Quantum Miniatures

Quantum-like theories with limited sets of observables

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Abstract

A framework is introduced for constructing models – called *quantum miniatures* – which structurally resemble standard quantum systems. To construct quantum miniatures, we consider an ordinary quantum system but restrict the set of physical observables, in order to examine the implications of the commonly made assumption that all self-adjoint operators represent observables.

We investigate various consequences of placing a restriction on observables, which allows for a natural extension of the state space of a miniature beyond the usual quantum states. Models are developed which maximally violate uncertainty relations both for spin variables and for position and momentum, allowing for simultaneous sharp prediction of non-commuting observables. Another model exhibits post-quantum non-locality, including PR box correlations.

The quantum miniature framework includes standard quantum theory as a special case, and allows us to define consistent foil theories to quantum theory, including providing an explicit physical setting for a well-known family of GPTs, the polygon models of Janotta et al.

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Author's declaration

I declare that the work presented in this thesis, except where otherwise stated, is based on my own research carried out at the University of York and has not been submitted previously for any degree at this or any other university. Sources are acknowledged by explicit references.

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Introduction

1.1 MOTIVATION AND OUTLINE

In standard quantum theory, a physical system is represented by a Hilbert space, and every physical observable is postulated to correspond to some self-adjoint operator on this space. The converse – that every self-adjoint operator represents some physical observable – was occasionally assumed by early pioneers of quantum theory. For example, in *Mathematical foundations of quantum mechanics*, von Neumann demonstrates that each observable corresponds to a Hermitian operator, but remarks in a footnote that the only operators really known to correspond to observables at that stage were those corresponding to position, momentum and energy [64, p. 200]. Seemingly as a matter of technical convenience, von Neumann proceeds a few pages later to assume that in fact every Hermitian operator corresponds to an observable. In *Principles of quantum mechanics* [24], Dirac effectively uses the word 'observable' as a synonym for 'self-adjoint operator',[1] and then poses the question

> can every observable be measured? The answer theoretically is yes. In practice it may be very awkward, or perhaps even beyond the ingenuity of the experimenter, to devise an apparatus which could measure some particular observable, but the theory always allows one to imagine that the measurement can be made [24].

Both von Neumann and Dirac therefore claim that every self-adjoint operator represents a physical observable, but neither puts forward a robust justification.

[1]Dirac identifies 'dynamical variables' with linear operators (p. 26), called 'real' if the operator is self-adjoint, and then writes 'we call a real dynamical variable whose eigenstates form a complete set an *observable*.' [24, p. 37]

Many more authors tacitly make this assumption, which we will term 'the operator assumption', or often avoid the question altogether.

Some doubt has been raised on the operator assumption. As Wigner points out,

> there is, however, no rule which would tell us which self-adjoint operators are truly observables, nor is there any prescription to know how the measurements are to be carried out, what apparatus to use, etc. [94]

and more strongly,

> for some observables, in fact for the majority of them (such as xyp_z), nobody seriously believes that a measuring apparatus exists [93].

Furthermore, there are positive reasons to reject the operator assumption. The existence of super-selection rules [91] is a widely cited example [92, 45, 28]. Another indication against the operator assumption comes from the field of computability theory, where it has been argued that this assumption leads to a contradiction with the Church-Turing thesis [65].

One indication in favour of a weaker version of the operator assumption is given by Swift & Wright, who have shown that, using suitable electric and magnetic fields compatible with Maxwell's equations, measurement of an observable corresponding to any self-adjoint operator in a finite-dimensional space can be implemented arbitrarily well by means of a generalised Stern-Gerlach apparatus [85]. However, no comparable result appears to be known for observables in infinite-dimensional systems.

In this thesis, we examine the possible consequences of removing the operator assumption, while retaining many other features of quantum theory. We consider quantum-like theories – called *quantum miniatures* – in which we explicitly assert which operators correspond to observables, motivated by the availability of experimental apparatus. This modification naturally leads to the possibility of extending the state space relative to standard quantum mechanics. We consider a range of restrictions. Particular attention is paid to models where the restriction is especially austere, leaving only a small finite number of observables. These models can serve as extreme examples which starkly illustrate the consequences of restricting the set of observables. A dual benefit is the relative simplicity of a comprehensive exposition.

The main motivation for studying quantum miniatures is to examine the role played by the operator assumption in standard quantum theory. To isolate the effect of the operator assumption, we consider foil theories which retain most of the formal structure of quantum theory – we retain the familiar Hilbert space setting, with states acting as linear operators, and we retain the Born rule for probability assignment.

The non-positive operators which naturally arise as states in quantum miniatures have previously been considered in isolated cases. For example, restricted spin measurements have been considered, with a non-quantum state space leading to a modified notion of entanglement. This has consequences for, e.g., the possibility of classical simulation of such a system [74]. Also, non-positive operators have been considered in connection with the non-local correlations achievable in various theories [1]. Quantum miniatures may provide a unified framework for a systematic study of these and other features. Another motivation is that quantum miniatures may provide a useful framework for defining non-quantum theories with continuous variables, as discussed in Chapter 4.

1.2 BACKGROUND

1.2.1 Foil theories

The quantum miniature framework is situated within a broad program of constructing *foil theories* for quantum mechanics. This approach aims to clarify our understanding of quantum mechanics by contrasting with other logically conceivable theories: 'ways the world *might have been*' [19]. The point of departure for construction of a foil theory may be either the operational content of quantum theory, or its formal structure.

Concerning the operational content, one may identify certain features of quantum theory (such as the strength of its non-local correlations, the appearance of uncertainty relations, its utility for some information theoretic protocol etc.), and construct a foil theory which mimics some of these features. This approach can help to clarify what is distinctive about quantum theory, and give hints towards possible reformulations (on an operational footing). A common framework for developing foil theories is that of *generalised probabilistic theories* (GPTs) [7, 48, 71]. The GPT framework aims to describe conceivable physical theories in terms of the statistics of some allowed class of measurements, and encompasses a broad range of qualitatively different theories, including quantum theory and classical probability theory. GPTs provide a useful setting to explore re-axiomatisations of quantum theory (e.g. [38, 62, 17]); in this context, foil theories can play a role in testing the sufficiency or the independence of certain possible axioms.

A prominent class of foil theories lying outside the GPT framework are *epistemically restricted* (or *epistricted*) theories, which start from a classical statistical theory and add an epistemic restriction [81, 82]. Epistricted theories reproduce many phenomena occurring in quantum theory [39]. However, the underlying ontology is essentially classical, so a primary aim is to elucidate those features of quantum theory which are *not* reproducible in any epistricted theory, and may thus be considered 'intrinsically quantum' [50].

Other approaches take a specific technical formulation of quantum theory and modify or replace this by a structurally similar formalism. The idea is that, by modifying some aspect of a theory and following through the logical consequences, one can gain insight into the connections between different aspects of the theory. For example, *modal quantum theory* [79] replaces complex Hilbert space by a vector space over a finite field. Rather than assigning probabilities to measurement outcomes, modal quantum theory prescribes only whether an outcome is possible or impossible. Remaining formally closer to the standard formulation, it was realised by Birkhoff and von Neumann that much of the logical structure of quantum theory is compatible with a replacement of the complex Hilbert space by a real or quaternionic alternative [12]. Foil theories involving such real [84, 16, 2] or quaternionic [30] Hilbert spaces have been constructed. The real version has been shown to be experimentally falsifiable [76].

Finally, one may consider foil theories where the basic setting remains the familiar complex Hilbert space formulation, but where individual postulates are altered. Weinberg has considered a foil theory with the possible dynamics extended to allow non-linear time evolution [89]. This extension of the possible dynamics has been shown to have radical implications, such as the possibility of arbitrarily fast communication [35]. A recent foil theory involves replacing the projection postulate of standard quantum theory with a 'passive' alternative, in which the state is unaffected by measurement [31]. This modification affects the interpretation of mixed states, and allows for state reconstruction on a single system, with repercussions for e.g. the computational power of this theory relative to standard quantum theory.

The quantum miniature framework is closest in spirit to these latter approaches. We maintain most of the formal structure of standard quantum theory, but alter one postulate, with the aim of seeing how this affects other aspects of the theory.

1.2.2 Algebraic Quantum Theory

The quantum miniature framework is based on restricting the set of observables relative to a reference quantum system. This restriction can result in sets of observables which are not closed under addition. This limitation marks a departure from standard quantum theory, where an important feature is that the collection of observables forms an algebra. Indeed, there is significant literature devoted to re-deriving (and generalising) the Hilbert space formulation of quantum theory from an abstraction of the algebraic structure of observables (herein referred to as *the algebraic approach*) [51, 80, 28].

The basic elements in the algebraic approach are a set of observables $\mathfrak{O}$, a set of states $\mathfrak{S}$, and a map $\langle\!\langle \cdot\, , \cdot \rangle\!\rangle : \mathfrak{O} \times \mathfrak{S} \to \mathbb{R}$, which is interpreted as providing expectation values for observables in any state. It is postulated that observables are entirely defined by expectation values so, for $X, Y \in \mathfrak{O}$,

$$\langle\!\langle X, \omega \rangle\!\rangle = \langle\!\langle Y, \omega \rangle\!\rangle \text{ for any } \omega \in \mathfrak{S} \implies X = Y. \tag{1.1}$$

Alternatively, this can be thought of as defining an equivalence relation, and then dealing with equivalence classes of observables.

Given any two observables, $X, Y \in \mathfrak{O}$, the sum $X \oplus Y$ is defined by the property that

$$\langle\!\langle X \oplus Y, \omega \rangle\!\rangle = \langle\!\langle X, \omega \rangle\!\rangle + \langle\!\langle Y, \omega \rangle\!\rangle \qquad \text{for all } \omega \in \mathfrak{S}. \tag{1.2}$$

It is then a postulate that the sum, $X \oplus Y$, is an element of $\mathfrak{O}$, i.e., an observable (Structure Axiom 3(iii) in [28]). Similarly, given an observable $X \in \mathfrak{O}$ and a real number λ, the scalar multiple λX is defined by the property that

$$\langle\!\langle \lambda X, \omega \rangle\!\rangle = \lambda \langle\!\langle X, \omega \rangle\!\rangle \qquad \text{for all } \omega \in \mathfrak{S}, \tag{1.3}$$

and it is postulated that any scalar multiple of an observable is also an observable. The map $\langle\!\langle \cdot\, , \cdot \rangle\!\rangle$ thus furnishes $\mathfrak{O}$ with the structure of a real vector space. Furthermore, $\mathfrak{O}$ is equipped with a partial order $\preceq$, defined by

$$X \preceq Y \text{ if and only if } \langle\!\langle X, \omega \rangle\!\rangle \leq \langle\!\langle Y, \omega \rangle\!\rangle \text{ for all } \omega \in \mathfrak{S}. \tag{1.4}$$

Further postulates furnish $\mathfrak{O}$ with the structure of a Jordan algebra – the full details are unnecessary for the present discussion but may be found in e.g. [28, 83].

It suffices to note that integer powers of any observable are also assumed to be observable.

In the algebraic approach, the expectation value of a series of measurements of an observable X is treated as fundamental. The possible outcomes of an individual measurement of X are then derived as the support of a probability measure which is defined by its moments $\langle\!\langle X^n \rangle\!\rangle$, for $n \in \mathbb{N}$. In contrast, the quantum miniature framework takes the possible outcomes of an individual measurement to be fundamental, with expectation values derivable from the probabilities for each outcome in a given state.

1.3 PRELIMINARIES

Before describing some examples of quantum miniatures, we introduce some useful concepts from standard quantum mechanics.

Notation Given a Hilbert space $\mathcal{H}$, the space of all self-adjoint linear operators $\mathcal{H} \to \mathcal{H}$ is denoted by $L^*(\mathcal{H})$. An operator $\rho : \mathcal{H} \to \mathcal{H}$ is called *positive* if the quadratic form $|\psi\rangle \mapsto \langle\psi|\,\rho\,|\psi\rangle$ is positive semi-definite. We adopt the following notation:

- $\mathrm{Tr}\,[X]$ denotes the trace of a trace-class operator $X \in L^*(\mathcal{H})$.
- $\langle X, Y\rangle_{\mathrm{HS}} = \mathrm{Tr}\left[X^\dagger Y\right]$ is the Hilbert-Schmidt inner product (used for finite-dimensional miniatures).
- $\langle X\rangle_\rho = \mathrm{Tr}\,[\rho X]$; when ρ is a state and X is an observable, this has the interpretation of the expectation value for X in the state ρ.
- $\Delta^2(X|\rho) = \langle X^2\rangle_\rho - \langle X\rangle_\rho^2$; when ρ is a state and X is an observable, this has the interpretation of the variance of X in the state ρ.
- $\Delta(X|\rho) = \sqrt{\Delta^2(X|\rho)}$.
- $[X, Y] = XY - YX$.

Given a vector space V and a subset $X \subset V$,

- $\mathrm{cone}(X) = \{\lambda x \,|\, x \in X, \lambda \in \mathbb{R}_+\}$.
- $\mathrm{conv}(X) = \{\lambda x + (1-\lambda)y \,|\, x, y \in X, \lambda \in [0,1]\}$.
- A cone $C \subset V$ is *pointed* if $C \cap (-C) = \{0\}$.
- A cone $C \subset V$ is *generating* if $C - C = V$.

1.3.1 Postulates for quantum theory

We adopt the following postulates for quantum theory, synthesised from e.g. [64, 29, 68, 86]:

Kinematics:

Q A physical system is represented by a complex Hilbert space, $\mathcal{H}$.

O Observables are represented by self-adjoint operators on $\mathcal{H}$.

S States of a system are represented by positive, unit-trace operators on $\mathcal{H}$.

C Given systems A and B, with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, the Hilbert space of the joint system AB is given by the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$.

Measurement: Given a system in state ρ, and an observable represented by $X \in L^*(\mathcal{H})$, with eigenvectors $|x\rangle$ and corresponding eigenvalues x,

E the possible outcomes of a measurement of X are the eigenvalues x;

B the probability to obtain outcome x is given by the Born rule,

$$\mathcal{P}(x|\rho) = \mathrm{Tr}\left[\rho\, |x\rangle\langle x|\right]. \tag{1.5}$$

Dynamics:

P If an observable represented by X is measured and the eigenvalue x is obtained, then the post-measurement state of the system is a projector onto the associated eigenvector $|x\rangle$.

T Between measurements, states ρ evolve via unitary operators U on $\mathcal{H}$ according to the map $\rho \mapsto U\rho U^\dagger$.

1.3.2 Uncertainty relations

Quantum theory predicts certain constraints on the uncertainties of two or more observables in a given state. ‘In its most general form, the uncertainty principle states that it is never possible to simultaneously predict the measurement outcomes for all observables of the system’ [42]. Quantitative statements to such an effect are known as uncertainty relations, and typically consist of an inequality involving the variances of two or more observables. The best known of these is the Heisenberg relation for position Q and momentum P [40, 52, 90]:

$$\Delta Q \Delta P \geq \frac{\hbar}{2}. \tag{1.6}$$

This follows immediately from the more general Robertson relation [77],

$$\Delta^2(A|\rho)\Delta^2(B|\rho) \geq \frac{1}{4} \langle [A, B] \rangle_\rho^2 . \tag{1.7}$$

This is universally valid, i.e., it holds for any pair of observables A and B, in any number of dimensions. The relation (1.7) is particularly useful in the case of position and momentum (or other canonical pairs), since $\langle [Q, P] \rangle$ is state-independent. However, in general, the Robertson relation does not provide a state-independent bound. Indeed, unless all eigenvalues of $[A, B]$ are strictly negative or strictly positive, then $\langle [A, B] \rangle_\rho = 0$ for some state ρ, so (1.7) gives no bound at all for the variances under consideration, and does not entail the general conclusion above.

In finite dimensions, for any collection of observables, A_i, which do not share a common eigenstate, there is no quantum state for which all of these observables may all be sharply predicted [42]. Hence, there is a non-zero lower bound on the sum of the variances:

$$\sum_i \Delta^2(A_i|\rho) \geq U. \tag{1.8}$$

The particular constant U will depend on algebraic relations between the observables A_i.

Several results are known in the case where the observables in question are spin components. For a spin s system,

$$\sum_{i=1,2,3} \Delta^2(S_{\boldsymbol{\eta}_i}|\rho) \geq s. \tag{1.9}$$

where $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\eta}_3$ are any three mutually orthogonal directions [9, 23]. For spins up to $s = 3/2$, results are also known for the sum of variances in any two directions [23], which may not be reduced to a truncated version of the 3 component bounds [42]:

$$\sum_{i=1,2} \Delta^2(S_{\boldsymbol{\eta}_i}|\rho) \geq c_2(s). \tag{1.10}$$

For spin $s = 1/2$ in particular, $c_2(1/2) = 1/4$ and the result (1.10) can be generalised to any pair of directions, not necessarily orthogonal to one another. Namely, if $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$ are a pair of directions separated by an angle of ϕ, then

$$\Delta^2(S_{\boldsymbol{\eta}}) + \Delta^2(S_{\boldsymbol{\eta}'}) \geq \frac{1}{2} \sin^2 (\phi/2) . \tag{1.11}$$

Uncertainty relations such as those discussed above place limitations on the set of possible values for the uncertainties achievable by some quantum state. A more complete description of this set is given by the *preparation uncertainty region* [14]. Given a collection of N observables, A_i, the quantum uncertainty region, $\mathrm{PUR}_Q(A_1, \ldots, A_N)$, is defined as the set of possible values that may be realised as the uncertainty in each observable in some quantum state:

$$\mathrm{PUR}_Q(A_1, \ldots, A_N) = \{(\Delta A_1, \ldots, \Delta A_N) \,|\, \exists \rho \in \Omega_Q : \Delta_\rho(A_i) = \Delta A_i\} \,. \tag{1.12}$$

We emphasise the word 'quantum' here, and add a subscript Q, since we later consider uncertainty regions for models other than quantum theory. For the case of position and momentum, the quantum uncertainty region is fully defined by the Heisenberg relation (1.6), i.e., [14]

$$\mathrm{PUR}_Q(Q, P) = \left\{ (\Delta Q, \Delta P) \,\middle|\, \Delta Q \Delta P \geq \frac{\hbar}{2} \right\}. \tag{1.13}$$

Similarly, for the case of spin $s = 1/2$, the quantum uncertainty region for a pair of spin components in orthogonal directions is fully defined by the additive uncertainty relation (1.10), along with the individual maxima $\Delta S_x, \Delta S_y \leq 1/2$:

$$\mathrm{PUR}_Q(S_x, S_y) = \left\{ (\Delta S_x, \Delta S_y) \,\middle|\, \Delta S_x, \Delta S_y \leq \frac{1}{2},\, \Delta S_x^2 + \Delta S_y^2 \geq \frac{1}{4} \right\}. \tag{1.14}$$

1.3.3 WIGNER FUNCTIONS

The usual description of a quantum state gives the probability densities for either position or momentum. However, given one of these it is not straightforward to visualise the other. It would be desirable to encode both within a single function, akin to a probability distribution over phase space.

This is the idea of the Wigner function [95], defined as the Wigner transform [41] of the density operator:

$$W_\rho(q, p) = \frac{1}{\pi\hbar} \int \mathrm{d}q' e^{-2ipq'/\hbar} \langle q + q' | \, \rho \, | q - q' \rangle \,. \tag{1.15}$$

Since ρ is Hermitian, the function W_ρ is real-valued, and since ρ is normalised, W_ρ is normalised. W_ρ is not a true probability distribution, since it will in general assume negative values at some points. However, the marginals for each variable do

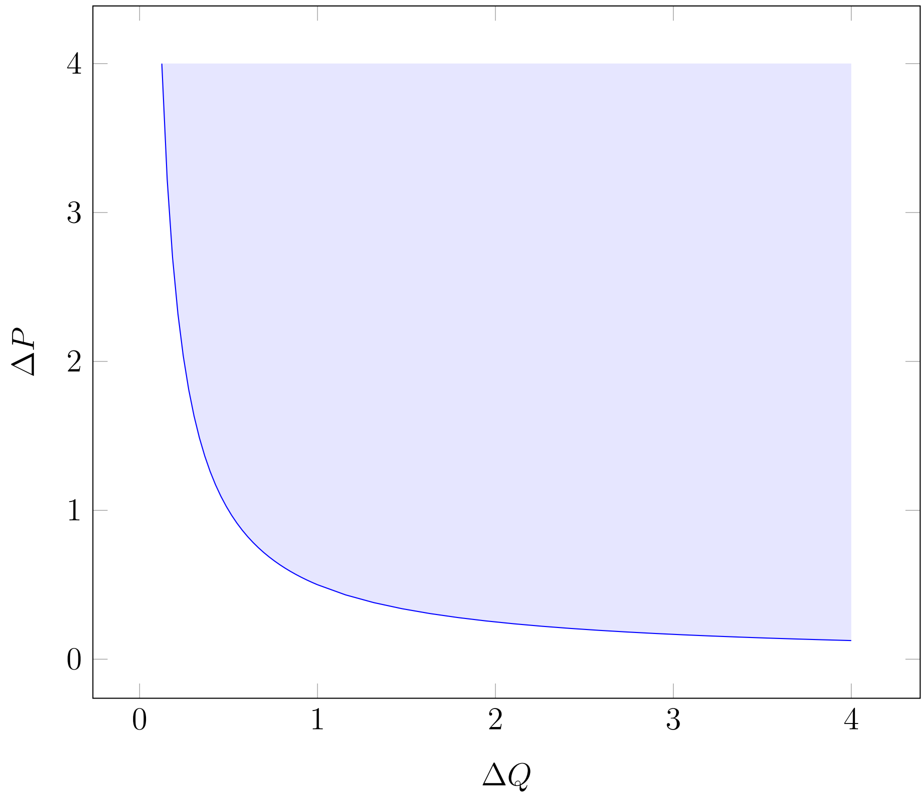


Figure 1.1: The quantum uncertainty region for position and momentum, $\mathrm{PUR}_Q(Q, P)$, as defined in (1.13).

constitute valid probability distributions, and agree with the distributions given by the original density operator:

$$\int_{\mathbb{R}} \mathrm{d}p \, W_\rho(q,p) = \mathrm{Tr}\left[\rho \, |q\rangle\langle q|\right], \tag{1.16}$$

$$\int_{\mathbb{R}} \mathrm{d}q \, W_\rho(q,p) = \mathrm{Tr}\left[\rho \, |p\rangle\langle p|\right]. \tag{1.17}$$

For any quantum state, ρ, the magnitude of the Wigner function is bounded:

$$|W_\rho(q,p)| \leq \frac{1}{\pi\hbar}. \tag{1.18}$$

Since W is normalised, this restricts how localised a Wigner function can be.

Given the Wigner function of a state, one can recover the density operator via the Weyl transform [90, 41]:

$$\rho = \frac{\hbar}{2\pi} \int_{\mathbb{R}^4} \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}p \, \mathrm{d}q \, W_\rho(q,p) \exp\left(\frac{ia(\hat{Q}-q) + ib(\hat{P}-p)}{\hbar}\right). \tag{1.19}$$

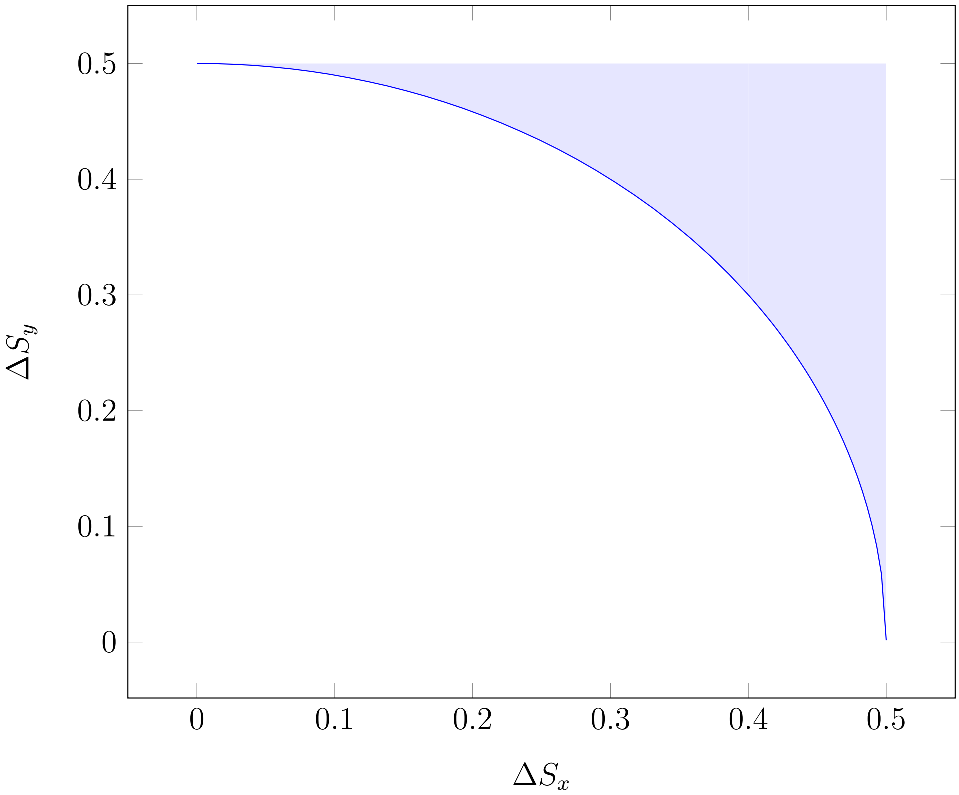


Figure 1.2: The quantum uncertainty region for an orthogonal pair of spin components, $\mathrm{PUR}_Q(S_x, S_y)$, defined in (1.14).

Another way to express the correspondence between Hilbert space operators and functions on phase space is given by the Wigner kernel [78, 33, 3]. The Wigner kernel, $\hat{\Delta}(q,p)$, may be expressed in terms of the parity operator, Π, and a two-parameter family of operators effecting phase-space translations,

$$\mathcal{T}(q,p) = \exp\left(\frac{ip\hat{Q} + iq\hat{P}}{\hbar}\right), \tag{1.20}$$

as

$$\hat{\Delta}(q,p) = \frac{1}{\pi\hbar}\mathcal{T}(q,p)\Pi\mathcal{T}^{\dagger}(q,p). \tag{1.21}$$

The Wigner transform (1.15) may then equivalently be written

$$W_A(q,p) = \mathrm{Tr}\left[A\,\hat{\Delta}(q,p)\right]. \tag{1.22}$$

Similarly, the Weyl transform (1.19) may be written

$$A = \int \mathrm{d}q\,\mathrm{d}p\,W_A(q,p)\hat{\Delta}(q,p). \tag{1.23}$$

The use of an operator kernel to establish a correspondence between Hilbert space operators and phase space functions admits a generalisation to quantum systems other than the non-relativistic particle, such as spin systems [87, 33]. Wigner functions for spin systems are discussed briefly in Appendix A.

1.3.4 Coherent states and squeezed states

Coherent states are eigenstates of the annihilation operator for the harmonic oscillator, $\hat{a} = (\hat{Q} + i\hat{P})/\sqrt{2}$. Labelling these states by their eigenvalue, α, we have

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n\in\mathbb{N}} \frac{\alpha^n}{\sqrt{n!}} |n\rangle . \tag{1.24}$$

Here, $|n\rangle$ are the harmonic oscillator eigenstates. These have position-basis wavefunctions

$$\langle q\,|n\rangle = \frac{1}{\sqrt{2^n n!}}(\pi\hbar)^{-1/4} \exp\left(-\frac{q^2}{2\hbar}\right) H_n\left(\frac{q}{\sqrt{\hbar}}\right), \tag{1.25}$$

where H_n are the Hermite polynomials. The harmonic oscillator ground state is itself a coherent state, with $\alpha = 0$.

The Wigner function of a coherent state is a Gaussian in q and p, centered at $(q_0, p_0) = (\sqrt{2}\mathrm{Re}\alpha, \sqrt{2}\mathrm{Im}\alpha)$:

$$W_{|\alpha\rangle}(q,p) = \frac{1}{\pi\hbar} \exp\left(-\frac{(q-q_0)^2 + (p-p_0)^2}{\hbar}\right). \tag{1.26}$$

The width of this Gaussian in each direction is directly related to the variances in q and p.

$$\Delta^2 Q = \frac{\hbar}{2}; \qquad \Delta^2 P = \frac{\hbar}{2} \tag{1.27}$$

Coherent states are states of minimum uncertainty in the sense that they saturate Heisenberg's uncertainty relation for position and momentum [40, 52, 90],

$$\Delta^2 Q \Delta^2 P \geq \frac{\hbar^2}{4}. \tag{1.28}$$

A more general class of minimum uncertainty states are the squeezed states. Squeezed states are obtained by applying the squeezing operator,

$$S(z) = \exp\left(\frac{z^* a^2 - z a^{\dagger 2}}{2}\right) \tag{1.29}$$

to the harmonic oscillator ground state $|0\rangle$. The parameter z is a complex number which describes how the distribution in phase space is altered. For real positive z, the distribution is contracted along the position axis and correspondingly stretched along the momentum axis, preserving the product of their variances. This may be seen in the Wigner function for such a state:

$$W_{|z\rangle}(q,p) = \frac{1}{\pi\hbar}\exp\left(-\frac{q^2/t + p^2 t}{\hbar}\right), \tag{1.30}$$

where $t = \exp(2z)$. The variances are

$$\Delta^2_{|z\rangle}Q = \frac{\hbar}{2\sqrt{t}}; \qquad \Delta^2_{|z\rangle}P = \frac{\hbar\sqrt{t}}{2}. \tag{1.31}$$

In the limit $t \to \infty$, we recover a position eigenstate, and in the opposite limit $t \to 0$, we recover a momentum eigenstate.

If z is not real, this distribution is also rotated in phase space by an angle given by the phase of z.

— 2 —

Quantum Miniatures

In this Chapter, we introduce the general framework of quantum miniatures. We begin with the central idea of taking a reference quantum system and restricting the set of available measurements. We then introduce the general description of states in quantum miniatures. This involves a natural extension of density operators familiar from usual quantum mechanics, and a discussion of equivalence classes.

We introduce postulates for quantum miniatures regarding each aspect of the theory in turn, using as a reference the postulates for standard quantum theory given in Section 1.3. The postulates for quantum miniatures are collected in Section 2.6.

2.1 REDUCED SETS OF OBSERVABLES

In describing observables within quantum miniatures, we take the approach that the set of observables in a given theory is directly informed by the experimental apparatus available. We further assume that any observable we can measure is correctly described by quantum theory. If, according to standard quantum theory, a given experimental setup may be used to measure a particular observable, then we assume that the same setup in a quantum miniature allows measurement of the *same* observable. This observable is represented by the same Hermitian operator, and we continue to assume that the possible outcomes of the measurement are the eigenvalues of this operator.

We therefore make the following postulates for quantum miniatures, cf. postulates **Q** and **O** in standard quantum theory:

M A physical system is represented by a complex Hilbert space, $\mathcal{H}$, and a set $\mathcal{O}_M \subseteq L^*(\mathcal{H})$.

$\mathbf{O_M}$ Observables are represented by elements of $\mathcal{O}_M$.

For each particular quantum miniature, we will present an experimental scenario, according to which we assert that some particular collection of observables may be measured. These are called the *basic observables*, which define the miniature. It is then implicit that any function of a single basic observable may be considered to be observable. In particular, we always implicitly assume that the identity operator on the relevant Hilbert space is an observable for every miniature. The set of basic observables for each miniature will be denoted $\mathcal{O}_M$, with a subscript specifying the particular miniature in question. In each case, $\mathcal{O}_M$ is a subset of the set, $\mathcal{O}_Q$, consisting all Hermitian operators: $\mathcal{O}_M \subseteq \mathcal{O}_Q$.

We do not, however, make any general assumptions about an experimenter's ability to measure combinations of observables, such as the sum or symmetric product. As noted in Section 1.2.2, this marks a departure from standard quantum theory, which is illustrated especially clearly by comparison with the algebraic approach. An essential component of the algebraic approach is the assumption that the sum of any two observables, defined by (1.2), is also an observable. The motivation for this postulate has been questioned [69], and we do not adopt any such postulate in general for quantum miniatures. As Bell points out (though in a different context),

> a measurement of a sum of noncommuting observables cannot be made by combining trivially the results of separate observations on the two terms – it requires a quite distinct experiment [10].

It is not guaranteed that such an experiment exists or, more weakly, that it is available to an experimenter. It has been pointed out that demanding compatibility of states with sums of observables as defined in (1.2), i.e., characterising states as positive linear functionals constitutes 'a non-trivial extrapolation over the strict physically motivated structure' [83, p. 19]. If we decline this extrapolation, then it is possible to find probability distributions for a pair of observables, X and Y, such that the value given by (1.2) has no tenable interpretation as the expectation value of their sum. For example, we will see in Section 3.1.1 an example of a miniature which provides consistent statistical predictions for spin components in orthogonal directions but, if linearly extended to account for spin in other directions, predicts expectation values lying outside the spectrum of possible outcomes, and negative 'probabilities' for supposed measurement outcomes.

2.2 STATES IN QUANTUM MINIATURES

We now turn to the definition of states in quantum miniatures. In the standard quantum case, we can think of Postulate **S** as a restriction on the types of operators which are admissible as states, in particular that these operators must be positive and have unit trace. These restrictions are related to Postulate **B** in that they describe conditions under which the Born rule produces valid probability distributions. In particular, positivity of the operator implies the distribution defined in (1.5) will be everywhere non-negative, and having unit trace implies the distribution is normalised. Assuming that all Hermitian operators constitute valid observables, we have also the converse, i.e., the distribution (1.5) being non-negative for all observables implies that ρ is positive, and normalisation of the distribution implies that ρ has unit trace. We may therefore alternatively classify states in quantum mechanics as exactly those operators which furnish valid probability distributions for measurements of all observables.

We carry over this alternative classification to quantum miniatures. Staying close to standard quantum theory, we postulate that states are described by linear operators acting on a Hilbert space, and that probabilities are given by a trace rule (1.5). However, the set of operators which give valid probability distributions for all observables may differ from that of standard quantum theory, since the set of observables is different. Since there are in general fewer observables than in quantum mechanics, the requirement of valid probability distributions will place fewer constraints on the states of a restricted quantum theory. We will still require the operators corresponding to states to be self-adjoint so that probability distributions are real-valued, and to have unit trace so that probability distributions are normalised. There is, however, no general requirement that states correspond only to positive operators.

We define a *quasi-density operator* to be a self-adjoint operator with unit trace. Given a quasi-density operator ρ and an observable X we say that ρ is *compatible with* X (and vice versa) exactly if

$$\mathrm{Tr}\left[\rho\,|x\rangle\langle x|\right] \geq 0 \tag{2.1}$$

for any eigenvector $|x\rangle$ of X. A quasi-density operator predicts non-negative, normalised probability distributions for any observable with which it is compatible. For a quantum miniature having a set of observables $\mathcal{O}_M$ we therefore require that any quasi-density operator representing a state in that miniature is compatible with all elements of $\mathcal{O}_M$. We also assume that *every* quasi-density operator compatible with all observables represents a state – this assumption is similar to the no-restriction hypothesis for generalised probabilistic theories [18, 49]. This leads to the following postulate on states in a miniature:

$\mathbf{S_M}$ States of a system are represented by quasi-density operators, ρ, on $\mathcal{H}$, subject to the requirement that

$$\mathrm{Tr}\left[\rho\,|x\rangle\langle x|\right] \geq 0 \tag{2.2}$$

for every eigenvector, $|x\rangle$, of every observable. Every quasi-density operator satisfying (2.2) represents a state.

We denote the space of all states in a given miniature by Ω_M. This set is determined by the set of observables $\mathcal{O}_M$; we can think of each observable as placing a set of constraints on the state space, i.e., each eigenvector of each observable places a linear inequality on the possible states, and Ω_M consists all quasi-density operators satisfying these inequalities. Note that compatibility of a quasi-density operator, ρ, with an observable X, implies that ρ is also compatible with any function $f(X)$, since the projectors onto eigenstates of $f(X)$ are a subset of the projectors for X itself. To define the state space of a miniature, it therefore suffices to demand compatibility with every basic observable.

There are some features common to the state spaces of all miniatures. Firstly, we note that Ω_M is convex for every miniature, since compatibility with any observable is preserved by convex combinations. Given any $|x\rangle \in \mathcal{H}$, we have

$$\mathrm{Tr}\left[(\lambda\rho + (1-\lambda)\rho')\,|x\rangle\langle x|\right] = \lambda\mathrm{Tr}\left[\rho\,|x\rangle\langle x|\right] + (1-\lambda)\mathrm{Tr}\left[\rho'\,|x\rangle\langle x|\right], \tag{2.3}$$

so, for any $0 \leq \lambda \leq 1$,

$$\mathrm{Tr}\left[\rho\,|x\rangle\langle x|\right] \geq 0 \text{ and } \mathrm{Tr}\left[\rho'\,|x\rangle\langle x|\right] \geq 0 \implies \mathrm{Tr}\left[(\lambda\rho + (1-\lambda)\rho')\,|x\rangle\langle x|\right] \geq 0. \tag{2.4}$$

Applying this result to each eigenvector of the observables within a given miniature, we find

$$\rho, \rho' \in \Omega_M \implies \lambda\rho + (1-\lambda)\rho' \in \Omega_M \tag{2.5}$$

for any $0 \leq \lambda \leq 1$, i.e., Ω_M is a convex set[1]. We call a state *extremal* if it can not be expressed as a convex combination of other states.

Another feature common to all miniatures is that Ω_M always contains Ω_Q as a subset, $\Omega_Q \subseteq \Omega_M$. This is because any positive operator ρ satisfies (2.1) for any $|x\rangle$, and is therefore compatible with every possible observable. Hence, these operators (corresponding to quantum states) are always elements of the state space of any quantum miniature. As a particular special case, we may consider the miniature where the set of observables consists all self-adjoint operators, so that $\mathcal{O}_M = \mathcal{O}_Q$. In this case, states must satisfy

$$\mathrm{Tr}\left[\rho\,|x\rangle\langle x|\right] \geq 0 \tag{2.6}$$

for every projector $|x\rangle\langle x|$ on $\mathcal{H}$, i.e., they must be positive operators. The state space is therefore equal to that of standard quantum theory, $\Omega_M = \Omega_Q$. The quantum miniature framework includes standard quantum theory as a special case.

There are also many cases where the inclusion $\Omega_Q \subset \Omega_M$ is strict, so that Ω_M contains non-quantum states represented by quasi-density operators with some negative eigenvalues. These negative eigenvalues only occur if a measurement containing a projector onto the associated eigenspace is impossible to perform using the apparatus available, so negative eigenvalues can never be observed in a miniature.

A broad class of miniatures which are readily seen to contain non-quantum states are those miniatures where the dimension of the Hilbert space and the number of basic observables are both finite. In this case, every basic observable introduces finitely many linear inequalities, and Ω_M is therefore a convex polytope (not necessarily bounded). Since Ω_Q is not a polytope, we immediately conclude that $\Omega_M \neq \Omega_Q$.

2.3 EQUIVALENCE CLASSES

In standard quantum mechanics, distinct density operators may always be distinguished by appropriate measurements[2]. Viewing quantum mechanics within the

[1]The fact that quantum miniatures always have a convex state space indicates compatibility with the GPT framework.

[2]For example, given distinct density operators ρ and ρ', the operator $D = \rho - \rho'$ is Hermitian and therefore an observable in quantum theory. D is non-zero since ρ and ρ' are distinct and hence has non-zero Hilbert-Schmidt norm. Noting that $\|D\|_{HS}^2 = \langle D\rangle_\rho - \langle D\rangle_{\rho'}$, we have that $\langle D\rangle_\rho \neq \langle D\rangle_{\rho'}$, so the density operators may be distinguished by repeated measurements of D. In a given quantum miniature, this measurement may not be performable.

framework of generalised probabilistic theories, this is essential in matching density operators to the more general notion of state: by definition, states in a generalised probabilistic theory are distinguishable on the basis of probability measurements.

In the case of quantum miniatures, the hypothetical measurement which could distinguish between a given pair of quasi-density operators may not correspond to an observable in the miniature. This means that, with the apparatus available, these quasi-density operators can not be distinguished on the basis of experimental data obtained from performable experiments.

We can instead define states as equivalence classes of operators, where two operators, ρ and ρ', are equivalent (denoted $\rho \sim \rho'$) if they generate identical probability distributions for all available measurements. Thus, $\rho \sim \rho'$ if and only if, for every observable $X \in \mathcal{O}$, and for every eigenvalue x of X, we have

$$\mathrm{Tr}\left[\rho\,|x\rangle\langle x|\right] = \mathrm{Tr}\left[\rho'\,|x\rangle\langle x|\right], \tag{2.7}$$

i.e., the probability for each outcome of every performable measurement is the same. We denote equivalence classes by square brackets:

$$[\rho] = \{\rho' \in \Omega \,|\, \rho' \sim \rho\}, \tag{2.8}$$

and the space of all equivalence classes by an overbar:

$$\bar{\Omega} = \{[\rho] \,|\, \rho \in \Omega\}. \tag{2.9}$$

In many cases, we define this set by a convenient choice of representative for each class.

The space $\bar{\Omega}$ inherits a natural convex structure from Ω if we define convex combinations of equivalence classes in the following way:

$$\lambda[\rho] + (1-\lambda)[\phi] = [\lambda\rho + (1-\lambda)\phi], \tag{2.10}$$

where $\lambda \in [0,1]$ and ρ and ϕ are any representatives of their respective equivalence classes. We can define probability maps for equivalence classes:

$$\mathcal{P}(x|[\rho]) = \mathcal{P}(x|\rho) \text{ for any } \rho \in [\rho]. \tag{2.11}$$

By construction this is independent of the choice of representative. The probability maps $[\rho] \mapsto \mathcal{P}(x|[\rho])$ are convex-linear.

We note that, while quantum miniatures generically contain additional non-quantum states, the property of observational indistinguishability is not a novel feature of these states. The indistinguishability rather arises as a result of miniatures not possessing the full set of measurements available in standard quantum theory. We will see examples in Section 3.1.1 of pairs of proper quantum states which are indistinguishable, as well as quantum states which are indistinguishable from non-quantum states. It is often of interest, within a particular miniature, to distinguish between those equivalence classes which contain at least one proper density operator, and those which do not. We will refer to equivalence which contain at least one proper density operator as quantum states.

2.4 SYMMETRIES

In many miniatures of interest, the set of observables has some symmetry, i.e. $\mathcal{O}_M$ is invariant under some collection of unitary maps, $\mathcal{G}$. The state space inherits this symmetry in the sense that, given a state $\rho \in \Omega_M$ and a unitary $U \in \mathcal{G}$, the operator $\rho' = U^\dagger \rho U$ is also a valid state. This is related to the covariance of the positivity constraints of $\mathbf{S_M}$. To see this, let ρ be a state of Ω_M. Given any $X \in \mathcal{O}$, denote $X' = UXU^\dagger$ so that, by assumption, $X' \in \mathcal{O}_M$. If $|x\rangle$ is an eigenvector of X with eigenvalue x, then $U|x\rangle$ is an eigenvector of X', also with eigenvalue x. Now,

$$\begin{aligned} \mathrm{Tr}\left[\rho' |x\rangle\langle x|\right] &= \mathrm{Tr}\left[U^\dagger \rho\, U |x\rangle\langle x|\right] && (2.12) \\ &= \mathrm{Tr}\left[\rho\, U|x\rangle\langle x|U^\dagger\right] && (2.13) \\ &= \mathcal{P}(X' = x|\rho). && (2.14) \end{aligned}$$

Since ρ is a state and X' is an observable, we have that

$$0 \le \mathcal{P}(X' = x|\rho) \le 1, \tag{2.15}$$

and so

$$0 \le \mathrm{Tr}\left[\rho' |x\rangle\langle x|\right] \le 1. \tag{2.16}$$

The quasi-density operator ρ' is therefore compatible with the observable X. Since X was chosen arbitrarily, ρ' must be compatible with all observables, so represents a state. Hence, the state space is invariant under the map $\rho \mapsto \rho' = U^\dagger \rho U$.

Furthermore this map is invertible and the inverse is convex-linear so that, if ρ is an extremal state, ρ' is also extremal.

We finally note that symmetry transformations acting on quasi-density operators preserve equivalence classes. More generally, we have the following:

Proposition 2.4.1. If the state space is closed under a linear map $\mathcal{U} : L(\mathcal{H}) \to L(\mathcal{H})$, then $\mathcal{U}$ preserves equivalence classes, i.e., if

$$\mathcal{U}\Omega \subseteq \Omega \tag{2.17}$$

then

$$\rho \sim \rho' \implies \mathcal{U}\rho \sim \mathcal{U}\rho' \text{ for all } \rho, \rho' \in \Omega. \tag{2.18}$$

Proof. Assume $\mathcal{U} : \rho \mapsto \mathcal{U}(\rho)$ is a linear map which does not preserve equivalence classes. There thus exists a pair of quasi-density operators, ρ and ρ', such that $\rho \sim \rho'$ and $\mathcal{U}(\rho) \nsim \mathcal{U}(\rho')$, so (without loss of generality)

$$\mathcal{P}(x|\mathcal{U}(\rho)) > \mathcal{P}(x|\mathcal{U}(\rho')) \tag{2.19}$$

for a measurement outcome x of some observable. Define

$$\sigma = (1+\lambda)\rho' - \lambda\rho; \tag{2.20}$$

this is a valid state for any value of λ since $\mathcal{P}(x|\sigma) = \mathcal{P}(x|\rho) \geq 0$ for all x.

Under $\mathcal{U}$, this maps to

$$\mathcal{U}(\sigma) = (1+\lambda)\mathcal{U}(\rho') - \lambda\mathcal{U}(\rho), \tag{2.21}$$

so that

$$\mathcal{P}(x|\mathcal{U}(\sigma)) = (1+\lambda)\mathcal{P}(x|\mathcal{U}(\rho')) - \lambda\mathcal{P}(x|\mathcal{U}(\rho)) \tag{2.22}$$

This, however, is negative for any value of λ satisfying

$$\lambda > \frac{\mathcal{P}(x|\mathcal{U}(\rho'))}{\mathcal{P}(x|\mathcal{U}(\rho)) - \mathcal{P}(x|\mathcal{U}(\rho'))}. \tag{2.23}$$

For such values of λ, the operator $\mathcal{U}(\sigma)$ is not a valid state and so the state space is not closed under the map $\mathcal{U}$. □

2.5 COMPOSITE SYSTEMS

In quantum miniatures, we maintain the postulate, **C**, that the Hilbert space of a joint system is given by the tensor product of the Hilbert spaces of each subsystem,

$$\mathcal{H}^{AB} = \mathcal{H}^A \otimes \mathcal{H}^B. \tag{2.24}$$

In standard quantum mechanics, with the additional postulate that all Hermitian operators are observable, this tensor product postulate is sufficient to determine the complete sets of observables and states for the joint system (since each is entirely determined by the Hilbert space). There are therefore general relations between $\mathcal{O}_Q^A$, $\mathcal{O}_Q^B$ and $\mathcal{O}_Q^{AB}$, and between Ω_Q^A, Ω_Q^B and Ω_Q^{AB}. The space of observables in this case is itself a Hilbert space and inherits the same tensor product structure[3]:

$$\mathcal{O}_Q^{AB} = \mathcal{O}_Q^A \otimes \mathcal{O}_Q^B. \tag{2.25}$$

We can also define a valid tensor product of convex sets, $\tilde{\otimes}$, by the property

$$\mathcal{D}(\mathcal{H}^A)\tilde{\otimes}\mathcal{D}(\mathcal{H}^B) = \mathcal{D}(\mathcal{H}^A \otimes \mathcal{H}^B), \tag{2.26}$$

where $\mathcal{D}(\mathcal{H})$ denotes the space of density operators on $\mathcal{H}$ (for details, see e.g. [71], sections 5.1 and 8.2), so that

$$\Omega_Q^{AB} = \Omega_Q^A \,\tilde{\otimes}\, \Omega_Q^B. \tag{2.27}$$

For a quantum miniature, the set of observables must be specified in its own right, and the state space is determined from the observables. Just as for single systems, the set of observables of a joint system is determined by the scope of the experimental apparatus available. This is decided not only by the individual capabilities of the measurement apparatuses for each subsystem, but also by the manner in which these apparatuses are able to interact and be used jointly. Hence, $\mathcal{O}^{AB}$ is not fully determined by $\mathcal{O}^A$ and $\mathcal{O}^B$; it is necessary also to give the full physical context.

For example, we may wish to describe an experimental set-up involving distant parties, A and B, where each party has access to a localised measurement apparatus. In this case we assume that A can measure observables from a set $\mathcal{O}^A$, and B can

[3]Note that it is more common in the GPT literature to refer to tensor products between the *effect space* of each party. In our context, the effect space is the convex hull of the set of projectors associated to each observable.

measure observables from a set $\mathcal{O}^B$. We further assume that there is no interaction between the apparatuses, and that A and B are both able to communicate the results of their measurement. This experimental setup allows the measurement of any observable which is the tensor product of observables for each party. Thus, the set of joint observables, $\mathcal{O}^{AB}$, consists exactly those of product type:

$$\mathcal{O}^{AB} = \left\{ X \otimes Y \,|\, X \in \mathcal{O}^A,\, Y \in \mathcal{O}^B \right\}. \tag{2.28}$$

Note that this is different from the minimal tensor product, which would also include convex combinations of product observables[4].

We may also imagine setups where the subsystems are more localised. In this case, there may be measurement devices capable of performing non-separable joint measurements and projecting onto non-separable elements of the joint Hilbert space. In all cases, the joint state space is determined by the joint observables in accordance with the postulate $\mathbf{S_M}$.

Reduced states Given a composite physical system, we may wish to describe the reduced state of a particular subsystem. In standard quantum mechanics, this reduced state is given by the partial trace. A defining property of the partial trace is that it correctly reproduces the statistics for measurements performed by a single party. Given a density operator $\rho \in \mathcal{D}(\mathcal{H}^{AB})$, denote the reduced state $\rho_A = \mathrm{Tr}_B(\rho)$, then for any projector E on $\mathcal{H}^A$,

$$\mathrm{Tr}\left[\rho(E \otimes \mathbb{1})\right] = \mathrm{Tr}\left[\rho_A E\right]. \tag{2.29}$$

This property is not dependent on the positivity of ρ, and continues to provide a useful notion of reduced state in the context of quantum miniatures. Consider a bipartite miniature which allows for measurements on a single subsystem, A, and denote the space of all observables measurable in this way by $\mathcal{O}^A$. Then the partial trace of any bipartite state produces a valid state within the single-system miniature defined by the observables $\mathcal{O}^A$, i.e., if

$$X \in \mathcal{O}^A \implies X \otimes \mathbb{1} \in \mathcal{O}^{AB} \tag{2.30}$$

then

$$\rho \in \Omega^{AB} \implies \mathrm{Tr}_B(\rho) \in \Omega^A. \tag{2.31}$$

This follows immediately from (2.29).

[4]The minimal tensor product would correctly describe the joint effect space.

2.6 POSTULATES FOR QUANTUM MINIATURES

We now summarise the postulates developed over the course of the preceding section.

Kinematics:

M A physical system is represented by a complex Hilbert space, $\mathcal{H}$, and a set $\mathcal{O}_M \subseteq L^*(\mathcal{H})$.

$\mathbf{O_M}$ Observables are represented by elements of $\mathcal{O}_M$.

$\mathbf{S_M}$ States of a system are represented by quasi-density operators ρ on $\mathcal{H}$, subject to the requirement that

$$\mathrm{Tr}\left[\rho \left|x\right\rangle\!\left\langle x\right|\right] \geq 0 \tag{2.32}$$

for every eigenvector, $|x\rangle$, of every observable. Every quasi-density operator satisfying (2.32) represents a state.

C Given systems A and B, with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, the Hilbert space of the joint system AB is given by the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$.

Measurement: Given a system in state ρ, and an observable represented by $X \in \mathcal{O}_M$, with eigenvectors $|x\rangle$ and corresponding eigenvalues x,

E the possible outcomes of a measurement of X are the eigenvalues x;

B the probability to obtain outcome x is given by the Born rule:

$$\mathcal{P}(x|\rho) = \mathrm{Tr}\left[\rho \left|x\right\rangle\!\left\langle x\right|\right]. \tag{2.33}$$

If we supplement these postulates with the assumption that all self-adjoint operators correspond to observables, the postulates above are equivalent to those presented in 1.3 for kinematics and measurement in standard quantum theory.

In this work, we do not treat dynamics in full generality for miniatures, so we do not adopt any general postulates. For individual models, it is possible to identify some possible options. For example, Section 4.1 describes a quantum miniature for a particle system; in this case we can define a consistent dynamics, and can also rule out certain seemingly reasonable alternatives, including evolution by the free Hamiltonian. In general, it is expected that miniatures having a state space larger than quantum theory, the dynamics will be restricted relative to the usual quantum case. At the minimum, we will require that dynamics must preserve the state space. In Section 3.1.1, we identify a possible source of ambiguity when seeking to define a rule for state update in quantum miniatures.

2.7 QUANTUM MINIATURES AS GPTS

In this section, we will show how quantum miniatures can be described within the framework of generalised probabilistic theories (GPTs). A particular theory in the GPT framework is specified by a collection of possible ways for systems to be prepared; a collection of possible measurements on a system, and a collection of possible outcomes for each measurement; the probability for a measurement on a system to return a specified outcome, given a specified preparation of the system. If two preparations of a system yield the same probability for any given measurement outcome to occur, these preparations are said to be operationally equivalent. A *state* is defined to be an equivalence class of such operationally equivalent preparations. Similarly, two measurement outcomes are said to be operationally equivalent if they occur with equal probability for any given preparation. Equivalence classes of measurement outcomes are called *effects*. For a comprehensive review of GPTs, see e.g. [48, 71].

There are several equivalent approaches to defining specific theories within the GPT framework, as discussed in [71]. One may start from the state space (with effects acting as probability-valued maps on states), or start from an effect algebra (with states acting on probability-valued maps on effects), or from a description of the collection of conditional probabilities which may theoretically be observed. In quantum miniatures, a specific model is defined by the set of observables, and the state space derived therefrom. The approach which most closely resembles this is to construct a GPT from an abstract effect algebra.

An *effect algebra* [32] consists of a set $\mathcal{E}$ with a partially defined binary operation $+$ and elements $0, 1 \in \mathcal{E}$. Given $a, b \in \mathcal{E}$ we write $a \perp b$ if $a + b$ exists. For any $a, b, c \in \mathcal{E}$ we require that:

(E1) If $a \perp b$ then $b \perp a$ and $a + b = b + a$.

(E2) If $a \perp b$ and $(a + b) \perp c$ then $b \perp c$, $a \perp (b + c)$ and $(a + b) + c = a + (b + c)$.

(E3) For every $a \in \mathcal{E}$ there exists $a' \in \mathcal{E}$ such that $a \perp a'$ and $a + a' = 1$.

(E4) If $a \perp 1$ then $a = 0$.

An effect algebra $\mathcal{E}$ is called a *convex effect algebra* [36] if for every $a \in \mathcal{E}$ and $\lambda \in [0, 1]$, there exists an element λa such that the following conditions hold for all $\lambda, \mu \in [0, 1]$, $a, b \in \mathcal{E}$:

(C1) $\mu(\lambda a) = \lambda(\mu a)$.

(C2) If $\lambda + \mu \leq 1$, then $\lambda a \perp \mu a$ and $\lambda a + \mu a = (\lambda + \mu)a$.

(C3) If $a \perp b$, then $\lambda a \perp \lambda b$ and $\lambda a + \lambda b = \lambda(a + b)$.

(C4) $1a = a$.

A representation theorem for convex effect algebras establishes a useful description in terms of convex cones. Given a vector space V and a convex, pointed cone $\mathcal{C} \subset V$, we may define a partial order $\preceq_{\mathcal{C}}$ on V such that, for $v, w \in V$,

$$v \preceq_{\mathcal{C}} w \iff w - v \in \mathcal{C}. \tag{2.34}$$

It has been shown that, for any convex effect algebra $\mathcal{E}$, there exists a vector space V and a pointed convex cone $\mathcal{C} \subset V$ such that $\mathcal{E}$ is affinely isomorphic to an interval with respect to the partial order $\preceq_{\mathcal{C}}$ [37],

$$E \cong \{v \in V \,|\, 0 \preceq_{\mathcal{C}} v \preceq_{\mathcal{C}} u\}, \tag{2.35}$$

where u is a non-zero element of V. Conversely, any convex pointed cone equips the interval (2.35) with the structure of a convex effect algebra [71].

Given an effect algebra characterised by a cone $\mathcal{C}$, we define the *pre-state space* $\mathcal{S}(\mathcal{E})$ by

$$\mathcal{S}(\mathcal{E}) = \{f \in \mathcal{C}^* \,|\, f(u) = 1\}, \tag{2.36}$$

where $\mathcal{C}^*$ is the dual cone to $\mathcal{C}$. If $\mathcal{C}$ is a generating cone, then this is a *state space* (cf. Definition 3.29 in [71]).

In standard quantum theory, effects are represented by positive operators, bounded by identity [32]:

$$\mathcal{E}_Q = \{X \in L^*(\mathcal{H}) \,|\, \mathbb{0} \leq X \leq \mathbb{1}\}. \tag{2.37}$$

Projection operators are extremal elements of this convex set [25].

Effect algebras for quantum miniatures We may define effect algebras for quantum miniatures in a similar way, by considering different partial orders on the space $L^*(\mathcal{H})$. Consider a given quantum miniature with set of observables $\mathcal{O}_M$ and denote by $\mathcal{P}_M$ the set consisting of all projectors onto the eigenvectors of each observable in $\mathcal{O}_M$. From the set $\mathcal{P}_M$, we construct the smallest cone containing the projectors in $\mathcal{P}_M$, and their convex combinations,

$$\mathcal{C}_M = \mathrm{cone}(\mathrm{conv}(\mathcal{P}_M)). \tag{2.38}$$

We will show that this cone is pointed and convex.

By definition, every element of $\mathcal{C}_M$ is a non-negative multiple of a convex combination of projectors in the set $\mathcal{P}_M$. Hence, given $X, X' \in \mathcal{C}_M$, we may write

$$X = \alpha \sum_{i\in I} \mu_i E_i, \qquad X' = \alpha' \sum_{j\in J} \mu'_j E'_j, \tag{2.39}$$

where $\alpha, \alpha' \in \mathbb{R}_+$, $\{\mu_i\}_{i\in I}, \{\mu'_j\}_{j\in J}$ are probability distributions and $E_i, E'_j \in \mathcal{P}_M$. Now, given $\lambda \in (0,1)$, we have

$$\lambda X + (1-\lambda)X' = \sum_{i\in I} \lambda\alpha\mu_i E_i + \sum_{j\in J}(1-\lambda)\alpha'\mu'_j E'_j \tag{2.40}$$

$$= \alpha'' \sum_{k\in I\dot\cup J} \mu''_k E''_k, \tag{2.41}$$

where $\alpha'' = \lambda\alpha + (1-\lambda)\alpha'$ and

$$\mu''_k = \begin{cases} \frac{\lambda\alpha}{\alpha''}\mu_k & k \in I \\ \\ \frac{(1-\lambda)\alpha'}{\alpha''}\mu'_k & k \in J \end{cases}, \qquad E''_k = \begin{cases} E_k & k \in I \\ \\ E'_k & k \in J \end{cases}. \tag{2.42}$$

Here, $\alpha'' \in R_+$, $\{\mu''_k\}_{k\in I\dot\cup J}$ is a probability distribution and $E''_k \in \mathcal{P}_M$, so $\lambda X + (1-\lambda)X' \in \mathcal{C}_M$, which demonstrates that $\mathcal{C}_M$ is convex.

Furthermore, note that $\mathcal{P}_M$ contains only positive operators (in particular, projection operators). Any convex combination of elements of $\mathcal{P}_M$, and any non-negative scalar multiple thereof, is also a positive operator, i.e., $\mathcal{C}_M$ contains only positive operators. Hence, $X \in \mathcal{C}_M \cap (-\mathcal{C}_M)$ only if $X = \mathbb{0}$, i.e., $\mathcal{C}_M$ is a pointed cone.

We can then define the effect algebra $\mathcal{E}_M$ of the quantum miniature in analogy with (2.37) as an interval with respect to the partial order $\preceq_M$,

$$\mathcal{E}_M = \{X \in L^*(\mathcal{H}) \,|\, \mathbb{0} \preceq_M X \preceq_M \mathbb{1}\}. \tag{2.43}$$

By Proposition 3.23 in [71], $\mathcal{E}_M$ has the structure of a convex effect algebra.

Since $\mathcal{C}_M$ contains only positive operators, we have

$$X \preceq_M Y \implies X \leq Y. \tag{2.44}$$

It follows that the effect algebra of a quantum miniature is always a subset of the quantum effect algebra $\mathcal{E}_Q$. Elements of $\mathcal{P}_M$ are extremal points in $\mathcal{E}_M$: being projection operators, they can not expressed as non-trivial convex combination of elements in $\mathcal{E}_Q$, in particular of elements in the subset $\mathcal{E}_M$. It also follows that the effect algebra of a quantum miniature is equal to the quantum effect algebra only if $\mathcal{P}_M$ contains all projectors.

Quasi-density operators as pre-states We now construct the naturally associated pre-state space $\mathcal{S}(\mathcal{E}_M)$, in the sense of (2.36). First we must construct the dual cone, $\mathcal{C}_M^*$. This is a cone in the dual vector space $[L^*(\mathcal{H})]^*$ of linear maps $L^*(\mathcal{H}) \to \mathbb{R}$, defined by

$$\mathcal{C}_M^* = \{f \in [L^*(\mathcal{H})]^* \mid f(X) \geq 0 \text{ for all } X \in \mathcal{C}_M\} \tag{2.45}$$

Since $L^*(\mathcal{H})$ is a real inner product space under the Hilbert-Schmidt inner product, the Riesz representation theorem [75] guarantees that, for every map $f \in [L^*(\mathcal{H})]^*$, there exists a unique operator $\rho \in L^*(\mathcal{H})$ such that[5]

$$f(X) = \mathrm{Tr}\,[\rho X] \text{ for all } X \in L^*(\mathcal{H}). \tag{2.46}$$

Hence, C_M^* is isomorphic to a set of self-adjoint operators:

$$\mathcal{C}_M^* \cong \{\rho \in L^*(\mathcal{H}) \mid \mathrm{Tr}\,[\rho X] \geq 0 \text{ for all } X \in \mathcal{C}_M\}. \tag{2.47}$$

The pre-state space associated to $\mathcal{E}_M$ consists of the normalised elements of the dual cone,

$$\mathcal{S}(\mathcal{E}_M) = \{f \in \mathcal{C}_M^* \mid f(\mathbb{1}) = 1\}. \tag{2.48}$$

This is isomorphic to a set of *quasi-density operators*:

$$\mathcal{S}(\mathcal{E}_M) \cong \{\rho \in L^*(\mathcal{H}) \mid \mathrm{Tr}\,[\rho] = 1, \mathrm{Tr}\,[\rho X] \geq 0 \text{ for all } X \in \mathcal{C}_M\}. \tag{2.49}$$

Note finally that the quasi-density operators in the set (2.49) are exactly those which are compatible with all projectors in $\mathcal{P}_M$, since

$$\mathrm{Tr}\,[\rho X] \geq 0 \text{ for all } X \in \mathcal{C}_M \iff \mathrm{Tr}\,[\rho X] \geq 0 \text{ for all } X \in \mathcal{P}_M. \tag{2.50}$$

The set of quasi-density operators in (2.49) is therefore exactly the set Ω_M described in Section 2.2.

Equivalence classes By definition, states in GPTs are distinguishable on the basis of measurement probabilities. Distinct quasi-density operators may generate identical probabilities for all available measurements in a miniature. To align with the GPT

[5] The operator ρ is specified uniquely by the action of f on the whole of $L^*(\mathcal{H})$; distinct operators may result in the same action on a given subspace of interest. This point is developed below in the discussion of equivalence classes.

definition of states, quasi-density operators may be sorted into equivalence classes, as discussed in Section 2.3. By construction, equivalence classes *are* distinguishable on the basis of measurement probabilities.

An alternative approach when describing miniatures in terms of convex cones is to recognise that $\mathcal{C}_M$ may be understood as a cone in the subspace of $L^*(\mathcal{H})$ spanned by the projectors onto eigenstates of the basis observables, $V_M = \mathrm{span}(\mathcal{P}_M)$, rather than the full vector space $L^*(\mathcal{H})$ if these differ. In this case, $\mathcal{C}_M$ is a generating cone in V_M. We may then understand the dual cone $\mathcal{C}_M^*$ as a cone in the dual space V_M^*. The Riesz representation theorem still holds, which shows that all equivalence classes have a unique representative in $\mathrm{span}(\mathcal{P}_M)$. Note that we always have $\mathbb{1} \in \mathrm{span}(\mathcal{P}_M)$ since $\mathcal{P}_M$ contains at least one measurement, i.e., a collection $\{E_i\}_i$ such that $\sum_i E_i = \mathbb{1}$.

— 3 —

Finite dimensions: quantum miniatures for spin systems

3.1 SPIN $s = 1/2$

3.1.1 Square-world

Set-up Consider an experimental set-up where one has access to a Stern-Gerlach apparatus to perform measurements of spin. This particular apparatus can not be rotated arbitrarily so as to measure spin components in an arbitrary direction, but rather is limited to a fixed pair of mutually orthogonal directions, which we label x and z respectively. It is observed that there are two possible outcomes from each measurement, which we label $+$ and $-$, respectively. We aim to construct a Quantum Miniature, which we call Square-world, to describe this set-up. It will be shown that this theory reproduces the predictions of Boxworld (also known as the g-bit or squit), a well-known example of a GPT, studied in detail in [7].

Following the principles laid out in Chapter 2, the theory is developed in terms of a Hilbert space, here $\mathcal{H} = \mathbb{C}^2$, and anything which can be measured by the experimental apparatus is to be considered an observable within the theory. Furthermore, we assume that any such observable is represented by the same Hermitian operator as appears in the usual quantum mechanical description. Any property which can not be measured is not treated as an observable in the theory.

In the case considered here, the apparatus allows measurements of spin in the directions x and z, so we include the usual operators σ_x and σ_z as observables in the theory. Since our apparatus is limited to these directions we do not, for example, take spin in other directions to be observable.

The full set of basic observables, $\mathcal{O}_{square}$, for this particular miniature is therefore

$$\mathcal{O}_{square} = \{\sigma_x, \sigma_z\}. \tag{3.1}$$

State space We now construct the state space of Square-world. In line with the general idea of Quantum Miniatures, we allow as a state any normalised Hermitian operator $\rho : \mathcal{H} \to \mathcal{H}$ which is compatible with all available measurements. Given any Hermitian operator, ρ, we may write

$$\rho = \frac{1}{2}\left(a\sigma_x + b\sigma_y + c\sigma_z + d\mathbb{1}\right) \tag{3.2}$$

for some $a, b, c, d \in \mathbb{R}$. Compatibility is defined by the condition that these generate valid probabilities for each possible measurement outcome of the available measurements σ_x and σ_z. The projectors onto the eigenstates of σ_x and σ_z are

$$E^x_+ = \frac{\mathbb{1} + \sigma_x}{2}, \qquad E^x_- = \frac{\mathbb{1} - \sigma_x}{2}, \tag{3.3}$$

$$E^z_+ = \frac{\mathbb{1} + \sigma_z}{2}, \qquad E^z_- = \frac{\mathbb{1} - \sigma_z}{2}, \tag{3.4}$$

and these define the probabilities for each outcome:

$$\mathcal{P}(\sigma_x = \pm|\rho) = \text{Tr}\left[\rho E^x_\pm\right], \qquad \mathcal{P}(\sigma_z = \pm|\rho) = \text{Tr}\left[\rho E^z_\pm\right]. \tag{3.5}$$

By direct computation,

$$\text{Tr}\left[\rho\right] = d, \tag{3.6}$$

and the predicted probabilities are

$$\text{Tr}\left[\rho E^x_\pm\right] = \frac{d \pm a}{2}, \qquad \text{Tr}\left[\rho E^z_\pm\right] = \frac{d \pm c}{2}. \tag{3.7}$$

We require that states have unit trace[1], and that predicted probabilities are non-negative. No other positivity constraints are enforced. Hence, ρ is a valid state if and only if $d = 1$, $-1 \leq a \leq 1$ and $-1 \leq c \leq 1$. This defines the space, Ω_{square}, of Square-world states:

$$\Omega_{square} = \left\{\frac{1}{2}\left(\mathbb{1} + a\sigma_x + b\sigma_y + c\sigma_z\right)\middle| -1 \leq a, c \leq 1, b \in \mathbb{R}\right\}. \tag{3.8}$$

[1] If we adopt the idea of unreliable preparations from the generalised probabilistic theories framework, then we take $0 \leq \text{Tr}\left[\rho\right] \leq 1$, and the state space is a square-faced pyramid rather than the square face itself. Throughout this work, we restrict attention to reliable preparations, i.e., all states considered have unit trace.

We can compare this with the space, Ω_Q, of states available in standard quantum theory:

$$\Omega_Q = \left\{ \frac{1}{2} \left(\mathbb{1} + a\sigma_x + b\sigma_y + c\sigma_z \right) \middle| (a^2 + b^2 + c^2)^{1/2} \leq 1 \right\}. \tag{3.9}$$

We see that Ω_Q is a strict subset of Ω_{square}, so that Square-world contains all quantum states, but also states which are not admissible in quantum theory with a full set of observables. In particular, there are states which predict negative probabilities if we apply the Born rule to operators which are observable in standard quantum theory but not in the miniature.

Consider, for example, the state $\rho = (\mathbb{1} + \sigma_x + \sigma_z)/2$. For a hypothetical measurement of the operator $(\sigma_x + \sigma_z)/\sqrt{2}$, we would predict, according to the Born rule, that

$$\mathcal{P}\left(\frac{\sigma_x + \sigma_z}{\sqrt{2}} = + \middle| \rho \right) = \frac{1 + \sqrt{2}}{2}, \qquad \mathcal{P}\left(\frac{\sigma_x + \sigma_z}{\sqrt{2}} = - \middle| \rho \right) = \frac{1 - \sqrt{2}}{2}. \tag{3.10}$$

Placing a restriction on the available measurements has thus allowed for a wider class of states to be admissible in the theory.

An alternative approach is to consider a construction in the manner of (1.2). Thus, we can define an operator A by the property that

$$\langle A|\rho\rangle = \frac{1}{\sqrt{2}}(\langle \sigma_x|\rho\rangle + \langle \sigma_z|\rho\rangle) \qquad \text{for all } \rho \in \Omega_{square}. \tag{3.11}$$

This uniquely identifies

$$A = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z). \tag{3.12}$$

However, $\langle A|\rho\rangle$ has no tenable interpretation as an expectation value if we assume that measurement outcomes are eigenvalues of the associated operator. Making this assumption, then a measurement of A has possible outcomes ± 1. A probability distribution over two outcomes is completely determined by its expectation value:

$$\text{“}\mathcal{P}(A = +1|\rho)\text{”} = \frac{1}{2}(1 + \langle A|\rho\rangle), \tag{3.13}$$

$$\text{“}\mathcal{P}(A = -1|\rho)\text{”} = \frac{1}{2}(1 - \langle A|\rho\rangle) \tag{3.14}$$

For some states, this ‘probability’ is negative, e.g. for $\rho = (\mathbb{1} + \sigma_x + \sigma_z)/2$,

$$\text{“}\mathcal{P}(A = -1|\rho)\text{”} = \frac{1 - \sqrt{2}}{2} < 0. \tag{3.15}$$

We therefore can not interpret A as being observable, even in principle. We see that compatibility with some collection of operators does not imply compatibility with linear (even convex) combinations thereof. Hence, the state space – being defined by compatibility with basic observables – is not necessarily compatible with an extension of $\mathcal{O}_{square}$ by closure under linear combinations.

Equivalence Classes So far, we have carried over the notion from quantum mechanics that (quasi-) density operators correspond to states one-to-one. In quantum mechanics, distinct density operators may always be distinguished by appropriate measurements. In Square-world and other miniatures, however, it is not guaranteed that the measurement which could distinguish the relevant operators is admissible in the theory.

It may be more appropriate then to define states in Square-world as equivalence classes of operators, where two operators are equivalent if they generate identical probability distributions for σ_x and σ_z measurements.

Two operators in Ω_{square},

$$\rho = \frac{1}{2}\left(\mathbb{1} + a\sigma_x + b\sigma_y + c\sigma_z\right), \qquad \rho' = \frac{1}{2}\left(\mathbb{1} + a'\sigma_x + b'\sigma_y + c'\sigma_z\right), \tag{3.16}$$

are equivalent if and only if $a = a'$ and $c = c'$. Hence, the equivalence class of an operator $\rho = (\mathbb{1} + a\sigma_x + b\sigma_y + c\sigma_z)/2$ is

$$[\rho] = \left\{ \frac{1}{2}\left(\mathbb{1} + a\sigma_x + b'\sigma_y + c\sigma_z\right) \middle| b' \in \mathbb{R} \right\}. \tag{3.17}$$

Denote by $\bar{\Omega}_{square}$ the space of equivalence classes:

$$\bar{\Omega}_{square} = \left\{ [\rho] \middle| \rho \in \Omega_{square} \right\}. \tag{3.18}$$

We can identify each equivalence class by any operator within the class, and for convenience we will choose representative operators for which $b = 0$:

$$\bar{\Omega}_{square} = \left\{ \frac{1}{2}\left(\mathbb{1} + a\sigma_x + c\sigma_z\right) \middle| -1 \leq a, c \leq 1 \right\}. \tag{3.19}$$

This set of representatives is closed under convex combinations, and gives a concrete realisation of the convex structure of $\bar{\Omega}_{square}$, in accordance with (2.10). Any state in $\bar{\Omega}_{square}$ may be written as a convex combination of the four extremal states

$$\tilde{\rho}_1 = \frac{\mathbb{1} + \sigma_x + \sigma_z}{2}, \qquad \tilde{\rho}_2 = \frac{\mathbb{1} + \sigma_x - \sigma_z}{2}, \tag{3.20}$$

$$\tilde{\rho}_3 = \frac{\mathbb{1} - \sigma_x - \sigma_z}{2}, \qquad \tilde{\rho}_4 = \frac{\mathbb{1} - \sigma_x + \sigma_z}{2}. \tag{3.21}$$

	$\tilde{\rho}_1$	$\tilde{\rho}_2$	$\tilde{\rho}_3$	$\tilde{\rho}_4$
E^x_+	1	1	0	0
E^x_-	0	0	1	1
E^z_+	1	0	0	1
E^z_-	0	1	1	0

Table 3.1: Table of fiducial probabilities for Square-world.

Using these and the projectors (3.3), we can generate the following table of fiducial probabilities: These reproduce the statistics for fiducial measurements in Boxworld [7]. The probabilities of any measurement outcome in any state may be calculated from these fiducial probabilities by linearity.

From the probability table 3.1 we see that the additional, non-quantum, states found in Square-world have properties different from any quantum state. For example, the extremal states all predict definite outcomes for a measurement of σ_x, and also predict definite outcomes for a measurement of σ_z. No quantum state has this property since the operators σ_x and σ_z have no eigenstates in common.

The extremal states, $\tilde{\rho}_n$, are examples of joint quasi-eigenstates. A state, ρ, is a quasi-eigenstate of an observable, X, if ρ predicts a definite outcome for a measurement of X, or equivalently if the variance vanishes,

$$\Delta^2_\rho(X) = 0. \tag{3.22}$$

We do not, however, require that $X\rho = \lambda\rho$.

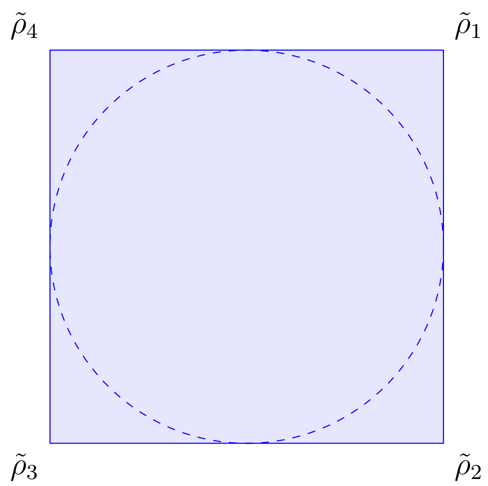


Figure 3.1: For a Square-world state $\rho = \frac{1}{2}(\mathbb{1} + \mathbf{r}\cdot\boldsymbol{\sigma})$, $\mathbf{r}$ may lie anywhere in the blue square shown, while quantum states are restricted to the dashed circle.

Uncertainty Relations Quantum theory predicts certain constraints on the uncertainties of two or more observables in a given state. The observables available in Square-world, S_x and S_z, have no eigenstates in common. Quantum mechanics therefore predicts that there is no state in which both of these observables can simultaneously be sharply predicted, i.e., in which the variance for both observables is zero. For spin-1/2, a quantitative bound is given by

$$\Delta^2_\rho(S_x) + \Delta^2_\rho(S_z) \geq \frac{1}{4}, \tag{3.23}$$

where ρ is any quantum state.

We have seen, however, that Square-world contains states which are not quantum states. The proof of relation (3.23) does not apply to these states, since the corresponding quasi-density operators are not positive-semidefinite. In fact, there is no non-trivial lower bound on the sum of the variances, since all four extremal states, $\tilde{\rho}_i$, predict zero variances for both measurements:

$$\Delta^2_{\tilde{\rho}_i}(S_x) + \Delta^2_{\tilde{\rho}_i}(S_z) = 0. \tag{3.24}$$

A general expression for the sum of variances in a state $\rho = (\mathbb{1} + a\sigma_x + c\sigma_z)/2$ is given by

$$\Delta^2_\rho(S_x) + \Delta^2_\rho(S_z) = \frac{1}{2} - \frac{a^2 + c^2}{4}, \tag{3.25}$$

which illustrates that the relation (3.23) is in fact violated by all non-quantum states of Square-world. The full Square-world uncertainty region, $\mathrm{PUR}_{square}(S_x, S_z)$, for the observables S_x and S_z is defined simply by independent upper and lower bounds, with the lower bounds themselves being trivial:

$$\mathrm{PUR}_{square}(S_x, S_z) = [0, 1/2] \times [0, 1/2]. \tag{3.26}$$

In moving to the quantum miniatures framework, we have not changed the description of observables compared to the ordinary quantum mechanical description. In particular, any relationship between these is preserved, including commutation relations. Thus, while uncertainty relations like (3.23) refer explicitly only to S_x and S_z, their validity is implicitly related to the ability (in principle) to measure other observables which do not feature directly in the relation.

Dynamics and state update A spin-1/2 system in standard quantum mechanics has spherical symmetry, related to the ability to (in principle) rotate the measurement apparatus in an arbitrary direction, or a passive equivalent. Restricting the measurements breaks this symmetry, leaving a state space which is symmetric only under permutations of the extremal states. This is related to a relabelling of the directions available to the measurement apparatus, and of the outcomes spin-up or spin-down. There are no non-trivial continuous dynamics which respect this symmetry.

Another novel feature of Square-world is that the state in which measurement of a given observable has definite outcome is non-unique. For example, in any state which is a convex combination of $\tilde{\rho}_1$ and $\tilde{\rho}_2$, a measurement of σ_x will result in the + outcome with probability 1. This may have implications for the post-measurement state.

3.1.2 POLYGON-WORLDS

It is possible to construct a wide variety of quantum miniatures in the same spirit as Square-world, by considering similar Stern-Gerlach apparatuses with different restrictions. We again assume we have access to an ensemble of spin-1/2 systems and a Stern-Gerlach apparatus, which may be directed in a certain set of directions. An interesting family of miniatures – including Square-world as a special case – arises if we take these allowed directions to exhibit discrete rotational symmetry. The state space of each miniature is a polygon which inherits the symmetry of the observables in each case, illustrating the general argument of 2.4. Disk-world may be seen as a limiting case where the number of directions approaches infinity, and reproduces the rebit, a subtheory of quantum mechanics [16, 2].

Observables The Hilbert space in each case is $\mathcal{H} = \mathbb{C}^2$. A particular miniature within the Polygon-world family is specified by a natural number, N, which specifies how many directions spin can be measured in. In each case, we take these directions, $\boldsymbol{\eta}_n$, to be evenly spaced around the xz-plane:

$$\boldsymbol{\eta}_n = \begin{pmatrix} \cos\phi_n \\ 0 \\ \sin\phi_n \end{pmatrix}, \qquad \text{where } \phi_n = \frac{2\pi(n-1)}{N} \tag{3.27}$$

for $n = 1, \ldots, N$. Our experimental apparatus allows measurements of spin components in each available direction, $S_{\boldsymbol{\eta}_n} = (\boldsymbol{\eta}_n \cdot \boldsymbol{\sigma})/2$. We note that there is some symmetry inherent in the set-up. This is because, for any direction $\boldsymbol{\eta}$, we have that $S_{-\boldsymbol{\eta}} = -S_{\boldsymbol{\eta}}$, so any apparatus capable of measuring $S_{\boldsymbol{\eta}}$ is also effectively capable of measuring $S_{-\boldsymbol{\eta}}$. The directions available to our apparatus therefore come in pairs, and it is natural to only consider cases where N is even. Furthermore, it suffices to take only those spin components within a given half-plane, say with $0 \leq \phi_n < \pi$. This gives the set of basic observables for an N-gon model:

$$\mathcal{O}_{N\text{-gon}} = \{\boldsymbol{\eta}_n \cdot \boldsymbol{\sigma}\}_{n=1,\dots,N/2} \,. \tag{3.28}$$

States States of each model are those quasi-density operators which predict valid probabilities for each outcome of each observable. The projectors onto to the $\pm$ eigenstates of $\boldsymbol{\eta}_n \cdot \boldsymbol{\sigma}$ are

$$E_n^{\pm} = \frac{\mathbb{1} \pm \boldsymbol{\eta}_n \cdot \boldsymbol{\sigma}}{2}. \tag{3.29}$$

Given any quasi-density operator $\rho = (\mathbb{1} + \mathbf{r} \cdot \boldsymbol{\sigma})/2$, we have

$$\operatorname{Tr}\left[\rho E_n^{\pm}\right] = \frac{1 \pm \boldsymbol{\eta}_n \cdot \mathbf{r}}{2}. \tag{3.30}$$

Thus, ρ is a state exactly if each of these probabilities is non-negative for every basic observable. Since the respective probabilities for the $+$ and $-$ outcomes sum to one, positivity also implies that each probability is no greater than 1.

The state space is thus constrained by N inequalities:

$$\begin{aligned} \boldsymbol{\eta}_n \cdot \mathbf{r} &\geq -1 \\ \boldsymbol{\eta}_n \cdot \mathbf{r} &\leq 1 \end{aligned} \qquad \text{for each } n = 1, \dots, \frac{N}{2}, \tag{3.31}$$

or, equivalently,

$$\boldsymbol{\eta}_n \cdot \mathbf{r} \leq 1 \qquad \text{for each } n = 1, \dots, N. \tag{3.32}$$

These inequalities define N lines bounding the state space. The lines indexed by n and m respectively are linearly independent if and only if $n \neq m$. The extremal states are at points where two of these lines meet, subject to satisfying the remaining inequalities. This occurs only for lines whose indices differ by one, intersecting at the N points, r_n, which satisfy $\boldsymbol{\eta}_n \cdot \mathbf{r}_n = \boldsymbol{\eta}_{n+1} \cdot \mathbf{r}_n = 1$:

$$\mathbf{r}_n = \sec\left(\frac{\pi}{N}\right) \begin{pmatrix} \cos\left(\frac{2\pi n}{N} - \frac{\pi}{N}\right) \\ \sin\left(\frac{2\pi n}{N} - \frac{\pi}{N}\right) \end{pmatrix}. \tag{3.33}$$

The extremal states, $\tilde{\rho}_n$, are then given by

$$\tilde{\rho}_n = \frac{1}{2}\left(\mathbb{1} + \mathbf{r}_n \cdot \boldsymbol{\sigma}\right). \tag{3.34}$$

These are joint quasi-eigenstates of $\boldsymbol{\eta}_n \cdot \boldsymbol{\sigma}$ and $\boldsymbol{\eta}_{n+1} \cdot \boldsymbol{\sigma}$, each having value $+1$ with probability 1. Extremal states therefore violate the quantum uncertainty relation (1.11).

The state space is the convex hull of the extremal points $\mathbf{r}_n$. This is a regular N-gon with radius $\mathcal{R}_N = \sec(\pi/N)$, defined as the maximum magnitude of the vector $\mathbf{r}$ compatible with (3.31). The edges of the polygon are normal to the directions in which spin measurements can be performed.

These models reproduce the predictions for single-party systems of the polygon models introduced by Janotta et al [47].

3.1.3 Disk-world and the Bloch Disk

We now consider an experimental setting where we again have a Stern-Gerlach apparatus which is limited to the xz-plane, but we now assume that this apparatus is able to rotate freely within the plane. Every spin component within this plane is therefore an observable. Denote by P_{xz} the set of unit vectors within the xz-plane, then

$$\mathcal{O}_{\text{disk}} = \{\boldsymbol{\eta} \cdot \boldsymbol{\sigma} \,|\, \boldsymbol{\eta} \in P_{xz}\} \tag{3.35}$$

The state space is constrained by inequalities of the form (3.32), related to every direction in the xz-plane:

$$\boldsymbol{\eta} \cdot \mathbf{r} \leq 1 \qquad\qquad \text{for all } \boldsymbol{\eta} \in P_{xz}. \tag{3.36}$$

These inequalities are collectively satisfied only if ρ is equivalent to a positive operator, after modding out r_y. Every state of Disk-world is therefore also a quantum state. We have an equality of equivalence classes,

$$\bar{\Omega}_{disk} = \bar{\Omega}_Q. \tag{3.37}$$

Here, $\bar{\Omega}_Q$ denotes the set of quantum states, modulo the equivalence relation

$$\rho \sim \rho' \iff r_y(\rho) = r_y(\rho'). \tag{3.38}$$

We can characterise $\bar{\Omega}_Q$ by a set of representatives which have $r_y = 0$:

$$\bar{\Omega}_Q = \left\{ \frac{1}{2} \left(\mathbb{1} + a\sigma_x + c\sigma_z \right) \,\middle|\, a^2 + c^2 \leq 1 \right\}. \tag{3.39}$$

These representatives constitute all positive semi-definite 2×2 matrices with real-valued entries and unit trace. This gives the Bloch disk, the state space of the rebit.

The state spaces of the n-gon models converge to the state space of Disk-world as $n \to \infty$, in the sense that $\Omega_{disk} \subset \Omega_n$ for all n and, given any point $\mathbf{r} \notin \Omega_{disk}$,

$$\exists N \in \mathbb{N} \text{ such that } n > N \implies \mathbf{r} \notin \Omega_n. \tag{3.40}$$

In particular,

$$\lim_{n\to\infty} \mathcal{R}_n = 1. \tag{3.41}$$

The equality of equivalence classes $\bar{\Omega}_{disk} = \bar{\Omega}_Q$ implies that the uncertainty region [14] for any tuple of observables in Disk-world is identical to the quantum uncertainty region: no quantum uncertainty relation concerning operators which represent Disk-world observables can be violated by any Disk-world state.

3.1.4 Cylinder-world

We have so far considered models in which the available directions for spin measurements are restricted to (a subset of) the plane. We now consider an experimental apparatus which can freely rotate within the xz-plane, similarly to Disk-world, and can also rotate discretely at an angle of $\pi/2$, in a manner similar to Square-world, to point in the y-direction.

Thus, with this apparatus one can measure spin components in all directions within the xz-plane, and in the y-direction, but not in any intermediate directions. The observables are

$$\mathcal{O}_{cyl} = \{ \boldsymbol{\eta} \cdot \boldsymbol{\sigma} \,|\, \boldsymbol{\eta} \in P_{xz} \} \cup \{ \sigma_y \}. \tag{3.42}$$

This model exhibits continuous symmetry under rotations in the xz-plane.

States Every Disk-world observable is also an observable within Cylinder-world. We therefore have all of the constraints imposed in Disk-world, (3.36), and additionally a pair of constraints on r_y, reflecting compatibility with the additional observable σ_y:

$$\mathrm{Tr}\left[\rho E_y^{\pm} \right] \geq 0 \iff -1 \leq r_y \leq 1. \tag{3.43}$$

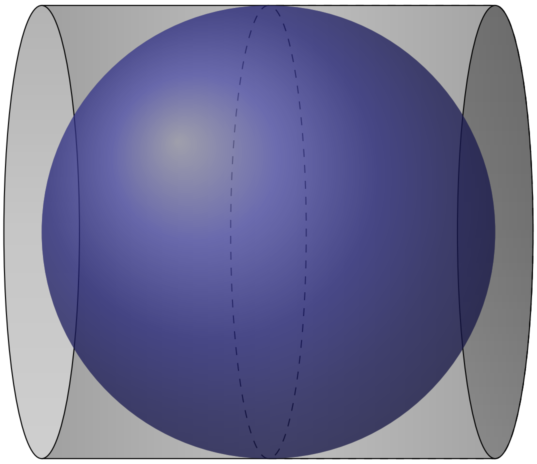


Figure 3.2: Illustration of the cylindrical state space of Cylinder-world, Ω_{cyl}. The state space of standard quantum mechanics is the inscribed sphere (the Bloch sphere), also shown.

These constraints are independent of the constraints (3.36), which only affect the components r_x and r_z. The space of quasi-density operators fitting these constraints can therefore be expressed in a factorisable form:

$$\Omega_{cyl} = \Omega_{disk} \times [-1, 1]. \tag{3.44}$$

The state space is a cylinder. This cylinder circumscribes the Bloch sphere, as illustrated in Figure 3.2.

The set of Disk-world observables $\mathcal{O}_{disk}$ is a strict subset of the set of Cylinder-world observables, $\mathcal{O}_{cyl}$. Accordingly, the set Ω_{cyl} of quasi-density operators compatible with $\mathcal{O}_{cyl}$ is a strict subset of Ω_{disk}, the set of quasi-density operators compatible with $\mathcal{O}_{disk}$, since all constraints from $\mathcal{O}_{disk}$ are imposed, as well as additional constraints.

However, the larger set of observables in $\mathcal{O}_{cyl}$ allows for finer distinction between quasi-density operators, impacting the structure of equivalence classes. Indeed, $\mathcal{O}_{cyl}$ is tomographically complete, so that any pair of quasi-density operators can be distinguished on the basis of available measurements (for example, the parameters a, b, c in (3.2) are equal to the expectation values of $\sigma_x, \sigma_y, \sigma_z$ respectively, so may be determined by these measurements which are all contained in $\mathcal{O}_{cyl}$). Hence, all equivalence classes in Cylinder-world are trivial, containing exactly one quasi-density operator. This means that $\Omega_{cyl} \subset \Omega_{disk}$, whereas $\bar{\Omega}_{cyl} \supset \bar{\Omega}_{disk}$. In particular,

$$\Omega_{cyl} = \Omega_{disk} \cap (\mathbb{R}^2 \times [-1, 1]) \tag{3.45}$$

and

$$\bar{\Omega}_{cyl} = \bar{\Omega}_{disk} \times [-1, 1]. \tag{3.46}$$

3.1.5 Recovering standard quantum theory

In general, any miniature in which every self-adjoint operator is observable has a state space equal to that of the corresponding quantum system. For spin-1/2, an arbitrary Hermitian operator X on $\mathcal{H} = \mathbb{C}^2$ can be written

$$X = a\sigma_x + b\sigma_y + c\sigma_z + d\mathbb{1} \tag{3.47}$$

with $a, b, c, d \in \mathbb{R}$. This expression can be equivalently written as a function of a single spin component,

$$X = A(\boldsymbol{\eta} \cdot \boldsymbol{\sigma}) + d(\boldsymbol{\eta} \cdot \boldsymbol{\sigma})^2, \tag{3.48}$$

where $A = \sqrt{a^2 + b^2 + c^2}$ and $\boldsymbol{\eta} \cdot \boldsymbol{\sigma}$ is the spin component in the direction

$$\boldsymbol{\eta} = \frac{1}{A} \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{3.49}$$

To recover the usual quantum state space, it is therefore sufficient to assume that spin components in every direction can be measured.

It is also sufficient to assume that any dense subset of the sphere correspond to spin components which can be measured. Let $\mathcal{D}$ be a dense subset of S^2, then given any direction $\boldsymbol{\eta} \in S^2$, there exists a Cauchy sequence $(\boldsymbol{\eta}_n)_{n \in \mathbb{N}}$ within $\mathcal{D}$ such that

$$\lim_{n \to \infty} \boldsymbol{\eta}_n = \boldsymbol{\eta}. \tag{3.50}$$

Given any vector $\mathbf{r}$ parameterising a quasi-density operator, the function $\mathbf{x} \mapsto \mathbf{x} \cdot \mathbf{r}$ is continuous, so

$$\lim_{n \to \infty} (\boldsymbol{\eta}_n \cdot \mathbf{r}) = \left(\lim_{n \to \infty} \boldsymbol{\eta}_n \right) \cdot \mathbf{r} = \boldsymbol{\eta} \cdot \mathbf{r}. \tag{3.51}$$

It then follows that

$$-1 \leq \boldsymbol{\eta}_n \cdot \mathbf{r} \leq 1 \text{ for all } n \in \mathbb{N} \implies -1 \leq \boldsymbol{\eta} \cdot \mathbf{r} \leq 1, \tag{3.52}$$

so any quasi-density operator which is compatible with $\boldsymbol{\eta}_n \cdot \boldsymbol{\sigma}$ for all $n \in \mathbb{N}$ is also compatible with the spin component $\boldsymbol{\eta} \cdot \boldsymbol{\sigma}$. Hence, any quasi-density operator which is compatible with spin components on a dense subset of the sphere is compatible with spin components in all directions, and is therefore a proper density operator.

3.2 SPIN $s = 1$

3.2.1 General setting

In this section, we consider quantum miniatures for spin-1 systems. The general setting is similar to that developed in Section 3.1 for spin-1/2 systems. For each miniature under consideration, we take the set of observables to be dictated by the capabilities of some Stern-Gerlach apparatus. In each case, we specify a set of directions, $\mathcal{D} \subseteq S^2$, which we assume are available to the measurement apparatus. Given a direction specified by a unit vector $\boldsymbol{\eta} \in S^2$, the spin component in that direction is

$$S_{\boldsymbol{\eta}} = \boldsymbol{\eta} \cdot \mathbf{S}. \tag{3.53}$$

The basic observables are then the spin components in each available direction:

$$\mathcal{O}_{\mathcal{D}} = \left\{ S_{\boldsymbol{\eta}} \,|\, \boldsymbol{\eta} \in \mathcal{D} \right\}. \tag{3.54}$$

Spin-1 systems provide more scope than spin-1/2 for the range of observables that may be considered. In spin-1/2, any Hermitian operator may be written as a function of a single spin component; in particular as a linear combination of a spin component and the identity operator. Spin-1, in contrast, allows for non-trivial quadratic expressions in the spin components. For example, given spins in mutually orthogonal directions, $S_{\boldsymbol{\eta}}$ and $S_{\boldsymbol{\eta}'}$, the symmetric product $(S_{\boldsymbol{\eta}} S_{\boldsymbol{\eta}'} + S_{\boldsymbol{\eta}'} S_{\boldsymbol{\eta}})/2$ is not expressible as a function of any single spin component.

States We can parameterise quasi-density operators for spin-1 systems in a similar way to those of spin-1/2 systems. Given any unit-trace Hermitian operator $\rho : \mathcal{H} \to \mathcal{H}$, we may write [85]

$$\rho = \frac{1}{3}\mathbb{1} + \sum_{i=1}^{8} r_i T_i = \frac{1}{3}\mathbb{1} + \mathbf{r} \cdot \mathbf{T}, \tag{3.55}$$

where $r_i \in \mathbb{R}$ and T_i are spin multipole operators:

$$T_1 = S_x, \qquad T_2 = S_y, \qquad T_3 = S_z; \tag{3.56}$$

$$T_4 = \frac{1}{2}(S_x S_y + S_y S_x), \quad T_5 = \frac{1}{2}(S_y S_z + S_z S_y), \quad T_6 = \frac{1}{2}(S_z S_x + S_x S_z); \tag{3.57}$$

$$T_7 = S_x^2 - \frac{2}{3}\mathbb{1}, \qquad T_8 = S_y^2 - \frac{2}{3}\mathbb{1}. \tag{3.58}$$

Each of these operators is traceless. With respect to the Hilbert-Schmidt inner product, $\langle A, B\rangle_{HS} = \mathrm{Tr}\left[A^\dagger B\right]$, each pair of basis elements is orthogonal, with the exception that $\langle T_7, T_8\rangle_{HS} = -1/3$. The Hilbert-Schmidt norms of each multipole operator are

$$|T_1|^2 = |T_2|^2 = |T_3|^2 = 2, \quad |T_4|^2 = |T_5|^2 = |T_6|^2 = \frac{1}{2}, \quad |T_7|^2 = |T_8|^2 = \frac{2}{3}. \tag{3.59}$$

Expressing as matrices in the S_z eigenbasis:

$$T_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad T_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \tag{3.60}$$

$$T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \tag{3.61}$$

$$T_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad T_5 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \tag{3.62}$$

$$T_6 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}; \tag{3.63}$$

$$T_7 = \frac{1}{6}\begin{pmatrix} -1 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & -1 \end{pmatrix}, \qquad T_8 = \frac{1}{6}\begin{pmatrix} -1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & -1 \end{pmatrix}. \tag{3.64}$$

By definition, states are required to give valid probabilities for each outcome of any available observable, i.e., for any $\boldsymbol{\eta} \in \mathcal{D}$,

$$0 \le \mathrm{Tr}\left[\rho E^+_{\boldsymbol{\eta}}\right] \le 1, \qquad 0 \le \mathrm{Tr}\left[\rho E^0_{\boldsymbol{\eta}}\right] \le 1 \qquad \text{and} \qquad 0 \le \mathrm{Tr}\left[\rho E^-_{\boldsymbol{\eta}}\right] \le 1. \tag{3.65}$$

These inequalities contain redundant information; states are normalised and the projectors sum to identity, so that (identically, independent of any positivity constraints)

$$\mathrm{Tr}\left[\rho E^+_{\boldsymbol{\eta}}\right] + \mathrm{Tr}\left[\rho E^0_{\boldsymbol{\eta}}\right] + \mathrm{Tr}\left[\rho E^-_{\boldsymbol{\eta}}\right] = \mathrm{Tr}\left[\rho \mathbb{1}\right] = 1. \tag{3.66}$$

It is therefore sufficient to require that probabilities be non-negative:

$$\mathrm{Tr}\left[\rho E^+_{\boldsymbol{\eta}}\right] \ge 0, \qquad \mathrm{Tr}\left[\rho E^0_{\boldsymbol{\eta}}\right] \ge 0 \qquad \text{and} \qquad \mathrm{Tr}\left[\rho E^-_{\boldsymbol{\eta}}\right] \ge 0 \tag{3.67}$$

and it follows that each of these must be less than 1.

For any direction $\boldsymbol{\eta}$, $S_{\boldsymbol{\eta}}$ has eigenvalues $+1, 0, -1$ and so we can write the spectral decomposition:

$$S_{\boldsymbol{\eta}} = E^{+}_{\boldsymbol{\eta}} - E^{-}_{\boldsymbol{\eta}}, \tag{3.68}$$

where $E^{\pm}_{\boldsymbol{\eta}}$ are the projectors onto the $+1$ and -1 eigenstates respectively. Hence, the projectors onto the eigenvalues are

$$E^{\pm}_{\boldsymbol{\eta}} = \frac{1}{2}\left(S^2_{\boldsymbol{\eta}} \pm S_{\boldsymbol{\eta}}\right) \tag{3.69}$$

$$E^{0}_{\boldsymbol{\eta}} = \mathbb{1} - S^2_{\boldsymbol{\eta}} \tag{3.70}$$

Given a quasi-density operator ρ, the probabilities for each outcome are given by $\mathcal{P}(+|\rho) = \mathrm{Tr}\left[\rho E^{+}_{\boldsymbol{\eta}}\right]$ etc., and ρ is compatible with the observable $S_{\boldsymbol{\eta}}$ if and only if each of these quantities is a valid probability. Compatibility is then expressed by the following inequalities:

$$\langle S^2_{\boldsymbol{\eta}}\rangle_\rho + \langle S_{\boldsymbol{\eta}}\rangle_\rho \geq 0 \tag{3.71}$$

$$\langle S^2_{\boldsymbol{\eta}}\rangle_\rho - \langle S_{\boldsymbol{\eta}}\rangle_\rho \geq 0 \tag{3.72}$$

$$1 - \langle S^2_{\boldsymbol{\eta}}\rangle_\rho \geq 0 \tag{3.73}$$

or, equivalently,

$$|\langle S_{\boldsymbol{\eta}}\rangle| \leq \langle S^2_{\boldsymbol{\eta}}\rangle \leq 1. \tag{3.74}$$

These can be expressed by linear inequalities on the vector $\mathbf{r} = (r_1, \ldots, r_8)$ appearing in (3.55):

$$2|\boldsymbol{\eta}\cdot\mathbf{r}| \leq \mathbf{V}_{\boldsymbol{\eta}}\cdot\mathbf{r} + \frac{2}{3} \leq 1, \tag{3.75}$$

where

$$\boldsymbol{\eta} = \begin{pmatrix} a \\ b \\ c \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{V}_{\boldsymbol{\eta}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ ab \\ bc \\ ca \\ a^2 - 1/3 \\ b^2 - 1/3 \end{pmatrix}. \tag{3.76}$$

Note that any value pair $\{\langle S_{\boldsymbol{\eta}}\rangle, \langle S_{\boldsymbol{\eta}}^2\rangle\}$ consistent with (3.74) is attainable by some quantum state. Consider

$$\rho = \alpha\left(\frac{1+\beta}{2}\,|1\rangle\langle 1| + \frac{1-\beta}{2}\,|-1\rangle\langle -1|\right) + (1-\alpha)\,|0\rangle\langle 0|\,; \tag{3.77}$$

this is a quantum state for any $0 \leq \alpha \leq 1$, $-1 \leq \beta \leq 1$, and

$$\langle S_z\rangle = \alpha\beta, \qquad\qquad \langle S_z^2\rangle = \alpha, \tag{3.78}$$

which exhausts all possible values consistent with (3.74). This suggests that the non-quantum character of any states which may appear in a spin-1 miniature can not be identified by the measurement statistics of any individual spin component, but rather in relationships between the statistics of distinct measurements. Uncertainty relations are an example of such relationships.

For each miniature we consider, we aim to identify the range of parameters r_i such that the associated quasi-density operator represents a state of that miniature. In standard quantum theory, the analogous problem has received much attention [15, 54, 11], and various bases have been used for parameterising density operators [46, 55, 11]. Even for standard quantum theory, the problem of identifying the space of parameters corresponding to valid states is known to be complicated. The basis we use, introduced in (3.55), does not have the property of a 'practical basis' [11], since not all basis elements are orthogonal, but is well adapted to specific consideration of spin measurements.

Overview We consider three models for spin-1 miniatures. One of these, Cube-world, has its state space completely characterised, with a complete list of extremal states given in (3.97). For the other two models, Cylinder-world and Sphere-world, partial results are available. In Cylinder-world, two distinct continuous families of extremal states have been identified, (3.127) and (3.140), but we are able to show that this list is incomplete. The full set of extremal states is unknown. For Sphere-world, no non-quantum states have been identified, and it is unknown whether or not the state space of Sphere-world is equal to that of standard quantum theory. Some implicit steps towards characterising the state space include a demonstration that Sphere-world states are compatible with symmetric products of orthogonal spin components, despite these not being observables in the model.

3.2.2 Cube-world

Observables Consider firstly a spin-1 system where only three orthogonal spin components S_x, S_y and S_z are assumed observable,

$$\mathcal{O}_{cube} = \{S_x, S_y, S_z\}. \tag{3.79}$$

Recall that this implies that any function of a single spin component such as $\alpha\mathbb{1} + \beta S_x + \gamma S_x^2$ is also observable, but joint functions such as $S_x + S_y$ or $S_x S_y + S_y S_x$ are not. We refer to this miniature as *Cube-world*.

States Given a quasi-density operator, $\rho = \frac{1}{3}\mathbb{1} + \mathbf{r} \cdot \mathbf{T}$,

$$\langle S_x \rangle_\rho = 2r_1, \qquad \langle S_x^2 \rangle_\rho = \frac{2}{3}r_7 - \frac{1}{3}r_8 + \frac{2}{3}; \tag{3.80a}$$

$$\langle S_y \rangle_\rho = 2r_2, \qquad \langle S_y^2 \rangle_\rho = -\frac{1}{3}r_7 + \frac{2}{3}r_8 + \frac{2}{3}; \tag{3.80b}$$

$$\langle S_z \rangle_\rho = 2r_3, \qquad \langle S_z^2 \rangle_\rho = -\frac{1}{3}r_7 - \frac{1}{3}r_8 + \frac{2}{3}. \tag{3.80c}$$

According to (3.74), ρ represents a Cube-world state if and only if

$$2|r_1| \leq \frac{2}{3} + \frac{2}{3}r_7 - \frac{1}{3}r_8 \leq 1, \tag{3.81a}$$

$$2|r_2| \leq \frac{2}{3} - \frac{1}{3}r_7 + \frac{2}{3}r_8 \leq 1, \tag{3.81b}$$

$$2|r_3| \leq \frac{2}{3} - \frac{1}{3}r_7 - \frac{1}{3}r_8 \leq 1. \tag{3.81c}$$

The parameters r_4, r_5 and r_6 do not feature in the probabilities, so factor out when considering equivalence classes. In the following, we therefore work with a reduced parameter vector, $\mathbf{r} = (r_1, r_2, r_3, r_7, r_8)$. By rearranging, the constraints (3.81) can be expressed as a set of nine linear inequalities,

$$\boldsymbol{v}_i \cdot \mathbf{r} \leq x_i \tag{3.82}$$

for each $i \in \bar{\mathcal{I}} = \{1a, 1b, 1c, 2a, 2b, 2c, 3a, 3b, 3c\}$, where

$$\boldsymbol{v}_{1a} = (-6, 0, 0, -2, 1)^\top, \qquad x_{1a} = 2, \tag{3.83}$$
$$\boldsymbol{v}_{1b} = (6, 0, 0, -2, 1)^\top, \qquad x_{1b} = 2, \tag{3.84}$$
$$\boldsymbol{v}_{1c} = (0, 0, 0, 2, -1)^\top, \qquad x_{1c} = 1, \tag{3.85}$$
$$\boldsymbol{v}_{2a} = (0, -6, 0, 1, -2)^\top, \qquad x_{2a} = 2, \tag{3.86}$$
$$\boldsymbol{v}_{2b} = (0, 6, 0, 1, -2)^\top, \qquad x_{2b} = 2, \tag{3.87}$$
$$\boldsymbol{v}_{2c} = (0, 0, 0, -1, 2)^\top, \qquad x_{2c} = 1, \tag{3.88}$$
$$\boldsymbol{v}_{3a} = (0, 0, -6, 1, 1)^\top, \qquad x_{3a} = 2, \tag{3.89}$$
$$\boldsymbol{v}_{3b} = (0, 0, 6, 1, 1)^\top, \qquad x_{3b} = 2, \tag{3.90}$$
$$\boldsymbol{v}_{3c} = (0, 0, 0, -1, -1)^\top, \qquad x_{3c} = 1. \tag{3.91}$$

Each inequality defines a half-space and the state space of Cube-world is the convex polytope given by the intersections of these half-spaces:

$$\Omega_{cube} = \{\mathbf{r} \in \mathbb{R}^5 \,|\, \forall i \in \bar{\mathcal{I}},\, \boldsymbol{v}_i \cdot \mathbf{r} \leq x_i\}. \tag{3.92}$$

The polytope Ω_{cube} is bounded by the hyperplanes $\boldsymbol{v}_i \cdot \mathbf{r} = x_i$, for $i \in \bar{\mathcal{I}}$.

Identifying extremal states, which are vertices of the polytope, represents a vertex enumeration problem [27], for which a wide range of algorithms are known [63, 26, 5]. In this simple case, the vertices can be enumerated by hand. In dimension d, any linearly independent collection of d hyperplanes intersect at a unique point, known as a *basic solution*. Such a point saturates d inequalities; if it also satisfies the remaining inequalities, it is an extremal point of the polytope, known as a basic *feasible* solution. If a basic solution for one collection of inequalities violates another inequality, it corresponds to a point lying outside the polytope.

Here, $d = 5$. Given a 5-element subset $\mathcal{I} \subset \bar{\mathcal{I}}$ we can form the matrix $V_{\mathcal{I}}$, whose rows are the vectors $\mathbf{v}_i$ for each $i \in \mathcal{I}$. We similarly define the vector, $\mathbf{x}_{\mathcal{I}}$, with entries x_i for each $i \in \mathcal{I}$. For example, if $\mathcal{I} = \{1b, 1c, 2b, 3a, 3b\}$,

$$V_{\mathcal{I}} = \begin{pmatrix} \boldsymbol{v}_{1b}^\top \\ \boldsymbol{v}_{1c}^\top \\ \boldsymbol{v}_{2a}^\top \\ \boldsymbol{v}_{3a}^\top \\ \boldsymbol{v}_{3b}^\top \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 6 & 0 & 1 & -2 \\ 0 & 0 & -6 & 1 & 1 \\ 0 & 0 & 6 & 1 & 1 \end{pmatrix}, \qquad \mathbf{x}_{\mathcal{I}} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 2 \end{pmatrix}. \tag{3.93}$$

The intersection, $\mathbf{r}_{\mathcal{I}}$, of the five hyperplanes in $\mathcal{I}$ satisfies

$$V_{\mathcal{I}}\mathbf{r}_{\mathcal{I}} = \mathbf{x}_{\mathcal{I}}. \tag{3.94}$$

Since we consider linearly independent sets of hyperplanes, the matrix $V_{\mathcal{I}}$ will in each case be non-singular, and can therefore be inverted to find $\mathbf{r}_{\mathcal{I}}$:

$$\mathbf{r}_{\mathcal{I}} = V_{\mathcal{I}}^{-1}\mathbf{x}_{\mathcal{I}}. \tag{3.95}$$

To identify all extremal points, it therefore suffices to construct $V_{\mathcal{I}}$ for all linearly independent, 5-element subsets $\mathcal{I}$, to solve (3.95) for the basic solutions, and finally to check whether each of these is compatible with the remaining inequalities.

To identify the linearly independent subsets, note that every such set must contain at least one of $\boldsymbol{v}_{ka}$ or $\boldsymbol{v}_{kb}$ for each $k \in \{1, 2, 3\}$. We can therefore eliminate from consideration sets which do not meet this requirement or which contain any of the following linearly dependent subsets:

$$\{1a, 1b, 1c\}, \{2a, 2b, 2c\}, \{3a, 3b, 3c\}, \{1c, 2c, 3c\}, \{1a, 1b, 2c\}, \{2a, 2b, 1c\}. \tag{3.96}$$

It may be verified by exhaustion that this eliminates all linearly dependent sets.

Table 3.2 lists all linearly independent 5-element subsets $\mathcal{I} \subset \bar{\mathcal{I}}$, and the associated basic solutions $\mathbf{r}_{\mathcal{I}}$. The table also indicates which of these are the feasible solutions: these correspond to the extremal states. The following is a complete list of the 12 extremal states of Cube-world:

$$\tilde{\rho}_{(\pm,\pm,0)} = \frac{1}{3}\mathbb{1} + \frac{1}{2}(\pm S_x \pm S_y) + \left(S_x^2 - \frac{2}{3}\mathbb{1}\right) + \left(S_y^2 - \frac{2}{3}\mathbb{1}\right), \tag{3.97a}$$

$$\tilde{\rho}_{(\pm,0,\pm)} = \frac{1}{3}\mathbb{1} + \frac{1}{2}(\pm S_x \pm S_z) + \left(S_x^2 - \frac{2}{3}\mathbb{1}\right) + \left(S_z^2 - \frac{2}{3}\mathbb{1}\right), \tag{3.97b}$$

$$\tilde{\rho}_{(0,\pm,\pm)} = \frac{1}{3}\mathbb{1} + \frac{1}{2}(\pm S_y \pm S_z) + \left(S_y^2 - \frac{2}{3}\mathbb{1}\right) + \left(S_z^2 - \frac{2}{3}\mathbb{1}\right). \tag{3.97c}$$

Each of these is a joint quasi-eigenstate of all three basic observables, S_x, S_y and S_z. Exactly one of these observables has eigenvalue zero in each case, while the remaining two are ± 1, with every combination being possible. The subscripts in (3.97) indicate the eigenvalues of (S_x, S_y, S_z) for each state.

A degenerate model A miniature related to Cube-world emerges if we consider an experimental set-up where only the squares of spin components can be measured, i.e.,

$$\mathcal{O}_{cube'} = \left\{S_x^2, S_y^2, S_z^2\right\}. \tag{3.98}$$

$\boldsymbol{v}_{1a}$	$\boldsymbol{v}_{1b}$	$\boldsymbol{v}_{1c}$	$\boldsymbol{v}_{2a}$	$\boldsymbol{v}_{2b}$	$\boldsymbol{v}_{2c}$	$\boldsymbol{v}_{3a}$	$\boldsymbol{v}_{3b}$	$\boldsymbol{v}_{3c}$	$\mathbf{r}_{\mathcal{I}}$
•	•	✓	•	•	✓	•	✓	X	$(0, 0, -1, -2, -2)$
•	•	✓	•	✓	•	•	✓	✓✓	**$(0, -1/2, -1/2, -1, 0)$**
•	•	✓	•	✓	X	•	•	✓	$(0, -1, 0, 0, 2)$
•	•	✓	•	✓	✓✓	•	✓	•	**$(0, -1/2, -1/2, -1, 0)$**
•	✓	•	•	•	✓	•	✓	✓✓	**$(-1/2, 0, -1/2, 0, -1)$**
•	✓	•	•	✓✓	✓	•	✓	•	**$(-1/2, 0, -1/2, 0, -1)$**
•	✓	•	•	✓	✓✓	•	•	✓	**$(-1/2, -1/2, 0, 1, 1)$**
•	✓	X	•	•	✓	•	•	✓	$(-1, 0, 0, 2, 0)$
•	✓	✓✓	•	•	✓	•	✓	•	**$(-1/2, 0, -1/2, 0, -1)$**
•	✓	✓✓	•	✓	•	•	•	✓	**$(-1/2, -1/2, 0, 1, 1)$**
•	✓✓	✓	•	✓	•	•	✓	•	**$(0, -1/2, -1/2, -1, 0)$**
✓	•	•	•	✓	✓✓	•	•	✓	**$(1/2, -1/2, 0, 1, 1)$**
✓	•	•	•	✓✓	✓	•	✓	•	**$(1/2, 0, -1/2, 0, -1)$**
✓	•	X	•	•	✓	•	•	✓	$(1, 0, 0, 2, 0)$
✓	•	✓✓	•	•	✓	•	✓	•	**$(1/2, 0, -1/2, 0, -1)$**
✓	•	✓✓	•	✓	•	•	•	✓	**$(1/2, -1/2, 0, 1, 1)$**
✓✓	•	✓	•	✓	•	•	✓	•	**$(0, -1/2, -1/2, -1, 0)$**
•	•	✓	✓	•	X	•	•	✓	$(0, 1, 0, 0, 2)$
•	•	✓	✓	•	✓✓	•	✓	•	**$(0, 1/2, -1/2, -1, 0)$**
•	✓	•	✓	•	✓✓	•	•	✓	**$(-1/2, 1/2, 0, 1, 1)$**
•	✓	•	✓✓	•	✓	•	✓	•	**$(-1/2, 0, -1/2, 0, -1)$**
•	✓	✓✓	✓	•	•	•	•	✓	**$(-1/2, 1/2, 0, 1, 1)$**
•	✓✓	✓	✓	•	•	•	✓	•	**$(0, 1/2, -1/2, -1, 0)$**
•	•	✓	•	•	✓	✓	•	X	$(0, 0, 1, -2, -2)$
•	•	✓	•	✓	✓✓	✓	•	•	**$(0, -1/2, 1/2, 0, -1)$**
•	✓	•	•	✓	•	✓✓	•	✓	**$(-1/2, -1/2, 0, 1, 1)$**
•	✓	•	•	✓✓	✓	✓	•	•	**$(-1/2, 0, 1/2, 0, -1)$**
•	✓	✓✓	•	•	✓	✓	•	•	**$(-1/2, 0, 1/2, 0, -1)$**
✓	•	•	✓	•	✓✓	•	•	✓	**$(1/2, 1/2, 0, 1, 1)$**
✓	•	•	✓✓	•	✓	•	✓	•	**$(1/2, 0, -1/2, 0, -1)$**
✓	•	✓✓	✓	•	•	•	•	✓	**$(1/2, 1/2, 0, 1, 1)$**
✓✓	•	✓	✓	•	•	•	✓	•	**$(0, 1/2, -1/2, -1, 0)$**
✓	•	•	✓✓	•	✓	✓	•	•	**$(1/2, 0, 1/2, 0, -1)$**
✓	•	✓✓	•	•	✓	✓	•	•	**$(1/2, 0, 1/2, 0, -1)$**
•	•	✓	✓	•	✓✓	✓	•	•	**$(0, 1/2, 1/2, -1, 0)$**
•	✓✓	✓	✓	•	•	✓	•	•	**$(0, 1/2, 1/2, -1, 0)$**
✓	•	•	✓✓	•	✓	✓	•	•	**$(1/2, 0, 1/2, 0, -1)$**
✓✓	•	✓	✓	•	•	✓	•	•	**$(0, 1/2, 1/2, -1, 0)$**

Table 3.2: Table with columns labelled by the vectors $\boldsymbol{v}_i$, and rows consisting all linearly independent, 5-element subsets $\mathcal{I} \subset \bar{\mathcal{I}}$. Dots indicate elements of $\mathcal{I}$ for each row, with other entries denoting whether the associated solution, $\mathbf{r}_{\mathcal{I}}$, is compatible (✓) or incompatible (X) with each inequality, $\boldsymbol{v}_i \cdot \mathbf{r}_{\mathcal{I}} \leq x_i$. Double checkmarks indicate that the inequality is saturated, i.e., $\boldsymbol{v}_i \cdot \mathbf{r}_{\mathcal{I}} = x_i$. Basic feasible solutions are marked in bold.

The possible outcomes of each measurement are $0, +1$. In this case, each basic observable is degenerate, having a 2-dimensional eigenspace associated to the outcome $+1$ and a 1-dimensional eigenspace associated to the outcome 0. Physically, such a measurement might be implemented by an apparatus similar to the Stern-Gerlach device which performs Cube-world measurements, but is unable to distinguish the sign of the outcome.

The rank-2 projector onto the $+1$-eigenspace of $S_{\boldsymbol{\eta}}^2$ is just $S_{\boldsymbol{\eta}}^2$ itself, and the rank-1 projector onto the 0-eigenspace is $\mathbb{1} - S_{\boldsymbol{\eta}}^2$. Hence, in a state ρ, the probabilities for each outcome are given by

$$\mathcal{P}(S_{\boldsymbol{\eta}}^2 = 1|\rho) = \mathrm{Tr}\left[\rho S_{\boldsymbol{\eta}}^2\right], \qquad \mathcal{P}(S_{\boldsymbol{\eta}}^2 = 0|\rho) = \mathrm{Tr}\left[\rho(\mathbb{1} - S_{\boldsymbol{\eta}}^2)\right], \tag{3.99}$$

so a quasi-density operator ρ is compatible with $S_{\boldsymbol{\eta}}^2$ if and only if

$$0 \leq \langle S_{\boldsymbol{\eta}}^2 \rangle_\rho \leq 1. \tag{3.100}$$

Applying this to $\boldsymbol{\eta} = x, y, z$, a quasi-density operator $\rho = \frac{1}{3}\mathbb{1} + \mathbf{r} \cdot \mathbf{T}$ is compatible with $\mathcal{O}_{cube'}$ if and only if

$$0 \leq \frac{2}{3} + \frac{2}{3} r_7 - \frac{1}{3} r_8 \leq 1, \tag{3.101a}$$

$$0 \leq \frac{2}{3} - \frac{1}{3} r_7 + \frac{2}{3} r_8 \leq 1, \tag{3.101b}$$

$$0 \leq \frac{2}{3} - \frac{1}{3} r_7 - \frac{1}{3} r_8 \leq 1. \tag{3.101c}$$

The parameters (r_7, r_8) are therefore limited to a triangle with vertices at $(-1, 0)$, $(0, -1)$ and $(1, 1)$. This provides a simpler model since the equivalence classes are 2-dimensional.

3.2.3 Cylinder-world: invariance under $U(1) \times Z_2$

Observables For the next miniature, Cylinder-world, we postulate that we can measure spin in the z-direction, as well as any direction in the xy-plane. We abbreviate $S_\phi = \cos\phi S_x + \sin\phi S_y$, and the basic observables are

$$\mathcal{O}_{cyl} = \{S_z\} \cup \{S_\phi \,|\, \phi \in [0, 2\pi)\}. \tag{3.102}$$

This model has cylindrical symmetry. Denote by $U_{\phi,\phi'}$ the unitary operator which maps S_ϕ to $S_{\phi'}$, while leaving S_z invariant:

$$S_{\phi'} = U_{\phi,\phi'} S_\phi U_{\phi,\phi'}^\dagger. \tag{3.103}$$

Define $\rho' = U_{\phi'\phi}\rho U^\dagger_{\phi'\phi}$, then

$$\langle S_{\phi'}\rangle_{\rho'} = \langle S_\phi\rangle_\rho \qquad \text{and} \qquad \langle S^2_{\phi'}\rangle_{\rho'} = \langle S^2_\phi\rangle_\rho \tag{3.104}$$

so that

$$\rho \text{ is a state } \implies \rho' \text{ is a state.} \tag{3.105}$$

We use the shorthand $U_\phi = U_{0,\phi}$, so that $U_{\phi,\phi'} = U_{\phi'}U^\dagger_\phi$ and

$$U_\phi \mathbf{T} U^\dagger_\phi = (R_\phi \oplus P_\phi)\mathbf{T} \tag{3.106}$$

where R_ϕ and P_ϕ are the actions of a rotation on the linear and quadratic basis elements respectively:

$$R_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.107}$$

$$P_\phi = \begin{pmatrix} \cos^2\phi - \sin^2\phi & 0 & 0 & -\cos\phi\sin\phi & \cos\phi\sin\phi \\ 0 & \cos\phi & -\sin\phi & 0 & 0 \\ 0 & \sin\phi & \cos\phi & 0 & 0 \\ \cos\phi\sin\phi & 0 & 0 & \cos^2\phi & \sin^2\phi \\ -\cos\phi\sin\phi & 0 & 0 & \sin^2\phi & \cos^2\phi \end{pmatrix}. \tag{3.108}$$

We can then express the action of an azimuthal rotation on an arbitrary state, ρ:

$$\rho \mapsto \rho' = \frac{1}{3}\mathbb{1} + \mathbf{r}' \cdot \mathbf{T} \tag{3.109}$$

where

$$\mathbf{r}' = (R^\top_\phi \oplus P^\top_\phi)\mathbf{r} = (R_{-\phi} \oplus P_{-\phi})\mathbf{r}. \tag{3.110}$$

The model also exhibits symmetry under reflection in the z-direction, Π_z. Under such a reflection $S_z \mapsto -S_z$ while leaving S_x, S_y invariant, we can again express the action on an arbitrary state via the parameters $\mathbf{r}$:

$$\mathbf{r} \mapsto \mathrm{diag}(1, 1, -1, 1, -1, -1, 1, 1)\mathbf{r}. \tag{3.111}$$

States A quasi-density operator ρ is a state if and only if

$$|\langle S_z\rangle_\rho| \leq \langle S_z^2\rangle_\rho \leq 1, \tag{3.112a}$$

$$|\langle S_\phi\rangle_\rho| \leq \langle S_\phi^2\rangle_\rho \leq 1 \text{ for all } \phi \in [0, 2\pi). \tag{3.112b}$$

Given a quasi-density operator, $\rho = \frac{1}{3}\mathbb{1} + \mathbf{r}\cdot\mathbf{T}$, the relevant expectation values are

$$\langle S_z\rangle_\rho = 2r_3, \tag{3.113a}$$

$$\langle S_z^2\rangle_\rho = -\frac{1}{3}r_7 - \frac{1}{3}r_8 + \frac{2}{3}; \tag{3.113b}$$

$$\langle S_\phi\rangle_\rho = 2(r_1\cos\phi + r_2\sin\phi), \tag{3.113c}$$

$$\langle S_\phi^2\rangle_\rho = r_4\cos\phi\sin\phi + r_7\left(\cos^2\phi - \frac{1}{3}\right) + r_8\left(\sin^2\phi - \frac{1}{3}\right) + \frac{2}{3}. \tag{3.113d}$$

The parameters r_5 and r_6 do not feature in the probabilities, so factor out when considering equivalence classes.

In the majority of miniatures we have considered so far, there have been finitely many basic observables, and the systems have all been finite-dimensional. This means that there are finitely many positivity constraints. The state space for each of these miniatures is a polytope, and therefore completely characterised by a finite number of extremal states, which can be identified from the positivity constraints by vertex enumeration. In the case of cylinder-world, there is an uncountable family of basic observables, leading to an uncountable family of positivity constraints, so techniques of vertex enumeration are not applicable in this case.

These previous miniatures do however give an indication of properties we may expect extremal states to have. In all miniatures considered so far, every extremal state is a joint quasi-eigenstate of some pair (or triplet) of non-commuting observables. Furthermore, if any joint quasi-eigenstates of a given pair of observables do exist, then there is an extremal state which is a joint quasi-eigenstate of these observables. In seeking the extremal states of Cylinder-world, we therefore aim to construct such states.

To this end, we first consider the case where one of these observables is the spin component in the z-direction, S_z. This does not commute with any of the azimuthal spin components, S_ϕ. If any joint quasi-eigenstate of S_z and S_ϕ exists for some value of ϕ, then this is connected by a unitary map to a joint quasi-eigenstate of S_z and $S_{\phi'}$ for any other angle ϕ', with the same pair of eigenvalues. It is therefore sufficient to consider joint quasi-eigenstates of S_z and S_x. Furthermore, it is sufficient to consider those having eigenvalues 0 or +1, since the −1 quasi-eigenstates of S_x are +1 quasi-eigenstates of S_{-x}, which means they are related by a unitary rotation map.

Recall that, for any state, ρ,

$$\langle S_x^2\rangle_\rho + \langle S_y^2\rangle_\rho + \langle S_z^2\rangle_\rho = 2. \tag{3.114}$$

It follows that there are no joint quasi-eigenstates having both values 0, since this would then imply that $\langle S_y^2\rangle = 2$, in contradiction with the constraints (3.112).

There are also no joint quasi-eigenstates having both values +1. Assume that ρ is a state such that

$$\langle S_x\rangle_\rho = \langle S_x^2\rangle_\rho = 1, \tag{3.115}$$
$$\langle S_z\rangle_\rho = \langle S_z^2\rangle_\rho = 1. \tag{3.116}$$

From (3.113), $r_1(\rho) = r_3(\rho) = 1/2$, and thus from (3.113c), $r_2(\rho) = 0$. Note that

$$\langle S_x^2\rangle = \frac{2}{3}r_7 - \frac{1}{3}r_8 + \frac{2}{3} \tag{3.117}$$
$$\langle S_z^2\rangle = -\frac{1}{3}r_7 - \frac{1}{3}r_8 + \frac{2}{3}, \tag{3.118}$$

so that $r_7(\rho) = 0$ and $r_8(\rho) = -1$. Thus, for this hypothetical joint quasi-eigenstate, ρ, we can simplify the expressions (3.113c) and (3.113d):

$$\langle S_\phi\rangle_\rho = \cos\phi, \tag{3.119}$$
$$\langle S_\phi^2\rangle_\rho = r_4(\rho)\cos\phi\sin\phi + \cos^2\phi. \tag{3.120}$$

In particular,

$$\left\langle S_{\pi/4}^2\right\rangle_\rho = \frac{1}{2}r_4(\rho) + \frac{1}{2}, \tag{3.121}$$
$$\left\langle S_{3\pi/4}^2\right\rangle_\rho = -\frac{1}{2}r_4(\rho) + \frac{1}{2}, \tag{3.122}$$

while

$$\left\langle S_{\pi/4}\right\rangle_\rho = \left\langle S_{3\pi/4}\right\rangle_\rho = \frac{\sqrt{2}}{2}, \tag{3.123}$$

so we require that

$$\frac{1}{2}r_4(\rho) + \frac{1}{2} \geq \frac{\sqrt{2}}{2} \tag{3.124}$$

and

$$-\frac{1}{2}r_4(\rho)+\frac{1}{2}\geq\frac{\sqrt{2}}{2}; \tag{3.125}$$

adding these we have

$$1\geq\sqrt{2}. \tag{3.126}$$

This is a contradiction, so the initial assumption $\langle S_x\rangle=\langle S_x^2\rangle=1$ and $\langle S_z\rangle=\langle S_z^2\rangle=1$ must be false, i.e., no joint quasi-eigenstate of S_x and S_z, having both values $+1$, exists.

There are, however, joint quasi-eigenstates of S_z and S_ϕ having definite values 0 for S_z and 1 for S_ϕ for some given angle ϕ. Denote

$$\tilde{\mathbf{r}}^0_\varphi=\left(\frac{1}{2}\cos\varphi,\frac{1}{2}\sin\varphi,0,0,1,1\right) \tag{3.127}$$

and the associated quasi-density operator, $\tilde{\rho}^0_\varphi=\frac{1}{3}\mathbb{1}+\tilde{\mathbf{r}}^0_\varphi\cdot\mathbf{T}$, then

$$\langle S_z\rangle_{\tilde{\rho}^0_\varphi}=0, \qquad \langle S_z^2\rangle_{\tilde{\rho}^0_\varphi}=0 \tag{3.128}$$

$$\langle S_\phi\rangle_{\tilde{\rho}^0_\varphi}=\cos(\phi-\varphi), \qquad \langle S_\phi^2\rangle_{\tilde{\rho}^0_\varphi}=1. \tag{3.129}$$

The requirements (3.112) are satisfied, so $\tilde{\rho}^0_\varphi$ is a state. This is a joint quasi-eigenstate of S_z, with eigenvalue 0, and S_φ, with eigenvalue 1 (and hence is a non-quantum state). Azimuthal rotations act on the parameter φ in this one-parameter family.

Furthermore, each state in the above family is extremal, i.e., for states ρ_1 and ρ_2,

$$\tilde{\rho}^0_\varphi=\lambda\rho_1+(1-\lambda)\rho_2 \text{ for some } \lambda\in(0,1) \implies \rho_1=\rho_2=\tilde{\rho}^0_\varphi. \tag{3.130}$$

To prove this implication, we will assume the equality on the left-hand side, and demonstrate that each of the parameters r_i must be equal for the states $\tilde{\rho}^0_\varphi$, ρ_1 and ρ_2, and hence that the states are equal. Firstly, we note that

$$\lambda\langle S_z^2\rangle_{\rho_1}+(1-\lambda)\langle S_z^2\rangle_{\rho_2}=\langle S_z^2\rangle_{\tilde{\rho}^0_\varphi}=0, \tag{3.131}$$

but $\langle S_z^2\rangle_{\rho_1}\geq 0$ and $\langle S_z^2\rangle_{\rho_2}\geq 0$, so equality is only possible if $\langle S_z^2\rangle_{\rho_1}=\langle S_z^2\rangle_{\rho_2}=0$.

This zero expectation value is sufficient to determine the parameters r_3, r_4, r_7 and r_8. Assume that ρ is any state for which $\langle S_z^2\rangle_\rho=0$. By compatibility with S_z, we have also that $\langle S_z\rangle_\rho=0$, and so $r_3(\rho)=0$. By (3.114) we have that $r_7(\rho)+r_8(\rho)=2$, and we can simplify the expression for S_ϕ^2 in any such state:

$$\langle S_\phi^2\rangle_\rho=r_4(\rho)\cos\phi\sin\phi+r_7(\rho)\cos^2\phi+r_8(\rho)\sin^2\phi. \tag{3.132}$$

In particular,

$$\left\langle S_0^2 \right\rangle_\rho = r_7(\rho) \text{ and } \left\langle S_{\pi/2}^2 \right\rangle_\rho = r_8(\rho). \tag{3.133}$$

Compatibility with these spin measurements then implies $r_7(\rho) \leq 1$ and $r_8(\rho) \leq 1$, so $r_7(\rho) + r_8(\rho) = 2$ only if $r_7(\rho) = r_8(\rho) = 1$. Thus, the parameters r_7 and r_8 are fixed and we can further simplify:

$$\langle S_\phi^2 \rangle_\rho = r_4(\rho) \cos\phi \sin\phi + 1. \tag{3.134}$$

It is clear that $\langle S_\phi^2 \rangle_\rho \leq 1$ for all $\phi \in [0, 2\pi)$ only if $r_4(\rho) = 0$. Combining these results, we obtain

$$\langle S_z^2 \rangle_\rho = 0 \implies r_3(\rho) = r_4(\rho) = 0,\, r_7(\rho) = r_8(\rho) = 1, \tag{3.135}$$

so that ρ_1, ρ_2 and $\tilde{\rho}_\varphi^0$ all agree in these parameters.

It remains to show that ρ_1, ρ_2 and $\tilde{\rho}_\varphi^0$ also agree in the parameters r_1 and r_2. Given a quasi-density operator, ρ, define $\mathbf{v}(\rho) = (r_1(\rho), r_2(\rho))$. Firstly note, since the expectation value of all azimuthal spins is bounded, $\langle S_\phi \rangle \leq 1$, we have from (3.113c) that

$$\mathbf{v}(\rho) \cdot \mathbf{u} \leq \frac{1}{2} \text{ for all unit vectors } \mathbf{u}, \tag{3.136}$$

i.e., $|\mathbf{v}(\rho)| \leq 1/2$. Hence, for any $\lambda \in (0, 1)$,

$$\lambda |\mathbf{v}(\rho_1)| + (1 - \lambda)|\mathbf{v}(\rho_2)| \leq \frac{1}{2}, \tag{3.137}$$

with equality only if $|\mathbf{v}(\rho_1)| = |\mathbf{v}(\rho_2)| = 1/2$. If $\tilde{\rho}_\varphi^0 = \lambda\rho_1 + (1 - \lambda)\rho_2$, then

$$\mathbf{v}(\tilde{\rho}_\varphi^0) = \lambda \mathbf{v}(\rho_1) + (1 - \lambda)\mathbf{v}(\rho_2). \tag{3.138}$$

By the triangle inequality,

$$\lambda |\mathbf{v}(\rho_1)| + (1 - \lambda)|\mathbf{v}(\rho_2)| \geq |\mathbf{v}(\tilde{\rho}_\varphi^0)| = \frac{1}{2}, \tag{3.139}$$

with equality only if $\mathbf{v}(\rho_1)$ is parallel to $\mathbf{v}(\rho_2)$. Combining (3.137) and (3.139), both inequalities must be saturated, so that $|\mathbf{v}(\rho_1)| = |\mathbf{v}(\rho_2)| = 1/2$, and $\mathbf{v}(\rho_1) \parallel \mathbf{v}(\rho_2)$. Along with (3.138), these imply that $\mathbf{v}(\rho_1) = \mathbf{v}(\rho_2) = \mathbf{v}(\tilde{\rho}_\varphi^0)$. We have therefore proved the result (3.130), i.e., that $\tilde{\rho}_\varphi^0$ is extremal.

Cylinder-world also contains joint quasi-eigenstates for $S_z = 1$ and $S_\phi = 0$: consider

$$\tilde{\mathbf{r}}^+_\varphi = \left(0, 0, \frac{1}{2}, -2\cos\varphi\sin\varphi, -\cos^2\varphi, -\sin^2\varphi\right) \tag{3.140}$$

and the associated quasi-density operator, $\tilde{\rho}^+_\varphi = \frac{1}{3}\mathbb{1} + \tilde{\mathbf{r}}^+_\varphi \cdot \mathbf{T}$. The expectation values predicted by this state are

$$\langle S_z\rangle_{\tilde{\rho}^+_\varphi} = 1, \qquad \langle S_z^2\rangle_{\tilde{\rho}^+_\varphi} = 1 \tag{3.141}$$

$$\langle S_\phi\rangle_{\tilde{\rho}^+_\varphi} = 0, \qquad \langle S_\phi^2\rangle_{\tilde{\rho}^+_\varphi} = \sin^2(\phi - \varphi). \tag{3.142}$$

Again, the conditions (3.112) are satisfied, so $\tilde{\rho}^+_\varphi$ is also a state.

We have thus constructed continuous families of joint quasi-eigenstates, $\tilde{\rho}^0_\varphi$, $\tilde{\rho}^0_{\varphi+\pi}$, $\tilde{\rho}^+_\varphi$ and $\Pi_z\tilde{\rho}^+_\varphi$, having eigenvalue pairs $(S_z, S_\varphi) = (0, +1)$, $(0, -1)$, $(+1, 0)$ and $(-1, 0)$ respectively, and ruled out the existence of joint quasi-eigenstates having any other eigenvalue pair. We can show, however, that these states do not exhaust the full set of extremal states in Cylinder-world. To this end, consider the quasi-density operator

$$\tilde{\rho} = \frac{1}{3}\mathbb{1} + \tilde{\mathbf{r}} \cdot \mathbf{T}, \text{ where } \tilde{\mathbf{r}} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right). \tag{3.143}$$

This operator lies outside the convex hull of the states identified so far, but is nevertheless a Cylinder-world state. We have

$$\langle S_z\rangle_{\tilde{\rho}} = 1, \tag{3.144}$$

$$\langle S_z^2\rangle_{\tilde{\rho}} = 1; \tag{3.145}$$

$$\langle S_\phi\rangle_{\tilde{\rho}} = \frac{1}{2}(\cos\phi + \sin\phi), \tag{3.146}$$

$$\langle S_\phi^2\rangle_{\tilde{\rho}} = \frac{1}{2}(\cos\phi\sin\phi + 1). \tag{3.147}$$

It is clear that that $\tilde{\rho}$ is compatible with S_z. To see that $\tilde{\rho}$ is also compatible with S_ϕ for all $\phi \in [0, 2\pi)$, we note that the function $f : [0, 2\pi) \to \mathbb{R}$ defined by

$$f : \phi \mapsto \langle S_\phi^2\rangle_{\tilde{\rho}} - |\langle S_\phi\rangle_{\tilde{\rho}}| \tag{3.148}$$

is non-negative:

$$f(\phi) = \frac{1}{2}(\cos\phi\sin\phi + 1 - |\cos\phi + \sin\phi|) = \frac{1}{4}\left(1 - |\cos\phi + \sin\phi|\right)^2 \geq 0. \tag{3.149}$$

Furthermore, $\sup_\phi(\cos\phi\sin\phi) = \frac{1}{2}$, so

$$\langle S_\phi^2\rangle_{\tilde\rho} \leq \frac{3}{4}. \tag{3.150}$$

Hence,

$$|\langle S_\phi\rangle_{\tilde\rho}| \leq \langle S_\phi^2\rangle_{\tilde\rho} \leq 1 \text{ for all } \phi \in [0, 2\pi), \tag{3.151}$$

i.e., $\tilde\rho$ is compatible with all available spin measurements and so is a state.

3.2.4 Sphere-world: invariance under $SU(2)$

Observables In Sphere-world, we assume that our Stern-Gerlach apparatus may be rotated freely, so that spin components in all directions may be measured.

$$\mathcal{O}_{\text{Sph}} = \left\{ S_{\boldsymbol\eta} \,\middle|\, \boldsymbol\eta \in S^2 \right\}. \tag{3.152}$$

States Similar to Cylinder-world, the state space of Sphere-world is defined by an uncountable collection of positivity constraints. Namely, for every direction $\boldsymbol\eta \in S^2$ we require states to be compatible with $S_{\boldsymbol\eta}$, so that

$$|\langle S_{\boldsymbol\eta}\rangle| \leq \langle S_{\boldsymbol\eta}^2\rangle \leq 1 \text{ for all } \boldsymbol\eta \in S^2. \tag{3.153}$$

Let us check that the positivity conditions are compatible with the symmetry of the model at hand. To do so, consider any Sphere-world state, ρ. Then, for any $\boldsymbol\eta$, with $\boldsymbol\eta'$ related according to $S_{\boldsymbol\eta} = U S_{\boldsymbol\eta'} U^\dagger$,

$$\langle S_{\boldsymbol\eta}\rangle_{U^\dagger\rho U} = \langle S_{\boldsymbol\eta'}\rangle_\rho, \qquad \langle S_{\boldsymbol\eta}^2\rangle_{U^\dagger\rho U} = \langle S_{\boldsymbol\eta'}^2\rangle_\rho. \tag{3.154}$$

Since ρ is a state, it is compatible with all spin components, so in particular is compatible with $S_{\boldsymbol\eta'}$, i.e.,

$$|\langle S_{\boldsymbol\eta'}\rangle_\rho| \leq \langle S_{\boldsymbol\eta'}^2\rangle_\rho \leq 1. \tag{3.155}$$

By the above, then

$$|\langle S_{\boldsymbol\eta}\rangle_{U^\dagger\rho U}| \leq \langle S_{\boldsymbol\eta}^2\rangle_{U^\dagger\rho U} \leq 1, \tag{3.156}$$

i.e., $U^\dagger\rho U$ is compatible with all spin components $S_{\boldsymbol\eta}$. Furthermore the map $\rho \mapsto U^\dagger\rho U$ is invertible and the inverse is convex so that, if ρ is an extremal state, $U^\dagger\rho U$ is also extremal.

A complete characterisation of the state space of Sphere-world is not yet known. However, numerical sampling indicates that Sphere-world contains non-quantum states, as discussed later in this section. We therefore have a quantum miniature with full spherical symmetry which differs from standard quantum theory. The existence of continuous reversible transformations between extremal states has been identified as a 'special feature' of quantum theory in contrast to commonly studied theories such as Boxworld [48]. Sphere-world shares this special feature.

As a step towards characterising the non-quantum states of Sphere-world, we show that compatibility with linear spin components implies compatibility with symmetric products of spin components in perpendicular directions. This is despite the fact that a symmetric product can not be expressed as a function of any of the basic observables, i.e., of a single spin component. We also show that non-quantum Sphere-world states do not allow the violation of the additive uncertainty relation (1.9).

The state space is bounded Before investigating non-quantum states, it is useful to establish quantitative bounds on the state space. We demonstrate boundedness component by component, using the parameterisation (3.55). From the inequalities (3.153), we have that $|\boldsymbol{\eta} \cdot \mathbf{r}| \leq 1/2$ for any $\boldsymbol{\eta} \in S_2$. This holds in particular for the spin components aligned with the axes x, y, z. Hence,

$$-\frac{1}{2} \leq r_i \leq \frac{1}{2} \qquad \text{for } i = 1, 2, 3. \tag{3.157}$$

We also have that $0 \leq \langle S_{\boldsymbol{\eta}}^2 \rangle \leq 1$ for the spin components aligned with the axes x and y. Hence,

$$-1 \leq r_7 \leq \frac{1}{2}, \qquad -1 \leq r_8 \leq \frac{1}{2}. \tag{3.158}$$

To see that the parameters r_i are bounded for $i = 4, 5, 6$, consider measurements of spin along the following directions:

$$\boldsymbol{\eta}_1 = \frac{1}{\sqrt{3}}(-1, 1, 1), \tag{3.159}$$

$$\boldsymbol{\eta}_2 = \frac{1}{\sqrt{3}}(1, -1, 1), \tag{3.160}$$

$$\boldsymbol{\eta}_3 = \frac{1}{\sqrt{3}}(1, 1, -1). \tag{3.161}$$

We have

$$\mathbf{V}_{\eta_1} \cdot \mathbf{r} = \frac{-r_4 + r_5 - r_6}{3}, \tag{3.162}$$

$$\mathbf{V}_{\eta_2} \cdot \mathbf{r} = \frac{-r_4 - r_5 + r_6}{3}, \tag{3.163}$$

$$\mathbf{V}_{\eta_3} \cdot \mathbf{r} = \frac{r_4 - r_5 - r_6}{3}, \tag{3.164}$$

where $\mathbf{V}_{veta}$ are as defined in (3.76). By adding these pairwise, (3.153) yields

$$-\frac{1}{2} \leq r_i \leq 1 \qquad \text{for } i = 4, 5, 6. \tag{3.165}$$

All components of $\mathbf{r}$ are therefore bounded for any Sphere-world state, lying within a hypercuboid $Q \subset \mathbb{R}^8$,

$$Q = [-1/2, 1/2]^3 \times [-1/2, 1]^3 \times [-1, 1/2]^2. \tag{3.166}$$

Sphere-world contains non-quantum states Numerical sampling of quasi-density operators indicates the existence of non-quantum states in Sphere-world. To identify such states, one can generate a collection of pseudo-random points in the parameter space Q. This collection of points can be mapped to a collection of quasi-density operators by the parameterisation (3.55). Non-quantum Sphere-world states can then be identified by applying a series of filters.

Firstly, we discard from the collection any quasi-density operators which are positive (i.e., proper density operators corresponding to quantum states). Next, we successively generate a series of pseudo-random directions $\boldsymbol{\eta} \in S^2$. For each $\boldsymbol{\eta}$, discard any quasi-density operators which are not compatible with $S_{\boldsymbol{\eta}}$ i.e., do not satisfy the conditions (3.153). This is checked by evaluation of the relevant expectation values.

At each stage of computation, we are left with a collection of quasi-density operators which are compatible with spin components in all directions sampled so far. Finally, we select a candidate $\tilde{\rho}$ from the remaining quasi-density operators. To check that $\tilde{\rho}$ is indeed a Sphere-world state, i.e., is compatible with *all* spin components (not just those in the directions sampled), we minimise the following functions:

$$f_0 : \boldsymbol{\eta} \mapsto 1 - \langle S_{\boldsymbol{\eta}}^2 \rangle_{\tilde{\rho}}, \tag{3.167}$$

$$f_1 : \boldsymbol{\eta} \mapsto \langle S_{\boldsymbol{\eta}}^2 \rangle_{\tilde{\rho}} + \langle S_{\boldsymbol{\eta}} \rangle_{\tilde{\rho}}, \tag{3.168}$$

$$f_2 : \boldsymbol{\eta} \mapsto \langle S_{\boldsymbol{\eta}}^2 \rangle_{\tilde{\rho}} - \langle S_{\boldsymbol{\eta}} \rangle_{\tilde{\rho}}. \tag{3.169}$$

The conditions (3.153) are satisfied if and only if each function is non-negative on the unit sphere, i.e.,

$$\min_{\boldsymbol{\eta}\in S^2} f_0(\boldsymbol{\eta}) \geq 0 \iff \langle S_{\boldsymbol{\eta}}^2\rangle \leq 1 \text{ for all } \boldsymbol{\eta} \in S^2, \tag{3.170}$$

$$\min_{\boldsymbol{\eta}\in S^2} f_1(\boldsymbol{\eta}) \geq 0 \iff -\langle S_{\boldsymbol{\eta}}\rangle \leq \langle S_{\boldsymbol{\eta}}^2\rangle \text{ for all } \boldsymbol{\eta} \in S^2, \tag{3.171}$$

$$\min_{\boldsymbol{\eta}\in S^2} f_2(\boldsymbol{\eta}) \geq 0 \iff \langle S_{\boldsymbol{\eta}}\rangle \leq \langle S_{\boldsymbol{\eta}}^2\rangle \text{ for all } \boldsymbol{\eta} \in S^2. \tag{3.172}$$

Note also that $f_1(\boldsymbol{\eta}) = f_2(-\boldsymbol{\eta})$, so $\min f_1 = \min f_2$ and it suffices to minimise just one of these functions.

An example of a quasi-density operator $\tilde{\rho}$ generated by the process described above is

$$\tilde{\rho} = \begin{pmatrix} 0.64491 & 0.03659 - 0.00434i & -0.00991 - 0.34284i \\ 0.03659 + 0.00434i & 0.29667 & -0.03659 + 0.00434i \\ -0.00991 + 0.34284i & -0.03659 - 0.00434i & 0.05842 \end{pmatrix}, \tag{3.173}$$

expressed as a matrix in the S_z eigenbasis. The operator $\tilde{\rho}$ is not positive, having eigenvalues 0.805179, 0.298345 and -0.103524, so $\tilde{\rho}$ is not a quantum state. Minimising the functions f_0 and f_1 (with expectation values evaluated with respect to $\tilde{\rho}$), we find that

$$\min_{\boldsymbol{\eta}\in S^2} f_0(\boldsymbol{\eta}) = 0.00307632, \tag{3.174}$$

occuring at $\boldsymbol{\eta} = (0.694883, 0.705908, 0.137231)$, and

$$\min_{\boldsymbol{\eta}\in S^2} f_1(\boldsymbol{\eta}) = 0.0553177, \tag{3.175}$$

occuring at $\boldsymbol{\eta} = (0.516326, 0.444366, 0.732083)$. The functions f_0 and f_1 are therefore both non-negative everywhere on S^2, so $\tilde{\rho}$ is indeed a Sphere-world state.

Compatibility with symmetric products We now show that all Sphere-world states are compatible (in the sense of (2.1)) with symmetric products of spin components, $(S_{\boldsymbol{\eta}}S_{\boldsymbol{\eta}'} + S_{\boldsymbol{\eta}'}S_{\boldsymbol{\eta}})/2$, for any pair of mutually perpendicular directions $\boldsymbol{\eta}, \boldsymbol{\eta}'$.

Consider first the symmetric product of S_x and S_y,

$$T_4 = \frac{1}{2}(S_xS_y + S_yS_x). \tag{3.176}$$

T_4 has eigenvalues $\pm 1/2$ and 0. The projectors onto eigenstates of T_4 are

$$E_4^+ = T_4 + 2T_4^2, \tag{3.177}$$

$$E_4^- = -T_4 + 2T_4^2, \tag{3.178}$$

$$E_4^0 = \mathbb{1} - 4T_4^2. \tag{3.179}$$

Note that $4T_4^2 = S_z^2$.

Consider a measurement of spin in the directions

$$\boldsymbol{\zeta}_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right), \tag{3.180}$$

$$\boldsymbol{\zeta}_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right). \tag{3.181}$$

We have, for any state $\rho \in \Omega_{Sph}$,

$$\left\langle S_{\boldsymbol{\zeta}_1}^2 \right\rangle_\rho = 1 - \left\langle E_4^+ \right\rangle_\rho, \tag{3.182}$$

$$\left\langle S_{\boldsymbol{\zeta}_2}^2 \right\rangle_\rho = 1 - \left\langle E_4^- \right\rangle_\rho \tag{3.183}$$

and also

$$\langle S_z^2 \rangle_\rho = 1 - \left\langle E_4^0 \right\rangle_\rho. \tag{3.184}$$

Since $S_{\boldsymbol{\zeta}_1}$, $S_{\boldsymbol{\zeta}_2}$ and S_z are observable in Sphere-world,

$$0 \leq \left\langle S_{\boldsymbol{\eta}_1}^2 \right\rangle_\rho \leq 1, \qquad 0 \leq \left\langle S_{\boldsymbol{\eta}_2}^2 \right\rangle_\rho \leq 1, \qquad 0 \leq \langle S_z^2 \rangle_\rho \leq 1, \tag{3.185}$$

so that

$$0 \leq \left\langle E_4^+ \right\rangle_\rho \leq 1, \qquad 0 \leq \left\langle E_4^- \right\rangle_\rho \leq 1, \qquad 0 \leq \left\langle E_4^0 \right\rangle_\rho \leq 1. \tag{3.186}$$

These are exactly the conditions that ρ be compatible with T_4.

Since the state space is invariant under SU(2) transformations, we also have that every state is compatible with the symmetric product $(S_{\boldsymbol{\eta}} S_{\boldsymbol{\eta}'} + S_{\boldsymbol{\eta}'} S_{\boldsymbol{\eta}})/2$ for each pair of mutually perpendicular directions, $\boldsymbol{\eta}$ and $\boldsymbol{\eta}'$.

Uncertainty relations The additive uncertainty relation (1.9) holds in Sphere-world. To see this, let $\rho \in \Omega_{Sph}$ be an arbitrary Sphere-world state and construct a vector,

$\boldsymbol{v}(\rho)$, whose components are the expectation values for spin along the coordinate axes x, y, z:

$$\boldsymbol{v}(\rho) = \left(\langle S_x \rangle_\rho , \langle S_y \rangle_\rho , \langle S_z \rangle_\rho \right) . \tag{3.187}$$

Note that

$$|\boldsymbol{v}(\rho)|^2 = \langle S_x \rangle_\rho^2 + \langle S_y \rangle_\rho^2 + \langle S_z \rangle_\rho^2 = |\boldsymbol{v}(\rho)| \mathrm{Tr} \left[\rho S_{\hat{\boldsymbol{v}}(\rho)} \right] \tag{3.188}$$

where $\hat{\boldsymbol{v}}(\rho)$ is a unit vector in the direction of $\boldsymbol{v}(\rho)$, and $S_{\hat{\boldsymbol{v}}}$ is the spin component in that direction. Since $S_{\hat{\boldsymbol{v}}}$ is an observable, we have from (3.153) that $\mathrm{Tr} \left[\rho S_{\hat{\boldsymbol{v}}(\rho)} \right] \leq 1$. Hence,

$$|\boldsymbol{v}(\rho)|^2 \leq |\boldsymbol{v}(\rho)| \qquad \text{so} \qquad |\boldsymbol{v}(\rho)| \leq 1 \, . \tag{3.189}$$

Note also that

$$\langle S_x^2 \rangle_\rho + \langle S_y^2 \rangle_\rho + \langle S_z^2 \rangle_\rho - \left(\langle S_x \rangle_\rho^2 + \langle S_y \rangle_\rho^2 + \langle S_z \rangle_\rho^2 \right) = 2 - |\boldsymbol{v}(\rho)|^2, \tag{3.190}$$

and so

$$\Delta_\rho^2(S_x) + \Delta_\rho^2(S_y) + \Delta_\rho^2(S_z) \geq 1. \tag{3.191}$$

This is the same bound attained for quantum states, and the bound is optimal – it is saturated by e.g. a highest weight eigenstate of S_z.

It remains open to determine whether other quantum uncertainty relations regarding spin may be violated by Sphere-world states.

A simpler model may be considered along the lines of the degenerate model described in Section 3.2.2, where the squares of spin components in every direction are taken as observables. This model would have a reduced parameter space, since the coefficients r_1, r_2 and r_3 of the linear terms in the multipole expansion (3.55) do not feature. However, this degenerate model would be of interest since it shares the symmetry properties of Sphere-world described above.

3.3 BIPARTITE SPIN MINIATURES

3.3.1 A pair of Square-world systems

Consider a situation where two parties, A and B, each have access to a spin-1/2 system and a Stern-Gerlach apparatus restricted in the manner described in Section

3.1.1. Each party can therefore measure spin components in either the x or z direction. We assume that A and B are widely separated, so that no non-separable measurements can be performed.

Hence, the experimental set-up allows spin measurements on either system individually, while performing no measurement on the other system – described by observables of the form $\sigma_i \otimes \mathbb{1}$ or $\mathbb{1} \otimes \sigma_j$. We also assume that the two parties can communicate the results of their measurements, so can perform joint spin measurements – described by observables of the form $\sigma_i \otimes \sigma_j$. This set-up is described by a bipartite quantum miniature, with the set of basic observables being

$$\mathcal{O}_{2SW} = \{\sigma_x \otimes \mathbb{1}, \sigma_z \otimes \mathbb{1}, \mathbb{1} \otimes \sigma_x, \mathbb{1} \otimes \sigma_z, \sigma_x \otimes \sigma_x, \sigma_x \otimes \sigma_z, \sigma_z \otimes \sigma_x, \sigma_z \otimes \sigma_z\}. \tag{3.192}$$

We will show that this miniature allows all bipartite non-signalling correlations.

State space For composite systems, we maintain the idea that the possible states of a theory are dictated by compatibility with all observables. States are therefore represented by quasi-density operators on the joint Hilbert space, subject to the requirement that states must generate valid probability distributions for any measurement which we are able to perform.

As such, we do not directly postulate any relationship between the state space, Ω^{AB}, of the composite system and the state spaces, Ω^A and Ω^B, of the constituent sub-systems; in particular we do not postulate any particular tensor product, such as the maximum or minimum tensor products described in [71]. Instead the composite state space is defined by the set of observables, $\mathcal{O}^{AB}$, and any tensor product structure which does appear in Ω^{AB} will be a consequence of the structure of $\mathcal{O}$.

The Hilbert space for the joint system in this case is $\mathcal{H}^{AB} = \mathcal{H}^A \otimes \mathcal{H}^B \sim \mathbb{C}^4$. Any quasi-density operator, ρ, on $\mathcal{H}^{AB}$ can be parameterised by the expectation values predicted for measurements of spin on each subsystem [43] – namely, $r_j = \langle \sigma_j \otimes \mathbb{1} \rangle$ and $s_k = \langle \mathbb{1} \otimes \sigma_k \rangle$ – and joint spin measurements, $t_{jk} = \langle \sigma_j \otimes \sigma_k \rangle$:

$$\rho = \frac{1}{4}\left(\mathbb{1} \otimes \mathbb{1} + \mathbf{r} \cdot \boldsymbol{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{s} \cdot \boldsymbol{\sigma} + \sum_{j,k} t_{jk} \sigma_j \otimes \sigma_k \right). \tag{3.193}$$

From the nine real parameters, $t_{jk} \in \mathbb{R}$, we form a matrix t.

Given a system in state ρ, having parameters $\mathbf{r}, \mathbf{s}, t$, we can express the probability that a joint measurement of spin in direction $j \in \{x, z\}$ by party A and in direction

$k \in \{x, z\}$ by party B will have the results m^A and $m^B \in \{\pm 1\}$:

$$\mathcal{P}(m^A, m^B | j, k; \rho) = \frac{1}{4}(1 + m^A r_j + m^B s_k + m^A m^B t_{jk}). \tag{3.194}$$

The probabilities for single-system measurements for each system are

$$\mathcal{P}(m^A | j; \rho) = \frac{1}{2}(1 + m^A r_j) \tag{3.195}$$

$$\mathcal{P}(m^B | k; \rho) = \frac{1}{2}(1 + m^B s_k). \tag{3.196}$$

The quasi-density operator ρ is a state exactly if all of these probabilities are non-negative for each outcome of each available measurement. Note that the probabilities for single-system measurements are equal to coarse-grainings of the probabilities for joint measurements:

$$\mathcal{P}(m^A | j; \rho) = \sum_{m^B} \mathcal{P}(m^A, m^B | j, k; \rho), \tag{3.197}$$

$$\mathcal{P}(m^B | k; \rho) = \sum_{m^A} \mathcal{P}(m^A, m^B | j, k; \rho). \tag{3.198}$$

Hence, we can characterise the state space entirely by the set of positivity constraints for joint measurements:

$$\mathcal{P}(m^A, m^B | j, k; \rho) \geq 0 \qquad \forall m^A, m^B \in \{\pm 1\}, j, k \in \{x, z\}, \tag{3.199}$$

and the positivity constraints for single-system measurements, e.g. $\mathcal{P}(m^A | j; \rho) \geq 0$, are implied.

Furthermore, the equations (3.197) demonstrate that quasi-density operators which differ only in the parameters r_y, s_y, t_{jy} and t_{yk} generate identical probability distributions. Hence, these parameters factor out in equivalence classes and, for convenience, we may describe any equivalence class by a representative for which each of these parameters is zero.

The state space is a convex polytope bounded by finitely many linear inequalities, and so is entirely specified by its extremal points. To identify the extremal states, we note that (3.194) defines a map, π, from the state space Ω_{2SW} to the non-signalling polytope with two inputs and two outputs, $\mathcal{NS}$ [8]. The non-signalling polytope is a convex subspace of $\mathbb{R}^{16}$, whose elements can be represented as vectors ('behaviours'), P, with 16 components indexed by the 16 total choices of inputs and outputs. In this context we identify the inputs as the choice by each party of which direction to measure the spin, $j, k \in \{x, z\}$, and the outputs as the results of those measurements, $m^A, m^B \in \{\pm 1\}$:

$$P = \begin{pmatrix} P(+1,+1|x,x) \\ \vdots \\ P(m^A, m^B|j,k) \\ \vdots \\ P(-1,-1|z,z) \end{pmatrix}. \tag{3.200}$$

We can thus define the map $\pi : \Omega_{2SW} \to \mathcal{NS}$ component-wise:

$$\pi : \rho \mapsto \pi(\rho) = \begin{pmatrix} \mathcal{P}(+1,+1|x,x;\rho) \\ \vdots \\ \mathcal{P}(m^A, m^B|j,k;\rho) \\ \vdots \\ \mathcal{P}(-1,-1|z,z;\rho) \end{pmatrix}, \tag{3.201}$$

where the components $\mathcal{P}(m^A, m^B|j,k;\rho)$ are as given in (3.194).

It is straightforwardly verified component by component that the map π is convex-linear, i.e., for all $\rho, \rho' \in \Omega_{2SW}$ and $\lambda \in (0,1)$,

$$\pi(\lambda\rho + (1-\lambda)\rho') = \lambda\pi(\rho) + (1-\lambda)\pi(\rho'). \tag{3.202}$$

Furthermore, π is invertible:

$$\pi^{-1}(P) = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \mathbf{r}^{(P)}\cdot\boldsymbol{\sigma}\otimes\mathbb{1} + \mathbb{1}\otimes\mathbf{s}^{(P)}\cdot\boldsymbol{\sigma} + \sum_{j,k} t_{jk}^{(P)}\sigma_j\otimes\sigma_k\right), \tag{3.203}$$

where the parameters $\mathbf{r}^{(P)}, \mathbf{s}^{(P)}$ and $t_{jk}^{(P)}$ are the expectation values generated by the behaviour P for random variables chosen to have inputs and outputs matching the relevant spin measurements. Concretely[2],

$$r_j^{(P)} = P(+1,+1|j,k) + P(+1,-1|j,k) - P(-1,+1|j,k) - P(-1,-1|j,k), \tag{3.204}$$
$$s_k^{(P)} = P(+1,+1|j,k) - P(+1,-1|j,k) + P(-1,+1|j,k) - P(-1,-1|j,k), \tag{3.205}$$
$$t_{jk}^{(P)} = P(+1,+1|j,k) - P(+1,-1|j,k) - P(-1,+1|j,k) + P(-1,-1|j,k). \tag{3.206}$$

[2]Note that the expressions (3.204) and (3.205) are well-defined due to P being non-signalling. This implies that the given expression for $r_j^{(P)}$ is independent of k, and the expression for $s_k^{(P)}$ is independent of j.

Hence, $\pi : \Omega_{2SW} \to \mathcal{NS}$ is a convex-linear bijection. This demonstrates that all non-signalling correlations may be achieved in the bipartite Square-world miniature, and the extremal states of Ω_{2SW} are exactly those which map to the extremal behaviours of the non-signalling polytope. There are 24 distinct extremal behaviours, characterised in [8]. These can be divided into two groups: 16 are local states and 8 are maximally non-local, in regards to the CHSH correlations discussed in Section 3.3.2.

The sixteen local states return determinate values for spin measurements by either party in both available directions. Local states can therefore be classified by these values: denote by $\tilde{\rho}^{m_x^A, m_z^A}_{m_x^B, m_z^B}$ the state for which a measurement of spin in direction j by party A and direction k by party B is certain to return the values m_j^A and m_k^B:

$$\tilde{\rho}^{m_x^A, m_z^A}_{m_x^B, m_z^B} = \frac{1}{4}\left(\mathbb{1} + m_x^A\sigma_x + m_z^A\sigma_z\right) \otimes \left(\mathbb{1} + m_x^B\sigma_x + m_z^B\sigma_z\right). \tag{3.207}$$

The 16 possible choices for $m_x^A, m_z^A, m_x^B, m_z^B$ give 16 distinct extremal states. The expectation values for each available measurement in the state $\tilde{\rho}^{m_x^A, m_z^A}_{m_x^B, m_z^B}$ are as follows:

$$\langle \sigma_j \otimes \mathbb{1} \rangle = m_j^A, \qquad \langle \mathbb{1} \otimes \sigma_k \rangle = m_k^B, \qquad \langle \sigma_j \otimes \sigma_k \rangle = m_j^A m_k^B. \tag{3.208}$$

The states (3.207) are all joint quasi-eigenstates of all available measurements, and therefore are non-quantum states. The reduced state for either party is an extremal state of single-system Square-world, e.g. the reduced state ρ^A for subsystem A is given by

$$\rho^A = \mathrm{Tr}_B\left[\tilde{\rho}^{m_x^A, m_z^A}_{m_x^B, m_z^B}\right] = \frac{1}{2}\left(\mathbb{1} + m_x^A\sigma_x + m_z^A\sigma_z\right). \tag{3.209}$$

The eight maximally non-local states may be labelled by three parameters, $\alpha \in \{\pm 1\}$ and $\ell, m \in \{x, z\}$:

$$\tilde{\rho}^{\alpha}_{\ell m} = \frac{1}{4}\mathbb{1} \otimes \mathbb{1} + \frac{\alpha}{4}\left(\sigma_\ell \otimes \sigma_m + \sigma_{\ell'} \otimes \sigma_m + \sigma_\ell \otimes \sigma_{m'} - \sigma_{\ell'} \otimes \sigma_{m'}\right), \tag{3.210}$$

where the directions ℓ', m' are defined to be different from ℓ, m.

The expectation values for each available measurement in the state $\tilde{\rho}^{\alpha}_{\ell m}$ are as follows:

$$\langle \sigma_j \otimes \mathbb{1} \rangle = \langle \mathbb{1} \otimes \sigma_k \rangle = 0, \qquad \langle \sigma_j \otimes \sigma_k \rangle = \begin{cases} -\alpha, & \text{if } j \neq \ell \text{ and } k \neq m \\ +\alpha, & \text{otherwise} \end{cases}. \tag{3.211}$$

If A, B measure in directions ℓ', m' respectively, it is guaranteed that the product of their measurement outcomes will be $-\alpha$, whereas if they measure in any other pair of directions, the product of their measurement outcomes is guaranteed to be $+\alpha$. The reduced state for either party is maximally mixed, i.e.,

$$\mathrm{Tr}_A[\tilde{\rho}^{\alpha}_{\ell m}] = \mathrm{Tr}_B[\tilde{\rho}^{\alpha}_{\ell m}] = \frac{1}{2}\mathbb{1}, \tag{3.212}$$

and individual outcomes are maximally unpredictable.

3.3.2 Correlations

Consider a Bell scenario [70, 13], where each party has a choice of two measurements – say A can measure either X or X' and B can measure either Y or Y' – with all measurements having ± 1 as their possible outcomes.

For any statistical theory assigning conditional probabilities to these measurements, the expectation value of their product is

$$E(X,Y) = P(++\,|X,Y) + P(--\,|X,Y) - P(+-\,|X,Y) - P(-+\,|X,Y). \tag{3.213}$$

The CHSH score is

$$\mathcal{C}_{XYX'Y'} = E(X,Y) + E(X,Y') + E(X',Y) - E(X',Y'). \tag{3.214}$$

There are fundamental bounds to the value that this quantity can obtain in different classes of theory. For any probabilistic theory obeying local causality, $-2 \leq \mathcal{C}_{XYX'Y'} \leq 2$ [21]. Quantum theory violates this bound, allowing CHSH scores up to $2\sqrt{2}$ (discussed below). Larger violations are possible in post-quantum theories such as the PR box, which satisfies relativistic causality but allows CHSH scores up to the algebraic maximum of 4 [73].

For a quantum miniature (including ordinary quantum mechanics), expectation values are determined by the state, ρ, via a trace rule:

$$E(X,Y) = \mathrm{Tr}\left[\rho X \otimes Y\right]. \tag{3.215}$$

For a given state, ρ, we can thus express the associated CHSH score, $\mathcal{C}(\rho)$, in a compact form:

$$\mathcal{C}_{XYX'Y'}(\rho) = \mathrm{Tr}\left[\rho\,\hat{\mathcal{C}}_{XYX'Y'}\right], \tag{3.216}$$

where $\hat{\mathcal{C}}_{XYX'Y'} = X \otimes Y + X \otimes Y' + X' \otimes Y - X' \otimes Y'$.

In standard quantum theory, the maximal CHSH score attainable in a given scenario is given by the operator norm of the associated operator [58]:

$$\max_{\rho \in \Omega_Q} \mathcal{C}_{XYX'Y'}(\rho) = \|\hat{\mathcal{C}}_{XYX'Y'}\|. \tag{3.217}$$

This norm may be calculated by means of the Khalfin-Tsirelson-Landau identity [57, 53],

$$\hat{\mathcal{C}}^2_{XYX'Y'} = 4\mathbb{1} \otimes \mathbb{1} - [X, X'] \otimes [Y, Y'], \tag{3.218}$$

from which we may derive

$$\|\hat{\mathcal{C}}_{XYX'Y'}\| = \sqrt{4 + \|[X, X'] \otimes [Y, Y']\|}. \tag{3.219}$$

Tsirelson's bound [20] is given by considering the maximum possible value of the right hand side of (3.219), and gives a scenario-independent limit on the possible values of the CHSH score obtainable by any quantum state:

$$-2\sqrt{2} \leq \mathcal{C}(\rho) \leq 2\sqrt{2}. \tag{3.220}$$

Note that this derivation relies on the positivity of all states under consideration, specifically

$$\rho \text{ is positive} \implies \mathrm{Tr}\left[\rho\hat{\mathcal{C}}\right] \leq \|\rho\|\|\hat{\mathcal{C}}\| \tag{3.221}$$

and also

$$\rho \text{ is positive and } \mathrm{Tr}\left[\rho\right] \implies \|\rho\| \leq 1. \tag{3.222}$$

In the context of a quantum miniature, we may have a state, ρ, which is not positive and so neither of the inequalities on the right are necessitated. Indeed, bipartite Square-world miniature contains states violating the first of these inequalities, and we see violations of Tsirelson's bound up to the algebraic maximum $\mathcal{C} = 4$.

In the bipartite Square-world system, the measurements for each party must be chosen from those performable by the apparatus, i.e., the choice is between σ_x and σ_z. There are thus four distinct CHSH scenarios we may consider (neglecting scenarios where $X = X'$ or $Y = Y'$), labelled by the directions, ℓ and m, chosen for the unprimed measurements, X and Y, for parties A and B respectively. To each scenario we associate an operator, $\hat{\mathcal{C}}_{\ell m}$:

$$\hat{\mathcal{C}}_{xx} = \sigma_x \otimes \sigma_x + \sigma_x \otimes \sigma_z + \sigma_z \otimes \sigma_x - \sigma_z \otimes \sigma_z, \tag{3.223}$$

$$\hat{\mathcal{C}}_{xz} = \sigma_x \otimes \sigma_z + \sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z - \sigma_z \otimes \sigma_x, \tag{3.224}$$

$$\hat{\mathcal{C}}_{zx} = \sigma_z \otimes \sigma_x + \sigma_z \otimes \sigma_z + \sigma_x \otimes \sigma_x - \sigma_x \otimes \sigma_z, \tag{3.225}$$

$$\hat{\mathcal{C}}_{zz} = \sigma_z \otimes \sigma_z + \sigma_z \otimes \sigma_x + \sigma_x \otimes \sigma_z - \sigma_x \otimes \sigma_x. \tag{3.226}$$

For the non-local extremal states (3.210) we have

$$\mathcal{C}_{\ell m}(\tilde{\rho}^{\alpha}_{pq}) = 4\alpha\delta_{\ell,p}\delta_{m,q} \tag{3.227}$$

Hence, each non-local extremal state saturates the algebraic bounds $-4 \leq \mathcal{C} \leq 4$ for exactly one of the four available CHSH scenarios. Note that $\|\tilde{\rho}^{\alpha}_{pq}\| = (1+\sqrt{5})/4$, and $\|\mathcal{C}_{\ell m}\| = \sqrt{5}$, so

$$\mathrm{Tr}\,[\tilde{\rho}^{\alpha}_{\ell m}\mathcal{C}_{\ell m}] > \|\tilde{\rho}^{\alpha}_{\ell m}\|\|\mathcal{C}_{\ell m}\| \simeq 1.809. \tag{3.228}$$

For the local extremal states (3.208),

$$\mathcal{C}_{\ell m}(\tilde{\rho}^{m^A_x m^A_z}_{m^B_x m^B_z}) = m^A_\ell(m^B_m + m^B_{m'}) + m^A_{\ell'}(m^B_m - m^B_{m'}). \tag{3.229}$$

This may take values ± 2, and therefore produces only local correlations (despite $\tilde{\rho}^{m^A_x m^A_z}_{m^B_x m^B_z}$ lying outside the set of quantum states).

3.3.3 Other bipartite Miniatures

In the bipartite Square-world miniature described in Section 3.3.1, each party was able to perform spin measurements in the x and z directions. We can generalise this construction by allowing each party to perform spin measurements in an arbitrary pair of fixed non-collinear directions; say party A can measure spin in the directions $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$, while party B can measure spin in the directions $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. We will refer to the resulting miniature as *Rhombus-world*.

The Hilbert space for each of these miniatures is the same, namely $\mathcal{H} = \mathbb{C}^4$, so in each case we can use the same parameterisation for states:

$$\rho = \frac{1}{4}\left(\mathbb{1} \otimes \mathbb{1} + \mathbf{r} \cdot \boldsymbol{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{s} \cdot \boldsymbol{\sigma} + \sum_{j,k} t_{jk}\sigma_j \otimes \sigma_k\right) \tag{3.230}$$

In terms of these parameters, the expectation values for each measurement are

$$\left\langle \boldsymbol{\eta}_j \cdot \boldsymbol{\sigma} \otimes \mathbb{1} \right\rangle = \boldsymbol{\eta}_j \cdot \mathbf{r}, \tag{3.231}$$

$$\langle \mathbb{1} \otimes \boldsymbol{\xi}_k \cdot \boldsymbol{\sigma} \rangle = \boldsymbol{\xi}_k \cdot \mathbf{s}, \tag{3.232}$$

$$\left\langle \boldsymbol{\eta}_j \cdot \boldsymbol{\sigma} \otimes \boldsymbol{\xi}_k \cdot \boldsymbol{\sigma} \right\rangle = \boldsymbol{\eta}_j \cdot t\boldsymbol{\xi}_k, \tag{3.233}$$

and the probabilities for joint measurements are

$$\mathcal{P}(m^A, m^B|\boldsymbol{\eta}_j, \boldsymbol{\eta}_k; \rho) = \frac{1}{4}\left(1 + m^A\boldsymbol{\eta}_j \cdot \mathbf{r} + m^B\boldsymbol{\xi}_k \cdot \mathbf{s} + m^A m^B\boldsymbol{\eta}_j \cdot t\boldsymbol{\xi}_k\right). \tag{3.234}$$

A slight modification of the argument in Section 3.3.1 shows that every miniature in this family allows all 2-2 non-signalling behaviours, i.e., given any $P \in \mathcal{NS}$, there exists $\rho^{(P)} \in \Omega$ such that

$$\mathcal{P}(m^A, m^B|\boldsymbol{\eta}_j, \boldsymbol{\xi}_k; \rho^{(P)}) = P(m^A, m^B|j, k). \tag{3.235}$$

To show that all non-signalling behaviours can be realised by some Rhombus-world state, it is sufficient to demonstrate that the conditional expectation values agree. Given any non-signalling behaviour, P, the conditional expectation values are

$$E^A(j) = P(+1,+1|j,k) + P(+1,-1|j,k) - P(-1,+1|j,k) - P(-1,-1|j,k), \tag{3.236}$$

$$E^B(k) = P(+1,+1|j,k) + P(+1,-1|j,k) - P(-1,+1|j,k) - P(-1,-1|j,k); \tag{3.237}$$

$$E^{AB}(j,k) = P(+1,+1|j,k) + P(-1,+1|j,k) - P(-1,+1|j,k) - P(-1,-1|j,k). \tag{3.238}$$

We aim to construct a state,

$$\rho^{(P)} = \frac{1}{4}(\mathbb{1} \otimes \mathbb{1} + \mathbf{r}^{(P)} \cdot \boldsymbol{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbf{s}^{(P)} \cdot \boldsymbol{\sigma} + \boldsymbol{\sigma} \cdot t^{(P)}\boldsymbol{\sigma}), \tag{3.239}$$

for which the expectation values expressed as in (3.231) agree with those calculated from P, i.e.,

$$\boldsymbol{\eta}_j \cdot \mathbf{r}^{(P)} = E^A(j), \tag{3.240}$$

$$\boldsymbol{\xi}_k \cdot \mathbf{s}^{(P)} = E^B(k) \tag{3.241}$$

$$\boldsymbol{\eta}_j \cdot t^{(P)}\boldsymbol{\xi}_k = E^{AB}(j,k). \tag{3.242}$$

As long as $\boldsymbol{\eta}_1 \nparallel \boldsymbol{\eta}_2$ and $\boldsymbol{\xi}_1 \nparallel \boldsymbol{\xi}_2$, both sets $\{\boldsymbol{\eta}_1, \boldsymbol{\eta}_2\}$ and $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\}$ span $\mathbb{R}^2$, and the set $\{\boldsymbol{\eta}_1\boldsymbol{\xi}_1^\top, \boldsymbol{\eta}_2\boldsymbol{\xi}_1^\top, \boldsymbol{\eta}_1\boldsymbol{\xi}_2^\top, \boldsymbol{\eta}_2\boldsymbol{\xi}_2^\top\}$ spans the space of 2×2 real-valued matrices. Hence, we can write

$$\mathbf{r}^{(P)} = a\boldsymbol{\eta}_1 + b\boldsymbol{\eta}_2, \tag{3.243}$$

$$\mathbf{s}^{(P)} = c\boldsymbol{\xi}_1 + d\boldsymbol{\xi}_2, \tag{3.244}$$

$$t^{(P)} = \alpha\boldsymbol{\eta}_1\boldsymbol{\xi}_1^\top + \beta\boldsymbol{\eta}_2\boldsymbol{\xi}_1^\top + \gamma\boldsymbol{\eta}_1\boldsymbol{\xi}_2^\top + \delta\boldsymbol{\eta}_2\boldsymbol{\xi}_2^\top. \tag{3.245}$$

Imposing the equalities (3.240) then establishes a set of linear relations between the coefficients $a, b, \ldots, \gamma, \delta$ and the conditional expectations $E^A(1), E^A(2), \ldots, E^{AB}(2,2)$. These relations are invertible and we find, defining $u = \boldsymbol{\eta}_1 \cdot \boldsymbol{\eta}_2$ and $w = \boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2$,

$$\mathbf{r}^{(P)} = \frac{E^A(1) - uE^A(2)}{1-u^2}\boldsymbol{\eta}_1 + \frac{E^A(2) - uE^A(1)}{1-u^2}\boldsymbol{\eta}_2, \tag{3.246}$$

$$\mathbf{s}^{(P)} = \frac{E^B(1) - wE^B(2)}{1-w^2}\boldsymbol{\xi}_1 + \frac{E^B(2) - wE^B(1)}{1-w^2}\boldsymbol{\xi}_2, \tag{3.247}$$

$$t^{(P)} = \alpha\boldsymbol{\eta}_1\boldsymbol{\xi}_1^\top + \beta\boldsymbol{\eta}_2\boldsymbol{\xi}_1^\top + \gamma\boldsymbol{\eta}_2\boldsymbol{\xi}_1^\top + \delta\boldsymbol{\eta}_2\boldsymbol{\xi}_2^\top, \tag{3.248}$$

where

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 & u & w & uw \\ u & 1 & uw & w \\ w & uw & 1 & u \\ uw & w & u & 1 \end{pmatrix}^{-1} \begin{pmatrix} E(1,1) \\ E(2,1) \\ E(1,2) \\ E(2,2) \end{pmatrix}. \tag{3.249}$$

Note that, if $u = w = 0$, these expressions simplify to the corresponding expressions (3.204)-(3.206) previously derived in the case that each party can perform measurements in a mutually orthogonal pair of directions.

Having thus defined $\mathbf{r}^{(P)}$, $\mathbf{s}^{(P)}$ and $t^{(P)}$, we define the quasi-density operator $\rho^{(P)}$ having these parameters, as in (3.230). This quasi-density operator is a state since, by construction, it satisfies (3.235) and therefore generates valid probabilities for every possible measurement. Since P was chosen arbitrarily from the set of non-signalling behaviours, it follows that any such behaviour may be realised as the collection of conditional probabilities produced by the available measurements on some state of the miniature.

Since every Rhombus-world miniature produces the same probabilities, they all produce the same range of CHSH scores, independent of which directions are chosen.

However, if we restrict the measurements to those performable in a given miniature but only have quantum states available, then the range of possible CHSH scores does depend on the directions. Specifically the commutators appearing in (3.219) depend on the angles between $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$, and between $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$.

The results of this Section and Section 3.3.1 resemble a result by Acìn et al., that every N-partite non-signalling behaviour with arbitrary inputs and outputs can be expressed in terms of local quantum measurements [1]. Specifically, they show that, given any N-partite non-signalling behaviour P with m inputs and r outputs, there exists

1. A Hilbert space $\mathcal{H}$ with dimension $\dim(\mathcal{H}) = \max\{r, m\}$,
2. For each set of inputs $\{x_i\}_{i=1,\ldots,N}$, a POVM on $\mathcal{H}$, i.e., a set of positive operators $\{M_{a_i}^{x_i}\}_{a_i=1,\ldots,r}$ on $\mathcal{H}$ such that

$$\sum_{a_i=1}^{r} M_{a_i}^{x_i} = \mathbb{1} \tag{3.250}$$

3. A unit-trace self-adjoint operator (i.e., a quasi-density operator) ρ on $\mathcal{H}^{\otimes N}$, such that, for each set of inputs and outputs,

$$\mathrm{Tr}\left[\rho\left(M_{a_1}^{x_1} \otimes \ldots \otimes M_{a_N}^{x_N}\right)\right] = P(a_1, \ldots, a_N | x_1, \ldots, x_N). \tag{3.251}$$

A constructive example is given for the case of PR box correlations.

On the basis of this result, [1] provides a common framework in which to discuss different classes of correlations by imposing different levels of constraint on the operator ρ appearing in (3.251). However, this work does not directly consider a physical scenario where *only* the measurements described by $\{M_{a_i}^{x_i}\}_{a_i=1,\ldots,r}$ can be performed, and does not give the interpretation of ρ as a physical state compatible with this scenario.

3.4 GENERAL FEATURES OF SPIN MINIATURES

3.4.1 Link between extremal states and quasi-eigenstates

In exploring the various quantum miniatures considered so far, identifying the extremal states is crucial in describing the state space of each miniature. We have seen that these extremal states demonstrate many of the novel features seen in quantum miniatures, such as the violation of quantum uncertainty relations, and the possibility of post-quantum correlations, for example. Related to the violation of uncertainty relations is the fact that extremal states are, in many cases, joint quasi-eigenstates of some pair of non-commuting observables.

For any quantum miniature with finitely many observables defined on a finite-dimensional Hilbert space, the state space is a convex polytope, as discussed in Section 2.2. As such, the state space can be defined as the interior of a collection of bounding hyperplanes, each defined by some linear inequality of the form

$$\mathcal{P}(\text{some measurement outcome}|\rho) \geq 0. \tag{3.252}$$

Quasi-eigenstates of an observable are states which predict definite outcomes for that observable (equivalently, those which predict zero variance). This means that the probability of obtaining any other outcome for that observable must be zero, i.e., must saturate one of these inequalities. In a spin s miniature, there are $2s+1$ possible outcomes for the measurement of any non-degenerate observable, so a quasi-eigenstate must saturate $2s$ inequalities pertaining to the same observable.

For a d-dimensional state space, extremal states lie at the intersection of d or more bounding hyperplanes (of which, d must be linearly independent), and therefore saturate d or more of these inequalities, though these need not all pertain to the same observable.

Both of these properties – extremality and being a joint quasi-eigenstate – are thus related to the saturation of multiple inequalities, and a simple counting argument establishes a sufficient criterion to conclude that all extremal states of a given miniature are quasi-eigenstates of some observable in that miniature. Consider a spin s miniature with N basic observables, having a d-dimensional state space. Extremal states saturate at least d bounding inequalities. In general, these inequalities may pertain to different observables. However, if $d > N(2s-1)$, then there must be at least one observable contributing $2s$ inequalities, and the state must therefore be a quasi-eigenstate of that observable. If $d > N(2s-1)+1$, then there must be two observables each contributing $2s$ inequalities, so the state is necessarily a joint quasi-eigenstate of both. This counting criterion is satisfied for many of the miniatures we have considered, as shown in Table 3.3. In particular, it is satisfied by all miniatures with finitely many observables in the case of spin $s = 1/2$.

A similar argument establishes that, if $d > N(2s-1)+k$, then all extremal states are joint quasi-eigenstates of at least $k+1$ observables. This condition is sufficient but not necessary – there may be linear dependences between different positivity constraints which impose the joint quasi-eigenstate condition without this inequality. These relationships may be used as a guide to determining extremal states in other miniatures, as well as helping to construct miniatures which maximally violate quantum uncertainty relations through the existence of joint quasi-eigenstates.

Miniature	Spin s	# basic observables N	Dimension d	$d > N(2s-1)+1$?
Square-world		2	2	✓
Cube-world	$\frac{1}{2}$	3	3	✓
n-gon world		$n/2$	2	✓
Square-world		2	4	✓
Cube-world	1	3	5	✓
n-gon world		$n/2$	5	for $n < 8$

Table 3.3

3.4.2 Useful properties of extremal states

We have established that, under certain conditions, extremal states must be joint quasi-eigenstates. Conversely, if joint-quasi eigenstates exist in a given miniature, they must be extremal. This result follows as a special case of a more general statement: if an additive uncertainty relation regarding a collection of observables $\{A_i\}_{i=1,\ldots,N}$,

$$\sum_{i=1}^{N} \Delta^2(A_i|\rho) \geq U, \tag{3.253}$$

is saturated by any state, then it is saturated by some extremal state. This is because – by the law of total variance – the map

$$\rho \mapsto \sum_i \Delta^2(A_i|\rho) \tag{3.254}$$

is concave. Thus, writing an arbitrary state ρ as a convex combination of extremal states,

$$\rho = \sum_j \lambda_j \tilde{\rho}_j, \tag{3.255}$$

we have

$$\sum_i \Delta^2(A_i|\rho) \leq \sum_j \lambda_j \left(\sum_i \Delta^2(A_i|\tilde{\rho}_j) \right) \tag{3.256}$$

$$\leq \min_j \left\{ \sum_i \Delta^2(A_i|\tilde{\rho}_j) \right\}. \tag{3.257}$$

Hence,

$$\sum_i \Delta^2(A_i|\rho) = U \implies \sum_i \Delta^2(A_i|\tilde{\rho}_j) = U \text{ for some } j. \tag{3.258}$$

If the collection $\{A_i\}$ has a joint quasi-eigenstate, then the constant U appearing in this uncertainty relation is zero, which establishes the special case.

Similarly, given a miniature with state space Ω_{min}, the existence of any state $\rho \in \Omega_{min}$ which is incompatible with a given Hermitian operator implies the existence of an extremal state which is incompatible with the same operator. It follows that the existence of any non-quantum state implies the existence of a non-quantum extremal state. In developing future models in the quantum miniature framework, these properties may help to reduce the problem of finding states with particular interesting properties to the problem of finding *extremal* states with those properties, which in many cases means consideration of a finite set.

— 4 —

Quantum miniatures for particle systems

In this chapter, we extend the ideas of the previous chapters to particle systems. We define miniatures in this setting by taking the Hilbert space to be that of a single non-relativistic quantum particle moving in one dimension, and by restricting the set of observables. Quantum miniatures thus provide a natural setting to consider non-quantum models with continuous variables, whereas GPTs are commonly restricted to consideration of finite-dimensional systems [48, 56, 71]. Other approaches to generalised theories for continuous variables include [72].

Some modifications are required when discussing an infinite-dimensional system compared with finite dimensions, just as is typical in standard quantum theory. For example, for observables with a continuous spectrum of possible outcomes, the probability map is replaced by a map describing the probability distribution over these possible outcomes.

4.1 CROSS-WORLD

4.1.1 Observables

The quantum miniature *Cross-world* is defined by specifying the set $\mathcal{O}_C$ of observables which are assumed to be accessible in experiments. This set is chosen to be a proper subset of the observables of a quantum particle with a single degree of freedom. We base our discussion on the Hilbert space $L^2(\mathbb{R})$ of square-integrable functions $\mathbb{R} \to \mathbb{C}$.

In order to have a notion of phase space, we assume position, Q, and momentum, P, to be observable. Given a function $\psi : \mathbb{R} \to \mathbb{C}$, the action of these operators is given by

$$Q\psi(q) = q\psi(q) \qquad \text{and} \qquad P\psi(q) = -i\hbar\frac{\partial}{\partial q}\psi(q). \tag{4.1}$$

For details relating to the domains of Q and P, see Appendix B. Thus, we define the set of basic observables to be

$$\mathcal{O}_C = \{Q, P\}. \tag{4.2}$$

We denote by E_q and E_p respectively the projectors onto generalised eigenstates of position and momentum (see Appendix B) :

$$E_q = |q\rangle\langle q| \qquad \text{and} \qquad E_p = |p\rangle\langle p|\,. \tag{4.3}$$

The probability distributions for measurements of position and momentum are given by a trace rule, as in standard quantum theory. Specifically, for a state ρ, the probability density at q for a position measurement is given by

$$\mathcal{P}(q|\rho) = \mathrm{Tr}\left[\rho E_q\right], \tag{4.4}$$

and similarly for momentum,

$$\mathcal{P}(p|\rho) = \mathrm{Tr}\left[\rho E_p\right]. \tag{4.5}$$

As a shorthand, we write

$$E_q(\rho) = \mathrm{Tr}\left[\rho E_q\right], \qquad E_p(\rho) = \mathrm{Tr}\left[\rho E_p\right]. \tag{4.6}$$

We can then define the state space of Cross-world to consist of exactly those quasi-density operators which produce valid probability distributions, i.e., which are non-negative and normalised:

$$\Omega_C = \{\rho : \mathcal{H} \to \mathcal{H} \,|\, \rho^\dagger = \rho,\ \mathrm{Tr}\left[\rho\right] = 1,\ E_q(\rho) \geq 0, E_p(\rho) \geq 0\ \forall q, p \in \mathbb{R}\}. \tag{4.7}$$

We also include eigenstates of position and momentum as states, viewing these as a limiting case of infinitely squeezed states (see Section 1.3.4).

All quantum states, i.e., all positive operators with unit trace, satisfy the constraints described in (4.7) and are therefore contained in Ω_C. In addition, there exist states in Cross-world which do not correspond to quantum states, since they are not positive; this is shown in Section 4.1.3.

4.1.2 Quasi-Wigner functions

An alternative to the density operator formulation of standard quantum theory is provided by the Wigner function (see Section 1.3.3). Extending the state space to include quasi-density operators also introduces new distribution functions which we call *quasi-Wigner functions*. Quasi-Wigner functions are obtained by applying the Wigner transform (1.15) to the quasi-density operators representing non-quantum states in Cross-world. Since quasi-density operators are Hermitian and have unit trace, quasi-Wigner functions are still real-valued and normalised. Furthermore, the identities

$$\int_{\mathbb{R}} \mathrm{d}p\, W_\rho(q,p) = \mathrm{Tr}\left[\rho\, |q\rangle\langle q|\right], \tag{4.8}$$

$$\int_{\mathbb{R}} \mathrm{d}q\, W_\rho(q,p) = \mathrm{Tr}\left[\rho\, |p\rangle\langle p|\right]. \tag{4.9}$$

still hold, so that probability densities for position and momentum are calculated in the same way as for usual Wigner functions, integrating along lines of constant q or constant p. However, quasi-Wigner functions can have different properties from standard Wigner functions corresponding to quantum states; for example, quasi-Wigner functions may be more localised than is possible for a standard Wigner function, and do not have to obey the condition (1.18). A quasi-Wigner function $W(q,p)$ describes a valid Cross-world state if and only if it generates valid probability distributions for position and momentum measurements, i.e.,

$$0 \leq \int_{\mathbb{R}} \mathrm{d}p\, W(q,p) \leq 1 \text{ for all } q, \tag{4.10}$$

$$0 \leq \int_{\mathbb{R}} \mathrm{d}q\, W(q,p) \leq 1 \text{ for all } p. \tag{4.11}$$

4.1.3 States and equivalence classes

We have seen in the case of Square-world that (quasi-) density operators may in some cases be indistinguishable by measurements, and so fall into equivalence classes. For Cross-world this raises the question of whether two operators may, in general, be distinguished based only on the statistics of position and momentum measurements. In quantum theory, this question gives rise to the Pauli problem for quantum states [66, 67], and has been answered in the negative: there are distinct quantum states generating identical probability distributions for both position and momentum [88, 22]. Another way of saying this is that the set $\{Q, P\}$ is tomographically incomplete within quantum theory.

We say two Cross-world operators, ρ and ρ', belong to the same equivalence class if and only if they generate the same probability distributions for both position and momentum measurements, i.e.,

$$\mathcal{P}(q|\rho) = \mathcal{P}(q|\rho') \qquad \text{and} \qquad \mathcal{P}(p|\rho) = \mathcal{P}(p|\rho') \tag{4.12}$$

for all p and q.

Proposition 4.1.1. Each equivalence class contains an element whose associated quasi-Wigner function is factorisable.

Proof. Given any Cross-world operator ρ, define the factorisable quasi-Wigner function $\widetilde{W}$ by

$$\widetilde{W}(q,p) = \mathcal{P}(q|\rho)\mathcal{P}(p|\rho), \tag{4.13}$$

where $\mathcal{P}(q|\rho)$ and $\mathcal{P}(p|\rho)$ are the marginal probability distributions for position and momentum generated by ρ. Note that $\widetilde{W}$ is not in general the Wigner function associated to the operator ρ. By construction, however, it does generate the same probability distributions for both position and momentum measurements. Define now the operator $\tilde{\rho}$ as the Weyl transform of $\widetilde{W}$. Since this operator generates the same probability distribution for position and momentum measurements as ρ, they belong to the same equivalence class. Since ρ was chosen arbitrarily, any equivalence class must therefore contain an operator whose associated Wigner function factorises. □

Proposition 4.1.2. Any pair of probability distributions over $\mathbb{R}$ may be realised as the respective probability densities for position and momentum measurements in some Cross-world state.

Proof. Given any pair of probability densities $\mathcal{P}_1$ and $\mathcal{P}_2$, define the quasi-Wigner function

$$W(q,p) = \mathcal{P}_1(q)\mathcal{P}_2(p). \tag{4.14}$$

This expression corresponds to a valid Cross-world state since its marginals in each variable are non-negative and normalised. □

Based on these results, we may completely characterise the space of equivalence classes, which we describe by convenient representatives. From the result 4.1.1, we may always take a representative with a factorisable quasi-Wigner function. The space of all such representatives, $\bar{\Omega}_C$, is closed under convex combinations, and by 4.1.2 is equivalent to the space of all pairs of probability distributions over $\mathbb{R}$:

$$\bar{\Omega}_C = \mathrm{Prob}(\mathbb{R}) \times \mathrm{Prob}(\mathbb{R}). \tag{4.15}$$

Hence, an equivalence class is extremal if and only if each factor in its factorisable representative is extremal in the convex space of probability distributions, i.e., if

$$\widetilde{W}_{[\rho]}(q,p) = \delta(q-q_0)\delta(p-p_0) \tag{4.16}$$

for some $q_0, p_0 \in \mathbb{R}$. Applying the Weyl transform (1.23) expressed in terms of the Wigner kernel, we see that the quasi-density operator associated to $\widetilde{W}$ is equal to the Wigner kernel defined in (1.21), evaluated at the point (q_0, p_0):

$$\tilde{\rho} = \int \mathrm{d}q\, \mathrm{d}p\, \delta(q-q_0)\delta(p-p_0)\hat{\Delta}(q,p) = \hat{\Delta}(q_0,p_0). \tag{4.17}$$

We thus have a complete characterisation of all extremal states in Cross-world. These form a continuous two-parameter family, parameterised one-one by points of phase space. Each extremal state is a joint quasi-eigenstate of position and momentum.

We see that Cross-world does allow quasi-Wigner functions beyond the usual Wigner functions associated to quantum states. Indeed, the delta distribution appearing in (4.16) has been presented as a specific example of a generalised function on phase space which does *not* represent a valid Wigner function in standard quantum theory [33, p. 235, 41, p. 127]. In Cross-world, this distribution is a valid quasi-Wigner function, describing an allowed state of the theory. Also, it has been shown that a pure quantum state has a non-negative Wigner function only if it is a coherent state [44]. An extension of this result to mixed quantum states establishes a bound on the non-Gaussianity of any state with a non-negative Wigner function [60, 61], which places limitations on the set of position probabilities. In Cross-world, on the other hand, no limitation can be placed since all position marginals are compatible with a positive overall quasi-Wigner function. In some cases, the non-quantum character of particular Cross-world states may be detected directly from experimental data, by the violation of uncertainty relations.

4.1.4 Uncertainty relations

For position and momentum, quantum mechanics provides a state-independent bound on the joint uncertainties, namely the Heisenberg uncertainty relation [40, 52],

$$\Delta^2 Q \Delta^2 P \geq \frac{\hbar^2}{4}. \tag{4.18}$$

All quantum states satisfy the relation (4.18), so any violation of this signifies that an operator is non-quantum. Violations will occur for Cross-world states for which the probability distributions in both position and momentum are highly localised.

An example of such a state has a quasi-Wigner function similar to the Wigner function of a coherent state, where now we rescale both variables by a common factor $1/\sqrt{\tau}$, with $0 < \tau < 1$:

$$W_\tau(q,p) = \frac{1}{\pi\hbar\tau} \exp\left(-\frac{q^2+p^2}{\hbar\tau}\right). \tag{4.19}$$

Note that $W_\tau(0,0) = 1/(\pi\hbar\tau)$, in violation of (1.18). Applying the Weyl transform (1.19), we find the operator corresponding to this quasi-Wigner function:

$$\rho_\tau = \frac{\hbar}{2\pi} \int_{\mathbb{R}^2} \mathrm{d}a\, \mathrm{d}b \, \exp\left(-\frac{\tau(a^2+b^2)}{4\hbar}\right) \exp\left(\frac{ia\hat{Q}+ib\hat{P}}{\hbar}\right). \tag{4.20}$$

Its matrix elements in the position basis are given by

$$\langle q_1 | \, \rho_\tau \, | q_2 \rangle = (\pi\hbar\tau)^{-1/2} \exp\left(-\frac{(\tau+1/\tau)(q_1^2+q_2^2) - 2(\tau-1/\tau)q_1q_2}{4\hbar}\right). \tag{4.21}$$

The uncertainties in position and momentum are

$$\Delta^2_{\rho_\tau} Q = \frac{\hbar\tau}{2}, \qquad \Delta^2_{\rho_\tau} P = \frac{\hbar\tau}{2}. \tag{4.22}$$

These violate Heisenberg's uncertainty relation for any $\tau < 1$, and can simultaneously get arbitrarily small as $\tau \to 0$. Hence, the operator ρ_τ cannot be the density operator of a quantum state, but it does correspond a valid state in Cross-world: it is a unit trace Hermitian operator furnishing positive probability distributions for both position and momentum. In the limit $\tau \to 0$, the quasi-Wigner function W_τ approaches that of an extremal state (4.16).

Furthermore, the Cross-world uncertainty region for position and momentum, $\mathrm{PUR}_C(Q,P)$, may be fully characterised. Since any pair of probability distributions may be realised as the respective probability densities for position and momentum measurements, it follows that any pair $(\Delta Q, \Delta P)$ may be realised as the respective uncertainties for these observables. Hence,

$$\mathrm{PUR}_C(Q,P) = \mathbb{R}_+ \times \mathbb{R}_+, \tag{4.23}$$

where $\mathbb{R}_+$ denotes the set of non-negative real numbers.

4.1.5 Symmetry and Dynamics

While we do not give a full general account of dynamics for an arbitrary quantum miniature, Cross-world provides an illustration of some novel features which are likely to appear in such an account compared to standard quantum mechanics.

Reversible transformations Motivated by standard quantum theory, we consider unitary evolutions of the form

$$\rho \mapsto \rho(t) = U(t)\rho U^\dagger(t), \tag{4.24}$$

where ρ is any initial state and $U(t)$ constitute a one-parameter family of unitary operators. However, in contrast to standard quantum theory, not every unitary operator will lead to a valid transformation.

We have seen that, generally speaking, restricting the set of observables allows for extended sets of states in miniatures. In a similar manner, these extended state spaces will restrict the possible dynamics of a theory. For example, evolution generated by the free Hamiltonian, where

$$U(t) = \exp\left(\frac{itP^2}{2m}\right) \tag{4.25}$$

is not permissible in Cross-world, as we now show.

Under this evolution, the quasi-Wigner function corresponding initially to an arbitrary state, ρ_0, evolves as follows:

$$\begin{aligned} W(q,p;t) &= \frac{1}{\pi\hbar}\int \mathrm{d}u\, e^{2iqu/\hbar} \langle p+u|\, U(t)\rho_0 U^\dagger(t)\, |p-u\rangle & (4.26)\\ &= \frac{1}{\pi\hbar}\int \mathrm{d}u\, \exp\left[\frac{2iu}{\hbar}\left(q - \frac{p}{m}t\right)\right] \langle p+u|\, \rho_0\, |p-u\rangle & (4.27)\\ &= W\left(q - \frac{p}{m}t, p; 0\right). & (4.28) \end{aligned}$$

We now show that this evolution is not admissible in Cross-world, in the sense that it does not preserve the state space, Ω_C. According to the arguments of Section 2.4, it suffices to show that evolution under the free Hamiltonian does not preserve equivalence classes.

Consider a pair of initial states, with initial Wigner functions $W^{\theta,\tau}$ and $W^{-\theta,\tau}$ respectively:

$$W^{\theta,\tau}(q,p;0) = C\exp\left[-\tau(q\cos\theta + p\sin\theta)^2 - \frac{1}{\tau}(q\sin\theta - p\cos\theta)^2\right]. \tag{4.29}$$

At time $t = 0$, the two Wigner functions $W^{\theta,\tau}(q, p; 0)$ and $W^{-\theta,\tau}(q, p; 0)$ correspond to squeezed coherent states, rotated equally in opposite directions. These represent valid Cross-world states (in particular, they are also valid quantum states) and they are Pauli partners, i.e., they generate identical probability distributions for position and momentum measurements. However, if we evolve each state under the free Hamiltonian, this relationship is not preserved.

Under the free Hamiltonian (4.25),

$$W^{\theta,\tau}(q, p; t) = C \exp\left[-\tau \left(\left(q - \frac{pt}{m} \right) \cos\theta + p \sin\theta \right)^2 - \frac{1}{\tau} \left(\left(q - \frac{pt}{m} \right) \sin\theta - p \cos\theta \right)^2 \right] \quad (4.30)$$

The probability density for position measurements at time t is

$$\int \mathrm{d}p \, W^{\theta,\tau}(q, p; t) = C \sqrt{\frac{\pi}{\beta}} \exp\left[- \left(\alpha - \frac{\gamma^2}{\beta} \right) q^2 \right] \quad (4.31)$$

where α, β and γ depend on t as well as on the parameters θ and τ:

$$\alpha = \tau \cos^2\theta + \frac{1}{\tau} \sin^2\theta \quad (4.32)$$

$$\beta = \tau \left(\sin\theta - \frac{t}{m} \cos\theta \right)^2 + \frac{1}{\tau} \left(\cos\theta + \frac{t}{m} \sin\theta \right)^2 \quad (4.33)$$

$$\gamma = \tau \left(\cos\theta \sin\theta - \frac{t}{m} \cos^2\theta \right) + \frac{1}{\tau} \left(-\cos\theta \sin\theta - \frac{t}{m} \sin^2\theta \right). \quad (4.34)$$

We see that, for non-zero times, the two Wigner functions $W^{\theta,\tau}$ and $W^{-\theta,\tau}$ give different probability distributions for position measurements. This demonstrates that evolution under the free Hamiltonian does not preserve equivalence classes, so does not preserve the state space of Cross-world.

It is nevertheless possible to define a consistent dynamics for Cross-world, which respects the symmetry of the observables and the state space. Similar to Square-world, the symmetry of the full quantum system is broken by the restriction on observables, leaving a smaller symmetry group. As the restriction on spin components leads to a preferred set of orientations, removing linear combinations of position and momentum from the available observables picks out a preferred class of Galilean frames. These frames are related by rigid translations parallel to the phase space axes – Galilean boosts, which map the position operator to a linear combination of

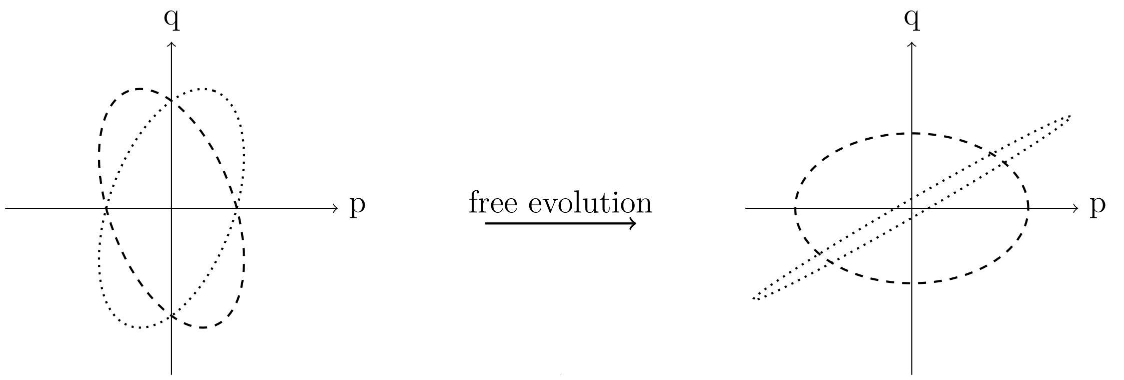


Figure 4.1: Schematic depiction of the evolution of two squeezed states under the free Hamiltonian. The states are initially Pauli partners, but do not remain as such under this evolution.

position and momentum, do not preserve the observables of Cross-world. We also have symmetry under the finite phase space rotation $Q \to P$, $P \to -Q$. Extremal states of Cross-world are mapped among each other under these transformations.

Rigid translations in phase space are generated by the position operator, for translations parallel to the momentum axis:

$$\rho \mapsto \rho' = e^{-iQt}\rho e^{iQt}, \qquad W_\rho(q,p) \mapsto W_{\rho'}(q,p) = W_\rho(q,p+t), \tag{4.35}$$

and the momentum operator, for translations parallel to the position axis:

$$\rho \mapsto \rho' = e^{-iPs}\rho e^{iPs}, \qquad W_\rho(q,p) \mapsto W_{\rho'}(q,p) = W_\rho(q+s,p). \tag{4.36}$$

These maps can be composed to produce any phase space translation,

$$\rho \mapsto \rho'' = \mathcal{T}^\dagger(s,t)\rho\mathcal{T}(s,t), \quad W_\rho(q,p) \mapsto W_{\rho''}(q,p) = W_\rho(q+s,p+t). \tag{4.37}$$

We have a continuous two-parameter family of symmetry transformations, each providing a one-one map from the set of extremal states onto itself.

Measurement In Square-world, we saw that each basic observable had two distinct extremal quasi-eigenstates, leading to a possible ambiguity in the post-measurement state. In Cross-world, there are now an uncountable family of quasi-eigenstates for both basic observables. One possible update rule is that, for a system initially described by the quasi-Wigner function W, after a position measurement returning the outcome $\bar{q}$, the quasi-Wigner function describing the system updates to $W^{\bar{q}}$, where

$$W^{\bar{q}}(q,p) = \delta(q-\bar{q}) \int \mathrm{d}q' W(q',p). \tag{4.38}$$

Immediate subsequent measurement of Q is then guaranteed to return $\bar{q}$, and immediate measurement of P has the same statistics as the original state. This rule is manifestly covariant under phase space translations, and respects equivalence classes. It also has the property that measurement of both basic observables leaves the system in a uniquely identified extremal state, i.e., the joint-quasi eigenstate (4.19) centred at the the measured values. The update rule described by (4.38) is not tenable in standard quantum theory, since an initially quantum state can be projected onto a non-quantum state.

4.2 STAR-WORLD

In a similar spirit to the polygon models for spin systems, we define a family of miniatures containing Cross-world as a special case. These miniatures arise by assuming particular sets of linear combinations of position and momentum to be observable – in particular we choose collections of linear combinations exhibiting discrete rotational symmetry in phase space.

Observables Let $\theta_n = n\pi/N$ for $N \geq 2$, and define the linear combinations

$$Q_n = Q\cos\theta_n + P\sin\theta_n \text{ for } n = 0, 1, \ldots, N-1. \tag{4.39}$$

For given N, these are taken to be the basic observables in a miniature we call N-pointed Star-world, $\mathcal{O}_{NSW} = \{Q_n\}$. For $N = 2$, we recover Cross-world.

The probability densities for each observable Q_n in a state ρ may be calculated by integrating its quasi-Wigner function W_ρ along lines of $q\cos\theta_n + p\sin\theta_n = \text{const.}$:

$$\mathcal{P}(Q_n = \alpha|\rho) = \int_{\mathbb{R}} \mathrm{d}t\, W_\rho(\alpha\cos\theta_n + t\sin\theta_n, \alpha\sin\theta_n - t\cos\theta_n). \tag{4.40}$$

States Each Star-world miniature admits quasi-Wigner functions outside the usual quantum Wigner functions. For example, the 'super-sharp' state (4.19) generates valid probability distributions for all observables in each case, despite being non-quantum.

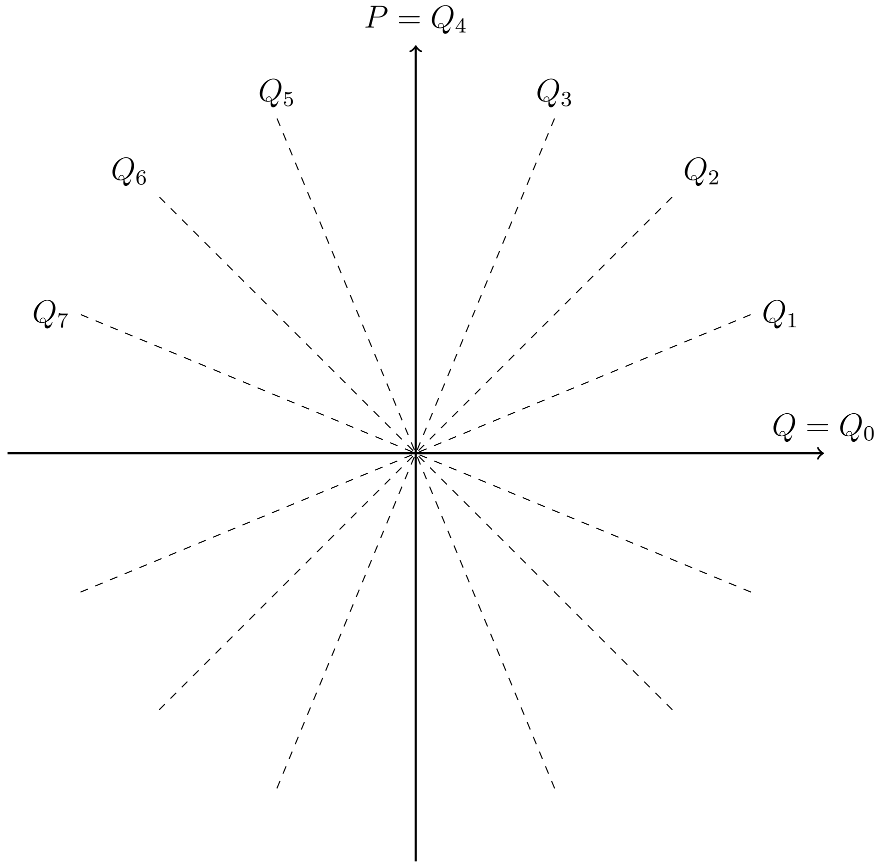


Figure 4.2: Illustration of the observables for 8-pointed Star-world.

Including additional observables relative to Cross-world does, however, imply additional constraints on admissible states. Consider an N-pointed Star-world miniature for some even number $N > 2$, so that the set of basic observables, $\mathcal{O}_{NSW}$, contains $\mathcal{O}_C = \mathcal{O}_{2SW}$ as a strict subset. Let ρ be a quantum state (which is therefore compatible with both $\mathcal{O}_{2SW}$ and $\mathcal{O}_{NSW}$), and let W_ρ be the corresponding Wigner function. Now consider the modified quasi-Wigner function, W', defined by adding positive perturbations centred at the points (q_0, p_0) and $(-q_0, -p_0)$ and corresponding negative perturbations centred at $(q_0, -p_0)$ and $(-q_0, p_0)$:

$$W'(q,p) = W_\rho(q,p) + f(q - q_0, p - p_0) - f(q - q_0, p + p_0) - f(q + q_0, p - p_0) + f(q + q_0, p + p_0). \quad (4.41)$$

By construction, this quasi-Wigner function provides the same marginal distributions for measurements of position and momentum as does the original Wigner function W_ρ, since the positive and negative perturbations exactly cancel out in these marginals.

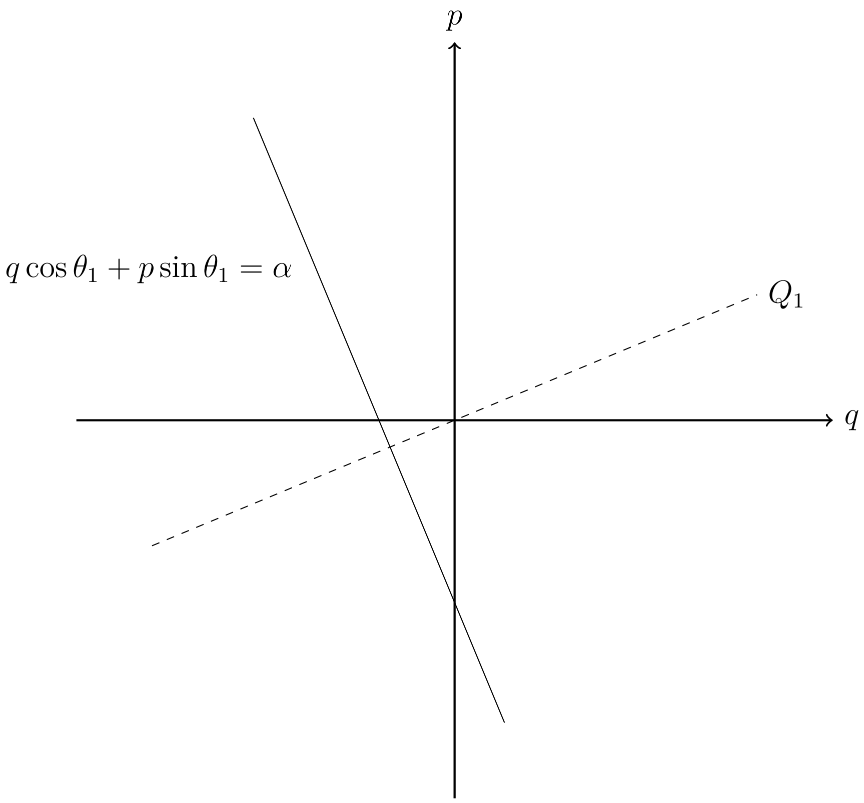
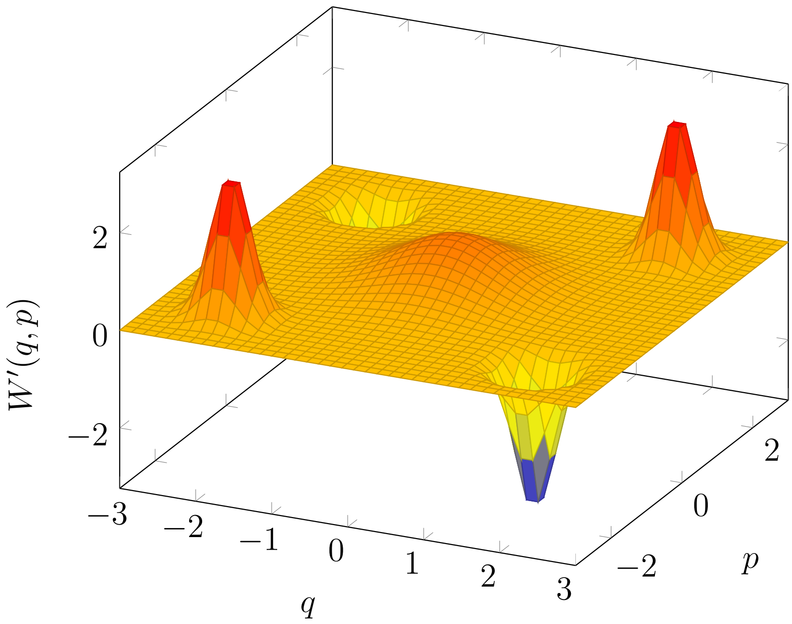


Figure 4.3: The probability density, $\mathcal{P}(Q_1 = \alpha|\rho)$, for the outcome α in a measurement of Q_1 is given by integrating $W_\rho(q,p)$ over the line $q\cos\theta_1 + p\sin\theta_1 = \alpha$.

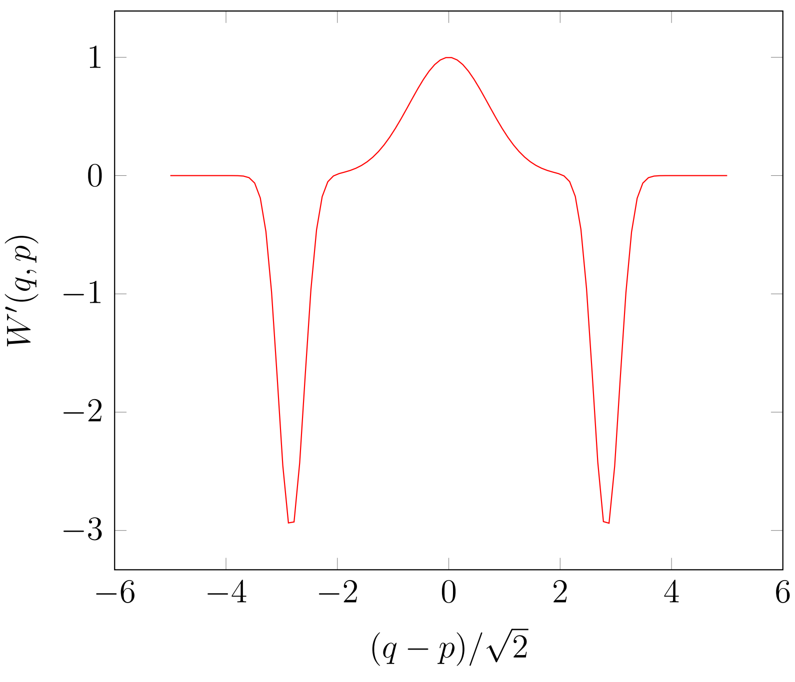
The quasi-density operator described by W' is therefore compatible with these observables, so represents a Cross-world state. However, it is possible to choose f in such a way that the integral along the line through (q_0, p_0) and $(-q_0, -p_0)$ is negative; for example, we may take f to a normalised Gaussian tightly peaked around zero. If q_0 and p_0 are chosen so this integral represents the probability for measurement of an observable, then W' would predict negative probabilities for this measurement: the corresponding quasi-density operator is thus incompatible with the observable and does not represent a state of N-star world.

More generally, if N is any integer multiple of M, then the M-pointed Star-world miniature admits quasi-Wigner functions which are incompatible with the N-pointed miniature. Examples may be constructed in a very similar manner to (4.41), by adding perturbations which are highly localised along the appropriate lines in phase space.

The structure of equivalence classes is also affected by adding extra observables. For example, the squeezed coherent states $W^{\theta,\tau}$ and $W^{-\theta,\tau}$ (4.29) are equivalent



(a) The perturbed quasi-Wigner function, $W'(q,p)$



(b) A projection of $W'(p,q)$ along the line $p=-q$

Figure 4.4: Plots of (a) the perturbed quasi-Wigner function, $W'(q,p)$, from (4.41) and (b) its projection along the line $p=-q$. The integral over $\mathbb{R}$ of this projection gives the predicted value for $\mathcal{P}(\frac{P+Q}{\sqrt{2}}=0|\rho')$; this value is negative, demonstrating that W' is incompatible with the observable $(P+Q)/\sqrt{2}$.

in Cross-world since they generate identical probability distributions for position and momentum, but are not equivalent in other Star-world miniatures, since the probability distributions for any rotated observable Q_n will differ.

4.3 PLANE WORLD

A final miniature, *Plane-world*, is given by assuming all linear combinations of position and momentum are observable:

$$\mathcal{O}_{PW} = \{Q\cos\theta + P\sin\theta \,|\, \theta \in [0,\pi)\}. \tag{4.42}$$

States Since every observable in N-pointed Star-world is also an observable of Plane-world, then the set of quasi-Wigner functions compatible with $\mathcal{O}_{PW}$ is a subset of those compatible with $\mathcal{O}_{NSW}$ for every N.

$$\mathcal{O}_{PW} = \bigcup_{N\in\mathbb{N}} \mathcal{O}_{NSW}, \qquad \Omega_{PW} = \bigcap_{N\in\mathbb{N}} \Omega_{NSW}. \tag{4.43}$$

Plane-world also admits non-quantum states, including super-sharp states. These are joint quasi-eigenstates of an uncountable family of mutually non-commuting observables. Specifically, given a super-sharp state $\tilde{\rho}$ centred at (q_0, p_0), a measurement of the observable $Q\cos\theta + P\sin\theta$ returns the value $q_0\cos\theta + p_0\sin\theta$ with certainty.

This model has a continuous non-Abelian symmetry group generated by phase space rotations and translations.

— 5 —

Summary and Conclusions

We have introduced the framework of quantum miniatures, aimed at examining the role played in quantum theory by the operator assumption, i.e., the assumption that all self-adjoint operators correspond to observables. This framework retains most of the formal structure of standard quantum theory except that, for each model, the operator assumption is supplanted by a specific list of operators which are taken to represent observables.

Quantum miniatures represent internally consistent foil theories to standard quantum theory which, importantly, demonstrates that the operator assumption is not a logical necessity. On the other hand, quantum miniatures display qualitatively different behaviour to standard quantum theory, which further clarifies the specific role of the operator assumption.

We developed several models for spin systems and particle systems, with specific assumptions as to the available observables in each case. We then derived the state spaces of each miniature, with the exception of Cylinder-world, Sphere-world, Star-world and Plane-world, where partial results were obtained.

We have investigated uncertainty relations, including a full determination of the uncertainty regions for all available observables in Square-world and Cross-world. The results clarify the implicit dependence of uncertainty relations upon observables which do not explicitly feature in the relation. For a bipartite Square-world system with local measurements, we investigated non-local correlations, and found that all non-signalling correlations are achievable.

Some results regarding dynamics have been obtained, but this remains the main area for further development of the miniature framework. This would involve updating the postulates to account for reversible transformations, and an investigation of possible state update rules.

Regarding kinematics, open questions also remain. For example: can we characterise the conditions for a miniature to have the same state space as usual quantum theory; is it required that all Hermitian operators are observable, or is there a subset which suffices? Can we characterise which GPTs may be realised as a quantum miniature? We have seen that polygon models, bipartite Square-world, and quantum theory can all be realised as such.

This work also indicates possible future directions for research. Quantum miniatures may provide a fruitful framework for defining GPTs with continuous variables, and a basis for investigating bipartite correlations, or other information-theoretic features of these. Another useful aspect of quantum miniatures is that realising different models in terms of operators on a common Hilbert space can allow for direct comparison between theories, such as identifying quantum states as a subset. For example, building upon results obtained so far, quantum miniatures may be used to investigate relationships between uncertainty relations and non-local correlations.

— A —

Wigner functions for spin

For a spin-s system, phase space functions can be defined over the sphere [87, 3] (alternative notions of phase space for finite-dimensional systems have also been put forward, including vector spaces over a finite field [96]). To each operator $A \in L^*(\mathcal{H})$, we assign the phase space function $W_A : S^2 \to \mathbb{R}$ given by

$$W_A(\boldsymbol{\eta}) = \mathrm{Tr}\left[A\hat{\Delta}_s(\boldsymbol{\eta})\right], \tag{A.1}$$

where $\hat{\Delta}_s$ is the Wigner kernel for spin. The kernel is not uniquely defined, but an option is [3]

$$\hat{\Delta}_s(\boldsymbol{\eta}) = D_s(S_{\boldsymbol{\eta}}) = \sum_{m=-s}^{s} D_s(m) \left|m, \boldsymbol{\eta}\right\rangle\!\left\langle m, \boldsymbol{\eta}\right|, \tag{A.2}$$

where $|m, \boldsymbol{\eta}\rangle$ is the eigenvector of $S_{\boldsymbol{\eta}}$ with eigenvalue m, and the function D_s can be expressed in terms of Clebsch-Gordan coefficients as

$$D_s(m) = \sum_{l=0}^{2s} \frac{2l+1}{2s+1} \left\langle \begin{matrix} s & l \\ m & 0 \end{matrix} \middle| \begin{matrix} s \\ m \end{matrix} \right\rangle. \tag{A.3}$$

Spin $s = 1/2$ For spin $s = 1/2$, we have $D(\pm 1/2) = (1 \pm \sqrt{3})/2$, so the kernel evaluted at a point $\boldsymbol{\eta} \in S^2$ is

$$\hat{\Delta}_{1/2}(\boldsymbol{\eta}) = \frac{1}{2}\mathbb{1} + \frac{\sqrt{3}}{2}\boldsymbol{\eta} \cdot \boldsymbol{\sigma}. \tag{A.4}$$

Given a spin-1/2 quasi-density operator $\rho = (\mathbb{1} + \mathbf{r} \cdot \boldsymbol{\sigma})/2$, the corresponding quasi-Wigner function given by (A.2) is

$$W_\rho(\boldsymbol{\eta}) = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{r} \cdot \boldsymbol{\eta}. \tag{A.5}$$

The quasi-Wigner function provides a simple criterion to identify non-quantum states in any spin-1/2 miniature. Specifically, ρ is a quantum state if and only if

$$\max_{\boldsymbol{\eta}\in S^2} |W_\rho(\boldsymbol{\eta})| \leq \frac{1+\sqrt{3}}{2}, \tag{A.6}$$

and equality holds if and only if ρ is a projector.

For spin-1/2 miniatures other than quantum theory, the modulus of the quasi-Wigner function may exceed the bound (A.6). In Square-world, for example,

$$\max_{\mathbf{m}\in S^2} |W_\rho(\mathbf{m})| \leq \frac{1+\sqrt{6}}{2}, \tag{A.7}$$

and equality holds if ρ is an extremal Square-world state.

Spin $s = 1$ For spin-1, the Wigner kernel is

$$\hat{\Delta}_1(\boldsymbol{\eta}) = \frac{1}{3}\mathbb{1} + \frac{1}{\sqrt{2}}\boldsymbol{\eta}\cdot\mathbf{T} + \frac{\sqrt{10}}{2}\mathbf{V}_{\boldsymbol{\eta}}\cdot\mathbf{T}, \tag{A.8}$$

with $V_{\boldsymbol{\eta}}$ defined in (3.76). Quasi-Wigner functions may provide a useful tool to help identify non-quantum states in spin miniatures, and to investigate their properties.

— B —

Rigged Hilbert space treatment of position and momentum

The position and momentum operators discussed in Chapter 4 are unbounded. This poses questions not encountered in finite dimensions such as the domain of definition and the existence of eigenvalues for such operators. These questions may be dealt with by the formalism of *rigged Hilbert space* [34], which gives rigorous meaning to the Dirac 'bra-ket' notation [4, 6, 59]. A rigged Hilbert space is a triplet

$$\Sigma \subset \mathcal{H} \subset \Sigma^{\times}, \tag{B.1}$$

where $\mathcal{H}$ is a Hilbert space, Σ is a dense subset of $\mathcal{H}$ and $\Sigma^{\times}$ is the space of antilinear functionals over Σ (for detailed requirements on the subspace Σ, see [34]). We will introduce some aspects of rigged Hilbert space theory relevant to the discussion of Chapter 4.

For a generic function $\psi \in L^2(\mathbb{R})$, the functions $Q\psi$ and $P\psi$ given by (4.1) may not belong to $L^2(\mathbb{R})$. Furthermore, we may wish to consider powers of these observables (e.g., when considering uncertainty relations). We therefore seek a subspace $\Sigma \subset \mathcal{H} = L^2(\mathbb{R})$ on which the action of both Q and P is well defined, as well as any of their powers. We take $\Sigma = \mathcal{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}$,

$$\mathcal{S}(\mathbb{R}) = \{f \in \mathcal{C}^{\infty}(\mathbb{R}, \mathbb{C}) \mid \|f\|_{n,m} < \infty \, \forall n, m \in \mathbb{N}\}, \tag{B.2}$$

where

$$\|f\|_{n,m} = \sup_{q \in \mathbb{R}} |q^n \partial^m f(q)|. \tag{B.3}$$

The Schwartz space is dense in $L^2(\mathbb{R})$ [75] and the triplets

$$\Sigma \subset \mathcal{H} \subset \Sigma', \qquad\qquad \Sigma \subset \mathcal{H} \subset \Sigma^{\times} \tag{B.4}$$

both constitute rigged Hilbert spaces [4, 59].

Given this structure, Dirac bras may be understood as elements of the dual space Σ', and kets as elements of the antidual space $\Sigma^{\times}$. Namely, to a continuous linear functional $F : \Sigma \to \mathbb{C}$ we associate a bra $\langle F|$, defined by its action on Σ,

$$\langle F|\psi\rangle = F(\psi) \text{ for any } \psi \in \Sigma. \tag{B.5}$$

To a continuous antilinear functional $G : \Sigma \to \mathbb{C}$ we associate a ket $|G\rangle$,

$$\langle \psi|G\rangle = G(\psi) \text{ for any } \psi \in \Sigma. \tag{B.6}$$

The bras $\langle q|$, $\langle p|$ and kets $|q\rangle$, $|p\rangle$ appearing in (4.3) are of particular relevance. For any $q \in \mathbb{R}$ and any $\psi \in \Sigma$,

$$\langle q|\psi\rangle = \psi(q) \text{ and } \langle \psi|q\rangle = \psi^*(q). \tag{B.7}$$

The Dirac delta may be understood as the kernel of the first (linear) functional [75]. For any $p \in \mathbb{R}$ and any $\psi \in \Sigma$,

$$\langle p|\psi\rangle = \hbar^{-1/2}\tilde{\psi}(p) \text{ and } \langle \psi|p\rangle = \hbar^{-1/2}\tilde{\psi}^*(p), \tag{B.8}$$

where $\tilde{\psi}$ is the Fourier transform of the function ψ [6, 59].

The operators Q and P, defined so far on Σ, may be extended to maps on $\Sigma^{\times}$. Given $|F\rangle \in \Sigma^{\times}$, the action of Q is defined by

$$Q\,|F\rangle = |F'\rangle \in \Sigma^{\times} \text{ such that } \langle \psi|F'\rangle = F(Q\psi) \text{ for any } \psi \in \Sigma, \tag{B.9}$$

and similarly for P. The ket $|q\rangle$ is a *generalised eigenvector* of Q, with eigenvalue q in the sense that [34, 4]

$$\langle \psi|\,Q|q\rangle = q\,\langle \psi|q\rangle \text{ for any } \psi \in \Sigma. \tag{B.10}$$

Similarly, $|p\rangle$ is a generalised eigenvector of P. A discussion of the status of $|q\rangle\langle q|$ as a projection operator may be found in [4].

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