Relational and semiclassical aspects of group field theory: from quantum gravity to cosmology

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ABSTRACT

Driven by the ambition and the fundamental need to reconcile quantum theory with general relativity, research in quantum gravity has grown steadily over the decades, with several candidates attacking the problem from different angles.

This thesis explores key aspects of the group field theory (GFT) approach to quantum gravity, a non-perturbative framework centred around background independence and discreteness which considers the familiar spacetime of general relativity as emerging from fundamental quantum geometric elements. After reviewing motivations and foundations of GFT, the thesis focusses on new research that spans three major themes: anisotropic cosmology, semiclassical states, and relational dynamics.

First, the thesis introduces anisotropic GFT cosmological models, broadening the current understanding beyond isotropic settings by incorporating new anisotropy degrees of freedom from quantum gravity considerations. These results enhance the applicability of GFT to more complex scenarios, which can lead to a variety of new phenomenological applications.

Next, a comprehensive semiclassical investigation provides a unified framework of Gaussian states for GFT cosmology, which generalises all previously studied quantum states and properties. Quantum fluctuations are shown to be under control, ensuring that models derived from GFT exhibit a semiclassical behaviour, crucial for bridging quantum gravity with large-scale cosmology.

Finally, the thesis addresses a critical conceptual challenge in quantum gravity: relational dynamics as a solution to the problem of time. By means of quantum clocks, the Page–Wootters formalism (here applied for the first time in non-perturbative quantum gravity) yields a covariant formulation of quantum dynamics in GFT. This allows for a coherent framework that relates to established canonical quantisation methods, strengthened by a conditional interpretation.

All such findings help address significant questions in GFT by offering novel phenomenological perspectives for cosmology, an exhaustive semiclassical analysis, and clear insights on the notion of (quantum) covariance. They also suggest many avenues for future research.

DECLARATION OF AUTHORSHIP

The contents of this thesis are the result of the author's own work, developed in collaboration with the supervisor, which gave rise to the following three papers. Two of these are published in peer reviewed journals, while the last one has been submitted for publication (and is available as a pre-print on arXiv). These papers stem from research conducted at the University of Sheffield between October 2020 and September 2024:

- A. Calcinari and S. Gielen, "Towards anisotropic cosmology in group field theory", *Class. Quant. Grav.* **40**, 085004 (2023), arXiv: 2210.03149;
- A. Calcinari and S. Gielen, "Generalised Gaussian states in group field theory and $\mathfrak{su}(1,1)$ quantum cosmology", *Phys. Rev. D* **109**, 066022 (2024), arXiv: 2310.08667;
- A. Calcinari and S. Gielen, "Relational dynamics and Page–Wootters formalism in group field theory" (2024), arXiv: 2407.03432 (submitted to *Quantum*).

The material in Part I mainly represents a review of the general area of research, its motivation and its recent developments. In particular, while Chapter 1 provides a non-technical introduction that contextualises the research of Part II, Chapter 2 and Chapter 3 review the background material in detail. These mainly follow the literature and essentially contain no new results, but can provide minimal novel insights from the author. Specifically, Chapter 3 is written in light of the author's recent work, and hence may raise new observations and include minor extensions to the existing studies.

The contents of Part II constitute original research, and provide a detailed exposition of the three papers mentioned above (respectively expounded in Chapter 4, Chapter 5 and Chapter 6). With the exception of ancillary paragraphs that review some additional material (as explicitly noted where necessary), this work represents new research that was conducted by the author under the guidance of the supervisor. Chapter 7 discusses the main results and future directions.

Finally, Part III contains supplementary material which includes a mixture of original research and review content. All figures and tables in this thesis were produced by the author.

The investigation carried out before (and during the initial stages of) the PhD:

- A. Calcinari, L. Freidel, E. Livine and S. Speziale, "Twisted geometries coherent states for loop quantum gravity", *Class. Quant. Grav.* **38**, 025004 (2021), arXiv: `2009.01125`;

will not be addressed in this thesis.

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1 INTRODUCTION

The problem of *quantum gravity* generically refers to the task of reconciling the geometrical description of general relativity (GR) with the principles of quantum mechanics – its pinnacle being quantum field theory (QFT) – which underpins all other fundamental interactions. While these frameworks describe their respective realm of physical phenomena to an impressive level of accuracy [1, 8, 150], their underlying principles appear to contradict one another [244, 340]. We do not observe any contradiction in experiments since gravity and quantum field theory operate at different scales; as we explain below, their effects become comparable at energies that are beyond our technological capabilities.

After nearly a century of research,[1] we still lack a complete theory for phenomena which are both quantum and gravitational: a theory capable of describing the quantum properties of spacetime geometry itself at very short distances, depicting the microstructure of spacetime and matter at the fundamental Planck scale. The Planck scale represents the energy level at which the effects of both quantum mechanics and general relativity become equally important, and it is defined by fundamental constants: the speed of light c, Planck's constant $\hbar$, and the gravitational (or Newton's) constant G. While it was Planck [322] who defined the celebrated fundamental units (length, time and mass)

In electronvolts, the Planck scale is usually expressed as $m_P c^2 \sim 10^{19}$ GeV.

$$
\begin{aligned}
l_\mathrm{P} &:= \sqrt{\frac{\hbar G}{c^3}}\,, \\
t_\mathrm{P} &:= \frac{l_\mathrm{P}}{c} = \sqrt{\frac{\hbar G}{c^5}}\,, \\
m_\mathrm{P} &:= \frac{\hbar}{l_\mathrm{P} c} = \sqrt{\frac{\hbar c}{G}}\,;
\end{aligned}
\tag{1}
$$

Bronstein was the one who noticed that they provide the scale for quantum gravity, imposing fundamental bounds on physical quantities [89, 90]. In particular, he realised that beyond this scale the

1 Arguably, research on quantum gravity started in 1930 with the papers by Rosenfeld [334, 335], followed by the work of Bronstein [89], and by Fierz and Pauli [156, 157] in the late thirties.

notion of length loses its meaning and the quantum uncertainty of spacetime becomes important [198, 344]. Such early investigations in some way still resonate in modern research programmes, where using novel strategies and ideas (such as the notions of higher dimensions or fundamental discreteness) the quantum gravity community still strives to develop a theory valid at scales defined by (1) (see, e.g., [76] for a recent overview of achievements and issues in a number of quantum gravity approaches).

In this chapter we briefly introduce various aspects related to the problem of quantum gravity, discuss what progress has been made in some candidate theories, and ultimately motivate the original research results that will be presented in Part II. In a sense, the whole of Part I delves into the technical details that are only mentioned here, by properly introducing the *group field theory* approach to quantum gravity, its cosmological applications and open questions.

DIFFEOMORPHISMS AND THE PROBLEM OF TIME. One of the reasons why the quantisation of gravity is particularly challenging is given by the absence of absolute structures. Indeed, a key concept in general relativity is that of spacetime diffeomorphisms (i.e., coordinate transformations), which highlights the lack of an fixed background geometry, as we will see in Chapter 2. It follows that the spacetime manifold – central object in Einstein's GR – does not have a physical meaning. This idea was already troubling Einstein, who came up with what is known as "hole argument" when trying to understand that this was not a problem of his formulation of GR, rather a feature of gravity [149, 355]. Note that diffeomorphism invariance can be obtained artificially by *parametrising* a theory with *ad hoc* coordinates (we will explicitly make use of this strategy in Chapter 6). In this way one can obtain a covariant framework, though it is always possible to deparametrise the theory to find the original non-covariant setting. In GR, spacetime is already a space of unphysical parameters, and one cannot reformulate it in a non-covariant way. In this sense, diffeomorphism invariance means that space and time are nothing but coordinates parametrising a covariant theory of geometry. As we will explain in Chapter 2, what is physical is the equivalence class of metric fields, where any two of them are called equivalent if they can be transformed into each other by a diffeomorphism.

Related to this is the fact that general relativity does not possess a conventional time evolution. Even though the Einstein field equations can be recast in Hamiltonian form, we will see in Chapter 2 that the canonical Hamiltonian of GR takes the form of a pure constraint, meaning that it generates the orbits of gauge symmetries and hence implements diffeomorphism invariance. This leads to the *problem of time* [233, 257, 340], which in the quantum theory is particularly worrying since one deals with quantum states that appear to be "frozen". As we will see, one then needs to use *relational* strategies to be able to describe dynamics; in simple terms, this means studying the evolution of some degrees of freedom with respect to the changes in another degree of freedom of the theory (normally required to be a monotonic quantity). These themes play an important role in the present thesis; we will provide a general introduction on background-independent quantum gravity approaches in Chapter 2 (see [195] for the link between background independence and diffeomorphism invariance), and emphasise in Chapter 3 their relevance for the group field theory formalism, explicitly expanding on the idea of relational dynamics. Furthermore, we will present in Chapter 6 new research results in the context of group field theory that are motivated by the general covariance of gravity.

QUANTUM GRAVITY CANDIDATES. As we mentioned, standard approaches to quantisation, which have been effective for the electromagnetic, weak, and strong forces, fail for gravity; this is due to Einstein's theory being perturbatively non-renormalisable [199, 372]. In particular, we will see in Chapter 2 that canonical quantisation as well as covariant (i.e., path integral) quantisation are not suited for GR, at least in their standard implementations. As a result, a wide range of alternative approaches has emerged (see [340] for a detailed historical discussion on quantum gravity ideas), each possessing distinct strengths and limitations [76]. Despite the variety of approaches, a consistent solution to the problem of quantum gravity is still not available as all frameworks face formal and conceptual issues, and notably, researchers continue to hold differing views (see [15] for a recent collection of insightful interviews with leading experts in quantum gravity). Here we simply classify the main approaches into two categories: unified theories and background-independent theories.

Unified approaches, like string theory (and its modern formulation known as M-theory), attempt to describe all interactions – including gravity – in terms of more fundamental objects such as strings and "branes", existing in higher-dimensional spacetimes [60]. M-theory is a framework for an interacting quantum theory that encompasses five known superstring theories, which are related to each other by dualities, and links their low energy regime to eleven-dimensional supergravity theories. While it offers an intriguing path towards unification, it is only known in certain limits, and its complete formulation is still under development.

In Chapter 2 we will provide a detailed overview of the main approaches that lead to the general framework of group field theory.

Background-independent approaches – such as loop quantum gravity (LQG) [340, 367] and group field theory (GFT) [160, 295] – focus on a key feature of GR, namely the lack of a fixed spacetime background (see [195, 354] for details on background independence). By insisting on this property at the quantum level, these approaches lead to a discrete notion of (quantum) geometry, which in turn has promising cosmological applications (especially in resolving the Big Bang singularity by predicting a bounce scenario, as we emphasise in the following). Especially relevant for this thesis will be the GFT framework, which extends LQG (and other approaches) by providing a QFT-like formalism in which geometry is made of fundamental quanta. More precisely, the GFT partition function defines a non-perturbative sum over simplicial geometries and topologies, which provides a combinatorial description of spacetime at the quantum level. As we will describe below (and expound in detail in Chapter 2 and Chapter 3), this will allow to interpret spacetime (and cosmology) as emerging from the collective behaviour of GFT quanta ("particles of geometry").

PROBING QUANTUM GRAVITY WITH COSMOLOGY. The difficulty of experimentally probing quantum gravity effects significantly hinders progress in the field (see [3] for recent developments and ideas). As mentioned, unlike the other fundamental forces which can be tested through high-energy particle experiments, the effects of quantum gravity are expected to become significant only at energy scales of the order of 10^{19} GeV (cf. (1)). These energy levels are far beyond the reach of current experimental capabilities, making direct observations of quantum gravity unattainable in our laboratories. In spite of this, one can still try to bridge the gap between the theoretical facets of quantum gravity and its phenomenological applications by identifying appropriate regimes where one expects quantum gravity effects to play a role. In this respect, the field of cosmology presents a unique and promising arena for testing quantum gravity: the early Universe, especially near the Big Bang, involves energy scales where quantum gravity effects are expected to dominate, offering a potential observational window into the theory [6, 7, 246, 247]. Moreover, from a theoretical point of view, quantum gravity might be able to resolve the puzzles of cosmology. It could provide motivations for – or alternatives to – the standard inflationary scenario; some examples are given by bouncing models [38, 88, 377], ekpyrotic and cyclic models [243, 259] or string gas cosmology models [57, 87]. Even more importantly, quantum gravity is expected to resolve the initial Big Bang singularity (see, e.g., [95–97, 168, 242] for string cosmology results and [35–37, 46, 78, 305, 306] for discrete and background-independent approaches). In particular, the singularity avoidance in group field theory cosmology (a very robust result, see, e.g., [5, 184, 186, 272, 379]) represents an important motivation for the work of this thesis; as such, we will provide a detailed review in Chapter 3.

In general, therefore, (quantum) cosmology has the potential to shed some light on the problem of quantum gravity. Note that in order to make contact with observations, which show no quantum features (such as geometry superpositions or uncertainties), identifying a semiclassical spacetime description is crucial for any quantum gravity and quantum cosmology model. Indeed, semiclassicality is an important aspect in traditional quantum cosmology [214, 245, 271] and in loop quantum cosmology [29, 359] (see also [98, 164, 362, 364] for kinematical states in the context of full LQG). As we will see, these considerations should also apply to group field theory, and motivate the work of Chapter 5, where we present original results regarding semiclassical states for GFT cosmology.

COSMOLOGY FROM GROUP FIELD THEORY. As we will thoroughly explain in Part I, group field theory is a background-independent and non-perturbative approach to quantum gravity that lies at the intersection of several quantum gravity frameworks. It generalises matrix and tensor models [86, 209, 293] by combining their discrete and combinatorial nature with group theoretic degrees of freedom typical of loop quantum gravity [299, 340]. Importantly, we will show in Chapter 2 that GFT provides a formal way to complete spin foam models [313] and make sense of a sum over triangulations of spacetime in terms of a path integral [160, 295]. In its canonical formulation [298], GFT describes quantum geometry as a many-body system, where the fundamental components are discrete building blocks of geometry akin to the quantum states of LQG. In this setting, as explained in Chapter 3, one can make use of some simplifying assumptions (meant to correspond to symmetry reduction) and obtain dynamical equations which allow to discuss simple phenomenological applications.

More precisely, while it is not generally easy to recover an effective continuum interpretation from GFT (or LQG) – which are characterised by discrete geometrical degrees of freedom – one can define relational dynamics for *global* observables such as the total (spatial) volume of a certain geometry, which can then be contrasted with globally homogeneous cosmological models. In such contexts, the idea of relational dynamics refers to the evolution of a geometrical observable (e.g., the volume of the Universe) with respect to the values taken by some other (physical) degree of freedom, such as a massless scalar field. In this picture, and in line with background independence, a cosmological spacetime is not a fundamental entity but emerges from the dynamics of the fundamental GFT degrees of freedom [297]. In other words, starting from the microscopic GFT quantum description, one can derive cosmology as the resulting macroscopic, coarse-grained theory, namely as "hydrodynamics of quantum gravity" [187, 300, 301]. As we will explicitly show in Chapter 3, the most striking result in this sector was the recovery of the classical Friedmann–Lemaître–Robertson–Walker (FLRW) dynamics with quantum corrections at high energies that replace the Big Bang singularity with a "Big Bounce" [305, 306]. From a phenomenological point of view, this immediately raises the question of whether GFT can reproduce the dynamics of more involved cosmological models. Also in light of the results obtained in loop quantum cosmology [39, 83], the next natural arena to explore would be the case of anisotropic cosmologies (in GR given by the classical Bianchi models as reviewed in Appendix A); we discuss new results in this direction in Chapter 4.

In cosmology, relational dynamics means asking how the volume changes as a function of (or, relatively to) the scalar field. See Appendix A and Chapter 3 for the implementation of these ideas (at the classical and quantum level).

AIM AND OUTLINE OF THE THESIS. This thesis will mainly address three questions in the context of group field theory and its cosmological sector, regarding phenomenology, semiclassicality, and relational dynamics. The work presented here extends the field of GFT in new directions, specifically aiming to: resolve ambiguities, clarify techniques and interpretations, and provide definitive insights, thereby offering a more coherent and structured understanding of the complex aspects appearing in the literature. While the study of Chapter 4 on anisotropic models seeks to extend the number of cosmological applications for GFT as a quantum gravity candidate, the

Gaussian-states analysis of Chapter 5 and the rigorous definition of quantum relational dynamics of Chapter 6 truly aim to incorporate various traits of the field into a consistent framework.

This thesis is comprised of three parts. Part I introduces various approaches to quantum gravity: in Chapter 2 we sketch the traditional attempts at quantising GR and explain in some detail the main path integral formalisms that adopt a discrete perspective; this leads to the definition of group field theory which is then rephrased in its canonical formulation in Chapter 3, where it is also applied to cosmology. Part II presents the main research results: these are new findings that generalise and expand the existing literature reviewed in Part I, and are based on [99–101]. Specifically, in Chapter 4 we define and analyse new models for GFT cosmology that go beyond the state of the art by introducing the notion of anisotropies, so as to characterise global anisotropic spacetimes emerging from quantum gravity; we hence extend the range of phenomenological applications of GFT. In Chapter 5 we tackle another critical issue of GFT cosmological models, that of semiclassical properties of quantum states. More precisely, we provide a unified framework that encompasses all previously studied semiclassical states (in all approaches); this is provided by the rich family of Gaussian states, for which quantum fluctuations are shown to be under control. Lastly, in Chapter 6 we clarify a pressing foundational aspect of group field theory, concerning a proper and formal definition of (quantum) relational dynamics. By applying the Page–Wootters formalism to non-perturbative quantum gravity for the first time, with this work we shed some light on the relational evolution of GFT geometric observables with respect to a "quantum clock" and obtain a manifestly covariant formulation of quantum dynamics, consistent with a Dirac quantisation. Part II concludes with an overall summary of the thesis in Chapter 7, which emphasises the main results and possible new research directions. Finally, Part III contains two appendices: Appendix A discusses the notion of relational dynamics in cosmology and Appendix B shows analytical expressions and computational tools used for Gaussian-states calculations.

Part I

THE ROAD TO GROUP FIELD THEORY COSMOLOGY

This part of the thesis provides an introduction to the general field of research. In particular, Chapter 2 explores the motivations behind the group field theory (GFT) approach to quantum gravity, and culminates with the definition of its path integral formulation. Chapter 3 introduces the canonical perspective for group field theories and outlines the framework of GFT cosmology, detailing the models used as starting points for the research described in Part II.

2 APPROACHES TO QUANTUM GRAVITY

In this chapter we review the approaches to quantum gravity that have led to the formulation of group field theory (GFT), which will be the underlying framework for the results of this thesis, discussed in Part II. In particular, we mainly focus on non-perturbative approaches that aim to incorporate the notions of discreteness and background independence.

First, we will give a brief (and somewhat historical) summary of the main ideas behind the canonical quantisation of general relativity (GR), leading to the so-called geometrodynamics perspective. Then, we will discuss formulations based on the path integral quantisation, highlighting difficulties in the continuum and delving into possible avenues to overcome such limitations in discrete settings. We will see that the key aspect of these theories is that the perturbative expansion of their path integral generates a sum over discrete geometries, showing that they can provide well-defined formulations for solving the problem of quantum gravity. Specifically, we will explore in some detail simplicial quantum gravity theories rooted in Regge calculus, matrix and tensor models, the covariant spin foam approach stemming from loop quantum gravity, and finally group field theory, which can be seen as a "larger embedding framework" that encompasses and generalises the previous ones.

2.1 CANONICAL QUANTUM GRAVITY

The canonical quantisation paradigm is a procedure for quantising a classical theory while trying to preserve its formal structures. To perform a canonical quantisation one starts from the Hamiltonian formulation of a classical theory, where the dynamics are described in terms of Poisson brackets, and promotes the phase space variables to quantum operators on a Hilbert space. The Poisson brackets between phase space variables are replaced by the canonical commutation relations as [133]

$$\{\cdot,\cdot\} \quad \to \quad \frac{1}{i\hbar}[\cdot,\cdot]\,, \tag{2}$$

and the quantum theory is described in terms of a wave function of the configuration variables (or, in field theories, by a wave functional of the fields [222]). The dynamics are then described by the quantum Hamiltonian operator $\hat{H}$, which governs the Schrödinger equation.

The canonical quantisation of gravity was firstly developed in [128] after the preliminary work of [16, 134]. It is based on the Hamiltonian formulation of GR which shows that it is a fully constrained theory, meaning that the total Hamiltonian can be expressed as a combination of constraints. More precisely, by assuming a $3+1$ foliation of spacetime, the familiar Einstein–Hilbert action[1] (using units in which the speed of light $c = 1$)

The $3+1$ splitting requires the spacetime to be globally hyperbolic, so that it can be foliated into Cauchy hypersurfaces.

$$S_{\mathrm{EH}} = \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g}\, R\,, \tag{3}$$

where G is Newton's constant of gravitation, $g = \det(g_{\mu\nu})$, $g_{\mu\nu}$ is the metric tensor and R denotes the Ricci scalar, can be cast as

$$S_{\mathrm{ADM}} = \frac{1}{16\pi G} \int \mathrm{d}t \int \mathrm{d}^3 x \left(\pi^{ab} \dot{q}_{ab} - N^a \mathcal{C}_a - N\mathcal{C} \right)\,, \tag{4}$$

which is known as Arnowitt–Deser–Misner (ADM) action. In addition to the metric tensor of three-dimensional spatial slices q_{ab} (the dot in $\dot{q}_{ab}$ denotes a derivative with respect to t) and its conjugate momentum π^{ab}, there are four Lagrange multipliers: the lapse function N and the shift vector field N^a. They multiply the (spatial-)diffeomorphism and Hamiltonian constraints, defined as

$$\begin{aligned} \mathcal{C}_a &= -2 D_b \pi^b{}_a\,, \\ \mathcal{C} &= \frac{16\pi G}{\sqrt{q}} \left(\pi_{ab}\pi^{ab} - \frac{1}{2} (\pi^a{}_a)^2 \right) - \frac{\sqrt{q}}{16\pi G}\, {}^{(3)}R\,, \end{aligned} \tag{5}$$

where $q = \det(q_{ab})$, ${}^{(3)}R$ is the Ricci scalar of q_{ab}, D_a is the spatial covariant derivative compatible with q_{ab} and indices are raised and lowered with q_{ab}. Varying (4) with respect to the Lagrange multipliers implies that the constraint equations $\mathcal{C}_a = 0$ and $\mathcal{C} = 0$ must be satisfied for physical configurations. These define what is known as the (physical) constraint hypersurface in the phase space of GR. As mentioned, the total Hamiltonian $H_{\mathrm{GR}} = \frac{1}{16\pi G} \int \mathrm{d}^3 x N^\mu \mathcal{C}_\mu$ vanishes for physical configurations, meaning that there are no dynamics with

1 If the spacetime manifold M has a boundary ∂M, one includes the Gibbons–Hawking–York (GHY) boundary term $S_{\mathrm{GHY}} = \frac{1}{8\pi G} \int_{\partial M} \mathrm{d}^3 x \sqrt{h}\, K$, where h is the determinant of the induced metric on the boundary and K is the trace of the associated extrinsic curvature [172, 382].

respect to the time parameter t. This is what leads to the *problem of time* in classical relativity (and quantum gravity) [233, 257, 340].

In essence, while the constraints $\mathcal{C}_a = 0$ generate usual diffeomorphisms on the three-dimensional spatial "slices", the Hamiltonian constraint $\mathcal{C} = 0$ generates evolution of such 3-manifolds so as to construct solutions (i.e., spacetimes) of the Einstein equations. Such solutions can be seen as trajectories on the constraint hypersurface of GR, akin to gauge orbits, since gauge transformations generated by the constraints (5) preserve the hypersurface. These themes will be important for the work presented in Chapter 6, where inspired by GR we will deal with similar types of systems working within the group field theory approach to quantum gravity.

In a generally covariant theory, time evolution is just like a gauge symmetry as time translations are coordinate transformations.

GEOMETRODYNAMICS. As we have mentioned, the fact that general relativity is a totally constrained theory (i.e., $H_{\mathrm{GR}} = 0$) implies that if the constraint equations are satisfied, then the Einstein field equations are satisfied. In other words, the whole dynamical content of GR is encoded in the constraints (5); hence one needs a canonical quantisation scheme for fully constrained theories. This is provided by the approach to quantisation proposed by Dirac, sometimes denoted constraint quantisation (or simply Dirac quantisation), where the physical states of the quantum theory are annihilated by the constraints [133, 224]. The strategy for a canonical quantisation of gravity (as described in the seminal work [128]) would be to find a representation of the phase space variables (q_{ab}, π^{ab}) as operators on a *kinematical* Hilbert space $\mathcal{H}_{\mathrm{kin}}$ that satisfy the standard commutation relations (2). By promoting the constraints (5) to operators $\hat{\mathcal{C}}$ and $\hat{\mathcal{C}}^a$ on $\mathcal{H}_{\mathrm{kin}}$, one would then define the space of physical solutions $\mathcal{H}_{\mathrm{phys}}$ by means of the quantum constraint equations

$$\hat{\mathcal{C}}^a \Psi[q] = 0\,, \tag{6}$$

and

$$\hat{\mathcal{C}} \Psi[q] = 0\,. \tag{7}$$

This procedure encounters a number of problems and is to be understood as a formal application of standard quantisation techniques to GR. Naively, one could work in the Schrödinger representation with $\hat{q}_{ab} = q_{ab}$ and $\hat{\pi}^{ab} = -\mathrm{i}\hbar \frac{\delta}{\delta q_{ab}}$ acting on the wave functionals $\Psi[q_{ab}(x)]$ of the three-metrics. The first issue one encounters is the definition of an inner product for the kinematical Hilbert space, which would require explicit knowledge of a measure on the space of metrics [194, 196]. Assuming one can impose the spatial-diffeomorphism constraint (6) and define an intermediate Hilbert space for spatially diffeomorphic metrics $\mathcal{H}_{\text{diff}}$ (ignoring that this would also lack a proper measure), one would still face the problem of the Hamiltonian constraint (7), which should define the space of physical solutions $\mathcal{H}_{\text{phys}}$ and is called Wheeler–DeWitt equation [128]. This is more complicated since, in addition to the absence of a suitable inner product on $\mathcal{H}_{\text{phys}}$, $\hat{\mathcal{C}}$ requires the definition of products of operators at the same point (cf. (5)), which are notoriously divergent objects.

This line of research was pursued in simplified scenarios where the space of metrics (called *superspace*) is reduced by considering spacetimes that are highly symmetric. This leads to the *minisuperspace* models of quantum cosmology, where one applies the canonical quantisation procedure to a symmetry-reduced version of the classical theory (e.g., considering homogeneous and/or isotropic metrics). In this manner, one can obtain quantum systems for which the Wheeler–DeWitt equation (7) can be studied explicitly (see, e.g., [179, 215, 244, 248, 380] for reviews on quantum cosmology).

LOOP QUANTUM GRAVITY. The task of canonically quantising GR was pushed further in the context of *loop quantum gravity* (LQG) [25, 340, 344]. This framework does not alter either the gravitational theory or the Dirac quantisation scheme, but rather focusses on different variables for gravity. In particular, in LQG one uses the so-called Ashtekar variables [21] to write GR as a theory similar to Yang–Mills gauge theories. The advantage of such a reformulation is that it simplifies the implementation of Dirac quantisation: one can explicitly define the kinematical Hilbert space of LQG, which is interpreted as the space of quantum geometry states (known as spin networks[2]) satisfying the spatial-diffeomorphism constraint.

Among the key results of LQG lies the fundamental discreteness of spatial geometry at the Planck scale. Indeed, by showing that geometric operators (such as area and volume) have discrete spectra [31, 32, 343], LQG describes geometry as a superposition of quantum states (depicted as graphs) where geometric quantities have minimal excitations proportional to powers of the Planck length (cf. (1)). It is due to this discreteness that the theory is expected to resolve the problem of

2 We will provide a more detailed exposition of spin network states in Section 2.2.4.

the classical singularities of GR, e.g., in cosmology [24, 116] and black holes [84, 287].

While one can impose the spatial-diffeomorphism constraint within the framework of LQG (leading to a proper definition of $\mathcal{H}_{\text{diff}}$), the dynamics of the theory are still not fully understood and implementing the Hamiltonian constraint remains a challenge. More precisely, despite having a well-defined Hamiltonian operator with finite action on spin networks [360, 361, 367] (which shows great improvements with respect to the traditional Hamiltonian constraint of geometrodynamics), the complete spectrum of the Hamiltonian operator and the specification of the physical Hilbert space $\mathcal{H}_{\text{phys}}$ are still unknown. Moreover, the Hamiltonian operator is subject to ambiguities and it is not unique; one could explore different prescriptions such as different regularisation schemes or different operator orderings [33].

$\mathcal{H}_{diff}$ is the space of spin network states that are defined on equivalence classes of graphs under diffeomorphisms, and it is dubbed the space of "knotted" spin networks.

The difficulties in obtaining the quantum dynamics of canonical LQG have prompted the formulation of the "master constraint programme" [365, 367], which seeks to implement the diffeomorphism and the scalar constraints simultaneously, and of the *spin foam* formalism, which offers a (somewhat covariant) path integral quantisation for loop quantum gravity [312, 313, 344]. We will give a brief overview of the spin foam formalism in Section 2.2.4, as this represents a natural arena to understand the foundations of group field theories, which will be presented in Section 2.3.

2.2 PATH INTEGRAL QUANTISATION

We now turn our attention to covariant methods of quantisation based on path integrals. In this formulation, firstly introduced by Feynman [154], one is concerned with transition amplitudes which describe the probability of having a quantum system in some final state, given its initial state. In quantum gravity, such an amplitude is expected to be interpreted as the transition between states of (quantum) geometry.

Rather than starting from classical dynamics as in (2), the connection of the path integral approach with the classical theory is somewhat subtle: the transition amplitude is calculated by taking an average over *all* intermediate states, weighted by the exponential of the action functional multiplied by the imaginary unit and divided by $\hbar$. This is consistent with the classical theory since in the limit $\hbar \to 0$ only classical solutions contribute to the path integral. For instance, in the simplest textbook example of a quantum particle going from an initial position q_i at the time t_i to a final position q_f at the time t_f, one takes the weighted average over all possible paths connecting the initial and final points as [155]

In the limit $\hbar \to 0$ the integral is dominated by the classical extrema of the action and hence leads to configurations that satisfy the classical equations of motion.

$$\langle q_f, t_f | q_i, t_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} \mathcal{D}q \, e^{\frac{i}{\hbar} S[q(t)]} \,, \tag{8}$$

where $\mathcal{D}q$ represents such an integration over all possible trajectories.

The path integral formalism provides a powerful and versatile framework that extends naturally to quantum field theory (QFT), where instead of summing over all possible trajectories of a particle, one sums over all possible field configurations. For a scalar field theory, one can write the general path integral [318]

$$Z = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar} S[\phi(x)]} \,, \tag{9}$$

where $\mathcal{D}\phi$ denotes integration over all possible field configurations and Z is sometimes called partition function. This nomenclature is used since, applying a Wick rotation and thus working with the Euclidean version of the field theory (i.e., sending $\mathrm{i}S \to -S_E$), the exponential in (9) is just like a statistical weight for the fluctuations in ϕ, and Z describes precisely the partition function of statistical mechanics [318].

In the following we will see that, while these ideas are not easy to implement in the case of gravity, one can tweak the path integral by considering discrete formulations (and hence discrete geometries) and obtain well-defined expressions for the partition function. This will naturally lead to the definition of group field theory at the end of the chapter, where we will show that a path integral much like (9) can be interpreted as a sum over "histories of quantum geometries" and thus provide a well-defined candidate to the problem of quantum gravity.

2.2.1 *From continuous to discrete path integrals*

The first prototype of background-independent covariant quantisation of gravity was formulated by Misner in [285], where a direct application of path integral techniques to general relativity was used to formally introduce the partition function

From now on we use conventions in which $\hbar = 1$.

$$Z = \int \mathcal{D}g \, e^{\mathrm{i}S_{\mathrm{GR}}[g]} \,. \tag{10}$$

Here $S_{\mathrm{GR}}[g]$ is the action of general relativity (the Einstein–Hilbert action (3) plus possible boundary terms, see footnote 1) and the notation $\mathcal{D}g$ formally represents an integration over all possible spacetime metrics g. Unfortunately, it is very difficult to make precise sense of the expression in (10) for a number of reasons. The most important issue one encounters in this setting is the definition of the superspace. In particular, the technical difficulty in defining a measure on an infinite-dimensional space of metrics is worsened by the fact that one here actually integrates over equivalence classes of metrics, which are related by diffeomorphisms.

Usually, path integrals are better behaved when defined in their Euclidean formulation, obtained by means of a Wick rotation from real to imaginary time. However, even performing a Wick rotation such that $\mathrm{i}S_{\mathrm{GR}} \to -S^{\mathrm{E}}_{\mathrm{GR}}$, one is faced with the obstacle that the Euclidean action $S^{\mathrm{E}}_{\mathrm{GR}}$ is not bounded from below, and therefore it leads to a divergent path integral [173, 283]. Nonetheless, Euclidean path integrals for gravity have been adopted in simplified scenarios by means of semiclassical approximations (where the partition function is approximated by the exponential of the negative Euclidean action evaluated at a stationary-phase point), and have led to important developments such as the recovery of black hole thermodynamics [172], or the *no boundary proposal* in quantum cosmology [221, 223].

One last concern is related to the fact that (10) does not implement a sum over topologies, and changes in topology[3] are heuristically expected (or at least conceivable) in a regime where spacetime is subject to strong quantum fluctuations [115, 175, 356].

Progress in the continuous path integral approach was mainly made thanks to perturbation theory. Indeed, expressing the metric as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \,, \tag{11}$$

where $\bar{g}_{\mu\nu}$ is a fixed classical background (for example the Minkowski flat metric $\eta_{\mu\nu}$) and $h_{\mu\nu}$ represents small perturbations, one can linearise the action (in particular choosing the de Donder gauge [286]) and obtain a quadratic kinetic term for the dynamical field $h_{\mu\nu}$. The advantage of this setting is that one can then employ the machinery of quantum field theories, and in particular use the background metric $\bar{g}_{\mu\nu}$ to define the Wick rotation. The (low energy) quanta of the gravi-

The de Donder (or harmonic) gauge is $\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h = 0$, where $h = h^\mu{}_\mu$ is the trace of $h_{\mu\nu}$.

3 Topology change is also not allowed by the canonical approach of Section 2.1, since the ADM formalism fixes a hyperbolic spacetime once and for all. We will return to this issue in later sections.

tational field associated to the metric perturbation $h_{\mu\nu}$ are interpreted as massless particles with spin 2, and are denoted *gravitons*.

Amongst the results of this paradigm is the modification to the Newtonian potential caused by quantum corrections [146], which provides a compelling model-independent quantum gravity prediction. However, such a perturbative and background-dependent approach only makes sense as an effective theory for a low energy description [145], and would be inconsistent if taken seriously at all energy scales. Indeed, we are assuming that the field $h_{\mu\nu}$ can be quantised on a fixed background as for any other field theory, but the backreaction of the perturbation will increase as the energy scale increases, so that it would be contradictory to treat the background as fixed and unperturbed. More precisely, the perturbative quantisation of general relativity fails because of non-renormalisable ultraviolet divergences (see, e.g., [199, 372]). These considerations point towards the idea that one might need to employ non-perturbative techniques, and that gravitons might not be satisfactory degrees of freedom to describe quantum gravity at high energies, and in particular at the Planck scale.

Another approach aiming to quantise gravity in the continuum formulation, in particular going beyond perturbative methods by using the functional renormalization group, is given by the *asymptotic safety* programme [114, 261, 291], initiated by Weinberg [375, 376]. This assumes the existence of a nontrivial fixed point of the renormalisation group flow, which controls the behaviour of the coupling constants in the ultraviolet regime and renders physical quantities safe from divergences. However, the asymptotic safety programme also employs an auxiliary background metric, and restoring background independence at the level of physical observables is a challenging task [148].

DISCRETENESS. Given the difficulties with the continuous path integral (10), and inspired by one of the core properties of quantum theory, some attention has been dedicated to the development of *discrete* formulations of covariant quantum gravity. By replacing the integration over geometries with a sum over *discretisations*, possibly adapted to also encode a sum over topologies, one can hope to avoid ultraviolet divergences, and recover general relativity by specifying a way of taking the continuum limit. There are several approaches that take these ideas seriously and we devote the rest of the chapter to explain some of them in detail.

In the context of simplicial gravity, a discretisation is called triangulation.

2.2.2 *Regge calculus and the Ponzano–Regge model*

An *ad hoc* discretisation of GR leads to what is known as Regge calculus [326]. We will focus only on *simplicial* gravity in what follows, which is based on triangulations of (generically d-dimensional) manifolds with dynamics given by a discretised version of the Euclidean Einstein–Hilbert action (also simply called Regge action).

Instead of the familiar continuous spacetime manifold, the base structure in "Regge gravity" is a piece-wise flat manifold to be thought as a collection of d-dimensional *simplices* glued together along their $(d-1)$-dimensional boundaries. An example of a triangulation of a smooth manifold in two dimensions can be seen in Figure 1.

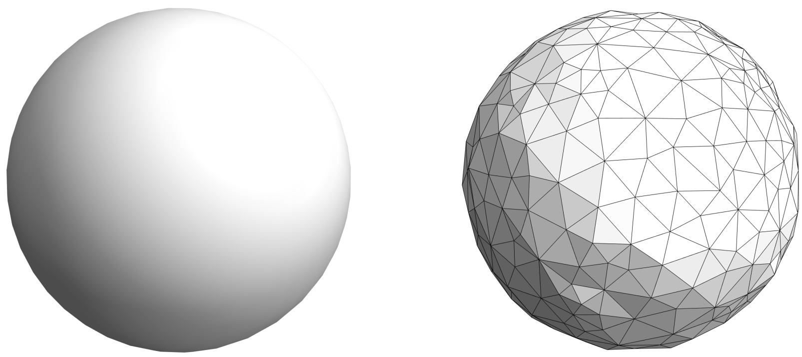


Figure 1: The surface of a sphere can be triangulated with 2-simplices, i.e., triangles.

A d-dimensional simplex (d-simplex in short) is an object with $d+1$ vertices and $d(d+1)/2$ edges connecting them, and its shape is fully specified by its edge lengths. A *simplicial complex* is defined as a set of simplices glued together along their boundaries, and hence a simplicial manifold is a simplicial complex where the boundary of the neighbourhood of any vertex is homeomorphic to a $(d-1)$-dimensional sphere[4] (see [216] for details).

For example: a 0-simplex is a point, a 1-simplex is an edge, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron.

The geometry of a simplex is completely specified by the lengths of its edges, so that the line element connecting neighbouring sites i and j is given by

$$l_{ij}^2 = \eta_{\mu\nu}(x_i - x_j)^\mu (x_i - x_j)^\nu \,, \qquad (12)$$

where $\eta_{\mu\nu}$ is the flat metric and x_i^μ (with $\mu = 1, \ldots, d$) denotes the μ-th coordinate of the i-th lattice site. In order to connect this de-

4 For example, the right panel of Figure 1 shows a triangulation of a 2-manifold in terms of 2-simplices, which have 3 edges and 3 vertices each. The neighbourhood of any such vertex has a boundary that is homeomorphic to a 1-sphere, namely a circle.

scription with (Euclidean) GR, we also need the notion of curvature for Regge gravity. Given that the manifold is piece-wise flat, curvature is described by going around loops which are dual to a $(d-2)$-dimensional subspace, also called a hinge h. Indeed, from the dihedral angles $\theta(s,h)$ associated with the faces of the simplices s meeting at a given hinge h, one can compute the deficit angle

$$\delta(h) = 2\pi - \sum_{s \supset h} \theta(s,h)\,, \tag{13}$$

which measures the curvature at h (see Figure 2).

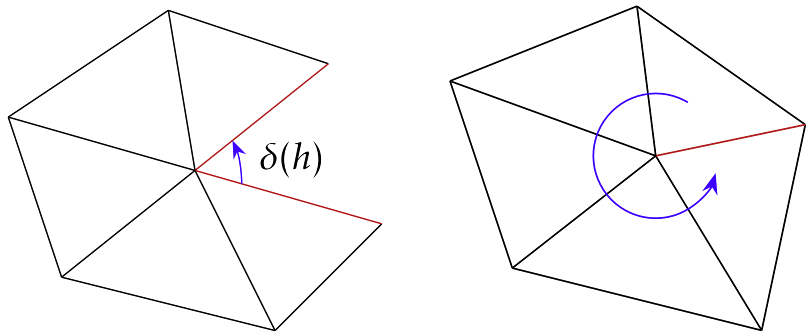


Figure 2: Illustration of the deficit angle in two dimensions, where several flat triangles share a vertex (hinge).

Then, one can show that by discretising the d-dimensional Einstein–Hilbert action (3) (here in its Euclidean form and with a cosmological constant Λ),

$$S^{\mathrm{E}}_{\mathrm{EH}} = -\frac{1}{16\pi G}\int \mathrm{d}^d x\,\sqrt{g}(R-2\Lambda)\,, \tag{14}$$

one obtains the Regge action[5] [216]

$$S_{\mathrm{Regge}} = -\frac{1}{8\pi G}\sum_h \delta(h)V(h)^{(d-2)} + \frac{\Lambda}{8\pi G}\sum_s V(s)^{(d)}\,. \tag{15}$$

Here $V(h)^{(d-2)}$ is the volume of a hinge (e.g., an area in four dimensions) and $V(s)^{(d)}$ is the volume of a d-simplex. Importantly, all the ingredients in (15) are independent of coordinates [326]. By varying the Regge action S_{Regge} with respect to the edge lengths, one obtains the simplicial analogue of Einstein's field equations. As the triangulation is refined, the Regge geometry transitions to a Riemannian ge-

5 One can also include the GHY boundary term (cf. footnote 1) in the Regge action as $\frac{1}{8\pi G}\sum_{h'}\psi(h')V(h')^{(d-2)}$, where $\psi(h') = \pi - \sum_{s\supset h'}\theta(s,h')$ is the *extrinsic* curvature angle for the hinges h' on the boundary (just like in (13), but with the flat angle of the half-plane given by π) [138, 326].

ometry, and S_{Regge} converges to the Euclidean Einstein–Hilbert action [108, 109].

At this stage, one can in principle define a path integral from the action (15), integrating over all discrete metrics (i.e., over all assignments of edge lengths in the bulk) and obtaining what is called "quantum Regge calculus" [333]. However, we will not review this general approach here. Rather, we now present the first application of Regge calculus to quantum gravity by reviewing the *Ponzano–Regge model* [56, 324]. This represents a simple illustrative example that naturally relates to quantum Regge calculus and, more importantly, leads to spin foam models and ultimately to group field theories, as we will explain in Section 2.2.4 and Section 2.3.

THE PONZANO–REGGE MODEL. The Ponzano–Regge model is a state-sum model that provides a discrete version of the path integral for three-dimensional Euclidean quantum gravity [56, 324]. Starting from a triangulation of a 3-manifold, the model relies on the assumption that lengths are heuristically quantised. Specifically, it assigns a discrete value $j + \frac{1}{2}$ to each edge of the triangulation, where the non-negative half-integer $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ is traditionally called spin and labels irreducible representations of the group $SU(2)$. Then, the geometry of each 3-simplex (or tetrahedron) is determined by its six edge lengths, and a weight given by a 6*j*-*symbol*[6] is assigned to each tetrahedron in a precise way as is illustrated in Figure 3.

We denote Wigner 6j-symbols as $\{6j\}$ in the following.

$$\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} \Leftrightarrow$$

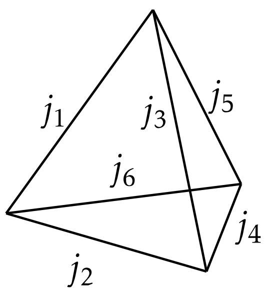


Figure 3: The spins of a 6*j*-symbol can be thought as labelling the edges of a tetrahedron. Indeed, a 6*j*-symbol vanishes unless the three spins defining each face satisfy the Clebsch–Gordan conditions (and heuristically, a tetrahedron is only defined if the sides of the triangles satisfy triangular inequalities).

6 This is a standard object from $SU(2)$ recoupling theory, constructed as a sum over products of four Clebsch–Gordan coefficients (or 3*j*-symbols); see, e.g., [371].

The partition function is then defined by a sum over the quantum amplitudes associated with every possible spin on every edge, and reads [56]

$$Z_{\mathrm{PR}} = \sum_{\{j_e\}} \prod_e (-1)^{2j_e}(2j_e+1) \prod_T \{6j\}_T \,, \tag{16}$$

where the labels e and T stand for edges and tetrahedra respectively, and $\sum_{\{j_e\}}$ means we are summing over all the spins associated with the interior of the manifold. Since the spins j_e define a quantisation of edge lengths in the triangulation, the Ponzano–Regge path integral (16) has a compelling interpretation in terms of quantum geometry where the probability amplitude of four triangles being glued together into a tetrahedron is given by the $6j$-symbol.

This interpretation is strengthened by the study of the semiclassical limit, namely by looking at the large spin asymptotic behaviour of a $6j$-symbol [331, 349]

The recent analysis of [236] examines 3d amplitudes that asymptotically behave as a single oscillating Regge exponential, tackling the "cosine problem" (see [111]).

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \sim \frac{1}{\sqrt{12\pi V}} \cos\left(\sum_{k=1}^{6} j_k\theta_k + \frac{\pi}{4}\right) , \tag{17}$$

where V is the volume of the tetrahedron and θ_k is defined as the external dihedral angles between the triangles sharing the k-th edge. The expression $\sum_{k=1}^{6} j_k\theta_k$ is a discretised analogue of a surface integral of the extrinsic curvature on the boundary of the tetrahedron. In other words, it is nothing but the Regge action $S^{(3d)}_{\mathrm{Regge}}$ for the tetrahedron (see (15), here without cosmological constant), so that the Ponzano–Regge model (16) provides an application of a variation of quantum Regge calculus (where lengths are quantised using $SU(2)$ representations) for discrete general relativity in three dimensions.

While in general the partition function (16) diverges, replacing the group $SU(2)$ of the Ponzano–Regge model with the quantum group $SU_q(2)$ leads to the Turaev–Viro model [368], which provides a finite path integral by introducing a nonvanishing cosmological constant. More precisely, when the deformation parameter q of the Turaev–Viro model is given by a root of unity, the representation theory of $SU_q(2)$ is truncated to a finite range of spins. Then, the path integral amplitude is formulated in terms of q-deformed $6j$-symbols [263] and their large spin asymptotics effectively mirror the cosmological constant term in the action. Heuristically, the presence of the cosmological constant introduces a physical infrared cut-off [71, 270, 353].

As we will see, the Ponzano–Regge model (or the regularised Turaev–Viro model) can be seen as the first realisation of the spin foam approach to quantum gravity, that we review in Section 2.2.4.

2.2.3 *Matrix and tensor models*

Other non-perturbative quantum gravity approaches based on the principles of discrete path integrals are given by matrix and tensor models [121, 129, 209]. In a nutshell, these models generate random discrete geometries as Feynman diagrams of theories where the fundamental object is a tensor (of rank 2 and 3 in two and three dimensions respectively), and can directly relate to discrete formulations of general relativity. We will only review matrix models for simplicity, and mention important aspects of the more general tensor models.

Before showing an example of a matrix model for two-dimensional quantum gravity, let us rewrite the Euclidean Einstein–Hilbert action as

$$\begin{aligned} S^{\mathrm{E}}_{\mathrm{EH}} &= \frac{1}{16\pi G}\int_\Sigma \mathrm{d}^2x\sqrt{g}\,(2\Lambda - R) \\ &= \frac{\Lambda}{8\pi G}A(\Sigma) - \frac{1}{4G}\chi(\Sigma)\,, \end{aligned} \tag{18}$$

where $A(\Sigma) = \int_\Sigma \mathrm{d}^2x\sqrt{g}$ is the total area of the $2d$ manifold Σ and $\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma \mathrm{d}^2x\sqrt{g}\,R$ is the Euler characteristic (due to the Gauss–Bonnet theorem). Importantly, for two-dimensional manifolds the Euler characteristic is related to the genus as $\chi = 2 - 2\mathfrak{g}$. By discretising the surface Σ in terms of equilateral triangles with area a (let us denote such a triangulation with the label $\triangle$), (18) can be cast as a Regge-like action (cf. (15))

We denote with $\mathfrak{g}$ the genus of closed and orientable surfaces.

$$S^{(2d)}_{\mathrm{Regge}} = \frac{\Lambda}{8\pi G}aF_\triangle - \frac{1}{4G}\chi_\triangle\,, \tag{19}$$

where the Euler characteristic relates to the number of vertices $V_\triangle$, edges $E_\triangle$ and faces $F_\triangle$ of the triangulation $\triangle$ as

$$\chi_\triangle = V_\triangle - E_\triangle + F_\triangle\,. \tag{20}$$

The discrete analogue of the usual gravitational path integral for the Regge action (19) reads

$$Z^{(2d)}_{\mathrm{Regge}} = \sum_\triangle \frac{1}{C_\triangle} e^{\frac{1}{4G}\chi_\triangle - \frac{\Lambda}{8\pi G}aF_\triangle}\,, \tag{21}$$

where $C_\triangle$ denotes the order of the automorphism group which leaves the triangulation invariant. $Z^{(2d)}_{\text{Regge}}$ is a discrete Feynman expansion for $2d$ quantum gravity where the sum over triangulations $\triangle$ captures both different geometries and topologies.

Let us now consider a nontrivial matrix model that can reproduce (21). The fundamental object of the theory is an $N \times N$ Hermitian matrix M, and we start from an action given by

$$\begin{aligned} S_{MM} &= N\left(\frac{1}{2}\,\mathrm{tr}(M^2) + \lambda\,\mathrm{tr}(M^3)\right) \\ &= N\left(\frac{1}{2}M_{ij}M_{ji} + \lambda\, M_{ij}M_{jk}M_{ki}\right), \end{aligned} \tag{22}$$

where λ is a coupling constant. In (22), the quadratic term provides (the inverse of) the propagator and is graphically represented by a two-stranded line (also called a *ribbon*, where each strand represents one index of the matrix). The cubic term describes an interaction between three matrices, and represents a three-valent vertex, as depicted in Figure 4.

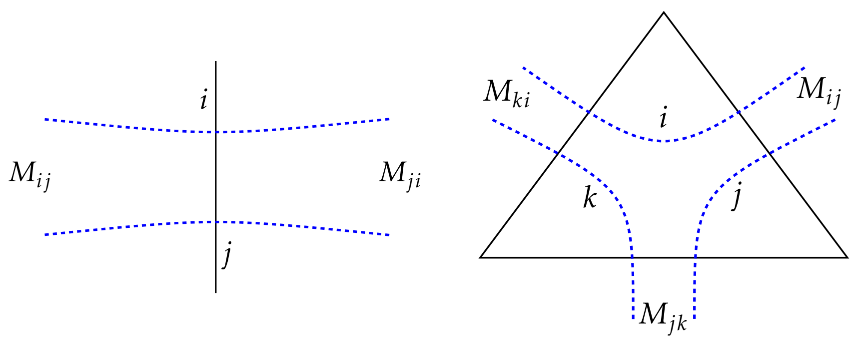


Figure 4: Propagator (left) and cubic interaction vertex (right). The blue dashed lines graphically represent matrix indices and show their contractions. The black solid lines define matrix elements in the dual picture, where in particular one can see that the cubic interaction generates a triangle.

With the interpretation of Figure 4, ribbon graphs are dual to a simplicial decomposition in terms of triangulated surfaces.

One can then define the partition function of the model as [129]

$$Z_{MM} = \int \mathrm{d}M\, e^{-S_{MM}}, \tag{23}$$

whose Feynman diagrams Γ are ribbon graphs (i.e., a set of vertices joined by two-stranded lines). By considering $\ln Z_{MM}$ we obtain the generating function of connected diagrams only, so that the free energy of the matrix model will correspond to the partition function of $2d$ gravity.

Closed loops in the ribbon graph (dual to vertices of triangles), propagators (dual to edges) and interactions (dual to triangles) contribute factors of N, N^{-1} and N respectively; then, one has that each diagram in (23) comes with an overall factor (cf. (20))

$$N^{V_\Gamma - E_\Gamma + F_\Gamma} = N^{\chi_\Gamma} = N^{2-2\mathfrak{g}}\,, \tag{24}$$

where V_Γ, E_Γ and F_Γ represent the number of vertices, edges and faces in the Feynman graph Γ, and χ_Γ is its Euler characteristic. Then, given that every interaction vertex also brings a factor of λ (cf. (22)), the free energy can be written as

$$\ln Z_{MM} = \sum_\Gamma \frac{1}{\mathrm{sym}_\Gamma} \lambda^{V_\Gamma} N^{\chi_\Gamma}\,, \tag{25}$$

where sym_Γ is a symmetry factor. Crucially, denoting $Z_\mathfrak{g} := \sum_\Gamma \frac{\lambda^{V_\Gamma}}{\mathrm{sym}_\Gamma}$, where the sum is over graphs with fixed $\mathfrak{g}$, one can write the free energy as an expansion in powers of N where the contributions of the various genera are neatly separated,

$$\ln Z_{MM} = \sum_\mathfrak{g} N^{2-2\mathfrak{g}} Z_\mathfrak{g}\,. \tag{26}$$

Finally, by recognising Γ as the dual graph to the triangulation $\triangle$ introduced in (19), one may identify λ with $e^{-\frac{\Lambda}{8\pi G}a}$ and N with $e^{\frac{1}{4G}}$ and observe that

$$\ln Z_{MM} = Z^{(2d)}_{\mathrm{Regge}}\,. \tag{27}$$

(27) relates the path integral for simplicial gravity with that obtained from the matrix model with action (22), and makes it clear that Feynman amplitudes are in correspondence with $2d$ simplicial complexes.

By studying the continuum limit of matrix models, which is called *double scaling limit* and is achieved by simultaneously sending $N \to \infty$ and setting the coupling constant λ to a critical value, one finds that the free energy provides a sum over all possible $2d$ simplicial complexes of all topologies. Intuitively, the large N limit restores the infinite degrees of freedom of continuous gravity and the tuning of the coupling λ corresponds to a phase transition where the matrix models describe infinitely refined triangulations.

TENSOR MODELS. Inspired by matrix models for $2d$ gravity, tensors of rank $d \geq 3$ were considered to develop models suitable to describe higher-dimensional random geometries and relate them to simplicial gravity [14, 203]. However, the naive implementation of the same idea (with tensors graphically represented by $(d-1)$-simplices and with interactions dictating their gluing to form a d-simplex) leads to some obstacles already for $d = 3$. In particular, Feynman diagrams generated by such tensor models can correspond to general topological spaces, including manifolds, pseudomanifolds and other more singular topologies [124, 204]. As a consequence, the partition function of tensor models cannot be expanded in terms of powers of N as for matrix models, and there is no way to reorganise the sum in terms of topological invariants (as in (26)).

To address these issues, coloured tensor models were introduced [205, 206, 211]. These models include additional combinatorial structures that ensure Feynman graphs are truly dual to simplicial complexes, excluding pathological configurations. Most importantly, coloured tensor models allow for a reorganisation of the perturbation series in terms of a $1/N$-expansion [207, 208, 210].

We will not review tensor models here and we refer to [209] for a recent overview. It is interesting to note that the shortcomings of the traditional tensor models gave birth to the group field theory programme, which started with the renowned Boulatov and Ooguri models [86, 293]. As we will see at the end of this chapter, GFTs are essentially tensor models enriched with additional data that encode metric degrees of freedom.

2.2.4 *Spin foam models*

With the aim of curing the pathologies of the traditional gravitational partition function (10), but without giving up the core ideas of a path integral, spin foam models were defined as a background-independent and non-perturbative framework for discrete quantum gravity [312, 313, 344] (for recent overviews, see [153, 265]). These models can relate to (and generalise) some of the approaches reviewed in the previous sections, and will serve as a stepping stone to introduce group field theories in Section 2.3. In short, spin foams describe spacetime histories and transition amplitudes for quantum states of geometry as defined in loop quantum gravity [329, 344], and provide a way of interpreting the Feynman path integral as a sum over "quantum spacetime geometries" in a precise way.

SPIN NETWORKS. In loop quantum gravity, spatial geometry at the Planck scale is defined in terms of quantum operators with discrete spectra representing areas, volumes and other geometrical quantities [31, 32, 343]. Quantum states of spatial geometry, called *spin networks*, are eigenstates of such operators and are defined on graphs embedded in $3d$ space where every link is associated with an element of the group $SU(2)$. Heuristically, the evolution of those states then weaves the fabric of the quantum $4d$ spacetime and defines spin foams.

More precisely, considering a closed graph Γ made of nodes connected by links, spin network states provide a basis for the Hilbert space of quantum geometry on that graph, $\mathcal{H}_\Gamma$. This, equipped with the $SU(2)$ Haar measure $\mathrm{d}\mu_{\mathrm{Haar}}$, is the space of square-integrable wavefunctions that are invariant under $SU(2)$ transformation at each node on the graph.[7] Then, spin network states $|\Gamma, j_l, \iota_n\rangle$ are labelled by a spin j_l for every link l and an intertwiner ι_n for every node n of the graph Γ, as explained below.

Just as in the quantum theory of angular momentum, the spin j refers to the irreducible representations of $SU(2)$ and the dimension of the associated Hilbert space $\mathcal{H}_j$ is

$$d_j = 2j + 1\,. \tag{28}$$

Denoting the generators of the $\mathfrak{su}(2)$ Lie algebra as $\vec{J} = (J_1, J_2, J_3)$, one can use the familiar basis of $\mathcal{H}_j$ which diagonalises both the Casimir $\vec{J}^2$ and the third generator J_3,

$$\begin{aligned} \vec{J}^2|j,m\rangle &= j(j+1)|j,m\rangle\,, \\ J_3|j,m\rangle &= m|j,m\rangle\,, \end{aligned} \tag{29}$$

where the magnetic index $m \in [-j, j]$. An intertwiner ι_n is an invariant tensor of $SU(2)$, which can be understood as a map from the tensor product of all the spins living on the links attached to n to the trivial representation. In general, the Hilbert space of an N-valent node (i.e., attached to N links with spins $j_1, \ldots, j_N$) is defined as

$$\mathcal{H}_N := \mathrm{Inv}\left[\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}\right]\,, \tag{30}$$

7 Since an element of $SU(2)$ represents the transport between the nodes of the graph (which are interpreted as space points), the invariance under $SU(2)$ transformations at the nodes represents the gauge invariance of the states under local change of reference frame at each point of space.

namely, as the subspace of the tensor product $\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_N}$ that is invariant under $SU(2)$ transformations. Intertwiners are elements of the space (30). Basic examples are given by the case of the trivial bivalent intertwiner between two spins j_1 and j_2, which only exists if the two spins are equal; and by the unique trivalent intertwiner between three spins $\iota^{j_1 j_2 j_3}$, which exists if and only if the spins satisfy

Note that in the notation $\iota^{j_1 j_2 j_3}$ the dependence on the magnetic indices is implicit; we will make this dependence explicit as $\iota^{j_1 j_2 j_3}_{m_1 m_2 m_3}$ only when necessary.

$$|j_1 - j_2| \leq j_3 \leq (j_1 + j_2)\,, \tag{31}$$

and is given by the 3j-symbol

$$\iota^{j_1 j_2 j_3} \equiv \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_3 + 1}} C^{j_3, -m_3}_{j_1 m_1\, j_2 m_2}\,, \tag{32}$$

where $C^{j_3,-m_3}_{j_1 m_1\, j_2 m_2} = \langle j_1, m_1; j_2, m_2 | j_3, -m_3 \rangle$ are Clebsch–Gordan coefficients [371]. For nodes with higher valency, the intertwiner space grows in dimension (and elements of $\mathcal{H}_{N>3}$ are usually indexed with an extra label κ). Note that N-intertwiners in (30) are interpreted as quantum N-simplices (e.g., states living on the space $\mathcal{H}_4$ can be understood as quantum tetrahedra, as firstly pointed out in [43, 51]).

Equipped with these ingredients, and thanks to the Peter–Weyl theorem,[8] a spin network wavefunction on a graph Γ can then be seen schematically as tensor products of representation matrices contracted with intertwiners

$$\psi_\Gamma^{\{j_l, \iota_n\}}(\{g_l\}) = \bigotimes_n \iota_n \bigotimes_l D^{(j_l)}(g_l)\,, \tag{33}$$

where $D^{(j_l)}(g_l)$ is the Wigner D-matrix representing the group element $g_l \in SU(2)$ in the spin j_l along the link l.[9] As mentioned, the geometrical interpretation of spin networks in loop quantum gravity comes from the quantisation of geometric observables: intertwiners represent excitations of the spatial volume that are "glued together" by the spins, which in turn are related to the quanta of area of the

8 The Peter–Weyl theorem states that a basis on the space $L^2(SU(2), \mathrm{d}\mu_{\mathrm{Haar}})$ is given by the matrix elements of the unitary irreducible representations of $SU(2)$, known as Wigner D-matrices [319].

9 Explicitly, using the basis $|j, m\rangle$ as in (29), one writes the matrix elements as $D^{(j)}_{mn}(g) = \langle j, m | g | j, n \rangle$.

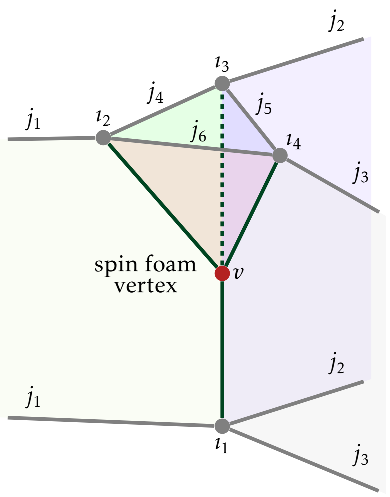
faces between neighbouring "chunks" of volume. More precisely, the spin j_l is related to the area of the surface dual to the link l as

$$A_l = l_0^2 \sqrt{j_l(j_l+1)}\,, \tag{34}$$

where l_0 is a fundamental length scale parameter related to the Planck length (cf. (1)). Similarly, the volume operator is well-defined and its eigenvalues can be found [32, 125, 342], but it is not as easy to diagonalise [68, 69, 94] (we will give more details on the volume operator in Chapter 4, see in particular Section 4.2.1).

Finally, also in light of the interpretation of intertwiners as quantised convex polyhedra [67, 163, 262], spin networks provide a compelling description for quantised discrete $3d$ geometries.

SPIN FOAMS. The evolution of spin networks is described in terms of spin foams. As exemplified in Figure 5 for the three-dimensional case, the nodes and links of a graph evolve along spin foam edges (akin to worldlines) and spin foam faces (akin to worldsheets) respectively, carrying intertwiner and spin labels. Spin foam edges and faces can meet at spin foam vertices where the graph changes; these vertices are interpreted as events where the geometry changes, and thus as spacetime points.



Triangulation	Dual graph
tetrahedron	vertex
triangle	edge
segment	face

Figure 5: A 2-complex with one vertex in three dimensions and the corresponding spin foam terminology. The figure represents the transition between $2d$ boundary graphs, showing the skeleton dual to the triangulation of $3d$ spacetime in the bulk. As we will see with the Ponzano–Regge model, the red vertex is dual to a tetrahedron.

Intuitively, a spin foam vertex plays the role that an interaction vertex plays in traditional field theories [42, 47, 48]. As we will see in more detail in Section 2.3, one can actually show that spin foams can be understood as Feynman diagrams of quantum field theories known as group field theories [86, 123, 160, 330]. The key point to stress here is that spin foams describe transition amplitudes associated with spin network histories. Formally, working with the group $SU(2)$, one defines the so-called local spin foam model by:

1. a 2-complex $\mathcal{C}$, i.e., a set of faces f, edges e and vertices v;
2. a set of representation labels j_f associated with the faces $f \in \mathcal{C}$;
3. a set of intertwiners ι_e associated with the edges $e \in \mathcal{C}$.

Note that j_f and ι_e are fixed for a given $\mathcal{A}_{SF}$, but the model is defined as a sum over all such labels, as explained below.

Then, a spin foam model is defined as a sum over j_f and ι_e of transition amplitudes

$$\mathcal{A}_{\text{SF}}[\mathcal{C}] = \prod_f \mathcal{A}_f(j_f) \prod_e \mathcal{A}_e(j_{f \ni e}, \iota_e) \prod_v \mathcal{A}_v(j_{f \ni v}, I_{e \ni v}) \,, \tag{35}$$

that are factorised in terms of face, edge and vertex amplitudes. The face and edge amplitudes $\mathcal{A}_f$ and $\mathcal{A}_e$ can be considered as the analogue of the path integral measure, defining the weights of spins and intertwiners. For the models of interest, the face amplitude is usually chosen to be $\mathcal{A}_f(j_f) = d_{j_f} = 2j_f + 1$. On the other hand, the vertex amplitude $\mathcal{A}_v$ contains the nontrivial dynamical information that encodes the evolution of spin networks (we will derive an explicit example later). While we use $SU(2)$ here, so as to connect spin foams with spin networks of canonical loop quantum gravity, the spin foam paradigm can be applied to (semi-simple) Lie groups and finite groups [137]. For example, the more general Barrett–Crane (BC) and Engle–Pereira–Rovelli–Livine (EPRL) models can be based on $SO(4)$ or $SL(2,\mathbb{C})$ (for the Euclidean and Lorentzian version of the theory, respectively). To embed spin networks in $SL(2,\mathbb{C})$-based spin foam models, one requires a special map between $SU(2)$ spins and $SL(2,\mathbb{C})$ representations [132, 147, 344]).

We will show some details of the Barrett–Crane (BC) model in later sections.

Somewhat as in a traditional path integral, one computes a transition amplitude for a given boundary state (in terms of spin networks with fixed spins and intertwiners), by summing over all possible spin and intertwiner labels in the bulk. Note, however, that while summing over all labelling of the 2-complex $\mathcal{C}$ encodes a sum over multiple geometries, it does not mean one is considering all possible geometries since $\mathcal{C}$ is fixed. Indeed, an expansion that more faithfully resembles a gravitational discrete path integral (similar in spirit to (10)) is given by

$$Z_{\rm SF} = \sum_{\mathcal{C}} \omega[\mathcal{C}] \sum_{j_f, \iota_e} \mathcal{A}_{\rm SF}[\mathcal{C}]\,, \tag{36}$$

where $\omega[\mathcal{C}]$ is a weight factor depending on the 2-complex. As we will see, (36) is exactly the type of expansion that one finds in the group field theory formalism. We stress that the spin foam paradigm usually tries to explicitly calculate a single contribution in the sum over 2-complexes in (36); in other words, it mostly focusses on a fixed triangulation.

THE PONZANO–REGGE MODEL AGAIN. Before mentioning aspects of the more general $4d$ spin foam models, we highlight here the connection between spin foams and the Ponzano–Regge model for $3d$ Euclidean quantum gravity introduced earlier in Section 2.2.2. To this aim, we will first cast general relativity as a BF theory,[10] namely a topological field theory which has no local degrees of freedom. Hence, we will derive a spin foam model by requiring topological invariance of the BF path integral, so that that the spin foam will not depend on the local details of the simplicial complex but only on its global topology and boundary. The paradigmatic case of $3d$ Euclidean GR enables to derive the vertex amplitude $\mathcal{A}_v$ of (35) explicitly, which will turn out to match with the one of the state-sum model proposed by Ponzano and Regge [324].

The BF theory for $3d$ gravity (here using the group $SU(2)$ for the Euclidean version, but see [159] for the Lorentzian version based on $SU(1,1)$) requires the introduction of Cartan fields. These are $\mathfrak{su}(2)$-valued 1-forms known as the triad $e = e^a J_a$ and the connection $\omega = \omega^a J_a$, with J_a denoting the usual basis of the algebra $\mathfrak{su}(2)$ such that $[J_a, J_b] = \mathrm{i}\epsilon^{abc} J_c$. Importantly, the metric is easily reconstructed from the triad as

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu\,, \tag{37}$$

where the internal indices $a, b, \ldots$ are raised and lowered with the flat metric $\eta_{ab} = \delta_{ab}$ (i.e., the Killing form of the Lie algebra $\mathfrak{su}(2)$). Then,

10 Even if general relativity is precisely a BF theory only in three dimensions, the Plebański formulation [323] of $4d$ gravity can be written as a BF theory subject to constraints [312, 313]. In practice, all spin foam amplitudes are generally derived by the quantisation of BF theories [42, 104]; we will give an example later.

In the language of differential forms, $\wedge$ denotes the wedge product, tr denotes the Killing form of $\mathfrak{su}(2)$ and d is the exterior derivative.

one can show that Euclidean GR in three dimensions (with $\Lambda = 0$ for simplicity) can be described by means of the action[11]

$$S_{\mathrm{EH}}^{(3d)}[e,\omega] = \int e^a \wedge F_a[\omega] = \int \mathrm{tr}(e \wedge F[\omega])\,, \tag{38}$$

where $F = \mathrm{d}\omega + \omega \wedge \omega$ is the curvature 2-form of the connection ω. Varying (38) with respect to the fields yields the equations of motion

$$\begin{aligned} &F[\omega] = 0\,,\\ &\mathrm{d}_\omega e = \mathrm{d}e + \omega \wedge e = 0\,, \end{aligned} \tag{39}$$

where d_ω denotes the exterior covariant derivative and $\mathrm{d}_\omega e$ is the definition of the torsion 2-form. Together, (39) are nothing but Einstein's field equations in vacuum, respectively indicating that the connection is flat and that it is the unique torsion-free connection associated with the triad e. In particular, the first equation in (39) is the analogue of $R = 0$, where R is the Ricci scalar familiar from GR. In a sense, the triad field plays the role of a Lagrange multiplier enforcing the vanishing of the curvature. Then, as anticipated, the theory described by (38) is topological in the sense that are no local degrees of freedom.

In the spin foam framework one considers the formal path integral of the BF theory (38),

$$Z_{\mathrm{BF}} = \int \mathcal{D}\omega \mathcal{D}e\, e^{\mathrm{i}\int e\wedge F[\omega]} = \int \mathcal{D}\omega \delta(F[\omega])\,, \tag{40}$$

and makes it well-defined by performing a discretisation on a 2-complex. More precisely, one now introduces a simplicial triangulation Δ of the manifold and its dual 2-complex $\mathcal{C}$. Then, one discretises the Cartan fields e and ω following lattice gauge theory approaches [327]. Crucially, because the theory is topological, discretisation does not imply any loss of information so that the discrete version of the path integral will yield an exact quantisation of the theory. Indeed, topological invariance is what guarantees that the resulting spin foam amplitude is independent of the triangulation Δ (in other words, the triangulated version of the theory has the same number of degrees of freedom of the full field theory).

In three dimensions, Δ represents a discretisation in terms of 3-simplices or tetrahedra.

The 1-form ω gets discretised in terms of $SU(2)$ elements g_e associated to the edges e of $\mathcal{C}$. Then, curvature is represented as the transport along a closed loop going around the spin foam faces f,

11 In the d-dimensional case, the triad is replaced by a $(d-2)$-form usually denoted B, so that the action (38) manifestly shows the origin of the name "BF theory".

$U_f := \prod_{e\in\partial f} g_e$. Using this, the discretised version of (40) takes the form [340]

$$Z_{\rm BF}^{\rm discrete}[\mathcal{C}] = \int \prod_{e\in\mathcal{C}} \mathrm{d}g_e \prod_{f\in\mathcal{C}} \delta(U_f)\,, \tag{41}$$

and describes a theory of flat (discrete) connections, reminiscent of the lattice gauge theory formulation of path integrals [327]. In order to finally obtain spin foam amplitudes, one can now make use of the Peter–Weyl theorem to expand the Dirac δ-distribution over the group manifold appearing in (41) as

$$\delta(g) = \sum_j d_j \,\mathrm{tr}\, D^{(j)}(g)\,, \tag{42}$$

where d_j is given in (28), the sum is over the unitary irreducible representations of $SU(2)$, and $D^{(j)}(g)$ denotes Wigner D-matrices (cf. (33)). Since every spin foam edge is shared by three faces (cf. Figure 5), by using (42) in (41) one obtains integrals of three Wigner D-matrices, which are nothing but projectors onto the invariant subspace of the tensor product of three irreducible representations, $\mathcal{H}_3 = \mathrm{Inv}\left[\mathcal{H}_{j_1}\otimes\mathcal{H}_{j_2}\otimes\mathcal{H}_{j_3}\right]$. As this is the space of intertwiners (cf. (30)), one can write such integrals as

$$\int \mathrm{d}g D^{j_1}_{m_1n_1}(g) D^{j_2}_{m_2n_2}(g) D^{j_3}_{m_3n_3}(g) = |\iota^{j_1j_2j_3}\rangle\langle\iota^{j_1j_2j_3}|\,, \tag{43}$$

where $|\iota^{j_1j_2j_3}\rangle$ represents the (unique) three-valent intertwiner state in $\mathcal{H}_3$ (i.e., the $3j$-symbol (32)).[12] Finally, since four edges meet at the vertex v, one ends up with the contraction of four $3j$-symbols yielding precisely the definition of a $6j$-symbol [371], so that (41) becomes

Figure 5 illustrates precisely this 3d model, where the 6j-symbol represents the tetrahedron (dual to the vertex), obtained by gluing four triangles (dual to the edges).

$$Z_{\rm BF}^{\rm discrete}[\mathcal{C}] = \sum_{j_f}\prod_f (-1)^{2j_f} d_{j_f} \prod_v \{6j\}_v\,. \tag{45}$$

This is exactly the partition function of the Ponzano–Regge model introduced in Section 2.2.2 (cf. (16)). The partition function (45) provides the first (and simplest) spin foam model in that it can be seen as a special case of (35), where $A_v = \{6j\}_v$ (and there is only one intertwiner available).

12 Note that in general one has

$$\int \mathrm{d}g \prod_{i=1}^N D^{(j_i)}(g) = \sum_\kappa |\iota_\kappa^{j_1\ldots j_N}\rangle\langle\iota_\kappa^{j_1\ldots j_N}|\,, \tag{44}$$

where the intertwiner basis on $\mathcal{H}_N$ for $N\geq 4$ is labelled by the index κ.

Spin foams in four dimensions

The Ponzano–Regge (toy) model served as an illustrative example for the development of four-dimensional spin foam models. The most promising spin foam candidates for $4d$ quantum gravity (famously the BC model [53] and the EPRL model [151]) are derived in a similar way, starting from the Plebański formulation of four-dimensional GR. The important difference with the $3d$ case is that the Plebański action is here made of two contributions: a BF term (based on $SO(4)$ for the Euclidean theory and $SL(2,\mathbb{C})$ for the Lorentzian theory) plus a "potential" for the B field, which crucially leads to local fluctuations of the curvature so that the theory is not topological (i.e., it allows to recover GR) [252].

For concreteness, let us consider the Euclidean Barrett–Crane model. We start from the Plebański action

$$S[B,\omega,\mu] = \int \left(B^{IJ} \wedge F_{IJ}[\omega] - \frac{1}{2}\mu_{IJKL} B^{KL} \wedge B^{IJ} \right) , \tag{46}$$

where the field B and the connection ω are respectively a $\mathfrak{so}(4)$-valued 2-form and 1-form, $F[\omega]$ is the curvature 2-form of ω, and the Lagrange multipliers μ_{IJKL} obey the symmetries $\epsilon^{IJKL}\mu_{IJKL} = 0$ and $\mu_{IJKL} = \mu_{KLIJ} = -\mu_{JIKL} = -\mu_{IJLK}$. Varying with respect to these forces the B field to be[13]

$$B^{IJ} = \epsilon^{IJ}{}_{KL} e^K \wedge e^L , \tag{47}$$

where the tetrad e^I field, typical of the Cartan formulation, generalises the triad e^a of the previous section. Specifically, thanks to the relation $g_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu$ (cf. (37)), the form of B in (47) allows to obtain precisely the (Palatini) action of GR from BF theory. In other words, while the $B \wedge F$ part of the action (46) is equivalent to the $3d$ case (cf. (39)), the potential reintroduces local (propagating) degrees of freedom into the theory by constraining B to be of the form (47) (the potential is said to impose *simplicity constraints*).

Following the strategy adopted with the Ponzano–Regge model, one first quantises the BF part of the Plebański action, hence defining a spin foam path integral for a $4d$ BF theory, and then imposes the simplicity constraints coming from the potential term at the quantum

13 Strictly speaking, the $SO(4)$ Plebański model consists of four different sectors (see [122] and references therein for details). For simplicity we focus here on the sector that allows to recover tetrad gravity, where B takes the form (47).

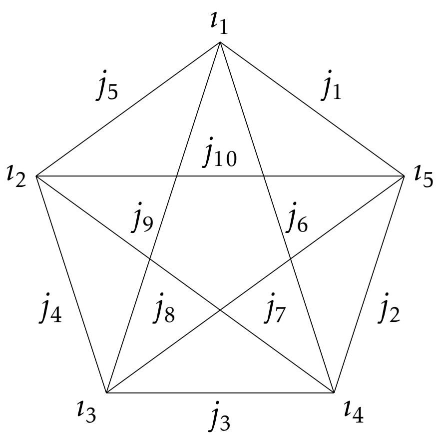
level. The quantisation of the BF term leads to a higher-dimensional analogue of (45): a topological (discrete) model on a 2-complex, which is now dressed with $SO(4)$ representations (we use boldface symbols $\mathbf{j}$ to distinguish from the $SU(2)$ case). Being in four dimensions, this is interpreted as the gluing of five tetrahedra (here dual to spin foam edges) into a 4-simplex (here dual to the spin foam vertex). The partition function takes the form [122, 312, 313, 340][14]

We do not specify the face amplitude $\mathcal{A}_f$ for now, and focus on the vertex amplitude.

$$Z^{(4d)}_{\rm BF}[\mathcal{C}] = \sum_{\mathbf{j}_f, \imath_e} \prod_f \mathcal{A}_f \prod_v \{15\mathbf{j}\}_v \,, \tag{48}$$

where $\{15\mathbf{j}\}$ is a $15j$-symbol [371], which generalises a $6j$-symbol by reproducing the structure of a 4-simplex. More precisely, just like $\{6j\}$ is defined as the contraction of four $3j$-symbols (heuristically representing a tetrahedron as the gluing of four triangles, cf. Figure 3), a $15j$-symbol is defined as the trace of the product of five "4-valent intertwiners" (elements of $\mathcal{H}_4$, see (30)). In four dimensions, a 4-simplex is dual to a spin foam vertex which bounds ten faces (equipped with representations $\mathbf{j}_1, \dots, \mathbf{j}_{10}$) and five intertwiners on the edges, which can be labelled with representations of internal virtual links $\imath_1, \dots, \imath_5$. The vertex amplitude in (48) is then a function of fifteen representation labels, graphically represented in Figure 6 (where we stick to the traditional notation for $SU(2)$ representations, which is how all the Wigner nj-symbols were originally defined [371]).

Another reason for adopting the usual spin notation ($j_1, \dots, j_{10}$) in Figure 6 will become clear shortly, since in the Barrett–Crane model the spin foam faces are indeed labelled with $SU(2)$ representations.



Triangulation	**Dual graph**
4-simplex	vertex
tetrahedron	edge
triangle	face

Figure 6: The pentagon diagram illustrates the skeleton of a 4-simplex, the simplest spin foam vertex in four dimensions. The five corners and the ten lines represent respectively the spin foam edges and faces (with such edges in their boundary) meeting at the vertex.

14 As it is, the state-sum (48) is generically divergent (just as the Ponzano–Regge model stemming from $3d$ BF theory); a regularised version defined in terms of the quantum group $SU_q(2) \times SU_q(2)$ was introduced in [118, 119].

THE BARRETT–CRANE SPIN FOAM MODEL. As in three dimensions, the model (48) stemming from a $4d$ BF theory is topologically invariant, the spin foam path integral is discretisation-independent and does not correspond to GR (where a choice of discretisation would reduce the number of degrees of freedom) [122, 312, 313, 340]. Indeed, the next step in the strategy laid out above is to obtain gravity by means of the simplicity constraints (cf. (46) and (47)). Imposing the quantum analogue of the simplicity constraints is what defines the BC model for quantum gravity. It turns out that this amounts to a restriction to a particular set of representations $\mathbf{j}$ of $SO(4) \approx SU(2) \times SU(2)$, generically labelled by two spins (j_+, j_-), where the spins of the two factors are equal, $j_+ = j_- \equiv j$.[15] This leads to the so-called *simple* (or balanced) representations (j, j) which are labelled by a single spin j. Restricting the sum over representation in (48) to simple representations leads to the state-sum models first proposed by Barrett and Crane [53]. More precisely, to make the state-sum model independent of the labels $\imath_1, \ldots, \imath_5$ associated with the spin foam edges (the corners of Figure 6), one includes a sum over such labels and consider special intertwiners (often called BC intertwiners [52, 328]) such that the vertex amplitude becomes a $10j$-symbol. As the name suggest, $\{10j\}$ is a function that only depends on the ten representations of the spin foam faces of a 4-simplex (the lines of Figure 6). The BC partition function then reads

Thanks to the BC intertwiners and the restriction to simple representations, $\{10j\}$ can be written as the product of $\{15j_+\}$ and $\{15j_-\}$ summed over the labels $\imath_1, \ldots, \imath_5$.

$$Z_{\mathrm{BC}}[\mathcal{C}] = \sum_{j_f, \imath_e} \prod_f \mathcal{A}_f \prod_e \mathcal{A}_e \prod_v \{10j\}_v \,, \tag{49}$$

where we left $\mathcal{A}_f$ and $\mathcal{A}_e$ generic. Different versions of the BC model [312, 313] can prescribe different choices of the face and edge amplitudes $\mathcal{A}_f$ and $\mathcal{A}_e$. For instance, given that the simple representations are given by (j, j) where j is an $SU(2)$ spin, the choice $\mathcal{A}_f = (2j+1)^2$ is often considered. There is also a variant of the BC model which takes

$$\mathcal{A}_e = \frac{\dim\left(\mathrm{Inv}[\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_4}]\right)}{\dim\left(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_4}\right)} \,, \tag{50}$$

where $\mathrm{Inv}[\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_4}]$ is the intertwiner space (30), instead of $\mathcal{A}_e = 1$ as in (48). This particular choice of edge amplitude naturally

15 To see this, one first notices that (47) implies the vanishing of the so-called pseudo-scalar Casimir, $\epsilon_{IJKL} B^{IJ} B^{KL} = 0$. In the spin representation (j_+, j_-), this becomes $j_+(j_+ + 1) - j_-(j_- + 1)$ which vanishes for $j_+ = j_-$ [312, 313, 340].

emerges in the group field theory formalism, see [312, 313, 340] for more details.

While we focussed on the Euclidean Barrett–Crane model because it is one of the most extensively studied spin foam models for 4*d* quantum gravity, the above construction can be generalised to the case of the Lorentzian theory, where the base group is chosen to be $SL(2,\mathbb{C})$ [53, 312, 313, 317] (see also [49, 234, 235, 238] for more recent developments on the BC model). Moreover, the same strategy can be applied to the EPRL model [151, 152], which was introduced to overcome some limitations of the BC model (see, e.g., [12, 131, 162, 267]) and has thus attracted particular interest in recent years. We will not review the details of these aspects here as they will not be important for the group field theory models used in this thesis (especially for the original work presented in Part II).

Regarding the semiclassical regime of spin foams, it is important to mention that studies on the large spin structure of vertex amplitudes show that the Regge action for (discrete) general relativity is recovered asymptotically (somewhat as for the Ponzano–Regge model (17)) [54, 55]. More recent developments have focused on effective spin foam models [19, 20, 139] investigating the field theory content of the semiclassical limit, on the saddle point analysis of the Lorentzian path integral [219], and on the case of more refined triangulations (corresponding to large spins and many vertices) [18, 143, 217].

To conclude, we simply stress that 4*d* spin foam models based on the Plebański formulation of GR can constrain the space of histories of the BF theory path integral to that of gravity (specifically, by implementing a suitable restriction on representation labels). Therefore, especially in light of the recent studies on their semiclassical limit mentioned above, spin foams provide a promising arena for 4*d* quantum gravity as they effectively realise a path integral over discretised metrics for general relativity.

2.3 GROUP FIELD THEORIES

With group field theories (GFTs) we refer to a class of quantum field theories defined by fields that live on group manifolds (e.g., the local gauge group of gravity), and that aim to describe spacetime in a combinatorial and algebraic way.

As mentioned in Section 2.2.3, GFTs were firstly developed to generalise matrix models to dimensions $d \geq 2$ by resolving the issues encountered in traditional definitions of tensor models [86, 293]. It was then realised that GFTs naturally emerge also in the context of spin foam models for quantum gravity, specifically as a tool to overcome the limitations of working with fixed spacetime triangulations [123, 160, 330]. Indeed, the perturbative expansion of a GFT path integral is labelled by Feynman diagrams which are dual to cellular complexes, as we will clarify in the following. As a result, GFTs bring together elements of matrix and tensor models, such as the combinatorial details of Feynman graphs, with the group theoretic data of spin foams, which encode geometric information corresponding to the elementary variables of LQG [340, 367]. As mentioned in Section 2.2.4, we will see that GFTs can provide exactly the sum over 2-complexes we have seen in (36), which is interpreted as a discrete definition of the covariant path integral for 4d quantum gravity. It is in this sense that GFTs can be seen as a formal completion of spin foam models [330, 344], providing a manifestly background-independent field theoretic framework for quantum gravity.

After briefly reviewing the classical theory and the connection to spin foams in Section 2.3.1 and Section 2.3.2, we will describe a more recent perspective which relies on a canonical quantisation of GFT in the next chapter. This connects in spirit with ideas from Section 2.1, and will be the framework mostly adopted in the rest of the thesis.

2.3.1 *Classical theory*

A classical GFT is the theory of a scalar field φ defined on d copies of a Lie group G as

Naively, what used to be discrete indices $i, j, \ldots$ of matrices and tensors (see Section 2.2.3) are now continuous labels represented by group elements $g_1, \ldots, g_d$.

$$\begin{aligned} \varphi : G^d &\to \mathbb{K}\,, \\ g_I &\mapsto \varphi(g_I) \equiv \varphi(g_1, \ldots, g_d)\,, \end{aligned} \tag{51}$$

where $\mathbb{K}$ can be $\mathbb{R}$ or $\mathbb{C}$ and g_I is a shorthand notation for all the group elements. Here we focus on theories that mimic the simplicial construction of the spin foam models of Section 2.2.4; thus, we consider GFTs that reproduce the combinatorics of a d-simplex by gluing $(d-1)$-simplices, just like in tensor models. Note that in general one can go beyond the choice of simplicial gravity in spin foams [237] and equivalently in GFTs [304] by considering models compatible with the canonical theory of LQG, where graphs have nodes of arbitrary valency (and hence represent general polyhedra, not just simplices).

In the specific case of GFTs applied to four-dimensional simplicial quantum gravity, $d = 4$ and one usually takes G to be the local gauge group of general relativity: G is typically $SO(3,1)$ or $SL(2,\mathbb{C})$ in the Lorentzian case, $SO(4)$ or $Spin(4)$ in the Euclidean case, or their rotation subgroup $SU(2)$ which is the gauge group of loop gravity (i.e., the Ashtekar–Barbero formulation of classical general relativity [50]). To implement a notion of discrete gauge invariance in the resulting simplicial gravity description, one usually requires invariance of the field under the right diagonal group action,

One can equivalently require invariance under the left action; this choice is purely conventional.

$$\varphi(g_I) \equiv \varphi(g_1, \dots, g_d) = \varphi(g_1 h, \dots, g_d h)\,, \qquad \forall h \in G\,. \tag{52}$$

This invariance is a way to ensure that the parallel transports g_I only encode gauge invariant data. Heuristically, interpreting $\varphi(g_I)$ as a $(d-1)$-simplex, the group elements are associated with its d boundary "faces" (of dimension $d-2$), and define a connection whose curvature is concentrated on the $(d-2)$-simplices as in Regge calculus.

A GFT is then specified by an action, which has the general form

$$S[\varphi, \bar{\varphi}] = \int \mathrm{d}^d g\, \mathrm{d}^d g'\; \bar{\varphi}(g_I) K(g_I, g'_I) \varphi(g'_I) + V[\varphi, \bar{\varphi}]\,, \tag{53}$$

where for a real field $\bar{\varphi} = \varphi$. Here $\int \mathrm{d}^d g$ stands for an integration over d copies of the group (i.e., over all the group elements g_I), using the Haar measure normalised to unity (we will only be interested in compact groups in this thesis).[16] The action is therefore split into a quadratic part and an interaction part $V[\varphi, \bar{\varphi}]$, which is in general a non-linear potential term. From the action (53) one can write down the classical equations of motion

$$\begin{aligned} \frac{\delta S[\varphi, \bar{\varphi}]}{\delta \varphi(g_I)} &= 0\,, \\ \frac{\delta S[\varphi, \bar{\varphi}]}{\delta \bar{\varphi}(g_I)} &= 0\,; \end{aligned} \tag{54}$$

then, specific choices for K and V in $S[\varphi, \bar{\varphi}]$ will yield different GFTs and hence different classical equations. Following tensor models, one could choose the simplest quadratic kernel, i.e., the identity kernel[17] $K(g_I, g'_I) = I(g_I, g'_I)$, so that the trivial kinetic term $\sim \int \mathrm{d}^d g\; \bar{\varphi}(g_I)\varphi(g_I)$

16 The Haar measure can also be defined for *locally compact* groups such as $SL(2,\mathbb{C})$. However, extra care is required since, while both left and right invariant, and unique up to scaling, the measure is not finite (see [346] for details); GFT models based on a non-compact group then need to be regularised in some way, as discussed in [183].

17 With identity kernel we mean $I(g_I, g'_I)$ such that $\int \mathrm{d}^d g\, I(g_I, g'_I) f(g'_I) = f(g_I)$.

ensures an identification of the group elements associated to the boundary $(d-2)$-simplices. However, GFT renormalisation studies [63, 64, 103] suggest that radiative corrections generate a Laplacian term which should be included in $K(g_I, g'_I)$ as

$$K(g_I, g'_I) = (m^2 + M^2\Delta) I(g_I, g'_I)\,, \tag{55}$$

where m^2 and M^2 are coupling constants and the Laplace–Beltrami operator Δ on G^d is given by

$$\Delta = \sum_{I=1}^{d} \Delta_{g_I}\,. \tag{56}$$

The interaction term $V[\varphi, \bar{\varphi}]$ in (53) is responsible for gluing the $(d-1)$-simplices described by the field φ to form a d-simplex, just as in tensor models. This means that an appropriate interaction term consists of products of fields that are paired according to a pattern which encodes the combinatorics of a d-simplex. For example, the potential of the prototype GFT for quantum gravity,[18] which deals with a real field φ in four dimensions (i.e., $I = 1, \ldots, 4$), reads

Here, $\mathrm{d}^{20}g$ stands for an integration over $5 \cdot 4 = 20$ group elements (recall that g_I is a shorthand notation for $g_1, \ldots, g_4$ (52)).

$$V[\varphi] = \frac{\lambda}{5!} \int \mathrm{d}^{20}g \; V(g_I^1, ..., g_I^5) \prod_{a=1}^{5} \varphi(g_I^a)\,, \tag{57}$$

where λ is a coupling constant and $V(g_I^1, ..., g_I^5)$ is a product of ten Dirac delta distributions on the group, imposing appropriate matching between group elements appearing as arguments of the fields $\varphi(g_I^i)$, in order to encode the pattern of gluings needed to form a 4-simplex out of five tetrahedra. Such an interaction should allow the structure of four-dimensional spacetime to emerge from the dynamics of the theory, as we will explicitly see in Section 2.3.2.

Except for where clearly stated, we will mostly focus on GFTs for simplicial gravity in four dimensions, and we will fix $G^d = SU(2)^4$ (this will be the main setup in Part II where the original results are presented). Then, the geometrical interpretation of GFT obtained by associating a 3-simplex (i.e., a tetrahedron) with the field $\varphi(g_I)$, is corroborated by the spin networks of canonical LQG (see Section 2.2.4). Indeed, in a dual picture, one can think equivalently of $\varphi(g_I)$ as an abstract node with four links labelled by $SU(2)$ arguments, which is

18 The first four-dimensional GFT was given by Ooguri in [293]; as we will show in Section 2.3.2 this corresponds to the BF theory (48), and can be used to define the proper GFT analogue of the Barrett–Crane spin foam model of Section 2.2.4.

equivalent to the way in which four-valent spin network nodes represent geometric tetrahedra in LQG. This is clarified by the Peter–Weyl theorem, which allows to deal with spin labels rather than group arguments (see Figure 7).

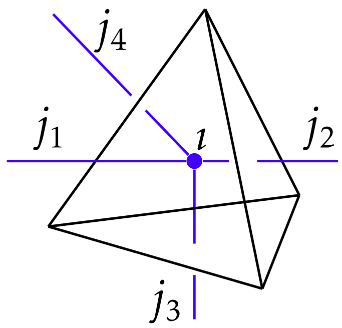


Figure 7: An excitation of the field φ can be interpreted as an open spin network vertex or a tetrahedron. This interpretation will play a central role in the canonical framework of Chapter 3.

PETER–WEYL DECOMPOSITION. As mentioned in Section 2.2.4, the group $SU(2)$ is compact and its irreducible unitary representations are finite-dimensional (cf. (28)) and labelled by half-integers j called spins. Then, it follows from the Peter–Weyl theorem[19] that $\varphi(g_I)$ can be expanded in field modes for a suitable choice of basis elements on $L^2(SU(2), d\mu_{\text{Haar}})$ as [319]

The mode expansion (58) is the analogue of the Fourier decomposition for standard field theories, and the "spin representation" can be interpreted as the momentum space for GFTs.

$$\begin{aligned}\varphi(g_I) &= \sum_J \varphi_J D_J(g_I)\,,\\ D_J(g_I) &= \sum_{n_I} \iota^{j_I,\kappa}_{n_I} \prod_{a=1}^{4} \sqrt{2j_a+1} D^{(j_a)}_{m_a n_a}(g_a)\,.\end{aligned} \tag{58}$$

Here φ_J are complex functions (possibly subject to reality conditions[20]) and the compact notation for the modes

$$\pm J = (j_I, \pm m_I, \kappa) \tag{59}$$

encodes representation (or spin) labels $j_I \in \mathbb{N}_0/2$, magnetic indices $m_I, n_I \in [-j_I, j_I]$ and intertwiner labels κ. In the convolution $D_J(g_I)$, $\iota^{j_I,\kappa}_{n_I} \in \mathcal{H}_4$ (cf. (30)) are intertwiners for the spins j_I (appearing in (58) because of property (52) and (44)), and $D^{(j)}_{mn}(g)$ are Wigner D-matrices for the irreducible unitary representations of $SU(2)$. One can picture

19 Recall that the Peter–Weyl theorem provides an orthonormal basis for $L^2(SU(2), d\mu_{\text{Haar}})$, specifically given by the Wigner D-matrices (cf. footnote 8).

20 If the field is real, the complex Peter–Weyl coefficients satisfy the reality condition [186, 293] $\bar{\varphi}_J = (-1)^{\sum_I (j_I - m_I)} \varphi_{-J}$.

the four spins j_I (and their corresponding magnetic indices) as living on the links emerging from the four-valent node, while $\imath$ lives on the node itself (cf. Figure 7).

The mode decomposition (58) shifts the focus from group variables to the more convenient spin labels, so that for example by denoting $K_{JJ'} = \int \mathrm{d}^d g\, \mathrm{d}^d g'\, \overline{D}_J(g_I) K(g_I, g'_I) D_{J'}(g'_I)$, the GFT action (53) can be written in a compact way as

$$S[\varphi, \bar{\varphi}] = \sum_{JJ'} \bar{\varphi}_J K_{JJ'} \varphi_{J'} + V[\varphi, \bar{\varphi}] \,. \tag{60}$$

For later convenience, we note that one can insert the expression (55) for $K(g_I, g'_I)$ in $K_{JJ'}$ and explicitly find (restricting to the free part of the action for simplicity)

$$S_{\text{free}}[\varphi, \bar{\varphi}] = \sum_J \bar{\varphi}_J K_J \varphi_J = \sum_J \bar{\varphi}_J \left(m^2 - M^2 \sum_{I=1}^{d} j_I(j_I + 1) \right) \varphi_J \,, \tag{61}$$

where we used the orthonormality of the basis elements in (58), namely $\int \mathrm{d}^d g\, \overline{D}_J(g_I) D_J(g_I) = 1$, and the fact that the $SU(2)$ Laplace–Beltrami operator acts as a Casimir on Wigner matrices, $\Delta_g D^{(j)}_{mn}(g) = -j(j+1) D^{(j)}_{mn}(g)$. The equations of motion obtained from the free action (61), $K_J \varphi_J = K_J \bar{\varphi}_J = 0$ (cf. (54)), will be useful in later chapters.

2.3.2 *Path integral quantisation*

Traditionally, the quantum dynamics for group field theories are defined by means of their partition function Z_{GFT}. Considering a general interaction term as $V = \sum \lambda_i V_i$ for some coupling constants λ_i, one can write the path integral for GFT as

$$Z_{\text{GFT}} = \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi}\; e^{-S[\varphi, \bar{\varphi}]} = \sum_{\Gamma} \frac{\prod_i \lambda_i^{n_i(\Gamma)}}{\text{sym}(\Gamma)} \mathcal{A}_\Gamma \,, \tag{62}$$

where in the case of a real-valued field (usually adopted to match with spin foam models) one drops the integration over $\bar{\varphi}$. Here, Γ denotes Feynman diagrams, $n_i(\Gamma)$ is the number of vertices of the interaction type i, $\mathcal{A}_\Gamma$ is the Feynman amplitude associated to Γ and the factor $\text{sym}(\Gamma)$ is the order of the symmetry group that leaves the graph Γ invariant.

The path integral for GFT (62) is conventionally defined in the literature (see, e.g., [295, 296]) as mimicking the statistical structure of matrix models (cf. (23)); however, one could also use $\mathrm{i}S[\varphi, \bar{\varphi}]$ at the exponent to define a proper quantum theory which is about transition amplitudes (and not mere probabilities). This choice is not expected to alter the formal perturbative expansion in a significant way (the amplitudes $\mathcal{A}_\Gamma$ would be unchanged), and aligns better with the canonical framework we introduce in Chapter 3. In this respect, note that the dichotomy between $-S[\varphi, \bar{\varphi}]$ and $\mathrm{i}S[\varphi, \bar{\varphi}]$ has little to do with an actual Wick rotation as there is no background time in this formalism, and the Lorentzian quantum gravity theory stemming out of GFT is obtained via the choice of the underlying group (e.g., $SL(2, \mathbb{C})$ rather than $SO(4)$, as mentioned in previous sections).

GFTS AND SPIN FOAMS. Remarkably, the sum (62) is over graphs that for suitable choices of V can be seen as discrete "histories of geometry", and whose Feynman amplitudes $\mathcal{A}_\Gamma$ are in correspondence with spin foam amplitudes defined in (35) [123, 330]. In this sense, the Feynman amplitudes of a group field theory can be associated with a discrete quantum gravity path integral, and the expansion (62) generates a sum over 2-complexes (or discrete spacetime histories), weighted by the couplings λ_i of the interaction terms. The equivalence between spin foam models and GFTs goes both ways. On the one hand, each Feynman graph of a group field theory is given by a spin foam, namely a 2-complex with faces dressed by representation labels (see Section 2.2.4). On the other hand, any local spin foam model (35) can be obtained from a GFT expansion [160, 330]. This implies that the GFT approach can be seen as a "larger framework" in which spin foam models are embedded, since a field theory contains more information and structure than its perturbative expansion [295].

More precisely, the Feynman diagrams Γ are dual graphs to the cellular complexes $\mathcal{C}$.

It is important to note that (62) implicitly involves multiple types of (informal) summation. We are summing over geometric data (i.e., group elements or representations, just like in spin foams) as well as over different Feynman graphs; in this sense, GFT automatically realises the sum we anticipated in (36). Crucially, we are not only summing over all triangulations for a given topology, but also over all topologies. This is guaranteed as any simplicial complex of arbitrary topology can be obtained by an appropriate (local) gluing of fundamental simplicial building blocks. In this sense, GFT provides a fully background-independent theory of (spacetime) geometry, generalising and unifying the ideas of all the models described in previous sections.

OOGURI MODEL. As a first instructive example, let us consider the case of a $4d$ BF theory and its corresponding GFT formulation, firstly introduced by Ooguri [293].[21] To link with the BC model, we here consider a variation of the Ooguri model where the real-valued field φ (51) is defined over the group manifold $SO(4)^4$ instead of $SU(2)^4$. Choosing the trivial kinetic term, the action (53) for the Ooguri model is given by

$$S[\varphi] = \frac{1}{2}\int \mathrm{d}^4 g\ \varphi^2(g_1, g_2, g_3, g_4) + V[\varphi]\,, \tag{63}$$

where the interaction term $V[\varphi]$ pairs the arguments of five copies of the field in such a way as to form a 4-simplex, namely

$$V[\varphi] = \frac{\lambda}{5!}\int \mathrm{d}^{10} g\ \varphi(g_1, g_2, g_3, g_4)\varphi(g_4, g_5, g_6, g_7)\times \\ \varphi(g_7, g_3, g_8, g_9)\varphi(g_9, g_6, g_2, g_{10})\varphi(g_{10}, g_8, g_5, g_1)\,. \tag{64}$$

This is an explicit expression for the potential in (57). Such a fifth-order interaction term has indeed the structure of a 4-simplex: if we represent each of the five field in the product as four strands (corresponding to $g_1, \dots, g_4$) and connect strands corresponding to the same group elements, we obtain Figure 8. This generalises the matrix model scenario depicted in Figure 4, where the propagator and the interaction vertex are visualised with multi-stranded lines (for simplicity we are only representing the ribbon graph and not the dual picture).

As anticipated in (62), since each of the four strands carries a representation of the group in momentum space (cf. (58)), the field theory defined by (63) is such that the Feynman expansion of the partition function is given by

$$Z_{\mathrm{Ooguri}} = \int \mathcal{D}\varphi\ e^{-S[\varphi]} = \sum_{\Gamma}\frac{\lambda^{n(\Gamma)}}{\mathrm{sym}(\Gamma)}\sum_{\mathbf{j}_f, \iota_e}\prod_{f\in\Gamma}\dim(\mathbf{j}_f)\prod_{v\in\Gamma}\{15\mathbf{j}\}_v\,, \tag{65}$$

meaning that for every given 2-complex Γ, the Feynman sum over momenta is precisely the spin foam model (48) defined on that 2-complex.

While the Ooguri mode clearly shows the relation between GFTs and spin foams, it still describes a topological BF theory, which can-

21 An even simpler model, called Boulatov model, considers a $3d$ BF theory. In [86], Boulatov points out for the first time the duality between spin foams and "quantum field theory on groups", later dubbed GFTs. Specifically, the Boulatov model clearly recasts the $3d$ Ponzano–Regge model (see previous sections) as a GFT over $SU(2)^3$.

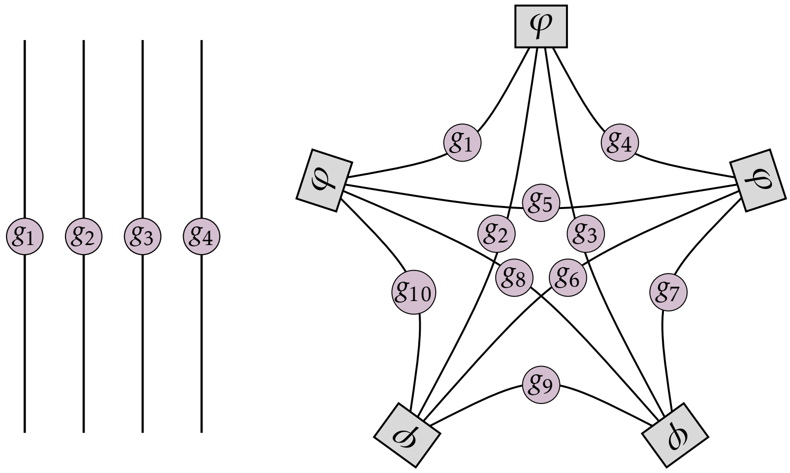


Figure 8: Left: the kinetic term dictates how a 3-simplex (tetrahedron) propagates and is represented by a collection of four strands. Right: Graph of five vertices, representing the skeleton of a 4-simplex as gluing of five tetrahedra (shared strands correspond to matching group arguments).

not correspond to a quantisation of general relativity. As explained in Section 2.2.4, one needs to reintroduce local degrees of freedom by implementing the simplicity constraints, which reduce the topological theory to gravity.

BARRETT–CRANE MODEL IN GFT. In the BC spin foam model, the simplicity constraints are imposed by restricting the state-sum corresponding to the $4d$ BF theory to simple representations of $SO(4)$. The analogue of this restriction can be formulated in the GFT language by considering the quotient group $SO(4)/SO(3)$, as explained below. Let us first consider the subgroup $SO(3)$ of $SO(4)$ and use the fact that a field $\psi(g)$ in $SO(4)$ is invariant under the action of $SO(3)$,

$$\psi(g) = \psi(gh)\,, \qquad \forall h \in SO(3)\,, \tag{66}$$

if and only if its mode expansion (cf. (58)) contains only simple irreducible representations [123, 340]. Then, adopting the compact notation for the (real-valued) field $\varphi(g_a, g_b, g_c, g_d) = \varphi_{abcd}$, we consider a GFT based on $SO(4)^4$ *without* requiring the property (52) and with action [123]

$$S[\varphi] = \frac{1}{2}\int \mathrm{d}^4 g\; P_g P_h \varphi_{1234}\; P_g P_h \varphi_{1234} + \frac{\lambda}{5!}\int \mathrm{d}^{10} g\; \mathcal{V}[\varphi]\,, \tag{67}$$

where

$$\mathcal{V}[\varphi] = P_g P_h \varphi_{1234}\, P_g P_h \varphi_{4567}\, P_g P_h \varphi_{7389}\, P_g P_h \varphi_{96210}\, P_g P_h \varphi_{10851}\,. \quad (68)$$

Note that these projectors do not commute, and different orderings correspond to different variations of the BC model [49].

The projectors P_h and P_g in (67) and (68) are defined as

$$P_h \varphi(g_I) := \int_{SO(3)} \mathrm{d}^4 h\, \varphi(g_I h_I)\,, \quad (69)$$

and

$$P_g \varphi(g_I) := \int_{SO(4)} \mathrm{d}g\, \varphi(g_I g)\,. \quad (70)$$

While P_h projects from the group $SO(4)$ to the quotient space given by $SO(4)/SO(3)$ (thus imposing the wanted constraints on the representations in momentum space thanks to the property (66)), P_g ensures that gauge invariance is maintained. Note that if P_h is removed one obtains the action (63) of the Ooguri model, with the field satisfying (52) thanks to P_g.

As mentioned in Section 2.2.4, different variations of the BC model exist; in GFT, they can be constructed by inserting the projectors (69) and (70) in different combinations in the action [315, 340]. Their Feynman expansion has the general structure

$$Z^{\mathrm{BC}}_{\mathrm{GFT}} = \sum_{\Gamma} \frac{\lambda^{n(\Gamma)}}{\mathrm{sym}(\Gamma)} \sum_{j_f, \iota_e} \prod_{f\in\Gamma} \mathcal{A}_f \prod_{e\in\Gamma} \mathcal{A}_e \prod_{v\in\Gamma} \{10j\}_v\,, \quad (71)$$

again showing that starting from a field theory over groups one can obtain a sum over 2-complexes of Feynman amplitudes corresponding to the spin foam models (49). $\mathcal{A}_f$ turns out to be exactly $(2j+1)^2$ using the projectors P_g and P_h as in (67) and (68), while the variant with $\mathcal{A}_e$ given in (50) is obtained by inserting only P_g in the kinetic term and keeping the potential as in (68). Importantly, this variation of the BC model has finite Feynman amplitudes [311, 316].

Let us summarise the key points of the models presented here and their importance in the context of $4d$ quantum gravity. The sum over Feynman graphs of the Ooguri model (65), corresponding to the topological BF theory in four dimensions, is trivial and unnecessary. This is because of triangulation invariance (implying all terms in the sum are equal), and because all the degrees of freedom of a topological theory are already captured by a single triangulation. On the other hand, in the case of BC models (49), fixing a 2-complex reduces the number of degrees of freedom of the theory. Thus, a sum over 2-complexes (71), where each term contributes in a different way, is necessary to obtain a quantum theory of gravity that is related to general relativity. It is in this sense that the expansion of the GFT partition function provides a completion of the spin foam programme for quantum gravity models, since there is no need to fix a triangulation (or its dual 2-complex) and there is no cut-off on the number of degrees of freedom.

From a computational point of view, GFT models for a full theory of quantum gravity still remain formal as it is not clear how to practically calculate expansions like (62). In fact, it is already very difficult to compute the individual transition amplitudes $\mathcal{A}_\Gamma$ between quantum geometries on a fixed 2-complex (this is done numerically, and optimising codes performing such evaluations is a current focus of research [13, 17, 142–144, 200, 218]). For this reason, using to good advantage the versatility of the field theoretic framework provided by GFTs, a lot of recent work has focused on a *canonical quantisation* of group field theories, which we introduce in Chapter 3.

Finally, let us stress again that the correspondence between spin foams and GFTs is a generic feature. Any local spin foam model of the form (35) can be interpreted as a Feynman graph of a group field theory [160, 330]. This includes the Lorentzian analogues of the models discussed above [53, 234], as well as the more involved EPRL model [65, 251].

3 CANONICAL FORMULATION OF GROUP FIELD THEORY AND COSMOLOGY

This chapter presents the main formalism underlying the research results of Part II: a canonical framework for GFT and its cosmological sector. As we discussed at the end of Chapter 2, the traditional way of thinking of quantum GFT is through the path integral as given in (62) and its expansion into Feynman graphs and amplitudes. This path integral directly connects to the covariant (spin foam) setting of LQG, and it generalises simplicial quantum gravity approaches such as matrix and tensor models and Regge-like gravity. More recently however, a lot of work has focused on the *canonical quantisation* of GFT, with the main goals of connecting to the canonical setting for LQG and – most importantly for us here – extracting effective cosmological dynamics (which are more easily defined in a canonical setting). As we will see, this shift in perspective allows to use "second quantisation" techniques that reinterpret the graph nodes of LQG as quanta of an underlying QFT of quantum gravity. This picture makes it possible to extract relatively simple dynamical equations, which can then be applied to symmetry-reduced settings to describe effective cosmology from a full quantum gravity perspective.

After reviewing how to couple matter degrees of freedom to serve as relational frames in GFT (a strategy that is fairly common in background-independent approaches to quantum gravity), we give an overview of the two main frameworks that have been established in the GFT literature, based on an algebraic quantisation and on a deparametrised quantisation. We will first explain how to extract dynamics from these approaches, and then discuss the emergence of effective cosmological equations in some detail. In particular, we will show how one can obtain a relational Friedmann-like equation from GFT, emphasising that this reproduces the correct dynamics of a flat Friedmann–Lamaître–Robertson–Walker (FLRW) Universe in a suitable semiclassical regime, while resolving the Big Bang singularity thanks to purely quantum corrections.

3.1 COUPLING TO MATTER

Canonical quantisation techniques were initially applied to GFT [298, 299] without considering any matter coupling. These ideas were immediately adopted in simplified settings (corresponding to the cosmological sector of GFT) with the goal of obtaining effective dynamical equations for global observables [183, 187]. Using tools typically belonging to "second quantised frameworks" (e.g., condensate-like states in a Fock space formulation of GFT, as we will describe later), the idea was to contrast the effective dynamics of observables such as the total (spatial) volume of a given geometry with globally homogeneous cosmological (vacuum) models of GR.[1]

A breakthrough in this line of research came thanks to the insight of *relational* dynamics, when it was realised that one could couple matter degrees of freedom to GFT and use them as relational frames (specifically clocks). In particular, by coupling a massless scalar field to serve as internal time, a relational volume observable (corresponding to the volume of space for a given value of the scalar field) was shown to satisfy the Friedmann dynamics of general relativity at low energies while also replacing the classical Big Bang singularity by a bounce [305, 306]. As we will explicitly see in Section 3.3, this is a very robust result of GFT, and similar findings have been obtained using different methods and from different starting points [5, 184, 186, 272, 379].

In approaches to quantum gravity where the continuum manifold is replaced by discrete structures (such as simplices and simplicial complexes), coupling to matter is usually achieved by enriching the nodes of the underlying graph (or their dual spatial building blocks, cf. Figure 7) with a label for the matter degrees of freedom. Then, one interprets the matter degrees of freedom as taking the values specified by such labels in the corresponding (discrete analogues of) space points. The idea of using matter fields as a reference frame was already adopted in other quantum gravity frameworks (see, e.g., [140, 189, 191, 231] for LQG references and [44, 167, 174, 278] for other approaches), and ultimately stems from models in classical GR and canonical quantum gravity [127], famously the Brown–Kuchař dust

1 Whether one can perform a canonical quantisation for pure gravity in GFT (i.e., without any coupling to matter) still remains an open question. We will return to these aspects in Chapter 4, where we introduce a new effective observable (corresponding to a cosmological anisotropy parameter) that might allow for GFT models that do not require extra matter fields to be used as relational clocks.

model of [91]. In the case of cosmological settings, it is common to add a massless scalar field to be used as relational time; in GFT, this is achieved by simply adding an extra argument to the field φ, as we detail below.

MASSLESS SCALAR FIELD AS RELATIONAL CLOCK. In this thesis we will focus on the simple scenario where the matter content is only given by a massless scalar field. A group field φ coupled to a massless scalar field χ is defined by generalising (51) as[2]

From now on, we will refer to φ as the "group field" to distinguish it from the (matter) scalar field χ.

$$\begin{aligned} &\varphi: G^d \times \mathbb{R} \to \mathbb{K}\,, \\ &\varphi(g_I, \chi) = \varphi(g_I h, \chi)\,, \qquad \forall h \in G\,, \end{aligned} \tag{72}$$

where φ can be complex-valued or real-valued by choosing $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ respectively, and the second line in (72) imposes the usual gauge invariance as in (52). Everything we defined in Section 2.3.1 is generalised here in a straightforward manner to accommodate the scalar field χ in the domain of the group field φ. Importantly, the action (53) becomes

$$S[\varphi, \bar{\varphi}] = \int \mathrm{d}g\, \mathrm{d}g'\, \mathrm{d}\chi\; \bar{\varphi}(g_I, \chi) K(g_I, g'_I) \varphi(g'_I, \chi) + V[\varphi, \bar{\varphi}]\,, \tag{73}$$

with a potential that generalise (57) as

As usual, one can choose the functions $V(g_I^1, ..., g_I^5)$ and $\overline{V}(g_I^1, ..., g_I^5)$ in (74) that reproduce the preferred spin foam model, e.g., the EPRL model or the BC model described in Section 2.3.2.

$$\begin{aligned} V[\varphi, \bar{\varphi}] = &\frac{\lambda}{5!} \int \mathrm{d}^{20}g\, \mathrm{d}\chi\; V(g_I^1, ..., g_I^5) \prod_{a=1}^{5} \varphi(g_I^a, \chi) \\ &+ \frac{\lambda}{5!} \int \mathrm{d}^{20}g\, \mathrm{d}\chi\; \overline{V}(g_I^1, ..., g_I^5) \prod_{a=1}^{5} \bar{\varphi}(g_I^a, \chi)\,, \end{aligned} \tag{74}$$

for a complex φ.

As mentioned, the scalar field χ takes values on the nodes of the underlying four-dimensional discrete structure. Then, just like in standard QFT, it is natural to assume that the potential $V[\varphi, \bar{\varphi}]$ describes a local interaction in χ (unlike in the geometric $SU(2)$ degrees of freedom) by gluing five tetrahedra associated with the same scalar field label, so that there is a single value of χ associated with any 4-simplex. On the other hand, the "gradients" of the scalar field are associated with the shared boundaries of neighbouring 4-simplices on the lattice. This means that the scalar field χ "propagates" along the links

2 One easily extends the formalism to multiple scalar fields by adding more arguments to the group field [185]. For instance, to couple four scalar fields one can define $\varphi: G^d \times \mathbb{R}^4 \to \mathbb{K}$ to obtain relational coordinates for time and space [177].

between adjacent building blocks of geometry, and gradients of the scalar field must then be encoded in the GFT kinetic term $K(g_I, g'_I)$.

With this interpretation in mind, the GFT action is assumed to respect the symmetries of the Lagrangian density associated to a non-interacting, minimally coupled massless scalar field on a background with metric $g_{\mu\nu}$,

The signature of the metric is $(-,+,+,+)$.

$$\mathcal{L}_{\text{scalar}} = -\frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\, \partial_\mu\chi\partial_\nu\chi\,, \tag{75}$$

namely shift and sign reversal symmetries,

$$\begin{aligned} \chi &\to \chi + c\,, \\ \chi &\to -\chi\,, \end{aligned} \tag{76}$$

with $c \in \mathbb{R}$. It immediately follows that $K(g_I, g'_I)$ and $V(g^1_I, ..., g^5_I)$ (together with its complex conjugate $\overline{V}(g^1_I, ..., g^5_I)$) cannot explicitly depend on the field values χ, as this would break the shift symmetry. However, given that the scalar field changes across neighbouring 4-simplices, one should generically consider $K(g_I, g'_I)$ as a differential operator in χ, without derivatives of odd powers (so as to respect the sign reversal symmetry) [260, 305, 306]. The simplest choice is therefore to assume the minimal form

$$K(g_I, g'_I) = K^{(0)}(g_I, g'_I) + K^{(2)}(g_I, g'_I)\partial^2_\chi\,, \tag{77}$$

which is commonly adopted in the literature, both when studying cosmological models (see, e.g., [186, 379]) and phase transitions in GFT [275, 276]. The form (77) mirrors the structure of the kinetic term (55), with a constant "mass term" suggested by the relation to spin foams [330] and a Laplacian term, but here with respect to χ.[3] Indeed, as seen in Section 2.3.1, quantum corrections coming from renormalisation [63] can dictate the specific forms of $K^{(0)}$ and $K^{(2)}$, which must include a Laplacian term in the group arguments. Then, for GFT models coupled to a matter scalar field, one is led to consider a symmetrical expression which consistently contains second derivatives with

3 Within a broader class of (non-local) models one can in principle have higher derivatives with respect to χ [260, 305, 306] and choose $K(g_I, g'_I)$ of the following form

$$K(g_I, g'_I) = \sum_{n=0}^{\infty} K^{(2n)}(g_I, g'_I)\frac{\partial^{2n}}{\partial\chi^{2n}}\,.$$

In this perspective, (77) would be seen as an approximation in which the field φ is assumed to vary slowly and the contribution of the higher-derivative terms is small, i.e., $|K^{(2n)}/K^{(0)}| \ll |K^{(2)}/K^{(0)}|^n$ for $n > 1$.

respect to all arguments. For concreteness, one usually generalises (55) by taking $K(g_I, g'_I)$ of the form (see, e.g., [176, 183, 275, 276])

$$K(g_I, g'_I) = \left(m^2 - \partial_\chi^2 + M^2 \sum_{I=1}^{d} \Delta_{g_I} \right) I(g_I, g'_I)\,, \tag{78}$$

which clearly corresponds to $K^{(0)}(g_I, g'_I) = (m^2 + M^2 \sum_I \Delta_{g_I}) I(g_I, g'_I)$ and $K^{(2)}(g_I, g'_I) = -I(g_I, g'_I)$ in (77).

An important point to stress here is that by coupling to a massless scalar field we obtained a GFT action that has the same form of a traditional field theory (with base manifold $SU(2)^4 \times \mathbb{R}$ instead of the familiar $\mathbb{R}^4$ of spacetime), where χ now plays the role of the standard time coordinate. This will allow to define relational observables which evolve with respect to the values of the relational clock time χ.

SPIN REPRESENTATION. As explained in Section 2.3.1, for compact groups one can obtain a mode expansion of the field φ thanks to the Peter–Weyl theorem. Using the field defined in (72) yields the same expression as in (58), with the difference that now the mode functions $\varphi_J(\chi)$ depend on the matter field. Fixing $G^d = SU(2)^4$, the interpretation of the field $\varphi_J(\chi)$ as a four-valent spin network node remains as in Figure 7, but the corresponding tetrahedron is enriched with an extra label representing the matter field χ which "sits" on the node (representing the discrete analogue of a space point).

The choices $d = 4$ and $G = SU(2)$ define simplicial models of gravity which connect with the spin network formalism of LQG, and will be the ones used in the chapters of Part II.

As emphasised in Section 2.3.1, the Peter–Weyl decomposition allows us to get rid of group elements and simplify some expressions by means of the spin representation provided by (58). In particular, for future convenience, we note that (77) can be written as[4]

$$K_J = K_J^{(0)} + K_J^{(2)} \partial_\chi^2\,, \tag{79}$$

and the specific form (78) becomes

$$K_J = m^2 - \partial_\chi^2 - M^2 \sum_I j_I(j_I + 1)\,. \tag{80}$$

These expressions are nothing but generalisations of the kinetic terms appearing in (61) to the case in which a scalar field is coupled to GFT, and will play a crucial role in later sections.

4 The kinetic term $K(g_I, g'_I)$ is in general proportional to the identity kernel $I(g_I, g'_I)$, even if one does not select the specific form of $K^{(0)}(g_I, g'_I)$ and $K^{(2)}(g_I, g'_I)$ as in (78). Then, as mentioned around (60) and (61), one can use the property in footnote 17 and the fact that $\int \mathrm{d}^d g\, \overline{D}_J(g_I) D_J(g_I) = 1$ to write (79) (and hence (80)).

3.2 CANONICAL QUANTISATION OF GROUP FIELD THEORIES

To be rigorous, such a quantisation would represent the counterpart of a GFT path integral defined with i*S at the exponent, see remarks below* (62).

We now turn to the task of defining a canonical quantisation scheme for GFT, which can be seen as a complementary formulation to that of path integrals (cf. Section 2.3.2), just like for traditional field theories. As mentioned, this Hilbert space quantisation of GFT is motivated by similarities with the canonical setting of LQG and, more practically, is used to extract effective cosmological dynamics [183, 187].

Two main approaches to canonical quantisation have been established in the GFT literature: one based on a kinematical Fock space of nondynamical spin network-like states on which dynamical equations are imposed in a suitable (usually mean-field) approximation [298, 299], and one where a time variable is selected before quantisation and used to directly obtain a physical Fock space and a physical (relational) Hamiltonian [186, 379].[5] We denote these quantisation schemes *algebraic approach* and *deparametrised approach*, respectively, and we devote the rest of this section to delve into their details. Then, we will describe in Section 3.3 how they lead to similar effective cosmological dynamics, setting the stage for Part II, where we present our main results.

3.2.1 *Algebraic approach*

This formulation led to the suggestive statement: "GFTs are QFTs of spacetime, not on spacetime" [295].

Guided by the analogy between GFT and the spin network states of LQG (cf. Figure 7), one can construct a "second quantised framework" for group field theories that is based on a Fock space, just as for non-relativistic field theories. This was firstly proposed in [298, 299], and leads to reinterpreting spin network vertices as *GFT quanta*, which are created and annihilated by operators and suggest a Hilbert space formulation of GFT. This approach is dubbed "algebraic" because it is based on the construction of a *kinematical* Hilbert space of abstract states, among which physical states are selected by demanding *a posteriori* that they satisfy a constraint coming from the underlying theory.

The algebraic approach requires a complex-valued group field (72), with domain given by $SU(2)^4 \times \mathbb{R}$. Starting from the Peter–Weyl de-

5 While in GFT there is no Hamiltonian constraint associated with diffeomorphisms in time (but see Chapter 6 for a novel proposal related to this), these two approaches share some similarities with the LQG strategies of either imposing dynamical equations on an abstract Hilbert space [9–11, 120, 188], or directly defining a physical Hilbert space by choosing a matter (usually "dust") clock [140, 189, 191, 231].

composition ([58]) and its complex conjugate, one defines the quantum theory by promoting the field modes to operators $\hat{\varphi}_J(\chi)$ and $\hat{\varphi}_J^\dagger(\chi)$ and postulating their commutation relations according to bosonic statistics,

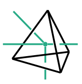
$$\left[\hat{\varphi}_J(\chi), \hat{\varphi}_{J'}^\dagger(\chi')\right] = \delta_{JJ'}\delta(\chi - \chi')\,. \tag{81}$$

One can then construct an abstract (unphysical) Fock space in the usual way, starting from a vacuum $|\varnothing\rangle$ (interpreted as "no geometry" state) satisfying

The Fock vacuum $|\varnothing\rangle$ is a state with no topological nor geometrical meaning and information.

$$\hat{\varphi}_J(\chi)|\varnothing\rangle = 0\,, \tag{82}$$

for all J and χ. The Fock space contains fundamental quanta (pictorially "atoms of space" [300]) created and annihilated by field operators, which are interpreted as equivalent to LQG (four-valent) spin network vertices. In particular, the one-quantum Hilbert space has states defined as excitations over the Fock vacuum



$$\Big| \quad \Big\rangle = |J,\chi\rangle = \hat{\varphi}_J^\dagger(\chi)|\varnothing\rangle\,, \tag{83}$$

representing (open) spin network vertices decorated with four Peter–Weyl labels (or group elements) and a scalar χ. From the one-quantum Hilbert space $\mathcal{H}_1^{\triangle} = L^2(SU(2)^4/SU(2) \times \mathbb{R})$,[6] one can construct the GFT Fock space as

$$\mathcal{F} = \bigoplus_{N=0}^{\infty} \text{sym}\left(\bigotimes_{i=1}^{N} \mathcal{H}_i^{\triangle}\right)\,, \tag{84}$$

where sym denotes symmetrisation, accounting for the choice of bosonic statistics of the field operators. One can also construct more complicated spin network states from (not uniquely defined) many-particle states; and hence, these Fock states can be related to kinematical states in loop quantum gravity [298, 299].

Let us stress that as in the kinematical Hilbert space of LQG, there is no notion of dynamics so far: this will only be implemented later by means of constraints, typically thanks to some approximations (e.g., in a mean-field regime [183, 187, 305], similarly to what happens in condensed matter physics). Indeed, the operators $\hat{\varphi}_J(\chi)$ and $\hat{\varphi}_J^\dagger(\chi)$

6 Forgetting about the $\mathbb{R}$ factor encoding the χ label, we point out that $L^2(SU(2)^4/SU(2)) = \text{Inv}\left[\bigotimes_I^4 \mathcal{H}_{j_I}\right]$ is precisely the space of quantum tetrahedra ([30]), where the quotient is due to property ([52]).

are not thought of as evolving in time, but as separate independent operators for each value of χ; furthermore, the states (83) should be interpreted as unphysical quantum tetrahedra (or spin network-like states), on which dynamical equations are yet to be imposed.

In this "second quantised" setup, one usually defines general one-body operators on the kinematical Hilbert space

$$\hat{\mathcal{O}}(\chi) = \sum_J \mathcal{O}^J \, \hat{\varphi}_J^\dagger(\chi)\hat{\varphi}_J(\chi) \,, \tag{85}$$

where $\mathcal{O}^J$ is the matrix element of the desired "first quantised" operator (usually imported from LQG) evaluated on a single spin network node. For example, the number operator is given by

$$\hat{N}(\chi) = \sum_J \hat{\varphi}_J^\dagger(\chi)\hat{\varphi}_J(\chi) \,, \tag{86}$$

and describes the number of "particles of geometry" (cf. (83)) *relationally*, in the sense that it is given as a function of the matter field χ associated to the creation and annihilation operators. Again: at this kinematical level, χ dependence does not represent time evolution yet (and the various $\hat{N}(\chi)$ defined for different χ are independent). Another important relational operator that encodes geometric information is the GFT volume operator,

$$\hat{V}(\chi) = \sum_J v_J \hat{\varphi}_J^\dagger(\chi)\hat{\varphi}_J(\chi) \,, \tag{87}$$

where the v_J's correspond to the volumes of quanta created by $\varphi_J^\dagger(\chi)$. These volume values can be formally obtained from a geometrical quantisation of tetrahedra in terms of $SU(2)$ recoupling theory [43, 51, 67], and have been thoroughly described in the LQG literature (see, e.g., [32, 125, 342] and [68, 69] for more recent methods).[7] The operator (87) heuristically describes a *global* notion of spatial volume seen as the sum of many discrete building blocks carrying their own (quantum) volume.

While the relational operators (86) and (87) describe the number and the volume at given values of the matter field χ, one can define the *total* number operator as

$$\hat{N} = \sum_J \int \mathrm{d}\chi \, \hat{\varphi}_J^\dagger(\chi)\hat{\varphi}_J(\chi) \,, \tag{88}$$

7 In Chapter 4 we will provide more details on the spectrum and eigenstates of the LQG volume operator, reviewing how it acts on four-valent spin network nodes.

and similarly the total volume operator as $\hat{V} = \sum_J \int \mathrm{d}\chi\, v_J \hat{\varphi}_J^\dagger(\chi)\hat{\varphi}_J(\chi)$. Albeit χ-independent (and hence not relational in the sense explained above), these "fully integrated" operators are not distributional in nature as opposed to (86) and (87). Indeed, while expectation values of (86) and (87) in mean-field coherent states appear well-behaved (and provide excellent agreement with usual cosmology [305, 306], as we will review in Section 3.3), their distributional nature becomes clear when studying higher powers as divergences appear in the computation of several physically relevant quantities [40, 41].[8] In light of these issues, operators of the form (88) play an important role in a novel strategy adopted within the algebraic approach, where relational dynamics is introduced by considering the special class of *coherent peaked states* [272, 273]. Such states provide a proposal for dealing with divergences and define relational dynamics in an alternative way, as we review at the end of Section 3.3.1.

Moreover, one can define the scalar field operator [272, 305, 306]

$$\hat{X} = \sum_J \int \mathrm{d}\chi\, \chi\, \hat{\varphi}_J^\dagger(\chi)\, \hat{\varphi}_J(\chi)\,, \tag{89}$$

and its momentum operator,

$$\hat{\Pi} = -\mathrm{i}\sum_J \int \mathrm{d}\chi\, \left(\hat{\varphi}_J^\dagger(\chi)\partial_\chi \hat{\varphi}_J(\chi)\right)\,. \tag{90}$$

These satisfy

$$[\hat{X}, \hat{\Pi}] = \mathrm{i}\hat{N}\,, \tag{91}$$

with $\hat{N}$ given in (88), meaning that they are not canonically conjugate. One can indeed make the observation that both operators are "extensive" whereas one would expect one intensive and one extensive quantity to form a canonical pair (see [181] for a related discussion). As we will see, these considerations (and in particular the commutator (91)) will lead to an intensive *effective* definition of the scalar field operator in the framework of coherent peaked states [272, 273].

DYNAMICS. Since GFTs are defined through an action functional (cf. (73)), second quantisation tools of quantum many-body systems suggest that the quantum dynamics be governed by *operator* equations obtained from the classical theory. Given the absence of a Hamil-

8 We will return to these points in Chapter 5, where we will see that for less simple states even the expectation values of $\hat{N}(\chi)$ and $\hat{V}(\chi)$ are divergent [100].

tonian formulation for GFT, one here directly promotes the Euler–Lagrange equations (54) to operators and views them as quantum constraints. Then, the equations of motion are implemented at the quantum level by requiring that physical states $|\Psi\rangle$ satisfy [183, 187, 305, 306]

As we will see, for a simple choice of action one can find exact solutions to this constraint equation [100] (see also [183] for an exact solution of an interacting theory).

$$\frac{\delta S[\hat{\varphi}, \hat{\varphi}^\dagger]}{\delta \hat{\varphi}_J^\dagger(\chi)} |\Psi\rangle = 0\,, \tag{92}$$

and a second equation obtained from the variation with respect to $\hat{\varphi}_J(\chi)$. Formally, one should define physical states as annihilated by both such constraints; however, since these are Hermitian conjugates, they are in general expected to be second-class constraints with no joint solution. In other words, the constraint in (92) and its Hermitian conjugate do not commute, and hence there is generically no common state that is annihilated by both. We point out that this issue is essentially ignored in the literature and physical states are usually defined as those who satisfy (92); we will do the same here, noting that this concern goes away in most applications where one considers expectation values, as explained below.[9]

(92) looks similar to the type of equation used in a Dirac quantisation of constrained systems. However, unlike in a Dirac quantisation, one here assumes that physical states are elements of the original Hilbert space.[10] This assumption is strictly speaking inconsistent since such states have infinite norm in the original inner product. This type of divergence is different from the one rising due to distributional operators encountered earlier; we postpone a proper account of such technical issues to Chapter 5, where we will provide all the details and extend the discussion beyond the literature. Note that in a theory with constraints such as (92), not every Hermitian operator automatically classifies as an observable; indeed, observables should preserve the physical state space. However, as we will see, these concerns are not taken into account when discussing cosmological models where operators such as the volume (87) or the matter scalar field (89) are simply assumed to be observables. We will construct a new way of imposing dynamics for constrained GFT models in Chapter 6,

9 Moreover, in a system with second-class constraints, say $\hat{c}$ and $\hat{c}^\dagger$, one could tweak the quantisation scheme and define physical states as those annihilated by the combination $\hat{c}^\dagger\hat{c}|\Psi\rangle = 0$. In our case, this would correspond to imposing the modulus squared of the quantum equation of motion.

10 For a more standard Dirac quantisation of (free) GFT that uses group averaging to define a physical Hilbert space, see [178].

where one can explicitly show that observables commute with the constraints of the theory.

Interestingly, working in a perturbative regime, one can bridge the operator formalism of this chapter with the path integral formulation of Section 2.3.2. In a path integral framework, the quantum dynamics can be given in terms of the Schwinger–Dyson equations for correlation functions [182, 183],

Working perturbatively means implicitly assuming that the GFT interaction is small. This is the regime in which cosmological dynamics can be extracted, as we will see later.

$$0 = \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \frac{\delta}{\delta \bar{\varphi}} \left(\mathcal{O}\, e^{-S[\varphi,\bar{\varphi}]} \right) = \left\langle \frac{\delta \mathcal{O}}{\delta \bar{\varphi}} - \mathcal{O} \frac{\delta S[\varphi, \bar{\varphi}]}{\delta \bar{\varphi}} \right\rangle , \tag{93}$$

where $\mathcal{O} = \mathcal{O}[\varphi, \bar{\varphi}]$ is any polynomial functional of the field and its complex conjugate (a similar equation is obtained by taking functional derivatives with respect to $\bar{\varphi}$). These are a formal way of encoding the quantum dynamics as they give an infinite number of relations, obtained by specifying $\mathcal{O}$ [289]. Since states that are solutions of the quantum theory must satisfy (93), the Schwinger–Dyson equations can be seen as an alternative way (other than (92)) of identifying physical states among the kinematical ones. In practice, one truncates this infinite tower of equations by considering a few very simple choices for $\mathcal{O}$, the most common being the identity. Indeed setting $\mathcal{O} = 1$ in (93) amounts to requiring that the operator version of the Euler–Lagrange equations hold on average,

One can convert (93) into an operator equation formally replacing the fields with $\hat{\varphi}$ and $\hat{\varphi}^\dagger$ (and choosing some operator ordering).

$$\left\langle \Psi \left| \frac{\delta S[\hat{\varphi}, \hat{\varphi}^\dagger]}{\delta \hat{\varphi}^\dagger} \right| \Psi \right\rangle = 0 , \tag{94}$$

which can be seen as an approximation (i.e., a weaker version) of (92).[11] As we will see, (94) allows to easily extract effective dynamical equations (in particular for cosmological settings), usually achieved thanks to suitable coherent states.

Let us stress that compared to requiring some expectation value to vanish, (92) provides a stronger condition for defining *exactly* physical states. We shall see in Chapter 5 that to be a solution of the constraint (92) (and hence physical) a generic state $|\Psi\rangle$ must satisfy strict conditions. We will refer to the states that satisfy one of the Schwinger–Dyson equations (93) (in particular the simplest one given by (94)) as *approximately* physical states.

11 Since (94) only requires expectation values to vanish, one can generically impose also the second equation with the Hermitian conjugate operator (i.e., where the derivative is taken with respect to $\hat{\varphi}_J(\chi)$), even if the constraints do not commute.

To give explicit examples that will be relevant for later applications, let us consider the GFT action (in the spin representation) with kinetic term (79). The first Schwinger–Dyson equation (94) then reads

$$\left\langle \Psi \left| \left(K_J^{(0)} + K_J^{(2)} \partial_\chi^2 \right) \hat{\varphi}_J(\chi) + \frac{\delta V[\hat{\varphi}, \hat{\varphi}^\dagger]}{\delta \hat{\varphi}_J^\dagger(\chi)} \right| \Psi \right\rangle = 0 \,. \tag{95}$$

In the case of weak interactions, such a non-linear equation can be dealt with by using *mean-field theory* techniques, typical of Bose–Einstein condensates. More precisely, one proceeds by approximating the operators by classical fields,

$$\begin{aligned} \hat{\varphi}_J(\chi) &= \varphi_J(\chi)\, \mathbb{I} + \delta\hat{\varphi}_J(\chi) \,, \\ \hat{\varphi}_J^\dagger(\chi) &= \bar{\varphi}_J(\chi)\, \mathbb{I} + \delta\hat{\varphi}_J^\dagger(\chi) \,, \end{aligned} \tag{96}$$

where $\mathbb{I}$ is the identity operator, and one assumes that the expectation values of the fluctuations $\delta\hat{\varphi}_J(\chi)$ and $\delta\hat{\varphi}_J^\dagger(\chi)$ over the mean field are small. After normal ordering, in such an approximation (95) reduces to the classical field equation (and corresponds to the GFT analogue of the Gross–Pitaevskii equation of condensed matter physics [320]). With the aim of extracting semiclassical dynamics (as we will see in Section 3.3 for the cosmological case), one usually justifies the mean-field approximation in a precise sense by adopting Fock coherent states, which effectively allow to replace the operators with their classical counterpart. Moreover, since the Gross–Pitaevskii approximation requires the interaction $V[\hat{\varphi}, \hat{\varphi}^\dagger]$ to be subdominant, one often neglects the potential and ends up with a simple equation of the form

$$(\partial_\chi^2 - \omega_J^2) \langle \Psi | \hat{\varphi}_J(\chi) | \Psi \rangle = 0 \,, \tag{97}$$

where we defined the coupling

$$\omega_J^2 = -K_J^{(0)} / K_J^{(2)} \,. \tag{98}$$

As we will see in Section 3.3, (97) will provide the setup for obtaining effective cosmological dynamics. In general, the task in this approach is to find equations for the functions defining $|\Psi\rangle$, namely conditions that ensure the states are (either exact or approximate) solutions of the quantum dynamics. Thanks to the use of coherent (condensate-like) states typical of matter physics, the approximations sketched here will be associated with the interpretation that quantum cosmology emerges as "hydrodynamics of quantum gravity" [187, 300] (see also [301] for recent developments of such ideas).

To conclude this section, we note that for the specific choice of kinetic term given by (80) (see also (78)), the coupling (98) can be given explicitly as

$$\omega_J = -\sqrt{m^2 - M^2 \sum_I j_I(j_I+1)}\,, \tag{99}$$

where the argument of the root must be positive in order to have a real coupling. This will be important for the anisotropic cosmological model studied in Chapter 4. It is worth pointing out that amongst all modes that contribute to the number and volume operator defined in (86) and (87), we will see that the dominant contribution is asymptotically given by the modes J for which $|\omega_J|$ takes a maximum, as proven in [176] for generic initial conditions. As can be seen from (99), these are the modes of lowest spins j_I. This is the reason why cosmological dynamics can be extracted from GFT by restricting to a single mode, as we will explain in Section 3.3.

3.2.2 *Deparametrised approach*

Given a classical GFT action, a canonical quantisation can be obtained by choosing a degree of freedom to parametrise the others *before* quantisation; here, the obvious candidate is the matter clock χ. In contrast to the algebraic approach, this "deparametrised" framework allows to write down a relational Hamiltonian, and hence perform a standard canonical quantisation (using the map (2) starting from a classical phase space structure). This approach was first developed in [186, 379], based on a real-valued GFT field whose Peter–Weyl modes satisfy $\bar{\varphi}_J(\chi) = (-1)^{\sum_I (j_I - m_I)} \varphi_{-J}(\chi)$, with $-J$ denoting sign reversal of magnetic indices (cf. footnote 20 and (59)).

Working in the spin representation and with a conventional $1/2$ factor as customary for real fields, the action (73) can be written as

$$S[\varphi] = \frac{1}{2} \int \mathrm{d}\chi \sum_J \varphi_{-J}(\chi) \left(K_J^{(0)} + K_J^{(2)} \partial_\chi^2 \right) \varphi_J(\chi) + V[\varphi]\,, \tag{100}$$

where we used the kinetic term (79). Note that $K_J^{(0)}$ and $K_J^{(2)}$ can in general be positive or negative. Without loss of generality, we take the kinetic term to be symmetric under $J \leftrightarrow -J$, i.e., $K_J^{(0)} = K_{-J}^{(0)}$ and $K_J^{(2)} = K_{-J}^{(2)}$ in the following. After integration by parts, one has

$$S[\varphi] = \frac{1}{2} \int \mathrm{d}\chi \sum_J \left(K_J^{(0)} \varphi_{-J}(\chi) \varphi_J(\chi) - K_J^{(2)} \partial_\chi \varphi_{-J}(\chi) \partial_\chi \varphi_J(\chi) \right) + V[\varphi] \,, \tag{101}$$

which is now just a function of field modes and their "time" derivatives, so that the Legendre transform to a relational Hamiltonian is straightforward. Indeed, one now introduces the conjugate momentum $\pi_J(\chi)$ to the group field $\varphi_J(\chi)$ as

$$\pi_J(\chi) = -K_J^{(2)} \partial_\chi \varphi_{-J}(\chi) \,, \tag{102}$$

and then performs the Legendre transform of the Lagrangian with respect to χ, which gives a relational Hamiltonian

$$H = -\frac{1}{2} \sum_J \left[\frac{\pi_J(\chi) \pi_{-J}(\chi)}{K_J^{(2)}} + K_J^{(0)} \varphi_J(\chi) \varphi_{-J}(\chi) \right] - V[\varphi] \,. \tag{103}$$

The GFT Hamiltonian (103) determines the dynamics of any observable $\mathcal{O}$ through Poisson brackets $\mathrm{d}\mathcal{O}/\mathrm{d}\chi = \{\mathcal{O}, H\}$, and indeed χ appears on the same footing as a background time parameter.

QUANTUM THEORY. Only at this stage, the group field and its momentum are promoted to operators with the canonical *equal-time* commutation relation

$$[\hat{\varphi}_J(\chi), \hat{\pi}_{J'}(\chi)] = \mathrm{i}\delta_{JJ'} \,. \tag{104}$$

The key difference with the previous approach is that these operators already satisfy dynamical equations, implemented through the Heisenberg equations of motion

We will be mainly working in the Heisenberg picture.

$$\mathrm{i}\frac{\mathrm{d}\hat{\mathcal{O}}}{\mathrm{d}\chi} = [\hat{\mathcal{O}}, \hat{H}] \,, \tag{105}$$

where $\hat{\mathcal{O}}$ is any operator and $\hat{H}$ is the quantum version of (103). This has a cost: we needed to specify our time variable once and for all from the very beginning to define the conjugate momentum of the field and the relational Hamiltonian.

One can now define creation and annihilation operators $\hat{a}_J^\dagger$ and $\hat{a}_J$ as in any bosonic field theory (denoting $\hat{\varphi}_J = \hat{\varphi}_J(0)$ and $\hat{\pi}_J = \hat{\pi}_J(0)$)

$$\hat{a}_J = \frac{1}{\sqrt{2\Omega_J}}(\Omega_J\,\hat{\varphi}_J + \mathrm{i}\epsilon_J\hat{\pi}_{-J})\,, \qquad \hat{a}_J^\dagger = \frac{1}{\sqrt{2\Omega_J}}(\epsilon_J\Omega_J\,\hat{\varphi}_{-J} - \mathrm{i}\hat{\pi}_J)\,, \tag{106}$$

Being only defined in reference to $\hat{\varphi}_J(0)$ and $\hat{\pi}_J(0)$, the ladder operators are χ-independent. In a sense, we are choosing $\chi = 0$ as the time when the Heisenberg (H) and the Schrödinger (S) pictures coincide, $\hat{a}_H(0) = \hat{a}_S = \hat{a}$.

with $\Omega_J = \sqrt{|K_J^{(0)}K_J^{(2)}|}$ and $\epsilon_J = (-1)^{\sum_I(j_I - m_I)}$. By construction these operators satisfy their own commutation relations,

$$\left[\hat{a}_J, \hat{a}_{J'}^\dagger\right] = \delta_{JJ'}\,. \tag{107}$$

Such ladder operators should not be confused with $\hat{\varphi}_J(\chi)$ and $\hat{\varphi}_J^\dagger(\chi)$ of Section 3.2.1 (see in particular (81)). Indeed, using $\hat{a}_J$ and $\hat{a}_J^\dagger$, one can construct a Fock space that is different from that introduced in the algebraic approach (84) since the states here are already interpreted as physical states, not subject to any constraints. Specifically, one again starts from a ground state $|0\rangle$ (interpreted as "no geometry state") such that

$$\hat{a}_J|0\rangle = 0\,, \tag{108}$$

for all J values, and excitations created by the ladder operators are interpreted as quanta of geometry. A one-particle state $|1\rangle = \hat{a}_J^\dagger|0\rangle$ represents a quantum tetrahedron (or four-valent node) with group-theoretic information J. The interpretation of quanta of geometry is on par with that of (83), but with the important difference that here the states are already physical.

The deparametrised approach allows to straightforwardly define relational observables analogous to (85). Indeed, working in the Heisenberg picture (cf. (105)), the χ-evolution of any operator $\hat{\mathcal{O}}$ is given by

$$\hat{\mathcal{O}}(\chi) = \hat{U}^\dagger(\chi)\,\hat{\mathcal{O}}(0)\,\hat{U}(\chi)\,, \tag{109}$$

where

$$\hat{U}(\chi) = e^{-\mathrm{i}\hat{H}\chi} \tag{110}$$

is the evolution operator and $\hat{H}$ is the Hamiltonian appearing in (105) (we will analyse the Hamiltonian in terms of ladder operators below).

To be rigorous, $\hat{a}_J(\chi) := \hat{U}^\dagger \hat{a}_J \hat{U}$ and $\hat{a}_J^\dagger(\chi) := \hat{U}^\dagger \hat{a}_J^\dagger \hat{U}$ should carry a label "H" (for Heisenberg), so as to distinguish them from the χ-independent ladder operators defining the Fock space; we omit this label for simplicity.

In particular, a time-dependent number operator is defined as

$$\hat{N}(\chi) = \sum_J \hat{a}_J^\dagger(\chi)\hat{a}_J(\chi); \tag{111}$$

similarly, the volume operator reads

$$\hat{V}(\chi) = \sum_J v_J \hat{a}_J^\dagger(\chi)\hat{a}_J(\chi)\,. \tag{112}$$

One could also simply define $\hat{N}(0) := \sum_J \hat{a}_J^\dagger \hat{a}_J$ and $\hat{V}(0) := \sum_J v_J \hat{a}_J^\dagger \hat{a}_J$ and use (109); we write the equivalent expressions (111) and (112) to make the resemblance to (86) and (87) more explicit.

While this quantisation is straightforward to obtain and interpret and the connection to the classical theory is clear throughout, one might raise the concern that χ appears only as a "classical" parameter with no quantum operators or fluctuations associated to it. This is a common concern with deparametrised approaches, as the expected covariance of the GFT formalism – the freedom to choose an arbitrary time parameter to express dynamics – might be broken by making a classical clock choice before quantisation (see, e.g., the general criticism of "tempus ante quantum" in [233, 257] and a more specific discussion for GFT in [272]). As we will see, the work presented in Chapter 6 resolves these concerns by embedding the deparametrised approach into a more covariant setting which allows for an arbitrary choice of evolution parameter.

RELATIONAL HAMILTONIAN. The quantum operator corresponding to the Hamiltonian (103),

$$\hat{H} = -\frac{1}{2}\sum_J \left(\frac{\hat{\pi}_J(\chi)\hat{\pi}_{-J}(\chi)}{K_J^{(2)}} + K_J^{(0)}\hat{\varphi}_J(\chi)\hat{\varphi}_{-J}(\chi) \right) - V[\varphi]\,, \tag{113}$$

is comprised of a sum-over-mode part (corresponding to the Hamiltonian of a free GFT) and an interaction potential. In this section we leave the potential $V[\varphi]$ generic, and we focus on the form that the first part in (113) can take in terms of the ladder operators (106) depending on the kinetic term K_J (cf. (79)) of the GFT action. In particular, we now show that it can be written as a sum of single-mode contributions.

We begin by noting that the first part in (113) is the sum of *two-mode* Hamiltonians, since every contribution to the sum depends on the modes J and $-J$. In other words, even in the free theory, the Peter–Weyl modes are *a priori* pairwise coupled such that the dynamics of the mode J is coupled with that of the mode $-J$. Since the term within brackets in (113) is symmetric with respect to exchanging J and $-J$ (recall that $K_J = K_{-J}$ by assumption), the sum receives the same contribution twice.[12] One thus obtains an overall factor of 2, so that we are interested in summing over two-mode components of the form

This section provides a somewhat novel extension of the literature as it deals with aspects regarding the Hamiltonian that were never clarified (see [101]).

$$\hat{H}_{(J,-J)} = -\frac{\hat{\pi}_J(\chi)\hat{\pi}_{-J}(\chi)}{K_J^{(2)}} - K_J^{(0)}\hat{\varphi}_J(\chi)\hat{\varphi}_{-J}(\chi)\,. \tag{114}$$

Excluding the rather fine-tuned modes for which $K_J^{(0)} = 0$, the contributions (114) can be of two different types, depending on the signs of $K_J^{(0)}$ and $K_J^{(2)}$. The modes for which these have the same sign (we denote this case with a single dash $'$) contribute with a two-mode Hamiltonian $\hat{H}'_{(J,-J)}$ that automatically decouples the mode J from $-J$ when working in the ladder operator basis. Indeed, by inverting (106) one can use

$$\begin{aligned} \hat{\varphi}_J &= \frac{1}{\sqrt{2\Omega_J}}\left(\epsilon_J \hat{a}^\dagger_{-J} + \hat{a}_J\right)\,, \\ \hat{\pi}_J &= \mathrm{i}\sqrt{\frac{\Omega_J}{2}}\left(\hat{a}^\dagger_J - \epsilon_J \hat{a}_{-J}\right)\,, \end{aligned} \tag{115}$$

to write the two-mode Hamiltonian (114) as

$$\begin{aligned} \hat{H}'_{(J,-J)} &= \epsilon_J\omega_J\left(\hat{a}^\dagger_J\hat{a}_J + \frac{1}{2}\right) + \epsilon_J\omega_J\left(\hat{a}^\dagger_{-J}\hat{a}_{-J} + \frac{1}{2}\right) \\ &= \hat{H}^{\mathrm{HO}}_J + \hat{H}^{\mathrm{HO}}_{-J}\,, \end{aligned} \tag{116}$$

where (cf. (98))

$$\omega_J = -\,\mathrm{sgn}\,(K_J^{(0)})\sqrt{|K_J^{(0)}/K_J^{(2)}|}\,. \tag{117}$$

That is, $\hat{H}'_{(J,-J)}$ decouples into the sum of two single-mode harmonic oscillator (HO) Hamiltonians, which have well-known properties (eigenvalue problem, spectral decomposition, *et cetera*).

12 The only exception is the case where $J = -J$, i.e., a mode with vanishing magnetic indices. This special case automatically contributes with a single-mode Hamiltonian so it does not need to be considered in the following discussion.

On the other hand, the modes for which $K_J^{(0)}$ and $K_J^{(2)}$ have opposite signs (we denote this case with a double dash $''$) contribute with a two-mode Hamiltonian $\hat{H}''_{(J,-J)}$ that takes the form

$$\hat{H}''_{(J,-J)} = \omega_J \left(\hat{a}_J^\dagger \hat{a}_{-J}^\dagger + \hat{a}_J \hat{a}_{-J} \right) , \tag{118}$$

where ω_J is given in (117). While this is still a two-mode operator coupling J and $-J$ (specifically, it is related to a two-mode *squeezing* operator), it is possible to diagonalise it as follows. First, we introduce the vector notation $\mathbf{a}^T = (\hat{a}_J, \hat{a}_{-J})$ and $(\mathbf{a}^\dagger)^T = (\hat{a}_J^\dagger, \hat{a}_{-J}^\dagger)$ which allows to rewrite (118) as

$$\hat{H}''_{(J,-J)} = \frac{1}{2}\omega_J \left((\mathbf{a}^\dagger)^T \sigma_1 (\mathbf{a}^\dagger) + (\mathbf{a})^T \sigma_1 (\mathbf{a}) \right) , \tag{119}$$

where here and in the following σ_i refers to the i-th Pauli matrix. Then, we perform a change of basis, $\mathbf{a} \to U\mathbf{a}$, by means of a unitary transformation

$$U = e^{-\mathrm{i}\frac{\pi}{4}\sigma_2} \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{i} \end{pmatrix} . \tag{120}$$

Finally, thanks to the property $e^{\mathrm{i}\frac{\pi}{4}\sigma_2}\sigma_1 e^{-\mathrm{i}\frac{\pi}{4}\sigma_2} = \sigma_3$, one can easily check that $U^T\sigma_1 U = \mathbb{I}_{2\times 2}$ so that $\hat{H}''_{(J,-J)}$ can be written in the diagonal form

$$\begin{aligned} \hat{H}''_{(J,-J)} &= \frac{1}{2}\omega_J \left((\hat{a}_J^\dagger)^2 + \hat{a}_J^2 \right) + \frac{1}{2}\omega_J \left((\hat{a}_{-J}^\dagger)^2 + \hat{a}_{-J}^2 \right) \\ &= \hat{H}_J^{\mathrm{SQ}} + \hat{H}_{-J}^{\mathrm{SQ}} , \end{aligned} \tag{121}$$

namely as the sum of two single-mode squeezing (SQ) Hamiltonians.

To summarise, all types of two-mode contributions $\hat{H}'_{(J,-J)}$ and $\hat{H}''_{(J,-J)}$ can be brought into a diagonal form ((116) and (121)); i.e., the Hamiltonians $\hat{H}_{(J,-J)}$ in (114) decouple into a sum of single-mode Hamiltonians for all modes. For the reasons shown above, it is convenient to define $\mathfrak{J}^{\mathrm{HO}}$ as the set of J such that $K_J^{(0)}$ and $K_J^{(2)}$ have the same sign and $\mathfrak{J}^{\mathrm{SQ}}$ as the set of J such that $K_J^{(0)}$ and $K_J^{(2)}$ have opposite signs. Then, it follows that the total GFT Hamiltonian (113) can be cast as

$$\hat{H} = \sum_J \hat{H}_J - V[\varphi] , \tag{122}$$

where, depending on the set J belongs to, $\hat{H}_J$ takes one of the two forms

$$\hat{H}_J = \begin{cases} \epsilon_J \omega_J \left(\hat{a}_J^\dagger \hat{a}_J + \frac{1}{2} \right) , & J \in \mathfrak{J}^{\text{HO}} \\ \frac{1}{2} \omega_J \left((\hat{a}_J^\dagger)^2 + \hat{a}_J^2 \right) , & J \in \mathfrak{J}^{\text{SQ}} . \end{cases} \tag{123}$$

As we will see in Section 3.3, the case of a squeezing Hamiltonian is particularly relevant for applications to cosmology, where the potential $V[\varphi]$ is usually neglected. In the case of a free GFT, the Hamiltonian for modes $J \in \mathfrak{J}^{\text{SQ}}$ does not leave the Fock vacuum invariant as squeezing leads to an exponentially growing number of particles under evolution in χ [5, 186, 379]. Since the volume operator (112) is closely related to the particle number, this provides a compelling picture for an expanding cosmological geometry arising from such an instability of the ground state, realising a type of "geometrogenesis" (the term was coined in [249] and then used in the GFT literature [183, 294, 297]). On the other hand, the particle number for modes $J \in \mathfrak{J}^{\text{HO}}$ is conserved under χ evolution, and this is interpreted as a volume that remains constant in time. Such stable modes quickly become insignificant compared to the unstable ones, which is why they ar usually ignored. Indeed, one can show that the modes associated with a squeezing Hamiltonian always dominate asymptotically [176]. In the next section we will explicitly see that the volume for a single mode $J \in \mathfrak{J}^{\text{SQ}}$ satisfies the correct effective Friedmann equation for *any* state in the theory [184]. This is in contrast with the way dynamics are encoded in the algebraic approach, since one there needs to use specific states that satisfy, e.g., (92) or (94) (and we will see that only coherent states seem to provide a viable option).[13]

In the literature one generally works with the two-mode Hamiltonian (118) adopting the "old basis" (prior to the transformation (120)). In this basis, particles of geometry are created in pairs with opposite values for the magnetic indices [186].

We remark that specialising to the kinetic term given by (80) the coupling (117) takes the form $\omega_J = -\sqrt{m^2 - M^2 \sum_I j_I(j_I+1)}$, just like in the algebraic approach (cf. (99)). In the deparametrised approach, $|\omega_J|$ plays the role of the rate of squeezing (often called squeezing parameter), which will in turn correspond to cosmological expansion; as mentioned, this is dominated by modes in which $|\omega_J|$ takes a maximum, namely those with the lowest spins j_I.

13 As we have seen in Section 3.2.1, the Fock space of the algebraic approach (84) is not physical, and it is natural to require states to satisfy some constraints. However, we will see that essentially no solutions to (92) or (94) are known beyond the simplest Fock coherent states, so that the physical state space is very limited.

3.3 EXTRACTING COSMOLOGICAL DYNAMICS

A major challenge for discrete approaches to quantum gravity is the derivation of an effective (emergent) continuum description which can be compared with classical GR or more general gravitational theories. The challenge arises on many levels, for instance in recovering the usual notions of a spacetime manifold from combinatorial structures [85]; recovering an effective description in terms of coordinates and restoring the continuum notion of diffeomorphisms or coordinate changes [136]; and understanding the intricate interplay between a continuum and semiclassical limit. Deriving such a description, however, is crucial for understanding the phenomenology of such quantum gravity theories and ensuring their compatibility with observation given that, e.g., new fifth-force degrees of freedom at low energies would have to be compatible with tight experimental bounds [4]. A common approach in this situation is to restrict to scenarios of high symmetry, in particular spatially homogeneous cosmology or spherically symmetric black holes. While by assumption they no longer include all degrees of freedom, symmetry-reduced models would be expected to capture at least some phenomena of the underlying theory (as they do in classical general relativity), while also connecting directly to phenomenology given the obvious relevance of cosmological and black hole spacetimes. An example is given by LQG, whose cosmological sector has been studied in loop quantum cosmology [37, 46, 78], and which allows to include discretisation effects in a number of effective black hole models [82, 166, 230, 240].

In this section we review the effective cosmological models that have been constructed in the GFT approach to quantum gravity, specifically working with the canonical quantisation schemes reviewed in Section 3.2. We will see that, mainly thanks to coupling the massless scalar field to be used as relational clock (cf. Section 3.1), GFT cosmology models can recover the dynamics of a flat FLRW Universe and include quantum corrections that allow to resolve the Big Bang singularity, replacing it with a bounce [5, 184, 186, 272, 305, 306, 379]. Interestingly, these results hold regardless of the approach followed, and thus provide strong insights regarding early cosmology from a quantum gravity perspective.

By contrast, note that very little has been achieved in the context of black holes from a GFT perspective, as the only existing work is purely kinematical [303, 307]. We will comment on new possibilities later on.

ASSUMPTIONS. In order to link GFT with cosmology, some simplifying assumptions are needed. First of all, the cosmological sector of GFTs often deals with regimes in which interactions can be neglected, so we will only consider the kinetic term of the action (i.e., the free theory). Since neglecting interactions between spin-network nodes (or correlations between the GFT quanta) can be thought as describing GFT configurations of high symmetry, this approximation is often interpreted as corresponding to macroscopic *homogeneous* spacetimes [183, 379]. Let us note that, because the theory describes a growing particle number, such an approximation can only hold for a finite amount of time before the number of quanta is too large and interactions become important [305, 306].

Moreover, we now focus on a single Peter–Weyl mode (cf. (58)). Indeed, it turns out that considering only a single J (somewhat representing excitations of the same "type") is enough to obtain the correct cosmological dynamics of a flat FLRW Universe [305, 306]. Other than by computational simplicity, this restriction is motivated by the fact that one mode quickly dominates dynamically (see discussion below (99) and (123)) so that this approximation becomes better and better with time. There is evidently a certain clash with the first approximation of negligible interactions, which gets worse over time, meaning that we expect GFT cosmological models to be valid in an intermediate regime sometimes called "mesoscopic".[14] The single-mode approximation means that a cosmological spacetime expands or contracts by modifications to the combinatorial structure of the spin network (i.e., by changing number of the GFT quanta), rather than by changing the spin labels on the network (i.e., transitioning between GFT quanta of different spin representations). Furthermore, one can use the insights gained from loop quantum cosmology, where all the spins are usually fixed to only one value [37, 46, 78], to motivate this single-mode restriction in GFT. Such assumptions behind loop quantum cosmology models are often motivated by a suggestion that cosmological expansion or contraction is indeed realised by a changing graph structure in full LQG [39, 83].

An effective restriction to a single mode would essentially emerge dynamically in any case, as shown in [176].

Finally, one then also needs to specify quantum states to calculate expectation values of the operators of interest (in particular, the volume operator). While neglecting interactions and restricting to a single mode are assumptions that apply to both the algebraic and the deparametrised quantisations described in Section 3.2, the choice of states plays a different role depending on the approach. Specifi-

14 In [305], such a regime is defined in analogy with condensed matter physics, where the number of GFT quanta is large enough to admit a global geometric interpretation but small enough to consider the GFT system as a "dilute non-interacting gas".

cally, since the notion of dynamics in the algebraic approach is implemented *through* the states, they represent a crucial ingredient for obtaining effective cosmology (as there is no Heisenberg picture). As we have described, one here needs (exactly or approximately) physical states that satisfy some constraint equations (cf. (92) and (94)), which however seem to be only solved by the quite specific class of coherent states;[15] while we will use coherent states in the following, we note that this represents quite an important restriction of the full physical state space. On the other hand, in the deparametrised approach dynamics are defined in the Heisenberg picture by (105) regardless of the choice of states. One can then discuss relational dynamics of operators (such as the volume (112)) thanks to (109), even before computing expectation values. In both approaches, one usually considers coherent states to implement a notion of semiclassicality similar to what is often done in quantum cosmology. This is because in a macroscopic Universe quantum fluctuations over expectation values should be small, as discussed in [273] and [184], where different choices of coherent states are examined with respect to this criterion. We will expand on the choice of states in Chapter 5, where we generalise previous results on semiclassicality to a broader family of states.

3.3.1 *Algebraic GFT cosmology*

As explained at the end of Section 3.2.1, one needs to specify a set of quantum states to extract dynamics in the algebraic quantisation of GFT. The prototypical state adopted in this formalism is given by a field coherent (or "condensate") state [183, 305]

In order to simplify the notation, we will drop the index J in our single-mode expressions.

$$|\sigma\rangle = \mathcal{N}_\sigma \exp\left(\int \mathrm{d}\chi\, \sigma(\chi)\hat{\varphi}^\dagger(\chi)\right)|\varnothing\rangle\,, \tag{124}$$

with

$$\mathcal{N}_\sigma = \exp\left(-\frac{1}{2}\int \mathrm{d}\chi |\sigma(\chi)|^2\right)\,. \tag{125}$$

The key property of such states,

$$\hat{\varphi}(\chi)|\sigma\rangle = \sigma(\chi)|\sigma\rangle\,, \tag{126}$$

15 We will explain this in more detail in Chapter 5, where we explicitly check the conditions necessary for states to be physical in the algebraic approach.

allows replacing (after normal ordering) all field operators by the collective variable $\sigma(\chi)$ (sometimes called condensate wavefunction), which explicitly depends on χ. Indeed, the coherent state (124) implements the idea of mean-field approximation discussed around (96), and the quantum equation of motion essentially reduces to the classical GFT equation of motion.

Since all quanta are characterised by a single quantum state and wavefunction, it has been conjectured that these states correspond (when coarse-grained) to homogeneous cosmological spacetimes [183].

While often described as an approximate solution in the literature (even for GFT models based on the free theory), we stress that states of the form (124) can solve (92) *exactly*. Indeed, due to the property (126), $|\sigma\rangle$ is a physical state provided that the function $\sigma(\chi)$ satisfies the classical free GFT equation of motion (cf. (97))

$$\left(\partial_\chi^2 - \omega^2\right)\sigma(\chi) = 0\,. \tag{127}$$

Note that for any σ solving (127) one has $\mathcal{N}_\sigma = 0$ in (125), meaning the state is non-normalisable. One way of regularising the state $|\sigma\rangle$ is by introducing an *ad hoc* cutoff in χ, which would represent an arbitrarily large (but finite) range of validity for the resulting effective relational dynamics.[16] Explicitly, the solution to (127) [176],

$$\sigma(\chi) = Ae^{\omega\chi} + Be^{-\omega\chi}\,, \tag{128}$$

where A and B are constants (which generically depend on the mode J), dictates how dynamics are implemented in the algebraic approach, as geometrical quantities inherit χ dependence through $\sigma(\chi)$. One can indeed derive effective dynamics for expectation values of the operators of interest, such as $\hat{N}(\chi)$ and in particular $\hat{V}(\chi)$, basically ignoring fluctuations. Starting from generic "second quantised" operators (85), and recalling that we are restricting to a single J, one calculates expectation values as

$$\langle\hat{\mathcal{O}}(\chi)\rangle_\sigma = \mathcal{O}\frac{\langle\sigma|\hat{\varphi}^\dagger(\chi)\hat{\varphi}(\chi)|\sigma\rangle}{\langle\sigma|\sigma\rangle} = \mathcal{O}|\sigma(\chi)|^2\,. \tag{129}$$

For the specific definitions (86) and (87) of the number and volume operators, one finds

$$\langle\hat{N}(\chi)\rangle_\sigma = |\sigma(\chi)|^2\,, \tag{130}$$

16 One might argue that this does not represent an issue as the free-theory approximation breaks down at some χ (when the interactions become important), so one should not trust the model for too large χ anyway.

and

$$\langle \hat{V}(\chi) \rangle_\sigma = v|\sigma(\chi)|^2 \,. \tag{131}$$

These expectation values can now be evaluated for a solution of the free theory (127). In particular, substituting the solution (128) into (131), one finds the volume

$$\langle V(\chi) \rangle_\sigma = v\left[\mathcal{S}\cosh(2\omega\chi) + \mathcal{D}\sinh(2\omega\chi) + 2\mathrm{Re}(A\bar{B})\right] , \tag{132}$$

where $\mathcal{S} = |A|^2 + |B|^2$ and $\mathcal{D} = |A|^2 - |B|^2$. Crucially, this form for the volume corresponds to a cosmological solution that satisfies Friedmann-like dynamics while also containing a cosmological bounce.[17] To make this explicit, one can introduce the conserved quantities [176, 305, 306]

$$\begin{aligned} E &= -4\mathrm{Re}(A\bar{B}) \,, \\ Q &= 2\mathrm{Im}(A\bar{B}) \,, \end{aligned} \tag{133}$$

respectively associated with χ translations and with the $U(1)$ symmetry of the complex GFT. With these, one finds that (132) satisfies an effective Friedmann equation

$$\left(\frac{1}{\langle \hat{V}(\chi) \rangle_\sigma} \frac{\mathrm{d}\langle \hat{V}(\chi) \rangle_\sigma}{\mathrm{d}\chi} \right)^2 = 4\omega^2 \left(1 + \frac{vE}{\langle \hat{V}(\chi) \rangle_\sigma} - \frac{v^2 Q^2}{\langle \hat{V}(\chi) \rangle_\sigma^2} \right) . \tag{134}$$

While the interpretation of E (sometimes called the "GFT energy") was never clarified, in [305] it was proposed that ωQ plays the role of conjugate momentum to the scalar field, so that by defining an effective energy density

$$\rho_{\mathrm{eff}} = \frac{\omega^2 Q^2}{2\langle \hat{V}(\chi) \rangle_\sigma^2} \,, \tag{135}$$

and a critical energy density $\rho_c = \omega^2/(2v^2)$, one can rewrite (134) in a form similar to loop quantum cosmology [359]

The term proportional to the GFT energy E is absent in loop quantum cosmology (see Appendix A).

$$\left(\frac{1}{\langle \hat{V}(\chi) \rangle_\sigma} \frac{\mathrm{d}\langle \hat{V}(\chi) \rangle_\sigma}{\mathrm{d}\chi} \right)^2 = 4\omega^2 \left(1 + \frac{vE}{\langle \hat{V}(\chi) \rangle_\sigma} - \frac{\rho_{\mathrm{eff}}}{\rho_c} \right) . \tag{136}$$

17 Even though the original work [305] was based on equilateral tetrahedra which were supposed to encode isotropy (meaning the spins j_I in J were assumed to be equal, cf. (59) and Figure 7), one can retain four different spins j_I for the faces of the building blocks; the key ingredient for this result lies in the single-mode restriction alone.

In (136), ρ_c would represent a universal upper bound for the density ρ_{eff}, which allows a bounce at a minimum nonsingular volume, just like in loop quantum cosmology. However, it is not clear why the $U(1)$ "charge" Q should be interpreted as the conjugate momentum of the matter field χ, as it would make more sense to associate this role to E (which is the conserved quantity associated with χ translations; see, e.g., [185] for more on this).

The large volume limit of (134) needs to be consistent with the classical Friedmann equation $(V'/V)^2 = 12\pi G$, which is easily found in classical GR by adopting a scalar field as a clock (we review its derivation in Appendix A, see in particular (405)). In this case, large volumes (or low energy densities) are obtained when $|\chi|$ is big enough so that we are away from the bounce. Hence, agreement with the classical theory requires the identification between the GFT coupling and Newton's constant as

Note that by identifying the GFT coupling with Newton's constant one has $\rho_c \sim \rho_{Planck}$ as $v \sim l_P^3$ (cf. (1)).

$$\omega^2 = 3\pi G\,. \tag{137}$$

This identification should be seen as the "emergence" of Newton's constant from more fundamental GFT parameters. We emphasise that while the first term in (134) recovers the (relational) Friedmann equation of GR, the two subleading contributions are seen as GFT (i.e., quantum gravity) corrections to classical cosmology, with the $1/\langle \hat{V}(\chi)\rangle_\sigma^2$ term crucially being responsible for the generic resolution of the Big Bang singularity. Furthermore, we remark that in order to obtain (134) we had to specifically use the state (124), as indicated by the index of $\langle \hat{V}(\chi)\rangle_\sigma$. In contrast, the effective Friedmann equation of the deparametrised approach holds in any state as we will see in the next section.

Finally, as we mentioned when defining the relational operator (87), note that even if the volume itself does not show any infinities, one can use the property

$$\langle \hat{\varphi}^\dagger(\chi)\hat{\varphi}(\chi)\hat{\varphi}^\dagger(\chi)\hat{\varphi}(\chi)\rangle = \delta(0)\langle \hat{\varphi}^\dagger(\chi)\hat{\varphi}(\chi)\rangle + \langle (\hat{\varphi}^\dagger(\chi))^2\hat{\varphi}^2(\chi)\rangle\,, \tag{138}$$

which follows from the commutator (81), to show that volume fluctuations diverge as

$$\frac{(\Delta\hat{V})_\sigma^2}{\langle \hat{V}\rangle_\sigma^2} = \frac{\delta(0)}{|\sigma(\chi)|^2}\,, \tag{139}$$

where $(\Delta\hat{V})^2_\sigma = \langle\hat{V}^2\rangle_\sigma - \langle\hat{V}\rangle^2_\sigma$. If one removes the distribution $\delta(0)$ by means of some regularisation procedure (e.g., replacing the Dirac delta in (81) with a Kronecker delta by considering smeared observables [40, 41, 250], or imposing the dynamics in a different way by working with the peaked coherent states of [272]), (139) gets automatically smaller and smaller over time as $\sigma(\chi)$ grows exponentially (cf. (128)). In this sense, one might then argue that these coherent states are semiclassical. In Chapter 5 we explore other states and study their semiclassicality.

COHERENT PEAKED STATES. As we mentioned in Section 3.2.1, one way of dealing with the issues of divergences (such as (139)) arising due to the distributional nature of relational operators is given by the coherent peaked states proposal [272]. This falls within the algebraic approach to GFT cosmology and focusses on specific types of states, defined as coherent states (cf. (124))

$$|\sigma_\epsilon; \chi_0, \pi_0\rangle := \mathcal{N}_{\sigma_\epsilon} \exp\left(\int \mathrm{d}\chi\, \sigma_\epsilon(\chi)\hat{\varphi}^\dagger(\chi)\right)|\varnothing\rangle \tag{140}$$

The Gaussian plays the role of a "peaking function" and allows to discuss states that are peaked at the relational time χ_0.

where $\mathcal{N}_{\sigma_\epsilon}$ is as in (125) with the difference that the mean field

$$\sigma_\epsilon(\chi; \chi_0, \pi_0) = e^{-\frac{(\chi-\chi_0)^2}{2\epsilon}} e^{\mathrm{i}\pi_0(\chi-\chi_0)}\tilde{\sigma}(\chi) \tag{141}$$

is given by the product of a Gaussian and a function $\tilde{\sigma}(\chi)$ (dubbed "reduced" wavefunction). The state carries various labels: ϵ is the standard deviation of the Gaussian, π_0 is a parameter needed to make sure quantum fluctuations are small [273], and crucially, χ_0 represents the value of the relational time at which the states are peaked. Requiring the Gaussian to be sharply peaked allows to interpret the internal clocks of the GFT quanta as "synchronised".[18] In other words, one can effectively interpret the expectation value of an operator on a coherent peaked state characterised by χ_0 as if it were computed at a relational time (i.e., on a slice labelled by) χ_0. As a possible criticism, note that (140) is actually a one-parameter family of states with respect to χ_0, which seems to play the role of an *ad hoc* external (background) time label, like in non-relativistic quantum mechanics.

In a sense, the states (140) can be considered as describing some kind of discrete counterpart of "leaves of a foliation" with respect to the massless scalar field.

18 On a technical level, some assumptions are needed for the states to behave well. Importantly: the standard deviation of the Gaussian is required to satisfy $\epsilon \ll 1$, and one also imposes the condition $\epsilon\pi_0^2 \gg 1$ so that quantum fluctuations are assured to be small. We refer to [272, 273] for the details and the approximations employed.

The wavefunction (141), required to satisfy certain equations as sketched below, is assumed to make the coherent peaked state normalisable in the Fock space inner product (since, naively, the integral in (125) can be rendered convergent by the peaking function of (141), in particular for small ϵ). To be more precise, since it is not clear whether (140) satisfies any constraint discussed in Section 3.2.1 (namely, the strong imposition (92) or even the approximate requirement (94)) for the commonly used kinetic term (79), a critical assumption of this formalism is that the action contains a kinetic term that can be expanded as

$$K = \sum_{n=0}^{\infty} \frac{K^{(2n)}}{(2n)!} \chi^{2n} \,. \tag{142}$$

In [272], this is justified by recalling that the kinetic term with (in principle infinite) even derivatives mentioned in footnote 3 admits a representation in terms of smooth functions. This is a weighty assumption which in particular means one is not able to specify an explicit GFT model. By means of some approximations on (142) (see [272]), one here simply considers the "instantaneous" averaged Euler–Lagrange equations (94),

$$\left\langle \sigma_\epsilon; \chi_0, \pi_0 \left| \frac{\delta S[\hat{\varphi}, \hat{\varphi}^\dagger]}{\delta \hat{\varphi}^\dagger(\chi_0)} \right| \sigma_\epsilon; \chi_0, \pi_0 \right\rangle = 0 \,, \tag{143}$$

This equation is essentially only required to hold at $\chi = \chi_0$, as indicated by the argument of the functional derivative.

where the coherent peaked state (140) is labelled by the same parameter χ_0 appearing in the argument of the functional derivative. Then, the difference with a standard coherent state (124) is that the small parameter $\epsilon \ll 1$ in (141) allows to adopt a generic (reduced) wavefunction $\tilde{\sigma}(\chi)$, which does not need to be slowly varying in χ (cf. footnote 3).

With all such assumptions, one can obtain an effective equation of motion for $\tilde{\sigma}(\chi)$ (viewed as approximations to the full equations of motion) which crucially has the same functional form of (127), but with generalised definitions of ω^2, E and Q (cf. (133)). In particular, such quantities now depend on the coherent peaked states parameters ϵ and π_0 (their explicit form is not important for our purposes, see [272, 273]). This is how this formalism reproduces the main results shown earlier, curing its (distributional) divergences.[19] To see it ex-

19 While the infinities related to distributional operators are dealt with in this approach (see [273] for the explicit computation of all the fluctuations, including in the clock field $\hat{\chi}$), the divergences in (the norm of) the states are bypassed altogether as the coherent peaked state is defined "by hand" to be normalisable (cf. (141) and (125)).

plicitly, we recall that this approach makes use of "totally integrated" operators instead of the χ-dependent ones (which are distributional). Indeed, using the number operator as an example, computing the expectation value of (88) one finds a function of the parameter χ_0 used in the definition of the state (140),

$$\langle \hat{N} \rangle_{\sigma_\epsilon,\pi_0,\chi_0} = \int \mathrm{d}\chi \, |\sigma_\epsilon(\chi;\chi_0,\pi_0)|^2 \simeq |\tilde{\sigma}(\chi_0)|^2 \,, \tag{144}$$

which is reminiscent of (130) but here with the state parameter χ_0 playing the role of relational time. Furthermore, given that the expectation value of the scalar field operator (89) in a coherent peaked state is $\langle \hat{X} \rangle_{\sigma_\epsilon,\pi_0,\chi_0} \simeq \chi_0 \langle \hat{N} \rangle_{\sigma_\epsilon,\pi_0,\chi_0}$, one now defines an intensive (effective) clock operator by

$$\hat{\chi} = \frac{\hat{X}}{\langle \hat{N} \rangle_{\sigma_\epsilon,\pi_0,\chi_0}} \,, \tag{145}$$

which is such that $[\hat{\chi}, \hat{\Pi}] = \mathrm{i}$ (see (91) and discussion thereafter). Then, in the given approximations, one has $\langle \hat{\chi} \rangle_{\sigma_\epsilon,\pi_0,\chi_0} \simeq \chi_0$, and χ_0 can indeed be seen as the expectation value of the clock $\hat{\chi}$ (which is why this approach classifies as a "tempus post quantum" formalism [272]).

Finally, one can define an *ad hoc* Hermitian operator representing an effective Hamiltonian that by construction describes evolution with respect to the parameter χ_0, and check that in the states (140) and with the mentioned approximations [272]

Note that the effective Hamiltonian is state-specific, so that it should more correctly be denoted by $\hat{H}_{\sigma_\epsilon,\pi_0,\chi_0}$, but we will not do so to lighten the notation.

$$\langle \hat{H} \rangle_{\sigma_\epsilon,\pi_0,\chi_0} \simeq \langle \hat{\Pi} \rangle_{\sigma_\epsilon,\pi_0,\chi_0} \,, \tag{146}$$

where $\hat{\Pi}$ is given in (90). This allows to obtain an effective Heisenberg-like equation for fully integrated operators $\hat{\mathcal{O}}$ such as the number (88) (or the volume),[20]

$$\frac{\mathrm{d}}{\mathrm{d}\chi_0} \langle \hat{\mathcal{O}} \rangle_{\sigma_\epsilon,\pi_0,\chi_0} = \mathrm{i} \langle [\hat{H}, \hat{\mathcal{O}}] \rangle_{\sigma_\epsilon,\pi_0,\chi_0} \,, \tag{147}$$

providing an alternative notion of relational dynamics in the algebraic approach, imposed *via* peaked coherent states. In this picture, dynamical equations emerge on the *kinematical* Hilbert space from considering the evolution of expectation values relative to the clock

20 Some subtleties arise regarding the fact that the expectation value of the effective Hamiltonian depends on χ_0 (specifically as $\langle \hat{H} \rangle_{\sigma_\epsilon,\pi_0,\chi_0} \simeq \pi_0 |\tilde{\sigma}(\chi_0)|^2$); these strange effects seem to be due to the granularity of geometry in GFT. See [272] for a discussion on these issues, their possible meaning and resolution.

expectation value $\langle\hat{\chi}\rangle_{\sigma_\epsilon,\pi_0,\chi_0}$, choosing rather specific states so that a simple relation between these expectation values can be derived.

Note that, as already mentioned in Section 3.2.1, the fact that the algebraic approach relies on constraints (cf. (92)) implies that observables should be identified as those operators which preserve the physical state space (and not just any Hermitian operators). This concern applies to the algebraic approach in general, regardless of the approximations used or the choice of coherent peaked states (140); however, it can be seen as more pressing here as the operators are central in defining relational dynamics. Indeed, it is not clear whether, for instance, the operator $\hat{\chi}$ in (145) (whose expectation value plays the role of relational time variable) preserves the physical state space.

Regarding cosmological dynamics, it clearly follows that just like (144) one can write down the volume operator expectation value as

$$\langle\hat{V}\rangle_{\sigma_\epsilon,\pi_0,\chi_0} = v\int \mathrm{d}\chi\,|\sigma_\epsilon(\chi;\chi_0,\pi_0)|^2 \simeq v|\tilde{\sigma}(\chi_0)|^2\,, \tag{148}$$

which represents the “coherent peaked states counterpart” of (131). As anticipated, (148) leads to effective cosmological dynamics that have the same functional form of (134), where now evolution is described in terms of the parameter χ_0 and the coefficients in the Friedmann equation depend on the labels defining the state (140) (i.e., ω^2 as well as E and Q depend on π_0 and ϵ, see [272]).

Ultimately, the coherent peaked states formalism provides a novel version of the algebraic approach which is not subject to the divergences typical of distributional operators (see discussion around (86) and (87)). It relies on different ways of approximating the full quantum dynamics and leads to the same results, supporting the robustness of the GFT cosmology approach overall. While providing significant insights on the possible quantum nature of the relational clock itself (which was treated as a classical time in previous studies), the proposal does not refer to a classical Hamiltonian notion of dynamics or to classical relational observables as in usual Dirac quantisation, and it is unclear whether constraints such as (92) or (94) play a fundamental role. We will return to these important points in Chapter 6, where we define a new way to impose dynamics in GFT inspired by the quantum time operator $\hat{\chi}$, but keeping in line with a traditional Dirac quantisation (and hence reducing from a kinematical to a physical Hilbert space by means of quantum constraints).

3.3.2 *Deparametrised GFT cosmology*

The idea of deparametrisation was introduced in GFT cosmology in [5] (see also [250] for related ideas) where an effective Friedmann equation similar to (134) was recovered in a simple toy model with a squeezing Hamiltonian generating evolution with respect to a preferred matter clock, without requiring a mean-field approximation. The same strategy was then adopted in full GFT in the Hamiltonian formalism of [186, 379], which we reviewed in Section 3.2.2. Initially, coherent states were used to obtain approximate solutions for expectation values of operators of interest, but this limitation was overcome in [184] where the dynamics were solved for operators in the Heisenberg picture, without requiring a specific state (cf. (109)). States still need to be chosen to compare expectation values with previous results; effective Friedmann equations could be found for many types of coherent states,[21] e.g., based on the $\mathfrak{su}(1,1)$ algebra generated by the volume and Hamiltonian operators [59], which we explain below.

Neglecting $V[\varphi]$ in the Hamiltonian (122) (see also (123) and discussion thereafter), the restriction to a single Peter–Weyl label for a free GFT practically means focusing on a Hamiltonian that describes single-mode squeezing. It is worth noting that even without the explicit reduction to single-mode contributions seen in Section 3.2.2, one could easily obtain a single-mode squeezing Hamiltonian starting from (118) by choosing "symmetric" initial conditions $\hat{a}_J(0) = \hat{a}_{-J}(0)$, which are shown to be preserved under time evolution [186, 379].[22]

As in the algebraic approach section, we simplify the notation by dropping the index J in our single-mode expressions. Hence, we now deal with a quantum system described by bosonic operators $\hat{a}(\chi)$ and $\hat{a}^\dagger(\chi)$ with $\left[\hat{a}(\chi), \hat{a}^\dagger(\chi)\right] = 1$ (cf. (107)). The main operators of interest for cosmological purposes are the squeezing Hamiltonian in (123), the number (111) and the volume (112), which reduce to

$$\begin{aligned} \hat{H} &= -\frac{\omega}{2}\left((\hat{a}^\dagger)^2 + \hat{a}^2\right), \\ \hat{V}(\chi) &= v\,\hat{N}(\chi) = v\,\hat{a}^\dagger \hat{a}, \end{aligned} \tag{149}$$

21 We will come back to the notion of semiclassical states for GFT cosmology in Chapter 5, where we expand the existing results to the general family of Gaussian states.

22 More directly, one could also consider a J with vanishing magnetic indices $m_I = 0$ (so that $J = -J$); this is a somewhat mild assumption as no geometrical observable depends on the values of the magnetic indices.

where again v represents the volume of a GFT quantum. As mentioned, the operators (149) generate the Lie algebra $\mathfrak{su}(1,1)$ extended by a central element [59, 184]. The algebra is closed by adding

$$\hat{C} = \mathrm{i}\frac{v}{2}\left((\hat{a}^\dagger)^2 - \hat{a}^2\right), \tag{150}$$

which would be related to the "Thiemann complexifier" in some analogous LQG models [62]. Here (150) does not have a direct physical interpretation, but determines whether the resulting cosmology has a time-reversal symmetry. The $\mathfrak{su}(1,1)$ algebra of these operators follows from their composition in terms of ladder operators: one traditionally defines the three possible quadratic combinations $\hat{K}_0 = \frac{1}{4}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)$, $\hat{K}_+ = \frac{1}{2}(\hat{a}^\dagger)^2$ and $\hat{K}_- = \frac{1}{2}\hat{a}^2$, which satisfy the $\mathfrak{su}(1,1)$ algebra

$$\begin{aligned}\left[\hat{K}_0, \hat{K}_\pm\right] &= \pm\hat{K}_\pm\,,\\ \left[\hat{K}_-, \hat{K}_+\right] &= 2\hat{K}_0\,.\end{aligned} \tag{151}$$

In our case, the GFT operators relate to these $\mathfrak{su}(1,1)$ generators as

$$\begin{aligned}\hat{H} &= -\omega(\hat{K}_+ + \hat{K}_-)\,,\\ \hat{V} &= 2v\hat{K}_0 - \frac{v}{2}\,,\\ \hat{C} &= \mathrm{i}v(\hat{K}_+ - \hat{K}_-)\,,\end{aligned} \tag{152}$$

and the algebra closes as

$$\begin{aligned}\left[\hat{V}, \hat{H}\right] &= 2\mathrm{i}\omega\hat{C}\,,\\ \left[\hat{C}, \hat{V}\right] &= 2\mathrm{i}\frac{v^2}{\omega}\hat{H}\,,\\ \left[\hat{C}, \hat{H}\right] &= 2\mathrm{i}\omega\left(\frac{v}{2} + \hat{V}\right),\end{aligned} \tag{153}$$

where we see the central element (identity operator) appearing in the third relation. We point out that the structure (153) (together with a cosmological interpretation) is generically found in other realisations of $\mathfrak{su}(1,1)$ quantum cosmology [59, 79, 80], for example in loop quantum cosmology (where this algebra commonly appears [73–75, 266]).

We can now turn to the dynamics of such operators. $\hat{H}$ determines the evolution of any other operator $\hat{\mathcal{O}}$ via the Heisenberg equation (105). From this, one can obtain the solutions [184]

$$\hat{V}(\chi) = -\frac{v}{2} + \left(\hat{V} + \frac{v}{2}\right)\cosh(2\omega\chi) + \hat{C}\sinh(2\omega\chi)\,, \tag{154}$$

$$\hat{C}(\chi) = \hat{C}\cosh(2\omega\chi) + \left(\hat{V} + \frac{v}{2}\right)\sinh(2\omega\chi)\,, \tag{155}$$

Note that (154) bears a strong (functional) similarity with the algebraic approach expression (132), though the former is an operator equation and the latter an expectation value.

where $\hat{V} = \hat{V}(0)$ and $\hat{C} = \hat{C}(0)$. We can now see that, while $\hat{V}$ represents the volume at $\chi = 0$, the presence of $\hat{C}$ determines whether the volume evolution (154) has a symmetry under $\chi \to -\chi$ (i.e., whether one has different pre- and post-bounce scenarios in the cosmological dynamics we derive below). Since we are working in the Heisenberg picture, the solutions (154) and (155) do not refer to any choice of quantum state; in fact, (154) is all one needs to obtain an effective Friedmann equation. Denoting by $\langle\hat{\mathcal{O}}(\chi)\rangle$ the expectation values in *any* state, one finds

$$\left(\frac{1}{\langle\hat{V}(\chi)\rangle}\frac{\mathrm{d}\langle\hat{V}(\chi)\rangle}{\mathrm{d}\chi}\right)^2 = 4\omega^2\left(1 + \frac{v}{\langle\hat{V}(\chi)\rangle} - \frac{1}{\langle\hat{V}(\chi)\rangle^2}\mathcal{I}_0\right)\,, \tag{156}$$

where $\mathcal{I}_0 = \langle\hat{V}\rangle^2 + v\langle\hat{V}\rangle - \langle\hat{C}\rangle^2$ essentially represents initial conditions (it is fully specified by the operators (154) and (155) at $\chi = 0$). (156) describes the same result obtained in the algebraic approach (cf. (134)), with only minor differences. To reiterate the main points: while for large volumes (or late times $\chi \to \pm\infty$) (156) is consistent with the classical Friedmann provided we identify the GFT coupling with G as in (137), the GFT quantum correction $1/\langle\hat{V}(\chi)\rangle^2$ is responsible for the generic resolution of the Big Bang singularity, which is replaced with a cosmological bounce through a minimal nonsingular volume.

Although the differences between (156) and (134) are minor, it is worth spelling out the analogies with the quantities E and Q appearing in (134) and defined in (133). First, it is immediate to see that the analogue of the "GFT energy" is automatically fixed to be 1 in the deparametrised approach. Moreover, by defining here again a critical energy density $\rho_c = \omega^2/(2v^2)$ and rewriting the last term inside the brackets of (156) in the suggestive form $-\rho_{\text{eff}}/\rho_c$, one finds a more appropriate interpretation in terms of the relational Hamiltonian (149) as conjugate momentum to the matter field. To be more precise, while the role of the conjugate momentum to χ was somehow played by the quantity ωQ in the algebraic approach (Q being the $U(1)$ charge associated with the complex GFT, see (133) and (135)), one here has that effective energy density ρ_{eff} takes the form [184]

$$\rho_{\text{eff}} = \rho_\chi(\chi) + \frac{1}{2\langle\hat{V}(\chi)\rangle^2}\tilde{\mathcal{I}}_0\,, \tag{157}$$

where $\tilde{\mathcal{I}}_0 = \omega^2\left(\langle\hat{N}\rangle^2 + \langle\hat{N}\rangle - \langle\hat{a}^\dagger\rangle^2\langle\hat{a}\rangle^2\right)$ again refers to initial conditions.[23] Crucially, the matter energy density $\rho_\chi(\chi)$ is now defined by identifying the Hamiltonian $H = \langle\hat{H}\rangle$ (cf. (149)) with the conjugate momentum of χ, namely

$$\rho_\chi(\chi) = \frac{H^2}{2\langle\hat{V}(\chi)\rangle^2}\,, \tag{158}$$

as one would expect (see (135) for comparison). Explicitly, using (157) and (158) one can rewrite (156) just like (136) as

$$\left(\frac{1}{\langle\hat{V}(\chi)\rangle}\frac{\mathrm{d}\langle\hat{V}(\chi)\rangle}{\mathrm{d}\chi}\right)^2 = 4\omega^2\left(1 + \frac{v}{\langle\hat{V}(\chi)\rangle} - \frac{\rho_{\text{eff}}}{\rho_c}\right)\,, \tag{159}$$

namely with a loop quantum cosmology-like term (cf. Appendix A).

To conclude, we also point out that (156) holds regardless of whether the quantum state one uses to compute expectation values is a pure or mixed state. All that is needed to obtain (156) is a (linear) operation mapping operators to their expectation values, and the density matrix expression $\langle\hat{V}(\chi)\rangle = \mathrm{tr}\left(\hat{\rho}\,\hat{V}(\chi)\right)$ is as good as the pure-state evaluation $\langle\hat{V}(\chi)\rangle = \langle\psi|\hat{V}(\chi)|\psi\rangle$. This point was not stressed in [184] where (156) was obtained, and will allow us to investigate semiclassical properties of mixed (in particular *thermal*) states in Chapter 5. Finally, while true for any quantum state, we emphasise that (156) is a relation between expectation values only, and does not take into account fluctuations (which, if large, may be problematic for a semiclassical interpretation). To claim that the effective Friedmann equation is a good description of the dynamics of cosmological observables, one needs to adopt quantum states that show some semiclassical features, another point that we address in Chapter 5.

23 The additional contributions to ρ_{eff} encoded in $\tilde{\mathcal{I}}_0$ also scale as $1/\langle\hat{V}(\chi)\rangle^2$, so they effectively simply shift the scalar field momentum compared to its classical value H.

Part II

PHENOMENOLOGY, QUANTUM STATES, RELATIONALISM

This part describes the research results of the thesis. These represent innovative developments and advancements of the general framework introduced in Part I, contributing significantly to the field of GFT and going beyond the existing state of the art along three distinct directions. Specifically, Chapter 4 extends the phenomenological results on GFT cosmology of Section 3.3 to anisotropic settings. Chapter 5 investigates semiclassical properties of quantum states, analysing the general class of Gaussian states for the cosmological models of interest. Finally, Chapter 6 develops a new rigorous way of defining relational dynamics for canonical group field theories (working with the free theory but not necessarily specialising to the cosmological sector), providing crucial insights on the notion of covariance and the problem of time in quantum gravity.

4

TOWARDS ANISOTROPIC COSMOLOGY

This chapter introduces anisotropic models for cosmology in group field theory. It is based on [99], and with the exception of a portion of Section 4.2.1 (where we review some quantum geometry notions imported form loop quantum gravity), it constitutes original research.

We have seen in Chapter 3 that the canonical formulation of group field theory provides a natural arena for studying effective cosmological models from a quantum gravity perspective. Indeed, in Section 3.3 we showed that the GFT cosmology literature is rich of ways to derive effective cosmological dynamics; more specifically, using a restriction to a single mode, one finds that a flat isotropic cosmology effectively emerges from GFT in a number of ways. An effective Friedmann equation that reproduces general relativity at late times and deviates from classical gravity at early times (with a quantum bounce replacing the initial singularity) is obtained following both the algebraic approach of Section 3.3.1 (which is how the GFT cosmology programme began) and adopting a deparametrised point of view as described in Section 3.3.2. Such dynamical equations are also similar to the effective dynamics of loop quantum cosmology, and describe an effective repulsive behaviour at high energies in a similar fashion.

While important for the phenomenology of GFT and for connecting to approaches such as loop quantum cosmology, these results have so far been restricted to the case of a flat homogeneous, isotropic Universe.[1] Some studies have included anisotropies perturbatively [105, 321] showing that they decay leading to isotropisation, but until recently there was no characterisation of, e.g., an anisotropic Bianchi cosmology.

In [99] we took the first steps towards the study of anisotropic cosmologies in group field theory; we devote this chapter to explaining those results, focussing on the simplest possible case of Bianchi I cosmology with local rotational symmetry so that two out of the three directional scale factors are taken to be equal. There are at least two, initially quite separate, challenges involved in this extension of past work. The first is to find a characterisation of anisotropies in group

1 Inhomogeneities can be included perturbatively as in [170, 274], and match physical expectations at least in a long-wavelength limit.

field theory, i.e., to define an observable that can distinguish isotropic and anisotropic geometries and quantify the amount of anisotropy. Here the key idea we use is the Misner parametrisation of Bianchi models (see, e.g., [81]; a review is also provided in Appendix A) in terms of a volume degree of freedom and two relative anisotropy variables, the Misner variables $\beta_\pm$. In a classical Bianchi I model $\beta_\pm$ behave as free, massless scalar fields in a flat FLRW geometry,[2] which have already been studied in GFT. On the other hand, the discreteness of geometry in GFT means that we cannot simply take over a continuum definition of anisotropy, so the construction of an analogue $\beta_\pm$ variable requires careful thought. The second challenge is to understand which simplifying approximations used in past work (see Section 3.3) need to be relaxed in order to allow for anisotropies in the effective description. For instance, while the work of [305, 306] only used "isotropic" states interpreted as describing simplicial building blocks for which all faces have equal area, it is known (see, e.g., [184]) that this microscopic restriction to isotropy is neither necessary nor sufficient to obtain the correct (flat FLRW) Friedmann dynamics: the more relevant assumption is to restrict to a single field mode in the Peter–Weyl expansion in representation data. Hence, in order to describe anisotropic geometries, multiple Peter–Weyl modes must be taken into account, but it is not clear how many (and which) modes are needed to capture physical anisotropies.

In this work we show how to tackle the first challenge; we define a $\beta_\pm$ analogue with a clear geometric interpretation, quantum ambiguities that disappear for large quanta, and correct physical properties – constant velocity and hence linear evolution – at least for a certain cosmological period of time, before the isotropisation observed in [105, 321] sets in and the anisotropy disappears. The second challenge is partially addressed, given that the $\beta_\pm$ dynamics partially match expectations from classical relativity, but the observed isotropisation does not correspond to a classically expected behaviour and, more importantly, anisotropies do not backreact on the effective Friedmann equation as expected. This suggests that while our constructions will be useful for future work, our model needs further refinement to reproduce the correct physics of a classical Bianchi Universe.

2 This property makes the Misner parametrisation particularly natural in classical GR. For comparison we should mention that in loop quantum cosmology the situation is different, as one quantises an LQG-corrected Hamiltonian constraint and the particular type of corrections makes Misner variables less convenient [39, 83].

We point out that any monotonically evolving quantity in a cosmological model can be used as a relational time: all other dynamical variables can be written, at least in principle, as functions of this "clock". In a vacuum Bianchi I model, the anisotropy variables $\beta_{\pm}$ have this property and hence, in contrast to what is often done in quantum cosmology and in particular in GFT, no coupling to matter would be needed to be able to express the dynamics in relational terms. The fact that we define a new quantity with monotonic evolution in GFT cosmological models hence raises the possibility of defining relational evolution in GFT without adding matter fields, which might help in understanding the problem of time [233, 257, 340], or possible dependence of dynamics on the choice of clock, in GFT (see [185] for some work on this issue in models with multiple possible clocks). We will return to this question in Chapter 6, where we develop the technical tools to potentially change relational clocks in the quantum cosmological models stemming out of GFT.

ANISOTROPIES IN GROUP FIELD THEORY. Classically, incorporating anisotropy means going from the FLRW Universe to the more general class of Bianchi models. We here aim to define a GFT model that can generalise the results presented in Section 3.3 at least to the simplest anisotropic cosmology (Bianchi I). To do this, we need to consider two new aspects: understanding how anisotropies modify the effective Friedmann equation, and understanding the dynamics of the anisotropies themselves.

As we will explain in the rest of the chapter, an anisotropic GFT model naively requires quanta of geometry which are non-equilateral tetrahedra. Moreover, in order to obtain a nontrivial time evolution for the anisotropies we will need to lift the single-mode restriction adopted in Section 3.3. Heuristically, this is because if the shape of non-equilateral tetrahedra is fixed there is no room for a dynamical notion of anisotropy. Multiple modes are required, so that the relative contributions of different shapes can change and a macroscopically "average" anisotropy can become dynamical.

Some preliminary work on anisotropic GFT cosmology was done in [105] where anisotropies were seen as perturbations. Furthermore, an anisotropic trirectangular tetrahedron was used as building block in [321] where a notion of "dynamical isotropisation" was found, but without details on the evolution of a (global) anisotropy parameter. This will be one of the main novelties in our work: we define a measure for anisotropies given by a parameter (corresponding to a Misner-like variable $\beta_\pm$ in classical cosmology) emerging from fundamental quantum gravity arguments.

OUTLINE OF THE CHAPTER. In Section 4.1 and Section 4.2 we set up the new model motivating the introduction of effective $\beta_\pm$ variables used to characterise anisotropies in GFT. After discussing initial conditions in Section 4.3, we propose scenarios based on a few Peter–Weyl modes in Section 4.4 and study the effective dynamics of the anisotropies and the spatial volume, comparing both with the dynamics of general relativity. In order to compute all quantities for the new model, we will need to review some aspects regarding the classical and quantum geometry of tetrahedra as used in LQG and GFT. This is done in Section 4.2.1, and represents the only review part of the chapter. We also propose a simplified "toy model" in Section 4.5, where some of our main results, in particular the linear growth in anisotropy which matches classical expectations, can be derived analytically rather than numerically. We conclude with a discussion on the possibility of including more Peter–Weyl modes and a brief summary of the main results in Section 4.6 and Section 4.7.

4.1 THE NEW ANISOTROPIC GFT MODEL

In order to study anisotropic cosmologies in GFT, we need to define a notion of "anisotropy variables", analogous to the Misner variables $\beta_\pm$. We will study the (relational) dynamics of these variables and observe how they affect the effective Friedmann equation for the volume $V(\chi)$ (cf. Chapter 3). We will compare these effective dynamical equations with those of Bianchi models (see Appendix A for the details), in particular with the simplest Bianchi I cosmology which would be the natural extension of the spatially flat FLRW Universe previously studied in GFT. As in the isotropic case reviewed in Section 3.3, we will follow both the algebraic and deparametrised approaches; we still work in the free theory and we consider the kinetic term K_J given in (80) so that the GFT coupling ω_J will be explicitly given by (99). We will once again see that the two approaches give slightly different behaviours close to the bounce, but basically match otherwise.

As usual, we consider a GFT based on $SU(2)^4 \times \mathbb{R}$ (see Section 3.2 and Section 3.3).

We will study two different observables, representing the degrees of freedom of a classical Bianchi I cosmology. The volume is defined as in Chapter 3; in particular, we will be interested in the expectation value

$$V(\chi) = \sum_J v_J N_J(\chi)\,, \tag{160}$$

In this chapter, expectation values are always taken with respect to Fock coherent states.

of the operators (87) or (112), depending on the approach followed (cf. Section 3.3.1 and Section 3.3.2). We use the notation for the quantities of interest without a hat to represent (coherent state) expectation values of the operators defined in Chapter 3; e.g., $N_J(\chi) = \langle \hat{N}_J(\chi) \rangle$ is the expectation value of the particle number in the mode J (where $\hat{N}_J(\chi)$ is either $\hat{\varphi}_J^\dagger(\chi)\hat{\varphi}_J(\chi)$ or $\hat{a}_J^\dagger(\chi)\hat{a}_J(\chi)$). Moreover, we introduce the new "average anisotropy parameters" $\beta_\pm(\chi)$, defined as

$$\beta_\pm(\chi) := \frac{1}{N(\chi)} \sum_J \beta_\pm^J N_J(\chi)\,, \tag{161}$$

where $\beta_\pm^J$ represents a mode-dependent "local anisotropy" measure associated to the GFT quanta in the mode J, analogous to how v_J in (160) is the quantum volume of the GFT building blocks. The definition of the functions $\beta_\pm^J$ constitutes a substantial part of the work for the new anisotropic GFT models; we will provide all the details on these in later sections.

Notice that the expression for $\beta_\pm(\chi)$ is different from the usual structure of operators in GFT (cf. (85)). We think of anisotropies as determined by the shape of our geometric building blocks; these variables should be "intensive" and not simply grow with the number of quanta, therefore we divide by the total number $N(\chi) = \sum_J N_J(\chi)$ (see, e.g., [180] for a similar concern related to a possible scalar matter quantum operator). $\beta_\pm(\chi)$ is not an expectation value, and we do not propose any definition of operators $\widehat{\beta}_\pm$ representing anisotropy; the overall $1/N(\chi)$ could not arise from taking an expectation value. Instead, the definition (161) introduces a semiclassical and effective notion of anisotropy only.

Since (161) is not an expectation value, the functions $\beta_\pm^J$ are not eigenvalues of an operator, though they heuristically play a similar role.

MULTIPLE PETER–WEYL MODES. In contrast to the single-mode assumption for isotropic GFT cosmology of Section 3.3, our anisotropic setup requires including multiple Peter–Weyl modes. Indeed, for only a single J in (161), $\beta_\pm(\chi)$ would clearly be constant, which is consistent with the idea that we would not incorporate any dynamical "shape" degrees of freedom. Moreover, we already know that the volume emerging from a single mode would give rise to an effective

Friedmann equation of the form of (134) or (156), which corresponds to an isotropic Universe. These two statements agree with the classical observation that the Bianchi I relational dynamics (426) do not differ from FLRW (405) if the anisotropy parameters are constant, as they only appear through derivatives (see Appendix A). Another way of seeing this is to observe that a Bianchi I metric for which the Misner variables are constant can be brought to FLRW form by rescaling coordinates. Hence, to make $\beta_\pm(\chi)$ dynamical one needs to allow for contributions coming from multiple shapes, i.e., modes with different values for $\beta^J_\pm$.

In order to compute the sum for the volume (160) and the anisotropies (161), we need to specify the single-mode expectation values $N_J(\chi)$, and the coefficients v_J and $\beta^J_\pm$. The volume eigenvalues v_J, as mentioned when defining the operators (87) and (112), are imported from LQG; a procedure to (numerically[3]) compute them is well-known [68, 69], and will be briefly recalled in Section 4.2.1. The expectation values $N_J(\chi)$ were discussed in Chapter 3: since for a single mode $V(\chi) = v_J N_J(\chi)$, we already have the explicit expressions for the two approaches in (132) and (154). The definitions of $\beta^J_\pm$, on the other hand, deserve some further elucidation, which we provide below.

4.2 DEFINING $\beta^J_\pm$: THE TRISOHEDRAL TETRAHEDRON

Unlike for quantities such as areas and volumes, there is no fundamental operator in LQG corresponding to Misner variables $\beta_\pm$. Our task is therefore to give a proposal for $\beta^J_\pm$ as functions of such more fundamental geometrical quantities, defined at the level of each tetrahedron. Areas and volume, in turn, are determined by the spins j_I and intertwiner ι, as depicted in Figure 7 and mentioned in the spin network overview of Section 2.2.4 (we will provide a more detailed exposition in Section 4.2.1). We think of the fundamental tetrahedra as embedded into a manifold with Bianchi I metric, and reconstruct parts of that metric from geometric quantities of a tetrahedron, as originally proposed for GFT in [183]. There is clearly some ambiguity in any such procedure, given that the required embedding is not part of the GFT formalism but additional input. Our proposal is a relatively direct extension of previous work in GFT cosmology:

Recall that for a given set of spins one has multiple intertwiners ι^κ labelled by κ (cf. (58) and (44)); the choice of the specific intertwiner is discussed in the following.

3 Given that these eigenvalues can only be computed numerically there is little or no hope to be able to solve the sums (161) in a closed form. We will have to truncate them and make some suitable choices for the modes.

we follow the idea [305, 306] that equal spins for a four-valent node (i.e., $j_I = (j, j, j, j)$) capture a discrete notion of isotropy; hence such a configuration should correspond to $\beta_{\pm} = 0$. Departures from this microscopic notion of isotropy are, in a sense, assumed to add up coherently to a macroscopic definition of anisotropy.

For any excitation associated to J (cf. (59)), the spins j_I determine the areas of faces of a tetrahedron (see Figure 7); more precisely, the eigenvalues of the area operator $\hat{A}_I$ associated to the I-th face of a quantum tetrahedron are given by formula (34), i.e., $l_0^2\sqrt{j_I(j_I+1)}$. However, these areas are not sufficient to determine the shape of a tetrahedron; there is a two-sphere's worth of different tetrahedra with given face areas [69], so additional assumptions are needed to identify a given quantum state with a configuration of a classical tetrahedron. The idea of [305, 306] is that for equal spins we should choose a *regular* tetrahedron, whose edges are all of equal length. To justify this assumption one fixes the intertwiner (implicitly appearing in the multi-index J, cf. (59)) to maximise the volume eigenvalue, as this would represent a situation which goes as close as possible to the classical picture. More generally, we decide to focus on *orthocentric* [169] simplices (discussed in some detail in Section 4.2.1). Importantly, orthocentric tetrahedra maximise the volume for given face areas, and the regular tetrahedron is a particular example of orthocentric tetrahedra. This motivates us to always choose the largest allowed volume eigenvalue for given face areas, and hence spins j_I; we will then interpret our states as orthocentric tetrahedra. For a few specific shapes, we explicitly show in Section 4.2.1 that the largest volume eigenvalue for a given mode is indeed the closest one to the classical volume of an orthocentric tetrahedron for the same face areas. In this sense, a choice for the intertwiner is dictated by the comparison that we make between GFT models and classical geometry.

A regular tetrahedron is the simplest platonic solid; it is specified by a single number (e.g., edge length, height, distance between opposite edges) and has fixed dihedral angles between its congruent faces.

The sum over $J = (j_I, m_I, \kappa)$ in (160) and (161) then reduces to spins j_I and magnetic indices m_I only, since for each choice of spins the intertwiner ι^κ (cf. (58)) is already fixed in all the sums from now on. Symbolically, we are left with

$$\sum_J = \sum_{j_I}\sum_{m_I}\sum_{\kappa} \qquad \Rightarrow \qquad \sum_{j_I}\sum_{m_I}, \tag{162}$$

since we always use the largest volume eigenvalues in such sums. Note that we still need to make sure that each combination of j_I included in the sum allows for a nonvanishing intertwiner.

A very minimal requirement for our definition of $\beta^J_\pm$ (now implicitly associated to the intertwiner fixed by the j_I) is that it should vanish for $j_I = (j, j, j, j)$, but be nonzero if at least one of the four spins differs from the others. The simplest "non-isotropic" (but still orthocentric) building block one can think of is a tetrahedron we will refer to as *trisohedral*. As one can see in Figure 9, this too is a quite particular shape, with three isosceles triangles (called "sides" with area A) and an equilateral one (called "base" of the tetrahedron, with area B).

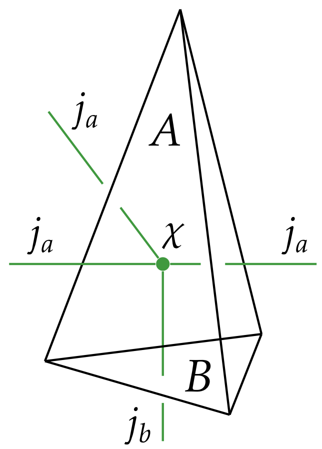


Figure 9: The trisohedral tetrahedron has two types of areas, edges, or dihedral angles (between sides and between a side and the base), *et cetera*. It generalises the regular tetrahedron, still remaining rather specific. We propose to associate a notion of anisotropy to such a non-equilateral shape.

Following the analogy from before, we then make the assumption that such a tetrahedron is represented by modes of the form

$$j_I = (j_a, j_a, j_a, j_b)\,, \tag{163}$$

where the spin j_a is associated with the area of the sides and j_b with the base (cf. (34)). Again, this assumption is partially justified by choosing an intertwiner that maximises the volume eigenvalue. It should be clear, however, that for given face areas this volume eigenvalue will not exactly match the classical volume of a trisohedral tetrahedron. We again refer to [69] and Section 4.2.1 for more discussion of this in the context of LQG.

In order to specify the numbers $\beta^J_\pm = \beta^{(j_a,j_a,j_a,j_b)}_\pm \equiv \beta^{j_a,j_b}_\pm$, we now think of the trisohedral tetrahedron of Figure 9 as embedded in a locally rotationally symmetric (LRS) Bianchi I spatial slice. This is a

Bianchi I geometry with only one preferred direction in which the expansion (or contraction) differs from the other two directions. The (spatial) line element follows from (418) and (419),

$$\begin{aligned} dl^2 &= a_1^2(dx^2 + dy^2) + a_2^2 dz^2 \\ &= V^{2/3}e^{2\beta_+}(dx^2 + dy^2) + V^{2/3}e^{-4\beta_+}dz^2 \,. \end{aligned} \tag{164}$$

Because of the symmetry of our building block, the model only needs two different scale factors, so we have set $\beta_- = 0$. Now consider a tetrahedron embedded in such a space, with one of its triangles lying on the $x - y$ plane. The tetrahedron is chosen such that it would be regular with respect to the background “fiducial" coordinates x, y and z; but its physical geometry depends on the dynamical variables.[4] In particular, the areas of the equilateral base and the isosceles sides are

$$\begin{aligned} B &= \frac{3\sqrt[6]{3}}{2}e^{2\beta_+}V^{2/3}\,, \\ A &= \frac{\sqrt[6]{3}}{2}e^{2\beta_+}\sqrt{1 + 8e^{-6\beta_+}}V^{2/3}\,. \end{aligned} \tag{165}$$

We identify B and A with the area eigenvalues associated to the spins j_b and j_a (see Figure 9). We then see that a configuration in which B and A are different would be interpreted as anisotropy ($\beta_+ \neq 0$) whereas $B = A$ corresponds to $\beta_+ = 0$, as anticipated.

Inverting either one of equations (165), one can express the anisotropy parameter as a function of face areas and volume, and define this to be the “anisotropy” associated to the tetrahedron. In the quantum theory, A, B and V are represented as eigenvalues determined by the spins (the volume eigenvalue also depends on the intertwiner, but we have fixed this as discussed above). This then leads us to possible proposals for how to define $\beta_+^{j_a,j_b}$ in (161). In order to compute these, we now review the geometrical notions needed for explicit calculations.

4.2.1 *Classical and quantum geometry of tetrahedra*

In this section we give a brief overview of important aspects regarding the geometry of tetrahedra, in particular recalling quantum properties coming from $SU(2)$ recoupling theory and adopted in loop quantum gravity. After a review part mainly focussing on the LQG

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4 We fix the edge length $l = \sqrt{2}\sqrt[3]{3}$ such that the physical volume of the tetrahedron is equal to V in (164).

volume eigenvalues, we conclude this section with the definition of our new "local anisotropy parameters" $\beta_+^{j_a,j_b}$; these will then allow us to continue the analysis of our anisotropic GFT model in later sections, where we will be studying its relational dynamics.

In general, the shape of a tetrahedron requires the knowledge of six quantities to be determined unambiguously; if only the areas of the four faces are known the space of possible configurations forms a two-sphere [69]. However, there is a unique *orthocentric* tetrahedron for given face areas, which is the one of maximal volume [169]. A tetrahedron is orthocentric if and only if all three pairs of opposite edges are perpendicular, or equivalently

$$e_1^2 + e_2^2 = e_3^2 + e_4^2 = e_5^2 + e_6^2 \,, \tag{166}$$

where $e_1, \ldots, e_6$ are the six edge lengths (such that e_1 is opposite to e_2, and so on). For such 3-simplices, an analogue of Heron's formula

$$16\Delta^2 = (a+b+c)(a-b+c)(a+b-c)(-a+b+c)\,, \tag{167}$$

which gives the area Δ of a triangle given its sides (a, b and c), can be derived. In the cases of interest for this chapter one finds that face areas A or (A, B) and the volume V are related by

$$\begin{aligned} V_{\text{reg}}^2 &= \frac{8}{27\sqrt{3}} A^3 \,, \\ V_{\text{tri}}^2 &= \frac{1}{27\sqrt{3}} B(9A^2 - B^2)\,, \end{aligned} \tag{168}$$

for the regular and the trisohedral tetrahedron of Figure 9, respectively. In the quantum theory, we take the analogue of an orthocentric tetrahedron to be the $SU(2)$ intertwiner with largest volume eigenvalue; we explain the concept of LQG volume eigenvalues in what follows.

LQG VOLUME EIGENVALUES. Following the terminology introduced in Chapter 2, we focus on a single four-valent node, with spins labelling its links denoted by $j_I = (j_1, j_2, j_3, j_4)$ (see Figure 7). To each representation j_I, with $I = 1, \ldots, 4$, we associate a $(2j_I + 1)$-dimensional vector space $\mathcal{H}_{j_I}$ that carries the action of the $SU(2)$ generators $\vec{J}_I$ (see (28) and discussion thereafter). The Hilbert space of the quantum tetrahedron then reads $\mathcal{H}_4 = \text{Inv}\left[\mathcal{H}_{j_1} \otimes \ldots \otimes \mathcal{H}_{j_4}\right]$, and objects that live in this space are called intertwiners (cf. (30)). A nonvanishing intertwiner can only exist if the j_I sum to an integer. We introduce a basis labelled by κ in the recoupling channel $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$, where the index κ ranges between $\kappa_{\text{min}} = \max\{|j_1 - j_2|, |j_3 - j_4|\}$ and $\kappa_{\text{max}} = \min\{j_1 + j_2, j_3 + j_4\}$ in integer steps. The Hilbert space $\mathcal{H}_4$ is d-dimensional with $d = \kappa_{\text{max}} - \kappa_{\text{min}} + 1$. States on this space can be understood as describing quantum tetrahedra, as firstly pointed out in [43, 51].

We refer to the literature (see, e.g., [125]) for the full derivations of the geometrical eigenvalues in the context of LQG. Here we only recall that the area operator takes the form $\widehat{A} = l_0^2 \sqrt{\vec{J} \cdot \vec{J}}$, where l_0 is a fundamental quantum gravity length scale (usually taken to depend on the Barbero–Immirzi parameter of LQG [50, 232]). States in $\mathcal{H}_4$ are eigenstates of the operator $\widehat{A}_I$, which measures the area of the I-th face of the quantum tetrahedron, with eigenvalues $l_0^2 \sqrt{j_I(j_I+1)}$ (cf. (34)). Moreover, the volume operator introduced for a tetrahedron in LQG reads [32, 342]

$$\widehat{V} = \frac{\sqrt{2}}{3} l_0^3 \sqrt{|\epsilon_{ijk} J_1^i J_2^j J_3^k|}\,. \tag{169}$$

We can focus on the radicand in (169). Without showing all the details of the calculation (which can be found in [69, 94]), one defines the operator $\widehat{Q} = \vec{J}_1 \cdot (\vec{J}_2 \times \vec{J}_3)$ with matrix element $Q_{\kappa,\kappa'} := \langle \kappa | \widehat{Q} | \kappa' \rangle$, where κ and κ' label intertwiner states as explained above. One can then show that nonvanishing matrix elements are obtained only if κ and κ' differ by 1. Thus one can denote $a_\kappa = \mathrm{i} Q_{\kappa,\kappa-1}$, so that

$$Q = \begin{pmatrix} 0 & \mathrm{i}a_1 & 0 & \cdots \\ -\mathrm{i}a_1 & 0 & \mathrm{i}a_2 & \\ 0 & -\mathrm{i}a_2 & 0 & \\ \vdots & & & \ddots \end{pmatrix} \tag{170}$$

is a $d \times d$ matrix. With the above conventions, the matrix elements are found to be [92, 93]

$$a_\kappa = \frac{1}{4} \frac{\sqrt{(j_1+j_2+\kappa+1)(-j_1+j_2+\kappa)(j_1-j_2+\kappa)(j_1+j_2-\kappa+1)}}{\sqrt{2\kappa+1}} \times \frac{\sqrt{(j_3+j_4+\kappa+1)(-j_3+j_4+\kappa)(j_3-j_4+\kappa)(j_3+j_4-\kappa+1)}}{\sqrt{2\kappa-1}}\,. \tag{171}$$

To obtain a compact expression, (171) can be cast in terms of Heron's formula (167) as

$$a_\kappa = \frac{4}{\sqrt{4\kappa^2-1}} \Delta\left(j_1+\frac{1}{2}, j_2+\frac{1}{2}, \kappa\right) \Delta\left(j_3+\frac{1}{2}, j_4+\frac{1}{2}, \kappa\right) . \quad (172)$$

Hence, for given spins j_I computing the spectrum of the volume operator amounts to finding the d eigenvalues of (170) (let us denote them $q^\kappa_{j_I}$) and then computing, according to (169),

The lighter notation v_J, hiding all the labels of $v^\kappa_{j_I}$ into the multi-index J, is adopted throughout the document. The only exception is given by this section, where all the details are explicitly shown.

$$v^\kappa_{j_I} = \frac{\sqrt{2}}{3} l_0^3 \sqrt{q^\kappa_{j_I}} . \quad (173)$$

Note that if $j_I = (j, j, j, j)$ the matrix elements (172) simplify to

$$a^j_\kappa = \frac{(2j\kappa+\kappa)^2 - \kappa^4}{4\sqrt{4\kappa^2-1}} , \quad (174)$$

while for $j_I = (j_a, j_a, j_a, j_b)$ (as in (163)) they read

$$\begin{aligned} a^{j_a,j_b}_\kappa &= \frac{\sqrt{(2j_a\kappa+\kappa)^2-\kappa^4}}{4\sqrt{4\kappa^2-1}} \\ &\times \sqrt{(j_b-j_a+\kappa)(j_a+j_b-\kappa+1)(j_a-j_b+\kappa)(j_a+j_b+\kappa+1)} . \end{aligned} \quad (175)$$

These matrix elements can be used to find the maximal volume eigenvalues $v^{\max}_j$ and $v^{\max}_{j_a,j_b}$ for the quantum shapes corresponding to regular and trisohedral tetrahedra, respectively; see Table 1 for examples.

For large spins, these volume eigenvalues show semiclassical behaviour (see, e.g., [67, 69, 94]), and so the "eigenvalue-counterparts" (using (34)) of equations (168),

$$\begin{aligned} (v^{\max}_j)^2/l_0^6 &\approx \frac{8}{27\sqrt{3}} [j(j+1)]^{3/2} , \\ (v^{\max}_{j_a,j_b})^2/l_0^6 &\approx \frac{1}{27\sqrt{3}} \sqrt{j_b(j_b+1)}\, [9j_a(j_a+1) - j_b(j_b+1)] , \end{aligned} \quad (176)$$

become more accurate as the spins grow. Here $v^{\max}$ means that we fix the intertwiner label κ so as to obtain the largest volume eigenvalue.

In Figure 10 and Figure 11 we show the possible volume eigenvalues v^κ_j and $v^\kappa_{j_a,j_b}$ in units of l_0^3 (dots), compared with the maximal classically allowed volume (curve and surface) for the same face areas (given by $l_0^2\sqrt{j(j+1)}$). We also show how the relative difference between this classical volume (for orthocentric tetrahedra) and the highest LQG volume eigenvalue decreases as the spins increase. Figure 10 focuses on the case of equal areas $j_a = j_b$, whereas Figure 11 shows various other combinations of j_a and j_b. Notice that both the classical and quantum volume of a tetrahedron are no longer well-defined for $j_b > 3j_a$. In the quantum theory, this constraint comes about because the dimension d of the Hilbert space $\mathcal{H}_4$ needs to be positive. The right panel of Figure 11 shows, as an illustrative example, the relative distances between the maximum eigenvalues and the surface along the plane $j_a = 2j_b$, but the qualitative behaviour is similar for other ratios.

In terms of (classical) geometry arguments, one easily sees that the trisohedral tetrahedron of Figure 9 cannot exist if $B > 3A$.

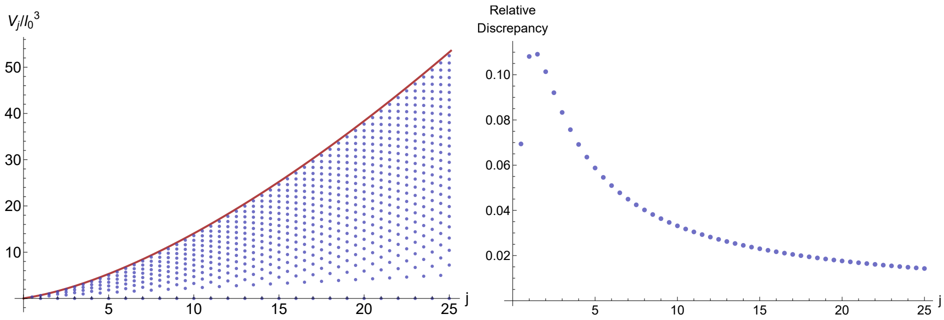

Figure 10: Left: comparison between LQG volume eigenvalues v_j^κ/l_0^3 (every dot for a given j corresponds to a different κ) and the classical volume of an equilateral tetrahedron as function of the area (curve). Right: relative difference between the classical volume and largest LQG eigenvalue. After two initial anomalies the mismatch decreases indefinitely: it becomes reasonably small ($\sim 1\%$) when $j \sim 40$, and goes down to $\sim 0.1\%$ when $j \sim 350$.

Figure 11 generalises the setting by depicting the dependence on two different spins j_a and j_b; indeed, the blue dots in the left panel of Figure 11 give the left panel of Figure 10.

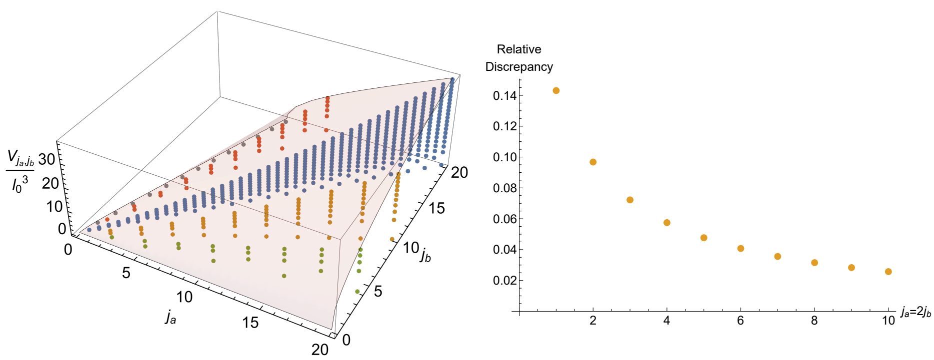

Figure 11: Left: comparison between some of the LQG volume eigenvalues $v^\kappa_{j_a,j_b}/l_0^3$ (dots) and the classical volume of a trisohedral tetrahedron (transparent surface) as function of the two areas. We plot eigenvalues along specific planes for which $j_a = j_b$ (blue), $j_a = 2j_b$ (orange), $j_a = 5j_b$ (green), $2j_a = j_b$ (red) and $3j_a = j_b$ (grey). The latter gives vanishing eigenvalues only and represents the limiting case as there are no tetrahedra (no nonzero intertwiners) for $3j_a < j_b$. Right: relative discrepancy along the plane $j_a = 2j_b$.

ANISOTROPIES. We now finally turn to the question of how to define a notion of anisotropy at the level of quantum geometry, in terms of a function $\beta_+^J = \beta_+^{j_a,j_b}$ for our trisohedral building blocks. While such a definition is not unique, we will identify the best candidate to play such a role.

Starting from (165), there are three ways to classically define the anisotropy of a tetrahedron in terms of two other geometric quantities: by inverting any one of (165) or combining them, one can obtain classical expressions such as $\beta_+(V,B)$, $\beta_+(V,A)$ or $\beta_+(A,B)$. Once we pick a favoured definition, the strategy is to replace classical expressions by LQG eigenvalues to obtain an effective (discrete) notion of anisotropy as a function of the spins associated to the faces of a tetrahedron.

For instance, inverting the first equation in (165), one can readily obtain

$$\beta_+(V,B) = \frac{1}{2}\log\left(\frac{2}{3\sqrt[6]{3}}\frac{B}{V^{2/3}}\right), \tag{177}$$

and replace the classical area B with $l_0^2\sqrt{j_b(j_b+1)}$ and the classical volume with the maximal eigenvalue $v^{\max}_{j_a,j_b}$ to get

$$\beta_+^{j_a,j_b} = \frac{1}{2}\log\left(\frac{2}{3\sqrt[6]{3}}\frac{l_0^2\sqrt{j_b(j_b+1)}}{(v^{\max}_{j_a,j_b})^{2/3}}\right). \tag{178}$$

Notice that by definition $\beta_+^{j_a,j_b} > 0$ when $j_a < j_b$ (naively, for volumes smaller than the equilateral ones). (178) could be used as a definition, but we can identify a number of issues. First, it cannot be written more explicitly, since the volume eigenvalue $v^{\max}_{j_a,j_b}$ is only obtained numerically, as outlined above. Moreover, (178) is not well-defined when the volume is exactly zero, and it is nonvanishing when $j_a = j_b$.

The same arguments would apply if we defined the anisotropy starting from the second relation in (165). In this case we would obtain an even more cumbersome expression $\beta_+(V,A)$, which could then be turned into a $\beta_+^{j_a,j_b}$ as a function of j_a, j_b and volume eigenvalues $v^{\kappa}_{j_a,j_b}$.

A third definition is obtained by taking the ratio of equations (165). The advantage of this choice is that one can get rid of the volume:

$$\frac{A}{B} = \frac{1}{3}\sqrt{1+8e^{-6\beta_+}}\,. \tag{179}$$

We can now rearrange to obtain a simpler definition of the anisotropy parameter as a function of A and B, which straightforwardly turn into the spins j_a and j_b. Indeed, starting from

$$\beta_{+}(A,B) = -\frac{1}{6}\ln\left(\frac{9A^2 - B^2}{8B^2}\right), \tag{180}$$

our candidate reads

$$\beta_{+}^{j_a,j_b} := -\frac{1}{6}\ln\left(\frac{9j_a(j_a+1) - j_b(j_b+1)}{8j_b(j_b+1)}\right). \tag{181}$$

This definition again (conventionally) gives a negative value for $j_a > j_b$, and crucially it also gives $\beta_{+}^{j_a,j_b} = 0$ for "isotropic" configurations, as we would like to demand. Moreover, it is always finite and well-defined regardless of the volume eigenvalue associated to the same pair (j_a, j_b).

Table 1 shows the maximum volume eigenvalues (cf. (173)), expressed in units of l_0^3, and the corresponding (dimensionless) anisotropies (181) for given spins $j_I = (j_a, j_a, j_a, j_b)$. Even though we may use larger spins in some of our calculations, we only give values up to $j_a = j_b = 7/2$.

j_a	j_b	$v^{\max}_{j_a,j_b}$	$\beta_{+}^{j_a,j_b}$
1/2	1/2	0.310	0
1/2	3/2	0	0.384
1	1	0.620	0
1	2	0.620	0.231
1	3	0	0.462
3/2	1/2	0.620	-0.284
3/2	3/2	0.993	0
3/2	5/2	1.075	0.172
3/2	7/2	0.931	0.324
2	1	1.075	-0.196
2	2	1.425	0
2	3	1.560	0.138

j_a	j_b	$v^{\max}_{j_a,j_b}$	$\beta_{+}^{j_a,j_b}$
5/2	1/2	0.931	-0.427
5/2	3/2	1.560	-0.153
5/2	5/2	1.910	0
5/2	7/2	2.086	0.116
3	1	1.520	-0.315
3	2	2.086	-0.126
3	3	2.444	0
7/2	1/2	1.241	-0.526
7/2	3/2	2.111	-0.254
7/2	5/2	2.653	-0.107
7/2	7/2	3.022	0
...			

Table 1: Maximal volume eigenvalues and anisotropy parameters for a range of spins.

Even though we consider the definition (181) to be well-motivated for GFT cosmology, the ambiguity we have highlighted here introduces, at least in principle, an additional uncertainty into the cosmological interpretation of GFT models. To quantify this uncertainty we can compare the various definitions we have discussed. To do this we plot in Figure 12 two-dimensional slices of the anisotropy dependence on the spins j_a and j_b, first for constant $j_b = \frac{1}{2}$ and along the plane $j_a = 2j_b$.

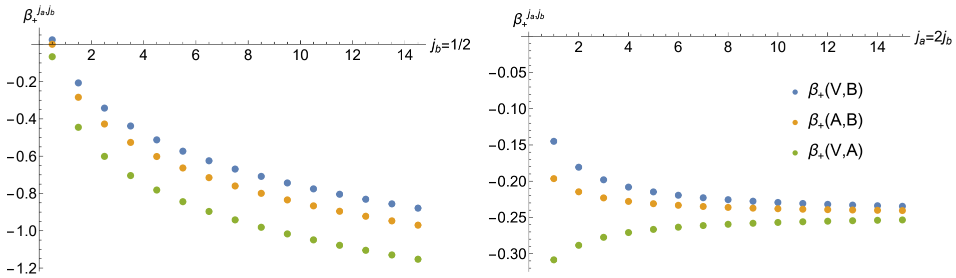

Figure 12: Left: For constant j_b, we see that the discrepancy between the three $\beta_+^{j_a,j_b}$ definitions is larger away from $j_a = j_b$. Right: Choosing $j_a = 2j_b$ shows how the three definitions agree more and more for increasing spin.

To summarise, we have compared different definitions for $\beta_+^{j_a,j_b}$ derived from (165), with the aim to obtain an effective local anisotropy associated to the mode (163) of the quantum tetrahedron of Figure 9. These definitions do not agree in general: this is because we fixed the volume of the tetrahedron to match with the quantity V in the metric assuming that we have an orthocentric tetrahedron, and there is no quantum eigenvalue satisfying exactly the same relations between volume and areas. Only one definition is simple and satisfies our desired property $\beta_+^{(j,j,j,j)} = 0$; this is given by (181). If we assume $j \geq \frac{1}{2}$, the expression for $\beta_+^{j_a,j_b}$ in (181) is always finite and well-defined, regardless of the volume eigenvalue for the given mode (as opposed to the other definitions) which may vanish for some spin configurations. For instance, $v_{(1,1,1,3)} = 0$.

With this definition, we are finally in a position to evaluate (161). We now focus on the initial conditions needed for tackling the sums.

4.3 INITIAL CONDITIONS

Independently of the approach one follows, the kinetic term (80) is characterised by the coupling (99), which for a mode specified by (163) reads

$$\omega_J^2 = m^2 - M^2\Big(3j_a(j_a+1) + j_b(j_b+1)\Big)\,. \tag{182}$$

Since we would like to include a number of Peter–Weyl modes in (161) for which $\omega_J^2 > 0$, the coupling M needs to be small relative to the "mass" m. For any given ratio M/m there will be maximum spins after which (182) becomes negative; for instance, choosing $M/m = 0.1$ means that $3j_a(j_a+1) + j_b(j_b+1) < 100$ for $\omega_J^2 > 0$. In the numerical analysis below, we will assume $M/m = 0.1$.

Given that we are not interested in the bounce or the connection between the contracting and expanding phases, we will be looking at χ-symmetric solutions, for which the minimum of the volume is at $\chi = 0$. Thus, in the general expressions of Section 3.3, we set $A_J = B_J =: \alpha_J$ in the algebraic approach (specifically in (128) and (132)) and $\hat{C}_J(0) = 0$ in the deparametrised formulation (cf. (154)). The expectation values for single-mode number operators then become

Recall that $A = A_J$ and $B = B_J$ in $\sigma_J(\chi)$ depend on J; this index was dropped in Section 3.3.1 since we were working with a single mode. Similar remarks hold for the operator $\hat{C} = \hat{C}_J$ (150) of the deparametrised approach of Section 3.3.2.

$$\begin{aligned} N_J^{\text{alg}}(\chi) &= |\sigma_J(\chi)|^2 \\ &= 2|\alpha_J|^2 \left(1 + \cosh(2\omega_J\chi)\right) = 4|\alpha_J|^2 \cosh^2\left(\omega_J\chi\right) , \end{aligned} \tag{183}$$

for the algebraic scheme with a mean-field approximation, and

$$N_J^{\text{dep}}(\chi) = -\frac{1}{2} + \left(N_J^{\text{dep}}(0) + \frac{1}{2}\right) \cosh(2\omega_J\chi) , \tag{184}$$

for the deparametrised approach. In (183), α_J is related to the number of quanta at $\chi = 0$ as $N_J^{\text{alg}}(0) = 4|\alpha_J|^2$. In the deparametrised approach, we then fix $N_J^{\text{dep}}(0) = 4|\alpha_J|^2$ so that the initial conditions are the same for both methods.

We then need to fix the range of the sums in (162). We are interested in modes of the form (163), specified by two spins j_a and j_b. In principle, the sums over j_a and j_b run from $\frac{1}{2}$ to ∞; but in practice, they have to be truncated because the eigenvalues are only computed numerically. Furthermore, not all combinations are allowed by $SU(2)$ recoupling theory, as not all of them allow for a nonvanishing intertwiner. Moreover, modes corresponding to zero volume are not really physically interesting and would not allow for a useful notion of anisotropy. In principle, all these considerations are to be taken into account. We will simplify these issues greatly by only keeping a few modes.

We also need to sum over magnetic indices m_I (cf. (162)). None of the geometric observables we are considering depend on m_I, so the m_I index corresponds to a degeneracy factor for physically indistinguishable modes. Given this, we will assume that the initial condition parameters $N_J(0) = N_{j_I,m_I}(0) = N_{j_I}(0)$ do not depend on m_I (i.e., the coefficients $\alpha_J = \alpha_{j_I,m_I} = \alpha_{j_I}$ in (183) and (184) only depend on the spins). The sums over m_I then just return additional multiplicative factors,

Recall (cf. (29)) that $m \in [-j, j]$ is the familiar magnetic index of $SU(2)$ recoupling theory.

$$\sum_{j_I}\sum_{m_a}\sum_{m_a}\sum_{m_a}\sum_{m_b} = \sum_{(j_a,j_b)} (2j_a+1)^3(2j_b+1)\,. \tag{185}$$

We then still need to truncate these sums to a few chosen values for j_a and j_b. One motivation for keeping only a small number of modes (but more than one) is the increasing arbitrariness coming from the need to specify initial conditions for each additional mode. This question of dependence on initial conditions generally affects cosmological models derived from fundamental quantum gravity, which tend to depend on some specific choice of (class of) initial states (see, e.g., [9–11, 120] for models from LQG).

There is then a choice of which modes to include, in particular whether j_a and j_b should be large or small. Here two main factors may come into play. At the *kinematical* level, the quantum properties of tetrahedra show semiclassical geometrical features for large spins [67, 69, 94]; large spins allow for more possible discrete (quantum) states, so that classical concepts such as orthocentric tetrahedra can be approximated closely. Indeed, the largest volume eigenvalues we choose are always smaller than the volume of the corresponding orthocentric tetrahedron, but the relative difference decreases as the spins grow (we have seen this feature in Section 4.2.1 for two specific examples). This would suggest keeping modes associated to large values of (j_a, j_b), since the geometric picture of orthocentric tetrahedra would be closer to the volume eigenvalues used.

On the other hand, we know that for the type of GFT dynamics considered here (namely based on the free theory with kinetic term (80)), the lowest spins will dominate the sum eventually [176], and larger spins give less important contributions to the dynamics at late times (see comments below (99) and at the end of Section 3.2.2). This is why in previous work the sum was often trivialised to only the smallest spins, or to a few values. This truncation is analogous to loop quantum cosmology where the spins are usually all fixed to be $j = \frac{1}{2}$ (but see [61] for an attempt at generalisation). Therefore, even though using large spins might make sense kinematically, when we include the dynamics we see that larger spins will quickly become insignificant for the evolution of the cosmological model. Finally, on general grounds one would expect that, for any fundamentally discrete (simplicial) approach to quantum gravity, a useful continuum limit is obtained for large simplicial complexes made of little simplices, rather than by magnified semiclassical building blocks.[5] Thus, we will only consider a few relatively small spins.

We now need to specify initial conditions given by the initial number of quanta $N_{j_I}(0) = N_{j_a,j_b}(0)$, which we then use for both the algebraic and deparametrised approaches. These initial conditions are of course in general arbitrary, but for concreteness we assume they follow a Gaussian distribution

$$N_{j_a,j_b}(0) = \mathcal{N} \exp\left(\frac{-(j_a - \overline{j_a})^2 - (j_b - \overline{j_b})^2}{\sigma^2}\right), \tag{186}$$

where we set $N_{j_a,j_b} = 0$ for modes with vanishing volume. For continuous parameters, such distributions allow for analytical integrations, as we will show in a toy model in Section 4.5. Here the spins are discrete, so we are considering a "discrete Gaussian distribution", assuming that the initial configuration is dominated by values around $(\overline{j_a}, \overline{j_b})$. By setting these away from $j_a = j_b = \frac{1}{2}$, we will see some initial (anisotropic) contribution coming from larger spins, before the lowest ones take over dynamically. Moreover, given that we effectively set to zero all terms after an arbitrary spin, it is reasonable to choose initial quanta having occupancy numbers that *gradually* go to zero, as modes approach the last one. This motivates using a Gaussian also in the discrete case. The initial conditions (186) are then determined by the peaks $(\overline{j_a}, \overline{j_b})$, standard deviation σ and normalisation factor $\mathcal{N}$.

4.4 THREE MODES WITH FIXED BASE

We now focus on a simple model in which we assume the spin associated to the base of the tetrahedron to be fixed to its minimum value $j_b = \frac{1}{2}$. We keep three modes associated to the lowest (allowed) spin values for the sides of the tetrahedron, $j_a = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\}$. This is a simple generalisation of the single-mode case reviewed in Section 3.3, since we consider two modes which encode anisotropy (as defined in Section 4.2) and one associated with equilateral tetrahedra (see Figure 13).

5 This resonates with the analysis of the semiclassical regime of spin foams mentioned at the end of Section 2.2.4, where recent work aims to include more refined triangulations (i.e., more vertices) other than studying large spin asymptotics [18, 143, 217].

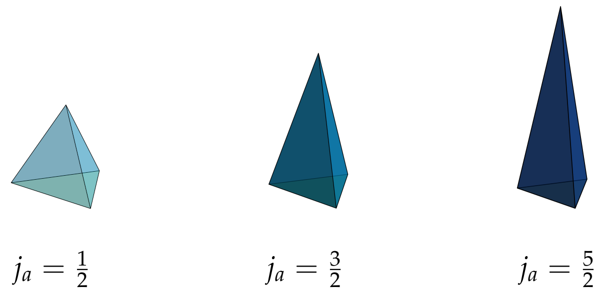

Figure 13: Three modes represent tetrahedra with the same base $j_b = \frac{1}{2}$ and three different area values for the sides. Note that the first mode corresponds to the regular tetrahedron.

Denoting the eigenvalues $v_J = v_{j_a,j_b}$ and the parameters $\beta_+^J = \beta_+^{j_a,j_b}$, and noticing that $(2j_b + 1) = 2$ in this case, we truncate (160) and (161) after the first three contributions obtaining

$$V(\chi) = 2\left(2^3\, v_{\frac{1}{2},\frac{1}{2}} N_{\frac{1}{2},\frac{1}{2}}(\chi) + 4^3\, v_{\frac{3}{2},\frac{1}{2}} N_{\frac{3}{2},\frac{1}{2}}(\chi) + 6^3\, v_{\frac{5}{2},\frac{1}{2}} N_{\frac{5}{2},\frac{1}{2}}(\chi)\right), \quad (187)$$

and

$$\beta_+(\chi) = \frac{2}{N(\chi)}\left(\cancel{2^3\, \beta_+^{\frac{1}{2},\frac{1}{2}} N_{\frac{1}{2},\frac{1}{2}}(\chi)} + 4^3\, \beta_+^{\frac{3}{2},\frac{1}{2}} N_{\frac{3}{2},\frac{1}{2}}(\chi) + 6^3\, \beta_+^{\frac{5}{2},\frac{1}{2}} N_{\frac{5}{2},\frac{1}{2}}(\chi)\right), \quad (188)$$

where $N(\chi) = 2\sum_{j_a=1/2}^{5/2}(2j_a+1)^3 N_{j_a,\frac{1}{2}}(\chi)$. Since $\beta_+^{\frac{1}{2},\frac{1}{2}} = 0$, the first term in (188) is zero. The other values of v_{j_a,j_b} and $\beta_+^{j_a,j_b}$ can be found in Table 1.

Our initial condition function (186) now only depends on j_a. We fix the peak to be at $\overline{j_a} = \frac{3}{2}$; σ is chosen to be around 1 so that the initial distribution is peaked at $j_a = \frac{3}{2}$ but also includes some quanta in the other modes. $\mathcal{N}$ is fixed by the requirement that the initial state should be reasonably semiclassical; given that we only focus on expectation values of operators, quantum fluctuations should not be too large (see [184] for a detailed analysis of relative fluctuations). As these fluctuations decrease with $N \gg 1$ (this is a generally expected behaviour also found, e.g., in [273]), we demand that initially all modes have an expected particle number of at least 25. Figure 14 shows different σ choices according to these criteria.

In Figure 15, we plot the effective Friedmann equation $[V'(\chi)/V(\chi)]^2$ and the anisotropy evolution $\beta_+(\chi)$ stemming from (187) and (188). We compare the effective Friedmann equation with the classical (relational) Bianchi dynamics reviewed in Appendix A (namely, solutions of (426) and (431)), where we set $\beta_- = 0$ because of the local

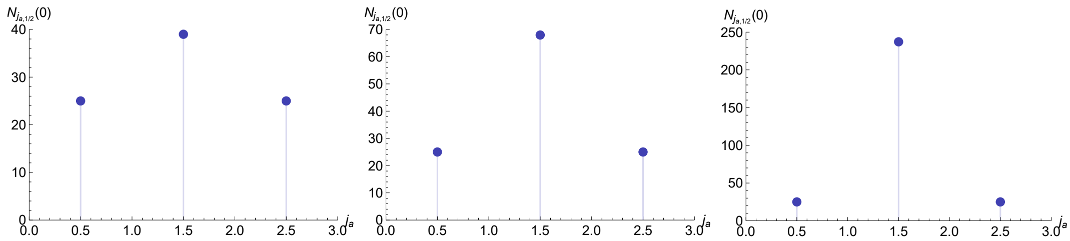

Figure 14: Initial conditions $N_{j_a,\frac{1}{2}}(0)$ for different standard deviations, $\sigma = \frac{3}{2}, 1, \frac{2}{3}$. $\mathcal{N}$ is chosen such that the number of quanta at $\chi = 0$ for the first and third mode is 25.

rotational symmetry of our GFT model. While the Bianchi I model is a natural comparison, we also compare with Bianchi II as this is a less simple classical anisotropic cosmology, which deviates from Bianchi I only at late times (which is what the GFT description of the anisotropy does too, albeit in a different way). The other free parameters of the classical plots are fixed as follows: Newton's constant G is determined by demanding that at late times we follow the isotropic solution $[V'(\chi)/V(\chi)]^2 = 12\pi G$; the slope $d\beta_+/d\chi$ is obtained from a linear fit of the initial part of the GFT anisotropy dynamics.

Note that we will not discuss in detail how the the geometry of the GFT modes (163) relates to the Bianchi II metric; we only show in Figure 15 a qualitative contrast with the Bianchi II model for broader comparison.

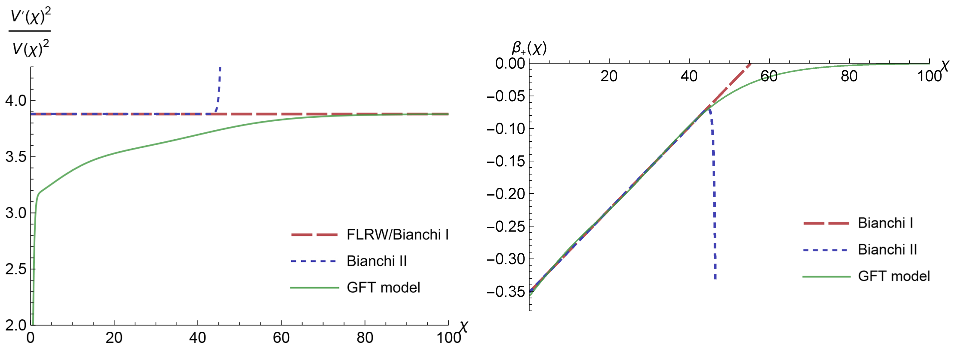

Figure 15: Effective Friedmann equation $[V'(\chi)/V(\chi)]^2$ and evolution of anisotropy $\beta_+(\chi)$. GFT couplings are set to $m = 1$, $M = 0.1$, while the Gaussian parameters are fixed as in Figure 14 with $\sigma = 1$. Different parameter choices do not change the qualitative behaviour of these plots.

The value of p_χ could, in principle, be fixed from the fundamental GFT model by using the quantity playing the role of a conjugate momentum of the scalar χ. In the algebraic approach one identifies

$$p_\chi = \sum_J \omega_J Q_J , \tag{189}$$

as discussed below (134) (with $Q_J = 2\mathrm{Im}(A_J \bar{B}_J)$ given in (133) but here generalising the expressions to the case of multiple Peter–Weyl modes). For our states with $A_J = B_J$, the quantities Q_J are actually zero, but one could introduce an arbitrary phase into A_J or B_J to obtain a nonvanishing p_χ. In the deparametrised formulation, the conjugate momentum of χ is given by the (relational) Hamiltonian

$$p_\chi = \sum_J \langle \hat{H}_J \rangle \,, \tag{190}$$

as explained below (157) (again here for multiple modes); (190) could then also be evaluated using coherent states. In either case, these expressions yield a relatively small p_χ and the Bianchi II curve in the plots would deviate from the linear Bianchi I behaviour very quickly, which is not what we see in the GFT model. In an attempt at obtaining a better fit, in Figure 15 we fix p_χ by assuming that the non-linear behaviour of Bianchi II appears roughly when the function $\beta_+(\chi)$ in GFT ceases to be linear.

Notice that the constant dashed red curve in the left panel represents Bianchi I but is indistinguishable from the asymptotic FLRW limit because the classical anisotropy backreaction (cf. (426)) would be very small: it scales as $(\mathrm{d}\beta_+/\mathrm{d}\chi)^2 \sim 4 \times 10^{-5}$ if we take $\mathrm{d}\beta_+/\mathrm{d}\chi$ as the slope of the initially linear part of the GFT expression for $\beta_+(\chi)$.

Figure 15 shows the dynamical "isotropisation" already mentioned in the literature [321]. As discussed below (99) and (123) for the two approaches of (3.3), this late-time limit is inevitable given that, for a model involving multiple Peter–Weyl modes, the mode with the largest value that ω_J can take will end up dominating the dynamics [176]. In our case this is achieved by the smallest spins $j_a = j_b = \frac{1}{2}$; this mode dominates at some point and will then dominate indefinitely (see Figure 16). When this happens, the effective Friedmann equation reaches a constant plateau (described by the first term in (187)) while the anisotropy gradually converges to zero (the crossed-out term in (188) now dominates the average). The constant value taken by $(V'/V)^2$ in this asymptotic limit is then the one we would have obtained for a single-mode model corresponding to the FLRW Universe. In the left panel of Figure 15 we label this constant value as "FLRW/Bianchi I" since, as we explained, the difference between FLRW and Bianchi I is too small to be noticeable in the plot.

The period before the asymptotic isotropisation (but after the bounce) can be compared with the classical Bianchi I model. In the left panel of Figure 15 we see that the GFT curve does not show a constant $[V'(\chi)/V(\chi)]^2$ (with value higher than the asymptotic FLRW value),

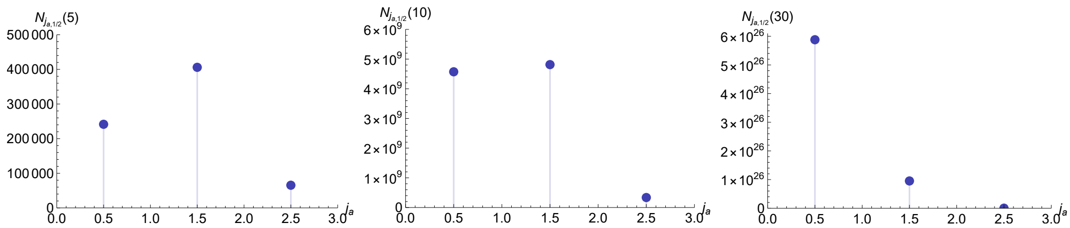

Figure 16: As time elapses, the peak moves towards the smallest spin until the mode with $j_a = j_b = \frac{1}{2}$ dominates forever. Soon after $\chi \sim 40$ the anisotropic contributions will become insignificant.

as the classical dynamics (426) would suggest. Instead, the green line shows the transition between different modes, which will eventually stop when the last mode takes over and dominates. This behaviour can be made more or less evident by changing the standard deviation σ in (186). We will see an example with a manifest transition between modes later on.

In the right panel of Figure 15 we see that $\beta(\chi)$, on the other hand, shows a much better agreement with classical (relational) cosmology. The anisotropy is approximately linear for a reasonably long time after the bounce, as would be expected from general relativity. Because of the tendency to isotropise, this agreement will obviously stop at some point as the anisotropy will approach zero; but before that point, $\beta(\chi)$ compares well with a classical Bianchi I model. It is worth reiterating here that in this construction we are ignoring GFT interactions (cf. Section 3.3), which become important when χ becomes sufficiently large.[6] Hence these plots are not to be trusted for too late times, when the weak-coupling assumption breaks down. In particular, we might never reach isotropisation but simply remain in a phase where $\beta(\chi)$ is approximatively linear, before interactions take over and change the picture completely.

Regardless of when $\beta_+(\chi)$ ceases to be linear, it is always monotonic; so in principle, as in classical cosmology, it could be used as a relational clock for GFT cosmology instead of the *ad hoc* introduced scalar field χ. Adding matter arbitrarily is often seen as an inescapable necessity plaguing any cosmological model coming from a background-independent quantum gravity theory, given the absence of (time) coordinates. In contrast, this model contains a gravitational degree of freedom which could potentially be used as relational time, and perhaps help addressing issues such as the problem of time or

6 When the total particle number is large, one expects correlations between the GFT quanta to be non-negligible. $N(\chi)$ grows quickly; with our choice of parameters, $N(5) \sim 10^7$, $N(25) \sim 10^{24}$ and $N(50) \sim 10^{45}$.

clock dependence in quantum cosmology (for more on this see, e.g., [272] and Chapter 6).

APPROACH INDEPENDENCE. So far, we have completely glossed over the question whether the time evolution of the particle number in each mode follows (183) in the algebraic approach, or (184) in the deparametrised approach. It turns out that, as one might have expected, the two approaches give identical results after a very short initial phase directly after the bounce, see Figure 17.

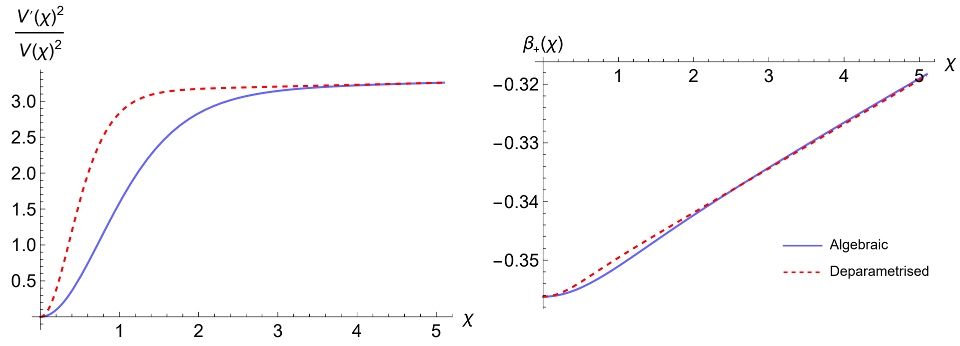

Figure 17: Effective Friedmann equation and anisotropy dynamics at high energies directly after the bounce, for both approaches and with the same parameters (fixed as in Figure 15).

The discrepancy between the two approaches can be traced back to the different effective Friedmann equations for a single mode, as discussed in Section 3.3 (see (134) and (156)). After the $1/V(\chi)^2$ contribution dominates (giving rise to the bounce), but before the constant term of the Friedmann equation takes over, there is a difference in the $1/V(\chi)$ term which in the algebraic approach depends on the arbitrary "GFT energy" E_J of (133) (which happens to be negative for (186) and our choice of parameters), whereas in the deparametrised approach it is fixed and positive. The plot for $\beta_+(\chi)$ also shows a minor difference between the two approaches. Changing initial conditions changes the details of these differences, but the underlying features are always the same; and given that we are interested in comparing later times (larger volumes) with classical models, these differences do not play a role in our analysis. Already for these small values of χ we observe an almost-constant $[V'(\chi)/V(\chi)]^2$ and quasi-linear $\beta_+(\chi)$.

4.5 mean-field continuous toy model

As discussed, the general expressions (160) and (161) must be evaluated numerically, even in simple cases such as the model of Section 4.4. In order to obtain some analytical result, one can consider toy models which allow explicit solution due to some simplifying assumptions. The model presented in this section uses the solutions in the mean-field approximation of the algebraic approach (183), but drops the assumption of discrete spins j_I.

Recall that the classical relations (165) relate the two types of area (for the base B and the sides A of the trisohedral tetrahedron) with the volume and the anisotropy parameter,

$$\begin{aligned} B &= B(V, \beta_+)\,, \\ A &= A(V, \beta_+)\,. \end{aligned} \tag{191}$$

Hence, at the classical level, the pair (V, β_+) is in one-to-one correspondence with the face areas (A, B); it would be easier to take these variables as the basic characterisation of a tetrahedron, expressing A and B as functions of them. We will do this here, and forget about the fact that in LQG (and GFT) the fundamental variables are discrete areas.

If we focus on V and β_+, we can make another simplifying assumption: we assume that the volume per tetrahedron is simply fixed to be $V = V_0$, and that only β_+ varies *continuously*. We follow the picture coming from single-mode truncations of GFT that the evolution of the total volume comes only from adding or removing building blocks, rather than changing their "size"; but we still have a range of different "shapes". This avoids the previously inevitable mixing of effects coming from modes with both different volume eigenvalues and different values of β_+.

We want to work with the same GFT kinetic term, i.e., the same expression (182) for the effective couplings for each mode. But given that we no longer have discrete spins, we need to rewrite this expression using V and β_+. We will do this by using the LQG relation $A_I^2 = l_0^4 j_I(j_I + 1)$ for area eigenvalues (cf. (34)) and the classical relations (165). We can then write down a relation between the discrete and toy models for modes of the form (163),

$$3j_a(j_a+1)+j_b(j_b+1) \doteq \frac{3A^2+B^2}{l_0^4} \\ = \frac{(3V_0)^{4/3}}{l_0^4}\left(e^{4\beta_+}+2e^{-2\beta_+}\right) . \tag{192}$$

The symbol $\doteq$ means that we are equating quantum eigenvalues with classical quantities using the LQG interpretation given to the spins. This relation is then substituted into the time-dependent expression of the average particle number (183), with initial conditions fixed by a suitable choice of α_J which we again take to be a Gaussian, $\alpha_{\beta_+} = \exp\{-(\beta_+ - \overline{\beta_+})^2/(2\sigma^2)\}$. Given that we do not have discrete modes, we do not have to worry about a specific normalisation factor. This is now a continuous normal distribution, the analogue of (186).

What used to be called α_J in (183) now gets renamed α_{β_+} as there are no spins here, and anisotropy is encoded directly in the continuous β_+ variable.

The "toy model counterparts" of (160) and (161) then read

$$V(\chi) = V_0 \int \mathrm{d}\beta_+ \, \alpha_{\beta_+}^2 \cosh^2(w_{\beta_+}\chi) \, , \tag{193}$$

and

$$\beta_+(\chi) = \frac{V_0}{V(\chi)} \int \mathrm{d}\beta_+ \, \beta_+ \alpha_{\beta_+}^2 \cosh^2(w_{\beta_+}\chi) \, , \tag{194}$$

where we defined the continuous analogue of (182) (via (192)),

$$w_{\beta_+}^2 = m^2 - M^2 \left(\frac{3V_0}{l_0^3}\right)^{4/3} \left(e^{4\beta_+}+2e^{-2\beta_+}\right) . \tag{195}$$

Numerical evaluation of $[V'(\chi)/V(\chi)]^2$ and $\beta_+(\chi)$ shows qualitatively similar behaviour to the plots in Figure 15 derived in the previous three-mode model; see Figure 18 for an example. Again we see an asymptotic "isotropisation" leading to the effective FLRW limit at late times, and an approximately linear evolution for the parameter β_+. We can now also strengthen these results by analytical approximations.

In order to find analytical solutions to the integrals (193) and (194) we assume small anisotropy, i.e., a Gaussian distribution with $\overline{\beta_+} \ll 1$. This justifies approximating the cosh function under the integral up to second order in β_+,

$$\cosh^2(w_{\beta_+}\chi) \approx \cosh^2(w_0\chi) - 6\beta_+^2 \frac{M^2(3V_0/l_0^3)^{4/3}\sinh(2w_0\chi)}{w_0}\chi \, , \tag{196}$$

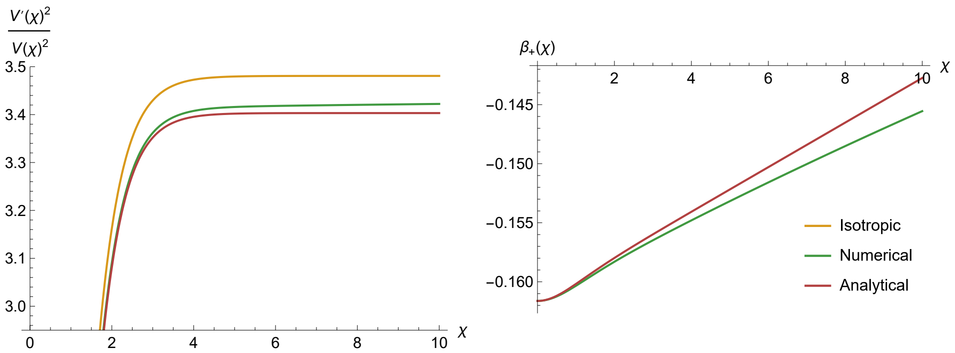

Figure 18: The effective Friedmann equation $(V'/V)^2$ quickly becomes constant in the analytical approximation and grows slowly in the numerics, but always gives values lower than the isotropic FLRW case (orange line). The right panel shows that the anisotropy is well approximated by the linear behaviour seen in general relativity. The late-time limit shows isotropisation: $(V'/V)^2$ tends to the orange constant and $\beta_+(\chi)$ goes to zero. Here we fixed $m = 1$, $M = 0.1$, $V_0/l_0^3 = 1$, $\overline{\beta_+} = -0.1616$ and $\sigma = 0.15$.

where $w_0 = \sqrt{m^2 - 3M^2(3V_0/l_0^3)^{4/3}}$. This approximation leads to integrals over β_+ that can be done immediately, and relatively simple expressions such as

$$V(\chi) \approx \sqrt{\pi}\, V_0\, \sigma \left(\cosh^2(w_0\chi) - \frac{3M^2(3V_0/l_0^3)^{4/3}(2\overline{\beta_+} + \sigma^2)\, \chi\, \sinh(2w_0\chi)}{w_0} \right) . \tag{197}$$

To simplify these further we then use the fact that M/m is small (see discussion below (182)) and expand $\beta_+(\chi)$ and $[V'(\chi)/V(\chi)]^2$ up to second order in M/m. This yields a simple analytical expression for the anisotropy,

The tanh *function quickly approaches* 1 *as* χ *elapses.*

$$\beta_+(\chi) \approx \overline{\beta_+} - \frac{12M^2(3V_0/l_0^3)^{4/3}\overline{\beta_+}\sigma^2}{m} \chi \tanh(m\chi) , \tag{198}$$

and the effective Friedmann equation

$$\begin{aligned} \left(\frac{V'(\chi)}{V(\chi)}\right)^2 &\approx 4m^2 \tanh^2(m\chi) \\ &\quad - 6M^2(3V_0/l_0^3)^{4/3} \left(4\overline{\beta_+}^2 + 2\sigma^2 + 1\right) \mathcal{F}(m\chi) , \end{aligned} \tag{199}$$

where $\mathcal{F}(m\chi) = \operatorname{sech}^2(m\chi)\,(2m\chi + \sinh(2m\chi))\tanh(m\chi)$. We can see that the anisotropy (198) is essentially a linear function of χ very soon after the bounce, with a slope dependent on parameters of our Gaussian such as $\overline{\beta_+}$ and σ. Moreover, noticing that $\mathcal{F}(m\chi) \overset{|\chi|\to\infty}{\to} 2$, we also obtain a constant late-time limit for the effective Friedmann equation,

$$\left(\frac{V'(\chi)}{V(\chi)}\right)^2 \overset{|\chi|\to\infty}{\sim} 4\left[m^2 - 3M^2(3V_0/l_0^3)^{4/3}\left(4\overline{\beta_+}^2 + 2\sigma^2 + 1\right)\right] . \tag{200}$$

By comparing with exact numerical results we can see that these approximations cannot be trusted for too large χ, see Figure 18.

In the limit in which we "switch off" anisotropic contributions, $\overline{\beta_+} \to 0$ and $\sigma \to 0$, (198) vanishes and in (200) we have $(V'/V)^2 \to 4m^2 - 12M^2(3V_0/l_0^3)^{4/3} = 4w_0$, which is nothing but the orange line in Figure 18. In (192) this corresponds to $4j(j+1) \doteq 3(3V_0/l_0^3)^{4/3}$ when the spins are all equal.

As a final comment, recall that in the classical Bianchi I model the Friedmann equation gets a constant contribution $(\mathrm{d}\beta_+/\mathrm{d}\chi)^2$ compared to the FLRW Universe (see Appendix A). We can ask whether the terms in (200) that depend on $\overline{\beta_+}$ and σ are related to the derivative of (198). But we see that

$$\begin{aligned}\left(\frac{\mathrm{d}\beta_+}{\mathrm{d}\chi}\right)^2 \overset{|\chi|\to\infty}{\sim} &\frac{144M^4(3V_0/l_0^3)^{8/3}\overline{\beta_+}^2\sigma^4}{m^2} \\ &\neq -\frac{4}{3}M^2\left(3V_0/l_0^3\right)^{4/3}\left(4\overline{\beta_+}^2 + 2\sigma^2\right) .\end{aligned} \tag{201}$$

As already noticed in the full GFT model of the previous section, contrary to what happens in general relativity, in our toy model the presence of anisotropy decreases $(V'/V)^2$. The two quantities we are comparing also are of different orders of magnitude, in particular different powers of the small ratio M/m.

Overall, our toy model could reproduce the main qualitative features seen in the full GFT analysis, in particular a nearly linear growth in the anisotropy for a range of χ and a negative contribution to the effective Friedmann equation.

4.6 INCLUDING MORE MODES INTO GFT MODELS

We now return to the discrete setting of full GFT, presenting results for models that go beyond the simple case described in Section 4.4. The easiest extension of what we showed before is to include more than three modes, but keep the assumption of a fixed base area (i.e., $j_b = \frac{1}{2}$). This means we add shapes to Figure 13 for greater j_a which are increasingly more stretched ("anisotropic"). We find that such an extension gives results that are not qualitatively different from the previous case, regardless of how many modes of this form we add. In Figure 19 we show the case of five modes, letting j_a range between $\frac{1}{2}$ and $\frac{9}{2}$.

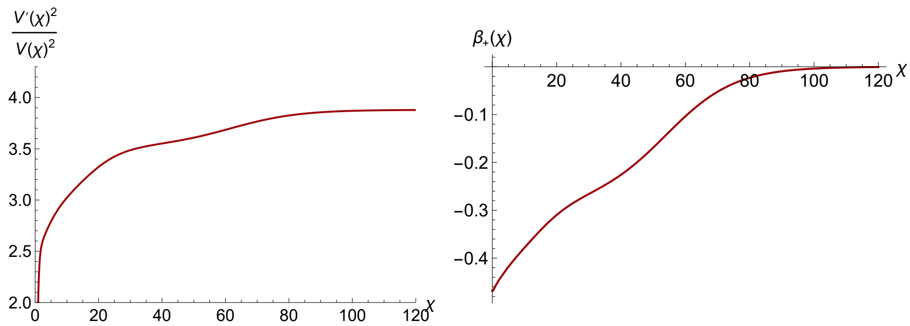

Figure 19: The main features encountered in Figure 15 are reproduced in this model: $[V'(\chi)/V(\chi)]^2$ shows shifts between modes before reaching the plateau for $j_a = j_b = \frac{1}{2}$, and $\beta_+(\chi)$ can be approximated by a linear function for some time before going to zero. Initial conditions and parameter choices are shown in Figure 20.

Since $j_b = \frac{1}{2}$ for all modes, we can see from (181) that the values of β_+ for the modes we consider all have the same sign, and are increasingly more negative as j_a grows. This is why $\beta_+(\chi)$ does not lose its monotonicity property in Figure 19. From the geometrical point of view, the shapes we are adding are progressively more stretched along one axis so that their local anisotropy never flips the direction in which it changes.

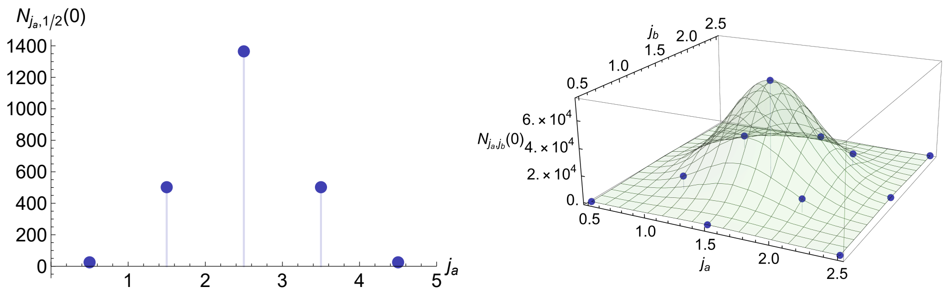

Figure 20: Initial conditions (defined as in (186)) for five and for eleven modes. As in Figure 14, $\mathcal{N}$ is such that the number of quanta is 25 in the modes which are the farthest from the peak. Left: $j_b = \frac{1}{2}$, $j_a \in \{\frac{1}{2}, \ldots, \frac{9}{2}\}$, $\overline{j_a} = \frac{5}{2}$ and $\sigma = 1$. Right: $(j_a, j_b) \in \{\frac{1}{2}, \ldots, \frac{5}{2}\}$, $\overline{j_a} = \overline{j_b} = \frac{3}{2}$ and $\sigma = \frac{1}{2}$. Combinations of spins with zero volume are not included.

A more important generalisation of our model can be obtained by relaxing the assumption that j_b is fixed to $\frac{1}{2}$. If j_b can vary, the first minor novelty comes from the fact that we now have some combinations of spins which need to be removed; these are spin configurations for which volume eigenvalues are zero. For instance, when $j_b = 3j_a$ the volume eigenvalue vanishes (see Table 1 and Section 4.2.1), as one might expect from classical arguments (the tetrahedron of Figure 9 flattens into a plane when $B = 3A$, and areas (34) scale linearly in j for large j). A second key novelty comes from the fact that not all the $\beta_+^{j_a,j_b}$ parameters have the same sign: in the dynamical evolution, we now no longer obtain a strictly monotonically increasing sequence of $\beta_+^{j_a,j_b}$ values associated to the modes dominating at different times. Hence, we generically find a non-monotonic $\beta_+(\chi)$. We show in Figure 21 an example with eleven generic modes, defined by letting both spins vary in the range $\{\frac{1}{2}, \ldots, \frac{5}{2}\}$ and excluding the ones not allowed by $SU(2)$ recoupling theory. See right panel of Figure 20 for a depiction of the initial number of quanta in the allowed modes. One could change initial conditions such that $\beta_+(\chi)$ is monotonic even with variable base spin j_b, by choosing *ad hoc* modes such that the succession of dominant $\beta_+^{j_a,j_b}$ values goes to zero monotonically.

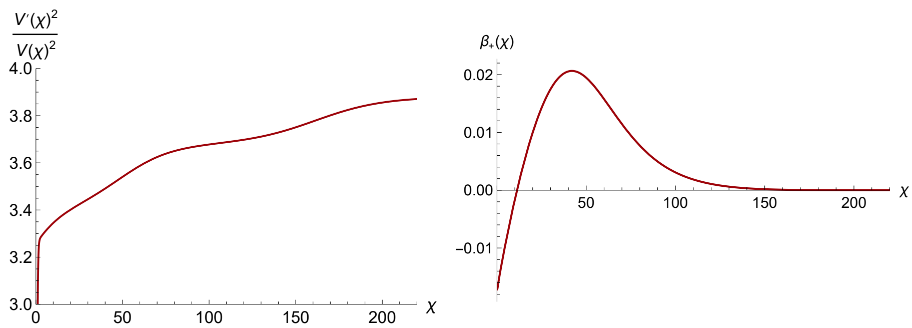

Figure 21: Shifts between modes in $[V'(\chi)/V(\chi)]^2$ are accentuated thanks to a smaller standard deviation in (186), $\sigma = \frac{1}{2}$. Anisotropy can decrease because these modes have a non-monotonic sequence of dominant $\beta_+^{j_a,j_b}$ values as χ elapses. Initial conditions and parameter choices are shown in Figure 20.

While any number of modes can be included without any computational obstacle, we do not report additional details on these many-mode scenarios because they are characterised by a larger arbitrariness encoded in further initial conditions, without introducing important novelties.

4.7 CONCLUSION AND OUTLOOK

The work described in this chapter represents a first attempt at characterising homogeneous but anisotropic (Bianchi) cosmologies within the framework of group field theory, focusing on the simplest Bianchi I model with an additional local rotational symmetry. Previous work in GFT cosmology has dealt with anisotropies perturbatively, demonstrating a decay process leading to isotropisation, but lacked a precise measure to quantify anisotropy. In this chapter we have proposed such a measure of anisotropy and studied its dynamics in simplified models.

Inspired by Misner's parametrisation of Bianchi models, we introduced GFT analogues of the anisotropy parameters $\beta_{\pm}$, which behave like free massless scalar fields on a flat FLRW background. We defined anisotropy in terms of expectation values of GFT operators, assigning a microscopic value of anisotropy at the level of each Peter–Weyl mode, and dividing by the total number of GFT quanta to create an "intensive" average quantity.

Having discussed various options for the effective notion of anisotropy at the local (quantum geometry) scale, and defined one which satisfies the demand that a tetrahedron with equal face areas is considered isotropic, we then showed the dynamics of the expectation value of the total volume and of the newly introduced anisotropy observable. Results indicate partial agreement with general relativity: while the volume dynamics differ from classical expectations, the anisotropy shows the expected linear evolution before isotropisation. We explored multiple scenarios, focussing in particular on a model with three Peter–Weyl modes. An analytical toy model, assuming anisotropy is a continuous parameter and fixing the volume per tetrahedron, could reproduce analytically the main features of the discrete models.

Future research directions stemming out of this work will be thoroughly described in Chapter 7. We only emphasise here that this work allows for further cosmological investigations and also entirely new lines of research. The first include: studying GFT interactions for Bianchi models, analysing the effects of anisotropy on the cosmological bounce (given that anisotropies can disrupt bounce scenarios [88]), and comparing GFT results with Bianchi models in loop quantum cosmology (in particular investigating whether the GFT anisotropic bounce corresponds to a Kasner transition [72, 378]). Moreover, novel

An extension of our work has already been suggested in [308], where the possibility of cosmological models retaining two anisotropies is considered.

research directions could also include the possibility of studying black hole dynamics in GFT (leveraging on the fact that the interior of a Schwarzschild black hole can be described by the anisotropic Kantowski–Sachs metric [239], which is very similar to Bianchi spacetimes), and exploring the use of anisotropy as a relational clock without coupling to an external scalar field (as we will stress in Chapter 6).

In the next chapter, we shift our focus to an equally important aspect of the GFT cosmology paradigm of Chapter 3: the quantum states used in the effective dynamical equations of the models of interest, and their semiclassical properties. Indeed, most of the literature (including the anisotropic models of this chapter) only considers (Fock) coherent states; we will introduce and investigate the general family of Gaussian states in Chapter 5.

5

GENERALISED GAUSSIAN STATES FOR $\mathfrak{su}(1,1)$ QUANTUM COSMOLOGY

This chapter defines and analyses Gaussian states for quantum cosmologies based on the $\mathfrak{su}(1,1)$ algebra, mainly focussing on the group field theory framework.[1] It is based on [100] and constitutes original research.

In any quantum theory based on a canonical formulation, one often requires semiclassical states to describe macroscopic phenomena. A prime example, closely related to what we will discuss in the following, is the description of a macroscopic electromagnetic field in terms of coherent or squeezed states in quantum optics. In general, an important question is whether coherence or semiclassical properties of an initially chosen state will be preserved under time evolution. This is famously the case for the harmonic oscillator (or free quantum fields) but not for more general interacting quantum systems.

In quantum gravity and quantum cosmology, identifying a semiclassical spacetime description is a rather crucial requirement, both conceptually and for making the link to the low-energy world in which we do not observe spacetime superpositions. In traditional Wheeler–DeWitt quantum cosmology, semiclassical spacetime was often identified in a WKB (Wentzel–Kramers–Brillouin) regime in which the wavefunction is assumed to be highly oscillating. This approximation is at the heart of applications to cosmological perturbation theory, in which the curved spacetime quantum field theory setting of inflationary cosmology emerges from the semiclassical limit of quantum cosmology (see, e.g., [214, 245, 271]). The notion of semiclassicality applied here is different from that of using coherent states or wavepackets; a WKB state is (by assumption) not localised in configuration space, but rather describes an entire family of classical trajectories "all at once".

Loop quantum gravity offers its own proposals for the semiclassical limit. In the canonical formulation of LQG, one works with quantum states living on superpositions of graphs (see the spin network

1 As we will explain, our results only rely on the $\mathfrak{su}(1,1)$ algebra generated by the operators of interest ((152) in GFT), and hence apply to more general models provided that one can give a cosmological interpretation to some of the generators.

paragraph in Section 2.2.4). A class of coherent states for LQG, which has found many applications in the literature, was proposed in [362, 364], whereas more recent proposals include [98, 164]. These states realise the traditional properties of coherent states, peakedness around a given classical configuration with small uncertainties. In general, given the rather complicated dynamics of full LQG (see comments at the end of Section 2.1), it is not clear whether these semiclassical properties would be preserved dynamically.

As we have seen in Chapter 3, the canonical framework of group field theory is closely related to the canonical formalism for LQG; Fock space quantisations of GFT (in both the algebraic and the deparametrised approaches) lead to state spaces that can be interpreted in terms of spin-network states of LQG (cf. Section 3.2). One may ask what kind of GFT quantum states could be used for a semiclassical, macroscopic limit of the theory, relevant in particular in the application to cosmology. As reviewed in Section 3.3, a particularly influential idea has been to use analogies with condensed matter physics and think of a "condensate" of quanta of geometry, or LQG spin-network vertices [183, 187, 294, 300]. Such a condensate can be characterised in a mean-field approximation, or equivalently using a Fock coherent state built on the fundamental field operators or the annihilation and creation operators of the theory. Many cosmological applications of GFT have focused exclusively on such coherent states in extracting a semiclassical limit, as reviewed in Section 3.3.

In [100] we broadened this perspective, and discussed a class of semiclassical states that go beyond the simplest choice of Fock coherent states. In particular, we could write down the most general *Gaussian* state associated with a single GFT (Peter–Weyl) field mode, and also discuss mixed states following the work of [40, 41, 250] on thermal states in GFT. This chapter provides an expositions of the results we found in [100]: our work builds on the work of [184], which had already discussed some more general types of coherent states built on the $\mathfrak{su}(1,1)$ algebra of observables most relevant for cosmology in the deparametrised approach (see Section 3.3.2), and extends all such results substantially, as explained below.

SEMICLASSICALITY CRITERIA. When discussing the relative merits of possible choices of semiclassical states, we need to be clear about what properties we require for a state to be considered semiclassical. Here we follow to a large extent the criteria set out in the context of

GFT in [184] and [273]; our main requirement for semiclassicality is that the relative uncertainty in the volume, $(\Delta\hat{V})/\langle\hat{V}\rangle$, can be made arbitrarily small, in particular at late times (large volumes) after dynamical evolution. This criterion is similar to what is often required for a semiclassical limit in loop quantum cosmology [29, 359]. In the deparametrised approach, we also require a small relative uncertainty in the Hamiltonian (associated with the matter coupled to gravity in GFT, see Section 3.2.2 and Section 3.3.2), which is time-independent. In contrast, one could also require that a semiclassical state saturate the lower bound on uncertainties implied by the uncertainty principle, in its stronger Robertson–Schrödinger form. We will argue that this second requirement seems less relevant physically, since the right-hand side of the uncertainty principle is in general state-dependent, and one can end up with an equality for which both sides are large. For the context of GFT and the most relevant cosmological observables, energy and volume, we will find a conflict between the two requirements: states with small relative uncertainties do not saturate the Robertson–Schrödinger inequality, while those that saturate the inequality do not have small relative uncertainties. This discrepancy was observed for squeezed states in [184]; again we generalise this discussion to general Gaussian states.

We will show that while general Gaussian states can be constructed using displacement, squeezing and thermality, semiclassical properties are mostly determined by the magnitude of displacement: squeezed or thermal states alone are not semiclassical in the sense we require, and hence do not lead to a good interpretation in terms of emergent cosmology. These results can be seen as justifying to an extent the emphasis on Fock coherent states in the GFT literature. While most of our analysis uses the deparametrised approach of Section 3.2.2 and Section 3.3.2, we also discuss generalised Gaussian states in the more commonly used algebraic approach of Section 3.2.1 and Section 3.3.1. In that setting, we find that generalisations of simple coherent states are difficult to construct, and only very simple versions of squeezing and thermality can be straightforwardly defined. We also encounter a number of technical issues related to divergences in the definition of states and observables. Ignoring these as much as possible, the general qualitative statements agree with those found in the deparametrised approach.

$\mathfrak{su}(1,1)$ QUANTUM COSMOLOGY. While we are mostly interested in GFT models, our results are much more generally applicable to any $\mathfrak{su}(1,1)$ cosmological scenario. Recall that the $\mathfrak{su}(1,1)$ Lie algebra naturally emerges in the deparametrised approach to GFT cosmology (see in particular (151)–(153)); however, this is not the only scenario with this feature. Indeed, as pointed out in [59], isotropic models of loop quantum cosmology and (bosonic) GFT cosmology can be seen as different realisations of the same underlying structure (sometimes called "harmonic cosmology" [79, 80]). The $\mathfrak{su}(1,1)$ Lie algebra was then also investigated in detail in the context of loop cosmology in [266], where it was realised that dynamics could be implemented as $SU(1,1)$ transformations, and it was later associated with the "Complexifier-Volume-Hamiltonian (CVH) algebra" in [62] (see also [73–75] for more recent work).

We have seen that states play a very different role in the two approaches described in Chapter 3; this is why we present their respective results separately here.

OUTLINE OF THE CHAPTER. In Section 5.1 we discuss different definitions of semiclassicality, and study the examples of coherent and squeezed states explicitly in Section 5.1.1 and Section 5.1.2. This analysis is then extended to general Gaussian states in Section 5.2. All such investigations deal with the deparametrised approach to GFT cosmology (cf. Section 3.3.2), where no technical or conceptual obstacle is encountered. Given that the algebraic approach to canonical quantisation is used in most of the GFT literature (cf. Section 3.3.1), in Section 5.3 we discuss our efforts at obtaining similar types of states in that approach. Specifically, we write down generalised Gaussian (condensate-like) states in Section 5.3.1, and study technical issues of condensates in Section 5.3.2 and Section 5.3.3. Section 5.4 summarises the main results. Some of the explicit analytics and computational techniques for this chapter are given in Appendix B.

5.1 SEMICLASSICAL PROPERTIES AND CANDIDATE STATES

Recall that in the general setup of (isotropic) GFT cosmology we deal with a single Peter–Weyl mode; this assumption is used throughout.

While true for any quantum state, the effective Friedmann equation (156), found in the deparametrised approach to GFT cosmology (as described in Section 3.3.2), is a relation between expectation values only. To claim that (156) is a good description of the dynamics of cosmological observables, one needs to adopt quantum states that show some semiclassical features, such as coherent states. More generally, one needs to specify criteria for any candidate state for cosmology to be considered as semiclassical. Here we focus on two criteria that are commonly used: the study of relative uncertainties and the Robertson–Schrödinger uncertainty principle. We define variances and covariances for any operators $\hat{A}$ and $\hat{B}$ as

$$(\Delta\hat{A})^2 = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 \,, \tag{202}$$

$$\Delta(\hat{A}\hat{B}) = \frac{1}{2}\langle \{\hat{A}, \hat{B}\} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle \,, \tag{203}$$

where $\{\cdot, \cdot\}$ is the anticommutator.

Our first criterion for semiclassicality would be to require that relative uncertainties $(\Delta\hat{A})^2/\langle\hat{A}\rangle^2$ be small at least in a large-volume or late-time regime where the classical theory is expected to emerge; here $\hat{A}$ could be either the Hamiltonian $\hat{H}$ or volume $\hat{V}$ of the deparametrised, single-mode theory of Section 3.3.2. One can also check what happens to the operator $\hat{C}$ defined in (150), even though its interpretation is less transparent, hence it is unclear whether this operator would need to be semiclassical.

There is another characterisation of semiclassical states that makes use of the quantities (202) and (203), namely the saturation of the Robertson–Schrödinger (RS) uncertainty principle [332, 348]. For the GFT operators (149) and (150), the uncertainty principle reads

$$(\Delta\hat{V})^2(\Delta\hat{H})^2 \geq |\Delta(\hat{V}\hat{H})|^2 + \omega^2\langle\hat{C}\rangle^2 \,. \tag{204}$$

For basic examples in standard quantum mechanics an inequality of this type is saturated (i.e., it becomes an equality) for canonically conjugate pairs when using coherent (or more generally Gaussian) states; but in general it is not guaranteed that there are states for which (204) can be minimised. As the volume evolves in time, the RS uncertainty principle (204) is a statement for each χ.

In the context of GFT cosmology or quantum cosmology in general, demanding small relative uncertainties seems physically more relevant than minimising uncertainties by demanding equality in (204); nothing in (204) requires both sides to be small in any sense, whereas the Universe appears to be sharp to observations, without quantum effects on large scales. Hence, we would say that a good candidate state for GFT cosmology models primarily needs to show small relative uncertainties. As we will see shortly, (Fock) coherent states have this property; we will also define more general states that are semiclassical in this sense.

From (154) we can derive the χ-dependent form of the volume variance as well as that of the covariance between the volume and the Hamiltonian,

$$(\Delta\hat{V}_\chi)^2 = (\Delta\hat{V})^2 \cosh^2_{2\omega\chi} + (\Delta\hat{C})^2 \sinh^2_{2\omega\chi} + \Delta(\hat{V}\hat{C}) \sinh_{4\omega\chi}\,, \tag{205}$$

$$\Delta(\hat{V}_\chi\hat{H}) = \Delta(\hat{V}\hat{H}) \cosh_{2\omega\chi} + \Delta(\hat{C}\hat{H}) \sinh_{2\omega\chi}\,, \tag{206}$$

Here and in the rest of this chapter we adopt a somewhat unusual notation f_α instead of $f(\alpha)$ for trigonometric and hyperbolic functions; this is to save space in lengthy expressions below.

where from now on we use subscripts to indicate time-dependent operators $\hat{\mathcal{O}}_\chi = \hat{\mathcal{O}(\chi)}$; operators with no subscript refer to initial conditions (i.e., to $\chi = 0$). We can then immediately derive the large-volume limit of relative uncertainties by taking $\chi \to \pm\infty$ in these expressions: $(\Delta\hat{H})^2/\langle\hat{H}\rangle^2$ does not evolve in time, but for the relative volume fluctuations we find using (205) and (154)

$$\frac{(\Delta\hat{V}_\chi)^2}{\langle\hat{V}_\chi\rangle^2} \overset{\chi\to\pm\infty}{\longrightarrow} \frac{(\Delta\hat{V})^2 + (\Delta\hat{C})^2 \pm 2\Delta(\hat{V}\hat{C})}{\left(\langle\hat{V}\rangle + \frac{v}{2} \pm \langle\hat{C}\rangle\right)^2}\,. \tag{207}$$

To verify the RS uncertainty principle, we would also need the limit

$$\frac{(\Delta(\hat{V}_\chi\hat{H}))^2}{\langle\hat{V}_\chi\rangle^2\langle\hat{H}\rangle^2} + \omega^2\frac{\langle\hat{C}_\chi\rangle^2}{\langle V_\chi\rangle^2\langle\hat{H}\rangle^2} \overset{\chi\to\pm\infty}{\longrightarrow} \frac{\left(\Delta(\hat{V}\hat{H}) \pm \Delta(\hat{C}\hat{H})\right)^2}{\left(\langle\hat{V}\rangle + \frac{v}{2} \pm \langle\hat{C}\rangle\right)^2\langle\hat{H}\rangle^2} + \frac{\omega^2}{\langle\hat{H}\rangle^2}\,. \tag{208}$$

(207) and (208) are crucial in order to explicitly obtain relatively simple expressions for all the states discussed in this chapter, see Appendix B for details.

All these quantities are determined by the initial conditions only. Notice that the late-time limit $\chi \to +\infty$ in general differs from the limit $\chi \to -\infty$ (as indicated by the $\pm$ notation) so that the asymmetry described by the quantity $\hat{C}$ (cf. (150)) is manifest here.

One might also be interested in the evolution of these quantities beyond the strict infinite-volume limit $\chi \to \pm\infty$. We will show some examples for general evolution of the RS inequality (204) and the relative uncertainties of the volume operator. For analytical results, we derive expansions in powers of the inverse volume as

$$\frac{(\Delta\hat{V}_\chi)^2}{\langle\hat{V}_\chi\rangle^2} = \mathcal{A} + \mathcal{B}\frac{v}{\langle\hat{V}_\chi\rangle} + \mathcal{C}\frac{v^2}{\langle\hat{V}_\chi\rangle^2} + \cdots\,, \tag{209}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... are functions of initial conditions. Such an expansion captures very well the full evolution as soon as we are in the macroscopic regime $\langle\hat{V}_\chi\rangle \gg v$.

We shall see that the saturation of (204) does not necessarily indicate that the states under question have small relative fluctuations; conversely, states such as the simplest Fock coherent states, which are semiclassical by looking at relative fluctuations, fail to minimise (204). In Section 5.1.1 and Section 5.1.2 we briefly review quantum states that have been investigated in the context of GFT cosmology, expanding the literature by explicitly checking whether the RS uncertainty principle is minimised and obtaining the exact dynamics of the relative uncertainties for the volume in closed form. These properties will serve as comparison for the new family of states presented in Section 5.2.

5.1.1 *Coherent states*

The most commonly used states in the GFT cosmology literature are Fock coherent states as we explained in Section 3.3 (these were introduced already in [183] and used in different ways in the deparametrised formalism [186, 379] and the algebraic approach [272, 305, 306]). As is customary in bosonic theories, coherent states can be defined via the action of the displacement operator $\hat{D}(\alpha)$ on the Fock vacuum as

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle\,, \tag{210}$$

with

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a}}\,, \qquad \alpha \in \mathbb{C}\,. \tag{211}$$

These have the key property that, at $\chi = 0$, $\hat{a}(0)|\alpha\rangle = \alpha|\alpha\rangle$. Appendix B contains a table with all the quantities of interest computed with the state (210) (see in particular Section B.1 and Table 2).

Recall that we work in the Heisenberg picture.

One can easily check that the RS uncertainty principle (204) for $\hat{V}$ and $\hat{H}$ is *not* saturated by coherent states. For instance (using Table 2 in appendix Appendix B), we see that at $\chi = 0$ (204) reads

$$\frac{v^2\omega^2}{2}|\alpha|^2 + v^2\omega^2|\alpha|^4 \geq v^2\omega^2|\alpha|^4\,. \tag{212}$$

This feature occurs because (149) and (150) are $\mathfrak{su}(1,1)$ compositions of the bosonic ladder operators, whereas the Fock coherent state is coherent with respect to $\hat{a}$ and $\hat{a}^\dagger$. One can in fact show that (204) is *never saturated* by coherent states. We refer to Appendix B where we report explicitly the analytical expressions representing the general case of (204); such a minimisation does not happen at any time and in particular not as $\chi \to \pm\infty$, where the system is meant to become semiclassical.

As one might expect, this does not really spoil the semiclassical nature of coherent states in the sense of relative uncertainties. Decomposing α into modulus and argument as $\alpha = |\alpha| \exp(i\vartheta)$, one can see that the relative uncertainties at $\chi = 0$ (again, see Appendix B),

$$\begin{aligned} \frac{(\Delta\hat{V})^2_C}{\langle\hat{V}\rangle^2_C} &= \frac{1}{|\alpha|^2}\,, \\ \frac{(\Delta\hat{H})^2_C}{\langle\hat{H}\rangle^2_C} &= \frac{4|\alpha|^2+2}{4|\alpha|^4\cos^2_{2\vartheta}}\,, \\ \frac{(\Delta\hat{C})^2_C}{\langle\hat{C}\rangle^2_C} &= \frac{4|\alpha|^2+2}{4|\alpha|^4\sin^2_{2\vartheta}}\,, \end{aligned} \tag{213}$$

can be made arbitrarily small by choosing appropriate $|\alpha|$ and avoiding parameters for which ϑ is a multiple of $\frac{\pi}{4}$ (or of the form $\frac{\pi}{4} + k\frac{\pi}{2}$ with $k \in \mathbb{Z}$, if we are only interested in small $(\Delta\hat{H})^2_C/\langle\hat{H}\rangle^2_C$). Away from $\chi = 0$, from (154) and (205) one finds the relative volume relative uncertainty

$$\frac{(\Delta\hat{V}_\chi)^2_C}{\langle\hat{V}_\chi\rangle^2_C} = \frac{(4|\alpha|^2+1)\cosh_{4\omega\chi} + 4|\alpha|^2\sin_{2\vartheta}\sinh_{4\omega\chi} - 1}{\left((2|\alpha|^2+1)\cosh_{2\omega\chi} + 2|\alpha|^2\sin_{2\vartheta}\sinh_{2\omega\chi} - 1\right)^2}\,. \tag{214}$$

By choosing $|\alpha|$ to be large, this can be made arbitrarily small at all times: consider the asymptotic behaviour of (214) for large $|\alpha|$,

$$\frac{(\Delta\hat{V}_\chi)^2_C}{\langle\hat{V}_\chi\rangle^2_C} \sim \frac{1}{|\alpha|^2}\frac{\sin_{2\vartheta}\sinh_{4\omega\chi} + \cosh_{4\omega\chi}}{(\sin_{2\vartheta}\sinh_{2\omega\chi} + \cosh_{2\omega\chi})^2}\,, \tag{215}$$

and notice that $1/|\alpha|^2$ multiplies a bounded function in χ. For late times, we recover the results of [184]

$$\frac{(\Delta\hat{V}_\chi)^2_C}{\langle\hat{V}_\chi\rangle^2_C} \xrightarrow{\chi\to\pm\infty} \frac{2(1+4|\alpha|^2(1\pm\sin_{2\vartheta}))}{(1+2|\alpha|^2(1\pm\sin_{2\vartheta}))^2} =: \mathcal{A}_C\,. \tag{216}$$

Again, this becomes arbitrarily small for large $|\alpha|$ and avoiding the values $\vartheta = \frac{\pi}{4} + k\frac{\pi}{2}$.

We can also expand (214) in inverse volume powers, finding

$$\frac{(\Delta\hat{V}_\chi)^2_C}{\langle\hat{V}_\chi\rangle^2_C} = \mathcal{A}_C\left(1 + \frac{v}{\langle\hat{V}_\chi\rangle}\right) + \mathcal{C}_C\frac{v^2}{\langle\hat{V}_\chi\rangle^2} + \mathcal{O}\left(\frac{1}{\langle\hat{V}_\chi\rangle^4}\right)\,, \tag{217}$$

with

$$\mathcal{C}_C = -\frac{|\alpha|^2 \pm 2|\alpha|^4(\sin_{2\vartheta} \pm 1)\left(|\alpha|^2(\cos_{4\vartheta} + 1) + 3\right)}{\left(2|\alpha|^2(\sin_{2\vartheta} \pm 1) \pm 1\right)^2}\,. \tag{218}$$

We see that the $1/\langle \hat{V}(\chi)\rangle$ correction is such that $\mathcal{B}_C = \mathcal{A}_C$ in (209), which is similar to the terms in the Friedmann equation (156). Higher contributions can be found, but they only minimally improve the expansion, whose key behaviour is already captured at the $1/\langle \hat{V}(\chi)\rangle^2$ order.

5.1.2 *Squeezed states*

Mimicking standard quantum mechanics notation, squeezed states can be defined via the action of the squeezing operator $\hat{S}(z)$ on the Fock vacuum as

$$|z\rangle = \hat{S}(z)|0\rangle\,, \tag{219}$$

with

$$\hat{S}(z) = e^{\frac{1}{2}\left(z\hat{a}^{\dagger 2} - \bar{z}\hat{a}^2\right)}\,, \qquad z \in \mathbb{C}\,. \tag{220}$$

We decompose z as $z = re^{i\psi}$, where r and ψ are real parameters.

These squeezed states can be seen as part of the Perelomov–Gilmore class of coherent states [192, 310] associated with $SU(1,1)$; this is how they were introduced for GFT in [184]. As described in [184], the volume operator (149) is bounded from below only in the $\mathfrak{su}(1,1)$ representations of the positive ascending series; when one restricts to the cases of interest for GFT,[2] the Perelomov–Gilmore coherent states coincide exactly with the squeezed states that we define in (219).

Contrary to coherent states, one can readily find that squeezed states *do* saturate the RS uncertainty principle (204) for the operators $\hat{V}$ and $\hat{H}$. Using again Table 3 in Appendix B, at $\chi = 0$ one explicitly has

$$\begin{aligned} &\frac{v^2\omega^2}{16}\sinh^2_{2r}\left(2\sinh^2_{2r}\cos_{2\psi} + \cosh_{4r} + 3\right) \\ &\quad = \frac{v^2\omega^2}{16}\cos^2_{\psi}\sinh^2_{4r} + \frac{v^2\omega^2}{4}\sin^2_{\psi}\sinh^2_{2r}\,. \end{aligned} \tag{221}$$

2 The representations of the positive discrete series are labelled by a real parameter k called Bargmann index. Using the bosonic realisation of $\mathfrak{su}(1,1)$ (151) and including the Fock vacuum among the eigenstates of the volume operator, one is led to choose $k = 1/4$, for which all the results of [184] coincide with the ones described here.

An analogous result for $SU(1,1)$ coherent states in loop quantum cosmology is reported in [266].

This minimisation happens because we are interested in uncertainties of the GFT operators (149) and (150), which form the $\mathfrak{su}(1,1)$ structure that squeezed states are built on (see Section 3.3.2). Turning on time dependence, we find that the uncertainty principle is indeed an exact equality throughout the whole evolution for the state in (219). Again we refer to Appendix B for the analytical expressions at generic times; there we show that the RS uncertainty principle is minimised for all values of χ and in particular in the late-time limit $\chi \to \pm\infty$.

The minimisation of the RS principle does not necessarily mean that relative uncertainties of cosmological observables are small, and indeed we find at $\chi = 0$ (see Table 3 in Appendix B)

$$\begin{aligned}\frac{(\Delta\hat{V})^2_S}{\langle\hat{V}\rangle^2_S} &= 2\coth^2_r\,,\\ \frac{(\Delta\hat{H})^2_S}{\langle\hat{H}\rangle^2_S} &= 2 + 2\sec^2_\psi\operatorname{csch}^2_{2r}\,,\\ \frac{(\Delta\hat{C})^2_S}{\langle\hat{C}\rangle^2_S} &= 2 + 2\csc^2_\psi\operatorname{csch}^2_{2r}\,.\end{aligned} \tag{222}$$

All these quantities are bounded from below by 2. One can still check whether the situation improves with time evolution; a minimal requirement for semiclassicality is that relative uncertainties are only small at large volume. Using (154) and (205) one can readily write down the exact time evolution of the relative uncertainties as

$$\begin{aligned}\frac{(\Delta\hat{V}_\chi)^2_S}{\langle\hat{V}_\chi\rangle^2_S} &= 2\,\frac{\sin_\psi\sinh_{2r}\sinh_{2\omega\chi} + \cosh_{2r}\cosh_{2\omega\chi} + 1}{\sin_\psi\sinh_{2r}\sinh_{2\omega\chi} + \cosh_{2r}\cosh_{2\omega\chi} - 1}\\ &= 2\left(1 + \frac{v}{\langle\hat{V}_\chi\rangle}\right)\,.\end{aligned} \tag{223}$$

Hence, the lower bound of 2 for the relative uncertainty holds at all times; a uniform large-volume limit of 2 was already found in [184].

DIPOLE STATES. As a final remark on squeezed states, we point out that a "dipole condensate" state defined as

$$|\xi\rangle = \exp\left(\frac{1}{2}\xi\,\hat{a}^\dagger\hat{a}^\dagger\right)|0\rangle\,, \qquad \xi \in \mathbb{C}\,, \tag{224}$$

is nothing else but a non-normalised squeezed state. States similar to (224) were introduced as possible condensate-like states in the early stages of GFT cosmology [183] (we will return to a discussion of these states in the algebraic approach in Section 5.3). Given the norm

$$\langle\xi|\xi\rangle = \frac{1}{\sqrt{1-|\xi|^2}}, \tag{225}$$

we should assume $|\xi| < 1$ in order to obtain a normalisable state.

To see that (224) is a squeezed state, we write the squeezing operator $\hat{S}(z)$ in "normal form" [158]

$$\begin{aligned}\hat{S}(z) = &\exp\left(\frac{z}{2|z|}\tanh|z|(\hat{a}^\dagger)^2\right)\exp\left(-\ln\cosh|z|\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)\right)\\ &\times\exp\left(-\frac{\bar{z}}{2|z|}\tanh|z|\,\hat{a}^2\right),\end{aligned} \tag{226}$$

so that one can write a squeezed state as

$$|z\rangle = \hat{S}(z)|0\rangle = \frac{1}{\sqrt{\cosh|z|}}\exp\left(\frac{z}{2|z|}\tanh|z|(\hat{a}^\dagger)^2\right)|0\rangle. \tag{227}$$

(227) shows that a dipole state (224) is a (rescaled) squeezed state (219), $|\xi\rangle = \sqrt{\cosh|z|}\,|z\rangle$, where the dipole parameter ξ and the squeezing parameter z are related by

$$\xi = \frac{z}{|z|}\tanh|z|. \tag{228}$$

Since they are just squeezed states, dipole condensates have no chance of being semiclassical according to the criterion of small relative uncertainties (cf. (222) and (223)).

5.2 THE GENERAL FAMILY OF GAUSSIAN STATES

Gaussian states can be defined in several equivalent ways. Traditionally, they are presented in quantum mechanics textbooks as states whose characteristic functions and quasi-probability distributions (also known as Wigner functions) are Gaussian functions. Equivalently, especially in the quantum optics and quantum information literature, Gaussian states are often described as states which are fully determined by the first and second canonical moments only [351]. Other characterisations are possible, both physical (as minimum uncertainty states) and mathematical (see [212] for connections to complex structures and symplectic forms).

We will focus on an equivalent but more operational definition of Gaussian states, given as Gibbs states of generic second-order Hamiltonians of bosonic fields [351]. Specifically, they can be defined as arising from the action of the displacement operator (211) and squeezing operator (220) on a thermal state [107, 277] (see Appendix B, and specifically Section B.2),

$$\hat{\rho}_G(\alpha, z, \beta) := \hat{D}(\alpha)\hat{S}(z)\,\hat{\rho}_\beta\,\hat{S}^\dagger(z)\hat{D}^\dagger(\alpha)\,, \tag{229}$$

In this chapter, the symbol β always refers to thermal effects and should not be confused with the anisotropies of Chapter 4.

where, denoting the usual Fock states by $|n\rangle = (n!)^{-1/2}(\hat{a}^\dagger)^n|0\rangle$,

$$\hat{\rho}_\beta := \frac{e^{-\beta\hat{a}^\dagger\hat{a}}}{\mathrm{tr}(e^{-\beta\hat{a}^\dagger\hat{a}})} = (1-e^{-\beta})\sum_n e^{-\beta n}|n\rangle\langle n|\,. \tag{230}$$

$\beta > 0$ is a free parameter, the analogue of the inverse temperature in the usual canonical ensemble.

A key property of Gaussian states is that (in the Schrödinger picture) they retain their Gaussian nature under time evolution; $\hat{U}\,\hat{\rho}_G\,\hat{U}^\dagger$ is also a Gaussian state if $\hat{U}$ is the unitary time evolution operator.[3] This property motivates studies of "Gaussian quantum mechanics", in which one restricts to Gaussian-preserving measurements and transformations and where quadratic Hamiltonians are fundamental [292, 374]. In this setting one avoids the difficulties that come with higher-order dynamics.

Of course, the family of *pure* Gaussian states is a subset of (229) obtained in the vanishing "temperature" limit,

$$\hat{\rho}_G \overset{\beta\to\infty}{\longrightarrow} \hat{D}(\alpha)\hat{S}(z)|0\rangle\langle 0|\hat{S}^\dagger(z)\hat{D}^\dagger(\alpha) =: |\alpha, z\rangle\langle\alpha, z|\,. \tag{231}$$

These states are the well-known displaced squeezed states, which relate nicely to the simpler states discussed in Section 5.1.

The general class of states (229) can straightforwardly be imported in the deparametrised GFT framework (and analogue $\mathfrak{su}(1,1)$ cosmologies) since, as detailed in Section 3.3.2, we deal with a bosonic system governed by a second-order Hamiltonian (149). The state (229) can in fact also be understood along the lines of [40, 41, 250], where GFT states are defined as statistical equilibrium states of exponential form $e^{-\hat{\mathcal{O}}}$ for some operator $\hat{\mathcal{O}}$. The parameter β in (230) is to

3 For a quadratic Hamiltonian the evolution operator can always be decomposed as $\hat{U} = e^{i\gamma}\hat{S}(z)\hat{D}(\alpha)\hat{R}(\phi)$ where $\hat{R}(\phi) = \exp(i\phi\,\hat{a}^\dagger\hat{a})$ is the rotation operator and $\exp(i\gamma)$ a phase factor [268, 269, 350]. While a rotation operator can in principle enter the definition of Gaussian states (229), it does not affect any result (see Section B.2 for details).

be taken formally (for instance as the periodicity in the 1-parameter flow of a Kubo–Martin–Schwinger state or as a Lagrange multiplier [40, 41, 250]), and does not necessarily relate to a physical notion of temperature. Effectively, given that $\hat{N} = \hat{a}^\dagger\hat{a}$ represents the number of quanta, β in (230) could be seen more akin to a chemical potential of a grandcanonical ensemble.

Equipped with the new and generalised family of states (229), we can now turn to the calculation of quantities of interest for harmonic cosmology; Section B.2 outlines helpful tools for using (229) to obtain the following results. First, we compute the expectation value of the three main operators for our models, given in (149) and (150). One finds

$$\begin{aligned}\langle\hat{V}\rangle_G &= v\left(|\alpha|^2 + N_\beta \cosh_{2r} + \sinh^2_r\right) , \\ \langle\hat{H}\rangle_G &= -\frac{\omega}{2}\left(2|\alpha|^2\cos_{2\vartheta} + (2N_\beta+1)\sinh_{2r}\cos_\psi\right) , \\ \langle\hat{C}\rangle_G &= \frac{v}{2}\left(2|\alpha|^2\sin_{2\vartheta} + (2N_\beta+1)\sinh_{2r}\sin_\psi\right) ,\end{aligned} \tag{232}$$

where we denote the thermal expectation value (computed with (230)) of the number operator as

$$N_\beta = \langle\hat{N}\rangle_{thermal} = \mathrm{tr}\left(\hat{\rho}_\beta\,\hat{a}^\dagger\hat{a}\right) = \frac{1}{e^\beta - 1} . \tag{233}$$

By means of (233), the reduction to pure states ($\beta \to \infty$) is achieved by setting $N_\beta = 0$. Next, we evaluate variances (202) and covariances (203). Incorporating the displacement and squeezing phases into a shorthand $\mathcal{F}_\pm = \cos_{2\vartheta}\cos_\psi \pm \sin_{2\vartheta}\sin_\psi$ and noticing that $2N_\beta + 1 = \coth_{\beta/2}$, one finds

$$\begin{aligned}(\Delta\hat{V})^2_G &= \frac{v^2}{4}\left[4|\alpha|^2\coth_{\frac{\beta}{2}}\left(\cosh_{2r} + \mathcal{F}_+\sinh_{2r}\right) + \coth^2_{\frac{\beta}{2}}\cosh_{4r} - 1\right] , \\ (\Delta\hat{H})^2_G &= \frac{\omega^2}{8}\left[8|\alpha|^2\coth_{\frac{\beta}{2}}\left(\cosh_{2r} + \mathcal{F}_-\sinh_{2r}\right)\right. \\ &\qquad\left. + \coth^2_{\frac{\beta}{2}}\left(1 + 2\sinh^2_{2r}\cos_{2\psi} + \cosh_{4r}\right) + 2\right] , \\ (\Delta\hat{C})^2_G &= \frac{v^2}{8}\left[8|\alpha|^2\coth_{\frac{\beta}{2}}\left(\cosh_{2r} - \mathcal{F}_-\sinh_{2r}\right)\right. \\ &\qquad\left. + \coth^2_{\frac{\beta}{2}}\left(1 + 2\cosh_{4r}\sin^2_\psi + \cos_{2\psi}\right) + 2\right] ,\end{aligned} \tag{234}$$

and

$$\begin{aligned} \Delta(\hat{V}\hat{H})_{\rm G} &= -\frac{v\omega}{4}\Big[4|\alpha|^2 \coth_{\frac{\beta}{2}}\left(\sinh_{2r}\cos_\psi + \cosh_{2r}\cos_{2\vartheta}\right) \\ &\qquad + \coth^2_{\frac{\beta}{2}}\sinh_{4r}\cos_\psi\Big]\,, \\ \Delta(\hat{V}\hat{C})_{\rm G} &= \frac{v^2}{4}\Big[4|\alpha|^2 \coth_{\frac{\beta}{2}}\left(\sinh_{2r}\sin_\psi + \cosh_{2r}\sin_{2\vartheta}\right) \\ &\qquad + \coth^2_{\frac{\beta}{2}}\sinh_{4r}\sin_\psi\Big]\,, \\ \Delta(\hat{H}\hat{C})_{\rm G} &= -\frac{v\omega}{4}\Big[4|\alpha|^2 \coth_{\frac{\beta}{2}}\sinh_{2r}(\sin_{2\vartheta}\cos_\psi + \cos_{2\vartheta}\sin_\psi) \\ &\qquad + \coth^2_{\frac{\beta}{2}}\sinh^2_{2r}\sin_{2\psi}\Big]\,. \end{aligned} \tag{235}$$

The quantities in (232), (234) and (235) combine in a nontrivial way coherent, squeezed and thermal contributions; they generalise the expressions for simple states reported Section B.1, being now (at the same time) functions of $\alpha = |\alpha|e^{i\vartheta}$, $z = re^{i\psi}$ and β.

Expectation values, variances and covariances are all the ingredients one needs to analyse the semiclassical criteria discussed in Section 5.1. For instance, using the first two expressions in (234), the first in (235) and the last in (232), it is straightforward (albeit tedious) to see that the Robertson–Schrödinger uncertainty principle (204) is *not* minimised by Gaussian states at $\chi = 0$. One can in fact prove that the inequality is never saturated for any χ, much like with coherent states. As expected, the inequality becomes an identity only when $\alpha = N_\beta = 0$, which is the case of a pure squeezed state (221). Details on the Robertson–Schrödinger principle for Gaussian states are given in Appendix B.

More importantly, we now show that Gaussian states can be chosen to have small quantum fluctuations. Even with such a large parameter space (spanned by α, z and β), one can notice from (234) and (232) that it is always possible to manipulate the displacement parameter α to make relative uncertainties arbitrarily small at $\chi = 0$. While squeezed and thermal states alone do not allow for such a feature, squeezing and thermal effects can lead to semiclassical Gaussian states as long as one uses a large enough displacement. To make this more explicit, we can expand the fluctuations stemming out of (234) and (232) for large $|\alpha|$, obtaining

$$\frac{(\Delta\hat{V})^2_{\mathrm{G}}}{\langle\hat{V}\rangle^2_{\mathrm{G}}} \sim \frac{1}{|\alpha|^2}\coth_{\frac{\beta}{2}}\left(\cosh_{2r} + \mathcal{F}_+ \sinh_{2r}\right), \tag{236}$$

$$\frac{(\Delta\hat{H})^2_{\mathrm{G}}}{\langle\hat{H}\rangle^2_{\mathrm{G}}} \sim \frac{1}{|\alpha|^2\cos^2_{2\vartheta}}\coth_{\frac{\beta}{2}}\left(\cosh_{2r} + \mathcal{F}_- \sinh_{2r}\right), \tag{237}$$

$$\frac{(\Delta\hat{C})^2_{\mathrm{G}}}{\langle\hat{C}\rangle^2_{\mathrm{G}}} \sim \frac{1}{|\alpha|^2\sin^2_{2\vartheta}}\coth_{\frac{\beta}{2}}\left(\cosh_{2r} - \mathcal{F}_- \sinh_{2r}\right). \tag{238}$$

These expressions still refer to $\chi = 0$, so they generalise (213) and (222). We now discuss the dynamics of quantum fluctuations, focussing on the volume operator.

Recall from Section 3.3.2 that the single-mode GFT Hamiltonian (149) makes the evolution operator $\hat{U}(\chi) = e^{-\mathrm{i}\hat{H}\chi}$ a squeezing operator (with purely imaginary squeezing parameter). Relations allowing a reordering of displacement and squeezing operators, or the composition of two squeezing operators into one, are well known (see appendix Section B.2), and it might be tempting to work in the Schrödinger picture and to define

$$\hat{\rho}_{\mathrm{G}}(\alpha, z, \beta; \chi) = \hat{U}(\chi)\hat{D}(\alpha)\hat{S}(z)\,\hat{\rho}_\beta\,\hat{S}^\dagger(z)\hat{D}^\dagger(\alpha)\hat{U}^\dagger(\chi), \tag{239}$$

as a time-dependent Gaussian state. However, one finds that using properties such as (433) and (434) on (239) leads to very complicated calculations, due to the mixing of parameters. We thus keep working in the Heisenberg picture, in which the explicit dynamical equations (154), (205) and (206) allow us to obtain the χ-evolution of all the quantities of interest. For example, the χ-dependent expression of the volume expectation value reads

$$\langle\hat{V}_\chi\rangle_{\mathrm{G}} = \frac{v}{2}\Big[2|\alpha|^2(\cosh_{2\omega\chi} + \sinh_{2\omega\chi}\sin_{2\vartheta}) - 1 \\ + \coth_{\frac{\beta}{2}}(\sinh_{2\omega\chi}\sinh_{2r}\sin_\psi + \cosh_{2\omega\chi}\cosh_{2r})\Big]. \tag{240}$$

While it is not useful to show all the other χ-dependent counterparts of (232), (234) and (235) in full, one can compute them in the same fashion (also using (155) for quantities containing $\hat{C}$). Crucially, we find that Gaussian states can be chosen to make the volume relative uncertainties, $(\Delta\hat{V}_\chi)^2_{\mathrm{G}}/\langle\hat{V}_\chi\rangle^2_{\mathrm{G}}$, arbitrarily small at all times. We can analytically show this again by considering the asymptotic behaviour of the evolving volume fluctuations for large $|\alpha|$, finding

$$\frac{(\Delta\hat{V}_\chi)^2_G}{\langle\hat{V}_\chi\rangle^2_G} \sim \frac{\coth_{\beta/2}}{|\alpha|^2}\frac{\Theta}{(\sin_{2\vartheta}\sinh_{2\omega\chi}+\cosh_{2\omega\chi})^2}, \tag{241}$$

with

$$\begin{aligned}\Theta = &\sinh_{4\omega\chi}(\sin_{2\vartheta}+\sinh_{2r}\sin_\psi)+\cosh_{4\omega\chi}(\sinh_{2r}\sin_{2\vartheta}\sin_\psi+1)\\ &+\sinh_{2r}\cos_{2\vartheta}\cos_\psi\ .\end{aligned} \tag{242}$$

Again avoiding the problematic values $\vartheta = \frac{\pi}{4}+k\frac{\pi}{2}$ with $k\in\mathbb{Z}$, this is the product of a bounded function (in χ) and a factor $1/|\alpha|^2$, which can hence be made arbitrarily small at all times. (241) exhibits similar features to the coherent-states case and generalises (215) to Gaussian states by also keeping squeezing and thermal contributions. The dynamics hence do not spoil the semiclassical behaviour of suitably chosen Gaussian states. To give some graphical intuition, we also plot in Figure 22 a few instances illustrating some interplay between the various state parameters, in particular exemplifying that $|\alpha|$ can make $(\Delta\hat{V}_\chi)^2_G/\langle\hat{V}_\chi\rangle^2_G$ as small as desired at all times. Compared to the choice of $|\alpha|$, the other parameters seem to have relatively minor impact on the relative uncertainty.

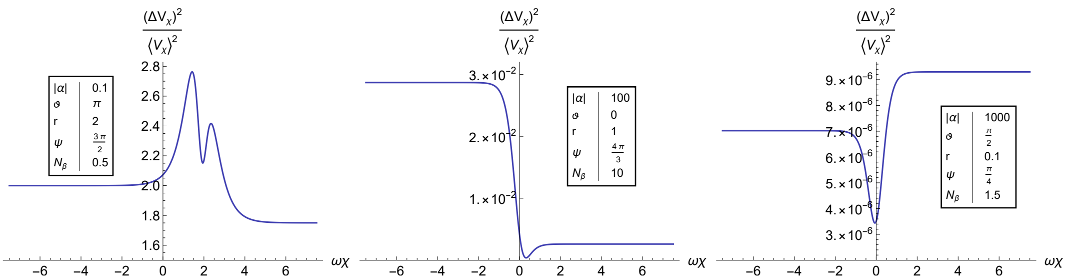

Figure 22: Volume relative uncertainties with Gaussian states for some state parameters.

Finally, we can explicitly find the asymptotic behaviour represented by the plateaus in Figure 22,

$$\frac{(\Delta\hat{V}(\chi))^2_G}{\langle\hat{V}(\chi)\rangle^2_G} \xrightarrow{\chi\to\pm\infty} 2-\frac{8|\alpha|^4(\sin_{2\vartheta}\pm1)^2}{\left[2|\alpha|^2(\sin_{2\vartheta}\pm1)\pm\coth_{\beta/2}(\cosh_{2r}\pm\sinh_{2r}\sin_\psi)\right]^2}\,. \tag{243}$$

which generalises (216) and shares the same properties. In particular, denoting $\mathcal{A}_G$ the right-hand side of (243), we can approximate it for large $|\alpha|$ and see that it can be made arbitrarily small:

$$\mathcal{A}_G \quad \sim \quad \frac{2}{|\alpha|^2}\frac{\coth_{\beta/2}(\cosh_{2r} \pm \sinh_{2r}\sin_\psi)}{(1 \pm \sin_{2\vartheta})} . \tag{244}$$

We can also expand $(\Delta\hat{V}_\chi)^2_G / \langle\hat{V}_\chi\rangle^2_G$ in inverse volume powers as per (209), finding

$$\frac{(\Delta\hat{V}_\chi)^2_G}{\langle\hat{V}_\chi\rangle^2_G} = \mathcal{A}_G\left(1 + \frac{v}{\langle\hat{V}_\chi\rangle}\right) + \mathcal{C}_G\frac{v^2}{\langle\hat{V}_\chi\rangle^2} + \mathcal{O}\left(\frac{1}{\langle\hat{V}_\chi\rangle^3}\right) , \tag{245}$$

where, using $h_\pm = \coth_{\frac{\beta}{2}}(\cosh_{2r} \pm \sinh_{2r}\sin_\psi)$ to encapsulate squeezing and thermal contributions,

$$\begin{aligned}\mathcal{C}_G = \frac{h_\pm\left(4|\alpha|^2(\sin_{2\vartheta} \pm 1) \pm h_\pm\right)}{8}&\left[\frac{2|\alpha|^2(\sin_{2\vartheta} \mp 1) \mp h_\mp}{h_\pm \pm 2|\alpha|^2(\sin_{2\vartheta} \pm 1)}\right. \\ &+ \frac{8|\alpha|^2\cos_{2\vartheta}\cot_\psi(h_\pm - h_\mp) \pm \cot^2_\psi(h_\pm - h_\mp)^2 \mp 4}{2h_\pm\left(h_\pm \pm 4|\alpha|^2(\sin_{2\vartheta} \pm 1)\right)} \\ &\left.\mp \frac{2|\alpha|^2\left(h_+ + h_- + 2|\alpha|^2\cos^2_{2\vartheta} \pm \sin_{2\vartheta}(h_\mp - h_\pm)\right) + h_-h_+ - 4}{\left(h_\pm \pm 2|\alpha|^2(\sin_{2\vartheta} \pm 1)\right)^2}\right] .\end{aligned} \tag{246}$$

Interestingly, here again $\mathcal{B}_G = \mathcal{A}_G$, as with all the other states. One can check that $\mathcal{C}_G$ reduces to (218) for $r = 0$ and $\beta \to \infty$ (since $h_\pm \to 1$) and vanishes for $\alpha = 0$ and $\beta \to \infty$ (cf. (223)).

To summarise, we have seen that the general family of Gaussian states contains states with small relative uncertainties and can thus be regarded as semiclassical. This property is realised for the volume, the Hamiltonian and the $\hat{C}$ operator in the context of GFT cosmology in the deparametrised approach, and it holds at all times (crucially at late times, where quantum fluctuations are actually expected to be small). Because all harmonic cosmologies [59, 79, 80] rely on the same underlying Lie algebra, these results actually hold for any general $\mathfrak{su}(1,1)$ (or CVH) quantum cosmological scenario. As for coherent states, Gaussian states do not saturate the Robertson–Schrödinger uncertainty principle, showing again that it is not clear whether such a criterion should be invoked to classify states as semiclassical.

5.3 ALGEBRAIC APPROACH AND SEMICLASSICAL STATES

In this section we explore the role and behaviour of semiclassical states in the original formalism of the GFT cosmology programme, namely the algebraic approach reviewed in Section 3.2.1 and applied to cosmology in Section 3.3.1. In particular, we investigate whether generalised Gaussian states can exist, and we highlight differences with the deparametrised approach adopted in previous sections.

We continue to adopt the single-mode restriction as with previous sections.

We have seen in Section 3.2.1 that the algebraic formalism for group field theory shares some similarities with a Dirac quantisation. It is based on the construction of a kinematical Hilbert space of abstract states, among which physical states are selected by demanding that they satisfy a constraint coming from the underlying theory. In this picture, states in the GFT kinematical Fock space (84) would be unphysical quantum tetrahedra (or spin network-like states, see (83)), on which dynamics are imposed *a posteriori*. As we have pointed out, since physical states live on the original Hilbert space, one finds that they have infinite norm (see, e.g., (124), (125) and (128)). Moreover, working in a "timeless" setting, one defines kinematical operators as relational observables (e.g., the volume as a function of the scalar field χ), which may also contain infinities. We will encounter divergences at many points in this chapter, so that our expressions need to be treated as formal and subject to some regularisation procedure (some ideas for dealing with these infinities include the coherent peaked states seen at the end of Section 3.3.1 [272] and the work on smeared observables of [40, 41, 250]). We are mainly interested in a general conceptual comparison with the analysis on Gaussian states of Section 5.2; regardless of divergences one can check whether Gaussian-like states can be defined in this formalism, in the sense of at least approximately physical states (see discussion around (93) and (94)).

COHERENT CONDENSATE STATES. The *only* class of states which has been successfully used to extract cosmological dynamics in the algebraic approach is given by field coherent states

The specific proposal given by the coherent peaked states of [272] is part of this class of states (cf. (140)).

$$|\sigma\rangle = \hat{\mathcal{D}}(\sigma)|\varnothing\rangle\,, \tag{247}$$

with

$$\hat{\mathcal{D}}(\sigma) = \exp\left(\int \mathrm{d}\chi\left[\sigma(\chi)\hat{\varphi}^\dagger(\chi) - \overline{\sigma}(\chi)\hat{\varphi}(\chi)\right]\right)\,, \tag{248}$$

which we define here in analogy with (210) and (211) using a displacement-like operator. Due to the Baker–Campbell–Hausdorff formula, (247) is equivalent to a single-particle condensate state of the type usually adopted in the literature, namely (124) (which explicitly shows a divergent norm, see comments below (127)). We have seen in Section 3.3.1 that states of the form (247) can solve (92) *exactly* provided that the displacement parameter $\sigma(\chi)$ satisfies the classical free GFT equation of motion (127), namely $\left(\partial_\chi^2 - \omega^2\right)\sigma(\chi) = 0$. The solution to this equation determines the dynamics of the algebraic approach since the volume depends on χ via $\sigma(\chi)$ (cf. (131)). In particular, one can obtain the volume expectation value $\langle \hat{V}(\chi)\rangle_\sigma$ given in (131) and show that it satisfies the Friedmann-like equation given in (134). Finally, let us recall that the states (247) are considered semiclassical in the sense that volume fluctuations, once regularised, decrease automatically over time as $\sigma(\chi)$ grows exponentially (cf. (139)).

5.3.1 *Gaussian-like states*

Because the effective Friedmann equation (134) seems to rely on coherent states, as indicated by the index of $\langle \hat{V}(\chi)\rangle_\sigma$, it is natural to ask whether one can use more general states – such as Gaussian states – to obtain a similar result. Given how dynamics are implemented in the algebraic approach, we shall see that it is not clear whether Gaussian states are a useful option for this framework. In order to define generalised Gaussian states we resort to the thermofield formalism since a well-defined procedure in terms of thermal-like density matrices is not directly available in the algebraic approach to GFT. The thermofield dynamics were developed in the context of GFT in [40, 41, 250] for thermal coherent states; this naturally extends to the case of Gaussian states following the strategy of Appendix B (see in particular Section B.2), with suitably generalised definitions. Explicitly, in analogy with (450), a Gaussian-like state in the algebraic approach can be defined as[4]

$$|\sigma, \zeta, \beta\rangle := \hat{\mathcal{D}}(\sigma)\hat{\mathcal{S}}(\zeta)|\emptyset_\beta\rangle\,, \tag{249}$$

4 Just like (247), the state (249) has a divergent norm. Again, one could impose a cutoff in the χ integrations, but in this case the divergences are even more severe. In Section 5.3.2 we provide more details on condensate states and their norms.

where we introduce a squeezing-like operator (cf. (220))

$$\hat{\mathcal{S}}(\zeta) = \exp\left(\frac{1}{2}\int \mathrm{d}\chi \left[\zeta(\chi)(\hat{\varphi}^\dagger(\chi))^2 - \overline{\zeta}(\chi)\hat{\varphi}^2(\chi)\right]\right), \tag{250}$$

and the algebraic counterpart of the thermal vacuum (444),

$$\begin{aligned} |\varnothing_\beta\rangle &= \hat{\mathcal{T}}(\theta_\beta)|\varnothing,\tilde{\varnothing}\rangle\,, \\ \hat{\mathcal{T}}(\theta_\beta) &= \exp\left(\int \mathrm{d}\chi\, \theta_\beta(\chi)\left[\hat{\varphi}^\dagger(\chi)\hat{\tilde{\varphi}}^\dagger(\chi) - \hat{\varphi}(\chi)\hat{\tilde{\varphi}}(\chi)\right]\right). \end{aligned} \tag{251}$$

The general construction of Section B.2 applies here: the state $|\varnothing,\tilde{\varnothing}\rangle$ is a product vacuum for the doublet *kinematical* Hilbert space (cf. (82)), the tilde operators $\hat{\tilde{\varphi}}(\chi)$ and $\hat{\tilde{\varphi}}^\dagger(\chi)$ satisfy the algebra in (81) (i.e., $[\hat{\tilde{\varphi}}(\chi),\hat{\tilde{\varphi}}^\dagger(\chi')] = \delta(\chi-\chi')$) while commuting with non-tilde operators, *et cetera*. In particular, one can make use of the relation $\sinh^2(\theta_\beta(\chi)) = (e^{\beta(\chi)}-1)^{-1}$ (cf. (449)) to express results in terms of the statistical parameter $\beta(\chi)$.

Such a Gaussian-like state is not physical as it does not solve the constraint (92). More precisely, focussing on the case of a *pure* Gaussian state for simplicity,

$$|\sigma,\zeta\rangle = \hat{\mathcal{D}}(\sigma)\hat{\mathcal{S}}(\zeta)|\varnothing\rangle\,, \tag{252}$$

(92) leads to the condition

$$(\partial_\chi^2 - \omega^2)\left(\sigma(\chi) + \frac{\zeta(\chi)}{|\zeta(\chi)|}\tanh(|\zeta(\chi)|)\left(\hat{\varphi}^\dagger(\chi) - \overline{\sigma}(\chi)\right)\right) = 0\,, \tag{253}$$

which cannot generically be solved for the displacement and squeezing functions $\sigma(\chi)$ and $\zeta(\chi)$. Including thermal contributions only results in a more complicated equation with no interesting novelties, as $\hat{\mathcal{T}}(\theta_\beta)$ is essentially a generalised squeezing operator just like $\hat{\mathcal{S}}(\zeta)$. While setting $\zeta = 0$ in (253) returns the classical equation of motion for $\sigma(\chi)$ (which makes the coherent-like state (247) physical), notice that setting $\sigma = 0$ shows that a purely squeezed-like state

$$|\zeta\rangle = \hat{\mathcal{S}}(\zeta)|\varnothing\rangle \tag{254}$$

is also unphysical as

$$(\partial_\chi^2 - \omega^2)\left(\frac{\zeta(\chi)}{|\zeta(\chi)|}\tanh(|\zeta(\chi)|)\,\hat{\varphi}^\dagger(\chi)\right) = 0 \tag{255}$$

cannot yield a solution for the squeezing function $\zeta(\chi)$.

The usual strategy in this situation is to shift the attention towards averages, and require that the state be only approximately physical. We can use the general Gaussian states (249) in (93), e.g. with $\hat{\mathcal{O}} = 1$, to determine the form of our state parameters as functions of χ. One finds that the first Schwinger–Dyson equation does not provide a condition for the squeezing and thermal functions:

$$\left\langle \frac{\delta S[\hat{\varphi}, \hat{\varphi}^\dagger]}{\delta \hat{\varphi}^\dagger} \right\rangle_{\sigma,\zeta,\beta} = \left(\partial_\chi^2 - \omega^2\right) \langle \hat{\varphi}(\chi) \rangle_{\sigma,\zeta,\beta} = \left(\partial_\chi^2 - \omega^2\right) \sigma(\chi) = 0\,. \tag{256}$$

As essentially observed in [40, 41, 250] for thermal coherent states, we can only determine the χ-dependence for the displacement parameter $\sigma(\chi)$, which in particular is the same as in the (pure) coherent-states scenario. This is due to the fact that squeezed states $|\zeta\rangle$, and thus also squeezed thermal states $|\zeta, \beta\rangle$, have a vanishing field expectation value $\langle \zeta, \beta | \hat{\varphi}(\chi) | \zeta, \beta \rangle = 0$. As a consequence, one cannot use (256) to extract dynamical information for ζ and β.

Since going to Schwinger–Dyson equations of higher order is rather complicated (see Section 5.3.2 for an attempt with dipole states and squeezed-like states), we can follow the idea of [40, 41, 250] and assume that the parameters ζ and β are constant. While the dynamics is still governed by the same function $\sigma(\chi)$, this simple generalisation does affect observable averages (such as the volume), and hence the resulting Friedmann equation, with new *static* contributions of squeezing and thermal nature. From the volume expectation value computed with (249),

$$\langle \hat{V}(\chi) \rangle_{\sigma,\zeta,\beta} = v \left[|\sigma(\chi)|^2 + \delta(0) \sinh^2(|\zeta|) + \frac{\delta(0)}{e^\beta - 1} \cosh(2|\zeta|) \right], \tag{257}$$

Of course, (257) reduces to (131) when squeezing and thermal contributions are switched off ($\zeta = 0$ and $\beta \to \infty$).

one finds the following effective Friedmann equation:

$$\left(\frac{1}{\langle \hat{V}(\chi) \rangle_{\sigma,\zeta,\beta}} \frac{\mathrm{d} \langle \hat{V}(\chi) \rangle_{\sigma,\zeta,\beta}}{\mathrm{d}\chi} \right)^2 = 4\omega^2 \left(1 + \frac{v}{\langle \hat{V}(\chi) \rangle_{\sigma,\zeta,\beta}} (E + \mathcal{E}) - \frac{v^2}{\langle \hat{V}(\chi) \rangle_{\sigma,\zeta,\beta}^2} (Q^2 + \mathcal{Q}^2) \right), \tag{258}$$

where E and Q are given in (133) and the squeezing and thermal contributions are encoded in

$$\begin{aligned} \mathcal{E} &= \delta(0)\left[1 - \coth(\beta/2)\cosh(2|\zeta|)\right] , \\ \mathcal{Q}^2 &= -\frac{1}{4}\mathcal{E}\left(\mathcal{E} + 2E\right) . \end{aligned} \tag{259}$$

As mentioned, we formally keep the Dirac delta distributions in these expressions assuming one can get rid of them by using, e.g., a smearing procedure [40, 41, 250]. Notice that such divergences affect the thermal and squeezing contributions already at the level of the volume expectation value (257) (contrary to the coherent-state case where they only appear at the level of fluctuations, see (139)).

Of course, when $\zeta = 0$ and $\beta \to \infty$, the generalised Friedmann equation (258) reduces to (134) since $\mathcal{E} = \mathcal{Q} = 0$. Similarly, one can check that the result of [40, 41, 250] with a "static thermal cloud" emerges by setting $\zeta = 0$. We remark however that both (258) and the modified Friedmann-like equation of [40, 41, 250] represent only a somewhat weak generalisation of (134) as the new contributions are assumed to be χ-independent; this is an arbitrary assumption that was made because the model is not predictive for ζ and β (cf. (256)).

Along the same lines, one can also find new constant contributions to the volume fluctuations for general Gaussian-like states. Since all the χ-dependence is encoded in the displacement parameter $\sigma(\chi)$, one can proceed to remove the divergences of the usual kind and find again that relative uncertainties are automatically tamed under time evolution. To give a concrete but concise example, the pure[5] Gaussian-like states $|\sigma, \zeta\rangle$ defined in (252) yield the following relative uncertainties

$$\frac{(\Delta \hat{V})^2_{\sigma,\zeta}}{\langle \hat{V}\rangle^2_{\sigma,\zeta}} = \frac{\delta(0)\,\Xi}{2|\zeta|\left(|\sigma(\chi)|^2 + \delta(0)\sinh^2(|\zeta|)\right)^2} , \tag{260}$$

with

$$\begin{aligned} \Xi = 2|\zeta||\sigma(\chi)|^2\cosh(2|\zeta|) + \sinh(2|\zeta|)\left(\sigma(\chi)^2\,\overline{\zeta} + \overline{\sigma}(\chi)^2\,\zeta\right) \\ + \delta(0)|\zeta|\sinh^2(2|\zeta|) . \end{aligned} \tag{261}$$

Since $\sigma(\chi)$ grows exponentially, one easily finds that at late times (260) reduces to an analogue of (236) (in this case with $\beta \to \infty$). Of

5 One can write down an expression including β, but it is not insightful to show this in full since thermal contributions are of squeezing type (see thermofield formalism in Section B.2) and assumed to be constant.

course, in the limit $\zeta \to 0$ one recovers (139). Notice that when $\sigma = 0$ one is left with an expression which incidentally has no divergences, namely

$$\frac{(\Delta\hat{V})^2_\zeta}{\langle\hat{V}\rangle^2_\zeta} = 2\coth^2(|\zeta|)\,, \tag{262}$$

just as in (222). We show a similar feature for dipole states (expected to be a type of squeezed state) in Section 5.3.2.

5.3.2 *A closer look at condensates: the dipole states*

In this section we provide more details on the (condensate) states that have been proposed in the algebraic approach to GFT [183, 187, 294, 300], and study their properties in relation to the questions addressed in this chapter.

As mentioned in Section 3.3.1, the single-particle condensate state ubiquitously adopted in the literature $|\sigma\rangle$ (cf. (124) or equivalently (247)), with $\sigma(\chi)$ solving the classical equations of motion (127), is an exact solution of the constraint (7) for a free GFT. After some regularisation (e.g., a cut-off in χ) is adopted to make the state well-defined, this yields compelling cosmological effective dynamics (134) as well as volume uncertainties (139) which are well-behaved if one considers smeared observables.

Next to $|\sigma\rangle$, two-particle (or *dipole*) condensate states

$$|\xi\rangle = \mathcal{N}_\xi \exp\left(\frac{1}{2}\int \mathrm{d}\chi\, \xi(\chi)\hat{\varphi}^\dagger(\chi)\hat{\varphi}^\dagger(\chi)\right)|\varnothing\rangle\,, \tag{263}$$

with

$$\mathcal{N}_\xi = \exp\left(-\sum_{k=1}^{\infty}\frac{1}{4k}\delta(0)\int \mathrm{d}\chi|\xi(\chi)|^{2k}\right)\,, \tag{264}$$

were also initially proposed for GFT cosmology [183], even though they were never used to obtain relational dynamics. Here we return to these states because of the connection with squeezed states (see Section 5.1.2, and in particular (228)), which makes them part of the Gaussian states family. Indeed, assuming one can make such two-particle states well-defined, dipoles $|\xi\rangle$ (263) and squeezed-like states $|\zeta\rangle$ (254) correspond to the same type of states and share the same properties.

An important difference between the normalisation factor $\mathcal{N}_\xi$ in (264) and the one given in [183] is the appearance of a Dirac delta function $\delta(0)$ due to the commutator (81), which does not appear for models without a matter scalar field. A cut-off in χ, while required to make sense of the integrals in $\mathcal{N}_\xi$, is not enough to make the state well-defined; some other regularisation would be needed to deal with the $\delta(0)$. As in Section 5.1.2, we will proceed to study properties of dipole states assuming that a way of regularising (263) is found (and hence leaving formal $\delta(0)$ factors in our expressions). We shall see that, even assuming they exist, dipole states are not viable candidates as semiclassical states for the class of GFT models discussed in this paper, for a number of reasons.

Formally keeping divergences in all our expressions below, one can "blindly" follow the usual strategy to derive a cosmological scenario by focussing on the volume operator. Computing its expectation value with respect to (263), one finds

These computations generalise the coherent-state ones of the literature, described in Section 3.3.1 (see, e.g., (129)).

$$\begin{aligned}\langle \hat{V}(\chi)\rangle_\xi &= v\frac{\langle \xi|\hat{\varphi}^\dagger(\chi)\hat{\varphi}(\chi)|\xi\rangle}{\langle \xi|\xi\rangle}\\ &= v\delta(0)\frac{|\xi(\chi)|^2}{1-|\xi(\chi)|^2}\,,\end{aligned} \tag{265}$$

where again the commutator (81) gives rise to a $\delta(0)$. Since the expectation value (265) is positive by construction, one concludes that the dipole function must satisfy the condition

$$|\xi(\chi)| < 1\,, \tag{266}$$

similarly to what we observed in Section 5.1.2. Next, we shall find that for any $\xi(\chi)$, quantum fluctuations of these states are never small. To see this, one first computes the volume variance (using the property (138)) as

$$\begin{aligned}(\Delta\hat{V}(\chi))^2_\xi &= v^2\left[\delta(0)\langle\hat{\varphi}^\dagger(\chi)\hat{\varphi}(\chi)\rangle_\xi + \langle(\hat{\varphi}^\dagger(\chi))^2\hat{\varphi}^2(\chi)\rangle_\xi - \langle\hat{\varphi}^\dagger(\chi)\hat{\varphi}(\chi)\rangle^2_\xi\right]\\ &= v^2\delta^2(0)\frac{2|\xi(\chi)|^2}{(|\xi(\chi)|^2-1)^2}\,,\end{aligned} \tag{267}$$

so that the relative uncertainty reads

$$\frac{(\Delta\hat{V}(\chi))^2_\xi}{\langle\hat{V}(\chi)\rangle^2_\xi} = \frac{2}{|\xi(\chi)|^2}\,. \tag{268}$$

Even though the $\delta(0)$ distributions fortuitously cancel (as already seen in (262)), such fluctuations can never be made small because of the condition (266). This means that dipole states never become semiclassical according to our main criteria.

One can now still turn to the task of finding the form of $\xi(\chi)$ using dynamical equations, but as anticipated for squeezed-like states in Section 5.3.2 (see in particular (255) and discussion thereafter), this does not seem to be possible. First of all, the dipole state (263) is not exactly physical since it does not solve the constraint (92). To be precise, using the property $\hat{\varphi}(\chi)|\xi\rangle = \xi(\chi)\hat{\varphi}^{\dagger}(\chi)|\xi\rangle$, from (92) one can obtain the condition

$$\left(\partial_{\chi}^{2}-\omega^{2}\right)\left(\xi(\chi)\hat{\varphi}^{\dagger}(\chi)\right)=0\,, \tag{269}$$

which cannot yield a solution for the dipole function $\xi(\chi)$. (269) is the analogue of (255) translated into the notation of dipole states via (228).

One could then try to find $\xi(\chi)$ as an approximate solution from expectation values of the type (93). As already assessed in Section 5.3.2 for more general states (cf. (256)), the first Schwinger–Dyson equation does not help in this respect since dipoles have vanishing field expectation value. Climbing the tower of Schwinger–Dyson equations would entail setting, e.g., $\hat{\mathcal{O}} = \hat{\varphi}$ in (93), which gives

$$\left(\partial_{\chi}^{2}-\omega^{2}\right)\langle\xi|\hat{\varphi}(\chi')\hat{\varphi}(\chi)|\xi\rangle=0\,. \tag{270}$$

Calculating explicitly the correlation function $\langle\xi|\hat{\varphi}(\chi')\hat{\varphi}(\chi)|\xi\rangle$ yields

$$\begin{aligned}&\left(\partial_{\chi}^{2}-\omega^{2}\right)\left(\delta(\chi-\chi')\frac{\xi(\chi)+\xi(\chi')|\xi(\chi)|^{2}}{1-|\xi(\chi')|^{2}|\xi(\chi)|^{2}}\right)\\&\qquad=\left(\partial_{\chi}^{2}-\omega^{2}\right)\left(\delta(\chi-\chi')\frac{\xi(\chi)}{1-|\xi(\chi)|^{2}}\right)=0\,,\end{aligned} \tag{271}$$

where in the last step we used the relation $\delta(\chi-\chi')f(\chi,\chi') = \delta(\chi-\chi')f(\chi)$. To show the connection with squeezed-like states one last time, one can write the same Schwinger–Dyson equation using the state $|\zeta\rangle = \hat{\mathcal{S}}(\zeta)|\varnothing\rangle$ defined in (254) with (250), finding

$$\left(\partial_{\chi}^{2}-\omega^{2}\right)\left(\delta(\chi-\chi')\frac{\zeta(\chi)}{2|\zeta(\chi)|}\sinh(2|\zeta(\chi)|)\right)=0\,. \tag{272}$$

Unfortunately, it seems that no nontrivial solution to either (271) or (272) exists. Schwinger–Dyson equations of higher order would be even harder to solve. In short, two-particle states such as squeezed-like states (254) or dipoles (263) do not seem to be suitable candidate states for the GFT models under investigation. Specifically, while they are clearly not semiclassical (cf. (268)), it is also not clear whether one can tackle the problem of finding a condition for the function $\xi(\chi)$ so to extract dynamics (in the same way that solving (127) gave rise to $\sigma(\chi)$ in (128) which led to (134)), in any meaningful way.

These observations are in conflict with some results of [183], where a detailed account on dipole states can be found. In particular, in the specific case of a GFT coupled to a matter scalar field χ, the condition on the dipole function is *not* given by the classical GFT equation of motion (97),

$$\left(\partial_\chi^2 - \omega^2\right)\xi(\chi) = 0\,, \tag{273}$$

as was claimed in [183]. Interestingly, plugging the solution of (273) (i.e., $\xi(\chi) = \alpha e^{\omega\chi} + \beta e^{-\omega\chi}$) into the volume expectation value (265) one finds new effective cosmological dynamics, where the volume diverges as $\langle V(\chi)\rangle_\xi \sim (\chi - \chi_0)^{-1}$ in a finite relational time $\chi = \chi_0$.[6] In particular, assuming α and β are small (so to satisfy (266)) and real for simplicity, one has the effective Friedmann equation

$$\left(\frac{1}{\langle\hat{V}(\chi)\rangle_\xi}\frac{\mathrm{d}\langle\hat{V}(\chi)\rangle_\xi}{\mathrm{d}\chi}\right)^2 = 4\omega^2\left(C_1 - \frac{vC_2}{\langle\hat{V}(\chi)\rangle_\xi} + \frac{C_3}{v}\langle\hat{V}(\chi)\rangle_\xi + \frac{C_4}{v^2}\langle\hat{V}(\chi)\rangle_\xi^2\right)\,, \tag{274}$$

where $C_1 = 1 - 12\alpha\beta$, $C_2 = 4\alpha\beta$, $C_3 = 2 - 12\alpha\beta$ and $C_4 = 1 - 4\alpha\beta$. Note that the last two terms in (274) *effectively* behave like matter components with equations of state typical of "dust" and "dark energy", respectively. To see this explicitly, we recall that in classical cosmology the relational Friedmann equation for the volume as a function of χ takes the form [184, 302]

$$\left(\frac{1}{V(\chi)}\frac{\mathrm{d}V(\chi)}{\mathrm{d}\chi}\right)^2 = \sum_i A_i V(\chi)^{-w_i+1}\,, \tag{275}$$

where A_i are constants and i labels the different types of perfect fluids (with equations of state $p_i = w_i\rho_i$) in the Universe. Comparing (274) and (275) one can readily find the "effective equation of state param-

6 If $(\alpha, \beta) \in \mathbb{R}$, one finds $\langle V(\chi)\rangle_\xi \sim \frac{v(1-4\alpha\beta)^{-1/2}}{2\omega(\chi-\chi_0)}$, with time of divergence given by $\omega\chi_0 = \ln\left(\frac{1+\sqrt{1-4\alpha\beta}}{2\alpha}\right)$.

eters" $w_3 = 0$ and $w_4 = -1$. Note that a term $\sim 1/\langle \hat{V}(\chi)\rangle_{\xi}^2$, usually characterising effective Friedmann equations coming from GFT, is absent in the "dipole cosmological dynamics" (274). While this path might have some interesting implications from a phenomenological point of view, it is not clear why one should assume (273) to begin with.

5.3.3 *Physicality conditions*

To shed some light on the more general question of finding physical states, we conclude by deriving the general solution to the constraint (92), here

$$\left(\partial_\chi^2 - \omega^2\right) \hat{\varphi}(\chi)|\Phi\rangle = 0\,. \tag{276}$$

Any element $|\Phi\rangle$ of the Fock space (84) can be written as

$$|\Phi\rangle = \sum_{n=0}^{\infty} \int d\chi_1 \ldots d\chi_n\, f_n(\chi_1, \ldots \chi_n) \hat{\varphi}^\dagger(\chi_1) \ldots \hat{\varphi}^\dagger(\chi_n)|\emptyset\rangle\,, \tag{277}$$

where the functions f_n are totally symmetric under exchange of their arguments. Substituting this form into (276) and using the commutator (81), it follows that we would need

$$\sum_{n=0}^{\infty} n \int d\chi_1 \ldots d\chi_{n-1}\, \left(\partial_\chi^2 - \omega^2\right) f_n(\chi, \chi_1, \ldots, \chi_{n-1}) \hat{\varphi}^\dagger(\chi_1) \ldots \hat{\varphi}^\dagger(\chi_{n-1})|\emptyset\rangle = 0\,, \tag{278}$$

which is true if and only if

$$\int d\chi_1 \ldots d\chi_{n-1}\, \left(\partial_\chi^2 - \omega^2\right) f_n(\chi, \chi_1, \ldots, \chi_{n-1}) \hat{\varphi}^\dagger(\chi_1) \ldots \hat{\varphi}^\dagger(\chi_{n-1}) = 0 \tag{279}$$

for all values of n. If such equations are satisfied by some f_n, then the state $|\Phi\rangle$ is physical. For example, forgetting about normalisation, the coherent state (124) corresponds to (277) with

$$f_n(\chi_1, \ldots, \chi_n) = \frac{1}{n!}\sigma(\chi_1) \ldots \sigma(\chi_n)\,. \tag{280}$$

The conditions (279) for the first few n's read

In (281) we are explicitly showing $n = 1, 2, 3$.

$$
\begin{aligned}
(\partial_\chi^2 - \omega^2)\sigma(\chi) &= 0\,, \\
\int \mathrm{d}\chi_1\, \sigma(\chi_1)\, \hat{\varphi}^\dagger(\chi_1)(\partial_\chi^2 - \omega^2)\sigma(\chi) &= 0\,, \\
\int \mathrm{d}\chi_1\, \mathrm{d}\chi_2\, \sigma(\chi_1)\sigma(\chi_2)\, \hat{\varphi}^\dagger(\chi_1)\hat{\varphi}^\dagger(\chi_2)(\partial_\chi^2 - \omega^2)\sigma(\chi) &= 0\,, \\
&\vdots
\end{aligned}
\tag{281}
$$

Clearly all these conditions are met if $\sigma(\chi)$ satisfies (127). Note that the constants $1/n!$ in (280) do not play a role in obtaining (281), so (280) can straightforwardly be generalised to $f_n(\chi_1,\ldots,\chi_n) = c_n\sigma(\chi_1)\ldots\sigma(\chi_n)$, where c_n are generic coefficients. In other words, one can define a slightly more general class of exact physical states,

$$
|F_\sigma\rangle = F\left(\int \mathrm{d}\chi\, \sigma(\chi)\hat{\varphi}^\dagger(\chi)\right)|\varnothing\rangle\,, \tag{282}
$$

where F can be any function that can be expressed in a power series, generalising the previously used exponential (cf. (124)). However, *all* these (physical) states are non-normalisable regardless of F; even the simple one-particle state $|\triangle\rangle = \int \mathrm{d}\chi\, \sigma(\chi)\hat{\varphi}^\dagger(\chi)|\varnothing\rangle$, for example, would require a cut-off as $\langle\triangle|\triangle\rangle = \int \mathrm{d}\chi|\sigma(\chi)|^2$. The only regular physical state (without any cut-off) seems to be the Fock vacuum.

As a last example, we can use the general expressions (279) to confirm that the two-particle condensate state (263) cannot be a physical state. Again forgetting about the normalisation, (277) reduces to a dipole state by choosing $f_n(\chi_1,\ldots,\chi_n) = 0$ for $n = 2m+1$ and

$$
f_n(\chi_1,\ldots,\chi_n) = \frac{1}{2^m m!}\delta(\chi_1-\chi_{m+1})\ldots\delta(\chi_m-\chi_{2m})\xi(\chi_{m+1})\ldots\xi(\chi_{2m})\,, \tag{283}
$$

for $n = 2m$, where $m \in \mathbb{N}$. The state would be physical if the conditions (279) are satisfied. However, the first few read

Excluding the trivial (odd) ones, in (284) we explicitly show $n = 2, 4, 6$.

$$
\begin{aligned}
(\partial_\chi^2 - \omega^2)\left(\xi(\chi)\hat{\varphi}^\dagger(\chi)\right) &= 0\,, \\
\int \mathrm{d}\chi_1\, \xi(\chi_1)(\hat{\varphi}^\dagger(\chi_1))^2(\partial_\chi^2 - \omega^2)\left(\xi(\chi)\hat{\varphi}^\dagger(\chi)\right) &= 0\,, \\
\int \mathrm{d}\chi_1\mathrm{d}\chi_2\, \xi(\chi_1)\xi(\chi_2)(\hat{\varphi}^\dagger(\chi_1))^2(\hat{\varphi}^\dagger(\chi_2))^2(\partial_\chi^2 - \omega^2)\left(\xi(\chi)\hat{\varphi}^\dagger(\chi)\right) &= 0\,, \\
&\vdots
\end{aligned}
\tag{284}
$$

which reduce to the condition (269).

In conclusion, it seems that only states built from iterations of the same single creation operator $\int d\chi\, \sigma(\chi)\hat{\varphi}^\dagger(\chi)$, generically given by (282) and in particular including the coherent state (124) (or equivalently (247)), are exact solutions of (276), and hence physical states. To the best of our knowledge, no other (nontrivial) state can be found such that the infinitely many conditions (279) are satisfied.

5.4 CONCLUSION AND OUTLOOK

In this chapter we constructed the wide class of (mixed) Gaussian states in the context of group field theory and $\mathfrak{su}(1,1)$ quantum cosmological models, and we analysed relevant properties for such states to be semiclassical and lead to a macroscopic cosmology. Drawing inspiration from quantum optics and quantum information theory, we defined Gaussian states in their most general form by applying displacement and squeezing operators to a thermal state. In a discrete setting such as GFT, the notion of particle number allows one to think of a statistical ensemble where adding or removing quanta might come with some intrinsic cost in "energy" (usually the chemical potential in thermodynamics). In this sense, the "thermal" effects appearing in our results are not inherently related to some physical temperature like the one adopted in standard cosmology, but a grand-canonical interpretation might be more meaningful.

Our work extends prior research in GFT cosmology, which mainly focused on coherent states, to the more general family of semiclassical states, in the context of a free GFT and a single field mode. The general family includes pure coherent and squeezed states as special cases, as well as thermal (or generally mixed) states. In the context of the deparametrised approach to canonical quantisation of GFT, we provide explicit analytical results for quantum fluctuations at all relational times. We find that while Gaussian states generally do not saturate the Robertson–Schrödinger inequality, their fluctuations can be minimised, especially at late times (where classical behaviour is expected to emerge). This result is mainly governed by the displacement parameter, which can be adjusted to include nonvanishing thermal and squeezing contributions. All such results rely on the algebra generated by the volume, the Hamiltonian and the $\hat{C}$ operator. For this reason, our findings apply to any general quantum system based on the $\mathfrak{su}(1,1)$ algebra, such as loop quantum cosmology.

Additionally, we investigated Gaussian-like states within the algebraic approach to GFT cosmology and found that, although we can define generalised Gaussian states assuming some regularisation is performed, no such states appear to be physical (even approximately), except for the pure coherent states considered in prior studies. In light of these findings, we analysed a simplified scenario where squeezing and thermality are time-independent, deriving an effective Friedmann equation that extends previous results, and observed that volume fluctuations decrease over time. We also found a class of physical states that are more general than coherent states, but these are not Gaussian states and are (in general) not semiclassical.

As we will emphasise in Chapter 7, future research could explore the connections between entanglement, entropy, and geometry in GFT (which has seen recent attention for example in LQG [58, 70, 264]), and extend our work to multi-mode Gaussian states, which may reveal new features for GFT cosmology that are not captured by the single-mode approach used here.

We stress that the algebraic approach and the deparametrised setting described in Chapter 3 were always considered to be two sides of the same underlying scenario, simply adopting different perspectives (more precisely, different quantisation strategies). After all, they were always found to give very similar results in all applications to cosmology (including the work on anisotropies of Chapter 4). The new discrepancies encountered here, specifically related to the implementation of dynamics and the discussion on quantum states, motivated the work of the next chapter. In Chapter 6, we will clarify some aspects of relational quantum dynamics for canonical GFT by means of a rigorous Dirac quantisation (cf. Chapter 2) and the Page–Wootters formalism.

6 RELATIONAL QUANTUM DYNAMICS AND PAGE–WOOTTERS FORMALISM

This chapter introduces a novel and rigorous definition of relational quantum dynamics for group field theory by means of well-established quantisation methods. With the exception of minor review sections (where we briefly recall necessary notions regarding, e.g., Dirac observables and the Page–Wootters formalism) it constitutes original research, and it is based on [101].

We have seen in Part I that a central theme in many approaches to quantum gravity is that of background independence. This principle stems directly from general relativity where the geometry of spacetime is not taken as a background structure for a given system, but rather is understood as a dynamical part of it. In particular, the absence of a background time poses a challenge for the definition of dynamics, which leads to the "problem of time" of classical and quantum gravity [233, 257]. More precisely, as we mentioned in Section 2.1, the canonical Hamiltonian of general relativity vanishes on shell and gravitational observables, required to have vanishing Poisson brackets with the constraints, appear to be "frozen" in coordinate time [16, 133, 224]. The most common proposal to bypass this problem is to adopt a relational strategy, where one picks a degree of freedom of the system to serve as internal time relative to which the remaining degrees of freedom evolve. For example, in the case of cosmological settings, the proposal is usually to add matter (commonly a massless scalar field) to describe the relational dynamics of the gravitational degrees of freedom.

As we will exemplify in this chapter, one can classically define relational Dirac observables of constrained systems following the general theory of [135, 190, 358] (see also [336–339] for earlier work and [197] for a recent extension of the concept of relational observables), which allows to deal with the freedom of choosing different clocks by means of "complete observables". These implement precisely the idea of encoding dynamics as a relation between phase space functions, without referring to any external structure (in particular any background time). The quantum theory for constrained systems can be obtained following the Dirac quantisation programme [133, 224, 282], which has the advantage of preserving the structures of the classical theory such as constraints. The Dirac programme provides a clear notion of relational observables for the quantum theory and importantly, because all the degrees of freedom are treated on the same footing, does not require any gauge-fixing or choice of time parameter before quantisation. This aspect is clarified in particular in the recent work of [227, 228] (see also [106] for a study on periodic clocks), where the Dirac quantisation programme is denoted "clock-neutral" as it describes physics before choosing a (temporal) reference frame.

In the case of gravity, the constraints are associated with the notion of covariance, see Section 2.1.

As explained in Section 2.3 and Chapter 3, GFT adheres to the paradigm of background independence, and describes spacetime as emerging from the collective behaviour of (possibly pre-geometric) quantum gravity degrees of freedom [160, 296]. While GFT is closely related to the covariant spin foam formulation of LQG and to tensor models (cf. Chapter 2), we showed that the canonical framework of Chapter 3 provides an excellent arena to explore cosmological applications (cf. Section 3.3). This is similar, in spirit, to the situation in canonical LQG (which inspired a Hilbert space quantisation for GFT in the first place [298]) and its cosmological sector loop quantum cosmology, as both settings offer relatively easy access to dynamical equations which can be related to the equations of cosmology. However, while LQG and loop quantum cosmology follow a more standard quantisation strategy (à la Dirac[1]), GFT is subject to quite unique dilemmas which are circumspectly addressed following the two approaches described in Section 3.2 and Section 3.3.

Distinctive complications for GFT. The somewhat peculiar challenge for the GFT framework is that one cannot directly apply the standard methods of canonical quantisation due to the absence of a Hamiltonian formulation of the theory. Since there is no background time and no immediate definition of a phase space structure at the classical level (cf. Section 2.3.1), the Hilbert space formalism of the algebraic approach to GFT is not derived from a canonical quantisation of a classical theory; rather, it is introduced via the kinematical structures of a Fock space, constructed along the lines of many-body quantum physics (cf. Section 3.2.1). As we have seen, this is a canon-

1 While the full Dirac quantisation programme for LQG faces significant technical challenges (see the end of Section 2.1), notable progress has been made in the context of loop quantum cosmology [37] where, due to symmetry reduction, there is only one constraint associated with the freedom to choose the time parameter.

ical formulation where the equations of motion are imposed at the quantum level as constraints (cf. (92)), so as to reduce from the postulated kinematical Hilbert space to a physical Hilbert space. As illustrated in Section 3.2.2, another way to define dynamics in GFT is to follow a deparametrised approach: this amounts to selecting a time parameter for the classical theory which allows to write down a relational Hamiltonian (cf. (103)), and hence perform a standard canonical quantisation. Deparametrisation, however, is subject to the general concerns and criticisms of "tempus ante quantum" frameworks [233, 257], as it is not clear whether the choice of a classical time label before quantisation breaks clock-covariance [272]. While both of these frameworks yield similar dynamics in the restriction to homogeneous cosmology (cf. Section 3.3 and Chapter 4), one would ideally like to leverage their complementary strengths by performing a genuine canonical quantisation of a classical theory in terms of constraints, without singling out an arbitrary time parameter at the classical level.

While the coherent peaked states formulation can be seen as a "tempus post quantum" framework (cf. (145)), the algebraic approach still does not refer to a Hamiltonian theory or to relational observables as in usual Dirac quantisation.

In this chapter we address these challenges by defining relational dynamics in GFT, in particular for models where a scalar field is used as relational clock, in a way that connects with the known methods for quantisation (such as the Dirac programme mentioned above and the *Page–Wootters formalism*, as explained below). We will only be interested in the free theory, where interactions are ignored. Following the parametrisation strategy adopted in quantum mechanics [133, 224] and quantum field theories [255, 256, 370], we reformulate GFT cosmology models as constrained systems where the constraint is associated with the notion of time reparametrisation invariance. This allows us to implement the programme of Dirac quantisation along the lines of other systems that are well understood (e.g., loop quantum cosmology), and to connect with the "trinity of relational quantum dynamics" of [227, 228]. Already for the classical theory, we can define relational Dirac observables for GFT in a precise way as those that Poisson-commute with the constraint [135, 190, 358]. Quantising the parametrised GFT à la Dirac, namely reducing from a kinematical Hilbert space to a physical space via group averaging techniques, makes clock covariance transparent for the GFT cosmological models of interest. In particular, we define the relational Dirac observable associated with the GFT number operator (the main observable for cosmology), and interpret its quantum dynamics by means of the Page–Wootters formalism [309, 381]. More precisely, thanks to the equivalence established in [227, 228], we describe the expectation value of the GFT number operator as conditional on the reading of the (quantum) clock associated with the matter scalar field. This is the first application of the Page–Wootters formalism in non-perturbative quantum gravity (a recent application to a perturbative quantum gravity setting was given in [126]), and enables us to discuss GFT cosmology in a "tempus post quantum" framework [233, 257, 272]. Remarkably, the relational dynamics turn out to match with the ones obtained in the deparametrised setting (where the clock is selected prior to quantisation), proving that deparametrisation in GFT cosmology is fully covariant. By defining a new variant of the Page–Wootters formalism for the case of multiple quantum clocks, we also generalise the setup to a situation with multiple Hamiltonian constraints (associated to an independent gauge invariance for each field mode), which realises the idea of "multi-fingered time evolution" [253, 254].

Ultimately, in this chapter we establish a framework that consistently describes the relational evolution of GFT geometric observables with respect to a "quantum time", here identified with the matter scalar field. Crucially, this is given by a canonical quantisation that does not require to single out the clock classically, and where dynamics are implemented by a Hamiltonian constraint (somewhat similar to the situation in general relativity). We thus ameliorate and somewhat clarify the situation as presented in the literature (so far in terms of the algebraic and deparametrised approaches of Section 3.2.1 and Section 3.2.2) by obtaining a manifestly covariant formulation of GFT relational quantum dynamics. This in particular is equipped with the conditional interpretation of the Page–Wootters formalism, and hence provides robust insights on relational dynamics for quantum gravity.

OUTLINE OF THE CHAPTER. We start in Section 6.1 by parametrising a simple GFT model, which relies on a single field mode, and we discuss the notion of classical Dirac observables. We then quantise the theory in Section 6.2, clearly distinguishing between kinematical aspects and relational quantum dynamics, obtained equivalently with the Dirac and the Page–Wootters formalisms. The construction is generalised to the case of multiple field modes in Section 6.3, where we showcase two scenarios: one where all the modes evolve with respect to one clock, and one where they evolve with respect to separate "single-mode times", respectively in Section 6.3.1 and Section 6.3.2. We provide a summary of the results in Section 6.4.

6.1 PARAMETRISATION OF CLASSICAL GROUP FIELD THEORY

Working with a group field $\varphi : SU(2)^4 \times \mathbb{R} \to \mathbb{R}$, we consider a free GFT defined by an action with kinetic term (79). We then start from the action (100) neglecting the interactions, which after integration by parts yields (101), reported here (without $V[\varphi]$) for convenience:

Just like in Chapter 3, in the following we always refer to GFT models coupled to the matter scalar field χ (cf. Section 3.1).

$$S_0[\varphi] = \frac{1}{2}\int \mathrm{d}\chi \sum_J \left(K_J^{(0)} \varphi_{-J}(\chi)\varphi_J(\chi) - K_J^{(2)} \partial_\chi \varphi_{-J}(\chi)\partial_\chi \varphi_J(\chi) \right) . \tag{285}$$

We now apply the general idea of deparametrisation (see, e.g., [133, 224]) as explained in the following. To start with a simple case, we restrict the formalism to a single field mode J with vanishing magnetic indices,[2] and we postpone the discussion of a quantum theory with multiple field modes to Section 6.3. The action then reads

$$S_0[\varphi] = \frac{1}{2}\,\mathrm{sgn}\left(K_J^{(0)}\right) \int \mathrm{d}\chi \left[|K_J^{(0)}|\varphi_J^2 + |K_J^{(2)}|(\partial_\chi \varphi_J)^2 \right] , \tag{286}$$

where we chose a mode for which $K_J^{(0)}$ and $K_J^{(2)}$ have opposite sign, meaning the dynamics for this mode are governed by a Hamiltonian of the form (cf. (103))

$$H_J(\varphi_J, \pi_J) = \frac{1}{2}\,\mathrm{sgn}\left(K_J^{(0)}\right) \left(\frac{{\pi_J}^2}{|K_J^{(2)}|} - |K_J^{(0)}|\varphi_J^2 \right) . \tag{287}$$

In the quantum theory, expressing the Hamiltonian (287) in the ladder operator basis, one finds the squeezing Hamiltonian in (123), as explained in Section 3.2.2.

Because we deal with this (specific) mode only, we drop the label J in the following discussion. Since the global sign of (287) is irrelevant,[3] from now on we choose $K^{(2)} < 0$ and $K^{(0)} > 0$ without loss of generality.

Following the standard parametrisation strategy [133, 224], we now introduce an arbitrary parameter τ to describe $\varphi(\chi)$ by means of two functions $\varphi(\tau)$ and $\chi(\tau)$, so that the group field and the matter field are treated parametrically on the same footing. In this manner we obtain a new action

$$S[\varphi, \chi] = \frac{1}{2}\int \mathrm{d}\tau \left[|K^{(0)}|\varphi^2(\partial_\tau \chi) + |K^{(2)}|\frac{(\partial_\tau \varphi)^2}{(\partial_\tau \chi)} \right] . \tag{288}$$

2 Recall that a mode (59) with $m_I = 0$ satisfies $J = -J$.
3 This becomes transparent by explicitly solving the eigenvalue problem for (the quantum version of) the Hamiltonian (287), as we do in Section 6.2.

The Hamiltonian theory derived from the parametrised action (288) has an extended phase space spanned by coordinates (φ, χ) and conjugate momenta (π_φ, p_χ) defined as usual,

$$\begin{aligned} \pi_\varphi &:= \frac{\partial \mathcal{L}}{\partial(\partial_\tau \varphi)} = \left|K^{(2)}\right| \frac{\partial_\tau \varphi}{\partial_\tau \chi}\,, \\ p_\chi &:= \frac{\partial \mathcal{L}}{\partial(\partial_\tau \chi)} = \frac{1}{2}\left|K^{(0)}\right|\varphi^2 - \frac{1}{2}\left|K^{(2)}\right| \frac{(\partial_\tau \varphi)^2}{(\partial_\tau \chi)^2}\,. \end{aligned} \tag{289}$$

It is easy to check that the Hamiltonian associated to the new action (288) vanishes. Indeed, the momenta (289) form a constraint

We denote $\mathcal{C}$ the constraint (290) because of the similarity with GR (cf. Section 2.1). Note that this is only an analogy; in this chapter, the symbol $\mathcal{C}$ should not be confused with the constraint (5) of GR. The same reasoning applies to the multiplier N.

$$\mathcal{C} = p_\chi + H(\varphi, \pi_\varphi) = 0\,, \tag{290}$$

where $H(\varphi, \pi_\varphi)$ is given in (287). (290) defines a constraint hypersurface in the extended phase space and generates trajectories on such surface according to

$$\partial_\tau f = \{f, N\mathcal{C}\}\,, \tag{291}$$

for any phase space function f, where N is a Lagrange multiplier defining the particular parametrisation of these trajectories. From (291) one can find the equations of motion

$$\begin{aligned} \partial_\tau \varphi &= \{\varphi, N\mathcal{C}\} = N\frac{\pi_\varphi}{\left|K^{(2)}\right|}\,, \\ \partial_\tau \chi &= \{\chi, N\mathcal{C}\} = N\,, \end{aligned} \tag{292}$$

while $\partial_\tau \pi_\varphi = \partial_\tau p_\chi = 0$. Of course, combining the equations (292) one finds the definition of π_φ in (289). The second equation in (292) shows that N gives the rate of change of χ with respect to the label τ, and we call it lapse function for this reason (borrowing the nomenclature from general relativity, see Section 2.1). Using (289), (290) and (292) one can formulate the same dynamics starting from the action

$$S[\varphi, \chi, \pi_\varphi, p_\chi, N] = \int d\tau \left(\pi_\varphi \partial_\tau \varphi + p_\chi \partial_\tau \chi - N\mathcal{C}\right)\,, \tag{293}$$

which explicitly shows that $N\mathcal{C}$ plays the role of the Hamiltonian (sometimes called super-Hamiltonian), and indeed yields the same equations of motion (292). Note that such a parametrised theory describes the same physics of the initial action $S_0[\varphi]$ (cf. (286)); while (293) contains one extra canonical pair (χ, p_χ), it also implies the constraint (290) (obtained by varying with respect to N). Since (290) is a first-class constraint, it eliminates two degrees of freedom so that the two actions describe the same number of independent degrees of freedom. We have now introduced a form of "general covariance", as (293) is invariant under τ-reparametrisation. This symmetry is reflected in the fact that N can be an arbitrary function, playing the role of a gauge field.

CLASSICAL DIRAC OBSERVABLES. The action (293) is comprised of two parts: one related to geometry described by the group field φ, and one for the matter scalar field χ. Since we will want to use the scalar field χ as internal dynamical clock to describe the GFT system (which corresponds to the choice of the lapse $N = 1$), we briefly review here the notions of classical relational dynamics and Dirac observables. Following [135, 190, 358], we begin by noticing that (291) defines a flow $\alpha^\tau_{\mathcal{C}}$ with parameter τ that transforms a phase space function f as

$$f \mapsto \alpha^\tau_{\mathcal{C}}(f) := \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \{f, \mathcal{C}\}_n\,, \tag{294}$$

where $\{f, \mathcal{C}\}_n$ denotes the iterated Poisson bracket defined as $\{f, \mathcal{C}\}_{n+1} := \{\{f, \mathcal{C}\}, \mathcal{C}\}_n$ with $\{f, \mathcal{C}\}_0 = f$. Given that the action (293) is invariant under τ-reparametrisation, the evolution with respect to the flow parameter τ is not physical as it is a gauge transformation on the constraint hypersurface defined by (290). Physical observables (known as Dirac observables) are defined as functions of canonical variables $F(\varphi, \pi_\varphi, \chi, p_\chi)$ that are invariant under τ evolution. Then, they satisfy

$$\{F, \mathcal{C}\} \approx 0\,, \tag{295}$$

where $\approx$ represents a "weak equality", meaning the equality holds on the constraint hypersurface. In other words, functions F satisfying (295) are constant along the trajectories (within the constraint hypersurface) generated by the constraint (290).

One can now use the strategy of "evolving constants of motion" [336–339] to define *relational Dirac observables*, which evolve with respect to another chosen observable along the flow generated by $\mathcal{C}$. These are also known as "complete" observables $F_{f,\chi}(\chi_0)$, and correspond to the value a partial observable f takes on $\mathcal{C}$ when another partial observable χ takes the value χ_0. The second partial observable

χ is thought as a dynamical "clock" degree of freedom, chosen to parametrise the flow in place of the unphysical parameter τ. In short, one can construct a complete observable satisfying (295) as [135, 190, 227, 228, 358]

$$F_{f,\chi}(\chi_0) \approx \sum_{n=0}^{\infty} \frac{(\chi_0 - \chi)^n}{n!} \left\{ f, \frac{\mathcal{C}}{\{\chi, \mathcal{C}\}} \right\}_n . \tag{296}$$

While (296) holds for (finite-dimensional) systems with a generic Hamiltonian constraint [135, 190, 358], our scenario belongs to the specific class of systems thoroughly analysed in [227, 228]; a simplification arises because of the partition of the classical constraint (290) into a χ component and a φ component

$$\mathcal{C} = H_\chi + H_\varphi = 0 , \tag{297}$$

where the so-called clock Hamiltonian is $H_\chi = p_\chi$ and the GFT Hamiltonian H_φ is given in (287). Thanks to the crucial fact that the clock Hamiltonian is canonically conjugate to χ,

$$\{\chi, H_\chi\} = \{\chi, p_\chi\} = 1 , \tag{298}$$

a relational Dirac observable associated to a function f_φ of the GFT phase space (i.e., a function of φ and π_φ) takes the simple form (using (297) and (298) in (296)) [135, 190, 227, 228, 358]

$$F_{f_\varphi,\chi}(\chi_0) \approx \sum_{n=0}^{\infty} \frac{(\chi_0 - \chi)^n}{n!} \{f_\varphi, H_\varphi\}_n . \tag{299}$$

In the context of our group field theory (cf. (293)), the observables $F_{f_\varphi,\chi}(\chi_0)$ are then interpreted as geometrical quantities (at this stage still described by classical phase space functions of GFT degrees of freedom) that evolve relationally with respect to the values that the matter scalar field takes. For instance, upon quantisation, this setting will be able to describe the evolution of the GFT number (or volume) operator as a function of the scalar field. In fact, the partition into a matter clock sector and a geometry sector will be exploited in the quantum theory to express the dynamics in the general framework of [227, 228]; this will allow to relate the parametrised theory introduced here to existing approaches to canonical quantisation of GFT.

6.2 QUANTUM THEORY FOR SINGLE MODE

6.2.1 *Kinematics*

We now quantise the theory described by $S[\varphi,\chi]$ in (293), promoting the canonical coordinates and momenta to operators with the commutation relations

$$\begin{aligned}[\hat{\varphi},\hat{\pi}_\varphi] &= \mathrm{i}\,,\\ [\hat{\chi},\hat{p}_\chi] &= \mathrm{i}\,,\end{aligned} \tag{300}$$

all the others being zero. Kinematically, the Hilbert space of such a quantum theory is the tensor product of two Hilbert spaces: one for the matter sector associated with the scalar field χ and one for the geometry sector associated with the group field φ, namely

$$\mathcal{H}_{\mathrm{kin}} = \mathcal{H}_\chi \otimes \mathcal{H}_\varphi\,. \tag{301}$$

Both $\mathcal{H}_\chi$ and $\mathcal{H}_\varphi$ are spaces of square-integrable functions over the real line, so that $\mathcal{H}_{\mathrm{kin}} = L^2(\mathbb{R}^2)$.

GEOMETRY SECTOR. The Hamiltonian for the (single-mode) GFT system living on the geometry sector is given by promoting (287) to an operator on $\mathcal{H}_\varphi$ as

$$\hat{H}_\varphi = \frac{1}{2}\,\mathrm{sgn}\left(K^{(0)}\right)\left(\frac{\hat{\pi}_\varphi^2}{|K^{(2)}|} - |K^{(0)}|\hat{\varphi}^2\right), \tag{302}$$

which resembles the Hamiltonian of a quantum particle with an inverted harmonic potential. The Schrödinger problem for the Hamiltonian (302) can be solved explicitly [112, 113]. Specifically, given the shape of the potential and by virtue of general properties for the one-dimensional Schrödinger equation (see, e.g., [258]), the energy spectrum is continuous, $\sigma(\hat{H}_\varphi) = (-\infty,\infty)$, and doubly-degenerate

$$\hat{H}_\varphi|\psi_\pm^E\rangle = E|\psi_\pm^E\rangle\,, \qquad E\in\mathbb{R}\,. \tag{303}$$

As mentioned below (287), note that the global sign factor $\mathrm{sgn}(K^{(0)})$ in (302) is irrelevant since switching sign amounts to a relabelling of the eigenvalue $E \in (-\infty,\infty)$. Thus we can set $\mathrm{sgn}(K^{(0)}) = 1$ for simplicity and without loss of generality in what follows. In terms of wavefunctions, we then want to solve $\hat{H}_\varphi \psi^E_\pm(\varphi) = E\,\psi^E_\pm(\varphi)$, that is

$$\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\psi^E_\pm(\varphi) + \left(\left|K^{(0)}K^{(2)}\right|\varphi^2 + 2\left|K^{(2)}\right|E\right)\psi^E_\pm(\varphi) = 0\,. \tag{304}$$

To that end, we introduce the variable

$$\zeta = \sqrt{2\mathrm{i}\sqrt{\left|K^{(0)}K^{(2)}\right|}}\;\varphi \tag{305}$$

so that (304) becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\psi^E_\pm(\zeta) + \left(\nu + \frac{1}{2} - \frac{\zeta^2}{4}\right)\psi^E_\pm(\zeta) = 0\,, \tag{306}$$

with

$$\nu = -\mathrm{i}\sqrt{\left|\frac{K^{(2)}}{K^{(0)}}\right|}\,E - \frac{1}{2}\,. \tag{307}$$

(306) is called Weber equation and has known solutions in terms of special functions known as parabolic cylinder functions [2, 201], denoted $D_\nu(\zeta)$. Specifically, the two independent solutions are given by $\psi^E_+(\zeta) = \mathcal{N}_+ D_\nu(\zeta)$ and $\psi^E_-(\zeta) = \mathcal{N}_- D_{-\nu-1}(\mathrm{i}\zeta)$, where $\mathcal{N}_\pm$ are normalisation constants and ζ and ν are given in (305) and (307). Interestingly, when ν is a non-negative integer n, a parabolic cylinder function simplifies to $D_n(\zeta) = 2^{-n/2}e^{-\zeta^2/4}H_n(\zeta/\sqrt{2})$, where H_n is a Hermite polynomial (which notably solves the differential equation representing the eigenvalue problem for the standard harmonic oscillator).

The first important property that we will need when discussing relational quantum dynamics in Section 6.2.2 is the orthonormality of the eigenstates $|\psi^E_\pm\rangle$ (switching to the bra-ket notation), which for suitable $\mathcal{N}_\pm$ reads [112, 113]

$$\langle\psi^E_m|\psi^{E'}_n\rangle = \delta_{mn}\delta(E - E')\,, \tag{308}$$

where m and n label the degeneracy ($+$ or $-$) and the Kronecker delta δ_{mn} indicates that "$+$ states" and "$-$ states" are orthogonal. (308) is sometimes called generalised orthonormality because of the distributional nature of the Dirac delta (one can rigorously deal with distributions by considering rigged Hilbert spaces; see, e.g., [45]). Moreover, one can decompose the Hamiltonian (302) and obtain a spectral resolution in the following form [112, 113]

$$\begin{aligned}\hat{H}_\varphi &= \int \mathrm{d}E\; E\, |\psi_+^E\rangle\langle\psi_+^E| + \int \mathrm{d}E\; E\, |\psi_-^E\rangle\langle\psi_-^E| \\ &= \int_\pm \mathrm{d}E\; E\, |\psi_\pm^E\rangle\langle\psi_\pm^E|\,,\end{aligned} \tag{309}$$

where we introduced the notation $\int_\pm := \sum_\pm \int$ to take into account the double degeneracy of the spectrum. Note that while a Schrödinger equation for GFT with the Hamiltonian (302) was introduced in [186, 379], there was no discussion of exact solutions (even for a single field mode).

Similarly to the deparametrised approach described in Section 3.2.2, one can now change basis and introduce ladder operators $\hat{\mathfrak{a}}$ and $\hat{\mathfrak{a}}^\dagger$ (which are defined just like in (106) in terms of the parameters appearing in (302)). These are to be considered *kinematical* operators at this stage: the dynamics of our (parametrised) theory are only defined when the quantum version of the constraint (297) is used, as we will do in the next section. In other words, one can build a *kinematical* Fock space starting from the Fock vacuum $|0\rangle$ defined by $\hat{\mathfrak{a}}|0\rangle = 0$, and building n-particle states in the usual way; for example, the one-particle state reads $|1\rangle = \hat{\mathfrak{a}}^\dagger|0\rangle$. Moreover, one can discuss kinematical operators built from $\hat{\mathfrak{a}}$ and $\hat{\mathfrak{a}}^\dagger$, such as the number operator

We stress that this is different from the deparametrised formalism of Section 3.2.2, which deals with physical states (cf. discussion around (108)).

$$\hat{\mathfrak{N}} = \hat{\mathfrak{a}}^\dagger\hat{\mathfrak{a}}\,, \tag{310}$$

and the volume operator $\hat{\mathfrak{V}} = v\hat{\mathfrak{N}}$. We use a new notation for these kinematical operators to emphasise that these are *a priori* different operators than the ones introduced in the deparametrised approach.

MATTER SECTOR. As already discussed at the classical level, we will want to use the matter field as relational clock. The quantum theory living on $\mathcal{H}_\chi$ is isomorphic to the Hilbert space of a particle on a line, so we have the following properties for the $\hat{\chi}$ operator

$$\begin{aligned}\hat{\chi}|\chi\rangle &= \chi|\chi\rangle\,, \\ \langle\chi|\chi'\rangle &= \delta(\chi-\chi')\,,\end{aligned} \tag{311}$$

and similarly for its conjugate momentum

$$\begin{aligned}\hat{p}_\chi|p_\chi\rangle &= p_\chi|p_\chi\rangle\,, \\ \langle p_\chi|p'_\chi\rangle &= \delta(p_\chi-p'_\chi)\,,\end{aligned} \tag{312}$$

where we choose conventions with $\langle p_\chi|\chi\rangle = \frac{1}{\sqrt{2\pi}}e^{-\mathrm{i}p_\chi\chi}$. It follows that one can write the identity on $\mathcal{H}_\chi$ as

$$\mathbb{I}_\chi = \int \mathrm{d}\chi|\chi\rangle\langle\chi| = \int \mathrm{d}p_\chi|p_\chi\rangle\langle p_\chi|\,, \tag{313}$$

and the spectral decomposition of $\hat{\chi}$ and $\hat{p}_\chi$ as

$$\hat{\chi} = \int \mathrm{d}\chi\, \chi\, |\chi\rangle\langle\chi|\,, \tag{314}$$

$$\hat{H}_\chi = \hat{p}_\chi = \int \mathrm{d}p_\chi\, p_\chi\, |p_\chi\rangle\langle p_\chi|\,. \tag{315}$$

Notice that interpreting $\hat{p}_\chi$ as the (quantum) clock Hamiltonian (cf. (297) and (298)) implies that the operator $\hat{\chi}$ in (314) will be interpreted as *time operator*.[4] The clock states correspond to eigenstates $|\chi\rangle$ of the time operator $\hat{\chi}$. Such states "evolve" under the action of the group generated by $\hat{p}_\chi$ as

Albeit in a very different way, $\hat{\chi}$ will conceptually play a similar role to that of the operator (145) in the coherent peaked states formulation of the algebraic approach.

$$\hat{U}_\chi(\alpha)|\chi'\rangle = |\chi' + \alpha\rangle\,, \tag{316}$$

where

$$\hat{U}_\chi(\alpha) = e^{-\mathrm{i}\hat{p}_\chi\alpha}\,, \qquad \alpha \in \mathbb{R}\,. \tag{317}$$

6.2.2 *Relational dynamics*

One now makes use of the quantum version of the classical constraint (290), $\hat{\mathcal{C}}$, to identify physical states among the kinematical ones. This is the first step of the Dirac quantisation procedure, which allows to discuss the notion of relational Dirac observables in a precise sense (via the quantum analogue of (295) and (299)). At the same time, the tensor product structure of (301) suggests that the Page–Wootters formalism (recalled below) could also be used to implement the notion of relational dynamics. For this reason, we explicitly write the quantum constraint as

$$\begin{aligned}\hat{\mathcal{C}} &= \hat{p}_\chi + \hat{H}_\varphi \\ &= \hat{p}_\chi \otimes \mathbb{I}_\varphi + \mathbb{I}_\chi \otimes \hat{H}_\varphi\,,\end{aligned} \tag{318}$$

where $\mathbb{I}_\varphi$ and $\mathbb{I}_\chi$ are the identity operators on $\mathcal{H}_\varphi$ and $\mathcal{H}_\chi$ respectively. It was shown in [227, 228] that with a constraint of the form (318),

4 More formally, one can define the time operator $\hat{\chi}$ as the first moment operator (314) of the "time observable" $E_\chi := |\chi\rangle\langle\chi|$, in turn defined via the most general notion of quantum observable as a positive operator-valued measure (POVM) [227, 228].

the Dirac algorithm for a constraint quantisation (for instance implemented using group averaging techniques) and the Page–Wootters formalism yield equivalent relational dynamics, as we will exemplify with our GFT model.

From now on, both operators and states will occasionally have a subscript χ or φ, to clarify (when necessary) in which sector of (301) they act or live.

DIRAC QUANTISATION. A generic state in the kinematical Hilbert space (301), $|\Psi_{\text{kin}}\rangle \in \mathcal{H}_{\text{kin}}$, can be written as

$$|\Psi_{\text{kin}}\rangle = \int_{\pm} \mathrm{d}E \int \mathrm{d}p_\chi \, \Psi_\pm(p_\chi, E) \, |p_\chi\rangle_\chi \otimes |\psi^E_\pm\rangle_\varphi \,. \tag{319}$$

Following the Dirac programme for quantising constrained systems, one defines physical states by demanding that they are annihilated by the constraint (318),

As mentioned, this is conceptually similar to the situation in general relativity (cf. (7)).

$$\hat{\mathcal{C}}|\Psi_{\text{phys}}\rangle = 0 \,. \tag{320}$$

As is well known [367], such physical states are not normalisable in $\mathcal{H}_{\text{kin}}$; one needs to introduce a new inner product since $\langle\Psi_{\text{phys}}|\Psi_{\text{phys}}\rangle_{\text{kin}}$ diverges, where $\langle\cdot|\cdot\rangle_{\text{kin}}$ is the inner product on (301). One way of doing this is by "projecting" a kinematical state onto a physical one by means of group averaging [279, 280, 367],

$$\delta(\hat{\mathcal{C}}) = \frac{1}{2\pi} \int \mathrm{d}\alpha \, e^{i\alpha\hat{\mathcal{C}}} \,, \tag{321}$$

as $|\Psi_{\text{phys}}\rangle = \delta(\hat{\mathcal{C}})|\Psi_{\text{kin}}\rangle$, and then defining a physical inner product

$$\langle\Psi_{\text{phys}}|\Psi_{\text{phys}}\rangle_{\text{phys}} := \langle\Psi_{\text{kin}}|\delta(\hat{\mathcal{C}})|\Psi_{\text{kin}}\rangle_{\text{kin}} \,. \tag{322}$$

Starting from (319) and using properties (303), (308) and (312), one explicitly finds

$$|\Psi_{\text{phys}}\rangle = \int_{\pm} \mathrm{d}E \, \Psi_\pm(-E, E) \, |-E\rangle_\chi \otimes |\psi^E_\pm\rangle_\varphi \,, \tag{323}$$

and

$$\langle\Psi_{\text{phys}}|\Psi_{\text{phys}}\rangle_{\text{phys}} = \int_{\pm} \mathrm{d}E \, |\Psi_\pm(-E, E)|^2 \,. \tag{324}$$

This norm defines $\mathcal{H}_{\text{phys}}$ as the space of solutions to (320). Physical states do not change under the flow of the total Hamiltonian $\hat{C}$,

$$\hat{U}_{\chi\varphi}(\alpha)|\Psi_{\text{phys}}\rangle = |\Psi_{\text{phys}}\rangle\,, \tag{325}$$

with

$$\hat{U}_{\chi\varphi}(\alpha) := e^{-\mathrm{i}\alpha\hat{C}} = e^{-\mathrm{i}\alpha\hat{p}_\chi} \otimes e^{-\mathrm{i}\alpha\hat{H}_\varphi}\,, \qquad \alpha \in \mathbb{R}\,, \tag{326}$$

and are sometimes called "timeless" or "frozen". More recently [227, 228], they have been denoted "clock-neutral" as they describe physics before choosing a temporal reference system. (325) is what gives rise to the problem of time [233, 257], which can be tackled by defining the quantum counterpart of Dirac relational observables (cf. (295) and (299)). Indeed, one can now choose the temporal reference system (namely, the clock) associated with the Hilbert space $\mathcal{H}_\chi$ with properties (311)–(316), and find the quantised version of (299) as [227, 228]

$$\hat{F}_{f_\varphi,\chi}(\chi_0) = \frac{1}{2\pi}\int \mathrm{d}\chi\,|\chi\rangle\langle\chi| \otimes \sum_{n=0}^{\infty}\frac{\mathrm{i}^n}{n!}(\chi-\chi_0)^n\left[\hat{f}_\varphi, \hat{H}_\varphi\right]_n\,, \tag{327}$$

where the commutator $[\hat{f}_\varphi, \hat{H}_\varphi]_n := [[\hat{f}_\varphi, \hat{H}_\varphi]_{n-1}, \hat{H}_\varphi]$ and $[\hat{f}_\varphi, \hat{H}_\varphi]_0 := \hat{f}_\varphi$. Thanks to the Baker–Campbell–Hausdorff formula, (327) can be equivalently recast as

$$\begin{aligned}\hat{F}_{f_\varphi,\chi}(\chi_0) &= \frac{1}{2\pi}\int \mathrm{d}\chi\,|\chi\rangle\langle\chi| \otimes \hat{U}_\varphi(\chi-\chi_0)\hat{f}_\varphi\hat{U}_\varphi^\dagger(\chi-\chi_0)\\ &= \frac{1}{2\pi}\int \mathrm{d}\alpha\,\hat{U}_{\chi\varphi}(\alpha)\left(|\chi_0\rangle\langle\chi_0| \otimes \hat{f}_\varphi\right)\hat{U}_{\chi\varphi}^\dagger(\alpha)\,,\end{aligned} \tag{328}$$

where $\hat{U}_\varphi(\alpha) = e^{-\mathrm{i}\hat{H}_\varphi\alpha}$, $\hat{U}_{\chi\varphi}(\alpha)$ is given in (326), and in the last line we changed integration variable, $\chi \to \alpha + \chi_0$. Crucially, quantum relational observables defined using the prescription (327) commute with the constraint operator $\hat{C}$,

$$\left[\hat{F}_{f_\varphi,\chi}(\chi_0), \hat{C}\right] = 0\,, \tag{329}$$

and are thus called (quantum) Dirac observables.[5]

5 Since our clock Hamiltonian is simply $\hat{H}_\chi = \hat{p}_\chi$, the commutator in (329) vanishes strongly (i.e., algebraically). In [227, 228] it is shown that for more complicated clock Hamiltonians one can still prove that a Dirac observable associated to a physical phase space function weakly commutes with the constraint $\hat{C}$, namely when applied to physical states.

The prototype Dirac observable for our GFT model is the number operator $\hat{f}_\varphi = \hat{\mathfrak{N}}$ (cf. (310)); thus, we will be specifically interested in the observable

The number operator is directly related to the volume operator usually adopted in GFT cosmology, as seen in Chapter 3.

$$\hat{N}_D(\chi_0) := \frac{1}{2\pi} \int d\alpha \, \hat{U}_{\chi\varphi}(\alpha) \left(|\chi_0\rangle\langle\chi_0| \otimes \hat{\mathfrak{N}} \right) \hat{U}^\dagger_{\chi\varphi}(\alpha) \, , \tag{330}$$

where the subscript D refers to the fact that this is a Dirac observable on the total Hilbert space. As opposed to the kinematical counterpart $\hat{\mathfrak{N}}$, the operator $\hat{N}_D(\chi_0)$ evolves as the parameter χ_0 runs (taking the values the time operator $\hat{\chi}$ can take), meaning that (330) truly defines relational quantum dynamics on $\mathcal{H}_{\text{phys}}$ for the GFT number operator. In particular, even if physical states do not transform under the action of the Hamiltonian constraint $\hat{\mathcal{C}}$, we will evaluate the relational Dirac observable (330) using the physical inner product (322) so as to obtain an expectation value for the number operator which indeed changes with respect to the matter scalar field.

page–wootters formalism. The framework introduced by Page and Wootters [309, 381] provides another way to define relational dynamics for systems subject to a quantum constraint of the form (318). Specifically, starting again with a kinematical Hilbert space (301) that is split into clock and system (our matter scalar field χ and single-mode GFT model, respectively), one selects physical states using the constraint equation (320). The apparent difference with the previous description arises when choosing an inner product on $\mathcal{H}_{\text{phys}}$ to completely specify the space of solutions of (320).

The conceptual idea behind the Page–Wootters formalism is to interpret quantum theory with conditional probabilities. More precisely, one defines the state of a system at a given instant of (relational) time as a solution to the constraint equation *conditioned* on a subsystem of the theory to be in a state corresponding to that time. Following this idea, we define the state of our single-mode GFT system at a given time, say χ_0, as a solution to the constraint (320) conditioned on the clock being in the state $|\chi_0\rangle$,

Figure 23 provides a pictorial representation of the physical (history) state (332) and the conditioning (331).

$$|\psi(\chi_0)\rangle_\varphi := \left(\langle\chi_0| \otimes \mathbb{I}_\varphi \right) |\Psi_{\text{phys}}\rangle \, . \tag{331}$$

It follows that a physical state can be expressed as a "history state", namely

$$|\Psi_{\text{phys}}\rangle = \int d\chi_0 \, |\chi_0\rangle_\chi \otimes |\psi(\chi_0)\rangle_\varphi \, , \tag{332}$$

since $\mathbb{I}_\chi \otimes \mathbb{I}_\varphi = \int d\chi_0 |\chi_0\rangle\langle\chi_0| \otimes \mathbb{I}_\varphi$. Indeed, the state (332) encodes information about the whole timeline. The usual formulation of quantum mechanics for the conditioned states can be recovered in terms of a Schrödinger equation in the clock time χ_0 [309, 381],

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\chi_0}|\psi(\chi_0)\rangle_\varphi = \hat{H}_\varphi |\psi(\chi_0)\rangle_\varphi\,, \tag{333}$$

which is easily found from the constraint equation by "conditioning" as $\left({}_\chi\langle\chi_0| \otimes \mathbb{I}_\varphi\right)\hat{\mathcal{C}}|\Psi_{\text{phys}}\rangle = 0$. Moreover, one introduces what is known as the Page–Wootters inner product [227, 228, 352]

$$\langle\Psi_{\text{phys}}|\Psi_{\text{phys}}\rangle_{\text{PW}} := \langle\Psi_{\text{phys}}|\Big(|\chi_0\rangle\langle\chi_0| \otimes \mathbb{I}_\varphi\Big)|\Psi_{\text{phys}}\rangle_{\text{kin}}\,, \tag{334}$$

which is consistent with the usual inner product on $\mathcal{H}_\varphi$ at all times since, using (311),

$$\begin{aligned}\langle\Psi_{\text{phys}}|\Psi_{\text{phys}}\rangle_{\text{PW}} &= \left(\int d\chi'\, {}_\chi\langle\chi'| \otimes {}_\varphi\langle\psi(\chi')|\right)\left(|\chi_0\rangle_{\chi\chi}\langle\chi_0| \otimes \mathbb{I}_\varphi\right) \\ &\quad\times \left(\int d\tilde{\chi}\, |\tilde{\chi}\rangle_\chi \otimes |\psi(\tilde{\chi})\rangle_\varphi\right) \\ &= {}_\varphi\langle\psi(\chi_0)|\psi(\chi_0)\rangle_\varphi\,,\end{aligned} \tag{335}$$

where ${}_\varphi\langle\psi(\chi_0)|\psi(\chi_0)\rangle_\varphi = {}_\varphi\langle\psi(0)|\psi(0)\rangle_\varphi$ is independent of χ_0 thanks to (333) and the fact that $\hat{H}_\varphi$ is self-adjoint (cf. (302)).

In what follows we will provide an application of the equivalence shown in [227, 228] between the relational dynamics defined using a Dirac quantisation and using the Page–Wootters formalism, mainly focussing on the number operator of our GFT model. To begin with, it is easy to check explicitly that

$$\langle\Psi_{\text{phys}}|\Psi_{\text{phys}}\rangle_{\text{PW}} = \int_\pm dE\, |\Psi_\pm(-E, E)|^2 \tag{336}$$

is the same as (324). Note that in the following calculations we use the generic expressions for the physical inner product (322) and the Page–Wootters inner product (334) instead of their explicit form (336) (or (324)).

NUMBER DYNAMICS. We can now turn to the calculation of the expectation value of the Dirac observable associated to the number operator (330), using the physical inner product. First, we find

$$\begin{aligned}\hat{N}_D(\chi_0)|\Psi_{\text{phys}}\rangle &= \frac{1}{2\pi}\int \mathrm{d}\alpha\, \hat{U}_{\chi\varphi}(\alpha)\left(|\chi_0\rangle\langle\chi_0|\otimes\hat{\mathfrak{N}}\right)|\Psi_{\text{phys}}\rangle \\ &= \frac{1}{2\pi}\int \mathrm{d}\alpha\, \hat{U}_{\chi}(\alpha)|\chi_0\rangle\langle\chi_0|\otimes\hat{U}_{\varphi}(\alpha)\hat{\mathfrak{N}}|\Psi_{\text{phys}}\rangle \\ &= \delta(\hat{\mathcal{C}})\left(|\chi_0\rangle\langle\chi_0|\otimes\hat{\mathfrak{N}}\right)|\Psi_{\text{phys}}\rangle\,,\end{aligned} \tag{337}$$

where we used $\hat{U}^{\dagger}_{\chi\varphi}(\chi)|\Psi_{\text{phys}}\rangle = |\Psi_{\text{phys}}\rangle$ (cf. (325)) and the definitions (326) and (321). Then, we calculate the expectation value in the physical inner product (322) as

$$\begin{aligned}N_D(\chi_0) &:= \langle\Psi_{\text{phys}}|\hat{N}_D(\chi_0)|\Psi_{\text{phys}}\rangle_{\text{phys}} \\ &= \langle\Psi_{\text{phys}}|\delta(\hat{\mathcal{C}})\left(|\chi_0\rangle\langle\chi_0|\otimes\hat{\mathfrak{N}}\right)|\Psi_{\text{phys}}\rangle_{\text{phys}} \\ &= \langle\Psi_{\text{kin}}|\delta(\hat{\mathcal{C}})\left(|\chi_0\rangle\langle\chi_0|\otimes\hat{\mathfrak{N}}\right)\delta(\hat{\mathcal{C}})|\Psi_{\text{kin}}\rangle_{\text{kin}} \\ &= \langle\Psi_{\text{phys}}|\left(|\chi_0\rangle\langle\chi_0|\otimes\hat{\mathfrak{N}}\right)|\Psi_{\text{phys}}\rangle_{\text{kin}} \\ &= \langle\Psi_{\text{phys}}|\hat{\mathfrak{N}}|\Psi_{\text{phys}}\rangle_{\text{PW}}\,.\end{aligned} \tag{338}$$

Of course, using the definition of conditioned states (331), it is easy to show that (338) gives back the result of the deparametrised approach of Section 3.2.2:

$$\begin{aligned}N_D(\chi_0) &= \langle\Psi_{\text{phys}}|\left(|\chi_0\rangle\langle\chi_0|\otimes\hat{\mathfrak{N}}\right)|\Psi_{\text{phys}}\rangle_{\text{kin}} \\ &= {}_{\varphi}\langle\psi(\chi_0)|\hat{\mathfrak{N}}|\psi(\chi_0)\rangle_{\varphi} \\ &= {}_{\varphi}\langle\psi|\,\hat{U}^{\dagger}_{\varphi}(\chi_0)\,\hat{\mathfrak{N}}\,\hat{U}_{\varphi}(\chi_0)\,|\psi\rangle_{\varphi}\,,\end{aligned} \tag{339}$$

where $|\psi\rangle_{\varphi} = |\psi(0)\rangle_{\varphi}$ and we switched from the Schrödinger picture to the Heisenberg picture in the last equality. Indeed, (339) shows that the expectation value of the relational Dirac observable $\hat{N}_D(\chi_0)$, computed with the physical inner product on $\mathcal{H}_{\text{phys}}$, is equivalent to the expectation value of what was denoted $\hat{N}(\chi) = \hat{U}^{\dagger}(\chi)\,\hat{a}^{\dagger}\hat{a}\,\hat{U}(\chi)$ in the deparametrised quantum theory (cf. (111) and (109), here for a single mode). Of course, the same holds true for the volume operator (and any other one built from ladder operators), so that all the GFT results and applications to cosmology are recovered in our parametrised theory.

In particular, it is straightforward to obtain the relational quantum dynamics of Dirac observables for the volume and the $\hat{C}$ operators of Section 3.3.2, i.e., (154) and (155).

The equivalence established in (339) strengthens the deparametrised approach described in Section 3.2.2 and Section 3.3.2, since the Dirac observable $\hat{N}_D(\chi_0)$ in (330) is defined relationally with respect to the eigenvalues of a time operator $\hat{\chi}$, in contrast with the deparametrised theory where the clock label is chosen before quantisation. In this sense, the construction of Section 6.2 belongs to the "tempus post quantum" type of relational dynamics discussed in [233].

Moreover, (338) shows that $N_D(\chi_0)$ is also equal to the Page–Wootters expectation values of the corresponding kinematical operator $\hat{\mathfrak{N}}$, which allows to reinterpret GFT observables as conditional on the clock. In particular, this suggests a canonical picture characterised by a splitting between quantum geometry and a constant-time "slice" on the history state (332), where the internal time takes a fixed value χ_0 (see left panel of Figure 23). This construction is similar in spirit to the $3+1$ splitting of canonical GR, where one describes evolution as a sequence of constant time hypersurfaces. While the physical state (332) of Page and Wootters is a superposition of all clock states and all GFT states, it allows to answer the relevant questions of what happens at any specific value of relational time. In other words, this approach justifies the interpretation of the GFT quanta as being associated to the same clock reading since the conditioned state (331), by definition, involves a projection onto the χ_0 slice (see right panel of Figure 23).

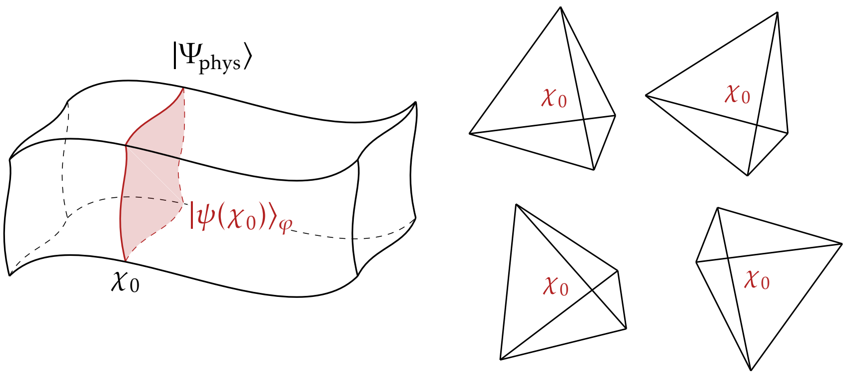

Figure 23: Left: the three-dimensional shape represents the history state $|\Psi_{\text{phys}}\rangle$ given in (332), where the Hilbert space of the clock is visualised horizontally. The quantum geometry (GFT) state $|\psi(\chi_0)\rangle_\varphi$ at the time χ_0 is obtained by conditioning $|\Psi_{\text{phys}}\rangle$ on the clock being in the state $|\chi_0\rangle$ (cf. (331)) and is pictorially represented by a two-dimensional slice. Right: the tetrahedra counted by the number expectation value $N_D(\chi_0)$ in (339) can be thought as labelled by the same value of the scalar field.

6.3 Extension to multiple field modes

In this section we show that our construction can be extended to any number of Peter–Weyl modes. While in principle a GFT contains an infinite number of modes J (see (58)), we will consider a truncation to a finite (but arbitrary) number of modes. Such a truncation could be elegantly implemented using the quantum group $SU_q(2)$,[6] where the deformation parameter q is related to a nonvanishing cosmological constant (see, e.g., [71, 161, 270, 353]), which leads to a maximum irreducible representation and hence a maximum value for the multi-index J. Similar ideas on q-deformation were recently applied in the context of three-dimensional group field theories in [193], and provide a way to implement a cut-off value $J_{\max}$ in the Peter–Weyl decomposition (58). We assume such a cut-off in the following sections.

6.3.1 *Single reparametrisation symmetry*

Without making any assumptions on magnetic indices and on the relative sign of $K_J^{(0)}$ and $K_J^{(2)}$ for the various modes, we start the analysis with the generic action given in (285). As a first generalisation of the procedure described in Section 6.1, we proceed here by introducing a parameter τ for the theory with multiple field modes. Then, all the $\varphi_J(\tau)$'s as well as $\chi(\tau)$ depend on τ, and by means of the chain rule one obtains the action

$$S[\varphi,\chi] = \frac{1}{2}\int \mathrm{d}\tau \sum_J \left(K_J^{(0)} \varphi_{-J}\varphi_J(\partial_\tau\chi) - K_J^{(2)} \frac{(\partial_\tau\varphi_{-J})(\partial_\tau\varphi_J)}{\partial_\tau\chi} \right) . \quad (340)$$

All the key points discussed for the single-mode scenario also apply here. In particular, while we have enlarged the phase space, the theory is now subject to a Hamiltonian constraint (cf. (290))

$$\mathcal{C} = H_\varphi^{\mathrm{tot}} + p_\chi = 0\,, \quad (341)$$

where, rewriting (103) (without interactions) here for convenience,

$$H_\varphi^{\mathrm{tot}} = -\frac{1}{2}\sum_J \left[\frac{\pi_J(\chi)\pi_{-J}(\chi)}{K_J^{(2)}} + K_J^{(0)} \varphi_J(\chi)\varphi_{-J}(\chi) \right] , \quad (342)$$

6 This is akin to how the Turaev–Viro model is obtained as a regularised Ponzano–Regge model (cf. Section 2.2.2), and similarly for the four-dimensional Barrett–Crane model (see footnote 14).

with momenta given by

$$\pi_J = -K_J^{(2)} \frac{\partial_\tau \varphi_{-J}}{\partial_\tau \chi}\,, \tag{343}$$

$$p_\chi = \frac{1}{2}\sum_J \left(K_J^{(0)} \varphi_{-J}\varphi_J + K_J^{(2)} \frac{(\partial_\tau \varphi_{-J})(\partial_\tau \varphi_J)}{(\partial_\tau \chi)^2} \right)\,. \tag{344}$$

The equations of motion for every group field mode, $\partial_\tau \varphi_J = -N\pi_{-J}/K_J^{(2)}$, and for the matter field, $\partial_\tau \chi = N$, can be obtained as in (292). They also follow from the action (in Hamiltonian form)

$$S = \int \mathrm{d}\tau \left(\sum_J \pi_J \partial_\tau \varphi_J + p_\chi \partial_\tau \chi - N\mathcal{C} \right)\,, \tag{345}$$

which generalises (293) for multiple modes.

Finally, in complete analogy with Section 6.1, one can adopt the strategy of relational Dirac observables and define quantities satisfying $\{F, \mathcal{C}\} \approx 0$ as

$$F_{f_\varphi,\chi}(\chi_0) \approx \sum_{n=0}^{\infty} \frac{(\chi_0 - \chi)^n}{n!} \{f_\varphi, H_\varphi^{\mathrm{tot}}\}_n\,, \tag{346}$$

where H_φ^{tot} is given in (342) and f_φ is now a function of multiple GFT modes. The complete observable $F_{f_\varphi,\chi}(\chi_0)$ associates values of f_φ to the specific relational time $\chi = \chi_0$ (see Section 6.1).

The quantum theory corresponding to the above construction is obtained in the usual manner by means of the commutators

$$\begin{aligned} [\hat{\varphi}_J, \hat{\pi}_{J'}] &= \mathrm{i}\delta_{JJ'}\,, \\ [\hat{\chi}, \hat{p}_\chi] &= \mathrm{i}\,, \end{aligned} \tag{347}$$

where the kinematical Hilbert space can be written as

$$\mathcal{H}_{\mathrm{kin}} = \mathcal{H}_\chi \otimes \mathcal{H}_\varphi^{\mathrm{tot}}\,, \tag{348}$$

with

$$\mathcal{H}_\varphi^{\mathrm{tot}} = \bigotimes_J \mathcal{H}_{\varphi_J}\,, \tag{349}$$

namely as a tensor product of a matter clock sector $\mathcal{H}_\chi$ (just as in (301)) and the *total* GFT Hilbert space given by the tensor product of single-mode Hilbert spaces $\mathcal{H}_{\varphi_J}$ for the various modes.

Even though the Hamiltonian $\hat{H}_\varphi^{\text{tot}}$ (obtained by quantising (342), cf. (113)) couples modes in pairs (J is coupled to $-J$), we showed in Section 3.2.2 that it can be written as the sum over modes of single-mode contributions (see discussion around (122)). As we have seen, one can distinguish between two cases based on the relative signs of $K_J^{(0)}$ and $K_J^{(2)}$, which lead to the harmonic oscillator (HO) and squeezing (SQ) Hamiltonians in (123); indeed, the total Hamiltonian for a free GFT can be written as

Recall that $\mathfrak{J}^{HO}$ is the set of J such that $K_J^{(0)}$ and $K_J^{(2)}$ have the same sign and $\mathfrak{J}^{SQ}$ is the set of J such that $K_J^{(0)}$ and $K_J^{(2)}$ have opposite signs.

$$\hat{H}_\varphi^{\text{tot}} = \sum_{J\in\mathfrak{J}^{\text{HO}}} \hat{H}_J + \sum_{J\in\mathfrak{J}^{\text{SQ}}} \hat{H}_J\,; \tag{350}$$

we refer to the explanation around (122) and (123) for all the details. In (350) we are suppressing the explicit mentioning of the identity operators acting on all the other factors of $\mathcal{H}_\varphi^{\text{tot}}$; i.e., a tensor product with $\bigotimes_{J'\neq J}\mathbb{I}_{\varphi_{J'}}$ is understood for every single-mode Hamiltonian $\hat{H}_J$. Both types of single-mode Hamiltonians in (350) (cf. (123)) admit a spectral decomposition.[7] Specifically, one can make use of (309) and the standard properties of harmonic oscillator-like Hamiltonians to obtain a total spectral decomposition for $\hat{H}_\varphi^{\text{tot}}$ of the form

$$\hat{H}_\varphi^{\text{tot}} = \sum_J \hat{H}_J = \sum_J \left(\sum\!\!\!\!\!\!\int_{E} E_J |\psi^{E_J}\rangle\langle\psi^{E_J}| \right), \tag{351}$$

where the sum-integral notation introduced in [227, 228] is used to take into account all modes in a compact way (it represents a sum for modes $J\in\mathfrak{J}^{\text{HO}}$ and an integral $\int_\pm$ for modes $J\in\mathfrak{J}^{\text{SQ}}$). Accordingly, the notation $|\psi^{E_J}\rangle$ refers to harmonic oscillator number eigenstates associated with *discrete* energy eigenvalues E_J for $J\in\mathfrak{J}^{\text{HO}}$,[8] and squeezing Hamiltonian eigenstates $|\psi_\pm^{E_J}\rangle$ introduced in Section 6.2.1, with *continuous* label E_J, for $J\in\mathfrak{J}^{\text{SQ}}$.

Since every $\hat{H}_J$ in the total Hamiltonian only acts on the corresponding $\mathcal{H}_{\varphi_J}$ (namely the factor of (348) with the same J), one can easily show that

$$\hat{H}_\varphi^{\text{tot}}\left(\bigotimes_J |\psi^{E_J}\rangle\right) = E_{\text{tot}}\left(\bigotimes_J |\psi^{E_J}\rangle\right), \tag{352}$$

7 While the harmonic oscillator case is solved in any quantum mechanics textbook, the case with $K_J^{(0)}$ and $K_J^{(2)}$ having opposite signs is explicitly given in Section 6.2.1 using the $(\hat{\varphi},\hat{\pi})$ basis, see (302) and discussion thereafter.

8 Specifically, $E_J = \epsilon_J\omega_J\left(n+\frac{1}{2}\right)$ for $J\in\mathfrak{J}^{\text{HO}}$, with $n\in\mathbb{N}_0$ (cf. (123)).

where $\bigotimes_J |\psi^{E_J}\rangle \in \mathcal{H}_\varphi^{\text{tot}}$ and $E_{\text{tot}} := \sum_J E_J$. Then, one can use the quantum constraint, $\hat{C} = \hat{p}_\chi \otimes \mathbb{I}_\varphi^{\text{tot}} + \mathbb{I}_\chi \otimes \hat{H}_{\text{tot}}$ with $\mathbb{I}_\varphi^{\text{tot}} = \bigotimes_J \mathbb{I}_{\varphi_J}$, to formally define a group averaging operation $\delta(\hat{C})$ (see (321)) to obtain physical states. As in the single-mode scenario, one starts with a kinematical state

$$|\Psi_{\text{kin}}\rangle = \sumint_{\{E_J\}} \int \mathrm{d}p_\chi \, \Psi\left(p_\chi, \{E_J\}\right) |p_\chi\rangle \otimes \left(\bigotimes_J |\psi^{E_J}\rangle\right), \tag{353}$$

where $\Psi(p_\chi, \{E_J\})$ depends on discrete E_J variables for all $J \in \mathfrak{J}^{\text{HO}}$ and on continuous E_J variables for all $J \in \mathfrak{J}^{\text{SQ}}$, which are respectively all summed and integrated over with the notation $\sumint_{\{E_J\}}$. Then, a physical state $|\Psi_{\text{phys}}\rangle = \delta(\hat{C})|\Psi_{\text{kin}}\rangle$ reads

$$|\Psi_{\text{phys}}\rangle = \sumint_{\{E_J\}} \Psi\left(-E_{\text{tot}}, \{E_J\}\right) |-E_{\text{tot}}\rangle \otimes \left(\bigotimes_J |\psi^{E_J}\rangle\right). \tag{354}$$

By suitably generalising the construction for multiple field modes, the Page–Wootters formalism described in Section 6.2 can be applied here. In particular, one can define the conditioned state $|\psi(\chi_0)\rangle_{\varphi_{\text{tot}}} \in \mathcal{H}_\varphi^{\text{tot}}$ as

$$|\psi(\chi_0)\rangle_{\varphi_{\text{tot}}} := \left(\langle\chi_0| \otimes \mathbb{I}_\varphi^{\text{tot}}\right)|\Psi_{\text{phys}}\rangle, \tag{355}$$

and check that both the Page–Wootters inner product (defined as in (334) with $\mathbb{I}_\varphi$ replaced by $\mathbb{I}_\varphi^{\text{tot}}$) and the physical inner product (322) evaluate to

$$\langle\Psi_{\text{phys}}|\Psi_{\text{phys}}\rangle_{\text{PW}} = \langle\Psi_{\text{phys}}|\Psi_{\text{phys}}\rangle_{\text{phys}} = \sumint_{\{E_J\}} \left|\Psi_\pm\left(-E_{\text{tot}}, \{E_J\}\right)\right|^2. \tag{356}$$

Since the procedure is the same as in the single-mode case, we can readily show the main results as follows. First, one defines the quantum version of (346) for the total number operator[9]

$$\hat{\mathfrak{N}}_{\text{tot}} := \sum_J \left[\hat{\mathfrak{a}}_J^\dagger \hat{\mathfrak{a}}_J \otimes \left(\bigotimes_{J' \neq J} \mathbb{I}_{\varphi_{J'}}\right)\right], \tag{357}$$

9 We are here being extra careful with notation for the sake of precision, for example mentioning the identity operators explicitly in (357).

generalising (330) as

$$\hat{N}_D^{\text{tot}}(\chi_0) := \frac{1}{2\pi}\int \mathrm{d}s \left[\hat{U}_\chi(s)\otimes\left(\bigotimes_J \hat{U}_{\varphi_J}(s)\right)\right] \times \left(|\chi_0\rangle\langle\chi_0|\otimes\hat{\mathfrak{N}}_{\text{tot}}\right)\left[\hat{U}_\chi^\dagger(s)\otimes\left(\bigotimes_J \hat{U}_{\varphi_J}^\dagger(s)\right)\right]. \tag{358}$$

In (358), the tensor product of operators

$$\hat{U}_\chi(s)\otimes\left(\bigotimes_J \hat{U}_{\varphi_J}(s)\right) := e^{-\mathrm{i}s\hat{p}_\chi}\otimes\left(\bigotimes_J e^{-\mathrm{i}s\hat{H}_J}\right) \tag{359}$$

leaves the physical state (354) invariant (cf. (325) and (326)).

Finally, exactly as in (338) and (339), one shows that

$$\begin{aligned} N_D^{\text{tot}}(\chi_0) &:= \langle\Psi_{\text{phys}}|\hat{N}_D^{\text{tot}}(\chi_0)|\Psi_{\text{phys}}\rangle_{\text{phys}} \\ &= \langle\Psi_{\text{phys}}|\hat{\mathfrak{N}}_{\text{tot}}|\Psi_{\text{phys}}\rangle_{\text{PW}} \\ &= {}_{\varphi_{\text{tot}}}\langle\psi(\chi_0)|\hat{\mathfrak{N}}_{\text{tot}}|\psi(\chi_0)\rangle_{\varphi_{\text{tot}}}, \end{aligned} \tag{360}$$

where we used the conditioned state of the Page–Wootters formalism (355) in the last equality. Recalling that the conditioned state satisfies a Schrödinger equation with respect to χ_0 (cf. (333)), we recover the results from the deparametrised approach for a GFT with multiple modes since, working in the Heisenberg picture, (360) is

$$\begin{aligned} N_D^{\text{tot}}(\chi_0) &= {}_{\varphi_{\text{tot}}}\langle\psi|\left[\bigotimes_J \hat{U}_{\varphi_J}^\dagger(\chi_0)\right]\hat{\mathfrak{N}}_{\text{tot}}\left[\bigotimes_J \hat{U}_{\varphi_J}(\chi_0)\right]|\psi\rangle_{\varphi_{\text{tot}}} \\ &= {}_{\varphi_{\text{tot}}}\langle\psi|\sum_J \hat{U}_{\varphi_J}^\dagger(\chi_0)\,\hat{\mathfrak{a}}_J^\dagger\hat{\mathfrak{a}}_J\,\hat{U}_{\varphi_J}(\chi_0)\,|\psi\rangle_{\varphi_{\text{tot}}}, \end{aligned} \tag{361}$$

namely the expectation value of (111) (cf. also (109)). Again note that a tensor product with $\bigotimes_{J'\neq J}\mathbb{I}_{\varphi_{J'}}$ is understood in the last line of (361).

In summary, this section simply shows that one can properly introduce both a Dirac quantisation and the Page–Wootters formalism for a parametrised GFT with any number of Peter–Weyl modes, and obtain a generalisation of all the results of Section 6.1 and Section 6.2. From a conceptual point of view, everything is analogous to the single-mode case: the quantum theory obtained from our parametrised GFT is still characterised by a Hilbert space split into clock and geometry (cf. (348)). By carefully adapting the formalism to accommodate multiple GFT modes, one finds a clear notion of physical relational observables representing quantum geometrical quantities (for instance, the total number of GFT quanta in the various modes (361)) for every value of relational time.

While we have assumed a maximum value for the Peter–Weyl label J_{max} in the above scenario, one might be able to remove the cut-off and consider the full GFT using the theory of *infinite tensor products* firstly developed in [290] and implemented in the context of loop quantum gravity in [347, 363]. In particular, extra care would be needed to make sure the inner product (356), which in principle can contain infinite sums and integrals, converges. Such questions require functional analysis techniques that are beyond the scope of the present work, and we leave them for future research.

6.3.2 *Many reparametrisation symmetries and multi-fingered time*

Here we go beyond the simple generalisation of Section 6.3.1, looking into the possibility of having different clocks for different GFT modes. In the quantum theory of Section 6.2, which focuses on a single (group field) mode, the Hilbert space (301) seems to have an "artificial" symmetry as it splits into two equal pieces, one for the matter clock and one for the GFT mode. This is not the case for the theory of Section 6.3.1, where we consider multiple modes on the GFT sector, but we still make use of one single clock (cf. (348)). In this section we restore such a symmetry so that we have a clock Hilbert space that is "as big" as the GFT Hilbert space. In other words, we study the case where every GFT mode J has its own relational time, which realises what is known as *multi-fingered time evolution* [253, 254], as we explain below. This relates to some of the ideas of [272] where the various GFT quanta are associated with different "single-quantum times"; however, in that scenario one encounters the problem of synchronisation since it is not clear how to find a unique time variable to describe the whole many-body system (see also the earlier work [250] for a classical picture). Here, we will show that by working with field modes – and "single-mode times" – rather than particles, one can still obtain well-defined Dirac observables and therefore pose meaningful relational dynamical questions.

In order to discuss "multiple times" and hence generalise the single reparametrisation invariance of Section 6.3.1, we employ the following trick. First, we need to rewrite the action (285) as a sum over single-J contributions, $S_0[\varphi] = \int \mathrm{d}\chi \sum_J \mathcal{L}_J$; namely, we separate the theory into uncoupled modes already from classical considerations in the Lagrangian formalism. In short, one can use the following classical field redefinitions that combine the J and $-J$ modes (somewhat as in (120)),

$$\begin{aligned}\varphi_J &\longrightarrow \tilde{\varphi}_J = \frac{1}{\sqrt{2}}\left(\varphi_J + \mathrm{i}\varphi_{-J}\right),\\ \varphi_{-J} &\longrightarrow \tilde{\varphi}_{-J} = \frac{1}{\sqrt{2}}\left(\varphi_J - \mathrm{i}\varphi_{-J}\right),\end{aligned} \tag{362}$$

together with the corresponding "velocities" $\partial_\chi\tilde{\varphi}_J$ and $\partial_\chi\tilde{\varphi}_{-J}$, to show that the summands of the Lagrangian in (285) can be written in the following form

$$\begin{aligned}\frac{1}{2}\left(K_J^{(0)}\tilde{\varphi}_{-J}\tilde{\varphi}_J - K_J^{(2)}\partial_\chi\tilde{\varphi}_{-J}\partial_\chi\tilde{\varphi}_J\right) &= \frac{1}{4}\Big(K_J^{(0)}\varphi_J^2 - K_J^{(2)}(\partial_\chi\varphi_J)^2\\ &\qquad + K_J^{(0)}\varphi_{-J}^2 - K_J^{(2)}(\partial_\chi\varphi_{-J})^2\Big)\\ &=: \frac{1}{2}\left(\mathcal{L}_J + \mathcal{L}_{-J}\right).\end{aligned} \tag{363}$$

Just as with the Hamiltonians (see discussion above (114)), the Lagrangians $\mathcal{L}_J$ and $\mathcal{L}_{-J}$ provide the same contribution to the sum $\sum_J$ so that one can write the action (285) as

The same decomposition was already reached in a different way in [185].

$$\begin{aligned}S_0[\varphi] &= \frac{1}{2}\int \mathrm{d}\chi \sum_J \left(K_J^{(0)}\varphi_J^2 - K_J^{(2)}(\partial_\chi\varphi_J)^2\right)\\ &= \int \mathrm{d}\chi \sum_J \mathcal{L}_J\,.\end{aligned} \tag{364}$$

Next, we move the integral sign under the summation sign in (364) and rename dummy integration variables such that, for every element in the sum over J, χ gets labelled with an index as χ_J. This allows to write the action equivalently as

$$S_0[\varphi] = \frac{1}{2}\sum_J \int \mathrm{d}\chi_J \left(K_J^{(0)}\varphi_J^2(\chi_J) - K_J^{(2)}\left(\partial_{\chi_J}\varphi_J(\chi_J)\right)^2\right). \tag{365}$$

Then, one can apply the parametrisation strategy adopted in previous sections, but here for every mode J. In other words, one can parametrise the theory multiple times (adding a symmetry for every J) by introducing a set of parameters τ_J. Just as in Section 6.1, this process essentially doubles the phase space and one can write the parametrised action as

$$\begin{aligned} S[\varphi, \chi] &= \frac{1}{2} \sum_J \int \mathrm{d}\tau_J \left(K_J^{(0)} \varphi_J^2 (\partial_{\tau_J} \chi_J) - K_J^{(2)} \frac{(\partial_{\tau_J} \varphi_J)^2}{\partial_{\tau_J} \chi_J} \right) \\ &= \frac{1}{2} \int \mathrm{d}\tau \sum_J \left(K_J^{(0)} \varphi_J^2 (\partial_\tau \chi_J) - K_J^{(2)} \frac{(\partial_\tau \varphi_J)^2}{\partial_\tau \chi_J} \right) . \end{aligned} \tag{366}$$

The first line of (366) explicitly shows multiple reparametrisation invariances (one for every mode) and in the second line we renamed the dummy integration variables τ_J to τ, so that we could write the action as a single integral. Note that while in the action (340) the same χ contributes to all the terms in the sum, in (366) we have a different χ_J for every J. With the usual steps, one can easily derive the Hamiltonian theory from (366). The conjugate momenta to φ_J and χ_J are defined for every mode J as

$$\begin{aligned} \pi_J &= -K_J^{(2)} \frac{\partial_\tau \varphi_J}{\partial_\tau \chi_J} , \\ p_{\chi_J} &= \frac{1}{2} \left(K_J^{(0)} \varphi_J^2 + K_J^{(2)} \frac{(\partial_\tau \varphi_J)^2}{(\partial_\tau \chi_J)^2} \right) . \end{aligned} \tag{367}$$

Moreover, the theory is subject to a set of *independent* first-class constraints (one for every mode),

$$\mathcal{C}_J = H_J + p_{\chi_J} = 0 , \tag{368}$$

with

$$H_J = -\frac{1}{2} \left[\frac{\pi_J^2(\chi)}{K_J^{(2)}} + K_J^{(0)} \varphi_J^2(\chi) \right] . \tag{369}$$

While the constraint (341) was associated with a single lapse function, one here has a mode-dependent lapse N_J for every constraint (368). Indeed, just like with (293) and (345), it is easy to see that this theory can be obtained from an action

$$S = \int \mathrm{d}\tau \sum_J \left(\pi_J \partial_\tau \varphi_J + p_{\chi_J} \partial_\tau \chi_J - N_J \mathcal{C}_J \right) , \tag{370}$$

which explicitly shows that the total super-Hamiltonian is given by $\sum_J N_J \mathcal{C}_J$. Since the modes are independent, one obtains the equations of motion

$$\begin{aligned} \partial_\tau \varphi_J &= \{\varphi_J, \sum_J N_J \mathcal{C}_J\} = -N_J \frac{\pi_J}{K_J^{(2)}}\,, \\ \partial_\tau \chi_J &= \{\chi_J, \sum_J N_J \mathcal{C}_J\} = N_J\,, \end{aligned} \tag{371}$$

clearly showing that the lapses N_J describe the rate of change of the various χ_J (which will be used as clocks) with respect to τ.

Following [135, 190, 358, 366], one can generalise the notion of relational Dirac observables to systems with multiple constraints. In particular, the scenario discussed here is easily tractable since the constraints form an Abelian algebra,[10] $\{\mathcal{C}_J, \mathcal{C}_{J'}\} = 0$. Because a gauge orbit is multi-dimensional, one needs to introduce as many dynamical clocks as there are constraints. Then, similarly to the single-clock case, a complete observable can be defined as a relation between a phase space function f_φ (of the φ degrees of freedom only) and a set of independent clocks χ_J. In short, the relational Dirac observable (generalising (299) and (346)) defined as [135, 190, 358, 366]

$$F_{f_\varphi,\{\chi_J\}}(\{\chi_J^0\}) := \sum_{n=0}^{\infty} \frac{1}{n!} \Big\{ f_\varphi, \sum_J \alpha_J H_J \Big\}_n \Big|_{\alpha_J \to (\chi_J^0 - \chi_J)} \tag{372}$$

is invariant under the flows generated by the constraints (368). Essentially, $F_{f_\varphi,\{\chi_J\}}(\{\chi_J^0\})$ gives the value of f_φ when the dynamical clocks χ_J take the values χ_J^0 for all J's (the curly brackets in (372) are meant to emphasise that we deal with a set of clocks and not with the J-th clock only). The linear combination of the single-mode GFT Hamiltonians $\sum_J \alpha_J H_J$ can be used to define a physical Hamiltonian[11] which generates evolution for the complete observables (372). Indeed, since one can show that $\partial_{\chi_J^0} F_{f_\varphi,\{\chi_J\}}(\{\chi_J^0\}) = \{F_{f_\varphi,\{\chi_J\}}(\{\chi_J^0\}), H_J\}$, the quantity $\sum_J \alpha_J H_J$ is the generator of multi-fingered time evolution; it evolves observables along the various arbitrary parameters α_J (related to the lapse functions in (371)) associated with the clocks χ_J (see [135, 190, 358, 366] for details).

10 Not many physical systems exhibit this property. An example with multiple constraints forming an Abelian algebra can be found in [102, 166], where loop quantum gravity techniques are applied to spherically symmetric settings.

11 Note that incorporating the J-dependent flow parameters into the definition of the Hamiltonian is equivalent to a rescaling of τ such that the initial and final configurations are separated by a time interval of length unity.

All the properties of the quantum theory corresponding to the action (370) easily follow by suitably generalising the constructions of previous sections. The canonical operators now satisfy

$$\begin{aligned} \left[\hat{\varphi}_J, \hat{\pi}_{J'}\right] &= \mathrm{i}\delta_{JJ'}\,, \\ \left[\hat{\chi}_J, \hat{p}_{\chi_{J'}}\right] &= \mathrm{i}\delta_{JJ'}\,, \end{aligned} \tag{373}$$

and the kinematical Hilbert space is defined as

$$\begin{aligned} \mathcal{H}_{\text{kin}} = \mathcal{H}_\chi^{\text{tot}} \otimes \mathcal{H}_\varphi^{\text{tot}} &= \Big(\bigotimes_J \mathcal{H}_{\chi_J}\Big) \otimes \Big(\bigotimes_J \mathcal{H}_{\varphi_J}\Big) \\ &= \bigotimes_J \left(\mathcal{H}_{\chi_J} \otimes \mathcal{H}_{\varphi_J}\right), \end{aligned} \tag{374}$$

where in the last equality we rearranged the single-mode Hilbert spaces to show that this theory can be seen as a tensor product over modes of the theory studied in Section 6.2. While we already dealt with a theory on $\mathcal{H}_\varphi^{\text{tot}}$ with multiple GFT modes in Section 6.3.1, it also follows from the structure in (374) that all the kinematical considerations for the "matter sector" (cf. (311)–(317)) described in Section 6.2 apply here. Thus, all single-mode quantities described there will simply obtain a J label here (for both the φ and χ sectors).

In order to follow Dirac's quantisation programme, we write down the quantum constraints (corresponding to (368)) on the Hilbert space (374) as

$$\hat{C}_J = \hat{p}_{\chi_J} \otimes \Big(\bigotimes_{J' \neq J} \mathbb{I}_{\chi_{J'}}\Big) \otimes \mathbb{I}_\varphi^{\text{tot}} + \mathbb{I}_\chi^{\text{tot}} \otimes \hat{H}_{\varphi_J} \otimes \Big(\bigotimes_{J' \neq J} \mathbb{I}_{\varphi_{J'}}\Big)\,, \tag{375}$$

where the various identity operators clarify that each $\hat{C}_J$ acts non-trivially only on the respective J-th piece of $\mathcal{H}_\chi^{\text{tot}}$ and $\mathcal{H}_\varphi^{\text{tot}}$ in (374). A physical state is annihilated by all the constraints separately,

$$\hat{C}_J |\Psi_{\text{phys}}\rangle = 0 \qquad \forall J\,, \tag{376}$$

and is defined via group averaging as $|\Psi_{\text{phys}}\rangle := \prod_J \delta(\hat{C}_J)|\Psi_{\text{kin}}\rangle$, where a generic state $|\Psi_{\text{kin}}\rangle$ on (374) reads

$$|\Psi_{\text{kin}}\rangle = \sum\!\!\!\!\!\!\!\int_{\{E_J\}} \int \prod_J \mathrm{d}p_{\chi_J}\, \Psi\left(\{p_{\chi_J}\}, \{E_J\}\right) \left[\bigotimes_J \left(|p_{\chi_J}\rangle \otimes |\psi^{E_J}\rangle\right)\right]. \tag{377}$$

Here we adopt the notation of Section 6.3.1 (see in particular (353)), and $\int \prod_J \mathrm{d}p_{\chi_J}$ means we integrate over the scalar field momenta for all modes. One obtains the physical states

$$|\Psi_{\text{phys}}\rangle = \sum\!\!\!\!\!\!\!\int_{\{E_J\}} \Psi(\{-E_J\},\{E_J\})\Big[\bigotimes_J \Big(|-E_J\rangle \otimes |\psi^{E_J}\rangle\Big)\Big], \tag{378}$$

which naturally generalises all the findings of previous sections (cf. (323) and (354)). Aiming to discuss again the equivalence between the physical inner product with multiple constraints (denoted with an M), which we define as

$$\langle\Psi_{\text{phys}}|\Psi_{\text{phys}}\rangle^{\text{M}}_{\text{phys}} := \langle\Psi_{\text{kin}}|\prod_J \delta(\hat{\mathcal{C}}_J)|\Psi_{\text{kin}}\rangle_{\text{kin}}, \tag{379}$$

and the Page–Wootters inner product, one needs to extend the Page–Wootters construction to the case of multiple clocks. In short, by introducing the "multi-conditioned" state in $\mathcal{H}^{\text{tot}}_{\varphi}$

$$|\psi(\{\chi^0_J\})\rangle^{\text{M}}_{\varphi_{\text{tot}}} := \Big[\Big(\bigotimes_J \langle\chi^0_J|\Big) \otimes \mathbb{I}^{\text{tot}}_{\varphi}\Big]|\Psi_{\text{phys}}\rangle, \tag{380}$$

which generalises (331) by "projecting" *all* clocks to the values χ^0_J,[12] one is lead to the following definition of the multi-fingered (M) time Page–Wootters inner product

$$\langle\Psi_{\text{phys}}|\Psi_{\text{phys}}\rangle^{\text{M}}_{\text{PW}} := \langle\Psi_{\text{phys}}|\Big[\Big(\bigotimes_J |\chi^0_J\rangle\langle\chi^0_J|\Big) \otimes \mathbb{I}^{\text{tot}}_{\varphi}\Big]|\Psi_{\text{phys}}\rangle_{\text{kin}}. \tag{381}$$

Then, as expected from the results for a single clock of [227, 228],

$$\langle\Psi_{\text{phys}}|\Psi_{\text{phys}}\rangle^{\text{M}}_{\text{phys}} = \langle\Psi_{\text{phys}}|\Psi_{\text{phys}}\rangle^{\text{M}}_{\text{PW}} = \sum\!\!\!\!\!\!\!\int_{\{E_J\}} |\Psi(\{-E_J\},\{E_J\})|^2. \tag{382}$$

We point out that the steps presented here represent the first explicit construction of the Page–Wootters formalism extended to the case of multiple (but finitely many) clocks, which corresponds to a Dirac quantisation with multiple Hamiltonian constraints. We note how-

12 Note that a physical state can be written as

$$|\Psi_{\text{phys}}\rangle = \Big(\bigotimes_J \int \mathrm{d}\chi^0_J |\chi^0_J\rangle\Big) \otimes |\psi(\{\chi^0_J\})\rangle^{\text{M}}_{\varphi_{\text{tot}}}$$

so that it encodes information about all the timelines in their entirety; we call it "state of histories", generalising (332).

ever that this could be seen as a special case of [226], where the Page–Wootters formalism is formally applied to field theories. As mentioned above, the limiting case of infinitely many modes (and thus clocks) could be related to infinite tensor product techniques, and indeed be dealt with wave-functional treatments for quantum field theories (which are employed in [226]).

We now proceed to the formal quantisation of the classical observable (372),

$$\hat{F}_{f_\varphi,\{\chi_J\}}(\{\chi_J^0\}) := \left[\bigotimes_J \left(\int \frac{\mathrm{d}\chi_J}{2\pi}|\chi_J\rangle\langle\chi_J|\right)\right] \otimes \sum_n^\infty \frac{\mathrm{i}^n}{n!}\left[\hat{f}_\varphi, \sum_J \alpha_J \hat{H}_{\varphi_J}\right]_n \Bigg|_{\alpha_J \to (\chi_J - \chi_J^0)}, \tag{383}$$

which represents a natural generalisation of the quantity (327) introduced in [227, 228]. In this sense, (383) extends the discussion of Dirac observables (and their connection to the Page–Wootters formalism) to the multi-fingered time scenario, which we now apply to our GFT setting.

From the physical Hamiltonian $\sum_J \alpha_J \hat{H}_{\varphi_J}$ one can define the multi-fingered evolution operator $\bigotimes_J \hat{U}_{\varphi_J}(\alpha_J) := e^{-\mathrm{i}\sum_J \alpha_J \hat{H}_{\varphi_J}}$ on $\mathcal{H}_\varphi^{\mathrm{tot}}$, where the factors $\hat{U}_{\varphi_J}(\alpha_J) = e^{-\mathrm{i}\alpha_J \hat{H}_J}$ generalise the evolution operators of previous sections (cf. (328)) by evolving along the mode-dependent time parameter α_J. Then, specialising the expression (383) to the GFT number operator with $\hat{f}_\varphi = \hat{\mathfrak{N}}_{\mathrm{tot}}$ (as in Section 6.3.1, cf. (357)), one can define

$$\begin{aligned}\hat{N}_D^{\mathrm{tot}}(\{\chi_J^0\}) &:= \left[\bigotimes_J \left(\int \frac{\mathrm{d}\chi_J}{2\pi}|\chi_J\rangle\langle\chi_J|\right)\right] \\ &\quad \otimes \left[\left(\bigotimes_J \hat{U}_{\varphi_J}(\chi_J - \chi_J^0)\right)\hat{\mathfrak{N}}_{\mathrm{tot}}\left(\bigotimes_J \hat{U}^\dagger_{\varphi_J}(\chi_J - \chi_J^0)\right)\right] \\ &= \int \prod_J \frac{\mathrm{d}\alpha_J}{2\pi}\, \hat{U}^{\mathrm{tot}}_{\chi\varphi}(\{\alpha_J\})\left[\left(\bigotimes_J |\chi_J^0\rangle\langle\chi_J^0|\right) \otimes \hat{\mathfrak{N}}_{\mathrm{tot}}\right](\hat{U}^{\mathrm{tot}}_{\chi\varphi})^\dagger(\{\alpha_J\}),\end{aligned} \tag{384}$$

where $\hat{U}^{\mathrm{tot}}_{\chi\varphi}(\{\alpha_J\}) := \bigotimes_J \left(\hat{U}_{\chi_J}(\alpha_J) \otimes \hat{U}_{\varphi_J}(\alpha_J)\right)$ is the generalisation of (326), we used the properties of the clock states introduced in Section 6.2 (which now apply to all the clocks χ_J), and we changed integration variables $\chi_J \to \alpha_J + \chi_J^0$ in the last line. We emphasise that $\hat{N}_D^{\mathrm{tot}}(\{\chi_J^0\})$ is different from (358) as it depends on multiple clocks and thus it represents a multi-fingered time version of the number operator. Recall that the subscript D means that (384) defines a true

Dirac observable, which strongly commutes with *all* the constraints, namely

$$[\hat{N}_D^{\text{tot}}(\{\chi_J^0\}), \hat{\mathcal{C}}_J] = 0 \qquad \forall J\,. \tag{385}$$

Finally, by means of the property (generalising (337))

$$\hat{N}_D^{\text{tot}}(\{\chi_J^0\})|\Psi_{\text{phys}}\rangle = \prod_J \delta(\hat{\mathcal{C}}_J)\big[\big(\bigotimes_J |\chi_J^0\rangle\langle\chi_J^0|\big)\otimes\hat{\mathfrak{N}}_{\text{tot}}\big]|\Psi_{\text{phys}}\rangle\,, \tag{386}$$

one can use the inner products (379) and (381), as well as the state (380), to show that

$$\begin{aligned}\langle\Psi_{\text{phys}}|\hat{N}_D^{\text{tot}}(\{\chi_J^0\})|\Psi_{\text{phys}}\rangle^{\text{M}}_{\text{phys}} &= \langle\Psi_{\text{phys}}|\hat{\mathfrak{N}}_{\text{tot}}|\Psi_{\text{phys}}\rangle^{\text{M}}_{\text{PW}}\\ &= {}_{\varphi_{\text{tot}}}^{\;\;\text{M}}\langle\psi(\{\chi_J^0\})|\hat{\mathfrak{N}}_{\text{tot}}|\psi(\{\chi_J^0\})\rangle^{\text{M}}_{\varphi_{\text{tot}}}\\ &= {}_{\varphi_{\text{tot}}}^{\;\;\text{M}}\langle\psi|\sum_J \hat{U}^\dagger_{\varphi_J}(\chi_J^0)\,\hat{\mathfrak{a}}_J^\dagger\hat{\mathfrak{a}}_J\,\hat{U}_{\varphi_J}(\chi_J^0)\,|\psi\rangle^{\text{M}}_{\varphi_{\text{tot}}}\,,\end{aligned} \tag{387}$$

where $|\psi\rangle^{\text{M}}_{\varphi_{\text{tot}}} = |\psi(0,\ldots,0)\rangle^{\text{M}}_{\varphi_{\text{tot}}}$ and we suppressed again the trivial factors $\bigotimes_{J'\neq J}\mathbb{I}_{\varphi_{J'}}$. As expected, the last line in (387) shows that every GFT mode in the total number operator evolves according to its own relational time parameter.

At this point we want to stress that since in a free GFT the various J modes are decoupled, a multi-fingered setting still allows to have well-defined Dirac observables, both classically and at the quantum level (in the sense that they commute with the constraints (385)). In other words, synchronisation between the different J-times is not really an issue here: while one can indeed reduce to the synchronised case of Section 6.3.1 by gauge-fixing all the χ_J to be the same χ (using the freedom coming from all the reparametrisation symmetries), this is not necessary in order to answer physically meaningful questions. Indeed, one can study the dynamics of the various modes of the number operator by means of mode-dependent clocks, and still be able to compute the total number operator combining all the information (i.e., summing all the separate contributions together). In a sense, it does not matter whether one uses a unique time or a different time for every mode (even with an infinite number of modes), as they are independent.

On this note, we also emphasise that the synchronisation problem[13] mentioned in the literature [272] relies on having different clocks for different GFT quanta, rather than field modes. Specifically, the issue arises because the number of particles is not conserved under time evolution (indeed, this is how the GFT cosmology framework explains the expansion of the Universe, cf. Section 3.3). Even in standard quantum field theory the number of particles is not a well-defined quantity, and the Fourier mode decomposition (here given by the Peter–Weyl decomposition (58)) represents the most natural way of reducing the theory to quantum-mechanical systems. Of course, one then interprets field excitations as particles so that the single-mode times of (387) describe the evolution of GFT quanta associated with the same J (naively, building blocks of quantum geometry with the same "shape", cf. Figure 7).

Finally, since adding reparametrisation symmetries does not change the physical content of the theory, we note that one could have investigated equivalent questions directly in the deparametrised setting of Section 3.2.2. While there one usually studies observables where every mode is associated with the same time label (cf. (111) and (109)), (387) suggests that one could have defined an observable as a sum of single-mode contributions *at different times*,

$$\hat{\mathcal{O}}_{\text{tot}}(\{\chi_J\}) = \sum_J e^{\mathrm{i}\hat{H}_J\chi_J}\,\hat{\mathcal{O}}_J(0)\,e^{-\mathrm{i}\hat{H}_J\chi_J}\,, \tag{388}$$

where every Peter–Weyl mode is associated with a different reading of the same clock. In the free theory, this is a well-defined observable for the deparametrised approach that was never investigated because of its somewhat unusual interpretation (where the modes are observed at different times), which in Section 6.3.2 is revisited in terms of multi-fingered time evolution. When $\hat{\mathcal{O}} = \hat{N}$, the expectation value of (388) is nothing but (387); in this sense, the articulated construction of Section 6.3.2 shows again the equivalence between deparametrisation and "post-quantum time" dynamics.

6.4 DISCUSSION AND OUTLOOK

The main result of this chapter is the construction and quantisation of parametrised group field theory models for quantum gravity coupled

13 The coherent peaked states of the algebraic approach try to address this question "by hand", see discussion below (140).

to a massless scalar field (to be used as a clock), and a proper definition of the corresponding relational quantum dynamics. By virtue of the equivalence between the quantum dynamics defined by relational Dirac observables and the Page–Wootters formalism [227, 228], we showed that using the *quantum* degree of freedom associated to the scalar field as dynamical clock yields the same results as choosing the scalar field as background time variable at the classical level (i.e., following the deparametrised approach of Section 3.2.2). In other words, our "tempus post quantum" theory proves that choosing a clock (here specifically the scalar field χ) and quantising are procedures that commute for free GFT models.

We first analysed a simple scenario restricted to a single field mode and examined the evolution of number operator relative to the scalar field; we used the strategy of "evolving constants of motion" at the classical level, and defined the quantum relational dynamics following both Dirac's method for constraint quantisation and the Page–Wootters formalism. This marks the first application of the Page–Wootters formalism to a non-perturbative quantum gravity theory, and provides a coherent GFT framework describing the evolution of (the expectation values of) the geometrical quantities of interest *conditional* on a quantum time operator reading a certain value.

We then generalised the construction to the more complicated case of free GFT models with an arbitrary (but finite) number of field modes. We distinguished between two scenarios: one where the system exhibits a single reparametrisation invariance (just like in the single-mode case) and one with many such invariances. While the first setting straightforwardly extends all the previously mentioned results, the case of many reparametrisation symmetries required a novel extension of the Page–Wootters formalism to the case of multiple quantum clocks.

We note in passing that since our construction is built upon a well-defined physical inner product in the quantum theory, no divergences arise as in formalisms that aim to describe relational dynamics at the kinematical level (cf. Section 3.2.1).

As we will stress in Chapter 7, this work opens up the possibility of investigating clock changes and obtaining a quantum notion of covariance [171, 225, 229, 369] in group field theory models. Given that the GFT literature relies to a great extent on a single massless scalar field, one can now explore other clock candidates with a proper treatment at the quantum level (e.g., building on the work of [185] for models with multiple scalar fields). Additionally, the tools developed here allow to explore what happens if one uses a degree of freedom on the geometry sector (e.g., a GFT mode) as a clock to describe relational dynamics of the matter scalar field. All such investigations would provide a stronger handle on general covariance questions in a quantum gravity framework such as GFT; questions that might be of interest for the communities of quantum information and foundations of quantum mechanics, other than quantum gravity.

Potentially, other geometrical observables in GFT could be used as a clock, for example the anisotropies of Chapter 4.

7 CONCLUSION

This chapter summarises the thesis and outlines potential future research directions that can stem from the results discussed herein.

Summary

In this thesis we presented the group field theory approach to the problem of quantum gravity and studied consequences in regards of its phenomenological applications in cosmology and its theoretical foundations. After introducing and motivating the field of research and the main techniques adopted in the literature, we focussed on three independent research directions: the definition of new anisotropic models for GFT cosmology, the analysis of semiclassical properties for general Gaussian-like quantum states, and the investigation of the formal tools underlying the idea of quantum relational dynamics.

More in detail, we first introduced the problem of quantum gravity in Chapter 1, emphasising that despite progress in several approaches, no comprehensive theory can describe the physics of the Planck scale in a consistent and satisfactory way. We stressed the complications caused by the problem of time, especially for background-independent approaches, and the importance of relational dynamics. We focussed in particular on the group field theory (GFT) approach, which describes spacetime as emerging from fundamental quanta and has promising applications in cosmology, especially in resolving the Big Bang singularity through a "Big Bounce" scenario. Since cosmology offers a potential testing ground for quantum gravity, particularly in early Universe contexts where quantum effects may have played a crucial role, we emphasised that the GFT framework can offer insights into fundamental questions about the structure of spacetime at the Planck scale and the nature of quantum gravity in general.

We then delved into the technical details in Chapter 2: here we briefly presented the obstacles encountered when applying the canonical and path integral quantisation strategies to general relativity, and hence we moved onto discrete perspectives. We then followed a somewhat historical route describing discrete path integral approaches to quantum gravity, starting from Regge calculus and illustrating the details of the Ponzano–Regge model, matrix models and spin foam models (often considered a covariant formulation of loop quantum gravity). These approaches set the incentive for the definition of the general framework of group field theory; indeed, we concluded the chapter by emphasising that GFT relates to (and generalises) the previously studied approaches, and focussed on its path integral quantisation. In particular, given that the Feynman graphs of some GFTs can be seen as simplicial complexes that are dual to a discretisation of spacetime, we motivated the GFT framework by showing that it can formally yield a discrete version of the path integral for $4d$ quantum gravity.

In Chapter 3 we switched gears and discussed the formalism underlying the research results of the thesis: the canonical formulation of GFT and the methods to effectively extract cosmology from such a quantum gravity candidate. This reformulation permits to use second-quantisation tools that connect with the canonical LQG framework, and allows to apply techniques typical of bosonic systems to quantum geometry (i.e., where ladder operators create and annihilate GFT quanta). Crucially, after having described the importance of matter coupling (specifically where a scalar field is used as relational clock), we highlighted that one can follow two distinct methods to obtain a canonical quantisation of GFT. These implement the notion of dynamics in different ways: the *algebraic approach* uses kinematical structures and imposes dynamical equations as constraints, and the *deparametrised approach* selects a time variable classically and directly obtains a physical Fock space in the quantum theory. Despite their disparate strategies, we showed that they lead to very similar effective dynamics when applied to cosmology, in particular yielding a bouncing FLRW model that resolves the Big Bang singularity by means of quantum corrections.

We then devoted the second part of the thesis to the exposition of our original research results. In Chapter 4, we tackled the question of describing anisotropic cosmologies in group field theory for the first time, focusing on the simplest case given by a locally rotationally symmetric (LRS) Bianchi I model. Since anisotropies in GFT were only considered (and shown to decay) perturbatively in previous work, our investigation breaks new ground by providing a precise definition of anisotropy that can discern between isotropic and anisotropic effective cosmologies (and quantify the degree of anisotropy). In essence, we defined a GFT-analogue of the classical Misner variables $\beta_\pm$ of GR, and studied their dynamics within the framework of GFT cosmology. After motivating the definition of a mode-dependent quantity capable of assigning "microscopic values of anisotropy" to the GFT quanta (in particular, ensuring that a building block of geometry with equal face areas can be considered isotropic), we constructed a global anisotropy observable by summing over the Peter–Weyl modes of the underlying GFT, multiplying by the number operator for that mode, and dividing by the total number of quanta to obtain an "intensive" measure. While our models partially agree with general relativity in terms of anisotropy evolution, they fail to reproduce the expected faster (compared to the isotropic case) expansion rate of the volume. Our findings suggest that interactions or different kinetic terms might be necessary to fully match classical behaviour, but show that the anisotropy observable follows the expected behaviour for a certain period after the bounce. Importantly, we could analytically confirm the linear behaviour of anisotropy in the context of a toy model with the help of some approximations.

Chapter 5 was devoted to a thorough analysis of Gaussian states in group field theory (and more generally $\mathfrak{su}(1,1)$-based) quantum cosmology, focussing in particular on two main semiclassical properties: the behaviour of relative uncertainties for the quantities of interest and the Robertson–Schrödinger uncertainty principle. Although in quantum optics and quantum information theory Gaussian states are naturally associated with a statistical interpretation, we examined them from a formal mathematical point of view in GFT; this is because it is not easy to give a precise physical meaning to statistical parameters for a background-independent quantum gravity theory, where spacetime is only seen as emergent. In the deparametrised approach to GFT, we showed analytically that the fluctuations of energy and volume can be made arbitrarily small, even though Gaussian states do not saturate the Robertson–Schrödinger inequality (which we argued to be a less relevant criterion). While we could also define generalised Gaussian states in the algebraic approach, where additional dynamical equations (constraints) are imposed on physical states, we found that only pure coherent states can be considered physical. In this context we analysed a simplified scenario with constant squeezing and thermal effects, finding a new effective Friedmann equation and decreasing volume fluctuations. Our constructions and results include all previous findings as subcases, and thus provide an exhaustive and complete framework for semiclassical states in GFT (working with a single-mode free theory).

Motivated by the dissonance between the algebraic approach and the deparametrised setting (hinted at in their basic formulations already in Chapter 3, but ultimately rendered manifest in the study of Chapter 5), in Chapter 6 we looked into the formalities behind the concept of quantum relational dynamics. Drawing inspiration from the "tempus post quantum" idea embraced by the peaked coherent states of [272], but aiming to implement a proper quantisation based on constraints, we constructed parametrised GFT models and quantised them following the Dirac programme as well as the Page–Wootters formalism. Already for a free parametrised GFT with a single field mode, we showed that using the scalar field as a dynamical clock in the quantum theory reproduces results from the deparametrised approach (specifically, regarding the relational evolution of the number operator). While the equivalence between Dirac quantisation and the Page–Wootters formalism was established in [228], the work of this chapter constitutes its first application to a non-perturbative quantum gravity setting; importantly, this enriches the notion of GFT dynamics with the conditional interpretation of the Page–Wootters formalism. In essence, these results resolve previous concerns about the loss of covariance due to classical deparametrisation, and provide a consistent description of the relational evolution of GFT geometric observables with respect to a "quantum time". We also generalised the construction to theories with multiple field modes: we were able to relate our findings to the idea of multi-fingered time evolution by extending the Page–Wootters formalism to handle multiple quantum clocks. This was done considering separate "mode times" for each Peter–Weyl mode rather than "particle times", so as to automatically avoid issues related to synchronisation between the GFT quanta.

Outlook

We now mention possible developments that may arise from the work presented in this thesis. While the general research programme of group field theory offers a vast range of unexplored directions – both in foundational aspects and in technical advancements (see, e.g., [288] for a recent philosophical take on GFT and [301] for a proposal suggesting connections between quantum gravity, quantum fluids and cosmology) – we limit ourselves to mentioning only those aspects that are directly related to the results of this thesis.

GAUSSIAN STATES. First, we point out that the work on Gaussian states provides the technical framework to explore questions typical of statistical mechanics (and statistical field theories), and hence opens up the possibility of connecting ideas such as that of entanglement entropy to quantum geometry. These concepts have seen attention for example in LQG [58, 70, 264]; thus, a natural direction for future work is to investigate similar questions in the context of GFT. In particular, in order to discuss entanglement, one could generalise the construction of Chapter 5 to multiple modes, specifically to more general squeezed states like those used in areas such as quantum optics [373] or cosmology [202]. One could explore whether our results hold for two-mode (or generically multi-mode) Gaussian states, at least in the deparametrised approach where such states can be easily defined. This extension may add new features to GFT cosmological scenarios which are not captured by our single-mode construction. Ultimately, these ideas align with the broad picture of investigating the (possibly holographic) behaviour of quantum geometry and the emergence of spacetime from more fundamental notions [110], such as the entanglement between GFT quanta.

ANISOTROPIES. The work on anisotropies conducted in Chapter 4 offers a broad spectrum of potential directions for future research. To begin with, one could ask whether the effective dynamics of anisotropies change if GFT interactions are taken into account. This would complicate the resulting calculations further, beyond the need for multiple Peter–Weyl modes (see, e.g., [184] for an already quite involved numerical study of a single mode with interactions).

A different direction would be to analyse the detailed effects of anisotropy on the bounce phase. In Chapter 4 we focused on a post-bounce regime because we aimed to compare with classical relativity, but in general one might ask whether the bounce itself could be spoiled (or in general modified) by anisotropies, as is often a main worry in bounce scenarios in which anisotropies dominate asymptotically on approach to the singularity [88]. In our model, we have a massless scalar field as matter, which would classically prevent the domination of anisotropies. We saw that anisotropies are present at early times and disappear at late times, but singularity avoidance does not seem affected by the inclusion of anisotropies. The details of the bounce may still be altered by their presence. Another line of investigation, closely related to the previous one, would be to compare the exact details of the cosmological bounce with the similar singularity resolution of loop quantum cosmology. In the Bianchi I context, this would mean to investigate whether the anisotropic nature of the model has the same influence (if any) on the bounce in loop quantum cosmology [39, 83] and in our work. Furthermore, a comparison with loop quantum cosmology [72, 378] suggests the potential to investigate whether the GFT anisotropic bounce corresponds to a Kasner transition.

A specific restriction of the work of Chapter 4 was that we were studying the GFT analogue of a Bianchi I Universe with an additional rotational symmetry, so that there is only one β variable rather than two. To lift this restriction, one could consider more general types of tetrahedra rather than trisohedral ones, discuss different proposals for $\beta_\pm$ variables in this more general context, and study their dynamics along the lines we have discussed. Inspired by our work, some advancements in this direction have already been proposed in [308].

Given the monotonic evolution of the new anisotropy observable β_+ in GFT, another significant line of research would be to try to use such a gravitational degree of freedom as relational clock. This would be similar to what happens at the classical level, where in a (LRS) Bianchi I Universe without matter the classical Friedmann equation can be written as[1]

$$\left(\frac{1}{3V}\frac{\mathrm{d}V}{\mathrm{d}\beta_+}\right)^2 = 1\,, \tag{389}$$

so that β_+ can be a relational clock with no need for a separate matter field. One might hope to incorporate this idea into GFT, and describe relational evolution without coupling to the somewhat arbitrary massless scalar field. Such a relational formalism would appear to require using an expectation value as a clock parameter, perhaps along the lines suggested in [272] (via peaked coherent states) or using the novel framework we developed in Chapter 6.

Finally, we highlight one last idea stemming out of the work in Chapter 4 that can potentially say something about the dynamics of black holes, thus pushing the canonical quantisation of GFT in

1 Following analogous steps to those in Appendix A, one can choose β_+ to be the clock by fixing the lapse appropriately (imposing $\dot{\beta}_+ = 1$, cf. (422)) and using the vanishing of (421); with $p_- = p_\chi = 0$ this implies $p_+^2 = 9V^2 p_V^2$, and yields (389).

genuinely new phenomenological directions. Specifically, one could leverage the fact that the interior of a (Schwarzschild) black hole can be described by the class of metrics known as Kantowski–Sachs (KS) [239], which are not so different from Bianchi models (an important difference being that the spatial topology of KS spacetimes is fixed to be $\mathbb{R} \times S^2$). At the classical level, using the KS metric $\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + V^{2/3}e^{-4\beta}\mathrm{d}r^2 + V^{2/3}e^{2\beta}\mathrm{d}\Omega^2$, with $\mathrm{d}\Omega^2$ the metric on the 2-sphere, one shows that the Hamiltonian constraint in the action $S = \int \mathrm{d}t \left(p_V \dot{V} + p_\beta \dot{\beta} + p_\chi \dot{\chi} - NC\right)$ (cf. (391)) can be written as

The KS spacetime only has one anisotropy β (with momentum p_β).

$$C = -6\pi G V p_V^2 + \frac{2\pi G}{3}\frac{p_\beta^2}{V} + \frac{p_\chi^2}{2V} - \frac{V^{1/3}}{8\pi G}e^{-2\beta}\,. \qquad (390)$$

This is very similar to the Hamiltonian constraint of a (generally curved) Bianchi model, such as the Bianchi II model reviewed in Appendix A (see in particular (428); the difference only lies in the last term of (390), which is due to curvature). The effective strategy employed in Chapter 4 may then allow to obtain the first *dynamical* GFT model for black holes. Indeed, the only existing work is kinematical [303, 307], and does not tell us anything about solutions of some dynamical equations, nor it predicts avoidance of the classical singularity. While nonsingular scenarios for black holes were explored in LQG from different perspectives, e.g. using loop quantum cosmology techniques [26], applying the "improved dynamics" scheme [165] or in spin foam models [66, 220, 281], these results still describe a plethora of different outcomes without a unique, clear physical description. A bounce-like scenario (similar to the one described in Chapter 4) is expected in GFT too, and could align with the results of LQG: the bounce could be seen as a transition from a black hole to a white hole (as in, e.g., [213]) or as a black hole generating a shock wave [230].

PAGE–WOOTTERS FORMALISM AND RELATIONAL DYNAMICS. To conclude, we want to stress the importance of the technical (and conceptual) results of Chapter 6, and their consequences.

A major repercussion of our work has to do with the general line of research on the notion of changing temporal reference frames [171, 225, 229, 369]. In light of the results of [227, 228], our work paves the way for exploring clock changes and achieving a quantum notion of covariance for group field theory. Specifically, it allows to go beyond the massless scalar field (ubiquitously used as relational time in the literature) by considering other clock candidates with a proper quantum treatment given by the Page–Wootters formalism. For instance, building on the models with multiple scalar fields discussed in [185], one could take advantage of the "clock-neutral" picture to explicitly show the equivalence between quantum dynamics with different choices of clocks in GFT. The methods of Chapter 6 also allow for the investigation of using geometric degrees of freedom – such as a GFT mode – as clocks to describe the relational dynamics of matter fields. In a sense, this reflects the situation in classical cosmology where for example the volume of an expanding Universe is expected to be a good clock. Potentially, these tools could be applied to other geometrical observables in GFT which may be considered as dynamical clocks (e.g., the anisotropy degree of freedom we developed in Chapter 4). These lines of research would contribute to a deeper understanding of the notion of general covariance in the GFT approach to quantum gravity, and can potentially spark discussions with other research programmes, such as quantum information [325] and foundation of quantum mechanics (see [126] for a recent result with implications for bridging the quantum gravity and quantum reference frame communities).

Finally, another natural line of research would be to extend the results of Chapter 6 to the case with infinite field modes, which requires functional analysis tools for the Page–Wootters formalism in the context of field theories [226], and might be related to infinite tensor product techniques [290, 347, 363].

Part III

APPENDICES

A

RELATIONAL DYNAMICS IN COSMOLOGY

In this appendix we give a brief overview on how to obtain a relational description of cosmological models in general relativity, and we mention similar relevant aspects in the context of effective loop quantum cosmology. Such a relational formulation of classical dynamics, in particular where a free massless scalar field χ is used as clock, is needed if one wants to compare the GFT results of Section 3.3 with GR (and loop quantum cosmology). Indeed, while the background independence of GFT forced us to consider relational cosmological dynamics, in GR one has the freedom to choose to work with different clocks (e.g., one can easily switch from the standard time coordinate t to the relational clock χ). This is essentially due to the Hamiltonian constraint (see Section 2.1) and more specifically, to the choice of the lapse function in the action (4), as we explain below.

We will first obtain a relational formulation of the dynamics of a (classical) Friedmann–Lamaître–Robertson–Walker (FLRW) Universe, which essentially represents the GR equivalent of the relational Friedmann equations (134) and (156) found in GFT. Such classical dynamics were mentioned in Section 3.3 and implicitly used to make the identification between the GFT coupling ω and Newton's constant G in (137). Then, we will give a summary of the related results obtained in what is called "effective loop quantum cosmology", which studies the evolution of the expectation values of the volume observable of loop quantum cosmology in semiclassical states. We will only focus on a flat isotropic scenario and describe the loop quantum cosmology effective Friedmann equation, which can also be compared with the GFT results (134) and (156). Finally, we will extend the discussion to the case of (classical) anisotropic cosmologies, specifically Bianchi I and Bianchi II models, which are important for a comparison with the results of Chapter 4. A relational Friedmann-like equation can indeed be written for these models, which additionally describe the dynamics of the anisotropies themselves with respect to the scalar field χ.

As seen in Section 2.1, the Hamiltonian formulation of general relativity is described by means of the ADM action (4). In this thesis, the models of interest deal with gravity coupled to a free massless scalar field χ (which will serve as relational clock) with conjugate momentum p_χ. Moreover, we are only interested in homogeneous (but possibly anisotropic) settings, therefore we do not have an energy gradient term for χ, and we can set $N^a = 0$ in (4). The action then reads

$$S = \int \mathrm{d}t \int \mathrm{d}^3x \left(\pi^{ab}\dot{q}_{ab} + p_\chi \dot{\chi} - NC \right) , \tag{391}$$

where a dot denotes a t derivative, and the constraint C for the total system has a gravitational part $\mathcal{C}$ and a matter part (denoted $\mathcal{C}_\chi$),

$$C = \mathcal{C} + \mathcal{C}_\chi = 0 . \tag{392}$$

The gravitational part $\mathcal{C}$ appearing in (392) is given in (5) and we rewrite it here for convenience

$$\mathcal{C} = \frac{16\pi G}{\sqrt{q}} \left(\pi_{ab}\pi^{ab} - \frac{1}{2}(\pi^a{}_a)^2 \right) - \frac{\sqrt{q}}{16\pi G}\, {}^{(3)}R , \tag{393}$$

while the matter part reads[1]

$$\mathcal{C}_\chi = \frac{p_\chi^2}{2\sqrt{q}} . \tag{394}$$

In a sense, we implicitly pick a compact integration region and fix its constant volume.

If all fields are assumed to be spatially homogeneous, the integral over space $\int \mathrm{d}^3x$ in (391) just gives a constant, which we can set to unity. This does not play any physical role in homogeneous settings.

A.1 FLRW AND LOOP QUANTUM COSMOLOGY

We now specialise the above framework to a spatially flat, homogeneous and isotropic FLRW (classical) Universe. Given the metric

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2 \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \right) , \tag{395}$$

the action (391) is simply expressed in terms of the scale factor a and its conjugate momentum p_a, as

$$S = \int dt \, \left(p_a \dot{a} + p_\chi \dot{\chi} - NC \right) , \tag{396}$$

1 To see this explicitly, one simply starts from the scalar field Lagrangian $L_\chi = \frac{1}{2}\frac{\sqrt{q}}{N}\dot{\chi}^2$ (cf. (75)) and switches to the Hamiltonian formulation by means of the momentum $p_\chi = \frac{\sqrt{q}}{N}\dot{\chi}$ and the Legendre transform $p_\chi\dot{\chi} - L_\chi = N\mathcal{C}_\chi$ with $\mathcal{C}_\chi$ given in (394).

with Hamiltonian [81]

$$NC = -\frac{2\pi G}{3}\frac{Np_a^2}{a} + \frac{Np_\chi^2}{2a^3} = 0\,. \tag{397}$$

As it is well known, the dynamics corresponding to the vanishing of the Hamiltonian (397) are encoded in the Friedmann equation, which is usually written as

Explicitly, from the Lagrangian in (3), reduced to a flat FLRW setting as $L = -\frac{3}{8\pi G}\frac{a\dot{a}^2}{N}$*, one finds* $p_a = -\frac{3}{4\pi G}\frac{a\dot{a}}{N}$*, which substituted in (397) yields (398).*

$$H^2 = \left(\frac{\dot{a}}{aN}\right)^2 = \frac{8\pi G}{3}\rho_\chi\,, \tag{398}$$

where H denotes the Hubble parameter, and the energy density of the scalar field χ is

$$\rho_\chi = \frac{p_\chi^2}{2a^6}\,. \tag{399}$$

Given the importance of the volume observable in GFT models for cosmology, we now work with the volume variable $V = a^3$ (and its conjugate momentum p_V) instead of the scale factor, as this will facilitate the comparison with the dynamics obtained in Section 3.3. Writing the symplectic part of the Lagrangian in (396) in the new variables as $p_V\dot{V}$ implies a relation between the momenta (namely, $p_a = 3a^2 p_V$) that leads to the Hamiltonian (cf. (397))

$$NC = -6\pi G N V p_V^2 + \frac{Np_\chi^2}{2V} = 0\,. \tag{400}$$

We can finally make use of the freedom associated with such a constraint to pick the scalar field χ as relational time variable. Recall that χ can act as clock because it is monotonic, since p_χ is a constant of motion as $\dot{p}_\chi = \{p_\chi, NC\} = 0$. Indeed, the equation of motion for χ,

$$\dot{\chi} = \{\chi, NC\} = \frac{Np_\chi}{V}\,, \tag{401}$$

can be arranged so as to relate the evolution in t with the rate of change in χ by the lapse function as

$$N\mathrm{d}t = \frac{V}{p_\chi}\mathrm{d}\chi\,. \tag{402}$$

Then, the equation of motion for the volume,

$$\dot{V} = \{V, NC\} = -12\pi G N V p_V\,, \tag{403}$$

can be reformulated using χ as clock thanks to the relation (402): substituting this in (403) one can express p_V as

$$p_V = -\frac{1}{12\pi G}\frac{p_\chi}{V^2}\frac{\mathrm{d}V}{\mathrm{d}\chi}\,. \tag{404}$$

Finally, plugging (404) in the vanishing of the Hamiltonian constraint (400), one finds the *relational* Friedmann equation

$$\left(\frac{1}{3V}\frac{\mathrm{d}V}{\mathrm{d}\chi}\right)^2 = \frac{4\pi G}{3}\,, \tag{405}$$

which describes the dynamics of the volume with respect to the values of the scalar field χ. The large-volume limit of the isotropic GFT models reviewed in Section 3.3 is compared to (405), which in particular suggests the identification (137).

Note that (405) can be equivalently obtained following different strategies. For instance, one could rearrange the constraint (400) for

$$p_\chi = \pm\sqrt{12\pi G}\,V p_V\,, \tag{406}$$

and directly evaluate the Poisson brackets

$$\frac{\mathrm{d}V}{\mathrm{d}\chi} = \{V, p_\chi\} = \pm\sqrt{12\pi G}\,V\,. \tag{407}$$

Essentially, (407) represents the evolution of V as dictated by the relational Hamiltonian p_χ that is conjugate to the chosen time variable χ. Clearly, squaring both sides of (407) yields (405). Yet another way to derive the relational Friedmann equation is given by explicitly making use of the lapse function: by setting $\dot{\chi} = 1$ in (401) (which means picking χ as time), one finds the lapse

$$N = \frac{V}{p_\chi}\,. \tag{408}$$

Given that the Hubble parameter in the volume variable reads

$$H^2 = \left(\frac{\dot{a}}{aN}\right)^2 = \left(\frac{\dot{V}}{3VN}\right)^2\,, \tag{409}$$

one can directly substitute (408) in the traditional Friedmann equation (398) to immediately find (405). We will also use the lapse (408) to describe dynamics with respect to χ in the case of loop quantum cosmology.

EFFECTIVE LOOP QUANTUM COSMOLOGY. Loop quantum cosmology uses the quantisation techniques of loop quantum gravity for models that are reduced by symmetry arguments at the classical level (i.e., one considers cosmological settings before quantisation). In particular, in this framework one can obtain a well-defined quantum theory for an isotropic FLRW Universe (coupled with a scalar field χ) which resolves the Big Bang singularity replacing it with a quantum bounce [28, 36, 77].

An important aspect in quantum cosmology is the peakedness of semiclassical states around classical solutions; this property is shown to be preserved during evolution in loop quantum cosmology, in particular for the volume observable we are interested in [345]. In such semiclassical states, the main features of the dynamics (in particular the bounce) are captured by the evolution of the expectation values of total volume and the momentum of the scalar field p_χ. Indeed, one can show that the so called *effective* approach to loop quantum cosmology provides semiclassical dynamics that are in excellent agreement with the quantum dynamics for sharply peaked states [35, 36, 130]. This is interpreted as an effective, semiclassical description of spacetime that incorporates quantum effects from loop quantum gravity, essentially due to a discretisation (sometimes called "polymerisation" [117]) of the Hamiltonian constraint, as explained below.

The first step in this scheme is to adopt a new variable representing the conjugate momentum of the volume V. Specifically, one usually defines the variable b as proportional to the Hubble parameter

$$b = kH = -4\pi G k p_V \tag{410}$$

where k is a numerical constant (usually identified with the Barbero–Immirzi parameter of LQG [50, 232]) and H is given in (409) (the last equality in (410) follows from (403)). Clearly, one can use the pair (b, V) with Poisson brackets $\{b, V\} = 4\pi G k$ to describe the same FLRW classical dynamics discussed above, simply expressing the Hamiltonian constraint (400) as

$$NC = -\frac{3NVb^2}{8\pi G k^2} + \frac{N p_\chi^2}{2V} = 0\,. \tag{411}$$

The "quantum effects" of the effective approach to loop quantum cosmology are incorporated in a semiclassical manner by means of the replacement

$$b \to \frac{\sin(\lambda b)}{\lambda}, \tag{412}$$

where λ is some length parameter (proportional to the Planck length (cf. (1))). (412) essentially represents a discretisation (or polymerisation) choice: this idea is imported from the full theory of LQG, where operators can only be defined for matrix elements of holonomies and not for the connection itself [34, 341]. As we will see, the cosmological dynamics resulting from the choice (412) modify the classical scenario with quantum gravity (holonomy) corrections [27]. Importantly, such holonomy modifications are responsible for a cosmological bounce in isotropic models: from (412) and (411) one simply obtains an effective Hamiltonian constraint

In LQG, gravity is seen as a theory of a $SU(2)$ connection (cf. Section 2.1). The holonomy of such a (generically curved) connection describes the extent to which parallel transport around closed loops fails to preserve the geometrical data.

$$NC^{(\text{eff})}_{\text{LQC}} = -\frac{3NV}{8\pi G k^2}\frac{\sin^2(\lambda b)}{\lambda^2} + \frac{Np_\chi^2}{2V}, \tag{413}$$

which can now be used to define effective dynamics. In particular, the equation of motion for the volume (cf. (403)) gets modified as

$$\dot{V} = \{V, NC^{(\text{eff})}_{\text{LQC}}\} = \frac{3NV}{k\lambda}\sin(\lambda b)\cos(\lambda b). \tag{414}$$

This, together with the vanishing of the Hamiltonian constraint (413), readily leads to a modified version of the classical Friedmann equation (398) which reads [359]

$$H^2 = \frac{8\pi G}{3}\rho_\chi\left(1 - \frac{\rho_\chi}{\rho_{\text{c}}}\right), \tag{415}$$

with H^2 and ρ_χ given in (409) and (399), and where

$$\rho_{\text{c}} = \frac{3}{8\pi G k^2 \lambda^2} \tag{416}$$

represents a maximum value for the energy density. The quantum gravity correction thus appears as a $-\rho_\chi^2/\rho_{\text{c}}$ modification to the classical Friedmann equation, meaning that the volume of the Universe bounces when the energy density approaches $\rho_{\text{c}} \sim \rho_{\text{Planck}}$.[2]

2 The precise value $\rho_{\text{c}} = 0.41\rho_{\text{Planck}}$ is found in LQG by fixing the length scale $k\lambda$ (related to what was earlier called l_0) so that the leading term in black hole entropy calculations matches the Bekenstein–Hawking formula [22, 23, 141, 284, 314]. This can also be seen as a free parameter for comparison with observations [25, 30].

Just like for the GR case, the loop quantum cosmology effective Friedmann equation (415) can be written in relational terms using χ as clock variable. As we have seen above, it is enough to choose the lapse given in (408); substituting this into (415) (and also recalling (399)) quickly yields

$$\left(\frac{1}{3V}\frac{\mathrm{d}V}{\mathrm{d}\chi}\right)^2 = \frac{4\pi G}{3}\left(1-\frac{\rho_\chi}{\rho_c}\right). \tag{417}$$

This is the type of equation that also emerges in GFT cosmology, albeit with minor differences and from completely different starting points. We refer the reader to Section 3.3 for all the details, where in particular we compare the GFT effective Friedmann equations (136) and (159) with the relational dynamics of GR (cf. (405)) and effective LQC (417).

A.2 BIANCHI I AND II

We now move our attention to a classical Bianchi I cosmology, with metric

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a_1(t)^2\mathrm{d}x^2 + a_2(t)^2\mathrm{d}y^2 + a_3(t)^2\mathrm{d}z^2 . \tag{418}$$

This generalises the flat FLRW metric (395) to the case with separate scale factor $a_i(t)$ in each Cartesian direction $i = 1, 2, 3$. We introduce the Misner parametrisation (see, e.g., [81]) with a volume variable $V = a_1 a_2 a_3$,

$$\begin{aligned} a_1 &= V^{1/3} e^{\beta_+ + \sqrt{3}\beta_-} , \\ a_2 &= V^{1/3} e^{\beta_+ - \sqrt{3}\beta_-} , \\ a_3 &= V^{1/3} e^{-2\beta_+} . \end{aligned} \tag{419}$$

The variables $\beta_\pm$ represent anisotropy parameters and have their own momenta $p_\pm$. Using this parametrisation, equations (391) and (392) become

$$S = \int \mathrm{d}t \, \left(p_V \dot{V} + p_+ \dot{\beta}_+ + p_- \dot{\beta}_- + p_\chi \dot{\chi} - NC\right) , \tag{420}$$

Just like for the momentum p_χ of the field χ, note that also the momenta of the anisotropies $\beta_\pm$ are conserved, i.e., $\dot{p}_\pm = 0$.

where

$$NC = -6\pi G N V p_V^2 + \frac{2\pi G}{3}\frac{N p_+^2}{V} + \frac{2\pi G}{3}\frac{N p_-^2}{V} + \frac{N p_\chi^2}{2V} = 0 . \tag{421}$$

Notice the similarity between the anisotropy variables and the scalar field χ. Their contribution to the Hamiltonian is basically the same, expect for numerical factors; at least classically, a Bianchi I Universe is not different from an FLRW Universe with free massless scalar fields. Even though we still make use of a matter clock in Chapter 4, this equivalence suggests that the anisotropy variables $\beta_\pm$ could play the role of a clock in GFT cosmological (anisotropic) models.

As before, we now use χ as a clock and we extract relational dynamics from the equations of motion as follows. Other than the equations of motion for the volume and the scalar field (which are also here given by (401) and (403)), we also have the dynamics for the anisotropies

$$\dot{\beta}_\pm = \{\beta_\pm, NC\} = \frac{4\pi G}{3}\frac{Np_\pm}{V}\,. \tag{422}$$

One can then use $\dot{V}$ and $\dot{\chi}$ to obtain p_V as in (404), which can be substituted into the vanishing of (421), to get

$$\left(\frac{1}{3V}\frac{\mathrm{d}V}{\mathrm{d}\chi}\right)^2 = \left(\frac{4\pi G}{3}\right)^2 \frac{p_+^2 + p_-^2}{p_\chi^2} + \frac{4\pi G}{3}\,. \tag{423}$$

This relational (generalised) Friedmann equation reduces to (405) in the isotropic case. We point out that the new anisotropic contribution is (also) constant in χ as the momenta $p_\pm$ and p_χ are constants of motion.

Moreover, we now have anisotropy degrees of freedom, whose relational dynamics are obtained in a similar fashion. We use (422) (together with $\dot{\chi}$) to write the momenta

$$p_\pm = \frac{3p_\chi}{4\pi G}\frac{\mathrm{d}\beta_\pm}{\mathrm{d}\chi}\,. \tag{424}$$

Importantly, since $p_\pm$ are constants of motion, we see that the anisotropies are linear in χ. Substituting the momenta (424) into the Hamiltonian constraint (421), we obtain

$$\left(\frac{\mathrm{d}\beta_\pm}{\mathrm{d}\chi}\right)^2 = \left(\frac{4\pi G}{3}\right)^2 \left(\frac{9V^2p_V^2}{p_\chi^2} - \frac{p_\mp^2}{p_\chi^2}\right) - \frac{4\pi G}{3}\,. \tag{425}$$

Albeit tedious one can also check from the right-hand side of (425) that $\mathrm{d}\beta_\pm/\mathrm{d}\chi$ is constant (specifically, one shows that other than p_χ and $p_\pm$, also the product Vp_V is conserved).

Conveniently, some algebraic manipulations show that (423) and (425) can also be combined into the relational equation

$$\left(\frac{1}{3V}\frac{\mathrm{d}V}{\mathrm{d}\chi}\right)^2 = \left(\frac{\mathrm{d}\beta_+}{\mathrm{d}\chi}\right)^2 + \left(\frac{\mathrm{d}\beta_-}{\mathrm{d}\chi}\right)^2 + \frac{4\pi G}{3}\,. \tag{426}$$

Note that constant anisotropies yield the same relational dynamics of a FLRW Universe (405). This is consistent with the fact that constant anisotropies can directly be removed from the metric by simply rescaling coordinates.

This form has the advantage that it only relies on derivatives with respect to χ rather than canonical momenta. This equation is used as a classical comparison for the anisotropic GFT cosmological model of Chapter 4 in the large volume limit.

A MODEL WITH CURVATURE. For the sake of completeness, we extend the above formalism to the Bianchi II case to see how curvature affects relational (classical) dynamics. We follow the same strategy as before, but we now also have the Bianchi II curvature term appearing in (392), which reads [81]

$$^{(3)}R = -\frac{1}{2}V^{-2/3}e^{4(\beta_+ + \sqrt{3}\beta_-)}\,. \tag{427}$$

The Hamiltonian constraint (392) then takes the form

$$C = -6\pi G V p_V^2 + \frac{2\pi G}{3}\frac{p_+^2}{V} + \frac{2\pi G}{3}\frac{p_-^2}{V} + \frac{V^{1/3}}{32\pi G}e^{4(\beta_+ + \sqrt{3}\beta_-)} + \frac{p_\chi^2}{2V} = 0\,. \tag{428}$$

Without repeating all the steps seen in the Bianchi I case, we recall that the strategy is to use the equations of motion to obtain the momenta p_V and $p_\pm$. Plugging them into $C = 0$ one obtains

$$\left(\frac{1}{3V}\frac{\mathrm{d}V}{\mathrm{d}\chi}\right)^2 = \left(\frac{4\pi G}{3}\right)^2\frac{p_+^2 + p_-^2}{p_\chi^2} + \frac{4\pi G}{3} + \frac{V^{4/3}}{12p_\chi^2}e^{4(\beta_+ + \sqrt{3}\beta_-)}\,, \tag{429}$$

and

$$\left(\frac{\mathrm{d}\beta_\pm}{\mathrm{d}\chi}\right)^2 = \left(\frac{4\pi G}{3}\right)^2\left(\frac{9V^2p_V^2}{p_\chi^2} - \frac{p_\mp^2}{p_\chi^2}\right) - \frac{4\pi G}{3} - \frac{V^{4/3}}{12p_\chi^2}e^{4(\beta_+ + \sqrt{3}\beta_-)}\,. \tag{430}$$

These are nothing but extensions of (423) and (425). Finally, putting everything together one can write a Bianchi II generalisation of the (relational) Friedmann equation,

$$\left(\frac{1}{3V}\frac{\mathrm{d}V}{\mathrm{d}\chi}\right)^2 = \left(\frac{\mathrm{d}\beta_+}{\mathrm{d}\chi}\right)^2 + \left(\frac{\mathrm{d}\beta_-}{\mathrm{d}\chi}\right)^2 + \frac{4\pi G}{3} + \frac{V^{4/3}}{12p_\chi^2}e^{4(\beta_+ + \sqrt{3}\beta_-)}\,, \tag{431}$$

which reduces to (426) when the last term (which comes from ${}^{(3)}R$) is zero. Notice that in an expanding Universe this term becomes dominant at later times: the dynamics are initially close to Bianchi I and deviate only when the exponential expansion of the volume takes over. Because of this, and given that the Bianchi II scenario is the simplest Bianchi model involving spatial curvature, we also compare our anisotropic GFT cosmology to these equations in Chapter 4.

B

COMPUTATIONS WITH GAUSSIAN STATES

In this appendix we provide all the details needed to achieve some of the main results of Chapter 5. Specifically, in Section B.1 we focus on the Robertson–Schrödinger uncertainty principle (cf. (204)) as a possible criterion for defining semiclassical states, and we prove some claims of Section 5.1 and Section 5.2 showing all the analytical expressions. Then, we briefly review the thermofield formalism in Section B.2, a computational tool that is used for most of the results of Chapter 5. While Section B.1 contains details for the results of Chapter 5 that focus on the deparametrised approach to GFT cosmology, Section B.2 introduces a framework that is used in both approaches (in a straightforward manned for the deparametrised approach, and with suitable generalisations described in Section 5.3 for the algebraic approach).

B.1 ROBERTSON–SCHRÖDINGER UNCERTAINTY PRINCIPLE

In this section we analyse in detail the Robertson–Schrödinger (RS) inequality (204) for all states described in Section 5.1 and Section 5.2. To begin with, we report in Table 2 and Table 3 the expectation values, variances and covariances for the operators of interest at $\chi = 0$, using coherent states (210) and squeezed states (219). Moreover, using the analytical expressions for the time evolution of variances and covariances (cf. (205) and (206)), we explicitly compute the dynamical behaviour of the left-hand side and right-hand side of the RS inequality (204). For coherent states we find

Just like in Chapter 5, the subscript of trigonometric and hyperbolic functions denotes their arguments in some lengthy equations.

$$(\Delta \hat{V}_\chi)^2_{\mathrm{C}} (\Delta \hat{H})^2_{\mathrm{C}} = \frac{v^2\omega^2}{8}\,(2|\alpha|^2+1) \times \left[4|\alpha|^2(\sinh_{4\omega\chi}\sin_{2\vartheta} + \cosh_{4\omega\chi}) + \cosh_{4\chi\omega} - 1\right],$$

and

$$(\Delta(\hat{V}_\chi \hat{H}))^2_{\mathrm{C}} + \omega^2 \langle \hat{C}_\chi \rangle^2_{\mathrm{C}} = \frac{v^2\omega^2}{4}\left[4|\alpha|^4 \cosh^2_{2\omega\chi}\cos^2_{2\vartheta} + \left(2|\alpha|^2(\cosh_{2\omega\chi}\sin_{2\vartheta} + \sin_{2\omega\chi}) + \sinh_{2\omega\chi}\right)^2\right],$$

	Coherent states evaluation
$\langle\hat{V}\rangle$	$v\|\alpha\|^2$
$\langle\hat{H}\rangle$	$-\frac{\omega}{2}\left(\bar{\alpha}^2+\alpha^2\right)=-\omega\|\alpha\|^2\cos(2\vartheta)$
$\langle\hat{C}\rangle$	$\mathrm{i}\frac{v}{2}\left(\bar{\alpha}^2-\alpha^2\right)=v\|\alpha\|^2\sin(2\vartheta)$
$(\Delta\hat{V})^2$	$v^2\|\alpha\|^2$
$(\Delta\hat{H})^2$	$\frac{\omega^2}{2}\left(1+2\|\alpha\|^2\right)$
$(\Delta\hat{C})^2$	$\frac{v^2}{2}\left(1+2\|\alpha\|^2\right)$
$\Delta(\hat{V}\hat{H})$	$-\frac{v\omega}{2}\left(\bar{\alpha}^2+\alpha^2\right)=-v\omega\|\alpha\|^2\cos(2\vartheta)$
$\Delta(\hat{V}\hat{C})$	$\mathrm{i}\frac{v^2}{2}\left(\bar{\alpha}^2-\alpha^2\right)=v^2\|\alpha\|^2\sin(2\vartheta)$
$\Delta(\hat{H}\hat{C})$	0

Table 2: Useful quantities computed with coherent states $|\alpha\rangle$ (cf. (210)), where the displacement parameter is decomposed as $\alpha=|\alpha|e^{\mathrm{i}\vartheta}$.

which can only be equal if $v\omega|\alpha|^2\cosh_{2\omega\chi}=0$. Since we exclude the trivial cases with vanishing GFT parameters and with $\alpha=0$ (which would reduce a coherent state $|\alpha\rangle$ to the vacuum $|0\rangle$), there is no χ for which the Robertson–Schrödinger relation is saturated. Conversely, for squeezed states we find that the uncertainty principle is minimised at all times:

$$
\begin{aligned}
(\Delta\hat{V}_\chi)^2_{\mathrm{S}}(\Delta\hat{H})^2_{\mathrm{S}} &= (\Delta(\hat{V}_\chi\hat{H}))^2_{\mathrm{S}}+\omega^2\langle\hat{C}_\chi\rangle^2_{\mathrm{S}} \\
&= \frac{v^2\omega^2}{16}\Big[\sinh^2_{2\omega\chi}\left(\sinh^4_{2r}\sin^2_{2\psi}+4\cosh^2_{2r}\right) \\
&\quad+\cosh^2_{2\omega\chi}\left(4\sinh^2_{2r}\sin^2_{\psi}+\sinh^2_{4r}\cos^2_{\psi}\right) \\
&\quad+\frac{1}{4}\sinh_{4\omega\chi}\sin_{\psi}\left(8\sinh^3_{2r}\cosh_{2r}\cos_{2\psi}+6\sinh_{4r}+\sinh_{8r}\right)\Big].
\end{aligned}
$$

In other words, χ-evolution does not change the statement of whether the RS uncertainty principle is saturated. Figure 24 shows this feature for some state parameters.

Other than studying generic intermediate times, we can in particular also evaluate the large-volume limits (207) and (208), which pro-

	Squeezed states evaluation
$\langle\hat{V}\rangle$	$v\sinh^2(r)$
$\langle\hat{H}\rangle$	$-\frac{\omega}{2}\sinh(2r)\cos(\psi)$
$\langle\hat{C}\rangle$	$\frac{v}{2}\sinh(2r)\sin(\psi)$
$(\Delta\hat{V})^2$	$\frac{v^2}{2}\sinh^2(2r)$
$(\Delta\hat{H})^2$	$\frac{\omega^2}{8}\left(3+\cosh(4r)+2\cos(2\psi)\sinh^2(2r)\right)$
$(\Delta\hat{C})^2$	$\frac{v^2}{8}\left(3+\cosh(4r)-2\cos(2\psi)\sinh^2(2r)\right)$
$\Delta(\hat{V}\hat{H})$	$-\frac{v\omega}{4}\cos(\psi)\sinh(4r)$
$\Delta(\hat{V}\hat{C})$	$\frac{v^2}{4}\sin(\psi)\sinh(4r)$
$\Delta(\hat{H}\hat{C})$	$-\frac{v\omega}{4}\sin(2\psi)\sinh^2(2r)$

Table 3: Useful quantities computed with squeezed states $|z\rangle$ (cf. (219)), where the squeezing parameter is decomposed as $z = re^{i\psi}$.

vide simpler expressions. Using the quantities in Table 2 and Table 3, we find

$$\frac{(\Delta\hat{V}_\chi)^2_C}{\langle\hat{V}_\chi\rangle^2_C}\frac{(\Delta\hat{H})^2_C}{\langle\hat{H}\rangle^2_C} \overset{\chi\to\pm\infty}{\longrightarrow} \frac{(2|\alpha|^2+1)[1+4|\alpha|^2(1\pm\sin_{2\vartheta})]}{|\alpha|^4\cos^2_{2\vartheta}(1+2|\alpha|^2(1\pm\sin_{2\vartheta}))^2},$$

$$\frac{(\Delta(\hat{V}_\chi\hat{H}))^2_C}{\langle\hat{V}_\chi\rangle^2_C\langle\hat{H}\rangle^2_C}+\omega^2\frac{\langle\hat{C}_\chi\rangle^2_C}{\langle V_\chi\rangle^2_C\langle\hat{H}\rangle^2_C} \overset{\chi\to\pm\infty}{\longrightarrow} \frac{1+4|\alpha|^2\left(2|\alpha|^2+1\right)(1\pm\sin_{2\vartheta})}{|\alpha|^4\cos^2_{2\vartheta}(1+2|\alpha|^2(1\pm\sin_{2\vartheta}))^2},$$

and

$$\frac{(\Delta\hat{V}_\chi)^2_S}{\langle\hat{V}_\chi\rangle^2_S}\frac{(\Delta\hat{H})^2_S}{\langle\hat{H}\rangle^2_S} \overset{\chi\to\pm\infty}{\longrightarrow} 4+4\mathrm{csch}^2_{2r}\sec^2_\psi,$$

$$\frac{(\Delta(\hat{V}_\chi\hat{H}))^2_S}{\langle\hat{V}_\chi\rangle^2_S\langle\hat{H}\rangle^2_S}+\omega^2\frac{\langle\hat{C}_\chi\rangle^2_S}{\langle V_\chi\rangle^2_S\langle\hat{H}\rangle^2_S} \overset{\chi\to\pm\infty}{\longrightarrow} 4+4\mathrm{csch}^2_{2r}\sec^2_\psi,$$

showing confirmation of the above statements for both classes of states.

In a similar fashion, one can also deal with the more general Gaussian states (229). As with coherent states, one finds that Gaussian states do not minimise the RS principle at any χ. For $\chi = 0$ one can quickly read off from the results in Section 5.2 (cf. (232), (234) and

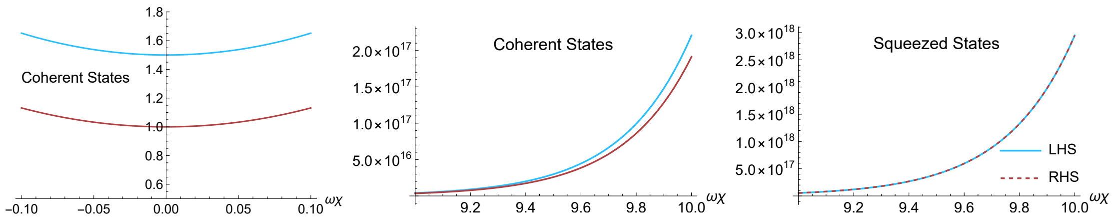

Figure 24: Left-hand side (LHS) and right-hand side (RHS) of the RS inequality (204) for coherent ($\alpha = 1$) and squeezed ($z = 1$) states, setting $v = 1$. The first two panels show that coherent states do not saturate the inequality at early or late times; the last panel shows that squeezed states saturate (204) at all times.

(235)) that the right-hand side and the left-hand side of the inequality (204) do not match, as

$$(\Delta\hat{V})^2_{\mathrm{G}}(\Delta\hat{H})^2_{\mathrm{G}} = \frac{v^2\omega^2}{32}\Big(4|\alpha|^2\mathcal{B}(\cosh_{2r} + \mathcal{F}_+ \sinh_{2r}) + \mathcal{B}^2\cosh_{4r} - 1\Big) \times \Big[8|\alpha|^2\mathcal{B}(\cosh_{2r} + \mathcal{F}_- \sinh_{2r}) + \mathcal{B}^2\Big(2\sinh^2_{2r}\cos_{2\psi} + \cosh_{4r} + 1\Big) + 2\Big] ,$$

and

$$(\Delta(\hat{V}\hat{H}))^2_{\mathrm{G}} + \omega^2\langle\hat{C}\rangle^2_{\mathrm{G}} = \frac{v^2\omega^2}{16}\Big[4\left(2|\alpha|^2\sin_{2\vartheta} + \mathcal{B}\sinh_{2r}\sin_{\psi}\right)^2 + \mathcal{B}^2\left(4|\alpha|^2(\cosh_{2r}\cos_{2\vartheta} + \sinh_{2r}\cos_{\psi}) + \mathcal{B}\sinh_{4r}\cos_{\psi}\right)^2\Big] ,$$

where $\mathcal{B} = \coth_{\beta/2}$ and $\mathcal{F}_\pm = \cos_{2\vartheta}\cos_{\psi} \pm \sin_{2\vartheta}\sin_{\psi}$. As with any other states, one can also use the time-dependent expressions (205) and (206) to compute the behaviour of the uncertainty principle under time evolution. Even though such results can be calculated analytically, we do not report the (lengthy) Gaussian state expressions because they are not insightful; we refer the reader to Figure 25 instead. On the other hand, we show explicitly that the inequality is not saturated for $\chi \to \pm\infty$. Using again the convenient large-volume limit (207), one can compute the product of the (late-time) volume and Hamiltonian relative uncertainties

$$\frac{(\Delta\hat{V}_\chi)^2_{\mathrm{G}}}{\langle\hat{V}_\chi\rangle^2_{\mathrm{G}}} \overset{\chi\to\pm\infty}{\longrightarrow} 2 - \frac{8|\alpha|^4(\sin_{2\vartheta} \pm 1)^2}{\mathcal{D}_1} ,$$

$$\frac{(\Delta\hat{H})^2_{\mathrm{G}}}{\langle\hat{H}\rangle^2_{\mathrm{G}}} = \frac{1}{2\mathcal{D}_2}\Big[8\mathcal{B}|\alpha|^2(\cosh_{2r} + \mathcal{F}_- \sinh_{2r}) + \mathcal{B}^2\Big(2\sinh^2_{2r}\cos_{2\psi} + \cosh_{4r} + 1\Big) + 2\Big] ,$$

where for convenience we defined

$$\mathcal{D}_1 = \left[2|\alpha|^2(\sin_{2\vartheta} \pm 1) + \mathcal{B}(\sinh_{2r} \sin_{\psi} \pm \cosh_{2r})\right]^2 ,$$
$$\mathcal{D}_2 = \left(2|\alpha|^2 \cos_{2\vartheta} + \mathcal{B} \sinh_{2r} \cos_{\psi}\right)^2 .$$

Comparing the product of $(\Delta\hat{V}_\chi)^2_{\mathrm{G}}/\langle\hat{V}_\chi\rangle^2_{\mathrm{G}}$ and $(\Delta\hat{H})^2_{\mathrm{G}}/\langle\hat{H}\rangle^2_{\mathrm{G}}$ with (208), which here reads

$$\frac{(\Delta(\hat{V}_\chi\hat{H}))^2_{\mathrm{G}}}{\langle\hat{V}_\chi\rangle^2_{\mathrm{G}}\langle\hat{H}\rangle^2_{\mathrm{G}}} + \omega^2\frac{\langle\hat{C}_\chi\rangle^2_{\mathrm{G}}}{\langle V_\chi\rangle^2_{\mathrm{G}}\langle\hat{H}\rangle^2_{\mathrm{G}}} \xrightarrow{\chi\to\pm\infty} \frac{\mathcal{B}^2}{\mathcal{D}_1\mathcal{D}_2}\Big\{\Big[\mathcal{B}\left(\sinh_{4r}\cos_\psi \pm \sinh^2_{2r}\sin_{2\psi}\right)$$
$$+ 4|\alpha|^2(\sinh_{2r}(\cos_\psi \pm \sin_{2\vartheta+\psi}) + \cosh_{2r}\cos_{2\vartheta})\Big]^2\Big\} + \frac{4}{\mathcal{D}_2},$$

one can see that the saturation of the RS relation does not occur at late times. Since the complexity of Gaussian states makes these expressions somewhat intransparent, we report in Figure 25 a graphical demonstration of the exact time evolution for some state parameters.

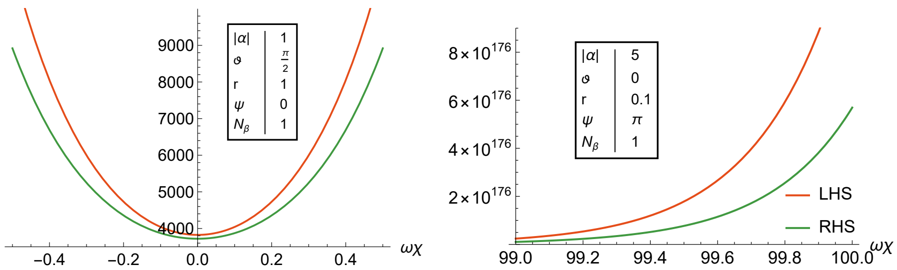

Figure 25: Left-hand side (LHS) and right-hand side (RHS) of the RS principle (204) for Gaussian states, setting $v = 1$. The inequality is not saturated at early or late times regardless of the choice of parameters.

B.2 GAUSSIAN STATES AND THERMOFIELD FORMALISM

In this appendix we present the features of Gaussian states that lead to the definition (229), as well as the tools used to obtain all the results of Section 5.2 (and generalised in Section 5.3). We start by means of a pivotal result, originally investigated in [268, 269, 350] (see also [212, 351] for modern perspectives), which states that when dealing with second-order bosonic Hamiltonians, the most general Gaussian state can always be expressed as three types of unitary operators acting on the thermal state $\hat{\rho}_\beta$ (230) (or on the Fock vacuum $|0\rangle$ for pure

Gaussian states). These so-called "fundamental Gaussian unitaries" are the squeezing, displacement and rotation operators

$$\begin{aligned} \hat{S}(z) &= e^{\frac{1}{2}(z\hat{a}^{\dagger 2}-\bar{z}\hat{a}^2)}\,, \\ \hat{D}(\alpha) &= e^{\alpha\hat{a}^\dagger-\bar{\alpha}\hat{a}}\,, \\ \hat{R}(\phi) &= e^{\mathrm{i}\phi\hat{a}^\dagger\hat{a}}\,, \end{aligned} \tag{432}$$

where $(z,\alpha)\in\mathbb{C}$, $z=re^{\mathrm{i}\psi}$ and $(r,\psi,\phi)\in\mathbb{R}$.

To prove that a general Gaussian state can be taken to be of the form (229), one can first notice that the operators in (432) satisfy

$$\begin{aligned} \hat{R}(\phi_1)\hat{R}(\phi_2) &= \hat{R}(\phi_1+\phi_2)\,, \\ \hat{D}(\alpha_1)\hat{D}(\alpha_2) &= e^{\frac{1}{2}(\overline{\alpha_2}\alpha_1-\alpha_2\overline{\alpha_1})}\hat{D}(\alpha_1+\alpha_2)\,, \\ \hat{S}(z_1)\hat{S}(z_2) &= e^{\mathrm{i}\phi/2}\hat{S}(z_3)\hat{R}(\phi)\,, \end{aligned} \tag{433}$$

where, defining $t_a=\tanh(r_a)\exp(\mathrm{i}\psi_a)$, ϕ and z_3 are determined by $e^{\mathrm{i}\phi}=\frac{1+t_1\overline{t_2}}{|1+t_1\overline{t_2}|}$ and $t_3=\frac{t_1+t_2}{1+\overline{t_1}t_2}$. Moreover, one finds that operators of different types "compose in a closed way", namely

$$\hat{S}^\dagger(z)\hat{D}(\alpha)\hat{S}(z) = \hat{D}(\alpha\cosh(r)-\bar{\alpha}e^{\mathrm{i}\psi}\sinh(r))\,, \tag{434}$$

$$\hat{R}^\dagger(\phi)\hat{S}(z)\hat{R}(\phi) = \hat{S}(e^{-2\mathrm{i}\phi}z)\,, \tag{435}$$

$$\hat{R}^\dagger(\phi)\hat{D}(\alpha)\hat{R}(\phi) = \hat{D}(e^{-\mathrm{i}\phi}\alpha)\,. \tag{436}$$

Since the state parameters are arbitrary, properties (433)–(436) show that one can produce a Gaussian state by acting on $\hat{\rho}_\beta$ with any number of the operators in (432), in any order. Finally, we can see that it is no loss of generality to define Gaussian states without using the rotation operator. Using (435) and (436), starting from an arbitrary definition of Gaussian states we could move the rotation operator until it acts on $\hat{\rho}_\beta$ (or on $|0\rangle$ in the case of zero temperature). These operations only change the free parameters z and α, which were arbitrary to being with, by a phase. But $\hat{R}(\phi)$ leaves both $\hat{\rho}_\beta$ and $|0\rangle$ invariant, and hence has no effect whatsoever.

Central to the computation of expectation values with (both pure and mixed) Gaussian states is the action of the displacement operator and squeezing operator on $\hat{a}$ and $\hat{a}^\dagger$:

$$\begin{aligned} \hat{S}^\dagger(z)\hat{a}\hat{S}(z) &= \cosh(r)\hat{a}+e^{\mathrm{i}\psi}\sinh(r)\hat{a}^\dagger\,, \\ \hat{S}^\dagger(z)\hat{a}^\dagger\hat{S}(z) &= \cosh(r)\hat{a}^\dagger+e^{-\mathrm{i}\psi}\sinh(r)\hat{a}\,, \end{aligned} \tag{437}$$

and

$$\begin{aligned} \hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) &= \hat{a}+\alpha\,, \\ \hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha) &= \hat{a}^\dagger+\bar{\alpha}\,. \end{aligned} \tag{438}$$

From these one can see that, for any function f of the ladder operators, expressions of the form $\hat{S}^\dagger(z)\hat{D}^\dagger(\alpha)f(\hat{a},\hat{a}^\dagger)\hat{D}(\alpha)\hat{S}(z)$ are in general simpler than $\hat{D}^\dagger(\alpha)\hat{S}^\dagger(z)f(\hat{a},\hat{a}^\dagger)\hat{S}(z)\hat{D}(\alpha)$. This is why, for instance, displaced squeezed states $|\alpha,z\rangle = \hat{D}(\alpha)\hat{S}(z)|0\rangle$ are usually adopted as pure Gaussian states instead of squeezed coherent states $|z,\alpha\rangle = \hat{S}(z)\hat{D}(\alpha)|0\rangle$ (the same applies to our general definition (229)); the property (434) effectively allows us to choose the most convenient ordering.

While by use of (437) and (438) one can obtain all the expressions of Section 5.2 using the density matrix $\hat{\rho}_G$ (229) and the standard techniques for calculating traces, we outline here the *thermofield formalism* as a very useful tool to extract the same results (in particular if one wishes to use a computational software, such as *Mathematica*). This also allows us to link the present paper with the work of [40, 41, 250], where such a formalism was adopted to introduce thermal effects in GFT.

Thermofield dynamics were introduced in [357] (see [241] for a recent detailed treatment) as a formalism to link ensemble averages of statistical mechanics to expectation values computed with a temperature-dependent vacuum state, dubbed *thermal vacuum* and denoted $|0_\beta\rangle$. In a nutshell, such a framework establishes a correspondence between a density matrix, which in our case is of thermal type $\hat{\rho}_\beta$ (230), and the pure vector state $|0_\beta\rangle$, such that

$$\mathrm{tr}\left(\hat{\rho}_\beta\hat{\mathcal{O}}\right) = \langle 0_\beta|\hat{\mathcal{O}}|0_\beta\rangle\,. \tag{439}$$

In [357] it is shown that one can define a thermal vacuum satisfying (439) only by enlarging the conventional Fock space. Specifically, one needs to double it by adding a fictitious system (denoted with a tilde) identical to the one under investigation. For simple theories such as our single-mode GFT cosmological models, this means introducing a new pair of bosonic ladder operators $(\hat{\tilde{a}},\hat{\tilde{a}}^\dagger)$ and constructing a second Fock space from a “tilde vacuum” $|\tilde{0}\rangle$ in the standard way, with

$$\left[\hat{\tilde{a}},\hat{\tilde{a}}^\dagger\right] = 1\,, \tag{440}$$

and

$$\hat{\tilde{a}}|\tilde{0}\rangle = 0\,. \tag{441}$$

The tilde operators commute with the non-tilde operators as they live in distinct spaces. The next step in the thermofield formalism is to define a "product vacuum" in the doublet Hilbert space as a *zero-temperature* ground state,

$$|0,\tilde{0}\rangle := |0\rangle \otimes |\tilde{0}\rangle\,, \tag{442}$$

such that $\hat{a}|0,\tilde{0}\rangle = \hat{\tilde{a}}|0,\tilde{0}\rangle = 0$, from which one can construct product states in the usual manner,

$$|n,\tilde{m}\rangle = \frac{(\hat{a}^\dagger)^n(\hat{\tilde{a}}^\dagger)^m}{\sqrt{n!}\sqrt{m!}}|0,\tilde{0}\rangle\,. \tag{443}$$

Finally, the correspondence with thermal states is established by introducing temperature through a Bogoliubov transformation that mixes the real and fictitious systems:

$$|0_\beta\rangle = \hat{T}(\theta_\beta)|0,\tilde{0}\rangle\,, \tag{444}$$

with

$$\hat{T}(\theta_\beta) = e^{\theta_\beta(\hat{a}^\dagger\hat{\tilde{a}}^\dagger - \hat{a}\hat{\tilde{a}})}\,. \tag{445}$$

The so-called thermalising operator $\hat{T}(\theta_\beta)$ is a *two-mode* squeezing operator and the real parameter θ_β encodes temperature in a way that will enable us to verify (439). One can easily check that the transformed, now *temperature-dependent*, ladder operators

$$\begin{aligned} \hat{a}_\beta &= \hat{T}(\theta_\beta)\hat{a}\hat{T}^\dagger(\theta_\beta) = \cosh(\theta_\beta)\hat{a} - \sinh(\theta_\beta)\hat{\tilde{a}}^\dagger\,, \\ \hat{\tilde{a}}_\beta &= \hat{T}(\theta_\beta)\hat{\tilde{a}}\hat{T}^\dagger(\theta_\beta) = \cosh(\theta_\beta)\hat{\tilde{a}} - \sinh(\theta_\beta)\hat{a}^\dagger\,, \end{aligned} \tag{446}$$

indeed annihilate the state (444)

$$\hat{a}_\beta|0_\beta\rangle = \hat{\tilde{a}}_\beta|0_\beta\rangle = 0\,. \tag{447}$$

The operators (446) and their adjoints still satisfy the bosonic algebra because Bogoliubov transformations are canonical. Therefore, one can construct a Fock space with respect to the β-dependent operators, where general states read

$$|n, \tilde{m}; \beta\rangle = \hat{T}(\theta_\beta)|n, \tilde{m}\rangle = \frac{(\hat{a}^\dagger_\beta)^n (\hat{\tilde{a}}^\dagger_\beta)^m}{\sqrt{n!}\sqrt{m!}} |0_\beta\rangle \,. \tag{448}$$

The association of the thermal vacuum $|0_\beta\rangle$ with a density matrix $\hat{\rho}_\beta$ as given in (230) is obtained by determining θ_β as a function of β. This is done by imposing the condition (439) for the number operator so that, thanks to (233), one has

$$\frac{1}{e^\beta - 1} = \text{tr}\left(\hat{\rho}_\beta \hat{a}^\dagger \hat{a}\right) = \langle 0_\beta | \hat{a}^\dagger \hat{a} | 0_\beta \rangle = \sinh(\theta_\beta)^2 \,, \tag{449}$$

where the right-hand side is computed using the thermal vacuum (444). The fictitious system is understood as unphysical and is only introduced as a useful tool. We are not interested in expectation values of tilde operators; this helps to define a simple thermofield counterpart of Gaussian states.

Starting from the thermal vacuum $|0_\beta\rangle$, we can construct states by analogy with (210) and (219) and hence use displacement and squeezing operators to define general Gaussian states in the thermofield formalism. Indeed, we can generalise the GFT coherent thermal states of [40, 41, 250], defining the thermofield analogue of $\hat{\rho}_G$ (cf. (229)) as

$$|\alpha, z, \beta\rangle = \hat{D}(\alpha)\hat{S}(z)\hat{T}(\theta_\beta)|0, \tilde{0}\rangle = \hat{D}(\alpha)\hat{S}(z)|0_\beta\rangle \,, \tag{450}$$

which clearly reduces to a pure Gaussian state when $\beta \to \infty$ (or equivalently $\theta_\beta \to 0$). Since the thermalising operator (444) is of squeezing type, the order in which the operators appear in (450) is in principle arbitrary (see discussion below (433)) and is chosen for convenience. There is however a subtlety: while $\hat{D}(\alpha)$ and $\hat{S}(z)$ only act on the non-tilde sector, $\hat{T}(\theta_\beta)$ is a two-mode operator which mixes the physical and fictitious parts. Thus, choosing $\hat{T}(\theta_\beta)$ to be to the left of $\hat{D}(\alpha)$ and/or $\hat{S}(z)$ requires the introduction of $\hat{\tilde{D}}(\tilde{\alpha})$ and/or $\hat{\tilde{S}}(\tilde{z})$, with $\tilde{\alpha} = \bar{\alpha}$ and $\tilde{z} = \bar{z}$ (see [241] for details).[1]

On the other hand, if the mixing between the physical and the fictitious sectors happens first (i.e., $\hat{T}(\theta_\beta)$ is to the right of every other operator as in (450)), one is free to displace and squeeze only the non-tilde component of the state. Of course, one can also include the

1 One can for example use the state $\hat{T}(\theta_\beta)\hat{D}(\alpha)\hat{\tilde{D}}(\bar{\alpha})\hat{S}(z)\hat{\tilde{S}}(\bar{z})|0, \tilde{0}\rangle$ or any other alternative definition achieved by changing the operator ordering; they will all be related to $|\Psi_G; \beta\rangle$ in (450) via (434).

tilde operators $\hat{\tilde{D}}(\bar{\alpha})$ and $\hat{\tilde{S}}(\bar{z})$ acting from the left in (450) (as done in [40, 41, 250] for coherent states), but this is not necessary; it would only be relevant if one were interested in the expectation values of tilde operators. In short, the state (450) is the simplest thermofield analogue of the Gaussian density matrix (229).

The explicit computational convenience of the thermofield formalism comes from the following transformation rules of our GFT ladder operators

$$\begin{aligned}\hat{T}^{\dagger}(\theta_{\beta})\hat{a}\hat{T}(\theta_{\beta}) &= \cosh(\theta_{\beta})\hat{a} + \sinh(\theta_{\beta})\hat{\tilde{a}}^{\dagger}\,,\\ \hat{T}^{\dagger}(\theta_{\beta})\hat{a}^{\dagger}\hat{T}(\theta_{\beta}) &= \cosh(\theta_{\beta})\hat{a}^{\dagger} + \sinh(\theta_{\beta})\hat{\tilde{a}}\,,\end{aligned} \tag{451}$$

which were tacitly used to compute the right-hand side of (449). The transformations (451), together with (437) and (438), allow us to forget entirely about Gaussian density matrices and taking traces. This makes calculations very mechanical: one first simply computes expectation values with (450) exactly as if using pure states; then, one makes use of the identification (449) to map the thermofield results to the standard (β-dependent) results of the density matrix approach.

While in the above we focused on techniques used for the deparametrised approach to GFT cosmology (i.e., for the results in Section 5.2), suitably generalised definitions and expressions can easily be applied to the algebraic approach; the details can directly be found in Section 5.3.

BIBLIOGRAPHY

[1] B. P. Abbott et al. "Observation of Gravitational Waves from a Binary Black Hole Merger". *Phys. Rev. Lett.* **116**, 061102 (2016), arXiv: 1602.03837 [gr-qc].

[2] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Dover Publications, New York, 1964).

[3] A. Addazi et al. "Quantum gravity phenomenology at the dawn of the multi-messenger era—A review". *Prog. Part. Nucl. Phys.* **125**, 103948 (2022), arXiv: 2111.05659 [hep-ph].

[4] E. G. Adelberger, J. H. Gundlach, B. R. Heckel, S. Hoedl, and S. Schlamminger. "Torsion balance experiments: A low-energy frontier of particle physics". *Prog. Part. Nucl. Phys.* **62**, 102–134 (2009).

[5] E. Adjei, S. Gielen, and W. Wieland. "Cosmological evolution as squeezing: a toy model for group field cosmology". *Class. Quant. Grav.* **35**, 105016 (2018), arXiv: 1712.07266 [gr-qc].

[6] I. Agullo, A. Ashtekar, and W. Nelson. "Quantum Gravity Extension of the Inflationary Scenario". *Phys. Rev. Lett.* **109**, 251301 (2012), arXiv: 1209.1609 [gr-qc].

[7] I. Agullo, A. Ashtekar, and W. Nelson. "Extension of the quantum theory of cosmological perturbations to the Planck era". *Phys. Rev. D* **87**, 043507 (2013), arXiv: 1211.1354 [gr-qc].

[8] K. Akiyama et al. "First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole". *Astrophys. J. Lett.* **875**, L1 (2019), arXiv: 1906.11238 [astro-ph.GA].

[9] E. Alesci and F. Cianfrani. "Quantum reduced loop gravity: Semiclassical limit". *Phys. Rev. D* **90**, 024006 (2014), arXiv: 1402.3155 [gr-qc].

[10] E. Alesci and F. Cianfrani. "Loop quantum cosmology from quantum reduced loop gravity". *EPL* **111**, 40002 (2015), arXiv: 1410.4788 [gr-qc].

[11] E. Alesci and F. Cianfrani. "Quantum reduced loop gravity: Universe on a lattice". *Phys. Rev. D* **92**, 084065 (2015), arXiv: 1506.07835 [gr-qc].

[12] E. Alesci and C. Rovelli. "Complete LQG propagator: Difficulties with the Barrett-Crane vertex". *Phys. Rev. D* **76**, 104012 (2007), arXiv: 0708.0883 [gr-qc].

[13] C. Allen, F. Girelli, and S. Steinhaus. "Numerical evaluation of spin foam amplitudes beyond simplices". *Phys. Rev. D* **105**, 066003 (2022), arXiv: 2201.09902 [gr-qc].

[14] J. Ambjørn, B. Durhuus, and T. Jónsson. "Three-dimensional simplicial quantum gravity and generalized matrix models". *Mod. Phys. Lett. A* **6**, 1133–1146 (1991).

[15] J. Armas. *Conversations on Quantum Gravity* (Cambridge University Press, Cambridge, 2021).

[16] R. L. Arnowitt, S. Deser, and C. W. Misner. "Republication of: The dynamics of general relativity". *Gen. Rel. Grav.* **40**, 1997–2027 (2008), arXiv: gr-qc/0405109.

[17] S. K. Asante, B. Dittrich, and H. M. Haggard. "Effective Spin Foam Models for Four-Dimensional Quantum Gravity". *Phys. Rev. Lett.* **125**, 231301 (2020), arXiv: 2004.07013 [gr-qc].

[18] S. K. Asante, B. Dittrich, and H. M. Haggard. "Discrete gravity dynamics from effective spin foams". *Class. Quant. Grav.* **38**, 145023 (2021), arXiv: 2011.14468 [gr-qc].

[19] S. K. Asante, B. Dittrich, and J. Padua-Argüelles. "Effective spin foam models for Lorentzian quantum gravity". *Class. Quant. Grav.* **38**, 195002 (2021), arXiv: 2104.00485 [gr-qc].

[20] S. K. Asante, B. Dittrich, and S. Steinhaus. "Spin Foams, Refinement Limit, and Renormalization", *Handbook of Quantum Gravity*, pp. 1–37. Ed. by C. Bambi, L. Modesto, and I. Shapiro (Springer, Singapore, 2023), arXiv: 2211.09578 [gr-qc].

[21] A. Ashtekar. "New Variables for Classical and Quantum Gravity". *Phys. Rev. Lett.* **57**, 2244–2247 (1986).

[22] A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov. "Quantum Geometry and Black Hole Entropy". *Phys. Rev. Lett.* **80**, 904–907 (1998), arXiv: gr-qc/9710007.

[23] A. Ashtekar, J. C. Baez, and K. Krasnov. "Quantum geometry of isolated horizons and black hole entropy". *Adv. Theor. Math. Phys.* **4**, 1–94 (2000), arXiv: gr-qc/0005126.

[24] A. Ashtekar. "Singularity resolution in loop quantum cosmology: A brief overview". *J. Phys. Conf. Ser.* **189**, 012003 (2009), arXiv: 0812.4703 [gr-qc].

[25] A. Ashtekar and E. Bianchi. "A short review of loop quantum gravity". *Rept. Prog. Phys.* **84**, 042001 (2021), arXiv: 2104.04394 [gr-qc].

[26] A. Ashtekar and M. Bojowald. "Quantum geometry and the Schwarzschild singularity". *Class. Quant. Grav.* **23**, 391–411 (2006), arXiv: gr-qc/0509075.

[27] A. Ashtekar, M. Bojowald, and J. Lewandowski. "Mathematical structure of loop quantum cosmology". *Adv. Theor. Math. Phys.* **7**, 233–268 (2003), arXiv: gr-qc/0304074.

[28] A. Ashtekar, A. Corichi, and P. Singh. "Robustness of key features of loop quantum cosmology". *Phys. Rev. D* **77**, 024046 (2008), arXiv: 0710.3565 [gr-qc].

[29] A. Ashtekar and B. Gupt. "Generalized effective description of loop quantum cosmology". *Phys. Rev. D* **92**, 084060 (2015), arXiv: 1509.08899 [gr-qc].

[30] A. Ashtekar and B. Gupt. "Quantum gravity in the sky: interplay between fundamental theory and observations". *Class. Quant. Grav.* **34**, 014002 (2017), arXiv: 1608.04228 [gr-qc].

[31] A. Ashtekar and J. Lewandowski. "Quantum theory of geometry: I. Area operators". *Class. Quant. Grav.* **14**, A55–A82 (1997), arXiv: gr-qc/9602046.

[32] A. Ashtekar and J. Lewandowski. "Quantum theory of geometry II: Volume operators". *Adv. Theor. Math. Phys.* **1**, 388–429 (1998), arXiv: gr-qc/9711031.

[33] A. Ashtekar and J. Lewandowski. "Background independent quantum gravity: a status report". *Class. Quant. Grav.* **21**, R53 (2004), arXiv: gr-qc/0404018.

[34] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao, and T. Thiemann. "Quantization of diffeomorphism invariant theories of connections with local degrees of freedom". *J. Math. Phys.* **36**, 6456–6493 (1995), arXiv: gr-qc/9504018.

[35] A. Ashtekar, T. Pawlowski, and P. Singh. "Quantum nature of the big bang: An analytical and numerical investigation". *Phys. Rev. D* **73**, 124038 (2006), arXiv: gr-qc/0604013.

[36] A. Ashtekar, T. Pawlowski, and P. Singh. "Quantum nature of the big bang: Improved dynamics". *Phys. Rev. D* **74**, 084003 (2006), arXiv: gr-qc/0607039.

[37] A. Ashtekar and P. Singh. "Loop quantum cosmology: a status report". *Class. Quant. Grav.* **28**, 213001 (2011), arXiv: 1108.0893 [gr-qc].

[38] A. Ashtekar and D. Sloan. "Loop quantum cosmology and slow roll inflation". *Phys. Lett. B* **694**, 108–112 (2011), arXiv: 0912.4093 [gr-qc].

[39] A. Ashtekar and E. Wilson-Ewing. "Loop quantum cosmology of Bianchi type I models". *Phys. Rev. D* **79**, 083535 (2009), arXiv: 0903.3397 [gr-qc].

[40] M. Assanioussi and I. Kotecha. "Thermal quantum gravity condensates in group field theory cosmology". *Phys. Rev. D* **102**, 044024 (2020), arXiv: 2003.01097 [gr-qc].

[41] M. Assanioussi and I. Kotecha. "Thermal representations in group field theory: squeezed vacua and quantum gravity condensates". *JHEP* **02**, 173 (2020), arXiv: 1910.06889 [gr-qc].

[42] J. C. Baez. "Spin foam models". *Class. Quant. Grav.* **15**, 1827–1858 (1998), arXiv: gr-qc/9709052.

[43] J. C. Baez and J. W. Barrett. "The quantum tetrahedron in 3 and 4 dimensions". *Adv. Theor. Math. Phys.* **3**, 815–850 (1999), arXiv: gr-qc/9903060.

[44] A. Baldazzi, K. Falls, and R. Ferrero. "Relational observables in asymptotically safe gravity". *Annals Phys.* **440**, 168822 (2022), arXiv: 2112.02118 [hep-th].

[45] L. E. Ballentine. *Quantum Mechanics: A Modern Development* (World Scientific, Singapore, 1998).

[46] K. Banerjee, G. Calcagni, and M. Martín-Benito. "Introduction to Loop Quantum Cosmology". *SIGMA* **8**, 016 (2012), arXiv: 1109.6801 [gr-qc].

[47] A. Baratin and L. Freidel. "Hidden quantum gravity in 3D Feynman diagrams". *Class. Quant. Grav.* **24**, 1993–2026 (2007), arXiv: gr-qc/0604016.

[48] A. Baratin and L. Freidel. "Hidden quantum gravity in 4D Feynman diagrams: emergence of spin foams". *Class. Quant. Grav.* **24**, 2027–2060 (2007), arXiv: hep-th/0611042.

[49] A. Baratin and D. Oriti. “Quantum simplicial geometry in the group field theory formalism: reconsidering the Barrett–Crane model”. *New J. Phys.* **13**, 125011 (2011), arXiv: 1108.1178 [gr-qc].

[50] J. F. Barbero G. “Real Ashtekar variables for Lorentzian signature space-times”. *Phys. Rev. D* **51**, 5507–5510 (1995), arXiv: gr-qc/9410014.

[51] A. Barbieri. “Quantum tetrahedra and simplicial spin networks”. *Nucl. Phys. B* **518**, 714–728 (1998), arXiv: gr-qc/9707010.

[52] J. W. Barrett and L. Crane. “Relativistic spin networks and quantum gravity”. *J. Math. Phys.* **39**, 3296–3302 (1998), arXiv: gr-qc/9709028.

[53] J. W. Barrett and L. Crane. “A Lorentzian signature model for quantum general relativity”. *Class. Quant. Grav.* **17**, 3101–3118 (2000), arXiv: gr-qc/9904025.

[54] J. W. Barrett, R. J. Dowdall, W. J. Fairbairn, H. Gomes, and F. Hellmann. “Asymptotic analysis of the Engle–Pereira–Rovelli–Livine four-simplex amplitude”. *J. Math. Phys.* **50**, 112504 (2009), arXiv: 0902.1170 [gr-qc].

[55] J. W. Barrett, R. J. Dowdall, W. J. Fairbairn, F. Hellmann, and R. Pereira. “Lorentzian spin foam amplitudes: graphical calculus and asymptotics”. *Class. Quant. Grav.* **27**, 165009 (2010), arXiv: 0907.2440 [gr-qc].

[56] J. W. Barrett and I. Naish-Guzman. “The Ponzano-Regge model”. *Class. Quant. Grav.* **26**, 155014 (2009), arXiv: 0803.3319 [gr-qc].

[57] T. Battefeld and S. Watson. “String gas cosmology”. *Rev. Mod. Phys.* **78**, 435–454 (2006).

[58] B. Baytaş, E. Bianchi, and N. Yokomizo. “Gluing polyhedra with entanglement in loop quantum gravity”. *Phys. Rev. D* **98**, 026001 (2018), arXiv: 1805.05856 [gr-qc].

[59] B. Baytas, M. Bojowald, and S. Crowe. “Equivalence of models in loop quantum cosmology and group field theory”. *Universe* **5**, 41 (2019), arXiv: 1811.11156 [gr-qc].

[60] K. Becker, M. Becker, and J. H. Schwarz. *String Theory and M-Theory: A Modern Introduction* (Cambridge University Press, Cambridge, 2006).

[61] J. Ben Achour, S. Brahma, and M. Geiller. “New Hamiltonians for loop quantum cosmology with arbitrary spin representations”. *Phys. Rev. D* **95**, 086015 (2017), arXiv: 1612.07615 [gr-qc].

[62] J. Ben Achour and E. R. Livine. “Thiemann complexifier in classical and quantum FLRW cosmology”. *Phys. Rev. D* **96**, 066025 (2017), arXiv: 1705.03772 [gr-qc].

[63] J. Ben Geloun. “On the finite amplitudes for open graphs in Abelian dynamical colored Boulatov–Ooguri models”. *J. Phys. A* **46**, 402002 (2013), arXiv: 1307.8299 [hep-th].

[64] J. Ben Geloun and V. Bonzom. “Radiative Corrections in the Boulatov-Ooguri Tensor Model: The 2-Point Function”. *Int. J. Theor. Phys.* **50**, 2819–2841 (2011), arXiv: 1101.4294 [hep-th].

[65] J. Ben Geloun, R. Gurau, and V. Rivasseau. “EPRL/FK group field theory”. *EPL* **92**, 60008 (2010), arXiv: 1008.0354 [hep-th].

[66] E. Bianchi, M. Christodoulou, F. D’Ambrosio, H. M. Haggard, and C. Rovelli. “White holes as remnants: a surprising scenario for the end of a black hole”. *Class. Quant. Grav.* **35**, 225003 (2018), arXiv: 1802.04264 [gr-qc].

[67] E. Bianchi, P. Donà, and S. Speziale. “Polyhedra in loop quantum gravity”. *Phys. Rev. D* **83**, 044035 (2011), arXiv: 1009.3402 [gr-qc].

[68] E. Bianchi and H. M. Haggard. “Discreteness of the volume of space from Bohr-Sommerfeld quantization”. *Phys. Rev. Lett.* **107**, 011301 (2011), arXiv: 1102.5439 [gr-qc].

[69] E. Bianchi and H. M. Haggard. “Bohr-Sommerfeld quantization of space”. *Phys. Rev. D* **86**, 124010 (2012), arXiv: 1208.2228 [gr-qc].

[70] E. Bianchi and R. C. Myers. “On the architecture of spacetime geometry”. *Class. Quant. Grav.* **31**, 214002 (2014), arXiv: 1212.5183 [hep-th].

[71] E. Bianchi and C. Rovelli. “Note on the geometrical interpretation of quantum groups and noncommutative spaces in gravity”. *Phys. Rev. D* **84**, 027502 (2011), arXiv: 1105.1898 [gr-qc].

[72] T. Blackmore and E. Wilson-Ewing. “Numerics of Bianchi type II and type IX spacetimes in effective loop quantum cosmology”. *Class. Quant. Grav.* **40**, 215010 (2023), arXiv: 2303.16747 [gr-qc].

[73] N. Bodendorfer, M. Han, F. Haneder, and H. Liu. "Path integral renormalization in loop quantum cosmology". *Phys. Rev. D* **103**, 126021 (2021), arXiv: 2012.02068 [gr-qc].

[74] N. Bodendorfer and F. Haneder. "Coarse graining as a representation change". *Phys. Lett. B* **792**, 69–73 (2019), arXiv: 1811.02792 [gr-qc].

[75] N. Bodendorfer and D. Wuhrer. "Renormalisation with SU(1, 1) coherent states on the LQC Hilbert space". *Class. Quant. Grav.* **37**, 185007 (2020), arXiv: 1904.13269 [gr-qc].

[76] J. de Boer et al. "Frontiers of Quantum Gravity: shared challenges, converging directions" (2022), arXiv: 2207.10618 [hep-th].

[77] M. Bojowald. "Absence of a Singularity in Loop Quantum Cosmology". *Phys. Rev. Lett.* **86**, 5227–5230 (2001), arXiv: gr-qc/0102069.

[78] M. Bojowald. "Loop Quantum Cosmology". *Living Rev. Rel.* **8**, 11 (2005), arXiv: gr-qc/0601085.

[79] M. Bojowald. "Large scale effective theory for cosmological bounces". *Phys. Rev. D* **75**, 081301 (2007), arXiv: gr-qc/0608100.

[80] M. Bojowald. "Harmonic cosmology: how much can we know about a universe before the big bang?" *Proc. Roy. Soc. Lond. A* **464**, 2135–2150 (2008), arXiv: 0710.4919 [gr-qc].

[81] M. Bojowald. *Canonical Gravity and Applications: Cosmology, Black Holes, and Quantum Gravity* (Cambridge University Press, Cambridge, 2010).

[82] M. Bojowald, S. Brahma, and D.-h. Yeom. "Effective line elements and black-hole models in canonical loop quantum gravity". *Phys. Rev. D* **98**, 046015 (2018), arXiv: 1803.01119 [gr-qc].

[83] M. Bojowald, D. Cartin, and G. Khanna. "Lattice refining loop quantum cosmology, anisotropic models, and stability". *Phys. Rev. D* **76**, 064018 (2007), arXiv: 0704.1137 [gr-qc].

[84] M. Bojowald, R. Goswami, R. Maartens, and P. Singh. "Black Hole Mass Threshold from Nonsingular Quantum Gravitational Collapse". *Phys. Rev. Lett.* **95**, 091302 (2005), arXiv: gr-qc/0503041.

[85] L. Bombelli, A. Corichi, and O. Winkler. "Semiclassical quantum gravity: obtaining manifolds from graphs". *Class. Quant. Grav.* **26**, 245012 (2009), arXiv: 0905.3492 [gr-qc].

[86] D. V. Boulatov. "A model of three-dimensional lattice gravity". *Mod. Phys. Lett. A* **7**, 1629–1646 (1992), arXiv: hep-th/9202074.

[87] R. H. Brandenberger. "String gas cosmology: progress and problems". *Class. Quant. Grav.* **28**, 204005 (2011), arXiv: 1105.3247 [hep-th].

[88] R. H. Brandenberger and P. Peter. "Bouncing Cosmologies: Progress and Problems". *Found. Phys.* **47**, 797–850 (2017), arXiv: 1603.05834 [hep-th].

[89] M. P. Bronstein. "Kvantovanie gravitatsionnykh voln [Quantization of gravitational waves]". *Zh. Eksp. Teor. Fiz.* **6**, 195–236 (1936).

[90] M. P. Bronstein. "Republication of: Quantum theory of weak gravitational fields". *Gen. Relativ. Gravit.* **44**, 267–283 (2012).

[91] J. D. Brown and K. V. Kuchař. "Dust as a standard of space and time in canonical quantum gravity". *Phys. Rev. D* **51**, 5600–5629 (1995), arXiv: gr-qc/9409001.

[92] J. Brunnemann and D. Rideout. "Properties of the volume operator in loop quantum gravity: I. Results". *Class. Quant. Grav.* **25**, 065001 (2008), arXiv: 0706.0469 [gr-qc].

[93] J. Brunnemann and D. Rideout. "Properties of the volume operator in loop quantum gravity: II. Detailed presentation". *Class. Quant. Grav.* **25**, 065002 (2008), arXiv: 0706.0382 [gr-qc].

[94] J. Brunnemann and T. Thiemann. "Simplification of the spectral analysis of the volume operator in loop quantum gravity". *Class. Quant. Grav.* **23**, 1289–1346 (2006), arXiv: gr-qc/0405060.

[95] R. Brustein and R. Madden. "Graceful exit and energy conditions in string cosmology". *Phys. Lett. B* **410**, 110–118 (1997), arXiv: hep-th/9702043.

[96] R. Brustein and R. Madden. "Model of graceful exit in string cosmology". *Phys. Rev. D* **57**, 712–724 (1998), arXiv: hep-th/9708046.

[97] R. Brustein and G. Veneziano. "The graceful exit problem in string cosmology". *Phys. Lett. B* **329**, 429–434 (1994), arXiv: hep-th/9403060.

[98] A. Calcinari, L. Freidel, E. Livine, and S. Speziale. "Twisted geometries coherent states for loop quantum gravity". *Class. Quant. Grav.* **38**, 025004 (2020), arXiv: 2009.01125 [gr-qc].

[99] A. Calcinari and S. Gielen. "Towards anisotropic cosmology in group field theory". *Class. Quant. Grav.* **40**, 085004 (2023), arXiv: 2210.03149 [gr-qc].

[100] A. Calcinari and S. Gielen. "Generalised Gaussian states in group field theory and $\mathfrak{su}(1,1)$ quantum cosmology". *Phys. Rev. D* **109**, 066022 (2024), arXiv: 2310.08667 [gr-qc].

[101] A. Calcinari and S. Gielen. "Relational dynamics and Page–Wootters formalism in group field theory" (2024), arXiv: 2407.03432 [gr-qc].

[102] M. Campiglia, R. Gambini, and J. Pullin. "Loop quantization of spherically symmetric midi-superspaces". *Class. Quant. Grav.* **24**, 3649–3672 (2007), arXiv: gr-qc/0703135.

[103] S. Carrozza. "Flowing in Group Field Theory Space: a Review". *SIGMA* **12**, 070 (2016), arXiv: 1603.01902 [gr-qc].

[104] A. S. Cattaneo, P. Cotta-Ramusino, J. Frohlich, and M. Martellini. "Topological *BF* theories in 3 and 4 dimensions". *J. Math. Phys.* **36**, 6137–6160 (1995), arXiv: hep-th/9505027.

[105] M. de Cesare, D. Oriti, A. G. A. Pithis, and M. Sakellariadou. "Dynamics of anisotropies close to a cosmological bounce in quantum gravity". *Class. Quant. Grav.* **35**, 015014 (2018), arXiv: 1709.00994 [gr-qc].

[106] L. Chataignier, P. A. Höhn, M. P. E. Lock, and F. M. Mele. "Relational Dynamics with Periodic Clocks" (2024), arXiv: 2409.06479 [quant-ph].

[107] S. Chaturvedi and V. Srinivasan. "Photon-number distributions for fields with Gaussian Wigner functions". *Phys. Rev. A* **40**, 6095–6098 (1989).

[108] J. Cheeger, W. Müller, and R. Schrader. "Lattice gravity or Riemannian structure on piecewise linear spaces", *Unified Theories of Elementary Particles*, pp. 176–188. Ed. by P. Breitenlohner and H. P. Dürr (Springer Berlin, Heidelberg, 1982).

[109] J. Cheeger, W. Müller, and R. Schrader. "On the curvature of piecewise flat spaces". *Commun. Math. Phys.* **92**, 405 (1984).

[110] G. Chirco. "Holographic Entanglement in Group Field Theory". *Universe* **5**, 211 (2019).

[111] M. Christodoulou, M. Långvik, A. Riello, C. Röken, and C. Rovelli. "Divergences and orientation in spinfoams". *Class. Quant. Grav.* **30**, 055009 (2013), arXiv: 1207.5156 [gr-qc].

[112] D. Chruściński. "Quantum mechanics of damped systems". *J. Math. Phys.* **44**, 3718–3733 (2003), arXiv: math-ph/0301024.

[113] D. Chruściński. "Quantum mechanics of damped systems. II. Damping and parabolic potential barrier". *J. Math. Phys.* **45**, 841–854 (2004), arXiv: math-ph/0307047.

[114] A. Codello, R. Percacci, and C. Rahmede. "Investigating the ultraviolet properties of gravity with a Wilsonian renormalization group equation". *Annals Phys.* **324**, 414–469 (2009), arXiv: 0805.2909 [hep-th].

[115] S. R. Coleman and F. De Luccia. "Gravitational effects on and of vacuum decay". *Phys. Rev. D* **21**, 3305 (1980).

[116] A. Corichi and P. Singh. "Geometric perspective on singularity resolution and uniqueness in loop quantum cosmology". *Phys. Rev. D* **80**, 044024 (2009), arXiv: 0905.4949 [gr-qc].

[117] A. Corichi, T. Vukašinac, and J. A. Zapata. "Polymer quantum mechanics and its continuum limit". *Phys. Rev. D* **76**, 044016 (2007), arXiv: 0704.0007 [gr-qc].

[118] L. Crane, L. H. Kauffman, and D. N. Yetter. "State-Sum Invariants of 4-Manifolds". *J. Knot Theory Ramif.* **06**, 177–234 (1997), arXiv: hep-th/9409167.

[119] L. Crane and D. N. Yetter. "A categorical construction of 4D topological quantum field theories", *Quantum Topology*, pp. 120–130. Ed. by L. H. Kauffman and R. A. Baadhio (World Scientific, Singapore, 1993), arXiv: hep-th/9301062.

[120] A. Dapor and K. Liegener. "Cosmological effective Hamiltonian from full loop quantum gravity dynamics". *Phys. Lett. B* **785**, 506–510 (2018), arXiv: 1706.09833 [gr-qc].

[121] F. David. "Planar diagrams, two-dimensional lattice gravity and surface models". *Nucl. Phys. B* **257**, 45 (1985).

[122] R. De Pietri and L. Freidel. "$so(4)$ Plebański action and relativistic spin-foam model". *Class. Quant. Grav.* **16**, 2187–2196 (1999), arXiv: gr-qc/9804071.

[123] R. De Pietri, L. Freidel, K. Krasnov, and C. Rovelli. "Barrett-Crane model from a Boulatov-Ooguri field theory over a homogeneous space". *Nucl. Phys. B* **574**, 785–806 (2000), arXiv: hep-th/9907154.

[124] R. De Pietri and C. Petronio. "Feynman diagrams of generalized matrix models and the associated manifolds in dimension four". *J. Math. Phys.* **41**, 6671–6688 (2000), arXiv: gr-qc/0004045.

[125] R. De Pietri and C. Rovelli. "Geometry eigenvalues and the scalar product from recoupling theory in loop quantum gravity". *Phys. Rev. D* **54**, 2664–2690 (1996), arXiv: gr-qc/9602023.

[126] J. De Vuyst, S. Eccles, P. A. Höhn, and J. Kirklin. "Gravitational entropy is observer-dependent" (2024), arXiv: 2405.00114 [hep-th].

[127] B. S. DeWitt. "The Quantization of Geometry", *Gravitation: An Introduction to Current Research*, pp. 266–381. Ed. by L. Witten (John Wiley & Sons, New York, 1963).

[128] B. S. DeWitt. "Quantum Theory of Gravity. I. The Canonical Theory". *Phys. Rev.* **160**, 1113–1148 (1967).

[129] P. Di Francesco, P. H. Ginsparg, and J. Zinn-Justin. "2D gravity and random matrices". *Phys. Rept.* **254**, 1–133 (1995), arXiv: hep-th/9306153.

[130] P. Diener, B. Gupt, and P. Singh. "Numerical simulations of a loop quantum cosmos: robustness of the quantum bounce and the validity of effective dynamics". *Class. Quant. Grav.* **31**, 105015 (2014), arXiv: 1402.6613 [gr-qc].

[131] Y. Ding, M. Han, and C. Rovelli. "Generalized spinfoams". *Phys. Rev. D* **83**, 124020 (2011), arXiv: 1011.2149 [gr-qc].

[132] Y. Ding and C. Rovelli. "Physical boundary Hilbert space and volume operator in the Lorentzian new spin-foam theory". *Class. Quant. Grav.* **27**, 205003 (2010), arXiv: 1006.1294 [gr-qc].

[133] P. A. M. Dirac. *Lectures on Quantum Mechanics* (Dover Publications, New York, 1964).

[134] P. A. M. Dirac. "The Theory of gravitation in Hamiltonian form". *Proc. Roy. Soc. Lond. A* **246**, 333–343 (1958).

[135] B. Dittrich. "Partial and complete observables for Hamiltonian constrained systems". *Gen. Rel. Grav.* **39**, 1891–1927 (2007), arXiv: gr-qc/0411013.

[136] B. Dittrich. "How to construct diffeomorphism symmetry on the lattice". *PoS* **QGQGS2011**, 012 (2011), arXiv: 1201.3840 [gr-qc].

[137] B. Dittrich, F. C. Eckert, and M. Martín-Benito. "Coarse graining methods for spin net and spin foam models". *New J. Phys.* **14**, 035008 (2012), arXiv: 1109.4927 [gr-qc].

[138] B. Dittrich, S. Gielen, and S. Schander. "Lorentzian quantum cosmology goes simplicial". *Class. Quant. Grav.* **39**, 035012 (2022), arXiv: 2109.00875 [gr-qc].

[139] B. Dittrich and A. Kogios. "From spin foams to area metric dynamics to gravitons". *Class. Quant. Grav.* **40**, 095011 (2023), arXiv: 2203.02409 [gr-qc].

[140] M. Domagala, K. Giesel, W. Kamiński, and J. Lewandowski. "Gravity quantized: Loop quantum gravity with a scalar field". *Phys. Rev. D* **82**, 104038 (2010), arXiv: 1009.2445 [gr-qc].

[141] M. Domagala and J. Lewandowski. "Black-hole entropy from quantum geometry". *Class. Quant. Grav.* **21**, 5233–5244 (2004), arXiv: gr-qc/0407051.

[142] P. Donà and P. Frisoni. "How-to Compute EPRL Spin Foam Amplitudes". *Universe* **8**, 208 (2022), arXiv: 2202.04360 [gr-qc].

[143] P. Donà, M. Han, and H. Liu. "Spinfoams and High-Performance Computing", *Handbook of Quantum Gravity*, pp. 1–38. Ed. by C. Bambi, L. Modesto, and I. Shapiro (Springer, Singapore, 2023), arXiv: 2212.14396 [gr-qc].

[144] P. Donà and G. Sarno. "Numerical methods for EPRL spin foam transition amplitudes and Lorentzian recoupling theory". *Gen. Rel. Grav.* **50**, 127 (2018), arXiv: 1807.03066 [gr-qc].

[145] J. F. Donoghue. "General relativity as an effective field theory: The leading quantum corrections". *Phys. Rev. D* **50**, 3874–3888 (1994), arXiv: gr-qc/9405057.

[146] J. F. Donoghue. "Leading quantum correction to the Newtonian potential". *Phys. Rev. Lett.* **72**, 2996–2999 (1994), arXiv: gr-qc/9310024.

[147] M. Dupuis and E. R. Livine. "Lifting SU(2) spin networks to projected spin networks". *Phys. Rev. D* **82**, 064044 (2010), arXiv: 1008.4093 [gr-qc].

[148] A. Eichhorn. "Status of the Asymptotic Safety Paradigm for Quantum Gravity and Matter". *Found. Phys.* **48**, 1407–1429 (2018), arXiv: 1709.03696 [gr-qc].

[149] A. Einstein and M. Grossmann. "Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation". *Mon. Math. Phys.* **26**, A38–A38 (1915).

[150] J. Ellis, V. Sanz, and T. You. "The effective Standard Model after LHC Run I". *JHEP* **03**, 157 (2015), arXiv: 1410.7703 [hep-ph].

[151] J. Engle, E. Livine, R. Pereira, and C. Rovelli. "LQG vertex with finite Immirzi parameter". *Nucl. Phys. B* **799**, 136–149 (2008), arXiv: 0711.0146 [gr-qc].

[152] J. Engle, R. Pereira, and C. Rovelli. "Flipped spinfoam vertex and loop gravity". *Nucl. Phys. B* **798**, 251–290 (2008), arXiv: 0708.1236 [gr-qc].

[153] J. Engle and S. Speziale. "Spin Foams: Foundations", *Handbook of Quantum Gravity*, pp. 1–40. Ed. by C. Bambi, L. Modesto, and I. Shapiro (Springer, Singapore, 2023), arXiv: 2310.20147 [gr-qc].

[154] R. P. Feynman. "Space-Time Approach to Non-Relativistic Quantum Mechanics". *Rev. Mod. Phys.* **20**, 367–387 (1948).

[155] R. P. Feynman and A. R. Hibbs. *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).

[156] M. Fierz. "Über die relativistische Theorie kräftefreier Teilchen mit beliebigem Spin". *Helv. Phys. Acta* **12**, 3–37 (1939).

[157] M. Fierz and W. Pauli. "On relativistic wave equations for particles of arbitrary spin in an electromagnetic field". *Proc. Roy. Soc. Lond. A* **173**, 211–232 (1939).

[158] R. A. Fisher, M. M. Nieto, and V. D. Sandberg. "Impossibility of naively generalizing squeezed coherent states". *Phys. Rev. D* **29**, 1107–1110 (1984).

[159] L. Freidel. "A Ponzano-Regge model of Lorentzian 3-dimensional gravity". *Nucl. Phys. B Proc. Suppl.* **88**, 237–240 (2000), arXiv: gr-qc/0102098.

[160] L. Freidel. "Group field theory: An Overview". *Int. J. Theor. Phys.* **44**, 1769–1783 (2005), arXiv: hep-th/0505016.

[161] L. Freidel and K. Krasnov. "Spin foam models and the classical action principle". *Adv. Theor. Math. Phys.* **2**, 1183–1247 (1999), arXiv: hep-th/9807092.

[162] L. Freidel and K. Krasnov. "A new spin foam model for 4D gravity". *Class. Quant. Grav.* **25**, 125018 (2008), arXiv: 0708.1595 [gr-qc].

[163] L. Freidel and E. R. Livine. "The fine structure of SU(2) intertwiners from U(N) representations". *J. Math. Phys.* **51**, 082502 (2010), arXiv: 0911.3553 [gr-qc].

[164] L. Freidel and E. R. Livine. "U(N) coherent states for loop quantum gravity". *J. Math. Phys.* **52**, 052502 (2011), arXiv: 1005.2090 [gr-qc].

[165] R. Gambini, J. Olmedo, and J. Pullin. "Spherically symmetric loop quantum gravity: analysis of improved dynamics". *Class. Quant. Grav.* **37**, 205012 (2020), arXiv: 2006.01513 [gr-qc].

[166] R. Gambini and J. Pullin. "Loop Quantization of the Schwarzschild Black Hole". *Phys. Rev. Lett.* **110**, 211301 (2013), arXiv: 1302.5265 [gr-qc].

[167] M. Gary and S. B. Giddings. "Relational observables in 2D quantum gravity". *Phys. Rev. D* **75**, 104007 (2007), arXiv: hep-th/0612191.

[168] M. Gasperini and G. Veneziano. "The pre-big bang scenario in string cosmology". *Phys. Rept.* **373**, 1–212 (2003), arXiv: hep-th/0207130.

[169] L. Gerber. "The orthocentric simplex as an extreme simplex". *Pacific J. Math.* **56**, 97–111 (1975).

[170] F. Gerhardt, D. Oriti, and E. Wilson-Ewing. "Separate universe framework in group field theory condensate cosmology". *Phys. Rev. D* **98**, 066011 (2018), arXiv: 1805.03099 [gr-qc].

[171] F. Giacomini, E. Castro-Ruiz, and Č. Brukner. "Quantum mechanics and the covariance of physical laws in quantum reference frames". *Nature Commun.* **10**, 494 (2019), arXiv: 1712.07207 [quant-ph].

[172] G. W. Gibbons and S. W. Hawking. "Action integrals and partition functions in quantum gravity". *Phys. Rev. D* **15**, 2752–2756 (1977).

[173] G. Gibbons, S. Hawking, and M. Perry. "Path integrals and the indefiniteness of the gravitational action". *Nucl. Phys. B* **138**, 141–150 (1978).

[174] S. B. Giddings, D. Marolf, and J. B. Hartle. "Observables in effective gravity". *Phys. Rev. D* **74**, 064018 (2006), arXiv: hep-th/0512200.

[175] S. B. Giddings and A. Strominger. "Axion-induced topology change in quantum gravity and string theory". *Nucl. Phys. B* **306**, 890–907 (1988).

[176] S. Gielen. "Emergence of a low spin phase in group field theory condensates". *Class. Quant. Grav.* **33**, 224002 (2016), arXiv: 1604.06023 [gr-qc].

[177] S. Gielen. "Group field theory and its cosmology in a matter reference frame". *Universe* **4**, 103 (2018), arXiv: 1808.10469 [gr-qc].

[178] S. Gielen. "Frozen formalism and canonical quantization in group field theory". *Phys. Rev. D* **104**, 106011 (2021), arXiv: 2105.01100 [hep-th].

[179] S. Gielen. "Quantum Cosmology" (2023), arXiv: 2309.01790 [gr-qc].

[180] S. Gielen, L. Marchetti, D. Oriti, and A. Polaczek. "Effective cosmology from one-body operators in group field theory". *Class. Quant. Grav.* **39**, 075002 (2022), arXiv: 2110.11176 [gr-qc].

[181] S. Gielen and D. Oriti. "Quantum cosmology from quantum gravity condensates: cosmological variables and lattice-refined dynamics". *New J. Phys.* **16**, 123004 (2014), arXiv: 1407.8167 [gr-qc].

[182] S. Gielen, D. Oriti, and L. Sindoni. "Cosmology from Group Field Theory Formalism for Quantum Gravity". *Phys. Rev. Lett.* **111**, 031301 (2013), arXiv: 1303.3576 [gr-qc].

[183] S. Gielen, D. Oriti, and L. Sindoni. "Homogeneous cosmologies as group field theory condensates". *JHEP* **06**, 013 (2014), arXiv: 1311.1238 [gr-qc].

[184] S. Gielen and A. Polaczek. "Generalised effective cosmology from group field theory". *Class. Quant. Grav.* **37**, 165004 (2020), arXiv: 1912.06143 [gr-qc].

[185] S. Gielen and A. Polaczek. "Hamiltonian group field theory with multiple scalar matter fields". *Phys. Rev. D* **103**, 086011 (2021), arXiv: 2009.00615 [gr-qc].

[186] S. Gielen, A. Polaczek, and E. Wilson-Ewing. "Addendum to "Relational Hamiltonian for group field theory"". *Phys. Rev. D* **100**, 106002 (2019), arXiv: 1908.09850 [gr-qc].

[187] S. Gielen and L. Sindoni. "Quantum Cosmology from Group Field Theory Condensates: a Review". *SIGMA* **12**, 082 (2016), arXiv: 1602.08104 [gr-qc].

[188] K. Giesel and T. Thiemann. "Algebraic quantum gravity (AQG): II. Semiclassical analysis". *Class. Quant. Grav.* **24**, 2499–2564 (2007), arXiv: gr-qc/0607100.

[189] K. Giesel and T. Thiemann. "Algebraic quantum gravity (AQG): IV. Reduced phase space quantization of loop quantum gravity". *Class. Quant. Grav.* **27**, 175009 (2010), arXiv: `0711.0119 [gr-qc]`.

[190] K. Giesel. "Introduction to Dirac observables". *Int. J. Mod. Phys. A* **23**, 1190–1199 (2008).

[191] K. Giesel and T. Thiemann. "Scalar material reference systems and loop quantum gravity". *Class. Quant. Grav.* **32**, 135015 (2015), arXiv: `1206.3807 [gr-qc]`.

[192] R. Gilmore. "Geometry of symmetrized states". *Ann. Phys.* **74**, 391–463 (1972).

[193] F. Girelli and M. Laudonio. "Group field theory on quantum groups" (2022), arXiv: `2205.13312 [hep-th]`.

[194] D. Giulini. "What is the geometry of superspace?" *Phys. Rev. D* **51**, 5630–5635 (1995), arXiv: `gr-qc/9311017`.

[195] D. Giulini. "Remarks on the Notions of General Covariance and Background Independence". *Lect. Notes Phys.* **721**, 105–120 (2007), arXiv: `gr-qc/0603087`.

[196] D. Giulini. "The Superspace of geometrodynamics". *Gen. Rel. Grav.* **41**, 785–815 (2009), arXiv: `0902.3923 [gr-qc]`.

[197] C. Goeller, P. A. Höhn, and J. Kirklin. "Diffeomorphism-invariant observables and dynamical frames in gravity: reconciling bulk locality with general covariance" (2022), arXiv: `2206.01193 [hep-th]`.

[198] G. Gorelik. "The drama of ideas in the history of quantum gravity: Niels Bohr, Lev Landau, and Matvei Bronstein". *Eur. Phys. J. H* **49**, 18 (2024).

[199] M. H. Goroff and A. Sagnotti. "Quantum gravity at two loops". *Phys. Lett. B* **160**, 81–86 (1985).

[200] F. Gozzini. "A high-performance code for EPRL spin foam amplitudes". *Class. Quant. Grav.* **38**, 225010 (2021), arXiv: `2107.13952 [gr-qc]`.

[201] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products* (Academic Press, Amsterdam, 2014).

[202] L. P. Grishchuk and Y. V. Sidorov. "Squeezed quantum states of relic gravitons and primordial density fluctuations". *Phys. Rev. D* **42**, 3413–3421 (1990).

[203] M. Gross. "Tensor models and simplicial quantum gravity in $>$ 2-D". *Nucl. Phys. B Proc. Suppl.* **25**, 144–149 (1992).

[204] R. Gurau. "Lost in translation: topological singularities in group field theory". *Class. Quant. Grav.* **27**, 235023 (2010), arXiv: 1006.0714 [hep-th].

[205] R. Gurau. "Topological Graph Polynomials in Colored Group Field Theory". *Ann. Henri Poincaré* **11**, 565–584 (2010), arXiv: 0911.1945 [hep-th].

[206] R. Gurau. "Colored Group Field Theory". *Commun. Math. Phys.* **304**, 69–93 (2011), arXiv: 0907.2582 [hep-th].

[207] R. Gurau. "The 1/N Expansion of Colored Tensor Models". *Ann. Henri Poincaré* **12**, 829–847 (2011), arXiv: 1011.2726 [gr-qc].

[208] R. Gurau. "The Complete 1/N Expansion of Colored Tensor Models in Arbitrary Dimension". *Ann. Henri Poincaré* **13**, 399–423 (2012), arXiv: 1102.5759 [gr-qc].

[209] R. Gurau. "Invitation to Random Tensors". *SIGMA* **12**, 094 (2016), arXiv: 1609.06439 [hep-th].

[210] R. Gurau and V. Rivasseau. "The 1/N expansion of colored tensor models in arbitrary dimension". *EPL* **95**, 50004 (2011), arXiv: 1101.4182 [gr-qc].

[211] R. Gurau and J. P. Ryan. "Colored Tensor Models - a Review". *SIGMA* **8**, 020 (2012), arXiv: 1109.4812 [hep-th].

[212] L. Hackl and E. Bianchi. "Bosonic and fermionic Gaussian states from Kähler structures". *SciPost Phys. Core* **4**, 025 (2021), arXiv: 2010.15518 [quant-ph].

[213] H. M. Haggard and C. Rovelli. "Quantum-gravity effects outside the horizon spark black to white hole tunneling". *Phys. Rev. D* **92**, 104020 (2015), arXiv: 1407.0989 [gr-qc].

[214] J. J. Halliwell and S. W. Hawking. "Origin of structure in the Universe". *Phys. Rev. D* **31**, 1777 (1985).

[215] J. J. Halliwell. "Introductory Lectures on Quantum Cosmology", *7th Jerusalem Winter School for Theoretical Physics: Quantum Cosmology and Baby Universes*, pp. 159–243. Ed. by S. Coleman, J. B. Hartle, T. Piran, and S. Weinberg (World Scientific, Singapore, 1991), arXiv: 0909.2566 [gr-qc].

[216] H. W. Hamber. *Quantum Gravitation: The Feynman Path Integral Approach* (Springer Berlin, Heidelberg, 2009).

[217] M. Han. "Einstein equation from covariant loop quantum gravity in semiclassical continuum limit". *Phys. Rev. D* **96**, 024047 (2017), arXiv: 1705.09030 [gr-qc].

[218] M. Han, Z. Huang, H. Liu, D. Qu, and Y. Wan. "Spinfoam on a Lefschetz thimble: Markov chain Monte Carlo computation of a Lorentzian spinfoam propagator". *Phys. Rev. D* **103**, 084026 (2021), arXiv: 2012.11515 [gr-qc].

[219] M. Han, H. Liu, and D. Qu. "Complex critical points in Lorentzian spinfoam quantum gravity: Four-simplex amplitude and effective dynamics on a double-Δ_3 complex". *Phys. Rev. D* **108**, 026010 (2023), arXiv: 2301.02930 [gr-qc].

[220] M. Han, C. Rovelli, and F. Soltani. "Geometry of the black-to-white hole transition within a single asymptotic region". *Phys. Rev. D* **107**, 064011 (2023), arXiv: 2302.03872 [gr-qc].

[221] J. B. Hartle and S. W. Hawking. "Wave Function of the Universe". *Phys. Rev. D* **28**, 2960–2975 (1983).

[222] B. Hatfield. *Quantum Field Theory of Point Particles and Strings* (Perseus Books, Cambridge, Massachusetts, 1992).

[223] S. W. Hawking. "The quantum state of the universe". *Nucl. Phys. B* **239**, 257 (1984).

[224] M. Henneaux and C. Teitelboim. *Quantization of Gauge Systems* (Princeton University Press, Princeton, 1992).

[225] P. A. Höhn. "Switching Internal Times and a New Perspective on the 'Wave Function of the Universe'". *Universe* **5**, 116 (2019), arXiv: 1811.00611 [gr-qc].

[226] P. A. Höhn, A. Russo, and A. R. H. Smith. "Matter relative to quantum hypersurfaces". *Phys. Rev. D* **109**, 105011 (2024), arXiv: 2308.12912 [quant-ph].

[227] P. A. Höhn, A. R. H. Smith, and M. P. E. Lock. "Equivalence of Approaches to Relational Quantum Dynamics in Relativistic Settings". *Front. in Phys.* **9**, 181 (2021), arXiv: 2007.00580 [gr-qc].

[228] P. A. Höhn, A. R. H. Smith, and M. P. E. Lock. "Trinity of relational quantum dynamics". *Phys. Rev. D* **104**, 066001 (2021), arXiv: 1912.00033 [quant-ph].

[229] P. A. Höhn and A. Vanrietvelde. "How to switch between relational quantum clocks". *New J. Phys.* **22**, 123048 (2020), arXiv: 1810.04153 [gr-qc].

[230] V. Husain, J. G. Kelly, R. Santacruz, and E. Wilson-Ewing. "Fate of quantum black holes". *Phys. Rev. D* **106**, 024014 (2022), arXiv: 2203.04238 [gr-qc].

[231] V. Husain and T. Pawłowski. "Time and a Physical Hamiltonian for Quantum Gravity". *Phys. Rev. Lett.* **108**, 141301 (2012), arXiv: 1108.1145 [gr-qc].

[232] G. Immirzi. "Quantum gravity and Regge calculus". *Nucl. Phys. B Proc. Suppl.* **57**, 65–72 (1997), arXiv: gr-qc/9701052.

[233] C. J. Isham. "Canonical Quantum Gravity and the Problem of Time", *Integrable Systems, Quantum Groups, and Quantum Field Theories.* pp. 157–287. Ed. by L. A. Ibort and M. A. Rodriguez (Springer, Dordrecht, 1993), arXiv: gr-qc/9210011.

[234] A. F. Jercher, D. Oriti, and A. G. A. Pithis. "Complete Barrett–Crane model and its causal structure". *Phys. Rev. D* **106**, 066019 (2022), arXiv: 2206.15442 [gr-qc].

[235] A. F. Jercher, D. Oriti, and A. G. A. Pithis. "Emergent cosmology from quantum gravity in the Lorentzian Barrett–Crane tensorial group field theory model". *JCAP* **01**, 050 (2022), arXiv: 2112.00091 [gr-qc].

[236] A. F. Jercher, J. D. Simão, and S. Steinhaus. "Partial absence of cosine problem in 3d Lorentzian spin foams" (2024), arXiv: 2404.16943 [gr-qc].

[237] W. Kamiński, M. Kisielowski, and J. Lewandowski. "Spin-foams for all loop quantum gravity". *Class. Quant. Grav.* **27**, 095006 (2010), arXiv: 0909.0939 [gr-qc]. [Corrigendum: *Class. Quant. Grav.* **29**, 049502 (2012)].

[238] W. Kamiński and S. Steinhaus. "The Barrett–Crane model: asymptotic measure factor". *Class. Quant. Grav.* **31**, 075014 (2014), arXiv: 1310.2957 [gr-qc].

[239] R. Kantowski and R. K. Sachs. "Some spatially homogeneous anisotropic relativistic cosmological models". *J. Math. Phys.* **7**, 443 (1966).

[240] J. G. Kelly, R. Santacruz, and E. Wilson-Ewing. "Effective loop quantum gravity framework for vacuum spherically symmetric spacetimes". *Phys. Rev. D* **102**, 106024 (2020), arXiv: 2006.09302 [gr-qc].

[241] F. C. Khanna, A. P. C. Malbouisson, J. M. C. Malbouisson, and A. E. Santana. *Thermal Quantum Field Theory: Algebraic Aspects and Applications* (World Scientific, Singapore, 2009).

[242] J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt, and N. Turok. "From big crunch to big bang". *Phys. Rev. D* **65**, 086007 (2002), arXiv: hep-th/0108187.

[243] J. Khoury, B. A. Ovrut, P. J. Steinhardt, and N. Turok. "Ekpyrotic universe: Colliding branes and the origin of the hot big bang". *Phys. Rev. D* **64**, 123522 (2001), arXiv: hep-th/0103239.

[244] C. Kiefer. *Quantum Gravity: Third Edition*, International Series of Monographs on Physics (Oxford University Press, Oxford, 2012).

[245] C. Kiefer. "Continuous measurement of mini-superspace variables by higher multipoles". *Class. Quant. Grav.* **4**, 1369 (1987).

[246] C. Kiefer and M. Krämer. "Can effects of quantum gravity be observed in the cosmic microwave background?" *Int. J. Mod. Phys. D* **21**, 1241001 (2012), arXiv: 1205.5161 [gr-qc].

[247] C. Kiefer and M. Krämer. "Quantum Gravitational Contributions to the Cosmic Microwave Background Anisotropy Spectrum". *Phys. Rev. Lett.* **108**, 021301 (2012), arXiv: 1103.4967 [gr-qc].

[248] C. Kiefer and B. Sandhoefer. "Quantum cosmology". *Z. Naturforsch. A* **77**, 543–559 (2022), arXiv: 0804.0672 [gr-qc].

[249] T. Konopka, F. Markopoulou, and L. Smolin. "Quantum Graphity" (2006), arXiv: hep-th/0611197.

[250] I. Kotecha and D. Oriti. "Statistical equilibrium in quantum gravity: Gibbs states in group field theory". *New J. Phys.* **20**, 073009 (2018), arXiv: 1801.09964 [gr-qc].

[251] T. Krajewski, J. Magnen, V. Rivasseau, A. Tanasa, and P. Vitale. "Quantum corrections in the group field theory formulation of the Engle-Pereira-Rovelli-Livine and Freidel-Krasnov models". *Phys. Rev. D* **82**, 124069 (2010), arXiv: 1007.3150 [gr-qc].

[252] K. Krasnov. "Gravity as BF theory plus potential". *Int. J. Mod. Phys. A* **24**, 2776–2782 (2009), arXiv: 0907.4064 [gr-qc].

[253] K. Kuchař. "A Bubble-Time Canonical Formalism for Geometrodynamics". *J. Math. Phys.* **13**, 768–781 (1972).

[254] K. Kuchař. "Canonical Quantization of Gravity", *Relativity, Astrophysics and Cosmology*, pp. 237–288. Ed. by W. Israel (Springer, Dordrecht, 1973).

[255] K. Kuchař. "Dirac constraint quantization of a parametrized field theory by anomaly-free operator representations of spacetime diffeomorphisms". *Phys. Rev. D* **39**, 2263–2280 (1989).

[256] K. Kuchař. "Parametrized scalar field on $\mathbb{R} \times S^1$: Dynamical pictures, spacetime diffeomorphisms, and conformal isometries". *Phys. Rev. D* **39**, 1579–1593 (1989).

[257] K. V. Kuchař. "Time and interpretations of quantum gravity". *Int. J. Mod. Phys. D* **20**, 3–86 (2011).

[258] L. D. Landau and E. M. Lifshits. *Quantum Mechanics: Non-Relativistic Theory*, Course of Theoretical Physics (Butterworth-Heinemann, Oxford, 1977).

[259] J.-L. Lehners. "Ekpyrotic and cyclic cosmology". *Phys. Rept.* **465**, 223–263 (2008), arXiv: `0806.1245 [astro-ph]`.

[260] Y. Li, D. Oriti, and M. Zhang. "Group field theory for quantum gravity minimally coupled to a scalar field". *Class. Quant. Grav.* **34**, 195001 (2017), arXiv: `1701.08719 [gr-qc]`.

[261] D. F. Litim. "Fixed Points of Quantum Gravity". *Phys. Rev. Lett.* **92**, 201301 (2004), arXiv: `hep-th/0312114`.

[262] E. R. Livine. "Deformations of polyhedra and polygons by the unitary group". *J. Math. Phys.* **54**, 123504 (2013), arXiv: `1307.2719 [math-ph]`.

[263] E. R. Livine. "3d Quantum Gravity: Coarse-Graining and q-Deformation". *Ann. Henri Poincaré* **18**, 1465–1491 (2017), arXiv: `1610.02716 [gr-qc]`.

[264] E. R. Livine. "Intertwiner entanglement on spin networks". *Phys. Rev. D* **97**, 026009 (2018), arXiv: `1709.08511 [gr-qc]`.

[265] E. R. Livine. "Spinfoam Models for Quantum Gravity: Overview" (2024), arXiv: `2403.09364 [gr-qc]`.

[266] E. R. Livine and M. Martín-Benito. "Group theoretical quantization of isotropic loop cosmology". *Phys. Rev. D* **85**, 124052 (2012), arXiv: `1204.0539 [gr-qc]`.

[267] E. R. Livine and S. Speziale. "Solving the simplicity constraints for spinfoam quantum gravity". *EPL* **81**, 50004 (2008), arXiv: `0708.1915 [gr-qc]`.

[268] X. Ma and W. Rhodes. "Squeezing in harmonic oscillators with time-dependent frequencies". *Phys. Rev. A* **39**, 1941–1947 (1989).

[269] X. Ma and W. Rhodes. "Multimode squeeze operators and squeezed states". *Phys. Rev. A* **41**, 4625–4631 (1990).

[270] S. Major and L. Smolin. "Quantum deformation of quantum gravity". *Nucl. Phys. B* **473**, 267–290 (1996), arXiv: gr-qc/9512020.

[271] G. Maniccia, M. De Angelis, and G. Montani. "WKB Approaches to Restore Time in Quantum Cosmology: Predictions and Shortcomings". *Universe* **8**, 556 (2022), arXiv: 2209.04403 [gr-qc].

[272] L. Marchetti and D. Oriti. "Effective relational cosmological dynamics from quantum gravity". *JHEP* **05**, 025 (2021), arXiv: 2008.02774 [gr-qc].

[273] L. Marchetti and D. Oriti. "Quantum Fluctuations in the Effective Relational GFT Cosmology". *Front. Astron. Space Sci.* **8**, 683649 (2021), arXiv: 2010.09700 [gr-qc].

[274] L. Marchetti and D. Oriti. "Effective dynamics of scalar cosmological perturbations from quantum gravity". *JCAP* **07**, 004 (2022), arXiv: 2112.12677 [gr-qc].

[275] L. Marchetti, D. Oriti, A. G. A. Pithis, and J. Thürigen. "Mean-Field Phase Transitions in Tensorial Group Field Theory Quantum Gravity". *Phys. Rev. Lett.* **130**, 141501 (2023), arXiv: 2211.12768 [gr-qc].

[276] L. Marchetti, D. Oriti, A. G. A. Pithis, and J. Thürigen. "Phase transitions in TGFT: a Landau–Ginzburg analysis of Lorentzian quantum geometric models". *JHEP* **02**, 074 (2023), arXiv: 2209.04297 [gr-qc].

[277] P. Marian and T. A. Marian. "Squeezed states with thermal noise. I. Photon-number statistics". *Phys. Rev. A* **47**, 4474–4486 (1993).

[278] D. Marolf. "Almost ideal clocks in quantum cosmology: a brief derivation of time". *Class. Quant. Grav.* **12**, 2469–2486 (1995), arXiv: gr-qc/9412016.

[279] D. Marolf. "Refined algebraic quantization: Systems with a single constraint" (1995), arXiv: gr-qc/9508015.

[280] D. Marolf. "Group averaging and refined algebraic quantization: Where are we now?", *The Ninth Marcel Grossmann Meeting: On Recent Developments in Theoretical and Experimental General Relativity, Gravitation and Relativistic Field Theories*, pp. 1348–1349. Ed. by V. G. Gurzadian, R. T. Jantzen, and R. Ruffini (World Scientific, Singapore, 2002), arXiv: gr-qc/0011112.

[281] P. Martin-Dussaud and C. Rovelli. "Evaporating black-to-white hole". *Class. Quant. Grav.* **36**, 245002 (2019), arXiv: 1905.07251 [gr-qc].

[282] H.-J. Matschull. "Dirac's Canonical Quantization Programme" (1996), arXiv: quant-ph/9606031.

[283] P. O. Mazur and E. Mottola. "The path integral measure, conformal factor problem and stability of the ground state of quantum gravity". *Nucl. Phys. B* **341**, 187–212 (1990).

[284] K. A. Meissner. "Black-hole entropy in loop quantum gravity". *Class. Quant. Grav.* **21**, 5245–5252 (2004), arXiv: gr-qc/0407052.

[285] C. W. Misner. "Feynman quantization of general relativity". *Rev. Mod. Phys.* **29**, 497–509 (1957).

[286] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation* (W. H. Freeman, San Francisco, 1973).

[287] L. Modesto. "Disappearance of the black hole singularity in loop quantum gravity". *Phys. Rev. D* **70**, 124009 (2004), arXiv: gr-qc/0407097.

[288] Á. Mozota Frauca. "Foundational Issues in Group Field Theory". *Found. Phys.* **54**, 33 (2024), arXiv: 2406.08501 [gr-qc].

[289] V. P. Nair. *Quantum Field Theory: A Modern Perspective* (Springer, New York, 2005).

[290] J. von Neumann. "On infinite direct products". *Compos. Math.* **6**, 1–77 (1939).

[291] M. Niedermaier and M. Reuter. "The Asymptotic Safety Scenario in Quantum Gravity". *Living Rev. Rel.* **9**, 5–173 (2006).

[292] S. Olivares. "Quantum optics in the phase space. A tutorial on Gaussian states". *Eur. Phys. J. Special Topics* **203**, 3–24 (2012).

[293] H. Ooguri. "Topological lattice models in four dimensions". *Mod. Phys. Lett. A* **7**, 2799–2810 (1992), arXiv: hep-th/9205090.

[294] D. Oriti. "Group field theory as the microscopic description of the quantum spacetime fluid: a new perspective on the continuum in quantum gravity". *PoS* **QG-PH**, 030 (2007), arXiv: 0710.3276 [gr-qc].

[295] D. Oriti. "The group field theory approach to Quantum Gravity", *Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter*, pp. 310–331. Ed. by D. Oriti (Cambridge University Press, Cambridge, 2009), arXiv: gr-qc/0607032.

[296] D. Oriti. "The microscopic dynamics of quantum space as a group field theory", *Foundations of Space and Time: Reflections on Quantum Gravity*, pp. 257–320. Ed. by G. Ellis, J. Murugan, and A. Weltman (Cambridge University Press, Cambridge, 2011), arXiv: 1110.5606 [hep-th].

[297] D. Oriti. "Disappearance and emergence of space and time in quantum gravity". *Stud. Hist. Phil. Sci. B* **46**, 186–199 (2014), arXiv: 1302.2849 [physics.hist-ph].

[298] D. Oriti. "Group field theory as the second quantization of loop quantum gravity". *Class. Quant. Grav.* **33**, 085005 (2016), arXiv: 1310.7786 [gr-qc].

[299] D. Oriti. "Group Field Theory and Loop Quantum Gravity", *Loop Quantum Gravity: The First 30 Years*, pp. 125–151. Ed. by A. Ashtekar and J. Pullin (World Scientific, Singapore, 2017), arXiv: 1408.7112 [gr-qc].

[300] D. Oriti. "The universe as a quantum gravity condensate". *C. R. Phys.* **18**, 235–245 (2017), arXiv: 1612.09521 [gr-qc].

[301] D. Oriti. "Hydrodynamics on (Mini)superspace or a Non-linear Extension of Quantum Cosmology: An Effective Timeless Framework for Cosmology from Quantum Gravity". *Fundam. Theor. Phys.* **216**, 221–252 (2024), arXiv: 2403.10741 [gr-qc].

[302] D. Oriti and X. Pang. "Phantom-like dark energy from quantum gravity". *JCAP* **12**, 040 (2021), arXiv: 2105.03751 [gr-qc].

[303] D. Oriti, D. Pranzetti, and L. Sindoni. "Black holes as quantum gravity condensates". *Phys. Rev. D* **97**, 066017 (2018), arXiv: 1801.01479 [gr-qc].

[304] D. Oriti, J. P. Ryan, and J. Thürigen. "Group field theories for all loop quantum gravity". *New J. Phys.* **17**, 023042 (2015), arXiv: 1409.3150 [gr-qc].

[305] D. Oriti, L. Sindoni, and E. Wilson-Ewing. "Emergent Friedmann dynamics with a quantum bounce from quantum gravity condensates". *Class. Quant. Grav.* **33**, 224001 (2016), arXiv: 1602.05881 [gr-qc].

[306] D. Oriti, L. Sindoni, and E. Wilson-Ewing. "Bouncing cosmologies from quantum gravity condensates". *Class. Quant. Grav.* **34**, 04LT01 (2017), arXiv: 1602.08271 [gr-qc].

[307] D. Oriti and Y.-L. Wang. "Spherically-symmetric geometries in a matter reference frame as quantum gravity condensate microstates". *Class. Quant. Grav.* **40**, 215011 (2023), arXiv: 2305.18902 [gr-qc].

[308] D. Oriti and Y.-L. Wang. "Effective anisotropic dynamics in group field theory cosmology". *Class. Quant. Grav.* **41**, 195006 (2024), arXiv: 2311.14377 [gr-qc].

[309] D. N. Page and W. K. Wootters. "Evolution without evolution: Dynamics described by stationary observables". *Phys. Rev. D* **27**, 2885–2892 (1983).

[310] A. M. Perelomov. "Coherent states for arbitrary Lie group". *Commun. Math. Phys.* **26**, 222–236 (1972), arXiv: math-ph/0203002 [math-ph].

[311] A. Perez. "Finiteness of a spinfoam model for Euclidean quantum general relativity". *Nucl. Phys. B* **599**, 427–434 (2001), arXiv: gr-qc/0011058.

[312] A. Perez. "Spin foam models for quantum gravity". *Class. Quant. Grav.* **20**, R43 (2003), arXiv: gr-qc/0301113.

[313] A. Perez. "The Spin-Foam Approach to Quantum Gravity". *Living Rev. Rel.* **16**, 3 (2013), arXiv: 1205.2019 [gr-qc].

[314] A. Perez. "Black holes in loop quantum gravity". *Rept. Prog. Phys.* **80**, 126901 (2017), arXiv: 1703.09149 [gr-qc].

[315] A. Perez and C. Rovelli. "(3+1)-dimensional spin foam model of quantum gravity with spacelike and timelike components". *Phys. Rev. D* **64**, 064002 (2001), arXiv: gr-qc/0011037.

[316] A. Perez and C. Rovelli. "A spin foam model without bubble divergences". *Nucl. Phys. B* **599**, 255–282 (2001), arXiv: gr-qc/0006107.

[317] A. Perez and C. Rovelli. "Spin foam model for Lorentzian general relativity". *Phys. Rev. D* **63**, 041501 (2001), arXiv: gr-qc/0009021.

[318] M. E. Peskin and D. V. Schroeder. *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, Massachusetts, 1995).

[319] F. Peter and H. Weyl. "Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe". *Mathematische Annalen* **97**, 737–755 (1927).

[320] L. Pitaevskii and S. Stringari. *Bose-Einstein Condensation and Superfluidity* (Oxford University Press, Oxford, 2016).

[321] A. G. A. Pithis and M. Sakellariadou. "Relational evolution of effectively interacting group field theory quantum gravity condensates". *Phys. Rev. D* **95**, 064004 (2017), arXiv: `1612.02456 [gr-qc]`.

[322] M. Planck. "Über irreversible Strahlungsvorgänge". *Ann. Phys.* **306**, 69–122 (1900).

[323] J. F. Plebański. "On the separation of Einsteinian substructures". *J. Math. Phys.* **18**, 2511–2520 (1977).

[324] G. Ponzano and T. Regge. "Semiclassical limit of Racah coefficients", *Spectroscopic and group theoretical methods in physics*, pp. 1–58. Ed. by F. Bloch (North-Holland, Amsterdam, 1968).

[325] X.-L. Qi. "Does gravity come from quantum information?" *Nat. Phys.* **14**, 984–987 (2018).

[326] T. Regge. "General relativity without coordinates". *Nuovo Cim.* **19**, 558–571 (1961).

[327] M. P. Reisenberger. "A lattice worldsheet sum for 4-d Euclidean general relativity" (1997), arXiv: `gr-qc/9711052`.

[328] M. P. Reisenberger. "On relativistic spin network vertices". *J. Math. Phys.* **40**, 2046–2054 (1999), arXiv: `gr-qc/9809067`.

[329] M. P. Reisenberger and C. Rovelli. "'Sum over surfaces' form of loop quantum gravity". *Phys. Rev. D* **56**, 3490–3508 (1997), arXiv: `gr-qc/9612035`.

[330] M. P. Reisenberger and C. Rovelli. "Spacetime as a Feynman diagram: the connection formulation". *Class. Quant. Grav.* **18**, 121–140 (2001), arXiv: `gr-qc/0002095`.

[331] J. Roberts. "Classical 6j-symbols and the tetrahedron". *Geom. Topol.* **3**, 21–66 (1999), arXiv: `math-ph/9812013`.

[332] H. P. Robertson. "The Uncertainty Principle". *Phys. Rev.* **34**, 163–164 (1929).

[333] M. Roček and R. M. Williams. "Quantum Regge calculus". *Phys. Lett. B* **104**, 31 (1981).

[334] L. Rosenfeld. "Zur Quantelung der Wellenfelder". *Ann. Phys.* **397**, 113–152 (1930).

[335] L. Rosenfeld. "Über die Gravitationswirkungen des Lichtes". *Z. Phys.* **65**, 589–599 (1930).

[336] C. Rovelli. "Quantum mechanics without time: A model". *Phys. Rev. D* **42**, 2638–2646 (1990).

[337] C. Rovelli. "Time in quantum gravity: An hypothesis". *Phys. Rev. D* **43**, 442–456 (1991).

[338] C. Rovelli. "What is observable in classical and quantum gravity?" *Class. Quant. Grav.* **8**, 297–316 (1991).

[339] C. Rovelli. "Partial observables". *Phys. Rev. D* **65**, 124013 (2002), arXiv: `gr-qc/0110035`.

[340] C. Rovelli. *Quantum Gravity*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 2004).

[341] C. Rovelli and L. Smolin. "Loop space representation of quantum general relativity". *Nucl. Phys. B* **331**, 80–152 (1990).

[342] C. Rovelli and L. Smolin. "Discreteness of area and volume in quantum gravity". *Nucl. Phys. B* **442**, 593–622 (1995), arXiv: `gr-qc/9411005`. [Erratum: *Nucl. Phys. B* **456**, 753–754 (1995)].

[343] C. Rovelli and L. Smolin. "Spin networks and quantum gravity". *Phys. Rev. D* **52**, 5743–5759 (1995), arXiv: `gr-qc/9505006`.

[344] C. Rovelli and F. Vidotto. *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory* (Cambridge University Press, Cambridge, 2014).

[345] C. Rovelli and E. Wilson-Ewing. "Why are the effective equations of loop quantum cosmology so accurate?" *Phys. Rev. D* **90**, 023538 (2014), arXiv: `1310.8654 [gr-qc]`.

[346] W. Rühl. *The Lorentz group and harmonic analysis*, Mathematical Physics Monograph Series (W. A. Benjamin, New York, 1970).

[347] H. Sahlmann, T. Thiemann, and O. Winkler. "Coherent states for canonical quantum general relativity and the infinite tensor product extension". *Nucl. Phys. B* **606**, 401–440 (2001), arXiv: `gr-qc/0102038`.

[348] E. Schrödinger. "Zum Heisenbergschen Unschärfeprinzip". *Sitzungsber. Preuss. Akad. Wiss.* **19**, 296–303 (1930).

[349] K. Schulten and R. G. Gordon. "Semiclassical approximations to 3j- and 6j-coefficients for quantum-mechanical coupling of angular momenta". *J. Math. Phys.* **16**, 1971–1988 (1975).

[350] B. L. Schumaker. "Quantum mechanical pure states with gaussian wave functions". *Phys. Rep.* **135**, 317–408 (1986).

[351] A. Serafini. *Quantum Continuous Variables: A Primer of Theoretical Methods* (CRC Press, Boca Raton, Florida, 2017).

[352] A. R. H. Smith and M. Ahmadi. "Quantizing time: Interacting clocks and systems". *Quantum* **3**, 160 (2019), arXiv: `1712.00081 [quant-ph]`.

[353] L. Smolin. "Quantum gravity with a positive cosmological constant" (2002), arXiv: `hep-th/0209079`.

[354] L. Smolin. "The Case for Background Independence", *The Structural Foundations of Quantum Gravity*, pp. 196–239. Ed. by D. Rickles, S. French, and J. Saatsi (Oxford University Press, Oxford, 2006), arXiv: `hep-th/0507235`.

[355] J. Stachel. "The Hole Argument and Some Physical and Philosophical Implications". *Living Rev. Relativ.* **17**, 1 (2014).

[356] A. Strominger. "Vacuum Topology and Incoherence in Quantum Gravity". *Phys. Rev. Lett.* **52**, 1733 (1984).

[357] Y. Takahashi and H. Umezawa. "Thermo field dynamics". *Int. J. Mod. Phys. B* **10**, 1755–1805 (1996). Reprinted from *Collective Phenomena* **2**, 55–80 (1975), Gordon and Breach Scientific Publishers Ltd.

[358] J. Tambornino. "Relational Observables in Gravity: a Review". *SIGMA* **8**, 017 (2012), arXiv: `1109.0740 [gr-qc]`.

[359] V. Taveras. "Corrections to the Friedmann equations from loop quantum gravity for a universe with a free scalar field". *Phys. Rev. D* **78**, 064072 (2008), arXiv: `0807.3325 [gr-qc]`.

[360] T. Thiemann. "Anomaly-free formulation of non-perturbative, four-dimensional Lorentzian quantum gravity". *Phys. Lett. B* **380**, 257–264 (1996), arXiv: `gr-qc/9606088`.

[361] T. Thiemann. "Quantum spin dynamics (QSD)". *Class. Quant. Grav.* **15**, 839–873 (1998), arXiv: `gr-qc/9606089`.

[362] T. Thiemann and O. Winkler. "Gauge field theory coherent states (GCS): II. Peakedness properties". *Class. Quant. Grav.* **18**, 2561–2636 (2001), arXiv: `hep-th/0005237`.

[363] T. Thiemann and O. Winkler. "Gauge field theory coherent states (GCS): IV. Infinite tensor product and thermodynamical limit". *Class. Quant. Grav.* **18**, 4997–5054 (2001), arXiv: hep-th/0005235.

[364] T. Thiemann. "Gauge field theory coherent states (GCS): I. General properties". *Class. Quant. Grav.* **18**, 2025–2064 (2001), arXiv: hep-th/0005233.

[365] T. Thiemann. "Quantum spin dynamics: VIII. The master constraint". *Class. Quant. Grav.* **23**, 2249–2266 (2006), arXiv: gr-qc/0510011.

[366] T. Thiemann. "Reduced phase space quantization and Dirac observables". *Class. Quant. Grav.* **23**, 1163–1180 (2006), arXiv: gr-qc/0411031.

[367] T. Thiemann. *Modern Canonical Quantum General Relativity*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, 2007).

[368] V. G. Turaev and O. Y. Viro. "State sum invariants of 3-manifolds and quantum 6j-symbols". *Topology* **31**, 865–902 (1992).

[369] A. Vanrietvelde, P. A. Höhn, F. Giacomini, and E. Castro-Ruiz. "A change of perspective: switching quantum reference frames via a perspective-neutral framework". *Quantum* **4**, 225 (2020), arXiv: 1809.00556 [quant-ph].

[370] M. Varadarajan. "Dirac quantization of parametrized field theory". *Phys. Rev. D* **75**, 044018 (2007), arXiv: gr-qc/0607068.

[371] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii. *Quantum Theory of Angular Momentum: Irreducible Tensors, Spherical Harmonics, Vector Coupling Coefficients, 3nj Symbols* (World Scientific, Singapore, 1988).

[372] A. E. M. van de Ven. "Two loop quantum gravity". *Nucl. Phys. B* **378**, 309–366 (1992).

[373] D. F. Walls. "Squeezed states of light". *Nature* **306**, 141–146 (1983).

[374] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd. "Gaussian quantum information". *Rev. Mod. Phys.* **84**, 621–669 (2012).

[375] S. Weinberg. "Critical Phenomena for Field Theorists", *Understanding the Fundamental Constitutents of Matter*, pp. 1–52. Ed. by A. Zichichi (Springer, Boston, Massachusetts, 1978).

[376] S. Weinberg. "Ultraviolet divergences in quantum theories of gravitation", *General Relativity: An Einstein Centenary Survey*, pp. 790–831. Ed. by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1980).

[377] E. Wilson-Ewing. "The matter bounce scenario in loop quantum cosmology". *JCAP* **03**, 026 (2013), arXiv: `1211.6269 [gr-qc]`.

[378] E. Wilson-Ewing. "The loop quantum cosmology bounce as a Kasner transition". *Class. Quant. Grav.* **35**, 065005 (2018), arXiv: `1711.10943 [gr-qc]`.

[379] E. Wilson-Ewing. "Relational Hamiltonian for group field theory". *Phys. Rev. D* **99**, 086017 (2019), arXiv: `1810.01259 [gr-qc]`.

[380] D. L. Wiltshire. "An Introduction to Quantum Cosmology", *8th Physics Summer School on Cosmology: The Physics of the Universe*, pp. 473–531. Ed. by B. Robson, N. Visvanathan, and W. S. Woolcock (World Scientific, Singapore, 1996), arXiv: `gr-qc/0101003`.

[381] W. K. Wootters. ""Time" replaced by quantum correlations". *Int. J. Theor. Phys.* **23**, 701–711 (1984).

[382] J. W. York. "Role of Conformal Three-Geometry in the Dynamics of Gravitation". *Phys. Rev. Lett.* **28**, 1082–1085 (1972).