leq R_{\tau_{n_k}}$ (since $(R_t)_{t\geq 0}$ is non-decreasing). Furthermore, (1.16) tells us that $R_{\tau_{n_k}} \to R_t$ $\mathbb{P}$-almost surely, and so $R_{D_t^\varepsilon} \leq R_t$ $\mathbb{P}$-almost surely. We conclude by recalling that $t \leq D_t^\varepsilon$ and that $(R_t)_{t\geq 0}$ is non-decreasing $\mathbb{P}$-almost surely, and so $R_t \leq R_{D_t^\varepsilon}$ $\mathbb{P}$-almost surely.

2° For indistinguishability, let $\hat{\Omega}$, with $\mathbb{P}(\hat{\Omega}) = 1$, be the set on which the processes $(V_t)_{t\geq 0}$, $(G_t)_{t\geq 0}$, and $(R_t)_{t\geq 0}$ are right-continuous, and such that for every $\omega \in \hat{\Omega}$ and every rational number

$q \geq 0$, $R_{D^\varepsilon_q(\omega)}(\omega) = R_q(\omega)$. For an arbitrary t, let $(q_n)_{n\in\mathbb{N}}$ be a sequence of rational numbers such that $q_n \geq t$ for all n, and $q_n \to t$. By the fact that $(R_t)_{t\geq 0}$ is non-decreasing we have

$$R_t(\omega) \leq R_{D^\varepsilon_t(\omega)}(\omega) \leq R_{D^\varepsilon_{q_n}(\omega)}(\omega) = R_{q_n}(\omega),$$

with the final equality holding by assumption of $\omega \in \hat{\Omega}$. Then, by right-continuity of $(R_t)_{t\geq 0}$ we conclude that for all $\omega \in \hat{\Omega}$, $R_t(\omega) = R_{D^\varepsilon_t(\omega)}(\omega)$ for all $t \geq 0$. ■

Theorem 1.3.21. *D^ε_τ is an ε-optimal stopping time.*

Proof. The right-continuity of both $(G_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ gives us that

$$G_{D^\varepsilon_\tau} \geq V_{D^\varepsilon_\tau} - \varepsilon. \tag{1.17}$$

Lemma 1.3.20 tells us that the processes $(R_{D^\varepsilon_t})_{t\geq 0}$ and $(R_t)_{t\geq 0}$ are indistinguishable; let $\tilde{\Omega}$ be the set—with full measure under $\mathbb{P}$—such that for $\omega \in \tilde{\Omega}$, $R_t(\omega) = R_{D^\varepsilon_t(\omega)}(\omega)$ for all $t \geq 0$. For any $(\mathscr{F}_t)$-stopping time τ, and for any $\omega \in \tilde{\Omega}$, denote by s the real number $\tau(\omega)$, and so we then have $R_s(\omega) = R_{D^\varepsilon_s(\omega)}(\omega)$. Hence we can conclude that $R_{D^\varepsilon_\tau} = R_\tau$ $\mathbb{P}$-almost surely. This immediately tells us—via the Doob-Meyer decomposition, and the martingale property of $(M_t)_{t\geq 0}$—that

$$\mathbb{E}V_{D^\varepsilon_\tau} = \mathbb{E}M_{D^\varepsilon_\tau} - \mathbb{E}R_{D^\varepsilon_\tau} = \mathbb{E}M_\tau - \mathbb{E}R_\tau = \mathbb{E}V_\tau. \tag{1.18}$$

Therefore, by taking expectations in (1.17) we can arrive at

$$\mathbb{E}G_{D^\varepsilon_\tau} \geq \mathbb{E}V_{D^\varepsilon_\tau} - \varepsilon = \mathbb{E}V_\tau - \varepsilon,$$

proving that D^ε_τ is ε-optimal. ■

We can now summarise the ε-optimal argument in the following theorem.

Theorem 1.3.22. *Let $(G_t)_{t\geq 0}$ be an $(\mathscr{F}_t)$-adapted, càdlàg, positive and bounded process with Snell envelope $(V_t)_{t\geq 0}$. For any $(\mathscr{F}_t)$-stopping time τ, and any $\varepsilon > 0$, denote by D^ε_τ the début time after τ of the set*

$$A^\varepsilon = \{(\omega, t) : G_t(\omega) \geq V_t - \varepsilon\},$$

then, for any stopping time $\sigma \geq \tau$,

$$V_\tau = \mathbb{E}[V_{D^\varepsilon_\tau \wedge \sigma} | \mathscr{F}_\tau] \quad \mathbb{P}\text{-}a.s.$$

Proof. Because $(V_t)_{t\geq 0}$ is a right-continuous supermartingale, we know that $(V_{t\wedge D^\varepsilon_\tau})_{t\geq 0}$ is also a right-continuous supermartingale (see Theorem II.77.4, [RW94]), and therefore to prove that $(V_{t\wedge D^\varepsilon_\tau})_{t\geq 0}$ is a martingale for any $\sigma \geq \tau$, it is sufficient to prove that it is constant in expectation for any $\sigma \geq \tau$ (Problem 1.3.25, [KS98a]).

Let σ be any $(\mathscr{F}_t)$-stopping time greater than or equal to τ, then by the optional sampling theorem (Theorem 1.3.22, [KS98a]), and since $\tau \leq D^\varepsilon_\tau$, we have that $V_\tau \geq \mathbb{E}\left[V_{\sigma\wedge D^\varepsilon_\tau} \middle| \mathscr{F}_\tau\right]$, and so $\mathbb{E}V_\tau \geq \mathbb{E}V_{\sigma\wedge D^\varepsilon_\tau}$. Next, recall that $\sigma \wedge D^\varepsilon_\tau$ is itself a stopping time, and that it is less than or equal to D^ε_τ, thus by another application of the optional sampling theorem we obtain $V_{\sigma\wedge D^\varepsilon_\tau} \geq \mathbb{E}\left[V_{D^\varepsilon_\tau} \middle| \mathscr{F}_{\sigma\wedge D^\varepsilon_\tau}\right]$, and so $\mathbb{E}V_{\sigma\wedge D^\varepsilon_\tau} \geq \mathbb{E}V_{D^\varepsilon_\tau}$. Combining the two inequalities in expectation from this paragraph and (1.18) yields

$$\mathbb{E}V_\tau \geq \mathbb{E}V_{\sigma\wedge D^\varepsilon_\tau} \geq \mathbb{E}V_{D^\varepsilon_\tau} = \mathbb{E}V_\tau,$$

allowing us to conclude. ■

Returning from our sojourn into ε-additive optimality, we can recall that Corollary 1.3.15 tells us that $(V_t)_{t\geq 0}$ is not only a supermartingale, but it is a martingale up until an optimal stopping time τ_D. We can obtain some more insight into the process with this understanding, and to do this we again make use of the Doob-Meyer decomposition.

For simplicity, let us assume that $(G_t)_{t\geq 0}$ actually has continuous paths—a stronger assumption that we will take to be true from Chapter 3 onwards. Notice that the general theory above still applies in this case. Theorem D.13 from Karatzas and Shreve, [KS98b], tells us that under this new continuity assumption, $(R_t)_{t\geq 0}$ is continuous and 'flat' outside of the set $\{t \geq 0 : V_t = G_t\}$, i.e.

$$\int_0^\infty \mathbb{1}_{\{V_t > G_t\}}\, dR_t = 0.$$

This result is very interesting because it gives us another view of why the set $D = \{V_t = G_t\}$ is so important. The process $(V_t)_{t\geq 0}$ behaves like a martingale until the first time we enter D, at this point $(R_t)_{t\geq 0}$ begins to increase, pulling $(V_t)_{t\geq 0}$ downwards, in a sense, reverting back to being only a supermartingale.

Remark 1.3.23. A vital point to realise is that, while we have framed the analysis above in the supremum problem formulation (as is typical in the field), there is no reason why we could not have presented it in the infimum problem form. In fact, the results for the infimum problem are identical, *mutatis mutandis*, however our Snell envelope is now the largest submartingale dominated by $(G_t)_{t\geq 0}$, and our ε and λ optimal sets A^{ε} and A^{λ} have their respective inequalities reversed; the optimal stopping time is still the hitting time of the set where the value and the payoff coincide.

1.4 STOCHASTIC GAMES

We have now come to the end of our study of optimal stopping; the methods and results described above are foundational in the field and will form the basis for much of the extended study to come. However, the problems seen so far have all been single-agent problems—one individual optimising a payoff process—so how does the analysis change when we introduce a competitor? This question opens up the world of stochastic games, a broad field of study with links to traditional, economic, game theory.

Let us begin with the rudiments of deterministic game theory as analysed through a classical problem, the 'Prisoner's Dilemma' (see Example 4.11 [MSZ13]). Two suspects have been arrested, however, the authorities are lacking any tangible evidence to obtain a prosecution. The only option they have is to try and persuade one, or both, detainees to implicate the other. Both prisoners are locked in cells, are unable to communicate, and are given the following choices:

1) if you cooperate and implicate the other, you will go free and the other will be sentenced to 10 years in prison,

2) if you both cooperate and implicate each other, then you will both serve a reduced sentence of six years,

3) if you both refuse to cooperate and implicate the other we will sentence you for a smaller crime and guarantee you both serve one year.

In traditional game theory, the outcome of the game is often shown in a table like so,

		Player 2: C	Player 2: D
Player 1	C	(6,6)	(0,10)
	D	(10,0)	(1,1)

where C is the cooperation strategy and D is the refusal to cooperate. From this we can begin to build up a mathematical framework for these types of games. Let $N = \{1, 2, ..., n\}$ be a finite set of players and S_i be the set of strategies of player $i \in N$. We define $S = S_1 \times S_2 \times ... \times S_N$, and a strategy vector $s = (s_i)_{i \in N}$ is an element of S. Moreover, let $u_i : S \to \mathbb{R}$ be a function mapping the strategy vector s to the payoff $u_i(s)$ for each player i in N. In the Prisoner's Dilemma, we have $N = 2$ and $S_1 = S_2 = \{C, D\}$ and, for simplicity, we keep the payoff function for each outcome equal to the respective jail times—hence the table above is also the table of payoffs.

This game is a prime example for the introduction of Nash equilibria, a stability concept in game theory named for John Nash, who helped popularise the idea in his seminal paper on 'non-cooperative games', [Nas51]. The concept had existed for a while before Nash's paper, in particular, in 1897, Cournot [Cou97] looked at a specific oligopoly model and arrived at an equilibrium. Equally, in 1944, Von Neumann and Morgenstern [VNM44] considered the concept of equilibrium using 'mixed strategies', but only for so called 'zero-sum' games (both are things that we will see later).

In principle, a Nash equilibrium is simple: a game is in a Nash equilibrium if, when any player deviates from their current strategy, their payoff gets worse. We can formalise this, introduce the notation s_{-i} to be the strategy vector not including the i^{th} player. In a game where all players are trying to maximise a payoff function, a Nash equilibrium is defined as follows.

Definition 1.4.1. (Definition 4.17, [MSZ13]) A strategy vector $s^* = (s_1^*, ..., s_n^*)$ is called a Nash equilibrium if, for each $i \in N$ and each strategy $s_i \in S_i$,

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*).$$

We will see through out the rest of this work that the Nash equilibrium plays a central role. Fortunately, we also have a related concept called 'best response' which is immediately linked to Nash equilibria.

Definition 1.4.2. (Definition 4.18, [MSZ13]) A strategy, s_i, for the i^{th} player is called a best response to s_{-i} if

$$u_i(s_i, s_{-i}) = \max_{t_i \in S_i} u_i(t_i, s_{-i}),$$

Definition 1.4.3. (Definition 4.19, [MSZ13]) The strategy vector $s^* = (s_1^*, \dots, s_n^*)$ is a Nash equilibrium if s_i^* is the best response to s_{-i}^* for every player i in N.

Best response arguments are incredibly powerful and will allow us great freedom when proving Nash equilibria without being tied to the original definition.

Returning to the Prisoner's Dilemma, we can find a unique Nash equilibrium, namely $s^* = (C, C)$ where both prisoners implicate the other. The payoff, $u_i(s^*) = 6$ for both $i = 1$ and $i = 2$. Considering best responses, if Player 1 was to deviate to $s_1 = D$, then Player 1 would receive 10 years in prison, $u_1((D, C)) = 10$, a worse outcome. Likewise, if Player 2 deviates, they too get 10 years in prison, $u_2((C, D)) = 10$. An interesting point here is that the Nash equilibrium is not strictly the best outcome for the two players, this would be (D, D), wherein both players serve only one year, $u_i((D, D)) = 1$ for $i \in \{1, 2\}$.

The existence and uniqueness of Nash equilibria is a major focus in game theory, uniqueness is rare, but existence has been shown widely in many set-ups. In fact, Nash equilibria are named for John Nash because he was the first to show the existence of these equilibria in 'non-zero sum games' with 'mixed strategies'.

'Zero-sum games' are games where the payoffs to all players sum to zero, i.e. they are games of conflict. In a two player game the game is zero-sum if $u_1(s) = -u_2(s)$ for any strategy vector $s \in S$. Seen through this lens, we can instead think of one payoff function $u(s) : S \to \mathbb{R}$, where Player 1 attempts to maximise and Player 2 to minimise.

The question then emerges as to how one actually plays these games. We have described before the set of strategies S_i for player i, and these are known as 'pure strategies'—in our stochastic games these will be stopping times. However, showing the existence of Nash equilibria in games with pure strategies is notoriously hard, and so we broaden our set of admissible strategies to 'mixed strategies'. These go by many names—and later we will describe the type we will use and the relationships between them all—but in classical game theory they are simply probability distributions over pure strategies.

What we have described up until now has been truly in the domain of economic game theory, and the games featured are very much one stage, deterministic games. As the need to control dynamical and stochastic systems grew, these ideas were extended, and one of the first instances of a stochastic game came from Dynkin [Dyn68] in 1968, however, since his original paper is only available in Russian, we will cite a modern paper by Ekström and Peskir [EP08].

Since their inception, multiple methods have been used to solve these problems; Dynkin originally used methods similar to those used by Snell in the single controller problem, [Sne52]; Bismut—a truly prolific writer in the area—addressed the Markovian version of the problem in [Bis77b], using methods of convex analysis that he had previously developed in [Bis77a]; Bensoussan and Friedman used variational inequalities [BF74] and [BF77]; and Stettner made use of penalisation methods, again in the Markovian framework, [Ste82b]. Martingale methods were then generalised in the work done by Lepeltier and Maingueneau [LM84], and it is this paper which will form the foundation of Chapter 2. For now, we will stay with Ekström and Peskir's work [EP08] as it provides a clear introduction to Markovian Dynkin games, which, while not the focus of this work, are arguably the nicer of the two formulations to grasp. This

To begin, let $(\mathbb{P}_x)_{x\in\mathbb{R}}$ be a family of probability measures and let $(X_t)_{t\geq 0}$ be a strong Markov process defined on the complete, filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}_x, (\mathscr{F}_t)_{t\geq 0})$, taking values in $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. It is assumed that $X_0 = x$ under $\mathbb{P}_x$ for any $x \in \mathbb{R}$. Moreover, let the trajectories of $(X_t)_{t\geq 0}$ be right-continuous and left-continuous over stopping times. Let $f(x)$, $g(x)$ and $h(x)$ be continuous payoff functions mapping $\mathbb{R}$ to $\mathbb{R}$, obeying the ordering $g(x) \leq h(x) \leq f(x)$, and satisfying the integrability conditions

$$\mathbb{E}_x\left[\sup_{t\geq 0}|f(X_t)|\right] < \infty \qquad \& \qquad \mathbb{E}_x\left[\sup_{t\geq 0}|g(X_t)|\right] < \infty$$

for all real x. Note that we do not require the same integrability condition for $h(x)$, since it is bounded between $f(x)$ and $g(x)$.

Arbitrarily, we say Player 1 is the minimiser and chooses the stopping time τ from $\mathscr{T}$, and likewise Player 2 is the maximiser and chooses a stopping time σ from $\mathscr{T}$. The Dynkin games that we are interested in are formulated as 'war of attrition' games, also called 'second mover advantage' games in the economics literature—these are games in which both players are incentivised to wait for the other to act first (i.e. to stop first). Due to the ordering of the payoff functions we formulate the expected payoff as follows

$$J_x(\tau,\sigma) := \mathbb{E}_x\big[f(X_\tau)\mathbb{1}_{\{\tau<\sigma\}} + g(X_\sigma)\mathbb{1}_{\{\tau>\sigma\}} + h(X_\tau)\mathbb{1}_{\{\tau=\sigma\}}\big]. \tag{1.19}$$

If the minimiser stops first, the payoff will be the higher amount and if the maximiser stops first, the payoff is the lower amount, hence both players have an incentive to stop second. Both players are incentivised to stop, however, since failing to do so will result in the payoff h at the terminal time (finite or infinite), which is suboptimal for both players.

In a traditional, single-agent, optimal stopping problem, we look to either find the supremum or infimum of the payoff function, however, in our case we need to find both simultaneously. It is well known that the order in which we take the infimum and supremum matters and so we cannot look for a single value, $V(x)$, immediately, instead we must define the upper and lower values:

$$V^+(x) = \inf_{\tau\in\mathscr{T}} \sup_{\sigma\in\mathscr{T}} J_x(\tau,\sigma), \qquad V^-(x) = \sup_{\sigma\in\mathscr{T}} \inf_{\tau\in\mathscr{T}} J_x(\tau,\sigma). \tag{1.20}$$

Naturally, due to the ordering of the infimum and supremum we have $V^-(x) \le V^+(x)$, i.e.

$$\sup_{\sigma\in\mathscr{T}} \inf_{\tau\in\mathscr{T}} J_x(\tau,\sigma) \le \inf_{\tau\in\mathscr{T}} \sup_{\sigma\in\mathscr{T}} J_x(\tau,\sigma).$$

We say that the game has a value if $V(x) := V^+(x) = V^-(x)$.

The concept of a Nash equilibrium maps over into this framework very nicely. Since any deviation from a Nash equilibrium produces worse outcomes for the deviating player, a Nash equilibrium in a stochastic game is a saddle point in the expected payoff.

Definition 1.4.4. A pair $(\tau^*,\sigma^*) \in \mathscr{T}\times\mathscr{T}$ is called a Nash equilibrium if, for any other choice of stopping times $(\tau,\sigma) \in \mathscr{T}\times\mathscr{T}$,

$$J_x(\tau^*,\sigma) \le J_x(\tau^*,\sigma^*) \le J_x(\tau,\sigma^*). \tag{1.21}$$

This is the basic set-up of a (Markovian) Dynkin game. Clearly, the way to proceed is to establish a value, and to prove the existence of a Nash equilibrium. In fact, if one can do the latter, then the former holds immediately. To see this, we first take the inequality $J_x(\tau^*,\sigma) \le J_x(\tau,\sigma^*)$ from (1.21), then take the supremum over σ of both sides, and then the infimum over τ of both sides to yield

$$\sup_{\sigma \in \mathcal{T}} J_x(\tau^*, \sigma) \leq \inf_{\tau \in \mathcal{T}} J_x(\tau, \sigma^*).$$

The left-hand side will always be greater than the infimum over τ, and the right-hand side will always be less than the supremum over σ, i.e.

$$\inf_{\tau \in \mathcal{T}} \sup_{\sigma \in \mathcal{T}} J_x(\tau, \sigma) \leq \sup_{\sigma \in \mathcal{T}} \inf_{\tau \in \mathcal{T}} J_x(\tau, \sigma^*),$$

hence $V^+(x) \leq V^-(x)$, implying $V(x) = J_x(\tau^*, \sigma^*)$.

What follows is a result from Ekström and Peskir wherein they prove the existence of both a value and a Nash equilibrium. Due to the Markovian set-up, their result bears a striking resemblance to what we have already seen.

Theorem 1.4.5. *(Theorem 2.1, [EP08]) Given the optimal stopping game defined in (1.20), if $(X_t)_{t\geq 0}$ is right-continuous, then the game has a value, $(V_t)_{t\geq 0}$. If $(X_t)_{t\geq 0}$ is additionally left-continuous over stopping times, then the pair (τ^*, σ^*) defined as*

$$\tau^* = \inf\{t \geq 0 : X_t \in D_1\} \quad \& \quad \sigma^* = \inf\{t \geq 0 : X_t \in D_2\},$$

where $D_1 = \{V = f\}$ and $D_2 = \{V = g\}$, is a Nash equilibrium.

The fact that we can characterise the optimal stopping times in this way is a feature of the Markovianity of the set-up. As we said earlier, we have presented this introduction to Dynkin games through the Markovian formulation, specifically because of its surface similarities to classical optimal stopping, however, our concern is with non-Markovian Dynkin games. Such games are more general, but the intuitions gained from the Markovian framework—such as the formulation of the optimal stopping times—will be useful for non-Markovian games, since these are significantly less well behaved.

We would like to conclude this chapter with a brief discussion on the applications of Dynkin games. As we mentioned at the start of this chapter, it is well known that a real-world use of optimal stopping is the pricing of 'American options'—financial market contracts that give the holder the right, but not the obligation, to buy or sell (call or put options respectively) some underlying asset before a given maturity date. The most famous application of Dynkin games was introduced by Yuri Kifer in two papers, one in 2000, [Kif00], and then a second in 2013, [Kif13].

In what he calls ‘game contingent claims’, or ‘Israeli options’, Kifer considers the American option, but allows the seller the right to cancel the option (at a cost) at any time before maturity. He shows that the pricing of such contracts is equivalent to the analysis of two-player Dynkin games of the form that we will study in Chapter 2.

Game contingent claims also inherit the structural nuances that normal options contracts have. For example consider the typical American option, but with the risk of default, or the Russian option (path dependent perpetual options), both apt for real world options trading. Allowing for the cancellation of these contracts again creates a game contingent claim. In 2012, Lempa and Matomäki, [LM13], derived necessary and sufficient conditions for the existence of a unique Nash equilibrium in Dynkin games with a random time horizon only seen by one of the players. This random time horizon can be seen in financial terms as a perpetual game option with default risk. In 2010, Dolinski, [Dol10], demonstrated a method for approximating the value of a continuous time Dynkin game with a sequence of values of discrete time Dynkin games, and then used this theory to approximate the value of Russian game options. Russian options are considered ‘exotic’ and are currently traded in the financial markets. The payoffs of these contracts are dependent on the entire history of the stock price, and typically this takes the form of the supremum of the stock price since the inception of the contract.

The prior two examples demonstrate the utility of game contingent claims; however, unfortunately game contingent claims, at this time, remain a mostly theoretical construction and have not seen adoption in the markets. However, they are not the only applications of Dynkin games; another comes from the area of ‘convertible bonds’—debt instruments already used in the market. A bond is a contract which facilitates the lending of monies from one agent to another; the receiver of capital promises to return the capital in the future at a fixed date, but will make regular interest payments (called coupons) in the intervening time. Convertible bonds are bonds with an option attached to convert the debt to equity (i.e. shares in the company) at the discretion of the bondholder. However, these convertible bonds are also ‘callable’, meaning the issuer of the convertible bond may, at their discretion, convert the bond to equity. When viewing the bond value itself, with the bondholder attempting to maximise it and the issuer attempting to minimise it, the result can be framed as a two-player Dynkin game. In their 2015 paper, Yan, Yi, Yang, and Liang, [Yan+15], prove the existence of a value for the bond and derive optimal strategies similar in form to those found in Theorem 1.4.5 under the Markovianity assumption.

A second application of Dynkin games that still uses the general concept of options theory without requiring the creation of tradable over-the-counter game options contracts are so-called 'real options'. The name is derived from the fact that real options are typically focussed on the right, but not the obligation, of the management of a business to pursue or abandon a business opportunity. Such options are more immediately applicable to the real-world. One example of such a contract was given in the paper by De Angelis, Gensbittel and Villeneuve, [DAGV21], where a company holds a concession to drill oil wells. A public authority would like to enter a contractual arrangement with the company whereby they may withdraw the concession rights at any time subject to a cancellation fee. The investor then views the decision to invest as an option on the price of the oil, or more simply the opportunity is profitable so long as the value of the extractable oil can yield more than the fixed costs to extract it (the equivalent of the strike price of the option). The ability of the public authority to cancel this option then yields a Dynkin game. The authors prove the existence of a value and a Nash equilibrium in the setting, and moreover characterise the optimal strategies as hitting times.

With these examples in mind, one can see the many possible applications of Dynkin games in the financial markets alone. The theory of these games allows us to assign values to—and optimal strategies for—valuable contracts which force competition between two parties.

CHAPTER 2

NON-MARKOVIAN GAMES

As we detailed in Chapter 1, the study of Dynkin games has a long history, with a significant portion of research being done into Markovian games. We have already seen the formulation for the payoff of a Markovian game in (1.19), however, this is a slightly more general setting to that which was used historically; there it is typical to only use two functions $f(x)$ and $g(x)$ such that $g(x) \leq f(x)$ for all $x \in \mathbb{R}$, and the payoff was

$$J_x(\tau,\sigma) := \mathbb{E}_x\big[f(X_\tau)\mathbb{1}_{\{\tau<\sigma\}} + g(X_\sigma)\mathbb{1}_{\{\tau\geq\sigma\}}\big]. \tag{2.1}$$

If, however, instead of having an underlying real-valued process $(X_t)_{t\geq 0}$ and then functions $f(x)$ and $g(x)$ mapping the reals to themselves, we could instead allow for these payoffs to be processes themselves, i.e. $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ (given under certain assumptions which we will discuss later). Our equivalent of (2.1) would then be

$$J(\tau,\sigma) := \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\tau\geq\sigma\}}\big].$$

Without giving too many details at this current moment, this is the non-Markovian formulation of a Dynkin game.

One can trace the development of non-Markovian games back to the study of Markovian games, when in 1977, Bismut analysed Markovian Dynkin games using convex analysis, [Bis77b]. In that paper (Remark II.3) Bismut mentions that in discussions with Mokobodzki, the latter had mentioned that if a certain assumption held then a key convergence result also held. This is one of the earliest mentions of what would become known as 'Mokobodzki's assumption'. The technical details of Mokobodzki's analysis are slightly involved and are not particularly pertinent here, so we will not dwell too long on them, but Mokobodzki later formalised this assumption in his 1978 lecture series. He analysed Bismut's work, and arrived at the following conclusion (Theorem 1, [Mok78]): given suitable measurability and boundedness properties on the space of functions C with which he works, the Markovian game studied in [Bis77b] has a solution if there exists functions p and q which satisfy (in our notation) $g \leq p-q \leq f$, and $p = \inf\{v \in C : v \geq g+q\}$ and $q = \inf\{v \in C : p-f\}$.

One of the first links between Markovian and non-Markovian theory can be found in Bismut's 1977 paper [Bis77c], which presents the results that the later paper, [Bis79b], then proves. The 1977 paper assumes a càdlàg stochastic process and considers a typical, single-agent, optimal stopping problem and solves it using martingale methods, however, in the final section he studies the Markovian game form from [Bis77b] and makes use of Mokobodzki's assumption.

In 1979, Bismut studied the problem using Mokobodzki's assumption with right-continuous, adapted and bounded processes, [Bis79a]. 1982 was a productive year; Alario-Nazaret completed his doctoral thesis on Dynkin games using these same assumptions, [AN82]; Stettner showed the existence of a value under these assumptions using a penalty method, [Ste82a]; and Alario-Nazaret, Lepeltier and Marchal, looked at the completely non-Markovian game, using optional and class D payoff processes. In the non-Markovian case, Mokobodzki's assumption looks extremely similar, and now states that the game has a value if there exist two positive supermartingales $(Y_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ of class D such that $g_t \leq Y_t - Z_t \leq f_t$ (see Theorem 2-3, [ANLM82]). They found a value and a Nash equilibrium for general continuity requirements on $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$, [ANLM82]. Finally, in 1984, Lepeltier and Maingueneau, [LM84], studied the fully general case of non-Markovian games without Mokobodzki's assumption.

Fascinatingly, the results shown by Lepeltier and Maingueneau were independently proven by Stettner just a few weeks later. The paper, [Ste84], utilises vastly different methodologies to [LM84], instead focusing on piecewise approximations of the processes $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$. He proves the right-continuity of the upper value, then using the techniques in Kifer's 1971 paper, [Kif71], shows the existence of a value. Moreover, Stettner's work also shows that a penalty method can be used to prove the existence of a value, and so does not rely on 'aggregation'.

This chapter will be a rigorous analysis of the work done by Lepeltier and Maingueneau as their method is foundational to our approach in later chapters. Large portions of this chapter are novel and will not be found in the original paper. For example, the section regarding the existence of a Nash equilibrium in [LM84] contains one theorem, Theorem 15, however, not only does Section 2.5 here contain a plethora of additional definitions, lemme and propositions designed to make fully rigorous the analysis of the Nash equilibrium, but we have also significantly altered the proof method of Theorem 15 itself (see Theorem 2.5.1) as we were unable to verify their original reasoning. Moreover, we have made; major changes to the proof of Corollary 12 (see Lemma 2.4.2); minor alterations to the proof methods for Lemma 5 (see Lemma 2.3.3), Theorem 9 (see Theorem 2.3.12), and Theorem 11 (see Theorem 2.4.1); contributed a novel section on minimality (see Section 2.6); and provided numerous other results throughout the chapter. For clarity, results taken directly from [LM84] will be labelled with their corresponding number from the original paper, all other results and remarks etcetera are unique to this work.

Their original work was also written in French, meaning that the translation and explanation of this work not only sets up the rest of our work, but also provides a useful source for an English language version of the important contributions from the authors. We will change the notation of the original paper so that it remains consistent with ours.

2.1 FORMULATION OF THE PROBLEM

Let $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\geq 0})$ be a filtered probability space, where the filtration satisfies the usual conditions, and $\mathscr{F}_0$ is the trivial σ-algebra. Recall that the usual conditions state that the filtration, $(\mathscr{F}_t)_{t\geq 0}$, is right-continuous, i.e. defining $\mathscr{F}_{t+} := \bigcap_{\varepsilon>0} \mathscr{F}_{t+\varepsilon}$, then $\mathscr{F}_t = \mathscr{F}_{t+}$ for all $t \geq 0$, and that $\mathscr{F}_0$ contains all the $\mathbb{P}$-null sets in $\mathscr{F}$.

The set of strategies for the two players is $\mathscr{T}$, which is the set of all $(\mathscr{F}_t)$-stopping times. Let $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ be two stochastic processes, and we make the following assumptions.

Assumption 2.1.1. *$(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ are:*

i) optional,

ii) right-continuous,

iii) bounded,

iv) defined at infinity by $f_\infty = g_\infty = 0$,

v) and satisfy $g_t \leq f_t$, for all $t \geq 0$, $\mathbb{P}$-almost surely.

We define the expected payoff, $J(\tau,\sigma)$, where $(\tau,\sigma) \in \mathscr{T} \times \mathscr{T}$, to be

$$J(\tau,\sigma) = \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}}\big]. \tag{2.2}$$

Player 1 chooses an $\mathscr{F}_t$ stopping time τ, and seeks to minimise $J(\tau,\sigma)$ in τ, and Player 2 chooses an $\mathscr{F}_t$ stopping time σ, and seeks to maximise $J(\tau,\sigma)$ in σ.

In the usual way, we will say that the game has a value $V := V^+ = V^-$ as soon as the expressions $V^+ = \inf_\tau \sup_\sigma J(\tau,\sigma)$ and $V^- = \sup_\sigma \inf_\tau J(\tau,\sigma)$ are equal. We will say that (τ^*,σ^*) is a Nash equilibrium for the game when

$$J(\tau^*,\sigma) \leq J(\tau^*,\sigma^*) \leq J(\tau,\sigma^*)$$

for all pairs $(\tau,\sigma) \in \mathscr{T} \times \mathscr{T}$. As mentioned in the previous chapter, if the game has a Nash equilibrium (τ^*,σ^*), then the game has a value, and more precisely

$$V = J(\tau^*,\sigma^*).$$

We have seen the pivotal role that the maximal conditional gain plays in optimal stopping, and to this end we define the concept of conditional upper and lower values.

Definition 2.1.2. (Definition 3, [LM84]) The upper (resp. lower) conditional value is the family of random variables indexed by $(\mathscr{F}_t)$-stopping times, defined for all $\rho \in \mathscr{T}$ by

$$\overline{V}(\rho) = \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_\rho} \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho} \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}} \big| \mathscr{F}_\rho\big],$$

$$\underline{V}(\rho) = \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho} \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_\rho} \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}} \big| \mathscr{F}_\rho\big].$$

Recall Definition 1.3.4. It is clear that the two families $\{\overline{V}(\rho), \rho \in \mathscr{T}\}$, $\{\underline{V}(\rho), \rho \in \mathscr{T}\}$ are $\mathscr{T}$-systems, and so we look to aggregate them into optional processes.

2.2 ON PATH REGULARITY

As we saw in Chapter 1, for example in Theorem 1.3.13, the path properties of the payoff processes (and, in this chapter, a small collection of additional processes) are at the heart of the study of optimal stopping problems. In Assumption 2.1.1 we have already specified the right-regularity of $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$—they are right-continuous—but we must also address their left-regularity.

In Chapter 1 (again see Theorem 1.3.13) our concept of left-regularity was left-continuity in expectation over stopping times, however, in this section we look to relax this further to what we call ‘previsible left semicontinuity’.

Definition 1.3.2 introduced the concept of the optional σ-algebra, and of optional processes, but there is a second, closely related, concept of previsibility.

Definition 2.2.1. (Definition IV.6 & Definition VI.12.1, [RW00]) The previsible σ-algebra $\mathscr{P}$ on $(0,\infty)\times\Omega$ is defined to be the smallest σ-algebra on $(0,\infty)\times\Omega$ such that every adapted càglàd (left-continuous with right limits) process is $\mathscr{P}$-measurable.

A process with time parameter set $(0,\infty)$ is called previsible if it is a $\mathscr{P}$-measurable map from $(0,\infty)\times\Omega$ to $\mathbb{R}$ (or to a more general state space).

A stopping time τ is called previsible if $\tau > 0$, and $[\tau,\infty) \in \mathscr{P}$.

Remark 2.2.2. We may now also comment on the assumption of optionality of $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ and why we must also assume right-continuity. Recall Definition 1.3.2, which provides the definition of optional processes. It is immediately evident that the set of such processes is larger than the set of càdlàg and adapted processes, but relatively simple processes can be optional and fail to be right-continuous. For any two stopping times $\sigma \leq \tau$, the process $\mathbb{1}_{(\sigma,\tau]}(\omega,t)$ is previsible (it is adapted with càglàd paths, see Example IV.62 (b), [DM78]) and therefore optional (see Theorem VI.4.7, [RW00]).

Previsible stopping times have an alternative definition. In fact, this second definition is of an object known as an 'announceable' stopping time, however, it has been proven that the two concepts are identical (see Theorem IV.71, [DM78]). First, however, we must introduce some notation. We say that a sequence $(s_n)_{n\in\mathbb{N}}$ strictly increases to a limit t, denoted by $s_n \uparrow\uparrow t$, if the sequence $(s_n)_{n\in\mathbb{N}}$ is monotonically increasing, convergent to t, and $s_n < t$ for all n. This is different from $s_n \uparrow t$, whereby s_n could equal t for all n greater than some n^*, say.

Definition 2.2.3. An $(\mathscr{F}_t)$-stopping time τ is said to be previsible if there exists a sequence of $(\mathscr{F}_t)$-stopping times $(\tau_n)_{n\in\mathbb{N}}$ such that $\tau_n \uparrow\uparrow \tau$.

Remark 2.2.4. In contrast to previsible stopping times we have 'totally inaccessible' stopping times. A stopping time τ is said to be totally inaccessible if for every previsible stopping time σ, $\mathbb{P}(\tau = \sigma < \infty) = 0$. These stopping times cannot be announced, and a classical example of such stopping times are the jump times of a Poisson process (see Theorem 4, [Pro04] for the general theorem which implies this).

Remark 2.2.5. The announceable definition of previsible stopping times is also rather intuitive, and as Rogers and Williams point out in [RW00], a previsible stopping time τ is one where, instantaneously before τ, an observer could have predicted its arrival. The term 'previsible' is anglicised from the French 'prévisible'—capable of being foreseen—and a direct translation to English is 'predictable', hence why some authors call them predictable stopping times.

As Remark 2.2.5 points out, previsibility encompasses an idea of anticipation, of being able to predict the next value instantaneously beforehand. Let $(X_t)_{t\geq 0}$ be a bounded and measurable process, but not necessarily adapted. Since the process is not adapted, at any time t, the random variable X_t need not be $\mathscr{F}_t$-measurable; since we commonly refer to a filtration $(\mathscr{F}_t)_{t\geq 0}$ as our 'knowledge', we do not 'know' anything about X_t a priori. However, we are able to estimate it given 'what we know', and this leads us to the concept of the previsible projection—the following intuition comes from Remark VI.45 c) in [DM82]. It is commonly understood that the best estimate of X_t given $\mathscr{F}_t$ is $\mathbb{E}[X_t|\mathscr{F}_t]$, and our best previsible estimate would be $\mathbb{E}[X_t|\mathscr{F}_{t-}]$, but are we able to estimate the entire path of $(X_t)_{t\geq 0}$? Of course, these conditional expectations are random variables, unique up to a null set, and we have an uncountable number of times to join, so can we find a process which combines these estimates? This is of course a question closely resembling that of aggregation. The answer is yes, and we are not simply restricted to times either, we can do this estimation at any previsible stopping time. We call the process which combines our best previsible estimations of X_t the 'previsible projection' of $(X_t)_{t\geq 0}$.

Theorem 2.2.6. *(Theorem VI.43, [DM82]) Let $(X_t)_{t\geq 0}$ be a positive or bounded measurable process. There exists a previsible process $(Z_t)_{t\geq 0}$ such that*

$$\mathbb{E}\big[X_\tau \mathbb{1}_{\{\tau<\infty\}}\big|\mathscr{F}_{\tau-}\big] = Z_\tau \mathbb{1}_{\{\tau<\infty\}} \quad \mathbb{P}\text{-}a.s.$$

for every previsible stopping time τ. The process $(Z_t)_{t\geq 0}$ is unique to within an evanescent set, is called the previsible projection of $(X_t)_{t\geq 0}$, and is denoted by $({}^pX_t)_{t\geq 0}$.

Remark 2.2.7. If we were to, instead, consider the same problem but using the typical estimate $\mathbb{E}[X_t|\mathscr{F}_t]$, we would obtain the object known as the 'optional projection' instead.

Theorem 2.2.6 makes use of the concept of evanescence, which is simply another way of discussing the uniqueness of stochastic processes. Of course, when we say uniqueness we actually mean that they differ only on a set of measure zero. We typically discuss this in terms of two processes being indistinguishable: let $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ be two stochastic processes, then X and Y are indistinguishable if $\mathbb{P}(X_t = Y_t, \forall t \geq 0) = 1$. This implies the existence of a universal null set (universal for all times), outside of which the paths of the processes are identical. Evanescence defines uniqueness in this manner by looking at the null set.

Dellacherie and Meyer, ([DM78], IV.8), explain the correspondence as follows. Let A be a subset of $[0,\infty)\times\Omega$ such that $\mathbb{1}_A(t,\omega)$ is a stochastic process, i.e. $\omega \mapsto \mathbb{1}_A(t,\omega)$ is a random variable for all $t \geq 0$. The set A is said to be evanescent if the process $\mathbb{1}_A$ is indistinguishable from zero, i.e. the projection of A onto Ω is contained in a $\mathbb{P}$-null set. If we take two processes X and Y as before, and define the set $A = \{(t,\omega) : X_t \neq Y_t\}$, then X and Y are indistinguishable if and only if the set A is evanescent.

Now that we are equipped with the necessary concept of previsibility, and are familiar with the previsible projection, we can finally state what we mean by 'previsible left semicontinuity' and why we choose this.

Definition 2.2.8. Let $(X_t)_{t\geq 0}$ be a real valued stochastic process such that for any previsible stopping time τ the integrability criteria $\mathbb{E}[|X_\tau|\mathbb{1}_{\{\tau<\infty\}}|\mathscr{F}_{\tau-}] < \infty$ holds. Then we say that $(X_t)_{t\geq 0}$ is previsibly left upper (resp. lower) semicontinuous if

$$ {}^{p}X_t \geq \limsup_{s \uparrow\uparrow t} X_s \qquad \left(\text{resp. } {}^{p}X_t \leq \liminf_{s \uparrow\uparrow t} X_s\right) $$

holds outside of an evanescent set.

Remark 2.2.9. The presence of an evanescent set means that if a processes is previsibly semicontinuous, then it is also previsibly semicontinuous over stopping times. This is because we have a global null set, outside of which the previsible semicontinuity property holds for all times.

At first glance this formulation may seem an odd choice, but it is in fact a natural generalisation from left-continuity in expectation over stopping times. We could have relaxed the left-regularity to be, say, upper (or lower) semicontinuity in expectation over stopping times, and this would arguably seem more natural, however, we can show that the two concepts are very closely linked. What follows in Theorem is a result from Bismut and Skalli, [BS97] which was repeated in more generality by Bank and Besslich, [BB19], and it provides this correspondence.

Theorem 2.2.10. *(Theorem II.1, [BS97] & Proposition 2.44, [BB19]) An optional positive process* $X : \Omega \times [0,\infty) \to \mathbb{R}$ *of class D satisfies*

$$ \mathbb{E}X_\tau \geq \limsup_{n\to\infty} \mathbb{E}X_{\tau_n} \tag{2.3} $$

for every monotone sequence (τ_n) *of stopping times converging to* τ*, if and only if* $(X_t)_{t\geq 0}$ *has right upper semicontinuous paths on* $[0,\infty)$ *and*

$$ {}^{p}X_t \geq \limsup_{s \uparrow\uparrow t} X_s \tag{2.4} $$

for $t \in (0,\infty)$ *outside of an evanescent set.*

Remark 2.2.11. One should notice that (2.3) needs to hold for both increasing and decreasing sequences of stopping times. This is a point of divergence with the original paper by Lepeltier and Maingueneau, [LM84], as they give this result as only needing to hold for decreasing sequences. The original authors do not give a reference for the result they use, however, Theorem 2.2.10 is the closest we were able to find, and this change in requiring the condition in expectation to hold for all monotonic sequences causes some quite noticeable further changes to proofs later.

Remark 2.2.12. The limit in (2.4) must have s strictly increase to t otherwise the result fails to hold. For example, take a simple probability space, $(\Omega, \mathscr{F}, \mathbb{P})$, with $\Omega = \{\omega_1, \omega_2\}$, and take the following filtration

$$\mathscr{F}_t := \begin{cases} \sigma(\omega_1, \omega_2) & \text{if } t \geq 1, \\ (\emptyset, \Omega) & \text{if } t < 1, \end{cases}$$

where $\sigma(\omega_1, \omega_2)$ is the σ-algebra generated by the events $\{\omega_1\}$ and $\{\omega_2\}$. Moreover, define the probability measure by $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = \frac{1}{2}$. On this space, define the process $(Y_t)_{t \geq 0}$ as

$$Y_t := \begin{cases} \mathbb{1}_{\{\omega = \omega_1\}} & \text{if } t \geq 1, \\ \frac{1}{2} & \text{if } t < 1, \end{cases}$$

then clearly $(Y_t)_{t \geq 0}$ is positive and bounded, the latter giving us class D. Moreover, it is right-continuous for every ω and adapted to the filtration $(\mathscr{F}_t)$, hence optional. Notice that $(Y_t)_{t \geq 0}$ is constant in expectation (trivially for $t < 1$, and for $t \geq 1$ we have $\mathbb{E}Y_t = \mathbb{P}(\omega = \omega_1) = \frac{1}{2}$), and so trivially obeys (2.3). However, let $(s_n)_{n \in \mathbb{N}}$ be an increasing sequence which converges to one, but stabilises at one for all n larger than some n^*, then

$$\limsup_{n \to \infty} Y_{s_n} = Y_1 = \mathbb{1}_{\{\omega = \omega_1\}},$$

but, the previsible projection of $(Y_t)_{t \geq 0}$ at time one is

$$^pY_1 = \mathbb{E}[Y_1 | \mathscr{F}_{1-}] = \mathbb{P}(\omega = \omega_1) = \frac{1}{2}.$$

Hence $^pY_1 < \limsup_{n \to \infty} Y_{s_n}$ on a set of positive measure—specifically $\{\omega_1\}$—contradicting the result of the theorem. Of course, this problem disappears if we insist that $(s_n)_{n \in \mathbb{N}}$ strictly converges to one since then $\limsup_{n \to \infty} Y_{s_n} = \frac{1}{2}$.

Remark 2.2.13. The process $(Y_t)_{t \geq 0}$ from Remark 2.2.12 can be trivially altered to demonstrate how processes can fail to be previsibly left upper semicontinuous. Simply take $(Y_t)_{t \geq 0}$ to be strictly larger than one half for $t < 1$, say $Y_t = 1$ for $t < 1$, then $^pY_1 = \frac{1}{2}$ as before, but $\limsup_{s \uparrow\uparrow 1} = 1$.

Remark 2.2.14. Theorem 2.2.10 is only concerned with the concept of upper semicontinuity, however, we can equally write this result out in terms of lower semicontinuity. Since the processes we will be dealing with are all bounded, we can always rescale them to ensure positivity is obtained. More precisely, let $(X_t)_{t\geq 0}$ satisfy all of the conditions of Theorem 2.2.10, and moreover let K be the upper bound on the process. Define a new process $\tilde{X}_t := K - X_t$, which is still positive. The interrelation between Theorem 2.2.10 and the version concerned with lower semicontinuity can then be seen via the following chain of reasoning:

$$\mathbb{E}\tilde{X}_\tau \leq \liminf_{n\to\infty} \mathbb{E}\tilde{X}_{\tau_n} \quad \Longleftrightarrow \quad K - \mathbb{E}X_\tau \leq K + \liminf_{n\to\infty} \mathbb{E}[-X_{\tau_n}] \quad \Longleftrightarrow \quad \mathbb{E}X_\tau \geq \limsup_{n\to\infty} \mathbb{E}X_{\tau_n}.$$

The same logic follows for the previsibility condition (2.4).

Since we have now introduced our concept of previsible left semicontinuity, we can formally provide the assumptions on the left-regularity of the payoff processes $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$.

Assumption 2.2.15.

i) $(f_t)_{t\geq 0}$ *is previsibly left lower semicontinuous,*

$$^p f_t \leq \liminf_{s\uparrow\uparrow t} f_s, \tag{A1}$$

ii) $(g_t)_{t\geq 0}$ *is previsibly left upper semicontinuous,*

$$^p g_t \geq \limsup_{s\uparrow\uparrow t} g_s. \tag{A2}$$

Due to the understanding of the correspondence between previsible left semicontinuity and semicontinuity in expectation over stopping times, the following proposition is now seemingly trivial, but is still very important.

Proposition 2.2.16. *The processes* $(f_t)_{t\geq 0}$ *and* $(g_t)_{t\geq 0}$ *are lower (resp. upper) semicontinuous in expectation over stopping times.*

Proof. Since both processes are optional, bounded (the latter implying class D from Remark 1.3.18, and we also adjust for positivity as mentioned in Remark 2.2.14), right-continuous, and satisfy (A1) and (A2), respectively, we can directly apply Theorem 2.2.10 to obtain that $\mathbb{E}g_\tau \geq \limsup_n \mathbb{E}g_{\tau_n}$ and $\mathbb{E}f_\tau \leq \liminf_n \mathbb{E}f_{\tau_n}$ for any monotone sequence (τ_n) of stopping times converging to τ. ■

Remark 2.2.17. It is also worthwhile to mention that these assumptions are not unexpected. A similar property can be found in the general set-up of the paper by De Angelis, Merkulov and Palczewski, [DAMP22], therein they allow for previsible jumps in the payoff processes, specifically for $(f_t)_{t\geq}$ to have downwards previsible jumps, and $(g_t)_{t\geq}$ to have upwards previsible jumps. Conditions (A1) and (A2) convey a similar idea, $(f_t)_{t\geq 0}$ can only jump downwards relative to its previsible projection and $(g_t)_{t\geq 0}$ can only jump upwards with respect to its previsible projection.

Before we move on to tackling the problem itself, we first must provide a handful of technical results which will be of use in the later stages of this analysis. Specifically, we must address the path regularity of certain variants of the payoff process $f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}}$, and their expectations. What follows makes use of some technical results found in Appendix B.

Theorem 2.2.18. *Let η be an $(\mathscr{F}_t)$-stopping time. The processes $(\tilde{Y}_t)_{t\geq 0}$ and $(\tilde{Z}_t)_{t\geq 0}$ defined as*

$$\tilde{Y}_t := f_\eta \mathbb{1}_{\{\eta\leq t\}} + g_t \mathbb{1}_{\{t<\eta\}}, \qquad \tilde{Z}_t := f_\eta \mathbb{1}_{\{\eta<t\}} + g_t \mathbb{1}_{\{t\leq\eta\}},$$

are previsibly left upper semicontinuous.

Proof. 1° Let $(s_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ be a strictly increasing sequence which converges to t. The limit supremum of $(\tilde{Y}_t)_{t\geq 0}$ can be split as follows

$$\begin{aligned}\limsup_{n\to\infty} \tilde{Y}_{s_n} &\leq \limsup_{n\to\infty} \left[f_\eta \mathbb{1}_{\{\eta\leq s_n\}}\right] + \limsup_{n\to\infty} \left[g_{s_n} \mathbb{1}_{\{s_n<\eta\}}\right] \\ &\leq f_\eta \limsup_{n\to\infty} \left[\mathbb{1}_{\{\eta\leq s_n\}}\right] + \limsup_{n\to\infty} \left[g_{s_n}\right] \limsup_{n\to\infty} \left[\mathbb{1}_{\{s_n<\eta\}}\right].\end{aligned}$$

The limit suprema of the indicator functions are directly calculable, so using the fact that $(g_t)_{t\geq 0}$ is previsibly left upper-semicontinuous, (A2), we can arrive at

$$\limsup_{n\to\infty} \tilde{Y}_{s_n} \leq f_\eta \mathbb{1}_{\{\eta<t\}} + {}^p g_t \mathbb{1}_{\{t\leq\eta\}}. \tag{2.5}$$

Next, we make the observation (see Theorem IV.56 in Dellacherie-Meyer, [DM78]) that the set $\{\eta < t\}$ belongs to $\mathscr{F}_{t-}$, and therefore so does $\{t \leq \eta\}$, meaning that the previsible projection in the second term in (2.5) can be written to include the indicator function too, i.e.

$$\limsup_{n\to\infty} \tilde{Y}_{s_n} \leq f_\eta \mathbb{1}_{\{\eta<t\}} + \mathbb{E}\left[g_t \mathbb{1}_{\{t\leq\eta\}} \middle| \mathscr{F}_{t-}\right].$$

We can then split the second indicator function over the sets of strict inequality and equality, then bound g_t from above by f_t to attain

$$\limsup_{n\to\infty} \tilde{Y}_{s_n} \leq f_\eta \mathbb{1}_{\{\eta<t\}} + \mathbb{E}\big[f_\eta \mathbb{1}_{\{t=\eta\}} + g_t \mathbb{1}_{\{t<\eta\}} \big| \mathscr{F}_{t-}\big].$$

Finally, we recall that the set $\{\eta < t\}$ not only belongs to $\mathscr{F}_{t-}$, but also to $\mathscr{F}_\eta$, and so $f_\eta \mathbb{1}_{\{\eta<t\}}$ is an $\mathscr{F}_\eta$-measurable random variable. Moreover, it is also measurable with respect to $\mathscr{F}_\eta|_{\{\eta<t\}}$ (see Definition B.2.1 for an explanation of this object), and Proposition B.2.3 tells us that it is, therefore, $\mathscr{F}_{t-}$-measurable. Thus we bring the first term inside of the conditional expectation, join the indicator functions and we are left with

$$\limsup_{n\to\infty} \tilde{Y}_{s_n} \leq \mathbb{E}\big[f_\eta \mathbb{1}_{\{\eta\leq t\}} + g_t \mathbb{1}_{\{t<\eta\}} \big| \mathscr{F}_{t-}\big] = {}^p\tilde{Y}_t.$$

2° Let us now consider $(\tilde{Z}_t)_{t\geq 0}$. The mappings $t \mapsto \mathbb{1}_{\{t\leq\eta\}}$ and $t \mapsto \mathbb{1}_{\{\eta<t\}}$ are left-continuous and adapted and so one can see that

$$\begin{aligned}
\limsup_{s\uparrow\uparrow t} \tilde{Y}_s &\leq f_\eta \limsup_{s\uparrow\uparrow t} \big\{\mathbb{1}_{\{\eta<s\}}\big\} + \limsup_{s\uparrow\uparrow t} \big\{g_s \mathbb{1}_{\{s\leq\eta\}}\big\} \\
&\leq f_\eta \mathbb{1}_{\{\eta<t\}} + ({}^p g_t)\mathbb{1}_{\{t\leq\eta\}} \\
&= f_\eta \mathbb{1}_{\{\eta<t\}} + {}^p\big(g_t \mathbb{1}_{\{t\leq\eta\}}\big),
\end{aligned}$$

with the final line holds using the same reasoning as in 1°. Likewise, we already know that $f_\eta \mathbb{1}_{\{\eta<t\}}$ is $\mathscr{F}_{t-}$-measurable, and so the result holds as required by bringing all terms inside of the previsible projection. ■

Corollary 2.2.19. *Let η be a $(\mathscr{F}_t)$-stopping time. The processes $(\hat{Y}_t)_{t\geq 0}$ and $(\hat{Z}_t)_{t\geq 0}$, defined as*

$$\hat{Y}_t := f_t \mathbb{1}_{\{t<\eta\}} + g_\eta \mathbb{1}_{\{\eta\leq t\}}, \qquad \hat{Z}_t := f_t \mathbb{1}_{\{t\leq\eta\}} + g_\eta \mathbb{1}_{\{\eta<t\}},$$

are previsibly left lower semicontinuous.

Proof. Notice that, because $-g_t \geq -f_t$, we can define $\hat{f}_t := -g_t$ and $\hat{g}_t := -f_t$ and so we see that $(-\hat{Y}_t)_{t\geq 0}$ and $(-\hat{Z}_t)_{t\geq 0}$ are identical to the processes $(\tilde{Y}_t)_{t\geq 0}$ and $(\tilde{Z}_t)_{t\geq 0}$ from Theorem 2.2.18. The processes $(-\hat{Y}_t)_{t\geq 0}$ and $(-\hat{Z}_t)_{t\geq 0}$ are therefore left upper semicontinuous, and since ${}^p(-\hat{Y}_t) = -{}^p\hat{Y}_t$, the claims hold. ■

Lemma 2.2.20. *Let ρ and η be stopping times such that $\eta \geq \rho$. Let $(\sigma^{\varepsilon})_{\varepsilon>0}$ and $(\tau^{\varepsilon})_{\varepsilon>0}$ be two families of stopping times which are greater than or equal to ρ for all $\varepsilon > 0$, which both increase to limits σ and τ respectively as ε decreases to zero. Recall the processes $(\tilde{Z}_t)_{t\geq 0}$ and $(\hat{Z}_t)_{t\geq 0}$ from Theorem 2.2.18 and Corollary 2.2.19 respectively. Then, $\mathbb{P}$-almost surely,*

$$\limsup_{\varepsilon\searrow 0}\mathbb{E}\big[\tilde{Z}_{\sigma^{\varepsilon}}\big|\mathscr{F}_{\rho}\big] \leq \mathbb{E}\big[\tilde{Z}_{\sigma}\big|\mathscr{F}_{\rho}\big], \quad \liminf_{\varepsilon\searrow 0}\mathbb{E}\big[\hat{Z}_{\tau^{\varepsilon}}\big|\mathscr{F}_{\rho}\big] \geq \mathbb{E}\big[\hat{Z}_{\tau}\big|\mathscr{F}_{\rho}\big].$$

Proof. 1° Consider the limit supremum first. Since we do not specify any requirements on the manner in which $(\sigma^{\varepsilon})_{\varepsilon>0}$ converges to σ, we need to split over the event $A = \{\sigma^{\varepsilon} < \sigma, \forall \varepsilon > 0\}$ and its compliment $A^c = \{\sigma^{\varepsilon} = \sigma, \text{ for some } \varepsilon > 0\}$. Using the reverse Fatou lemma then gives

$$\limsup_{\varepsilon\searrow 0}\mathbb{E}\big[\tilde{Z}_{\sigma^{\varepsilon}}\big|\mathscr{F}_{\rho}\big] \leq \mathbb{E}\left[\mathbb{1}_A \limsup_{\varepsilon\searrow 0}\tilde{Z}_{\sigma^{\varepsilon}} + \mathbb{1}_{A^c}\limsup_{\varepsilon\searrow 0}\tilde{Z}_{\sigma^{\varepsilon}}\middle|\mathscr{F}_{\rho}\right] \quad \mathbb{P}\text{-a.s.}$$

On the right-hand side, we know how the first term will behave. Theorem 2.2.18 gives us previsible left upper semicontinuity up to an evanescent set, so therefore—as per Remark 2.2.9—$(\tilde{Z}_t)_{t\geq 0}$ is previsibly left upper semicontinuous over stopping times too. For the second term we see that since σ^{ε} equals σ for some $\varepsilon > 0$, there is no requirement for left-continuity conditions; it is a constant sequence. Therefore, we arrive at

$$\limsup_{\varepsilon\searrow 0}\mathbb{E}\big[\tilde{Z}_{\sigma^{\varepsilon}}\big|\mathscr{F}_{\rho}\big] \leq \mathbb{E}\big[\mathbb{1}_A{}^p\big(\tilde{Z}_{\sigma}\big) + \mathbb{1}_{A^c}\tilde{Z}_{\sigma}\big|\mathscr{F}_{\rho}\big] \quad \mathbb{P}\text{-a.s.}$$

Regarding the set A, we can write it as follows if we take a countable intersection over $\varepsilon > 0$

$$A = \{\sigma^{\varepsilon} < \sigma, \forall \varepsilon > 0\} = \bigcap_{\varepsilon\in(0,\infty)\cap\mathbb{Q}} \{\sigma^{\varepsilon} < \sigma\}.$$

According to Dellacherie and Meyer (Theorem IV.56, [DM78]), the set $\{\sigma^{\varepsilon} < \sigma\}$ belongs to $\mathscr{F}_{\sigma-}$, and, therefore, so does the countable intersection. Thus A belongs to $\mathscr{F}_{\sigma-}$. Using this, and the conditional expectation form of the previsible projection (see Theorem 2.2.6), we arrive at

$$\limsup_{\varepsilon\searrow 0}\mathbb{E}\big[\tilde{Z}_{\sigma^{\varepsilon}}\big|\mathscr{F}_{\rho}\big] \leq \mathbb{E}\big[\mathbb{E}\big[\mathbb{1}_A\tilde{Z}_{\sigma}\big|\mathscr{F}_{\sigma-}\big] + \mathbb{1}_{A^c}\tilde{Z}_{\sigma}\big|\mathscr{F}_{\rho}\big] \quad \mathbb{P}\text{-a.s.}$$

For the first term, since we are on the set A, $\sigma^{\varepsilon} < \sigma$ for all $\varepsilon > 0$, this implies that $\rho < \sigma$, and furthermore that $\mathscr{F}_{\rho}|_A \subseteq \mathscr{F}_{\sigma-}|_A$ since $\rho < \sigma$ on every event in both σ-algebras. Hence taking

$\mathscr{F} = \mathscr{F}_\rho$ and $\mathscr{G} = \mathscr{F}_{\sigma-}$ in Lemma B.2.4, we can use the tower property and obtain the following

$$\limsup_{\varepsilon \searrow 0} \mathbb{E}\big[\tilde{Z}_{\sigma^\varepsilon} \big| \mathscr{F}_\rho\big] \leq \mathbb{E}\big[\tilde{Z}_\sigma \big| \mathscr{F}_\rho\big] \quad \mathbb{P}\text{-a.s.}$$

2° Much as before, introduce the set $B = \{\tau^\varepsilon < \tau, \forall \varepsilon > 0\}$ and its complement $B^c = \{\tau^\varepsilon = \tau, \text{ for some } \varepsilon > 0\}$. The result for $(\hat{Z}_t)_{t\geq 0}$ holds *mutatis mutandis*, however, this time we use Corollary 2.2.19. ■

2.3 AGGREGATION OF VALUES

The first step for analysing this game is to aggregate the families of values found in Definition 2.1.2. We make use of a theorem from Dellacherie-Lenglart [DL82], which gives us practical conditions for when a $\mathscr{T}$-system can be aggregated.

Definition 2.3.1. A $\mathscr{T}$-system, $\{X(\tau), \tau \in \mathscr{T}\}$, is said to be right upper (resp. lower) semicontinuous if for any decreasing sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ with limit τ, we have

$$\limsup_{n\to\infty} X(\tau_n) \leq X(\tau), \qquad \Big(\text{resp. } \liminf_{n\to\infty} X(\tau_n) \geq X(\tau)\Big).$$

Theorem 2.3.2. *(Theorem 4, [DL82]) Under the usual conditions, any $\mathscr{T}$-system, $\{X(\tau), \tau \in \mathscr{T}\}$, which is right upper semicontinuous (resp. right lower semicontinuous) can be aggregated into a unique optional process with right upper semicontinuous (resp. right lower semicontinuous) trajectories.*

Before establishing that the $\mathscr{T}$-system $\{\overline{V}(\rho), \rho \in \mathscr{T}\}$ is right upper semicontinuous (and therefore aggregable), we will need the following generalisation of a result from Bismut [Bis79b]: we do not change the values, $\overline{V}(\rho)$ and $\underline{V}(\rho)$, when we swap the strictness of inequalities in the expression of the conditional payoff.

Lemma 2.3.3. *(Lemma 5, [LM84]) For all $(\mathscr{F}_t)$-stopping times ρ:*

$$\overline{V}(\rho) = \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_\rho} \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho} \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}} \big| \mathscr{F}_\rho\big] \quad \mathbb{P}\text{-}a.s.$$

$$\underline{V}(\rho) = \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho} \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_\rho} \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}} \big| \mathscr{F}_\rho\big] \quad \mathbb{P}\text{-}a.s.$$

Proof. We will do the proof for $\overline{V}(\rho)$, as the result for $\underline{V}(\rho)$ is identical. First, define the following two families

$$K(\tau,\sigma) = \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}} \big| \mathscr{F}_\rho\big],$$

$$\tilde{K}(\tau,\sigma) = \mathbb{E}[f_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}} \big| \mathscr{F}_\rho].$$

It then suffices to show that for all $\sigma \geq \rho$:

$$\operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho} K(\tau,\sigma) = \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho} \tilde{K}(\tau,\sigma) \quad \mathbb{P}\text{-a.s.}$$

On the one hand, since $g_t \leq f_t$ for all $t \geq 0$, we have

$$\begin{aligned} f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}} &= f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}} + g_\sigma \mathbb{1}_{\{\sigma=\tau\}}, \\ &\leq f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}} + f_\tau \mathbb{1}_{\{\sigma=\tau\}}, \\ &= f_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}}, \end{aligned}$$

and so, by the monotonicity of conditional expectation, we have

$$K(\tau,\sigma) \leq \tilde{K}(\tau,\sigma) \quad \mathbb{P}\text{-a.s.}$$

Therefore,

$$\operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho} K(\tau,\sigma) \leq \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho} \tilde{K}(\tau,\sigma) \quad \mathbb{P}\text{-a.s.}$$

On the other hand, for any $\sigma \geq \rho$, and σ_n defined by:

$$\sigma_n := \begin{cases} \sigma & \text{if} \quad \sigma < \tau, \\ \sigma + \frac{1}{n} & \text{if} \quad \sigma \geq \tau, \end{cases}$$

we either have $\sigma_n < \tau$, $\sigma_n > \tau$ or $\sigma_n = \tau = \sigma = +\infty$. Then, since $g_\infty = f_\infty = 0$, we see that $g_{\sigma_n}\mathbb{1}_{\{\sigma_n=\tau\}} = f_\tau \mathbb{1}_{\{\sigma_n=\tau\}} = 0$, and so we have

$$f_\tau \mathbb{1}_{\{\tau<\sigma_n\}} + g_{\sigma_n} \mathbb{1}_{\{\sigma_n\leq\tau\}} = f_\tau \mathbb{1}_{\{\tau<\sigma_n\}} + g_{\sigma_n} \mathbb{1}_{\{\sigma_n<\tau\}}. \tag{2.6}$$

We notice that, because of the definition of $(\sigma_n)_{n\in\mathbb{N}}$, it clearly converges downwards to σ, and the right-hand side of (2.6) can be rewritten using the following two observations: 1) $\{\tau < \sigma_n\} = \{\tau \leq \sigma\}$, and 2) $\{\sigma_n < \tau\} = \{\sigma < \tau\}$. Notably, since f and g are bounded processes, so is $(f_\tau \mathbb{1}_{\{\tau<\sigma_n\}} + g_{\sigma_n}\mathbb{1}_{\{\sigma_n\leq\tau\}})_n$ for any n. We also know that $g_{\sigma_n} \to g_\sigma$ by right-continuity of g and so

$$f_\tau \mathbb{1}_{\{\tau<\sigma_n\}} + g_{\sigma_n}\mathbb{1}_{\{\sigma_n<\tau\}} = f_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + g_{\sigma_n}\mathbb{1}_{\{\sigma<\tau\}} \to f_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + g_{\sigma}\mathbb{1}_{\{\sigma<\tau\}}, \tag{2.7}$$

and so we take the limit of $K(\tau,\sigma_n)$ and, using the dominated convergence theorem, we can move the limit inside of the conditional expectation, and then, using both (2.6) and (2.7), we can obtain

$$\begin{aligned}
\lim_{n\to\infty} K(\tau,\sigma_n) &= \lim_{n\to\infty} \mathbb{E}\left[f_\tau \mathbb{1}_{\{\tau<\sigma_n\}} + g_{\sigma_n}\mathbb{1}_{\{\sigma_n\leq\tau\}} \middle| \mathscr{F}_\rho\right] \\
&= \mathbb{E}\left[\lim_{n\to\infty}\left(f_\tau \mathbb{1}_{\{\tau<\sigma_n\}} + g_{\sigma_n}\mathbb{1}_{\{\sigma_n\leq\tau\}}\right) \middle| \mathscr{F}_\rho\right] \\
&= \mathbb{E}\left[f_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + g_{\sigma}\mathbb{1}_{\{\sigma<\tau\}} \middle| \mathscr{F}_\rho\right] \\
&= \tilde{K}(\tau,\sigma).
\end{aligned}$$

Furthermore, we know that, for any given $n \in \mathbb{N}$, $\operatorname{ess\,sup}_{\sigma\in\mathscr{T}_\rho} K(\tau,\sigma) \geq K(\tau,\sigma_n)$ $\mathbb{P}$-almost surely, and so the essential supremum must also bound the limit too, i.e. $\operatorname{ess\,sup}_{\sigma\in\mathscr{T}_\rho} K(\tau,\sigma) \geq \tilde{K}(\tau,\sigma)$ $\mathbb{P}$-almost surely. Then, by taking the essential supremum over both sides we get

$$\operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho} K(\tau,\sigma) \geq \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho} \tilde{K}(\tau,\sigma) \quad \mathbb{P}\text{-a.s.}$$

Hence, we have equality $\mathbb{P}$-almost surely. ∎

Remark 2.3.4. It should be noted that the proof above deviates from the method used in the original paper (see Lemma 5, [LM84]) after equation (2.6), this is because the original method uses Fatou's lemma in a way which is unclear and—as shown above—unneeded, so we have altered the proof for clarity.

We are now in a position to show that our $\mathscr{T}$-systems, $\{\overline{V}(\rho), \rho \in \mathscr{T}\}$ and $\{\underline{V}(\rho), \rho \in \mathscr{T}\}$, are aggregable using Theorem 2.3.2, and as stated earlier, the key to that is the following result.

Theorem 2.3.5. *(Theorem 6, [LM84]) The $\mathscr{T}$-system $\{\overline{V}(\rho), \rho \in \mathscr{T}\}$ (resp. $\{\underline{V}(\rho), \rho \in \mathscr{T}\}$) is right upper semicontinuous (resp. right lower semicontinuous).*

Proof. We will only show the proof for $\{\overline{V}(\rho), \rho \in \mathscr{T}\}$, since the method for $\{\underline{V}(\rho), \rho \in \mathscr{T}\}$ is identical, *mutatis mutandis*.

Due to Lemma 2.3.3 we know that $\overline{V}(\rho)$ can be written as

$$\overline{V}(\rho) = \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_\rho} \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho} \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau\le\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}}\big|\mathscr{F}_\rho\big] \quad \mathbb{P}\text{-a.s.}$$

The set of stopping times, τ, which are greater than or equal to ρ is identical to the set of stopping times $\tau\vee\rho$ when $\tau\in\mathscr{T}$, so we can write

$$\overline{V}(\rho) = \operatorname*{ess\,inf}_{\tau\in\mathscr{T}} \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho} \mathbb{E}\big[f_{\tau\vee\rho} \mathbb{1}_{\{\tau\vee\rho\le\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\vee\rho\}}\big|\mathscr{F}_\rho\big] \quad \mathbb{P}\text{-a.s.}$$

Now, for $\tau\in\mathscr{T}$ we obtain the following by decomposing over the value of τ

$$\begin{aligned}&\operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho} \mathbb{E}\big[f_{\tau\vee\rho} \mathbb{1}_{\{\tau\vee\rho\le\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\vee\rho\}}\big|\mathscr{F}_\rho\big]\\ &\qquad = \mathbb{1}_{\{\tau>\rho\}} \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho} \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau\le\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}}\big|\mathscr{F}_\rho\big] + \mathbb{1}_{\{\tau\le\rho\}} \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho} \mathbb{E}[f_\rho|\mathscr{F}_\rho].\end{aligned}$$

Notice that we have moved the indicator functions outside of both the conditional expectation, and the essential supremum, because the sets $\{\tau>\rho\}$ and $\{\tau\le\rho\}$ are $\mathscr{F}_\rho$-measurable, and do not depend on σ.

If we denote, for any given $\tau\in\mathscr{T}$, by $(Z_t^\tau)_{t\ge0}$ the (right-continuous) Snell envelope of the right-continuous process $X_t^\tau = f_\tau \mathbb{1}_{\{\tau\le t\}} + g_t \mathbb{1}_{\{t<\tau\}}$, we get:

$$\overline{V}(\rho) = \operatorname*{ess\,inf}_{\tau\in\mathscr{T}} \Big(\mathbb{1}_{\{\tau>\rho\}} Z_\rho^\tau + \mathbb{1}_{\{\tau\le\rho\}} f_\rho\Big).$$

The final term above is obtained since $(f_t)_{t\ge0}$ is optional, thus progressive (Theorem VI.4.7, [RW00]), and so f_ρ is $\mathscr{F}_\rho$-measurable. If we denote, for any given $\tau\in\mathscr{T}$, by $(K_t^\tau)_{t\ge0}$ the right-continuous process:

$$K_t^\tau = \mathbb{1}_{\{\tau>t\}} Z_t^\tau + \mathbb{1}_{\{\tau\le t\}} f_t,$$

we finally get

$$\overline{V}(\rho) = \operatorname*{ess\,inf}_{\tau\in\mathscr{T}} K_\rho^\tau$$

for all $\rho\in\mathscr{T}$.

Let (ρ_n) be a sequence of stopping times decreasing to ρ. By the definition of the essential infimum, there exists a countable subset $J \subset \mathscr{T}$ such that

$$\overline{V}(\rho) = \inf_{\tau \in J} K^\tau_\rho,$$

and moreover, countable subsets $(J_n)_{n \geq 1} \subset \mathscr{T}$ such that

$$\overline{V}(\rho_n) = \operatorname*{ess\,inf}_{\tau \in \mathscr{T}} K^\tau_{\rho_n} = \inf_{\tau \in J_n} K^\tau_{\rho_n} \quad \text{for all } n.$$

If we set $\mathscr{J} = J \cup (\bigcup_n J_n)$ (which is still countable), we still have

$$\overline{V}(\rho) = \inf_{\tau \in \mathscr{J}} K^\tau_\rho,$$

$$\overline{V}(\rho_n) = \inf_{\tau \in \mathscr{J}} K^\tau_{\rho_n} \quad \text{for all } n.$$

Since, for any $\tau \in \mathscr{T}$, the map $t \mapsto K^\tau_t(\omega)$ is $\mathbb{P}$-almost surely right-continuous, and $\mathscr{J}$ is a countable subset of $\mathscr{T}$, then we can take a universal $\mathbb{P}$-null set outside of which the mapping $t \mapsto K^\tau_t(\omega)$ is right-continuous for all $\tau \in \mathscr{J}$, i.e. the set $\{\omega : K^\tau_t(\omega)$ is right-continuous for all $\tau \in \mathscr{J}\}$ has probability one. Next, since the infimum of a family of right-continuous functions is right upper semicontinuous (see Definition 2.8 and the subsequent point (c), [Rud87]), we deduce that $\inf_{\tau \in \mathscr{J}} K^\tau_t(\omega)$ is right upper semicontinuous. So if $\rho_n(\omega)$ decreases to $\rho(\omega)$

$$\big(\overline{V}(\rho)\big)(\omega) = \inf_{\tau \in \mathscr{J}} K^\tau_{\rho(\omega)} \geq \limsup_n \left(\inf_{\tau \in \mathscr{J}} K^\tau_{\rho_n(\omega)} \right) = \limsup_n \big(\overline{V}(\rho_n)\big)(\omega)$$

and since the map $t \mapsto K^\tau_t(\omega)$ is right-continuous for all $\tau \in \mathscr{J}$ $\mathbb{P}$-almost surely, we get

$$\overline{V}(\rho) \geq \limsup_n \overline{V}(\rho_n) \quad \mathbb{P}\text{-a.s.}$$

In the same way, one would show that the $\mathscr{T}$-system $\{\underline{V}(\rho), \rho \in \mathscr{T}\}$ is right lower semicontinuous, except there is no need for the use of Lemma 2.3.3 in this case. ■

Using Theorem 2.3.2 we then immediately obtain the aggregated processes.

Theorem 2.3.6. *(Theorem 7, [LM84]) There exists an optional process $(\overline{V}_t)_{t \geq 0}$ (resp. $(\underline{V}_t)_{t \geq 0}$) which is right upper semicontinuous (resp. right lower semicontinuous), and which aggregates the $\mathscr{T}$-system $\{\overline{V}(\rho), \rho \in \mathscr{T}\}$ (resp. $\{\underline{V}(\rho), \rho \in \mathscr{T}\}$).*

The rest of this section will be focused on establishing the right-continuity of the conditional values. Take $(\underline{V}_t)_{t\geq 0}$ for example, since we already know it is right lower semicontinuous, all we need is to show that it is also right upper semicontinuous. We approach this via Bismut and Skalli's theorem, Theorem 2.2.10, which as we have described, connects right semicontinuity with left-previsible semicontinuity and semicontinuity in expectation over stopping times. To obtain the desired result it will suffice to show that $(\underline{V}_t)_{t\geq 0}$ (resp. $(\overline{V}_t)_{t\geq 0}$) is upper (resp. lower) semicontinuous in expectation over stopping times.

Remark 2.3.7. In the original paper by Lepeltier and Maingueneau, [LM84], they state that it is sufficient to show only right semicontinuity in expectation over stopping times, and they state a result akin to Theorem 2.2.10 but only requiring the sequence (τ_n) to be non-increasing and converging to τ. We were unable to find such a result, and the closest we could obtain was Theorem 2.2.10 above. Because of this, while [LM84] only concerns itself with proving right semicontinuity in expectation over stopping times of the corresponding upper and lower value processes, we must grapple with the regularity from the left too. This results in a marked change to the original method, and is one of our major contributions to this work.

Our main tool for achieving this will be Lemma 2.3.10, which is a continuous-time version of the same result proven by Neveu (see VI.6, [Nev75]) for discrete-time stochastic games. Its proof requires properties of upwards and downwards directed sets (recall Lemma 1.2.4), and so first we present the following lemma, which is an expanded proof of Proposition A.2. from [EK79]. Here we deal with upwards directed families, but the proof for downward directed ones is identical, however, it uses the essential infimum instead.

Lemma 2.3.8. *Let $(Y^i)_{i\in I}$ be a family of $\mathscr{F}$-measurable random variables, for some σ-algebra $\mathscr{F}$, taking values in $[0,\infty]$, and which is upwards directed and bounded below, then for all sub-σ-algebras $\mathscr{G}\subset\mathscr{F}$,*

$$\mathbb{E}\left[\operatorname*{ess\,sup}_{i\in I} Y^i \middle| \mathscr{G}\right] = \operatorname*{ess\,sup}_{i\in I}\mathbb{E}\left[Y^i\middle|\mathscr{G}\right] \quad \mathbb{P}\text{-a.s.}$$

Proof. First, since $Y^i \leq \operatorname{ess\,sup}_{i\in I} Y^i$ for all $i\in I$, we know that $\mathbb{E}[Y^i|\mathscr{G}] \leq \mathbb{E}[\operatorname{ess\,sup}_{i\in I} Y^i|\mathscr{G}]$. Then, since the essential supremum is the smallest random variable dominating $\mathbb{E}[Y^i|\mathscr{G}]$, we know that

$$\operatorname*{ess\,sup}_{i\in I}\mathbb{E}\left[Y^i\middle|\mathscr{G}\right] \leq \mathbb{E}\left[\operatorname*{ess\,sup}_{i\in I} Y^i \middle| \mathscr{G}\right] \quad \mathbb{P}\text{-a.s.} \tag{2.8}$$

Next, using Lemma 1.2.4, since $(Y^i)_{i\in I}$ is upwards directed we may choose the countable subset

$J = \{i_n : n \geq 1\} \subset I$ in such a way so that $\operatorname{ess\,sup}_{i \in I} Y^i = \lim_{n\to\infty} Y^{i_n}$, where $Y^{i_1} \leq Y^{i_2} \leq \ldots$ $\mathbb{P}$-almost surely. The following holds by the monotone convergence theorem:

$$\mathbb{E}\left[\operatorname*{ess\,sup}_{i\in I} Y^i \middle| \mathscr{G}\right] = \mathbb{E}\left[\lim_{n\to\infty} Y^{i_n} \middle| \mathscr{G}\right] = \lim_{n\to\infty} \mathbb{E}\left[Y^{i_n} \middle| \mathscr{G}\right]. \tag{2.9}$$

Since $\mathbb{E}\left[Y^{i_n} \middle| \mathscr{G}\right] \leq \operatorname{ess\,sup}_{i\in I} \mathbb{E}\left[Y^i \middle| \mathscr{G}\right]$, we then combine (2.8) with (2.9) to conclude. ■

Additionally, we need a simple result about constructing stopping times.

Lemma 2.3.9. *Let $(\mathscr{G}_t)_{t\geq 0}$ be a filtration, σ_1, σ_2 and ρ be $(\mathscr{G}_t)$-stopping times such that σ_1, $\sigma_2 \geq \rho$, and A be a $\mathscr{G}_\rho$-measurable set, then the random variable $\sigma := \sigma_1 \mathbb{1}_A + \sigma_2 \mathbb{1}_{A^c}$ is a $(\mathscr{G}_t)$-stopping time greater than or equal to ρ.*

Proof. First, notice that by construction $\sigma \geq \rho$, and because of this fact, $\{\sigma \leq t\} = \emptyset$ for all $t < \rho$. Thus $\{\sigma \leq t\} \in \mathscr{G}_t$ for all $t < \rho$.

Next, take $t \geq \rho$, then we write

$$\{\sigma \leq t\} = (\{\sigma \leq t\} \cap A) \cup (\{\sigma \leq t\} \cap A^c) = (\{\sigma_1 \leq t\} \cap A) \cup (\{\sigma_2 \leq t\} \cap A^c).$$

Since A and A^c belong to $\mathscr{G}_\rho$, they belong to $\mathscr{G}_t$ for all $t \geq \rho$. Moreover, since σ_1 and σ_2 are stopping times, the sets $\{\sigma_1 \leq t\}$ and $\{\sigma_2 \leq t\}$ also belong to $\mathscr{G}_t$. Thus $\{\sigma \leq t\} \in \mathscr{G}_t$ for all $t \geq \rho$.

The two results give $\{\sigma \leq t\} \in \mathscr{G}_t$ for all $t \geq 0$. ■

Lemma 2.3.10. *(Lemma 8, [LM84]) If ρ_1 and ρ_2 are any two $\mathscr{F}_t$-stopping times such that $\rho_1 \leq \rho_2$, then we have*

$$\mathbb{E}\left[\overline{V}_{\rho_2} \middle| \mathscr{F}_{\rho_1}\right] = \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_{\rho_2}} \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \mathbb{E}\left[f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}} \middle| \mathscr{F}_{\rho_1}\right] \quad \mathbb{P}\text{-}a.s.,$$

$$\mathbb{E}\left[\underline{V}_{\rho_2} \middle| \mathscr{F}_{\rho_1}\right] = \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_{\rho_2}} \mathbb{E}\left[f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}} \middle| \mathscr{F}_{\rho_1}\right] \quad \mathbb{P}\text{-}a.s.$$

Proof. Denote for all stopping times τ and σ

$$R(\tau, \sigma) = f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}}.$$

For all stopping times $\tau \geq \rho_2$, the family $\{\mathbb{E}[R(\tau,\sigma)|\mathscr{F}_{\rho_2}], \sigma \geq \rho_2\}$ is upwards directed. Indeed, using Lemma 2.3.9, if σ_1 and σ_2 are two stopping times greater than or equal to ρ_2, the formula

$$\sigma := \begin{cases} \sigma_1 & \text{if} \quad \mathbb{E}\big[R(\tau,\sigma_1)\big|\mathscr{F}_{\rho_2}\big] \geq \mathbb{E}\big[R(\tau,\sigma_2)\big|\mathscr{F}_{\rho_2}\big], \\ \sigma_2 & \text{if} \quad \mathbb{E}\big[R(\tau,\sigma_1)\big|\mathscr{F}_{\rho_2}\big] < \mathbb{E}\big[R(\tau,\sigma_2)\big|\mathscr{F}_{\rho_2}\big], \end{cases}$$

defines a stopping time greater than or equal to ρ_2, such that

$$\mathbb{E}\big[R(\tau,\sigma)\big|\mathscr{F}_{\rho_2}\big] = \mathbb{E}[R(\tau,\sigma_1)|\mathscr{F}_{\tau_2}] \vee \mathbb{E}\big[R(\tau,\sigma_2)\big|\mathscr{F}_{\rho_2}\big].$$

Hence by Lemma 2.3.8 we have

$$\begin{aligned} \mathbb{E}\left[\operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \mathbb{E}\big[R(\tau,\sigma)\big|\mathscr{F}_{\rho_2}\big]\middle|\mathscr{F}_{\rho_1}\right] &= \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \mathbb{E}\big[\mathbb{E}\big[R(\tau,\sigma)\big|\mathscr{F}_{\rho_2}\big]\big|\mathscr{F}_{\rho_1}\big] \\ &= \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \mathbb{E}\big[R(\tau,\sigma)\big|\mathscr{F}_{\rho_1}\big] \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{2.10}$$

On the other hand, the family $\{\operatorname{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \mathbb{E}[R(\tau,\sigma)|\mathscr{F}_{\rho_2}], \tau \geq \rho_2\}$ is downwards directed; again using Lemma 2.3.9, let τ_1 and τ_2 be two stopping times greater than or equal to ρ_2, then

$$\tau = \begin{cases} \tau_1 & \text{if} \quad \operatorname{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \mathbb{E}\big[R(\tau_1,\sigma)\big|\mathscr{F}_{\rho_2}\big] < \operatorname{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \mathbb{E}\big[R(\tau_1,\sigma)\big|\mathscr{F}_{\rho_2}\big], \\ \tau_2 & \text{if} \quad \operatorname{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \mathbb{E}\big[R(\tau_1,\sigma)\big|\mathscr{F}_{\rho_2}\big] \geq \operatorname{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \mathbb{E}\big[R(\tau_2,\sigma)\big|\mathscr{F}_{\rho_2}\big], \end{cases}$$

defines a stopping time which is greater than or equal to ρ_2. Then we see that

$$\operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \mathbb{E}\big[R(\tau,\sigma)\big|\mathscr{F}_{\rho_2}\big] = \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \mathbb{E}\big[R(\tau,\sigma_1)\big|\mathscr{F}_{\rho_2}\big] \wedge \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \mathbb{E}\big[R(\tau_2,\sigma)\big|\mathscr{F}_{\rho_2}\big].$$

In the same way as above we can write, using the explicit form of $\overline{V}_{\rho_2}$

$$\begin{aligned} \mathbb{E}\big[\overline{V}_{\rho_2}\big|\mathscr{F}_{\rho_1}\big] &= \mathbb{E}\left[\operatorname*{ess\,inf}_{\tau\in\mathscr{T}_{\rho_2}} \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \mathbb{E}\big[R(\tau,\sigma)\big|\mathscr{F}_{\rho_2}\big]\middle|\mathscr{F}_{\rho_1}\right] \\ &= \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_{\rho_2}} \mathbb{E}\left[\operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \mathbb{E}\big[R(\tau,\sigma)\big|\mathscr{F}_{\rho_2}\big]\middle|\mathscr{F}_{\rho_1}\right] \\ &= \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_{\rho_2}} \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{\rho_2}} \mathbb{E}\big[R(\tau,\sigma)\big|\mathscr{F}_{\rho_1}\big] \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

where the final line follows from (2.10). The proof for the lower conditional value is identical *mutatis mutandis*. ∎

Remark 2.3.11. Upon inspection of the proof above, one can see that changing the definition of $R(\tau,\sigma)$ to be $R(\tau,\sigma) = f_\tau \mathbb{1}_{\{\tau\le\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}}$, does not change the validity of the lemma.

We are now able to give the final result of this section, namely that the values of the game are right-continuous. This result is a corollary of the following theorem, Theorem 2.3.12, which proves the semicontinuity in expectation.

Theorem 2.3.12. *(Theorem 9, [LM84]) The upper (resp. lower) value process $(\overline{V}_t)_{t\ge0}$ (resp. $(\underline{V}_t)_{t\ge0}$) is lower (resp. upper) semicontinuous in expectation over stopping times.*

The proof of this theorem can be split into two Lemmas. The reader will notice that both proofs are very similar, but differ enough to warrant the presentation of both in full. Remark 2.3.7 is now extremely pertinent as our proof of Theorem 2.3.12 deviates from the original found in [LM84]. Lemma 2.3.13 contains the result found in Lepeltier and Maingueneau (albeit with a slightly different proof method), however, due to the additional left-regularity required for the application of Theorem 2.2.10, Lemma 2.3.14 is also required.

Lemma 2.3.13. *The upper (resp. lower) value process $(\overline{V}_t)_{t\ge0}$ (resp. $(\underline{V}_t)_{t\ge0}$) is right lower (resp. upper) semicontinuous in expectation over stopping times.*

Proof. We will show the proof for $(\overline{V}_t)_{t\ge}$ since the proof for $(\underline{V}_t)_{t\ge0}$ is identical.

Let ρ be an $\mathscr{F}_t$ stopping time and $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of stopping times decreasing to ρ. By virtue of Lemma 2.3.10 (and since $\mathscr{F}_0$ is the trivial σ-algebra)

$$\mathbb{E}\overline{V}_\rho = \inf_{\tau\in\mathscr{T}_\rho}\sup_{\sigma\in\mathscr{T}_\rho}\mathbb{E}\big[f_\tau\mathbb{1}_{\{\tau<\sigma\}} + g_\sigma\mathbb{1}_{\{\sigma\le\tau\}}\big] \le \inf_{\tau\in\mathscr{T}_{\rho_n}}\sup_{\sigma\in\mathscr{T}_\rho}\mathbb{E}\big[f_\tau\mathbb{1}_{\{\tau<\sigma\}} + g_\sigma\mathbb{1}_{\{\sigma\le\tau\}}\big]. \tag{2.11}$$

For all $n\in\mathbb{N}$. The last inequality holds because $\mathscr{T}_{\rho_n}$ is a subset of $\mathscr{T}_\rho$ for all n, since $\rho_n\downarrow\rho$.

Next, for an arbitrary $\tau\ge\rho_n$, we can decompose the terms inside of the expectation over the sets $\{\sigma<\rho_n\}$ and $\{\sigma\ge\rho_n\}$ to obtain

$$\begin{aligned}\mathbb{E}[f_\tau\mathbb{1}_{\{\tau<\sigma\}} + g_\sigma\mathbb{1}_{\{\sigma\le\tau\}}] = \mathbb{E}[\mathbb{1}_{\{\sigma<\rho_n\}}(f_\tau\mathbb{1}_{\{\tau<\sigma\}} + g_\sigma\mathbb{1}_{\{\sigma\le\tau\}})]\\ +\mathbb{E}[\mathbb{1}_{\{\sigma\ge\rho_n\}}(f_\tau\mathbb{1}_{\{\tau<\sigma\}} + g_\sigma\mathbb{1}_{\{\sigma\le\tau\}})].\end{aligned}$$

Beginning with the left-most expectation on the right-hand side, we see that, because $\rho_n \leq \tau$, we must have that $\{\sigma < \rho_n\} \subset \{\sigma \leq \tau\}$, which then implies that $\{\sigma < \rho_n\} \cap \{\sigma \leq \tau\} = \{\sigma < \rho_n\}$. Additionally it is obvious that $\{\sigma < \rho_n\} \cap \{\tau < \sigma\} = \emptyset$. Finally, for the second expectation, on $\{\sigma \geq \rho_n\}$ it is trivially true that $\sigma \vee \rho_n = \sigma$, allowing us to write the following

$$\begin{aligned}\mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}}\big] &= \mathbb{E}\big[g_\sigma \mathbb{1}_{\{\sigma<\rho_n\}} + \big(f_\tau \mathbb{1}_{\{\tau<\sigma\vee\rho_n\}} + g_{\sigma\vee\rho_n} \mathbb{1}_{\{\sigma\vee\rho_n\leq\tau\}}\big)\mathbb{1}_{\{\sigma\geq\rho_n\}}\big] \\ &= \mathbb{E}\big[g_\sigma \mathbb{1}_{\{\sigma<\rho_n\}} + f_\tau \mathbb{1}_{\{\tau<\sigma\vee\rho_n\}} + g_{\sigma\vee\rho_n} \mathbb{1}_{\{\sigma\vee\rho_n\leq\tau\}} - g_{\rho_n} \mathbb{1}_{\{\sigma<\rho_n\}}\big]\end{aligned}$$

where, in the second line, we have used the fact that $\mathbb{1}_{\{\sigma\geq\rho_n\}} = 1 - \mathbb{1}_{\{\sigma<\rho_n\}}$. Thus,

$$\mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}}\big] = \mathbb{E}\big[(g_\sigma - g_{\rho_n})\mathbb{1}_{\{\sigma<\rho_n\}} + (f_\tau \mathbb{1}_{\{\tau<\sigma\vee\rho_n\}} + g_{\sigma\vee\rho_n} \mathbb{1}_{\{\sigma\vee\rho_n\leq\tau\}})\big].$$

Taking the supremum of both sides, and using the fact that for any functions u and v mapping into the reals, $\sup(u+v) \leq \sup u + \sup v$, we obtain

$$\begin{aligned}\sup_{\sigma\in\mathscr{T}_\rho} \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}}\big] \leq &\sup_{\sigma\in\mathscr{T}_\rho} \mathbb{E}\big[(g_\sigma - g_{\rho_n})\mathbb{1}_{\{\sigma<\rho_n\}}\big] \\ &+ \sup_{\sigma\in\mathscr{T}_\rho} \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau<\sigma\vee\rho_n\}} + g_{\sigma\vee\rho_n} \mathbb{1}_{\{\sigma\vee\rho_n\leq\tau\}}\big]\end{aligned}$$

For the first term, see that on the event $\{\sigma \geq \rho_n\}$ the indicator function is zero, and so we are effectively concerned with values of σ which lie in the range $\rho \leq \sigma < \rho_n$. We can rewrite this payoff as $g_{\sigma\wedge\rho_n} - g_{\rho_n}$ since it coincides with $(g_\sigma - g_{\rho_n})\mathbb{1}_{\{\sigma<\rho_n\}}$. For the second supremum we notice that for any $\sigma \geq \rho$, the stopping time $\sigma \vee \rho_n$ is always greater than or equal to ρ_n. Thus we can, with an abuse of notation, reindex the supremum over stopping times $\sigma \geq \rho_n$. Combining these two ideas yields

$$\sup_{\sigma\in\mathscr{T}_\rho} \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}}\big] \leq \sup_{\sigma\in\mathscr{T}_\rho} \mathbb{E}\big[g_{\sigma\wedge\rho_n} - g_{\rho_n}\big] + \sup_{\sigma\in\mathscr{T}_{\rho_n}} \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}}\big]. \quad (2.12)$$

Using (2.11) and (2.12) gives:

$$\begin{aligned}
\mathbb{E}\overline{V}_\rho &\leq \inf_{\tau\in\mathscr{T}_{\rho_n}} \left\{ \sup_{\sigma\in\mathscr{T}_\rho} \mathbb{E}[g_{\sigma\wedge\rho_n} - g_{\rho_n}] + \sup_{\sigma\in\mathscr{T}_{\rho_n}} \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}}\big] \right\} \\
&= \sup_{\sigma\in\mathscr{T}_\rho} \mathbb{E}g_{\sigma\wedge\rho_n} - \mathbb{E}g_{\rho_n} + \mathbb{E}\overline{V}_{\rho_n} \\
&= \sup_{\sigma\in\mathscr{T}_\rho^{\rho_n}} \mathbb{E}g_\sigma - \mathbb{E}g_{\rho_n} + \mathbb{E}\overline{V}_{\rho_n},
\end{aligned}$$

where we repurpose the concept of $\mathscr{T}_\rho^{\rho_n}$ from discrete time optimal stopping, (1.3), for continuous time, meaning stopping times between ρ and ρ_n.

Let $\varepsilon > 0$ be given. For any n we can choose $\rho \leq \sigma_n \leq \rho_n$ such that

$$\mathbb{E}\overline{V}_\rho < \mathbb{E}g_{\sigma_n} - \mathbb{E}g_{\rho_n} + \mathbb{E}\overline{V}_{\rho_n} + \varepsilon.$$

(because we can choose σ_n such that $\mathbb{E}g_{\sigma_n} > \sup_{\sigma\in\mathscr{T}_\rho} \mathbb{E}g_\sigma - \varepsilon$). Since ρ_n decreases to ρ we must have that $\sigma_n \to \rho$ from the right. Since the process $(g_t)_{t\geq 0}$ is $\mathbb{P}$-almost surely right-continuous, we obtain $g_{\sigma_n} \to g_\rho$ and $g_{\rho_n} \to g_\rho$ $\mathbb{P}$-almost surely. Moreover, since $(g_t)_{t\geq 0}$ is bounded by assumption, we can use the bounded convergence theorem to get $\mathbb{E}g_{\sigma_n} \to \mathbb{E}g_\rho$ and $\mathbb{E}g_{\rho_n} \to \mathbb{E}g_\rho$. Then, for sufficiently large n, we know $\mathbb{E}g_{\sigma_n} - \mathbb{E}g_{\rho_n} \leq \varepsilon$, and as a result, again for a sufficiently large n

$$\mathbb{E}\overline{V}_\rho < \mathbb{E}\overline{V}_{\rho_n} + \varepsilon + \varepsilon.$$

Thus, taking the limit infimum we have

$$\mathbb{E}\overline{V}_\rho \leq \liminf_{n\to\infty} \big(2\varepsilon + \mathbb{E}\overline{V}_{\rho_n}\big) = 2\varepsilon + \liminf_{n\to\infty} \mathbb{E}\overline{V}_{\rho_n}.$$

Since ε was arbitrary,

$$\mathbb{E}\overline{V}_\rho \leq \liminf_{n\to\infty} \mathbb{E}\overline{V}_{\rho_n}.$$

∎

Lemma 2.3.14. *The upper (resp. lower) value process $(\overline{V}_t)_{t\geq 0}$ (resp. $(\underline{V}_t)_{t\geq 0}$) is left lower (resp. upper) semicontinuous in expectation over stopping times.*

Proof. Again, we will only show the proof for $(\overline{V}_t)_{t\geq}$ since the proof for $(\underline{V}_t)_{t\geq 0}$ is identical.

Let ρ be an $\mathscr{F}_t$-stopping time and $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of $\mathscr{F}_t$-stopping times increasing to ρ. From Lemma 2.3.3, Lemma 2.3.10, and the fact that $\mathscr{F}_0$ is the trivial σ-algebra, we can obtain

$$\mathbb{E}\overline{V}_{\rho_n} = \inf_{\tau\in\mathscr{T}_{\rho_n}} \sup_{\sigma\in\mathscr{T}_{\rho_n}} \mathbb{E}[f_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}}] \geq \inf_{\tau\in\mathscr{T}_{\rho_n}} \sup_{\sigma\in\mathscr{T}_{\rho}} \mathbb{E}[f_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}}]. \tag{2.13}$$

for all $n \in \mathbb{N}$. The last inequality holds because $\mathscr{T}_{\rho_n} \supseteq \mathscr{T}_\rho$ for all n, since $\rho_n \uparrow \rho$. Take any $\sigma \geq \rho$, then we can decompose the expectation over whether τ lies above or below ρ:

$$\mathbb{E}[f_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}}] = \mathbb{E}\big[\big(f_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}}\big)\mathbb{1}_{\{\tau<\rho\}} + \big(f_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}}\big)\mathbb{1}_{\{\tau\geq\rho\}}\big].$$

In a similar way to the previous proof we note the following observations. First, on the set $\{\tau < \rho\}$, since $\sigma \geq \rho$, we must also be on the set $\{\tau < \sigma\}$, which is a subset of $\{\tau \leq \sigma\}$. Second, on the set $\{\tau \geq \rho\}$, we can trivially write $\tau = \tau \vee \rho$. Third, we can write $\mathbb{1}_{\{\tau\geq\rho\}} = 1 - \mathbb{1}_{\{\tau<\rho\}}$. Using these and the fact that $\sigma \geq \rho$ we can obtain

$$\mathbb{E}[f_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}}] = \mathbb{E}\big[\big(f_\tau - f_\rho\big)\mathbb{1}_{\{\tau<\rho\}} + \big(f_{\tau\vee\rho} \mathbb{1}_{\{\tau\vee\rho\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\vee\rho\}}\big)\big].$$

Taking the supremum we get

$$\begin{aligned}\sup_{\sigma\in\mathscr{T}_\rho} \mathbb{E}[f_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}}] &= \mathbb{E}\big[\big(f_\tau - f_\rho\big)\mathbb{1}_{\{\tau<\rho\}}\big] + \sup_{\sigma\in\mathscr{T}_\rho} \mathbb{E}\big[f_{\tau\vee\rho} \mathbb{1}_{\{\tau\vee\rho\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\vee\rho\}}\big] \\ &= \mathbb{E}\big[f_{\tau\wedge\rho} - f_\rho\big] + \sup_{\sigma\in\mathscr{T}_\rho} \mathbb{E}\big[f_{\tau\vee\rho} \mathbb{1}_{\{\tau\vee\rho\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\vee\rho\}}\big]\end{aligned}$$

We can then take the infimum over all $\tau \geq \rho_n$. Using (2.13), and the fact that for any functions u and v mapping into the reals, $\inf(u+v) \geq \inf u + \inf v$, we obtain

$$\begin{aligned}\mathbb{E}\overline{V}_{\rho_n} &\geq \inf_{\tau\in\mathscr{T}_{\rho_n}} \mathbb{E}\big[f_{\tau\wedge\rho} - f_\rho\big] + \inf_{\tau\in\mathscr{T}_{\rho_n}} \sup_{\sigma\in\mathscr{T}_\rho} \mathbb{E}\big[f_{\tau\vee\rho} \mathbb{1}_{\{\tau\vee\rho\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\vee\rho\}}\big] \\ &= \inf_{\tau\in\mathscr{T}_{\rho_n}} \mathbb{E}\big[f_{\tau\wedge\rho}\big] - \mathbb{E}\big[f_\rho\big] + \inf_{\tau\in\mathscr{T}_\rho} \sup_{\sigma\in\mathscr{T}_\rho} \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau\leq\sigma\}} + g_\sigma \mathbb{1}_{\{\sigma<\tau\}}\big] \\ &= \inf_{\tau\in\mathscr{T}_\rho^{\rho^n}} \mathbb{E}[f_\tau] - \mathbb{E}\big[f_\rho\big] + \mathbb{E}\overline{V}_\rho, \end{aligned} \tag{2.14}$$

where again we repurpose the notation of $\mathscr{T}_\rho^{\rho_n}$ from discrete time optimal stopping, (1.3), meaning stopping times between ρ and ρ_n.

By the definition of the infimum, we can choose, for any given ε, a sequence (τ_n) such that $\rho_n \leq \tau_n \leq \rho$ for all $n \in \mathbb{N}$, and that

$$\inf_{\tau \in \mathscr{T}_\rho^{\rho_n}} \mathbb{E}[f_\tau] > \mathbb{E}[f_{\tau_n}] - \varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{2.15}$$

Recall that at this point in the proof of Lemma 2.3.13 we were content with finding a sequence $(\sigma_n)_{n\in\mathbb{N}}$ which simply converged to ρ from the right with no concern about monotonicity. However, since $(f_t)_{t\geq 0}$ is assumed to be previsibly left lower semicontinuous, we must make use of Theorem 2.2.10 in order to obtain convergence results. There is no reason, a priori, for the sequence (τ_n) above to be non-decreasing, and as a result we will need to construct one which is.

Let us consider the infimum above as a typical optimal stopping problem. First, we deal with the stochastic time horizon by defining $\hat{f}_t := f_{t\wedge\rho}$, and so we can now consider the optimal stopping problem on the interval $[0,\infty]$. Since $(\hat{f}_t)_{t\geq 0}$ is bounded (therefore we can ensure positivity via the methods already explained) and right-continuous, Theorem 1.3.7 gives us the Snell envelope of $(\hat{f}_t)_{t\geq 0}$ (recall that Remark 1.3.23 tells us that the results of the general theory—when suitably modified—apply to infimum problems too), which we will denote by $(U_t)_{t\geq 0}$, and which satisfies

$$U_\tau = \operatorname*{ess\,inf}_{\sigma\in\mathscr{T}_\tau} \mathbb{E}\big[\hat{f}_\sigma \big| \mathscr{F}_\tau\big] \quad \mathbb{P}\text{-a.s.}$$

Notice that for all $\tau \geq \rho$ we have $\hat{f}_\tau = f_\rho$ and so $U_\tau = f_\rho$. Moreover, defining as usual the ε-optimal stopping time (Theorem 1.3.21 and Remark 1.3.23) $D^\varepsilon_{\rho_n}(\omega) = \inf\{t \geq \rho_n : U_t(\omega) \geq \hat{f}_t(\omega) - \varepsilon\}$, we can easily see that $D^\varepsilon_{\rho_n} \leq D^\varepsilon_{\rho_{n+1}}$ if $\rho_n \leq \rho_{n+1}$, and specifically $D^\varepsilon_\rho(\omega) = \inf\{t \geq \rho : U_t(\omega) \geq f_\rho(\omega) - \varepsilon\} = \rho$ since the inequality is always true for all $t \geq \rho$. Thus we take the sequence $(\tau_n)_{n\in\mathbb{N}}$ to be $\tau_n := D^\varepsilon_{\rho_n}$ which is non-decreasing and converges to ρ by the squeeze rule. Moreover, since $(U_t)_{t\geq 0}$ is a martingale until $D^\varepsilon_{\rho_n}$, we know that

$$\mathbb{E}U_{D^\varepsilon_{\rho_n}} = \mathbb{E}U_{\rho_n} = \inf_{\sigma\in\mathscr{T}_{\rho_n}} \mathbb{E}\hat{f}_\sigma = \inf_{\sigma\in\mathscr{T}_\rho^{\rho_n}} \mathbb{E}f_\sigma.$$

Since $D^\varepsilon_{\rho_n}$ is ε-optimal, we know that

$$\inf_{\sigma\in\mathscr{T}_\rho^{\rho_n}} \mathbb{E}f_\sigma = \mathbb{E}U_{\rho_n} \geq \mathbb{E}\hat{f}_{D^\varepsilon_{\rho_n}} - \varepsilon,$$

for all $n \in \mathbb{N}$. Using (2.14) and (2.15) we arrive at

$$\mathbb{E}\overline{V}_{\rho_n} > \mathbb{E}[f_{\tau_n}] - \varepsilon - \mathbb{E}\left[f_\rho\right] + \mathbb{E}\overline{V}_\rho. \tag{2.16}$$

Now, since $(f_t)_{t\geq 0}$ is optional, bounded, right-continuous and satisfies (A1), then Theorem 2.2.10 tells us that for any monotone sequence (τ_n) of stopping times converging to ρ

$$\liminf_{n\to\infty} \mathbb{E}[f_{\tau_n}] \geq \mathbb{E}[f_\rho]. \tag{2.17}$$

Thus, taking the limit infimum in (2.16) and using (2.17), we get

$$\liminf_{n\to\infty} \mathbb{E}\overline{V}_{\rho_n} > \liminf_{n\to\infty} \mathbb{E}[f_{\tau_n}] - \varepsilon - \mathbb{E}\left[f_\rho\right] + \mathbb{E}\overline{V}_\rho \geq \mathbb{E}\overline{V}_\rho - \varepsilon.$$

Since ε was arbitrary,

$$\liminf_{n\to\infty} \mathbb{E}\overline{V}_{\rho_n} \geq \mathbb{E}\overline{V}_\rho.$$

■

Corollary 2.3.15. *The upper and lower value process, $(\overline{V}_t)_{t\geq 0}$ and $(\underline{V}_t)_{t\geq 0}$, are right-continuous.*

Proof. Consider the lower value process $(\underline{V}_t)_{t\geq 0}$. First, by definition, the family $\{\underline{V}(\rho), \rho \in \mathscr{T}\}$ is bounded (since both $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ are), hence $\underline{V}_\rho$ is bounded $\mathbb{P}$-almost surely for any stopping time ρ. This boundedness gives us class D, and as we have mentioned in Remark 2.2.14, can be used to comply with positivity. Secondly, Theorem 2.3.12 tells us that $(\underline{V}_t)_{t\geq 0}$ is upper semicontinuous in expectation over stopping times. Thirdly, Theorem 2.3.6 gives us optionality. These three points allow us to apply Theorem 2.2.10 to deduce that $(\underline{V}_t)_{t\geq 0}$ is right upper semicontinuous. Moreover, Theorem 2.3.6 also tells us that $(\underline{V}_t)_{t\geq 0}$ is right lower semicontinuous and so, therefore, $(\underline{V}_t)_{t\geq 0}$ is actually right-continuous. The case of the upper value $(\overline{V}_t)_{t\geq 0}$ is identical *mutatis mutandis*. ■

2.4 EXISTENCE OF A VALUE

Now that we have aggregated the conditional values and shown that they are right-continuous, the first major goal in the study of stochastic games is to establish the existence of a value. In order to do this we will use the classical results of optimal stopping which we have seen earlier in Chapter 1, and most importantly Theorem 1.3.22.

For any $\varepsilon > 0$, and for any $(\mathscr{F}_t)$-stopping time ρ, let us denote by D^ε_ρ (resp. D'^ε_ρ) the début time after ρ of the set:

$$A^\varepsilon = \{(t,\omega) : g_t \geq \overline{V}_t - \varepsilon\}, \qquad \big(\text{resp.} \quad A'^\varepsilon = \{(t,\omega) : f_t \leq \underline{V}_t + \varepsilon\}\big).$$

A key result in this section is to show that $(\overline{V}_t)_{t\geq 0}$ (resp. $(\underline{V}_t)_{t\geq 0}$), starting from ρ, behaves like a submartingale (resp. supermartingale) until D^ε_ρ (resp. D'^ε_ρ).

Theorem 2.4.1. *(Theorem 11, [LM84]) For any $(\mathscr{F}_t)$-stopping time ρ we have:*

$$g_\rho \leq \overline{V}_\rho \leq f_\rho \quad \mathbb{P}\text{-a.s.}, \qquad \Big(\text{resp.} \quad g_\rho \leq \underline{V}_\rho \leq f_\rho \quad \mathbb{P}\text{-a.s.}\Big).$$

Moreover, for any $(\mathscr{F}_t)$-stopping time $\eta \geq \rho$, we have:

$$\overline{V}_\rho \leq \mathbb{E}\Big[\overline{V}_{D^\varepsilon_\rho \wedge \eta}\Big|\mathscr{F}_\rho\Big] \quad \mathbb{P}\text{-a.s.}, \qquad \Big(\text{resp.} \quad \underline{V}_\rho \geq \mathbb{E}\Big[\underline{V}_{D'^\varepsilon_\rho \wedge \eta}\Big|\mathscr{F}_\rho\Big] \quad \mathbb{P}\text{-a.s.}\Big)$$

Proof. 1° By using Lemma 2.3.3, and then choosing $\tau = \rho$, we have

$$\overline{V}_\rho \leq \operatorname*{ess\,sup}_{\sigma \in \mathscr{T}_\rho} \mathbb{E}\big[f_\rho \mathbb{1}_{\{\rho \leq \sigma\}} + g_\sigma \mathbb{1}_{\{\sigma < \rho\}}\big|\mathscr{F}_\rho\big] \quad \mathbb{P}\text{-a.s.}$$

Notice that the final term disappears trivially because σ is taken to be greater than or equal to ρ. As a result

$$\overline{V}_\rho \leq \operatorname*{ess\,sup}_{\sigma \in \mathscr{T}_\rho} \mathbb{E}[f_\rho|\mathscr{F}_\rho] = f_\rho \quad \mathbb{P}\text{-a.s.}$$

On the other hand, if we take $\sigma = \rho$ in the original definition of $\overline{V}(\rho)$ (see Definition 2.1.2), we have the following identity for the aggregated process:

$$\overline{V}_\rho \geq \operatorname*{ess\,inf}_{\tau \in \mathscr{T}_\rho} \mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau < \rho\}} + g_\rho \mathbb{1}_{\{\rho \leq \tau\}}\big|\mathscr{F}_\rho\big] \quad \mathbb{P}\text{-a.s.}$$

Thus,

$$\overline{V}_\rho \geq \operatorname*{ess\,inf}_{\tau \in \mathscr{T}_\rho} \mathbb{E}[g_\rho|\mathscr{F}_\rho] = g_\rho \quad \mathbb{P}\text{-a.s.}$$

2° Recall the process $(\tilde{Y}_t)_{t\geq 0}$ from Theorem 2.2.18, and choose $\eta = \tau$ for any stopping time

$\tau \geq D^{\varepsilon}_{\rho}$. Since τ is a stopping time, both indicator functions are adapted, thus since the process $(\tilde{Y}_t)_{t\geq 0}$ is càdlàg, it is optional. Denote by $(Z^{\tau}_t)_{t\geq 0}$ its Snell envelope, and let $\mathfrak{D}^{\varepsilon,\tau}_{\rho}$ be the début time after ρ of the set:

$$\mathfrak{A}^{\varepsilon,\rho} := \{(t,\omega) : \tilde{Y}_t \geq Z^{\tau}_t - \varepsilon\}.$$

We can now bound the upper value from above by the Snell envelope in the following way. Let ν be any stopping time such that $\nu \leq D^{\varepsilon}_{\rho}$, then

$$\begin{aligned}
\overline{V}_{\nu} &= \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_{\nu}} \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{\nu}} \mathbb{E}\left[f_{\tau}\mathbb{1}_{\{\tau\leq\sigma\}} + g_{\sigma}\mathbb{1}_{\{\sigma<\tau\}}\middle|\mathscr{F}_{\nu}\right] \\
&= \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_{\nu}} Z^{\tau}_{\nu} \qquad (2.18)\\
&\leq Z^{\tau}_{\nu} \quad \mathbb{P}\text{-a.s.} \qquad (2.19)
\end{aligned}$$

for all $\tau \geq \nu$, therefore, a fortiori for all $\tau \geq D^{\varepsilon}_{\rho}$.

We want to show that $\mathfrak{D}^{\varepsilon,\tau}_{\rho} \geq D^{\varepsilon}_{\rho}$, and to this end let $\rho \leq t < D^{\varepsilon}_{\rho}$. Notice that if $D^{\varepsilon}_{\rho} = \rho$ (i.e. $t = \rho$) then $\mathfrak{D}^{\varepsilon,\tau}_{\rho} \geq D^{\varepsilon}_{\rho} = \rho$ holds trivially from the definition of D^{ε}_{ρ}, so we are only concerned with the case where t can be found strictly less than D^{ε}_{ρ}. Under this assumption we have that $\overline{V}_t > g_t + \varepsilon$ since we have not entered A^{ε} yet, and moreover, that $\tilde{Y}_t = g_t$ since $t < D^{\varepsilon}_{\rho} \leq \tau$. Combining this with (2.19) we get

$$Z^{\tau}_t \geq \overline{V}_t > g_t + \varepsilon = \tilde{Y}_t + \varepsilon \quad \mathbb{P}\text{-a.s.}$$

and therefore, for all $\tau \geq D^{\varepsilon}_{\rho}$, we have that $\mathfrak{D}^{\varepsilon,\tau}_{\rho} \geq D^{\varepsilon}_{\rho}$ $\mathbb{P}$-almost surely.

We can now look to apply Theorem 1.3.22. First note that $(\tilde{Y}_t)_{t\geq 0}$ is not necessarily positive, but Remark 1.3.14 explains how the fact that it is bounded can be used to compensate for this. Moreover, $(\tilde{Y}_t)_{t\geq 0}$ is $(\mathscr{F}_t)$-adapted and càdlàg and so a direct application of Theorem 1.3.22 tells us that for any stopping time $\eta \geq \rho$

$$Z^{\tau}_{\rho} = \mathbb{E}\left[Z^{\tau}_{\mathfrak{D}^{\varepsilon,\tau}_{\rho}\wedge\eta}\middle|\mathscr{F}_{\rho}\right] \quad \mathbb{P}\text{-a.s.}$$

This says that the process $(Z^{\tau}_{\mathfrak{D}^{\varepsilon,\tau}_{\rho}\wedge t})_{t\geq\rho}$ is a martingale, and since $\mathfrak{D}^{\varepsilon,\tau}_{\rho} \geq D^{\varepsilon}_{\rho}$ $\mathbb{P}$-almost surely, then the process $(Z^{\tau}_{D^{\varepsilon}_{\rho}\wedge t})_{t\geq\rho}$ must also be a martingale, i.e. $Z^{\tau}_{\rho} = \mathbb{E}[Z^{\tau}_{D^{\varepsilon}_{\rho}\wedge\eta}|\mathscr{F}_{\rho}]$ $\mathbb{P}$-almost surely. More precisely, this is because a right-continuous martingale stopped at a stopping time is still a martingale (Theorem VI.12, [DM82]).

Using the definition of $Z^{\tau}_{D^{\varepsilon}_{\rho}\wedge\eta}$ as an essential supremum we may show that this is equivalent to the following

$$Z^{\tau}_{\rho} = \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{D^{\varepsilon}_{\rho}\wedge\eta}} \mathbb{E}\Big[\mathbb{E}\Big[\tilde{Y}_{\sigma}\Big|\mathscr{F}_{D^{\varepsilon}_{\rho}\wedge\eta}\Big]\Big|\mathscr{F}_{\rho}\Big] = \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{D^{\varepsilon}_{\rho}\wedge\eta}} \mathbb{E}\big[f_{\tau}\mathbb{1}_{\{\tau\leq\sigma\}} + g_{\sigma}\mathbb{1}_{\{\sigma<\tau\}}\big|\mathscr{F}_{\rho}\big] \tag{2.20}$$

for all $\eta \geq \rho$ and all $\tau \geq D^{\varepsilon}_{\rho}$. The first equality above follows from the fact that the family $\{\mathbb{E}[\tilde{Y}_{\sigma}|\mathscr{F}_{D^{\varepsilon}_{\rho}\wedge\eta}], \sigma \geq D^{\varepsilon}_{\rho}\wedge\eta\}$ is upwards directed (see the proof of Lemma 2.3.10), and the second follows from the tower property of conditional expectation, since $\rho \leq D^{\varepsilon}_{\rho}\wedge\eta$. Recall that (2.18) holds for any $\nu \leq D^{\varepsilon}_{\rho}$ and any $\tau \geq \nu$, thus if we take $\nu = \rho$, and notice that $\eta \geq \rho$ and $\rho \leq D^{\varepsilon}_{\rho}\wedge\eta \leq D^{\varepsilon}_{\rho}$, then, not only do we know trivially that (2.18) holds for all $\tau \geq \rho$, but we also know that it holds for all $\tau \geq D^{\varepsilon}_{\rho}\wedge\eta$ a fortiori. Utilising this and (2.20), we can then deduce that

$$\overline{V}_{\rho} \leq \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_{D^{\varepsilon}_{\rho}\wedge\eta}} \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_{D^{\varepsilon}_{\rho}\wedge\eta}} \mathbb{E}\big[f_{\tau}\mathbb{1}_{\{\tau\leq\sigma\}} + g_{\sigma}\mathbb{1}_{\{\sigma<\tau\}}\big|\mathscr{F}_{\rho}\big] = \mathbb{E}\Big[\overline{V}_{D^{\varepsilon}_{\rho}\wedge\eta}\Big|\mathscr{F}_{\rho}\Big].$$

Note that the inequality is obtained since we move from an essential infimum over stopping times greater than or equal to ρ in the definition of $\overline{V}_{\rho}$, to one over stopping times greater than or equal to $D^{\varepsilon}_{\rho}\wedge\eta$—a smaller set. The final equality holds by virtue of Lemma 2.3.10 and Remark 2.3.11 (taking $\rho_1 = \rho$ and $\rho_2 = D^{\varepsilon}_{\rho}\wedge\eta$).

The result for the lower conditional value is done in the same way, *mutatis mutandis*. ■

From this theorem we can deduce the indistinguishability of $(\overline{V}_t)_{t\geq 0}$ and $(\underline{V}_t)_{t\geq 0}$, and the existence of a value.

Corollary 2.4.2. *(Corollary 12, [LM84]) For all $(\mathscr{F}_t)$-stopping times ρ*

$$\overline{V}_{\rho} = \underline{V}_{\rho} \quad \mathbb{P}\text{-}a.s. \tag{2.21}$$

In particular, we have $\overline{V}_0 = \underline{V}_0$, i.e.

$$\inf_{\tau\in\mathscr{T}} \sup_{\sigma\in\mathscr{T}} J(\tau,\sigma) = \sup_{\sigma\in\mathscr{T}} \inf_{\tau\in\mathscr{T}} J(\tau,\sigma),$$

and, therefore, the game has a value.

Proof. Due to Theorem 2.4.1, we have that, for every stopping time $\eta \geq \tau$

$$\overline{V}_\rho \leq \mathbb{E}\Big[\overline{V}_{D^\varepsilon_\rho \wedge \eta}\Big|\mathscr{F}_\rho\Big] \quad \mathbb{P}\text{-a.s.}$$

Writing the left-hand side over the events $\{\eta < D^\varepsilon_\rho\}$ and $\{D^\varepsilon_\rho \leq \eta\}$ yields

$$\overline{V}_\rho \leq \mathbb{E}\Big[\overline{V}_\eta \mathbb{1}_{\{\eta < D^\varepsilon_\rho\}} + \overline{V}_{D^\varepsilon_\rho} \mathbb{1}_{\{D^\varepsilon_\rho \leq \eta\}}\Big|\mathscr{F}_\rho\Big] \quad \mathbb{P}\text{-a.s.} \tag{2.22}$$

Since $(\overline{V}_t - g_t)_{t\geq 0}$ is right-continuous, and since D^ε_ρ is an infimum, we can say that we must have

$$\overline{V}_{D^\varepsilon_\rho} \leq g_{D^\varepsilon_\rho} + \varepsilon.$$

On the other hand we know the upper value is bounded above by $(f_t)_{t\geq 0}$ from Theorem 2.4.1 and so $\overline{V}_\eta \leq f_\eta$ by Theorem 2.4.1, and hence we have the upper bound

$$\overline{V}_\rho \leq \mathbb{E}\Big[f_\eta \mathbb{1}_{\{\eta < D^\varepsilon_\rho\}} + g_{D^\varepsilon_\rho} \mathbb{1}_{\{D^\varepsilon_\rho \leq \eta\}}\Big|\mathscr{F}_\rho\Big] + \varepsilon \quad \mathbb{P}\text{-a.s.} \tag{2.23}$$

If we use the same logic, *mutatis mutandis*, for the lower conditional value $(\underline{V}_t)_{t\geq 0}$, we can obtain the following for all stopping times $\eta' \geq \rho$

$$\underline{V}_\rho \geq \mathbb{E}\Big[f_{D'^\varepsilon_\rho} \mathbb{1}_{\{D'^\varepsilon_\rho < \eta'\}} + g_{\eta'} \mathbb{1}_{\{\eta' \leq D'^\varepsilon_\rho\}}\Big|\mathscr{F}_\rho\Big] - \varepsilon \quad \mathbb{P}\text{-a.s.} \tag{2.24}$$

If we take $\eta = D'^\varepsilon_\tau$ in (2.23) and $\eta' = D^\varepsilon_\tau$ in (2.24) we are left with

$$\begin{aligned}
\overline{V}_\rho &\leq \mathbb{E}\Big[f_{D'^\varepsilon_\rho} \mathbb{1}_{\{D'^\varepsilon_\rho < D^\varepsilon_\rho\}} + g_{D^\varepsilon_\rho} \mathbb{1}_{\{D^\varepsilon_\rho \leq D'^\varepsilon_\rho\}}\Big|\mathscr{F}_\rho\Big] + \varepsilon, \\
\underline{V}_\rho &\geq \mathbb{E}\Big[f_{D'^\varepsilon_\rho} \mathbb{1}_{\{D'^\varepsilon_\rho < D^\varepsilon_\rho\}} + g_{D^\varepsilon_\rho} \mathbb{1}_{\{D^\varepsilon_\rho \leq D'^\varepsilon_\rho\}}\Big|\mathscr{F}_\rho\Big] - \varepsilon,
\end{aligned}$$

i.e. $\overline{V}_\rho \leq \underline{V}_\rho + 2\varepsilon$. Letting $\varepsilon \to 0$, we find $\overline{V}_\rho \leq \underline{V}_\rho$ implying that $\underline{V}_\rho = \overline{V}_\rho$ $\mathbb{P}$-almost surely. ■

Remark 2.4.3. Note that the proof above can be altered to swap the strictness of the indicator function inequalities in (2.22), and therefore in every subsequent step in the proof. This is of no immediate use, but will be important during the next section.

Equation (2.21) tells us that a 'value'—not in the strict sense at time zero, but as equality between the upper and lower values—exists at all stopping times ρ. From this we can define the value process as

$$V_t := \underline{V}_t = \overline{V}_t,$$

for all $t \geq 0$. We can now characterise this process.

Theorem 2.4.4. *(Theorem 13, [LM84]) The conditional value process $(V_t)_{t\geq 0}$ is the unique right-continuous process satisfying the following,*

i) $g_t \leq V_t \leq f_t$ *for all* $t \geq 0$,

ii) if D^ε_ρ (resp. D'^ε_ρ) denotes the début time after ρ of the set

$$A^\varepsilon = \{(t,\omega) : g_t \geq V_t - \varepsilon\}, \qquad (\textit{resp. } A'^\varepsilon = \{(t,\omega) : f_t \leq V_t + \varepsilon\}),$$

then

$$\mathbb{E}\Big[V_{D'^\varepsilon_\rho \wedge \sigma}\Big|\mathscr{F}_\rho\Big] \leq V_\rho \leq \mathbb{E}\Big[V_{D^\varepsilon_\rho \wedge \tau}\Big|\mathscr{F}_\rho\Big] \quad \mathbb{P}\text{-}a.s.$$

for all $(\mathscr{F}_t)$-stopping times ρ, σ, τ such that τ, $\sigma \geq \rho$ and for all $\varepsilon > 0$.

Proof. In the same way as we did for Corollary 2.4.2, we establish that for a right-continuous $(Y_t)_{t\geq 0}$, if it satisfies *i)* and *ii)* then

$$\mathbb{E}\Big[f_{D'^\varepsilon_\rho}\mathbb{1}_{\{\sigma > D'^\varepsilon_\rho\}} + g_\sigma \mathbb{1}_{\{\sigma \leq D'^\varepsilon_\rho\}}\Big|\mathscr{F}_\rho\Big] \leq Y_\rho \leq \mathbb{E}\Big[f_\tau \mathbb{1}_{\{\tau < D^\varepsilon_\rho\}} + g_{D^\varepsilon_\rho}\mathbb{1}_{\{\tau \geq D^\varepsilon_\rho\}}\Big|\mathscr{F}_\rho\Big] \quad \mathbb{P}\text{-a.s.} \tag{2.25}$$

for all $(\mathscr{F}_t)$-stopping times ρ, η', τ, such that η',$\tau \geq \rho$, and for all $\varepsilon > 0$.

Notice that, by taking the essential infimum over τ on both sides of the right-most inequality in (2.25) we get

$$Y_\rho \leq \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_\rho}\mathbb{E}\Big[f_\tau \mathbb{1}_{\{\tau < D^\varepsilon_\rho\}} + g_{D^\varepsilon_\rho}\mathbb{1}_{\{\tau \geq D^\varepsilon_\rho\}}\Big|\mathscr{F}_\rho\Big] \leq \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho}\operatorname*{ess\,inf}_{\eta\in\mathscr{T}_\rho}\mathbb{E}\big[f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\tau\geq\sigma\}}\big|\mathscr{F}_\rho\big] = \underline{V}_\rho,$$

and then (2.21) yields $Y_\rho \leq V_\rho$. Likewise, taking the essential supremum over σ on the left-most inequality in (2.25) yields

$$\begin{aligned} Y_\rho &\geq \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho}\mathbb{E}\Big[f_{D'^\varepsilon_\rho}\mathbb{1}_{\{\sigma > D'^\varepsilon_\rho\}} + g_\sigma \mathbb{1}_{\{\sigma \leq D'^\varepsilon_\rho\}}\Big|\mathscr{F}_\rho\Big] \\ &\geq \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_\rho}\operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho}\mathbb{E}\big[f_\tau \mathbb{1}_{\{\sigma>\tau\}} + g_\sigma \mathbb{1}_{\{\sigma\leq\tau\}}\big|\mathscr{F}_\rho\big] = \overline{V}_\rho. \end{aligned}$$

Again, (2.21) yields $Y_\rho \geq V_\rho$, and hence $Y_\rho = V_\rho$ $\mathbb{P}$-almost surely. We know that since $(Y_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ have right-continuous paths, they must be indistinguishable (Problem 1.1.5, [KS98a]). ■

This characterisation of the value of the game has much in common with that of the Snell envelope in the single-agent problem, and Theorem 2.4.4 can be compared to Theorem 1.3.22.

2.5 EXISTENCE OF A NASH EQUILIBRIUM

The stopping times D^ε_ρ and D'^ε_ρ are ε-optimal, and as we have seen in the single-agent case, the natural idea is then to make ε tend towards zero and we then see the stopping times D^ε_ρ and D'^ε_ρ increase respectively towards two stopping times $\overline{D}_\rho$ and $\overline{D}'_\rho$. In Chapter 1.4 we demonstrated the need for some left-regularity to attain an optimal stopping time in the single-agent problem, and we have already seen what regularity we will be assuming.

Recall from Assumption 2.2.15 that we assume $(f_t)_{t\geq 0}$ is previsibly left lower semicontinuous, and that $(g_t)_{t\geq 0}$ is previsibly left upper semicontinuous. In this section, we will show that under this left-regularity, the pair $(\overline{D}_0, \overline{D}'_0)$ is a Nash equilibrium.

The proof of Theorem 2.5.1 is markedly different from the corresponding proof found in Lepeltier and Maingueneau, [LM84]. They arrive at (in our numbering) equation (2.33) and simply substitute $\rho = \overline{D}_0 \wedge \overline{D}'_0$ to arrive at their conclusion. We could not verify their reasoning here, as they seem to imply that, for example, $\overline{D}_{\overline{D}_0 \wedge \overline{D}'_0} = \overline{D}_0$. The proof of Theorem 2.5.1 is split into six parts, and parts two to six are dedicated to our attempt at justifying the claims in the original proof.

Theorem 2.5.1. *(Theorem 15, [LM84]) If the processes $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ are right-continuous, as per Assumption 2.1.1, and obey the previsible left semicontinuity assumptions (A1) and (A2), and if the pair $\overline{D}_0$ and $\overline{D}'_0$ are the non-decreasing limits of the sequences $(D^\varepsilon_0)_{\varepsilon>0}$ and $(D'^\varepsilon_0)_{\varepsilon>0}$ respectively, then the pair $(\overline{D}_0, \overline{D}'_0)$ is a Nash equilibrium for the game.*

Moreover, if D_0 and D'_0 denote the début times of the sets $\{(t,\omega) : V_t(\omega) = g_t(\omega)\}$ and $\{(t,\omega) : V_t(\omega) = f_t(\omega)\}$ respectively, we have

$$\overline{D}_0 \leq D_0 \qquad \& \qquad \overline{D}'_0 \leq D'_0,$$

and

$$\overline{D}_0 = D_0, \ \text{on} \ \{\overline{D}_0 \leq \overline{D}'_0\} \qquad \& \qquad \overline{D}'_0 = D'_0, \ \text{on} \ \{\overline{D}'_0 < \overline{D}_0\}.$$

Proof. 1° First, we prove that the pair $(\overline{D}_0, \overline{D}'_0)$ is a Nash equilibrium. Using (2.23), (2.24) and the fact that $\overline{V}_\rho = \underline{V}_\rho =: V_\rho$, we have for all stopping times η, $\eta' \geq \rho$ and all $\varepsilon > 0$:

$$\mathbb{E}\Big[f_{D'^{\varepsilon}_\rho}\mathbb{1}_{\{D'^{\varepsilon}_\rho<\eta'\}}+g_{\eta'}\mathbb{1}_{\{\eta'\leq D'^{\varepsilon}_\rho\}}\Big|\mathscr{F}_\rho\Big]-\varepsilon\leq V_\rho\leq\mathbb{E}\Big[f_\eta\mathbb{1}_{\{\eta<D^\varepsilon_\rho\}}+g_{D^\varepsilon_\rho}\mathbb{1}_{\{D^\varepsilon_\rho\leq\eta\}}\Big|\mathscr{F}_\rho\Big]+\varepsilon \quad (2.26)$$

$\mathbb{P}$-almost surely. We therefore obtain, by taking the limit supremum of the right-hand inequality,

$$V_\rho \leq \limsup_{\varepsilon\searrow 0}\mathbb{E}\Big[f_\eta\mathbb{1}_{\{\eta<D^\varepsilon_\rho\}}+g_{D^\varepsilon_\rho}\mathbb{1}_{\{D^\varepsilon_\rho\leq\eta\}}\Big|\mathscr{F}_\rho\Big] \quad \mathbb{P}\text{-a.s.}$$

Recall that $\overline{D}_\rho$ is the limit of the non-decreasing sequence $(D^\varepsilon_\rho)_{\varepsilon>0}$, then, using Lemma 2.2.20 with $\sigma^\varepsilon = D^\varepsilon_\rho$ and $\sigma = \overline{D}_\rho$, we have

$$V_\rho \leq \mathbb{E}\Big[f_\eta\mathbb{1}_{\{\eta<\overline{D}_\rho\}}+g_{\overline{D}_\rho}\mathbb{1}_{\{\overline{D}_\rho\leq\eta\}}\Big|\mathscr{F}_\rho\Big] \quad \mathbb{P}\text{-a.s.} \quad (2.27)$$

Setting $\eta = \overline{D}'_\rho$ we get

$$V_\rho \leq \mathbb{E}\Big[f_{\overline{D}'_\rho}\mathbb{1}_{\{\overline{D}'_\rho<\overline{D}_\rho\}}+g_{\overline{D}_\rho}\mathbb{1}_{\{\overline{D}_\rho\leq\overline{D}'_\rho\}}\Big|\mathscr{F}_\rho\Big] \quad \mathbb{P}\text{-a.s.} \quad (2.28)$$

In the same way one could begin with the left-hand inequality in (2.26), however, there is one small technical difference. From Remark 2.4.3 we know that the strictness of the inequalities in the indicator functions in (2.26) can be swapped, i.e. it is also true that

$$\mathbb{E}\Big[f_{D'^{\varepsilon}_\rho}\mathbb{1}_{\{D'^{\varepsilon}_\rho\leq\eta'\}}+g_{\eta'}\mathbb{1}_{\{\eta'<D'^{\varepsilon}_\rho\}}\Big|\mathscr{F}_\rho\Big]-\varepsilon\leq V_\rho\leq\mathbb{E}\Big[f_\eta\mathbb{1}_{\{\eta\leq D^\varepsilon_\rho\}}+g_{D^\varepsilon_\rho}\mathbb{1}_{\{D^\varepsilon_\rho<\eta\}}\Big|\mathscr{F}_\rho\Big]+\varepsilon.$$

Taking the limit infimum of the left inequality we arrive at

$$V_\rho \geq \liminf_{\varepsilon\searrow 0}\mathbb{E}\Big[f_{D'^{\varepsilon}_\rho}\mathbb{1}_{\{D'^{\varepsilon}_\rho\leq\eta'\}}+g_{\eta'}\mathbb{1}_{\{\eta'<D'^{\varepsilon}_\rho\}}\Big|\mathscr{F}_\rho\Big].$$

Once again, recall that $\overline{D}'_\rho$ is the limit of the non-decreasing sequence $(D'^{\varepsilon}_\rho)_{\varepsilon>0}$, and so using Lemma 2.2.20 with $\tau^\varepsilon = D'^{\varepsilon}_\rho$ and $\tau = \overline{D}'_\rho$, we have

$$V_\rho \geq \mathbb{E}\Big[f_{\overline{D}'_\rho}\mathbb{1}_{\{\overline{D}'_\rho\leq\eta'\}}+g_{\eta'}\mathbb{1}_{\{\eta'<\overline{D}'_\rho\}}\Big|\mathscr{F}_\rho\Big] \quad \mathbb{P}\text{-a.s.} \quad (2.29)$$

Furthermore, by setting $\eta' = \overline{D}_\rho$ we get

$$V_\rho \geq \mathbb{E}\Big[f_{\overline{D}'_\rho}\mathbb{1}_{\{\overline{D}'_\rho \leq \overline{D}_\rho\}} + g_{\overline{D}_\rho}\mathbb{1}_{\{\overline{D}_\rho < \overline{D}'_\rho\}}\Big|\mathscr{F}_\rho\Big] \quad \mathbb{P}\text{-a.s.} \tag{2.30}$$

Consolidating (2.28) and (2.30), we have that

$$\mathbb{E}\Big[f_{\overline{D}'_\rho}\mathbb{1}_{\{\overline{D}'_\rho \leq \overline{D}_\rho\}} + g_{\overline{D}_\rho}\mathbb{1}_{\{\overline{D}_\rho < \overline{D}'_\rho\}}\Big|\mathscr{F}_\rho\Big] \leq V_\rho \leq \mathbb{E}\Big[f_{\overline{D}'_\rho}\mathbb{1}_{\{\overline{D}'_\rho < \overline{D}_\rho\}} + g_{\overline{D}_\rho}\mathbb{1}_{\{\overline{D}_\rho \leq \overline{D}'_\rho\}}\Big|\mathscr{F}_\rho\Big]. \tag{2.31}$$

It is simple to see that on the right-hand side we have

$$g_{\overline{D}_\rho}\mathbb{1}_{\{\overline{D}_\rho = \overline{D}'_\rho\}} \leq f_{\overline{D}_\rho}\mathbb{1}_{\{\overline{D}_\rho = \overline{D}'_\rho\}} = f_{\overline{D}'_\rho}\mathbb{1}_{\{\overline{D}_\rho = \overline{D}'_\rho\}} \quad \mathbb{P}\text{-a.s.}, \tag{2.32}$$

meaning that equality actually holds in (2.31). Notably, we could equally use the fact that $g_t \leq f_t$ for all $t \geq 0$ on the left-hand side of (2.31) to show equality, moreover, doing so tells us that

$$\mathbb{E}\Big[f_{\overline{D}'_\rho}\mathbb{1}_{\{\overline{D}'_\rho < \overline{D}_\rho\}} + g_{\overline{D}_\rho}\mathbb{1}_{\{\overline{D}_\rho \leq \overline{D}'_\rho\}}\Big|\mathscr{F}_\rho\Big] = V_\rho = \mathbb{E}\Big[f_{\overline{D}'_\rho}\mathbb{1}_{\{\overline{D}'_\rho \leq \overline{D}_\rho\}} + g_{\overline{D}_\rho}\mathbb{1}_{\{\overline{D}_\rho < \overline{D}'_\rho\}}\Big|\mathscr{F}_\rho\Big]. \tag{2.33}$$

This does not need to stop here either, using the fact that $g_t \leq f_t$ for all $t \geq 0$ we see that (2.27) can be bounded below by

$$V_\rho \geq \mathbb{E}\Big[f_{\overline{D}'_\rho}\mathbb{1}_{\{\overline{D}'_\rho < \eta'\}} + g_{\eta'}\mathbb{1}_{\{\eta' \leq \overline{D}'_\rho\}}\Big|\mathscr{F}_\rho\Big] \quad \mathbb{P}\text{-a.s.}$$

It then suffices to unite (2.27), (2.29) and (2.33) by making $\rho = 0$ to obtain that $(\overline{D}_0, \overline{D}'_0)$ is a Nash equilibrium for the game.

2° We now wish to show that when the stopping times $\overline{D}_\rho$ and $\overline{D}'_\rho$ coincide, the payoff processes $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ coincide at that time. To this end, we consider the set $\{\overline{D}'_\rho = \overline{D}_\rho\}$. By taking expectation in (2.33) we can immediately see that

$$\mathbb{E}\Big[g_{\overline{D}_\rho}\mathbb{1}_{\{\overline{D}_\rho = \overline{D}'_\rho\}}\Big] = \mathbb{E}\Big[f_{\overline{D}'_\rho}\mathbb{1}_{\{\overline{D}'_\rho = \overline{D}_\rho\}}\Big]. \tag{2.34}$$

Consider again equation (2.32). Since $g_t \leq f_t$, the set on which $g_{\overline{D}_\rho}\mathbb{1}_{\{\overline{D}_\rho = \overline{D}'_\rho\}} < f_{\overline{D}'_\rho}\mathbb{1}_{\{\overline{D}'_\rho = \overline{D}_\rho\}}$ must be a null set, otherwise it would violate (2.34), therefore we must have

$$g_{\overline{D}_\rho}\mathbb{1}_{\{\overline{D}_\rho = \overline{D}'_\rho\}} = f_{\overline{D}_\rho}\mathbb{1}_{\{\overline{D}'_\rho = \overline{D}_\rho\}} \quad \mathbb{P}\text{-a.s.} \tag{2.35}$$

3° We now consider an auxiliary single-agent problem faced by the maximiser, i.e. the best response to $\overline{D}_0'$. We utilise the methods of classical optimal stopping to construct the optimal stopping time $\overline{D}$.

It is clear that the set A^{ε} (resp. A'^{ε}) must contain the set $\{(t,\omega) : V_t(\omega) = g_t(\omega)\}$ (resp. $\{(t,\omega) : V_t(\omega) = f_t(\omega)\}$). Moreover, since D_0^{ε} is non-decreasing it is obvious that $D_0^{\varepsilon} \leq D_0$, thus its limit point must also obey this inequality, $\overline{D}_0 \leq D_0$. In the same way we see $\overline{D}_0' \leq D_0'$.

Before continuing, recall Remark 1.3.14 and notice that shifting both payoff processes by a constant has no effect on the value of the game (except for shifting it by that constant), and moreover, any optimal stopping times for the shifted game are still optimal for the original game. Because of this fact, we can assume that $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ are positive without loss of generality.

Let us begin by considering the following optimal stopping problem

$$\tilde{V}_0 = \sup_{\eta \in \mathscr{T}} \mathbb{E}\Big[f_{\overline{D}_0'} \mathbb{1}_{\{\overline{D}_0' < \eta\}} + g_\eta \mathbb{1}_{\{\eta \leq \overline{D}_0'\}}\Big].$$

Lemma 2.1.3 tells us that we can instead use

$$\tilde{V}_0 = \sup_{\eta \in \mathscr{T}} \mathbb{E}\Big[f_{\overline{D}_0'} \mathbb{1}_{\{\overline{D}_0' \leq \eta\}} + g_\eta \mathbb{1}_{\{\eta < \overline{D}_0'\}}\Big]. \tag{2.36}$$

Recall the process $(\tilde{Y}_t)_{t\geq 0}$ defined in Theorem 2.2.18 and choose the stopping time η in Theorem 2.2.18 to be $\eta = \overline{D}_0'$ (not to be confused with the stopping time η used throughout this proof). The process $(\tilde{Y}_t)_{t\geq 0}$ then takes the form $\tilde{Y}_t := f_{\overline{D}_0'} \mathbb{1}_{\{\overline{D}_0' \leq t\}} + g_t \mathbb{1}_{\{t < \overline{D}_0'\}}$ which is bounded, positive, adapted, right-continuous, and (due to Theorem 2.2.18) previsibly left upper semicontinuous. We solve the problem

$$\tilde{V}_0 = \sup_{\eta \in \mathscr{T}} \mathbb{E}\big[\tilde{Y}_\eta\big]. \tag{2.37}$$

If we denote by $\tilde{V}_\rho$ the maximal conditional gain, written as follows

$$\tilde{V}_\rho = \operatorname*{ess\,sup}_{\eta \in \mathscr{T}_\rho} \mathbb{E}\big[\tilde{Y}_\eta \big| \mathscr{F}_\rho\big],$$

then the classical theory of optimal stopping (see Theorem 1.3.7) gives us the existence of the Snell envelope, $(\tilde{V}_t)_{t\geq 0}$, which is a unique, right-continuous, and bounded supermartingale.

We cannot directly use the classical theory of optimal stopping described in Chapter 1 since $(\tilde{Y}_t)_{t\geq 0}$ is not left-continuous in expectation over stopping times. As a result, we need to re-prove the existence of an optimal stopping time.

First, recall that we assume, without loss of generality, that $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ are positive. Denote by $A^\lambda := \{(\omega,t) : \tilde{Y}_t \geq \lambda \tilde{V}_t\}$ for any $\lambda \in (0,1)$, and let D^λ be the début of A^λ, then Corollary 2.17 from El Karoui [EK79] gives us

$$\sup_{\eta\in\mathscr{T}} \mathbb{E}\big[\tilde{Y}_\eta\big] \leq \frac{1}{\lambda}\mathbb{E}\big[\tilde{Y}_{D^\lambda}\big]. \tag{2.38}$$

Consider an increasing sequence of numbers $(\lambda_n)_{n\in\mathbb{N}} \subset [0,1)$ which tends to one. The sets A^{λ_n} are decreasing and their début times are increasing. Denote by $\overline{D}$ the non-decreasing limit of D^{λ_n}. Taking the limit supremum on both sides of (2.38) we have

$$\sup_{\eta\in\mathscr{T}} \mathbb{E}\big[\tilde{Y}_\eta\big] \leq \limsup_{n\to\infty} \frac{1}{\lambda_n}\mathbb{E}\big[\tilde{Y}_{D^{\lambda_n}}\big] \leq \limsup_{n\to\infty} \mathbb{E}\big[\tilde{Y}_{D^{\lambda_n}}\big]. \tag{2.39}$$

Since $(\tilde{Y}_t)_{t\geq 0}$ is previsibly left upper semicontinuous, we can use Theorem 2.2.10 in (2.39) to see

$$\sup_{\eta\in\mathscr{T}} \mathbb{E}\big[\tilde{Y}_\eta\big] \leq \mathbb{E}\big[\tilde{Y}_{\overline{D}}\big].$$

Clearly the strict inequality would make no sense, as $\overline{D}$ itself belongs to the set of stopping times we are maximising over on the left-hand side, so equality must hold, meaning that $\overline{D}$ is optimal.

4° We now create a new stopping time D, and show that it must be equal to $\overline{D}$ and, therefore, optimal in (2.37). We then prove the equality between D, D_0 and $\overline{D}_0$ on $\{\overline{D}_0 < \overline{D}_0'\}$.

Let D be the début of the set $\{(t,\omega) : \tilde{V}_t(\omega) = \tilde{Y}_t(\omega)\}$, and notice that this set belongs to A^λ for all λ. As a result we must have that $D^\lambda \leq D$ for all λ, implying that $\overline{D} \leq D$. However, since $\overline{D}$ is optimal for the problem in (2.37), we must have that $\tilde{V}_{\overline{D}} = \tilde{Y}_{\overline{D}}$ $\mathbb{P}$-almost surely by aggregation and Theorem 1.3.6. Moreover, since D is the first time at which this occurs, we must have that $D \leq \overline{D}$; hence equality and furthermore, the optimality of D, i.e.

$$\tilde{V}_0 = \sup_{\eta\in\mathscr{T}} \mathbb{E}\big[\tilde{Y}_\eta\big] = \mathbb{E}\big[\tilde{Y}_D\big].$$

Since $(\tilde{V}_t)_{t\geq 0}$ and $(\tilde{Y}_t)_{t\geq 0}$ are both right-continuous, we know that $\tilde{Y}_D = \tilde{V}_D$ by definition of D

being a début time, meaning that

$$\tilde{V}_D = f_{\overline{D}_0'} \mathbb{1}_{\{\overline{D}_0' \leq D\}} + g_D \mathbb{1}_{\{D < \overline{D}_0'\}}. \tag{2.40}$$

Therefore we know that, on $\{D < \overline{D}_0'\}$, we have $\tilde{V}_D = g_D$.

We now need to draw a relation between D, $\overline{D}_0$ and D_0. As stated earlier, we already know:

$$\overline{D}_0 \leq D_0. \tag{2.41}$$

However, strictly speaking, $\tilde{V}$ is not the value of the game we are studying. For clarity, we have the value arising from the game,

$$V_\rho = \operatorname*{ess\,inf}_{\tau \in \mathscr{T}_\rho} \operatorname*{ess\,sup}_{\sigma \in \mathscr{T}_\rho} \mathbb{E}\left[f_\tau \mathbb{1}_{\{\tau \leq \sigma\}} + g_\sigma \mathbb{1}_{\{\sigma < \tau\}} \middle| \mathscr{F}_\rho \right],$$

and the value arising from the optimal stopping problem,

$$\tilde{V}_\rho = \operatorname*{ess\,sup}_{\eta \in \mathscr{T}_\rho} \mathbb{E}\left[f_{\overline{D}_0'} \mathbb{1}_{\{\overline{D}_0' \leq \eta\}} + g_\eta \mathbb{1}_{\{\eta < \overline{D}_0'\}} \middle| \mathscr{F}_\rho \right].$$

Specifically, $D_0 = \inf\{t \geq 0 : V_t = g_t\}$, and $D = \inf\{t \geq 0 : \tilde{V}_t = \tilde{Y}_t\}$ with the latter taking the form of $D = \inf\{t \geq 0 : \tilde{V}_t = g_t\}$ on the set $\{D < \overline{D}_0'\}$. Moreover, it is apparent that while these values may not inherently coincide, they are equal at time zero, i.e. $V_0 = \tilde{V}_0$ using the fact that $\overline{D}_0'$ is optimal for the minimiser in the game formulation.

Clearly it is true that $V_\rho \leq \tilde{V}_\rho$ for all $\rho \in \mathscr{T}$, meaning that

$$D_0 \leq D, \text{ on } \{D < \overline{D}_0'\}, \tag{2.42}$$

since on this set we have $g_D \leq V_D \leq \tilde{V}_D = g_D$, implying that $V_D = g_D$. Combining (2.41) and (2.42) we have that

$$\overline{D}_0 \leq D_0 \leq D, \text{ on } \{D < \overline{D}_0'\}.$$

Finally, we require that $D \leq \overline{D}_0$ to obtain equality. This is a standard result of optimal stopping problems. Since the pair $(\overline{D}_0, \overline{D}_0')$ is a Nash equilibrium, $\overline{D}_0$ must be the best response to $\overline{D}_0'$, and so, therefore, it must also be optimal in the single-agent problem $\tilde{V}_0$, (2.37). However, we have already proved that $D = \inf\{t \geq 0 : \tilde{V}_t = \tilde{Y}_t\}$ is optimal in (2.37), and, moreover, this début time is minimal (see Corollary 1.3.15), and so it must be the case that $D \leq \overline{D}_0$ $\mathbb{P}$-almost surely. Therefore,

$$\overline{D}_0 = D_0 = D, \;\; on \;\; \{D < \overline{D}'_0\}. \tag{2.43}$$

In general, since $D \leq \overline{D}_0$, we have the inclusion that $\{\overline{D}_0 < \overline{D}'_0\} \subseteq \{D < \overline{D}'_0\}$, and so equality between the optimal times still holds on the smaller set,

$$\overline{D}_0 = D_0 = D, \;\; \text{on} \;\; \{\overline{D}_0 < \overline{D}'_0\}. \tag{2.44}$$

5° The arguments above can be followed *mutatis mutandis* to solve the infimum problem

$$\hat{V}_0 = \inf_{\eta \in \mathscr{T}} \mathbb{E}\Big[f_\eta \mathbb{1}_{\{\eta < \overline{D}_0\}} + g_{\overline{D}_0} \mathbb{1}_{\{\overline{D}_0 \leq \eta\}}\Big],$$

however, in this case we use the process $(\hat{Y}_t)_{t\geq 0}$ from Corollary 2.2.19 instead of $(\tilde{Y}_t)_{t\geq 0}$ from Theorem 2.2.18. We therefore show that (for the logically defined D')

$$\overline{D}'_0 = D'_0 = D', \;\; \text{on} \;\; \{\overline{D}'_0 < \overline{D}_0\}. \tag{2.45}$$

6° We are left with the case when $\{\overline{D}_0 = \overline{D}'_0\}$. We have already shown in (2.35) that, on $\{\overline{D}_\rho = \overline{D}'_\rho\}$, the payoff functions are equal, i.e. for $\rho = 0$, $f_{\overline{D}_0} = g_{\overline{D}_0}$. Moreover, since $g_t \leq V_t \leq f_t$ for all $t \geq 0$, we know that $g_{\overline{D}_0} = V_{\overline{D}_0} = f_{\overline{D}_0}$ (of course, this still holds for $\overline{D}'_0$). From this we see that $D_0 \leq \overline{D}_0$ and $D'_0 \leq \overline{D}'_0$. Coupled with the fact that $\overline{D}_0 \leq D_0$ and $\overline{D}'_0 \leq D'_0$, we have that

$$D_0 = \overline{D}_0 = \overline{D}'_0 = D'_0, \;\; \text{on} \;\; \{\overline{D}'_0 = \overline{D}_0\}.$$

Clearly, this means we can add this result to either (2.44) or (2.45), but to stay in keeping with the original paper, we place the equality with (2.44) to yield

$$\overline{D}_0 = D_0, \;\; \text{on} \;\; \{\overline{D}_0 \leq \overline{D}'_0\}, \qquad \& \qquad \overline{D}'_0 = D'_0, \;\; \text{on} \;\; \{\overline{D}'_0 < \overline{D}_0\}.$$

■

2.6 MINIMALITY

Finally, we present an adaptation of the classical result of optimal stopping we saw in Corollary 1.3.15, namely that the first time the Snell envelope equals the payoff process is the minimal optimal stopping time. Here we adapt this concept for Dynkin games.

To begin with we consider the concept of interchangeability. This idea has existed since Nash's original paper on non-cooperative games in 1951 [Nas51], but more recent analysis on the topic can be found in the 2013 paper by Naumov and Nicholls [NN13]. We interpret Nash's original definition here for our setting.

Denote by $\mathscr{E}$ the set of all Nash equilibrium points. A game is said to be solvable if when (τ^*,σ^*) and (τ,σ) belong to $\mathscr{E}$, then so do (τ^*,σ) and (τ,σ^*). Nash calls this the interchangeability condition, and moreover, if this condition is met then we say the game is solvable, and $\mathscr{E}$ is called the solution to the game. Here we present the analogue of this concept in stochastic games of this form.

Lemma 2.6.1. *(Solvability) Let (τ_1,σ_1) and (τ_2,σ_2) be two Nash equilibria, then both (τ_1,σ_2) and (τ_2,σ_1) are Nash equilibria. Moreover,*

$$J(\tau_1,\sigma_1) = J(\tau_2,\sigma_2) = J(\tau_1,\sigma_2) = J(\tau_2,\sigma_1). \tag{2.46}$$

Proof. Since both (τ_1,σ_1) and (τ_2,σ_2) are Nash equilibria we have a closed loop of inequalities:

$$J(\tau_1,\sigma_2) \le J(\tau_1,\sigma_1) \le J(\tau_2,\sigma_1),$$
$$J(\tau_2,\sigma_1) \le J(\tau_2,\sigma_2) \le J(\tau_1,\sigma_2).$$

This proves (2.46).

For the primary result, first consider the pair (τ_1,σ_2). Using (2.46), and the fact that (τ_1,σ_1) and (τ_2,σ_2) are Nash equilibria, we see

$$J(\tau_1,\sigma_2) = J(\tau_1,\sigma_1) = \sup_{\sigma\in\mathscr{T}} J(\tau_1,\sigma),$$
$$J(\tau_1,\sigma_2) = J(\tau_2,\sigma_2) = \inf_{\tau\in\mathscr{T}} J(\tau,\sigma_2),$$

i.e. σ_2 is the best response to τ_1, and τ_1 is the best response to σ_2. The same method holds for showing (τ_2, σ_1) is a Nash equilibrium. ■

We can now present two concepts of minimality.

Theorem 2.6.2. *Let $(\overline{D}_0, \overline{D}'_0)$ be the Nash equilibrium from Theorem 2.5.1, and let (τ^*, σ^*) be a second Nash equilibrium of the game. Then*

$$\overline{D}_0 \leq \sigma^* \quad \text{on } \{\overline{D}_0 < \overline{D}'_0\}, \tag{2.47}$$

$$\overline{D}'_0 \leq \tau^* \quad \text{on } \{\overline{D}'_0 < \overline{D}_0\}, \tag{2.48}$$

and, moreover,

$$\overline{D}_0 \leq \sigma^* \quad \text{on } \{\overline{D}_0 < \tau^*\}, \tag{2.49}$$

$$\overline{D}'_0 \leq \tau^* \quad \text{on } \{\overline{D}_0 < \sigma^*\}. \tag{2.50}$$

Proof. Equation (2.47) holds immediately from equation (2.44) as follows: (2.44) tells us that on $\{\overline{D}_0 < \overline{D}'_0\}$, $\overline{D}_0$ coincides with the smallest optimal stopping time, D, of the problem $\tilde{V}_0$ in (2.37). Since $\tilde{V}_0 = V_0$, D is also the minimal stopping time for the game at time zero, and so, therefore, $\overline{D}_0$ is also minimal in the game. Thus, for any other optimal stopping time σ^*, it must be true that $\overline{D}_0 \leq \sigma^*$ on $\{\overline{D}_0 < \overline{D}'_0\}$. The same holds for proving (2.48).

Equations (2.49) and (2.50) have the same proof method, so we only present the case for (2.49). Theorem 2.5.1 tells us that $(\overline{D}'_0, \overline{D}_0)$ is a Nash equilibrium for the game, couple this with Lemma 2.6.1 and we see that $(\tau^*, \overline{D}_0)$ is also a Nash equilibrium, i.e. they are best responses to each other. We introduce the single-agent problem, $\hat{V}_0$ as follows

$$\hat{V}_0 = \sup_{\eta \in \mathscr{T}} \mathbb{E}\big[f_{\tau^*} \mathbb{1}_{\{\tau^* \leq \eta\}} + g_\eta \mathbb{1}_{\{\eta < \tau^*\}}\big] =: \sup_{\eta \in \mathscr{T}} \mathbb{E}\big[\hat{Y}_\eta\big],$$

which is similar to (2.36). An important note to make is that since $\overline{D}_0$ and σ^* are optimal stopping times for the game V_0, they are also optimal for the single-agent problem $\hat{V}_0$.

We may follow the exact same reasoning found in the proof of Theorem 2.5.1 to yield a stopping time $\hat{D}$ which is the increasing limit of the respective approximating sequence $\hat{D}^\lambda$, and is also the debut of the set $\{(t,\omega) : \hat{V}_t(\omega) = \hat{Y}_t(\omega)\}$. Specifically, we know that $\hat{D} = \inf\{t \geq 0 : \hat{V}_t =$

$g_t\}$ on the set $\{\hat{D} < \tau^*\}$. Also, in much the same way, we can say that $\hat{V}_{\hat{D}} = \hat{Y}_{\hat{D}}$, implying that (see (2.40) for relevant arguments), on the set $\{\hat{D} < \tau^*\}$, we have $\hat{V}_{\hat{D}} = g_{\hat{D}}$.

Recall that $\overline{D}_0 \leq D_0$ (see part three of the proof of Theorem 2.5.1). Since it is true that $V_\rho \leq \hat{V}_\rho$ for all $\rho \in \mathscr{T}$, we must have that $D_0 \leq \hat{D}$ on the set $\{\hat{D} < \tau^*\}$. From classical optimal stopping (see Corollary 1.3.15), we know that $\hat{D}$ is the minimal stopping time for the $\hat{V}_0$ problem, meaning that $\hat{D} \leq \overline{D}_0$. The chains of inequalities in this paragraph yield

$$\hat{D} = \overline{D}_0 = D_0 \quad \text{on } \{\hat{D} < \tau^*\}. \tag{2.51}$$

Moreover, since $\hat{D} \leq \overline{D}_0$ in general—since $\hat{D}$ is the minimal optimal stopping time for $\hat{V}_0$—the inclusion $\{\hat{D} < \tau^*\} \supset \{\overline{D}_0 < \tau^*\}$ holds, thus (2.51) holds specifically on the smaller set

$$\hat{D} = \overline{D}_0 = D_0 \quad \text{on } \{\overline{D}_0 < \tau^*\}.$$

Clearly, both $\overline{D}_0$ and σ^* are best responses to τ^* in $\hat{V}_0$ as a result of Lemma 2.6.1, but on $\{\overline{D}_0 < \tau^*\}$ the stopping time $\overline{D}_0$ is minimal, hence

$$\overline{D}_0 \leq \sigma^* \quad \text{on } \{\overline{D}_0 < \tau^*\}.$$

■

Theorem 2.6.3. *Let $(\overline{D}'_0, \overline{D}_0)$ be the Nash equilibrium from Theorem 2.5.1, and let (τ^*, σ^*) be any other Nash equilibrium of the game, then*

$$\tau^* \wedge \sigma^* \geq \overline{D}_0 \wedge \overline{D}'_0.$$

Proof. In this proof we will use terminology from the proof of Theorem 2.5.1, notably that $\tilde{V}_0$ is a single-agent problem which coincides with V_0. $(\tilde{Y}_t)_{t\geq 0}$ is defined as $\tilde{Y}_t := f_{\overline{D}'_0} \mathbb{1}_{\{\overline{D}'_0 \leq t\}} + g_t \mathbb{1}_{\{t < \overline{D}'_0\}}$, and then the problem is given by

$$\tilde{V}_0 = \sup_{\eta \in \mathscr{T}} \mathbb{E}\left[f_{\overline{D}'_0} \mathbb{1}_{\{\overline{D}'_0 \leq \eta\}} + g_\eta \mathbb{1}_{\{\eta < \overline{D}'_0\}} \right].$$

This single-agent problem has a minimal optimal stopping time given by $D = \inf\{t \geq 0 : \tilde{V}_t = \tilde{Y}_t\}$, where $(\tilde{V}_t)_{t\geq 0}$ is the Snell envelope of $(\tilde{Y}_t)_{t\geq 0}$ (see the proof of Theorem 2.5.1 for full details).

Since both $(\overline{D}_0', \overline{D}_0)$ and (τ^*, σ^*) are Nash equilibria, Lemma 2.6.1 tells us that we can interchange them, specifically $(\overline{D}_0', \sigma^*)$ is a Nash equilibrium. This means that both $\overline{D}_0$ and σ^* are best responses to $\overline{D}_0'$, and as a result are solutions to $\tilde{V}_0$ too.

Akin to how we argued at the beginning of the proof of Theorem 2.6.2, we already know from (2.43) that $D = \overline{D}_0$ on $\{D < \overline{D}_0'\}$. Since D (and therefore $\overline{D}_0$) is minimal for the game V_0, we must have that $\sigma^* \geq \overline{D}_0$. If, however, we are on $\{D \geq \overline{D}_0'\}$, then D is still minimal for the game V_0, i.e. $\sigma^* \geq D$, and, therefore, $\sigma^* \geq \overline{D}_0'$. Hence $\sigma^* \geq \overline{D}_0$ on $\{D < \overline{D}_0'\}$ (i.e. $\overline{D}_0 = D < \overline{D}_0'$), and $\sigma^* \geq \overline{D}_0'$ on $\{D \geq \overline{D}_0'\}$ (i.e. $\overline{D}_0 \geq D \geq \overline{D}_0'$), meaning that

$$\sigma^* \geq \overline{D}_0 \wedge \overline{D}_0'.$$

This argument can be made, *mutatis mutandis*, to show that $\tau^* \geq \overline{D}_0 \wedge \overline{D}_0'$. Therefore the result follows immediately. ■

* * *

CONCLUDING REMARKS

As we have mentioned previously, this chapter contains some extremely important ideas which we will build upon in the subsequent chapters. A critical result from this Chapter is the sub- and supermartingale results in Theorem 2.4.4. This result captures the dynamics of the value process, behaving like a submartingale until the maximiser's optimal stopping time, and likewise, behaving as a supermartingale until the minimiser's optimal stopping time. Of course the major result in this chapter is Theorem 2.5.1 which allows us to characterise the optimal strategies of both players on certain sets. While such an explicit finding will not be possible for us later, the results that we find in Section 4.4 will share similarities to the form of these strategies $(\overline{D}_0, \overline{D}_0')$. Another important result to note is that we prove in Corollary 2.4.2 that there exists a value at all stopping times ρ. This result is present in the original paper by Lepeltier and Maingueneau [LM84], however, they only use it as a means to an end to find the value at time zero, as is customary. The existence of a value at any stopping time will be at the centre of our work, allowing us to construct sub-game strategies (see Section 4.2) and formulate martingales (see Sections 4.1 and 4.3).

We must also stress that this chapter was not simply a translation of the paper [LM84] into English. The work in the preceding pages both expands upon and extends the original research. The discussion on the link between previsibly left upper-semicontinuous trajectories and being upper-semicontinuous in expectation was all but absent in the original paper. This is most prevalent in Section 2.5 where we have had to drastically alter the proof of Theorem 2.5.1 as we could not find a way to verify their reasoning. It becomes painfully obvious during the reading of Section 2.5 that the nuances of the chosen left-regularity, and how this affects the way we prove the form of the optimal stopping times, are highly technical and that the additional work we have done in studying this has led to a more rigorous conclusion.

Section 2.6 is entirely novel work and provides the most opportunity for further study. The original idea for this came from work done under the formulation of later chapters, where we wanted to consider the payoff of the game when one player used the minimum of two equilibrium strategies. Naturally this lead to us wondering if it was possible to consider a 'minimal' equilibrium at all, and so we pursued this question in the original formulation of Lepeltier and Maingueneau. Theorem 2.6.3 gives us minimality in the sense that either $\overline{D}_0$ or $\overline{D}'_0$ will be the first optimal stopping time and Theorem 2.6.2 gives us minimality on an individual level, but only subject to the condition of relative minimality. We believe there is much more to explore in this area, let alone the implications this has on the more general games studied in the following chapters.

PART II

GAMES WITH ASYMMETRIC INFORMATION

CHAPTER 3

FUNDAMENTALS OF THE PROBLEM

This chapter is an introduction to the more general set-up of Dynkin games, around which the remaining work will be focused. The problem that we will address is derived from a more general formulation; here we will introduce both problems and describe how ours emerges from the wider one. Moreover, we will then discuss the surrounding literature on the topic, drawing parallels to existing works, and discussing the purpose of this research and its place within the body of literature. Finally, we will close this chapter with a generalisation of some work done in [DAMP22], wherein we show the existence of a value at any point during the game, rather than just at time zero. At the end of this chapter, the groundwork will have been set in order to present our key results, namely necessary and sufficient conditions for the existence of a value and Nash equilibria.

3.1 RANDOMISED STOPPING TIMES

As we will soon see, the main tools that we require to handle games with information asymmetry are 'randomised stopping times'. There are many instances of games where a value in pure stopping times fails, one instance of this is with the introduction of information asymmetry, and to combat this we are forced to widen our scope of possible strategy sets in order to obtain a value. In classical game theory players can choose from two types of strategies: pure and mixed, where the latter is simply a probability distribution over the former.

In Dynkin games the nomenclature of these strategies is somewhat non-standard, however, one frequently sees two main names: 'mixed strategies' and 'randomised strategies'. Occasionally one will see the name 'behavioural strategies', and all three have distinct game theoretical definitions outside of Dynkin games (see [MSZ13]).

In their paper, [TV02], Touzi and Vieille describe the move to randomised strategies (what they call 'mixed strategies') as 'convexifying' the set of stopping times, or 'defining a measurable structure on the set of stopping times'. Thankfully, there is little reason to be caught up in the differences between the different definitions of such 'convexified' sets of stopping times because all three of the classical formulations are equivalent, as shown in [TV02]. In this section we will introduce how we define randomised stopping times (deviating from the naming conventions of [TV02]), before demonstrating the equivalence of the different definitions. This will not only be important for our understanding of a core concept in information asymmetry, but will also allow us to make connections to work outside of our own. For instance, we will undertake a detailed look at the work done by Grün in her paper [Grü13], as our own set-up is similar to hers, however, she utilises a different formulation of randomised stopping times.

We begin with our definition of a randomised stopping time. Let T be a deterministic time-horizon. Given a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$ satisfying the usual conditions and a sub-filtration $(\mathscr{G}_t) \subset (\mathscr{F}_t)$, let

$$\begin{aligned}\mathscr{A}(\mathscr{G}_t) := \{(\chi_t)_{t\in[0,T]} : (\chi_t)_{t\in[0,T]} \text{ is } (\mathscr{G}_t)\text{-adapted with } t \mapsto \chi_t(\omega) \text{ càdlàg,}\\ \text{non-decreasing, } \chi_{0-}(\omega) = 0 \text{ and } \chi_T(\omega) = 1 \text{ for all } \omega \in \Omega\}. \quad (3.1)\end{aligned}$$

We often omit the full notation $(\chi_t)_{t\in[0,T]} \in \mathscr{A}(\mathscr{G}_t)$ in favour of $\chi \in \mathscr{A}(\mathscr{G}_t)$ for convenience.

One can see why we could think of these as randomised cumulative distribution functions as they are non-decreasing and begin and end at zero and one respectively. We can now consider the full definition of a randomised stopping time.

Definition 3.1.1. Let $(\mathscr{G}_t) \subset (\mathscr{F}_t)$ be a sub-filtration. A random variable η is called a $(\mathscr{G}_t)$-randomised stopping time if there exists a random variable Z with uniform distribution $U(0,1)$, independent of $\mathscr{F}_T$, and a process $\chi \in \mathscr{A}(\mathscr{G}_t)$ such that

$$\eta = \eta(\chi, Z) := \inf\{t \in [0,T] : \chi_t > Z\}, \quad \mathbb{P}\text{-a.s.} \tag{3.2}$$

The random variable Z is called a randomisation device for the randomised stopping time η, and the process $(\chi_t)_{t\in[0,T]}$ is called the generating process. The set of $(\mathscr{G}_t)$-randomised stopping times is denoted by $\mathscr{T}^R(\mathscr{G}_t)$. It is assumed that randomisation devices of different stopping times are independent.

Remark 3.1.2. Notice that for any given realisation of Z, say u, in $(0,1)$, the randomised stopping time $\eta(\chi,u)$ becomes a stopping time. Lemma II.74.3 from [RW94] tells us that the first entrance time of a right-continuous, $(\mathscr{G}_t)$-adapted process, $(\chi_t)_{t\in[0,T]}$, into the open set, $(u,+\infty)$, is a $(\mathscr{G}_{t+})$ stopping time. However, under the assumption of usual conditions, $(\mathscr{G}_{t+}) = (\mathscr{G}_t)$.

Proposition 3.1.3. *For a given and fixed $\chi \in \mathscr{A}(\mathscr{G}_t)$, the mapping $\eta(\omega,u) : \Omega \times (0,1) \to [0,T]$ given by $(\omega,u) \mapsto \eta(\omega,u) := \inf\{t \in [0,T] : \chi_t(\omega) > u\}$ is $\mathscr{F} \otimes \mathscr{B}(0,1)$-measurable.*

Proof. For any $s \in [0,T]$ let $A_s := \{(\omega,u) : \chi_s(\omega) > u\}$. Notice three things: first, $(A_s)_{s\in[0,T]}$ is increasing. For any $s_1 < s_2$, if $(\omega,u) \in A_{s_1}$ then $\chi_{s_1}(\omega) > u$, and by the fact that $(\chi_t)_{t\in[0,T]}$ is non-decreasing we have that $\chi_{s_2}(\omega) > u$, hence $A_{s_1} \subset A_{s_2}$. Second, if $s < \eta(\omega,u)$ for any pair (ω,u), then $A_s = \emptyset$. Third, for any $s \in [0,T]$, the function $s \mapsto F_s(\omega,u)$, where $F_s(\omega,u) : [0,T] \times \Omega \times (0,1) \to (-1,1)$ is defined as $F_s(\omega,u) := \chi_s(\omega) - u$, is $(\mathscr{F} \otimes \mathscr{B}(0,1))$-measurable, hence the set of all (ω,u) such that $F_s(\omega,u) > 0$ is also $(\mathscr{F} \otimes \mathscr{B}(0,1))$-measurable.

We claim the following equality holds. For a given t in $[0,T]$,

$$\{(\omega,u) : \eta(\omega,u) < t\} = \bigcup_{\substack{s\in\mathbb{Q} \\ s<t}} A_s. \tag{3.3}$$

Notice that we choose s to be rational, this is because $(A_s)_{s\in[0,T]}$ is increasing. If s were irrational, and such that $\eta(\omega,u) < s < t$, then, since $\eta(\omega,u) < s$, we know that $\chi_s(\omega) > u$, and so $(\omega,u) \in A_s \subset A_{\bar{s}}$ where $\bar{s}$ is a rational between s and t.

To justify (3.3), first take a pair (ω,u) belonging to the left-hand side set, then since η is the infimum of all times t such that $\chi_t(\omega) > u$, there must exist (by the fact that $(\chi_t)_{t\in[0,T]}$ is non-decreasing) an $s \in \mathbb{Q}$ such that $\eta(\omega,u) < s < t$ and $\chi_s(\omega) > u$. Therefore, $(\omega,u) \in A_s$. For the opposite inclusion, take (ω,u) in the union on the right-hand side. There must exist, at least one $s \in \mathbb{Q}$ such that $(\omega,u) \in A_s$ and so $\chi_s(\omega) > u$. Thus, s belongs to the set $\{r \in [0,T] : \chi_r(\omega) > u\}$ and so $\eta(\omega,u) \leq s < t$ and the proposed equality is proven.

Finally, the countable union of the $(\mathscr{F} \otimes \mathscr{B}(0,1))$-measurable sets A_s is itself $(\mathscr{F} \otimes \mathscr{B}(0,1))$-measurable. The codomain $[0,T]$ is equipped with the Borel σ-algebra, which is generated by sets of the form $[0,b)$ for any $b \in [0,T]$, therefore, the mapping $(\omega,u) \mapsto \eta(\omega,u)$ is $(\mathscr{F} \otimes \mathscr{B}(0,1))$-measurable if the set $\{(\omega,u) : \eta(\omega,u) < t\}$ is $(\mathscr{F} \otimes \mathscr{B}(0,1))$-measurable for every t in $[0,T]$, and hence we conclude. ■

Going forward, we will no longer introduce the filtrations we are working with; $(\mathscr{G}_t)$ is always assumed to be an arbitrary sub-filtration of $(\mathscr{F}_t)$, and any other filtrations will be written explicitly.

It is simple to see that the set of stopping times are contained within the set of randomised stopping times, i.e. $\mathscr{T}(\mathscr{G}_t) \subset \mathscr{T}^R(\mathscr{G}_t)$. Let τ be a stopping time, then setting

$$\xi_t(\omega) = \mathbb{1}_{\{\tau \leq t\}}(\omega),$$

we see that $\xi \in \mathscr{A}(\mathscr{G}_t)$. Defining $\eta(\xi,Z)$ using (3.2) we see that it is now equivalent to

$$\eta = \inf\{t \in [0,T] : \xi_t > 0\}, \quad \mathbb{P}\text{-a.s.},$$

which, as $(\xi_t)_{t\in[0,T]}$ is clearly càdlàg, is equal to τ for each $\omega \in \Omega$. Therefore, stopping times are the trivial randomised stopping times.

This is only one of the common definitions of randomised stopping times. What follows will be an introduction to the two alternative definitions and a demonstration as to the equivalence of all three. We will be closely following the work done by Touzi and Vieille, [TV02].

We begin with 'mixed strategies'. We enlarge our probability space from $(\Omega, \mathscr{F}, \mathbb{P})$ to $(\Omega \times (0,1), \mathscr{F} \otimes \mathscr{B}(0,1), \mathbb{P} \times \lambda_1)$, where λ_1 is the Lebesgue measure on the open interval $(0,1)$. It is worthy of note that we are adapting many of these concepts in order to fit a problem with a finite time horizon. We define a mixed strategy as follows.

Definition 3.1.4. Let ϕ be an $(\mathscr{F} \otimes \mathscr{B}(0,1))$-measurable function mapping $\Omega \times (0,1)$ into $[0,T]$ such that, for λ_1-almost every u in $(0,1)$,

$$\sigma_u := \phi(\cdot, u)$$

is a $(\mathscr{G}_t)$-stopping time in $[0,T]$. Denote by $\Phi(\mathscr{G}_t)$ the space of mixed strategies.

This definition only covers a single player using mixed strategies, but if we want a second player too we will need a second, independent, copy $((0,1), \mathscr{B}(0,1), \lambda_2)$ of the probability space $((0,1), \mathscr{B}(0,1), \lambda_1)$. Hence we will be left with a working probability space of $(\Omega \times (0,1) \times (0,1), \mathscr{F} \otimes \mathscr{B}(0,1) \otimes \mathscr{B}(0,1), \mathbb{P} \times \lambda_1 \times \lambda_2)$. The following proposition—a modified version of Proposition 7.1, [TV02]—shows the link between these two classes of randomised stopping times.

Proposition 3.1.5. *There exists a surjective mapping H from $\Phi(\mathscr{G}_t)$ to $\mathscr{A}(\mathscr{G}_t)$.*

Proof. We construct the map $H : \Phi(\mathscr{G}_t) \to \mathscr{A}(\mathscr{G}_t)$ as follows. For any $\phi \in \Phi(\mathscr{G}_t)$, define

$$H_t(\phi)(\omega) = \int_0^1 \mathbb{1}_{\{\sigma_u \leq t\}}(\omega)\, du \quad \text{for } t \in [0,T], \qquad H_{0-}(\phi)(\omega) := 0. \tag{3.4}$$

Clearly, for λ_1-almost every u and for every ω:

i) H_t is non-decreasing, since if $t_1 \leq t_2$, then $\mathbb{1}_{\{\sigma_u \leq t_1\}} \leq \mathbb{1}_{\{\sigma_u \leq t_2\}}$,

ii) H_t is also right-continuous by the dominated convergence theorem and the fact that the mapping $t \mapsto \mathbb{1}_{\{\sigma_u \leq t\}}$ is right-continuous,

iii) $H_T = 1$ since $\sigma_u \leq T$,

iv) $H_{0-} = 0$ by definition,

v) $(H_t)_{t \in [0,T]}$ is adapted since σ_u is a stopping time.

Therefore, $(H_t(\phi))_{t \in [0,T]}$ belongs to $\mathscr{A}(\mathscr{G}_t)$.

To prove that the map H is surjective we define

$$\phi^\chi(\omega, u) := \inf\{s \geq 0 : \chi_s(\omega) > u\} \quad \text{for } \chi \in \mathscr{A}(\mathscr{G}_t).$$

Proposition 3.1.3 tells us that $(\omega,u)\mapsto\phi^\chi(\omega,u)$ is $(\mathscr{F}\otimes\mathscr{B}(0,1))$-measurable, and Remark 3.1.2 tells us that $\phi^\chi(\omega,u)$ is a stopping time for any ω and a fixed u, since $(\chi_t)_{t\in[0,T]}$ is right-continuous and adapted. Thus ϕ^χ belongs to $\Phi(\mathscr{G}_t)$.

Denote by $\sigma_u(\cdot):=\phi^\chi(\cdot,u)$, then, for any $t\in[0,T]$,

$$H_t(\phi^\chi)(\omega)=\int_0^1 \mathbb{1}_{\{\sigma_u\leq t\}}(\omega)\,du=\int_0^1 \mathbb{1}_{\{\chi_t\geq u\}}(\omega)\,du,$$

since $(\chi_t)_{t\in[0,T]}$ is non-decreasing and right-continuous (thus $\chi_{\sigma_u}\geq u$), and therefore $H_t(\phi^\chi)(\omega)=\chi_t(\omega)$. Since $(\chi_t)_{t\in[0,T]}$ was arbitrary, and $\phi^\chi\in\Phi(\mathscr{G}_t)$, the result follows. ■

Remark 3.1.6. Proposition 3.1.5 gives a surjection, but not a bijection as we would like. However, it is possible to create a bijection if we alter Definition 3.1.4. Take $H:\Phi(\mathscr{G}_t)\to\mathscr{A}(\mathscr{G}_t)$ as defined in (3.4), and say that two elements ϕ and ψ from $\Phi(\mathscr{G}_t)$ are equivalent if they both map to the same element of $\mathscr{A}(\mathscr{G}_t)$ under H. $\hat{\Phi}(\mathscr{G}_t)$ is then defined as the quotient space of $\Phi(\mathscr{G}_t)$ under this equivalence relation, and this should be the proper definition of the set of mixed strategies.

As already mentioned, these two definitions of randomised stopping times are not the only ones. In 1979, Bismut introduced randomised stopping times through a third method, [Bis79b]. This formulation is less commonly used; however, it makes use of a functional analytic definition, and there is a clear bijection between this definition and ours, hence why it is worth introducing.

Let E be the space of optional processes $(Y_t)_{t\in[0,T]}$ which are right-continuous on $[0,T)$, with left limits on $(0,T]$, and are such that

$$\|Y_t\|_E:=\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|\right]<\infty. \tag{3.5}$$

E is a Banach space with the norm $\|\cdot\|_E$. We denote by $E^\star$ the dual space of E (i.e. the space of all linear functionals on E which are continuous with respect to $\|\cdot\|_E$). We write $(Y_t^-)_{t\in[0,T]}$ for the left limit process of $(Y_t)_{t\in[0,T]}$, i.e. $Y_t^-=\lim_{s\uparrow\uparrow t}Y_s$.

Proposition 3.1.7. *(Proposition 1.3, [Bis79b]) Let μ be an element of $E^\star$. There exists two adapted, right-continuous processes of finite variation $(A_t)_{t\in[0,T]}$ and $(B_t)_{t\in[0,T]}$ taking values in $\mathbb{R}\cup\{\infty\}$ such that*

i) $(A_t)_{t\in[0,T]}$ *is not necessarily zero at* 0, *ii)* $(B_t)_{t\in[0,T]}$ *is zero at* 0*, previsible, and is a pure jump process with countably many jumps at countably many times.*

For any $(Y_t)_{t\in[0,T]}$ *in* E *we have the natural pairing:*

$$\langle \mu, Y\rangle = \mathbb{E}\left(\int_0^T Y_t\, dA + \int_0^T Y_t^-\, dB\right). \tag{3.6}$$

The representation in (3.6) is unique. For μ *to be non-negative, it is necessary and sufficient for* $(A_t)_{t\in[0,T]}$ *and* $(B_t)_{t\in[0,T]}$ *to be non-decreasing.*

Incidentally, Proposition 3.1.7 is a nice use of the Hahn-Banach theorem. Utilising a theorem from Meyer (Theorem 27, [Mey76]), which gives the same decomposition (3.6) for càdlàg processes satisfying (3.5), one uses the fact that these processes are dense in E, and so we have the extension of (3.6) to E.

Definition 3.1.8. (Definition 1.3, [Bis79b]) K' is the set of non-negative $\mu \in E^\star$ such that $A_T + B_T = 1$. K is the set of elements of K' such that $B = 0$. K is the set of randomised stopping times.

Proposition 3.1.9. *There exists a bijective map from* K *to* $\mathscr{A}(\mathscr{G}_t)$.

Proof. By Proposition 3.1.7, any element μ of K has the form

$$\langle \mu, Y\rangle = \mathbb{E}\left[\int_0^T Y_t\, dA_t\right],$$

for any $(Y_t)_{t\in[0,T]}$ belonging to E and where $(A_t)_{t\in[0,T]}$ is unique. We define the map $G : K \to \mathscr{A}(\mathscr{G}_t)$ as $\mu \mapsto (A_t)_{t\in[0,T]}$.

To prove the map is surjective, let $(\chi_t)_{t\in[0,T]}$ be a given element of $\mathscr{A}(\mathscr{G}_t)$, then it is easy to see that the mapping

$$\hat{\mu} : Y_t \mapsto \mathbb{E}\left[\int_0^T Y_t\, d\chi_t\right],$$

belongs to $E^\star$ since it is a continuous linear functional of elements of E. Proposition 3.1.7 tells us that this mapping $\hat{\mu}$ has a unique representation (3.6), hence we have $A_t = \chi_t$ and $B_t = 0$ for all $t \in [0,T]$. Since $(\chi_t)_{t\in[0,T]}$ is non-decreasing, we know that $\hat{\mu}$ is non-negative, and so coupled with the fact that $\chi_T = 1$ we have that $\hat{\mu}$ belongs to K, and $G(\hat{\mu}) = \chi_t$.

To prove that the map is bijective, take any two elements of K, μ^1 and μ^2. Proposition 3.1.7 gives us the unique decompositions with processes $(\chi_t^1)_{t\in[0,T]}$ and $(\chi_t^2)_{t\in[0,T]}$ and such that

$G(\mu^i) = \chi_t^i$ for $i \in \{1,2\}$. If $\chi_t^1 = \chi_t^2$ $\mathbb{P}$-almost surely, then by uniqueness of the representation we must have $\mu^1 = \mu^2$. ■

The results given here allow us to be comfortable with the alternate definitions for randomised stopping times, or 'mixed strategies', since they are all equivalent. As we have mentioned, we will be using Definition 3.1.1 for our randomised stopping times, however, we will soon consider research which utilised the 'mixed strategy' definition, but we can now confidently compare our work to this knowing the equivalence of our sets of strategies.

Before moving on, we make a generalisation of Definition 3.1.1. We have only been considering games starting from time zero here, and we will want to be able to analyse those games starting from some stopping time ρ in $[0,T]$. This coincides with the concept of subgames and will be very important for determining relevant super- and submartingales to characterise our Nash equilibria. The extension of the definition is somewhat trivial. Given an $(\mathscr{F}_t)$-stopping time ρ taking values in $[0,T]$, we consider a new set, $\mathscr{A}_\rho(\mathscr{G}_t)$, for a given sub-filtration $(\mathscr{G}_t) \subset (\mathscr{F}_t)$ as follows

$$\mathscr{A}_\rho(\mathscr{G}_t) := \big\{(\chi_t)_{t\in[0,T]} \in \mathscr{A}(\mathscr{G}_t) : \chi_{\rho-}(\omega) = 0 \text{ for all } \omega \in \Omega\big\}. \tag{3.7}$$

These are simply those generating processes which still have left limit of zero at ρ. Notice that choosing $\rho = 0$ yields the original definition of $\mathscr{A}(\mathscr{G}_t)$, and that $\mathscr{A}_\rho(\mathscr{G}_t) \subset \mathscr{A}(\mathscr{G}_t)$.

Definition 3.1.10. Let $(\mathscr{G}_t) \subset (\mathscr{F}_t)$ be a sub-filtration and let ρ be a $(\mathscr{G}_t)$-stopping time. A random variable η is called a $(\mathscr{G}_t)$-randomised stopping time after ρ if there exists a random variable Z with uniform distribution $U(0,1)$, independent of $\mathscr{F}_T$, and a process $\chi \in \mathscr{A}_\rho(\mathscr{G}_t)$ such that

$$\eta = \eta(\chi, Z) := \inf\{t \in [\rho,T] : \chi_t > Z\}, \quad \mathbb{P}\text{-a.s.}$$

The random variable Z is called a randomisation device for the randomised stopping time η, and the process $(\chi_t)_{t\in[0,T]}$ is called the generating process. The set of $(\mathscr{G}_t)$-randomised stopping times after ρ is denoted by $\mathscr{T}_\rho^R(\mathscr{G}_t)$. It is assumed that randomisation devices of different stopping times are independent.

Remark 3.1.11. We often say that a $(\mathscr{G}_t)$-randomised stopping time τ is admissible at time ρ if it belongs to $\mathscr{T}_\rho^R(\mathscr{G}_t)$.

3.2 THE GENERAL FORMULATION

In the previous chapter we have seen the most general result for non-Markovian Dynkin games with pure strategies (stopping times), and this leads us to the main topic of this work, namely non-Markovian Dynkin games with asymmetric information.

In classical (economic) game theory, letting players use mixed strategies allows us to find values and Nash equilibria in a wider collection of games, and the same is also true for stochastic games. Additionally, another generalisation we can make is to allow the players to know differing amounts of information about the game itself. In fact, both of these concepts go hand-in-hand; mixed strategies are often necessary for the value of the game to exist [DAMP22], and can lead to better payoffs than when players play naively [Grü13].

The foundations for zero-sum games of partial and asymmetric information were laid out in the 2022 paper by De Angelis, Merkulov and Palczewski, [DAMP22], which showed the existence of a value and a Nash equilibrium under very general assumptions. We begin by presenting their problem, and will follow this with our own.

Let $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in[0,T]})$ be a complete filtered probability space, where $T \in (0,\infty]$ is the time horizon of the problem. Denote by $\mathscr{L}_b$ the Banach space of all càdlàg, $(\mathscr{B}([0,T]) \otimes \mathscr{F})$-measurable processes with the norm

$$\|X\|_{\mathscr{L}_b} := \mathbb{E}\left[\sup_{t\in[0,T]} |X_t|\right] < \infty.$$

We say that a stochastic process $(X_t)_{t\in[0,T]}$ is regular if

$$\mathbb{E}[X_\eta - X_{\eta-} | \mathscr{F}_{\eta-}] = 0 \quad \mathbb{P}\text{-a.s.}$$

for all previsible $(\mathscr{F}_t)$-stopping times η. Natural questions arise from the inclusion of an infinite time horizon, specifically on the definitions of càdlàg processes and of regularity at infinity. These are addressed in [DAMP22] and in Chapter 1 (see Theorem 1.3.1 and the discussion following). We will be taking a finite time horizon for this work.

As in Chapter 2, we will be studying two-player Dynkin games on the interval $[0,T]$. The presence of partial and asymmetric information is found in the filtrations of the two players. Consider two sub-filtrations $(\mathscr{F}_t^1) \subset (\mathscr{F}_t)$, and $(\mathscr{F}_t^2) \subset (\mathscr{F}_t)$, both of which satisfy the usual conditions, whereby the former is the filtration containing the information known by Player 1, and the latter containing the information known to Player 2. Different restrictions on these filtrations can generate different set-ups of information asymmetry. In the trivial case, where $(\mathscr{F}_t^1) = (\mathscr{F}_t) = (\mathscr{F}_t^2)$, the game has full information and becomes the framework of Chapter 2. If we have $(\mathscr{F}_t^1) = (\mathscr{F}_t^2)$, but $(\mathscr{F}_t^1) \neq (\mathscr{F}_t)$, then there is partial, but symmetric information, i.e. both players have less information than exists about the game, but they both have access to the same information. However, if we impose that $(\mathscr{F}_t^1) \neq (\mathscr{F}_t^2) \neq (\mathscr{F}_t)$, then the game has both partial and asymmetric information, in other words, neither player is fully informed, but have access to two different streams of information. For example, if in fact $(\mathscr{F}_t^1) = (\mathscr{F}_t)$ and $(\mathscr{F}_t^1) \neq (\mathscr{F}_t^2)$, then Player 1 is fully informed, while Player 2 is 'uninformed', or if $(\mathscr{F}_t^1) \neq (\mathscr{F}_t)$ and $(\mathscr{F}_t^1) \neq (\mathscr{F}_t^2)$, then there exists an asymmetry of information between the players, neither of whom are fully informed. Pure stopping times alone are no longer enough to guarantee the existence of a value and a Nash equilibrium in games of this type, however, as was shown by De Angelis, Merkulov and Palczewski [DAMP22], the extension of the strategy set to randomised stopping times is enough for existence.

At the onset of the game, Player 1 chooses $\tau \in \mathscr{T}_0^R(\mathscr{F}_t^1)$ and Player 2 chooses $\sigma \in \mathscr{T}_0^R(\mathscr{F}_t^2)$. The game finishes at $\tau \wedge \sigma \wedge T$, at which time a payment $P(\tau,\sigma)$ is made from Player 1 to Player 2, the form of which is as follows. Let $(f_t)_{t\in[0,T]}$, $(h_t)_{t\in[0,T]}$ and $(g_t)_{t\in[0,T]}$ be such that

Assumption 3.2.1.

i) $(f_t)_{t\in[0,T]}$ *and* $(g_t)_{t\in[0,T]}$ *belong to* $\mathscr{L}_b$*,*

ii) $(f_t)_{t\in[0,T]}$ *and* $(g_t)_{t\in[0,T]}$ *are* $(\mathscr{F}_t)$*-adapted regular processes,*

iii) $g_t \leq h_t \leq f_t$ *for all* $t \in [0,T]$*,* $\mathbb{P}$*-a.s.,*

iv) $(h_t)_{t\in[0,T]}$ *is an* $(\mathscr{F}_t)$*-adapted, measurable process.*

Notice that $(h_t)_{t\in[0,T]}$ is not assumed to be regular or càdlàg. The payoff of the game is then

$$P(\tau,\sigma) = f_\tau \mathbb{1}_{\{\tau<\sigma\}} + g_\sigma \mathbb{1}_{\{\tau>\sigma\}} + h_\tau \mathbb{1}_{\{\tau=\sigma\}}. \tag{3.8}$$

This payoff will look familiar (see the expected payoffs in (1.19) and (2.2)) as it is the normal, war of attrition, payoff for Dynkin games. In the traditional game, players interact with the game through the expected payoff

$$J(\tau,\sigma) = \mathbb{E}[P(\tau,\sigma)], \tag{3.9}$$

however, we can consider a game starting at some $(\mathscr{F}_t^1) \cap (\mathscr{F}_t^2)$-stopping time ρ, rather than at time zero. We call these 'subgames' and players pick randomised stopping times in the sets $\mathscr{T}_\rho^R(\mathscr{F}_t^1)$ and $\mathscr{T}_\rho^R(\mathscr{F}_t^2)$. The natural extension of the expected payoff is then the conditional expected payoff, conditional on any σ-algebra, $\mathscr{G}_\rho \subseteq \mathscr{F}_\rho^1 \cap \mathscr{F}_\rho^2$,

$$J_\rho(\tau,\sigma) = \mathbb{E}\big[P(\tau,\sigma)\big|\mathscr{G}_\rho\big].$$

We take the τ-player to be the minimiser, and the σ-player to be the maximiser, leading to two distinct objects

$$\underline{V}(\rho) = \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho^R(\mathscr{F}_t^2)} \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_\rho^R(\mathscr{F}_t^1)} J_\rho(\tau,\sigma), \qquad \overline{V}(\rho) = \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_\rho^R(\mathscr{F}_t^1)} \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho^R(\mathscr{F}_t^2)} J_\rho(\tau,\sigma), \tag{3.10}$$

which we call the lower and upper conditional values respectively. Due to the natural ordering of infima and suprema, these values are also ordered, namely $\underline{V}(\rho) \le \overline{V}(\rho)$, and we say that the (sub)game has a value if

$$\underline{V}(\rho) = \overline{V}(\rho) =: V(\rho).$$

The value of a game is also called a Stackelberg equilibrium in the literature, but, as we have seen, it is not the only type of equilibrium we are concerned about. The (sub)game is said to have a Nash equilibrium $(\tau^*,\sigma^*) \in \mathscr{T}_\rho^R(\mathscr{F}_t^1) \times \mathscr{T}_\rho^R(\mathscr{F}_t^2)$ if, for every pair of of randomised stopping times (τ,σ), the following holds

$$J_\rho(\tau^*,\sigma) \le J_\rho(\tau^*,\sigma^*) \le J_\rho(\tau,\sigma^*).$$

Remark 3.2.2. If a pair of strategies (τ^*,σ^*) are a Nash equilibrium, then we know from Definition 1.4.3 that they are best responses to each other, i.e.

$$V(\rho) = \operatorname*{ess\,sup}_{\sigma\in\mathscr{T}_\rho^R(\mathscr{F}_t^2)} J_\rho(\tau^*,\sigma) = \operatorname*{ess\,inf}_{\tau\in\mathscr{T}_\rho^R(\mathscr{F}_t^1)} J_\rho(\tau,\sigma^*).$$

From this it is simple to see that $V(\rho) = J_\rho(\tau^*, \sigma^*)$ because

$$J_\rho(\tau^*, \sigma^*) \leq \operatorname*{ess\,sup}_{\sigma \in \mathscr{T}^R_\rho(\mathscr{F}^2_t)} J_\rho(\tau^*, \sigma) = \operatorname*{ess\,inf}_{\tau \in \mathscr{T}^R_\rho(\mathscr{F}^1_t)} J_\rho(\tau, \sigma^*) \leq J_\rho(\tau^*, \sigma^*).$$

Remark 3.2.3. Akin to Remark 2.1 in [DAMP22], we notice that our game has a useful symmetry. Due to the lack of restrictions on the signs of the payoff processes $(f_t)_{t\in[0,T]}$, $(g_t)_{t\in[0,T]}$ and $(h_t)_{t\in[0,T]}$, if a value exists for the game with payoff $P(\tau, \sigma)$, then a value must also exist for the game with payoff $-P(\tau, \sigma)$. In this latter formulation, the roles of maximiser and minimiser are reversed since

$$\sup_\sigma \inf_\tau \mathbb{E}\big[P(\tau,\sigma)\big|\mathscr{G}_\rho\big] = -\inf_\sigma \sup_\tau \mathbb{E}\big[-P(\tau,\sigma)\big|\mathscr{G}_\rho\big].$$

The set-up described here is an extended version of that found in the paper by De Angelis, Merkulov and Palczewski, [DAMP22], and their original set-up is the case when we choose $\rho = 0$, i.e. we always look at the game from the beginning. They prove two important results (in fact they prove three, but one is a generalisation of another whereby they relax the assumptions on the processes $(f_t)_{t\in[0,T]}$, $(g_t)_{t\in[0,T]}$ and $(h_t)_{t\in[0,T]}$, which we will not need to consider here), namely that a value and a Nash equilibrium exist for the unconditional game, as well as when were consider an expected payoff conditional on a σ-algebra $\mathscr{G} \subset \mathscr{F}^1_0 \cap \mathscr{F}^2_0$.

As currently formulated, the analysis of this problem is hampered by the relative difficulty of working with randomised stopping times. The underlying generating processes of these randomised stopping times are far easier to handle as they are simply bounded, non-decreasing càdlàg processes with defined start and end points. Fortunately, it is possible to link these generating processes to the indicator functions of their respective randomised stopping times, and these results are crucial in creating an expected payoff which is simple to work with.

Lemma 3.2.4. *(Lemma 4.1, [DAMP22]) Let $\eta \in \mathscr{T}^R(\mathscr{F}_t)$ with generating process $(\xi_t)_{t\in[0,T]}$. Then, for any $\mathscr{F}_T$-measurable random variable κ with values in $[0,T]$,*

$$\mathbb{E}\big[\mathbb{1}_{\{\eta\leq\kappa\}}\big|\mathscr{F}_T\big] = \xi_\kappa, \qquad \mathbb{E}\big[\mathbb{1}_{\{\eta>\kappa\}}\big|\mathscr{F}_T\big] = 1-\xi_\kappa,$$
$$\mathbb{E}\big[\mathbb{1}_{\{\eta<\kappa\}}\big|\mathscr{F}_T\big] = \xi_{\kappa-}, \qquad \mathbb{E}\big[\mathbb{1}_{\{\eta\geq\kappa\}}\big|\mathscr{F}_T\big] = 1-\xi_{\kappa-}.$$

Remark 3.2.5. Lemma 3.2.4 opens up a new intuition on randomised stopping times; throughout this work we will use wording which hints towards an idea of gradual stopping. This is not strictly true, but gives us a vocabulary for discussing the evolution of generating processes. We say that a player with generating process $(\chi_t)_{t\in[0,T]}$ has 'stopped completely' at time t if $1-\chi_t = 0$, and if $1-\chi_t > 0$ then the player is said to have 'not stopped yet'. Of course, in practice, players have either stopped or they have not, but by considering the 'progress' of their generating processes, we can begin to talk about their actions more clearly.

It is relatively straightforward to see how this lemma is useful when paired with equations (3.8) and (3.9). We will skip the complete derivation here as it is done in [DAMP22], but we may use Lemma 3.2.4 to show that the expected payoff can be written as follows. Let $(\mathscr{G}_t)$ be filtration so that $\mathscr{G}_t \subseteq \mathscr{F}_t^1 \cap \mathscr{F}_t^2$ for all $t \in [0,T]$, let the generating processes for τ and σ be $(\xi_t)_{t\in[0,T]}$ and $(\zeta_t)_{t\in[0,T]}$ respectively, and then for any $(\mathscr{G}_t)$-stopping time ρ

$$J_\rho(\xi,\zeta) = \mathbb{E}\left[\int_{[\rho,T)} f_t(1-\zeta_t)\,d\xi_t + \int_{[\rho,T)} g_t(1-\xi_t)\,d\zeta_t + \sum_{t\in[\rho,T]} h_t\Delta\zeta_t\Delta\xi_t \,\middle|\, \mathscr{G}_\rho\right]. \tag{3.11}$$

There are two important things to notice with this formulation: first is the abuse of notation $J_\rho(\xi,\zeta)$, but the two families of random variables $J_\rho(\xi,\zeta)$ and $J_\rho(\tau,\sigma)$ are equivalent; and the second is that the integrals and summations are taken from ρ onwards. This need not be written this way as the strategies τ and σ are taken from $\mathscr{T}_\rho^R(\mathscr{F}_t^1)$ and $\mathscr{T}_\rho^R(\mathscr{F}_t^2)$ respectively, meaning that $(\xi_t)_{t\in[0,T]}$ and $(\zeta_t)_{t\in[0,T]}$ are taken from $\mathscr{A}_\rho(\mathscr{F}_t^1)$ and $\mathscr{A}_\rho(\mathscr{F}_t^1)$. In other words, the generating processes $(\xi_t)_{t\in[0,T]}$ and $(\zeta_t)_{t\in[0,T]}$ are zero on the interval $[0,\rho)$.

3.3 THE SPECIFIC PROBLEM

In this section we will introduce the problem which will be the focus of this research. It is a case of the wider problem introduced above, and is one which has already been considered by Grün in her 2013 paper [Grü13], but we will discuss this and more in the next section.

The genesis of this problem comes from adding an exogenous source of information asymmetry, namely a random variable which can take a finite number of values, the outcome of which is known only to one player. We say that this game has only asymmetric information as there is a difference between what the two players know, but the informed player will be completely informed. Information asymmetry of this kind necessitates the existence of a belief process, a stochastic process which will be the best guess of the uninformed player of the outcome of the random variable, based on the information available to him and the fact that the informed player has yet to stop.

Let us now formalise this set-up. First, we pick a finite time horizon T. Let Θ be an $(\mathscr{F}_t)$-random variable, taking values in $[\mathscr{I}] := \{0, 1, ..., \mathscr{I}\}$, with corresponding probability mass function, $\mathbb{P}(\Theta = i) = \pi_i$, for all $i \in [\mathscr{I}]$. As before, the way we formalise the information asymmetry is by using filtrations: the maximiser will have the least information, and their filtration will be denoted by $(\mathscr{F}_t^2) \subset (\mathscr{F}_t)$; the minimiser will know the outcome of Θ at the onset of the game, and so his filtration will be $(\mathscr{F}_t^1) \subset (\mathscr{F}_t)$, defined as $\mathscr{F}_t^1 := \mathscr{F}_t^2 \vee \sigma(\Theta)$.

Assumption 3.3.1. *The filtrations $(\mathscr{F}_t^1)_{t\in[0,T]}$ and $(\mathscr{F}_t^2)_{t\in[0,T]}$ are assumed to satisfy the usual conditions, and moreover, we take $\mathscr{F}_0^2$ to be the trivial σ-algebra, $\{\emptyset, \Omega\}$.*

Lemma 3.3.2 below is a slightly extended version of Lemma 3.1 found in [DAMP22]—itself taken from Lemma 3.3 in the paper by Esmaeeli and Imkeller [EI18]—and it tells us that instead of thinking of this game as a two-player game, we can see it as a $\mathscr{I}+1+1$ player game.

Lemma 3.3.2. *Let ρ be an $(\mathscr{F}_t^2)$-stopping time, then any $\tau \in \mathscr{T}_\rho^R(\mathscr{F}_t^1)$ has a representation*

$$\tau = \sum_{i=0}^{\mathscr{I}} \tau^i \mathbb{1}_{\{\Theta = i\}},$$

where $\tau^0, ..., \tau^{\mathscr{I}} \in \mathscr{T}_\rho^R(\mathscr{F}_t^2)$, with generating processes $\xi^1, ..., \xi^{\mathscr{I}} \in \mathscr{A}_\rho(\mathscr{F}_t^2)$ and a common randomisation device Z_τ.

It is easy to see that, since we have $\mathscr{I}+1$ minimisers, we have an $\mathscr{I}+1+1$ player game.

A crucial point of this observation is the fact that while the minimiser's strategy belongs to $\mathscr{T}_\rho^R(\mathscr{F}_t^1)$, its decomposition uses τ^i belonging to $\mathscr{T}_\rho^R(\mathscr{F}_t^2)$. This is saying something very intuitive, that given any fixed outcome of Θ, the minimiser is using no more information to play the game than the maximiser is. This decomposition will also be incredibly useful for navigating some measurability problems later.

Due to the imposition of a finite time horizon, we say that the game terminates at the minimum of the players' times, τ and σ, and the terminal time, i.e. at $\tau \wedge \sigma \wedge T$, and the minimiser will pay the maximiser the payoff

$$P(\tau, \sigma) = f_\tau^\Theta \mathbb{1}_{\{\tau < \sigma\}} + g_\sigma^\Theta \mathbb{1}_{\{\tau > \sigma\}} + h_\tau^\Theta \mathbb{1}_{\{\tau = \sigma\}}.$$

Naturally, this payoff resembles (3.8), however, our payoff processes are allowed to change depending on the outcome of Θ.

Assumption 3.3.3. *For all $i \in [\mathscr{I}]$, the payoff processes $(f_t^i)_{t\in[0,T]}$, $(g_t^i)_{t\in[0,T]}$, $(h_t^i)_{t\in[0,T]}$ are $(\mathscr{F}_t^2)$-adapted. The processes $(f_t^\Theta)_{t\in[0,T]}$, $(g_t^\Theta)_{t\in[0,T]}$ and $(h_t^\Theta)_{t\in[0,T]}$, defined as*

$$f_t^\Theta := \sum_{i=0}^{\mathscr{I}} f_t^i \mathbb{1}_{\{\Theta=i\}}, \quad h_t^\Theta := \sum_{i=0}^{\mathscr{I}} h_t^i \mathbb{1}_{\{\Theta=i\}}, \quad g_t^\Theta := \sum_{i=0}^{\mathscr{I}} g_t^i \mathbb{1}_{\{\Theta=i\}}, \tag{3.12}$$

for all $t \in [0,T]$, satisfy the assumptions in Assumption 3.2.1 (with $(\mathscr{F}_t) \equiv (\mathscr{F}_t^1)$), and they are taken to be continuous.

Remark 3.3.4. The processes defined in (3.12) are $(\mathscr{F}_t^1)$-adapted by construction.

The specific form of the expected payoff of the game is the first time we see how this example of information asymmetry manifests in the mathematics. Notice that $f_{\tau^i}^i$ is $\mathscr{F}_T^2$-measurable, but τ^i and σ are not since they are randomised, however, they are $(\mathscr{F}_T^2 \vee \sigma(Z_\tau) \vee \sigma(Z_\sigma))$-measurable, where Z_τ and Z_σ are the randomisation devices for the randomised stopping times τ and σ respectively. Therefore, we can deduce from Lemma 3.3.2 that

$$\begin{aligned}\mathbb{E}\left[f_\tau^\Theta \mathbb{1}_{\{\tau<\sigma\}}\right] &= \mathbb{E}\left[\mathbb{E}\left[\sum_{i=0}^{\mathscr{I}} f_{\tau^i}^i \mathbb{1}_{\{\tau^i<\sigma\}} \mathbb{1}_{\{\Theta=i\}} \middle| \mathscr{F}_T^2 \vee \sigma(Z_\tau) \vee \sigma(Z_\sigma)\right]\right] \\ &= \mathbb{E}\left[\sum_{i=0}^{\mathscr{I}} f_{\tau^i}^i \mathbb{1}_{\{\tau^i<\sigma\}} \mathbb{E}\left[\mathbb{1}_{\{\Theta=i\}} \middle| \mathscr{F}_T^2 \vee \sigma(Z_\tau) \vee \sigma(Z_\sigma)\right]\right].\end{aligned}$$

Since the random variable Θ is independent from the σ-algebra $\mathscr{F}_T^2 \vee \sigma(Z_\tau) \vee \sigma(Z_\sigma)$ for all $t \in [0,T]$, the conditional expectation becomes a normal expectation and we get

$$\mathbb{E}\left[f_\tau^\Theta \mathbb{1}_{\{\tau<\sigma\}}\right] = \mathbb{E}\left[\sum_{i=0}^{\mathscr{I}} f_{\tau^i}^i \mathbb{1}_{\{\tau^i<\sigma\}} \mathbb{E}\left[\mathbb{1}_{\{\Theta=i\}}\right]\right] = \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\left[f_{\tau^i}^i \mathbb{1}_{\{\tau^i<\sigma\}}\right],$$

where we have used that $\mathbb{E}\left[\mathbb{1}_{\{\Theta=i\}}\right] = \mathbb{P}(\Theta = i) = \pi_i$. Using this same logic for the remaining payoffs, we can arrive at the expected payoff

$$J(\tau,\sigma) = \mathbb{E}[P(\tau,\sigma)] = \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\left[f_{\tau^i}^i \mathbb{1}_{\{\tau^i<\sigma\}} + g_\sigma^i \mathbb{1}_{\{\tau^i>\sigma\}} + h_{\tau^i}^i \mathbb{1}_{\{\tau^i=\sigma\}}\right],$$

for any $(\tau,\sigma) \in \mathscr{T}^R(\mathscr{F}_t^1) \times \mathscr{T}^R(\mathscr{F}_t^2)$. We already know that it is possible to integrate out the randomisation devices of both players in the expected payoff using Lemma 3.2.4, leaving only the generating processes. The outcome is intuitive for this problem formulation and looks very similar to (3.11), however, we have the decomposition over values of Θ. As before, let $\xi \in \mathscr{A}(\mathscr{F}_t^1)$ and $\xi^i, \zeta \in \mathscr{A}(\mathscr{F}_t^2)$ be the generating processes for $\tau \in \mathscr{T}^R(\mathscr{F}_t^1)$ and $\tau^i, \sigma \in \mathscr{T}^R(\mathscr{F}_t^2)$ respectively, then the expected payoff is

$$J(\xi,\zeta) = \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\left[\int_{[0,T)} f_t^i(1-\zeta_t)\,d\xi_t^i + \int_{[0,T)} g_t^i(1-\xi_t^i)\,d\zeta_t + \sum_{t\in[0,T]} h_t^i \Delta\zeta_t \Delta\xi_t^i\right]. \tag{3.13}$$

Occasionally we will still refer to the payoff on its own, and so, with a slight abuse of notation, we denote it $P(\xi,\zeta)$

$$P(\xi,\zeta) := \sum_{i=0}^{\mathscr{I}} \pi_i \left(\int_{[0,T)} f_t^i(1-\zeta_t)\,d\xi_t^i + \int_{[0,T)} g_t^i(1-\xi_t^i)\,d\zeta_t + \sum_{t\in[0,T]} h_t^i \Delta\zeta_t \Delta\xi_t^i\right). \tag{3.14}$$

Of course, we then see that $J(\xi,\zeta) = \mathbb{E}[P(\xi,\zeta)]$. Moreover, we will also need to consider the payoff for the specific case of $\Theta = i$, and we denote this by

$$P^i(\xi^i,\zeta) := \int_{[0,T)} f_t^i(1-\zeta_t)\,d\xi_t^i + \int_{[0,T)} g_t^i(1-\xi_t^i)\,d\zeta_t + \sum_{t\in[0,T]} h_t^i \Delta\zeta_t \Delta\xi_t^i. \tag{3.15}$$

Remark 3.3.5. Note that, despite the similar notation, $P(\xi,\zeta) : \mathscr{A}(\mathscr{F}_t^1) \times \mathscr{A}(\mathscr{F}_t^2) \to \mathbb{R}$, however, $P^i(\xi^i,\zeta) : \mathscr{A}(\mathscr{F}_t^2) \times \mathscr{A}(\mathscr{F}_t^2) \to \mathbb{R}$.

The pair (ξ_t, ζ_t) corresponding to the pair (τ,σ) will be used throughout this work, and we will often refer to $(\xi_t)_{t\in[0,T]}$ and $(\zeta_t)_{t\in[0,T]}$ as strategies. Just like in Remark 3.1.11, we will say that a generating process is admissible if it belongs to the correct $\mathscr{A}(\cdot)$.

There is no change to our equilibrium concepts, so a value $V(0)$ exists if

$$V(0) := \inf_{\xi\in\mathscr{A}(\mathscr{F}_t^1)} \sup_{\zeta\in\mathscr{A}(\mathscr{F}_t^2)} J(\xi,\zeta) = \sup_{\zeta\in\mathscr{A}(\mathscr{F}_t^2)} \inf_{\xi\in\mathscr{A}(\mathscr{F}_t^1)} J(\xi,\zeta),$$

and a pair of strategies (ξ^*,ζ^*) is a Nash equilibrium if, for any other pair of strategies (ξ,ζ),

$$J(\xi^*,\zeta) \le J(\xi^*,\zeta^*) \le J(\xi,\zeta^*).$$

To take this idea further, however, we want to consider subgames. A Dynkin game is fundamentally played at time zero, this is where the strategies are decided and, in this setting, is when the information of the outcome of Θ is revealed to one player, however, one can imagine the original game starting at some non-zero time (assuming the game has not stopped by that point already). We already have a concept of randomised strategies for subgames, i.e. randomised stopping times after a stopping time ρ, so all that is left is to extend our understanding of the expected payoff.

In our formulations of $J(\xi,\zeta)$ in (3.13), we have seen the appearance of π_i. This comes around as an effect of splitting the payoff $P(\tau,\sigma)$ over the events $\{\Theta = i\}$ and using $\mathbb{P}(\Theta = i) = \pi_i$. However, starting the game at some stopping time ρ requires us to have an understanding of, intuitively speaking, what $\mathbb{P}(\Theta = i|\mathscr{F}^2_\rho \cap \{\tau > \rho\})$ looks like. This is one example of what we could fashion into a belief process $(\Pi^i_t)_{t\in[0,T]}$, the exact form of which will be defined later, but the idea is that this process should represent what probability the uniformed player assigns to the event $\{\Theta = i\}$ given that the informed player has not yet stopped the game.

Taking this belief process into account, we can then define the conditional expected payoff at a stopping time ρ in the following way (this will be done rigorously in Chapter 4). For any $(\xi,\zeta) \in \mathscr{A}_\rho(\mathscr{F}^1_t)\times\mathscr{A}_\rho(\mathscr{F}^2_t)$

$$J^\Pi_\rho(\xi,\zeta) := \sum_{i=0}^{\mathscr{I}} \Pi^i_{\rho-}\mathbb{E}\left[\int_{[\rho,T)} f^i_t(1-\zeta_t)\,d\xi^i_t + \int_{[\rho,T)} g^i_t(1-\xi^i_t)\,d\zeta_t + \sum_{t\in[\rho,T]} h^i_t\Delta\zeta_t\Delta\xi^i_t \middle| \mathscr{F}^2_\rho\right]. \tag{3.16}$$

There are two important things to notice. First is that we choose to condition with respect to $(\mathscr{F}^2_t)_{t\geq 0}$, and this can be understood by recalling the structure of the information asymmetry. This conditional payoff is the sum of all payoffs over all possible outcomes of Θ, where in each outcome i, both players use the common knowledge, only (as we will see in Chapter 4) the informed player sees the individual i^{th} term in the payoff, whereas the uninformed player only sees the sum total.

Secondly, the belief process is evaluated at 'time' $\rho-$. What we are saying here is that decisions are made at time ρ (based on the conditional expected payoff) before any jumps are seen in the payoff processes.

As one would expect, the same concept of a value exists for a subgame starting from ρ, however, since our expected payoff is now conditional—and hence a random variable—we must take essential suprema and infima. In other words, we say the subgame has a value if at any $(\mathscr{F}^2_t)$-

stopping time ρ we have

$$V^{\Pi}(\rho) = \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)} \operatorname*{ess\,inf}_{\xi \in \mathscr{A}_\rho(\mathscr{F}_t^1)} J_\rho^{\Pi}(\xi,\zeta) = \operatorname*{ess\,inf}_{\xi \in \mathscr{A}_\rho(\mathscr{F}_t^1)} \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)} J_\rho^{\Pi}(\xi,\zeta).$$

We are left with a family of random variables, $\{V^{\Pi}(\rho) : \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$, indexed by stopping times. This will look familiar from Chapter 2, and one might fairly expect that we should attempt to aggregate this family into a stochastic process, but this direct approach encounters some technical issues. Instead, we will construct a second family of random variables $\{M(\rho) : \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ which we can aggregate instead, then reason that this implies the aggregation of the value.

3.4 EXISTENCE OF A VALUE

In this section we present an extension of the work done by De Angelis, Merkulov and Palczewski, [DAMP22], and show that the value of a game, and a pair of optimal strategies, exist at every stopping time ρ in $[0,T]$. We will do this by essentially the same methodology, however, we will show that we can adapt some technical lemmas so that we can consider strategies from the set $\mathscr{A}_\rho$ rather than just $\mathscr{A}$ (with respect to the relevant filtrations). The main result used here, and in [DAMP22], is Sion's theorem ([Sio58], Corollary 3.3), which we recall here.

Theorem 3.4.1. *Let A and B be convex subsets of a linear topological space, one of which is compact. Let $\varphi(\mu,\nu)$ be a function $A \times B \mapsto \mathbb{R}$ that is quasi-concave and upper semi-continuous in μ for each $\nu \in B$, and quasi-convex and lower semi-continuous in ν for each $\mu \in A$. Then*

$$\sup_{\mu \in A} \inf_{\nu \in B} \varphi(\mu,\nu) = \inf_{\nu \in B} \sup_{\mu \in A} \varphi(\mu,\nu).$$

The work that follows requires some understanding of the technicalities of topology and functional analysis, and while we choose to not include them here, we have compiled them in Appendix B for completeness.

Our concern will be with the Banach space $\mathscr{S}$, given in [DAMP22], and defined as

$$\mathscr{S} := L^2([0,T] \times \Omega, \mathscr{B}([0,T]) \otimes \mathscr{F}, \lambda \times \mathbb{P}),$$

where λ is the Lebesgue measure on $[0,T]$. Another main focus for us will be the sets $\mathscr{A}$ and $\mathscr{A}_\rho$

for any $(\mathscr{F}_t^2)$-stopping time ρ (with relevant filtrations omitted from the notation). Of course these are subsets of $\mathscr{S}$—since generating processes are bounded—however, they are also contained inside a ball in $\mathscr{S}$. This is simple to see, take any two processes $(\chi_t^1)_{t\in[0,T]}$ and $(\chi_t^2)_{t\in[0,T]}$ belonging to $\mathscr{A}$, then

$$\|\chi^1 - \chi^2\|_{\mathscr{S}} = \mathbb{E}\left[\int_0^T |\chi_t^1 - \chi_t^2|^2 \, dt\right] \leq \mathbb{E}\left[\int_0^T 1 \, dt\right] = T.$$

Hence for a finite time horizon T, all generating processes lie inside an open ball in $\mathscr{S}$. (Notice that we do not lose any generality with the finite time horizon statement due to the one-point compactification argument).

For reasons we will see in Lemma 3.4.5, we need to introduce a subtle, yet important variation on these sets. Recall the definitions of $\mathscr{A}(\mathscr{G}_t)$ in (3.1) and $\mathscr{A}_\rho(\mathscr{G}_t)$ in (3.7), and notice that they hold for all $\omega \in \Omega$. Here we introduce two new sets, $\mathscr{A}^\circ(\mathscr{G}_t)$ and $\mathscr{A}_\rho^\circ(\mathscr{G}_t)$ defined as follows

$$\mathscr{A}^\circ(\mathscr{G}_t) := \{\chi \in \mathscr{S} : \exists \hat{\chi} \in \mathscr{A}(\mathscr{G}_t) \text{ such that } \chi_t = \hat{\chi}_t \text{ for } (\lambda \times \mathbb{P})\text{-a.e.}(t,\omega) \in [0,T] \times \Omega\}, \quad (3.17)$$

$$\mathscr{A}_\rho^\circ(\mathscr{G}_t) := \{\chi \in \mathscr{A}^\circ(\mathscr{G}_t) : \chi_{\rho-} = 0 \text{ for } (\lambda \times \mathbb{P})\text{-a.e.}(t,\omega) \in [0,T] \times \Omega\}. \quad (3.18)$$

Remark 3.4.2. When comparing our definitions to those of [DAMP22], the reader will find that the notation of $\mathscr{A}(\mathscr{G}_t)$ and $\mathscr{A}^\circ(\mathscr{G}_t)$ has been switched around here. We have chosen to do this as the sets defined in (3.17) and (3.18) are only used in this section, so we give them the circle notation to keep our notation simpler throughout the rest of the document.

Remark 3.4.3. The element $(\hat{\chi}_t)_{t\in[0,T]}$ in these definitions is said to be the 'càdlàg representative' of $(\chi_t)_{t\in[0,T]}$. Such representatives are not unique, but they are indistinguishable (see Lemma 5.6, [DAMP22]). Importantly, due to this indistinguishability, all càdlàg representatives define the same measure on $[0,T]$, and so the resulting Lebesgue-Stieltjes integral does not depend on the choice of representative.

Lemma 3.4.4. *Let $(\mathscr{G}_t)_{t\in[0,T]}$ be a filtration satisfying the usual conditions, then $\mathscr{A}_\rho^\circ(\mathscr{G}_t)$ is convex.*

Proof. Choose any two $(\chi_t^1)_{t\in[0,T]}$ and $(\chi_t^2)_{t\in[0,T]}$ belonging to $\mathscr{A}_\rho^\circ(\mathscr{G}_t)$ and let $\alpha \in [0,1]$. Define the process $\chi_t := \alpha\chi_t^1 + (1-\alpha)\chi_t^1$. Clearly $(\chi_t)_{t\in[0,T]}$ is adapted and càdlàg by construction, $\chi_{0-} = \alpha \cdot 0 + (1-\alpha)\cdot 0 = 0$, $\chi_T = 1$, and since both $(\chi_t^1)_{t\in[0,T]}$ and $(\chi_t^2)_{t\in[0,T]}$ are non-decreasing, so is $(\chi_t)_{t\in[0,T]}$ for any fixed $\alpha \in [0,1]$. Defining $(\hat{\chi}_t)_{t\in[0,T]}$ in the same manner as $(\chi_t)_{t\in[0,T]}$, using the càdlàg representatives of $(\chi_t^1)_{t\in[0,T]}$ and $(\chi_t^2)_{t\in[0,T]}$, we see first that $\hat{\chi} \in \mathscr{A}_\rho(\mathscr{G}_t)$, and secondly that $\chi_t = \hat{\chi}_t$ for $(\lambda \times \mathbb{P})$-almost every (t,ω) in $[0,T]\times\Omega$. Thus, $(\chi_t)_{t\in[0,T]}$ belongs to $\mathscr{A}_\rho^\circ(\mathscr{G}_t)$. ■

Lemma 3.4.5. *For any $(\mathscr{F}_t^2)$-stopping time ρ, and any filtration $(\mathscr{G}_t) \subseteq (\mathscr{F}_t)$ satisfying the usual conditions, the set $\mathscr{A}_\rho^\circ(\mathscr{G}_t)$ is weakly compact in $\mathscr{S}$.*

Proof. We will drop the notational dependence on the filtration $(\mathscr{G}_t)_{t\in[0,T]}$, and so we write $\mathscr{A}_\rho$ for $\mathscr{A}_\rho(\mathscr{G}_t)$, $\mathscr{A}$ for $\mathscr{A}(\mathscr{G}_t)$, and $\mathscr{A}_\rho^\circ$ for $\mathscr{A}_\rho^\circ(\mathscr{G}_t)$. The set $\mathscr{A}_\rho^\circ$ is a subset of $\mathscr{A}^\circ$, which itself is a subset of a ball in $\mathscr{S}$. Kakutani's theorem (Theorem 3.17, [Bre11]) says that since $\mathscr{S}$ is a reflexive Banach space, this ball is weakly compact, therefore, since any weakly closed subspace of a weakly compact space is weakly compact, we only need to show that $\mathscr{A}_\rho^\circ$ is weakly closed. Since $\mathscr{A}_\rho^\circ$ is convex by Lemma 3.4.4, it is enough to show that $\mathscr{A}_\rho^\circ$ is strongly closed ([Bre11], Theorem 3.7).

Take a sequence of generating processes $(\chi^n)_{n\in\mathbb{N}} \subset \mathscr{A}_\rho^\circ$ that converges strongly in $\mathscr{S}$ to $(\chi_t)_{t\in[0,T]}$. We will prove that $\chi \in \mathscr{A}_\rho^\circ$ by constructing a càdlàg, non-decreasing, and adapted process $(\hat{\chi}_t)_{t\in[0,T]}$ such that $\hat{\chi}_t = 0$ for all $t < \rho$, $\hat{\chi}_{\rho-} = 0$, $\hat{\chi}_T = 1$, and $\hat{\chi}_t(\omega) = \chi_t(\omega)$ $(\lambda \times \mathbb{P})$-almost everywhere. With no loss of generality we can pass to the càdlàg representatives $(\hat{\chi}^n)_{n\in\mathbb{N}} \subset \mathscr{A}_\rho$ which, since the norm on the space $\mathscr{S}$ is unaffected by changes on $(\lambda\times\mathbb{P})$-null sets, also converge to $(\chi_t)_{t\in[0,T]}$ in $\mathscr{S}$. Theorem 4.9, [Bre11], tells us that there is a subsequence $(n_k)_{k\geq 1}$ such that $\hat{\chi}_t^{n_k}(\omega) \to \chi_t(\omega)$ $(\lambda\times\mathbb{P})$-almost everywhere.

Define the following

$$\Omega_t := \left\{ \omega \in \Omega : \lim_{k\to\infty} \hat{\chi}_t^{n_k}(\omega) = \chi_t(\omega) \right\},$$

and $f(s,\omega) := \mathbb{1}_{A^c} : [0,T]\times\Omega \to \mathbb{R}$. Fubini's theorem guarantees the existence of a set $\hat{D} \subseteq [0,T]$, such that $\lambda([0,T]\setminus\hat{D}) = 0$, and, for any $t \in \hat{D}$, the section $f(t,\cdot) = \mathbb{1}_{\{\cdot\in\Omega_t\}}$ is integrable and that

$$\int_0^T \mathbb{E}[\mathbb{1}_{\Omega_t}]\,dt = (\lambda\times\mathbb{P})(A^c),$$

therefore,

$$\int_0^T \mathbb{P}(\Omega_t)\,dt = T. \tag{3.19}$$

This implies that for any $t \in \hat{D}$, $\mathbb{P}(\Omega_t) = 1$ since if it was strictly less than one for any non-null subset of $\hat{D}$, we would have strict inequality in the right-hand side of (3.19). Because $\hat{D}$ is a subset of $[0,T]$, we can take a countable dense subset $D \subset \hat{D}$ and define $\Omega_0 := \bigcap_{t\in D} \Omega_t$. It is easy to verify using De Morgan's laws that $\mathbb{P}(\Omega_0) = 1$. Moreover, we then have

$$\lim_{k\to\infty} \hat{\chi}_t^{n_k}(\omega) = \chi_t(\omega) \quad \text{for all } (t,\omega) \in D \times \Omega_0.$$

Since the functions $t \mapsto \hat{\chi}_t^{n_k}(\omega)$ are non-decreasing for every $\omega \in \Omega$, the pointwise limit, $D \ni t \mapsto \chi_t(\omega)$ for all $\omega \in \Omega_0$, must also be non-decreasing. We extend this mapping to $[0,T]$ by defining $\hat{\chi}_t(\omega) := \chi_t(\omega)$ for $t \in D$ and, for all $\omega \in \Omega_0$;

$$\hat{\chi}_t(\omega) := \lim_{\substack{s\in D \\ s\downarrow t}} \chi_s(\omega), \qquad \hat{\chi}_t(\omega) := 0 \quad \text{for all } t < \rho, \tag{3.20}$$

$$\hat{\chi}_{\rho-}(\omega) := 0, \qquad \hat{\chi}_T(\omega) := 1. \tag{3.21}$$

Note the limits exist as $(\chi_t)_{t\in[0,T]}$ is monotonic, i.e. $\chi_t(\omega)$ is always a bound on the sequence. For $\omega \in \mathscr{N} := \Omega \setminus \Omega_0$, we set $\hat{\chi}_t(\omega) = 0$ for $t < T$ and $\hat{\chi}_T(\omega) = 1$. Notice that since we assume the filtration satisfies the usual conditions, and $\mathbb{P}(\mathscr{N}) = 0$, we have that $\mathscr{N} \in \mathscr{G}_0$, and therefore $\Omega_0 \in \mathscr{G}_0$. We can then write $\hat{\chi}_t = \mathbb{1}_{\mathscr{N}} \cdot 0 + \mathbb{1}_{\Omega_0} \cdot \chi_t$; of course the first term is $\mathscr{G}_0$-measurable, but for the second we know for all $t \in D$ and $\omega \in \Omega_0$, that $\lim_{k\to\infty} \hat{\chi}_t^{n_k}(\omega) = \chi_t(\omega)$, thus, as the limit of $\mathscr{G}_t$-measurable functions, it is $\mathscr{G}_t$-measurable for all $t \in D$. Thus $(\hat{\chi}_t)_{t\in[0,T]}$ is $(\mathscr{G}_t)$-measurable for all $t \in D$. To consider all $t \in [0,T]$ we notice from the first relation in (3.20) that $\hat{\chi}_t$ is measurable with respect to $\bigcap_{s\in D, s>t} \mathscr{G}_s = \mathscr{G}_{t+} = \mathscr{G}_t$ for each $t \in [0,T]$ by the right-continuity of the filtration. Hence $(\hat{\chi}_t)_{t\in[0,T]}$ is $(\mathscr{G}_t)$-adapted and $\hat{\chi} \in \mathscr{A}_\rho$.

The subsequence $(\hat{\chi}_t^{n_k})_{k\in\mathbb{N}}$ still, of course, converges to $(\chi_t)_{t\in[0,T]}$ in $\mathscr{S}$, and so it remains to show that $(\hat{\chi}_t^{n_k})_{k\in\mathbb{N}}$ also converges to $(\hat{\chi}_t)_{t\in[0,T]}$ in $\mathscr{S}$ so that $\hat{\chi}_t(\omega) = \chi_t(\omega)$ $(\lambda \times \mathbb{P})$-almost everywhere, and therefore $\chi \in \mathscr{A}_\rho^\circ$. It is sufficient to show that $(\hat{\chi}_t^{n_k})_{k\in\mathbb{N}}$ converges to $(\hat{\chi}_t)_{t\in[0,T]}$ $(\lambda \times \mathbb{P})$-almost everywhere and then use the dominated convergence theorem to obtain that $\hat{\chi}^{n_k} \to \hat{\chi}$ in $\mathscr{S}$. For each $\omega \in \Omega_0$, due to Theorem A.2.1, the process $t \mapsto \hat{\chi}_t(\omega)$ has at no more than countably many points of discontinuity, i.e. jumps, on any bounded interval. In other words we say $\lambda([0,T] \setminus C_{\hat{\chi}}(\omega)) = 0$, where, for any càdlàg process $(X_t)_{t\in[0,T]}$ and any $\omega \in \Omega$, $C_X(\omega)$ is the set of continuity points of the trajectory $t \mapsto X_t(\omega)$,

$$C_X(\omega) := \{t \in [0,T] : X_{t-}(\omega) = X_t(\omega)\}.$$

Fix $\omega \in \Omega_0$ and let $t \in C_{\hat{\chi}}(\omega) \cap (0,T)$. Since D is a dense subset of a set of full measure we can take an increasing sequence $(t_m^1)_{m\geq 1} \subset D$ and a decreasing one $(t_m^2)_{m\geq 1} \subset D$, both converging to t as $m \to \infty$. For each $m \geq 1$ we have

$$\hat{\chi}_t(\omega) = \lim_{m\to\infty} \hat{\chi}_{t_m^2}(\omega) = \lim_{m\to\infty} \lim_{k\to\infty} \hat{\chi}_{t_m^2}^{n_k}(\omega) \geq \limsup_{k\to\infty} \hat{\chi}_t^{n_k}(\omega). \tag{3.22}$$

The first equality comes from the left-hand side of (3.20), the second equality by the fact that $t_m^2 \in D$ and $\omega \in \Omega_0$ and so $\hat{\chi}_{t_m^2}(\omega) := \chi_{t_m^2}(\omega) = \lim_{k\to\infty} \hat{\chi}_{t_m^2}^{n_k}(\omega)$. The inequality comes from the fact that $\hat{\chi}_{t_m^2}^{n_k}(\omega) \geq \hat{\chi}_t^{n_k}(\omega)$ by monotonicity. More specifically, the previous inequality trivially yields $\limsup_{n\to\infty} \hat{\chi}_{t_m^2}^{n_k}(\omega) \geq \limsup_{n\to\infty} \hat{\chi}_t^{n_k}(\omega)$, however, the left-hand side is simply the regular limit since we know it exists. Taking limits over m then yields the final result in (3.22). By analogous arguments we also obtain

$$\hat{\chi}_t(\omega) = \lim_{m\to\infty} \hat{\chi}_{t_m^1}(\omega) = \lim_{m\to\infty} \lim_{k\to\infty} \hat{\chi}_{t_m^1}^{n_k}(\omega) \leq \liminf_{k\to\infty} \hat{\chi}_t^{n_k}(\omega),$$

where, in contrast with (3.22), the first equality holds because $t \in C_{\hat{\chi}}(\omega)$. Hence we conclude that

$$\hat{\chi}_t(\omega) = \lim_{n\to\infty} \hat{\chi}_t^{n_k}(\omega) \quad \text{for all } t \in C_{\hat{\chi}}(\omega) \cap (0,T) \text{ and } \omega \in \Omega_0.$$

Convergence is trivial for $t = T$, regardless of whether it belongs to $C_{\hat{\chi}}(\omega)$ or not, since $\hat{\chi}_T^{n_k}(\omega) = \hat{\chi}_T(\omega) = 1$ for all $\omega \in \Omega$. If $0 \in C_{\hat{\chi}}(\omega)$, then $\hat{\chi}_0(\omega) = \hat{\chi}_{0-}(\omega) = 0$. Inequality (3.22) reads $0 = \hat{\chi}_0(\omega) \geq \limsup_{k\to\infty} \hat{\chi}_0^{n_k}(\omega)$. Since $\hat{\chi}_0^{n_k}(\omega) \geq 0$, we in fact have equality. Moreover, noticing that we also have $\liminf_{k\to\infty} \hat{\chi}_0^{n_k}(\omega) \geq 0$ due to positivity, we have proved that $\hat{\chi}_0^{n_k}(\omega) \to \hat{\chi}_0(\omega) = 0$. Therefore we are left with

$$\lim_{k\to\infty} \hat{\chi}_t^{n_k}(\omega) = \hat{\chi}_t(\omega) \quad \text{for all } t \in C_{\hat{\chi}}(\omega) \text{ and all } \omega \in \Omega_0.$$

Since $C_{\hat{\chi}}(\omega)$ is a set of full measure then $\hat{\chi}^{n_k} \to \hat{\chi}$ in $\mathscr{S}$ by the dominated convergence theorem, and therefore $\mathscr{A}_\rho^\circ$ is strongly closed in $\mathscr{S}$. ■

Next, we establish the functional with which we will work. Let ρ be an $(\mathscr{F}_t^2)$-stopping time and define $\mathfrak{J}(\xi,\zeta;\rho) : \mathscr{A}_\rho^\circ(\mathscr{F}_t^1) \times \mathscr{A}_\rho^\circ(\mathscr{F}_t^2) \to \mathbb{R}$ as follows

$$\mathfrak{J}(\xi,\zeta;\rho) = \mathbb{E}\left[\sum_{i=0}^{\mathscr{I}} p^i \left(\int_{[0,T)} f_t^i (1-\zeta_t)\, d\xi_t^i + \int_{[0,T)} g_t^i (1-\xi_t^i)\, d\zeta_t + \sum_{t\in[0,T]} h_t^i \Delta\zeta_t \Delta\xi_t^i \right)\right], \quad (3.23)$$

where, for now, the random variable $p = (p^0, ..., p^{\mathscr{I}})$ is any $(\mathscr{F}_\rho^1)$-measurable random variable on the simplex $\Delta(\mathscr{I})$, such that p^i is $(\mathscr{F}_t^2)$-measurable for any i. Define the upper and lower values of the game in randomised strategies, $V^*(\rho)$ and $V_*(\rho)$ respectively, as

$$V^*(\rho) = \inf_{\xi\in\mathscr{A}_\rho^\circ(\mathscr{F}_t^1)} \sup_{\zeta\in\mathscr{A}_\rho^\circ(\mathscr{F}_t^2)} \mathfrak{J}(\xi,\zeta;\rho), \qquad V_*(\rho) = \sup_{\zeta\in\mathscr{A}_\rho^\circ(\mathscr{F}_t^2)} \inf_{\xi\in\mathscr{A}_\rho^\circ(\mathscr{F}_t^1)} \mathfrak{J}(\xi,\zeta;\rho). \quad (3.24)$$

Remark 3.4.6. A crucial point to note here is that while for our original game we would expect to take (ξ,ζ) from $\mathscr{A}_\rho(\mathscr{F}_t^1) \times \mathscr{A}_\rho(\mathscr{F}_t^2)$, Remark 3.4.3 tells us that, due to the indistinguishability of the càdlàg representatives—and, therefore, the fact that the Lebesgue-Stieltjes integral are unchanged—the game described in (3.24) is identical to that where (ξ,ζ) is taken from $\mathscr{A}_\rho(\mathscr{F}_t^1) \times \mathscr{A}_\rho(\mathscr{F}_t^2)$. In other words, (3.24) is equivalent to the game

$$V^*(\rho) = \inf_{\xi\in\mathscr{A}_\rho(\mathscr{F}_t^1)} \sup_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)} \mathfrak{J}(\xi,\zeta;\rho), \qquad V_*(\rho) = \sup_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)} \inf_{\xi\in\mathscr{A}_\rho(\mathscr{F}_t^1)} \mathfrak{J}(\xi,\zeta;\rho). \quad (3.24')$$

Due to the issues raised in Remark 5.3 [DAMP22], we will need to 'smoothen' the control strategy of one of the players in order to attain some additional regularity in the payoff. To this end we introduce the class of absolutely continuous generating processes after a stopping time ρ defined as follows. Given a filtration $(\mathscr{G}_t) \subseteq (\mathscr{F}_t)$ and a $(\mathscr{G}_t)$-stopping time ρ

$$\mathscr{A}_{\rho,ac}(\mathscr{G}_t) := \{\chi \in \mathscr{A}_\rho(\mathscr{G}_t) : t \mapsto \chi_t(\omega) \text{ is absolutely continuous on } [0,T) \text{ for all } \omega\in\Omega\},$$

$$\mathscr{A}_{\rho,ac}^\circ(\mathscr{G}_t) := \{\chi \in \mathscr{S} : \exists\hat{\chi} \in \mathscr{A}_{\rho,ac}(\mathscr{G}_t) \text{ such that } \chi_t = \hat{\chi}_t \text{ for } (\lambda\times\mathbb{P})\text{-a.e.}(t,\omega) \in [0,T]\times\Omega\}.$$

Notice that these processes may still have a jump at the terminal time.

We define a new game in which the minimiser uses these 'smoothed' controls. Define the upper and lower values as

$$W^*(\rho) := \inf_{\xi \in \mathscr{A}^\circ_{\rho,ac}(\mathscr{F}^1_t)} \sup_{\zeta \in \mathscr{A}^\circ_\rho(\mathscr{F}^2_t)} \mathfrak{J}(\xi, \zeta; \rho), \qquad W_*(\rho) := \sup_{\zeta \in \mathscr{A}^\circ_\rho(\mathscr{F}^2_t)} \inf_{\xi \in \mathscr{A}^\circ_{\rho,ac}(\mathscr{F}^1_t)} \mathfrak{J}(\xi, \zeta; \rho). \quad (3.25)$$

Theorem 3.4.7. *The game defined in (3.25) has a value, i.e.* $W(\rho) := W^*(\rho) = W_*(\rho)$*. Moreover, the maximiser has an optimal strategy,* $\zeta^* \in \mathscr{A}^\circ_\rho(\mathscr{F}^2_t)$ *such that*

$$\inf_{\xi \in \mathscr{A}^\circ_{\rho,ac}(\mathscr{F}^1_t)} \mathfrak{J}(\xi, \zeta^*; \rho) = W(\rho).$$

Proof. Since $\mathscr{A}^\circ_{\rho,ac}$ is a subset of $\mathscr{A}^\circ_{ac}$ (i.e. when $\rho = 0$), then Lemma 5.20 from [DAMP22] still holds, giving us that for any $(\xi, \zeta) \in \mathscr{A}^\circ_{\rho,ac}(\mathscr{F}^1_t) \times \mathscr{A}^\circ_\rho(\mathscr{F}^2_t)$, the functionals $\mathfrak{J}(\xi, \cdot; \rho) : \mathscr{A}^\circ_\rho(\mathscr{F}^2_t) \to \mathbb{R}$ and $\mathfrak{J}(\cdot, \zeta; \rho) : \mathscr{A}^\circ_{\rho,ac}(\mathscr{F}^1_t) \to \mathbb{R}$ are, respectively, upper semicontinuous and lower semicontinuous in the strong topology of $\mathscr{S}$. The proof for convexity and semi-continuity with respect to the weak topology is identical to what is written in the original proof of Theorem 5.4 in [DAMP22], again due to $\mathscr{A}^\circ_{\rho,ac}$ being a subset of $\mathscr{A}^\circ_{ac}$, and so we omit it here. Sion's theorem, Theorem 3.4.1, is now directly applicable and we obtain the value.

For the optimal strategy, we—just like the authors of [DAMP22]—point to a version of Theorem 3.4.1 found in the paper by Komiya [Kom88] which allows us to write a maximum instead of a supremum in (3.25), i.e.

$$\sup_{\zeta \in \mathscr{A}^\circ_\rho(\mathscr{F}^2_t)} \inf_{\xi \in \mathscr{A}^\circ_{\rho,ac}(\mathscr{F}^1_t)} \mathfrak{J}(\xi, \zeta; \rho) = \max_{\zeta \in \mathscr{A}^\circ_\rho(\mathscr{F}^2_t)} \inf_{\xi \in \mathscr{A}^\circ_{\rho,ac}(\mathscr{F}^1_t)} \mathfrak{J}(\xi, \zeta; \rho) = \inf_{\xi \in \mathscr{A}^\circ_{\rho,ac}(\mathscr{F}^1_t)} \mathfrak{J}(\xi, \zeta^*; \rho),$$

where $\zeta^* \in \mathscr{A}^\circ_\rho(\mathscr{F}^2_t)$ attains the maximum. ■

In order to attain a value for the game without the need for absolutely continuous controls for the minimiser, we need to be able to approximate processes (in a suitable way) belonging to $\mathscr{A}^\circ_\rho(\mathscr{F}^1_t)$ with sequences of processes in $\mathscr{A}^\circ_{\rho,ac}(\mathscr{F}^1_t)$. This is the same result as Proposition 5.5 in [DAMP22], however, we alter the construction in the proof to account for the inclusion of the $(\mathscr{F}^2_t)$-stopping time ρ.

Proposition 3.4.8. *For any* $\zeta \in \mathscr{A}^\circ_\rho(\mathscr{F}^2_t)$ *and* $\xi \in \mathscr{A}^\circ_\rho(\mathscr{F}^1_t)$*, there is a sequence of generating* *processes* $(\xi^n_t)_{n\in\mathbb{N}}$ *belonging to* $\mathscr{A}^\circ_{\rho,ac}(\mathscr{F}^1_t)$ *such that*

$$\limsup_{n\to\infty} \mathfrak{J}(\xi^n,\zeta;\rho) \le \mathfrak{J}(\xi,\zeta;\rho). \tag{3.26}$$

Proof. Fix $\zeta \in \mathscr{A}_\rho(\mathscr{F}^2_t)$. We will explicitly construct the sequence of absolutely continuous processes $(\xi^n_t)_{n\in\mathbb{N}}$ that approximate $(\xi_t)_{t\in[0,T]}$ in a suitable sense. The choice of càdlàg representatives does not affect the expected payoff, $\mathfrak{J}(\xi,\zeta;\rho)$, and so, without loss of generality, we assume that $\xi \in \mathscr{A}_\rho(\mathscr{F}^1_t)$ and $\zeta \in \mathscr{A}_\rho(\mathscr{F}^2_t)$. We define a process $\phi^n_t(\omega) := n(t-\rho(\omega)) \wedge 1 \vee 0$, and let $\xi^{i,n}_t(\omega) = \int_{[0,t]} \phi^n_{t+\rho-s}(\omega)\, d\xi^i_s$ for $t \in [0,T)$, and $\xi^{i,n}_T = 1$. We shall show that $(\xi^{i,n}_t)$ is n-Lipschitz and hence absolutely continuous on $[0,T)$. Since $\phi^n_{t+\rho-s}(\omega) \equiv 0$ for every $\omega \in \Omega$ and $s \in (t,T]$, we must have that $\xi^{i,n}_t(\omega) = \int_{[0,T]} \phi^n_{t+\rho-s}(\omega)\, d\xi^i_s$ for $t \in [0,T)$. For any two arbitrary $t_1, t_2 \in [0,T)$:

$$\begin{aligned}|\xi^{i,n}_{t_1} - \xi^{i,n}_{t_2}| &= \left| \int_{[0,T]} \left(\phi^n_{t_1+\rho-s} - \phi^n_{t_2+\rho-s}\right) d\xi^i_s \right| \le \int_{[0,T]} \left|\phi^n_{t_1+\rho-s} - \phi^n_{t_2+\rho-s}\right| d\xi^i_s \\ &\le \int_{[0,T]} n|(t_1-s)-(t_2-s)|\, d\xi^i_s = \int_{[0,T]} n|t_1-t_2|\, d\xi^i_s = n|t_1-t_2|,\end{aligned}$$

where the first inequality is Jensen's inequality, and the second inequality follows by the definition of $(\phi^n_t)_{t\in[0,T]}$. Moreover, since $(\xi_t)_{t\in[0,T]}$ belongs to $\mathscr{A}_\rho(\mathscr{F}^1_t)$, we have that $\xi^i_t(\omega) = 0$ for all $t < \rho$ and all $\omega \in \Omega$, so therefore $\xi^{i,n}_t(\omega) = 0$ for all $t < \rho$ and all $\omega \in \Omega$ as the measure induced by $(\xi^i_t)_{t\in[0,T]}$ places no mass here.

We will verify the assumptions of Proposition 5.14 ([DAMP22]), which states that for a given filtration $(\mathscr{G}_t) \subseteq (\mathscr{F}_t)$, two processes $(\chi_t)_{t\in[0,T]}$ and $(\psi_t)_{t\in[0,T]}$ in $\mathscr{A}(\mathscr{G}_t)$, a sequence $(\psi^n)_{n\in\mathbb{N}} \subset \mathscr{A}(\mathscr{G}_t)$, and an $(\mathscr{F}_t)$-adapted and regular processes $(X_t)_{t\in[0,T]}$ belonging to $\mathscr{L}_b$; assuming the sequence $(\psi^n_t)_{n\in\mathbb{N}}$ is non-decreasing and, for $\mathbb{P}$-almost every $\omega \in \Omega$, $\lim_{n\to\infty} \psi^n_t(\omega) = \psi_{t-}(\omega)$ for all $t \in [0,T)$, then

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T)} X_t(1-\chi_{t-})\, d\psi^n_t\right] = \mathbb{E}\left[\int_{[0,T)} X_t(1-\chi_t)\, d\psi_t\right],$$

and, for $\mathbb{P}$-almost every $\omega \in \Omega$, $\lim_{n\to\infty} \psi^n_{t-}(\omega) = \psi_{t-}(\omega)$ for all $t \in [0,T]$.

Clearly the sequence $(\xi^{i,n}_t)_{n\in\mathbb{N}}$ is non-decreasing in n, as first, the sequence $(\phi^n_t)_{n\in\mathbb{N}}$ is non-decreasing, and second, the measure induced by $(\xi^i_t)_{t\in[0,T]}$ is positive for each $\omega \in \Omega$. Moreover, we see from the construction of $(\xi^{i,n}_t)_{n\in\mathbb{N}}$ that $\xi^{i,n}_\rho = 0 \to \xi^i_{\rho-}$ as $n \to \infty$, and furthermore, for any

$\omega \in \Omega$, $t \in (\rho(\omega), T)$ and $n > \frac{1}{t-\rho(\omega)}$,

$$\xi_t^{i,n} = \int_{[0,t)} \phi^n_{t+\rho-s}\, d\xi_s^i = \int_{[0,t-\frac{1}{n}]} \phi^n_{t+\rho-s}\, d\xi_s^i + \int_{(t-\frac{1}{n},t)} \phi^n_{t+\rho-s}\, d\xi_s^i,$$

where the first equality holds because $\phi^n_\rho = 0$, so jumps of $(\xi_s^i)_{s\in[0,T]}$ at $s=t$ have zero contribution. For $s \in [0, t-\frac{1}{n}]$, clearly $n(t-s) \geq n(t-t+\frac{1}{n}) = 1$, and so $\phi^n_{t+\rho-s} = 1$, and moreover, for any $s \in (t-\frac{1}{n}, t)$ we have both $n(t-s) < 1$ by similar reasoning, and $n(t-s) > n(t-t) = 0$, and so $\phi^n_{t+\rho-s} = n(t-s)$. These two facts mean that

$$\xi_t^{i,n} = \xi^i_{t-\frac{1}{n}} + \int_{(t-\frac{1}{n},t)} n(t-s)\, d\xi_s^i. \tag{3.27}$$

By letting $n \to \infty$, we see that $\xi_t^{i,n} \to \xi^i_{t-}$. This is because the second term in (3.27) vanishes:

$$0 \leq \int_{(t-\frac{1}{n},t)} n(t-s)\, d\xi_s^i \leq \xi^i_{t-} - \xi^i_{t-\frac{1}{n}} \to 0.$$

Having verified that Proposition 5.14 from [DAMP22] holds, we use the additional information of the continuity of $(\xi_t^{i,n})_{t\in[0,T]}$ on $[0,T)$ to yield

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T)} f_t^i(1-\zeta_t)\, d\xi_t^{i,n}\right] = \lim_{n\to\infty} \mathbb{E}\left[\int_{[0,T)} f_t^i(1-\zeta_{t-})\, d\xi_t^{i,n}\right] = \mathbb{E}\left[\int_{[0,T)} f_t^i(1-\zeta_t)\, d\xi_t^i\right] \tag{3.28}$$

and $\lim_{n\to\infty} \xi^{i,n}_{T-} = \xi^i_{T-}$. Therefore,

$$\lim_{n\to\infty} \Delta\xi_T^{i,n} = \Delta\xi_T^i, \tag{3.29}$$

since $\xi_T^{i,n} = 1$ for all $n \geq 1$. Applying (3.28), (3.29), and the dominated convergence theorem to the expected payoff $\mathfrak{J}(\xi^n, \zeta; \rho)$ gives

$$\begin{aligned}\lim_{n\to\infty} \mathfrak{J}(\xi^n, \zeta; \rho) &= \lim_{n\to\infty} \mathbb{E}\left[\sum_{i=0}^{\mathscr{I}} p^i \left(\int_{[0,T)} f_t^i(1-\zeta_t)\, d\xi_t^{i,n} + \int_{[0,T)} g_t^i(1-\xi_t^{i,n})\, d\zeta_t + h_T^i \Delta\zeta_T \Delta\xi_T^{i,n}\right)\right] \\ &= \mathbb{E}\left[\sum_{i=0}^{\mathscr{I}} p^i \left(\int_{[0,T)} f_t^i(1-\zeta_t)\, d\xi_t^i + \int_{[0,T)} g_t^i(1-\xi_{t-}^i)\, d\zeta_t + h_T^i \Delta\zeta_T \Delta\xi_T^i\right)\right]. \end{aligned} \tag{3.30}$$

Note that we can trivially rewrite $\mathfrak{J}(\xi, \zeta; \rho)$ as follows (see Section A.2 for reference on when the integrals below become summations)

$$
\begin{aligned}
\mathfrak{J}(\xi,\zeta;\rho) &= \mathbb{E}\left[\sum_{i=0}^{\mathscr{I}} p^i \left(\int_{[0,T)} f_t^i(1-\zeta_t)\,d\xi_t^i + \int_{[0,T)} g_t^i(1-\xi_t^i)\,d\zeta_t + \sum_{t\in[0,T]} h_t^i \Delta\zeta_t \Delta\xi_t^i\right)\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{\mathscr{I}} p^i \left(\int_{[0,T)} f_t^i(1-\zeta_t)\,d\xi_t^i + \int_{[0,T)} g_t^i(1-\xi_{t-}^i)\,d\zeta_t \right.\right. \\
&\quad \left.\left. + \sum_{t\in[0,T)} (h_t^i - g_t^i)\Delta\zeta_t \Delta\xi_t^i + h_T^i \Delta\zeta_T \Delta\xi_T^i\right)\right] \\
&\geq \mathbb{E}\left[\sum_{i=0}^{\mathscr{I}} p^i \left(\int_{[0,T)} f_t^i(1-\zeta_t)\,d\xi_t^i + \int_{[0,T)} g_t^i(1-\xi_{t-}^i)\,d\zeta_t + h_T^i \Delta\zeta_T \Delta\xi_T^i\right)\right], \qquad (3.31)
\end{aligned}
$$

where the last inequality is due to $g_t^i \leq h_t^i$ for all $t \in [0,T]$ $\mathbb{P}$-almost surely. Combining (3.31) with (3.30) gives the desired result. ∎

Corollary 3.4.9. *For any* $\zeta \in \mathscr{A}_\rho^\circ(\mathscr{F}_t^2)$*,*

$$
\inf_{\xi\in\mathscr{A}_{\rho,ac}^\circ(\mathscr{F}_t^1)} \mathfrak{J}(\xi,\zeta;\rho) = \inf_{\xi\in\mathscr{A}_\rho^\circ(\mathscr{F}_t^1)} \mathfrak{J}(\xi,\zeta;\rho). \qquad (3.32)
$$

Proof. Since $\mathscr{A}_{\rho,ac}^\circ(\mathscr{F}_t^1) \subset \mathscr{A}_\rho^\circ(\mathscr{F}_t^1)$ we immediately have that

$$
\inf_{\xi\in\mathscr{A}_{\rho,ac}^\circ(\mathscr{F}_t^1)} \mathfrak{J}(\xi,\zeta;\rho) \geq \inf_{\xi\in\mathscr{A}_\rho^\circ(\mathscr{F}_t^1)} \mathfrak{J}(\xi,\zeta;\rho).
$$

However, using Lemma 3.4.8 we reason thusly. Trivially we know that

$$
\inf_{\xi\in\mathscr{A}_{\rho,ac}^\circ(\mathscr{F}_t^1)} \mathfrak{J}(\xi,\zeta;\rho) \leq \mathfrak{J}(\xi^n,\zeta;\rho),
$$

and so by taking limit supremum of both sides and utilising (3.26) we have

$$
\inf_{\xi\in\mathscr{A}_{\rho,ac}^\circ(\mathscr{F}_t^1)} \mathfrak{J}(\xi,\zeta;\rho) \leq \mathfrak{J}(\xi,\zeta;\rho),
$$

where $(\xi_t)_{t\in[0,T]}$ on the right-hand side is the limit of the sequence $(\xi_t^n)_{t\in[0,T]}$, and belongs to $\mathscr{A}_\rho^\circ(\mathscr{F}_t^1)$. Taking the infimum over $\xi \in \mathscr{A}_\rho^\circ(\mathscr{F}_t^1)$ of both sides yields

$$
\inf_{\xi\in\mathscr{A}_{\rho,ac}^\circ(\mathscr{F}_t^1)} \mathfrak{J}(\xi,\zeta;\rho) \leq \inf_{\xi\in\mathscr{A}_\rho^\circ(\mathscr{F}_t^1)} \mathfrak{J}(\xi,\zeta;\rho),
$$

and so equality holds. ■

Theorem 3.4.10. *The game defined in (3.24) has a value, $\hat{V}(\rho) := V_*(\rho) = V^*(\rho)$, in randomised stopping times, and there exists a pair of optimal strategies.*

Proof. Again, since $\mathscr{A}^\circ_{\rho,ac}(\mathscr{F}^1_t) \subset \mathscr{A}^\circ_\rho(\mathscr{F}^1_t)$ we have that $V_*(\rho) \leq W_*(\rho)$ and $V^*(\rho) \leq W^*(\rho)$. An immediate consequence of Corollary 3.4.9 is that $W_*(\rho) = V_*(\rho)$. The existence of a value, $\hat{V}(\rho) := V_*(\rho) = V^*(\rho)$, then follows from Theorem 3.4.7, i.e.

$$W(\rho) = W_*(\rho) = V_*(\rho) \leq V^*(\rho) \leq W^*(\rho) = W(\rho).$$

For the optimal strategies, Theorem 3.4.7 guarantees the existence of an optimal strategy for the maximiser $\zeta^* \in \mathscr{A}^\circ_\rho(\mathscr{F}^2_t)$. Choosing this process in (3.32) yields

$$\hat{V}(\rho) = V_*(\rho) = \inf_{\xi \in \mathscr{A}^\circ_\rho(\mathscr{F}^1_t)} \mathfrak{J}(\xi, \zeta^*; \rho).$$

Due to the symmetry of the game mentioned in Remark 3.2.3, we can apply Theorem 3.4.7 and Proposition 3.4.8 on the game with payoff $-P(\xi,\zeta)$ (with ξ the maximiser and ζ the minimiser) to obtain

$$-\hat{V}(\rho) =: \hat{V}'(\rho) = \inf_{\zeta \in \mathscr{A}^\circ_\rho(\mathscr{F}^2_t)} \mathfrak{J}(\xi^*, \zeta; \rho),$$

where $(\xi^*_t)_{t\in[0,T]}$ is the optimal for the maximiser in the game with $W'(\rho) := -W(\rho)$. Therefore, $(\xi^*_t)_{t\in[0,T]}$ is optimal for the minimiser in the game with value $\hat{V}(\rho)$, and so the pair $(\xi^*, \zeta^*) \in \mathscr{A}^\circ_\rho(\mathscr{F}^1_t) \times \mathscr{A}_\rho(\mathscr{F}^2_t)$ is a Nash equilibrium. ■

We are now very close to our desired result, however, we are still considering an unconditional expected payoff. To this end, define the conditional expected payoff as follows.

Definition 3.4.11. Let ρ be an $(\mathscr{F}^2_t)$-stopping time, $p = (p^0, ..., p^{\mathscr{I}})$ be any $(\mathscr{F}^2_\rho)$-measurable random variable on the simplex $\Delta(\mathscr{I})$, and let (ξ,ζ) be a pair of generating processes belonging to $\mathscr{A}_\rho(\mathscr{F}^1_t) \times \mathscr{A}_\rho(\mathscr{F}^2_t)$. The family of random variables $\{J(\xi,\zeta;\rho) : \rho \in \mathscr{T}(\mathscr{F}^2_t)\}$ given by

$$J(\xi,\zeta;\rho) = \sum_{i=0}^{\mathscr{I}} p^i \mathbb{E}\left[\int_{[0,T)} f^i_t(1-\zeta_t)\, d\xi^i_t + \int_{[0,T)} g^i_t(1-\xi^i_t)\, d\zeta_t + \sum_{t\in[0,T]} h^i_t \Delta\zeta_t \Delta\xi^i_t \middle| \mathscr{F}^2_\rho\right]$$

is called the conditional expected payoff at ρ.

It is clear that $\mathbb{E}J(\xi,\zeta;\rho)=\mathfrak{J}(\xi,\zeta;\rho)$. We redefine our upper and lower values to recognise that the payoff is now a random variable, hence we need to take essential infima and suprema:

$$\overline{V}(\rho)=\operatorname*{ess\,inf}_{\xi\in\mathscr{A}_\rho(\mathscr{F}^1_t)}\operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}^2_t)}J(\xi,\zeta;\rho),\qquad \underline{V}(\rho)=\operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}^2_t)}\operatorname*{ess\,inf}_{\xi\in\mathscr{A}_\rho(\mathscr{F}^1_t)}J(\xi,\zeta;\rho). \tag{3.33}$$

Before continuing, we provide three additional short but useful properties of the conditional expected payoff. Lemma 3.4.12 shows that it is nicely behaved when generating processes coincide; this will be an important result when we discuss directedness of families of the conditional expected payoff shortly. Secondly, Proposition 3.4.13 proves that the conditional payoff is a bilinear map. Finally, Lemma 3.4.14 gives us an integrability result.

Lemma 3.4.12. *Let ρ be an $(\mathscr{F}^2_t)$-stopping time in $[0,T]$, and A be an $\mathscr{F}^2_\rho$-measurable set. Let $\chi_1,\chi_2\in\mathscr{A}_\rho(\mathscr{F}^2_t)$ and define $\chi:=\chi_1\mathbb{1}_A+\chi_2\mathbb{1}_{A^c}$, then $\chi\in\mathscr{A}_\rho(\mathscr{F}^2_t)$ and*

$$\mathbb{1}_A J(\xi,\chi;\rho)=\mathbb{1}_A J(\xi,\chi_1;\rho),$$
$$\mathbb{1}_{A^c} J(\xi,\chi;\rho)=\mathbb{1}_{A^c} J(\xi,\chi_2;\rho).$$

The same results hold for $(\chi_t)_{t\in[0,T]}$ in the first argument of J also.

Proof. First we show that $(\chi_t)_{t\in[0,T]}$ is a valid generating process. We immediately see that $\chi_\rho=\chi_{1,\rho}\mathbb{1}_A+\chi_{2,\rho}\mathbb{1}_{A^c}$ is $(\mathscr{F}^2_\rho)$-measurable. For any $t>\rho$ we have $\chi_t=\chi_{1,t}\mathbb{1}_A+\chi_{2,t}\mathbb{1}_{A^c}$ which is $(\mathscr{F}^2_t)$-measurable. For any $t<\rho$, $(\chi_t)_{t\in[0,T]}$ is trivially zero. Thus the new process $(\chi_t)_{t\in[0,T]}$ is $(\mathscr{F}^2_t)$-adapted. Secondly, for each $\omega\in\Omega$ we know $\chi_{\rho-}=0$ and $\chi_T=1$. The process is also clearly non-decreasing and càdlàg by construction, completing the proof for validity.

Finally, we show the second result directly. Since A is $(\mathscr{F}^2_t)$-measurable we have

$$\mathbb{1}_A J(\cdot,\chi;\rho)=\sum_{i=0}^{\mathscr{I}}p^i\mathbb{E}\Big[\mathbb{1}_A P(\cdot,\chi)\Big|\mathscr{F}^2_\rho\Big]=\sum_{i=0}^{\mathscr{I}}p^i\mathbb{E}\Big[\mathbb{1}_A P(\cdot,\chi_1)\Big|\mathscr{F}^2_\rho\Big]=\mathbb{1}_A J(\cdot,\chi_1;\rho).$$

See Appendix A for a discussion on Lebesgue-Stieltjes integrals and equality between measures. The reason the above logic holds is because the restriction to A of the measures induced by χ and χ_1 are equal, therefore the integrals are not affected by switching measures on the set A.

Regarding these results for the first argument of $J(\cdot,\cdot;\rho)$, we notice that a generating processes $(\chi_t)_{t\in[0,T]}$ in this argument belongs to $\mathscr{A}(\mathscr{F}^1_t)$, and therefore has the decomposition $\chi^0,...,\chi^{\mathscr{I}}$, belonging to $\mathscr{A}(\mathscr{F}_t^2)$. For any two χ_1 and χ_2 in $\mathscr{A}(\mathscr{F}_t^1)$, we define $\chi^i := \chi_1^i \mathbb{1}_A + \chi_2^i \mathbb{1}_{A^c}$, and the same proof method as above holds. ■

Proposition 3.4.13. *For any $\rho \in \mathscr{T}(\mathscr{F}_t^2)$, the mapping $J(\xi,\zeta;\rho) : \mathscr{A}_\rho(\mathscr{F}_t^1) \times \mathscr{A}_\rho(\mathscr{F}_t^2) \to \mathbb{R}$ is a bilinear map.*

Proof. Without loss of generality, choose $(\xi_t)_{t\in[0,T]}$ fixed and define $\zeta_t := \alpha\zeta_t^1 + (1-\alpha)\zeta_t^2$ for any $(\zeta_t^1)_{t\in[0,T]}$ and $(\zeta_t^2)_{t\in[0,T]}$ in $\mathscr{A}(\mathscr{F}_t^2)$. Theorem 6.1.2 in [CB00] tells us that

$$\int_{[\rho,T)} g_t^i(1-\xi_t^i)\,d\zeta_t = \alpha \int_{[\rho,T)} g_t^i(1-\xi_t^i)\,d\zeta_t^1 + (1-\alpha)\int_{[\rho,T)} g_t^i(1-\xi_t^i)\,d\zeta_t^2.$$

It is also easy to see that the following hold,

$$\begin{aligned} f_t^i(1-\zeta_t) &= \alpha f_t^i(1-\zeta_t^1) + (1-\alpha) f_t^i(1-\zeta_t^2), \\ \Delta\zeta_t &= \alpha\Delta\zeta_t^1 + (1-\alpha)\Delta\zeta_t^2. \end{aligned}$$

It is then trivial to verify that $J(\xi,\zeta;\rho) = \alpha J(\xi,\zeta^1;\rho) + (1-\alpha)J(\xi,\zeta^2;\rho)$. ■

Lemma 3.4.14. *Let ρ be an $(\mathscr{F}_t^2)$-stopping time, then for any $\xi \in \mathscr{A}_\rho(\mathscr{F}_t^1)$ we have that the Snell envelope of the conditional expected payoff, $J(\xi,\zeta;\rho)$, is integrable,*

$$\mathbb{E}\left| \operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi,\zeta;\rho) \right| < \infty.$$

Proof. Let us look for a family of random variables which dominates $J(\xi,\zeta;\rho)$. Firstly, we notice that the random variable p^i belongs on the simplex $\Delta(\mathscr{I})$, and is therefore always bounded above by one—as are the generating processes—and so

$$\begin{aligned} J(\xi,\zeta;\rho) &\le \sum_{i=0}^{\mathscr{I}} \mathbb{E}\left[\int_{[0,T)} f_t^i(1-\zeta_t)\,d\xi_t^i + \int_{[0,T)} g_t^i(1-\xi_t^i)\,d\zeta_t + \sum_{t\in[0,T]} h_t^i \Delta\zeta_t \Delta\xi_t^i \middle| \mathscr{F}_\rho^2 \right] \\ &\le \sum_{i=0}^{\mathscr{I}} \mathbb{E}\left[\int_{[0,T)} |f_t^i|\,d\xi_t^i + \int_{[0,T)} |g_t^i|\,d\zeta_t + \sum_{t\in[0,T]} |h_t^i|\Delta\zeta_t \middle| \mathscr{F}_\rho^2 \right], \end{aligned}$$

where we have used that the payoff processes are bounded above by their absolute value.

Since $(f_t^i)_{t\in[0,T]}$ and $(g_t^i)_{t\in[0,T]}$ are the only two processes we assume to be in the Banach space $\mathscr{L}_b$, we need to bound $(|h_t^i|)_{t\in[0,T]}$ by those processes. Note, for any $t \in [0,T]$, that since $g_t \le$

$h_t \leq f_t$, this implies that $|h_t| \leq \max\{|f_t|, |g_t|\} \leq |f_t| + |g_t|$. Moreover, we may take the suprema over $t \in [0,T]$ of $|f_t^i|$, $|g_t^i|$ and $|f_t^i| + |g_t^i|$ and remove them from the integrals and summations. Additionally, notice that $\sum_{s\in[0,T]} \Delta\zeta_s \leq 1$, and for any $\chi \in \mathscr{A}(\mathscr{F}_t^2)$, $\int_{[0,T)} d\chi_s = \chi_{T-} \leq 1$. Thus,

$$J(\xi,\zeta;\rho) \leq \sum_{i=0}^{\mathscr{I}} \mathbb{E}\left[\sup_{t\in[0,T]} |f_t^i| + \sup_{t\in[0,T]} |g_t^i| + \sup_{t\in[0,T]} (|f_t^i| + |g_t^i|) \middle| \mathscr{F}_\rho^2 \right] =: \sum_{i=0}^{\mathscr{I}} Z^i(\rho), \tag{3.34}$$

and

$$\mathbb{E}Z^i(\rho) = \mathbb{E}\left[\sup_{t\in[0,T]} |f_t^i| + \sup_{t\in[0,T]} |g_t^i| + \sup_{t\in[0,T]} (|f_t^i| + |g_t^i|) \right] < \infty. \tag{3.35}$$

Notice, that since $J(\xi,\zeta;\rho)$ is dominated by this integrable random variable, then its essential supremum, $\operatorname{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi,\zeta;\rho)$, (and its absolute value) must also be dominated by this integrable random variable, by definition. ■

Remark 3.4.15. Note that Lemma 3.4.14 would also hold for the essential infimum, this is because $J(\xi,\zeta;\rho)$ itself is dominated by the integrable random variable $Z^i(\rho)$, and therefore so is its essential infimum by definition.

The next results are somewhat expected as they correspond directly to Lemma 5.26 and Lemma 5.27 in [DAMP22], namely that the conditional expected payoff, and related families, have certain directedness properties. Here we present the minimal structure we require on these families, however, one will see that it is trivial to adapt these proofs to show that all families are in fact lattices.

Lemma 3.4.16. *Let ρ be an $(\mathscr{F}_t^2)$-stopping time in $[0,T]$, then:*

1. *for any $\xi \in \mathscr{A}_\rho(\mathscr{F}_t^1)$, the family $\{J(\xi,\zeta;\rho), \zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)\}$ is upwards-directed,*
2. *for any $\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)$, the family $\{J(\xi,\zeta;\rho), \xi \in \mathscr{A}_\rho(\mathscr{F}_t^1)\}$ is downwards-directed,*
3. *the family $\{\operatorname{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi,\zeta;\rho), \xi \in \mathscr{A}_\rho(\mathscr{F}_t^1)\}$ is downwards-directed,*
4. *the family $\{\operatorname{ess\,inf}_{\xi\in\mathscr{A}_\rho(\mathscr{F}_t^1)} J(\xi,\zeta;\rho), \zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)\}$ is upwards-directed.*

Proof. The proofs of these claims are all highly similar, therefore we will only prove point one and three here and one will see that the techniques used are trivially adapted for the opposite directedness claim.

1. Let ζ_1 and ζ_2 both belong to $\mathscr{A}_\rho(\mathscr{F}_t^2)$ and define $A := \{J(\xi,\zeta_1;\rho) \geq J(\xi,\zeta_2;\rho)\} \in \mathscr{F}_\rho^2$. Moreover, let $\zeta = \zeta_1 \mathbb{1}_A + \zeta_2 \mathbb{1}_{A^c}$. By Lemma 3.4.12, we know that $\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)$ and $\mathbb{1}_A J(\xi,\zeta;\rho) = \mathbb{1}_A J(\xi,\zeta_1;\rho)$, and likewise for A^c. Thus, we may say that

$$J(\xi,\zeta;\rho) = J(\xi,\zeta_1;\rho)\mathbb{1}_A + J(\xi,\zeta_2;\rho)\mathbb{1}_{A^c} = J(\xi,\zeta_1;\rho) \vee J(\xi,\zeta_2;\rho).$$

3. Let ξ_1 and ξ_2 both belong to $\mathscr{A}_\rho(\mathscr{F}_t^1)$ and define

$$A := \left\{ \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi_1,\zeta;\rho) < \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi_2,\zeta;\rho) \right\} \in \mathscr{F}_\rho^2.$$

Moreover, since both ξ_1 and ξ_2 admit a decomposition over the values of Θ, we can define $\xi^i = \xi_1^i \mathbb{1}_A + \xi_2^i \mathbb{1}_{A^c}$. We know from Lemma 3.4.12 that $\xi^i \in \mathscr{A}_\rho(\mathscr{F}_t^2)$ and $\mathbb{1}_A J(\xi,\zeta;\rho) = \mathbb{1}_A J(\xi_1,\zeta;\rho)$, and likewise for A^c. From earlier, we know the family $\{J(\xi,\zeta;\rho), \zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)\}$ is upwards-directed and so its essential supremum is attained by a limit of process $(\zeta^n)_{n\in\mathbb{N}} \subset \mathscr{A}_\rho(\mathscr{F}_t^2)$,

$$\mathbb{1}_A \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi,\zeta;\rho) = \lim_{n\to\infty} \mathbb{1}_A J(\xi,\zeta^n;\rho) = \lim_{n\to\infty} \mathbb{1}_A J(\xi_1,\zeta^n;\rho) = \mathbb{1}_A \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi_1,\zeta;\rho),$$

and likewise for A^c. Thus, we may say that

$$\begin{aligned}
\operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi,\zeta;\rho) &= \mathbb{1}_A \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi_1,\zeta;\rho) + \mathbb{1}_{A^c} \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi_2,\zeta;\rho) \\
&= \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi_1,\zeta;\rho) \wedge \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi_2,\zeta;\rho).
\end{aligned}$$

■

We can now return our focus to the value of the game. Immediately from their definitions we see that the upper and lower conditional values (3.33) are ordered such that $\underline{V}(\rho) \leq \overline{V}(\rho)$. If we prove equality in expectation then we have that $\underline{V}(\rho) = \overline{V}(\rho)$ $\mathbb{P}$-almost surely as needed.

Lemma 3.4.17. *Recall $V_*(\rho)$ and $V^*(\rho)$ as in (3.24). We have*

$$\mathbb{E}\overline{V}(\rho) = V^*(\rho) \qquad \mathbb{E}\underline{V}(\rho) = V_*(\rho).$$

Proof. Fix $\xi \in \mathscr{A}(\mathscr{F}_t^1)$. Since the family $\{J(\xi,\zeta;\rho), \zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)\}$ is upwards-directed, its essential supremum is attained by a sequence $(\zeta^n)_{n\in\mathbb{N}} \subset \mathscr{A}_\rho(\mathscr{F}_t^2)$, so using the monotone convergence theorem gives

$$\mathbb{E}\left[\operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi,\zeta;\rho)\right] = \lim_{n\to\infty}\mathbb{E}[J(\xi,\zeta^n;\rho)] = \lim_{n\to\infty}\mathfrak{J}(\xi,\zeta^n;\rho) \le \sup_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)}\mathfrak{J}(\xi,\zeta;\rho).$$

The opposite inequality follows from the fact that $\operatorname{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi,\zeta;\rho) \ge J(\xi,\zeta;\rho)$ for any choice of $(\zeta_t)_{t\in[0,T]}$ in $\mathscr{A}_\rho(\mathscr{F}_t^2)$. Therefore, we have that

$$\mathbb{E}\left[\operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi,\zeta;\rho)\right] = \sup_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)}\mathfrak{J}(\xi,\zeta;\rho). \tag{3.36}$$

Furthermore, since the family $\{\operatorname{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi,\zeta;\rho), \xi \in \mathscr{A}_\rho(\mathscr{F}_t^1)\}$ is downwards directed, its essential infimum (i.e. $\overline{V}(\rho)$) is attained by a sequence $(\xi^n)_{n\in\mathbb{N}} \subset \mathscr{A}_\rho(\mathscr{F}_t^1)$. Using similar arguments as above we obtain that

$$\mathbb{E}\overline{V}(\rho) = \inf_{\xi\in\mathscr{A}_\rho(\mathscr{F}_t^1)}\mathbb{E}\left[\operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J(\xi,\zeta;\rho)\right]. \tag{3.37}$$

Combining (3.36) and (3.37) completes the proof that $\mathbb{E}\overline{V}(\rho) = V^*(\rho)$. The second part of the statement requires analogous arguments. ■

Corollary 3.4.18. *The game defined in (3.33) has a value,* $V(\rho) := \underline{V}(\rho) = \overline{V}(\rho)$.

Proof. We know from Theorem 3.4.10 that the game defined in (3.24) has a value $\hat{V}(\rho) = V^*(\rho) = V_*(\rho)$, and so using Lemma 3.4.17 we obtain that $\mathbb{E}\underline{V}(\rho) = \mathbb{E}\overline{V}(\rho)$. Therefore, since $\underline{V}(\rho) \le \overline{V}(\rho)$ $\mathbb{P}$-almost surely, we obtain $V(\rho) := \underline{V}(\rho) = \overline{V}(\rho)$ $\mathbb{P}$-almost surely. ■

3.5 SURROUNDING RESEARCH

The literature around Dynkin games with asymmetric information is relatively new, but the inclusion of randomised strategies can be traced back a bit further. We begin with a brief summary, before expanding on some of the important research that aligns closely with this work.

In 1985, Yasuda [Yas85] introduced randomisation into the discrete-time, multi-stage stopping game considered by Neveu in 1975 [Nev75]. In this context, Yasuda defined randomisation as having a certain probability of stopping at any stage in the game. In 2002 and 2005, Touzi and Vieille [TV02] and Laraki and Solan [LS05] respectively published papers on continuous-time Dynkin games, and used randomised stopping times to allow them to drop the ordering of the payoff processes. As we saw in Section 3.1, the paper by Touzi and Vieille discusses the relationships between different formulations of randomised stopping times, and in 2012, Solan, Tsirelson and Vieille did the same, [STV12], and showed the equivalence of three types of randomised stopping times defined by Yasuda [Yas85], Chalasani and Jha [CJ01] and Aumann [Aum64]. Finally, in 2012, Kifer [Kif13] utilised randomised stopping times in order to price (Dynkin) game options in incomplete markets with transaction costs, and in 2017, Riedel and Steg used them to construct subgame-perfect Nash equilibria (a concept we will come back to later in this work).

Broadly speaking, research into games with asymmetric information can be parsed into two categories, one which uses a 'guess-and-verify' approach, and a second which utilises variational PDEs. The roots of the latter are significantly easier to trace than those of the former. The method of utilising viscosity solutions to PDEs took form with the advent of differential games, a field which itself has a fascinating history. In 1925 Charles Roos wrote a paper, [Roo25], which is one of the earliest records of a differential game. In it he considers two competing manufacturers, both of whom are trying to maximise their profit by controlling their production amount subject to a cost constraint. Demand for the products is assumed to depend on the price, and the rate of change of the price, of the product, and this is where we see the concept of a differential game appearing. Simply put, a differential game is a two-player game wherein both players attempt to control an underlying dynamical system (one driven by a system of differential equations) to optimise a given payoff function. The field became of great importance when, in 1951 and 1965 respectively, Rufus Isaacs published his report on *Games of Pursuit* [Isa51] for the RAND Corporation, and his book *Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization* [Isa65], which frames these types of problems in terms of pursuer and pursued. Most notably, he introduced the so-called 'homicidal chauffeur game' which sees a never tiring pedestrian attempt to outrun a pursuing car. While macabre, games of pursuit were taken up in the field, with Bellman describing them in his 1961 book *Adaptive Control Processes*, [Bel61], and unsurprisingly this problem became vital to the study of combat situations such as missile guidance and aeroplane dog-fighting, with the former still being a relevant topic of research to this day (see [Far17] and [Xi+23] for example).

Two major contributors to the differential games approach to problems of asymmetric information are Pierre Cardaliaguet and Catherine Rainer who released a trio of papers around 2008 tackling this exact issue. The pair's main work is arguably [CR09a], wherein they prove the existence of a value for two-player, zero-sum stochastic differential games, and provide a characterisation of the value as a 'viscosity solution' to a PDE. In a concurrent solo paper, [Car09], Cardaliaguet explicitly linked the problem studied in [CR09a] to what is known as a 'double obstacle' problem. The duo also published [CR09b], which contains an approachable set-up, and draws on the work done in the previous two papers. They study the continuous-time version of a repeated game first introduced by Auman and Maschler [AM95] in 1995. In the simplest form, they consider a game on a time horizon $[0,T]$, with payoffs $\ell^i : [0,T] \times U \times V \to \mathbb{R}$ for $i \in [\mathscr{I}]$, where the spaces U and V are the spaces of controls of the two players. There is a probability vector $\pi = (\pi_0, ..., \pi_{\mathscr{I}})$ on the simplex $\Delta(\mathscr{I})$ for the outcomes of i—of course this is the equivalent of Θ in our framework. At time zero Player 1 learns the outcome of Θ and then attempts to minimise a payoff

$$\int_0^T \ell^i(s, u(s), v(s))\, ds,$$

while the second player tries to maximise it. The choice of controls is public knowledge, but the maximiser does not know which payoff they are maximising. The value of this game is shown in [CR09a] if 'Isaacs' condition' holds, namely that

$$H(t,\pi) := \inf_{u \in U} \sup_{v \in V} \sum_{i=0}^{\mathscr{I}} \pi_i \ell^i(t,u,v) = \sup_{v \in V} \inf_{u \in U} \sum_{i=0}^{\mathscr{I}} \pi_i \ell^i(t,u,v)$$

for all (t,π) belonging to $[0,T] \times \Delta(\mathscr{I})$. They prove that the game has a value in randomised strategies. For them, a randomised strategy is akin to a vector of pure strategies and a vector of probabilities which correspond to the pure strategies. This is this the same as our 'mixed strategies' in Definition 3.1.4, and so we know they are equivalent to our formulation. Moreover, they use [Car09] to show that the value V is the unique viscosity solution to the following Hamilton–Jacobi obstacle problem

$$\min\left\{ w_t + H(t,\pi), \lambda_{\min}\left(\frac{\partial^2 w}{\partial \pi^2}\right)\right\} = 0 \qquad \text{in } [0,T] \times \Delta(\mathscr{I}). \tag{3.38}$$

We will define what $\lambda_{\min}$ means, and what problems of this form represent, later in this section.

The paper [CR09b] then goes on to characterise the informed player's optimal strategy using a dual representation, a technique we will see again in Grün's work, [Grü13].

It is arguable that Christine Grün first introduced information asymmetry to Markovian Dynkin games in 2013, [Grü13]—a paper we will revisit soon. Her framework saw one player know the payoff process and the other only an initial distribution of possible payoff processes, and she utilised the above methods of differential games. In a subsequent paper, with her co-author Fabien Gensbittel [GG18], players observe their own stochastic process and the payoff depends on both. It is clear to see the influence that differential games have had on the field, even if we will not be taking this approach in our work.

The origins of the 'guess-and-verify' approach are significantly harder to trace, since it relies on cumulative understanding of similar problems and classical solutions to single-agent problems. We refer the reader to Chapter 1 for the surrounding theory, but we will highlight three specific papers. In 2019, De Angelis, Gensbittel and Villeneuve published a paper, [DAGV21], considering a Dynkin game arising from the pricing of an option on an asset whose return is unknown to both players, in effect, a partial and asymmetric information set-up. As they assume a Markovian dynamic, they are able to characterise the existence of a Nash equilibrium in stopping times dependent on a stopping region whose boundary is free-moving—these so-called free-boundary problems are covered extensively in the book by Peskir and Shiryaev [PS06]. In 2017, Ekström, Glover and Leniec's paper, [EGL17], provided a verification result based on (sub/ super-)martingales, however, they study a nonzero-sum game. Finally, in 2020, De Angelis, Ekström and Glover, [DAEG21], looked at a Markovian Dynkin game where the drift of the underlying diffusion is known only to one player. They show the game admits a value with the uninformed player using stopping times, while the informed player uses randomised stopping times to 'hide their informational advantage'. Their verification result depends on solving 'suitable quasi-variational inequalities with some non-standard constraints'.

This guess-and-verify method can be frustrating since the use of an ansatz seemingly requires us to already know the answer before attempting the problem. Once one is sufficiently familiar with the methodologies used in single-agent optimal control and Markovian and non-Markovian Dynkin games—see Chapters 1 and 2 respectively—the solution methods often begin to form a pattern. It is this pattern which formed the initial guess as to what our own necessary and sufficient conditions should look like, and is why we have structured our work in the way we have.

* * *

We have breezed through some extremely technical results here. As we have already stated, in her 2013 paper, [Grü13], Christine Grün introduced information asymmetry to Markovian Dynkin games and utilised the preexisting literature of differential games. The work done in this paper is one of the closest to our own problem, and thus requires a greater level of inspection so that we may draw comparisons to our own work.

Grün's work is done in a canonical probability space, the details of which are not important for our purposes, so we will omit many of the technical aspects in order to present her results. Her work focuses on Markovian Dynkin games driven by a stochastic differential equation. Let $b(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $\sigma(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be bounded and Lipschitz continuous with respect to (t,x). The dynamics of a diffusion for any $t \in [0,T]$ and $x \in \mathbb{R}^d$ are given by

$$dX_s^{t,x} = b(s, X_s^{t,x})\, ds + \sigma(s, X_s^{t,x})\, dW_s, \qquad X_t^{t,x} = x, \quad s \in [t,T]. \tag{3.39}$$

The two filtrations are as follows: $(\mathscr{F}_t^2)_{t\in[0,T]}$ is the filtration generated by the Brownian motion, and, like our own set-up, $(\mathscr{F}_t^1)_{t\in[0,T]}$ is such that $\mathscr{F}_t^1 := \mathscr{F}_t^2 \vee \sigma(\Theta)$ for all $t \in [0,T]$. There is a subtle difference to our set-up in how the payoff processes are defined. For a given outcome $\Theta = i$, $h^i(x) : \mathbb{R}^d \to \mathbb{R}$ is now simply the terminal payoff, however, if players stop before the time horizon then the payoff is one of two different functions: $f^i(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ or $g^i(t,x) : [0,T] \times \mathbb{R}^d \to \mathbb{R}$. The effect of this set-up can be seen in the demonstrative expected payoff:

$$\mathbb{E}\big[f^i(\tau, X_\tau^{t,x})\mathbb{1}_{\{\tau\le\sigma,\tau<T\}} + g^i(\sigma, X_\sigma^{t,x})\mathbb{1}_{\{\tau>\sigma,\sigma<T\}} + h^i(X_T^{t,x})\mathbb{1}_{\{\tau=\sigma=T\}}\big],$$

where each outcome $i \in [\mathscr{I}]$ has probability π_i. One can see that, in the case of both players stopping simultaneously before the time horizon, the payoff is f^i, rather that h^i (as we would expect in our set-up). In other words, the equivalent formulation in our framework (adjusting for Markovianity) would be $h^i(t, X_t^{t,x}) := f^i(t, X_t^{t,x})\mathbb{1}_{t\in[0,T)} + h(X_t^{t,x})\mathbb{1}_{t=T}$.

The second mover advantage conditions are still present, however, they look slightly different due the differences with payoff processes. Specifically, we assume that, for all $i \in [\mathscr{I}]$, $t \in [0,T]$ and $x \in \mathbb{R}^d$,

$$g^i(t,x) \le f^i(t,x) \qquad \& \qquad g^i(T,x) \le h^i(x) \le f^i(T,x).$$

The strategies of the two players are handled very similarly, namely if we consider playing with pure stopping times, we can reframe the game as an $\mathscr{I}+1+1$ player game, with $\mathscr{I}+1$ minimisers and one maximiser. For illustration, fix a triple $(t,x,\pi) \in [0,T]\times\mathbb{R}^d\times\Delta(\mathscr{I})$ and take a family of stopping times, $(\tau^i)_{i\in\{0,\dots,\mathscr{I}\}}$ for the minimiser, and a stopping time σ for the maximiser, both greater than or equal to t. The expected payoff for a game beginning at time t can be written for each i as follows

$$J_t^i(x,\tau^i,\sigma) = \mathbb{E}\big[f^i(\tau_i, X_{\tau_i}^{t,x})\mathbb{1}_{\{\tau_i\le\sigma,\tau_i<T\}} + g^i(\sigma, X_\sigma^{t,x})\mathbb{1}_{\{\tau_i>\sigma,\sigma<T\}} + h^i(X_T^{t,x})\mathbb{1}_{\{\tau_i=\sigma=T\}}\big]. \tag{3.40}$$

Grün, as one would expect, actually utilises randomised stopping times to attain her results. Specifically, she uses mixed strategies, which we saw in Proposition 3.1.5 and Remark 3.1.6 are equivalent to our randomised stopping times. For now, we simply say that the minimiser chooses a family of randomised stopping times $(\eta^i)_{i\in\{0,\dots,\mathscr{I}\}}$, and the maximiser chooses the randomised stopping time γ.

Due to the presence of a uniformly distributed randomisation device, it is possible to connect the i^{th}-expected payoff under randomised stopping times to the i^{th}-expected payoff using regular stopping times (3.40). Recall from Remark 3.1.2 that, for any specific value u of the randomisation device of a randomised stopping time η, $\eta(u)$ is a stopping time (notice that we have dropped the dependence on the generating process for ease of notation). Hence we may integrate over all possible values of u and obtain

$$J_t^i(x,\eta^i,\gamma) = \int_0^1\int_0^1 J_t^i(x,\eta^i(u_1),\gamma(u_2))\,du_1du_2, \tag{3.41}$$

and thus

$$J_t(x,\eta,\gamma) = \sum_{i=0}^{\mathscr{I}} \pi_i J_t^i(x,\eta^i,\gamma).$$

Remark 3.5.1. Notice that $J_t^i(x,\eta^i,\gamma)$ in (3.41) should not be confused with $J_t^i(x,\tau^i,\sigma)$ in (3.40); we integrate the latter over all values of u_1 and u_2, and in this way define the expected payoff over randomised stopping times using the expected payoff over regular stopping times.

The upper and lower values of the game are then defined in exactly the same way as (3.10),

$$\underline{V}(t,x,\pi) = \sup_{\gamma \in \mathscr{T}_t^R(\mathscr{F}_s^2)} \inf_{\eta \in \mathscr{T}_t^R(\mathscr{F}_s^1)} J_t(x,\eta,\gamma),$$
$$\overline{V}(t,x,\pi) = \inf_{\eta \in \mathscr{T}_t^R(\mathscr{F}_s^1)} \sup_{\gamma \in \mathscr{T}_t^R(\mathscr{F}_s^2)} J_t(x,\eta,\gamma),$$

and a value is said to exist if $V(t,x,\pi) := \underline{V}(t,x,\pi) = \overline{V}(t,x,\pi)$.

As we previously mentioned, the approach that Grün takes to this problem is vastly different to the one taken by ourselves, specifically, she uses the concept of viscosity solutions of PDEs to prove the existence of a value—see Theorem 3.5.3 below. As this broader field is not within the scope of our work, we will not give classical definitions of solutions of this kind, however, Grün does provide specific definitions for this problem due to the additional constraint that π must lie on the simplex $\Delta(\mathscr{I})$. For further reference, the solution concept was created by Crandall and Lions in the 1980's, and their paper (along with coauthor Ishii) [CIL92], contains the fundamentals of viscosity solutions.

Equation (3.38) introduced the 'Hamilton–Jacobi obstacle problem', and the PDE studied by Grün—an example of a 'double obstacle problem'—is given below in (3.42). Let $w(t,x,\pi)$ be a function (later a candidate value), and take the Euclidean inner product, i.e. $\langle \phi(t,x), \pi \rangle = \sum_{i=0}^{\mathscr{I}} \pi_i \phi^i(t,x)$, then the central problem in [Grü13] is given by

$$\max\left\{ \max\left\{ \min\left\{ \left(-\frac{\partial}{\partial t} - \mathcal{L}_X\right)[w], w - \langle g(t,x),\pi \rangle \right\}, w - \langle f(t,x),\pi \rangle \right\}, -\lambda_{\min}\left(\pi, \frac{\partial^2 w}{\partial \pi^2}\right) \right\} = 0. \tag{3.42}$$

The inclusion of the term $-\lambda_{\min}$ can be somewhat nebulous, however, it is routinely referred to as the 'smallest eigenvalue of the symmetric matrix $\frac{\partial^2 w}{\partial \pi^2}$ in the direction of the tangent cone to the simplex' and is related to the convexity of $w(t,x,\pi)$ (see [Car09], [CR09b], and [BFR23]). For completeness, let π belong to the simplex $\Delta(\mathscr{I})$ and A be a symmetric matrix, then we define the tangent cone, $T_{\Delta(\mathscr{I})(\pi)}$, and $\lambda_{\min}(\pi, A)$ as follows

$$T_{\Delta(\mathscr{I})(\pi)} := \overline{\bigcup_{\lambda>0} \frac{\Delta(\mathscr{I}) - \pi}{\lambda}}, \qquad \lambda_{\min}(\pi, A) := \min_{z \in T_{\Delta(\mathscr{I})(\pi)} \setminus \{0\}} \frac{\langle Az, z \rangle}{|z|^2}.$$

Remark 3.5.2. The role of these two objects becomes much clearer in the 2-dimensional case as the tangent cone is simply the set of all vectors of the form $z = (z_1, -z_1)$. The Hessian matrix of second order partial derivatives is then

$$\frac{\partial^2 w}{\partial \pi^2} = \begin{bmatrix} \frac{\partial^2 w}{\partial \pi_1^2} & \frac{\partial^2 w}{\partial \pi_1 \partial \pi_2} \\ \frac{\partial^2 w}{\partial \pi_1 \partial \pi_2} & \frac{\partial^2 w}{\partial \pi_2^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 w}{\partial \pi_1^2} & -\frac{\partial^2 w}{\partial \pi_1^2} \\ -\frac{\partial^2 w}{\partial \pi_1^2} & \frac{\partial^2 w}{\partial \pi_1^2} \end{bmatrix},$$

implying that $\lambda_{\min}(\pi, \frac{\partial^2 w}{\partial \pi^2}) = 2\frac{\partial^2 w}{\partial \pi_1^2}$. If one was to find the eigenvalues for this Hessian in the classical way, one would find $\lambda = 2\frac{\partial^2 w}{\partial \pi_1^2}$ from before, and $\lambda = 0$. The eigenvector for the former is tangent to the tangent cone, and is orthogonal for the latter. Additionally, requiring $-\lambda_{\min} \leq 0$ is easily now seen as a convexity constraint on w.

Theorem 3.5.3. *(Theorem 3.4, [Grü13]) For any $(t,x,\pi) \in [0,T] \times \mathbb{R}^d \times \Delta(\mathscr{I})$ the game has a value $V(t,x,\pi)$ given by*

$$V(t,x,\pi) := \overline{V}(t,x,\pi) = \underline{V}(t,x,\pi),$$

where $V(t,x,\pi) : [0,T] \times \mathbb{R}^d \times \Delta(\mathscr{I}) \to \mathbb{R}$ is characterised as the unique viscosity solution to (3.42) in the class of bounded uniformly continuous functions, which are Lipschitz in π.

Theorem 3.5.3 above is, of course, one of her major results in the paper—providing the existence of a value, and characterising it as the solution to the PDE (3.42)—however, Grün's work does not stop at the existence of a value, she also characterises the informed player's optimal strategy via a 'dual representation'. First, she enlarges her canonical probability space so as to introduce a new process $(p_t)_{t\in[0,T]}$ representing the beliefs of the uninformed player. For a given π on the simplex, she denotes by $\mathscr{P}(t,\pi)$ the set of all probability measures which make $(p_t)_{t\in[0,T]}$ a martingale and keep the underlying process, $(W_t)_{t\in[0,T]}$ (see (3.39)), a Brownian motion. A new payoff is constructed to weight the payoffs with respect to the belief process—much like we do in (3.16)—and is given by

$$\begin{aligned} J_{t-}(x,\tau,\sigma,P) := \mathbb{E}_P\Big[&\langle p_\tau, f(\tau, X_\tau^{t,x})\rangle \mathbb{1}_{\{\tau\leq\sigma,\tau<T\}} \\ &+ \langle p_\sigma, g(\sigma, X_\sigma^{t,x})\rangle \mathbb{1}_{\{\sigma<\tau\leq T\}} + \langle p_T, h(X_T^{t,x})\rangle \mathbb{1}_{\{\tau=\sigma=T\}} \Big| \mathscr{F}_{t-}^2 \Big], \end{aligned}$$

where the notation $\mathbb{E}_P$ denotes the expectation with respect to the probability measure $P \in \mathscr{P}(t,\pi)$.

With some further technical considerations for probability measures, she defines the following

$$W(t,x,\pi) = \operatorname*{ess\,inf}_{P\in\mathscr{P}(t,\pi)} \operatorname*{ess\,inf}_{\eta} \operatorname*{ess\,sup}_{\gamma} J_{t-}(x,\tau,\sigma,P). \tag{3.43}$$

We omit the sets to which the randomised stopping times η and γ belong to due to the technicalities of the expanded canonical space. She then proves that this function $W(t,x,\pi)$ coincides with the value $V(t,x,\pi)$, and this is what she calls the 'dual representation' of the game.

Grün gives an intuitive explanation as to how she arrives at the optimal strategies for the informed player which we will repeat here. Let us begin by assuming that there exists an optimal $\overline{P}$ in $\mathscr{P}(t,\pi)$ which attains the outermost essential infimum in (3.43)—of course this is a big assumption, and Grün herself uses ε-optimal methods in her proofs, but the intuition is still valid. Much like Theorems 1.4.5 and 2.5.1, Grün formulates her optimal strategy to be the first entry time of a given set. Define

$$D = \Big\{(t,x,\pi) \in [0,T]\times\mathbb{R}^d\times\Delta(\mathscr{I}) : V(t,x,\pi) \geq \langle f(t,x),\pi\rangle\Big\}$$

which is closed by the continuity of $V(t,x,\pi)$ and $f(t,x)$. Then the optimal strategy of the informed player is given by

$$\tau^* = \inf\big\{s\in[t,T] : (s,X_s^{t,x},p_s)\in D\big\}.$$

* * *

To summarise, Grün's work can be considered to be the Markovian equivalent to our set-up, except she approaches the problem via PDEs. She is able to prove the existence of a value as the solution to the PDE (3.42), and, moreover, she characterises the optimal strategies of the informed player. However, as will become clearer during Chapter 4, her analysis is limited to only a characterisation of the informed player's optimal strategy due to the structure of the PDE. We will characterise optimal strategies for both players. The key tool which allows us to do this, and which differentiates our analysis from hers, is that we will consider a set of values $\{\mathcal{V}^i(\rho), \rho\in\mathscr{T}(\mathscr{F}_t^2)\}$, which describe the value of the game to the i^{th} incarnation of the informed player. This does not make any strict or rigorous sense now, however, we will return to this sentiment in the concluding remarks of Chapter 4. For now, we can say that the value described by Grün only uses the full payoff $J(x,\eta,\gamma)$, which is a weighted average of all payoffs $J^i(x,\eta^i,\gamma)$. In effect this is the value of the game as seen by the uninformed player—someone who cannot tell the different payoffs apart. If we consider the individual $J^i(x, \eta^i, \gamma)$ payoffs themselves, we can construct the value as seen by the informed player. In Chapter 4 we will show that these constructions are vital to allow us to analyse the optimal behaviour of the uninformed player.

CHAPTER 4

NECESSARY CONDITIONS

In Chapter 3 we proved the existence of a value at any $(\mathscr{F}_t^2)$-stopping time ρ, leaving us with a collection of random variables $\{V(\rho) : \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$. Of course, we would like to aggregate this family into a value process, however, the direct approach to aggregation has some technical challenges. To circumvent this, we will consider a second family of random variables, $\{M(\rho) : \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$, which combines the value of the game at ρ with the cumulative payoff along the optimal strategies up until ρ. Not only will we aggregate this second family—and, by proxy, aggregate the value—but we will also prove it is martingale. This formulation is incredibly powerful as we will show that we can 'truncate' the time-zero optimal strategies at any ρ to obtain a new pair of optimal strategies for the subgame (as long as the game is still ongoing using the optimal strategies).

Moreover, we show that we can construct two further families, $\{M(\rho;\xi^*,\zeta),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$ and $\{M^i(\rho;\xi^i,\zeta^*),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$ (for $i\in[\mathscr{I}]$), related to the original martingale family which can be aggregated and are super- and submartingales. These three martingales are then enough for us to analyse the properties of the optimal strategies of the game.

4.1 A MARTINGALE ARISING FROM THE VALUE

To begin with, recall that we know optimal strategies for both players exist at time zero, as guaranteed by taking $\rho=0$ in Theorem 2.5 3.4.10 and recalling that $\mathscr{F}_0^2$ is trivial under Assumption 3.3.1. Let $\tau^*\in\mathscr{T}(\mathscr{F}_t^1)$ be the optimal strategy for the (informed) minimiser, and let $\xi^*\in\mathscr{A}(\mathscr{F}_t^1)$ be its generating process. Let us consider the following probability, $\mathbb{P}(\Theta=i|\mathscr{F}_t^2\cap\{\tau^*>t\})$. What we are asking ourselves is, given that the informed player is playing optimally, what do we think the outcome of Θ was given that the informed player has not stopped before time t? In Proposition 4.1.1 we use Bayes' theorem and Lemma 3.2.4, to get a more usable formulation.

Proposition 4.1.1. *Let $\tau^*\in\mathscr{T}(\mathscr{F}_t^1)$ be the minimiser's optimal strategy at time zero and let $\xi^*\in\mathscr{A}(\mathscr{F}_t^1)$ be the corresponding generating process, then*

$$\mathbb{P}\big(\Theta=i\big|\mathscr{F}_t^2\cap\{\tau^*>t\}\big)=\frac{\pi_i(1-\xi_t^{i,*})}{\sum_{j=0}^{\mathscr{I}}\pi_j(1-\xi_t^{j,*})}. \tag{4.1}$$

Proof. A version of Bayes' theorem exists for conditioning on two events. In fact this can be derived using the definition of conditional probability, however, for brevity we simply point to [KF09] Section 2.1.2.2. For three events A, B, and C we have

$$\mathbb{P}(A|B\cap C)=\frac{\mathbb{P}(B|A\cap C)\mathbb{P}(A|C)}{\mathbb{P}(B|C)}.$$

We can take $B=\{\tau^*>t\}$ and $C=\mathscr{F}_t^2$, and then, utilising the law of total probability to say that $\mathbb{P}(\tau^*>t|\mathscr{F}_t^2)=\sum_{j=0}^{\mathscr{I}}\mathbb{P}(\tau^*>t|\mathscr{F}_t^2\cap\{\Theta=j\})\mathbb{P}(\Theta=j)$, we get

$$\mathbb{P}\big(\Theta=i\big|\mathscr{F}_t^2\cap\{\tau^*>t\}\big)=\frac{\mathbb{P}\big(\tau^*>t\big|\mathscr{F}_t^2\cap\{\Theta=i\}\big)\mathbb{P}\big(\Theta=i\big|\mathscr{F}_t^2\big)}{\sum_{j=0}^{\mathscr{I}}\mathbb{P}\big(\tau^*>t\big|\mathscr{F}_t^2\cap\{\Theta=j\}\big)\mathbb{P}\big(\Theta=j\big)}. \tag{4.2}$$

Let us consider the probability $\mathbb{P}\big(\tau^*>t\big|\mathscr{F}_t^2\cap\{\Theta=i\}\big)$. Notice that $\mathscr{F}_t^2\cap\{\Theta=i\}\subset\mathscr{F}_T^1$ for all

$i \in [\mathscr{I}]$. As a result, we can use the tower property of conditional expectation to say

$$\mathbb{P}\big(\tau^* > t \big| \mathscr{F}_t^2 \cap \{\Theta = i\}\big) = \mathbb{E}\big[\mathbb{1}_{\{\tau^*>t\}} \big| \mathscr{F}_t^2 \cap \{\Theta = i\}\big] = \mathbb{E}\big[\mathbb{E}\big[\mathbb{1}_{\{\tau^*>t\}} \big| \mathscr{F}_T^1\big] \big| \mathscr{F}_t^2 \cap \{\Theta = i\}\big].$$

Lemma 3.2.4 then gives us that $\mathbb{E}\big[\mathbb{1}_{\{\tau^*>t\}} \big| \mathscr{F}_T^1\big] = 1 - \xi_t^*$. Moreover, Lemma 3.3.2 tells us that $\tau^* = \tau^{i,*}$ on $\{\Theta = i\}$, and so

$$\mathbb{P}\big(\tau^* > t \big| \mathscr{F}_t^2 \cap \{\Theta = i\}\big) = \mathbb{E}\big[1 - \xi_t^* \big| \mathscr{F}_t^2 \cap \{\Theta = i\}\big] = 1 - \xi_t^{i,*},$$

since $1 - \xi_t^{i,*}$ is $(\mathscr{F}_t^2 \cap \{\Theta = i\})$-measurable. Thus, (4.2) becomes

$$\mathbb{P}\big(\Theta = i \big| \mathscr{F}_t^2 \cap \{\tau^* > t\}\big) = \frac{(1 - \xi_t^{i,*})\mathbb{P}\big(\Theta = i \big| \mathscr{F}_t^2\big)}{\sum_{j=0}^{\mathscr{I}} (1 - \xi_t^{i,*})\mathbb{P}(\Theta = j)}.$$

Of course, $\mathbb{P}(\Theta = j) = \pi_j$ by definition, and $\mathbb{P}\big(\Theta = i \big| \mathscr{F}_t^2\big) = \pi_i$ since Θ is independent of $\mathscr{F}_t^2$. Thus the desired claim holds. ■

Notice that this probability $\mathbb{P}(\Theta = i | \mathscr{F}_t^2 \cap \{\tau^* > t\})$ is really what the object p^i was in (3.23). If we were to simply construct a belief process in this way it would be somewhat naïve, this is because it does not account for what happens at the exact end of the game, or more precisely, when the minimiser has stopped. In this case the denominator of (4.1) becomes zero and we need this process to take values on a simplex at all times. This motivates the following definition which utilises the Kronecker delta, δ_{ij}. Recall that $\delta_{ij} = 1$ if $i = j$, and equals zero otherwise.

Definition 4.1.2. Let $\xi^* \in \mathscr{A}(\mathscr{F}_t^1)$ be the minimiser's optimal generating process, then the process $\Pi_t^* = (\Pi_t^{0,*}, \Pi_t^{1,*}, ..., \Pi_t^{\mathscr{I},*})$ defined as

$$\Pi_t^{i,*} := \begin{cases} \frac{\pi_i(1-\xi_t^{i,*})}{\sum_{j=0}^{\mathscr{I}} \pi_j(1-\xi_t^{j,*})} & \text{if } \sum_{j=0}^{\mathscr{I}} \pi_j(1-\xi_t^{j,*}) > 0 \\ \delta_{i0} & \text{if } \sum_{j=0}^{\mathscr{I}} \pi_j(1-\xi_t^{j,*}) = 0 \end{cases} \tag{4.3}$$

is called the belief process.

Remark 4.1.3. If $\sum_{j=0}^{\mathscr{I}} \pi_j(1-\xi_t^{j,*}) = 0$, then $\Pi_t^* = (1, 0, ..., 0) \in \Delta^{\mathscr{I}}$, otherwise, it is clear that its elements are all non-negative and sum to one. Moreover, we see that $\Pi_{0-}^{i,*} = \pi_i$ for all $i \in [\mathscr{I}]$.

Proposition 4.1.4. *The belief process $(\Pi_t^*)_{t\in[0,T]}$, and its left-limit process $\Pi_{t-}^* := \lim_{s\uparrow\uparrow t}\Pi_s^*$, are $(\mathscr{F}_t^2)$-adapted and, for any $(\mathscr{F}_t^2)$-stopping time ρ, $\Pi_{\rho-}^*$ is $\mathscr{F}_\rho^2$-measurable.*

Proof. To begin, notice that that terms π_i, $\xi_t^{i,*}$, and δ_{i0} are $(\mathscr{F}_t^2)$-measurable; therefore, so are the sets $\{\sum_{j=0}^{\mathscr{I}}\pi_j(1-\xi_t^{j,*})>0\}$ and $\{\sum_{j=0}^{\mathscr{I}}\pi_j(1-\xi_t^{j,*})=0\}$. Writing (4.3) as

$$\Pi_t^{i,*} = \frac{\pi_i(1-\xi_t^{i,*})}{\sum_{j=0}^{\mathscr{I}}\pi_j(1-\xi_t^{j,*})}\mathbb{1}_{\{\sum_{j=0}^{\mathscr{I}}\pi_j(1-\xi_t^{j,*})>0\}} + \delta_{i0}\mathbb{1}_{\{\sum_{j=0}^{\mathscr{I}}\pi_j(1-\xi_t^{j,*})=0\}}, \tag{4.4}$$

we see it is a measurable by construction. Moreover, (4.4) shows that $(\Pi_t^*)_{t\in[0,T]}$ is càdlàg and $(\mathscr{F}_t^2)$-adapted because the process $(\xi_t^{i,*})_{t\in[0,T]}$ is càdlàg and $(\mathscr{F}_t^2)$-adapted.

The left-limit process $(\Pi_{t-}^{i,*})_{t\in[0,T]}$ is well-defined, and again without loss of generality we may take this left-limit over strictly increasing sequences of rational numbers, since $\mathbb{Q}$ is dense in $\mathbb{R}$. Theorem IV.17 from [DM78] then tells us that $(\Pi_{t-}^{i,*})_{t\in[0,T]}$ is progressively measurable with respect to $(\mathscr{F}_t^2)$. Since it is progressively measurable, it is also adapted. Finally, Proposition 1.2.18 from [KS98a] then tells us that $\Pi_{\rho-}^{i,*}$ is $\mathscr{F}_\rho^2$-measurable. ■

Now that we have a formal concept of a belief process, our definition of the conditional expected payoff from Definition 3.4.11 can be re-stated here, but now in its proper context.

Definition 4.1.5. Let ρ be an $(\mathscr{F}_t^2)$-stopping time, $(\Pi_t^*)_{t\in[0,T]}$ be the belief process and let (ξ,ζ) be a pair of generating processes belonging to $\mathscr{A}_\rho(\mathscr{F}_t^1)\times\mathscr{A}_\rho(\mathscr{F}_t^2)$. The family of random variables $\{J_\rho^{\Pi^*}(\xi,\zeta):\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$ given by

$$J_\rho^{\Pi^*}(\xi,\zeta) := \sum_{i=0}^{\mathscr{I}}\Pi_{\rho-}^{i,*}\mathbb{E}\left[\int_{[\rho,T)} f_t^i(1-\zeta_t)\,d\xi_t^i + \int_{[\rho,T)} g_t^i(1-\xi_t^i)\,d\zeta_t + \sum_{t\in[\rho,T]} h_t^i\Delta\zeta_t\Delta\xi_t^i \,\middle|\, \mathscr{F}_\rho^2\right] \tag{4.5}$$

is called the conditional expected payoff at time ρ.

Remark 4.1.6. Notice that by taking $\rho=0$ we retrieve (3.13).

Proposition 4.1.7. *Let ρ be an $(\mathscr{F}_t^2)$-stopping time. Define the upper and lower conditional values of a game as*

$$\overline{V}^{\Pi^*}(\rho) = \operatorname*{ess\,inf}_{\xi\in\mathscr{A}_\rho(\mathscr{F}_t^1)}\operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J_\rho^{\Pi^*}(\xi,\zeta), \qquad \underline{V}^{\Pi^*}(\rho) = \operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)}\operatorname*{ess\,inf}_{\xi\in\mathscr{A}_\rho(\mathscr{F}_t^1)} J_\rho^{\Pi^*}(\xi,\zeta).$$

The value of this game exist and we denote it by $V^{\Pi^*}(\rho) := \overline{V}(\rho) = \underline{V}(\rho)$.

Proof. This follows immediately from Corollary 3.4.18. ■

We now need to introduce a new concept, one which will be prolific in this work and which allows us to adapt strategies to still be valid at future times. Up until now we have had a concept of a strategy being admissible at a stopping time, but we would like a way to take a generating process admissible at the beginning of the game, and construct a new one which is admissible at ρ. Note that we need not actually stop there; why not have a strategy admissible at ρ and construct one admissible at $\rho' > \rho$? This is related to the concept of time-consistency, and we will see this in Section 4.2.

Definition 4.1.8. Let ρ be an $(\mathscr{F}_t^2)$-stopping time and $\chi \in \mathscr{A}(\mathscr{F}_t^i)$, for either $i \in \{1,2\}$, a generating process. Define

$$\chi_t^\rho := \frac{\chi_t - \chi_{\rho-}}{1 - \chi_{\rho-}} \mathbb{1}_{[\rho,T]}(t),$$

with the convention that, if $\chi_{\rho-} = 1$, then $\chi_t^\rho := \mathbb{1}_{\{t=T\}}(t)$—this is called the cemetery point. We call this new process $(\chi_t)_{t\in[0,T]}$ truncated at ρ, or a subgame strategy.

Remark 4.1.9. We use the specific notation of $(\overline{\chi}_t^\rho)_{t\in[0,T]}$ if $(\chi_t)_{t\in[0,T]}$ is an optimal strategy.

Proposition 4.1.10. *For any $\chi \in \mathscr{A}(\mathscr{F}_t^2)$ and $(\mathscr{F}_t^2)$-stopping time ρ, the truncated strategy at ρ is an admissible strategy at ρ, i.e. $\chi^\rho \in \mathscr{A}_\rho(\mathscr{G}_t)$.*

Proof. Since $(\chi_t)_{t\in[0,T]}$ is $(\mathscr{F}_t^2)$-adapted, non-decreasing and càdlàg, so is $(\chi_t^\rho)_{t\in[0,T]}$. The left limit at ρ is zero since $\chi_t^\rho = 0$ for all $t < \rho$. Finally, $\chi_T = 1$ which implies $\chi_T^\rho = 1$. ■

As we can see from Definition 4.1.8, the value of $1 - \chi_{\rho-}$ is important. Recall Remark 3.2.5 wherein we discuss the concept of a player 'gradually stopping'; it is here where this type of language gives us more freedom of understanding.

One can consider this an indicator of whether the game is still being played, for instance, if $1 - \chi_{\rho-} = 0$, then the player using $(\chi_t)_{t\in[0,T]}$ has stopped completely—there is no more room for stopping in their generating process—however, if $1 - \chi_{\rho-} > 0$, then the game has not finished strictly before ρ.

Because of this observation, we introduce a family of sets which indicate which players have stopped, or not, at any given stopping time ρ.

Definition 4.1.11. Let ρ be an $(\mathscr{F}_t^2)$-stopping time and (ξ^*, ζ^*) be the generating processes of the optimal randomised stopping times at time zero. Define the following sets

1. $\Gamma_\rho := \{\omega \in \Omega : 1 - \zeta^*_{\rho-}(\omega) > 0\}$,

2. $\Lambda^i_\rho := \{\omega \in \Omega : \pi_i(1 - \xi^{i,*}_{\rho-}(\omega)) > 0\}$,

3. $\Omega^i_\rho := \Lambda^i_\rho \cap \Gamma_\rho = \{\omega \in \Omega : \pi_i(1 - \xi^{i,*}_{\rho-}(\omega))(1 - \zeta^*_{\rho-}(\omega)) > 0\}$,

4. $\Omega_\rho := \{\omega \in \Omega : \sum_{i=0}^{\mathscr{I}} \pi_i(1 - \xi^{i,*}_{\rho-}(\omega))(1 - \zeta^*_{\rho-}(\omega)) > 0\}$.

Remark 4.1.12. We do not need to define these sets specifically for the optimal generating processes, and in fact we will use sets of this form for general generating processes later when considering subgame-perfect Nash equilibria, but for now we use the optimal processes since these are the ones required for this chapter.

As mentioned, these sets all have a physical intuition: on Γ_ρ, the maximiser has not stopped before ρ; on Λ^i_ρ, the i^{th} minimiser has not stopped before ρ; on Ω^i_ρ, neither the i^{th} minimiser nor the maximiser have stopped before ρ; and on Ω_ρ, neither player has stopped before ρ. These sets can also be related to each other.

Proposition 4.1.13. *Under the assumptions of Definition 4.1.11, we have*

1. $\Lambda_\rho := \bigcup_{i\in[\mathscr{I}]} \Lambda^i_\rho \equiv \{\omega \in \Omega : \sum_{i=0}^{\mathscr{I}} \pi_i(1 - \xi^{i,*}_{\rho-}) > 0\}$,

2. $\Omega_\rho = \bigcup_{i\in[\mathscr{I}]} \Omega^i_\rho$,

3. $\Omega_\rho = \Lambda_\rho \cap \Gamma_\rho$.

Proof. 1. Let $\omega \in \Lambda_\rho$, then $\omega \in \Lambda^i_\rho$ for at least one $i \in [\mathscr{I}]$, say $\omega \in (\Lambda^k_\rho)_{k\in I}$ for some $I \subset [\mathscr{I}]$. Thus, $\pi_k(1 - \xi^{k,*}_{\rho-}) > 0$ for all $k \in I$, implying that $\sum_{i=0}^{\mathscr{I}} \pi_i(1 - \xi^{i,*}_{\rho-}) > 0$, since all terms in this sum are non-negative. Hence $\Lambda_\rho \subseteq \{\sum_{i=0}^{\mathscr{I}} \pi_i(1 - \xi^{i,*}_{\rho-}) > 0\}$. Next, let $\omega \in \{\sum_{i=0}^{\mathscr{I}} \pi_i(1 - \xi^{i,*}_{\rho-}) > 0\}$, thus at least one term in this sum must be positive, say $\pi_k(1 - \xi^{k,*}_{\rho-}) > 0$ for all $k \in J \subset [\mathscr{I}]$. Therefore for any $k \in J$, $\omega \in \Lambda^k_\rho \subset \Lambda_\rho$.

2. The exact same reasoning allows us to conclude the second result.

3. Simply, for $\omega \in \Lambda_\rho \cap \Gamma_\rho$, both $\sum_{i=0}^{\mathscr{I}} \pi_i(1 - \xi^{i,*}_{\rho-}) > 0$ and $1 - \zeta^*_{\rho-} > 0$, meaning $\omega \in \Omega_\rho$. Finally, if $\omega \in \Omega_\rho$ then, again, both $\sum_{i=0}^{\mathscr{I}} \pi_i(1 - \xi^{i,*}_{\rho-}) > 0$ and $1 - \zeta^*_{\rho-} > 0$.

■

Remark 4.1.14. Proposition 4.1.13 tells us that we can formulate the left-limit process $(\Pi^{i,*}_{t-})_{t\in[0,T]}$ (which plays an important role in (4.5)) explicitly as

$$\Pi^{i,*}_{t-}(\omega) = \begin{cases} \frac{\pi_i(1-\xi^{i,*}_{t-})}{\sum_{k=0}^{\mathscr{I}} \pi_k(1-\xi^{k,*}_{t-})} & \text{if } \omega \in \Lambda_t, \\ \delta_{i0} & \text{if } \omega \in (\Lambda_t)^c. \end{cases}$$

We are now in a position to construct a martingale, and to use it to aggregate the value. Before commenting further, let us introduce a new family of random variables.

Definition 4.1.15. Let (ξ^*, ζ^*) be the generating processes of the optimal strategies at time zero, and let ρ be an $(\mathscr{F}^2_t)$-stopping time. Define the following family of random variables $\{M(\rho) : \rho \in \mathscr{T}(\mathscr{F}^2_t)\}$ by

$$M(\rho) := \sum_{i=0}^{\mathscr{I}} \pi_i \Bigg(V^{\Pi^*}(\rho)(1-\xi^{i,*}_{\rho-})(1-\zeta^*_{\rho-}) + \int_{[0,\rho)} f^i_s(1-\zeta^*_s)\, d\xi^{i,*}_s + \int_{[0,\rho)} g^i_s(1-\xi^{i,*}_s)\, d\zeta^*_s + \sum_{s\in[0,\rho)} h^i_s \Delta\zeta^*_s \Delta\xi^{i,*}_s \Bigg).$$

Sometimes we may use the notation $M(\rho; \xi^*, \zeta^*)$ to emphasis the dependence on the strategies.

We will prove in what follows that this family of random variables is a $\mathscr{T}(\mathscr{F}^2_t)$-martingale system and can be aggregated. Before that, we begin by showing that the family $\{M(\rho) : \rho \in \mathscr{T}(\mathscr{F}^2_t)\}$ is constant in expectation. In order to do this we present a technical lemma which introduces constructed generating processes—designed to be optimal up until ρ and then generic afterwards—and writes the expected payoff in terms of this construction.

Lemma 4.1.16. *Let ρ be an $(\mathscr{F}^2_t)$-stopping time and (ξ^*, ζ^*) be the generating processes of the optimal randomised stopping times at time zero belonging to $\mathscr{A}(\mathscr{F}^1_t) \times \mathscr{A}(\mathscr{F}^2_t)$. Define, for any $\eta \in \mathscr{A}_\rho(\mathscr{F}^1_t)$ and $\chi \in \mathscr{A}_\rho(\mathscr{F}^2_t)$,*

$$\hat{\xi}_t := \xi^*_t \mathbb{1}_{[0,\rho)}(t) + \Big[\xi^*_{\rho-} + (1-\xi^*_{\rho-})\eta_t\Big] \mathbb{1}_{[\rho,T]}(t), \tag{4.6}$$

$$\hat{\zeta}_t := \zeta^*_t \mathbb{1}_{[0,\rho)}(t) + \Big[\zeta^*_{\rho-} + (1-\zeta^*_{\rho-})\chi_t\Big] \mathbb{1}_{[\rho,T]}(t). \tag{4.7}$$

Then $\hat{\xi} \in \mathscr{A}(\mathscr{F}_t^1)$, $\hat{\zeta} \in \mathscr{A}(\mathscr{F}_t^2)$, and the following two equalities hold:

$$\sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}[P^i(\hat{\xi}^i, \zeta^*)] = \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\Bigg[\int_{[0,\rho)} f_s^i(1-\zeta_s^*)\, d\xi_s^{i,*} + \int_{[0,\rho)} g_s^i(1-\xi_s^{i,*})\, d\zeta_s^* + \sum_{s\in[0,\rho)} h_s^i \Delta\zeta_s^* \Delta\xi_s^{i,*} \\ + (1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*}) J_\rho^{\Pi^*}(\eta, \overline{\zeta}^\rho) \Bigg], \quad (4.8)$$

and

$$\mathbb{E}[P(\xi^*, \hat{\zeta})] = \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\Bigg[\int_{[0,\rho)} f_s^i(1-\zeta_s^*)\, d\xi_s^{i,*} + \int_{[0,\rho)} g_s^i(1-\xi_s^{i,*})\, d\zeta_s^* + \sum_{s\in[0,\rho)} h_s^i \Delta\zeta_s^* \Delta\xi_s^{i,*} \\ + (1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*}) J_\rho^{\Pi^*}(\overline{\xi}^\rho, \chi) \Bigg]. \quad (4.9)$$

Proof. We will only present the proof for the case of $\hat{\xi}_t$ and (4.8), because the proof for the remainder is identical. First we show that the constructed strategy is a valid strategy. Clearly $\hat{\xi}_{0-} = 0$ and $\hat{\xi}_T = 1$ (since $\eta_T = 1$). Also, $\hat{\xi}_t$ is $(\mathscr{F}_t^1)$-adapted and càdlàg by construction. Finally, $(\hat{\xi}_t)_{t\in[0,T]}$ is clearly increasing on both $[0,\rho)$ and $(\rho, T]$, and then at the point ρ we see

$$\hat{\xi}_\rho = \xi_{\rho-}^* + (1-\xi_{\rho-}^*)\eta_\rho \geq \xi_{\rho-}^*,$$

and so $(\hat{\xi}_t)_{t\in[0,T]}$ is non-decreasing on $[0,T]$. Thus $\hat{\xi} \in \mathscr{A}(\mathscr{F}_t^1)$.

Next, we move to prove the main statement of the theorem. Using the definition of $(\hat{\xi}_t^i)_{t\in[0,T]}$, and splitting the integrals and the summation in the payoff around ρ, we find

$$\sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}[P^i(\hat{\xi}^i, \zeta^*)] = \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\Bigg[\int_{[0,\rho)} f_s^i(1-\zeta_s^*)\, d\xi_s^{i,*} + \int_{[0,\rho)} g_s^i(1-\xi_s^{i,*})\, d\zeta_s^* + \sum_{s\in[0,\rho)} h_s^i \Delta\zeta_s^* \Delta\xi_s^{i,*} \\ + (1-\xi_{\rho-}^{i,*}) \Bigg(\int_{[\rho,T)} f_s^i(1-\zeta_s^*)\, d\eta_s^i + \int_{[\rho,T)} g_s^i(1-\eta_s^i)\, d\zeta_s^* + \sum_{s\in[\rho,T]} h_s^i \Delta\zeta_s^* \Delta\eta_s^i \Bigg) \Bigg],$$

where, for the jump terms, we use that: if $0 \leq s < \rho$, $\Delta\hat{\xi}_s^i = \Delta\xi_s^{i,*}$; and if $s \geq \rho$, $\Delta\hat{\xi}_s^i = (1-\xi_{\rho-}^{i,*})\Delta\eta_s^i$.

We can trivially introduce $(\mathbb{1}_{\Omega_\rho} + \mathbb{1}_{(\Omega_\rho)^c})$ to the start of the second term. Notice that all terms multiplied by $\mathbb{1}_{(\Omega_\rho)^c}$ will disappear from the equation. This is because, on $(\Omega_\rho)^c$, we have two cases; the first is that $\zeta_{\rho-}^* = 1$, meaning that ζ^* will continuously approach one at ρ, and so $\zeta_t^* = 1$ for all $t \geq \rho$, moreover, there is no jump at ρ either, and hence both integrals and the summation will be zero. The second is if $\sum_{i=0}^{\mathscr{I}} \pi_i(1-\xi^{i,*}_{\rho-}) = 0$, then, since every term in this sum is non-negative, it implies that every term must be zero, hence all integrals and sums are nullified by the prefactor $\pi_i(1-\xi^{i,*}_{\rho-})$. Of course, both cases can happen together, but this changes nothing in the argument. We are then left with

$$\sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}[P^i(\hat{\xi}^i,\zeta^*)] = \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\bigg[\int_{[0,\rho)} f^i_s(1-\zeta^*_s)\,d\xi^{i,*}_s + \int_{[0,\rho)} g^i_s(1-\xi^{i,*}_s)\,d\zeta^*_s + \sum_{s\in[0,\rho)} h^i_s \Delta\zeta^*_s \Delta\xi^{i,*}_s$$
$$+ \mathbb{1}_{\Omega_\rho}(1-\xi^{i,*}_{\rho-})\bigg(\int_{[\rho,T)} f^i_s(1-\zeta^*_s)\,d\eta^i_s + \int_{[\rho,T)} g^i_s(1-\eta^i_s)\,d\zeta^*_s + \sum_{s\in[\rho,T]} h^i_s\Delta\zeta^*_s\Delta\eta^i_s\bigg)\bigg].$$

Moreover, due to the presence of $\mathbb{1}_{\Omega_\rho}$, we know that $1-\zeta^*_{\rho-} > 0$, and so we are free to truncate $(\zeta^*_t)_{t\in[0,T]}$. If we then define the following payoff

$$P^i(\eta^i,\overline{\zeta}^\rho) = \int_{[\rho,T)} f^i_s(1-\overline{\zeta}^\rho_s)\,d\eta^i_s + \int_{[\rho,T)} g^i_s(1-\eta^i_s)\,d\overline{\zeta}^\rho_s + \sum_{s\in[\rho,T]} h^i_s\Delta\overline{\zeta}^\rho_s\Delta\eta^i_s.$$

we can obtain

$$\sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}[P^i(\hat{\xi}^i,\zeta^*)] = \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\bigg[\int_{[0,\rho)} f^i_s(1-\zeta^*_s)\,d\xi^{i,*}_s + \int_{[0,\rho)} g^i_s(1-\xi^{i,*}_s)\,d\zeta^*_s + \sum_{s\in[0,\rho)} h^i_s \Delta\zeta^*_s \Delta\xi^{i,*}_s$$
$$+ \mathbb{1}_{\Omega_\rho}(1-\xi^{i,*}_{\rho-})(1-\zeta^*_{\rho-})P^i(\eta^i,\overline{\zeta}^\rho)\bigg]. \quad (4.10)$$

We can split this expectation into two and bring the summation and π_i inside. Notice that in the final term of (4.10), all of the prefactors $\mathbb{1}_{\Omega_\rho}(1-\xi^{i,*}_{\rho-})(1-\zeta^*_{\rho-})$ are $(\mathscr{F}^2_\rho)$-measurable, so we can use the tower property of conditional expectation to get

$$\sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}[P^i(\hat{\xi}^i,\zeta^*)] = \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\bigg[\int_{[0,\rho)} f^i_s(1-\zeta^*_s)\,d\xi^{i,*}_s + \int_{[0,\rho)} g^i_s(1-\xi^{i,*}_s)\,d\zeta^*_s + \sum_{s\in[0,\rho)} h^i_s \Delta\zeta^*_s \Delta\xi^{i,*}_s\bigg]$$
$$+ \mathbb{E}\bigg[\mathbb{1}_{\Omega_\rho}(1-\zeta^*_{\rho-})\sum_{i=0}^{\mathscr{I}} \pi_i(1-\xi^{i,*}_{\rho-})\mathbb{E}\Big[P^i(\eta^i,\overline{\zeta}^\rho)\Big|\mathscr{F}^2_\rho\Big]\bigg]. \quad (4.11)$$

Since $\Omega_\rho \subset \Lambda_\rho$, Remark 4.1.14 tells us that on Ω_ρ, the belief process, $(\Pi^{i,*}_t)_{t\in[0,T]}$, takes a specific form which we can rearrange to obtain

$$\pi_i(1-\xi^{i,*}_{\rho-}) = \Pi^{i,*}_{\rho-}\sum_{k=0}^{\mathscr{I}} \pi_k(1-\xi^{k,*}_{\rho-}).$$

As a result, we can write the second expectation term in (4.11) as

$$\mathbb{E}\left[\left(\mathbb{1}_{\Omega_\rho}(1-\zeta^*_{\rho-})\sum_{k=0}^{\mathscr{I}}\pi_k(1-\xi^{k,*}_{\rho-})\right)J^{\Pi^*}_\rho(\eta,\overline{\zeta}^\rho)\right].$$

Hence the result follows. ■

We are now in a position to prove that the family $\{M(\rho),\rho\in\mathscr{T}(\mathscr{F}^2_t)\}$ is constant in expectation. While this property is necessary for the martingale property, it is not sufficient; however, this result, which we show in Theorem 4.1.17, will be enough to later prove the martingale property.

First, recall that the time zero optimal generating processes (ξ^*,ζ^*) are a Nash equilibrium, and are thus best responses to each other. Remark 3.2.2 tells us that $V^{\Pi^*}(0)=J^{\Pi^*}_0(\xi^*,\zeta^*)$, and therefore $V^{\Pi^*}(0)\le J^{\Pi^*}_0(\xi^*,\zeta^*)$ and $V^{\Pi^*}(0)\ge J^{\Pi^*}_0(\xi^*,\zeta^*)$ for any generating processes $(\xi_t)_{t\in[0,T]}$ and $(\zeta_t)_{t\in[0,T]}$ belonging to $\mathscr{A}(\mathscr{F}^1_t)$ and $\mathscr{A}(\mathscr{F}^2_t)$ respectively.

Theorem 4.1.17. *The family of random variables $\{M(\rho),\rho\in\mathscr{T}(\mathscr{F}^2_t)\}$ has constant expectation, i.e. for any $\rho\in\mathscr{T}(\mathscr{F}^2_t)$*

$$\mathbb{E}M(\rho)=M(0).$$

Proof. Let ρ be any $(\mathscr{F}^2_t)$-stopping time. Notice that $M(0)=V^{\Pi^*}(0)$, and recall $(\hat{\xi}_t)_{t\in[0,T]}$ from (4.6). Since $(\hat{\xi}_t)_{t\in[0,T]}$ may be suboptimal for the minimiser we know that $V^{\Pi^*}(0)\le\mathbb{E}[P(\hat{\xi},\zeta^*)]$. From this, we can use (4.8) and bound the conditional expected payoff by the essential supremum over ζ. Thus, for any $\eta\in\mathscr{A}_\rho(\mathscr{F}^1_t)$, we have

$$V^{\Pi^*}(0)\le\sum_{i=0}^{\mathscr{I}}\pi_i\mathbb{E}\Bigg[\int_{[0,\rho)}f^i_s(1-\zeta^*_s)\,d\xi^{i,*}_s+\int_{[0,\rho)}g^i_s(1-\xi^{i,*}_s)\,d\zeta^*_s+\sum_{s\in[0,\rho)}h^i_s\Delta\zeta^*_s\Delta\xi^{i,*}_s$$
$$+(1-\zeta^*_{\rho-})(1-\xi^{i,*}_{\rho-})\operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}^2_t)}J^{\Pi^*}_\rho(\eta,\zeta)\Bigg].\quad(4.12)$$

Lemma 3.4.16 tells us that the family $\{\operatorname{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}^2_t)}J^{\Pi^*}_\rho(\eta,\zeta),\eta\in\mathscr{A}_\rho(\mathscr{F}^1_t)\}$ is downwards directed, and so its essential infimum is reached, pointwise, by a sequence $(\eta^n)_{n\in\mathbb{N}}\subset\mathscr{A}_\rho(\mathscr{F}^1_t)$. Since (4.12) holds for any $\eta\in\mathscr{A}_\rho(\mathscr{F}^1_t)$ it must also hold for any η^n. Using Lemma 3.4.14, and the dominated convergence theorem, we can pass the limit as $n\to\infty$ inside of the expectation to obtain

$$V^{\Pi^*}(0) \leq \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\Big[\int_{[0,\rho)} f_s^i (1-\zeta_s^*)\, d\xi_s^{i,*} + \int_{[0,\rho)} g_s^i (1-\xi_s^{i,*})\, d\zeta_s^* + \sum_{s\in[0,\rho)} h_s^i \Delta\zeta_s^* \Delta\xi_s^{i,*} + (1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*}) \operatorname*{ess\,inf}_{\eta\in\mathscr{A}_\rho(\mathscr{F}_t^1)} \operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J_\rho^{\Pi^*}(\eta,\zeta) \Big].$$

The final term contains the definition of the value $V^{\Pi^*}(\rho)$ and one can then see that the right-hand side is equal to $\mathbb{E}M(\rho)$. Since $M(0) = V^{\Pi^*}(0)$, this means that $M(0) \leq \mathbb{E}M(\rho)$.

We now repeat the same argument for the reverse inequality. Recall the process $(\hat{\zeta}_t)_{t\in[0,T]}$ from (4.36). Using (4.9), the fact that $(\hat{\zeta}_t)_{t\in[0,T]}$ may be suboptimal for the maximiser, and bounding below by the essential infimum over ξ, we can say that, for any $\chi \in \mathscr{A}_\rho(\mathscr{F}_t^2)$,

$$V^{\Pi^*}(0) \geq \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\Big[\int_{[0,\rho)} f_s^i (1-\zeta_s^*)\, d\xi_s^{i,*} + \int_{[0,\rho)} g_s^i (1-\xi_s^{i,*})\, d\zeta_s^* + \sum_{s\in[0,\rho)} h_s^i \Delta\zeta_s^* \Delta\xi_s^{i,*} + (1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*}) \operatorname*{ess\,inf}_{\xi\in\mathscr{A}_\rho(\mathscr{F}_t^1)} J_\rho^{\Pi^*}(\xi,\chi) \Big]. \quad (4.13)$$

Lemma 3.4.16 tells us that the family $\{\operatorname{ess\,inf}_{\xi\in\mathscr{A}_\theta(\mathscr{F}_t^1)} J_\theta^{\Pi^*}(\xi,\chi), \chi \in \mathscr{A}_\rho(\mathscr{F}_t^2)\}$ is upwards directed thus there exists a sequence $(\chi^n)_{n\in\mathbb{N}} \subset \mathscr{A}_\rho(\mathscr{F}_t^2)$ which attains the essential supremum. Since (4.13) holds for all $\chi \in \mathscr{A}_\rho(\mathscr{F}_t^2)$, we know it must also hold for any χ^n. Thus, again using the dominated convergence theorem and Lemma 3.4.14, we can pass the limit into the expectation. The essential supremum is then attained by the limit:

$$V^{\Pi^*}(0) \geq \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\Big[\int_{[0,\rho)} f_s^i (1-\zeta_s^*)\, d\xi_s^{i,*} + \int_{[0,\rho)} g_s^i (1-\xi_s^{i,*})\, d\zeta_s^* + \sum_{s\in[0,\rho)} h_s^i \Delta\zeta_s^* \Delta\xi_s^{i,*} + (1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*}) \operatorname*{ess\,sup}_{\chi\in\mathscr{A}_\rho(\mathscr{F}_t^2)} \operatorname*{ess\,inf}_{\xi\in\mathscr{A}_\rho(\mathscr{F}_t^1)} J_\rho^{\Pi^*}(\xi,\chi) \Big] \quad (4.14)$$

Again, we see that the right-hand side is the same as $\mathbb{E}M(\rho)$, yielding $M(0) \geq \mathbb{E}M(\rho)$. Putting both results together we have that $M(0) = \mathbb{E}M(\rho)$ for any $(\mathscr{F}_t^2)$-stopping time ρ. ■

At the moment, the object of our focus is a family of random variables, but we would prefer to have stochastic processes. In much the same way as we saw in Chapter 2, the way to do this is through $\mathscr{T}(\mathscr{F}_t^2)$-systems; recall the first part of Definition 1.3.4.

Our next task will be to show that the family $\{M(\rho), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-martingale system, but in order to do that we first must show that $\{V^{\Pi}(\rho), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-system, since this appears in our definition of $M(\rho)$. The main proof method here will revolve around stopping times of the form $\rho_A := \rho \mathbb{1}_A + T \mathbb{1}_{A^c}$ for any $\mathscr{F}_\rho^2$-measurable set A (recall that these are stopping times by Lemma 2.3.9).

Lemma 4.1.18. *Let ρ be a $(\mathscr{G}_t)$-stopping time, $A \in \mathscr{G}_\rho$, and define the stopping time $\rho_A := \rho \mathbb{1}_A + T \mathbb{1}_{A^c}$. Then, for any process $(X_t)_{t\in[0,T]}$ belonging to $\mathscr{L}_b$,*

$$\mathbb{1}_A \mathbb{E}[X|\mathscr{G}_{\rho_A}] = \mathbb{1}_A \mathbb{E}[X|\mathscr{G}_\rho], \qquad \& \qquad \mathbb{1}_{A^c} \mathbb{E}[X|\mathscr{G}_{\rho_A}] = \mathbb{1}_{A^c} \mathbb{E}[X|\mathscr{G}_T].$$

Proof. Begin by noticing that on A, $\rho_A = \rho$, and on A^c, $\rho_A = T$, meaning that $\rho \leq \rho_A$ for all $\omega \in \Omega$. Thus, Lemma 1.2.15 from [KS98a] gives that $\mathscr{G}_\rho \subset \mathscr{G}_{\rho_A}$.

Since we now know that $A \in \mathscr{G}_{\rho_A}$, we realise that $\mathbb{1}_A \mathbb{E}[X|\mathscr{G}_{\rho_A}] = \mathbb{E}[\mathbb{1}_A X|\mathscr{G}_{\rho_A}]$. The object $\mathbb{E}[\mathbb{1}_A X|\mathscr{G}_{\rho_A}]$ is the only $\mathscr{G}_{\rho_A}$-measurable random variable that satisfies the following

$$\int_H \mathbb{E}[\mathbb{1}_A X|\mathscr{G}_{\rho_A}]\, d\mathbb{P} = \int_H \mathbb{1}_A X\, d\mathbb{P}, \tag{4.15}$$

for all $H \in \mathscr{G}_{\rho_A}$. Likewise, $\mathbb{E}[X|\mathscr{G}_\rho]$ is the only $\mathscr{G}_\rho$-measurable random variable that satisfies

$$\int_{H'} \mathbb{E}[X|\mathscr{G}_\rho]\, d\mathbb{P} = \int_{H'} X\, d\mathbb{P}. \tag{4.16}$$

for all $H' \in \mathscr{G}_\rho$. We claim that, for any $H \in \mathscr{G}_{\rho_A}$, $H \cap A \in \mathscr{G}_\rho$. This is simple to see. Recall two facts: one, a set B belongs to $\mathscr{G}_\rho$ if $B \cap \{\rho \leq t\} \in \mathscr{G}_t$ for all $t \in [0,T]$; two, for any $\omega \in A$, $\rho_A = \rho$. The latter tells us that $A \cap \{\rho \leq t\} = A \cap \{\rho_A \leq t\}$. Therefore,

$$(H \cap A) \cap \{\rho \leq t\} = (H \cap A) \cap \{\rho_A \leq t\}. \tag{4.17}$$

Since both H and A belong to $\mathscr{G}_{\rho_A}$, so does $H \cap A$, and so $(H \cap A) \cap \{\rho_A \leq t\} \in \mathscr{G}_t$ for all $t \in [0,T]$. Equation (4.17) then tells us that $(H \cap A) \cap \{\rho \leq t\} \in \mathscr{G}_t$ for all $t \in [0,T]$ and therefore $H \cap A \in \mathscr{G}_\rho$. Using this fact, along with both (4.15) and (4.16), we see that

$$\int_H \mathbb{E}[\mathbb{1}_A X|\mathscr{G}_{\rho_A}]\, d\mathbb{P} = \int_{H\cap A} X\, d\mathbb{P} = \int_H \mathbb{1}_A \mathbb{E}[X|\mathscr{G}_\rho]\, d\mathbb{P}. \tag{4.18}$$

Since $A \in \mathscr{G}_\rho$, we see that on the right-hand side $\mathbb{1}_A \mathbb{E}[X|\mathscr{G}_\rho] = \mathbb{E}[\mathbb{1}_A X|\mathscr{G}_\rho]$. Thus, using (4.18), it is true that $\mathbb{E}[\mathbb{1}_A X|\mathscr{G}_\rho]$ is another $\mathscr{G}_{\rho_A}$-measurable random variable (since it is $\mathscr{G}_\rho$ measurable, it is also $\mathscr{G}_{\rho_A}$-measurable) that satisfies (4.15) for any $H \in \mathscr{G}_{\rho_A}$. Since the conditional expectation is $\mathbb{P}$-almost surely unique, we must have $\mathbb{E}[\mathbb{1}_A X|\mathscr{G}_{\rho_A}] = \mathbb{E}[\mathbb{1}_A X|\mathscr{G}_\rho]$ $\mathbb{P}$-almost surely. The other result is done in exactly the same way. ■

Proposition 4.1.19. *The family of random variables $\{V^{\Pi^*}(\rho), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-system.*

Proof. Consider Definition 1.3.4: for the definition of a $\mathscr{T}(\mathscr{F}_t^2)$-system, point ii) is simple. The conditional expectations are $\mathscr{F}_\rho^2$-measurable and so is their sum. Remark 4.1.4 tells us that $\Pi^*_{\rho-}$ is $\mathscr{F}_\rho^2$-measurable, and so we know $J_\rho^{\Pi^*}(\xi,\zeta)$ is $\mathscr{F}_\rho^2$-measurable for any $\xi \in \mathscr{A}_\rho(\mathscr{F}_t^1)$ and $\zeta \in \mathscr{A}(\mathscr{F}_t^2)$. By definition, the essential supremum (resp. infimum) of a family of $\mathscr{F}_\rho^2$-measurable random variables is itself $\mathscr{F}_\rho^2$-measurable. Since the family $\{J_\rho^{\Pi^*}(\xi,\zeta) : \zeta \in \mathscr{A}(\mathscr{F}_t^2)\}$ is $\mathscr{F}_\rho^2$-measurable for any $\xi \in \mathscr{A}_\rho(\mathscr{F}_t^1)$, so is $\{\operatorname{ess\,sup}_{\zeta \in \mathscr{A}(\mathscr{F}_t^2)} J_\rho^{\Pi^*}(\xi,\zeta) : \xi \in \mathscr{A}(\mathscr{F}_t^1)\}$; therefore, $\overline{V}^{\Pi^*}(\rho)$ is $\mathscr{F}_\rho^2$-measurable and so is $V^{\Pi^*}(\rho)$. Of course, we could have constructed the measurability of $\underline{V}^{\Pi^*}(\rho)$ instead to get the same result.

For point i) let $A \subset \mathscr{F}_\rho^2$ and define $\rho_A = \rho \mathbb{1}_A + T\mathbb{1}_{A^c}$, then notice that (using Lemma 4.1.18) $V^{\Pi^*}(\rho_A) = \mathbb{1}_A V^{\Pi^*}(\rho) + \mathbb{1}_{A^c} \sum_{i=0}^{\mathscr{I}} \Pi^i_{T-} \mathbb{E}[h_T^i|\mathscr{F}_T^2]$. If we select $A = \{\rho = \rho'\}$, then $\rho_A = \rho'_A$, i.e. the two stopping times $\rho_A(\omega)$ and $\rho'_A(\omega)$ are identical for every $\omega \in \Omega$. Thus,

$$V^{\Pi^*}(\rho_A) = V^{\Pi^*}(\rho'_A),$$

implying that

$$\mathbb{1}_A V^{\Pi^*}(\rho) + \mathbb{1}_{A^c} \sum_{i=0}^{\mathscr{I}} \Pi^i_{T-} \mathbb{E}\big[h_T^i \big| \mathscr{F}_T^2\big] = \mathbb{1}_A V^{\Pi^*}(\rho') + \mathbb{1}_{A^c} \sum_{i=0}^{\mathscr{I}} \Pi^i_{T-} \mathbb{E}\big[h_T^i \big| \mathscr{F}_T^2\big].$$

This yields $\mathbb{1}_A V^{\Pi^*}(\rho) = \mathbb{1}_A V^{\Pi^*}(\rho')$. ■

Theorem 4.1.20. *The family of random variables $\{M(\rho), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-martingale system.*

Proof. 1° We know from Proposition 4.1.19, that $\{V^{\Pi^*}(\rho), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-system. All remaining terms are $\mathscr{F}_\rho^2$-measurable and agree almost surely on the set $\{\rho = \rho'\}$. Therefore $\{M(\rho), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-system.

2° We now want to prove integrability. Let ρ be an $(\mathscr{F}_t^2)$-stopping time. By the triangle inequality we have

$$\mathbb{E}|M(\rho)| \leq \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\bigg| V^{\Pi^*}(\rho)(1-\xi_{\rho-}^{i,*})(1-\zeta_{\rho-}^*) + \int_{[0,\rho)} f_s^i (1-\zeta_s^*)\, d\xi_s^{i,*} + \int_{[0,\rho)} g_s^i (1-\xi_s^{i,*})\, d\zeta_s^* + \sum_{s\in[0,\rho)} h_s^i \Delta\zeta_s^* \Delta\xi_s^{i,*} \bigg|.$$

We can make the observation that any term of the form $1-\chi_t$ or $\Delta\chi_t$, for $\chi \in \mathscr{A}(\mathscr{F}_t^2)$, is bounded above by one. We may then use the triangle inequality once more, along with Jensen's inequality:

$$\mathbb{E}|M(\rho)| \leq \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\left[\left| V^{\Pi^*}(\rho) \right| + \int_{[0,\rho)} |f_s^i|\, d\xi_s^{i,*} + \int_{[0,\rho)} |g_s^i|\, d\zeta_s^* + \sum_{s\in[0,\rho)} |h_s^i| \Delta\zeta_s^* \right].$$

We can make similar arguments to those in the proof of Lemma 3.4.14 to bound the integrals and summation by the relevant suprema. We then arrive at

$$\mathbb{E}|M(\rho)| \leq \sum_{i=0}^{\mathscr{I}} \mathbb{E}\left[\left| V^{\Pi^*}(\rho) \right| + \sup_{s\in[0,T]} |f_s^i| + \sup_{s\in[0,T]} |g_s^i| + \sup_{s\in[0,T]} (|f_s^i| + |g_s^i|) \right].$$

Since $(f_t)_{t\in[0,T]}$ and $(g_t)_{t\in[0,T]}$ belong to $\mathscr{L}_b$, we know that

$$\mathbb{E}\left[\sup_{s\in[0,T]} |f_s^i| + \sup_{s\in[0,T]} |g_s^i| + \sup_{s\in[0,T]} (|f_s^i| + |g_s^i|) \right] < \infty.$$

3° It remains to show that $\mathbb{E}|V^{\Pi^*}(\rho)| < \infty$. Recall the definition of $J_\rho^{\Pi^*}(\xi,\zeta)$ from Definition 4.1.5, and notice that we can apply the same bounding method seen in Theorem 3.4.14, similar to 2°. Recall equations (3.34) and (3.35) tell us that $J_\rho^{\Pi^*}(\xi,\zeta)$ is dominated by an integrable random variable—which we called $\sum_{i=0}^{\mathscr{I}} Z^i(\rho)$—for any choice of $(\xi,\zeta) \in \mathscr{A}(\mathscr{F}_t^1) \times \mathscr{A}(\mathscr{F}_t^2)$, thus

$$\mathbb{E}\left| V^{\Pi^*}(\rho) \right| = \mathbb{E}\left| \operatorname*{ess\,inf}_{\xi\in\mathscr{A}_\rho(\mathscr{F}_t^1)} \operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J_\rho^{\Pi^*}(\xi,\zeta) \right| \leq \mathbb{E}\left| \sum_{i=0}^{\mathscr{I}} Z^i(\rho) \right|.$$

Equation (3.35) then gives the final result.

4° Finally, we want to prove the martingale property. We know from Theorem 4.1.17 that for any $(\mathscr{F}_t^2)$-stopping time ρ, $\mathbb{E}[M(\rho)] = M(0)$. From this we will deduce the martingale property using the solution method for Exercise 5.6 in [Bal17]. Let $S \leq S'$ be two $(\mathscr{F}_t^2)$-stopping times, fix

$A \in \mathscr{F}_S^2$, and define a bounded stopping time τ as follows

$$\tau(\omega) := S\mathbb{1}_A(\omega) + S'\mathbb{1}_{A^c}(\omega).$$

This is a stopping time by Lemma 2.3.9. Now, $M(\tau) = M(S)\mathbb{1}_A + M(S')\mathbb{1}_{A^c}$ $\mathbb{P}$-a.s. since we have already shown $\{M(\rho), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-system. Since $\mathbb{E}[M(\rho)] = M_0$ for any $(\mathscr{F}_t^2)$-stopping time ρ, we can say that $\mathbb{E}[M(\tau)] = \mathbb{E}[M(S')]$. Thus,

$$\mathbb{E}[M(S)\mathbb{1}_A] + \mathbb{E}[M(S')\mathbb{1}_{A^c}] = \mathbb{E}[M(\tau)] = \mathbb{E}[M(S')] = \mathbb{E}[M(S')\mathbb{1}_A] + \mathbb{E}[M(S')\mathbb{1}_{A^c}].$$

This equality implies that $\mathbb{E}[M(S)\mathbb{1}_A] = \mathbb{E}[M(S')\mathbb{1}_A]$, i.e. the partial averaging definition of conditional expectation. To see this explicitly, we know that the equation characterising the conditional expectation $\mathbb{E}[M(S')|\mathscr{F}_S^2]$ is $\mathbb{E}[\mathbb{E}[M(S')|\mathscr{F}_S^2]\mathbb{1}_A] = \mathbb{E}[M(S')\mathbb{1}_A]$ for all $A \in \mathscr{F}_S^2$. We have shown that $M(S)$ satisfies this, i.e. we have $\mathbb{E}[M(S')|\mathscr{F}_S^2] = M(S)$. ∎

All that remains now is to aggregate the family $\{M(\rho), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ into a unique, right-continuous martingale, however, we need one final technical lemma to ensure that our reasoning is correct. We will need to utilise rational valued stopping times, and will construct a specific sequence of them. We need to ensure this sequence is decreasing and convergent.

Lemma 4.1.21. *Let $S(\omega)$ be any stopping time. Denote by D_n the subset of the dyadic rationals of the form $p2^{-n}$ for some $p \in \mathbb{N}$. Define*

$$S_n(\omega) := \begin{cases} S(\omega) & \text{if } S(\omega) = +\infty, \\ 2^{-n}k_n & \text{for } 2^{-n}(k_n - 1) \le S(\omega) < 2^{-n}k_n, \end{cases}$$

then $S_n \searrow S$ as $n \to \infty$.

Proof. Trivially, if $S(\omega) = +\infty$, then $S_n(\omega)$ converges to $S(\omega)$.

Next, take $\omega \in \Omega$ such that $S(\omega) < +\infty$. For ease of notation, define $I_n := [2^{-n}(k_n - 1), 2^{-n}k_n)$. Define $M_n = 2^{-(n+1)}(2k_n - 1)$, this is the midpoint of I_n and clearly $M_n \in D_{n+1}$. If $S(\omega) \in [2^{-n}(k_n - 1), M_n)$, then setting $k_{n+1} = 2k_n - 1$ we get $S(\omega) \in [2^{-(n+1)}(k_{n+1} - 1), 2^{-(n+1)}k_{n+1}) = I_{n+1} \subset I_n$, and $S_{n+1}(\omega) < S_n(\omega)$. If $S(\omega) \in [M_n, 2^{-n}k_n)$, then by setting $k_{n+1} = 2k_n$ gives us that $S(\omega) \in [2^{-(n+1)}(k_{n+1} - 1), 2^{-(n+1)}k_{n+1}) = I_{n+1} \subset I_n$, and in this case $S_{n+1}(\omega) = S_n(\omega)$. Thus, $(S_n(\omega))_{n \ge 1}$ is a non-increasing sequence bounded below by $S(\omega)$.

We know that $(S_n(\omega))_{n\in\mathbb{N}}$ converges to its infimum. Assume $S(\omega) < \hat{S}(\omega) := \inf_n S_n(\omega)$. By the density of the dyadic rationals in $\mathbb{R}$, it is possible to choose $m \in \mathbb{N}$, with corresponding k_m, such that $S(\omega) \in [2^{-m}(k_m - 1), 2^{-m}k_m)$ and that $|S(\omega) - 2^{-m}k_m| < |S(\omega) - \hat{S}(\omega)|$. Hence, for all $n \geq m$, $S(\omega) \leq S_n(\omega) < \hat{S}(\omega)$, meaning $\hat{S}(\omega)$ is not a lower bound on $(S_n(\omega))$. Thus, $S(\omega) = \inf_n S_n(\omega)$ and so $S_n(\omega) \searrow S(\omega)$ as $n \to \infty$. ■

Theorem 4.1.22. *There exists a unique $(\mathscr{F}_t^2)$-martingale, $(M_t)_{t\in[0,T]}$, which is right-continuous and aggregates the family of random variables $\{M(\rho), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$.*

Proof. The process $M_r := (M(r))_{r\in(\mathbb{Q}\cap[0,T])\cup\{T\}}$ is a martingale over the rationals which is (trivially) continuous in expectation. We can then choose a modification which is right-continuous over the rationals (see Theorem 1.3.13, [KS98a]) and which we can extend to the reals using right-continuity (notice we include the final time T in our definition of $(M_r)_{r\in(\mathbb{Q}\cap[0,T])\cup\{T\}}$, since if $T \notin \mathbb{Q}$ then our approximation from the right breaks down). We call this process $(M_t)_{t\in[0,T]}$.

Let $S \in \mathscr{T}(\mathscr{F}_t^2)$ be a stopping time. Define $(S_n)_{n\in\mathbb{N}}$ as per Lemma 4.1.21, and as such we know that $S_n \searrow S$. We cannot guarantee that $S_n \in [0,T]$, so we consider $(S_n \wedge T)_{n\in\mathbb{N}}$. Consequently, $(S_n \wedge T)_{n\in\mathbb{N}}$ takes a countable number of values in $(\mathbb{Q}\cap[0,T])\cup\{T\}$. For a given $\omega \in \Omega$, say $(S_n \wedge T)(\omega)$ takes the value $q_n(\omega)$ on the set $A_n(\omega)$, then

$$(S_n \wedge T)(\omega) = \sum_{n=1}^{\infty} q_n(\omega)\mathbb{1}_{A_n(\omega)}.$$

As a direct result of this, we can say

$$\mathbb{E}[M(S_n \wedge T)] = \sum_{n=1}^{\infty} \mathbb{E}[M(q_n)\mathbb{1}_{A_n}] = \sum_{n=1}^{\infty} \mathbb{E}[M_{q_n}\mathbb{1}_{A_n}] = \mathbb{E}[M_{S_n\wedge T}]$$

with the first equality holding because $\{M(\rho), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-system, and the second because $M(q)$ and M_q agree on $(\mathbb{Q}\cap[0,T])\cup\{T\}$ $\mathbb{P}$-almost surely. Taking limits on both sides:

$$\mathbb{E}[M(S \wedge T)] = \lim_{n\to\infty} \mathbb{E}[M(S_n \wedge T)] = \lim_{n\to\infty} \mathbb{E}[M_{S_n\wedge T}] = \mathbb{E}[M_{S\wedge T}]$$

where the first equality holds since $\{M(\rho), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ is continuous in expectation and the last equality holds by the dominated convergence theorem and the right-continuity of $(M_t)_{t\in[0,T]}$.

Let $A \in \mathscr{F}^2_{S\wedge T}$ and recall that the stopping time $(S \wedge T)_A$ is defined as $(S \wedge T)_A := S \wedge T\,\mathbb{1}_A +$

$T\mathbb{1}_{A^c}$, then

$$\mathbb{E}[M_{S\wedge T}\mathbb{1}_A + M_T\mathbb{1}_{A^c}] = \mathbb{E}[M_{(S\wedge T)_A}] = \mathbb{E}[M((S\wedge T)_A)] = \mathbb{E}[M(S\wedge T)\mathbb{1}_A + M(T)\mathbb{1}_{A^c}]. \quad (4.19)$$

Since $M(T) = M_T$ by definition, we can deduce from (4.19) that $\mathbb{E}[M_{S\wedge T}\mathbb{1}_A] = \mathbb{E}[M(S\wedge T)\mathbb{1}_A]$, i.e. $\mathbb{E}[M(S\wedge T)|\mathscr{F}^2_{S\wedge T}] = M_{S\wedge T}$. Since $M(S\wedge T)$ is $\mathscr{F}^2_{S\wedge T}$-measurable, the conditional expectation equality implies that $M(S\wedge T) = M_{S\wedge T}$ $\mathbb{P}$-almost surely. ■

4.2 NASH EQUILIBRIA IN SUBGAMES

The current view we have of the game is rather static, with choices being made at the onset, and the game playing out from there, however, one can take a more dynamic approach to understanding the game, where at any time we consider how to play the game solely from that point on. The paper by Riedel and Steg, [RS17], provides a detailed discussion on the formulation of subgames and strategies, albeit in a slightly different setting, so we adapt their concepts here.

For our purposes, we consider subgames as choosing an $(\mathscr{F}^2_t)$-stopping time ρ, and considering the payoff processes $(f_t)_{t\in[0,T]}$, $(g_t)_{t\in[0,T]}$ and $(h_t)_{t\in[0,T]}$ restricted to the stochastic interval $[\rho, T]$. Players must choose subgame strategies as $(\xi,\zeta) \in \mathscr{A}_\rho(\mathscr{F}^1_t)\times\mathscr{A}_\rho(\mathscr{F}^2_t)$. It is in linking these subgames together that we obtain our dynamic view of the game. In order to construct a strategy at time zero, we require that the subgame strategies chosen are consistent in some way, specifically we need them to be 'time-consistent'.

Definition 4.2.1. Let $(\mathscr{G}_t)_{t\in[0,T]}$ be a filtration satisfying Assumption 3.3.1 such that $(\mathscr{F}^2_t)\subset(\mathscr{G}_t)$. Let $(\chi^\rho)_{\rho\in\mathscr{T}(\mathscr{F}^2_t)}$ be a family of generating process such that, for any $(\mathscr{F}^2_t)$-stopping time ρ, $\chi^\rho \in \mathscr{A}_\rho(\mathscr{G}_t)$. The family is said to be time-consistent if, for every pair of $(\mathscr{F}^2_t)$-stopping times $\sigma \le \rho \le T$,

$$\chi^\sigma_t = \chi^\sigma_{\rho-} + (1-\chi^\sigma_{\rho-})\chi^\rho_t \quad (4.20)$$

for all $t\in[\rho,T]$.

Recall from Lemma 3.2.4 that the generating processes evaluated at stopping times are themselves conditional probabilities. In this way we see that (4.20) is a requirement for consistent conditional probabilities when moving between subgames.

A very important family of generating processes was given implicitly in Definition 4.1.8, namely that of the truncated strategies. The following result is crucial as it tells us that the act of truncation can create a family of time-consistent strategies. In fact, we should not be surprised that this result is true since we define a truncated strategy quite literally using the definition of time-consistency—albeit with a graveyard point for retaining a well-defined object.

Proposition 4.2.2. *Let $(\mathscr{G}_t)_{t\in[0,T]}$ be a filtration satisfying Assumption 3.3.1 such that $(\mathscr{F}_t^2) \subset (\mathscr{G}_t)$. Let $\chi \in \mathscr{A}(\mathscr{G}_t)$ be a generating process and define the family $(\chi^\rho)_{\rho\in\mathscr{T}(\mathscr{F}_t^2)}$ as*

$$\chi_t^\rho := \begin{cases} \frac{\chi_t - \chi_{\rho-}}{1-\chi_{\rho-}} \mathbb{1}_{[\rho,T]}(t) & \text{if } \chi_{\rho-} < 1, \\ \mathbb{1}_{\{t=T\}}(t) & \text{if } \chi_{\rho-} = 1. \end{cases}$$

Then $(\chi^\rho)_{\rho\in\mathscr{T}(\mathscr{F}_t^2)}$ is time-consistent.

Proof. Let $\eta(\omega) := \inf\{t \in [0,T] : \chi_t(\omega) = 1\}$. By right-continuity of $(\chi_t)_{t\in[0,T]}$, we know that $\chi_\eta = 1$, and for any stopping time $\sigma < \eta$, we have $\chi_\sigma < 1$. Notice that there is no issue with the definition of η since the set $\{t \in [0,T] : \chi_t(\omega) = 1\}$ always at least contains T.

We can consider three distinct events for two ordered stopping times $\sigma \le \rho$: 1) $\{\sigma(\omega) \le \rho(\omega) < \eta(\omega)\}$, 2) $\{\sigma(\omega) < \eta(\omega) \le \rho(\omega)\}$, and 3) $\{\eta(\omega) \le \sigma(\omega) \le \rho(\omega)\}$.

1) For all $t \ge \rho$,

$$\begin{aligned} \chi_{\rho-}^\sigma + (1-\chi_{\rho-}^\sigma)\chi_t^\rho &= \frac{\chi_{\rho-} - \chi_{\sigma-}}{1-\chi_{\sigma-}} \mathbb{1}_{[\sigma,T]} + \left(1 - \frac{\chi_{\rho-} - \chi_{\sigma-}}{1-\chi_{\sigma-}} \mathbb{1}_{[\sigma,T]}\right) \frac{\chi_t - \chi_{\rho-}}{1-\chi_{\rho-}} \mathbb{1}_{[\rho,T]} \\ &= \frac{\chi_t - \chi_{\sigma-}}{1-\chi_{\sigma-}} \\ &= \chi_t^\sigma. \end{aligned}$$

2) First, take the case of $\rho = \eta$. If $\chi_{\rho-} < 1$, then for all $t \ge \rho$, the same reasoning as above holds since both truncated strategies are of the same form. However, if $\chi_{\rho-} = 1$, then

$$\chi_{\rho-}^\sigma + (1-\chi_{\rho-}^\sigma)\chi_t^\rho = 1 + 0 \cdot \mathbb{1}_{\{t=T\}} = \chi_t^\sigma.$$

Now consider $\rho > \eta$. We know $\chi_{\rho-} = 1$, meaning the same reasoning holds as above.

3) Consider the case when $\eta = \sigma$. If $\chi_{\sigma-} < 1$, then the same logic from 2) holds. If $\chi_{\sigma-} = 1$,

then for all $t \geq \rho$

$$\chi^{\sigma}_{\rho-} + (1-\chi^{\sigma}_{\rho-})\chi^{\rho}_t = \mathbb{1}_{\{t=T\}} + \Big(1-\mathbb{1}_{\{t=T\}}\Big)\mathbb{1}_{\{t=T\}} = \mathbb{1}_{\{t=T\}} = \chi^{\sigma}_t.$$

Next, let $\eta < \sigma$. Again, the same reasoning holds because both truncated strategies are the indicator function of the terminal time.

■

This dynamic view of the game leads to another, stronger solution concept for games known as subgame perfect Nash equilibria. In essence, a pair of strategies (ξ^*, ζ^*) are a subgame perfect Nash equilibrium if, for every stopping time ρ, the pair $(\overline{\xi}^{\rho}, \overline{\zeta}^{\rho})$ is a Nash equilibrium regardless of the prior history of the game—it is simple to see that a subgame perfect Nash equilibrium is also a Nash equilibrium.

Unfortunately, we are not quite able to achieve subgame perfection, however, we can partially. We introduce the solution concept of a 'partial' subgame perfect Nash equilibrium, which depends only on the initial strategy paths.

To begin, we need to expand on the sets given in Definition 4.1.11. The sets Ω^i_{ρ} and Ω_{ρ} are defined using the optimal time-zero strategies (ξ^*, ζ^*), however, we define the wider concepts $\hat{\Omega}^i_{\rho}(\xi,\zeta)$ and $\hat{\Omega}_{\rho}(\xi,\zeta)$ for any general generating processes.

Definition 4.2.3. Let ρ be an $(\mathscr{F}^2_t)$-stopping time and (ξ,ζ) be any generating processes belonging to $\mathscr{A}(\mathscr{F}^1_t) \times \mathscr{A}(\mathscr{F}^2_t)$. Define

$$\hat{\Omega}^i_{\rho}(\xi,\zeta) := \Big\{\omega \in \Omega : \pi_i(1-\xi^i_{\rho-}(\omega))(1-\zeta_{\rho-}(\omega)) > 0\Big\},$$
$$\hat{\Omega}_{\rho}(\xi,\zeta) := \left\{\omega \in \Omega : \sum_{i=0}^{\mathscr{I}} \pi_i(1-\xi^i_{\rho-}(\omega))(1-\zeta_{\rho-}(\omega)) > 0\right\}.$$

Definition 4.2.4. Let $(\xi^{\rho},\zeta^{\rho})_{\rho\in\mathscr{T}(\mathscr{F}^2_t)}$ be a pair of time-consistent strategies, where $(\xi^{\rho},\zeta^{\rho}) \in \mathscr{A}_{\rho}(\mathscr{F}^1_t) \times \mathscr{A}_{\rho}(\mathscr{F}^2_t)$ for every $\rho \in \mathscr{T}(\mathscr{F}^2_t)$. Denote by (ξ,ζ) the pair (ξ^0,ζ^0) and write $\hat{\Omega}_{\rho} := \hat{\Omega}_{\rho}(\xi,\zeta)$. We say that the pair $(\xi^{\rho},\zeta^{\rho})$ is a partial subgame perfect Nash equilibrium if, for any $(\mathscr{F}^2_t)$-stopping time ρ and any pair $(\chi,\eta) \in \mathscr{A}_{\rho}(\mathscr{F}^1_t) \times \mathscr{A}_{\rho}(\mathscr{F}^2_t)$,

$$\mathbb{1}_{\hat{\Omega}_{\rho}} J^{\Pi}_{\rho}(\xi^{\rho},\eta) \leq \mathbb{1}_{\hat{\Omega}_{\rho}} J^{\Pi}_{\rho}(\xi^{\rho},\zeta^{\rho}) \leq \mathbb{1}_{\hat{\Omega}_{\rho}} J^{\Pi}_{\rho}(\chi,\zeta^{\rho}).$$

Remark 4.2.5. We will often deal with the special case of this definition where we take the optimal time-zero strategies (ξ^*, ζ^*) and create the time-consistent family $(\overline{\xi}^\rho, \overline{\zeta}^\rho)$ as per Proposition 4.2.2. In this case we see that $\hat{\Omega}_\rho(\xi, \zeta) = \Omega_\rho$, and then, for any pair $(\xi, \zeta) \in \mathscr{A}_\rho(\mathscr{F}_t^1) \times \mathscr{A}_\rho(\mathscr{F}_t^2)$, the partial subgame perfect Nash equilibrium condition looks like

$$\mathbb{1}_{\Omega_\rho} J_\rho^{\Pi^*}(\overline{\xi}^\rho, \zeta) \leq \mathbb{1}_{\Omega_\rho} J_\rho^{\Pi^*}(\overline{\xi}^\rho, \overline{\zeta}^\rho) \leq \mathbb{1}_{\Omega_\rho} J_\rho^{\Pi^*}(\xi, \overline{\zeta}^\rho).$$

In other words, a pair of strategies is a 'partial subgame perfect Nash equilibrium' if their truncated forms are a Nash equilibrium in every subgame, provided that the initial strategies have not yet stopped.

Of course, we have shown that the family of generating strategies created by truncating is time-consistent, so the following theorem will show that the initial Nash equilibrium (ξ^*, ζ^*) is itself a partial subgame perfect Nash equilibrium.

Theorem 4.2.6. *Let $\rho \in \mathscr{T}(\mathscr{F}_t^2)$ and (ξ^*, ζ^*) be a Nash equilibrium at time zero, then*

$$\mathbb{1}_{\Omega_\rho} V^{\Pi^*}(\rho) = \operatorname*{ess\,sup}_{\chi \in \mathscr{A}_\rho(\mathscr{F}_t^2)} \mathbb{1}_{\Omega_\rho} J_\rho^{\Pi^*}(\overline{\xi}^\rho, \chi), \quad \mathbb{P}\text{-a.s.} \tag{4.21}$$

$$\mathbb{1}_{\Omega_\rho} V^{\Pi^*}(\rho) = \operatorname*{ess\,inf}_{\eta \in \mathscr{A}_\rho(\mathscr{F}_t^1)} \mathbb{1}_{\Omega_\rho} J_\rho^{\Pi^*}(\eta, \overline{\zeta}^\rho), \quad \mathbb{P}\text{-a.s.} \tag{4.22}$$

Proof. 1° Take any $\chi \in \mathscr{A}(\mathscr{F}_t^2)$ and define $(\hat{\zeta}_t)_{t\in[0,T]}$ by (4.36). From (4.9) in Lemma 4.1.16, and the fact that $(\hat{\zeta}_t)_{t\in[0,T]}$ is suboptimal for the maximiser, we have for any $\chi \in \mathscr{A}_\rho(\mathscr{F}_t^2)$,

$$\begin{aligned} V^{\Pi^*}(0) \geq \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\bigg[\int_{[0,\rho)} f_s^i (1-\zeta_s^*)\, d\xi_s^{i,*} + \int_{[0,\rho)} g_s^i (1-\xi_s^{i,*})\, d\zeta_s^* + \sum_{s\in[0,\rho)} h_s^i \Delta\zeta_s^* \Delta\xi_s^{i,*} \\ + (1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*}) J_\rho^{\Pi^*}(\overline{\xi}^\rho, \chi) \bigg]. \end{aligned} \tag{4.23}$$

Notice that in the final term, $\sum_{i=0}^{\mathscr{I}} \pi_i (1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*}) J_\rho^{\Pi^*}(\overline{\xi}^\rho, \chi)$, we may freely introduce the indicator function $\mathbb{1}_{\Omega_\rho}$, since on (Ω_ρ^c) the term is zero regardless. We require this indicator function so we can later remove the prefactors $\sum_{i=0}^{\mathscr{I}} \pi_i (1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*})$.

Since, by Lemma 3.4.16, the family $\{J_\rho^{\Pi^*}(\xi, \chi), \chi \in \mathscr{A}_\rho(\mathscr{F}_t^2)\}$ is upwards-directed, we know that the essential supremum over $\chi \in \mathscr{A}_\rho(\mathscr{F}_t^2)$ is attained by a sequence $(\chi^n)_{n\in\mathbb{N}}$. We now apply (4.23) with $\chi = \chi^n$ and so by taking limits, and applying the dominated convergence theorem, we arrive at

$$V^{\Pi^*}(0) \geq \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\bigg[\int_{[0,\rho)} f_s^i (1-\zeta_s^*)\, d\xi_s^{i,*} + \int_{[0,\rho)} g_s^i (1-\xi_s^{i,*})\, d\zeta_s^* + \sum_{s\in[0,\rho)} h_s^i \Delta\zeta_s^* \Delta\xi_s^{i,*} \\ + \mathbb{1}_{\Omega_\rho}(1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*}) \operatorname*{ess\,sup}_{\chi\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J_\rho^{\Pi^*}(\overline{\xi}^\rho, \chi) \bigg]. \quad (4.24)$$

Specifically choosing $(\overline{\zeta}_t^\rho)_{t\in[0,T]}$, we can get a lower bound on the essential supremum:

$$V^{\Pi^*}(0) \geq \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\bigg[\int_{[0,\rho)} f_s^i (1-\zeta_s^*)\, d\xi_s^{i,*} + \int_{[0,\rho)} g_s^i (1-\xi_s^{i,*})\, d\zeta_s^* + \sum_{s\in[0,\rho)} h_s^i \Delta\zeta_s^* \Delta\xi_s^{i,*} \\ + \mathbb{1}_{\Omega_\rho}(1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*}) J_\rho^{\Pi^*}(\overline{\xi}^\rho, \overline{\zeta}^\rho) \bigg]. \quad (4.25)$$

From this, and we can deduce that

$$V^{\Pi^*}(0) \geq \mathbb{E}[P(\xi^*, \zeta^*)] = V^{\Pi^*}(0). \quad (4.26)$$

Recall the form of $P(\xi, \zeta)$ given in (3.14). (4.26) holds because, for $\omega \in \Omega_\rho$, the right-hand side of (4.25) is clearly $\mathbb{E}[P(\xi^*, \zeta^*)]$ (using Definition 4.1.5). For $\omega \in (\Omega_\rho)^c$, however, the right-hand side of (4.25) is now only the integral and summation terms over the interval $[0, \rho)$; we demonstrate that, in this case, $P(\xi^*, \zeta^*)$ is zero on the interval $[\rho, T]$. We know that, since we are on $(\Omega_\rho)^c$, it must be that $\pi_i(1-\xi_{\rho-}^{i,*})(1-\zeta_{\rho-}^*) = 0$ for all $i \in [\mathscr{I}]$. It cannot be that $\pi_i = 0$ for all i, so at a minimum, one of the players must have stopped. First take the case where $\pi_i > 0$ for all i, then for each i, either $(1-\zeta_{\rho-}^*) = 0$ or $(1-\xi_{\rho-}^{i,*}) = 0$, implying that either $(\xi_t^{i,*})_{t\in[0,T]}$ or $(\zeta_t^*)_{t\in[0,T]}$ is equal to one for all $t \in [\rho, T]$. Either way we see

$$P(\xi^*, \zeta^*) := \sum_{i=0}^{\mathscr{I}} \pi_i \left(\int_{[0,\rho)} f_t^i (1-\zeta_t^*)\, d\xi_t^{i,*} + \int_{[0,\rho)} g_t^i (1-\xi_t^{i,*})\, d\zeta_t^* + \sum_{t\in[0,\rho)} h_t^i \Delta\zeta_t^* \Delta\xi_t^{i,*} \right).$$

Next, if $\pi_i = 0$ for at least one value of i, then the terms in the sum corresponding to those values of i will all be zero. The remaining terms then yield the correct result using the same logic as before since at least one of the players must stop.

The inequalities in (4.24)–(4.26) imply that

$$V^{\Pi^*}(0) = \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\left[\int_{[0,\rho)} f_s^i(1-\zeta_s^*)\,d\xi_s^{i,*} + \int_{[0,\rho)} g_s^i(1-\xi_s^{i,*})\,d\zeta_s^* + \sum_{s\in[0,\rho)} h_s^i \Delta\zeta_s^* \Delta\xi_s^{i,*}\right] + \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\left[\mathbb{1}_{\Omega_\rho}(1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*}) \operatorname*{ess\,sup}_{\chi\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J_\rho^{\Pi^*}(\overline{\xi}^\rho,\chi)\right], \quad (4.27)$$

where we have intentionally split the expectation for clarity later. Moreover, we know from (4.14), and the fact that $M(0) = V^{\Pi^*}(0) = \mathbb{E}[M(\rho)]$, that

$$V^{\Pi^*}(0) = \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\left[\int_{[0,\rho)} f_s^i(1-\zeta_s^*)\,d\xi_s^{i,*} + \int_{[0,\rho)} g_s^i(1-\xi_s^{i,*})\,d\zeta_s^* + \sum_{s\in[0,\rho)} h_s^i \Delta\zeta_s^* \Delta\xi_s^{i,*}\right] + \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\left[\mathbb{1}_{\Omega_\rho}(1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*}) V^{\Pi^*}(\rho)\right]. \quad (4.28)$$

Equating (4.27) and (4.28) yields

$$\mathbb{E}\left[\left(\sum_{i=0}^{\mathscr{I}} \pi_i (1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*})\right) \mathbb{1}_{\Omega_\rho} \operatorname*{ess\,sup}_{\chi\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J_\rho^{\Pi^*}(\overline{\xi}^\rho,\chi)\right] = \mathbb{E}\left[\left(\sum_{i=0}^{\mathscr{I}} \pi_i (1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*})\right) \mathbb{1}_{\Omega_\rho} V^{\Pi^*}(\rho)\right], \quad (4.29)$$

since $V^{\Pi^*}(\rho)$ and $\operatorname{ess\,sup}_{\chi\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J_\rho^{\Pi^*}(\overline{\xi}^\rho,\chi)$ do not depend on i.

Recall that Proposition 4.1.7 gives us the form of the value $V^{\Pi^*}(\rho)$, and from it we can see

$$V^{\Pi^*}(\rho) = \operatorname*{ess\,inf}_{\xi\in\mathscr{A}_\rho(\mathscr{F}_t^1)} \operatorname*{ess\,sup}_{\chi\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J_\rho^{\Pi^*}(\xi,\zeta) \le \operatorname*{ess\,sup}_{\chi\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J_\rho^{\Pi^*}(\overline{\xi}^\rho,\chi) \quad \mathbb{P}\text{-a.s.} \quad (4.30)$$

Since we remain on Ω_ρ in (4.29), we know that the prefactor terms are positive, i.e. $\sum_{i=0}^{\mathscr{I}} \pi_i (1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*}) > 0$, and so (4.29) and (4.30) imply that

$$\mathbb{P}\left[\operatorname*{ess\,sup}_{\chi\in\mathscr{A}_\rho(\mathscr{F}_t^2)} \mathbb{1}_{\Omega_\rho} J_\rho^{\Pi^*}(\overline{\xi}^\rho,\chi) \ge \mathbb{1}_{\Omega_\rho} V^{\Pi^*}(\rho)\right] = 1.$$

If it was the case that the strict inequality held on some set with positive probability, then from

(4.28) we can see that

$$V^{\Pi^*}(0) < \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\Bigg[\int_{[0,\rho)} f_s^i(1-\zeta_s^*)\,d\xi_s^{i,*} + \int_{[0,\rho)} g_s^i(1-\xi_s^{i,*})\,d\zeta_s^* + \sum_{s\in[0,\rho)} h_s^i \Delta\zeta_s^* \Delta\xi_s^{i,*} + \mathbb{1}_{\Omega_\rho}(1-\zeta_{\rho-}^*)(1-\xi_{\rho-}^{i,*}) \operatorname*{ess\,sup}_{\chi\in\mathscr{A}_\rho(\mathscr{F}_t^2)} J_\rho^{\Pi^*}(\overline{\xi}^\rho,\chi)\Bigg].$$

The right-hand side is equal to $V^{\Pi^*}(0)$ by (4.27)—a contradiction. Therefore we must have that

$$\mathbb{P}\left[\operatorname*{ess\,sup}_{\chi\in\mathscr{A}_\rho(\mathscr{F}_t^2)} \mathbb{1}_{\Omega_\rho} J_\rho^{\Pi^*}(\overline{\xi}^\rho,\chi) = \mathbb{1}_{\Omega_\rho} V^{\Pi^*}(\rho)\right] = 1.$$

2° The proof of (4.22) is identical *mutatis mutandis*. The alternations needed are: we now take $(\hat{\xi}_t^i)_{t\in[0,T]}$ from (4.6) to be our suboptimal strategy, but now for the minimiser; and the family of random variables we obtain is now $\{J_\rho^{\Pi^*}(\xi,\zeta), \xi \in \mathscr{A}_\rho(\mathscr{F}_t^1)\}$ which is downwards-directed. We omit further details of the proof. ■

4.3 SUB- AND SUPERMARTINGALES ARISING FROM SUBOPTIMAL PLAY

We have shown that, given a Nash equilibrium at time zero (as guaranteed by [DAMP22]), we can construct a martingale which encapsulates the game when both players are playing optimally, however, we want to be able to construct other objects which cover suboptimal play too. These processes will be sub- and supermartingales and we will show that these will reduce to the constructed martingale when both players play optimally. Due to the structure of the problem, i.e. there is only one maximiser, we begin by considering what happens when they play suboptimally.

In this section we introduce a new collection of random variables, specifically at any $(\mathscr{F}_t^2)$-stopping time ρ, the best response of the maximiser to truncation. Recall from Lemma 3.3.2 that, due to the presence of information asymmetry, we say there are $\mathscr{I}+1+1$ players in this game—$\mathscr{I}+1$ minimisers, and one maximiser—who all see the evolution of the game differently. The minimisers know the outcome of the random variable Θ, and so their understanding of the game will not need a belief process, however, the maximiser does not know, and therefore the best response must include it. Define, for all $(\mathscr{F}_t^2)$-stopping times ρ, the best response for the maximiser as follows:

$$\mathcal{V}(\rho) := \mathbb{1}_{\Lambda_\rho} \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)} J_\rho^{\Pi^*}(\overline{\xi}^\rho, \zeta). \tag{4.31}$$

Inherently this best response family is not related to the value of the game, however, as a corollary of Theorem 4.2.6 we can relate the two. The proof of the following result is trivial.

Corollary 4.3.1. *For any $(\mathscr{F}_t^2)$-stopping time ρ, the random variables $V^{\Pi^*}(\rho)$ and $\mathcal{V}(\rho)$ agree on the set Ω_ρ, i.e.*

$$\mathbb{1}_{\Omega_\rho} V^{\Pi^*}(\rho) = \mathbb{1}_{\Omega_\rho} \mathcal{V}(\rho).$$

Much as we did for the martingale, we now define a family of random variables using the best response, and then prove that it is a $\mathscr{T}(\mathscr{F}_t^2)$-supermartingale system.

Definition 4.3.2. Let ρ be an $(\mathscr{F}_t^2)$-stopping time. Let $(\zeta_t)_{t\in[0,T]}$ belong to $\mathscr{A}(\mathscr{F}_t^2)$ and define the family of random variables $\{M(\rho;\xi^*,\zeta), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ in the following manner

$$\begin{aligned} M(\rho;\xi^*,\zeta) := \sum_{i=0}^{\mathscr{I}} \pi_i \Bigg(& \mathcal{V}(\rho)(1-\xi_{\rho-}^{i,*})(1-\zeta_{\rho-}) \\ & + \int_{[0,\rho)} f_s^i (1-\zeta_s)\, d\xi_s^{i,*} + \int_{[0,\rho)} g_s^i (1-\xi_s^{i,*})\, d\zeta_s + \sum_{s\in[0,\rho)} h_s^i \Delta\zeta_s \Delta\xi_s^{i,*} \Bigg). \end{aligned} \tag{4.32}$$

Proposition 4.3.3. *The family of random variables $\{M(\rho;\xi^*,\zeta), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-system for any $\zeta \in \mathscr{A}(\mathscr{F}_t^2)$.*

Proof. For ease of notation, denote by $R^i_{[\rho_1,\rho_2)}(\xi^i,\zeta)$, for any two $(\mathscr{F}_t^2)$-stopping times ρ_1 and ρ_2, the expression

$$R^i_{[\rho_1,\rho_2)}(\xi^i,\zeta) := \int_{[\rho_1,\rho_2)} f_s^i(1-\zeta_s)\, d\xi_s^i + \int_{[\rho_1,\rho_2)} g_s^i(1-\xi_s^i)\, d\zeta_s + \sum_{s\in[\rho_1,\rho_2)} h_s^i \Delta\zeta_s \Delta\xi_s^i.$$

By construction $R^i_{[0,\rho)}(\xi^{i,*},\zeta)$ is $\mathscr{F}_\rho^2$-measurable, and $R^i_{[0,\rho)}(\xi^{i,*},\zeta) = R^i_{[0,\rho')}(\xi^{i,*},\zeta)$ on the event $\{\rho = \rho'\}$ for any two $(\mathscr{F}_t^2)$-stopping times ρ and ρ'.

The set Λ_ρ is $\mathscr{F}_\rho^2$-measurable by definition, as is $J_\rho^{\Pi^*}(\overline{\xi}^\rho,\zeta)$ as it is a conditional expectation with respect to $\mathscr{F}_\rho^2$, and therefore so is its essential supremum. Thus $\mathcal{V}(\rho)$ is $\mathscr{F}_\rho^2$-measurable.

Let A be a set belonging to $\mathscr{F}_\rho^2$ and define the $(\mathscr{F}_t^2)$-stopping time $\rho_A := \rho\,\mathbb{1}_A + T\,\mathbb{1}_{A^c}$. Since the family $\{J_\rho^{\Pi^*}(\overline{\xi}^\rho,\zeta), \zeta \in \mathscr{A}_{\rho_A}(\mathscr{F}_t^2)\}$ is upwards directed (Lemma 3.4.16), the essential supremum in $\mathcal{V}(\rho_A)$ is attained by the limit of a sequence $(\zeta^n)_{n\in\mathbb{N}}$ belonging to $\mathscr{A}_{\rho_A}(\mathscr{F}_t^2)$. Therefore, we can expand the form of $\mathcal{V}(\rho_A)$ on the set A as follows

$$\mathbb{1}_A\mathcal{V}(\rho_A) = \mathbb{1}_{\Lambda_{\rho_A}} \lim_{n\to\infty} \sum_{i=0}^{\mathscr{I}} \mathbb{1}_A \Pi^i_{\rho_A-} \mathbb{E}\Big[R^i_{[\rho_A,T)}(\overline{\xi}^{i,\rho_A},\zeta^n) + h^i_T \Delta\zeta^n_T \Delta\overline{\xi}^{i,\rho_A}_T \Big| \mathscr{F}^2_{\rho_A}\Big]. \tag{4.33}$$

Let ρ' be another $(\mathscr{F}_t^2)$-stopping time, and let $A = \{\rho = \rho'\}$, then $\rho_A = \rho'_A$ by construction. It is clear to see that $\Lambda_{\rho_A} \cap A = \Lambda_\rho \cap A$ since, for any $\omega \in \Lambda_{\rho_A} \cap A$, we have $0 < \sum_{i=0}^{\mathscr{I}} \pi_i(1-\xi^{i,*}_{\rho_A-}) = \sum_{i=0}^{\mathscr{I}} \pi_i(1-\xi^{i,*}_{\rho-})$ as $\rho_A = \rho$ on A, and so $\omega \in \Lambda_\rho \cap A$. Obtaining $\Lambda_{\rho_A} \cap A \supseteq \Lambda_\rho \cap A$ is similar. In the same way, since the belief process is defined using the same processes, $\mathbb{1}_A\Pi^i_{\rho_A-} = \mathbb{1}_A\Pi^i_{\rho-}$. Finally, Lemma 4.1.18 shows that the conditional expectation in (4.33) with respect to $\mathscr{F}^2_{\rho_A}$ agrees with that with respect to $\mathscr{F}^2_\rho$ on A, and agrees with the conditional expectation with respect to $\mathscr{F}^2_T$ on A^c. This implies that

$$\mathbb{1}_A\mathcal{V}(\rho_A) = \mathbb{1}_{\Lambda_\rho} \lim_{n\to\infty} \sum_{i=0}^{\mathscr{I}} \mathbb{1}_A \Pi^i_{\rho-} \mathbb{E}\Big[R^i_{[\rho_A,T)}(\overline{\xi}^{i,\rho_A},\zeta^n) + h^i_T \Delta\zeta^n_T \Delta\overline{\xi}^{i,\rho_A}_T \Big| \mathscr{F}^2_{\rho}\Big].$$

Moreover, since A belongs to $\mathscr{F}^2_\rho$, the indicator function $\mathbb{1}_A$ can be moved inside the conditional expectation to yield

$$\mathbb{1}_A\Big(R^i_{[\rho_A,T)}(\overline{\xi}^{i,\rho_A},\zeta^n) + h^i_T \Delta\zeta^n_T \Delta\overline{\xi}^{i,\rho_A}_T\Big) = \mathbb{1}_A\Big(R^i_{[\rho,T)}(\overline{\xi}^{i,\rho},\zeta^n) + h^i_T \Delta\zeta^n_T \Delta\overline{\xi}^{i,\rho}_T\Big).$$

Thus we can finally say that

$$\mathbb{1}_A\mathcal{V}(\rho_A) = \mathbb{1}_{\Lambda_\rho}\mathbb{1}_A \operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_{\rho_A}(\mathscr{F}_t^2)} J^{\Pi^*}_\rho(\overline{\xi}^\rho,\zeta).$$

For any $\omega \in A$, a process $(\zeta_t)_{t\in[0,T]}$ belonging to $\mathscr{A}_{\rho_A}(\mathscr{F}_t^2)$ must be zero for all $t < \rho_A$, but therefore is also zero for all $t < \rho$ by definition, and so therefore

$$\mathbb{1}_{\Lambda_\rho}\mathbb{1}_A \operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_{\rho_A}(\mathscr{F}_t^2)} J^{\Pi^*}_\rho(\overline{\xi}^\rho,\zeta) \le \mathbb{1}_{\Lambda_\rho}\mathbb{1}_A \operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_{\rho}(\mathscr{F}_t^2)} J^{\Pi^*}_\rho(\overline{\xi}^\rho,\zeta). \tag{4.34}$$

Likewise, we can argue that for $\omega \in A$, a process $\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)$ is zero for all $t < \rho$, but therefore is also zero for all $t < \rho_A$, yielding the reverse inequality of (4.34); therefore, the essential supremum may be taken over $\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)$. Hence

$$\mathbb{1}_A \mathcal{V}(\rho_A) = \mathbb{1}_A \mathcal{V}(\rho).$$

Since $\rho_A = \rho'_A$ everywhere, we must have $\mathcal{V}(\rho_A) = \mathcal{V}(\rho'_A)$. Using the same arguments as above for A^c we have the following, akin to (4.33),

$$\mathbb{1}_{A^c} \mathcal{V}(\rho_{A^c}) = \mathbb{1}_{\Lambda_{\rho_{A^c}}} \lim_{n\to\infty} \sum_{i=0}^{\mathscr{I}} \mathbb{1}_{A^c} \Pi^i_{T-} \mathbb{E}\big[h^i_T \big| \mathscr{F}^2_T\big], \tag{4.35}$$

since both $\Delta\zeta^n_T$ and $\Delta\overline{\xi}^{i,T}_T$ equal one.

Splitting both sides of $\mathcal{V}(\rho_A) = \mathcal{V}(\rho'_A)$ over the indicator functions of A and A^c, and using both (4.33) and (4.35) and the fact that $\rho_A = \rho'_A$ everywhere, we get

$$\mathbb{1}_A \mathcal{V}(\rho) + \mathbb{1}_{A^c} \sum_{i=0}^{\mathscr{I}} \Pi^i_{T-} \mathbb{E}[h^i_T | \mathscr{F}^2_T] = \mathbb{1}_A \mathcal{V}(\rho') + \mathbb{1}_{A^c} \sum_{i=0}^{\mathscr{I}} \Pi^i_{T-} \mathbb{E}[h^i_T | \mathscr{F}^2_T],$$

implying that $\mathbb{1}_A \mathcal{V}(\rho) = \mathbb{1}_A \mathcal{V}(\rho')$. The prefactors $(1-\xi^{i,*}_{\rho-})(1-\zeta_{\rho-})$ are also $\mathscr{F}^2_\rho$-measurable and $(1-\xi^{i,*}_{\rho-})(1-\zeta_{\rho-}) = (1-\xi^{i,*}_{\rho'-})(1-\zeta_{\rho'-})$ on $\{\rho = \rho'\}$, thus completing the proof. ■

Theorem 4.3.4. *The family $\{M(\rho;\xi^*,\zeta), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-supermartingale system for any $\zeta \in \mathscr{A}(\mathscr{F}_t^2)$.*

Proof. 1° Integrability follows in the same manner as it did in the proof of Theorem 4.1.20. All we need to show is that $\mathbb{E}|\mathcal{V}(\rho)| < \infty$, however, this was shown in Lemma 3.4.14.

2° Let ν be an $(\mathscr{F}_t^2)$-stopping time such that $\nu \le \rho$. Expand the first term of (4.32) using (4.31) to obtain

$$\sum_{i=0}^{\mathscr{I}} \pi_i \mathcal{V}(\nu)(1-\xi^{i,*}_{\nu-})(1-\zeta_{\nu-}) = \mathbb{1}_{\Lambda_\nu} \sum_{i=0}^{\mathscr{I}} \pi_i (1-\xi^{i,*}_{\nu-})(1-\zeta_{\nu-}) \operatorname*{ess\,sup}_{\chi \in \mathscr{A}_\nu(\mathscr{F}_t^2)} J^{\Pi^*}_\nu(\overline{\xi}^\nu, \chi).$$

Lemma 3.4.16 tells us that the family $\{J^{\Pi^*}_\rho(\overline{\xi}^\rho, \chi), \chi \in \mathscr{A}_\rho(\mathscr{F}_t^2)\}$ is upwards directed, meaning that the essential supremum at ρ is reached by a sequence $(\chi^n)_{n\in\mathbb{N}} \in \mathscr{A}_\rho(\mathscr{F}_t^2)$. Let us define a strategy for the maximiser as follows

$$\hat{\zeta}_t := \zeta^\nu_t \mathbb{1}_{[\nu,\rho)}(t) + \Big(\zeta^\nu_{\rho-} + (1-\zeta^\nu_{\rho-})\chi^n_t\Big) \mathbb{1}_{[\rho,T]}(t). \tag{4.36}$$

It is easy to show that $(\hat{\zeta}_t)_{t\in[0,T]}$ belongs to $\mathscr{A}_\nu(\mathscr{F}_t^2)$ (see Lemma 4.1.16). This process may be suboptimal for $\mathcal{V}(\nu)$, and so we have

$$\sum_{i=0}^{\mathscr{I}} \pi_i \mathcal{V}(\nu)(1-\xi_{\nu-}^{i,*})(1-\zeta_{\nu-}) \geq \mathbb{1}_{\Lambda_\nu} \sum_{i=0}^{\mathscr{I}} \pi_i (1-\xi_{\nu-}^{i,*})(1-\zeta_{\nu-}) J_\nu^{\Pi^*}(\overline{\xi}^\nu, \hat{\zeta}). \tag{4.37}$$

Expanding the conditional payoff, and then using Remark 4.1.14 to use the explicit form of the belief process, the right-hand side of (4.37) can be rewritten as

$$\mathbb{1}_{\Lambda_\nu}(1-\zeta_{\nu-}) \sum_{i=0}^{\mathscr{I}} \pi_i (1-\xi_{\nu-}^{i,*}) \mathbb{E}\Bigg[\int_{[\nu,T)} f_t^i (1-\hat{\zeta}_t)\, d\overline{\xi}_t^{i,\nu} + \int_{[\nu,T)} g_t^i (1-\overline{\xi}_t^{i,\nu})\, d\hat{\zeta}_t + \sum_{t\in[\nu,T]} h_t^i \Delta\hat{\zeta}_t \Delta\overline{\xi}_t^{i,\nu} \Bigg| \mathscr{F}_\nu^2 \Bigg].$$

Due to the presence of Λ_ν we can write $\overline{\xi}^{i,\nu}$ in its truncated form. Strictly speaking we can only do this on a non-empty subset of $[\mathscr{I}]$. For the remaining values of i, the integral and summation (or π_i) terms are zero. Cancelling out the truncation with the prefactor $(1-\xi_{\nu-}^{i,*})$ we have

$$\mathbb{1}_{\Lambda_\nu}(1-\zeta_{\nu-}) \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\Bigg[\int_{[\nu,T)} f_t^i (1-\hat{\zeta}_t)\, d\xi_t^{i,*} + \int_{[\nu,T)} g_t^i (1-\xi_t^{i,*})\, d\hat{\zeta}_t + \sum_{t\in[\nu,T]} h_t^i \Delta\hat{\zeta}_t \Delta\xi_t^{i,*} \Bigg| \mathscr{F}_\nu^2 \Bigg].$$

Splitting everything over the subintervals $[\nu,\rho)$ and $[\rho,T)$ (resp. $[\rho,T]$) and using the form of $(\hat{\zeta}_t)_{t\in[0,T]}$ from (4.36) on those intervals we have

$$\begin{aligned}\mathbb{1}_{\Lambda_\nu}(1-\zeta_{\nu-}) \sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\Bigg[&\int_{[\nu,\rho)} f_t^i (1-\zeta_t^\nu)\, d\xi_t^{i,*} + \int_{[\nu,\rho)} g_t^i (1-\xi_t^{i,*})\, d\zeta_t^\nu + \sum_{t\in[\nu,\rho)} h_t^i \Delta\zeta_t^\nu \Delta\xi_t^{i,*} \\ &+ (1-\zeta_{\rho-}^\nu)\left(\int_{[\rho,T)} f_t^i (1-\zeta_t^n)\, d\xi_t^{i,*} + \int_{[\rho,T)} g_t^i (1-\xi_t^{i,*})\, d\zeta_t^n + \sum_{t\in[\rho,T]} h_t^i \Delta\zeta_t^n \Delta\xi_t^{i,*} \right) \Bigg| \mathscr{F}_\nu^2 \Bigg].\end{aligned}$$

Moving the prefactor $(1-\zeta_{\nu-})$ inside of the conditional expectation and distributing it across both terms, it is simple to show that $(1-\zeta_{\nu-})(1-\zeta_{\rho-}^\nu) = (1-\zeta_{\rho-})$. Notice that the presence of the prefactor $(1-\zeta_{\nu-})$ on the first term essentially acts as the indicator function $\mathbb{1}_{\{\zeta_{\nu-}(\omega)<1\}}$. When $1-\zeta_{\nu-}=0$ the first term above is trivially zero, so we can use the explicit form of the truncated strategy $(\zeta_t^\nu)_{t\in[0,T]}$ and we get

$$\mathbb{1}_{\Lambda_\nu}\sum_{i=0}^{\mathscr{I}}\pi_i\mathbb{E}\Bigg[\int_{[\nu,\rho)}f_t^i(1-\zeta_t)\,d\xi_t^{i,*}+\int_{[\nu,\rho)}g_t^i(1-\xi_t^{i,*})\,d\zeta_t+\sum_{t\in[\nu,\rho)}h_t^i\Delta\zeta_t\Delta\xi_t^{i,*}$$
$$+(1-\zeta_{\rho-})\left(\int_{[\rho,T)}f_t^i(1-\zeta_t^n)\,d\xi_t^{i,*}+\int_{[\rho,T)}g_t^i(1-\xi_t^{i,*})\,d\zeta_t^n+\sum_{t\in[\rho,T]}h_t^i\Delta\zeta_t^n\Delta\xi_t^{i,*}\right)\Bigg|\mathscr{F}_\nu^2\Bigg].$$

Notice that the last line is still on the indicator function $\mathbb{1}_{\Lambda_\nu}$, however, all terms are clearly zero on $\mathbb{1}_{(\Lambda_\nu)^c}$, meaning we need not have the indicator function explicitly there. Moreover, using the same logic we see that we can freely introduce the indicator function $\mathbb{1}_{\Lambda_\rho}$ to the last line, since on $\mathbb{1}_{(\Lambda_\rho)^c}$ all terms are zero too, meaning we have

$$\sum_{i=0}^{\mathscr{I}}\pi_i\mathbb{E}\Bigg[\mathbb{1}_{\Lambda_\nu}\left(\int_{[\nu,\rho)}f_t^i(1-\zeta_t)\,d\xi_t^{i,*}+\int_{[\nu,\rho)}g_t^i(1-\xi_t^{i,*})\,d\zeta_t+\sum_{t\in[\nu,\rho)}h_t^i\Delta\zeta_t\Delta\xi_t^{i,*}\right)$$
$$+\mathbb{1}_{\Lambda_\rho}(1-\zeta_{\rho-})\left(\int_{[\rho,T)}f_t^i(1-\zeta_t^n)\,d\xi_t^{i,*}+\int_{[\rho,T)}g_t^i(1-\xi_t^{i,*})\,d\zeta_t^n+\sum_{t\in[\rho,T]}h_t^i\Delta\zeta_t^n\Delta\xi_t^{i,*}\right)\Bigg|\mathscr{F}_\nu^2\Bigg]. \tag{4.38}$$

Given that the last line is on the set Λ_ρ, we can truncate $(\xi_t^{i,*})_{t\in[0,T]}$ and gain the prefactor $(1-\xi_{\rho-}^{i,*})$. Moreover, recall from Remark 4.1.14 that on Λ_ρ we have

$$\pi_i(1-\xi_{\rho-}^{i,*})=\Pi_{\rho-}^{i,*}\sum_{k=0}^{\mathscr{I}}\pi_k(1-\xi_{\rho-}^{k,*}),$$

meaning we may reintroduce the belief process as follows. Using the formula above and the tower property of conditional expectation—using the fact that $\mathbb{1}_{\Lambda_\rho}\Pi_{\rho-}^{i,*}\sum_{k=0}^{\mathscr{I}}\pi_k(1-\xi_{\rho-}^{k,*})(1-\zeta_{\rho-})$ is $\mathscr{F}_\rho^2$-measurable—we get from (4.38), and some simple algebra,

$$\sum_{i=0}^{\mathscr{I}}\pi_i\mathbb{E}\Bigg[\mathbb{1}_{\Lambda_\nu}\left(\int_{[\nu,\rho)}f_t^i(1-\zeta_t)\,d\xi_t^{i,*}+\int_{[\nu,\rho)}g_t^i(1-\xi_t^{i,*})\,d\zeta_t+\sum_{t\in[\nu,\rho)}h_t^i\Delta\zeta_t\Delta\xi_t^{i,*}\right)$$
$$+\mathbb{1}_{\Lambda_\rho}(1-\xi_{\rho-}^{i,*})(1-\zeta_{\rho-})J_\rho^{\Pi^*}(\overline{\xi}^\rho,\zeta^n)\Bigg|\mathscr{F}_\nu^2\Bigg].$$

Of course this is still all just the right-hand side of (4.37), so taking limits on both sides and using the dominated convergence theorem, we realise that since $(\zeta_t^n)_{n\in\mathbb{N}}$ is the sequence which realises

$\operatorname{ess\,sup}_{\zeta\in\mathscr{A}_\nu(\mathscr{F}_t^2)} J_\nu^{\Pi^*}(\overline{\xi}^\nu,\zeta)$ we get

$$
\begin{aligned}
&\sum_{i=0}^{\mathscr{I}}\pi_i\mathcal{V}(\nu)(1-\xi_{\nu-}^{i,*})(1-\zeta_{\nu-})\\
&\qquad\geq\sum_{i=0}^{\mathscr{I}}\pi_i\mathbb{E}\Bigg[\mathbb{1}_{\Lambda_\nu}\bigg(\int_{[\nu,\rho)}f_t^i(1-\zeta_t)\,d\xi_t^{i,*}+\int_{[\nu,\rho)}g_t^i(1-\xi_t^{i,*})\,d\zeta_t+\sum_{t\in[\nu,\rho)}h_t^i\Delta\zeta_t\Delta\xi_t^{i,*}\bigg)\\
&\qquad\qquad+\mathbb{1}_{\Lambda_\rho}(1-\zeta_{\rho-})(1-\xi_{\rho-}^{i,*})\operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)}J_\rho^{\Pi^*}(\overline{\xi}^\rho,\zeta)\bigg|\mathscr{F}_\nu^2\Bigg].
\end{aligned}
$$

Additionally, like before, we can remove the indicator function from the first line because all terms are zero on Λ_ρ^c, yielding,

$$
\begin{aligned}
&\sum_{i=0}^{\mathscr{I}}\pi_i\mathcal{V}(\nu)(1-\xi_{\nu-}^{i,*})(1-\zeta_{\nu-})\\
&\qquad\geq\sum_{i=0}^{\mathscr{I}}\pi_i\mathbb{E}\Bigg[\int_{[\nu,\rho)}f_t^i(1-\zeta_t)\,d\xi_t^{i,*}+\int_{[\nu,\rho)}g_t^i(1-\xi_t^{i,*})\,d\zeta_t+\sum_{t\in[\nu,\rho)}h_t^i\Delta\zeta_t\Delta\xi_t^{i,*}\\
&\qquad\qquad+(1-\zeta_{\rho-})(1-\xi_{\rho-}^{i,*})\mathcal{V}(\rho)\bigg|\mathscr{F}_\nu^2\Bigg].
\end{aligned}
\tag{4.39}
$$

Finally, using (4.32), (4.39) tells us that

$$
\begin{aligned}
M(\nu;\xi^*,\zeta)&\geq\sum_{i=0}^{\mathscr{I}}\pi_i\mathbb{E}\Bigg[\int_{[0,\rho)}f_t^i(1-\zeta_t)\,d\xi_t^{i,*}+\int_{[0,\rho)}g_t^i(1-\xi_t^{i,*})\,d\zeta_t+\sum_{t\in[0,\rho)}h_t^i\Delta\zeta_t\Delta\xi_t^{i,*}\\
&\quad+(1-\zeta_{\rho-})(1-\xi_{\rho-}^{i,*})\mathcal{V}(\rho)\bigg|\mathscr{F}_\nu^2\Bigg]\\
&=\mathbb{E}\big[M(\rho;\xi^*,\zeta)\big|\mathscr{F}_\nu^2\big]
\end{aligned}
$$

as required. ■

Remark 4.3.5. As we mentioned prior to the theorem above, we would expect that the family $\{M(\rho;\xi^*,\zeta),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$ should obey the martingale property when the maximiser plays optimally. In this case this is trivial to see since the family of random variables $\{M(\rho;\xi^*,\zeta^*),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$ coincides with $\{M(\rho),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$. Because of this, it is clear that the same process, $(M_t)_{t\in[0,T]}$, from Theorem 4.1.22 that aggregates $\{M(\rho),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$ also aggregates the family $\{M(\rho;\xi^*,\zeta^*),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$.

Corollary 4.3.6. *There exists a process, $(Z_t)_{t\in[0,T]}$, which aggregates the family of random variables* $\{\sum_{i=0}^{\mathscr{I}} \pi_i \mathcal{V}(\rho)(1-\xi^{i,*}_{\rho-})(1-\zeta^*_{\rho-}), \rho \in \mathscr{T}(\mathscr{F}^2_t)\}$.

Proof. We construct this aggregating process explicitly. Recall the process $(M_t)_{t\in[0,T]}$ from Theorem 4.1.22, then define

$$Z_t := M_t - \sum_{i=0}^{\mathscr{I}} \pi_i \left(\int_{[0,t)} f^i_s(1-\zeta^*_s)\, d\xi^{i,*}_s + \int_{[0,t)} g^i_s(1-\xi^{i,*}_s)\, d\zeta^*_s + \sum_{s\in[0,t)} h^i_s \Delta\zeta^*_s \Delta\xi^{i,*}_s \right).$$

To prove aggregation, take any $(\mathscr{F}^2_t)$-stopping time ρ, then since $(M_t)_{t\in[0,T]}$ aggregates the family $\{M(\rho;\xi^*,\zeta^*), \rho \in \mathscr{T}(\mathscr{F}^2_t)\}$ we have, $\mathbb{P}$-almost surely,

$$\begin{aligned} Z_\rho &= M(\rho;\xi^*,\zeta^*) - \sum_{i=0}^{\mathscr{I}} \pi_i \left(\int_{[0,\rho)} f^i_s(1-\zeta^*_s)\, d\xi^{i,*}_s + \int_{[0,\rho)} g^i_s(1-\xi^{i,*}_s)\, d\zeta^*_s + \sum_{s\in[0,\rho)} h^i_s \Delta\zeta^*_s \Delta\xi^{i,*}_s \right) \\ &= \sum_{i=0}^{\mathscr{I}} \pi_i \mathcal{V}(\rho)(1-\xi^{i,*}_{\rho-})(1-\zeta^*_{\rho-}). \end{aligned}$$

■

* * *

We now turn our attention to the minimisers of the game. We seek similar results to those proved above, however, we will be finding $\mathscr{T}(\mathscr{F}^2_t)$-submartingale systems. Notably, as mentioned before, we are now dealing with each incarnation of the minimiser—who knows the outcome of Θ—and therefore when we define the best response variable akin to (4.31), we do not need to take into account the belief process. In the following lemma we introduce the i^{th} best response variable and prove its relation to the value of the game and Theorem 4.2.6.

Lemma 4.3.7. *For $i \in [\mathscr{I}]$ define the family of random variables $\{\mathcal{V}^i(\rho), \rho \in \mathscr{T}(\mathscr{F}^2_t)\}$ by*

$$\mathcal{V}^i(\rho) := \mathbb{1}_{\Gamma_\rho} \operatorname*{ess\,inf}_{\xi^i \in \mathscr{A}_\rho(\mathscr{F}^2_t)} \mathbb{E}\left[\int_{[\rho,T)} f^i_s(1-\overline{\zeta}^\rho_s)\, d\xi^i_s + \int_{[\rho,T)} g^i_s(1-\xi^i_s)\, d\overline{\zeta}^\rho_s + \sum_{s\in[\rho,T]} h^i_s \Delta\overline{\zeta}^\rho_s \Delta\xi^i_s \,\middle|\, \mathscr{F}^2_\nu \right].$$

Then

1. $\mathbb{1}_{\Omega_\rho} V^{\Pi^*}(\rho) = \mathbb{1}_{\Omega_\rho} \sum_{i=0}^{\mathscr{I}} \Pi^{i,*}_{\rho-} \mathcal{V}^i(\rho)$, $\quad \mathbb{P}$-*a.s.*,
2. $\mathbb{1}_{\Omega^i_\rho} \mathcal{V}^i(\rho) = \mathbb{1}_{\Omega^i_\rho} \mathbb{E}[P^i(\overline{\xi}^{i,\rho}, \overline{\zeta}^\rho) | \mathscr{F}^2_\rho]$, $\quad \mathbb{P}$-*a.s.*

Proof.

1. Recall the notation of $P^i(\xi^i,\zeta)$ from (3.15). From Theorem 4.2.6 we have the equality

$$\mathbb{1}_{\Omega_\rho} V^{\Pi^*}(\rho) = \operatorname*{ess\,inf}_{\xi\in\mathscr{A}_\rho(\mathscr{F}_t^1)} \mathbb{1}_{\Omega_\rho} \sum_{i=0}^{\mathscr{I}} \Pi^{i,*}_{\rho-}\mathbb{E}\Big[P^i\Big(\xi^i,\overline{\zeta}^\rho\Big)\Big|\mathscr{F}^2_\rho\Big] \quad \mathbb{P}\text{-a.s.} \tag{4.40}$$

Since any $\xi \in \mathscr{A}_\rho(\mathscr{F}_t^1)$ has a decomposition into processes $\xi^i \in \mathscr{A}_\rho(\mathscr{F}_t^2)$ for each $i \in [\mathscr{I}]$, and the essential infimum in (4.40) is of a sum over the values of i, the optimisation over each individual i is independent of the rest; we can consider the sum of optimisations instead of the optimisation of the sum, yielding

$$\mathbb{1}_{\Omega_\rho} V^{\Pi^*}(\rho) = \mathbb{1}_{\Omega_\rho} \sum_{i=0}^{\mathscr{I}} \Pi^{i,*}_{\rho-} \operatorname*{ess\,inf}_{\xi^i\in\mathscr{A}_\rho(\mathscr{F}_t^2)} \mathbb{E}\Big[P^i\Big(\xi^i,\overline{\zeta}^\rho\Big)\Big|\mathscr{F}^2_\rho\Big] = \mathbb{1}_{\Omega_\rho} \sum_{i=0}^{\mathscr{I}} \Pi^{i,*}_{\rho-}\mathcal{V}^i(\rho).$$

2. Theorem 4.2.6 gives us that the truncated strategies are optimal at time ρ, and so

$$\mathbb{1}_{\Omega_\rho} \sum_{i=0}^{\mathscr{I}} \Pi^{i,*}_{\rho-}\mathcal{V}^i(\rho) = \mathbb{1}_{\Omega_\rho} V^{\Pi^*}(\rho) = \mathbb{1}_{\Omega_\rho} \sum_{i=0}^{\mathscr{I}} \Pi^{i,*}_{\rho-}\mathbb{E}\Big[P^i\Big(\overline{\xi}^{i,\rho},\overline{\zeta}^\rho\Big)\Big|\mathscr{F}^2_\rho\Big] \quad \mathbb{P}\text{-a.s.}$$

with the first equality following from 1. above. Thus,

$$\mathbb{1}_{\Omega_\rho} \sum_{i=0}^{\mathscr{I}} \Pi^{i,*}_{\rho-}\Big(\mathcal{V}^i(\rho) - \mathbb{E}\Big[P^i\Big(\overline{\xi}^{i,\rho},\overline{\zeta}^\rho\Big)\Big|\mathscr{F}^2_\rho\Big]\Big) = 0 \quad \mathbb{P}\text{-a.s.}$$

If we introduce $(\mathbb{1}_{\Omega^i_\rho} + \mathbb{1}_{(\Omega^i_\rho)^c})$ inside of the summation we can use the fact that $\Omega^i_\rho \subset \Omega_\rho$ and $\Omega_\rho \cap (\Omega^i_\rho)^c = \Omega_\rho \setminus \Omega^i_\rho$ to get

$$\begin{aligned} 0 = \sum_{i=0}^{\mathscr{I}} \Big\{ &\mathbb{1}_{\Omega^i_\rho}\Pi^{i,*}_{\rho-}\Big(\mathcal{V}^i(\rho) - \mathbb{E}\Big[P^i\Big(\overline{\xi}^{i,\rho},\overline{\zeta}^\rho\Big)\Big|\mathscr{F}^2_\rho\Big]\Big) \\ &+ \mathbb{1}_{\Omega_\rho\setminus\Omega^i_\rho}\Pi^{i,*}_{\rho-}\Big(\mathcal{V}^i(\rho) - \mathbb{E}\Big[P^i\Big(\overline{\xi}^{i,\rho},\overline{\zeta}^\rho\Big)\Big|\mathscr{F}^2_\rho\Big]\Big)\Big\} \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

For the first term we know that, for $\omega \in \Omega^i_\rho$, we have $\pi_i(1-\xi^{i,*}_{\rho-})(1-\zeta^*_{\rho-}) > 0$ (and therefore $\sum_{i=0}^{\mathscr{I}} \pi_i(1-\xi^{i,*}_{\rho-})(1-\zeta^*_{\rho-}) > 0$), thus the belief process, $\Pi^{i,*}_{\rho-}$, must also be positive. For the second term, for any $\omega \in \Omega_\rho \setminus \Omega^i_\rho$, we have $\sum_{i=0}^{\mathscr{I}} \pi_i(1-\xi^{i,*}_{\rho-})(1-\zeta^*_{\rho-}) > 0$ and $\pi_i(1-\xi^{i,*}_{\rho-})(1-\zeta^*_{\rho-}) = 0$; therefore, we must have $1-\zeta^*_{\rho-} > 0$ and $\pi_i(1-\xi^{i,*}_{\rho-}) = 0$, so the belief process must be equal to zero. As a result we know that

$$\sum_{i=0}^{\mathscr{I}} \mathbb{1}_{\Omega^i_\rho} \Pi^{i,*}_{\rho-} \Big(\mathcal{V}^i(\rho) - \mathbb{E}\Big[P^i\Big(\overline{\xi}^{i,\rho}, \overline{\zeta}^{\rho}\Big) \Big| \mathscr{F}^2_\rho \Big] \Big) = 0 \quad \mathbb{P}\text{-a.s.}$$

Since the belief process must be positive on Ω^i_ρ and the sum is equal to zero, it must be that

$$\mathbb{1}_{\Omega^i_\rho} \mathcal{V}^i(\rho) = \mathbb{1}_{\Omega^i_\rho} \mathbb{E}\Big[P^i\Big(\overline{\xi}^{i,\rho}, \overline{\zeta}^{\rho}\Big) \Big| \mathscr{F}^2_\rho \Big] \quad \mathbb{P}\text{-a.s.}$$

■

In much the same way as we saw in Corollary 4.3.1, this new family of best response random variables coincides with the value of the game so long as neither player has stopped completely. We obtain slightly more information from this result, however, since we explicitly show that, for the problem starting at ν, $(\overline{\xi}^{i,\nu}_t)_{t\in[0,T]}$ is the best response to $(\overline{\zeta}^{\nu}_t)_{t\in[0,T]}$ on Ω^i_ν for each $i \in [\mathscr{I}]$.

Once again, let us define a new family of random variables, akin to those in Definition 4.3.2, however, we will show that this family is a $\mathscr{T}(\mathscr{F}^2_t)$-submartingale system instead.

Definition 4.3.8. Let $(\xi_t)_{t\in[0,T]}$ belong to $\mathscr{A}(\mathscr{F}^1_t)$ and define, for each $i \in [\mathscr{I}]$, the family of random variables $\{M^i(\rho;\xi^i,\zeta^*), \rho \in \mathscr{T}(\mathscr{F}^2_t)\}$ in the following manner

$$M^i(\rho;\xi^i,\zeta^*) := \mathcal{V}^i(\rho)(1-\xi^i_{\rho-})(1-\zeta^*_{\rho-}) + \int_{[0,\rho)} f^i_s(1-\zeta^*_s)\,d\xi^i_s + \int_{[0,\rho)} g^i_s(1-\xi^i_s)\,d\zeta^*_s + \sum_{s\in[0,\rho)} h^i_s \Delta\zeta^*_s \Delta\xi^i_s. \quad (4.41)$$

Theorem 4.3.9. *The family $\{M^i(\rho;\xi^i,\zeta^*), \rho \in \mathscr{T}(\mathscr{F}^2_t)\}$ is a $\mathscr{T}(\mathscr{F}^2_t)$-submartingale system for any $\xi \in \mathscr{A}(\mathscr{F}^1_t)$.*

Proof. 1° The fact that the family is a $\mathscr{T}(\mathscr{F}^2_t)$-system can be shown in the same way, *mutatis mutandis*, as in Proposition 4.3.3.

2° Integrability is proven in the same way as in Theorem 4.1.20. We need to show that $\mathbb{E}|\mathcal{V}^i(\rho)| < \infty$, which can be done, *mutatis mutandis*, in the same manner as in Lemma 3.4.14.

3° Using Lemma 3.4.16, we know that the essential infimum in $\mathcal{V}^i(\rho)$ can be attained in a non-increasing manner using a sequence $(\xi^{i,n})_{n\in\mathbb{N}} \subset \mathscr{A}_\rho(\mathscr{F}_t^2)$, i.e.

$$\mathcal{V}^i(\rho) = \mathbb{1}_{\Gamma_\rho} \lim_{n\to\infty} \mathbb{E}\left[\int_{[\rho,T)} f_s^i(1-\overline{\zeta}_s^\rho)\, d\xi_s^{i,n} + \int_{[\rho,T)} g_s^i(1-\xi_s^{i,n})\, d\overline{\zeta}_s^\rho + \sum_{s\in[\rho,T]} h_s^i \Delta\overline{\zeta}_s^\rho \Delta\xi_s^{i,n} \middle| \mathscr{F}_\rho^2\right]. \tag{4.42}$$

For any $n \geq 1$, define the following

$$\hat{\xi}_t^i := \xi_t^{i,\nu} \mathbb{1}_{[\nu,\rho)}(t) + \left(\xi_{\rho-}^{i,\nu} + (1-\xi_{\rho-}^{i,\nu})\xi_t^{i,n}\right)\mathbb{1}_{[\rho,T]}(t).$$

Notice that, trivially, we have $\hat{\xi}_{\nu-}^i = 0$, $\hat{\xi}_T^i = 1$ and that $(\hat{\xi}_t^i)_{t\in[0,T]}$ is càdlàg. We can see that $(\hat{\xi}_t^i)_{t\in[0,T]}$ is increasing, because $\hat{\xi}_\rho^i \geq \xi_{\rho-}^{i,\nu} = \hat{\xi}_{\rho-}^i$, thus we know $\hat{\xi}^i \in \mathscr{A}_\nu(\mathscr{F}_t^1)$. Since $(\hat{\xi}_t^i)_{t\in[0,T]}$ is suboptimal for the minimiser at ν, we have that

$$\mathcal{V}^i(\nu) \leq \mathbb{1}_{\Gamma_\nu}\mathbb{E}\left[\int_{[\nu,T)} f_s^i(1-\overline{\zeta}_s^\nu)\, d\hat{\xi}_s^i + \int_{[\nu,T)} g_s^i(1-\hat{\xi}_s^i)\, d\overline{\zeta}_s^\nu + \sum_{s\in[\nu,T]} h_s^i \Delta\overline{\zeta}_s^\nu \Delta\hat{\xi}_s^i \middle| \mathscr{F}_\nu^2\right],$$

and this allows us to write

$$\begin{aligned} M^i(\nu;\xi,\zeta^*) \leq \mathbb{1}_{\Gamma_\nu}\mathbb{E}\Bigg[&(1-\xi_{\nu-}^i)(1-\zeta_{\nu-}^*)\bigg(\int_{[\nu,T)} f_s^i(1-\overline{\zeta}_s^\nu)\, d\hat{\xi}_s^i + \int_{[\nu,T)} g_s^i(1-\hat{\xi}_s^i)\, d\overline{\zeta}_s^\nu \\ &+ \sum_{s\in[\nu,T]} h_s^i \Delta\overline{\zeta}_s^\nu \Delta\hat{\xi}_s^i\bigg)\bigg|\mathscr{F}_\nu^2\Bigg] \\ &+ \int_{[0,\nu)} f_s^i(1-\zeta_s^*)\, d\xi_s^i + \int_{[0,\nu)} g_s^i(1-\xi_s^i)\, d\zeta_s^* + \sum_{s\in[0,\nu)} h_s^i \Delta\zeta_s^* \Delta\xi_s^i. \end{aligned}$$

In the same way as we have done before, we can split the interval $[\nu,T)$ into $[\nu,\rho)$ and $[\rho,T)$ (respectively right-closed for the summation term) and use the definition of $(\hat{\xi}_t^i)_{t\in[0,T]}$ to see that

$$\begin{aligned} &M^i(\nu;\xi,\zeta^*) \\ &\leq \mathbb{1}_{\Gamma_\nu}\mathbb{E}\Bigg[(1-\xi_{\nu-}^i)(1-\zeta_{\nu-}^*)(1-\xi_{\rho-}^{i,\nu-})\bigg(\int_{[\rho,T)} f_s^i(1-\overline{\zeta}_s^\nu)\, d\xi_s^{i,n} + \int_{[\rho,T)} g_s^i(1-\xi_s^{i,n})\, d\overline{\zeta}_s^\nu \\ &+ \sum_{s\in[\rho,T]} h_s^i \Delta\overline{\zeta}_s^\nu \Delta\xi_s^{i,n}\bigg) + (1-\xi_{\nu-}^i)(1-\zeta_{\nu-}^*)\bigg(\int_{[\nu,\rho)} f_s^i(1-\overline{\zeta}_s^\nu)\, d\xi_s^{i,\nu} + \int_{[\nu,\rho)} g_s^i(1-\xi_s^{i,\nu})\, d\overline{\zeta}_s^\nu \\ &+ \sum_{s\in[\nu,\rho)} h_s^i \Delta\overline{\zeta}_s^\nu \Delta\xi_s^{i,\nu}\bigg)\bigg|\mathscr{F}_\nu^2\Bigg] \\ &+ \int_{[0,\nu)} f_s^i(1-\zeta_s^*)\, d\xi_s^i + \int_{[0,\nu)} g_s^i(1-\xi_s^i)\, d\zeta_s^* + \sum_{s\in[0,\nu)} h_s^i \Delta\zeta_s^* \Delta\xi_s^i. \end{aligned}$$

It is straightforward to show that $(1-\xi^i_{\nu-})(1-\xi^{i,\nu}_{\rho-}) = (1-\xi^i_{\rho-})\mathbb{1}_{[\nu,T]}$. Moreover, using the relevant prefactors, we can cancel out the truncations, and furthermore, since Γ_ν is $\mathscr{F}^2_\nu$-measurable, we may move the indicator function inside the conditional expectation and be left with

$$\begin{aligned}
&M^i(\nu;\xi,\zeta^*) \\
&\leq \mathbb{E}\Bigg[\mathbb{1}_{\Gamma_\nu}(1-\xi^i_{\rho-})\bigg(\int_{[\rho,T)} f^i_s(1-\zeta^*_s)\,d\xi^{i,n}_s + \int_{[\rho,T)} g^i_s(1-\xi^{i,n}_s)\,d\zeta^*_s + \sum_{s\in[\rho,T]} h^i_s\Delta\zeta^*_s\Delta\xi^{i,n}_s\bigg) \\
&+ \mathbb{1}_{\Gamma_\nu}\bigg(\int_{[\nu,\rho)} f^i_s(1-\zeta^*_s)\,d\xi^{i,*}_s + \int_{[\nu,\rho)} g^i_s(1-\xi^{i,*}_s)\,d\zeta^*_s + \sum_{s\in[\nu,\rho)} h^i_s\Delta\zeta^*_s\Delta\xi^{i,*}_s\bigg)\bigg|\mathscr{F}^2_\nu\Bigg] \\
&+ \int_{[0,\nu)} f^i_s(1-\zeta^*_s)\,d\xi^i_s + \int_{[0,\nu)} g^i_s(1-\xi^i_s)\,d\zeta^*_s + \sum_{s\in[0,\nu)} h^i_s\Delta\zeta^*_s\Delta\xi^i_s.
\end{aligned} \tag{4.43}$$

In order to combine the last two lines together, we use the same argument that we have repeated frequently: notice that the terms inside the brackets in (4.43) are zero on $(\Gamma_\nu)^c$. As a result we can remove the indicator function from that line, move the last line into the conditional expectation and join the terms, yielding

$$\begin{aligned}
&M^i(\nu;\xi,\zeta^*) \\
&\leq \mathbb{E}\Bigg[\mathbb{1}_{\Gamma_\nu}(1-\xi^i_{\rho-})\bigg(\int_{[\rho,T)} f^i_s(1-\zeta^*_s)\,d\xi^{i,n}_s + \int_{[\rho,T)} g^i_s(1-\xi^{i,n}_s)\,d\zeta^*_s + \sum_{s\in[\rho,T]} h^i_s\Delta\zeta^*_s\Delta\xi^{i,n}_s\bigg) \\
&+ \int_{[0,\rho)} f^i_s(1-\zeta^*_s)\,d\xi^i_s + \int_{[0,\rho)} g^i_s(1-\xi^i_s)\,d\zeta^*_s + \sum_{s\in[0,\rho)} h^i_s\Delta\zeta^*_s\Delta\xi^i_s\bigg|\mathscr{F}^2_\nu\Bigg].
\end{aligned}$$

We may use the same reasoning to justify changing the indicator function to $\mathbb{1}_{\Gamma_\rho}$, and so we can now trivially truncate the maximiser's strategy in the first conditional expectation, by introducing $(1-\zeta^*_{\rho-})$ as a prefactor:

$$\begin{aligned}
M^i(\nu;\xi,\zeta^*) \leq \mathbb{E}\Bigg[&\mathbb{1}_{\Gamma_\rho}(1-\xi^i_{\rho-})(1-\zeta^*_{\rho-})\bigg(\int_{[\rho,T)} f^i_s(1-\overline{\zeta}^\rho_s)\,d\xi^{i,n}_s + \int_{[\rho,T)} g^i_s(1-\xi^{i,n}_s)\,d\overline{\zeta}^\rho_s \\
&+ \sum_{s\in[\rho,T]} h^i_s\Delta\overline{\zeta}^\rho_s\Delta\xi^{i,n}_s\bigg) \\
&+ \int_{[0,\rho)} f^i_s(1-\zeta^*_s)\,d\xi^i_s + \int_{[0,\rho)} g^i_s(1-\xi^i_s)\,d\zeta^*_s + \sum_{s\in[0,\rho)} h^i_s\Delta\zeta^*_s\Delta\xi^i_s\bigg|\mathscr{F}^2_\nu\Bigg].
\end{aligned}$$

Using the tower property of conditional expectation with respect to $\mathscr{F}^2_\rho$, and pulling out all of the

$\mathscr{F}^2_\rho$-measurable terms, we can then take limits on both sides and using the dominated convergence theorem to get

$$\begin{aligned}M^i(\nu;\xi,\zeta^*) \leq \mathbb{E}\Bigg[&\mathbb{1}_{\Gamma_\rho}(1-\xi^i_{\rho-})(1-\zeta^*_{\rho-})\lim_{n\to\infty}\mathbb{E}\bigg[\int_{[\rho,T)} f^i_s(1-\overline{\zeta}^\rho_s)\,d\xi^{i,n}_s \\ &+\int_{[\rho,T)} g^i_s(1-\xi^{i,n}_s)\,d\overline{\zeta}^\rho_s + \sum_{s\in[\rho,T]} h^i_s\Delta\overline{\zeta}^\rho_s\Delta\xi^{i,n}_s\bigg|\mathscr{F}^2_\rho\bigg] \\ &+\int_{[0,\rho)} f^i_s(1-\zeta^*_s)\,d\xi^i_s + \int_{[0,\rho)} g^i_s(1-\xi^i_s)\,d\zeta^*_s + \sum_{s\in[0,\rho)} h^i_s\Delta\zeta^*_s\Delta\xi^i_s\bigg|\mathscr{F}^2_\nu\Bigg],\end{aligned}$$

the right-hand side of which can easily be seen to be $\mathbb{E}[M^i(\rho;\xi,\zeta^*)|\mathscr{F}^2_\nu]$ from (4.42). ■

Before moving on, we present two further quick results, both of which are related to when the family $\{M^i(\rho;\xi^i,\zeta^*),\rho\in\mathscr{T}(\mathscr{F}^2_t)\}$ acts as a $\mathscr{T}(\mathscr{F}^2_t)$-martingale system. In the same way that we mentioned that $\{M(\rho;\xi^*,\zeta^*),\rho\in\mathscr{T}(\mathscr{F}^2_t)\}$ coincides with $\{M(\rho),\rho\in\mathscr{T}(\mathscr{F}^2_t)\}$, we can relate $\{M^i(\rho;\xi^{i,*},\zeta^*),\rho\in\mathscr{T}(\mathscr{F}^2_t)\}$ to the latter also. Furthermore, we can specifically show that for each choice of i in $[\mathscr{I}]$, the family $\{M^i(\rho;\xi^{i,*},\zeta^*),\rho\in\mathscr{T}(\mathscr{F}^2_t)\}$ is a $\mathscr{T}(\mathscr{F}^2_t)$-martingale system.

Lemma 4.3.10. *For any $(\mathscr{F}^2_t)$-stopping time ρ we have*

$$\mathbb{1}_{\Omega_\rho}V^{\Pi^*}(\rho)\sum_{i=0}^{\mathscr{I}}\pi_i(1-\xi^{i,*}_{\rho-}) = \mathbb{1}_{\Omega_\rho}\sum_{i=0}^{\mathscr{I}}\pi_i\mathcal{V}^i(\rho)(1-\xi^{i,*}_{\rho-}),$$

and therefore $M(\rho)=\sum_{i=0}^{\mathscr{I}}\pi_iM^i(\rho;\xi^{i,*},\zeta^*)$.

Proof. Begin by recalling that $\Omega^i_\rho\subset\Omega_\rho$, and so using part one of Lemma 4.3.7 gives us that

$$\mathbb{1}_{\Omega_\rho}V^{\Pi^*}(\rho) = \mathbb{1}_{\Omega_\rho}\sum_{i=0}^{\mathscr{I}}\Pi^{i,*}_{\rho-}\mathcal{V}^i(\rho), \quad \mathbb{P}\text{-a.s.}$$

Using (4.3) to expand the belief process on the set Ω_ρ yields the first result:

$$\mathbb{1}_{\Omega_\rho}V^{\Pi^*}(\rho)\sum_{i=0}^{\mathscr{I}}\pi_i(1-\xi^{i,*}_{\rho-}) = \mathbb{1}_{\Omega_\rho}\sum_{i=0}^{\mathscr{I}}\pi_i\mathcal{V}^i(\rho)(1-\xi^{i,*}_{\rho-}). \tag{4.44}$$

The second result comes from recalling the definitions of both families $\{M(\rho),\rho\in\mathscr{T}(\mathscr{F}^2_t)\}$ and

$\{M^i(\rho;\xi^{i,*},\zeta^*),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$ (the latter being (4.41) evaluated at $\xi_t^{i,*}$), and that

$$\sum_{i=0}^{\mathscr{I}}\pi_i V^{\Pi^*}(\rho)(1-\xi_{\rho-}^{i,*})(1-\zeta_{\rho-}^*)=\mathbb{1}_{\Omega_\rho}\sum_{i=0}^{\mathscr{I}}\pi_i V^{\Pi^*}(\rho)(1-\xi_{\rho-}^{i,*})(1-\zeta_{\rho-}^*)$$
$$\sum_{i=0}^{\mathscr{I}}\pi_i \mathcal{V}^i(\rho)(1-\xi_{\rho-}^{i})(1-\zeta_{\rho-}^*)=\mathbb{1}_{\Omega_\rho}\sum_{i=0}^{\mathscr{I}}\pi_i \mathcal{V}^i(\rho)(1-\xi_{\rho-}^{i})(1-\zeta_{\rho-}^*),$$

then using (4.44). ■

Corollary 4.3.11. *If the i^{th} minimiser plays optimally, i.e. chooses $(\xi_t^{i,*})_{t\in[0,T]}$, then the family of random variables $\{M^i(\rho;\xi^{i,*},\zeta^*),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-martingale system.*

Proof. Due to Theorem 4.3.9 we only need to prove the martingale property. Let $\nu\le\rho$ be two $(\mathscr{F}_t^2)$-stopping times, and fix $i\in[\mathscr{I}]$. Define the following two sets

$$N:=\big\{\omega\in\Omega:\mathbb{E}\big[M^i(\rho;\xi^{i,*},\zeta^*)\big|\mathscr{F}_\nu^2\big]<M^i(\nu;\xi^{i,*},\zeta^*)\big\},$$
$$A^i:=\big\{\omega\in\Omega:\mathbb{E}\big[M^i(\rho;\xi^{i,*},\zeta^*)\big|\mathscr{F}_\nu^2\big]>M^i(\nu;\xi^{i,*},\zeta^*)\big\}.$$

By the submartingale property (see Theorem 4.3.9), N is a null set. The set A^i is the set on which the 'strict' submartingale property holds; we will show this is also a null set. Assume towards a contradiction that $\mathbb{P}(A^i)>0$, and notice that on $(A^i)^c$ we, of course, still have the usual submartingale property.

Lemma 4.3.10 allows us to write

$$\mathbb{E}\big[M(\rho)\big|\mathscr{F}_\nu^2\big]=\sum_{j\in[\mathscr{I}]\setminus\{i\}}\pi_j\mathbb{E}\big[M^j(\rho;\xi^{j,*},\zeta^*)\big|\mathscr{F}_\nu^2\big]+\pi_i\mathbb{E}\big[M^i(\rho;\xi^{i,*},\zeta^*)\big|\mathscr{F}_\nu^2\big],$$

and we can then split the second term on the right-hand side over A^i and $(A^i)^c$ to see that

$$\begin{aligned}\mathbb{E}[M^i(\rho;\xi^{i,*},\zeta^*)|\mathscr{F}_\nu^2]&=\mathbb{1}_{A^i}\mathbb{E}[M^i(\rho;\xi^{i,*},\zeta^*)|\mathscr{F}_\nu^2]+\mathbb{1}_{(A^i)^c}\mathbb{E}[M^i(\rho;\xi^{i,*},\zeta^*)|\mathscr{F}_\nu^2]\\&>\mathbb{1}_{A^i}M^i(\nu;\xi^{i,*},\zeta^*)+\mathbb{1}_{(A^i)^c}M^i(\nu;\xi^{i,*},\zeta^*)\\&=M^i(\nu;\xi^{i,*},\zeta^*).\end{aligned}$$

It then follows immediately from the submartingale property that $\mathbb{E}[M(\rho)|\mathscr{F}_\nu^2]>M(\nu)$, a contradiction on Theorem 4.1.20. ■

Theorem 4.3.12. *There exists a unique* $(\mathscr{F}_t^2)$*-martingale,* $(M_t^i)_{t\in[0,T]}$*, which is right-continuous and aggregates the family of random variables* $\{M^i(\rho;\xi^{i,*},\zeta^*),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$.

Proof. The proof of this theorem is identical to that of Theorem 4.1.22. ■

Corollary 4.3.13. *There exists a process,* $(Z_t^i)_{t\in[0,T]}$*, which aggregates the family of random variables* $\{\mathcal{V}^i(\rho)(1-\xi_{\rho-}^{i,*})(1-\zeta_{\rho-}^*),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$.

Proof. In much the same was a Corollary 4.3.6, we construct this aggregating process explicitly. Recall the process $(M_t^i)_{t\in[0,T]}$ from Theorem 4.3.12, then define

$$Z_t^i := M_t^i - \int_{[0,t)} f_s^i(1-\zeta_s^*)\,d\xi_s^{i,*} - \int_{[0,t)} g_s^i(1-\xi_s^{i,*})\,d\zeta_s^* - \sum_{s\in[0,t)} h_s^i\Delta\zeta_s^*\Delta\xi_s^{i,*}.$$

Take any $(\mathscr{F}_t^2)$-stopping time ρ, then $\mathbb{P}$-almost surely

$$\begin{aligned} Z_\rho^i &= M^i(\rho;\xi^{i,*},\zeta^*) - \int_{[0,\rho)} f_s^i(1-\zeta_s^*)\,d\xi_s^{i,*} - \int_{[0,\rho)} g_s^i(1-\xi_s^{i,*})\,d\zeta_s^* - \sum_{s\in[0,\rho)} h_s^i\Delta\zeta_s^*\Delta\xi_s^{i,*} \\ &= \mathcal{V}^i(\rho)(1-\xi_{\rho-}^{i,*})(1-\zeta_{\rho-}^*). \end{aligned}$$

■

4.4 PROPERTIES OF THE OPTIMAL STRATEGIES

A final topic to cover for necessary conditions are the properties of the optimal strategies themselves. The work done by De Angelis, Merkulov and Palczewski, [DAMP22], proved the existence of optimal strategies, but did not characterise them. As we discussed in Section 1.4, specifically in Theorem 1.4.5, in the Markovian framework, the optimal controls can be explicitly formulated as first hitting times. Recall that for the optimal stopping game defined in (1.20) with the driver, $(X_t)_{t\geq 0}$, being a strong Markov process which is right-continuous and left-continuous over stopping times, then the optimal pair of stopping times (τ^*,σ^*) take the form

$$\tau^* = \inf\{t\geq 0 : X_t\in\{V=f\}\} \quad \& \quad \sigma^* = \inf\{t\geq 0 : X_t\in\{V=g\}\}.$$

We would like to draw similar conclusions for our framework, however, due to the use of randomised stopping times, one can think of players stopping with a 'given intensity' rather than stopping absolutely, meaning this formulation of hitting times breaks down. We still expect that the times at which both players stop (with some intensity) will be related to times when the value equals one of the payoff processes.

We will begin by focusing on the strategy of the minimiser. We know that, since the minimiser knows the outcome of Θ, their minimisation problem depends on i. From Corollary 4.3.11 we know that $\{M^i(\rho;\xi^*,\zeta^*),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-martingale system, and so we can look to rewrite it in terms of the minimiser's randomised stopping time (see Proposition 4.4.3).

Define the following stopping time (recall Remark 3.1.2)

$$\tau^{i,*}(u) := \inf\{s\geq 0 : \xi_s^{i,*} > u\}, \quad u\in(0,1). \tag{4.45}$$

Definition 4.4.1. Define the family $\{m^i(\rho;u,\zeta^*),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$, for any $i\in[\mathscr{I}]$ and any $u\in(0,1)$,

$$\begin{aligned} m^i(\rho;u,\zeta^*) := {}&\mathbb{1}_{\{\tau^{i,*}(u)\geq\rho\}}(1-\zeta^*_{\rho-})\mathcal{V}^i(\rho) + \mathbb{1}_{\{\tau^{i,*}(u)<\rho\}} f^i_{\tau^{i,*}(u)}(1-\zeta^*_{\tau^{i,*}(u)}) \\ &+ \int_{[0,T)} \mathbb{1}_{\{s<\rho\}} g^i_s \mathbb{1}_{\{\tau^{i,*}(u)>s\}}\, d\zeta^*_s + \mathbb{1}_{\{\tau^{i,*}(u)<\rho\}} h^i_{\tau^{i,*}(u)}\Delta\zeta^*_{\tau^{i,*}(u)}, \end{aligned} \tag{4.46}$$

Proposition 4.4.2. *The mapping $(\omega,u)\mapsto m^i(t;u,\zeta^*)$ is $(\mathscr{F}_t^2\otimes\mathscr{B}(0,1))$-measurable for any $t\in[0,T]$ and $i\in[\mathscr{I}]$.*

Proof. This follows almost immediately from (4.46). Given that $\tau^{i,*}(u)$ is an $(\mathscr{F}_t^2)$-stopping time, and $\mathcal{V}^i(t)$ is $(\mathscr{F}_t^2)$-measurable by its construction (see Lemma 4.3.7), it is clear that $(\omega,u)\mapsto m^i(t;u,\zeta^*)$ is comprised entirely of $(\mathscr{F}_t^2\otimes\mathscr{B}(0,1))$-measurable objects. ■

Proposition 4.4.3. *Take any $u\in(0,1)$ and let $(\hat{\xi}^i_t)_{t\in[0,T]}$ be defined such that $\hat{\xi}^i_t := \mathbb{1}_{\{t\geq\tau^{i,*}(u)\}}(t)$, then $M^i(\rho,\hat{\xi}^i,\zeta^*) = m^i(\rho;u,\zeta^*)$, and*

$$M^i(\rho;\xi^*,\zeta^*) = \int_0^1 m^i(\rho;u,\zeta^*)\,du, \tag{4.47}$$

for any $i\in[\mathscr{I}]$ and any $(\mathscr{F}_t^2)$-stopping time ρ.

Proof. To prove $M^i(\rho,\hat{\xi}^i,\zeta^*) = m^i(\rho;u,\zeta^*)$ we first consider the value of $\hat{\xi}^i_{t-}$ as follows

$$\hat{\xi}^i_{t-} = \lim_{s\uparrow\uparrow t}\hat{\xi}^i_s = \lim_{s\uparrow\uparrow t}\mathbb{1}_{\{s\geq\tau^{i,*}(u)\}}(s) = \mathbb{1}_{\{t>\tau^{i,*}(u)\}}(t).$$

As a result it is true that $1-\hat{\xi}^i_{t-} = \mathbb{1}_{\{t\leq\tau^{i,*}(u)\}}(t)$, and therefore also that $1-\hat{\xi}^i_{\rho-} = \mathbb{1}_{\{\rho\leq\tau^{i,*}(u)\}}(t)$. In the same manner—though trivial—we see that $1-\hat{\xi}^i_t = \mathbb{1}_{\{t<\tau^{i,*}(u)\}}(t)$. Moreover, it is clear that $(\hat{\xi}^i_t)_{t\in[0,T]}$ is the generating process of the stopping time $\tau^{i,*}(u)$, and therefore we have

$$\int_{[0,\rho)} f^i_s(1-\zeta^*_s)\,d\hat{\xi}^i_s = \mathbb{1}_{\{\tau^{i,*}(u)<\rho\}} f^i_{\tau^{i,*}(u)}(1-\zeta^*_{\tau^{i,*}(u)}),$$

$$\sum_{s\in[0,\rho)} h^i_s\Delta\zeta^*_s\Delta\hat{\xi}^i_s = \int_{[0,\rho)} h^i_s\Delta\zeta^*_s\,d\hat{\xi}^i_s = \mathbb{1}_{\{\tau^{i,*}(u)<\rho\}} h^i_{\tau^{i,*}(u)}\Delta\zeta^*_{\tau^{i,*}(u)}.$$

The fact that $M^i(\rho,\hat{\xi}^i,\zeta^*) = m^i(\rho;u,\zeta^*)$ then follows immediately from (4.41).

To prove (4.47) we make repeated use of Proposition A.3.3. First, we recall the definition of $\{M^i(\rho;\xi^{i,*},\zeta^*),\rho\in\mathscr{T}(\mathscr{F}^2_t)\}$ as (4.41) evaluated at $\xi^i=\xi^{i,*}$,

$$M^i(\rho;\xi^{i,*},\zeta^*) := \mathcal{V}^i(\rho)(1-\xi^{i,*}_{\rho-})(1-\zeta^*_{\rho-}) + \int_{[0,\rho)} f^i_s(1-\zeta^*_s)\,d\xi^{i,*}_s + \int_{[0,\rho)} g^i_s(1-\xi^{i,*}_s)\,d\zeta^*_s + \sum_{s\in[0,\rho)} h^i_s\Delta\zeta^*_s\Delta\xi^{i,*}_s. \quad (4.48)$$

We can use (4.46) to obtain an expression for the integral in (4.47) as follows

$$\int_0^1 m^i(\rho;u,\zeta^*)\,du = (1-\zeta^*_{\rho-})\mathcal{V}^i(\rho)\int_0^1 \mathbb{1}_{\{\tau^{i,*}(u)\geq\rho\}}\,du + \int_0^1 \mathbb{1}_{\{\tau^{i,*}(u)<\rho\}} f^i_{\tau^{i,*}(u)}(1-\zeta^*_{\tau^{i,*}(u)})\,du + \int_0^1\int_{[0,T)} \mathbb{1}_{\{s<\rho\}} g^i_s \mathbb{1}_{\{\tau^{i,*}(u)>s\}}\,d\zeta^*_s\,du + \int_0^1 \mathbb{1}_{\{\tau^{i,*}(u)<\rho\}} h^i_{\tau^{i,*}(u)}\Delta\zeta^*_{\tau^{i,*}(u)}\,du.$$

The second and last terms on the right-hand side can be dealt with via Proposition A.3.3 (and one can note that this is the same method used for these terms during the proof of Lemma 4.2 in [DAMP22]), yielding

$$\int_0^1 m^i(\rho;u,\zeta^*)\,du = (1-\zeta^*_{\rho-})\mathcal{V}^i(\rho)\int_0^1 \mathbb{1}_{\{\tau^{i,*}(u)\geq\rho\}}\,du + \int_{[0,\rho)} f^i_s(1-\zeta^*_s)\,d\xi^{i,*}_s + \int_0^1\int_{[0,T)} \mathbb{1}_{\{s<\rho\}} g^i_s \mathbb{1}_{\{\tau^{i,*}(u)>s\}}\,d\zeta^*_s\,du + \sum_{s\in[0,\rho)} h^i_s\Delta\zeta^*_s\Delta\xi^{i,*}_s. \quad (4.49)$$

We take the remaining two terms one at a time. For the first, recall (4.45) and take $f(t)=\xi^{i,*}_t$,

$f^*(u) = \tau^{i,*}(u)$ and $g(t) = \mathbb{1}_{\{t<\rho\}}$ in Proposition A.3.3—applying it ω-wise—to get

$$\int_0^1 \mathbb{1}_{\{\tau^{i,*}(u)\geq\rho\}}\,du = 1 - \int_0^1 \mathbb{1}_{\{\tau^{i,*}(u)<\rho\}}\,du = 1 - \int_{[0,T]} \mathbb{1}_{\{t<\rho\}}\,d\xi_t^{i,*} = \int_{[0,\rho)} d\xi_t^{i,*} = 1 - \xi_{\rho-}^{i,*}. \tag{4.50}$$

For the remaining term, we can use Fubini's theorem to switch the order of the integrals, and we show (in the same manner as above) that

$$\int_0^1 \mathbb{1}_{\{\tau^{i,*}(u)>s\}}\,du = 1 - \xi_s^{i,*}. \tag{4.51}$$

Combining (4.48) and (4.49)–(4.51) gives the final result. ■

Corollary 4.4.4. *For any $u \in (0,1)$ and any $i \in [\mathscr{I}]$, the family $\{m^i(\rho;u,\zeta^*), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-submartingale system.*

Proof. This follows immediately from Theorem 4.3.9 and the first point of Proposition 4.4.3. ■

Lemma 4.4.5. *For λ-almost every u in $(0,1)$, the family $\{m^i(\rho;u,\zeta^*), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-martingale system.*

Proof. Corollary 4.4.4 and tells us that $\mathbb{E}[m^i(\rho;u,\zeta^*)|\mathscr{F}_\nu^2] \geq m^i(\nu;u,\zeta^*)$ for any two $(\mathscr{F}_t^2)$-stopping times ρ and ν. Moreover, Corollary 4.3.11 and the first point of Proposition 4.4.3 give

$$\mathbb{E}\left[\int_0^1 m^i(\rho;u,\zeta^*)\,du \middle| \mathscr{F}_\nu^2\right] = \int_0^1 m^i(\nu;u,\zeta^*)\,du. \tag{4.52}$$

Rearranging (4.52) and taking expectation yields

$$\mathbb{E}\left[\int_0^1 m^i(\rho;u,\zeta^*) - m^i(\nu;u,\zeta^*)\,du\right] = 0.$$

The measurability of $(\omega,u) \mapsto m^i(t;u,\zeta^*)$ is guaranteed by Proposition 4.4.2, and we also have integrability, hence we are free to use Fubini's theorem to switch the integral and the expectation:

$$\int_0^1 \mathbb{E}\big[m^i(\rho;u,\zeta^*) - m^i(\nu;u,\zeta^*)\big]\,du = 0. \tag{4.53}$$

The fact that $\mathbb{E}[m^i(\rho;u,\zeta^*)|\mathscr{F}_\nu^2] \geq m^i(\nu;u,\zeta^*)$ implies that the expectation in (4.53) must be non-negative, and so we conclude that $\mathbb{E}\big[m^i(\rho;u,\zeta^*) - m^i(\nu;u,\zeta^*)\big] = 0$ for λ-almost all $u \in (0,1)$.

Moreover, for those values of u outside of the null set, since $\mathbb{E}[m^i(\rho;u,\zeta^*)-m^i(\nu;u,\zeta^*)|\mathscr{F}^2_\nu]\geq 0$ and yet its expectation is zero, we must have that

$$\mathbb{E}[m^i(\rho;u,\zeta^*)-m^i(\nu;u,\zeta^*)|\mathscr{F}^2_\nu]=0 \quad \mathbb{P}\text{-a.s.}$$

■

We would like to have that this martingale property holds for the specific choice of $u=1$, since that corresponds to the last time that the minimiser can stop, i.e. $\tau^{i,*}(1)=\inf\{s\geq 0:\xi^{i,*}_s=1\}$ (see Lemma 4.4.6 below). We therefore need to approximate it using values of u belonging to the set of λ-full measure. For future reference define this set to be

$$\mathfrak{U}^i := \big\{u\in(0,1):\mathbb{E}\big[m^i(T;u,\zeta^*)\big]=\mathbb{E}[m^i(0;u,\zeta^*)]\big\}. \tag{4.54}$$

Notice that we choose specifically $\rho=T$ and $\nu=0$ so that $\mathfrak{U}^i$ considers those u such that we have martingality over the whole trajectory. As we made clear in the proof of Lemma 4.4.5, the set of u such that $\mathbb{E}\big[m^i(T;u,\zeta^*)\big]=\mathbb{E}[m^i(0;u,\zeta^*)]$, is not only of full measure, but also corresponds directly to those u such that the martingale property holds. Moreover, due to Proposition 4.4.2, we can see that $\mathfrak{U}^i$ is Borel. Since we are dealing with the Lebesgue measure on $[0,1]$, Borel sets of full measure are dense, therefore $\mathfrak{U}^i$ is dense in $[0,1]$. Hence we can always take a sequence of $(u_n)_{n\in\mathbb{N}}\subset\mathfrak{U}^i$ which increases to one.

Lemma 4.4.6. *Let $\chi\in\mathscr{A}(\mathscr{G}_t)$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $[0,1]$ converging to one, then the sequence of stopping times $(\eta(u_n))_{n\in\mathbb{N}}$, defined by $\eta(u):=\inf\{t\geq 0:\chi_t>u\}$, is non-decreasing, i.e. for any $u_1\leq u_2$, we have $\eta(u_1)\leq\eta(u_2)$, and $(\eta(u_n))_{n\in\mathbb{N}}$ converges to $\eta(1):=\inf\{t\geq 0:\chi_t=1\}$.*

Proof. Take $u_1<u_2$ in $[0,1]$ (since equality is trivial), then, since $(\chi_t)_{t\in[0,T]}$ is non-decreasing, we know that $\{t\geq 0:\chi_t>u_1\}\supset\{t\geq 0:\chi_t>u_2\}$, therefore $\eta(u_1)\leq\eta(u_2)$. The convergence to $\eta(1)$ holds by Proposition A.3.2 (see Appendix A) for each $\omega\in\Omega$. ■

The power of this reformulation in terms of $m^i(\rho;u,\zeta^*)$ comes from the fact that we are able to reduce the impact of the minimiser down to stopping times rather than generating processes. We now construct a process which, at any time, compares the value to either of the payoff processes $(f_t)_{t\in[0,T]}$ and $(h_t)_{t\in[0,T]}$ depending on the manner of which the maximiser stops.

Theorem 4.4.7. *Define* $(Y_t^i)_{t\in[0,T]}$ *as* $Y_t^i := (\mathcal{V}^i(t) - f_t^i)(1-\zeta_t^*) + (\mathcal{V}^i(t) - h_t^i)\Delta\zeta_t^*$*, then,*

1. $Y_t^i \le 0\ \forall t \in [0,T]$ $\mathbb{P}$*-a.s.,*

2. $\int_{[0,T]} Y_t^i \, d\xi_t^{i,*} = 0$ $\mathbb{P}$*-a.s.*

Proof. To begin, recall that

$$\mathcal{V}^i(0) \;=\; \mathbb{E}\left[\int_{[0,T)} f_s^i(1\;-\;\zeta_s^*)\; d\xi_s^{i,*} \;+\; \int_{[0,T)} g_s^i(1\;-\;\xi_s^{i,*})\; d\zeta_s^* \;+\; \sum_{s\in[0,T]} h_s^i \Delta\zeta_s^* \Delta\xi_s^{i,*}\right]. \quad (4.55)$$

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{U}^i$, converging to one. Pick any $v\in[0,u_n]$, and in $m^i(\rho;u_n,\zeta^*)$ choose the stopping time $\rho = \tau^{i,*}(v)$. Lemma 4.4.6 implies that $\tau^{i,*}(v) \le \tau^{i,*}(u_n)$. Taking expectation in equation (4.46), and noting that $\{\tau^{i,*}(v) > s\} \subset \{\tau^{i,*}(u_n) > s\}$, we can arrive at

$$\mathbb{E}\big[m^i(\tau^{i,*}(v);u_n,\zeta^*)\big] = \mathbb{E}\left[(1-\zeta^*_{\tau^{i,*}(v)-})\mathcal{V}^i(\tau^{i,*}(v)) + \int_{[0,T)} g_s^i \mathbb{1}_{\{s<\tau^{i,*}(v)\}}\, d\zeta_s^*\right]. \quad (4.56)$$

Now we can integrate over v, but we must be careful that what we integrate is correctly measurable. Specifically, we need to consider $(1-\zeta^*_{\tau^{i,*}(v)-})\mathcal{V}^i(\tau^{i,*}(v))$. Recall from Corollary 4.3.13 that there exists a process $(Z_t^i)_{t\in[0,T]}$ which aggregates $\{\mathcal{V}^i(\rho)(1-\xi^{i,*}_{\rho-})(1-\zeta^*_{\rho-}), \rho\in\mathscr{T}(\mathscr{F}_t^2)\}$. We argue that $1-\xi^{i,*}_{\tau^{i,*}(v)-}$ is strictly positive: first, note that if $\tau^{i,*}(v) < \tau^{i,*}(1)$ then $\xi^{i,*}_{\tau^{i,*}(v)} < 1$. Second, notice that $v<1$ (since $v\le u_n$ and $u_n\in\mathfrak{U}^i$); therefore, if $\tau^{i,*}(v) = \tau^{i,*}(1)$ then there must be a jump to one, i.e. $\Delta\xi^{i,*}_{\tau^{i,*}(v)} > 0$. Both conditions imply that $1-\xi^{i,*}_{\tau^{i,*}(v)-} > 0$. Because of this, we can define the object

$$(1-\zeta^*_{\tau^{i,*}(v)-})\mathcal{V}^i(\tau^{i,*}(v)) := \frac{Z^i_{\tau^{i,*}(v)}}{1-\xi^{i,*}_{\tau^{i,*}(v)-}},$$

which is jointly measurable in (ω,v), well-defined, and integrable by construction. Therefore, we can freely integrate (4.56) over v, and moreover, we can use Fubini's theorem (twice for the second term) to get

$$\int_0^{u_n} \mathbb{E}\big[m^i(\tau^{i,*}(v);u_n,\zeta^*)\big]\, dv = \mathbb{E}\Bigg[\int_0^{u_n}(1-\zeta^*_{\tau^{i,*}(v)-})\mathcal{V}^i(\tau^{i,*}(v))\, dv + \int_{[0,T)} g_s^i \int_0^{u_n} \mathbb{1}_{\{s<\tau^{i,*}(v)\}}\, dv\, d\zeta_s^*\Bigg]. \quad (4.57)$$

Recall that u_n belongs to $\mathfrak{U}$, and so, for each $v \in [0, u_n]$, we have

$$\mathbb{E}\big[m^i(\tau^{i,*}(v); u_n, \zeta^*)\big] = \mathbb{E}\big[m^i(0; u_n, \zeta^*)\big],$$

and the right-hand side can be easily shown to be equal to $\mathbb{E}\mathcal{V}^i(0)$, which is trivially $\mathcal{V}^i(0)$ (see equation (4.55)). Combining this with equation (4.57) yields

$$\mathcal{V}^i(0)u_n = \mathbb{E}\left[\int_0^{u_n}(1-\zeta^*_{\tau^{i,*}(v)-})\mathcal{V}^i(\tau^{i,*}(v))\,dv + \int_{[0,T)} g^i_s \int_0^{u_n} \mathbb{1}_{\{s<\tau^{i,*}(v)\}}\,dv\,d\zeta^*_s\right].$$

Since all terms contained inside the integrals with respect to v contain no dependence on n (of course the bounds of the integrals do), we can repeatedly use the dominated convergence theorem:

$$\mathcal{V}^i(0) = \mathbb{E}\left[\int_0^{1}(1-\zeta^*_{\tau^{i,*}(v)-})\mathcal{V}^i(\tau^{i,*}(v))\,dv + \int_{[0,T)} g^i_s \int_0^{1} \mathbb{1}_{\{s<\tau^{i,*}(v)\}}\,dv\,d\zeta^*_s\right].$$

Notice than when we take limits on the right-hand side, we use that $\mathbb{1}_{\{v\le u_n\}} \to \mathbb{1}_{\{v<1\}}$. Of course, the strict inequality makes no difference for us as the integral is only different on a set of measure zero to that over all v in $[0,1]$.

Using (4.51) and the typical change of variables (see Proposition A.3.3), we arrive at

$$\mathcal{V}^i(0) = \mathbb{E}\left[\int_{[0,T]}(1-\zeta^*_{s-})\mathcal{V}^i(s)\,d\xi^{i,*}_s + \int_{[0,T)} g^i_s(1-\xi^{i,*}_s)\,d\zeta^*_s\right]. \tag{4.58}$$

Combining (4.58) with (4.55) yields

$$\mathbb{E}\left[\int_{[0,T]}(1-\zeta^*_{s-})\mathcal{V}^i(s)\,d\xi^{i,*}_s - \int_{[0,T)} f^i_s(1-\zeta^*_s)\,d\xi^{i,*}_s - \sum_{s\in[0,T]} h^i_s \Delta\zeta^*_s \Delta\xi^{i,*}_s\right] = 0. \tag{4.59}$$

Notice that using the suboptimal strategy, $\hat{\xi}^i_t = \mathbb{1}_{\{t\ge\rho\}}(t)$, in the definition of $\mathcal{V}^i(\rho)$ causes most terms to vanish except for the possible jumps at ρ, giving us

$$\begin{aligned}(1-\zeta^*_{\rho-})\mathcal{V}^i(\rho) &\le (1-\zeta^*_{\rho-})\mathbb{1}_{\Gamma_\rho}\mathbb{E}\left[f^i_\rho(1-\overline{\zeta}^{\rho-}_\rho)\Delta\hat{\xi}^i_\rho + h^i_\rho\Delta\overline{\zeta}^{\rho-}_\rho\Delta\hat{\xi}^i_\rho \,\middle|\, \mathscr{F}^2_\rho\right] \quad \mathbb{P}\text{-a.s.},\\ &= f^i_\rho(1-\zeta^*_\rho) + h^i_\rho\Delta\zeta^*_\rho, \end{aligned} \tag{4.60}$$

where we can remove the indicator function in the last line since all terms are zero on $(\Gamma_\rho)^c$.

Notice, from (4.60), we know that

$$f_s^i(1-\zeta_s^*)+h_s^i\Delta\zeta_s^* \geq (1-\zeta_{s-}^*)\mathcal{V}^i(s) = (1-\zeta_s^*)\mathcal{V}^i(s)+\Delta\zeta_s^*\mathcal{V}^i(s) \quad \mathbb{P}\text{-a.s.}$$

In other words,

$$Y_s^i = \big(\mathcal{V}^i(s)-f_s^i\big)(1-\zeta_s^*)+\big(\mathcal{V}^i(s)-h_s^i\big)\Delta\zeta_s^* \leq 0,$$

proving point 1.

For the second point, we notice that (4.60) of course holds for deterministic times, and so

$$\int_{[0,T]}(1-\zeta_{s-}^*)\mathcal{V}^i(s)\,d\xi_s^{i,*}-\int_{[0,T)}f_s^i(1-\zeta_s^*)\,d\xi_s^{i,*}-\sum_{s\in[0,T]}h_s^i\Delta\zeta_s^*\Delta\xi_s^{i,*}\leq 0. \tag{4.61}$$

Here, from (4.59) and (4.61), we see that we have a non-positive random variable with zero expectation, and so we must have

$$\int_{[0,T]}(1-\zeta_{s-}^*)\mathcal{V}^i(s)\,d\xi_s^{i,*}=\int_{[0,T)}f_s^i(1-\zeta_s^*)\,d\xi_s^{i,*}+\sum_{s\in[0,T]}h_s^i\Delta\zeta_s^*\Delta\xi_s^{i,*} \quad \mathbb{P}\text{-a.s.}$$

Expanding the prefactor $1-\zeta_{s-}^*$ of $\mathcal{V}^i(s)$ as $1-\zeta_{s-}^* = 1-\zeta_s^*+\Delta\zeta_s^*$, and rearranging the terms onto one side gives

$$0=\int_{[0,T)}\big(\mathcal{V}^i(s)-f_s^i\big)(1-\zeta_s^*)\,d\xi_s^{i,*}+\sum_{s\in[0,T]}\big(\mathcal{V}^i(s)-h_s^i\big)\Delta\zeta_s^*\Delta\xi_s^{i,*}=\int_{[0,T]}Y_s^i\,d\xi_s^{i,*} \quad \mathbb{P}\text{-a.s.}$$

The last equality follows because we may write the sum as an integral with respect to the measure induced by $(\xi_t^{i,*})_{t\in[0,T]}$. ■

* * *

The work up until now is only half of the problem; we still need to do the same for the maximiser, i.e. using our supermartingale family $\{M(\rho;\xi^*,\zeta),\rho\in\mathcal{T}(\mathcal{F}_t^2)\}$. Trivially, we know from Remark 4.3.5 that the family $\{M(\rho;\xi^*,\zeta^*),\rho\in\mathcal{T}(\mathcal{F}_t^2)\}$ is a $\mathcal{T}(\mathcal{F}_t^2)$-martingale system since it coincides with $\{M(\rho),\rho\in\mathcal{T}(\mathcal{F}_t^2)\}$. We will look to write this system in terms of the maximiser's randomised stopping time; define

$$\sigma^*(u):=\inf\{s\geq 0:\zeta_s^*>u\}.$$

Definition 4.4.8. Define the family $\{m(\rho;u,\xi^*),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$ for any $u\in(0,1)$ as

$$m(\rho;u,\xi^*):=\sum_{i-0}^{\mathscr{I}}\pi_i\bigg(\mathbb{1}_{\{\sigma^*(u)\geq\rho\}}(1-\xi^{i,*}_{\rho-})\mathcal{V}(\rho)+\int_{[0,T)}\mathbb{1}_{\{s<\rho\wedge\sigma^*(u)\}}f^i_s\,d\xi^{i,*}_s$$
$$+\mathbb{1}_{\{\sigma^*(u)<\rho\}}g^i_{\sigma^*(u)}(1-\xi^{i,*}_{\sigma^*(u)})+\mathbb{1}_{\{\sigma^*(u)<\rho\}}h^i_{\sigma^*(u)}\Delta\xi^{i,*}_{\sigma^*(u)}\bigg). \quad (4.62)$$

Proposition 4.4.9. *The mapping $(\omega,u)\mapsto m(t;u,\xi^*)$ is $(\mathscr{F}_t^2\otimes\mathscr{B}(0,1))$-measurable for any $t\in[0,T]$.*

Proof. The proof is nearly identical to that of Proposition 4.4.2, specifically, (4.62) is comprised entirely of $(\mathscr{F}_t^2\otimes\mathscr{B}(0,1))$-measurable objects. ■

Proposition 4.4.10. *Take any $u\in(0,1)$ and let $(\hat{\zeta}_t)_{t\in[0,T]}$ be defined such that $\hat{\zeta}_t=\mathbb{1}_{\{\sigma^*(u)\leq t\}}$, then $M(\rho;\xi^*,\hat{\zeta})=m(\rho;u,\xi^*)$, and*

$$M(\rho;\xi^*,\zeta^*)=\int_0^1 m(\rho;u,\xi^*)\,du.$$

Proof. The proof follows the proof of Proposition 4.4.3 *mutatis mutandis*. The inclusion of a sum in (4.62) makes no material difference to the steps involved. ■

Corollary 4.4.11. *For any $u\in(0,1)$, the family of random variables $\{m(\rho;u,\xi^*),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-supermartingale system.*

Proof. This follows immediately from Theorem 4.3.4 and the first point of Proposition 4.4.10. ■

Lemma 4.4.12. *For λ-almost every $u\in(0,1)$, the family $\{m(\rho;u,\xi^*),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-martingale system.*

Proof. This proof follows in the same manner as Lemma 4.4.5 *mutatis mutandis*. Of course, in this case we utilise the $\mathscr{T}(\mathscr{F}_t^2)$-supermartingale system property from Theorem 4.3.4. ■

Again, we need to approximate $u=1$ with values for which the martingale property holds as per Lemma 4.4.12. This time we define the following set

$$\mathfrak{U}:=\{u\in(0,1):\mathbb{E}[m(T;u,\xi^*)]=\mathbb{E}[m(0;u,\xi^*)]\}. \quad (4.63)$$

Again, we can see that this set is Borel, and since it is also of full measure, it is dense in $[0,1]$.

Theorem 4.4.13. *Define the process* $(Y_t)_{t\in[0,T]}$ *as*

$$Y_t := \sum_{i=0}^{\mathscr{I}} \pi_i \Big(\big(\mathcal{V}(t) - g_t^i\big)(1-\xi_t^{i,*}) + \big(\mathcal{V}(t) - h_t^i\big)\Delta\xi_t^{i,*} \Big),$$

then

1. $Y_t \geq 0 \; \forall t \in [0,T]$ $\mathbb{P}$*-a.s.,*

2. $\int_{[0,T]} Y_t \, d\zeta_t^* = 0$ $\mathbb{P}$*-a.s.*

Proof. The proof follows the proof of Theorem 4.4.7 *mutatis mutandis*. The differences include beginning with $\mathcal{V}(0)$ instead of $\mathcal{V}^i(0)$, and to obtain the equivalent version of (4.60) for this proof we take the strategy $\hat{\zeta}_t = \mathbb{1}_{\{t\geq\rho\}}(t)$. Finally, in the proof of Theorem 4.4.7 we had to take careful consideration of the integrability of terms with respect to u in (4.56), and we also need to do the same here. The arguments, however, are identical. We need to guarantee that $\sum_{i=0}^{\mathscr{I}} \pi_i(1-\xi^{i,*}_{\sigma^*(u)-})\mathcal{V}(\sigma^*(u))$ is a jointly measurable map in (ω,u). The exact same logic holds to show that $1-\zeta^*_{\sigma^*(u)-} > 0$ as to show $1-\xi^{i,*}_{\tau^{i,*}(u)-} > 0$; therefore, we define

$$\sum_{i=0}^{\mathscr{I}} \pi_i(1-\xi^{i,*}_{\sigma^*(u)-})\mathcal{V}(\sigma^*(u)) := \frac{Z_{\sigma^*(u)}}{1-\zeta^*_{\sigma^*(u)-} > 0},$$

which is a jointly measurable map in (ω,u). ■

We mentioned at the outset of this section that we wanted a result characterising the optimal strategies—akin to those found in the Markovian framework, i.e. Theorem 1.4.5—and Theorems 4.4.7 and 4.4.13 are such results. It may not look like it at first glance, since we do not have a nice closed form description of the optimal strategies; however, our results do tell us about the behaviour of the optimal strategies in relation to each other, the values, and the payoff processes.

For the sake of illustration, consider Theorem 4.4.7 and imagine that the maximiser plays a strategy $(\zeta_t^*)_{t\in[0,T]}$ which is continuous and is such that $\zeta_t^* < 1$ for all $t \in [0,T)$; therefore, our process $(Y_t^i)_{t\in[0,T]}$ takes the form $Y_t^i = (\mathcal{V}^i(t) - f_t^i)(1-\zeta_t^*)$, and $1-\zeta_t^* > 0$ for all $t \in [0,T)$. Now, imagine the minimiser plays a strategy $(\xi_t^{i,*})_{t\in[0,T]}$ which is piecewise constant, i.e. an increasing step function. This means that the measure induced by $(\xi_t^{i,*})_{t\in[0,T]}$ can only put a strictly positive mass at a countable number of points in $[0,T]$ (see Theorem A.2.1). From Theorem 4.4.7 we can then conclude that, because $Y_t^i \leq 0$ for all $t \in [0,T]$ $\mathbb{P}$-almost surely, and its integral with respect to the measure induced by $(\xi_t^{i,*})_{t\in[0,T]}$ is zero, $Y_t^i = 0$ for all jump times of $(\xi_t^{i,*})_{t\in[0,T]}$; in other words, for all $t \in [0,T]$ such that $\Delta\xi_t^{i,*} > 0$

$$(\mathcal{V}^i(t) - f_t^i)(1-\zeta_t^*) = 0.$$

Since $1-\zeta_t^* > 0$ for all $t \in [0,T)$ we can conclude that $\mathcal{V}^i(t) = f_t^i$ for all $t \in [0,T)$ such that $\Delta\xi_t^{i,*} > 0$. In plain English: in this example, the minimiser only acts when $\mathcal{V}^i(t) = f_t^i$.

This example is admittedly contrived and illustrated in a hand-wavy manner, however, it demonstrates the core of the argument; our results in Theorems 4.4.7 and 4.4.13 are the logical equivalents of those characterisations found in Theorem 1.4.5. Players' decisions are fundamentally tied, not just to each other, but to the relationship between the respective values ($\mathcal{V}(\rho)$ and $\mathcal{V}^i(\rho)$) and the payoff processes. A practical and rigorous example of this can be found in Chapter 6 (see Theorems 6.1.7 and 6.1.8) which is concerned with a simple example of a game.

* * *

CONCLUDING REMARKS

This chapter is one of the largest in this work, and we would like to conclude it by drawing some comparisons to the work done by Grün [Grü13], which we detailed in Section 3.5. Of course, we must point out immediately that the two are not directly comparable since Grün chooses a Markovian framework—being driven by an underlying diffusion process $(X_s^{t,x})$—whereas ours is non-Markovian. The two approaches lead to different sets of results, such as Grün able to give a PDE which can be explicitly solved to give the value function (see [BFR23] for a numerical approximation), whereas we cannot due to the non-Markovianity of our problem. That being said, we are able to characterise the strategies of both players, whereas she can only characterise the informed player's strategy. Theorems 4.3.4 and 4.3.9 characterise the optimal strategies of both players; the former telling us that $(\xi_t^*)_{t\in[0,T]}$ is such that for any action of the uninformed player, the family $\{M(\rho;\xi^*,\zeta),\rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-supermartingale system, and the latter saying that $(\zeta_t^*)_{t\in[0,T]}$ is such that $\{M^i(\rho;\xi^i,\zeta^*),\rho \in \mathscr{T}(\mathscr{F}_t^2)\}$

is a $\mathscr{T}(\mathscr{F}_t^2)$-submartingale system for any choice of strategy from the informed player. It is in this second result that we see why it might not have been possible for Grün to formulate the optimal strategy of the uninformed player; she was missing the equivalent submartingale. Moreover, the object $\mathcal{V}^i$ also allows us to obtain a second characterisation of $(\xi_t^*)_{t\in[0,T]}$ (see Theorem 4.4.7).

It should also be noted how important the sets defined in Definition 4.1.11 are. As we have already said, randomisation leads to an idea of gradual stopping, or stopping with some intensity, which makes the end of the game less immediately graspable. In a traditional game of stopping, the game ends immediately upon the action of a player, meaning, at any given time, the game has either stopped, or it has not. This is not fully true under randomisation, the game may still be being played at any given instant, but it may be in the process of stopping, and that can come from either player. On a purely technical level, the presence of these sets allows us to truncate strategies, which allows us to proceed with the proofs, but on a broader sense they incorporate statements about the current state of the game—to what degree has it stopped—into our work. The key examples of this being both a blessing and a curse are found in Definition 4.2.4 and Theorem 4.2.6. In the former we cannot obtain typical 'perfection' wherein, at any instant, it does not matter how either player had previously played, the structure of the optimal solution is the same from that point onwards, whereas we have the restriction that the optimally played game must still be going on, hence why we need to introduce the concept of 'partial subgame perfection'. In the latter, the theorem makes for a wonderfully simple intuition; if one is to adopt the family of strategies obtained by truncating the initial optimal strategy, of course these only remain optimal so long as the game is still being played.

CHAPTER 5

SUFFICIENT CONDITIONS

5.1 INTRODUCTION

The previous chapter focused on the results obtainable when we know there exists a value and a Nash equilibrium; however, we would like a set of sufficient conditions which guarantee the existence of a value and a Nash equilibrium—the resulting theorem is known as a verification theorem. More precisely, we look to show that the necessary conditions found in Chapter 4 are those sufficient conditions, and in this way we can characterise the game.

We often find that verification theorems are prevalent when explicit solutions are found via the guess and verify approach. Such explicit solutions are rare, but whenever they are found they are almost always followed by a verification theorem. In optimal stopping games, explicit solutions and verification theorems are typically more complex and slightly rarer than in single-agent optimal stopping problems, since establishing a verification theorem may be omitted when a primary result showing the existence of a value and a Nash equilibria has already been given. That being said, we still find these types of results in the literature. Three instances are in 2017 by Ekström, Glover and Leniec [EGL17], 2018 by Gensbittel and Grün [GG18], and in 2020 by De Angelis, Ekström and Glover [DAEG21], however, the first of the three is a nonzero-sum game, the second is Markovian, and the third—while also being Markovian—concerns itself with 'quasi-variational inequalities', a topic that is far out of the scope of this work.

Our approach to verification theorems will be as follows. In Chapter 4 we took our time zero optimal strategies, (ξ^*, ζ^*), constructed the families of random variables $\{M(\rho;\xi^*,\zeta), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ and $\{M^i(\rho;\xi^i,\zeta^*), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$, and then proved that they are sub- and super- (and, in certain situations, martingale-) $\mathscr{T}(\mathscr{F}_t^2)$-systems. However, in this chapter we do not assume the existence of (ξ^*, ζ^*), so instead we will construct processes $(M_t^i)_{t\in[0,T]}$ and $(M_t)_{t\in[0,T]}$ and show that if they obey the martingale properties then we obtain a value and a Nash equilibrium.

Definition 5.1.1. Let $(\xi_t^0)_{t\in[0,T]}$, $(\xi_t^1)_{t\in[0,T]}$, $(\zeta_t)_{t\in[0,T]}$ belong to $\mathscr{A}(\mathscr{F}_t^2)$. Let $(\mathscr{U}_t^i)_{t\in[0,T]}$, for $i \in [\mathscr{I}]$, and $(\mathscr{U}_t)_{t\in[0,T]}$ be $(\mathscr{F}_t^2)$-progressively measurable stochastic processes. Define

$$M_t^i(\xi^i,\zeta) := \mathscr{U}_t^i(1-\xi_{t-}^i)(1-\zeta_{t-}) + \int_{[0,t)} f_s^i(1-\zeta_s)\,d\xi_s^i + \int_{[0,t)} g_s^i(1-\xi_s^i)\,d\zeta_s + \sum_{s\in[0,t)} h_s^i \Delta\zeta_s \Delta\xi_s^i, \tag{5.1}$$

$$M_t(\xi,\zeta) := \sum_{i=0}^{\mathscr{I}} \pi_i \Big(\mathscr{U}_t(1-\xi_{t-}^i)(1-\zeta_{t-}) + \int_{[0,t)} f_s^i(1-\zeta_s)\,d\xi_s^i + \int_{[0,t)} g_s^i(1-\xi_s^i)\,d\zeta_s + \sum_{s\in[0,t)} h_s^i \Delta\zeta_s \Delta\xi_s^i \Big). \tag{5.2}$$

There are some subtleties surrounding the conditional payoff as well. Since we no longer have the existence of an optimal pair (ξ_t^*, ζ_t^*), there is no belief process $(\Pi_t^*)_{t\in[0,T]}$. Recall Definition 3.4.11, where the conditional payoff was defined for any $(\mathscr{F}_\rho^2)$-measurable random variable, p, with values on the simplex $\Delta(\mathscr{I})$. We now rely on the same concept, however, now we allow for a 'generic belief process', i.e. an $(\mathscr{F}_t^2)$-progressively measurable process, $(\Pi_t)_{t\in[0,T]}$, with values in $\Delta(\mathscr{I})$. So, our conditional payoff now takes the following form

$$J^{\Pi}_{\rho}(\xi,\zeta)=\sum_{i=0}^{\mathscr{I}}\Pi^{i}_{\rho-}\mathbb{E}\left[\int_{[\rho,T)}f^{i}_{t}(1-\zeta_{t})\,d\xi^{i}_{t}+\int_{[\rho,T)}g^{i}_{t}(1-\xi^{i}_{t})\,d\zeta_{t}+\sum_{t\in[\rho,T]}h^{i}_{t}\Delta\zeta_{t}\Delta\xi^{i}_{t}\middle|\mathscr{F}^{2}_{\rho}\right],$$

for any $(\mathscr{F}^{2}_{t})$-stopping time ρ.

Moreover, we introduce new notation for a concept that we have been using throughout the previous chapter. Namely, for any $(\xi^{i}_{t})_{t\in[0,T]}$ and $(\zeta_{t})_{t\in[0,T]}$ belonging to $\mathscr{A}_{\rho}(\mathscr{F}^{2}_{t})$ we write

$$J^{i}_{\rho}(\xi^{i},\zeta)=\mathbb{E}\left[\int_{[\rho,T)}f^{i}_{s}(1-\zeta_{s})\,d\xi^{i}_{s}+\int_{[\rho,T)}g^{i}_{s}(1-\xi^{i}_{s})\,d\zeta_{s}+\sum_{s\in[\rho,T]}h^{i}_{s}\Delta\zeta_{s}\Delta\xi^{i}_{s}\middle|\mathscr{F}^{2}_{\rho}\right].$$

Of course, we could have introduced this notation earlier, and while it would have cut down on the length of our equations, it would have obfuscated many of the techniques we used. In this chapter we are not so much concerned with these problems, and as such opt for a more condensed notation. Of course, one thing that is obvious from the previous work, but has never been written in these terms, is that J^{Π}_{ρ} is the convex sum of J^{i}_{ρ}, i.e.

$$J^{\Pi}_{\rho}(\xi,\zeta)=\sum_{i=0}^{\mathscr{I}}\Pi^{i}_{\rho-}J^{i}_{\rho}(\xi^{i},\zeta). \tag{5.3}$$

5.2 VERIFICATION THEOREM A

As already mentioned, the aim of these verification theorems is to give conditions such that a given pair of randomised strategies is optimal. We will always denote this given pair by $(\hat{\xi},\hat{\zeta})$, and they are the analogue to (ξ^{*},ζ^{*}).

On a technical note, as before we always assume that we are working on a complete, filtered probability space $(\Omega,\mathscr{F},\mathbb{P},(\mathscr{F}_{t})_{t\in[0,T]})$. Partial subgame perfect Nash equilibria are central to these verification theorems—see Definition 4.2.4. Moreover, since the pair $(\hat{\xi},\hat{\zeta})$ will be our main focus in this chapter, we recall the definitions of $\hat{\Omega}^{i}_{\rho}(\xi,\zeta)$ and $\hat{\Omega}_{\rho}(\xi,\zeta)$ from Definition 4.2.3 and state our simplified notation of

$$\hat{\Omega}^{i}_{\sigma}:=\hat{\Omega}^{i}_{\sigma}(\hat{\xi},\hat{\zeta}),\qquad\qquad \hat{\Omega}_{\sigma}:=\hat{\Omega}_{\sigma}(\hat{\xi},\hat{\zeta}),$$

for any $(\mathscr{F}^{2}_{t})$-stopping time σ.

Before giving our first verification theorem, we prove a simple lemma which gives a condition for when we can interchange the indicators of the sets $\hat{\Omega}^i_\sigma$ and $\hat{\Omega}_\sigma$.

Lemma 5.2.1. *Let $(\hat{\xi},\hat{\zeta}) \in \mathscr{A}(\mathscr{F}^1_t) \times \mathscr{A}(\mathscr{F}^2_t)$ and let $\pi = (\pi_0, \pi_1, ..., \pi_{\mathscr{I}}) \in \Delta^{\mathscr{I}}$. Then, for any $i \in [\mathscr{I}]$, and any $(\mathscr{F}^2_t)$-stopping time σ,*

$$\mathbb{1}_{\hat{\Omega}^i_\sigma} \pi_i (1 - \xi^{i,*}_{\sigma-}) = \mathbb{1}_{\hat{\Omega}_\sigma} \pi_i (1 - \xi^{i,*}_{\sigma-}).$$

Proof. Since $\hat{\Omega}^i_\sigma \subset \hat{\Omega}_\sigma$, we can consider three cases. The first is when $\omega \in \hat{\Omega}^i_\sigma \cap \hat{\Omega}_\sigma$. In this case $\pi_i(1 - \xi^{i,*}_{\sigma-}) > 0$ and the equality holds. Secondly, if $\omega \in \hat{\Omega}_\sigma \setminus \hat{\Omega}^i_\sigma$, then the left-hand side is zero due to the indicator function. To show the right-hand side is also equal to zero we notice that, on $\hat{\Omega}_\sigma \setminus \hat{\Omega}^i_\sigma$, we have $\pi_i(1 - \xi^{i,*}_{\sigma-})(1 - \zeta^*_{\sigma-}) = 0$ and $\sum_{i=0}^{\mathscr{I}} \pi_i(1 - \xi^{i,*}_{\sigma-})(1 - \zeta^*_{\sigma-}) > 0$. The latter implies that $(1 - \zeta^*_{\sigma-}) > 0$ and this, together with the former statement, implies that $\pi_i(1 - \xi^{i,*}_{\sigma-}) = 0$, so the equality holds. Lastly, if $\omega \in (\hat{\Omega}_\sigma)^c$, then both sides are zero. ■

Notice that, in the following Theorem, we need not assume any left or right-continuity of the payoff processes $(f_t)_{t\in[0,T]}$, $(g_t)_{t\in[0,T]}$, and $(h_t)_{t\in[0,T]}$, and moreover, their ordering of $g_t \leq h_t \leq f_t$ for all $t \in [0,T]$ is not required either. Later, in Corollary 5.2.5, we demonstrate that this additional condition allows us to bound the processes $(\mathscr{U}_t)_{t\in[0,T]}$ and $(\mathscr{U}^i_t)_{t\in[0,T]}$ for all $i \in [\mathscr{I}]$.

Theorem 5.2.2. *Let $(f^i_t)_{t\in[0,T]}$, $(g^i_t)_{t\in[0,T]}$ and $(h^i_t)_{t\in[0,T]}$ be $(\mathscr{F}^2_t)$-progressively measurable and belong to the Banach space $\mathscr{L}_b$ for $i \in [\mathscr{I}]$. Let the pair $(\hat{\xi},\hat{\zeta}) \in \mathscr{A}(\mathscr{F}^1_t) \times \mathscr{A}(\mathscr{F}^2_t)$ be given. For $i \in [\mathscr{I}]$, define*

$$\Pi^i_t := \begin{cases} \frac{\pi_i(1-\hat{\xi}^i_t)}{\sum_{k=0}^{\mathscr{I}} \pi_k(1-\hat{\xi}^k_t)} & \text{if } \sum_{k=0}^{\mathscr{I}} \pi_k(1-\hat{\xi}^k_t) > 0, \\ \delta_{i0} & \text{if } \sum_{k=0}^{\mathscr{I}} \pi_k(1-\hat{\xi}^k_t) = 0. \end{cases} \tag{5.4}$$

Let $(\mathscr{U}^i_t)_{t\in[0,T]}$ and $(\mathscr{U}_t)_{t\in[0,T]}$ be $(\mathscr{F}^2_t)$-progressively measurable and such that $(\mathscr{U}^i_t)_{t\in[0,T]}$ and $(\mathscr{U}_t)_{t\in[0,T]}$ obey the following conditions:

$$\mathscr{U}^i_T = h^i_T, \tag{5.5}$$

$$\sum_{i=0}^{\mathscr{I}} \pi_i \Delta\hat{\xi}^i_T \mathscr{U}_T = \sum_{i=0}^{\mathscr{I}} \pi_i \Delta\hat{\xi}^i_T h^i_T, \tag{5.6}$$

$$\mathscr{U}_0 = \sum_{i=0}^{\mathscr{I}} \pi_i \mathscr{U}^i_0, \tag{5.7}$$

and for any $\zeta \in \mathscr{A}(\mathscr{F}_t^2)$, $(M_t(\hat{\xi},\zeta))_{t\in[0,T]}$ *is an* $(\mathscr{F}_t^2)$*-supermartingale, and for any* $i \in [\mathscr{I}]$ *and any* $\xi^i \in \mathscr{A}(\mathscr{F}_t^2)$, $(M_t^i(\xi^i,\hat{\zeta}))_{t\in[0,T]}$ *is an* $(\mathscr{F}_t^2)$*-submartingale.*

Finally, let $(\hat{\xi}^\sigma,\hat{\zeta}^\sigma)_{\sigma\in\mathscr{T}(\mathscr{F}_t^2)}$ *be the family of time-consistent strategies obtained from truncating* $(\hat{\xi},\hat{\zeta})$*. Then for any* $(\mathscr{F}_t^2)$*-stopping time* σ*,*

1. *the game has a value at* σ*, i.e.*

$$\mathbb{1}_{\hat{\Omega}_\sigma}\mathscr{U}_\sigma = \operatorname*{ess\,inf}_{\xi\in\mathscr{A}_\sigma(\mathscr{F}_t^1)}\operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_\sigma(\mathscr{F}_t^2)}\mathbb{1}_{\hat{\Omega}_\sigma}J_\sigma^\Pi(\xi,\zeta) = \operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_\sigma(\mathscr{F}_t^2)}\operatorname*{ess\,inf}_{\xi\in\mathscr{A}_\sigma(\mathscr{F}_t^1)}\mathbb{1}_{\hat{\Omega}_\sigma}J_\sigma^\Pi(\xi,\zeta), \tag{5.8}$$

2. $(\mathscr{U}_t)_{t\in[0,T]}$ *and* $(\mathscr{U}_t^i)_{t\in[0,T]}$ *are related through the following convex combination*

$$\mathbb{1}_{\hat{\Omega}_\sigma}\mathscr{U}_\sigma = \mathbb{1}_{\hat{\Omega}_\sigma}\sum_{i=0}^{\mathscr{I}}\Pi_{\sigma-}^i\mathscr{U}_\sigma^i, \tag{5.9}$$

3. *the pair* $(\hat{\xi}^\sigma,\hat{\zeta}^\sigma)$ *is a partial subgame-perfect Nash equilibrium, in the sense that for any* $\xi \in \mathscr{A}_\sigma(\mathscr{F}_t^1)$ *and* $\zeta \in \mathscr{A}_\sigma(\mathscr{F}_t^2)$*,*

$$\mathbb{1}_{\hat{\Omega}_\sigma}J_\sigma^\Pi(\hat{\xi}^\sigma,\zeta) \le \mathbb{1}_{\hat{\Omega}_\sigma}J_\sigma^\Pi(\hat{\xi}^\sigma,\hat{\zeta}^\sigma) \le \mathbb{1}_{\hat{\Omega}_\sigma}J_\sigma^\Pi(\xi,\hat{\zeta}^\sigma). \tag{5.10}$$

Proof. Take $\sigma \in \mathscr{T}(\mathscr{F}_t^2)$ and let $\zeta \in \mathscr{A}_\sigma(\mathscr{F}_t^2)$. Define

$$\overline{\zeta}_t := \hat{\zeta}_t\mathbb{1}_{\{t<\sigma\}}(t) + \left[\hat{\zeta}_{\sigma-} + \zeta_t(1-\hat{\zeta}_{\sigma-})\right]\mathbb{1}_{\{t\ge\sigma\}}(t).$$

Clearly $\overline{\zeta} \in \mathscr{A}(\mathscr{F}_t^2)$. By the supermartingale property we must have

$$M_\sigma(\hat{\xi},\zeta) \ge \mathbb{E}\left[M_T(\hat{\xi},\zeta)\middle|\mathscr{F}_\sigma^2\right] \quad \mathbb{P}\text{-a.s.}$$

Using this, we obtain

$$\begin{aligned}\sum_{i=0}^{\mathscr{I}}\pi_i\mathscr{U}_\sigma(1-\hat{\xi}_{\sigma-}^i)(1-\overline{\zeta}_{\sigma-}) \ge \sum_{i=0}^{\mathscr{I}}\pi_i\mathbb{E}\Bigg[&\mathscr{U}_T(1-\hat{\xi}_{T-}^i)(1-\overline{\zeta}_{T-}) + \int_{[\sigma,T)}f_s^i(1-\overline{\zeta}_s)\,d\hat{\xi}_s^i \\ &+ \int_{[\sigma,T)}g_s^i(1-\hat{\xi}_s^i)\,d\overline{\zeta}_s + \sum_{s\in[\sigma,T)}h_s^i\Delta\overline{\zeta}_s\Delta\hat{\xi}_s^i\Bigg|\mathscr{F}_\sigma^2\Bigg].\end{aligned} \tag{5.11}$$

Since $(\overline{\zeta}_t)_{t\in[0,T]}$ and $((1-\hat{\zeta}_{\sigma-})\zeta_t)_{t\in[0,T]}$ define the same measure on $[\sigma,T)$, the Lebesgue-Stieltjes integrals in (5.11) with respect to $d\overline{\zeta}_t$ and $(1-\hat{\zeta}_{\sigma-})\,d\zeta_t$ are equal too (see Section A.2). Moreover, by the definition of $(\overline{\zeta}_t)_{t\in[0,T]}$, we have that $(1-\overline{\zeta}_s)=(1-\hat{\zeta}_{\sigma-})(1-\zeta_s)$, and $\Delta\overline{\zeta}_s=(1-\hat{\zeta}_{\sigma-})\Delta\zeta_s$, for $s\geq\sigma$. Additionally, recall that $\overline{\zeta}_{\sigma-}=\hat{\zeta}_{\sigma-}$. We can now see that

$$\sum_{i=0}^{\mathscr{I}}\pi_i\mathscr{U}_\sigma(1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-})\geq\sum_{i=0}^{\mathscr{I}}\pi_i\mathbb{E}\Bigg[(1-\hat{\zeta}_{\sigma-})\bigg(\mathscr{U}_T(1-\hat{\xi}^i_{T-})(1-\zeta_{T-})+\int_{[\sigma,T)}f^i_s(1-\zeta_s)\,d\hat{\xi}^i_s$$
$$+\int_{[\sigma,T)}g^i_s(1-\hat{\xi}^i_s)\,d\zeta_s+\sum_{s\in[\sigma,T)}h^i_s\Delta\zeta_s\Delta\hat{\xi}^i_s\bigg)\bigg|\mathscr{F}^2_\sigma\Bigg].$$

Using (5.6) yields

$$\sum_{i=0}^{\mathscr{I}}\pi_i\mathscr{U}_\sigma(1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-})$$
$$\geq\sum_{i=0}^{\mathscr{I}}\pi_i(1-\hat{\zeta}_{\sigma-})\mathbb{E}\bigg[\int_{[\sigma,T)}f^i_s(1-\zeta_s)\,d\hat{\xi}^i_s+\int_{[\sigma,T)}g^i_s(1-\hat{\xi}^i_s)\,d\zeta_s+\sum_{s\in[\sigma,T]}h^i_s\Delta\zeta_s\Delta\hat{\xi}^i_s\bigg|\mathscr{F}^2_\sigma\bigg].\quad(5.12)$$

Recall $\hat{\Omega}^i_\sigma:=\{\omega\in\Omega:\pi_i(1-\hat{\xi}^i_{\sigma-}(\omega))(1-\hat{\zeta}_{\sigma-}(\omega))>0\}$. On both sides of (5.12)—and inside of the summations—we can trivially introduce $\mathbb{1}_{\hat{\Omega}^i_\sigma}+\mathbb{1}_{(\hat{\Omega}^i_\sigma)^c}$. Notice that, when expanding this, again on both sides, all terms on the set $(\hat{\Omega}^i_\sigma)^c$ become zero. To further demonstrate this, if either $\pi_i=0$ or $\hat{\zeta}_{\sigma-}=1$, then all terms on $(\hat{\Omega}^i_\sigma)^c$ in the above equation become zero. If, however, we have $\hat{\xi}^i_{\sigma-}=1$, this implies that $\hat{\xi}^i_t=1$ for all $t\in[\sigma,T]$, and the jump at σ is zero also. Again, all terms on $(\hat{\Omega}^i_\sigma)^c$ are zero. We are therefore left with only the indicator function of $\hat{\Omega}^i_\sigma$, and this allows us to truncate $(\hat{\xi}^i_t)_{t\in[0,T]}$:

$$\sum_{i=0}^{\mathscr{I}}\mathbb{1}_{\hat{\Omega}^i_\sigma}\pi_i\mathscr{U}_\sigma(1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-})\geq\sum_{i=0}^{\mathscr{I}}\mathbb{1}_{\hat{\Omega}^i_\sigma}\pi_i(1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-})\mathbb{E}\bigg[\int_{[\sigma,T)}f^i_s(1-\zeta_s)\,d\hat{\xi}^{i,\sigma}_s$$
$$+\int_{[\sigma,T)}g^i_s(1-\hat{\xi}^{i,\sigma}_s)\,d\zeta_s+\sum_{s\in[\sigma,T]}h^i_s\Delta\zeta_s\Delta\hat{\xi}^{i,\sigma}_s\bigg|\mathscr{F}^2_\sigma\bigg].\quad(5.13)$$

Lemma 5.2.1 tells us that $\mathbb{1}_{\hat{\Omega}^i_\sigma}\pi_i(1-\hat{\xi}^i_{\sigma-})=\mathbb{1}_{\hat{\Omega}_\sigma}\pi_i(1-\hat{\xi}^i_{\sigma-})$, hence we may rewrite (5.13) as

$$\mathbb{1}_{\hat{\Omega}_\sigma}\sum_{i=0}^{\mathscr{I}}\pi_i\mathscr{U}_\sigma(1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-})\geq\mathbb{1}_{\hat{\Omega}_\sigma}\sum_{i=0}^{\mathscr{I}}\pi_i(1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-})\mathbb{E}\bigg[\int_{[\sigma,T)}f^i_s(1-\zeta_s)\,d\hat{\xi}^{i,\sigma}_s$$
$$+\int_{[\sigma,T)}g^i_s(1-\hat{\xi}^{i,\sigma}_s)\,d\zeta_s+\sum_{s\in[\sigma,T]}h^i_s\Delta\zeta_s\Delta\hat{\xi}^{i,\sigma}_s\bigg|\mathscr{F}^2_\sigma\bigg].$$

Dividing through by $\sum_{i=0}^{\mathscr{I}} \pi_i(1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-})$ is now possible due to the presence of $\mathbb{1}_{\hat{\Omega}_\sigma}$:

$$\mathbb{1}_{\hat{\Omega}_\sigma}\mathscr{U}_\sigma \geq \mathbb{1}_{\hat{\Omega}_\sigma}\sum_{i=0}^{\mathscr{I}}\Pi^i_{\sigma-}\mathbb{E}\bigg[\int_{[\sigma,T)} f^i_s(1-\zeta_s)\,d\hat{\xi}^{i,\sigma}_s + \int_{[\sigma,T)} g^i_s(1-\hat{\xi}^{i,\sigma}_s)\,d\zeta_s + \sum_{s\in[\sigma,T]} h^i_s\Delta\zeta_s\Delta\hat{\xi}^{i,\sigma}_s\bigg|\mathscr{F}^2_\sigma\bigg]. \tag{5.14}$$

In other words, we have shown that for $\sigma \in \mathscr{T}(\mathscr{F}^2_t)$ and for any $\zeta \in \mathscr{A}_\sigma(\mathscr{F}^2_t)$,

$$\mathbb{1}_{\hat{\Omega}_\sigma}\mathscr{U}_\sigma \geq \mathbb{1}_{\hat{\Omega}_\sigma}J^{\Pi}_\sigma(\hat{\xi}^\sigma,\zeta). \tag{5.15}$$

Likewise we can consider $(M^i_t(\xi^i,\hat{\zeta}))_{t\in[0,T]}$. Let $\xi \in \mathscr{A}_\sigma(\mathscr{F}^1_t)$ and define

$$\overline{\xi}^i_t := \hat{\xi}^i_t\mathbb{1}_{\{t<\sigma\}} + \Big[\hat{\xi}^i_{\sigma-} + \xi^i_t(1-\hat{\xi}^i_{\sigma-})\Big]\mathbb{1}_{\{t\geq\sigma\}}, \quad i \in [\mathscr{I}].$$

Since $(M^i_t(\xi^i,\hat{\zeta}))_{t\in[0,T]}$ is a submartingale, then in particular the submartingale property holds for the pair of stopping times $\sigma \leq T$. Importantly, we know from (5.5) that $\mathscr{U}^i_T = h^i_T$ so we can incorporate it as follows, $\mathbb{P}$-almost surely, that

$$\mathscr{U}^i_\sigma(1-\overline{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-}) \leq \mathbb{E}\bigg[\int_{[\sigma,T)} f^i_s(1-\hat{\zeta}_s)\,d\overline{\xi}^i_s + \int_{[\sigma,T)} g^i_s(1-\overline{\xi}^i_s)\,d\hat{\zeta}_s + \sum_{s\in[\sigma,T]} h^i_s\Delta\hat{\zeta}_s\Delta\overline{\xi}^i_s\bigg|\mathscr{F}^2_\sigma\bigg]. \tag{5.16}$$

Just like before (again, see Section A.2), we may replace $d\overline{\xi}^i_s$ with $(1-\hat{\xi}^i_{\sigma-})\,d\xi^i_s$ in the Lebesgue-Stieltjes integrals. Recall $\hat{\Omega}_\sigma$. We multiply both sides of (5.16) by its indicator, since this will allow us to truncate $(\hat{\zeta}_t)_{t\in[0,T]}$. This all yields

$$\mathbb{1}_{\hat{\Omega}_\sigma}\mathscr{U}^i_\sigma(1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-}) \leq \mathbb{1}_{\hat{\Omega}_\sigma}(1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-})\mathbb{E}\bigg[\int_{[\sigma,T)} f^i_s(1-\hat{\zeta}^\sigma_s)\,d\xi^i_s + \int_{[\sigma,T)} g^i_s(1-\xi^i_s)\,d\hat{\zeta}^\sigma_s + \sum_{s\in[\sigma,T]} h^i_s\Delta\hat{\zeta}^\sigma_s\Delta\xi^i_s\bigg|\mathscr{F}^2_\sigma\bigg]. \tag{5.17}$$

Again, since we are on $\hat{\Omega}_\sigma$, we can divide through by $\sum_{i=0}^{\mathscr{I}} \pi_i(1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-})$, and since it is positive, we may replace the resulting fraction, after cancelling $(1-\hat{\zeta}_{\sigma-})$, with $\Pi^i_{\sigma-}$ to obtain

$$\mathbb{1}_{\hat{\Omega}_\sigma}\Pi^i_{\sigma-}\mathscr{U}^i_\sigma \leq \mathbb{1}_{\hat{\Omega}_\sigma}\Pi^i_{\sigma-}J^i_\sigma(\xi^i,\hat{\zeta}^\sigma). \tag{5.18}$$

Since (5.18) holds for all $i \in [\mathscr{I}]$, we sum over them and get that

$$\mathbb{1}_{\hat{\Omega}_\sigma} \sum_{i=0}^{\mathscr{I}} \Pi^i_{\sigma-} \mathscr{U}^i_\sigma \leq \mathbb{1}_{\hat{\Omega}_\sigma} J^\Pi_\sigma(\xi, \hat{\zeta}^\sigma). \tag{5.19}$$

Taking $\zeta_t = \hat{\zeta}^\sigma_t$ in (5.15) and $\xi_t = \hat{\xi}^\sigma_t$ in (5.19) we obtain

$$\mathbb{1}_{\hat{\Omega}_\sigma} \sum_{i=0}^{\mathscr{I}} \Pi^i_{\sigma-} \mathscr{U}^i_\sigma \leq \mathbb{1}_{\hat{\Omega}_\sigma} J^\Pi_\sigma(\hat{\xi}^\sigma, \hat{\zeta}^\sigma), \tag{5.20}$$

$$\mathbb{1}_{\hat{\Omega}_\sigma} \mathscr{U}_\sigma \geq \mathbb{1}_{\hat{\Omega}_\sigma} J^\Pi_\sigma(\hat{\xi}^\sigma, \hat{\zeta}^\sigma). \tag{5.21}$$

We see from (5.20) and (5.21) that

$$\mathbb{1}_{\hat{\Omega}_\sigma} \sum_{i=0}^{\mathscr{I}} \Pi^i_{\sigma-} \mathscr{U}^i_\sigma \leq \mathbb{1}_{\hat{\Omega}_\sigma} \mathscr{U}_\sigma. \tag{5.22}$$

Now we show the reverse inequality. It is clear from (5.7) that

$$M_0(\hat{\xi}, \hat{\zeta}) = \sum_{i=0}^{\mathscr{I}} \pi_i M^i_0(\hat{\xi}, \hat{\zeta}).$$

From this, and both the sub- and supermartingale properties, we find that

$$\mathbb{E}[M_\sigma(\hat{\xi}, \hat{\zeta})] \leq M_0(\hat{\xi}, \hat{\zeta}) = \sum_{i=0}^{\mathscr{I}} \pi_i M^i_0(\hat{\xi}, \hat{\zeta}) \leq \mathbb{E}\left[\sum_{i=0}^{\mathscr{I}} \pi_i M^i_\sigma(\hat{\xi}, \hat{\zeta})\right].$$

Using the definitions of M_σ and M^i_σ, we see that a lot of terms cancel out. This leaves us with

$$\mathbb{E}\left[\sum_{i=0}^{\mathscr{I}} \pi_i \mathscr{U}_\sigma (1 - \hat{\xi}^i_{\sigma-})(1 - \hat{\zeta}_{\sigma-}) - \sum_{i=0}^{\mathscr{I}} \pi_i \mathscr{U}^i_\sigma (1 - \hat{\xi}^i_{\sigma-})(1 - \hat{\zeta}_{\sigma-})\right] \leq 0.$$

As before, we can introduce $\mathbb{1}_{\hat{\Omega}^i_\sigma} + \mathbb{1}_{(\hat{\Omega}^i_\sigma)^c}$ to all terms in the summations. On the latter set we see that all terms are zero. We, again, use Lemma 5.2.1 to then finally arrive at

$$\mathbb{E}\left[\mathbb{1}_{\hat{\Omega}_\sigma} \sum_{i=0}^{\mathscr{I}} \pi_i \mathscr{U}_\sigma (1 - \hat{\xi}^i_{\sigma-})(1 - \hat{\zeta}_{\sigma-}) - \mathbb{1}_{\hat{\Omega}_\sigma} \sum_{i=0}^{\mathscr{I}} \pi_i \mathscr{U}^i_\sigma (1 - \hat{\xi}^i_{\sigma-})(1 - \hat{\zeta}_{\sigma-})\right] \leq 0. \tag{5.23}$$

Recall (5.22). Due to the presence of $\hat{\Omega}_\sigma$, we can multiply both sides by $1 - \hat{\zeta}_{\sigma-}$, expand the belief process as per (5.4), and rearrange to get

$$\mathbb{1}_{\hat{\Omega}_\sigma} \sum_{i=0}^{\mathscr{I}} \pi_i \mathscr{U}_\sigma (1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-}) - \mathbb{1}_{\hat{\Omega}_\sigma} \sum_{i=0}^{\mathscr{I}} \pi_i \mathscr{U}^i_\sigma (1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-}) \geq 0.$$

Thus, (5.23) shows that the expectation of a non-negative random variable is non-positive; therefore, we must have

$$\mathbb{1}_{\hat{\Omega}_\sigma} \sum_{i=0}^{\mathscr{I}} \pi_i \mathscr{U}_\sigma (1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-}) = \mathbb{1}_{\hat{\Omega}_\sigma} \sum_{i=0}^{\mathscr{I}} \pi_i \mathscr{U}^i_\sigma (1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-}) \quad \mathbb{P}\text{-a.s.},$$

and so by the same logic as before

$$\mathbb{1}_{\hat{\Omega}_\sigma} \mathscr{U}_\sigma = \mathbb{1}_{\hat{\Omega}_\sigma} \sum_{i=0}^{\mathscr{I}} \Pi^i_{\sigma-} \mathscr{U}^i_\sigma, \tag{5.24}$$

proving (5.9). This implies that, from (5.20), (5.21) and (5.24), we get

$$\mathbb{1}_{\hat{\Omega}_\sigma} \mathscr{U}_\sigma = \mathbb{1}_{\hat{\Omega}_\sigma} J^\Pi_\sigma (\hat{\xi}^\sigma, \hat{\zeta}^\sigma), \tag{5.25}$$

and moreover, we can combine (5.15), (5.19) and (5.25), to give

$$\mathbb{1}_{\hat{\Omega}_\sigma} J^\Pi_\sigma (\hat{\xi}^\sigma, \zeta) \leq \mathbb{1}_{\hat{\Omega}_\sigma} J^\Pi_\sigma (\hat{\xi}^\sigma, \hat{\zeta}^\sigma) \leq \mathbb{1}_{\hat{\Omega}_\sigma} J^\Pi_\sigma (\xi, \hat{\zeta}^\sigma), \tag{5.26}$$

for any $(\xi, \zeta) \in \mathscr{A}_\sigma(\mathscr{F}^1_t) \times \mathscr{A}_\sigma(\mathscr{F}^2_t)$. This proves (5.10). Equation (5.8) follows immediately; observe from (5.26) that

$$\operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\sigma(\mathscr{F}^2_t)} \mathbb{1}_{\hat{\Omega}_\sigma} J^\Pi_\sigma (\hat{\xi}^\sigma, \zeta) \leq \mathbb{1}_{\hat{\Omega}_\sigma} J^\Pi_\sigma (\hat{\xi}^\sigma, \hat{\zeta}^\sigma) \leq \operatorname*{ess\,inf}_{\xi \in \mathscr{A}_\sigma(\mathscr{F}^1_t)} \mathbb{1}_{\hat{\Omega}_\sigma} J^\Pi_\sigma (\xi, \hat{\zeta}^\sigma).$$

We can bound the left-most term below by its essential infimum over $\xi \in \mathscr{A}_\sigma(\mathscr{F}^1_t)$, and the right-most term by its essential supremum over $\zeta \in \mathscr{A}_\sigma(\mathscr{F}^2_t)$, giving

$$\operatorname*{ess\,inf}_{\xi \in \mathscr{A}_\sigma(\mathscr{F}^1_t)} \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\sigma(\mathscr{F}^2_t)} \mathbb{1}_{\hat{\Omega}_\sigma} J^\Pi_\sigma (\xi, \zeta) \leq \mathbb{1}_{\hat{\Omega}_\sigma} J^\Pi_\sigma (\hat{\xi}^\sigma, \hat{\zeta}^\sigma) \leq \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\sigma(\mathscr{F}^2_t)} \operatorname*{ess\,inf}_{\xi \in \mathscr{A}_\sigma(\mathscr{F}^1_t)} \mathbb{1}_{\hat{\Omega}_\sigma} J^\Pi_\sigma (\xi, \zeta).$$

The reverse inequality holds as a natural ordering of the essential infimum and supremum. Thus,

$$\mathbb{1}_{\hat{\Omega}_\sigma} J^{\Pi}_{\sigma}(\hat{\xi}^{\sigma}, \hat{\zeta}^{\sigma}) = \operatorname*{ess\,inf}_{\xi \in \mathscr{A}_\sigma(\mathscr{F}^1_t)} \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\sigma(\mathscr{F}^2_t)} \mathbb{1}_{\hat{\Omega}_\sigma} J^{\Pi}_{\sigma}(\xi, \zeta) = \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\sigma(\mathscr{F}^2_t)} \operatorname*{ess\,inf}_{\xi \in \mathscr{A}_\sigma(\mathscr{F}^1_t)} \mathbb{1}_{\hat{\Omega}_\sigma} J^{\Pi}_{\sigma}(\xi, \zeta),$$

and $\mathbb{1}_{\hat{\Omega}_\sigma} \mathscr{U}_\sigma$ is the value of the game. ■

We have a few simple corollaries from this result. They are concerned with when we can obtain both a typical, and a subgame-perfect, Nash equilibrium.

Corollary 5.2.3. *Let $\sigma = 0$, then $\hat{\Omega}_0 = \Omega$, and at time zero, the Nash equilibrium holds for all $\omega \in \Omega$, i.e.*

$$J^{\Pi^*}_0(\hat{\xi}, \zeta) \le J^{\Pi^*}_0(\hat{\xi}, \hat{\zeta}) \le J^{\Pi^*}_0(\xi, \hat{\zeta}).$$

Corollary 5.2.4. *If, for all $s \in [0,T]$, we have $\hat{\zeta}_{s-} < 1$ and $\sum_{i=0}^{\mathscr{I}} \pi_i (1 - \hat{\xi}^i_{s-}) > 0$, then $\hat{\Omega}_\sigma = \Omega$ for all $\sigma \in \mathscr{T}(\mathscr{F}^2_t)$, and so the Nash equilibrium found in Theorem 5.2.2 holds everywhere, i.e.*

$$J^{\Pi^*}_\sigma(\hat{\xi}^\sigma, \zeta) \le J^{\Pi^*}_\sigma(\hat{\xi}^\sigma, \hat{\zeta}^\sigma) \le J^{\Pi^*}_\sigma(\xi, \hat{\zeta}^\sigma).$$

Finally, we know from Chapter 4 that the value processes are bounded by the payoff processes. Theorem 5.2.2 makes no claims about this since we need the additional information that the payoff processes are ordered.

Corollary 5.2.5. *Under the set-up for Theorem 5.2.2, if the payoff processes are ordered as $g^i_t \le h^i_t \le f^i_t$ for all i in $[\mathscr{I}]$ and $t \in [0,T]$ $\mathbb{P}$-almost surely, then the following bounds on the processes $(\mathscr{U}^i_t)_{t\in[0,T]}$ and $(\mathscr{U}_t)_{t\in[0,T]}$ hold*

$$\mathbb{1}_{\hat{\Omega}^i_\sigma} \mathscr{U}^i_\sigma \le \mathbb{1}_{\hat{\Omega}^i_\sigma} f^i_\sigma,$$
$$\mathbb{1}_{\hat{\Omega}_\sigma} \mathscr{U}_\sigma \ge \mathbb{1}_{\hat{\Omega}_\sigma} \sum_{i=0}^{\mathscr{I}} \Pi^i_{\sigma-} g^i_\sigma.$$

Proof. Choose $\zeta_t = \mathbb{1}_{\{t \ge \sigma\}}(t)$ in (5.14), then using the fact that $g^i_t \le h^i_t$ for all $t \in [0,T]$ $\mathbb{P}$-almost surely we get

$$\begin{aligned}\mathbb{1}_{\hat{\Omega}_\sigma}\mathscr{U}_\sigma &\geq \mathbb{1}_{\hat{\Omega}_\sigma}\sum_{i=0}^{\mathscr{I}}\Pi^i_{\sigma-}\mathbb{E}\Big[g^i_\sigma(1-\hat{\xi}^{i,\sigma}_\sigma)+h^i_\sigma\Delta\hat{\xi}^{i,\sigma}_\sigma\Big|\mathscr{F}^2_\sigma\Big]\\ &\geq \mathbb{1}_{\hat{\Omega}_\sigma}\sum_{i=0}^{\mathscr{I}}\Pi^i_{\sigma-}g^i_\sigma(1-\hat{\xi}^{i,\sigma}_{\sigma-})\\ &= \mathbb{1}_{\hat{\Omega}_\sigma}\sum_{i=0}^{\mathscr{I}}\Pi^i_{\sigma-}g^i_\sigma.\end{aligned}$$

Notice that (5.17) still holds if we change the indicator function to be of $\hat{\Omega}^i_\sigma$ instead of $\hat{\Omega}_\sigma$, because $\hat{\Omega}^i_\sigma \subset \hat{\Omega}_\sigma$. Choose $\xi^i_t = \mathbb{1}_{\{t\geq\sigma\}}(t)$ and, since $h^i_t \leq f^i_t$ for all $t\in[0,T]$ $\mathbb{P}$-almost surely:

$$\begin{aligned}\mathbb{1}_{\hat{\Omega}^i_\sigma}\mathscr{U}^i_\sigma(1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-}) &\leq \mathbb{1}_{\hat{\Omega}^i_\sigma}(1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-})\mathbb{E}\Big[f^i_\sigma(1-\hat{\zeta}^\sigma_\sigma)+h^i_\sigma\Delta\hat{\zeta}^\sigma_\sigma\Big|\mathscr{F}^2_\sigma\Big]\\ &\leq \mathbb{1}_{\hat{\Omega}^i_\sigma}(1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-})f^i_\sigma,\end{aligned}$$

implying that $\mathbb{1}_{\hat{\Omega}^i_\sigma}\mathscr{U}^i_\sigma \leq \mathbb{1}_{\hat{\Omega}^i_\sigma}f^i_\sigma$ since $(1-\hat{\xi}^i_{\sigma-})(1-\hat{\zeta}_{\sigma-})>0$ on $\hat{\Omega}^i_\sigma$.

■

5.3 VERIFICATION THEOREM B

In this section we present a second verification theorem which highlights the roles of the processes $(\mathscr{U}_t)_{t\in[0,T]}$ and $(\mathscr{U}^i_t)_{t\in[0,T]}$, specifically we will assume that they take the form of best responses to truncations. This is somewhat natural as this structure mirrors that found in (4.31) and Lemma 4.3.7. However, we only need to assume that $M_t(\hat{\xi},\hat{\zeta})_{t\in[0,T]}$ and $M^i_t(\hat{\xi},\hat{\zeta})_{t\in[0,T]}$ are martingales.

Theorem 5.3.1. *Let $(f^i_t)_{t\in[0,T]}$, $(g^i_t)_{t\in[0,T]}$ and $(h^i_t)_{t\in[0,T]}$ be $(\mathscr{F}^2_t)$-progressively measurable processes which belong to $\mathscr{L}_b$ for $i\in[\mathscr{I}]$. Let the pair $(\hat{\xi},\hat{\zeta})\in\mathscr{A}(\mathscr{F}^1_t)\times\mathscr{A}(\mathscr{F}^2_t)$ be given. Define*

$$\Pi^i_t := \begin{cases}\frac{\pi_i(1-\xi^i_t)}{\sum_{k=0}^{\mathscr{I}}\pi_k(1-\xi^k_t)} & \text{if } \sum_{k=0}^{\mathscr{I}}\pi_k(1-\xi^k_t)>0,\\ \delta_{i0} & \text{if } \sum_{k=0}^{\mathscr{I}}\pi_k(1-\xi^k_t)=0.\end{cases} \tag{5.27}$$

Let $(\mathscr{U}^i_t)_{t\in[0,T]}$ and $(\mathscr{U}_t)_{t\in[0,T]}$ be $(\mathscr{F}^2_t)$-progressively measurable processes such that:

$$\mathscr{U}^i_\rho := \operatorname*{ess\,inf}_{\xi^i\in\mathscr{A}_\rho(\mathscr{F}^2_t)} J^i_\rho(\xi^i,\hat{\zeta}^\rho),\quad \textit{for } i\in[\mathscr{I}] \textit{ and } \rho\in\mathscr{T}(\mathscr{F}^2_t), \tag{5.28}$$

$$\mathscr{U}_\rho := \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)} J_\rho^\Pi(\hat{\xi}^\rho, \zeta) \quad \rho \in \mathscr{T}(\mathscr{F}_t^2). \tag{5.29}$$

If $(M_t(\hat{\xi}, \hat{\zeta}))_{t\in[0,T]}$ and $(M_t^i(\hat{\xi}, \hat{\zeta}))_{t\in[0,T]}$, as defined in (5.1) and (5.2), are martingales, then, for any $\rho \in \mathscr{T}(\mathscr{F}_t^2)$, the random variable $\mathbb{1}_{\hat{\Omega}_\rho}\mathscr{U}_\rho$ is a value. Moreover, let the family $(\hat{\xi}^\rho, \hat{\zeta}^\rho)_{\rho\in\mathscr{T}(\mathscr{F}_t^2)}$ be the time-consistent family of strategies created by truncating $(\hat{\xi}, \hat{\zeta})$, then the pair $(\hat{\xi}^\rho, \hat{\zeta}^\rho)$ is a partial subgame-perfect Nash equilibrium, i.e. for any pair $(\xi, \zeta) \in \mathscr{A}_\rho(\mathscr{F}_t^1) \times \mathscr{A}(\mathscr{F}_t^2)$

$$\mathbb{1}_{\hat{\Omega}_\rho} J_\rho^\Pi(\hat{\xi}^\rho, \zeta) \leq \mathbb{1}_{\hat{\Omega}_\rho} J_\rho^\Pi(\hat{\xi}^\rho, \hat{\zeta}^\rho) \leq \mathbb{1}_{\hat{\Omega}_\rho} J_\rho^\Pi(\xi, \hat{\zeta}^\rho).$$

Additionally,

$$\mathbb{1}_{\hat{\Omega}_\rho}\mathscr{U}_\rho = \mathbb{1}_{\hat{\Omega}_\rho}\sum_{i=0}^{\mathscr{I}} \Pi_{\rho-}^i \mathscr{U}_\rho^i.$$

Proof. 1° To begin, take $\rho = T$ in (5.29) and we see that $\mathscr{U}_T = \sum_{i=0}^{\mathscr{I}} \Pi_{T-}^i h_T^i$, therefore

$$M_T(\hat{\xi}, \hat{\zeta}) = \sum_{i=0}^{\mathscr{I}} \pi_i \Bigg(\Delta\hat{\xi}_T^i \Delta\hat{\zeta}_T \sum_{j=0}^{\mathscr{I}} \Pi_{T-}^j h_T^j + \int_{[0,T)} f_s^i (1-\hat{\zeta}_s)\, d\hat{\xi}_s^i + \int_{[0,T)} g_s^i (1-\hat{\xi}_s^i)\, d\hat{\zeta}_s + \sum_{s\in[0,T)} h_s^i \Delta\hat{\zeta}_s \Delta\hat{\xi}_s^i \Bigg).$$

It is possible to simplify the first term in the sum using (5.27). In order to do this we must introduce $\hat{\Omega}_T$ and its compliment, however, notice that on $(\hat{\Omega}_T)^c$ we know that $\sum_{i=0}^{\mathscr{I}} \pi_i \Delta\hat{\xi}_T^i \Delta\hat{\zeta}_T = 0$, meaning all terms are zero. As a result we are left with only the terms on $\hat{\Omega}_T$:

$$\begin{aligned}
\mathbb{1}_{\hat{\Omega}_T} \sum_{i=0}^{\mathscr{I}} \pi_i \Delta\hat{\zeta}_T \Delta\hat{\xi}_T^i \sum_{j=0}^{\mathscr{I}} \Pi_{T-}^j h_T^j &= \mathbb{1}_{\hat{\Omega}_T} \sum_{i=0}^{\mathscr{I}} \pi_i \Delta\hat{\zeta}_T \Delta\hat{\xi}_T^i \sum_{j=0}^{\mathscr{I}} \frac{\pi_j \Delta\hat{\xi}_T^j}{\sum_{k=0}^{\mathscr{I}} \pi_k \Delta\hat{\xi}_T^k} h_T^j \\
&= \mathbb{1}_{\hat{\Omega}_T} \frac{\sum_{i=0}^{\mathscr{I}} \pi_i \Delta\hat{\xi}_T^i}{\sum_{k=0}^{\mathscr{I}} \pi_k \Delta\hat{\xi}_T^k} \Delta\hat{\zeta}_T \sum_{j=0}^{\mathscr{I}} \pi_j \Delta\hat{\xi}_T^j h_T^j \\
&= \mathbb{1}_{\hat{\Omega}_T} \sum_{i=0}^{\mathscr{I}} \pi_i \Delta\hat{\zeta}_T \Delta\hat{\xi}_T^i h_T^i,
\end{aligned}$$

with the final line having the sum relabelled. Of course, this equality also holds trivially on $(\hat{\Omega}_T)^c$ and this means that we may write

$$M_T(\hat{\xi}, \hat{\zeta}) = \sum_{i=0}^{\mathscr{I}} \pi_i \Bigg(\int_{[0,T)} f_s^i (1-\hat{\zeta}_s)\, d\hat{\xi}_s^i + \int_{[0,T)} g_s^i (1-\hat{\xi}_s^i)\, d\hat{\zeta}_s + \sum_{s\in[0,T]} h_s^i \Delta\hat{\zeta}_s \Delta\hat{\xi}_s^i \Bigg). \tag{5.30}$$

By the martingale property, $\mathbb{E}[M_T(\hat{\xi},\hat{\zeta})|\mathscr{F}^2_\rho] = M_\rho(\hat{\xi},\hat{\zeta})$ for all $\rho \in \mathscr{T}(\mathscr{F}^2_t)$, and (5.30), we get

$$\sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\left[\int_{[0,T)} f^i_s(1-\hat{\zeta}_s)\,d\hat{\xi}^i_s + \int_{[0,T)} g^i_s(1-\hat{\xi}^i_s)\,d\hat{\zeta}_s + \sum_{s\in[0,T]} h^i_s \Delta\hat{\zeta}_s \Delta\hat{\xi}^i_s \middle| \mathscr{F}^2_\rho\right]$$
$$= \sum_{i=0}^{\mathscr{I}} \pi_i \left(\mathscr{U}_\rho(1-\hat{\xi}^i_{\rho-})(1-\hat{\zeta}_{\rho-}) + \int_{[0,\rho)} f^i_s(1-\hat{\zeta}_s)\,d\hat{\xi}^i_s + \int_{[0,\rho)} g^i_s(1-\hat{\xi}^i_s)\,d\hat{\zeta}_s + \sum_{s\in[0,\rho)} h^i_s \Delta\hat{\zeta}_s \Delta\hat{\xi}^i_s\right).$$

Since the integrals on the right-hand side are all $\mathscr{F}^2_\rho$-measurable, this implies that

$$\sum_{i=0}^{\mathscr{I}} \pi_i \mathscr{U}_\rho(1-\hat{\xi}^i_{\rho-})(1-\hat{\zeta}_{\rho-}) =$$
$$\sum_{i=0}^{\mathscr{I}} \pi_i \mathbb{E}\left[\int_{[\rho,T)} f^i_s(1-\hat{\zeta}_s)\,d\hat{\xi}^i_s + \int_{[\rho,T)} g^i_s(1-\hat{\xi}^i_s)\,d\hat{\zeta}_s + \sum_{s\in[\rho,T]} h^i_s \Delta\hat{\zeta}_s \Delta\hat{\xi}^i_s \middle| \mathscr{F}^2_\rho\right].$$

Notice that we can introduce the indicator of $\hat{\Omega}_\rho$ on both sides. This allows us to truncate the strategies on the right-hand side. Recall that for a generating process $(\chi_t)_{t\in[0,T]}$, $(1-\chi_{\rho-})$ is $\mathscr{F}^2_\rho$-measurable, then we may remove those terms from the conditional expectation, i.e.

$$\mathbb{1}_{\hat{\Omega}_\rho} \sum_{i=0}^{\mathscr{I}} \pi_i \mathscr{U}_\rho(1-\hat{\xi}^i_{\rho-})(1-\hat{\zeta}_{\rho-}) =$$
$$\mathbb{1}_{\hat{\Omega}_\rho} \sum_{i=0}^{\mathscr{I}} \pi_i (1-\hat{\xi}^i_{\rho-})(1-\hat{\zeta}_{\rho-}) \mathbb{E}\left[\int_{[\rho,T)} f^i_s(1-\hat{\zeta}^\rho_s)\,d\hat{\xi}^{i,\rho}_s + \int_{[\rho,T)} g^i_s(1-\hat{\xi}^{i,\rho}_s)\,d\hat{\zeta}^\rho_s \right.$$
$$\left. + \sum_{s\in[\rho,T]} h^i_s \Delta\hat{\zeta}^\rho_s \Delta\hat{\xi}^{i,\rho}_s \middle| \mathscr{F}^2_\rho\right].$$

Cancelling the $(1-\hat{\zeta}_{\rho-})$ terms and dividing through by $\sum_{j=0}^{\mathscr{I}} \pi_j(1-\hat{\xi}^j_{\rho-})$ we arrive at

$$\mathbb{1}_{\hat{\Omega}_\rho} \sum_{i=0}^{\mathscr{I}} \Pi^i_{\rho-} \mathscr{U}_\rho = \mathbb{1}_{\hat{\Omega}_\rho} \sum_{i=0}^{\mathscr{I}} \Pi^i_{\rho-} \mathbb{E}\left[\int_{[\rho,T)} f^i_s(1-\hat{\zeta}^\rho_s)\,d\hat{\xi}^{i,\rho}_s + \int_{[\rho,T)} g^i_s(1-\hat{\xi}^{i,\rho}_s)\,d\hat{\zeta}^\rho_s \right.$$
$$\left. + \sum_{s\in[\rho,T]} h^i_s \Delta\hat{\zeta}^\rho_s \Delta\hat{\xi}^{i,\rho}_s \middle| \mathscr{F}^2_\rho\right]. \quad (5.31)$$

Using the fact that $\sum_{i=0}^{\mathscr{I}} \Pi^i_{\rho-} = 1$, (5.31) can be written as

$$\mathbb{1}_{\hat{\Omega}_\rho} \mathscr{U}_\rho = \mathbb{1}_{\hat{\Omega}_\rho} J^\Pi_\rho(\hat{\xi}^\rho, \hat{\zeta}^\rho), \quad (5.32)$$

i.e. from (5.29) we have that

$$\mathbb{1}_{\hat{\Omega}_\rho} J_\rho^\Pi(\hat{\xi}^\rho, \hat{\zeta}^\rho) = \operatorname*{ess\,sup}_{\zeta \in \mathscr{A}_\rho(\mathscr{F}_t^2)} \mathbb{1}_{\hat{\Omega}_\rho} J_\rho^\Pi(\hat{\xi}^\rho, \zeta).$$

2° The proof for the essential infimum is highly similar, so we will present a less detailed version here. Much in the same way, take $\rho = T$ in (5.28) and we get that $\mathscr{U}_T^i = h_T^i$. Therefore,

$$M_T^i(\hat{\xi}^i, \hat{\zeta}) = \int_{[0,T)} f_s^i (1 - \hat{\zeta}_s)\, d\hat{\xi}_s^i + \int_{[0,T)} g_s^i (1 - \hat{\xi}_s^i)\, d\hat{\zeta}_s + \sum_{s \in [0,T]} h_s^i \Delta\hat{\zeta}_s \Delta\hat{\xi}_s^i.$$

Arguing in the exact same way as above, using the martingale property, we get

$$\mathscr{U}_\rho^i (1 - \hat{\xi}_{\rho-}^i)(1 - \hat{\zeta}_{\rho-}) = \mathbb{E}\left[\int_{[\rho,T)} f_s^i (1 - \hat{\zeta}_s)\, d\hat{\xi}_s^i + \int_{[\rho,T)} g_s^i (1 - \hat{\xi}_s^i)\, d\hat{\zeta}_s + \sum_{s \in [\rho,T]} h_s^i \Delta\hat{\zeta}_s \Delta\hat{\xi}_s^i \,\middle|\, \mathscr{F}_\rho^2 \right],$$

Trivially we can multiply both sides by $\mathbb{1}_{\hat{\Omega}_\rho^i} \pi_i$. Like before, due to the presence of $\mathbb{1}_{\hat{\Omega}_\rho^i}$ we can truncate the strategies on the right-hand side to yield

$$\begin{aligned} \mathbb{1}_{\hat{\Omega}_\rho^i} \pi_i \mathscr{U}_\rho^i (1 - \hat{\xi}_{\rho-}^i)(1 - \hat{\zeta}_{\rho-}) = \mathbb{1}_{\hat{\Omega}_\rho^i} \pi_i (1 - \hat{\xi}_{\rho-}^i)(1 - \hat{\zeta}_{\rho-}) \mathbb{E}\Bigg[& \int_{[\rho,T)} f_s^i (1 - \hat{\zeta}_s^\rho)\, d\hat{\xi}_s^{i,\rho} \\ & + \int_{[\rho,T)} g_s^i (1 - \hat{\xi}_s^{i,\rho})\, d\hat{\zeta}_s^\rho + \sum_{s \in [\rho,T]} h_s^i \Delta\hat{\zeta}_s^\rho \Delta\hat{\xi}_s^{i,\rho} \,\Bigg|\, \mathscr{F}_\rho^2 \Bigg]. \end{aligned}$$

Lemma 5.2.1 then allows us to then change the indicator functions to $\mathbb{1}_{\hat{\Omega}_\rho}$, and so we can then finally divide through by $\pi_i (1 - \hat{\xi}_{\rho-}^i)(1 - \hat{\zeta}_{\rho-})$ to obtain

$$\mathbb{1}_{\hat{\Omega}_\rho} \mathscr{U}_\rho^i = \mathbb{1}_{\hat{\Omega}_\rho} \mathbb{E}\left[\int_{[\rho,T)} f_s^i (1 - \hat{\zeta}_s^\rho)\, d\hat{\xi}_s^{i,\rho} + \int_{[\rho,T)} g_s^i (1 - \hat{\xi}_s^{i,\rho})\, d\hat{\zeta}_s^\rho + \sum_{s \in [\rho,T]} h_s^i \Delta\hat{\zeta}_s^\rho \Delta\hat{\xi}_s^{i,\rho} \,\middle|\, \mathscr{F}_\rho^2 \right],$$

implying that

$$\mathbb{1}_{\hat{\Omega}_\rho} \mathscr{U}_\rho^i = \mathbb{1}_{\hat{\Omega}_\rho} J_\rho^i(\hat{\xi}^{i,\rho}, \hat{\zeta}^\rho). \tag{5.33}$$

Moreover, we can use (5.33), (5.3), and (5.28) to show

$$\begin{aligned} \mathbb{1}_{\hat{\Omega}_\rho} J_\rho^\Pi(\hat{\xi}^\rho, \hat{\zeta}^\rho) = \mathbb{1}_{\hat{\Omega}_\rho} \sum_{i=0}^{\mathscr{I}} \Pi_{\rho-}^i J_\rho^i(\hat{\xi}^{i,\rho}, \hat{\zeta}^\rho) &= \mathbb{1}_{\hat{\Omega}_\rho} \sum_{i=0}^{\mathscr{I}} \Pi_{\rho-}^i \operatorname*{ess\,inf}_{\xi^i \in \mathscr{A}_\rho(\mathscr{F}_t^2)} J_\rho^i(\xi^i, \hat{\zeta}^\rho) \\ &= \operatorname*{ess\,inf}_{\xi \in \mathscr{A}_\rho(\mathscr{F}_t^1)} \mathbb{1}_{\hat{\Omega}_\rho} J_\rho^\Pi(\xi, \hat{\zeta}^\rho), \end{aligned}$$

with the final equality following from the structure of $\mathscr{A}_\rho(\mathscr{F}_t^1)$—see Lemma 3.3.2. Hence, we have the partial subgame perfect Nash equilibrium:

$$\operatorname*{ess\,sup}_{\zeta\in\mathscr{A}_\rho(\mathscr{F}_t^2)} \mathbb{1}_{\hat{\Omega}_\rho} J_\rho^\Pi(\hat{\xi}^\rho,\zeta) = \mathbb{1}_{\hat{\Omega}_\rho} J_\rho^\Pi(\hat{\xi}^\rho,\hat{\zeta}^\rho) = \operatorname*{ess\,inf}_{\xi\in\mathscr{A}_\rho(\mathscr{F}_t^1)} \mathbb{1}_{\hat{\Omega}_\rho} J_\rho^\Pi(\xi,\hat{\zeta}^\rho).$$

3° Trivially, it follows from (5.3), (5.32), and (5.33) that

$$\mathbb{1}_{\hat{\Omega}_\rho}\mathscr{U}_\rho = \mathbb{1}_{\hat{\Omega}_\rho} J_\rho^\Pi(\hat{\xi}^\rho,\hat{\zeta}^\rho) = \mathbb{1}_{\hat{\Omega}_\rho}\sum_{i=0}^{\mathscr{I}}\Pi^i_{\rho-}J^i_\rho(\hat{\xi}^{i,\rho},\hat{\zeta}^\rho) = \mathbb{1}_{\hat{\Omega}_\rho}\sum_{i=0}^{\mathscr{I}}\Pi^i_{\rho-}\mathscr{U}^i_\rho.$$

■

* * *

CONCLUDING REMARKS

We would like to briefly compare and contrast the two verification theorems found in Theorem 5.2.2 (‘verification theorem A’) and Theorem 5.3.1 (‘verification theorem B’). An important fact to notice is that neither result follows directly from the other; they are, in fact, two distinct theorems despite some common assumptions and results. While verification theorem A assumes less about the processes $(\mathscr{U}_t)_{t\in[0,T]}$ and $(\mathscr{U}_t^i)_{t\in[0,T]}$, it requires a slightly wider requirement for the processes $(M_t^i)_{t\in[0,T]}$ and $(M_t)_{t\in[0,T]}$, namely that they are sub- and supermartingales respectively. Contrast this against verification theorem B where we assume the explicit form of the processes $(\mathscr{U}_t)_{t\in[0,T]}$ and $(\mathscr{U}_t^i)_{t\in[0,T]}$, but now only require one specific assumption of the martingality of $(M_t^i)_{t\in[0,T]}$ and $(M_t)_{t\in[0,T]}$.

Since these results are different, it is difficult to draw direct correspondence between them, however, we can make some intuitive observations as to why we can get similar results with distinctly different assumptions. Consider the conditions on $(\mathscr{U}_t)_{t\in[0,T]}$ and $(\mathscr{U}_t^i)_{t\in[0,T]}$ found in (5.5)–(5.7) in verification theorem A; taken together these equations imply that

$$M_T(\hat{\xi},\hat{\zeta}) = \sum_{i=0}^{\mathscr{I}}\pi_i M_T^i(\hat{\xi},\hat{\zeta}), \qquad \& \qquad M_0(\xi,\zeta) = \sum_{i=0}^{\mathscr{I}}\pi_i M_0^i(\xi,\zeta).$$

Taking the latter with the specific pair $(\hat{\xi},\hat{\zeta})$ we see that we are effectively 'pinning' the sub- and supermartingales, $(M_t^i(\hat{\xi},\hat{\zeta}))_{t\in[0,T]}$ and $(M_t(\hat{\xi},\hat{\zeta}))_{t\in[0,T]}$, together at $t=0$ and $t=T$. By contrast, verification theorem B does not require this. From (5.28) and (5.29) we do see that $M_T(\hat{\xi},\hat{\zeta}) = \sum_{i=0}^{\mathscr{I}} \pi_i M_T^i(\hat{\xi},\hat{\zeta})$ (see proof of Theorem 5.3.1), but there is no guarantee that the same relation at time zero holds. Intuitively, we see that the lack of 'pinning' of the sub- and supermartingales in verification theorem B is therefore somewhat compensated for by the requirement of martingality with the pair $(\hat{\xi},\hat{\zeta})$.

A second point worthy of note is that verification theorem B mirrors the construction of Chapter 4. In that work, we constructed a martingale using the optimal time-zero pair (ξ^*,ζ^*), and this was all we needed to prove the best response results found in Theorem 4.2.6. As we pointed out in Section 5.3, the definitions of the processes $(\mathscr{U}_t)_{t\in[0,T]}$ and $(\mathscr{U}_t^i)_{t\in[0,T]}$ closely matche the forms of the families of random variables $\{\mathcal{V}(\rho),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$ and $\{\mathcal{V}^i(\rho),\rho\in\mathscr{T}(\mathscr{F}_t^2)\}$, so it seems natural that a verification theorem that assumes the structure of $(\mathscr{U}_t)_{t\in[0,T]}$ and $(\mathscr{U}_t^i)_{t\in[0,T]}$ to be best responses should only need a martingale condition.

The final point we would like to raise is the fact that the regularity of the payoff processes $(f_t)_{t\in[0,T]}$, $(g_t)_{t\in[0,T]}$ and $(h_t)_{t\in[0,T]}$ is not required in either of the verification theorems. Moreover, neither is the ordering of said payoff processes—a defining factor in the set-up of the game. Corollary 5.2.5, therefore, shows us that, while the ordering is not needed to give a value or Nash equilibrium, it does allow us to derive conditions on the processes $(\mathscr{U}_t)_{t\in[0,T]}$ and $(\mathscr{U}_t^i)_{t\in[0,T]}$ that a value of the game would also obey. Note that we do not need the information that $(\mathscr{U}_t)_{t\in[0,T]}$ is a value to derive these conditions.

CHAPTER 6

EQUILIBRIUM IN A SIMPLE GAME

This chapter will be dedicated to studying an example, allowing us to fully solve the game and utilise the verification theorems from Chapter 5. The example will be the one given in Section 6.2 in the paper by De Angelis, Merkulov and Palczewski, [DAMP22], which they used to show the necessity of randomisation. Here we will explicitly compute the optimal strategies, the value, and verify the martingale conditions. Later, we will extend this example by introducing a single parameter which will vary a part of the payoff structure, and we will see how the equilibria change.

To begin with, let us introduce the set-up of the game. Let the filtration of the uninformed player, $(\mathscr{F}_t^2)$, be the trivial filtration; usefully, this implies that all payoff processes are deterministic functions and stopping times become constants, however, we have decided to keep the descriptors ‘processes’ and ‘stopping times’ in this chapter to be in keeping with the previous material. We also set $T = 1$, choose $\mathscr{I} = 1$, and restrict the effect of Θ such that $f_t^0 = f_t^1 =: f_t$ and $g_t^0 = g_t^1 =: g_t$ for all $t \in [0,1]$, which we define explicitly as follows

$$f_t := (10t+4)\mathbb{1}_{\{0\le t<\frac{1}{10}\}} + 5\mathbb{1}_{\{\frac{1}{10}\le t\le 1\}},$$
$$g_t := (15t-6)\mathbb{1}_{\{\frac{2}{5}\le t<\frac{1}{2}\}} + (9-15t)\mathbb{1}_{\{\frac{1}{2}\le t<\frac{3}{5}\}},$$

and let $h^0 = 0$, $h^1 = 5$, and define $(h_t)_{t\in[0,1]}$ as $h_t := f_t\mathbb{1}_{\{0\le t<1\}} + h_t^{\Theta}\mathbb{1}_{\{t=1\}}$. The figure below shows the form of these functions.

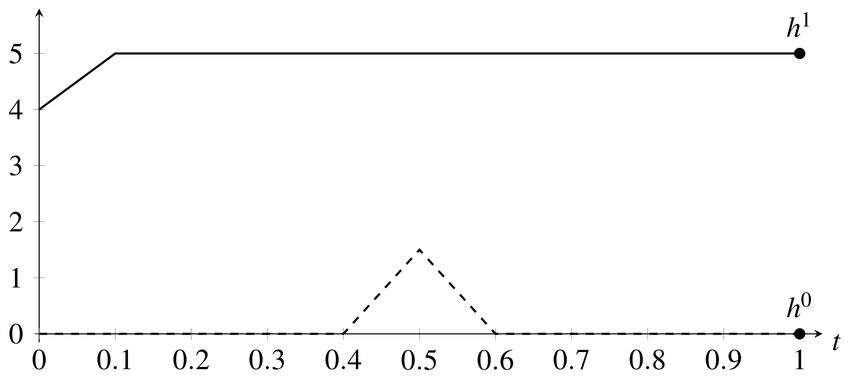


Figure 6.1: — f_t, - - - g_t.

Remark 6.1. This game is almost deterministic, but not quite; the only source of randomness comes from our random variable Θ. Because of this, our processes $(h_t)_{t\in[0,1]}$ and $(\xi_t^*)_{t\in[0,1]}$ depend on ω, however, their decompositions over $i \in [\mathscr{I}]$—$(h_t^i)_{t\in[0,1]}$ and $(\xi_t^{i,*})_{t\in[0,1]}$—are deterministic.

6.1 CONSTRUCTION OF OPTIMAL STRATEGIES

The intuition that [DAMP22] gives is that an informed player can ‘gradually reveal’ their additional information in order to induce the uninformed player to act in a desirable way. However, as they show, just because the maximiser is uninformed and so has no additional information to reveal, does not always mean that they should just play pure strategies.

The main methodology we use here will be guess and verify, however, we will include some preliminary results which cast light, not only on the structure of the game and its related processes, but also on how we arrived at our final guesses for the optimal strategies using the intuition we have gained from Chapter 4. The following result is a natural first step; since the payoffs $(f_t)_{t\in[0,1]}$ and $(g_t)_{t\in[0,1]}$ are constant for all $t \geq \frac{3}{5}$, then we would expect the value of the game to be simple, and in fact, we would expect that the value would be the weighted payoff of the terminal values of the game given the current value of the belief process.

First, recall that Corollary 3.4.18 guarantees the existence of a value of this game for every stopping time in $[0,1]$, i.e. in this setting, the value $V^{\Pi^*}(t)$ exists for every $t \in [0,1]$. An optimal pair (ξ^*, ζ^*) exists at time zero and we use it to define $(\Pi^*_t)_{t\in[0,T]}$ in the same manner as Definition 4.1.2. It is worth noting that, despite knowing the form of the belief process, we do not know how it behaves (since we do not know $(\xi^*_t)_{t\in[0,T]}$); as a result some of our analysis will revolve around analysing the different values $(\Pi^*_t)_{t\in[0,T]}$ can take to give an insight into the structure of $(\xi^*_t)_{t\in[0,T]}$.

Proposition 6.1.1. *Let $t \in [\frac{3}{5}, 1]$, then $V^{\Pi^*}(t) = 5\Pi^{1,*}_{t-}$. Moreover, the optimal strategies of all players, with the game starting from $t = \frac{3}{5}$, are $\xi^{i,*}_s = \zeta^*_s = \mathbb{1}_{\{s=1\}}(s)$, $s \geq t$.*

Proof. Because we have a trivial filtration, all conditional expectations become standard expectations. Moreover, since all objects Using the definition of $V^{\Pi^*}(t)$ we have

$$V^{\Pi^*}(t) = \inf_{\xi\in\mathscr{A}_t(\mathscr{F}^1_s)} \sup_{\zeta\in\mathscr{A}_t(\mathscr{F}^2_s)} \sum_{i=0}^{1} \Pi^{i,*}_{t-}\left(\int_{[t,1)} f_s(1-\zeta_s)\,d\xi^i_s + \sum_{s\in[t,1]} h^i_s \Delta\zeta_s \Delta\xi^i_s\right), \tag{6.1}$$

since $(g_t)_{t\in[0,1]}$ is zero on $[\frac{3}{5}, 1]$. Take $\xi^i_s = \mathbb{1}_{\{s=1\}}$, which is suboptimal for the minimiser, so we get the upper bound

$$V^{\Pi^*}(t) \leq \sup_{\zeta\in\mathscr{A}_t(\mathscr{F}^2_s)} \sum_{i=0}^{1} \Pi^{i,*}_{t-}(h^i_1 \Delta\zeta_1) = 5\Pi^{1,*}_{t-} \sup_{\zeta\in\mathscr{A}_t(\mathscr{F}^2_s)} \Delta\zeta_1 = 5\Pi^{1,*}_{t-},$$

where the final supremum is attained by $\zeta_s = \mathbb{1}_{\{s=1\}}$. Next, since $V^{\Pi^*}(t)$ is a value, we can swap the infimum and supremum in (6.1). Moreover, we see from Remark 4.5 in [DAMP22] that the inner optimisation can always be taken over stopping times, hence if we take $\zeta_s = \mathbb{1}_{\{s=1\}}$, we arrive at the lower bound

$$\begin{aligned} V^{\Pi^*}(t) &\geq \inf_{\tau^i\in[t,1]} \sum_{i=0}^{1} \Pi^{i,*}_{t-}\left(f_{\tau^i}\mathbb{1}_{\{\tau^i<1\}} + h^i_1 \mathbb{1}_{\{\tau^i=1\}}\right) \\ &= 5\Pi^{0,*}_{t-} \inf_{\tau^0\in[t,1]} \mathbb{1}_{\{\tau^0<1\}} + 5\Pi^{1,*}_{t-} \inf_{\tau^1\in[t,1]} \mathbb{1}_{\{\tau^1\leq 1\}} \\ &= 5\Pi^{1,*}_{t-}, \end{aligned}$$

where the first infimum is attained by choosing $\tau^0 = 1$ and the second takes the value 1 regardless of the choice of stopping time, so taking $\tau^1 = 1$ attains the infimum too. ■

Next we show that we can bound the values $\mathcal{V}^i(t)$ and $V^{\Pi^*}(t)$. Recall the set Γ_t from Definition 4.1.11. In this set up we see that $\mathbb{1}_{\Gamma_t} \equiv \mathbb{1}_{\{\zeta^*_{t-}<1\}}\mathcal{V}^i(t)$ from Lemma 4.3.7 which, in the set-up of this chapter is as follows; for any $i \in \{0,1\}$, and any time $t \in [0,1]$,

$$\mathcal{V}^i(t) := \mathbb{1}_{\{\zeta^*_{t-}<1\}} \operatorname*{ess\,inf}_{\xi^i \in \mathscr{A}_t(\mathscr{F}^2_s)} \mathbb{E}\left[\int_{[t,1)} f_s(1-\overline{\zeta}^t_s)\,d\xi^i_s + \int_{[t,1)} g_s(1-\xi^i_s)\,d\overline{\zeta}^t_s + \sum_{s\in[t,1]} h^i_s \Delta\overline{\zeta}^t_s \Delta\xi^i_s\right]. \quad (6.2)$$

Proposition 6.1.2. *For any $t \in [0,1]$ we have*

$$\mathcal{V}^0(t) \le \mathbb{1}_{\{\zeta^*_{t-}<1\}} \sup_{s\in[t,1)} g_s, \qquad \mathcal{V}^1(t) \le \mathbb{1}_{\{\zeta^*_{t-}<1\}} f_t, \qquad V^{\Pi^*}(t) \ge g_t.$$

Proof. Take $\xi^0_s = \mathbb{1}_{\{s=1\}}$ in the definition of $\mathcal{V}^0(t)$, we then get the upper bound

$$\mathcal{V}^0(t) \le \mathbb{1}_{\{\zeta^*_{t-}<1\}} \int_{[t,1)} g_s\,d\overline{\zeta}^t_s \le \mathbb{1}_{\{\zeta^*_{t-}<1\}} \left(\sup_{s\in[t,1)} g_s\right) \nu^{\overline{\zeta}^t}([t,1)) \le \mathbb{1}_{\{\zeta^*_{t-}<1\}} \sup_{s\in[t,1)} g_s,$$

where $\nu^{\overline{\zeta}^t}$ is the measure induced by $(\overline{\zeta}^t_s)_{s\in[0,1]}$. Likewise, take $\xi^1_s = \mathbb{1}_{\{s\ge t\}}$ in the definition of $\mathcal{V}^1(t)$ to find

$$\mathcal{V}^1(t) \le \mathbb{1}_{\{\zeta^*_{t-}<1\}} \Big(f_t(1-\overline{\zeta}^{t-}_t) + h^1_t \Delta\overline{\zeta}^{t-}_t\Big) = \mathbb{1}_{\{\zeta^*_{t-}<1\}} \Big(f_t(1-\overline{\zeta}^{t-}_{t-})\Big) = \mathbb{1}_{\{\zeta^*_{t-}<1\}} f_t.$$

Finally, taking $\zeta_s = \mathbb{1}_{\{s\ge t\}}$ in the definition of $V^{\Pi^*}(t)$, we see that, since $g_t \le h^i_t$,

$$V^{\Pi^*}(t) \ge \inf_{\xi^i \in \mathscr{A}_t(\mathscr{F}^1_s)} \sum_{i=0}^{1} \Pi^{i,*}_{t-}\big(g_t(1-\xi^i_t) + h^i_t \Delta\xi^i_t\big) \ge \inf_{\xi^i \in \mathscr{A}_t(\mathscr{F}^1_s)} \sum_{i=0}^{1} \Pi^{i,*}_{t-}\big(g_t(1-\xi^i_{t-})\big) = g_t,$$

since $\xi^i \in \mathscr{A}_t(\mathscr{F}^1_s)$ we know that $\xi^i_{t-} = 0$. ■

Corollary 6.1.3. *If $\zeta^*_{t-} < 1$ for all $t \in [0,1]$ then the optimal strategy for the $i = 0$ incarnation of the minimiser is to wait until the end of the game, i.e. for all $t \in [0,1]$,*

$$\xi^{0,*}_t = \mathbb{1}_{\{t=1\}}(t).$$

Proof. Since $\sup_{s\in[t,1)} g_s < f_t$ for all $t \in [0,1]$, then a direct result of Proposition 6.1.2 is that

$$\mathcal{V}^0(t) \leq \sup_{s\in[t,1)} g_s < f_t \tag{6.3}$$

for all $t \in [0,1]$. Theorem 4.4.7 tells us that

$$\begin{aligned} 0 &= \int_{[0,1]} \big((\mathcal{V}^0(t) - f_t)(1-\zeta_t^*) + (\mathcal{V}^0(t) - h_t^0)\Delta\zeta_t^* \big)\, d\xi_t^{0,*} \\ &= \int_{[0,1)} (\mathcal{V}^0(t) - f_t)(1-\zeta_{t-}^*)\, d\xi_t^{0,*} + \mathcal{V}^0(1)\Delta\zeta_1^*\Delta\xi_1^{0,*}, \end{aligned} \tag{6.4}$$

with the second equality following from the structure of the game. Moreover, because $h^0 = 0$, (6.2) tells us that $\mathcal{V}^0(1) = 0$, and so (6.4) becomes

$$0 = \int_{[0,1)} (\mathcal{V}^0(t) - f_t)(1-\zeta_{t-}^*)\, d\xi_t^{0,*}.$$

Finally, since we know $1-\zeta_{t-}^* > 0$ for all $t \in [0,1]$, and $\mathcal{V}^0(t) < f_t$ for all $t \in [0,1]$ from (6.3), we have a strictly negative integrand whose integral is zero. This can only be true if the measure (induced by $(\xi_t^{0,*})_{t\in[0,1]}$) of the set $[0,1)$ is zero. In other words, if $\xi_{1-}^{0,*} - \xi_{0-}^{0,*} = \xi_{1-}^{0,*} = 0$. Therefore the result stands. ■

We now have more of an insight into the structure of the optimal strategies. Let us formulate them generally in order to attain some conditions on their optimality. Let c_1 and c_2 be constants in $[0,1)$, and z_t^1 and z_t^2 be two non-decreasing functions such that $z_{0-}^1 = z_{0-}^2 = 0$ and $z_{\frac{3}{5}-}^1 \leq c_1$ and $z_{\frac{3}{5}-}^2 \leq c_2$. Our optimal strategies then look like the following:

$$\xi_t^{0,*} = \mathbb{1}_{\{t=1\}}(t), \tag{6.5}$$

$$\xi_t^{1,*} = z_t^1 \mathbb{1}_{\{0\leq t<\frac{3}{5}\}} + c_1 \mathbb{1}_{\{\frac{3}{5}\leq t<1\}} + \mathbb{1}_{\{t=1\}}, \tag{6.6}$$

$$\zeta_t^* = z_t^2 \mathbb{1}_{\{0\leq t<\frac{3}{5}\}} + c_2 \mathbb{1}_{\{\frac{3}{5}\leq t<1\}} + \mathbb{1}_{\{t=1\}}. \tag{6.7}$$

Note that the restriction of the constants c_1 and c_2 to the half-open interval $[0,1)$ guarantees that when we truncate these processes at $t = \frac{3}{5}$, we still recover the strategies shown in Proposition 6.1.1. Recall that we denote the truncated versions of the optimal processes, at some stopping time τ, by $(\overline{\xi}_t^{\tau})_{t\in[0,1]}$ and $(\overline{\zeta}_t^{\tau})_{t\in[0,1]}$.

Proposition 6.1.4. *Denote $\tau = \frac{3}{5}$. The strategies defined in (6.5)–(6.7) are such that, for any $\xi \in \mathscr{A}_\tau(\mathscr{F}_t^1)$ and $\zeta \in \mathscr{A}_\tau(\mathscr{F}_t^2)$, we have*

1. $J_\tau^{\Pi^*}(\overline{\xi}^\tau, \zeta) \leq J_\tau^{\Pi^*}(\overline{\xi}^\tau, \overline{\zeta}^\tau) \iff (\Delta\overline{\xi}_1^{1,\tau} > 0 \Longrightarrow \Delta\overline{\zeta}_1^\tau = 1)$.

2. $(\zeta_t^*)_{t\in[0,1]}$ *is continuous at τ, i.e. $z_{\tau-}^2 = c_2$.*

Proof. 1° First notice that, from the definition of $J_t^{\Pi^*}$ in Definition 4.1.5, and (6.5)–(6.7), that

$$J_\tau^{\Pi^*}(\overline{\xi}^\tau, \overline{\zeta}^\tau) = \sum_{i=0}^{1} \Pi_{\tau-}^{i,*}\Big(f_\tau(1-\overline{\zeta}_\tau^\tau)\Delta\overline{\xi}_\tau^{i,\tau} + h_\tau^i \Delta\overline{\zeta}_\tau^\tau \Delta\overline{\xi}_\tau^{i,\tau} + h_1^i \Delta\overline{\zeta}_1^\tau \Delta\overline{\xi}_1^{i,\tau}\Big)$$

Since $h_t^i = f_t$ for all $t \in [0,1)$ we can absorb the jump term $\Delta\overline{\zeta}_\tau^\tau$ into $(1-\overline{\zeta}_\tau^\tau)$ and get

$$J_\tau^{\Pi^*}(\overline{\xi}^\tau, \overline{\zeta}^\tau) = \Pi_{\tau-}^{1,*}\Big(f_\tau \Delta\overline{\xi}_\tau^{1,\tau} + h_1^1 \Delta\overline{\zeta}_1^\tau \Delta\overline{\xi}_1^{1,\tau}\Big). \tag{6.8}$$

Notice the summation disappears since there is no jump at $t = \tau$ for $(\xi_t^{0,*})_{t\in[0,1]}$, and $h_1^0 = 0$.

Next, for a general $\zeta \in \mathscr{A}_\tau(\mathscr{F}_t^2)$, we can, in much the same way, see that

$$J_\tau^{\Pi^*}(\overline{\xi}^\tau, \zeta) = \Pi_{\tau-}^{1,*}\Big(f_\tau \Delta\overline{\xi}_\tau^{1,\tau} + h_1^1 \Delta\zeta_1 \Delta\overline{\xi}_1^{1,\tau}\Big). \tag{6.9}$$

Here we can show, after some algebra, that

$$J_\tau^{\Pi^*}(\overline{\xi}^\tau, \zeta) \leq J_\tau^{\Pi^*}(\overline{\xi}^\tau, \overline{\zeta}^\tau) \iff \Big(\Delta\overline{\xi}_1^{1,\tau} > 0 \Longrightarrow \Delta\overline{\zeta}_1^\tau = 1\Big).$$

The 'if' direction of the if and only if statement is easy to show, and follows trivially from (6.8) and (6.9). The only if direction can be shown by proving the contrapositive. Assume that $\Delta\overline{\xi}_1^{1,\tau} > 0$ and $\Delta\overline{\zeta}_1^\tau < 1$. We can define $(\hat{\zeta}_t)_{t\in[0,1]}$ such that $\Delta\hat{\zeta}_1^\tau = 1$, then using (6.9) we obtain

$$J_\tau^{\Pi^*}(\overline{\xi}^\tau, \hat{\zeta}) > \Pi_{\tau-}^{1,*}\Big(f_\tau \Delta\overline{\xi}_\tau^{1,\tau} + h_1^1 \Delta\overline{\zeta}_1^\tau \Delta\overline{\xi}_1^{1,\tau}\Big) = J_\tau^{\Pi^*}(\overline{\xi}^\tau, \overline{\zeta}^\tau).$$

2° Let the $\Theta = 1$ player play suboptimally with $\xi^1 \in \mathscr{A}_\tau(\mathscr{F}_t^2)$, then

$$J_\tau^{\Pi^*}((\overline{\xi}^{0,\tau}, \xi), \overline{\zeta}^\tau) = \Pi_{\tau-}^{1,*}\Bigg(\int_{[\tau,1)} f_t(1-\overline{\zeta}_t^\tau)\, d\xi_t + h_\tau^1 \Delta\overline{\zeta}_\tau^\tau \Delta\xi_\tau + h_1^1 \Delta\overline{\zeta}_1^\tau \Delta\xi_1\Bigg).$$

The potential jump term in the integral at $t = \tau$ is $f_\tau(1-\overline{\zeta}_\tau^\tau)\Delta\xi_\tau$. If we use the fact that $h_t^1 = f_t$

for all $t \in [0,1]$ we can combine this term with $h^1_\tau \Delta\overline{\zeta}^\tau_\tau \Delta\xi_\tau$ to obtain $f_\tau \Delta\xi_\tau$ (using the fact that $\overline{\zeta}^\tau_{\tau-} = 0$), yielding

$$J^{\Pi^*}_\tau((\overline{\xi}^{0,\tau}_t, \xi_t), \overline{\zeta}^\tau_t) = \Pi^{1,*}_{\tau-}\left(f_\tau \Delta\xi_\tau + \int_{(\tau,1)} f_t(1-\overline{\zeta}^\tau_t)\, d\xi_t + h^1_1 \Delta\overline{\zeta}^\tau_1 \Delta\xi_1\right).$$

The remaining terms in the integral are constant and equal to $f_\tau(1-\overline{\zeta}^\tau_\tau)$, so the integral becomes $f_\tau(1-\overline{\zeta}^\tau_\tau)(\xi_{1-}-\xi_\tau)$. Considering $h^1_1\Delta\overline{\zeta}^\tau_1\Delta\xi_1$, the value of $\Delta\overline{\zeta}^\tau_1$ can be seen to be $1-\overline{\zeta}^\tau_\tau$ by definition, and of course $\Delta\xi_1 = 1-\xi_{1-}$. Again, by using the fact that $h^1_t = f_t$ for all $t \in [0,1]$, and that $f_1 = f_\tau$, we get

$$J^{\Pi^*}_\tau((\overline{\xi}^{0,\tau}, \xi), \overline{\zeta}^\tau) = \Pi^{1,*}_{\tau-}\Big(f_\tau \Delta\xi_\tau + f_\tau(1-\overline{\zeta}^\tau_\tau)(1-\xi_\tau)\Big).$$

Let us imagine that we have $\overline{\zeta}^\tau_\tau > 0$, then since $\Delta\xi_\tau = \xi_\tau$, and $f_\tau > f_\tau(1-\overline{\zeta}^\tau_\tau)$, we find the minimal value of the expectation, i.e. the best response to $\overline{\zeta}^\tau_t$, by choosing $\xi_\tau = 0$. Thus, since in an equilibrium we know $(\xi_t)_{t\in[0,1]}$ must be flat on $[\tau,1)$, then $\Delta\xi_1 = 1$. However, we have already shown that for optimal $(\zeta_t)_{t\in[0,1]}$, $\Delta\xi_1 = 1$ implies that $\Delta\overline{\zeta}^\tau_1 = 1$, which contradicts the fact that $\overline{\zeta}^\tau_\tau > 0$, hence we must have $\overline{\zeta}^\tau_\tau = 0$, i.e. continuity at $t = \tau$. ■

Now that we have some understanding of what our optimal strategies look like, we can consider the game as started at a time earlier than $t = \frac{3}{5}$, since this is where we see non-triviality in the payoff functions.

Proposition 6.1.1 hints that our value process may well end up being $5\Pi^{1,*}_{t-}$. At time $t = \frac{1}{2}$, $(g_t)_{t\in[0,1]}$ is at its peak, so intuitively the maximiser is faced with the decision of whether to stop now, or to wait until the end of the game (we already know the $i=0$ minimiser always waits, and for the $i=1$ minimiser the payoff is constant and so it really makes no difference how they act).

We know that the maximiser can only act if the value process is equal to $(g_t)_{t\in[0,1]}$, and so we consider two cases in the following proposition which will illuminate the issue.

Proposition 6.1.5. *Beginning the game at $t = \frac{1}{2}$ we have the following:*

1) If $5\Pi^{1,*}_{\frac{1}{2}-} > g_{\frac{1}{2}}$, then the strategies $\xi^{i,*}_t = \zeta^*_t = \mathbb{1}_{\{t=1\}}$ for $i \in \{0,1\}$, are a Nash equilibrium.

2) If $5\Pi^{1,*}_{\frac{1}{2}-} = g_{\frac{1}{2}}$, then the both sets of strategies $\xi^{0,*}_t = \xi^{1,*}_t = \zeta^*_t = \mathbb{1}_{\{t=1\}}$, and $\xi^{0,*}_t = \xi^{1,*}_t = \mathbb{1}_{\{t=1\}}$, $\zeta^*_t = \mathbb{1}_{\{t\geq\frac{1}{2}\}}$ are Nash equilibria, i.e. the maximiser is indifferent between stopping immediately and waiting until the end.

Proof. 1° We will prove that $\xi_t^{i,*} = \zeta_t^* = \mathbb{1}_{\{t=1\}}$ is a Nash equilibrium directly for the general case of $5\Pi^{1,*}_{\frac{1}{2}-} \geq g_{\frac{1}{2}}$. Using the definition of $J_t^{\Pi^*}$ (Definition 4.1.5), we see

$$J^{\Pi^*}_{\frac{1}{2}}((\xi^{0,*},\xi^{1,*}),\zeta^*) = 5\Pi^{1,*}_{\frac{1}{2}-}. \tag{6.10}$$

Next, let the maximiser play any strategy $\zeta \in \mathscr{A}_{\frac{1}{2}}(\mathscr{F}_t^2)$, then the payoff is directly calculable as

$$\begin{aligned} J^{\Pi^*}_{\frac{1}{2}}((\xi^{0,*},\xi^{1,*}),\zeta) &= \int_{[\frac{1}{2},1)} g_t \, d\zeta_t + 5\Pi^{1,*}_{\frac{1}{2}-}\Delta\zeta_1 \\ &\leq g_{\frac{1}{2}}(\zeta_{1-} - \zeta_{\frac{1}{2}-}) + 5\Pi^{1,*}_{\frac{1}{2}-}\Delta\zeta_1 \\ &\leq 5\Pi^{1,*}_{\frac{1}{2}-}, \end{aligned} \tag{6.11}$$

where the first inequality is found using the supremum upper bound on g_t and the measure of $[\frac{1}{2},1)$ under the measure induced by $(\zeta_t)_{t\in[0,1]}$ is simply $\zeta_{1-} - \zeta_{\frac{1}{2}-}$. Next, letting the minimiser play any $\xi \in \mathscr{A}_{\frac{1}{2}}(\mathscr{F}_t^1)$ we see

$$\begin{aligned} J^{\Pi^*}_{\frac{1}{2}}((\xi^{0},\xi^{1}),\zeta^*) &= \Pi^{0,*}_{\frac{1}{2}-}\left(\int_{[\frac{1}{2},1)} f_t \, d\xi_t^0\right) + \Pi^{1,*}_{\frac{1}{2}-}\left(\int_{[\frac{1}{2},1)} f_t \, d\xi_t^1 + h_1^1\Delta\xi_t^1\right) \\ &= \Pi^{0,*}_{\frac{1}{2}-}\left(\int_{[\frac{1}{2},1)} f_t \, d\xi_t^0\right) + 5\Pi^{1,*}_{\frac{1}{2}-} \\ &\geq 5\Pi^{1,*}_{\frac{1}{2}-}, \end{aligned} \tag{6.12}$$

since the integral term is non-negative. Thus, by combining (6.10), (6.11) and (6.12), we find the Nash equilibrium.

2° Next we assume that $5\Pi^{1,*}_{\frac{1}{2}-} = g_{\frac{1}{2}}$ and take our strategies to be $\xi_t^{i,*} = \mathbb{1}_{\{t=1\}}$ for $i \in \{0,1\}$, and $\zeta_t^* = \mathbb{1}_{\{t\geq\frac{1}{2}\}}$. Again, it is easy to see that

$$J^{\Pi^*}_{\frac{1}{2}}((\xi^{0,*},\xi^{1,*}),\zeta^*) = g_{\frac{1}{2}}.$$

Equally, just as before we have that for any strategy $\zeta \in \mathscr{A}_{\frac{1}{2}}(\mathscr{F}_t^2)$, (6.11) still holds, however, since $5\Pi^{1,*}_{\frac{1}{2}-} = g_{\frac{1}{2}}$ we equivalently have

$$J^{\Pi^*}_{\frac{1}{2}}((\xi^{0,*},\xi^{1,*}),\zeta) \leq g_{\frac{1}{2}}.$$

Next, taking any $\xi \in \mathscr{A}_{\frac{1}{2}}(\mathscr{F}_t^1)$, we arrive at

$$J_{\frac{1}{2}}^{\Pi^*}((\xi^0,\xi^1),\zeta^*) = \Pi_{\frac{1}{2}-}^{0,*}\left(g_{\frac{1}{2}}(1-\xi_{\frac{1}{2}}^0) + f_{\frac{1}{2}}\Delta\xi_{\frac{1}{2}}^0\right) + \Pi_{\frac{1}{2}-}^{1,*}\left(g_{\frac{1}{2}}(1-\xi_{\frac{1}{2}}^1) + f_{\frac{1}{2}}\Delta\xi_{\frac{1}{2}}^1\right).$$

Using the fact that $g_t \le f_t$ for all $t \in [0,1]$, we see that

$$J_{\frac{1}{2}}^{\Pi^*}((\xi^0,\xi^1),\zeta^*) \ge g_{\frac{1}{2}}.$$

Hence we have the Nash equilibrium. ∎

This indifference result is very important. We expect that a likely place for the maximiser to act will be at $t = \frac{1}{2}$, given that it is the maximal point of $(g_t)_{t\in[0,1]}$, however, should the minimiser act in such a way as to push the value process down to the peak of $(g_t)_{t\in[0,1]}$ (note that the minimiser has direct control over the value process since the belief process depends solely on $(\xi_t)_{t\in[0,1]}$), then it would not be shocking to see the maximiser's strategy be to randomise between the two points $t = \frac{1}{2}$ and $t = 1$.

However, we still do not have an understanding of how the maximiser should behave in the time interval before $t = \frac{1}{2}$. The next proposition will solve this problem and, therefore, give us our first true intuition as to the formulation of $(\zeta_t^*)_{t\in[0,1]}$.

Proposition 6.1.6. *For all $t \in [0,\frac{1}{2}]$, the value process is bounded below, $V^{\Pi^*}(t) \ge g_{\frac{1}{2}}$, and therefore $\zeta_t^* = 0$ for all $t < \frac{1}{2}$.*

Proof. Let $t \in [0,\frac{1}{2}]$, then, since we have already found the explicit optimal strategy $(\xi_t^{0,*})_{t\in[0,1]}$, we can express the value in terms of its original formulation

$$V^{\Pi^*}(t) = \sup_{\zeta\in\mathscr{A}_t(\mathscr{F}_s^2)} \inf_{\xi^1\in\mathscr{A}_t(\mathscr{F}_s^2)} J_t^{\Pi^*}((\xi^{0,*},\xi^1),\zeta)$$

We can write the payoff function on the right-hand side explicitly as follows

$$J_t^{\Pi^*}((\xi^{0,*},\xi^1),\zeta) = \Pi_{t-}^{0,*}\int_{[t,1)} g_s\,d\zeta_s + \Pi_{t-}^{1,*}\Bigg(\int_{[t,1)} f_s(1-\zeta_{s-})\,d\xi_s^1 + \int_{[t,1)} g_s(1-\xi_s^1)\,d\zeta_s + f_1\Delta\zeta_1\Delta\xi_1^1\Bigg),$$

where we have absorbed the summation term $\sum_{s\in[t,1)} h_s^1\Delta\zeta_s\Delta\xi_s^1$ into the integral with respect to

$d\xi^1_s$, since $h^1_s = f_s$ (see Section A.2), this explains the presence of $(1-\zeta_{s-})$—the left limit, ζ_{s-}, being present due to the inclusion of the jump terms.

If we choose $\zeta_s = \mathbb{1}_{\{s\geq\frac{1}{2}\}}$, this is suboptimal for the maximiser and we attain the following lower bound:

$$\begin{aligned}
V^{\Pi^*}(t) &\geq \inf_{\xi^1\in\mathscr{A}_t(\mathscr{F}^2_s)} \left\{ \Pi^0_{t-} g_{\frac{1}{2}} + \Pi^1_{t-}\left(\int_{[t,\frac{1}{2}]} f_s\, d\xi^1_s + g_{\frac{1}{2}}(1-\xi^1_{\frac{1}{2}}) \right) \right\} \\
&= g_{\frac{1}{2}} + \inf_{\xi^1\in\mathscr{A}_t(\mathscr{F}^2_s)} \left\{ \Pi^1_{t-}\left(\int_{[t,\frac{1}{2}]} f_s\, d\xi^1_s - g_{\frac{1}{2}}\xi^1_{\frac{1}{2}} \right) \right\}. \qquad (6.13)
\end{aligned}$$

We can get a lower bound on the integral term in (6.13) by using the infimum of $(f_t)_{t\in[0,1]}$ over $[t,\frac{1}{2}]$, and since $(\xi^1_s)_{t\in[0,1]}$ belongs to $\mathscr{A}_t(\mathscr{F}^2_s)$, we know its left limit at t is zero, therefore,

$$\begin{aligned}
V^{\Pi^*}(t) &\geq g_{\frac{1}{2}} + \inf_{\xi^1\in\mathscr{A}_t(\mathscr{F}^2_s)} \left\{ \Pi^1_{t-}\left(f_0(\xi^1_{\frac{1}{2}} - \xi^1_{t-}) - g_{\frac{1}{2}}\xi^1_{\frac{1}{2}} \right) \right\} \\
&= g_{\frac{1}{2}} + \inf_{\xi^1\in\mathscr{A}_t(\mathscr{F}^2_s)} \left\{ \Pi^1_{t-}(f_0 - g_{\frac{1}{2}})\xi^1_{\frac{1}{2}} \right\}.
\end{aligned}$$

This is minimised by taking any strategy such that $\xi^1_{\frac{1}{2}} = 0$, yielding $V^{\Pi}(t) \geq g_{\frac{1}{2}}$.

This is a uniform lower bound on $V^{\Pi^*}(t)$ on the interval $[0,\frac{1}{2}]$, meaning, since $g_t < g_{\frac{1}{2}}$ for all $t<\frac{1}{2}$, then $V^{\Pi^*}(t) \neq g_t$ in $[0,\frac{1}{2})$, and so the maximiser must not act before $t=\frac{1}{2}$. ■

In light of Propositions 6.1.5 and 6.1.6, we now have a worthwhile candidate for the maximiser's optimal strategy: it will be zero before $t=\frac{1}{2}$ and will then randomise between $t=\frac{1}{2}$ and $t=1$. In other words, for some $a\in[0,1]$, our candidate is

$$\zeta^*_t = a\mathbb{1}_{\{t\geq\frac{1}{2}\}} + (1-a)\mathbb{1}_{\{t=1\}}. \qquad (6.14)$$

Let us try and use this to gain some more insight into how the minimiser might play as a response. We already know that if $\Theta=0$, the minimiser will wait until the end of the game, however, if $\Theta=1$, the payoff function for the minimiser is now increasing. A best response to $(\zeta^*_t)_{t\in[0,1]}$ from the minimiser will avoid any action at $t=\frac{1}{2}$, since that would result in a payoff of 5, the highest value. Moreover, it would make sense for the minimiser to at least partially stop immediately, since $f_0 \equiv \inf_{t\in[0,1]} f_t$, enough so that the value process is pushed down to $g_{\frac{1}{2}}$ in order for the maximiser to act. Then, since the payoff for $t\geq\frac{1}{10}$ is constant for the minimiser, we can take their final action to be partially stopping at the end of the game too. Hence, we can theorise a potential candidate strategy for the $\Theta = 1$ minimiser as, for some $b \in [0,1]$,

$$\xi_t^{1,*} = b\mathbb{1}_{\{t\geq 0\}} + (1-b)\mathbb{1}_{\{t=1\}}. \tag{6.15}$$

Theorem 6.1.7. *If it is optimal for the maximiser to jump at $t = \frac{1}{2}$ and $t = 1$ (see (6.14)), and it is optimal for the $\Theta = 1$ minimiser to jump at $t = 0$ and $t = 1$ (see (6.15)), then the optimal strategy for the maximiser is*

$$\zeta_t^* = \frac{2}{7}\mathbb{1}_{\{t\geq\frac{1}{2}\}}(t) + \frac{5}{7}\mathbb{1}_{\{t=1\}}(t). \tag{6.16}$$

Proof. 1° Since the minimiser is assumed to jump at both $t = 0$ and $t = 1$ we can say $b \in (0,1)$, and likewise we can say $a \in (0,1)$. Notice that for the case of $\Theta = 1$, $(f_t)_{t\in[0,1]}$ and $(h_t)_{t\in[0,1]}$ coincide. Coupling this with Theorem 4.4.7 tells us that

$$\int_{[0,1]} (\mathcal{V}^i(t) - f_t)(1-\zeta_{t-}^*)\, d\xi_t^{1,*} = 0.$$

Because $(\xi_t^{1,*})_{t\in[0,1]}$ only jumps at $t = 0$ and $t = 1$, and $b \in (0,1)$, the measure induced by $(\xi_t^{i,*})_{t\in[0,1]}$ only puts a (strictly positive) mass at the points $t = 0$ and $t = 1$. Moreover, Theorem 4.4.7 tells us the integrand is non-positive, meaning the integrand itself must be zero at both $t = 0$ and $t = 1$ giving us

$$\mathcal{V}^1(0) = f_0, \qquad \& \qquad (\mathcal{V}^1(1) - f_1)\Delta\zeta_1^* = 0,$$

Since we assume $\Delta\zeta_1^* > 0$, we must have that $\mathcal{V}^1(1) = f_1$.

2° Recall the system $\{m^i(\rho; u, \zeta^*), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$, and that Lemma 4.4.5 tells us it is a $\mathscr{T}(\mathscr{F}_t^2)$-martingale system for λ-almost every $u \in [0,1]$. Moreover, recall $\mathfrak{U}^i$ from (4.54) and that this set is dense in $[0,1]$. Since we assume $b < 1$, we can choose a $u \in \mathfrak{U}^i$ such that $b < u < 1$. Finally, notice that, due to the structure of (6.15), it is true that $\tau^{1,*}(v) = \mathbb{1}_{\{v\geq b\}}$, and therefore $\tau^{1,*}(u) = 1$. Since $u \in \mathfrak{U}^i$ we know that $\{m^i(\rho; u, \zeta^*), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$ is a $\mathscr{T}(\mathscr{F}_t^2)$-martingale system, and using Definition 4.4.1 we see

$$m^i(t; u, \zeta^*) = (1-\zeta_{t-}^*)\mathcal{V}^i(t) + \int_{[0,t)} g_s^i\, d\zeta_s^*.$$

3° Next, since $(\zeta_t^*)_{t\in[0,1]}$ only stops at two distinct points, $t = \frac{1}{2}$ and $t = 1$, we can write

$(m^i(t;u,\zeta^*))_{t\in[0,1]}$ as

$$m^i(t;u,\zeta^*) = \mathcal{V}^1(t)\mathbb{1}_{\{t\le\frac{1}{2}\}} + \Big(\mathcal{V}^1(t)(1-\zeta^*_{t-}) + g_{\frac{1}{2}}\Delta\zeta^*_{\frac{1}{2}}\Big)\mathbb{1}_{\{t>\frac{1}{2}\}}.$$

We have already shown that $\mathcal{V}^1(0) = f_0 = 4$, and so

$$m^i(0;u,\zeta^*) = \mathcal{V}^1(0) = 4,$$

thus, by the martingale property (i.e. it is constant), $m^i(t;u,\zeta^*) = 4$ for all $t \in [0,1]$.

We also proved that $\mathcal{V}^1(1) = f_1 = 5$ and so we can say, again since $(m^i(t;u,\zeta^*))_{t\in[0,1]}$ is constant, and that ζ^*_t is zero before the jump at $t = \frac{1}{2}$,

$$m^i(1;u,\zeta^*) = 4 = \mathcal{V}^1(1)(1-\zeta^*_{1-}) + g_{\frac{1}{2}}\Delta\zeta^*_{\frac{1}{2}} = 5(1-\zeta^*_{\frac{1}{2}}) + \frac{3}{2}\zeta^*_{\frac{1}{2}}.$$

This implies that $\Delta\zeta^*_{\frac{1}{2}} = \frac{2}{7}$, and the final result holds. ∎

Now we have an explicit guess of the maximiser's optimal strategy, we can do the same to attain one for the minimiser.

Theorem 6.1.8. *If the maximiser's strategy is of the form (6.16), and it is optimal for the $\Theta = 1$ minimiser to jump at $t = 0$ and $t = 1$ (see (6.15)), then the optimal strategy for the minimiser is*

$$\xi^{1,*}_t = \left(1 - \frac{3\pi_0}{7\pi_1}\right)\mathbb{1}_{\{t\ge 0\}}(t) + \frac{3\pi_0}{7\pi_1}\mathbb{1}_{\{t=1\}}(t). \tag{6.17}$$

Proof. 1° Recall the system $\{m(\rho;u,\xi^*), \rho \in \mathscr{T}(\mathscr{F}^2_t)\}$ from Definition 4.4.8 and that it is a $\mathscr{T}(\mathscr{F}^2_t)$-martingale system for λ-almost every $u \in (0,1)$. Moreover, recall $\mathfrak{U}$ from (4.63) and recall it is dense in $[0,1]$. Notice that $\sigma^*(v) = \frac{1}{2}\mathbb{1}_{\{u<\frac{2}{7}\}} + \mathbb{1}_{\{u\ge\frac{2}{7}\}}$, then choose $u \in \mathfrak{U}$ such that $\frac{1}{2} < u < 1$; then $\sigma^*(u) = 1$. Since $u \in \mathfrak{U}$, we can use Definition 4.4.8 to see that the process $(m(t;u,\xi^*))_{t\in[0,1]}$ takes the form

$$m(t;u,\xi^*) := \sum_{i-0}^{\mathscr{I}} \pi_i\left((1-\xi^{i,*}_{t-})\mathcal{V}(t) + \int_{[0,t)} f^i_s\, d\xi^{i,*}_s\right),$$

and is a martingale. Because of this, we must have that $m(1;u,\xi^*) = m(\frac{1}{2};u,\xi^*)$, which results in

$$\pi_0\mathcal{V}(1)+\pi_1\left(\mathcal{V}(1)\Delta\xi_1^{1,*}+4\Delta\xi_0^{1,*}\right)=\pi_0\mathcal{V}\left(\frac{1}{2}\right)+\pi_1\left(\left(1-\xi_{\frac{1}{2}-}^{1,*}\right)\mathcal{V}\left(\frac{1}{2}\right)+4\Delta\xi_0^{1,*}\right).$$

This follows when one recalls that $\xi_t^{0,*} = \mathbb{1}_{\{t=1\}}(t)$ from Corollary 6.1.3. We can simplify this slightly by noticing that $\Delta\xi_0^{1,*} = \xi_0^{1,*}$ and, since $\xi_t^{1,*} = b$ for all $t \in [0,1)$, $(1-\xi_{\frac{1}{2}-}^{1,*}) = \Delta\xi_1^{1,*}$:

$$\pi_0\mathcal{V}(1)+\pi_1\left(\mathcal{V}(1)\Delta\xi_1^{1,*}+4\xi_0^{1,*}\right)=\pi_0\mathcal{V}\left(\frac{1}{2}\right)+\pi_1\left(\Delta\xi_1^{1,*}\mathcal{V}\left(\frac{1}{2}\right)+4\xi_0^{1,*}\right). \tag{6.18}$$

2° Recall that Theorem 4.4.13 tells us that

$$\int_{[0,1]}\sum_{i=0}^{\mathscr{I}}\pi_i\Big((\mathcal{V}(t)-g_t)(1-\xi_t^{i,*})+\big(\mathcal{V}(t)-h_t^i\big)\Delta\xi_t^{i,*}\Big)\,d\zeta_t^*=0.$$

Since $(\zeta_t^*)_{t\in[0,1]}$ only jumps at $t=\frac{1}{2}$ and $t=1$ (see Theorem 6.1.7) the measure induced by it only puts a (strictly positive) mass at these times. Moreover, Theorem 4.4.13 tells us that the integrand is non-negative and so we must have that the integrand is zero at both $t=\frac{1}{2}$ and $t=1$. This leads to the following pair of equations

$$0=\pi_0\left(\mathcal{V}\left(\frac{1}{2}\right)-\frac{3}{2}\right)+\pi_1\left(\mathcal{V}\left(\frac{1}{2}\right)-\frac{3}{2}\right)\Delta\xi_1^{i,*}, \tag{6.19}$$

$$0=\pi_0\mathcal{V}(1)+\pi_1(\mathcal{V}(1)-5)\Delta\xi_1^{1,*}, \tag{6.20}$$

where we have used explicit values of the payoff processes. Equations (6.18)–(6.20) comprise a system of three linear equations in three variables—$\mathcal{V}(1)$, $\mathcal{V}(\frac{1}{2})$, and $\Delta\xi_1^{1,*}$—which can be easily solved by eliminating $\mathcal{V}(1)$ and $\mathcal{V}(\frac{1}{2})$, yielding

$$\Delta\xi_1^{1,*}=\frac{3\pi_0}{7\pi_1},$$

which completes the proof. ■

Remark 6.1.9. Section A.3 has been frequently referenced throughout this work and is where one can find our 'change of variables' formula. Interestingly, the section handles the idea of so-called 'generalised inverses', and it is stated in [DAMP22] that, for example, $u \mapsto \sigma^*(u)$ is the generalised inverse of $t \mapsto \zeta_t^*(\omega)$. On its own this statement is quite general, however, this setting allows us to see this directly. Recall that during the proofs of Theorem 6.1.7 and Theorem 6.1.8 we showed that $\tau^{1,*}(u) = \mathbb{1}_{\{u\geq b\}}$ and $\sigma^*(u) = \frac{1}{2}\mathbb{1}_{\{u<\frac{2}{7}\}} + \mathbb{1}_{\{u\geq\frac{2}{7}\}}$; these are exactly the inverse functions of the processes $(\zeta_t^*)_{t\in[0,1]}$ and $(\xi_t^{1,*})_{t\in[0,1]}$ found in (6.16) and (6.17).

One thing to notice is that since we know $\xi^{1,*} = 1 - \frac{3\pi_0}{7\pi_1}$ on $[0,1)$, we can see that in order for this term to remain non-negative, $\pi_1 \geq \frac{3}{10}$. This has actually always been with us, and we could have shown this earlier, specifically if we recall that we stipulated that we wanted $5\Pi^{1,*}_{\frac{1}{2}-} \geq g_{\frac{1}{2}}$ in Proposition 6.1.5. Using the explicit form of the belief process in (4.3) and that $\xi_t^{0,*} = \mathbb{1}_{\{t=1\}}$, we see that

$$5\Pi^{1,*}_{\frac{1}{2}-} \geq g_{\frac{1}{2}} \quad \Longleftrightarrow \quad \xi^{1,*}_{\frac{1}{2}-} \leq 1 - \frac{3\pi_0}{7\pi_1}, \tag{6.21}$$

with the condition on the right-hand side holding only if $\pi_1 \geq \frac{3}{10}$. Of course, equality is attained in (6.21), meaning we could have deduced b solely from our guess (6.15) and the condition that $5\Pi^{1,*}_{\frac{1}{2}-} = g_{\frac{1}{2}}$.

However, while our guesses of the optimal strategies seem to be self-consistent, they are still currently guesses until we show they are optimal. This is straightforward to do and we will use best response arguments.

Theorem 6.1.10. *For $\pi_1 \geq \frac{3}{10}$, the pair $(\xi^*, \zeta^*) \in \mathscr{A}(\mathscr{F}_t^1) \times \mathscr{A}(\mathscr{F}_t^2)$, defined as*

$$\begin{aligned}
\xi_t^{0,*} &= \mathbb{1}_{\{t=1\}}(t),\\
\xi_t^{1,*} &= \left(1 - \frac{3\pi_0}{7\pi_1}\right)\mathbb{1}_{\{t\geq 0\}}(t) + \frac{3\pi_0}{7\pi_1}\mathbb{1}_{\{t=1\}}(t),\\
\zeta_t^* &= \frac{2}{7}\mathbb{1}_{\{t\geq\frac{1}{2}\}}(t) + \frac{5}{7}\mathbb{1}_{\{t=1\}}(t),
\end{aligned}$$

is a Nash equilibrium.

Proof. Recall from Corollary 6.1.3 that the form chosen for $(\xi_t^{0,*})_{t\in[0,1]}$ is optimal.

Let us assume the maximiser plays $(\zeta_t^*)_{t\in[0,1]}$, then we will show that $(\xi_t^{1,*})_{t\in[0,1]}$ is optimal. Recall that the inner optimisation can be taken over stopping times, so the game takes the form

$$\sup_{\zeta\in\mathscr{A}(\mathscr{F}_t^2)} \inf_{(\tau_0,\tau_1)\in[0,1]^2} J_0^{\Pi^*}((\tau_0,\tau_1),\zeta) = \inf_{(\tau_0,\tau_1)\in[0,1]^2} J_0^{\Pi^*}((\tau_0,\tau_1),\zeta^*).$$

When we choose τ_i to be a stopping time, the form of the payoff function becomes:

$$J_0^{\Pi^*}((\tau_0,\tau_1),\zeta^*) = \sum_{i=0}^{1} \pi_i \left(f_{\tau_i}(1-\zeta^*_{\tau_i}) + \int_{[0,\tau_i)} g_s \, d\zeta^*_s + h^i_{\tau_i}\Delta\zeta^*_{\tau_i} \right).$$

Because the payoff $J_0^{\Pi^*}$ is decomposed into $i=0$ and $i=1$ parts, we see that this infimum can be split and simplified into

$$\inf_{(\tau_0,\tau_1)\in[0,1]^2} J_0^{\Pi^*}((\tau_0,\tau_1),\zeta^*) = \inf_{\tau_0\in[0,1]} \pi_0 \left(f_{\tau_0}\mathbb{1}_{\{\tau_0<\frac{1}{2}\}} + 5\mathbb{1}_{\{\tau_0=\frac{1}{2}\}} + 4\mathbb{1}_{\{\frac{1}{2}<\tau_0<1\}} + \frac{3}{7}\mathbb{1}_{\{\tau_0=1\}} \right)$$
$$+ \inf_{\tau_1\in[0,1]} \pi_1 \left(f_{\tau_1}\mathbb{1}_{\{\tau_1<\frac{1}{2}\}} + 5\mathbb{1}_{\{\tau_1=\frac{1}{2}\}} + 4\mathbb{1}_{\{\frac{1}{2}<\tau_1\le 1\}} \right).$$

Here, the first infimum is attained by taking $\tau_0 = 1$ and the second infimum is attained by taking either $\tau_1 = 0$ or $\tau_1 \in (\frac{1}{2}, 1]$. So $(\tau_0,\tau_1) = (1,0)$ and $(\tau_0,\tau_1) = (1,1)$ are best responses to $(\zeta^*_t)_{t\in[0,1]}$. Proposition 3.4.13 implies that the convex combination of two optimal strategies is also optimal, thus $(\xi^{i,*}_t)_{t\in[0,1]}$ as given is a best response to $(\zeta^*_t)_{t\in[0,1]}$.

Next, we show that $(\zeta^*_t)_{t\in[0,1]}$ is a best response to $(\xi^*_t)_{t\in[0,1]}$. Our optimisation problem is the corresponding supremum problem

$$\sup_{\sigma\in[0,1]} J_0^{\Pi^*}((\xi^{0,*},\xi^{1,*}),\sigma) = \sup_{\sigma\in[0,1]} \sum_{i=0}^{1} \pi_i \left(\int_{[0,\sigma)} f_s \, d\xi^{i,*}_s + g_\sigma(1-\xi^{i,*}_\sigma) + h^i_\sigma \Delta\xi^{i,*}_\sigma \right). \tag{6.22}$$

In a similar manner, we can split this problem along values of σ resulting, after some trivial simplifying, in the following

$$4\pi_1\Delta\xi^{1,*}_0 + \sup_{\sigma\in[0,1]} \left\{ \left(g_\sigma\pi_0 + g_\sigma\pi_1(1-\xi^{1,*}_0) \right) \mathbb{1}_{\{0<\sigma<1\}} + 5\pi_1\Delta\xi^{1,*}_1 \mathbb{1}_{\{\sigma=1\}} \right\}. \tag{6.23}$$

Picking $\sigma* = \frac{1}{2}$ clearly maximises the first term in the supremum and one can show that, in fact,

$$g_{\sigma*}\pi_0 + g_{\sigma*}\pi_1(1-\xi^{1,*}_0) = 5\pi_1\Delta\xi^{1,*}_1 = \frac{15}{7}\pi_0,$$

meaning that both $\sigma* = \frac{1}{2}$ and $\sigma* = 1$ are best responses to $(\xi^{0,*}_t, \xi^{1,*}_t)$. ■

Having found a Nash equilibrium, we are now free to explicitly calculate all of the processes we have constructed and which are fundamental to the game. Fortunately, the Nash equilibrium strategies we have just found behave very nicely in subgames. Trivially, we see, for any stopping time $\rho \in [0,1]$, that $\overline{\xi}_t^{0,\rho} = \xi_t^{0,*}$. More interestingly, the $\Theta = 1$ minimiser's strategy becomes to wait for the end of the game for any $\rho \in (0,1]$:

$$\overline{\xi}_t^{1,\rho} = \begin{cases} \mathbb{1}_{\{t=1\}}(t) & \forall \rho \in (0,1], \\ \xi_t^{1,*} & \rho = 0. \end{cases} \tag{6.24}$$

Finally, the maximiser's strategy remains unchanged until we truncate past $t = \frac{1}{2}$:

$$\overline{\zeta}_t^{\rho} = \begin{cases} \zeta_t^* & \forall \rho \in [0, \frac{1}{2}], \\ \mathbb{1}_{\{t=1\}}(t) & \forall \rho \in (\frac{1}{2}, 1]. \end{cases} \tag{6.25}$$

This nice behaviour in subgames allows us to compute, not just the value $(V_t^{\Pi^*})_{t\in[0,1]}$ and the martingale $(M_t)_{t\in[0,1]}$, but the belief process, $(\Pi_t^*)_{t\in[0,1]}$, and the values $(\mathcal{V}_t^0)_{t\in[0,1]}$ and $(\mathcal{V}_t^1)_{t\in[0,1]}$.

Theorem 6.1.11. *For $\pi_1 \geq \frac{3}{10}$:*

1) the value of the game takes the following form

$$V_t^{\Pi^*} = \left(4\pi_1 + \frac{3}{7}\pi_0\right)\mathbb{1}_{\{t=0\}}(t) + \frac{3}{2}\mathbb{1}_{\{0<t\leq 1\}}(t), \tag{6.26}$$

2) the belief process is

$$\Pi_{t-}^{1,*} = \pi_1 \mathbb{1}_{\{t=0\}}(t) + \frac{3}{10}\mathbb{1}_{\{t>0\}}(t), \tag{6.27}$$

3) the martingales $(M_t(\xi^*, \zeta^*))_{t\in[0,1]}$, $(M_t^0(\xi^{0,*}, \zeta^*))_{t\in[0,1]}$ and $(M_t^1(\xi^{1,*}, \zeta^*))_{t\in[0,1]}$ take the values $4\pi_1 + \frac{3}{7}\pi_0$, $\frac{3}{7}$ and 4 respectively,

4) the value processes for the minimisers are left-continuous functions of the form

$$\mathcal{V}_t^0 = \frac{3}{7}\mathbb{1}_{\{0\leq t\leq \frac{1}{2}\}}(t), \qquad \mathcal{V}_t^1 = 4\mathbb{1}_{\{0\leq t\leq \frac{1}{2}\}}(t) + 5\mathbb{1}_{\{\frac{1}{2}<t\leq 1\}}(t). \tag{6.28}$$

Proof. $1°$ and $2°$ can be solved together. First, recall the definition of $V_t^{\Pi^*}$:

$$V_t^{\Pi^*} = \sum_{i=0}^{1} \Pi_{t-}^{i,*}\left(\int_{[t,1)} f_s(1-\overline{\zeta}_s^t)\,d\overline{\xi}_s^{i,t} + \int_{[t,1)} g_s(1-\overline{\xi}_s^{i,t})\,d\overline{\zeta}_s^t + \sum_{s\in[t,1]} h_s^i \Delta\overline{\zeta}_s^t \Delta\overline{\xi}_s^{i,t}\right).$$

Let $t > \frac{1}{2}$. From (6.24) and (6.25), we know that all subgame strategies become indicator functions for the terminal time. As a result, the definition of the value $V_t^{\Pi^*}$ reduces down to

$$V_t^{\Pi^*}\mathbb{1}_{\{t>\frac{1}{2}\}}(t) = 5\Pi_{t-}^{1,*}\mathbb{1}_{\{t>\frac{1}{2}\}}(t).$$

Usefully, because both minimisers' strategies are constant on the interval $(0,1)$, we can show that the belief process is also constant on $(0,1]$. Let $s > 0$, then $\xi_{s-}^{0,*} = 0$ and $\xi_{s-}^{1,*} = 1 - \frac{3\pi_0}{7\pi_1}$. Moreover, it then follows that

$$\Pi_{s-}^{1,*}\mathbb{1}_{\{s>0\}}(s) = \frac{\pi_1(1-\xi_{s-}^{1,*})}{\sum_{i=0}^{1}\pi_i(1-\xi_{s-}^{i,*})}\mathbb{1}_{\{s>0\}}(s) = \frac{3}{10}\mathbb{1}_{\{s>0\}}(s).$$

Put another way, we can fully describe the belief process as

$$\Pi_{s-}^{1,*} = \pi_1\mathbb{1}_{\{s=0\}}(s) + \frac{3}{10}\mathbb{1}_{\{s>0\}}(s),$$

and so

$$V_t^{\Pi^*}\mathbb{1}_{\{t>\frac{1}{2}\}}(t) = \frac{3}{2}\mathbb{1}_{\{t>\frac{1}{2}\}}(t).$$

An interesting point for the value function is at $t = \frac{1}{2}$, since that is where we know the value must coincide with the peak of $(g_t)_{t\in[0,1]}$. We can verify this too, using the definition of the value and the subgame strategies from above, we see

$$V_{\frac{1}{2}}^{\Pi^*} = g_{\frac{1}{2}}\Delta\zeta_{\frac{1}{2}}^*\Pi_{\frac{1}{2}-}^{0,*} + \left(g_{\frac{1}{2}}\Delta\zeta_{\frac{1}{2}}^* + h_1^1\Delta\zeta_1^*\right)\Pi_{\frac{1}{2}-}^{1,*} = \frac{3}{2}.$$

Next, let $t \in (0, \frac{1}{2})$, then in much the same way we can write

$$V_t^{\Pi^*} = \sum_{i=0}^{1}\Pi_{t-}^{i,*}\left(g_{\frac{1}{2}}\Delta\zeta_{\frac{1}{2}}^* + h_1^i\Delta\zeta_1^*\right) = \frac{3}{2}.$$

Finally, let us compute the value at time zero. Much in the same way as before, we can write

$$V_0^{\Pi^*} = g_{\frac{1}{2}}\Delta\zeta_{\frac{1}{2}}^*\pi_0 + \left(f_0\Delta\xi_0^{1,*} + g_{\frac{1}{2}}(1-\xi_{\frac{1}{2}}^{1,*})\Delta\zeta_{\frac{1}{2}}^* + h_1^1\Delta\xi_1^{1,*}\Delta\zeta_1^*\right)\pi_1 = 4\pi_1 + \frac{3}{7}\pi_0.$$

3° Simply take $t = 0$, and since martingales are trivially constant in this set-up,

$$M_t = M_0 = V_0^{\Pi^*} = 4\pi_1 + \frac{3}{7}\pi_0. \tag{6.29}$$

The values of $(M_t^0(\xi^{0,*},\zeta^*))_{t\in[0,1]}$ and $(M_t^1(\xi^{1,*},\zeta^*))_{t\in[0,1]}$ follows immediately from (6.29) and Lemma 4.3.10.

4° Because of the simplicity of $(\xi_t^{0,*})_{t\in[0,1]}$, we can write $(\mathcal{V}_t^0)_{t\in[0,1]}$ as

$$\mathcal{V}_t^0 = \int_{[t,1)} g_s\, d\overline{\zeta}_s^t = \int_{[t,1)} g_s\, d\zeta_s^* \mathbb{1}_{\{0\le t\le \frac{1}{2}\}}(t) + 0\cdot \mathbb{1}_{\{\frac{1}{2}<t\le 1\}}(t) = \frac{3}{7}\mathbb{1}_{\{0\le t\le \frac{1}{2}\}}(t).$$

Next, recall the definition of $(\mathcal{V}_t^1)_{t\in[0,1]}$,

$$\mathcal{V}_t^1 := \int_{[t,1)} f_s(1-\overline{\zeta}_s^t)\, d\overline{\xi}_s^{1,t} + \int_{[t,1)} g_s(1-\overline{\xi}_s^{1,t})\, d\overline{\zeta}_s^t + \sum_{s\in[t,1]} h_s^1 \Delta\overline{\zeta}_s^t \Delta\overline{\xi}_s^{1,t}.$$

The three main regions for differing subgame strategies are $t = 0$, $t \in (0,\frac{1}{2}]$ and $t \in (\frac{1}{2},1]$. For the first we see

$$\mathcal{V}_0^1 = \int_{[0,1)} f_s(1-\zeta_s^*)\, d\xi_s^{1,*} + \int_{[0,1)} g_s(1-\xi_s^{1,*})\, d\zeta_s^* + \sum_{s\in[0,1]} h_s^1 \Delta\zeta_s^* \Delta\xi_s^{i,*} = 4.$$

For the second region, $(\xi_t^{1,*})_{t\in[0,1]}$ becomes an indicator function for the end of the game, so for $t \in (0,\frac{1}{2}]$ we have

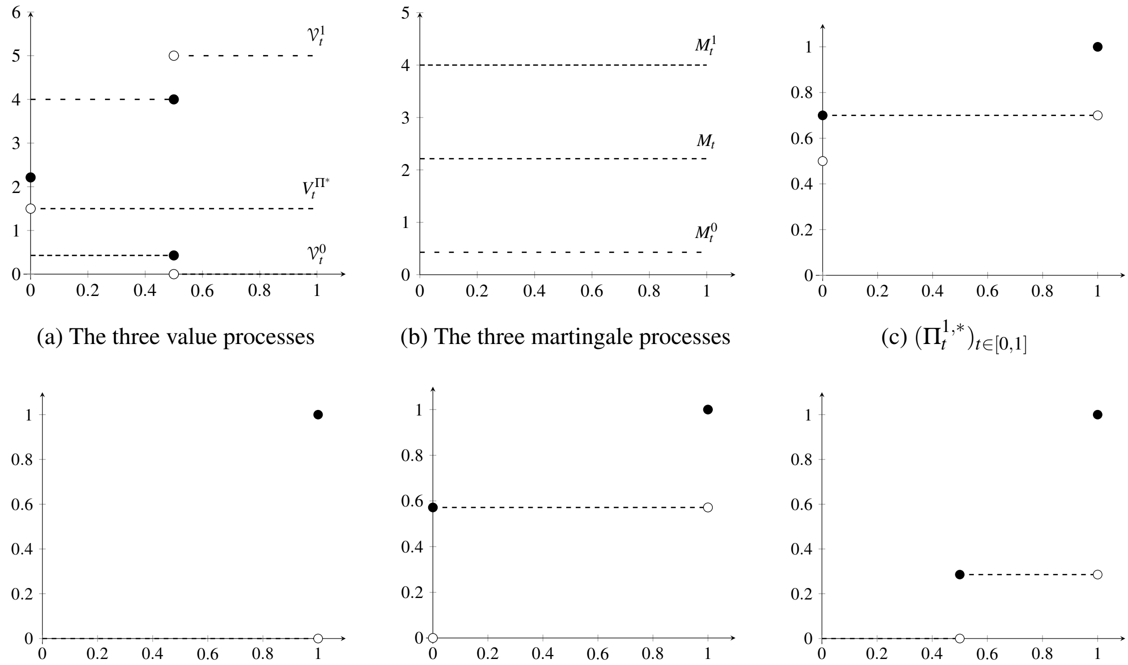
$$\mathcal{V}_t^1 = \int_{[t,1)} g_s\, d\zeta_s^* + h_1^1 \Delta\zeta_1^* = 4.$$

Finally, for the last region, all subgame strategies become waiting for the end of the game, so

$$\mathcal{V}_t^1 = h_1^1 = 5.$$

■

For completeness, we include a sketch below, Figure 6.2, of the optimal strategies, the associated values, and martingales.



(a) The three value processes (b) The three martingale processes (c) $(\Pi_t^{1,*})_{t\in[0,1]}$

(d) $(\xi_t^{0,*})_{t\in[0,1]}$ (e) $(\xi_t^{1,*})_{t\in[0,1]}$ (f) $(\zeta_t^*)_{t\in[0,1]}$

Figure 6.2: Example values, martingales, sub- and super-martingales, and generating processes under $\pi_1 = \frac{1}{2}$.

Before we continue, we can address a small detail that appears in the proof of Theorem 6.1.10, namely that it seems we would be able to let the $\Theta = 1$ incarnation of the minimiser stop at any time $t \geq \frac{3}{5}$ since all of these times are optimal responses to ζ^*. In fact, this is true.

Theorem 6.1.12. *For $\pi_1 \geq \frac{3}{10}$, the pair $(\xi^*, \zeta^*) \in \mathscr{A}(\mathscr{F}_t^1)\mathscr{A}(\mathscr{F}_t^2)$, defined as*

$$\begin{aligned}
\xi_t^{0,*} &= \mathbb{1}_{\{t=1\}}(t), \\
\xi_t^{1,*} &= \left(1 - \frac{3\pi_0}{7\pi_1}\right)\mathbb{1}_{\{t\geq 0\}}(t) + \frac{3\pi_0}{7\pi_1}\mathbb{1}_{\{k\leq t\leq 1\}}(t), \qquad \text{for any } k \in \left[\frac{3}{5}, 1\right] \\
\zeta_t^* &= \frac{2}{7}\mathbb{1}_{\{t\geq\frac{1}{2}\}}(t) + \frac{5}{7}\mathbb{1}_{\{t=1\}}(t),
\end{aligned}$$

is a Nash equilibrium.

The proof of this theorem is identical to that of Theorem 6.1.10, and so we omit it here.

Remark 6.1.13. The only real change in the omitted proof is that equation (6.23) becomes

$$4\pi_1\Delta\xi_0^{1,*} + \sup_{\sigma\in[0,1]}\Big\{g_\sigma\Big(\pi_0+\pi_1(1-\xi_0^{1,*})\Big)\mathbb{1}_{\{0<\sigma<k\}} + 5\pi_1\Delta\xi_k^{1,*}\mathbb{1}_{\{k\le\sigma\le1\}}\Big\}.$$

Given that $\sigma^* = \frac{1}{2}$ attains the supremum, notice the necessity of choosing $\sigma^* = 1$ as the second point in this maximisation despite any choice of $\sigma^* \in [k,1]$ also attaining the supremum. If the maximiser randomised between two points, $t = \frac{1}{2}$ and $t = \sigma^* \in [k,1)$, say, then the obvious best response to this strategy by the informed minimiser in the $\Theta = 1$ case is simply to wait until the end of the game since the maximiser stops first and the payoff is zero. Hence only $\sigma^* = 1$ yields the correct best response argument.

Remark 6.1.14. A final remark is that in the proofs of both Theorem 6.1.10 and 6.1.12, we state that the second infimum is attained by $\tau_1 \in (\frac{1}{2},1]$, but then restrict the value of k to be in the interval $[\frac{3}{5},1]$. The reason for this is simple. Imagine k belonged to $(\frac{1}{2},\frac{3}{5})$, then the maximiser would deduce the following: if they stopped with full intensity at time $t = k$, then, if $\Theta = 1$, they just stopped simultaneously with the minimiser and so the payoff would be $f_k = 5$. If $\Theta = 0$, then they just stopped first and so would receive the payoff g_k. Hence they expect the payoff $5\pi_1 + g_k\pi_0$ which is strictly larger than $5\pi_1$ since $g_k > 0$. This is the best payoff they can achieve. If they were to stop at $\sigma > \frac{3}{5} > k$, then the expected payoff is $5\pi_1$. If they stop before k, then the payoff is at most $\frac{3}{2}(\pi_1+\pi_0+0) = \frac{3}{2}$. If $\sigma \in (k,\frac{3}{5}]$ then the payoff is $5\pi_1 + g_\sigma\pi_0$ which is strictly less than $5\pi_1 + g_k\pi_0$ since $(g_t)_{t\in[0,1]}$ is strictly decreasing on the interval $[\frac{1}{2},\frac{3}{5}]$. This is not the best response $(\zeta_t^*)_{t\in[0,1]}$ that we are looking for, hence why we need to restrict k to the interval $[\frac{3}{5},1]$.

We do not yet have a full picture of this game, since the Nash equilibrium found above is only valid for $\pi_1 \ge \frac{3}{10}$. This value of π_1 will soon be seen as somewhat of a threshold for the minimiser where if the probability that $\Theta = 1$ is 'sufficiently low', i.e. for $\pi_1 < \frac{3}{10}$, then the maximiser will not be swayed by the minimisers randomisation—it is 'sufficiently likely' that the final payoff will be zero.

Because of this, we would expect the maximiser to play it safe and stop fully half way through the game. This is, of course, a logical thing to do as the maximiser is simply stopping at the global maximum of their payoff function. The best way for a minimiser to respond to this is to let the maximiser stop, since $\sup_t g_t < \inf_t f_t$.

Theorem 6.1.15. *For $\pi_1 < \frac{3}{10}$, the pair $(\xi^*, \zeta^*) \in \mathscr{A}(\mathscr{F}_t^1) \times \mathscr{A}(\mathscr{F}_t^2)$, defined as*

$$\xi_t^{0,*} = \xi_t^{1,*} = \mathbb{1}_{\{t=1\}}(t),$$
$$\zeta_t^* = \mathbb{1}_{\{t \geq \frac{1}{2}\}}(t),$$

is a Nash equilibrium.

Proof. Let us assume the maximiser plays $(\zeta_t^*)_{t\in[0,1]}$, then we will show that $(\xi_t^{0,*}, \xi_t^{1,*})$ is optimal. As is now standard, we see that this infimum can be split, and simplified

$$\inf_{(\tau_0,\tau_1)\in[0,1]^2} J_0^{\Pi^*}\Big(\zeta^*, (\tau_0, \tau_1)\Big) = \inf_{(\tau_0,\tau_1)\in[0,1]^2} \pi_i \bigg(f_{\tau_i} \mathbb{1}_{\{\tau_i < \frac{1}{2}\}} + 5\mathbb{1}_{\{\tau_i = \frac{1}{2}\}} + \frac{3}{2}\mathbb{1}_{\{\frac{1}{2} < \tau_i \leq 1\}} \bigg).$$

Both infima are attained by taking $(\tau_0, \tau_1) \in (\frac{1}{2}, 1] \times (\frac{1}{2}, 1]$. So $(\tau_0, \tau_1) = (1,1)$ is a best responses to $(\zeta_t^*)_{t\in[0,1]}$.

Next, we show that $(\zeta_t^*)_{t\in[0,1]}$ is a best response to $(\xi_t^{0,*}, \xi_t^{1,*})$. Our optimisation problem is the same as in (6.22) and in a similar manner, we split this problem along values of σ resulting in the following problem

$$\sup_{\sigma\in[0,1]} \big\{ g_\sigma \mathbb{1}_{\{0<\sigma<1\}} + 5\pi_1 \mathbb{1}_{\{\sigma=1\}} \big\}.$$

Since $\pi_1 < \frac{3}{10}$, it is true that $5\pi_1 < \frac{3}{2} = g_{\frac{1}{2}}$, meaning that taking $\sigma* = \frac{1}{2}$ is the best response to $(\xi^{0,*}, \xi^{1,*})$. ■

Remark 6.1.16. Notice that this proof still holds if we allow $\pi_1 = \frac{3}{10}$. In Section 6.3 we consider an extension of this game set-up and we will discuss why we have two Nash equilibria for this critical point of π_1 (see Corollary 6.3.2 and Remark 6.3.3), but for now we keep the distinction between the regions for π_1 for simplicity.

Corollary 6.1.17. *For $\pi_1 < \frac{3}{10}$, the value process, for $t \in [0, \frac{1}{2}]$, takes the following form*

$$V_t^{\Pi^*} = \frac{3}{2}\mathbb{1}_{\{0\leq t\leq \frac{1}{2}\}}(t) + 5\pi_1 \mathbb{1}_{\{\frac{1}{2}<t\leq 1\}}(t).$$

Proof. All subgame strategies in $[0, \frac{1}{2}]$ remain unchanged from the original strategies, thus,

$$V_t^{\Pi^*} \mathbb{1}_{\{0\leq t\leq \frac{1}{2}\}}(t) = g_{\frac{1}{2}} \mathbb{1}_{\{0\leq t\leq \frac{1}{2}\}}(t) = \frac{3}{2}\mathbb{1}_{\{0\leq t\leq \frac{1}{2}\}}(t).$$

For any subgame starting after $t=\frac{1}{2}$ we see that the strategy of the maximiser, in alignment with Definition 4.1.8, becomes $\mathbb{1}_{\{s=1\}}(s)$. Moreover, due to the simplicity of the minimisers' strategies, the belief process is trivial, $\Pi^{1,*}_{t-}=\pi_1$ for all $t\in[0,1]$. Thus we see that for $t>\frac{1}{2}$,

$$V_t^{\Pi^*}=h_1^1\Pi^{1,*}_{t-}=5\pi_1.$$

■

6.2 VERIFICATION

In this section we will formally verify that the solutions found in Theorem 6.1.10 are valid through the use of Theorem 5.2.2. The main objects needed here are the processes $(M_t(\xi^*,\zeta))_{t\in[0,1]}$, $(M_t^0(\xi^0,\zeta^*))_{t\in[0,1]}$ and $(M_t^1(\xi^1,\zeta^*))_{t\in[0,1]}$ whose formulations are found in Theorem 4.3.4 and Theorem 4.3.9. The remaining strategies from Theorem 6.1.15 for the case of $\pi_1<\frac{3}{10}$ can be done in the same manner.

Verification of the sub- and supermartingale properties is very simple in the deterministic case, as submartingales are non-decreasing functions and supermartingales are non-increasing ones.

Proposition 6.2.1. *For any $\xi^0\in\mathscr{A}(\mathscr{F}_t^2)$, the process $(M_t^0(\xi^0,\zeta^*))_{t\in[0,1]}$ is a submartingale.*

Proof. On the interval $[0,\frac{1}{2}]$ we have

$$M_t^0(\xi^0,\zeta^*)=\frac{3}{7}(1-\xi^0_{t-})+\int_{[0,t)}f_s\,d\xi^0_s. \tag{6.30}$$

Let $s\in[0,\frac{1}{2}]$ and $\varepsilon>0$ be given such that the interval $[s,s+\varepsilon]\subset[0,\frac{1}{2}]$. Then

$$\begin{aligned}M^0_{s+\varepsilon}(\xi^0,\zeta^*)-M^0_s(\xi^0,\zeta^*)&=\frac{3}{7}(\xi^0_{s-}-\xi^0_{(s+\varepsilon)-})+\int_{[s,s+\varepsilon)}f_s\,d\xi^0_s\\&\geq\frac{3}{7}(\xi^0_{s-}-\xi^0_{(s+\varepsilon)-})+4(\xi^0_{(s+\varepsilon)-}-\xi^0_{s-})\\&\geq 0.\end{aligned}$$

Since the interval $[s,s+\varepsilon]$ is arbitrary, we know that $(M_t^0(\xi^i,\zeta^*))_{t\in[0,1]}$ is non-decreasing on $[0,\frac{1}{2}]$.

On $(\frac{1}{2},1]$ we have

$$M_t^0(\xi^0,\zeta) = \int_{[0,t)} f_s(1-\zeta_s^*)\,d\xi_s^0 + \frac{3}{7}(1-\xi_{\frac{1}{2}}^0) + \frac{10}{7}\Delta\xi_{\frac{1}{2}}^0 \tag{6.31}$$
$$= \int_{[0,\frac{1}{2})} f_s\,d\xi_s^0 + \frac{25}{7}(\xi_{t-}^0 - \xi_{\frac{1}{2}-}^0) + \frac{3}{7}(1-\xi_{\frac{1}{2}-}^0) + \Delta\xi_{\frac{1}{2}}^0$$
$$= \frac{25}{7}\xi_{t-}^0 + C,$$

where C is a constant encompassing all non-time-dependent terms. This is clearly non-decreasing.

Finally, consider the jump at time $\frac{1}{2}$. Let $0<\varepsilon\le\frac{1}{2}$ be given, then using (6.30) and (6.31) gives

$$M_{\frac{1}{2}+\varepsilon}^0(\xi^0,\zeta) - M_{\frac{1}{2}}^0(\xi^0,\zeta) = \frac{25}{7}(\xi_{(\frac{1}{2}+\varepsilon)-}^0 - \xi_{\frac{1}{2}-}^0) + \Delta\xi_{\frac{1}{2}}^0 \ge 0.$$

■

Proposition 6.2.2. *For any $\xi^1 \in \mathscr{A}(\mathscr{F}_t^2)$, the process $(M_t^1(\xi^1,\zeta^*))_{t\in[0,1]}$ is a submartingale.*

Proof. Take $0<\varepsilon\le\frac{1}{2}$, then by definition of $(M_t^1(\xi^1,\zeta^*))_{t\in[0,1]}$ and $(\zeta_t^*)_{t\in[0,1]}$ we have

$$M_{\frac{1}{2}+\varepsilon}^1(\xi^1,\zeta^*) = \frac{25}{7}(1-\xi_{(\frac{1}{2}+\varepsilon)-}^1) + \int_{[0,\frac{1}{2})} f_s\,d\xi_s^1 + \frac{25}{7}(\xi_{(\frac{1}{2}+\varepsilon)-}^1 - \xi_{\frac{1}{2}-}^1) + \frac{3}{7}(1-\xi_{\frac{1}{2}}^1) + \frac{10}{7}\Delta\xi_{\frac{1}{2}}^1,$$

which we can easily simplify down to

$$M_{\frac{1}{2}+\varepsilon}^1(\xi^1,\zeta^*) = 4(1-\xi_{\frac{1}{2}-}^1) + \int_{[0,\frac{1}{2})} f_s\,d\xi_s^1 + \Delta\xi_{\frac{1}{2}}^1.$$

This tells us that, for all $t\in(\frac{1}{2},1]$, $(M_t^1(\xi^1,\zeta^*))_{t\in[0,1]}$ is a constant. Additionally, it is simple to show that

$$M_{\frac{1}{2}}^1(\xi^1,\zeta^*) = 4(1-\xi_{\frac{1}{2}-}^1) + \int_{[0,\frac{1}{2})} f_s\,d\xi_s^1,$$

and so clearly, $M_{\frac{1}{2}}^1(\xi^1,\zeta^*) \le M_t^1(\xi^1,\zeta^*)$ for all $t\in(\frac{1}{2},1]$. Moreover, for $t\in[0,\frac{1}{2}]$ we have

$$M_t^1(\xi^i,\zeta^*) = 4(1-\xi_{t-}^1) + \int_{[0,t)} f_s\,d\xi_s^1. \tag{6.32}$$

Let $s \in [0, \frac{1}{2}]$ and $\varepsilon > 0$ be arbitrary such that the interval $[s, s+\varepsilon] \subset [0, \frac{1}{2}]$. Then, using (6.32):

$$\begin{aligned} M^1_{s+\varepsilon}(\xi^i, \zeta^*) - M^1_s(\xi^i, \zeta^*) &= 4(\xi^1_{s-} - \xi^1_{(s+\varepsilon)-}) + \int_{[s,s+\varepsilon)} f_t \, d\xi^1_t \\ &\geq 4(\xi^1_{s-} - \xi^1_{(s+\varepsilon)-}) + 4(\xi^1_{(s+\varepsilon)-} - \xi^1_{s-}) \\ &= 0. \end{aligned}$$

So, again, since the interval $[s, s+\varepsilon]$ is arbitrary, we know that $(M^1_t(\xi^i, \zeta^*))_{t\in[0,1]}$ is non-decreasing on $[0, \frac{1}{2}]$. ■

Due to the similarity of these proofs, we present the following in an abbreviated manner.

Proposition 6.2.3. *For any $\zeta \in \mathscr{A}(\mathscr{F}^2_t)$, the process $(M_t(\xi^*, \zeta))_{t\in[0,1]}$ is a supermartingale.*

Proof. To begin, consider $(M_t(\xi^*, \zeta))_{t\in[0,1]}$ on the interval $(0, 1]$, here we can write it explicitly as

$$M_t(\xi^*, \zeta) = \frac{3\pi_0}{2}(1 - \zeta_{t-}) + \frac{10\pi_0}{7}\int_{[0,t)} g_s \, d\zeta_s + \frac{9\pi_0}{14}(1 - \zeta_{t-}) + f_0\left(\pi_1 - \frac{3\pi_0}{7}\right),$$

Next, rearranging to factor the π_i terms we get

$$M_t(\xi^*, \zeta) = 4\pi_1 + \frac{2}{7}\pi_0\left(\frac{15}{2}(1 - \zeta_{t-}) + 5\int_{[0,t)} g_s \, d\zeta_s - 6\right).$$

First, consider $\varepsilon > 0$, then using the fact that $g_t \leq \frac{3}{2}$ for all $t \in [0, 1]$ we can say

$$M_\varepsilon(\xi^*, \zeta) \leq 4\pi_1 + \frac{2}{7}\pi_0\left(\frac{15}{2}(1 - \zeta_{\varepsilon-}) + \frac{15}{2}(\zeta_{\varepsilon-} - \zeta_{0-}) - 6\right) = 4\pi_1 + \frac{3}{7}\pi_0 = M_0(\xi^*, \zeta).$$

Finally, let $s \in (0, 1)$ and $\varepsilon > 0$ be given such that the interval $[s, s+\varepsilon] \subset (0, 1]$, then

$$M_{s+\varepsilon}(\xi^*, \zeta) - M_s(\xi^*, \zeta) \leq \frac{2}{7}\pi_0\left(\frac{15}{2}(\zeta_{s-} - \zeta_{(s+\varepsilon)-}) + \frac{15}{2}(\zeta_{(s+\varepsilon)-} - \zeta_{s-})\right) = 0.$$

The interval $[s, s+\varepsilon]$ is arbitrary, so $(M_t(\xi^*, \zeta))_{t\in[0,1]}$ is non-increasing on $(0, 1]$. ■

Theorem 6.2.4. *Let (ξ^*, ζ^*) be the processes given in Theorem 6.1.10 and let $(\overline{\xi}^\rho, \overline{\zeta}^\rho)_{\rho\in\mathscr{T}(\mathscr{F}^2_t)}$ be the time-consistent family of strategies found by truncating (ξ^*, ζ^*), then the pair $(\overline{\xi}^\rho, \overline{\zeta}^\rho)$ is a partial subgame-perfect Nash equilibrium.*

Proof. We make use of Theorem 5.2.2; we must verify that its assumptions are fulfilled.

i) Since the processes $(f_t)_{t\in[0,1]}$, $(g_t)_{t\in[0,1]}$, $(h_t^0)_{t\in[0,1]}$, and $(h_t^1)_{t\in[0,1]}$ are bounded, continuous, and deterministic, they are trivially $(\mathscr{F}_t^2)$-progressively measurable and elements of $\mathscr{L}_b$.

ii) The pair (ξ^*, ζ^*) can easily be seen to belong to $\mathscr{A}(\mathscr{F}_t^1) \times \mathscr{A}(\mathscr{F}_t^2)$.

iii) The belief process $(\Pi_t^*)_{t\in[0,1]}$ is defined as per (5.4) and takes the value seen in (6.27).

iv) We must now check that the values $(V_t^{\Pi^*})_{t\in[0,1]}$, $(\mathcal{V}_t^0)_{t\in[0,1]}$ and $(\mathcal{V}_t^1)_{t\in[0,1]}$ found in (6.26) and (6.28) (which are also trivially $(\mathscr{F}_t^2)$-progressively measurable) obey the conditions (5.5)–(5.7). For (5.5), it is obvious that $\mathcal{V}_T^0 = 0 = h^0$ and $\mathcal{V}_T^1 = 5 = h^1$. Equation (5.6) can be verified as
$$\sum_{i=0}^{1} \pi_i \Delta\xi_T^{i,*} V_T^{\Pi^*} = \frac{15\pi_0}{7} = \sum_{i=0}^{1} \pi_i \Delta\xi_T^{i,*} h_T^i,$$
and (5.7) holds because $V_0^{\Pi^*} = 4\pi_1 + \frac{3}{7}\pi_0 = \sum_{i=0}^{1} \pi_i V_0^i$ trivially.

v) Propositions 6.2.1–6.2.3 show that for any $\zeta \in \mathscr{A}(\mathscr{F}_t^2)$, the process $(M_t(\xi^*, \zeta))_{t\in[0,T]}$ is an $(\mathscr{F}_t^2)$-supermartingale, and likewise, for any $i \in [\mathscr{I}]$ and any $\xi^i \in \mathscr{A}(\mathscr{F}_t^2)$, the processes $(M_t^i(\xi^i, \zeta^*))_{t\in[0,T]}$ is an $(\mathscr{F}_t^2)$-submartingale.

Therefore, the conclusion follows by Theorem 5.2.2. ■

6.3 AN EXTENSION OF THE GAME

We finish this chapter with a study of an extension of the game presented in the previous section. Through this we will see how the optimal way to play the game not only depends on the value of π_1, but on the critical regions of the terminal payoffs.

For this we keep the same setting as before, however, instead of fixing $h_1^1 = f_1 = 5$, we allow $h_1^1 = 5 - \delta$ for some $\delta \in [0, 5]$. We will omit many of the proofs here, since they are done identically to those in the previous sections, however, we will share some intuition as to why we make the adjustments we do.

Let us begin by considering $0 \leq \delta \leq 1$. Here, the terminal payoff for stopping together lies between the infimum and supremum of f. The maximiser, as in the section before, should act in such a way as to make the $\Theta = 1$ minimiser indifferent between stopping immediately, and waiting for the end of the game. We can achieve this by adjusting $(\zeta_t^*)_{t\in[0,1]}$ as follows:

$$\zeta_t^* = \left(\frac{2}{7} - \varepsilon\right)\mathbb{1}_{\{t\geq\frac{1}{2}\}}(t) + \left(\frac{5}{7} + \varepsilon\right)\mathbb{1}_{\{t=1\}}(t),$$

where, of course, we expect ε to depend on the value of δ. Using the same best response arguments, we can show that the ε which causes indifference in the $\Theta = 1$ minimiser is

$$\varepsilon = \frac{10\delta}{7(7-2\delta)}, \tag{6.33}$$

leading to a new optimal strategy

$$\zeta_t^* = \frac{2(\delta-1)}{2\delta-7}\mathbb{1}_{\{t\geq\frac{1}{2}\}}(t) + \frac{5}{7-2\delta}\mathbb{1}_{\{t=1\}}(t). \tag{6.34}$$

We can see that when $\delta = 0$ we obtain our original strategy back, and when $\delta = 1$, the strategy becomes $\mathbb{1}_{\{t=1\}}$.

We may do the exact same trick to the $\Theta = 1$ minimiser's strategy to keep the maximiser indifferent between stopping half way through, and waiting until the end. We adjust $(\xi_t^{1,*})_{t\in[0,1]}$ with a factor $\hat{\varepsilon}$ akin to (6.33)

$$\xi_t^{1,*} = \left(1 - \frac{3\pi_0}{7\pi_1} - \hat{\varepsilon}\right)\mathbb{1}_{\{t\geq 0\}}(t) + \left(\frac{3\pi_0}{7\pi_1} + \hat{\varepsilon}\right)\mathbb{1}_{\{t=1\}}(t).$$

In this case $\hat{\varepsilon}$ takes the value

$$\hat{\varepsilon} = \frac{6\delta\pi_0}{7(7-2\delta)\pi_1},$$

resulting in a new optimal strategy

$$\xi_t^{1,*} = \left(1 - \frac{3\pi_0}{\pi_1(7-2\delta)}\right)\mathbb{1}_{\{t\geq 0\}}(t) + \frac{3\pi_0}{\pi_1(7-2\delta)}\mathbb{1}_{\{t=1\}}(t). \tag{6.35}$$

This strategy is only admissible if

$$\frac{3\pi_0}{\pi_1(7-2\delta)} \leq 1,$$

or, after some algebra, we see that $(\xi_t^{1,*})_{t\in[0,1]}$ is admissible for

$$\frac{3}{2(5-\delta)} \leq \pi_1 \leq 1, \quad \delta \in [0,1]. \tag{6.36}$$

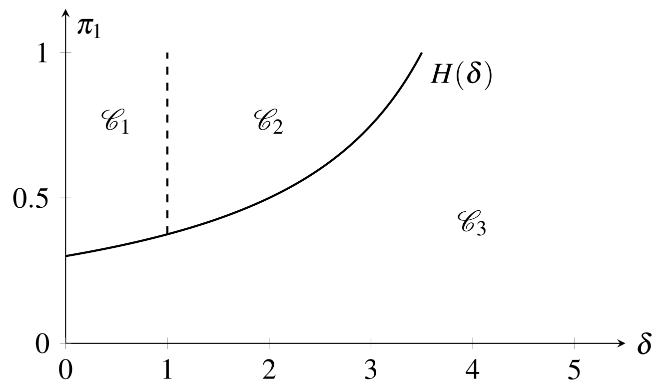
This is sensible, since for $\delta = 0$ we regain our original range for randomisation, $\pi_1 \geq \frac{3}{10}$.

The results here hint at a space such that $(\delta, \pi_1) \in [0,5] \times [0,1]$, on which the game's optimal strategies are different. We already know that for $\delta = 0$, our original randomised strategies hold for $\pi_1 \geq \frac{3}{10}$ (see Theorem 6.1.10), and for $\pi_1 < \frac{3}{10}$ the minimisers wait until the end of the game and the maximiser stops halfway through (see Theorem 6.1.15). Moreover, there is a randomisation region at least as big as that described in (6.36). This region is bounded by the hyperbola

$$H(\delta) = \frac{3}{2(5-\delta)}.$$

which intersects the π_1 axis at $\pi_1 = \frac{3}{10}$ and increases until $\delta = \frac{7}{2}$ with $\pi_1 = 1$. Our logic then splits this plane into three regions defined as

$$\begin{aligned}
\mathscr{C}_1 &:= \{(\delta, \pi_1) : 0 \leq \delta \leq 1, H(\delta) \leq \pi_1 \leq 1\}, \\
\mathscr{C}_2 &:= \left\{(\delta, \pi_1) : 1 < \delta \leq \frac{7}{2}, H(\delta) \leq \pi_1 \leq 1\right\}, \\
\mathscr{C}_3 &:= \{(\delta, \pi_1) : 0 \leq \delta \leq 5, 0 \leq \pi_1 < H(\delta)\}.
\end{aligned}$$



We already know that the strategies given in (6.34) and (6.35) are optimal in $\mathscr{C}_1$, and so we only have the remaining two sections to consider.

Theorem 6.3.1. *For $(\delta, \pi_1) \in \mathscr{C}_2$, the optimal strategies are*

$$\xi_t^{i,*} = \zeta_t^* = \mathbb{1}_{\{t=1\}}(t), \quad i \in \{0,1\},$$

and for $(\delta, \pi_1) \in \mathscr{C}_3$, the optimal strategies are

$$\xi_t^{i,*} = \mathbb{1}_{\{t=1\}}(t), \quad i \in \{0,1\}, \tag{6.37}$$

$$\zeta_t^{*} = \mathbb{1}_{\{t \geq \frac{1}{2}\}}(t). \tag{6.38}$$

Proof. The proof of this corollary is near-identical to the proof of Theorem 6.1.15. ■

Originally, for $\delta = 0$, we found a second Nash equilibrium when $\pi_1 < \frac{3}{10}$, but what Theorem 6.3.1 tells us is that, in fact, for any choice of (δ, π_1) below the hyperbola the game is trivially non-random and the optimal way to play is for the minimiser to wait until the end, and the maximiser to stop fully at time one half. Essentially, in $\mathscr{C}_3$, either the terminal payoff is too low, the probability of $\Theta = 1$ is too low, or both simultaneously, meaning the game is simply not worth playing for the maximiser, so they stop at the maximum point of their payoff function.

The hyperbola, $H(\delta)$, is the 'indifference curve', and on it the maximiser can either stop at the end of the game or at time one half and receive the same payoff. Moreover, despite the Nash equilibria in $\mathscr{C}_1$ and $\mathscr{C}_2$ holding on $H(\delta)$, the optimal strategies in the $\mathscr{C}_3$ region are also Nash equilibria on the hyperbola.

Corollary 6.3.2. *For any pair (δ, π_1) such that $\pi_1 = H(\delta)$, then the pair of strategies (6.37) and (6.38) is a Nash equilibrium.*

Remark 6.3.3. Consider the proof of Theorem 6.1.15. If π_1 is allowed to be equal to $\frac{3}{10}$, then the result still holds, we just end up with two values of σ^*, namely $\sigma^* = \frac{1}{2}$ as before, and also $\sigma^* = 1$, hence choosing $\sigma^* = \frac{1}{2}$ is still optimal. We keep the strict bound on π_1 in the statement of Theorem 6.1.15 so as to not induce confusion.

CHAPTER 7

CONCLUSION

Our goal for this work was to develop a martingale theory for, and characterisation of, a class of Dynkin games with asymmetric information. Since many of the preceding chapters require a lot of technical work to develop these results, we look to provide a summary of our findings here as well as to highlight potential avenues for future work.

7.1 KEY CONTRIBUTIONS

We begin with our contributions to Dynkin games with full information, studied in Chapter 2. These games provide a foundational insight into the game studied from Chapter 3 onwards, and our work in the area extends far beyond a simple translation into English of the work of Lepeltier and Maingueneau, [LM84].

As stated in Chapter 2, we found it necessary to make alterations to the proof methods of a suite of theorems as we were unable to justify the original authors' arguments. The most notable modifications are found in Theorems 2.3.12 and 2.5.1, proving the semicontinuity in expectation over stopping times of the conditional value processes, and the existence of a Nash equilibrium respectively. The former result is based on Theorem 2.2.10 which connects the concept of previsible left semicontinuity to semicontinuity in expectation over stopping times. We found that Theorem 2.2.10 requires semicontinuity in expectation over stopping times from both the right *and the left*, contrasting with [LM84] which states only that it is required from the right. We provide the results required to apply Theorem 2.2.10, but this is emblematic of the wider work we did on the study of path regularity. In this work we have provided a rigorous analysis of such technical details and demonstrated how they alter the original methods.

The proof presented for Theorem 2.5.1 in this work is substantially different from that given in [LM84] as we were unable to justify one argument made in the original proof. The new method we chose to use is based on the fundamental ideas of optimal stopping laid out in Chapter 1. The result of our work becomes evident when cross referencing Chapter 2 with [LM84]; we provide a valuable supplement to the original work, expanding on important areas left undiscussed, and providing additional justifications for the results.

Finally, before moving onto our main focus in this thesis, we highlight Section 2.6 on the minimality of optimal strategies in games with full information. This section is novel work and is not found in Lepeltier and Maingueneau's paper. Inspired by the classical minimality results in single-agent optimal stopping, we provide two concepts of minimality for the Nash equilibrium found in Theorem 2.5.1.

Part II of this work contains our main contributions: the characterisation of our game via the establishment of a set of necessary and sufficient conditions for the existence of a value and a Nash equilibrium. Prior to this work we knew the existence of both a value and a Nash equilibrium in our game from the work done by De Angelis, Merkulov and Palczewski, [DAMP22], however, their methods were non-constructive, giving us no insight into the structure of the optimal strategies or the value.

An important contribution we make in this work is to demonstrate the role of the families $\{M(\rho) : \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$, $\{M(\rho;\xi^*,\zeta), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$, and $\{M^i(\rho;\xi^i,\zeta^*), \rho \in \mathscr{T}(\mathscr{F}_t^2)\}$. The classical results describe the behaviour of the value process itself, i.e. it behaves like a submartingale (resp. supermartingale) up until the maximiser's (resp. minimiser's) optimal stopping time (see

Theorem 2.4.4). We instead describe the existence of the afformentioned families, constructed from the value and encapsulating the payoff of the game, which themselves are (sub-/super-) martingales. The deviation from the standard approach of directly characterising the value to one which instead constructs these three distinct families of random variables is what allows us to characterise our game, and moreover differentiates our work from the existing literature.

In Section 4.2 we consider Nash equilibria in subgames. Akin to the use of dynamic programming in the single-agent optimal control problem, the use of subgames gives us insight into the optimal strategies for the game. Since our game is only ever played at time zero, the idea of subgames is theoretical and contingent on the original (time-zero game) still being underway. This idea explains the prevalence of sets such as Ω_ρ in our work, and the requirement for us to provide a concept of a 'partial' subgame perfect Nash equilibrium. However, we prove that in such subgames, the optimal way to play is to use a sufficiently scaled (i.e. 'truncated') version of the optimal strategy at time zero.

While the understanding of how our optimal strategies remain optimal in subgames gives us some insight into their structure, we go further by characterising them in Section 4.4. The results in Theorems 4.4.7 and 4.4.13 describe when the two players act by relating the value to the payoff processes. Since we are dealing with randomised stopping times (for which we have described a concept of 'gradual stopping'), we cannot formulate the optimal strategies as first hitting times like we can in the case of full information, but our results still provide this connection to the times at which the value coincides with the payoff processes.

This concludes the brief summary of the key contributions for our necessary conditions, however, we also prove in Chapter 5 that these conditions are sufficient. Specifically, we provide two distinctly different verification theorems, both requiring different martingality conditions and different assumptions on the explicit form (or lack thereof) of our value process candidates. The utility of Chapter 5 is demonstrated in Chapter 6 whereby we begin with an example found in Section 6.2 of [DAMP22], utilise the necessary conditions of Chapter 4 to deduce our candidates for the optimal strategies, and then apply our verification result from Chapter 5 to conclude that they are in fact optimal. In this way we demonstrate all that we have contributed in this work; the example was known to have a value and a Nash equilibrium from [DAMP22], however, the theory that we have developed in this work now allows us to not just characterise the optimal strategies, but directly construct and verify them in an explicit example.

7.2 FUTURE WORK

We would be remiss not to discuss the future directions in which this research can go. The immediate next step is to generalise the results into the setting found in Section 3.2. Our game is concerned with information asymmetry generated by an exogenous random variable Θ which is known to only one player, however, such games need not be restricted in this way. Instead we may use two σ-algebras, $(\mathscr{F}_t^1)_{t\in[0,T]}$ and $(\mathscr{F}_t^2)_{t\in[0,T]}$—one for each player—and the relationship between them and the wider σ-algebra $(\mathscr{F}_t)_{t\in[0,T]}$ on the probability space to provide not just information asymmetry, but also to introduce partial information, i.e. where neither of the players are fully informed. This is the setting of the paper by De Angelis, Merkulov and Palczewski, [DAMP22], and since this more general game formulation encompasses most, if not all of the games found in the surrounding literature, the extension of our work to this setting would provide very powerful results.

The second direction of work could be into concepts of minimality for optimal strategies. As already mentioned, we have considered this problem in the setting of non-Markovian games with full information in Chapter 2 wherein we provide two concepts of a minimal Nash equilibrium. Of course, such games use stopping times as strategies, and so the extension to randomised strategies provides a non-trivial problem.

BIBLIOGRAPHY

[AN82] M. Alario-Nazaret. Jeux de Dynkin. PhD thesis. l'Universitéde Franche-Comté, 1982.

[ANLM82] M. Alario-Nazaret, J.P. Lepeltier, and B. Marchal. *Dynkin Games*. Stochastic Differential Systems. Springer, 1982.

[Ale24] P. Alexandroff. Über die Metrisation der im Kleinen kompakten topologischen Räume. *Mathematische Annalen* 92 (1924), pp. 294–301.

[AM95] R.J. Auman and M.B. Maschler, With the collaboration of Richard E. Stearns. *Repeated Games with Incomplete Information*. MIT Press, 1995.

[Aum64] R.J. Aumann. *Advances in Game Theory*. Princeton University Press, 1964.

[Bal17] P. Baldi. *Stochastic Calculus. An Introdution Through Theory and Exercises*. Springer, 2017.

[BB19] P. Bank and D. Besslich. On Lenglart's theory of Meyer-sigma-fields and El Karoui's theory of optimal stopping. *arXiv: 1810.08485v2* (2019).

[BFR23] L. Baňas, G. Ferrari, and T.A. Randrianasolo. Numerical approximation of Dynkin Games with Asymmetric Information. *arXiv: 2312.01847v1* (2023).

[Bel57] R. Bellman. *Dynamic Programming*. Princeton University Press, 1957.

[Bel61] R. Bellman. *Adaptive Control Processes*. Princeton University Press, 1961.

[BF74] A. Bensoussan and A. Friedman. Nonlinear variational inequalities and differential games with stopping times. *Journal of Functional Analysis* 16:3 (1974), pp. 305–352.

[BF77] A. Bensoussan and A. Friedman. Nonzero-sum stochastic differential games with stopping times and free boundary problems. *Transactions of the American Mathematical Society* 231:2 (1977), pp. 275–327.

[Ber76] D.P. Bertsekas. *Dynamic Programming and Stochastic Control*. Academic Press, 1976.

[Bis77a] J.M. Bismut. Dualité convexe, temps d'arrêt optimal et contrôle stochastique. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 38 (1977), pp. 169–198.

[Bis77b] J.M. Bismut. Sur un problème de Dynkin. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 39 (1977), pp. 31–53.

[Bis77c] J.M. Bismut. Temps d'arrêt optimal et quasi-temps d'arrêt. *Comptes rendus de l'Académie des Sciences*. A 284:23 (1977), pp. 1519–1521.

[Bis79a] J.M. Bismut. Contrôle de processus alternants et applications. *Z. Wahrscheinlichkeitstheorie* 47 (1979), pp. 241–288.

[Bis79b] J.M. Bismut. Temps d'arrêt optimal, quasi-temps d'arrêt et retournement du temps. *The Annals of Probability* 7:6 (1979), pp. 933–964.

[BS97] J.M. Bismut and B. Skalli. Temps d'arrêt optimal, théorie générale des processus et processus de Markov. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 39 (1997), pp. 301–313.

[Bre11] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 2011.

[Car09] P. Cardaliaguet. A double obstacle problem arising in differential game theory. *Journal of Mathematical Analysis and Applications* 360 (2009), pp. 95–107.

[CR09a] P. Cardaliaguet and C. Rainer. Stochastic differential games with asymmetric information. *Applied Mathematics and Optimization* 59:1 (2009), pp. 1–36.

[CR09b] Pierre Cardaliaguet and Catherine Rainer. On a Continuous-Time Game with Incomplete Information. *Mathematics of Operations Research* 34:4 (2009), pp. 769–794.

[CB00] M. Carter and B. van Brunt. *The Lebesgue-Stieltjes Integral*. Springer, 2000.

[CJ01] P. Chalasani and S. Jha. Randomized stopping times and American option pricing with transaction costs. *Mathematical Finance* 11:1 (2001), pp. 33–77.

[Coh80] D.L. Cohn. *Measure Theory*. Birkhäuser, 1980.

[Cou97] A. Cournot. *Researches into the Mathematical Principles of the Theory of Wealth*. The Macmillain Company, 1897.

[CIL92] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society* 27 (1992), pp. 1–67.

[DAEG21] T. De Angelis, E. Ekström, and K. Glover. Dynkin games with incomplete and asymmetric information. *Mathematics of Operations Research* 47:1 (2021), pp. 560–586.

[DAGV21] T. De Angelis, F. Gensbittel, and S. Villeneuve. A Dynkin game on assets with incomplete information on the return. *Mathematics of Operations Research* 46:1 (2021), pp. 28–60.

[DAMP22] T. De Angelis, N. Merkulov, and J. Palczewski. On the value of non-Markovian Dynkin games with partial and asymmetric information. *The Annals of Applied Probability* 32:3 (2022), pp. 1774–1813.

[DL82] C. Dellacherie and E. Lenglart. Sur des problèmes de régularisation, de recollement et d'interpolation en théorie des processus. *Séminaire de probibilités XVI 1980/81*. Ed. by J. Azéma, M. Yor, A. Dold, and B. Eckmann. Vol. 920. Institut de Mathématiques, Université de Strasbourg. Springer, 1982.

[DM78] C. Dellacherie and P-.A. Meyer. *Probabilities and Potential A*. Mathematical Studies 29. North Holland, 1978.

[DM82] C. Dellacherie and P-.A. Meyer. *Probabilities and Potential B*. Mathematical Studies 72. North Holland, 1982.

[Dol10] Yan Dolinsky. Applications of weak convergence for hedging of game options. *The Annals of Applied Probability* 20:5 (2010), pp. 1891 –1906. DOI: 10.1214/09-AAP675. URL: https://doi.org/10.1214/09-AAP675.

[Dyn68] E.B. Dynkin. A game-theoretic version of an optimal stopping problem. *Doklady Akademii Nauk SSSR* 185:1 (1968), pp. 16–19.

[EGL17] E. Ekström, K. Glover, and M. Leniec. Dynkin games with heterogenous beliefs. *Journal of Applied Probability* 54:1 (2017), pp. 236–251.

[EP08] E. Ekström and G. Peskir. Optimal stopping games for Markov processes. *SIAM Journal on Control and Optimization* 47:2 (2008), pp. 684–702.

[EK79] N. El Karoui. Les Aspects Probabilistes du Contrôle Stochastique. *Ecole d'Eté de Probabilités de Saint-Flour IX*. Ed. by P.L. Hannequin, A. Dold, and B. Eckmann. Vol. 876. Lecture Notes in Mathematics. Springer, 1979.

[Eng89] R. Engelking. *General Topology*. Heldermann Verlag Berlin, 1989.

[EI18] N. Esmaeeli and P. Imkeller. American options with asymmetric information and reflected BSDE. *Bernoulli* 24:2 (2018), pp. 1394–1426.

[Far17] F. A. Faruqi. *Differential Game Theory and Applications to Missile Guidance*. John Wiley and Sons, 2017.

[FR75] W. Flemming and R. Rishel. *Deterministic and Stochastic Optimal Control*. Springer New York, 1975.

[GG18] F. Gensbittel and C. Grün. Zero-sum stopping games with asymmetric information. *Mathematics of Operations Research* 44:1 (2018), pp. 277–302.

[Grü13] C. Grün. On Dynkin games with incomplete information. *SIAM Journal on Control and Optimization* 51 (2013), pp. 4039–4065.

[Isa51] R. Isaacs. *Games of Pursuit*. Santa Monica, CA: RAND Corporation, 1951.

[Isa65] R. Isaacs. *Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization*. John Wiley and Sons, 1965.

[KS98a] I. Karatzas and S.E. Shreve. *Brownian Motion and Stochastic Calculus*. Springer, 1998.

[KS98b] I. Karatzas and S.E. Shreve. *Methods of Mathematical Finance*. Springer, 1998.

[Kif71] Y. Kifer. Optimal stopping in games with continuous time. *Theory of Probability and its Applications* 16:3 (1971).

[Kif00] Y. Kifer. Game options. *Finance and Stochastics* 4 (2000), pp. 443–463.

[Kif13] Y. Kifer. Dynkin's Games and Israeli Options. *International Scholarly Research Notices* 2013:1 (2013), p. 856458.

[KF09] D. Koller and N. Friedman. *Probabilistic Graphical Models: Principles and Techniques*. MIT Press, 2009.

[KF75] A.N. Kolmogorov and S.V. Fomin. *Introductory Real Analysis*. Dover, 1975.

[Kom88] H. Komiya. Elementary proof for Sion's minimax theorem. *Kodai Mathematical Journal* 11:1 (1988), pp. 5–7.

[Lam09] D. Lamberton. *Optimal stopping and American options*. 2009.

[LS05] R. Laraki and E. Solan. The value of zero-sum stopping games in continuous times. *SIAM Journal on Control and Optimization* 43:5 (2005), pp. 1913–1922.

[LM13] Jukka Lempa and Pekka Matomäki. A Dynkin game with asymmetric information. *Stochastics* 85:5 (2013), pp. 763–788.

[LM84] J.P. Lepeltier and M.A. Maingueneau. Le jeu de Dynkin en théorie générale sans l'hypothèse de Mokobodzki. *Stochastics* 13 (1984), pp. 25–44.

[Mai78] M.A. Maingueneau. Temps d'arrêt optimaux et théorie générale. *Séminaire de probibilités XII, Univeristé de Strasbourg 1976/77*. Ed. by C. Dellacherie, P-.A. Meyer, A. Dold, and B. Eckmann. Vol. 649. Institut de Mathématiques, Université de Strasbourg. Springer, 1978.

[MSZ13] M. Maschler, E. Solan, and S. Zamir. *Game Theory*. Cambridge University Press, 2013.

[Mey76] P-.A. Meyer. Un cours sur les intégrales stochastiques. *Séminaire de probibilités X*. Ed. by A. Dold and B. Eckmann. Vol. 511. Institut de Mathématiques, Université de Strasbourg. Springer, 1976.

[Mok78] G. Mokobodzki. Sur l'operateur de reduite Remarques sur un travail de J.M. Bismut. *Séminaire de Théorie du Potentiel, Paris, 1976-1977, No. 3*. Ed. by F. Hirsch, G. Mokobodzki, A. Dold, and B. Eckmann. Vol. 681. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 1978.

[Nas51] J. Nash. Non-cooperative games. *Annals of Mathematics* 54:2 (1951), pp. 286–295.

[NN13] P. Naumov and B. Nicholls. On interchangeability of Nash equilibria in multi-player strategic games. *Synthese* 190 (2013), pp. 57–78.

[Nev75] J. Neveu. *Discrete-Parameter Martingales*. North Holland, 1975.

[PS06] G. Peskir and A. Shiryaev. *Optimal Stopping and Free-Boundary Problems*. Birkhäuser, 2006.

[Pro04] P.E. Protter. *Stochastic Integration and Differential Equations*. Second. Springer, 2004.

[RY05] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Springer, 2005.

[RS17] F. Riedel and J-H. Steg. Subgame-perfect equilibria in stochastic timing games. *Journal of Mathematical Economics* 72 (2017), pp. 36–50.

[RW94] L.C.G. Rogers and D. Williams. *Diffusions, Markov Processes and Martingales Volume 1: Foundations*. 1994.

[RW00] L.C.G. Rogers and D. Williams. *Diffusions, Markov Processes and Martingales Volume 2: Ito Calculus*. Cambridge University Press, 2000.

[Roo25] C. F. Roos. A mathematical theory of competition. *American Journal of Mathematics* 47:3 (1925), pp. 163–175.

[Ros78] J.S. Rose. *A Course on Group Theory*. Cambridge University Press, 1978.

[Rud87] W. Rudin. *Real and Complex Analysis*. Third. McGraw-Hill, 1987.

[Sio58] M. Sion. On general minimax theorem. *Pacific Journal of Mathematics* 8:1 (1958), pp. 171–176.

[Sne52] J. L. Snell. Applications of martingale system theorems. *Transactions of the American Mathematical Society* 73:2 (1952), pp. 293–312.

[STV12] E. Solan, B. Tsirelson, and N. Vieille. Random Stopping Times in Stopping Problems and Stopping Games. *arXiv:1211.5802v1* (2012).

[Ste82a] Ł. Stettner. On a general zero-sum stochastic game with optimal stopping. *Probability and Mathematical Statistics* 3:1 (1982), pp. 103–112.

[Ste82b] Ł. Stettner. Zero-sum Markov games with stopping and impulsive strategies. *Applied Mathematics and Optimization* 9 (1982), pp. 1–24.

[Ste84] Ł. Stettner. On closedness of general zero-sum stopping game. *Bulletin of the Polish Academy of Sciences Mathematics* 32:5–6 (1984), pp. 351–361.

[TV02] N. Touzi and N. Vieille. Continuous-time Dynkin games with mixed strategies. *SIAM Journal on Control and Optimization* 41 (2002), pp. 1073–1088.

[VNM44] J. Von Neumann and O. Morgenstern. *Theory of Games and Economic Behaviour*. Princeton University Press, 1944.

[Xi+23] A. Xi, Y. Cai, Y. Deng, and H. Jiang. Zero-sum differential game guidance law for missile interception engagement via neuro-dynamic programming. *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering* 237:14 (2023), pp. 3352–3366.

[Yan+15] Huiwen Yan, Fahuai Yi, Zhou Yang, and Gechun Liang. Dynkin game of convertible bonds and their optimal strategy. *Journal of Mathematical Analysis and Applications* 426:1 (2015), pp. 64–88.

[Yas85] M. Yasuda. On randomized strategy in Neveu's stopping problem. *Stochastic Processes and their Applications* 21:1 (1985), pp. 159–166.

APPENDICES

APPENDIX A

LEBESGUE-STIELTJES THEORY

The following appendix contains some technical results relating to Lebesgue-Stieltjes theory, specifically with regard to integration theory. The act of 'integrating out the randomisation device' using Lemma 3.2.4 leaves us with integrals such as those found in (3.11) and we are concerned with some properties of these integrals which make them easier to work with. This appendix is not designed to give an understanding of the wider field of Lebesgue-Stieltjes theory, rather it assumes a knowledge of said theory along with measure theory.

Section A.1 covers a brief discussion on the equivalence of measure induced by both a generating process, and that generating process truncated at some $(\mathscr{F}_t^2)$-stopping time ρ. Section A.2 then covers two topics: one, that we can freely convert between summations of the form

$\sum_{t\in[0,T]} h_t \Delta\zeta_t \Delta\xi_t$ and integrals of the form $\int_{[0,T]} h_t \Delta\zeta_t \, d\xi_t$; second that integrals with respect to a truncated generating process can be, in some sense, 'untruncated' into integrals with respect to the original generating process. Finally, Section A.3 introduces generalised inverses and time changes. It is in this section where we find our 'change of variable' theorem which is highly prevalent throughout this work.

A.1 ON THE EQUIVALENCE OF MEASURES

To begin, we present a corollary from Dynkin's π-λ theorem which provides a way of proving the equivalence of two finite measures. Recall that a π-system is a class of sets $\mathscr{C}$ which is stable under finite intersections.

Corollary A.1.1. *(Corollary 1.6.2, [Coh80]) Let $(X,\mathscr{A})$ be a measurable space, and let $\mathscr{C}$ be a π-system on X such that $\mathscr{A}=\sigma(\mathscr{C})$. If μ and ν are finite measures on $\mathscr{A}$ that satisfy $\mu(X)=\nu(X)$ and also satisfy $\mu(C)=\nu(C)$ for each C in $\mathscr{C}$, then $\mu=\nu$.*

We define the class $\mathscr{C}_0 := \{(a,b] : 0\le a\le b\le T\}$; clearly it is a π-system. If we take $(X,\mathscr{A}) = ([0,T],\mathscr{B}[0,T])$, then our π-system $\mathscr{C}_0$ does in fact generate the Borel σ-algebra on $[0,T]$.

Let us take a generating process $\chi \in \mathscr{A}(\mathscr{F}_t^2)$ and a stopping time $\rho \in \mathscr{T}(\mathscr{F}_t^2)$, and assume that $\chi_{\rho-}<1$, then we know that the truncated process takes the form

$$\chi_t^\rho = \frac{\chi_t-\chi_{\rho-}}{1-\chi_{\rho-}}\mathbb{1}_{[\rho,T]}(t).$$

Denote by μ_{χ^ρ} the measure induced by $(\chi_t^\rho)_{t\in[0,T]}$. We can then consider $\mu_{\chi^\rho}(a,b]$. Trivially, if $a\le b<\rho$ then $\mu_{\chi^\rho}(a,b]=0$, likewise if $a<\rho\le b$ then $\mu_{\chi^\rho}(a,b]=\chi_b^\rho$. In the case when $\rho\le a\le b$ we have something more interesting:

$$\mu_{\chi^\rho}(a,b] = \frac{\chi_b-\chi_a}{1-\chi_{\rho-}} = \frac{\mu_\chi(a,b]}{1-\chi_{\rho-}}.$$

This leads us to the conclusion that we have equivalence of measures on a subspace, specifically $([\rho,T],\mathscr{B}[\rho,T])$. Notice that the measures we have constructed so far are fine to be constructed on a subspace, this is due to Carathéoory's extension theorem.

Theorem A.1.2. *Let $([\rho,T],\mathscr{B}[\rho,T])$ be a measurable space. Let $\mu:=\mu_\chi$ and $\nu:=(1-\chi_{\rho-})\mu_{\chi^\rho}$, then $\mu=\nu$.*

Proof. Let $\mathscr{C} = \{(a,b] : \rho \le a \le b \le T\}$, then $\mathscr{C}$ is a π-system such that $\mathscr{B}[\rho,T] = \sigma(\mathscr{C})$. Both measures μ and ν are finite and they agree on the measure of the space by definition, namely $\nu[\rho,T] = (1-\chi_{\rho-})\mu_{\chi^\rho}[\rho,T] = \mu[\rho,T] = 1-\chi_{\rho-}$, which is finite by assumption. Next, take any $(a,b] \in \mathscr{C}$ and we see that $\nu(a,b] = (1-\chi_{\rho-})\mu_{\chi^\rho}(a,b] = \mu(a,b]$. Thus the result follows from Corollary A.1.1. ■

Theorem A.1.2 tells us what may seem intuitively obvious, that when we induce a measure from a scaled function, it is a scaled version of the measure induced by the original function.

A.2 INTEGRATION

A common trick that we use during this work is switching integrals of the form $\int_{[0,T]} h_t \Delta\zeta_t \, d\xi_t$ into summations such as $\sum_{s\in[0,T]} h_t \Delta\zeta_t \Delta\xi_t$. The first question is whether or not this sum is even well defined, it would seem that we could have an uncountable number of terms here, however, we have a simple result that allows us to ensure this sum is actually countable.

Theorem A.2.1. *(Theorem 31.3, [KF75]) A non-decreasing function can have no more than countably many points of discontinuity.*

Given an integral like $\int_{[0,T]} h_t \Delta\zeta_t \, d\xi$, we know that $(\zeta_t)_{t\in[0,T]}$, being a generating process, can only have a countable number of jump points, meaning $\Delta\zeta_t > 0$ on some countable subset $A \subset [0,T]$. Thus we could write $\int_A h_t \Delta\zeta_t \, d\xi$. In order to see that this integral becomes a sum we need to consider the properties of the measure induced by $(\xi_t)_{t\in[0,T]}$.

The typical approach here is to use the Lebesgue decomposition of a measure with respect to another measure. This, however, will require us to grapple with the concepts of 'singular' measures, which is beyond the scope of this appendix. However, by decomposing our generating function and inducing a measure from this decomposition we can attain the same results. This is not the typical way of formulating the Lebesgue decomposition—since that holds much more generally for any two finite measures—but in our case it gives a more intuitive understanding.

Theorem A.2.2. *(Theorem 31.5 and Problem 31.9, [KF75]) Let f be a non-decreasing function, then f is the sum of a continuous, non-decreasing function φ and a jump function ψ.*

Remark A.2.3. Since this result does not come directly from Kolmogorov and Fomin, i.e. it needs proving through Problem 31.9, the solution is to define the jump process ψ as follows. Let

$(t_n)_{n\in\mathbb{N}}$ be a countable family of points of discontinuity of f, guaranteed by Theorem A.2.1, and $h_n := f_{t_n+} - f_{t_n}$ and $h'_n := f_{t_n} - f_{t_n-}$. The notation of $f_{t_n\pm}$ corresponds to the left (-) and right (+) limits of f. The jump function is then defined as

$$\psi_t = \sum_{t_n<t} h_n + \sum_{t_n\leq t} h'_n.$$

The jump function ψ simplifies for our case since $(\xi_t)_{t\in[0,T]}$ is right-continuous—meaning $h_n = 0$—so the generating process $(\xi_t)_{t\in[0,T]}$ can by decomposed as follows

$$\xi_t = \varphi_t + \sum_{t_n\leq t} \Delta\xi_{t_n},$$

where φ_t is a continuous, non-decreasing function. For ease of notation later, denote by D_ξ, the set of all points of discontinuity of $(\xi_t)_{t\in[0,T]}$.

The parallel here is that we will be able to decompose the measure μ_ξ into the sum of a continuous and a discrete measure. The following, again, comes from Kolmogorov and Fomin [KF75], but they use charges (i.e. signed measures); we restate their definitions here simply for measures on a Lebesgue measure space. A measure μ is said to be concentrated on a set $A \in \mathscr{A}$ if $\mu(E) = 0$ for every measurable set $E \subset X \setminus A$. A measure is continuous if $\mu(E) = 0$ for every single element set $E \subset X$ of Lebesgue measure zero, and a measure is discrete if μ is concentrated on a finite or countable set with Lebesgue measure zero.

Theorem A.2.4. *Let $\psi := \sum_{t_n\leq t} \Delta\xi_{t_n}$ and let μ_ψ be the measure induced by ψ, then $\mu_\psi(B) = \sum_{t_n\in B} \Delta\xi_{t_n}$ for any $B \in \mathscr{B}[0,T]$.*

Proof. Denote $\nu(B) := \sum_{t_n\in B} \Delta\xi_{t_n}$. Begin with the measure of the whole space, $\mu_\psi[0,T] = \psi_T - \psi_{0-} = \sum_{t_n\leq T} \Delta\xi_{t_n}$, which is true because $\sum_{t_n\leq 0-\varepsilon} \Delta\xi_{t_n} = 0$ for all $\varepsilon > 0$. Thus we clearly have $\nu[0,T] = \mu_\psi[0,T]$.

Next, let $a \in [0,T]$, then

$$\mu_\psi(a,T] = \sum_{t_n\leq T} \Delta\xi_{t_n} - \sum_{t_n\leq a} \Delta\xi_{t_n} = \sum_{a<t_n\leq t} \Delta\xi_{t_n}.$$

So we have $\nu(a,T] = \mu_\psi(a,T]$. Thus Corollary A.1.1 gives $\mu_\psi(B) = \sum_{t_n\in B} \Delta\xi_{t_n}$ for any Borel set B. ∎

It is easy to see that μ_ψ is discrete, taking $A = \{t_n : n \in \mathbb{N}\}$ is enough. It is also trivial to see that μ_φ is continuous, since single element sets in $[0,T]$ are the singletons $\{t\}$, and $\mu_\varphi\{t\} = \varphi_t - \varphi_{t-} = 0$. It is true that since we can decompose $(\xi_t)_{t\in[0,T]}$ into two functions φ and ψ, that we can also decompose μ_ξ into the corresponding measures induced by φ and ψ. This is not immediately obvious from the work shown here, but this can be seen to be true, either with the classical approach to the Lebesgue decomposition, or through another application of Corollary A.1.1, to show $\mu_\varphi + \mu_\psi$ is in fact equal to μ_ξ for every Borel set.

This decomposition is very important because we can now see that, at jump times of $(\xi_t)_{t\in[0,T]}$, the measure μ_φ puts no mass there, whereas the measure μ_ψ only puts mass there. Going back to our original phrasing, we said that the integral $\int_{[0,T]} h_t \Delta\zeta_t \, d\xi_t$ could be written as $\int_A h_t \Delta\zeta_t \, d\xi_t$ for some countable subset $A \subset [0,T]$. We now know that $A \equiv D_\zeta := \{t \in [0,T] : \Delta\zeta_t > 0\}$. The continuous part of the measure μ_ξ puts no mass on these points, leaving only the discrete measure μ_ψ, and so

$$\int_{[0,T]} h_t \Delta\zeta_t \, d\xi_t = \int_{D_\zeta} h_t \Delta\zeta_t \, d\xi_t = \int_{D_\zeta} h_t \Delta\zeta_t \, d\mu_\psi = \sum_{t_n \in D_\zeta} h_{t_n} \Delta\zeta_{t_n} \Delta\xi_{t_n}.$$

We can just as well write $\sum_{t\in[0,T]} h_t \Delta\zeta_t \Delta\xi_t$ for the final term, since any $t \notin D_\zeta$ will cause $\Delta\zeta_t = 0$, and any $t \notin D_\xi$ (recall that D_ξ is the set of all points of discontinuity of $(\xi_t)_{t\in[0,T]}$) will cause $\Delta\xi_t = 0$. Put another way, the only time we have non-zero terms in these sums is when $t \in D_\zeta \cap D_\xi$.

One final result that is frequently used is the fact that we may 'untruncate' a Lebesgue-Stieltjes integral. For example, let χ be a generating process and ρ a stopping time. Assume that $1 - \chi_{\rho-} > 0$, then we say that

$$(1 - \chi_{\rho-}) \int_{[\rho,T]} f_t \, d\overline{\chi}_t^\rho = \int_{[\rho,T]} f_t \, d\chi_t. \tag{A.1}$$

Without loss of generality, take f to be positive (if not then decompose it as $f = f^+ - f^-$), then by definition, the left-hand side of (A.1) is simply a supremum, and we may move constants inside:

$$(1 - \chi_{\rho-}) \int_{[\rho,T]} f_t \, d\overline{\chi}_t^\rho = \sup\left\{ (1 - \chi_{\rho-}) \int_{[\rho,T]} f \, d\overline{\chi}_t^\rho : g \leq f, \, g \text{ is a simple function} \right\}.$$

Theorem A.1.2 tells us that on the measurable space $([\rho,T], \mathscr{B}[\rho,T])$, the measures μ_χ and $(1 - \chi_{\rho-})\mu_{\chi^\rho}$ are equivalent. Thus for any simple function g—with canonical form $g = \sum_{i=0}^N a_i \mathbb{1}_{A_i}$—

such that $g \leq f$,

$$(1-\chi_{\rho-}) \int_{[\rho,T]} g_t \, d\overline{\chi}_t^{\rho} = \sum_{i=1}^{N} a_i (1-\chi_{\rho-}) \mu_{\chi^{\rho}}(A_i) = \sum_{i=1}^{N} a_i \mu_{\chi}(A_i) = \int_{[\rho,T]} g_t \, d\chi_t.$$

A.3 GENERALISED INVERSES AND TIME CHANGES

We know that it is not always guaranteed that a given function $f: X \to Y$ has an inverse. If f is bijective then it is invertible, but if we extend our concept of an inverse, then we can still consider the 'inverse' of non-bijective functions. We say that f has a right (resp. left) inverse h (resp. g) if $f \circ h = \mathrm{id}_Y$ (resp. if $g \circ f = \mathrm{id}_X$) where id_Y (resp. id_X) is the identity function on the set Y (resp. X). If f has both a left and a right inverse, then f has an inverse f^{-1}. It is true that (see chapter 0, [Ros78]) f has a right inverse if and only if it is surjective, and has a left inverse if and only if it is injective—note the reference given is a text on group theory, and as a result uses the notation of $x\psi\phi := \phi(\psi(x))$.

The problem we face is that this still is not a wide enough generalisation. A generating process $\chi : [0,T] \to [0,1]$ is guaranteed to neither be injective nor surjective: the former failing due to the presence of 'flat sections', and the latter failing due to the presence of jumps. Here we introduce the concept of a 'generalised inverse'; the route we will take follows that of Section 0.4 in [RY05].

Let f be a non-decreasing, possibly infinite, right-continuous function with domain $\mathbb{R}^+$, then for $s \geq 0$ we define $f^*(s) := \inf\{t \geq 0 : f(t) > s\}$, under the convention that $\inf\{\emptyset\} = +\infty$.

Lemma A.3.1. *(Lemma 0.4.8, [RY05]) The function f^* is right-continuous. Moreover $f(f^*(s)) \geq s$ and $f(t) = \inf\{s \geq 0 : f^*(s) > t\}$.*

We call f^* the generalised inverse of f. This result highlights the interplay between the two functions. One can see from the definition of f^* that it must be non-decreasing, and since the left and right limits always exist (they may be infinite) for monotonic functions, we have $f^*(t-) := \lim_{s \nearrow t} f^*(s)$ is well defined for every $t \geq 0$.

Proposition A.3.2. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a right-continuous, non-decreasing function, then*

$$f^*(s-) = \lim_{u \nearrow s} f^*(u) = \inf\{t \geq 0 : f(t) \geq s\}.$$

Proof. To begin, notice that $\{t \geq 0 : f(t) \geq s\} = \bigcap_{u<s} \{t \geq 0 : f(t) > u\} \subset \{t \geq 0 : f(t) > u'\}$

for any $u' < s$. Thus $f^*(u) \leq \inf\{t \geq 0 : f(t) \geq s\}$ for any $u < s$, implying that $f^*(s-) \leq \inf\{t \geq 0 : f(t) \geq s\}$. Assume that the strict inequality held, then there would exist a t' between the two numbers, implying that 1) since $t' < \inf\{t \geq 0 : f(t) \geq s\}$ then $f(t') < s$ and 2) that $f^*(u) < t'$ for all $u < s$. The latter implies that $t' \in \{t \geq 0 : f(t) > u\}$, meaning that $f(t') > u$. Combining the two inequalities and taking the limit as u goes to s gives $s < f(t') < s$, an obvious contradiction. ■

This explicit form allows us to say that $\Delta f^*(s) > 0$ if and only if f has a flat section at the level s. In fact, Lemma A.3.1 and Proposition A.3.2 together tell us that jumps in f correspond to flat sections of f^*, and jumps of f^* correspond to flat sections of f. As a result f^* is continuous if and only if f is strictly increasing. If f is continuous, then we do not necessarily get that f^* is continuous; in fact it remains right-continuous. Imagine that the level s is a flat section of f, then f^* jumps at s, but $f^*(f(s)) > s$; if there is no flat section at s then $f^*(f(s)) = s$.

We can now consider the 'change of variable' formula found in the book *Continuous Martingale and Brownian Motion* by Daniel Revuz and Marc Yor. This result is vital for switching between integrals with respect to generating processes and ones indexed by stopping times. Recall that we have frequently written equations akin to the following. Let $(\chi_t)_{t\in[0,T]}$ be a generating process of the randomised stopping time σ, and define $\sigma(u) := \inf\{t \in [0,T] : \chi_t > u\}$, then

$$\int_{[0,T]} f_t(1-\zeta_t)\, d\chi_t = \int_0^1 f_{\sigma(u)}(1-\zeta_{\sigma(u)})\, du, \tag{A.2}$$

for some generating process $(\zeta_t)_{t\in[0,T]}$. We now see that $\sigma^*(u)$ is a generalised inverse of the generating process $(\chi_t)_{t\in[0,T]}$, and the change of variable formula justifies (A.2).

Proposition A.3.3. *(Proposition 0.4.9, [RY05]) If g is a positive Borel function on $\mathbb{R}^+$ then*

$$\int_{\mathbb{R}^+} g(u)\, d(f(u)) = \int_0^\infty g(f^*(s))\mathbb{1}_{\{f^*(s)<\infty\}}\, ds.$$

APPENDIX B

A SUPPLEMENT ON PROBABILITY THEORY AND FUNCTIONAL ANALYSIS

Throughout the preceding work, we have made use of some basic results in probability theory and functional analysis. In Lemma 2.2.20 we needed to carefully consider the use of the tower property of conditional expectation on particular subsets. Section 3.4 follows in the same vein as [DAMP22], and requires an understanding of the surrounding theory and results in functional analysis. We chose to omit these considerations from the main body of the text as it was not the proper place for them, however, we do not want to leave them from this work entirely, so we dedicate this appendix to them.

Section B.1 provides a brief introduction to some of the basic key concepts in probability theory such as stopping times, filtrations, and martingales. Section B.2 deals with the trace of a σ-algebra on a set, and proves a simple extension of the tower property of conditional expectation. Section B.3 is then concerned with the technical background for Section 3.4.

B.1 RUDIMENTS OF PROBABILITY THEORY

We will assume that the reader is already familiar with measure theory, namely regarding measure and probability spaces, σ-algebras, and measurable functions. Given this, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $(X_t)_{t\geq 0}$ be a stochastic process.

Definition B.1.1. A family $(\mathscr{F}_t)_{t\geq 0}$ of sub-σ-algebras of $\mathscr{F}$ which is non-decreasing, i.e. $\mathscr{F}_s \subseteq \mathscr{F}_t$ for all $s \leq t$, is called a filtration.

Definition B.1.2. The stochastic process $(X_t)_{t\geq 0}$ is said to be adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$ if, for each $t \geq 0$, X_t is $\mathscr{F}_t$-measurable.

We can now consider the idea of a 'random time', which we know as a stopping time. Dellacherie and Meyer called the stopping time the 'cornerstone of the general theory of processes' and likened its importance to the field to that of the derivative in calculus ([DM78], pg. 115-IV). Its definition is exceedingly simple; however, it plays an outsized role in the field.

Definition B.1.3. A random variable T on Ω with values in $[0,\infty]$ is called a stopping time of the filtration $(\mathscr{F}_t)_{t\geq 0}$ if for all $t \in [0,\infty]$, the event $\{T \leq t\}$ belongs to $\mathscr{F}_t$.

Finally, we give the definition of a martingale. These processes are considered to be so important in the field that 'no probabilist can any longer afford to ignore martingale theory' ([DM82], pg. 2).

Definition B.1.4. The stochastic process $(X_t)_{t\geq 0}$ is called a submartingale (resp. supermartingale) with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ if $\mathbb{E}|X_t| < \infty$ for each $t \geq 0$, and for every $s,t \geq 0$ such that $s \leq t$,

$$X_s \leq \mathbb{E}[X_t|\mathscr{F}_s] \qquad (\text{resp. } X_s \geq \mathbb{E}[X_t|\mathscr{F}_s]).$$

The process $(X_t)_{t\geq 0}$ is called a martingale if it is both a super- and a submartingale, i.e.

$$X_s = \mathbb{E}[X_t|\mathscr{F}_s].$$

We often require that sub/ supermartingales are right-continuous so that we may make use of the 'optional sampling theorem'. This result shows that the (sub/super-) martingale property holds over stopping times for right-continuous (sub/super-) martingales.

Theorem B.1.5. *(Theorem 3.22, [KS98a]) Let $(X_t)_{t\geq 0}$ be a right-continuous submartingale (resp. supermartingale) with a last element X_∞, and let $\sigma \leq \tau$ be two $(\mathscr{F}_t)$-stopping times. We have*

$$X_\sigma \leq \mathbb{E}[X_\tau|\mathscr{F}_\sigma] \quad \mathbb{P}\text{-a.s.} \qquad (\text{resp. } X_\sigma \geq \mathbb{E}[X_\tau|\mathscr{F}_\sigma] \quad \mathbb{P}\text{-a.s.})$$

B.2 ON THE TOWER PROPERTY OF CONDITIONAL EXPECTATION

We begin with the concept of the trace of a σ-algebra. This is a restriction of a σ-algebra to an event, and it is a classical result of measure theory that the resulting object is again a σ-algebra.

Definition B.2.1. Let X be a set, $\mathscr{F}$ a σ-algebra on X, and A a subset of X. The trace of $\mathscr{F}$ on A is defined to be

$$\mathscr{F}|_A := \{F \cap A : F \in \mathscr{F}\}.$$

Proposition B.2.2. *$\mathscr{F}|_A$ is a σ-algebra on A.*

Proposition B.2.3. *Let η be an $(\mathscr{F}_t)$-stopping time, then for any $t \in [0,T]$ we have*

$$\mathscr{F}_\eta|_{\{\eta<t\}} \subset \mathscr{F}_{t-}.$$

Proof. First, we must prove that $\mathscr{F}_\eta = \mathscr{F}_{\eta+}$. Recall that a set F belongs to $\mathscr{F}_{\eta+}$ if $F \cap \{\eta \leq t\} \in \mathscr{F}_{t+}$ for all $t \geq 0$. Since the filtration is right-continuous under the usual conditions, i.e. $\mathscr{F}_{t+} = \mathscr{F}_t$ for all $t \geq 0$, we see that this definition aligns with that of $\mathscr{F}_\eta$.

Next we can prove the main result. Problem 1.2.21 from Karatzas and Shreve [KS98a] tells us that F belongs to $\mathscr{F}_{\eta+}$ (and therefore to $\mathscr{F}_\eta$) if $F \cap \{\eta < t\} \in \mathscr{F}_t$ for all $t \geq 0$. Let $(t_n)_{n\in\mathbb{N}}$ be an increasing sequence converging to t. Since F belongs to $\mathscr{F}_\eta$ we know that

$$F \cap \{\eta < t_n\} \in \mathscr{F}_{t_n} \subset \mathscr{F}_{t-}.$$

Letting $n \to \infty$ we obtain $F \cap \{\eta < t\} \in \mathscr{F}_{t-}$, and so, by definition of the trace of $\mathscr{F}_\eta$ on $\{\eta < t\}$ we must have that $\mathscr{F}_\eta|_{\{\eta<t\}} \subset \mathscr{F}_{t-}$. ∎

Lemma B.2.4. *Let $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ be σ-algebras such that $\mathscr{H} \supseteq \sigma(\mathscr{F} \cup \mathscr{G})$. Let A be a set in $\mathscr{G}$. If $\mathscr{F}|_A \subseteq \mathscr{G}|_A$, then for any integrable random variable X*

$$\mathbb{E}[\mathbb{1}_A \mathbb{E}[X|\mathscr{G}]|\mathscr{F}] = \mathbb{E}[\mathbb{1}_A X|\mathscr{F}].$$

Proof. Using the definition of conditional expectation we can write two equations, one for each side of the desired equality, i.e. for all $F \in \mathscr{F}$,

$$\mathbb{E}[\mathbb{1}_F \mathbb{E}[\mathbb{1}_A \mathbb{E}[X|\mathscr{G}]|\mathscr{F}]] = \mathbb{E}[\mathbb{1}_F \mathbb{1}_A \mathbb{E}[X|\mathscr{G}]] \tag{B.1}$$

$$\mathbb{E}[\mathbb{1}_F \mathbb{E}[\mathbb{1}_A X|\mathscr{F}]] = \mathbb{E}[\mathbb{1}_F \mathbb{1}_A X]. \tag{B.2}$$

Given the definition of conditional expectation in (B.2), and the relation given in (B.1), we seek to prove the equality $\mathbb{E}[\mathbb{1}_F \mathbb{1}_A \mathbb{E}[X|\mathscr{G}]] = \mathbb{E}[\mathbb{1}_F \mathbb{1}_A X]$, yielding the desired result. It stands to reason that if $\mathbb{1}_F \mathbb{1}_A \mathbb{E}[X|\mathscr{G}] = \mathbb{E}[\mathbb{1}_F \mathbb{1}_A X|\mathscr{G}]$, then we can conclude. To this end, notice that since $A \in \mathscr{G}$, then $\mathscr{G}|_A \subseteq \mathscr{G}$, and since $\mathscr{F}|_A \subseteq \mathscr{G}|_A$ then $\mathscr{F}|_A \subseteq \mathscr{G}$, meaning $\mathbb{1}_{F\cap A}$ is $\mathscr{G}$-measurable. ■

B.3 FUNCTIONAL ANALYSIS

There are some specific topological concepts and results that are vital for the application of Sion's theorem, Theorem 3.4.1 and we outline them here. We follow the book by Brezis, [Bre11].

Let E be a Banach space (a complete normed vector space), and let $f \in E^\star$ (the dual space of E, namely the space of all linear functionals that are continuous with respect to the norm on E). We define the linear functional $\varphi_f : E \to \mathbb{R}$ by $\varphi_f(x) = f(x)$ for any x in E. Collating the family of these functionals $(\varphi_f)_{f\in E^\star}$ we can define a new topology on E.

Definition B.3.1. The weak topology on E is the coarsest topology associated to the collection of functionals $(\varphi_f)_{f\in E^\star}$ such that every functional is continuous with respect to that topology.

Remark B.3.2. When commenting on properties of a space equipped with the weak topology, such as compactness or closedness, we often say weakly compact or weakly closed for clarity.

Here we have discarded the more natural topology that one can induce on a Banach space, and created the minimal topology to ensure continuity of the maps $(\varphi_f)_{f\in E^\star}$. However, the natural topology is still one we need to consider and it is the one induced by the norm on the Banach space. Any norm, $\|\cdot\|_E$, on a Banach space E induces a metric on E by $d_E(x,y) = \|x-y\|_E$ for x and y in E, however, one can induce a topology from a metric. We denote by $B(x_0,r)$ the open ball, with centre x_0, to be the set $\{x \in E : d_E(x_0,x) < r\}$ for any positive r. Moreover, letting $\mathscr{B}(x) = \{B(x,r) : r > 0\}$ for every x in E defines a collection of families of subsets of E. These can be shown to generate a topology (see [Eng89], Chapter 4.1).

Definition B.3.3. The topology induced by the norm of a Banach space is called the strong topology.

Remark B.3.4. (Proposition 3.6, [Eng89]) When E is finite-dimensional, the weak and strong topologies on E are identical.

Remark B.3.5. The dual space $E^\star$ is defined to be the continuous linear functionals on E, but specifically we mean continuous with respect to the strong topology. Therefore the strong topology is one of the topologies which makes all elements of $E^\star$ continuous, but for the weak topology we are looking for the coarsest topology for which that is true.

The L^2 space is particularly well behaved: it follows from the Riesz representation theorem (see Remark 4.4, [Bre11]) that L^2 is its own dual space and moreover, it is the only L^p space to also be a Hilbert space, meaning the L^2 norm is induced by an inner product. This gives us a set-up to look at how the choice of topology changes the concept of convergence—a simple way of examining the difference between the topologies.

Consider the specific case of $L^2(\mathbb{R})$, these are functions which map the reals to themselves. Convergence in the strong topology is nothing but convergence in the L^2 norm, namely for any sequence of functions $(f_n)_{n\in\mathbb{N}}$ belonging to $L^2(\mathbb{R})$, they converge to a function $f \in L^2(\mathbb{R})$ if

$$\int_{\mathbb{R}} |f_n - f|^2 \, dx \to 0 \quad \text{as } n \to \infty.$$

Contrast this with convergence in the weak topology which says that for every test function ϕ belonging to the dual space, the inner product of f_n and ϕ converges to the inner product of f and ϕ. In our case, since L^2 is its own dual, and we are taking real functions, this is very simple. For a any test function $\phi \in L^2(\mathbb{R})$, $(f_n)_{n\in\mathbb{N}}$ converges to f in the weak topology if

$$\int_{\mathbb{R}} f_n \phi \, dx \to \int_{\mathbb{R}} f \phi \, dx.$$

One can show (Proposition 3.5, [Bre11]) that convergence in the strong topology implies convergence in the weak.

One of the key properties of L^p spaces (for $1 < p < \infty$) is that they are reflexive. In order to define this we need the concept of the bidual space of E denoted by $E^{\star\star}$. The bidual is the dual of the dual space, i.e. it is the space of all linear functionals mapping $E^\star$ to $\mathbb{R}$, which are continuous with respect to $\|\cdot\|_{E^\star}$, and it is equipped with the norm $\|\cdot\|_{E^{\star\star}}$. Reflexivity is a property that relates the dual to the bidual as follows. Pick any $x \in E$, then the map $F(x) : E^\star \to \mathbb{R}$, defined as $F(x)(f) = f(x)$, is a continuous linear functional on $E^\star$, meaning $F(x) \in E^{\star\star}$. From this, one defines the map $F : E \to E^{\star\star}$. The Hahn-Banach theorem tells us that this canonical mapping is injective. The space E is said to be reflexive if F is surjective, in other words, E is reflexive if there exists an isomorphism between the Banach spaces E and the bidual $E^{\star\star}$. Fortunately, we do not need to directly prove the reflexivity of $\mathscr{S}$ since it is an L^p space, and it is a well known that L^p spaces are reflexive for any $1 < p < \infty$ ([Bre11], Theorem 4.10).

INDICES

INDEX

Index

INDEX OF NOTATION

INDEX OF NOTATION